

**INTERNATIONAL STUDY GROUP
ON THE RELATIONS BETWEEN**

HISTORY AND PEDAGOGY OF MATHEMATICS

**Proceedings of the
2024 ICME Satellite Meeting**



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PRESENTATION

The eleventh Congress of the International Group on the Relations between History and Pedagogy of Mathematics (HPM) took place at the University of New South Wales, Sydney, from Monday, July 1 to Friday, July 5, 2024. This Group was created as a Study Group at the 2^d International Congress on Mathematical Education (ICME) in 1972, and was accepted as an affiliated Group by the International Commission on Mathematical Instruction (ICMI) in 1976. Information about history and activities of HPM can be found on the HPM website¹, in particular in a paper about its first 25 years of the Group written by Florence Fasanelli and John Fauvel in 2004. Fasanelli & Fauvel, 2004). During the HPM 2024 meeting, a panel on “The HPM group: past, present and future” was held to commemorate its 50th anniversary.

HPM Congresses are held every four years as satellites of ICME conferences. The first one was held in 1984 in Adelaide (Australia). Since then, HPM congresses have been held in different places all over the world: Firenze (Italy), 1988; Toronto (Canada), 1992; Braga (Portugal), 1996; Taipei (Taiwan), 2000; Uppsala (Sweden), 2004; Mexico DF (Mexico), 2008; Daejeon (Korea), 2012; Montpellier (France), 2016. In 2020, HPM10 planned for Macao could not be held in (China) because of the COVID-19 pandemic

Quadrennial meetings are a major activity of HPM Group to bring together individuals and teams with a keen interest in the relationship between the history of mathematics and mathematics education. They include:

- Researchers in mathematics education who are interested in the history of mathematics and mathematical thinking;
- Mathematics teachers at all levels who are eager to gain insights into how the history of mathematics can be integrated into teaching and how they can help students to learn mathematics;
- Historians of mathematics who wish to talk about their research;

¹ <https://hpm.sites.uu.nl/>

- Mathematicians who want to learn about new possibilities to teach their discipline; and
- All those with an interest in the history of mathematics and pedagogy.

The theme of HPM 2024 was “Mathematics of Australia and the Indo-Pacific.” While this theme serves as a main focus for the meeting, the program and activities are structured around the following seven general topics:

1. Theoretical and/or conceptual frameworks for integrating history in mathematics education.
2. History and epistemology in students’ and teachers’ mathematics education: Classroom experiments and teaching materials.
3. Original sources in the classroom and their educational effects.
4. Mathematics and its relation to science, technology, and the arts: Historical issues and interdisciplinary teaching and learning.
5. Cultures and history of mathematics fruitfully interwoven.
6. Topics in the history of mathematics education.
7. History of mathematics in Australia and the Indo-Pacific.

Each part of the Proceedings concerns one of these topics. Producing the Proceedings of the HPMs has always been a major task and a great responsibility since they have become standard references in its domain. The Proceedings of HPM form a collection that constitutes a source used by teachers and researchers. The papers of all HPM proceedings are available on the HPM website.

The present volume collects papers and abstracts stemming from all types of activities that were accepted and included in the scientific program of ESU-9. For each main theme, one plenary lecture was delivered. There are papers based on oral presentations and workshops that focused on specific historical, epistemological, or pedagogical topics. Two papers are also devoted to the two panels of the program.

Acknowledgments

We begin by thanking all the participants who attended this meeting and all those who contributed to the Proceedings. We also extend our warmest thanks to the members of the Scientific Committee who reviewed the papers and helped improve them.

We thank all members of the Local Organizing Committee: Jim Pettigrew (chair), Merryn Horrocks and Donald Shearman and the University of the University of New South Wales for hosting and supporting HPM 2024.

The Editors

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AN ETHICAL PERSPECTIVE ON THE HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION

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ABSTRACT

In this paper, I introduce a perspective in which ethics is at the core of the reflection on the history of mathematics in mathematics education. Considering and discussing various empirical and theoretical studies in the HPM literature, I present a theoretical positioning inspired by the writings of Bakhtin and Levinas, along with our recent works with pre- and in-service mathematics teachers grounded in this approach. Focusing primly on teachers' education, this ethical perspective invites to see the engagement with the history of mathematics as an opportunity to experience *otherness*, in which sensitivity to epistemology, to fundaments and purposes of mathematics education and to social and political issues in mathematics education develops. In this exploration, I discuss especially the themes of responsibility, engagement and dialogicality. Moreover, I highlight how pedagogical goals, ways of engaging with the history, teachers' and learners' roles, and challenges appear from this light, as well as some possible paths for research in terms of research problems and methodology.

1 Introduction

The first HPM meeting as a Satellite meeting of ICME took place at the University of Adelaide, 40 years ago in 1984. At that moment, Ubiratan D'Ambrosio (from Brazil) and Christian Houzel (from France) were elected co-chairs for the next four years. ICME-5 was also held at Adelaide five days later. D'Ambrosio gave there a plenary talk that was entitled *Socio-Cultural Bases for Mathematical Education* (1986). We can easily find the proceedings on the ICMI website.

In his talk, D'Ambrosio addressed very large points of discussion, stating that "mathematics education was going through one of the most critical periods in its long history", that this current era is "witnessing both the emergence of what might be called the electronic era and profound changes in the social political and economic texture of the world" (*id.*, p. 1). He also emphasised the necessity to examine "the role of mathematics education in bringing a new human dimension into relations between individuals, societies and cultures"

(*ibid.*). This speech resonates very much today at it seems very from actuality. But what is effectively the actuality of those questions? What is the new in the novelty of today regarding those questions, regarding today's research development and today's ways of being in research?

Let's remember that one of the first objective of the HPM group is to "further a deeper understanding of the way mathematics evolves, and the forces which contribute to this evolution", "relate the teaching of mathematics and the history of mathematics teaching to the development of mathematics in ways which assist the improvement of instruction and the development of curricula", "produce materials which can be used by teachers of mathematics to provide perspectives and to further the critical discussion of the teaching of mathematics" (see HPM group 2024). We can see clear links between D'Ambrosio's questions, and his ethnomathematics program, and the HPM group one. But what are exactly those links?

In the paper, I will emphasise how *ethics* can be a link between the two and can be a way to understand the role and the potential of the history of mathematics in mathematics education, encompassing the historical, social, cultural and political dimensions of mathematics and mathematics education.

2 The ethnomathematics program and the HPM perspective

Ethnomathematics is a research program about the history and philosophy of mathematics, with obvious implications for teaching (D'Ambrosio 2006). The main objective of the program is to develop an epistemology and an anchored education program. It is characterised by the fact that it emphasises the constitutive role of mathematics in society. This role arises from a performative view of mathematics – that is, a view of mathematics as a form of social action, rather than a picture of reality. By doing mathematics, humans create their world. While mathematics as social action can be seen at an individual level in everyday uses of mathematics, mathematics comes to shape social reality in much broader ways. The ethnomathematics research program gives importance to engaging with different forms of mathematics, coming from different cultures and eras, emphasizing a critical posture (political stance) and creativity (esthetical engagement and reflexivity). Education for peace embrace these aspects in larger way.

At ESU 9 in Salerno in the summer of 2022, Pedro Palhares gave a wonderful plenary talk on the legacy of Ubiratan D'Ambrosio and reminded few of the links between ethnomathematics program and the HPM perspective and objectives. He emphasises that school mathematical knowledge informs us about societal problems for which it is intended to be a response, it tells us about the ways in which it is culturally and ideologically expressed (2022).

Indeed, for D'Ambrosio (2006), mathematics has always been shaped by the economic, social and symbolic structures within which it was practiced and developed. He suggests that the original source of abstraction (mathematical and otherwise) is to be found in the material production of life and cultural ways to engage with others. The argument in question reappears in many contemporary works on ethnomathematics. It is within the economic, social and symbolic structures of the society that the needs to be satisfied in educational terms are defined. And it is within these same structures that the "ways" of satisfying these needs are also defined. These "ways" are in fact "ways of doing things": they appear as "knowledge".

Considering this perspective and the HPM one, we could say that, within the historical and cultural dimension of mathematics, could appear a repositioning of *mathematics education as a social and political project*. Indeed, mathematics education has always been a societal project, influenced by political concerns. The problem is that we tend to forget the social nature of mathematics education, due to perceptions of mathematics as ahistorical and universal. The political here, of course, has to be understood as *governmentality*, that is to say, the force that can determine the behavior or the conduct of the others. This led me to formulate the hypotheses that ethics, through the question of *otherness*, could be a link between the historical, cultural, social and political, a link between ethnomathematics and the HPM perspective. Let's think about the very nature of this link.

3 Ethics and ethical perspectives in mathematics education

Ethics could be understood as a philosophy of morality, as its etymology told us, and as it was, in fact, understood and developed in many philosophical traditions. We can, of course, think about the original Greek sense of *ta ēthē*, the particular ways in which human beings act in the world and conduct themselves toward one another and toward themselves (Fried 2021a).

This said, ethics has evolved more largely, especially within the phenomenological tradition in the 20th century, as a fundamental field of reflection around the thematic of Otherness. The main thematic of this investigation is the relation between Self and Other, interiority and exteriority, the social and affective experience of the Other, something that has to do much more with Being and Being-with-others. Many elements of this thematic of Otherness have been developed in the phenomenological tradition in which I globally grounded my research. For instance, our capacity to acquire a narrative identity as a condition of the responsibility of an ethical life (*cf.* Paul Ricoeur), the ethics of the neighbor and of the distant opened to social and environmental questions (*cf.* Hans Jonas) or the search for a first philosophy in the ethics (*cf.* Emmanuel Levinas).

Experience of Otherness, in terms of mathematics and education, is particularly significant when engaging with the history of mathematics. In the context of teacher education, for instance, it seems to me that what is called *dépaysement* [disorientation] (Barbin 1997, Guillemette 2017) when engaging with the past, the search of metadiscursive rules (Kjeldsen 2012), the Self-knowledge (Fried 2007), the learning to listen (Arcavi and Isoda 2007) through history... All seems to have something to do with a special experience of Otherness. The idea here is that engaging with the history of mathematics in educational context could have ethical implications and that should at least be addressed, explored and maybe developed.

Ethical questions seem to occupy little place in the field of mathematics education. But there are few studies that have explored mathematics education in relation to ethics. De Freitas (2008) speaks of *ethics of the use of mathematics* in which students and teachers must reflect on the ethical consequences of their actions. There is an attempt to assist, through the use of these powerful tools, those who really need it, so that the time spent in mathematics class around ethical reflections will serve the intentions of social justice inherent in critical pedagogy. Neyland (2010) sees mathematics as a case scenario in his call for an *ethical critique* of our education systems. To make it fast, he accuses technocratic ideology of sabotaging the ethical dimension of the relationship between teacher and students in favor of an evaluation system that focuses on results and which deprives the learners of the pleasure of doing mathematics. Ernest's *ethics of conversation* (2002) in mathematics classroom seeks to promote the presence of multiple voices and where students are

called to learn to *live together* mathematically, immersed in the heart of a practice where mathematics is done by explaining, arguing, validating the emergent understandings and meanings. It is mathematical argumentation which is here the object of investigation, and the place where a relational ethic of mutual respect develops and is exercised.

Without disagreeing with these perspectives, I believe that within the HPM perspective, ethics must be considered and explored differently due to history itself and the voices from the past with which we seek to engage in dialogue. There exists, in a sense, a unique third party in our context. In this regard, my research with pre- and in-service teachers has prompt-ed me to reflect on ethical questions in a new light within the HPM perspective. A different way that comes from two influences: (1) the historico-cultural perspective in mathematics education in which I ground my research and (2) the phenomenological and dialogical description of the lived experience of the learners confronted with historical texts that I obtained in the different studies that I conducted. A different way led to think about the history of mathematics, in educational context, as a *topos* where special *experiences of otherness* occur, particularly in the form of a dialogue.

Within the perspective that I will try to sketch here, this thematic of the place, the experience, the encounter, the Other and the dialogue, brings me to think about ethics not in terms of morality – even if the question of morality or virtue always surge and get back – but in terms of *being, being-with-others, being-in-relation*. This will have implications in terms of pedagogical goals, ways of engaging with the history, teachers' and learners' roles, challenges, and, of course, in terms of research.

As Michael Fried has putted it, if we want to develop a theoretical perspective on the history of mathematics in mathematics education, this perspective should at least problematize the very nature of mathematics, and of mathematics teaching and learning, but, above all, should ask what it means to face the past in the context of mathematics education (see Fried *et al.* 2016). What it means to face the past as a mathematician, as a historian, but also as an *educator*, because I'm more and more convinced that mathematics educators has their own way to relate to the past where, for instance, the thematic of interpretation is maybe not central, the hermeneutical stance, I would say, is not the first dimension of their experience (see Guillemette and Radford 2022).

The ethical perspective that I will try to sketch arose principally from two studies that I will briefly present. The first was conducted with pre-services teachers and the other with in-service teachers. Both tried, in a way, to give the voice to the teachers and to understand how they relate to the past in educational context.

4 A first study with pre-service teachers

The first study is an empirical research conduct with six preservice teachers engaged with the reading of historical texts (see Guillemette 2017 for a more detailed account). The participants were taking part of a history of mathematics courses (45 hours) in which were integrated seven historical reading activities (A'h'mose, Euclid, Archimedes, al-Khwārizmī, Chuquet, Roberval, Fermat). The objective of research was to describe the lived experience of *dépaysement* (disorientation) (Barbin 1997) of the participants. The study was grounded in historico-cultural perspective in mathematics education (Radford 2021). Phenomenological and dialogical approach were adopted for the analyses of data obtain from group reading activities.

4.1 Investigating *dépaysement*: fragility, adversity and empathy

Three main thematics emerged from the analyses: Fragility, Adversity and Empathy. Phenomenologically, mathematics were perceived in their beginning and unstable aspects. Fragile and plurals, mathematics appeared in opposition to academic mathematics perceived as formal, and rigid. The scholar and academic mathematics are perceived as sophisticated and refined; school mathematical knowledge is needed to be unencapsulated, unravelled. Furthermore, readings of historical text were demanding and gruelling for the participants. Implicit argument, singular reasoning, unusual approach, particular calligraphy, special vocabulary and syntax, give a feeling of vagueness, incompleteness, confusion, and lead the future teachers to a reflection on affectivity in the learning and teaching mathematics.

Empathy, the third thematic, is an effort of openness and flexibility were deployed. Participants, for instance, made explicit references to a past experience of disorientation, immersion in a foreign context, feeling of missing things, sense of a void, and insecurity. Attention is given to the experience of the pupils whom “suffer” objects of knowledge, an attention to the subjectivi-

ty of the pupils - and, more generally, of the learners - and an openness to marginal reasoning and to the lived experience of the learners in affective emerged in this experience of empathy.

The engagement with the past in the context of teachers' training brings a difficult experience of otherness characterized by acts of violence, violence against the authors that are deprived of their singularity, that are translated, resumed, reified. In his phenomenology of otherness, Levinas called violence the "thematization of the Other" (Levinas 1971/2010, p. 37, my translation). Empathy here is understood as an effort for establishing a non-violent relation with the Other, an effort to keep free and alive his subjectivity, "a relation with the others like with a Mystery" (Levinas 1979/2011, p. 63, my translation). In other terms, there is, in the Levinasian ethics a violence of the ontological gesture. He formulated an ethics based on what he called the Face of the Other [*le visage de l'autre*], something that cannot be filled by intuition, a fundamental authority. If I understand the Other, is it not ultimately to suppress it as an other and reduce it to a simple form of self-concern? This is the main ethical question of Levinas. Because of the Other, I can no longer exist as a force that only cares about itself.

Demattè (2019), in the context of secondary school teaching, has also discussed in terms of Otherness and violence (with references to Levinas) the encounter with the past within the secondary school classroom. He focusses particularly on the importance, not only on the dialogue of students between the past, but on the joint responsibility and commitment (the fundamental promise and togetherness toward a common understanding) of students and teachers toward the past, and the inherent asymmetrical responsibilities on both sides. I think that this is a very good point and idea to focus on for future research in this perspective.

This is where maybe ethical questions surge and makes sense when thinking about the history of mathematics in mathematics education, because there is the possibility of a transfer of this empathic relation in the classroom. A new responsibility appears here for the preservice teachers, that of welcoming the pupils and their reasoning in a non-violent manner. This is why I suggested that, in the context of teachers education, the readings of historical text,

through *dépaysement*, can support a non-violent mathematics education (Guillemette 2017).

4.2 Ethics and the question of otherness

But let's go further with this encounter with the past in educational context, and with this ethical perspective, and the question of otherness. For that, we will need some theoretical groundings and tools.

In his phenomenology of the practical act, Mikhaïl Bakhtin (1895-1975) developed a dialogical perspective on human being, and human sciences. Bakhtin defends a more dialectical and critical orientation of modernism. Like Walter Benjamin, and other critical theorists, Bakhtin combines an awareness of the deleterious effects of modernity – the reification, exploitation and alienation of man – with an appreciation of its potential to promote the cause of human freedom and emancipation. In this perspective, being-in-the-world is dialogical through and through, we are co-participants in a world where other subjects exist and that our very identity is constituted of these relationships. The Other has, for Bakhtin, a “surplus of vision” on us, and vice versa, making possible a “reversal of perspectives” which is capital here in the human intersubjective experience and the formation of a proper ethical consciousness. The relation I-Other is necessarily a leitmotif in the Bakhtinian corpus, as he insisted on the situated and embodied nature of lived existence and experience and its implications for ethics. Because, for Bakhtin, from this dialogical perspective, there are always two subjects that are participating in the act of meaning. Reading is perceived here as “the meeting of two texts—of the ready-made and of the reactive text being created—and, consequently, the meeting of the two subjects and two authors” (Bakhtin 1986, p. 107). The focal point is the emergent dialogue, on levels of ideas, ways-to-engage-in-the-world or forms of consummation. In other terms, for Bakhtin, individual voices and understandings cannot make themselves heard without integrating themselves into the complex chorus of the already existing discourses. This simple but profound dialogical perspective led him to develop a totally new way of interpreting human subjectivity and, therefore, the entire field of human sciences. As Brandist emphasized around Bakhtinian dialogism: “Intersection of horizons is precisely what characterises the dialogic relations between discourses: double-voiced or 'hybridised' dis-

courses are its manifestation and as such, an index of the type of intersubjective interaction” (2002, p. 101). Indeed, for Bakhtin, there is an architectonic of the Self, a kind of constant upsurge of the Other in our acts of consciousness, making different responsive evaluative accents when engaging with others.

Back to the results of our study, students’ focus is not properly on the past mathematicians’ idiosyncrasies, style, or representativeness of their era, their works are hardly situated in the continuum of the development of mathematics. We can hardly see a form of conciliation or a form of appreciation of mathematicians work on their own terms and an appreciation of the text within the historical evolution of mathematics. They do not read the text completely within the present synchronic plan as the mathematician would possibly do. Students are, of course, embedded in their teacher training context and are obviously more inclined to put forward their pedagogical or didactic concerns, and what the preservice teachers read is already imbued with their responsive evaluative accents. They are not driven by translating into modern ways, nor by situating the text in a historical continuum. What I see is that it is a *dialogical reading* that brings with it, its potential estrangement. Furthermore, this dialogical reading questions the self in its theoretical assumptions and offers vicarious aspects around the ways of being in mathematics and ways of doing mathematics. They *engage* differently, having their own way of reading historical text, which is not that of the historian nor the mathematician, but that of the *educator* (see Guillemette & Radford 2022).

For Bakhtin, to understand something means to enter responsively in a dialogue; that is to say, to bring *engagement* and *answerability* within the encounter with the Other. Indeed, the appearance of otherness throws into relief elements that are indispensable to any genuine dialogue. It is not the positive or constructive exchange that is crucial here, but rather a recognition of the ‘Otherness’ of ‘the Other’. They engage (etymologically speaking, they *bind* themselves) with the text in a way that brings different nuances and evaluative accents. While some students seek connections through modern formula and technique, others focus on new ways to tackle mathematical problems. They answer through the attending way their subjectivity shows itself in the reception (*accueil*) of the Other. The students’ polyphonic engagements are inhabited by centripetal and centrifugal

forces that shape how students answer to the text. In Bakhtinian terms. centripetal forces represent the drive for unitary language, standardisation and linguistic hegemony, centrifugal forces represent the presence of heteroglossia, multiplicity, stratification and decentralisation.

The tension between those opposite forces are significant when engaging with historical artefact, as voices from the school mathematics and mathematics from the past are in interaction within the engagement of the students. This is where the ethical and dialogical perspective that I introduce here takes on its concrete aspects.

4.3 An ethical perspective – implications for teachers' education

This ethical perspective has brought us to a certain openness to the educators way to relate the past, which seems to have its own orientation, and I think is still to be described more finely in research.

One way to do it would be to focus on the engagement/answerability when relating to the past, by emphasizing the voices manifesting and interacting (voices from the classroom, from the past and from the actual historical and sociopolitical environment), but maybe also, for research concerns, voices coming from research, voices from the HPM community for instance. The difficulty experienced by the preservice teachers resides in the adversity that comes with the experience of otherness due to historical and cultural distance, but, above all, in the adversity in the welcoming of the alterity of what is Other, welcoming without transforming the Other into something which is a *mine*; a modernized or reified Other as Levinas would say, instead of an authentic interlocutor. As Radford (2021) stated it:

Responsibility appears here as an unconditional undertaking towards others. It realises a union, a link, a connection, and a bond between self and others. Responsibility is expressed in the answerability we make to the call of the Other, a call that comes not necessarily from a linguistic or semiotic formulation, as Levinas shows, but from the mere presence of what we ourselves are not (p. 225).

Maybe we rejoin here, from another point of view, the moderator in dialogue attitude that Fried described in the last HPM meeting (see Fried 2021b).

5 *A second study with in-service teachers*

In this section, I will present another study that focus more on teachers' practices around the history of mathematics in mathematics education with the objective of fostering and developing this ethical perspective.

We conducted a participatory study (Desgagné and Bednarz 2005) involving four in-service teachers exploring the introduction of the history of mathematics in secondary school classrooms (Guillemette 2023). The participants included two women and two men, all of whom were mid-career teachers with 10 to 20 years of experience. We held an initial meeting to discuss the objectives and modalities of our collaboration. Over the course of a month, we organised three additional meetings, each lasting 150 minutes, at the Department of Mathematics at the Université du Québec à Montréal. The study was not framed as a professional development initiative; instead, it served as a platform for teachers and researchers to come together and reflect on incorporating the history of mathematics into secondary school curricula. Grounded in the same theoretical framework, our research objectives were to (1) dialogically describe the epistemological, didactic, and pedagogical reflections of secondary school teachers regarding the introduction of the history of mathematics in their classrooms, (2) collaboratively develop didactic and pedagogical tools for their teaching practices, and (3) identify the constraints and challenges encountered or anticipated in the teaching process.

During the three meetings, I introduced "historical situations" to the group. These consisted of very brief excerpts from historical texts and images sourced from research literature, presented without detailed historical or pedagogical explanations—just a short introduction from me. The purpose of these historical situations was to spark discussion. The idea was to put these situations into action, explore them, and collaboratively brainstorm possible applications for the classroom. This approach aimed to foster dialogue and mitigate pre-formed conclusions. The historical situations I presented included the following: Babylonian cuneiform numerals, Egyptian numerals and arithmetic, Lunes of Hippocrates of Chios, Nicomedes and the problem of the box and the ladder, Ptolemy trigonometry, Al-Khwārizmī's algebra, Fibonacci and the two towers problem, Cavalieri and the volume of the sphere, Jacob's staff. We didn't have the time to work on all of those but the group chose : Babylonian cuneiform numerals, Ptolemy trigonometry, Al-Khwārizmī's algebra, Cavalieri and the volume of the sphere, Jacob's staff. Transcription of

video recordings of the meetings, productions of the group and a research diary constituted the data here.

With the research team, we refined and applied our ethical and dialogical perspective on the introduction of the history of mathematics in mathematics education in the context of this study. To make it short, the idea was to get in a dialogue with the participants and the mathematicians from the past. To describe how together researcher, participants, voices from the past and actual mathematical culture come into a dialogical interaction and to give access to a share meaning, emergent ideas and anticipations.

Concretely for each collaborative meeting, we first get the transcriptions of the video recordings with the help of professional services, then we highlight groups of utterances related to the objectives of research. Those groups of utterances were then put on a spreadsheet in order to work on each utterance and to describe the literal sense and the dialogical sense (its situation in the dialogue) of each utterance. The last stage of analyses was to write short dialogical narrations. In these narrations, we defend and highlight the position of participants, the voices from the past, voice coming from research especially. These dialogical narrations are enriched by the research diary and the group productions. The idea is to continue to think with the participants through writings. At the moment of the production of these proceedings, the dialogical narrations were ongoing.

5.1 Investigating teachers' reflections: *ways to relate to mathematics, hospitality and reflexivity*

Three main thematic emerged from the analyses: ways to relate to mathematics, hospitality and reflexivity.

The first thematic focuses on epistemological questions. The group emphasises the cultural, historical and contextual nature of what it means to do mathematics through the different explored examples. The group was also sketching the idea that mathematical activity produces mathematics. It seems obvious, but the idea is that, in time, mathematics evolve and develop through the situated activity of the practitioners. Another related idea was that mathematical Knowledge is not in books, nor in our head, knowledge is in interaction. Again, another related observation is how mathematics affect us as a historical product and how we affect mathematics through our activity. This was beginning to be evident for the group.

The second thematic is related more to the life of the classroom when engaging with historical elements. Few ideas emerged around what can be related to the thematic of *hospitality* and *care*. For the participants, to be a meaningful mediator, the history of mathematics needs to appear as something problematic, history has to become an *enquiry*. Such a task, of course, is extremely difficult. With in-service teachers, we can only vaguely perceive its contours. Concretely for the participants, when referring to the history of mathematics, the gaze turns more unto the teacher, as he becomes a kind of storyteller. This brings special responsibility, different from the usual position of the mathematics teachers. When teachers talk about how they introduce the history of mathematics or how they project themselves, teachers see themselves at center physically. They find it hard to maintain this exploration of the history in time (the time of the school), they don't feel comfortable with the history appearing punctually in the classroom. There is a need to create an intrigue, a plot, different from the intrigue that they are creating with school mathematics. Plus, a special attitude toward novelty needs to be enacted, communicated and shared, novelty of ways of being and doing in mathematics. New manners that are not the same of that is usually awaited. The presence of the history of mathematics in the classroom push usual organisation of the teachers in time and space, putting them in a particular narrative stance.

The third thematic is reflexivity. Fundamental questions related to mathematics education emerged and were grounded in social and political dimension: Why mathematics education? Where do I want to go with my students? There was for the group a reconsideration of school mathematical activity, mathematical school knowledge appears, in this context, as linked with social demands, that can be critically assessed, and we can feel globally a repositioning of mathematics education as a social and political project. Indeed, mathematics education too has an history and that mathematics education in embedded in its cultural, social and political context. Considering these elements of reflection, I think that it is interesting to see and to foster how the historical, the epistemological, the social and the political intertwine when the history of mathematics enters the picture in mathematics education. I think that this could be another aspect that could be explored more profoundly in research.

6 Conclusions

In his essay on exoticism, the French writer, Victor Segalen, introduced the notion of *exote*. The *exote* is this subject:

[w]ho strangely, discovers himself there, so that his own culture becomes foreign, at least strange too. The *exote* is the one who, born traveller, in worlds of marvelous diversity, feels all the flavour of diversity, seeking his disorientation in time and space, in the Other and its difference (1986, p. 42, my translation)

When thinking about the history of mathematics in mathematics education, the metaphor of travelling and this idea of searching for ourselves in the Other seems very relevant. There is something there to think about *Otherness*, its valorisation and its defence and protection in a way.

This idea of the *exote*, makes me think about the history of mathematics in mathematics education, its role and potential, in particular, its power to make the learners live a distance and to figure out the mediation of mathematics by the surrounding culture, to perceive learning mathematics as learning a certain manner to be-with-others, to learn to travel (to give oneself to otherness of the Other) which is, in a way, the only way to understand ourselves. I would say with Hanna Arendt (1998/2019), to get ready for *action*, that is to say to belong to the world through speaking and acting which take place in the public sphere, and that is, for Arendt, the very possibility to initiate something new, the possibility to begin, with all the educational and political implications that this novelty can drive with it.

In other terms, we have to acknowledge the power of the experiences of otherness in mathematics education, experiences for which the history of mathematics is not a reservoir of interesting situations, but the necessary *topos* where those experiences could emerge with possibility to restore to mathematical activity its most precious ontological force, its recreation against the backdrop of culture and history.

The critical and difficult engagement with the history of mathematics in this perspective can be seen as a condition of freedom in mathematics and mathematics education. Maybe we rejoin here the history of HPM community and D'Ambrosio and his quest, announced 40 years ago, for a new human dimension into relations between individuals, societies and cultures, through and with, mathematics education.

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ON VISUALIZATION IN HISTORY OF MATHEMATICS AND TEACHING

Signs, diagrams and graphics

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ABSTRACT

In this paper, we study types and roles of visualization in four historic key moments: Greek geometry with Euclid's *Elements* and Hérigone; Algebra in Renaissance with Jerome Cardan and Jacques Peletier du Mans; Cartesian geometry in 17th century with René Descartes and Bernard Lamy; Logic with Gottfried Leibniz and John Venn. For this purpose, we use the classification of signs and the design of diagrams of Charles Sanders Peirce. It leads us to a semiotic history which is rich in reflections on teaching practices with the aim, in particular, of enabling teachers to explain to students the role and meaning of signs, but also that of diagrams in mathematical practice.

1 Introduction

Our aim is to emphasize the role of visualization in teaching. This requires, not only to know when and how signs were introduced in history, but to present different meaning of signs, patterns and writings in teaching of mathematics such as to hold a discourse on figures, to give an operating status to objects, to generalize procedures, to represent relations between objects, or to transport practices and knowledge from one mathematical field into another. We begin by presenting the semiotic of Charles Sanders Peirce before using it to analyze four historic key moments.

2 The division of signs by Charles Sanders Peirce

Peirce (1839-1914) was an American logician and philosopher, son of the mathematician Benjamin Peirce. His numerous philosophical and scientific papers on semiotics or sign theory appeared in the *Collected Papers*. His division of signs in three kinds, which he called “icons”, “indices” and “symbols”, had been given in a paper of 1885. Peirce, 1998, 460-461).

“Icons” are used to represent the objects they resemble, as the “icon” for the square figure (Fig. 1, left), the “indices” represent objects to indicate objects, such as A, B, C, D for the vertices of a square (Fig. 1, center), “symbols” represent objects arbitrarily, as $\sqrt{2}$ for the magnitude of BC (Fig. 1, right).

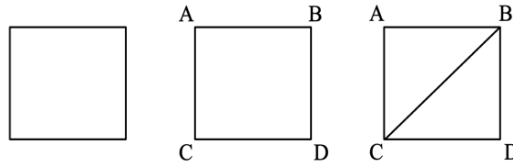


Figure 1. The three kinds of signs by Peirce

The roles of “icons” and “indices” are related to their uses. “Icons”, which look like objects, can be used to observe them with a certain intention, such as the three equal triangles (Fig. 2, top). While the “indices” can indicate connections to be observed. “Indices” 1, 2 and 3 invite us to establish a link between the angles of the triangles, to see the three angles of the triangle joined together to form a flat angle (Fig. 2, bottom). They are used here to show a visual phenomenon. Peirce thus specified the function of an index (Peirce, 1885, p. 362): “I call such a sign an index, a pointing finger being the type of class. The index asserts nothing; it only says ‘there!’. It takes hold of our eyes as it were, and forcibly directs them to a particular object, and there it stops.”.

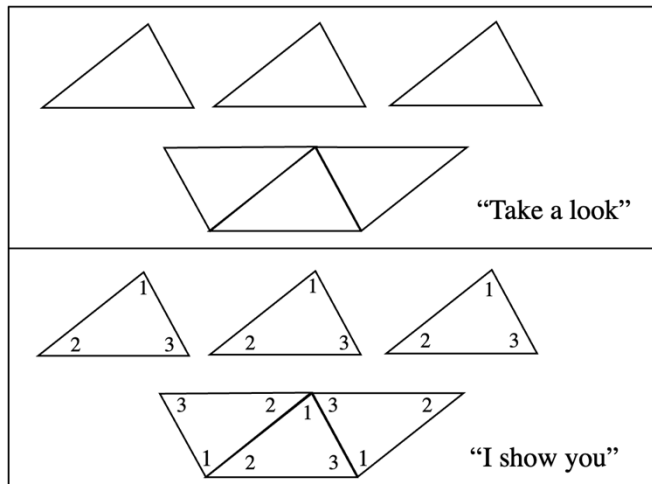


Figure 2. Roles of icons and indices

Peirce’s “diagram” is also linked to the idea of resemblance, to represent not an object but a relation between objects. He wrote: “The pure diagram is

designed to represent and to render intelligible, the form of relation merely. Consequently, diagrams are restricted to the representation of a certain class of relations; namely, those that are intelligible” (Peirce, 1906, p. 314).

2 Signs and diagrams in geometry: Euclid and Hérigone

We examine the role of “signs” and “diagrams” with the Proposition 32 of Book I in Euclid's *Elements* and then in a textbook of Pierre Hérigone (1639).

2.1 Signs and diagrams in Euclid's *Elements*

Euclid began with two statements. The first statement gives the proposition without letter to designate the elements of the figure, while the second statement uses “indices” that refer to a drawing (Euclid, 1956, p. 316):

In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles. Let ABC be a triangle, and let one side of it BC be produced to D . I say that the exterior angle ACD equals the sum of the two interior and opposite angles CAB and ABC , and the sum of the three interior angles of the triangle ABC , BCA , and CAB equals two right angles.

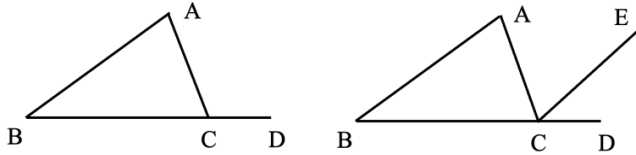


Figure 3. Proposition 32 of Book I

This statement introduces an “icon” for the triangle ABC , which “resembles” a triangle. But is not a triangle, since its segments are more or less thick lines, whereas a line is defined by Euclid as a length without width (definition 2). The “indices” A , B , C , and D allow us to follow and to understand the discourse of the first statement (Fig. 3, left). The proof begins after construction of CE parallel to AB (Fig. 3, right) (Euclid, 1956, p. 317):

Then, since AB is parallel to CE , and AC has fallen upon them, the alternate angles BAC and ACE are equal to one another (I. 29). Again, since AB is parallel to CE , and BD has fallen upon them, the exterior angle ECD is equal to the interior and opposite angle ABC (I. 29). But the angle ACE was also proved equal to the angle BAC ; therefore, the whole angle ACD is equal to the two interior and opposite angles BAC and ABC .

The two first sentences have a common discursive pattern which forms a disposition to be observed and serves to associate them. This pattern is a “dia-

gram” in the sense of Peirce, that makes intelligible a relationship between lines and angles (Peirce, 1885, p. 363-364):

The truth, however, appears to be that all deductive reasoning, even simple syllogism, implies an element of observation. Indeed, deduction consists to construct an icon or a diagram such as relations between parts of this icon have to present a complete analogy with parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts.

2.2 Signs and diagrams in Hérigone’s *Elements*

The French professor of mathematics and algebraist Pierre Hérigone (Massa Esteve, 2006) edited *The first six Books of the Elements of Euclid demonstrated by notes, with a very brief and intelligible method*. He wrote in his Preface (Hérigone, 1639, np.): “seeing that the greatest difficulties were in proofs, on whose intelligence depends the knowledge of all parts of mathematics: I invented a new method of making proofs, brief and intelligible, without the use of any language.” We took his proposition 32 of Book I of Euclid (Fig. 4).

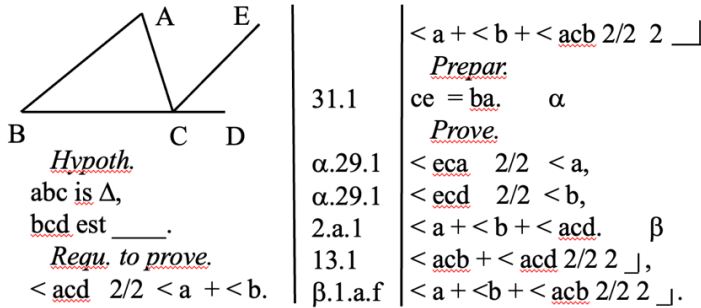


Figure 4. Proposition 32 (Hérigone, 1639, p. 74)

signs	types	things
Δ	icone	triangle
—	icone	line
$<$	icone	angle
abc	index	triangle
bcd	index	line
acd	index	angle
a, b	indices	angles
2/2	symbol	equality
=	symbol	parallel
$\perp$	symbol	right angle

Table 1. “Icons”, “indices” and “symbols” in Hérigone

Hérigone used “icons”, “indices” and “symbols” (Barbin, 2011) in the sense of Peirce (Table 1). The use of “symbols”, that means arbitrary signs, permits to observe “diagrams” better and to render them to be more “intelligible”.

3 Symbols and diagrams in algebra: Cardan and Jacques Peletier

Most of the signs of algebra were introduced in the 15th and 16th centuries in books on arithmetic and algebra. In this period, each author has his own ways of writing arithmetic and algebraic signs. For our purpose, it is interesting to observe and compare some writings of the equation $2x^2 - 5x = 23$ (Plane, 2006), to find out some types of symbols and their roles (Table 2).

Stiffel (1525)	$2z \text{ equal } 5x + 23$	Girard (1629)	$2(2) - 5(1) = 23(0)$
Gosselin (1577)	$2Q \text{ M } 5L \text{ equal } 23$	Viète (1600)	$2a_q - 5a \text{ æq. } 23$
Bombelli (1572)	$\overset{\curvearrowright}{2} m \overset{\cup}{5} \text{ equal } a \text{ } 23$	Harriot (1631)	$2aa - 5a = 23$
Viète (1580)	$2Q - 5N \text{ equal } 23$	Hérigone (1634)	$2a_2 \sim 5a \text{ } ^z/_z 23$
Ramus (1586)	$2q - 5l \text{ equal } 23$	Descartes (1637)	$2xx - 5x \propto 23$
Buteo (1559)	$2\Diamond M 5p = 23$	All 18 th century	$2xx - 5x = 23$

Table 2. Writings of an equation (Plane, 2006, p. 28)

For arithmetic operations plus and minus, Jerome Cardan used signs p and m in 1545, that are abbreviations. While the German algebraist used $+$ and $-$, introduced by Johannes Widmann in 1489, that are “symbols” because they seem arbitrary. For the unknown of an algebraic problem and its power, the French Guillaume Gosselin also used abbreviations L (Ligne) for the unknown and Q (Quarré) for its square, like François Viète and Ramus, but Johannes Buteo (alias Jean Borel) used a geometric “icon” with the form of a lozenge for the square of the unknown. Rafael Bombelli and Albert Girard used numerical signs, that allowed them to make obvious, for example, the rule on the multiplication of powers of numbers. These last signs have to be considered as “indices”, that means signs to observe and for showing.

In 1559, Buteo adopted the sign for equality introduced by the Robert Recorde who wrote: “to avoid the tedious repetition of these words: is equal to: I will set a pair of parallels, 2 lines of one length thus $=$ because no 2 thinks can be more equal” (Recorde, 1557, p. 235). His sign is both a geomet-

ric “icon” and a “symbol”, a kind of “metaphor”. For several decades, algebraists preferred to write in full the word “equal”, which designates the primary relation of algebra but which is also the most frequent word in algebra.

The introduction of symbolism in algebra made it possible to write algorithms for solving algebraic equations in symbolic form, thus giving rise to what Peirce called the “icons” of algebra. Indeed, he also used the term “icon” for the representation of formulas, because they are used for observing calculations and finding solutions (Peirce, 1885, p. 364):

As for algebra, the very idea of the art is that it presents formulae which can be manipulated, and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries which are embodied in general formulae. These are patterns which we have the right to imitate in our procedure, and are the *icons par excellence* of algebra.

Thus, the solution of an equation can be given by a formula, as we are used today. Cardan in his *Ars Magna* (1545) wrote a solution of the equation of degree 3, $x^3 + 6 = 20$, as an “algebraic icon” (Fig. 5, left). This transition from arithmetic to algebra led him to write and observe an imaginary solution for an equation of degree 2. It seemed to him “sophisticated”, but Bombelli would admit imaginary solutions with profit to solve equations of degrees 3 and 4.

cubus p. 6. rebus æqualis 20.

2. 10.

8. 108.

108.

p. 108. p. 10.

m. 108. m. 10.

p. 10.

m. 108. m. 10.

p. 10.

m. 108. m. 10.

√24 m. √6

√818 p. √2

√8432 m. √812

m. √8108 p. √848.

Figure 5. Formulas in algebra (Cardan, 1545, p. 250) (Peletier, 1554, p. 159)

As a result, too, the algebraists and geometers of the 17th century considered what they called, not numbers, but “irrational quantities” and “negative quantities”, because symbols made it possible to observe that arithmetic operations could be extended to these quantities (Barbin, 1995).

In the first algebra book in French, edited in 1554, Jacques Peletier considered that irrationals can be considered as real numbers because “they have their algorithm, their order and infallible rules, as well as the rational ones ” (Peletier, 1554, p. 131-132). “Symbols” permit him to show the similarity of the rules and the operations for rational and for irrational numbers, for instance in a multiplication between $(\sqrt{24} - \sqrt{6})$ by $(\sqrt{18} + \sqrt{2})$ (Fig. 5, right).

4 Signs and diagrams in geometry: Descartes and Lamy

In *The geometry* of 1637, that is a part of his *Discourse of the method*, Descartes began by explaining how to use the operations of arithmetic in geometry. For example, the product of two segments is still a segment for Descartes, and not a rectangle, as in Euclid's *Elements*. That is why, he introduced a segment that he called “unit”, by analogy with the unit of arithmetic and which he denoted by the symbol “1”. This “symbol” is not, like for Peirce, an arbitrary sign. Taking AB the unit, the product of BD by BC is BE , thanks to Thales’ theorem (Fig. 6).

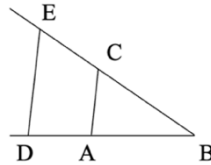


Figure 6. The product of two segments (Descartes, 1637, p. 334)

Descartes proposed a method that made it possible to solve “all geometric problems”. It is based on an algebraic calculation of segments with the symbols of arithmetic operations and the square root. It proceeds in five steps: 1) the problem must be assumed to be solved; 2) designate each known and unknown segment by a “symbol”, x , y , z , etc. for the unknown segments and a , b , c , etc. for known segments; 3) translate the problem into relationships between these letters; 4) obtain one or more equations; 5) solve the equation(s).

If we write, like Descartes, each segment by one italic letter only (Fig. 6), if $BD = a$, $BC = b$, $BE = d$, $AB = 1$, then $d = ab$. These letters are “symbols” that represent segments. The transition from the geometric problem to the algebraic one is therefore based on a new use of “symbols” in geometry, that means a symbolic arithmetization of geometry. This translation has the effect, as Descartes liked to remark, of no longer having to contemplate the figures of geometry, but to observe calculations. Therefore, there is a shift from “geometric icons” to the diagrams, that are “algebraic icons”.

Let us examine the transition from “geometric icons” to “algebraic icons” by reading the Cartesian method in Bernard Lamy’s textbook *Elements of geometry* (1734). Lamy took the proposition 4 of Book II of Euclid’s *Elements*. It must be proved that, given any point D of a given segment AB , the area of the square constructed on AB is equal (in area) to (the sum of) those of the squares on AD and DB and twice the rectangle between AD and DB (Fig. 7).

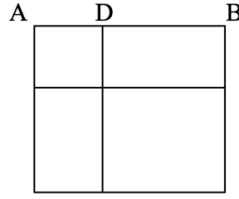


Figure 7. Proposition 4 of Euclid's Book II (Euclid, 1956, p. 379).

After the construction of the “geometric icons”, Euclid's proof is based on the observation of “geometric icons”. While, after having stated the Cartesian method, Lamy wrote (Lamy, 1734, p. 138):

Let a line AB be cut in two parts by a point D . We have to prove that $AB^2 = AD^2 + 2 AD \times DB + DB^2$.

A D B

Let $AD = b$, $DB = d$. Therefore $AB = b + d$. But the square of $b + d$ is $b^2 + 2 bd + d^2$. But $AD^2 = bb$, $DB^2 = dd$, $2 AD \times DB = 2bd$. Therefore $AB^2 = bb + 2 bd + dd$.

It is noteworthy that Euclid's “geometric icons” have disappeared. Lamy only represented a segment with three points. The transition from geometry to algebra requires the Cartesian translation of the problem, namely:

$$AB^2 = AD^2 + 2 AD \times DB + DB^2.$$

Its symbolic script, where each segment is represented by a letter, is:

$$(b + d)^2 = b^2 + 2 bd + d^2.$$

Lamy concluded by observing this algebraic icon, “an icon par excellence of algebra”, which is true as the result of an algebraic calculation. In the seventeenth century, evidence of the calculation seemed more enlightening than Euclidean discourse, which seemed more likely to convince (Barbin, 1992).

5 Graphics and diagrams in Logic: Leibniz and Venn

From Aristotle, logic appears as a means of verifying propositions by using schemas (Barbin, 2024). Peirce considered Aristotelian syllogisms as cases of what he called diagrams: “For instance, take the syllogistic formula,

All M is P

All S is M

(so) All S is P.

This is really a diagram of the relations of S, M and P” (Peirce, 1885, p. 364).

Leibniz introduced two geometric graphics to represent Aristotle's premises and syllogisms in a Latin manuscript of 1686, only known at the beginning of the 20th century, thanks to its edition by Louis Couturat. In the graphics, the

affirmative universal premise “All B are C” is represented by two nested segments *B* and *C*. In the second, the premise is represented by nested circles. For the other ones, segments and circles are disjoint or overlapping (Fig. 8). Syllogisms are represented in the same way and, according to Leibniz's manuscript, they are thus “verified” (Fig. 9).

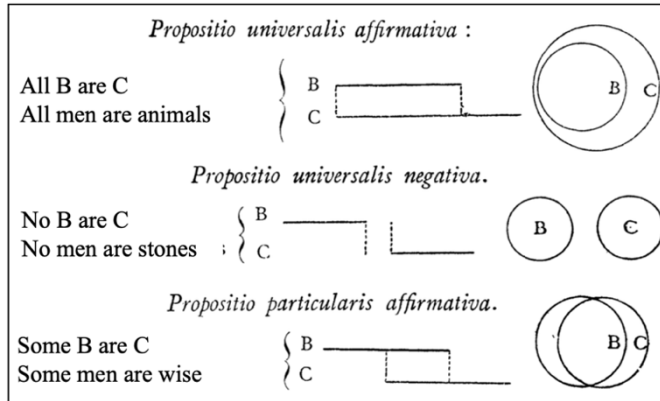


Figure 8. The premises in Leibniz (Leibniz, 1686, p. 292-293)

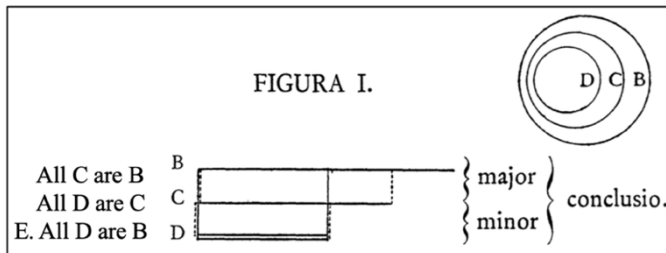


Figure 9. The syllogism “Barbara” in Leibniz (Leibniz, 1686, p. 294)

Representation of premises and syllogisms by means of circles appears in Leonhard Euler's *Letters to a Princess of Germany* (Euler, 1802, p. 397):

These circles, or rather these spaces, for it is of no importance of what figure they are of, are extremely commodious for facilitating our reflections on this subject, and for unfolding all the boasted mysteries of logic, which that art finds it so difficult to explain; whereas, by means of these signs, the whole is rendered sensible to the eye.

He did not mention Leibniz and the success of his *Letters* led to the circles being called “Euler's circles”. Peirce studied them under the name of “Euler diagrams” in a paper of 1911. He explained how a syllogism is “illustrated by means of circles” on an example of the same type as Leibniz. He wrote that after Euler, there were several attempts to improve his system, but they were all failures until the publications of the logician John Venn.

In his 1881 *Symbolic Logic*, Venn associated syllogisms with “diagrams”: “This will set before the eye, at a glance, the whole import of the propositions collectively” (Venn, 1881, p. 123). He took care to distinguish the “Euler’s circles”, from his own diagrams. He took the syllogism “Celarent”:

No y is z
 All x is y
 (so) No x is z

With Euler, the circles are drawn one after the other. First, we must draw two disjoint circles y and z corresponding to the premise “No y is z ”, then the circle x is inscribed in the circle y , to conclude that “No x is z ” (fig. 10, left). On the contrary, with Venn diagrams, the circles are drawn from the beginning in a general position and then the areas corresponding to the premises are removed one by one. Here, we first remove (or hatch) all of the part of the circle y that is in the circle z . Then we hatch the part of x that is not in y . We can then see that all the remaining (unhatched) part of the x -circle is not in the rest of the z -circle (fig. 10, right). For Venn, diagrams were a “visual aid”: “One main source of aid which diagrams can afford is worth noticing here. It is that sort of visual aid which is their especial function” (Venn, 1881, p. 118).

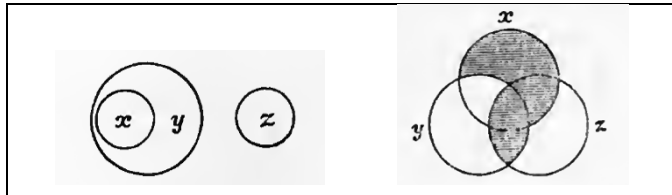


Figure 10. Euler’s circles and Venn’s diagram (Venn, 1881, p. 115-116)

Peirce wrote that Venn had made a marked improvement and he criticized Euler’s original proposal on the scope of “diagrams” (Peirce, 1911, p. 354):

What is it, then, that these diagrams are supposed to accomplish? Is it to prove the validity of the syllogistic formula? That sounds rather ridiculous [...] Suppose we ask ourselves *why* it is that, if a circle P wholly encloses a circle M which itself wholly encloses a circle S , the circle P necessarily wholly encloses the circle S .

Then, he made several improvements to the Euler’s diagrams.

6. What a semiotic history can teach us

The extreme attention of Peirce to the “philosophy” of notations, signs, and diagrams in mathematics and in logic explains why his work can be so valuable to mathematicians and teachers, because it linked to the meaning of the signs and graphics, to their usefulness in the practice of mathematics.

Semiotic history à la Peirce allows not to confine to a merely chronological history of signs, but to make a way into an epistemological history, as we show in this paper, that leads us: 1) to better see and understand the difficulties of the students with signs, especially in the introduction of algebraic symbols; 2) in general, to be careful with the students when introducing “symbols” by clarifying their arbitrary and their pronunciation; 3) to explicit them the roles and meanings of signs: signs are not simply notations, they permit to hold a discourse on figures in geometry, to represent relations between objects and to generalize procedures in arithmetic, to give an operating status to irrational and negative numbers; 4) to transport practices and knowledge from arithmetic to algebra and then to geometry, especially in the introduction of cartesian geometry ; 5) in general, to be careful with the students to the passage from one meaning to another for the same symbol.

Semiotic history also leads us to emphasize visualization in mathematics. Peirce broadened his discourse on signs in “An Essay toward Improving our reasoning in security and in uberty”, by situated them in the general framework of the space, on the sheet of a paper (Peirce, 1998, p. 472):

Reasoning is dependent on Graphical Signs. By “graphical” I mean capable of being written or drawn, so as to be spatially arranged [...]. I do not believe one can go very deeply into any important and considerably large subject of discussion without using space as a field in which to arrange mental processes and images of objects.

Mathematics is writing, and what we see, and which everyone can recognize as mathematics, is writing. I like to write “I think with my pen” with Wittgenstein, and “I think with my inkwell” with Peirce. In the mathematics class, a specific time to write and to see what is thought in writing therefore seems necessary 6) to encourage students to visualize (not just to see); 7) to give students time to recognize patterns (diagrams) and to know how to use them with a full understanding; 8) to resort to useful diagrams to represent reasoning; 9) to encourage students to freely represent their ideas.

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EVOLUTIONARY ARGUMENTS FOR HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION: A CRITICAL AND CONTRUCTIVE DISCUSSION

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ABSTRACT

In research done in the field of history and pedagogic of mathematics, theories from other areas of didactics of mathematics are often an integral part. The vice versa is not the case, and when history of mathematics finds its way into general mathematics education research, it is often in forms related to so-called evolutionary arguments. In this paper, we discuss this by analyzing three influential theories in didactics of mathematics, which have been informed by history of mathematics: works by Anna Sfard (1991, 1995), Guy Brousseau (1997), and Guershon Harel and Larry Sowder (2007). We analyze their work with respect to how they use the history of mathematics and for what purposes in order to invite a discussion of the potential influence of HPM in more general mathematics education research.

1 Introduction

When we look at theories and theoretical frameworks and constructs within the HPM research area, which deal with the roles of history of mathematics in mathematics education and the significance of history for the teaching and learning of mathematics, a variety of theories from other areas of mathematics education form an integral part of much of the research. If we look at it from the other side, from mathematics education research at large, the history of mathematics does not play a significant role, and when it finds its way into the more general mathematics education research literature it is often in a form which is related to so-called “evolutionary arguments” (Jankvist, 2009). In this paper, we take a closer look at this by displaying some examples from mathematics education. We look at three firmly rooted and influential theories in didactics of mathematics, which have been informed by history of mathematics: Anna Sfard’s model for learning of mathematical concepts, Guy Brousseau’s work on epistemic obstacles, and Guershon Harel’s and Larry Sowder’s development of students’ proof schemes. We analyze their work

with respect to how they use the history of mathematics and for what purposes. Before presenting our analyses, we provide a brief introduction to evolutionary arguments resorting to the history of mathematics in mathematics education. Upon the analysis of the three cases, we invite a critical and constructive discussion of the role of history of mathematics in mathematics education research and point towards other ways in which history (may) inform theoretical developments in the field.

2 Evolutionary arguments in mathematics education research

The idea that “ontogenesis recapitulating phylogenesis” permeated educational thoughts, by transferring the idea from biology to psychology and cognitive development, and philosophies from the turn of the nineteenth century and well into the twentieth century—eventually known as the “genetic principle”. In mathematics, the German mathematician Felix Klein¹ advocated for the genetic principle in teaching, that (quoted from Schubring, 2011, p. 82):

... teaching should, by tying to the natural disposition of the youth, lead them slowly to higher things and eventually even to abstract formulation, by following the same path on which the entire mankind struggled to climb from its naïve primitive state upwards to more developed insight.

In the early 1960s, leading mathematicians from North America published a memorandum “On the Mathematics Curriculum of the High School”, formulating “fundamental principles and practical guidelines”, such as their 5th principle, labeled the “Genetic method” (Ahlfors et al., 1962, p. 190-191):

It is of great advantage to the student of any subject to read the original memoirs on that subject, for science is always most completely assimilated when it is in the nascent state” wrote James Clerk Maxwell. There were some inspired teachers, such as Ernst Mach, who in order to explain an idea referred to its genesis and retraced the historical formation of the idea. This may suggest a general principle: The best way to guide the mental development of the individual is to let him retrace the mental development of the race—retrace its great lines, of course, and not the thousand errors of detail. This genetic principle may safeguard us from a common confusion: If A is logically prior to B in a certain system, B may still justifiably precede A in teaching, especially if B has preceded A in history. On the whole, we may expect greater success by following suggestions from the genetic principle than from the purely formal approach to mathematics.

¹ For a discussion on Felix Klein and the genetic principle, see Jahnke et al. (2022).

This plea for the genetic method was commented on by the director of the School Mathematics Study Group (SMSG), Edward G. Begle from Stanford University, who made it clear that most of the guidelines formulated in the Memorandum agreed with the texts of the SMSG, except, probably

... the “Genetic Principle” stated in the fifth guideline. This principle, as stated, would, for example, deny to our students the efficiency of using algebra in the first course in geometry and would require children to learn to compute with Egyptian, Babylonian, Greek and Roman numerals before being introduced to the historically later but far more efficient place-value decimal system. (Begle, 1962, p. 426).

Nonetheless, the genetic principle was put forward by at least the 65 mathematicians in the USA and Canada, who signed the memorandum that was published in *The American Mathematical Monthly* in 1962.

3 Three cases of theory building in mathematics education drawing on the history of mathematics

The genetic principle re-entered the discourse in mathematics education in the 1980s with Jean Piaget and his studies on the relation between psychogenesis and history of science (Schubring, 2011, p. 84). In the following, we will look at three cases from the past 30 years where evolution and genesis in the history of mathematics is used to advocate theoretical constructs.

3.1 A model for concept development informed by history

In a well-known series of papers, Anna Sfard (e.g. 1991, 1995) presented a theoretical model for concept development. The model is based on observations from the history of development of mathematical concepts and is applied to the individual learning of mathematical concepts. Sfard (1995, p. 15) wanted to find the “roots of the difficulties experienced by students” when they are confronted with abstract mathematical concepts.

She distinguishes between two different ways of conceiving mathematical objects: operational conception and structural conception. Her thesis is that there is an ontological gap between the two, which she found to be an explanation for the difficulties experienced by learners of mathematics in conceiving mathematical objects. By structural conception she meant being able to see and understand a mathematical entity as an (abstract) object, which means being able to refer to it as a “real” thing. In contrast, or rather dual, to this conception, by operational conception she referred to processes, algorithms and actions, meaning, she explained: “interpreting a

notion as a process implies regarding it as a potential rather than an actual entity, which comes into existence upon request in a sequence of actions” (Sfard, 1991, p. 4). She argued that “*In the process of concept formation, operational conceptions would precede the structural*” and claimed that “this statement is basically true whether historical development or individual learning is concerned” (Sfard, 1991, p. 10, italics in the original).

Sfard took this as a basic conjecture from which she deduced her model for concept acquisition. It led to a model describing a cyclic process consisting of three phases: (1) The preconceptual stage, where mathematicians accustomed themselves to certain operations on already known (i.e. constructurally conceived) objects. (2) A period of predominantly operational approach. In this phase the coming, new object begins to emerge. (3) The structural phase, where what has been emerging in the previous phases, becomes recognized as a full-fledged new mathematical object (Sfard, 1991, p. 13).

She connected her model to students’ individual learning, quoting Piaget: “the [mathematical] abstraction is drawn not from the object that is acted on, but from the action itself.” (Sfard, 1991, p. 17). Comparably to the three phases above, she distinguished between three stages in individual learners’ concept formation. (1) Interiorization: the stage where “a learner gets acquainted with the processes which will eventually give rise to a new concept [...] These processes are operations performed on lower-level mathematical objects.” (2) Condensation: the stage where “a person becomes more and more capable of thinking about a given process as a whole, without feeling an urge to go into details. [...] The condensation phase lasts as long as a new entity remains tightly connected to a certain process.” (3) Reification: happens “when a person becomes capable of conceiving the notion as a fully-fledged object. [...] Reification, therefore, is defined as an ontological shift.” (Sfard, 1991, p. 18-19)

In her model, reification requires that one have tried to make operations with the notion as a whole. This is exactly, she argued, why reification is so difficult for mathematics learners, and this is her answer to her initial question of why students experience such difficulties when they are confronted with abstract mathematical concepts. Sfard called it a vicious circle: In order to reify a mathematical object, one must already have used it as a (reified) object in higher-level interiorization processes.

In 1995, she further elaborated on her claims of historical and psychological parallels (Sfard, 1995, p. 17). She collected instances from history of algebra and from classroom experiments to argue that it is inherently difficult to transition from an operational to a structural approach to mathematical thinking (Sfard, 1995, p. 22). Because of this, she concluded,

for those who teach, therefore, familiarity with the history of mathematics is not just optional; rather, it seems indispensable to make them alert to the deeply hidden difficulties concerned with new concepts. (Sfard, 1995, p. 34).

3.2 Unavoidable epistemological obstacles

Our next case stems from Guy Brousseau's (1997) well-known Theory of Didactical Situations, where history of mathematics comes into play when talking about *epistemological obstacles*—a notion Brousseau has borrowed from the French philosopher, Gaston Bachelard. Obstacles are often identified through students' difficulties with or errors related to certain concepts. According to Brousseau, an obstacle is to be considered as a piece of knowledge, wrong as it may be, and not as a lack of knowledge.

Brousseau distinguishes between obstacles of ontogenetic, didactical, or epistemological origin. The first ones are those due to limitations (e.g. neurophysiological ones) of a student at a given time. The second ones are those which depend on choices made within an educational system. The third kind is the ones of most interest to us here. According to Brousseau, these obstacles play a formative role and should not be avoided. They may, he wrote, be identified in the history of the concept itself. Brousseau's general hypothesis was that "certain of the students' difficulties can be grouped around obstacles attested to by history" (Brousseau, 1997, p. 96):

It is in the analysis of resistance and in the debate [...] one must look for elements which will allow the identification of obstacles for the students. In any case, it will never be enough to tack—to apply without modification—historical study onto didactical study. It is from this origin, too, that we must draw arguments in order to choose a genesis of a concept suitable for use in schools and to construct or 'invent' teaching situations that will provide this genesis.

An important element is that historical studies may not be applied directly in a didactical situation; a modification must take place. Although it is not direct ontogenesis-phylogenesis that Brousseau argues for, it still contains central elements of evolutionary argumentation. Here, it may be relevant to also notice Brousseau's concept of "*genèse fictive*" (fictive genesis), relat-

ed to the “true functioning of science,” in the context of his distinction between *savoir* (knowledge as a body of content) and *connaissance* (personal understanding) (Brousseau, 1986).

3.3 Historical-epistemological factors of students’ proof schemes

According to Guershon Harel and Larry Sowder (2007, p. 809), “A person’s (or a community’s) proof scheme consists of what constitutes ascertaining and persuading for that person (or community)”. They categorize a proof scheme into one of three main classes, each containing various subclasses. The first class is what they refer to as ‘external conviction proof’ schemes. These can appear as an authoritarian proof scheme, where something is believed to be true because an authority figure or textbook says so. Their second class is the ‘empirical proof’ schemes, which includes inductive proof schemes, where one is convinced by specific empirical examples or a “crucial” generic example. Their third class is deductive proof schemes from mathematics, based on deduction from a set of premises.

Upon describing their construct of proof schemes, Harel and Sowder turn to a discussion of mathematical and historical-epistemological factors in relation to proof schemes. They provide an analysis of the proof scheme constructed across three historical periods of mathematics: “Greek mathematics, post-Greek mathematics (approximately from the 16th to the 19th century), and modern mathematics” (p. 811). The reasoning that they provide is that proof schemes are used to validate assertions within specific contexts. Therefore, it is essential to consider the nature of these (historical) contexts when discussing proof schemes. Additionally, they argue, the motivation or intellectual need driving conceptual changes over time is important, hence their discussion focuses on three interconnected aspects of historical and epistemological development: (a) the context of proving, (b) the methods of proving (proof schemes), and (c) the motivation behind conceptual changes. An understanding of these elements can provide insights into key aspects of learning and teaching proof, they assert, and this also with reference to evolutionary arguments:

It is still an open question whether the development of a mathematical concept within an individual student or a community of students parallels the development of that concept in the history of mathematics, though cases of parallel developments have been documented (e.g., Sfard, 1995). If this is the case, one would expect that the path of development would vary from culture to culture. (p. 816)

Harel and Sowder are interested in what extent the history of mathematics may reveal the motivation for the shift from more empirical proof schemes (and even external proof schemes) to deductive proof schemes:

To what extent did the practice of mathematics in the 16th and 17th centuries reflect global epistemological positions that can be traced back to Aristotle's specifications for perfect science? These are important questions, if we are to draw a parallel between the individual's epistemology of mathematics and that of the community. (p. 818)

Although several of the references to evolutionary arguments that Harel and Sowder provide, are phrased as questions, e.g., as in the quotes above, their very endeavor of trying to find parallels between students' learning of mathematics and the coming into being of mathematical constructs reflects the authors' conception of the potential role of the history of mathematics in mathematics education.

4 Discussion

To be sure, the recapitulation thesis has been subject to criticism from the various domains of history, psychology and biology. Here we will only mention one aspect of the discussions within the field of history and didactics of mathematics, namely the underlying theory of history. As has also been pointed out by Schubring (2011) and by Furinghetti and Radford (2002), taken strictly, the recapitulation theory in the learning of mathematics leaves no room for a genuine history of mathematics. It presupposes a view of the history of mathematics as a subject that in essence only has a history in the sense that definitions and theorems are articulated and written down in historical contexts. What is missing is sensitivity to the contexts in which the development of mathematics took place by people who lived and acted under specific historical circumstances.

Harel and Sowder draw no conclusions regarding the evolutionary nature of their investigation. Although they do not argue for or against the ontogenesis-phylogenesis thesis, it is still saying that when they refer to the history of mathematics, it is this thesis that underlies and permeates their investigation. An investigation that at its heart is "onto-phylo" since it searches for parallels between the historical, mathematical communities' proof schemes and that of modern-day individuals. They are not blind, though, to the critiques (Harel & Sowder, 2007, p. 816):

Are there common elements or phases to different paths of development across cultures? Did the development of the concept of proof in, for example, China and India follow a similar path to that of the Western world or was there a leap in time from using perceptual proof schemes to modern axiomatic proof schemes?

Brousseau suggested that history serves a role in the teaching of mathematics due to the inevitability of epistemological obstacles, although it is not to serve in a direct manner. Rather it must be considered from a didactical point of view and adapted to the teaching and learning situation of today. In that sense, it is perhaps not too different from the way Freudenthal saw history as a “guide” to guided reinvention (see Jahnke et al., 2022). Brousseau (1997, p. 101) concluded that “historical arguments can intervene in choices of teaching under the surveillance of a Theory of Didactical Situations”.

Sfard wrote against what she called a structural way of teaching new concepts to students, which she found was in custom in high school teaching at that time. She is using historical observations to construct a model for the formation of mathematical concepts. She uses the model to understand students’ cognitive behavior, and this is where her second thesis comes into play:

what has been said (about the formation of concepts learned from history [in the 1992 paper with the function concept as case, p. 65]) applies also to cognitive development of mathematics students.

She wants to make a case for another way of introducing mathematical concepts to students. She uses her model to say that we should begin operationally—and she uses traits from the history of mathematics to show instances where former mathematicians, according to her, have worked in that way. If we look at the practice of teaching, she is advocating for history of mathematics as part of pre-service teacher education, so they can teach with history of mathematics, which, as we interpret her writings, may have similarities to the distinction of teaching *with* and *about* the nature of science in science teaching from Abd-El-Khalick’s (2013) framework (see also Kjeldsen, 2014).

Our three cases are influential theories in didactics of mathematics, and they are examples of theory building that are informed by history of mathematics—and for all three, it is in a form, which is related to evolutionary arguments. On the one hand, in the huge amount of research done in the HPM community over the past decades (see e.g., Chorlay et al., 2022), we find inspiration and opportunities for research where the history of mathematics has potential to inform theory building in general mathematics education without

reference to evolutionary arguments. On the other hand, there are other theories in didactics of mathematics where history is a coherent domain to look for in order to inform further theory building, and there are (at least a few) ‘voices’ in mathematics education research outside the HPM-group, pointing towards history of mathematics having such a role.

One such theory is Anna Sfard’s (2008) theory of commognition. She did not herself make the connection, but within that theory it is possible to make a theoretical argument for using history of mathematics to make students aware of meta-discursive rules in mathematics and make them explicit objects of students’ reflections (Kjeldsen & Blomhøj, 2012; Kjeldsen & Petersen, 2014). This has been further investigated in teaching practice within the impressive TRIUMPHS project (e.g., Barnett, 2022), see also Bernardes and Roque (2015).

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AN INTERTWINING OF HISTORY, EPISTEMOLOGY, MATHEMATICS AND DIDACTICS

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ABSTRACT.

This paper explores the intricate intertwining of history, epistemology, mathematics, and didactics through an examination of overlooked connections between the work of the Chinese philosopher Shao Yong and contemporary Western mathematical and scientific thought. Furthermore, it discusses the implications of Shao Yong's work for innovative a-didactic situations, as implemented with diverse learners in Chile

1 Introduction

We argue that History, Epistemology, Mathematics, and Didactics¹ can intertwine, metaphorically speaking, to form a four-stranded rope.

We intend to present and discuss a multifaceted example of this intertwining. This example involves activities carried out with a broad spectrum of learners at the University of Chile over the last ten years, ranging from undergraduate students to in-service primary school teachers.

To this end, we will first comment sequentially on some concrete instances of our four strands, while allowing for some revisiting. First, our historical strand begins with a string of random sevents starting with the Jesuit China Mission (1552-1715). Remarkable epistemological phenomena follow, involving sophisticated contemporary mathematical notions and the work of the neo-Confucianist philosopher Shao Yong (Birdwhistell, 1989; Kang, 2011; Ryan, 1966; Zhang, 2020; Zhang & Feng, 2008; Zhu, 2018). We highlight some relations and connections between the work of Shao Yong and later Western mathematics and science, relations which are often neglected in contemporary Western and Chinese scholarship (Li, 2019; Needham, 1956). We exemplify their implications for didactics, from an enactivist perspective, by discussing activities and *a-didactic situations* (Brousseau, 2002), suggested by

¹ The term "didactics" is used in the continental sense of "the art of teaching."

Shao Yong's work. We report on their implementation with a diverse group of learners. We conclude with a discussion, a caveat regarding the Whiggish approach to the history of mathematics, and a few questions.

2 *The historical strand*

This discussion commences with a significant historical and religious phenomenon: since its foundation, the Society of Jesus sought the global propagation of Christianity. China represented a pivotal mission field (Mariani, 2012). Their strategic approach involved first establishing linkages between Christian tenets, Western scientific advancements, and Chinese traditional culture, which they held in high regard. To this end, their missionaries undertook extensive study of the Chinese language and culture.

Among them was Joachim Bouvet (1656–1730), a prominent mathematician and astronomer, who arrived in China in 1685 and encountered the Yi Jing (Wilhelm, 1997), an intriguing ancient oracle based on combinatorial objects (the 64 hexagrams). He was particularly struck by the arrangement of these 64 hexagrams in an 8×8 tableau, attributed to the distinguished Chinese philosopher Shao Yong (1011–1077). Bouvet was a close friend to the German polymath Gottfried Wilhelm Leibniz (1646–1716), who, in 1689, had conceived the binary system, driven by his quest for a minimalist universal language capable of articulating all human culture. Leibniz conceptualized the fundamental elements of this system as the "Void" or "Nothingness" and the "Whole" or "One" (representing God), written as 0 and 1. Bouvet then immediately recognized the isomorphic correspondence between Shao Yong's 8×8 tableau of the hexagrams (when interpreted akin to a Western text) and the numbers 0 to 63 expressed in Leibniz's binary numeration.

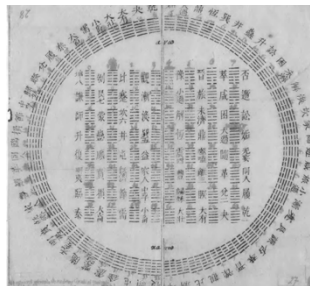


Figure 1. Shao Yong's original 8×8 tableau annotated by Leibniz.

Leibniz was enthused by Bouvet's discovery, as he had previously en-

countered difficulty in persuading his fellow mathematicians in Europe of the significance of his binary system. This finding allowed him to assert the profound importance of the binary system, evidenced by its independent development in China millennia earlier. Furthermore, this discovery aligned with his dream to bridge Chinese "natural theology" with Christianity. Bouvet was similarly captivated by the various diagrammatic renderings of the 64 hexagrams created by Shao Yong, whose intricate and metaphorical philosophical system integrated Confucian, Taoist, and Buddhist thought (Birdwhistell, 1989). Figure 2 illustrates two distinct synthetic representations of the 64 hexagrams: the rectangular and the circular Primeval Diagrams, *Xiantian Tu*² ((先天圖) (Marshall, 2015). The latter plays a pivotal role in Shao Yong's magnum opus, the *Huangji Jingshi* (皇極經世) or "Supreme Principles Governing the World" (Zhang & Feng, 2008). This complex and overarching work of cosmology and metaphysics endeavours to elucidate the entire universe and its unfolding through a comprehensive system of numerical cycles and correlations.

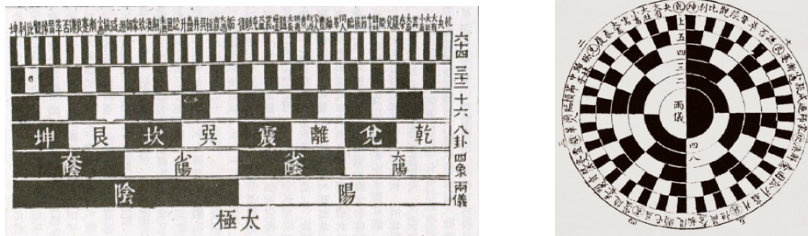


Figure 2. Rectangular and circular Xiantian Tu (Primeval Diagram)

Indulging in a degree of "whiggish overcoding" (Butterfield, 1931; Fried, 2011; Sinclair & de Freitas, 2019), a mathematician would readily identify a six-generation binary tree within the Xiantian Tu. Indeed, upon being presented with the rectangular *Xiantian* diagram, Bourbaki³ himself exclaimed: The

² *Xiantian* (先天) literally translates as "before sky" (rather than "before heaven"), a metaphor for "inborn", "primeval", the primordial state of wholeness and unity of the cosmos prior to the emergence of duality (the differentiation into Yin and Yang). *Tu* (圖) means diagram.

³ Or rather, one of his foremost avatars. See Corry (2004) for the history of Bourbaki.

binary tree! The historical-epistemological question arises then as to which are the different avatars of the binary tree throughout human history. Might the Xiantian Tu be the earliest one?

As previously noted in Soto-Andrade et al. (2022), another pertinent example emerges around 1600 in Dahomey, in the figure of Legba, the God of Chaos, also known as the God of the Crossroads. Legba's "forked path" is metaphorically represented by the iron fork in Figure 5 (Egash, 1999, p. 144). As explained by a Vodun divination priest (Ibid., p. 143), this fork embodies Legba because his responses to inquiries are binary—either "Yes" or "No"—with the specific path remaining unknown. Consequently, a series of such questions generates a sequence of bifurcations, forming a binary tree.

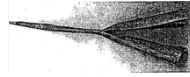


Figure 3. The forked path of Legba, the Vodun God of the Crossroads
Much later, in 2012, a four-step Xiantian diagram unexpectedly emerged in the context of “eternal inflation” within Western cosmology (Harlow et al., 2012). This diagram serves as a finite avatar of the "eternal symmetree," proposed by Stanford physicists as a *discrete combinatorial model* for a multiverse. See Figure 4.



Figure 4. (Left): A finite causal tree and (Right): the causal future of a in the infinite eternal “symmetree” with its boundary at infinity (Harlow et al. 2012).

3 The epistemological strand

From an epistemological perspective, it is indeed remarkable that both the ancient Chinese thinkers and Leibnitz shared a same insane and ambitious dream: to represent and manage the world's complexity through a language composed solely of binary elements (0s and 1s, or Yin and Yang, broken and unbroken lines). It is worth recalling that the Yi Jing oracle was supposed to give guidance and inspiration for any real-life problem using its hexagrams. This “insane dream” has, in fact, become a reality in contemporary times.

We also contend that a typical Western scientist, like Bouvet or Leibnitz,

does not feel the need for Xiantian like diagrams, once the 8x8 tableau is available. Nor do our students and in-service teachers, whom we have interviewed in this respect. Shao Yong, however, was profoundly attuned to change and *processes* within the Universe rather than static *structures*, an approach largely uncharacteristic of Western thought (with the possible exception of Heraclitus). As Birdwhistell (1989, p. 52) notes, he was “an alien, of another place and time”. Apparently, Shao Yong intended to present the 64 hexagrams as the outcome of a (binary) branching process, rather than merely a diverse collection of species of diagrams. Concurrently, he aimed to construct a synthetic (as opposed to analytic) view of the 64 hexagrams, one that could be comprehended at a glance. He thus emerges as a proto-scientific forerunner to Darwin, who conceptualized the diverse species of living beings on Earth as the outcome of a branching process, specifically, as the latest cohort of “the tree of life,” a metaphor whose profound inspirational power Darwin himself acknowledged (Gould, 2010). See Figure 5.

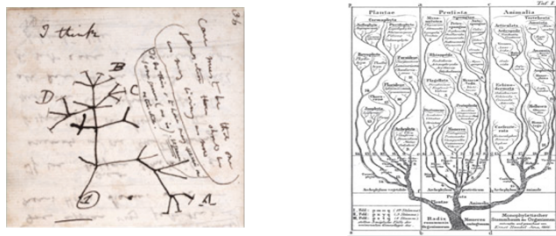


Figure 5. Sketch of a tree depicting phylogenetic relationships among species (Darwin, 1837) and a Tree of Life (Haeckel 1866).

The extent to which Shao Yong comprehended the binary system and the binary tree is a contentious issue in contemporary Chinese scholarship (Zhao, 2008). Some researchers posit that his construction of the 8x8 tableau relied on a recursive procedure rather than a direct binary expansion of numbers (Zhu, 2018). (Cf. Section 5.2).

4 The mathematical strand

Mathematically, the Xiantian Tu is a six-generation binary tree, hosting all whole numbers from 0 to 63. Since in mathematics “sky is the limit”, we would like to accommodate there *all* whole numbers. So, we just extend the six-generation tree into a binary tree with infinitely many generations, thereby yielding the “eternal symmetree” depicted in Figure 4(b). The rectangu-

lar *Xiantian Tu* further suggests a novel metaphor for integers: "Numbers are finite ascending paths within the infinite binary tree." Naturally, infinite ascending paths also exist, and any finite path can be canonically extended to an infinite one by consistently choosing the 'left' branch (representing *Yin*, or the *absence* of the corresponding power of 2). Notably, negative integers appear as infinite ascending paths that choose *right* from a certain node onwards. This suggests an amazing generalisation of the integers, as infinite ascending paths within the binary tree, or equivalently, as points on its *boundary at infinity* (Figure 4(b)). This offers a visual-geometric construction of Hensel's p -adic^[L]_{SEP} integers (Dickson, 1910), for $p = 2$, providing an intuitive alternative to his purely analytical algebraic approach, which cleverly manipulated infinite series of powers of 2 (Ibid.).

5 The didactical strand: Illustrative examples of related classroom activities

Our classroom activities, directly inspired by Shao Yong's work, are designed as *a-didactic situations*⁴ (Brousseau, 2020) and have been implemented since 2015 with the following cohorts of learners at the University of Chile:

- (a) First-year social sciences and humanities students enrolled in a one-semester mathematics course;
- (b) Prospective mathematicians, physicists, and mathematics-physics teachers undertaking a history and epistemology of mathematics course;
- (c) In-service primary school teachers participating in a professional development program designed to strengthening their mathematical formation.

All learners worked autonomously on pre-assigned or self-constructed problems within randomly formed small groups of three or four monitored by the instructor and teaching assistants. They uploaded a summary of their work, open questions and conjectures included, at the conclusion of each 90-minute session. It is important to observe that certain activities listed below preclude others, thereby naturally structuring a "didactical decision tree".

5.1 Discovery approach to the binary system and the hexagrams.

A class of students, without initially counting themselves, tried to figure out how to code their quantity, assuming they had no previous arithmetical knowledge. The idea emerged of *pairing*, where an "odd man out" may even-

⁴ In an *a-didactic situation* the teacher's didactic intent remains implicit: she steps aside to let the students explore and discover on their own.

tually remain. They continued the process by pairing the resulting pairs, potentially yielding a solitary pair, and so forth. They discussed then how to code the presence or absence of these residual individuals, pairs, quadruples, and so on. Through this exploration, they observed that a Chinese system instead of 1 and 0 would employ Yang and Yin for such encoding, thereby ultimately arriving at a representation of their quantity coded by hexagrams. We observed that students from group (b) did not have an embodied experience of the binary system, they had learned it in a purely formal way.

5.2 Archaeologic exploration of Yi Jing.

Learners received a copy of Shao Yong's 8×8 tableau, presented as an undated and unlocalised archeologic object, and tried to make sense of it. Several students from group (b) immediately saw it as an 8×8 matrix indexed by tri-grams! Others, from groups (a) and (c), slowly recognised the binary sequence of numbers by interpreting "unbroken" or "broken" in the n -th line as the *presence* or *absence* of 2^n , as Leibnitz did. Interestingly, we observed that some "somatically educated" learners (e.g., martial arts, tai chi or Feldenkrais practitioners, or dancers), despite being unfamiliar with the binary system, were able to intuitively grasp the binary sequence of the hexagrams. This understanding stemmed from feeling the "rhythm" established by the first four hexagrams: a pattern suggesting that after two forward steps, a return and "recapitulation" are necessary before a third step forward is taken. From this embodied insight, they could reconstruct the entire sequence or predict the subsequent hexagram when presented with any given one.

5.3 Synthetic view of the 8×8 square?

We asked the learners whether they were happy with the 8×8 tableau. Maybe they missed a more synthetic rendering, one that they could grasp in a glimpse? Learners from group (c), who are accustomed to working with concrete materials, spent approximately thirty minutes working in groups, cutting out paper printouts of the hexagrams and attempting to arrange them contiguously, side by side, in the sequence given by the 8×8 tableau (read as a Western book page). During this process, they observed that placing two broken lines side-by-side inadvertently created misleading unbroken lines. This observation led them to conceive the idea of coding broken lines as black

(Yin) and unbroken lines as white (Yang), within the hexagrams, thereby generating diagrams akin to the *Xiantian Tu*, and outwitting the other learners.

5.4 Grasping Xiantian in a glimpse

We asked our students to attempt to reconstruct the rectangular Xiantian diagram, after a brief viewing of approx. 200 milliseconds, and without any prior knowledge of its structure. Figure 6 shows the students' productions. The first diagram is due to a humanities student, who later went to Law school. The fourth one is due to a prospective math teacher. We did not observe a better performance among mathematically inclined students here. We have found that this reconstruction task is hard even for professional mathematicians.



Figure 6. Students' reconstruction of the rectangular *Xiantian* diagram

5.5 Numbers as ascending paths in a rooted binary tree

This was proposed to learners (b), who metaphorised numbers from 0 to 63 as ascending waks in the six-generation binary tree. They went further to the infinite binary tree (Needham, 1956, p. 276) and its *boundary*, subsequently re-discovering Hensel's notation of 2-adic numbers as series of powers of 2 (Dickson, 1910). Within this framework, they interpreted the far-right path (which consistently chooses the right branch) as representing the number -1.

5.6 A dice-based positional system (invented nowhere on earth)

Some islanders, somewhere on earth, invented dice, but did not develop the concept of 0. Remarkably, they manage quite well to efficiently record substantial quantities and perform arithmetic. Their primary school students, it seems, exhibit greater contentment than our own. The question then arises: how did they achieve this? This was proposed to learners (c), who, working in small random groups, were able to rediscover the islanders' dice-based positional system in less than an hour through hands-on manipulation of dice.

Figure 7 illustrates a sample of their work, showing the following “dice expansions” (beginning at the upper left):

$7=1\times6+1\times1$ =“one-one”, $30=4\times6+6\times$ =“four-six”, $42=6\times6+6\times1$ =“six-six”,
 $12=1\times6+6\times1$ =“one-six”, $36=5\times6+6\times1$ =“five-six” and
 $2024=1\times6^4+3\times6^3+2\times6^2+1\times6+2\times1$ = “one-three-two-one-two”.

Furthermore, the addition of 1 to 42 is demonstrated, by simply rearranging and flipping the dice to obtain “one-one-one”. The primary and friendly rule is that *a die showing a six can be transferred one column to the left by simultaneously flipping it to show a one*.



Figure 7. In-service teachers’ work on the dice-based positional system

A compelling epistemological question, often overlooked by cognitive archaeologists (Overmann, 2023), is why such a system apparently never emerged in any known terrestrial culture.

5 Discussion, caveats and open questions

We have merely scratched the surface of Shao Yong’s work and its mathematical, epistemological and didactical implications. Nevertheless, we have highlighted specific connections between his work and contemporary Western mathematics and science that seem to have been overlooked by contemporary scholarship. For instance, the *Xiantian Tu* can be interpreted as a diagram depicting the *Origin of the (64) Species* of hexagrams, conceived as the outcome of a *branching process*. Contemporary Chinese scholars, however, refer only in very general terms to the analogy between Shao Yong’s and Darwin’s evolutionary thinking. Also, Needham (1956, p. 276) specifically notes that “the [binary branching] process continues until the sixty-four hexagrams are formed, and could naturally go on *ad infinitum*” but does not connect this with Hensel’s analytic-algebraic construction of *p*-adic numbers. Moreover, the “symmetree” appeared in cosmology (one of Shao Yong’s main concerns) almost a century later. Students from group (b), interestingly, saw the transition from one *Xiantian* diagram to another as a change of coordinate system (e.g. rectangular to polar). Significantly, Shao Yong’s work emphasises syn-

thetic over analytical, as well as evolutionary over taxonomic, perspectives. Shao Yong's *Xiantian* diagrams inspired the design of various divergent a-didactic situations for our students, especially for those from group (b) following a distinct historical track (Fried, 2001). Their engagement in these history-track activities enhanced their mathematical understanding of subjects typically taught in an abstract and disconnected manner. Specifically, they constructed a novel metaphor for integers, as finite ascending paths in the infinite binary tree. Learners from the less mathematically inclined groups (a) and (c) reported having, for the first time, experientially understood the binary system, of which they previously held only vague recollections.

One limitation of our historical-pedagogical approach is that it requires ample time for students to explore and undertake the 'twist of mind' inherent in our divergent a-didactic activities and situations. One activity can take at least 60 minutes within one of our classes, which last 90 minutes. Especially first-year university students often come from a secondary school mathematics education that trained them primarily to merely execute prescribed tasks. Consequently, they often struggle to adapt to open-ended, problematic a-didactic situations where they are expected to explore, formulate questions, and configure their own problems.

Furthermore, our study needs an epistemic caveat. We wonder whether *Xiantian Tu* represents the first emergence of the binary tree in human history. To us, however, the binary tree is not a pre-existing mathematical object "standing out there", with various avatars throughout human history. Rather, we view it as a typical Western mathematical construction (hierarchical, per Deleuze & Guattari, 1980), which we automatically *project*, from a Whiggish perspective (Butterfield, 1931; Fried, 2001), onto the (rather *process* oriented) *Xiantian Tu*. Epistemological disorientation (Clark et al., 2018) might work as an antidote to this stance, which may be aptly metaphorised by Friedrich's (1817) 'Wanderer above the Sea of Fog' painting. In this picture the gentleman could represent Bourbaki⁵ retrospectively looking down on the history of mathematics from the vantage point of current structuralist mathematics. In

⁵ See Corry (2004) for the history of the mythical structuralist mathematician Bourbaki.

Soto-Andrade et al. (2022) we evoked the “Brownian mirage” of perceiving direction in our path backwards, which from the perspective of the past is indeed the path of a random walker (Fried, 2001).

Some open questions:

- Can concepts or methods that never emerged in human history nevertheless emerge in an enactivist classroom under minimal prompting? A cognitive archaeology fiction of sorts?
- In what degree is the history of mathematics a random walk?
- Is *epoché* (“bracketing”) genuinely possible in a contemporary classroom?
- To what extent can the enactivist historical-pedagogical approach exemplified in this paper be integrated into secondary or tertiary mathematics education as an antidote to the prevailing cognitive abuse (Watson, 2000) often observed in mathematics teaching for non-mathematically inclined learners?

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APPLIED MATHEMATICS FIRST, PURE SECOND

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Mathematics is taught on a Platonist schema. According to a Platonist philosophy, mathematics is about *abstract* world of numbers, sets and vector spaces, and the pure mathematical results about them are, down the track, “applied” to questions arising in physics, biology, finance and so on. Most students’ mathematical education stops before any serious engagement with those areas, so students are often trapped in a world of unmotivated abstractions. That has inevitable negative impact on interest and understanding, while employers complain that mathematics graduates know little about how to use their knowledge.

An Aristotelian philosophy of mathematics, on the other hand, sees mathematics as inherently about certain aspects of the real (non-abstract) world – structural and quantitative aspects such as symmetry, continuity and ratio. It sees mathematics as arising from study of those aspects of the world, and pure mathematics as being an intensive study of the harder topics that are found to apply in great generality (or whose motivation has been forgotten over generations).

The talk examines a few of the historical classics in real-world mathematics: Archimedes’ derivation of the law of the lever from symmetry, Euler’s work on the Bridges of Königsberg, and the exponential model of population growth. In each case it is explained how a naturally-arising problem about a structural aspect of the real world is expressed mathematically so as to reveal the necessities in the world – “Why it must be so”. These cases also show that some of the deepest themes of mathematics, such as discrete versus continuous and local versus global, span the pure/applied division.

The talk concludes with a brief look at one contemporary success in incorporating a direct “applied” perspective in teaching, the COMAP Mathematical Contest in Modeling.

LEARNING PROBABILITIES BY PROBLEMS AND PARADOXES: THE ORGANIZATION OF JOSEPH BERTRAND'S TEXTBOOK (1889)

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Joseph Bertrand's textbook *Calcul des probabilités* had been published in 1889. This textbook is the result of Bertrand's teaching at the École polytechnique and the Collège de France, and it strongly influenced French mathematicians such as Gaston Darboux, Henri Poincaré and Émile Borel. Thereafter, until the 1940s, it was often referred to.

This textbook is actually a kind of textbook for learning how to learn probability, and today teachers can find in it a rich material to build their approach, and in particular to get rid of possible errors. Moreover, it proposes to learn probabilities by problem solving. Indeed, the author proceeded from numerous "concrete" problems, old and new, in which he illustrated the basic notions of probability calculation, and especially the difficulties and paradoxes to which their misuse can lead.

After a slightly historical reflection on the "laws of chance", in a well-chosen order he addressed the question of equipartition of chances and the composition of probabilities, then the calculation of expectations, the law of large numbers and the ruin of gamblers, the question of the probability of causes, and the law of errors of observation and statistics.

We only illustrate our point with some problems concerning the two most basic questions of equipartition of chances and the ways of combining probabilities between them. In particular, we will examine the famous "Bertrand's paradox".

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HISTORY IN TEACHERS' MATHEMATICS EDUCATION

Problematising concepts, views on mathematics, its learning and teaching

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ABSTRACT

In recent decades, we have seen the integration of history into students' and teachers' mathematics education flourish, yielding numerous teaching proposals and materials. In this paper, I explore some examples of how I have promoted this integration within teacher education, grounded in the theory of commognition. Through the notions of commognitive conflict and metadiscursive rules, mathematical practices from the past are explored to problematise concepts, views on mathematics (e.g., mathematics is universal, immutable, only brilliant minds can engage in it), and views on how it is learned and taught. Within this framework, the use of primary sources is deemed essential. To conclude, I share the findings of a recent investigation into mathematics textbooks and highlight some challenges we must address to further promote this integration in Brazil.

1 *Setting the stage*

The field of the history of mathematics in mathematics education has made significant progress in recent decades, particularly in proposing various ways to integrate history into mathematics teaching at all levels, as well as in developing diverse instructional materials. Many examples can be found in the proceedings of the meetings and working groups organized by the International Study Group on the relations between History and Pedagogy of Mathematics (HPM group),¹ as well as in many journals (e.g., *Convergence*).² In Brazil, such examples can be found in the proceedings of the national meetings organized by the Brazilian Society for the History of Mathematics (SBHMat),³ which have been held biennially in odd-numbered years. Additionally,

¹Information about all the meetings organized by the HPM group can be found on the website <https://hpm.sites.uu.nl/>.

²Convergence is an open-access online journal about the history of mathematics and its use in teaching, promoted by the Mathematical Association of America.

³<https://www.sbhmat.org/>

SBHMat publishes a journal dedicated specifically to the history of mathematics (HM) for teachers—*Revista de História da Matemática para Professores (RHMP)*.⁴

Given the breadth of discussions on classroom experiments and teaching materials involving HM, I have made choices in organizing my plenary lecture, which inevitably mean leaving many aspects aside. I chose to focus on mathematics teacher education, drawing on my experiences at this level since I began studying and researching within the domain of the history of mathematics in mathematics education. I then share how I have been approaching this integration, along with the considerations that guide me when designing teaching materials.

I have been exploring how HM could play a significant role in problematising concepts and views about mathematics (e.g., mathematics is universal, immutable, and produced exclusively by geniuses) by using past mathematical practices to provide a counterpoint to present-day mathematics (Bernardes, 2016; Moustapha-Corrêa et al., 2021; Moustapha-Corrêa et al., 2022).

In the next section, I briefly present the theoretical framework that underpins my taking to integrate HM and mathematics education. Following this, I present three examples along with the teaching materials developed in each situation. After the examples, I share some findings from research conducted by our research group in Rio de Janeiro, which examined how HM is presented in Brazilian mathematics textbooks. Finally, I conclude by discussing some challenges.

2 Framing the discussion

The theoretical foundations that support my approaches are organized in this section into three key parts. The first part brings some concepts of *commognition theory*, which frames mathematics as a *discourse*. The second part addresses how *commognition* supports the integration of history and teaching. The third part briefly discusses the historical interpretation described by Grattan-Guinness (2004) as “history”.

⁴<https://rhmp.com.br/index.php/RHMP/>

2.1 Some words about *commognition*

Commognition is a rich theory for investigating learning scenarios in mathematics education, particularly, for supporting the integration of the history and the teaching of mathematics. The term commognition is derived from the combination of the words *communication* and *cognition*. Sfard (2008) has coined the term to highlight the idea that learning occurs through communication with other participants in the discourse (interpersonal), as well as through self-communication, which is essentially thinking (intrapersonal).

The theory is quite broad, so I focus on the concepts that I have been using to design activities and analyse data in my research, such as *metarules* and *commognitive conflicts*. Metarules are the rules that shape the patterns in the production and legitimization of discursive actions (or *routines*) (Sfard, 2008, p. 201). For example: Which actions are considered legitimate ways to validate a mathematical argument? How do we define a mathematical object? How do we validate a proof or a definition? Commognitive conflict, on the other hand, refers to an encounter with a new discourse—a situation in which different participants are acting according to different metarules (Sfard, 2008, p. 256). In such situations, the involved discourses are considered *incommensurable*, since “they do not share criteria for deciding whether a given narrative should be endorsed” (Sfard, 2008, p. 257). Learning within this framework occurs through changes in the discourse—that is, changes in the learners' metarules and/or the narratives they endorse. These changes in metarules may be triggered by commognitive conflicts, and the learning that arises from such changes is referred to as *meta-level learning*.

Mathematics is viewed as a type of communication, with its own features and rules. What differentiates it from other discourses—such as those of history, biology, and others—are the keywords (e.g. function, matrix), the visual mediators used (e.g. graphs, tables), the routines (sets of metarules), and the narratives endorsed (Sfard, 2021, p. 575).

2.2 Integrating History and Teaching through Commognition

Kjeldsen and collaborators (e.g., Kjeldsen & Blomhøj, 2012; Kjeldsen & Petersen, 2014) pointed out that historical discourses on mathematics help make students' metarules explicit: “history can be used in mathematics education to reveal metadiscursive rules and make them explicit objects of reflection and—ultimately—to provoke commognitive conflicts” (Kjeldsen &

Blomhøj, 2012, p. 330). To clarify, metarules are usually implicit in the discourse, which makes them difficult to perceive. Therefore, the use of primary and secondary historical sources can facilitate this process, as they present and discuss past mathematical practices.

Aligned with the ideas of Kjeldsen and Blomhøj (2012), my interest has been in using HM to promote commognitive conflicts, which then foster reflection on metarules. By being exposed to other discourses and realising the existence of different metarules, mathematics teachers may become aware of their own metarules. The idea is that the commognitive conflicts may challenge teachers' comfort zones, since the discourses of past mathematical practices are rather different from those with which teachers are usually acquainted. Consequently, this confrontation allows them to question concepts, views of mathematics, and teaching approaches. Through this process, we can problematise concepts, views, and other aspects of mathematics and its teaching.

However, it is important that mathematical practices be located within their own historical, social, and cultural context. If history is used with an anachronistic approach, that is, if practices are interpreted through the current lenses of mathematics, commognitive conflicts may not emerge—as pointed out by Kjeldsen and Blomhøj (2012). In other words, the differences in practices need to be noticed. Commognitive conflicts arise precisely from the differences between practices. It is this contrast that leads to problematisation. Thus, in my taking to integrate HM into mathematics education, commognition influences the entire process, from the selection of historical sources to the design of tasks and data analysis.

2.3 Historiographical perspectives

Regarding the historiographical discussions, one of the references I have drawn upon is Grattan-Guinness' (2004) distinction between two ways of interpreting the mathematics of the past: *history* and *heritage*. The heritage perspective focuses on tracing the influence of mathematical ideas on subsequent developments, often interpreting past mathematics through a modern lens. This approach tends to view the past in terms of its contribution to present-day mathematics, suggesting a sense of continuity and a potentially deterministic perspective on the development of mathematical ideas. In contrast, the history perspective seeks to explore how and why mathematical developments

occurred (or not), offering a more nuanced and contextualized view of the past.

Although Grattan-Guinness acknowledges both interpretations as legitimate, the history perspective seems more appropriate for contrasting past practices with present ones, as it aims to paint a more faithful picture of both what happened and what did not happen in the past.

3 Integrating History into Teaching: Three Examples

In this section, I share three experiences in which I conducted or collaborated on research involving the elaboration of teaching materials.

3.1 Exploring a memoir on matrices

In this example, I share some ideas about teaching material I developed during my doctoral research (Bernardes, 2016), in which the concept of a matrix was problematised, along with how this mathematical object was established. On this occasion, I conducted a mini-course aimed at undergraduates taking their first linear algebra course, and the teaching material was used as part of the mini-course.

The teaching material focused on a memoir written by the British mathematician Arthur Cayley, published in the *Philosophical Transactions of the Royal Society of London* in 1858 (Forsyth, 1895). In the memoir, Cayley introduced a theory of matrices. He defined a matrix as a suitable notation for representing a linear system and from this definition he proceeded to define the operations on matrices. The memoir is an interesting historical source for realising the origins of matrix operations, including matrix multiplication, which is often intriguing for the students.

A key aspect in selecting Cayley's memoir as a primary source was the contrast between his approach to matrices and the way they are typically introduced in Brazilian Linear Algebra courses. Additionally, the memoir provides context for matrix multiplication. In the mid-19th century, matrices were emerging as a mathematical object, and their association with linear systems as a suitable notation was used to define matrix operations (Breckenmacher, 2006; Bernardes & Roque, 2016).

In Brazil, Linear Algebra courses typically begin with matrices, then move on to determinants, followed by linear systems, vector spaces, linear trans-

formations, and so on. As a result, both practices—Cayley’s practice and those in Brazilian courses—are shaped by different metarules. To explore this, I engaged in an interpretative process through a commognitive lens to identify the metarules and routines underlying Cayley’s practice. In doing so, I relied on Frédéric Brechenmacher’s (2006) work as a secondary source on the history of matrices, as well as my own analysis of Cayley’s memoir. Brechenmacher’s work provided valuable insights into how Cayley’s approach differed from modern practices, helping to highlight the distinct metarules at play during his time.

With those ideas in mind, I prepared material featuring a presentation of the mathematician Arthur Cayley, along with a translation of the first pages of his memoir from English to Portuguese, accompanied by some activities. At the end of the material, I proposed mathematical exercises designed to help participants understand the mathematics presented in the source, as well as more theoretical, open-ended questions intended to encourage reflection on Cayley’s practice with matrices. My aim with the broader questions was to stimulate reflections on the metarules.

In this context, the experience provided an opportunity for participants to perceive and reflect on their own metarules when using matrices, as well as the metarules underlying Cayley’s practice (Bernardes & Roque, 2018). By engaging with these reflections, they were able to challenge the common, non-problematized view of mathematics, which often presents it as consisting of immutable, universal truths. Instead, they encountered a more problematized perspective, seeing mathematics as knowledge that is socially produced and historically situated (Giraldo & Roque, 2021). In particular, the participants reflected on what a matrix is and how mathematics develops. For instance, they learned that matrices emerged after determinants and linear systems, which differs from the order in which they had previously learned them. Additionally, the participants were able to learn that matrices have a history, demonstrating that mathematical concepts can change and, therefore, are not fixed or immutable entities.

3.2 The ZIZO approach

I outline the main concepts of ZIZO approach here, while a more comprehensive explanation can be found in Moustapha-Corrêa, Bernardes, and Giraldo (2022). The ZIZO approach, developed for mathematics teacher education, is

built around specific *history-focused tasks* that engage teachers with historical sources and collective discussions about these sources and their contexts. These tasks fall into two categories: i) *Immersion task*, where teachers explore historical excerpts, as similar to the original as possible, and ii) *Overview task*, where teachers study secondary historical sources, usually textbooks or videos, to situate the historical practices within their social and cultural context.

An important aspect is that the history-focused tasks are designed to provoke commognitive conflicts. As the original sources often present discourses shaped by metarules unfamiliar to teachers, engaging with these sources may challenge their comfort zones. This confrontation with incommensurable discourses enables teachers to reflect on their practices and assumptions. In addition to the tasks, the ZIZO approach is also structured into three interconnected moments: i) *Immersion in primary sources*, when the immersion tasks are proposed; ii) *Historical overview*, which starts with the overview task and continues with a plenary discussion to place the practice or the primary source in its historical, social, and cultural context; and iii) *Unveiling the source*, which involves returning to the primary source and unveiling it after the overview task.

Due to these movements — first, an immersion, zooming in; then an overview moment, zooming out; and finally, an unveiling moment, zooming in again—this approach was named ZIZO. I will illustrate the ZIZO approach through an application in a master's degree course, focusing on Euclid's process of quadrature and the use of Pythagoras' Theorem in Books 1 and 2 of *The Elements* (Moustapha-Corrêa, Bernardes & Giraldo, 2022).

In Brazil, the teaching of geometry in basic education primarily emphasises formula application, so the goal was to contrast this approach with Euclid's method, which does not rely on numerical measurements.

The immersion moment began with analysing Proposition II.14 (PII.14), which involves constructing a square equal to a given rectilinear figure (polygon). The participants then analysed Proposition I.47 (PI.47), a version of the Pythagoras' Theorem, used to add areas of two squares through geometric constructions. The participants were challenged to identify the use of Pythagoras' Theorem in the context of polygon quadrature in PII.14. They were asked to identify the fundamental steps of both propositions and raise questions during the immersion task.

For the overview task, participants read a chapter from a Brazilian HM book (Roque, 2012) and watched some Brazilian video lectures. During the plenary session (historical overview), the structure of *The Elements*, its axiomatic method, and the role of geometric constructions with ruler and compass were discussed. In the unveiling moment, the teachers revisited the propositions with another perspective.

An unexpected issue arose during this process. In the Brazilian translation of *The Elements* (Bicudo, 2009), the term "rectilineal" was used to refer to a polygon, but some participants interpreted it as referring to a straight line. This misunderstanding caused confusion and made it difficult for several participants to follow the proof of PII.14. Some participants admitted that they could only understand the proof by relying on the metric relations in a right triangle—essentially interpreting it through their own metarules. This situation highlights that the commognitive conflicts we plan are not always the ones that emerge. Unexpected conflicts may arise, and this shows the challenges in recognizing the metarules underlying historical practices. Nevertheless, I believe it is crucial to incorporate these types of tasks in teacher education, as they allow us to perceive the diversity with which mathematics is produced. In other words, having contact with different ways of doing mathematics—through original sources—helps to problematise the universality of mathematics.

3.3 Workshops for teachers

The initiative presented here is part of the work developed by the CHEMat (Coletivo de História no Ensino de Matemática) research group,⁵ to which I belong. The group consists of researchers, mathematics teachers, and undergraduates from different institutions across the state of Rio de Janeiro in Brazil. In this section, I describe one of the workshops we designed for mathematics teachers, in which we explore the Pythagorean School's practice with figurate numbers.

Our goal is to contrast this historical practice with the modern application of the Pythagoras' Theorem, a result widely known and studied in Brazilian Basic Education. It is usually introduced with a geometric background, becoming an algebraic result that consists of finding the length of one side of a

⁵ All CHEMat publications are accessible at <https://chemat.uniriotec.br/>.

right triangle, given the lengths of the other two sides. Our motivations for this workshop are twofold: to share our perspective on integrating history into mathematics teaching and to address how history is currently presented in mathematics textbooks. Below is an excerpt from a textbook used in the first year of *Ensino Médio* (High School) in Brazil:

Evidence suggests that Pythagoras was born around 572 BCE on the island of Samos. In Croton, in southern Italy, he founded the Pythagorean school, a centre for the study of philosophy, mathematics, and natural sciences, among other activities. Although the Babylonians knew about the Pythagorean Theorem over a thousand years earlier, the theorem is attributed to Pythagoras because, as it appears, he was the first to provide a general proof of the theorem. (Balestri, 2016, p. 246; translated from the original Brazilian book)

We are concerned that such textbook narratives may lead students and teachers to believe that the Babylonians and the Pythagoreans solved the same problems as we do today. Such narratives reflect the heritage perspective (Grattan-Guinness, 2004), as modern concepts are retrojected onto the past practices. Therefore, one of the objectives of this workshop is to critically examine such anachronisms and to deconstruct myths surrounding figures like Pythagoras, Euclid, and ancient Greek mathematics. Often, historical narratives in textbooks portray ancient Greek scholars as though they were almost contemporaries, with one passing the baton to the next, which oversimplifies and distorts the development of mathematical ideas.

We start by discussing some historical insertions selected from mathematical textbooks, and then we propose an immersion into the practice of the Pythagorean School, even though we do not use original sources directly with the participants. The teaching material consists of a handout with activities designed to guide them in exploring the Pythagorean School practice with figurate numbers (triangular and square numbers) as described by Roque (2012), whose discussion builds upon Burkert's (1972) seminal study. To concretely represent these numbers, we employ plastic bottle caps as physical manipulatives.

The goal is to contrast the practice of determining Pythagorean triples with the Pythagoras' Theorem as we know it today. Specifically, we want participants to come to how some triples can be determined through the geometric rearrangement of two square numbers to form a third square (Figure 1).

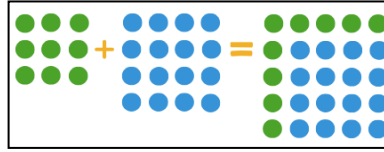


Figure 1 - The square number 25 formed by combining the square numbers 9 and 16 (Roque, 2012).

After exploring some examples of Pythagorean triples, we expect participants to realise that there is no right triangle in this practice of deducing relationships from numbers represented by dots (Roque, 2012). Therefore, we discuss the problem of asserting that the Pythagorean School knew the theorem as we know today, much less that Pythagoras was the first to prove it. In addition, we also highlight the fact that the Pythagorean School left no written records, which complicates making definitive historical claims.

As in previous examples, Sfard's discursive framework inspired our selection of a practice governed by metarules distinct from contemporary approaches to the theorem. Here, we explored an arithmetic-based practice—one that contrasts sharply with the algebraic treatment predominant in the mathematics textbooks. We faced some unique challenges due to the absence of extant original sources from the Pythagorean School. Nonetheless, this example is still interesting since the circumstances challenged us to adapt our methods and our previous examples.

We have not collected or analysed workshop data so far, but the commognitive framework could offer valuable insights for examining how participants might negotiate mathematical meanings. For instance, it could reveal whether (and how) they perceived tensions between numerical patterns and geometric expectations—particularly regarding the right triangle's absence in Pythagorean school practice.

4. The history of mathematics in Brazilian Textbooks

Having discussed three examples, I now share another CHEMat work, focusing on how HM has been addressed in Brazilian mathematics textbooks (Teixeira, 2023; Amadeo et al., in press).

CHEMat group embarked on an investigation with the guiding question in mind: Which HM reaches teachers and students in Brazilian schools? To answer this question, we analysed 36 mathematics textbooks (2015-2018 edi-

tions) designed for the Middle and High School levels (ages 10-17), in which we identified 653 insertions of HM.

Our research group documented extensive details about the historical insertions, including historical periods, civilizations mentioned in them, mathematical content, iconography etc. Data collection employed a 60-question form. A more comprehensive explanation of the methods and analysis can be found in Amadeo et al. (in press).

I will present one key research finding from an analysis of 24 High School textbooks, in which were identified 434 historical insertions. Each insertion was classified according to three sets of categories. One of them employs the three themes proposed by Fried (2014) for grouping initiatives that connect the history of mathematics with mathematics education. The *motivational* theme encompasses uses of history aimed at making mathematics more engaging by adding flavour to lessons. This includes anecdotes or “stories” that help students perceive mathematicians as human beings prone to errors, just like anyone else.

The *curricular* theme includes initiatives that use history to teach standard curriculum topics like functions, equations, and trigonometry. Here, concepts are approached from a present perspective, with historical treatments adapted to align with contemporary approaches. Consequently, there's a tendency to project concepts as we know them today onto past practices, resulting in a historical distortion. Nevertheless, the use of HM in this theme sounds more mathematically significant for learning than the motivational one. Finally, the *cultural* theme stems from the understanding that mathematics and culture are inseparable. Mathematics is conceived as an “*essentially* human activity”—“an expression of culture” (Fried, 2014, p. 688, italics in the original). Thus, history is viewed as an integral part of mathematics itself.

Fried’s themes were adapted using some indicators to classify the historical insertions, considering the context of textbooks. The motivational theme includes biographies, curiosities, and anecdotes (often in small boxes) that could be removed without compromising the mathematical content. The curricular theme addresses historical problems, original demonstrations, or the genesis of concepts, often with a touch of Whiggism. The cultural theme includes the insertions that explore historical sources, mentioning their purposes and other aspects; as well as insertions that bring elements from the sociocultural context of mathematical practices, including

the aims and forms of knowledge production within specific communities. These non-exclusive indicators served as guidelines for analysis, acknowledging that many inserts may combine multiple aspects.

The results indicated that the motivational theme was the most prevalent, accounting for 71% of the insertions. The curricular theme comprised 24% of the insertions, while the cultural theme represented only 3% (Amadeo et al., in press). This disparity between the motivational theme and the others can be explained by the relative ease of creating insertions that simply include historical curiosities or anecdotes about mathematicians. Most of insertions classified as motivational were short biographies, occasionally listing only the names and birth/death years of historical figures. Thus, they tend to be shorter and therefore occupy less textbook space.

While such insertions—where the history of mathematics (HM) serves primarily to add flavour to instruction—may have some value, their overwhelming predominance becomes problematic, as it is ultimately restrictive. Moreover, in my view, this approach contributes minimally (if any) to the learning of the mathematical concepts.

The study did not include interviews with teachers. It remains unclear if and how the historical information presented in textbooks is actually used by teachers in their classrooms. Therefore, my reflections are based solely on the research findings from the textbook analysis.

Building on the panorama revealed by this study, our research group has developed proposals for more meaningful integration of history with mathematics education, aligned with the *history* interpretation (Grattan-Guinness, 2004) and with ongoing efforts to approximate Fried's cultural theme (Fried, 2014). This remains a challenging endeavour, particularly due to the constraints imposed by an overloaded curriculum that teachers must deal in Brazilian Basic Education.

The workshop presented in Section 3.3 represents one such initiative, designed for teacher education. The aim is to develop a critical perspective on how mathematics history is presented in textbooks, while empowering teachers to move beyond the historical information contained in the insertions.

5 Final reflections and bridging the gap

Throughout this plenary, I presented three examples of integrating history into mathematics education at the level of mathematics teacher education,

sharing insights into the design of the teaching materials used. In two of these cases, original sources were directly incorporated to provoke commognitive conflicts, stimulate reflections on metarules, and lead to problematisations of concepts, views on mathematics, and its learning and teaching. In the third case, the teaching material explored the practice of the Pythagorean School by organizing numbers into spatial configurations using physical manipulatives, thereby creating opportunities to problematize Pythagoras' Theorem.

In connection with the broader discussion on teaching materials that integrate history into mathematics education, I shared some findings from a study that reveal a concerning scenario regarding how HM is presented in Brazilian textbooks. Despite textbook authors' efforts to include a considerable amount of historical information (which is commendable), in most cases, historical elements are included merely to motivate the teaching of mathematical content. Besides this, the authors still rely on outdated historiographical references (Amadeo et al., in press). This scenario does not reflect the excellence of Brazilian research in the field of the history of mathematics in mathematics education. A significant gap remains between cutting-edge research in this area and the historical content presented in textbooks.

Regarding mathematics teacher education, it is crucial to develop a critical perspective on the historical information presented in mathematics textbooks. Our research group CHEMat has been engaged in collaborative efforts to design workshops aimed at promoting a more meaningful integration between history and mathematics education. This has been pursued by contrasting mathematical practices from the past with those of the present, a perspective grounded in commognition. Through this contrast, it becomes possible to problematize mathematical concepts, as well as views on mathematics, its learning, and its teaching.

I hope that the examples presented, and the discussion have sparked ideas and provided inspiration for future work in the field.

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CALCULUS WITH HISTORY-BASED MANIPULATIVES: PROBLEMATIZING THE TRACTRIX

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ABSTRACT

We share our attempts in reviving historical aspects of calculus through the exploration of a geometric-mechanical artifact. This allows for the experience of significant steps through hands-on activities, from Leibniz's early conception of calculus (trac-tional motion, 17th century) to the Enlightenment (demonstration machines, 18th century) and the realization of practical mathematics instruments ("integragraphs," 19th-20th century). Although the adopted history-based artifact has already been introduced in the literature, its effective use in education remains unclear. To explore this, we will focus on the first construction of a curve by the inverse-tangent problem, the tractrix. Despite its significant historical value, this task left participants quite unsatisfied during some previous workshops. Therefore, the need to define useful and engaging problems related to this construction has emerged. We will reflect on how we adopted such an approach in teacher training.

1 Introduction and background

In this work, we provide an example of how to integrate history and historical instruments into a workshop on infinitesimal calculus, coherently with Italian research on workshop activities using mathematical mechanical tools, as described in the literature as *mathematical machines* (Maschietto & Bartolini Bussi, 2011). In such a context, manipulatives are named machines and not instruments to suggest something that takes an input, performs a process, and produces an output. We adopted a new device that is deeply rooted in historical ideas related to the mechanical implementation of the solution to inverse tangent problems, which analytically corresponds to the resolution of ODEs.

Unlike contemporary teaching approaches, our device aims to facilitate the introduction of various calculus concepts through geometric constructions without limits, reminiscent of Leibniz's initial approach to calculus. Although

geometry is considered today mainly a visualization tool, from ancient Greece to the 17th century, geometric constructions were essential for justifying the existence of mathematical objects. From this perspective, it was necessary to define the primitive class of acceptable tools in geometric constructions: this is the problem of “exactness.” In this context, machines (also beyond the ruler and compass) offer an approach in which the concept of a curve is treated in a way that is new to modern students: curves are seen not as a set of points, but as a continuous geometric trace. Such a notion of exactness was crucial in Descartes to justify the introduction of algebra in geometry (cf. Bos, 2001). By extending Descartes’ definition of acceptable curves, Leibniz validated the inclusion of transcendental curves through machines, and a key content was the possibility of constructing curves given the property of their tangent, i.e., as the solution of inverse tangent problems (see Blåsjö, 2017).

Inverse tangent problems are compelling for several reasons. Beyond their historical significance in the development of calculus (also considering problems in physics), these problems maintain a strong connection with material implementations, such as the creation of scientific instruments for demonstration, education, and practical application. A rich overview concerning both theoretical content and material devices is visible in Tournès (2009).

2 A history-based device

Our device collects the legacy of instruments for the inverse tangent problem, summarizing them with a simple design. Although the main ideas behind the device have been evident since its first version (cf. Maschietto & Milici, 2024), improvements have been proposed based on user feedback (e.g., see Milici et al., in press). The aim is to make users focus on the mathematical role of the components, minimizing technical complications. The prototype of such an instrument has been built using typical FabLab tools, including laser cutting and 3D printing, along with some additional hardware components, such as screws, ball bearings, and rubber O-rings. Let us name the components of the device according to the numbering in Figure 1. With these components, the device can be assembled in different configurations (Section 3).

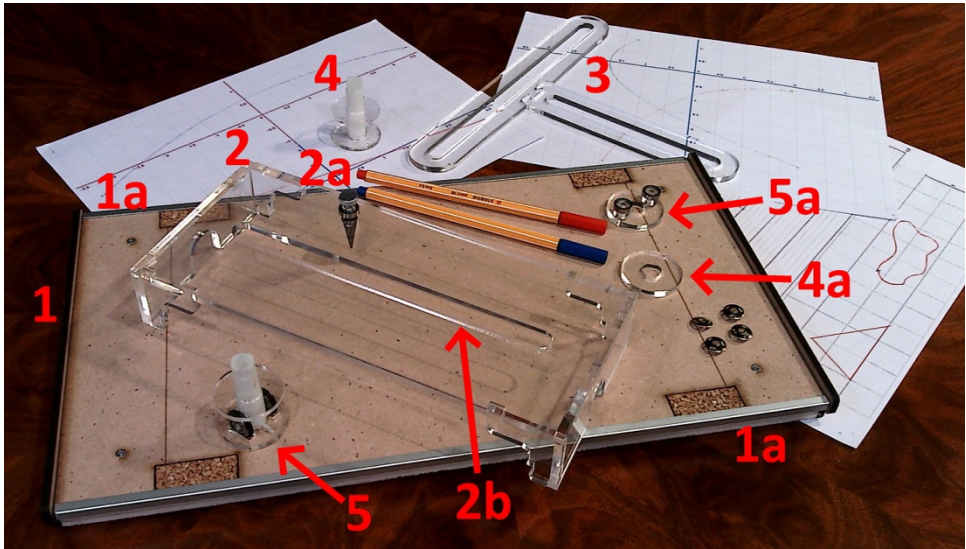


Figure 1. The pieces of the device.

- (1) The “base.” It is a wooden base with four cork rectangles to fix the corners of a paper sheet with thumbtacks. It has two “rails” (1a) along its long borders.
- (2) The “plate.” It is a rectangular plate of transparent plexiglass that can slide along the rails (1a). The “peg” (2a) is fixed to the plate. On its top, there is a ball bearing to pass through one of the rods (3), and on the bottom, there is a spike. The “slot” (2b) is carved in the middle of the plate.
- (3) “Rods.” They are linear guides to be put on the plate. The rods can be joined to form a T in which a rod is the perpendicular bisector of the other.
- (4) The “simple pointer.” It is a piece that can slide inside the slot (2b) and a rod (3). On its top, one can put the “head” (4a), which helps the user move the piece. It features a hole that can be used as a viewfinder to move the pointer along a curve and as a marker to leave a trace.
- (5) The “wheeled pointer.” It is a simple pointer (4) with two parallel wheels at its bottom that can rotate at different speeds. It has a “head” (5a): unlike (4a), it has two ball bearings to constrain the direction of the rod (3) through which it passes, ensuring it is parallel or perpendicular to the direction of the wheels (the “head” can be right-angle rotated).

3 Possible activities

Coherently with our previous research, we adopt the theoretical framework of the Theory of Semiotic Potential (Bartolini Bussi & Mariotti, 2008) to set hands-on activities on the history and epistemology of inverse tangent problems. The fundamental idea of the Theory of Semiotic Mediation is that an instructor uses a specific artifact as a tool of semiotic mediation for constructing mathematical meanings. Activities are organized within didactical cycles, including group activities with the artifact, individual activities, and collective mathematical discussions. In this framework, a crucial notion is the semiotic potential of an artifact; its analysis is fundamental for conceiving tasks and is at the basis of interaction with participants. The semiotic potential of our device got deepened in Maschietto & Milici (2024, §6). To provide an idea, we introduce some possible activities.

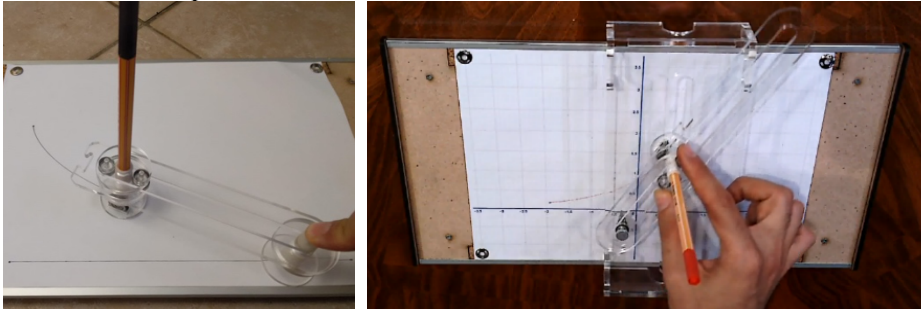


Figure 2. The device is assembled to trace Perrault's curve (left) and an exponential curve (right). Cf. <https://www.youtube.com/watch?v=LMLt90R8zHA> 0:14-0:50 and <https://www.youtube.com/watch?v=kqtU9GpcN78> 0:00-0:15.

We can introduce the problem of the existence of curves in the 17th century, specifically Perrault's construction of the tractrix (cf. Section 4.1; for more references about the various activities, see Crippa & Milici, 2023, §2, or Maschietto & Milici, 2024, §2). By introducing the device assembled as shown on the left side of Figure 2, we can propose tracing the curve. The aim of this activity is to link the mechanical properties of the device and the geometrical properties of the tractrix, thus reinterpreting the idea of tangent (in this case, the tangent exists before the curve).

Machines tracing an exponential curve by solving the inverse tangent problem implement the geometrical property of having a constant subtangent, a property that is generally neglected by modern students. The request for the

second activity is to invent a machine for the exponential function (the reconstruction of an exponential machine starting from historical sources in a math education workshop is illustrated in Plantevin & Milici, 2022). The assembled machine is visible on the right of Figure 2. As a historical counterpart, we can recall the related historical machines of the 18th century (cf. Poleni, 1729). To foster this construction, we propose focusing on the geometrical properties of the tangent to the exponential using GeoGebra. Then, we suggest reflecting on the components of the device (e.g., which component can we use to guide the tangent?) to trace the sought curve.

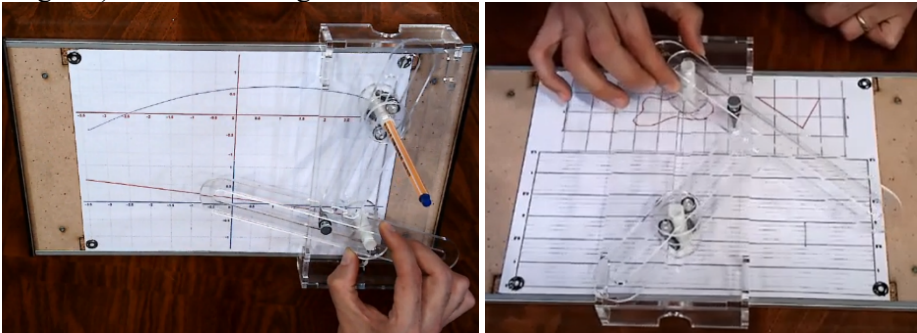


Figure 3. The device for derivatives and anti-derivatives, featuring two reference frames (left) and an integragraph-mode sheet (right) – cf. <https://www.youtube.com/watch?v=TyxCAR317HE>, 0:00-1:05 and 1:07-3:45.

After introducing the role of the tangent and practicing a bit with the components of the device, we are ready to propose an activity related to the Fundamental Theorem of Calculus by introducing the device in the configuration as in the left of Figure 3 (with a background sheet with two Cartesian references). To recognize that the device is performing derivatives and antiderivatives (considering participants who already know Calculus), we propose focusing on the gestures necessary to move the wheeled pointer along straight and general curves (to move this pointer along a curve, the direction of the wheels has to be parallel to the tangent to the curve). Then, focus on simple pointer displacement when moving the wheeled pointer on straight segments. That should help in generalizing when moving along the graph of a function, thus realizing that the simple pointer describes the derivative. Similar reasoning, starting from moving the simple pointer on a horizontal segment and generalizing to the motion along the graph of a function, should lead to the fact that the wheeled pointer moves along one of the anti-derivatives. A crucial el-

ement is to notice that the device transmits the direction of the wheeled pointer (i.e., the direction of the tangent) to the ordinate of the simple pointer. Furthermore, by simply changing the paper sheet (Figure 3, right), if we move the simple pointer along the perimeter of a figure, we can calculate its area by measuring the displacement of the wheeled pointer, as in the historical instruments called “integrators” (cf. Abdank-Abakanowicz, 1886).

Even though we marginally tested such a part, we can also use the device to introduce the concept of differential equations. We can note that, from a historical perspective, Leibniz’s geometrical insight into calculus originated from the inverse tangent problem, which analytically corresponds to solving differential equations. We propose to explore the device assembled as shown on the right side of Figure 2, but with the wheeled pointer rotated by a right angle (cf. <https://www.youtube.com/watch?v=kqtU9GpcN78> 0:17-0:50, but without the Cartesian system). That defines a parabola by implementing the property of having a constant subnormal (parabolas are orthogonal trajectories of exponential curves). To analytically explore the trace, we propose that users introduce a Cartesian system and convert the mechanical constraint into a simple differential equation.

4 *Problematizing the tractrix*

The tractrix holds a significant place in the history of mathematics as a paradigmatic example of a transcendental curve. Indeed, its construction comes from the first mechanical solution of an inverse tangent problem. First emerging in the 17th century, the tractrix became a focal point for early modern geometers, including Huygens and Leibniz. The tractrix was not only a theoretical curiosity but also a practical challenge: constructing it through continuous motion required innovative mechanical devices.

4.1 **Historical sources**

As a starting point for understanding the historical and conceptual development of mathematical instruments for transcendental curves, Giovanni Poleni’s 1728 *Letter to Hermann* (published in Poleni, 1729) offers a rich and revealing case. This letter is not merely a theoretical reflection but introduces tangible, functional mathematical instruments (not just idealized mechanisms or prototypes) for solving the inverse tangent problem, par-

with the plane by its weight or other physical properties, this point moves along a trajectory in which the taut string or the rod is always tangent to the described curve.

Lastly, Gottfried Wilhelm Leibniz is considered the third discoverer. Although his publication came after the others, he was the first to mathematically formalize the curve's nature using the tools of modern analysis and differential calculus.

Poleni, however, emphasized that none of these three thinkers succeeded in building a fully functional and accurate machine to trace the tractrix. Each of their devices had limitations or flaws. This perceived shortcoming left an opening for a fourth contributor, Poleni himself, who believed he had finally achieved what the others had not: a precise and theoretically sound method for constructing the curve. Perrault's demonstration with a pocket watch (left of Figure 4), Huygens' rigid rod mechanism, and Leibniz's use of a taut string all attempted to embody the curve's defining property: a constant-length tangent to a fixed axis. However, Poleni critiques these earlier efforts as incomplete or imprecise, particularly in their reliance on physical forces like gravity or friction, which compromised the geometrical purity of the construction. His innovation, the use of a wheel to implement tangent direction, offered a more stable and reproducible solution. This idea, however, had a precedent in the work of John Perks, a British schoolteacher whose 1706 and 1715 papers in the *Philosophical Transactions* described similar mechanisms. Although Poleni never cited Perks, the conceptual overlap is striking and has been explored in detail in (Crippa & Milici, preprint).

Perhaps most significantly, Poleni's letter contains what is likely the first formal justification of why a wheel's direction can be used to guide the tangent to a curve. He draws on Leibniz's notion that a curve can be seen as composed of infinitesimal straight segments, each tangent to the curve at a point. Poleni explains that a wheel rolling along a curve naturally aligns its direction with these tangents. As shown in his diagram (right of Figure 4), the wheel's contact point traces the curve such that the direction of motion at each instant is tangent to the path. (Notice that the wheel is represented in various positions and, due to the 2-dimensionality of the diagram, is drawn on the plane although it is perpendicular to the plane.) This insight not only grounded the mechanical construction in sound mathematical reasoning but also marked a conceptual leap in the design of mathematical instruments.

4.2 Teachers' training proposals

Perrault's construction of the tractrix holds historical importance precisely because it does not rely on advanced mathematical knowledge (Perrault himself was not a mathematician). Yet, his mechanical approach offered a compelling solution to a new mathematical problem: defining a curve through the inverse tangent problem. From an epistemological perspective, this marks a pivotal shift. Traditionally, geometry dealt with direct tangent problems, where the curve is given and the tangent is derived. In contrast, Perrault's construction reverses this logic: the curve is generated based on the behavior of its tangents. This shift required a conceptual rethinking of what a tangent is and how it can define a curve. Such conceptual changes often seem obvious in hindsight, but they are difficult to achieve without a fundamental reorientation: in our previous workshop activities participants struggled to grasp the tangent-based reasoning without guided intervention.

Building on previous feedback, we integrated the device, particularly in its tractrix configuration (left side of Figure 2), into the course "Critique of the Principles" for pre-service mathematics teachers (Palermo, Fall 2024). Together with other mathematical machines, our device was adopted to introduce approaches to mathematics that differ from today's mainstream view. One of the aims was to remind that, for a long period, geometric constructions were at the basis of the foundation of mathematics. After introducing the "issue of exactness in geometric constructions" in antiquity and for Descartes, we proceeded to Perrault's construction without providing the mathematical reflections on the role of the tangent. Participants firstly used the device to draw Perrault's curve and were then given a sheet with another pre-drawn Perrault's curve traced using an unknown arm length (the constant distance between the pointer on the directrix and the one tracing the curve). The drawing included the directrix (base segment) but not the cusp point. We asked:

- How long is the arm of the drawn curve? How did you find it and why?
- In general, what mechanical constraints of our instrument and what geometric conditions/invariants define this curve?

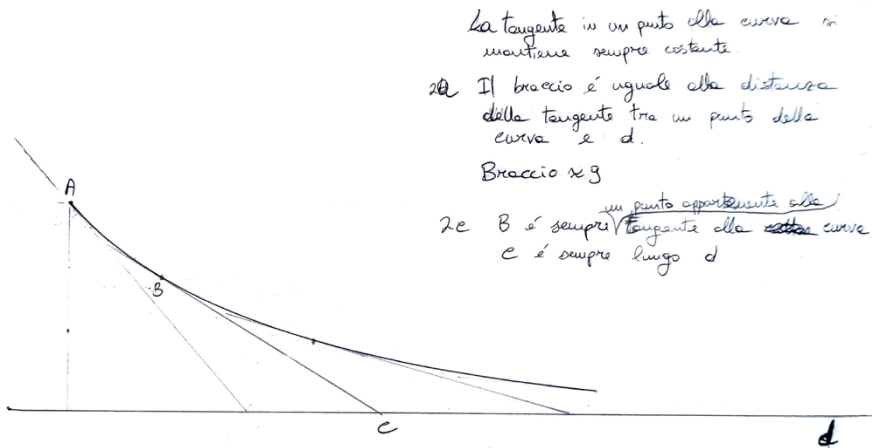


Figure 5. Problem: Given a part of the tractrix (without the cusp) and the directrix, find the length of the “arm.”

Participants worked in little groups of about four people, and were encouraged to freely use the tools provided, which, in addition to our device, included a ruler. After some reasoning, many naturally drew the estimated tangents and made measures with a ruler, thereby gaining an intuitive understanding of the tangent’s role in the construction. For example, in the answer of Figure 5, we can see some traced tangents (note that their first attempted solution consisted of drawing the perpendicular to the directrix d at endpoint A). To find the length of the arm, they clearly recall the role of the tangent: the solution is given by the distance of the tangent line from any point on the curve and the intersection with the directrix (points B and C in the figure). This activity enabled participants to justify the construction (that they materially performed before this problem) by linking mechanical components to geometric constraints with minimal instructor intervention.

It is important to emphasize the significance of the material components in this reasoning. Indeed, even though it is sufficient to trace lines with the ruler to find the length of the arm, the relation with the device tracing the tractrix is essential to provide concrete significance to the curve. For example, some participants walked along the printed curve using the tractrix device by roughly adjusting the arm length. (To materially implement a different arm, they used a finger to keep the distance between the pointers constant, even though one of the pointers was no longer at the extremity of the connecting rod.) Such a modification of the device was used to gain insight into a solution or to

verify the exactness of their ideas. In such a setting, the tangent to the curve is not only a theoretical mathematical object but becomes the modelling of a material object (representing the direction of the wheeled pointer).

After this activity, it was possible to introduce the history of the tractional motion without making the passage from direct to the inverse tangent problem appear obvious. The discovery of a strong connection between mechanical components and geometric reasoning (e.g., the direction of the wheeled pointer) laid the groundwork for further activities, such as those described in Section 3 (exponential and derivatives/antiderivatives).

5 Conclusions

In this paper, we focus on the problematization of the tractrix in hands-on activities that adopt a historically based artifact. We aimed to minimize instructor intervention and allow participants to grasp the underlying geometric principles independently. The use of the physical device proved to be a natural and effective way to introduce historical concepts through tangible, hands-on activities. This approach not only facilitated engagement but also offered a different perspective on the foundations of calculus, bridging historical insight with conceptual understanding. People interested can reproduce the activity by building the adopted device using digital factory tools. Indeed, we freely share online the sources to reconstruct the device, including assembly instructions and printable sheets, at <https://www.thingiverse.com/thing:5532958>. Some videos are visible at <https://www.machines4math.com/>.

ACKNOWLEDGEMENTS

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EDUCATORS NAVIGATING THE INTERSECTION OF ELEMENTARY MATHEMATICS, STORYTELLING AND HISTORY: ILLUSTRATIONS FROM THE ISLAMIC CONTEXT*

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ABSTRACT

This study aimed to develop elementary curricular materials incorporating storytelling elements from the history of mathematics (HM) and investigate student and teacher perceptions of these materials. Using participatory action research methodology, we created curriculum materials featuring mathematical stories from the Islamic historical context and examined their reception among elementary students (n=90), preservice teachers (n=90), and in-service teachers (n=15). Data were collected through student interest surveys, focus groups, and teacher reflections on sample interdisciplinary units, including one featuring al-Kindī's contributions to cryptography, which we highlight in this paper. Findings reveal that elementary students expressed high interest in narrative-based mathematics instruction. Students demonstrated clear preferences for story-based learning that supported understanding and retention, while focus groups revealed gaps in their knowledge of mathematicians from their cultural backgrounds. Both preservice and in-service teachers affirmed the educational value of culturally responsive materials that use historical narratives as authentic contexts for mathematical reasoning. However, some raised concerns about balancing storytelling with mathematical rigor.

1 Introduction

Integrating the History of Mathematics (HM) into instruction can make lessons more engaging (Agterberg et al., 2022), yet teachers, especially at the elementary level, struggle to locate and apply suitable historical content (Girit-Yıldız & Ulusoy, 2024; Haydar & Durmaz, 2022; Moyon, 2022). As such, targeted professional development is needed to support classroom integration of HM (Agterberg et al., 2022). Storytelling drawn from HM can

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enhance both teacher practice and students' mathematical identities by embedding mathematics in culturally meaningful contexts. When paired with culturally relevant pedagogy, such stories can shift from being "window stories" to "mirror stories" that reflect students' lives (Sims-Bishop, 2011). Scholars further argue that HM helps convey mathematics as a dynamic field (Henningsen & Stein, 1997), reveals the human challenges behind mathematical discovery (Furinghetti, 2007), and highlights the discipline's evolution across cultures (Jankvist, 2009). Despite recognizing HM's value, teachers face a lack of curricular materials (Moyon, 2022; Tzanakis & Thomaidis, 2012). In centralized systems like Türkiye's, textbooks are often the primary resource, but their treatment of HM is limited. Studies of these materials have examined cultural diversity, representation of mathematicians, and mathematical tasks (Girit-Yıldız & Ulusoy, 2024; Moyon, 2022); however, as Moyon (2022) observes, "most of these studies are descriptive... didactic devices or mathematical tasks associated with the HM in textbooks are not really developed" (pp. 1617–1618). This study aimed to develop a sample of elementary curricular materials incorporating storytelling elements with HM and investigate student and teacher perceptions of these materials.

2 Teaching with the History of Mathematics

Literature discusses various benefits of using HM in mathematics teaching. Classroom applications based on ancient Egyptian mathematics and Khwarizmi's quadratic equations can attract student interest at primary and high school levels (Barbin et al., 2018; Kafoussi & Margaritidou, 2023). Gifted students reported that HM-embedded videos expanded their knowledge of mathematicians' work, increased curiosity about mathematical concepts, enhanced motivation for invention, and developed positive attitudes toward mathematics and learning (Karataş-Aydın & Işıksal-Bostan, 2022).

Various studies examine in-service and preservice teachers' views on HM and its classroom implementation. Prospective mathematics teachers' (PMTs') evaluations of HM in textbooks served as a bridge for incorporating HM into lesson plans. When PMTs integrated HM to engage students, the integration reached higher cognitive demand levels, though they performed less well with pedagogical instructions and student thinking (Girit-Yıldız & Ulusoy, 2024; Moyon, 2022). PMTs provided mostly negative evaluations of HM quality in

textbooks, criticizing its relationship with mathematics assignments and presentation regarding content dimension (Girit-Yıldız & Ulusoy, 2024). They stated that HM content explanations were insufficient for establishing strong relationships with instructional text, lacked satisfactory explanations, and included limited information about concept origins and evolution. PMTs critiqued instances where HM was used merely as anecdotes, failed to foster mathematical thinking, or served only as motivation, since creating high cognitive-level tasks does not guarantee execution at that level (Davis et al., 2016). Efficient approaches to using HM in mathematics instruction align with the HPM perspective (Clark et al., 2016), which encourages educators to view mathematics both as a polished, logically deduced body of knowledge as well as a dynamic process of discovery shaped by historical, cultural, and interdisciplinary contexts. Teaching mathematics through this dual lens enables students to engage in authentic mathematical inquiry while recognizing its human dimensions. Integrating these perspectives can foster deeper conceptual understanding, more meaningful learning, and greater interest in mathematics, especially among students who may not identify with mathematics as it is traditionally presented. As Robertson (2022) reflects, “My interest in history came through teaching. I wanted to emphasize that mathematics was the product of people, ‘like you and me.’” This perspective humanizes mathematics and helps students view it as an evolving field shaped by diverse thinkers across time and cultures. Given these documented benefits of HM alongside the identified limitations in current textbook implementations, storytelling presents a promising pedagogical approach that could address the content gaps while maintaining the engaging and motivational aspects that make HM effective in mathematics education. Based on this rationale, this study investigated how in-service and pre-service teachers can effectively use stories from HM to enhance elementary students' mathematics learning.

3 *Methods*

3.1 *Design of the study*

To answer our research questions, we adopted a participatory, proactive action research methodology. Action research provides for educators “a framework

that guides [their] energies toward a better understanding of why, when, and how students become better learners (Miller, 2007). Using participatory action research in the context of curriculum development was advocated by Plomp and Nieveen (2009) and modeled and refined by Eilks and Feierabend (2013).

3.2 Participants

Consistent with the participatory action research methodology, this study engaged multiple stakeholder groups to gather comprehensive feedback on the curriculum materials and their implementation. Three distinct participant groups were involved: elementary school students ($n = 90$), preservice teachers ($n = 90$), and in-service teachers ($n = 15$). The elementary school participants ranged from Grades 2 to 6, including 10 second graders, 20 third graders, 20 fourth graders, 23 fifth graders, and 17 sixth graders. The preservice teachers were undergraduate students enrolled in a mathematics education program, distributed across three academic years: 26 sophomores, 51 juniors, and 13 seniors. All had completed a History of Mathematics course in their first year, which provided relevant background knowledge for engaging with the historical content embedded in the curriculum. First-year preservice teachers were excluded from the study as they had not yet completed this prerequisite coursework. The in-service teacher group consisted of 15 practicing educators, including general classroom teachers, mathematics specialists, bilingual educators, and special education teachers working across Grades 4 through 9. To ensure confidentiality, preservice teachers were assigned pseudonyms (PT1, PT2, etc.) and in-service teachers were labeled similarly (T1, T2, etc.), with each abbreviation representing a unique participant within their respective groups.

3.3 Data sources

Multiple data sources were used to inform the design, implementation, and evaluation of the curricular materials. First, we developed a set of screening criteria grounded in historical accuracy, developmental appropriateness, and curricular alignment to guide the selection of mathematical stories. To capture student perspectives, a Student Interest Survey was administered to measure students' levels of interest in each story and to collect their rankings of the ten historical contexts. To gain deeper insight into students' experiences and

thinking, we conducted five focus groups with students from Grades 2 through 6. These focus groups explored students' general attitudes toward mathematics, included reflection questions tied to the Student Interest Instrument, and incorporated prompts aligned with the Culturally Relevant Pedagogy framework to examine issues of identity, representation, and engagement. In addition, we collected written reflections from both preservice and in-service teachers in response to the sample curricular materials. These reflections examined the use of the HM through multiple lenses, including mathematical content, curricular coherence, developmental appropriateness, interdisciplinary connections, and instructional design. Together, these varied data sources provided a robust foundation for evaluating the role of HM-based storytelling in supporting culturally responsive and developmentally appropriate mathematics instruction.

Student Interest Instrument: 10 Historical Contexts

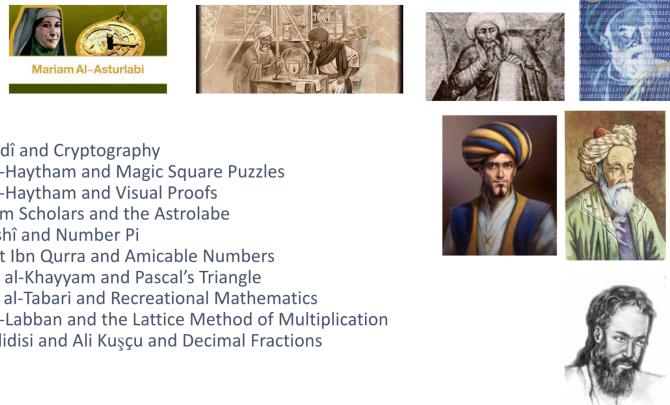


Figure 1. Cover of the Student Interest Instrument

3.4 The sample curriculum material:

In this paper, we present findings from preservice and in-service teacher reflections on a sample interdisciplinary unit: “*al-Kindî and Cryptography*”. The unit begins with Notice and Wonder routines that introduce students to the concept of code breaking, followed by an online game that engages them

in basic substitution ciphers. This leads into a historical narrative about Julius Caesar and his use of letter-shifting techniques to encode military messages, which in turn supports mini-lessons on inverse operations and functional rules that help students explore the mathematical structure of encoding and decoding. The second literacy component introduces students to the House of Wisdom and the Arab philosopher and mathematician al-Kindî. A curated reading highlights al-Kindî's early contributions to what is now known as cryptography, particularly his use of letter frequency patterns in Arabic texts, an approach foundational to modern frequency analysis. The unit concludes with social studies connections to the history of Baghdad and the scientific and mathematical advancements of the House of Wisdom. These interdisciplinary insights prepare students for a whole-class mathematical investigation in which they reconstruct al-Kindî's method for deciphering encoded messages through letter frequency patterns.

4 Findings

4.1 Elementary students' interest in the HM stories

Analysis of the Student Interest Instrument revealed that elementary students expressed a high level of interest in all ten historical mathematics (HM) story contexts drawn from the Islamic tradition. On a 5-point Likert scale (1 = Not interested at all, 5 = Very interested), mean interest ratings across grade levels were consistently above 3.0, with many ratings in the 4.0–4.7 range, indicating strong engagement with the materials. Notably, the activity involving Ibn al-Haytham's *Magic Square Puzzles* received the highest average rating among second graders ($M = 4.7$), while Hasib al-Tabari's *Recreational Mathematics* was particularly popular among fifth and fourth graders ($M = 4.57$ and 4.5 , respectively). Al-Kindî's and *Cryptography* also emerged as a top-rated context across multiple grades, reflecting students' interest in puzzle-like, game-based mathematical applications.

In addition to rating each story, students were asked to rank their top three favorite activities. These rankings revealed clear developmental patterns: younger students (Grades 2–3) tended to prefer game-like and narrative-rich contexts, such as *Magic Squares* and the *Astrolabe*, while older students (Grades 5–6) favored content that aligned more directly with their

mathematical skill development, including the *Lattice Method of Multiplication* and the calculation of π (al-Kashî). It is interesting to note that these preferences were consistent with the national mathematics curriculum standards (Ministry of National Education [MoNE], 2013, 2018), reflecting grade-level expectations in areas such as multiplication fluency and number sense.

4.2 Student Focus Groups

Our analysis of the student focus group interviews is showing the following emerging themes:

Storytelling in Mathematics Instruction: Students expressed a clear preference for story-based mathematics instruction, noting that narratives supported both understanding and retention. One student reflected, “*I would have loved it if [teacher] had explained it that way, it would have been nice. The stories in the book would stay in my mind, and I would have done the math more easily.*” Another described mathematics as “like a human being... like a teacher,” influenced by the children’s story *How Many Souls Does Mathematics Have?*, which portrayed a child interacting playfully with numbers. These accounts support how storytelling offers students both affective and cognitive entry points into mathematical thinking.

Cultural Representation in Mathematics When asked to name mathematicians from their own cultural backgrounds, younger students were unable to name any, while older students mentioned a few Islamic figures (e.g., Ibn Sina, Ali Kuşçu) alongside Western inventors such as Bell and Edison. This reflects previous findings on the limited representation of Turkish-Islamic mathematicians in the curriculum (Yıldız et al., 2016). Such gaps may hinder students’ development of culturally grounded mathematical identities and growth mindsets (Kaya & Haydar, 2021). Notably, some students demonstrated a nuanced understanding of cultural affiliation in mathematics. One student remarked, “*When we use their inventions or have fun with their inventions, I feel close to them.*” This response suggests that cultural affiliation alone is not sufficient; for students to feel genuinely connected, stories must also resonate with their lived experiences.

Mathematical Mindset: Stories of mathematicians served as powerful tools for cultivating a growth mindset among students. One student shared, “*They’ve done it, I can do it too. I give myself the determination to do better,*”

reflecting how exposure to the struggles and achievements of real mathematicians can motivate perseverance and self-belief.

4.3 Preservice teachers' views about the sample curriculum material

Our initial analysis of PST's reflections over the sample curricular materials revealed several emergent themes that both support and extend findings from the student focus groups and provide valuable insights for curriculum refinement and teacher preparation.

Cultural Representation and Interdisciplinary Connections: PSTs affirmed the curriculum's culturally responsive design, highlighting the value of including mathematicians from diverse cultures and periods. As one noted, "Including different people from different cultures would provide students with cultural richness." The use of Islamic scholars was seen as affirming students' identities, echoing Sims-Bishop's (2011) "mirror stories." PSTs also appreciated the interdisciplinary integration of mathematics with history, language, and technology, praising stories like those of Caesar and al-Kindi for fostering cross-disciplinary thinking.

Balancing Context and Content: Some PSTs raised concerns that storytelling could overshadow mathematical reasoning. One noted that the content risked drifting from core objectives, while another felt the narratives diluted mathematical rigor. These reflections echo Realistic Mathematics Education principles (Freudenthal, 1991), emphasizing the need for the story to be used as a context for rigorous mathematizing.

Bridging Past and Present: A recurring suggestion was to connect historical content with contemporary relevance. PSTs proposed including modern figures like Alan Turing and highlighting current applications such as cryptography in cybersecurity and finance. As one remarked, "Only the past is connected with the past, and the future is not connected with the future." Such feedback emphasizes the need for "window stories" that link history to students' present and future realities (Singh, 2020).

Engagement and Developmental Appropriateness: PSTs recommended strategies to enhance engagement and developmental alignment, such as gamified activities and digital tools. Suggestions included encryption games tied to story chapters and platforms like Khan Academy's Journey to Cryptography. These proposals reflect an effort to align content with students' digital learning habits and developmental needs.

4.4 Teachers' reflections on the curriculum design

In-service teachers evaluated the “*al-Kindî and Cryptography*” unit across several dimensions, including content, context, interdisciplinary integration, technological use, and historical framing. Their feedback revealed three core themes related to the use of HM:

Historical Narratives and Origins of Mathematical Ideas: Teachers valued the use of historical storytelling to introduce mathematical ideas, noting it appealed to students who are more engaged by narrative or humanities content. One teacher observed, “*Students who may be more interested in, say, humanities or history, will find this very engaging and accessible.*” Historical context was praised for clarifying the real-world origins of mathematical ideas and promoting purposeful learning. Another participant noted, “*Students can understand how our everyday, common math ideas were developed out of necessity.*” The narrative format was seen as a subtle yet effective way to teach concepts like inverse operations, rules, and patterns.

Interdisciplinary Strength: Teachers highlighted the seamless integration of math with history, literacy, and social studies. This approach countered the perception of mathematics as isolated, instead showing it as a discipline embedded in broader knowledge systems. One teacher remarked, “*It’s great to see math concepts in other subjects and not in isolation.*” The curriculum also supported co-teaching and collaborative planning across subjects. As one suggested, “*The history teacher could teach the background, the ELA teacher could support reading and writing, and the math teacher could explore decoding strategies.*”

Cultural and Cross-Cultural Representation: Teachers appreciated the curriculum’s emphasis on mathematical contributions from diverse traditions and times, particularly through the lens of cryptography. The juxtaposition of Caesar and al-Kindî offered students multiple cultural entry points into mathematics. One participant explained, “*We learned how Julius Caesar used coded messages... and how al-Kindî developed a system to decode them, still in use today.*” This representation was seen as key to inspiring students and fostering a more inclusive understanding of mathematics. As one teacher concluded, “*It’s important to have representation of cultures... to inspire them.*” These observations reflect a commitment to connecting mathematics with historically diverse contributions across civilizations.

4 Conclusion

This study reveals both the promise and complexity of integrating historical mathematical narratives into elementary curricula using storytelling. While students demonstrated strong engagement with story-based mathematics instruction featuring Islamic mathematicians, their responses revealed that cultural connection extends beyond just representation; stories must resonate with lived experiences to truly function as meaningful contexts for learning. The developmental patterns we observed, with younger students preferring game-like narratives and older students gravitating toward skill-aligned content, suggest that effective HM integration requires careful attention to both mathematical objectives and age-appropriate storytelling approaches. Teacher feedback highlighted a critical tension in curriculum design. While both preservice and in-service educators valued the interdisciplinary richness and cultural responsiveness of historical narratives, they also emphasized the need to maintain mathematical rigor and avoid letting stories overshadow core learning objectives. This indicates that effectively integrating the history of mathematics requires more than featuring diverse historical figures; it involves developing meaningful narratives that both support mathematical reasoning and link historical developments to present-day applications. These findings point toward the need for teacher preparation programs that help educators navigate the complexities of using storytelling as a vehicle for meaningful mathematical learning rather than simply as motivational content.

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LEARNING MATHEMATICS, LEARNING ABOUT MATHEMATICS IN PRIMARY SCHOOL: A LONGITUDINAL PILOT EXPERIENCE ON THE HISTORICAL PERSPECTIVE

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ABSTRACT

Since 2006, action research at Roma Tre University (Italy) has explored the introduction of the history of mathematics into primary education. Topics from the history and anthropology of mathematics were integrated into mathematics courses in the Master's Degree in Primary Education (Gil Clemente, Millán Gasca, 2016). This encouraged both students and in-service teachers from the Primary School Math Research Lab to explore how these historical aspects could be presented to children.

Following a suggestion by Karen Fuson in 2012, a history of mathematics learning path was developed for integration into the existing mathematics curriculum. As part of this broader research, a five-year pilot study (2018–2023) was conducted across all five grades of primary school in Rome. Contents were selected based on a structured learning path (Millán Gasca et al., 2017) and included children's books on the history of mathematics. Assessment focused on standard learning outcomes in mathematics and history as well as “flourishing” personal growth outcomes (Millán Gasca 2016; see Su 2020).

1. Theoretical background and pedagogical framework

Two main research questions were addressed:

- Is it suitable to introduce aspects of the historical evolution of mathematics to children? Can it be cognitively accessible, age-appropriate, and scalable across age groups? What added value does it bring to learning mathematics beyond basic numeracy in primary school? What general educational benefits are expected?
- Is it feasible to integrate history of mathematics into primary curricula and classroom practice (Tournès & Moyon, 2018)? Can it offer educational settings and approaches for children's initiation into mathematics? These questions relate to whether history of mathematics should be used as a tool or pursued as a goal in itself (Jankvist, 2009).

Two counterarguments concerning the suitability and feasibility of this approach address the alleged 'erratic sense of the past,' especially in early grades

(Clark et al., 2019, p. 5) and a possible complication involved in addressing numeracy skills while devoting attention to “sophisticated” aspects of cultural contextualization. What is at stake is thus the more general question of primary comprehension. Yet since long ago Piagetian views on deficiencies in the logical abilities of young children have been overcome in studies on cognitive competence when reasoning occurs in an embedded context—one that aligns with their lived experiences: children possess logical abilities, but these emerge only when the context is meaningful and the communicative intent is clear (Donaldson 1978). Kieran Egan, a historian and scholar in education, describes children’s comprehension of causality (in time) as what holds stories together and moves them along (Egan 1997).

Mathematics in primary school is usually associated with practical skills with numbers. Our framework instead addresses the overarching question “what is mathematics?” which is often set aside, thus demanding children to silently accept it as school subject. Such a perspective on mathematics in the early school grades as a gateway to a cultural universe “belonging to each of us as human being,” crucial in the access to the scientific attitude (Eshach & Fried, 2005), were already present in Friedrich Fröbel and still more distinctly in Mary Everest Boole’s “preparation of the child for science”. Charles-Ange Laisant’s views on “mathematical initiation” (v. Millán Gasca 2015, Magrone, Millán Gasca, eds 2018), and Federico Enriques’s reflection on precocity and the need to combine utilitarian (functional) and formative goals (in the sense of *paideia*, see Fried 2018) since early childhood (Enriques 1938) also reflect this perspective.

Central to our approach is the primacy of human experience to unveil conceptual meaning of elementary objects and relationships having ancestral roots—one, successive, more than, straight line and betweenness, geometrical congruence -(Guzmán 2007); to start the exploration of the *terra incognita* of mathematics as a sphere of human activity and expression (Doxiadis 2003); as a journey in which mathematics contributes to human growth by providing avenues for beauty, joy, and discovery (Su 2017).-At early ages, the teacher's role is to nurture a sense of wonder: “our purpose is to expose the wonder and ingenuity of counting. Our first task, then, is to show that what we take for granted, what is such a casual routine these days, is a product of amazing achievements” (Egan, 1986, p. 77).

Current emphasis in HPM domain-related literature on the importance of monitoring and disseminating implementation in the classroom regarding history of mathematics topics (Clark et al., 2019; Chorlay et al., 2022) is particularly critical in primary and early childhood education, where topics on the history of mathematics are still rarely.

2. Exploring the introduction of history of mathematics in primary school. Action research methodology

Action research methodology follows a cyclical structure involving feedback and iterative improvement regarding educational objectives, teaching methods and the choice of historical content (see Table 1).

Early observational phase 2009-2011	Gathering of field data, through single teacher interviews, focus groups, and scattered school sessions by in-service and prospective teachers (the origin of numerals in Mesopotamia, ancient Greek mathematicians, De Castro 2011/2022, Usai 2012)
Field work A (2012-2016)	Design of prototypes of educational activities and itineraries (preschool (5 years old)/primary school grades 1-5) • by schoolteachers belonging to the community of practice of the Roma Tre research lab on mathematics in primary education • by prospective teachers in the Roma Tre Department of Education practicum
2017	Design of a learning path in history of mathematics for Italian primary school (Mazzitelli, Millán Gasca, Neri, Spagnoletti Zeuli 2017, see Table 2)
Field work B (2018-2023)	Longitudinal study of a single primary school class group (Rome, 2018-2023)

Table 1. Roma Tre project on history of mathematics in primary school. Action research diagram

The findings and conclusions from Fieldwork B presented here are intended to further improve the learning path (Table 2). Data gathering and analysis follow Max van Manen's phenomenological methodology, which focus on the individual and community lived experiences (Van Manen, 1990). Raw data includes reflexive writing (by researchers, teachers, trainees), “close” interviews, anecdotes, focus groups on single issues, audio and video in the classroom, children’s individual productions (utterances, logbook, worksheets, collective posters, artefacts, oral and written tests), and general discussions with pupils.

Grade	Historical/cultural topics	Related contents
1	Prehistory: The origins of number words (spoken language) Numerals in other languages Prehistory: The origins of geometry (shaping materials, the symbolic representation on cave walls) From prehistory to the invention of writing: written numeration	Number words, figures, decimal positional numeration system Geometrical objects and relationships. Shapes. Addition, additive decomposition of numbers
2	Mathematics in Egypt: hieroglyphic numbers; land surveyor. The measurements of the ancient peoples of the East: length and its units. The Indians invented the writing of numbers that we use today Numbers in China and Central America Hypatia of Alexandria	Number symbolism, additive decomposition of numbers Primordial/primitive concepts of geometry. Polygons: their elements Geometrical bases of measurement Number symbolism, positional principle, base 10. Zero Calculations and measurements in the world, numerical symbolism: base 10-20 Women and mathematics
3	The calendar and the ages of history Babylonian astronomers The Greek mathematicians: Pythagoras. First approximation to the invention of the metric system.	Ordinal numbers, the centuries as segments on a chronological axis The mystery of cyclic motions and omens Base 60 in the measurement of time and angle Philosophy and mathematics: the search for the principles of the world Connection geometry-arithmetic Conversion of measures and money, measurement systems
4	Roman numbers Mathematics was born in the ancient Greece The Tale of Herodotus. Thales The Theorem of Pythagoras. Euclid and Archimedes Mathematicians who wrote in Arabic in the countries of Islam Leonardo Fibonacci Chinese mathematicians	Numerical symbolism and additive decomposition Calculations and measurements in technical activities and crafts Our idea of mathematics Relations between segments, between angles, between plane figures, between solids Equalities between areas, the first equations Mathematical demonstrations The circle and π, the diagonal of the square, prime numbers
5	Mathematics in the birth of modern science in Europe	The ellipse Proportionality

	Kepler, Galileo and Newton Games of chance: probability The Enlightenment: Euler; Monge. The origins of the metric system and other ancient systems From the first calculators to the modern computer	Mathematics and the advancement of sci- ence and technology Projections of solid figures The geometric basis of measurement
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Table 2. A learning path for history of mathematics in (Italian) primary school (Millán Gasca et al 2017). In grades 1-2, storytelling with some elements of general historical context. In grades 3-5, the Italian curriculum starts with prehistory.

3. *The longitudinal study (2018-2023): contents, educational methodology, and findings*

Over a period of five years (2018-2023), a pilot experience was developed in a primary state public school class in the city centre of Rome, Italy, consisting of 19 children from different socio-economic backgrounds, including 7 with various special needs, 3 of whom were non-native speakers. The first author was the mathematics teacher (from grade 3 also Italian language teacher). In addition to incorporating the history of mathematics, a second pilot experience was developed, on the integration of dance, music, art and storytelling for an aesthetic experience of mathematics (Millán Gasca, A. M., & Neri Macchiaverna, F., in press). Problem solving and math talk were pivotal in the overall educational methodology of mathematics, especially with regard to the teacher's *stance* (George Pólya): to *dramatize* with children the process of discovery, encouraging curiosity and reflection, supporting their reasoning without taking over, and valuing mistakes.

The introduction of history of mathematics was implemented thanks to an integrated methodology: stories in selected picture books (see below), workshop or *atelier*, group work, oral discussion and debate, individual log-books to document their learning and related mathematical concepts, production of material exhibitions, and stories. During the reading, the teacher focused on expressive reading through the use of pauses, clear pronunciation, modulation of voice, movement within the space, and interaction with the children. During the reading, to foster mimesis, the teacher emphasized moments when questions were posed and asked the children to reason and respond. The reading of each book could last more than one lesson. In workshops or ateliers and during group work, the key points of the stories were re-

worked both at the narrative and conceptual levels: questions and exercises (sometimes involving concrete, even self-made, tools) were proposed on the emerging mathematical topics (table 2).



Figure 1 From left to right: Children build “fingers and balls” inspired by the book *Uri, the little Sumerian*; Group work: three girls decide to use the kit to work on horizontal abacus.

During discussions and debates, children shared their responses to questions, problems, or exercises. Starting from the collective discussion, they then wrote their logbooks to record and remember what they experienced. In grades 1-2, the logbook was initially designed by the teacher: it served to reconsider the classroom experience, to recall and sequence the various phases of the narrative—often rich in temporal and causal structures—to become aware of mathematical concepts to engage in reflective thinking inspired by hands-on activities.

As the children progressed in their handwriting skills, they were provided only with guiding questions to help them narrate their experiences; further, they independently created graphical expressions—combining words, drawings, diagrams—to represent their learning exploration. The logbook included exercises. In grades 3-5, pupils conducted research through selected books from the classroom library, presenting their findings through “lectures” on topics such as π , the Rhind papyrus, and biographies of mathematicians. Thus, writing and oral presentations were synergical to encourage reflection and understanding.

Thus, somatic-mimetic, as well as mythic and romantic understanding, were put into play, following Egan’s perspective on primary understanding and his storytelling-based approach to lesson planning (“What is the most important thing about this topic or story? Why should it matter to the children? What would engage them affectively?” Egan 1987).

UNDU	$1+2=3$	
DUDU	$2+2=4$	
UNDUDU	$1+2+2=5$	
DUDUDU	$2+2+2=6$	
UNDUDUDU	$1+2+2+2=7$	
DUDUDUDU	$2+2+2+2=8$	
UNDUDUDUDU	$1+2+2+2+2=9$	

Figure 2 Grade 1 logbook: An exercise on the number words invented in *Awa teches numbers*, involving addition, and representation of number with figures and fingers.

In grade 1, the following two books were read: Anna Cerasoli's *The Great Invention of Bubal* (Cerasoli, 2006) on a prehistoric shepherdess who invents counting methods, and Enrico Giusti's *Awa Teaches Numbers*, based on ethnomathematical research (Giusti, 2019).

Books for grade 2 were: Rafaella Petti's *Uri the Little Sumerian* (Petti, 2025). and Anna Cerasoli's *Pharaoh's Geometry* (Cerasoli, A. 2009) whose main character Ames, is the son of a land surveyor who learns to use ropes for technical tasks. These stories helped the students to grasp the act of counting and cardinality, the symbolic representation of quantities, decomposition, primordial geometrical concepts, and the generations of shapes.

In grades 3-5, following the pupils' interests and needs, the choice was to delve deeper into the already considered historical topics, also in connection with the Italian history curriculum, which usually starts in grade 3 with pre-history, progressing to Greece and Rome in grade 5. We expanded the study of ancient number systems and mathematical achievements from various cultures, including Egyptian, Greek, and Roman civilizations and examined the origin of measurement and ratio thanks to *Mama Khanyi and the Pots* (Vale et al 2018). Further historical topics included in the designed learning path (Table 2) were chosen by the pupils—for example, an in-depth study of Fibonacci's figure through the reading of the book *The Boy Who Invented Zero* (Feni-

ello, A., & Folì, G., 2014). The teacher provided a historical frame of reference and considered the issue of sources.

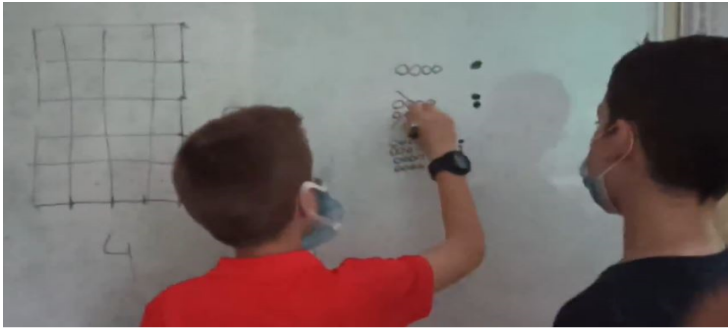


Figure 3 During a lecture on Egyptian mathematics, two children explain the multiplication method.

Writing played a fundamental role in working on the personal elaboration of experiences and the understanding of mathematical concepts.

Rome, October 14, 2021 (5th grade)

Yesterday, together with our teachers, we reviewed everything we have studied and understood about our number system.

While re-reading our logbook, we identified three key words: positional, decimal, and ordered. These terms helped us recall all that we have learned together over the past five years.

Our system is decimal because we group by tens. But why group by ten? We found a simple explanation: we have ten fingers. Since the time of early humans, our ancestors counted and calculated using their fingers. In this way, fingers were like a natural calculator. That's why even today, it feels intuitive to count in fives or tens.

The character Awa taught us that numbers and number systems were born from the body. For example, we remembered that to say "ten," Awa used the word everything, and to say "four," they said one hand minus one.

In first and second grade, we also read the story of Uri, who—through his adventures and challenges—helped us understand how useful and essential grouping is.

Another important discovery was the invention of number writing. Uri first invented clay fingers, then small balls, followed by tokens, and finally clay tablets on which numbers were inscribed.

Our system is also positional: the value of a digit changes depending on its position within a number. All of this is made possible by the invention of zero.

Zero has a long history. It was invented by Indian mathematicians, who called it śūnya. In Sanskrit, the ancient Indian language from which many European languages—including Italian—derive, śūnya means "empty." Thanks to zero, we understand when to leave an empty place, and we can distinguish numbers such as 40 from 4 or 400. (...).

The phenomenological observation and analysis of the lived experience in the classroom led us to reconsidering and adapting the original learning path, because each class is unique and is composed of unique individuals.

To analyse the qualitative raw data offered by the interviews, the diaries, and the teacher's notes on classroom conversations, we adopted a combination of three approaches: holistic, selective, and detailed (Van Manen, 1990). Several key findings related to children's attitudes and knowledge emerged:

- Most pupils expressed the feeling of being “as if” they were directly facing the problems that the characters in the stories had to solve and “as if” they were living in the same historical context as those characters;
- A strong desire to understand and address the problems and questions emerged. All pupils, including those with special needs, showed increasing curiosity and engagement, their imagination expanded and their creativity was enhanced by the combination of historical narratives and mathematical problems;
- Pupils developed new strategies to explain and narrate mathematical concepts and find new ways to explain their learning;
- Logbooks and interviews show that the pupils were able to create meaningful connections between different mathematical concepts interdisciplinary connection to the subject matter.

Individual tests, based on the assessment objectives selected by the school from the “National Guidelines” for grades 1 to 3 and 4 to 5, were conducted to monitor individual pupil progress. At the end of the fifth grade, out of 19 children (7 boys and 12 girls), 12 (4 boys and 8 girls) achieved excellent results in mathematics, 2 (one boy and one girl) very good results, 4 (2 girls and two boys) good results, and 1 (one girl) more than sufficient results. The programme contents had been simplified for one student while tests were modified (slightly simplified) and additional time was given for 3 pupils from the third group and for 1 pupil from the first group (just additional time) in accordance with their personalized educational plans. These evaluations also reflect the raw numerical data available from the testing activity and national tests in 5 that were analysed using qualitative phenomenological methods.

4. Final remarks

From a practical standpoint, the pilot experience confirmed that the integration of the history of mathematics into classroom practice is *feasible* through storytelling, an approach that is accessible and scalable across different ages. Pupils showed an intense appreciation of history of mathematics since grade 1, and in grade 3-5 they wanted to turn back to the same historical periods to obtain a better, more mature understanding, before turning to new issues. The pilot experience let us widely explore the issue of *suitability*.

First, considering aspects of the historical evolution of mathematics offers mathematical experience soaked by human intentions and goals, thus enhancing understanding and connections. The stories showed mathematics as an evolving body of knowledge, shaped by human experience, and help children understand the origins and nature of mathematical concepts. This approach can bridge the gap between abstract principles and lived experience.

Moreover, it offers a cultural richness regarding anthropological and philosophical aspects of number and form that extends maths learning beyond basic numeracy: pupils showed a growing awareness of the specificity of mathematical universes as linked to human life but involving imagination and the “invisible”. By placing mathematics within its cultural and historical contexts, pupils gain a deeper and more global understanding of the subject, seeing it not just as a disembodied set of principles/procedures but as a dynamic and evolving “worlds”, including speculative aspects. Historical narratives foster an appreciation for mathematics, presenting it as a vivid and immersive activity

Finally, there are overall educational goals involved in the pilot experience, regarding the building of a “scientific attitude” (objectivity, argumentation, research prompted by problems). By exploring figures such as Ahmes, Fibonacci, Al-Khwarizmi, and Awa, actual mathematicians or fictional characters inspired by historical research, pupils deeply shared their mathematical struggles. Engaging with historical narratives nurtures cognitive and emotional development, enabling children to cope with challenges (mathematical problems) involving analytical skills, fostering perseverance and courage. Historical contexts provide an appropriate setting for problem-solving, encouraging students to persevere individually and in a cooperative manner. Learning about the challenges that mathematicians faced led pupils to approach mathematical tasks as opportunities for discovery and growth, thus enhancing their commitment to the subject. In this way, a historical perspective can be the

driving force for presenting mathematical content in primary school. The historical framework adds an aesthetic and feeling dimension to mathematics, inviting students to encounter mathematical concepts through wonder, pleasure, play, fear, curiosity, helping children to relate to these feelings in both the mathematical and broader life scenarios.

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THE MENTAL TELESCOPE: THE NON-EUCLIDEAN GEOMETRY CASE STUDY

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The story of the birth of non-Euclidean geometries is a very interesting chapter in the history of mathematics, for a number of reasons. First of all, the current systematization was reached only with a troubled path that lasted many centuries, which suggests that perhaps there was an epistemological obstacle hiding, which the historical perspective highlights. Moreover, the history of the development of these new geometries is fascinating because it provides an example of how, although starting from very remote roots, mathematics is a subject in constant evolution, alive and susceptible to change, against the stereotype that sees it instead as a “dead” topic. Finally, analysing the development of this branch of geometry serves to combat the stereotype that mathematics is the creation of isolated geniuses, presenting a clear example of how it is instead a wonderful collective cultural creation.

Our intent is to show that non-Euclidean geometries can be a tool to promote the understanding of the modern axiomatic method in mathematics, to stimulate students’ aptitudes to logical thinking and to allow students to consolidate the knowledge of Euclidean geometry by developing it in a critical way.

Planned structure of the Workshop:

- Introduction to spherical geometry (laboratorial session).
- Formalization: what is non-Euclidean geometry (with historical glance), axiomatization of geometries.
- The mental telescope: how non-Euclidean mathematical ideas have provided tools for physics and fertilised art, showing how the interdisciplinary approach is successful not only in teaching, but also in the development of culture as a whole.

Polystyrene balls, rubber bands, drawing pins, small wooden trains, felt-tip pens, inflatable balls (all provided by the speaker).

HOW TO CONSTRUCT AND USE INSTRUMENTS WITH THE PUPILS, SO THAT THEY APPRECIATE WHAT MATHEMATICS ARE FOR

**Description of two projects (2021-22 and 2023-24)
in a Brussels secondary ‘active school’**

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During two academic years, we proposed to the Lycée intégral Roger Lallemand (LiRL), a Brussels secondary school based on active pedagogy, two different projects relying mathematics with surveying and music.

The LiRL works on three weeks (‘tripleette’) multidisciplinary themes, like *Climate*, *Navigation*, *Coal*, ... or *Angela Davis*, for instance. During the tripleette mornings (‘modules’), the pupils attend courses in different fields, and in the afternoons (‘workshops’), they have the opportunity, thanks to the existence of workshop-classrooms in the LiRL, to build their own instruments, with the help of a technology teacher.

Our first project, titled *En vrai, la géométrie ça sert à quoi ?* and subsidized by the Brussels Regional Government (Cocof) after a contest called *La culture a de la classe*, aimed at teaching third secondary grade (14 years old) pupils how some ancient instruments were made and used in order to elaborate precise maps and plans. The utility of these instruments (Geometric Square, Sector) relies on proportions, which were studied during the mathematical modules. Other modules (French, History) investigated the origins of these instruments, of which some were developed in Belgium, by Michel Coignet (1549-1623) in Antwerp and by Gemma Frisius (1508-1555) in Louvain, for instance. The argument of the tripleette was more generally defined as *Reconstruction after a disaster*, and the final task of the pupils was to measure with the instruments a square (Place Morichar) nearby their school, and to imagine how to rebuild it.

The second project, also subsidized by the Cocof, was titled *Accords et ondes*, and aimed at teaching fifth secondary grade (16 years old) pupils wave characteristics and their relation to the musical theory. Here, the morning modules were devoted to the study of physical and mathematical concepts, such as wave lengths, frequencies, speeds in different environments, or proportions, geometrical progressions, roots and their properties. In the afternoon workshops, the pupils tested wave propagation, calculated the speed of sound,

listened to sounds of different frequencies, in tune or not, with the help of a musician participating in the project. They also construct a simple instrument, called *Epinette des Vosges*, comparable to the *Dulcimer*, in order to understand the importance of the musical scales and intervals.

During the workshop, we will distribute to the participants cardboard models of the instruments and show some pictures and films shot during the projects. We will also present historical texts (translated from French) that induce the pupils, and the participants at the workshop, to use the instruments and appreciate mathematics.

ELEMENTARISATION OF MATHEMATICS FOR UNDERGRADUATES BY INTEGRATING HISTORICAL STANCES WITH TRIGONOMETRIC FUNCTIONS TOWARDS FOURIER METHODS.

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As mathematics advances, a mathematical idea once considered complex become elementary in the sense that it becomes fundamental for higher mathematics. From the perspective of undergraduate mathematics education, a mathematical idea becomes essential for upper-level courses for applied mathematics and STEM majors. Building on Felix Klein's conception, the elementarisation provides a dynamic perspective describing the history of mathematics and its relationship of mathematics education and undergraduate mathematics. The concept of hysteresis, as emphasized by Klein and Schurbring, addresses the process of elementarisation as mathematical ideas become more elementary and foundational based on recent advances and practices with mathematics and its applications. A mathematical idea or practice such as combining trigonometric functions becomes elementary in building and applying mathematics for its further development of mathematics and its applications. New stances on a mathematical idea are gained as these ideas are advanced, recontextualized, reclarified, and redefined in the advancement of mathematics and mathematical sciences. Through historical restructuring of applied mathematics found in upper-level courses in STEM disciplines, students are compelled to grasp a higher conception of a mathematical idea or a practice, for example Fourier methods related to trigonometric functions in their courses such as control systems, signal analysis. The hysteresis orients learner to build a higher stance on a mathematical concept such as trigonometric functions to understand the progressive restructuring of its elements in building mathematics as experienced in the historical development of mathematics and its applications in STEM disciplines. As mathematics advances while applying its concepts and practices, these mathematical practices, such as related to trigonometric functions, are reclarified through their progressive restructuring reorienting mathematical ideas facilitating their advancement and applications. This paper provides an exemplification of the elementarisation process by presenting, discussing and analyzing multiple historical stances on trigonometric functions including circular, hyperbolic, elliptical, com-

plex, finite, higher dimensional spaces, and geometric algebra. The historical approach behind Fourier analysis stance will be emphasized with its orientation towards trigonometric functions as bases of function spaces. Adopting the concept of hysteresis on trigonometric functions, renewed foundational developments on trigonometric functions will be argued to enter the undergraduate mathematics and school mathematics after this process of elementarisation. The implications of delayed hysteresis and neglect will be discussed from the perspectives of engineering mathematics education and physics education. Recommendations for learning trajectories of elementarised mathematics will be presented for their integration in core undergraduate mathematics courses for STEM majors. This hysteresis process informs the curriculum restructuring by revisiting what fundamental mathematical ideas and practices are for the advancement of mathematics and mathematical sciences. This process puts the undergraduate mathematics for STEM disciplines into a productive relation with the progress of mathematics and its fundamental applications in STEM disciplines.

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THE IMPORTANCE OF HISTORY OF MATHEMATICS IN THE CLASSROOM (BY CHATGPT): A FIRST REFLECTION ABOUT THE USE OF CHATGPT IN HPM

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On 15th of January (2024), a CNN headline pointed out that «'Jobs may disappear': Nearly 40% of global employment could be disrupted by AI» (statement from the IMF [1]). In Mathematics education, the advancement and widespread application of AI are expected to introduce new challenges, which are currently under investigation and will require some time to be thoroughly comprehended [2, 3]. The question is to know if historians of mathematics will be among these newly unemployed? And the teachers of mathematics? And the teachers that use History of Mathematics? Will they be replaced by AI too? It should be noted that the debate on the implications that AI has/will have on mathematics education is still in its infancy, although it is already recognized by some authors that “ChatGPT is a useful educational tool, but caution is needed when using it, and guidelines for safe usage should be developed” [4].

As an experiment, we ask ChatGPT to do an abstract for HPM 2024 with only six iterations: «Made me an abstract about the importance of History of Mathematics in classroom (250-400 words)»; «Can you add some references?»; « I want the references inside the text and a list in the final.»; «Are there critics about the use of the history of mathematics in the classroom?»; «Can you insert these critics in the abstract above?»; and «With more references in the text please». The result it's clearly not perfect (for instance, one given reference does not exist) but it is a good example from what a student could do in just a few minutes. Below is the beginning of the ChatGPT's abstract:

The study of the history of mathematics holds immense importance in the classroom, serving as a powerful tool to enhance students' understanding of mathematical concepts and fostering a deeper appreciation for the discipline (Dunham, 1990; Stedall, 2012). However, this approach is not without its critics, who highlight potential challenges associated with its implementation. While the integration of historical narratives into the curriculum is intended to enrich the learning experience and nurture critical thinking skills (Dunham, 1990), some critics argue that the historical content might not always

align with modern curriculum goals and the practical needs of students. They contend that time spent on historical context could be better utilized for covering more contemporary and applicable mathematical concepts. (...)

In this workshop we intend that the participants use ChatGPT to create materials for teaching and for the History of Mathematics, in order to provoke reflection on the potentialities and dangers that may arise for the HPM field from this kind of applications. It is intended to think about the major questions about the use of AI in math education, particularly, about the conceptualization of historical knowledge as well as the role of the teacher in historically oriented mathematics education.

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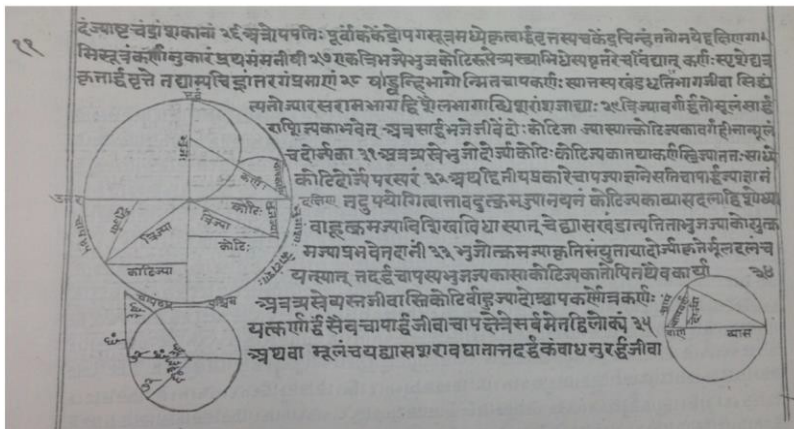
HISTORICAL TANGENTS! ORIGINAL SOURCES FROM TRIGONOMETRY TEXTS THROUGHOUT HISTORY

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Trigonometry, as we understand it today, carries a rich and intricate backstory that spans across various civilizations over more than two thousand years. Despite its neat presentation in modern mathematics curricula, it is often a challenging topic for students who meet it as a set of abstract analytic functions ultimately deriving from ratios of quantities. However, the history of trigonometry reveals a mathematical concept that, originally conceived of to address the demands of ancient astronomy, has undergone profound transformations and was central to numerous groundbreaking scientific endeavors in a wide variety of contexts. We will consider various original historical sources, which testify to a more creative and geometric approach to this field, offering new perspectives to appeal to the imagination of learners to deepen their grasp of the topic.



A DIALOGUE ON THE EDUCATOR'S WAY TO RELATE TO MATHEMATICAL HISTORICAL TEXTS

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ABSTRACT

There is not only one way to read a mathematical historical text. For instance, there is the mathematician who looks at the past from a modern synchronic plan, or the historian who looks at the past from both a synchronic and a diachronic plan. But there is also the educator's way to read a mathematical historical text, which we consider to be different. Indeed, in dialogue with the past, interlocutors assume a certain attitude of *responsibility* that, in educational context, takes a special meaning. Particularly, the educators must manage a certain complexity with learners in relation to the past which implies realizing conditions so that the relation of learners with the author takes place and accompanying them with inclusive gestures in their interpretation enterprise. This special role singles out considerably the educator's way of relating to the text. On the scene, we have voices from the past, from the classroom, and from a larger sphere of communication - such as those active in the actual sociopolitical environment - that are manifest and that, we think, should be considered in order to understand better the role and potential of the history of mathematics in mathematics education. In this communication, with reference to HPM literature, and focusing on the notion of *otherness* from Levinas and Bakhtin, we will try to push further the investigation of the educator's way to relate to historical mathematical texts. Our paper will maintain the form of a dialogue as a report and expansion of our email correspondence.

1 Introduction

This paper includes the messages of our email exchange, with minimal modifications. For convenience of the reader, these regard erasing of some redundancies or repetitions, typos, and reformatting of the text.

We made these choices as we want to maintain the features of a dialogue, in search to clarify and to develop some ideas that were brought up in the article *History of mathematics in the context of mathematics teachers' education: a dialogical/ethical perspective* (Guillemette and Radford, 2022) where the authors, focusing on the use of original sources in mathematics education with

emphasis on preservice teachers' education and the practice of exploring historical texts, attempt to develop a specific theoretical positioning to think more finely the way preservice teachers engage with the past. They develop a dialogical/ethical perspective that emphasizes on the emergent dialogue in the encounter with the past. Assuming that dialogues always occur in specific forms of activity, with their own object-oriented specificities, the authors highlight how preservice teachers seem to read historical texts in a manner that is neither the mathematician's (looking at the past from a modern synchronic plan) nor the historian's (both synchronously by considering a given historical system of mathematical concepts and diachronically by considering concepts' evolution in time and space). This different manner is characterized by acts that bring with the "potential estrangement" of the text, offering "vicarious aspects around the ways of *being* in mathematics and ways of *doing* mathematics" (id., p. 1501).

In the present contribution, as the reader will see, the focus is principally on ethics and on the notion of *otherness* developed by Levinas and Bakhtin. We try to understand better, in these terms, the educators' way of engaging with the past. Searching for emerging ideas and a theoretical emulation, we decide that our paper will principally report our email exchange prior to the 2024 HPM congress.

2 *Transcription of the email exchange*

Our exchange reported here begun just after Adriano's reading of (Guillemette and Radford, 2022). He considered the opportunity to contribute to giving methodological value to the use of original historical documents (texts) with students. He shares with various authors (for example, with Jahnke et al. (2002)) the idea that these have a great value in mathematics education. However, the educator's choice to use the originals lends itself to critical considerations. One concerns the rigor and reliability of the interpretation (a joint work with mathematicians and historians of mathematics seems necessary). Another concerns the approach to the original document by the students (what can be their involvement and on what can it be based?) and the characteristics of the proposal by the teacher (how does it fit into the educational path? for what objectives?). David proposed to share our reflections with participants in HPM Conference through an oral presentation. Here it is the email exchange:

A. I make reference to a resume of your article (Guillemette and Radford, 2022) but with a part regarding Levinas¹ contributions, a not trivial conjunction, maybe. The perspective to express the point of view of a secondary school teacher intrigues me. Moreover, how can we tackle the objection of “whig history²”?

D. The objection of “whig history” is interesting... There is also a risk of “using” the history of mathematics for other goal than history itself. We could go back to Fried’s (2001) article *Can mathematics education and history of mathematics coexist?*. Whig history is part of the historian voice and perspective, the idea is that the educator does not talk from the same topos, as we understand it in Guillemette and Radford 2022. If we situated ourselves in teachers’ education context, the voice of the historian *could* maybe be invoked, and orient the appreciative orientation of the group (preservice teachers and teachers’ educator)... there is maybe something to develop there... But, I don’t know where we could put Levinas’s perspective in it actually... What do you think?

A. When I write “We discuss...” I want to make reference to Levinas. I imply that an educator has a different "community" to make reference to, that of students (not of researchers, within which specific categories of research questions are shared). I would like to use the expression “today’s di-

¹ For Levinas, the I-Other, face-to-face, relation is the origin of ethics, as responsibility for the Other. This relation is “asymmetrical” (Levinas, 1979, p. 215). This Asymmetry is the ethical core of relation I-Other, where there is neither reciprocity nor measurement. It means that the other and I are never in interchangeable positions, and my role is never transferable, because the other’s life is infinitely more important than mine (“Thou shalt not kill”). Neither moral euphoria, since this act is not voluntary, nor any expectation of reciprocity involved here, as the Face of the other, is the upmost indisputable authority. With reference to mathematics education, the learner is under educator’s authority but also the educator is under learner’s authority. The educators make a work proposal to the learners, but then the ethical relation of responsibility requires a response to the learners, and the educators orient their choices, consciously but also unconsciously.

² I think about the negative connotation (for us) of “Whig history” (that focuses on the successes that led to present-day mathematics, while ignoring failures, to go too directly to look for the present in the past, as would say Butterfield) considering that educators are usually oriented to today’s mathematics.

achrony”³ to mean that in teaching we have not to “say immediately everything” of a historical document to the students instead, what we have to say depends on particular educational needs, so that we may postpone some considerations. So, Levinas helps us to highlight the interactional aspects. Of course, historians, and mathematicians, are inevitable references.

D. Yes, OK, but I wonder how Levinas can help us to describe the interactional (interesting!) aspects as you said... It seems easier with Bakhtin⁴, as we did with Luis. Maybe we could get back to *trace* and *face* concepts, but how do they get special meaning when describing the educator perspective?

A. Levinas can help to “sustain or justify” our point of view of educators, considering the ethical relations of the educator with the students. Face and everything in which there is trace of the Other produce the establishing of those relations, but also of that with the author of the document. I believe that Levinas can enter where you and Luis speak of “Engagement and answerability” (I would like to add *responsibility*, for the document). So, I refer to what I say in Demattè (2022). I suppose that the relations with the

³ By using “diachrony” I underline the development over time of interpretation by students of a historical text. For example, about interpretation of a passage from Euler’s *Introductio in Analysin Infinitorum* on logarithm (Demattè, 2022): I introduced Euler as an eminent character of 18th century (in a multidisciplinary project); the class faced exercises and problems taken from Euler’s originals concerning the concepts of exponential and logarithm (from both *Introductio in analysin infinitorum* and *Vollständige Anleitung zur Algebra*); individually, each student read the passage and wrote a report; these reports were briefly discussed; more exercises and problems inspired by the passage were solved... I do not pretend that all this exhausts the interpretation of the passage (for instance, I did not deepen the term “imaginaris” - which has been translated as “imaginary” and as “complex” by different translators - with students, a couple of examples excluded).

⁴ In his phenomenology of the practical act, Mikhaïl Bakhtin (1895-1975) developed a dialogical perspective on human being, and human sciences. In his perspective, being-in-the-world is dialogical through and through, as we are co-participants in a world where other subjects exist and that our very identity is constituted of these relationships. The Other has, for Bakhtin, a “surplus of vision” on us, and vice versa, making possible a “reversal of perspectives” which is capital here in the human intersubjective experience and the formation of a proper ethical consciousness.

students and with the author of the document characterize the educators' role and are different from those of historians and mathematicians.

D. I think that it could be interesting to develop about the *responsibility* in the triad (teacher/students/mathematician from the past), as we insist more on engagement with Luis. Levinas would surely help us here indeed. I'm interested in your idea of bridging the educator roles and its relation to the educator way to relate with the past... It makes me think that we assume that the educator way to relate with the past is different from the mathematician and the historian way - different... but maybe not entirely, historian and mathematician also have their "third party" (*tiers*) interlocutor. Plus, and I don't know exactly how to fit it, but the responsibility is also shared with to the students, as you clearly pointed it in your article in DdM (*ibid.*)... What do you think?

A. The educator roles and its relation to the educator's way to relate with the past, I think, depends on the different people that the educator refers to. I agree: the educator way to relate with the past is different from the mathematician and the historian way. Do you think that historians and mathematicians can have as their "third party" (*tiers*) the educators? Where? I try to answer: in the HPM group!

Achieving the sharing of responsibility with the students through an educational initiative... It risks to be, I think, an oxymoron - being the responsibility based on an ethical relation (in order to preserve it, that relation must be lived without making it an object of reflection...). As educators, I think we can only present us as "models" of responsibility with the mathematician of the past, in front of the student (I consider an imposition as an institutional act that would preserve "form but not substance", that is the exteriority required by the Third Party instead aiming at the I-Other relation; but I don't exclude that this imposition could leave space for an I-Other relation)

D. Of course, the educator lives a kind of tension in relation to the truthiness of the interpretation of the text, but it is not the same as the historian one (his entanglement in the hermeneutical circle – the tiers here is the History itself as this never-ending, never-achieving, always constructing narrative) and the mathematician one (the search for the expansion of the mathematical realms– the tiers here is Mathematics itself as a science). For

the educator, I think it is much more a kind of prudence (*prudentia* - the cardinal virtue with all its ethical deepness) that is different and that has to be explored more finely (the tiers here is for me, mathematics education as a social and political project). Lacan would say a different big Other (*grand Autre*)⁵. You are right, the educator role brings different way to relate the text and its interpretation. I would say that the educator role is oriented generally on *giving sense*. But this *giving sense* is taking a special sense (!) when classroom engage with the history of mathematics. Indeed, mathematics educators seek, within the exploration of the history of mathematics, to restore to mathematical activity its most precious ontological force, its recreation against the backdrop of culture and history... For me, *responsibility* of both teacher and pupils takes on special meaning here also... This said, as I said, in my paper in ESU 9 proceeding, criticizing Fried's notion of "moderator in dialogue attitude", that it is not enough to simply have this position in the presence of learners – to enact this attitude *in front of them*. One could say that, as educators, we should situate ourselves as moderators between the past and present *in preparation* for the encounter with the past in our classroom. But again, this dialogue between the old and new must engage the entire classroom if we want learners' horizons to expand. We would like the whole class, teachers and students, to dialogue with the past...

⁵ Lacanian psychoanalysis describes the subject as split between different interpellations, more or less authoritarian voices anchored in the symbolic order, sometimes called *le grand Autre* [the big Other]. In our discussion, we try to describe the ethical relation to the historical text as *responsibility*, a responsibility beyond any ontological condition, a responsibility that is not coming from a decision or a freedom, but in the authority of the Face of the other. This ethical relation of responsibility is never a simple isolated face-to-face. Indeed, for Levinas, "the disquieting imprecision of third parties shrouds the face-to-face encounter with the other signifies an insistence on disturbing, on preventing the two from remaining within the two" (Bensussan, 2006, p. 36, our translation). In the investigation of the educator's way to relate to the past, one of the main objective could then be to determine the sense of this Third Party and its manifestation in educational context. The ethical relation of responsibility when reading an historical text is something always shaped by a Third Party, other voices than the one of the reader and the readed, determining the appreciative orientation in the encounter with the past and ways to relate to it.

A. Here my comments. Prior to the truthiness of the interpretation, I see the meaning the educator gives to the text (depending on the personal experience/competence but, first of all, the aim to involve students, in a sort of “today's diachrony” of the text inserted in the process of students’ learning: a never-ending process, abandoning the myth of a grasp “hic et nunc”/in “exhaustive” form/without “possibility of remediation”... see also Freire (2000). Why not consider the tiers (Third Party) only as human subjects dealing with a common object (see Radford’s theory of objectification) - that is the mathematical content/the text itself or more generally history - for a radical humanization of mathematics? An object belonging to the teacher (as the proponent of school activities), therefore – hopefully – desired by the students (see Levinas quotation in my ESU9 workshop), and progressively constructed in teacher/student interaction? Why this reference to “prudential”? This reference to Lacan reminds the Levinasian Third (but again, why mathematics education instead of teachers and other people involved in mathematics education?). Could you explain with reference to learning your reference to “ontological force, its recreation against the backdrop of culture and history”?

D. Indeed, there is the meaning and the truthiness of this meaning. There is empirical evidence about the fact that teachers do not introduce history of math in classroom because they don’t feel enough competent. Another problem is that they are interested in introducing it, but they don’t know precisely why. There’s a double problem.

It is the case of Fried who problematizes clearly the differences between the two communities considering their relations to the past.

Bakhtin would say that when someone talks (interacts), he always has a third interlocutor (*surdestinataire* ideally just) who is also responsive (ideally). I do not consider the tier as human subject, because the Third is voiceless concretely (yet it is ideally). In fact, for Lacan, the truth is that there is no big Other, we are alone, but this is another story. My idea is that if we want to explore the relation that educators have with the past, maybe we should try to understand from *where* they stand.

This is an attempt to understand the educators’ way to relate to the past, taking in consideration where they stand. My first intuition is to situate the educator as embedded in a political and social project that is mathematics education, which demands constant care, judgement, and adjustment.

3 *Educators and historical texts: synthesis and concluding reflections*

For some mathematics educators, the reading of historical texts is a privileged way for learners to encounter history, in school and in the context of teachers' education (e.g., Arcavi and Isoda, 2007; Demattè, 2021; Demattè, 2022; Guillemette, 2017; Jahnke, 2014). Several studies have emphasized the experience of “dissonance” or “alienation” (e.g., Arcavi and Isoda, 2007; Guillemette, 2017), which students can feel when reading historical texts. Others have suggested that these reading activities could help the prospective teachers to learn something about their own mathematics knowledge and positions by experiencing and “reflecting on the *contrast* between modern concepts and their historical counterparts” (Fried et al., 2016, p. 218).

This said, as pointed above, the different ways to read and to encounter mathematics from the past in the context of mathematics education are still keenly discussed. There is a need to understand better how educators relate to the past, both when referring to teachers' classroom implementations related to the history of mathematics and to teachers' educators. Fried (2018) observes a variety of relationships to the mathematics of the past. He focuses and discusses, among others, mathematics educators' way to engage with the history of mathematics, highlighting their own specificities. We would like to suggest that this educators' way to relate to the past should be investigated, both theoretically and empirically.

A fundamental point concerns the choice of texts and the implementation of activities with the learners, for interpretation. The choice is completely up to the educator, it is part of his pedagogical authority, and is conditioned by his epistemological beliefs regarding the construction of knowledge. For example, wanting to propose to the learners a reflection on algebraic symbolism, the educator could choose a text by Descartes (in which the symbolism is completely analogous to the current one) or one by Bombelli (which has many aspects of diversity compared to the current one). We think that in the second case the educator can be aware that the discussion with the learners will be broader and will lead to highlighting more historical-epistemological aspects. We suppose that these could concern algebraic symbolism as a tool (choosing Descartes) or as a special form of language (choosing Bombelli). We express ourselves in terms of suppositions, aware of how the educator's personal, different beliefs can influence the classroom discussion.

In the context of a classroom, the educator reacts to the text in terms of an agentic culturally and historically situated positioning through which his subjectivity shows itself polyphonically in the reception of the text that carries its unavoidable estrangement. Pointing at future empirical studies in the HPM literature, the way this subjectivity is showing itself still has to be described.

We have suggested, in Levinasian terms, that there is possibly a Third Party that is engaged in this emerging dialogue when referring to educators confronting with the past, a Third Party that is not the same of the mathematicians nor of the historians. We agree with Fried when he says that “the way teachers of mathematics teach mathematics is not only determined by their understanding of the nature of mathematics, historical or otherwise, but also by their own teaching goals and the kinds of problems they hope their teaching practices will solve” (2018, p.18). This is where the Saussurean perspective, used to describe and to differentiate the mathematician’s and the historian’s way to relate to the past, loses its efficiency. We think that the lack of a sustained discussion around de Saussure’s perspective has obscured the understanding of the empirically observed problems surrounding synchronic and diachronic readings of historical texts, therefore, there is a need that in HPM literature a discussion of these issues to take place⁶.

In this paper, we think about the ideal role of educators. We consider that those who act in their role as educators refer to, for example, the special institution constituted by the HPM study group, a special Third Party that indicates (considering the work of its members) the need for an ethical relation with both the author of the historical text and with the students, to be managed according to justice. However, we problematize the subjectivity of the educators and consider that it may happen that they refer, for example, to the community of mathematicians or that they refer only to some aspects of the HPM proposal. We ask ourselves then if, in this way, the conditions exist for the right coexistence of the ethical relation with the student and the ethical rela-

⁶ This Saussurean perspective on language, focusing on interpretation and relying on the concepts of diachrony and synchrony, has been frequently discussed in HPM literature (e.g., Bråting, 2019; Rodriguez & Lopez Fernandez, 2010). What has been less discussed is the very conception of language offered by de Saussure and its implication for the understanding of the history of mathematics in mathematics education (see Guillemette and Radford, 2022).

tion with the author to be realized. Evidently, other communities of reference (for example that of educators in the broad sense) will be able to provide the prerequisites for this coexistence to be achieved.

Indeed, the school mathematics culture may enter in the scene here, as the mathematics knowledge take on original forms in educational context. From a historico-socio-cultural perspective, the source of mathematical abstraction is to be found in the material production of life and cultural ways to engage with others. It is within the economic, social and symbolic structures (for instance: values, governmentality, historical significances) of the society that the needs to be satisfied in educational terms are defined, mediated by ethical relations with the learners. And it is within these same structures that the “ways” of satisfying these needs are also defined, and to which educators can inspire for getting to take into account the needs deriving from learners’ subjectivities. These “ways” are in fact “ways of doing things”: they appear as “knowledge” (D’Ambrosio, 2006). This is where we could find the genetic roots – focal point of emergence and growing – of the dialogue between voices from the school context and voices from the past, and maybe the focal point of the investigation of the educator way to relate to the past.

Another way to characterize this engagement is to take into consideration the inherent responsibility of the educator, who proposed the text, in the emergent relation between the learners and the mathematicians from the past. The educator experienced relations with both the learner and the mathematicians from the past, to make coexist. To promote these relations, the educator has to make choices that are just. *Justice* becomes the key concept (Levinas, 1991) and has to inspire the answers to questions, such as: In which way can the text support personal learners’ needs? What have I to say to the learner about the author, and the context in which the text has been produced? About the mathematical content of the text, how can author’s reasons and learner’s needs meet⁷? Is the text an opportunity to tackle learners’ difficulties or does it introduce supplementary difficulties? In this case, what educational oppor-

⁷ Let’s consider, for example, that the text might be part of a larger work - such as a book with its own purposes and structure - where it is part of a discourse or a deductive reasoning, instead the learners can find in it the opportunity to deepen a specific content or to experience the “dissonance” between their own and author’s understanding.

tunities can justify an investment of time and resources to tackle these difficulties? Inside what I say to the learners, is there something that can stimulate them in deepening author's works or life context?

In making choices conditioned by justice, the educator focuses on, thematizes and compares different needs of the author and the learner (such as those mentioned inside the previous questions). It is a new perspective, different from that of a relation with an Other (the author or the learner, separately, to relate ethically to, in an unthematized form). It is the prelude of an entity ethically constituted, whose principles are developed from justice and could be even formalized as a system of criteria to be followed in classroom.

The HPM community is a social symbolic structure whose members made choices to try to answer those questions; it is bearer of knowledge, as ways of doing things, regarding original historical documents in teaching and learning mathematics. In any case, it is not excluded that the educators do not refer to each of the different subjects, that is: the learners (as a group or as singles), the author, the members of a community devoted to pedagogy and mathematics. It is not in the present paper that we want to discuss the negative educational consequences deriving from this.

This said, educators' way to relate to the past in educational contexts still needs to be described in theoretical and empirical terms. The idea is not to discredit the mathematician's and historian's way to relate to the past, but to signify how fundamentally different could be the educator's one, and how this investigation could foster new ways to understand how educators conceive the role of the history of mathematics in mathematics education, how they manage and envision the introduction of the history in the classroom and the difficulties they are expecting and living. The pedagogical and didactical role, and potential of the history of mathematics could then be discussed in more refined and intimate dialogue with teachers' and teachers' educators' voices.

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NEWTON'S DE ANALYSI VS FUNDAMENTAL THEOREM OF CALCULUS

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In *De Analysi*, Newton derives three primary achievements of modern calculus: the area under the curve $y(x) = x^{\frac{m}{n}}$ equals $\frac{n}{m+n} x^{\frac{m+n}{n}}$ (Rule I), the power series of arcsine, and the power series of sine.

Two further rules introduced without proof reinforce Rule I. Rule II and III state that the area under finitely or infinitely many curves equals the sum of areas under each curve.

The standard interpretation of *De Analysi* runs through calculus: adopting the Riemann integral, it presents Rule I as the Fundamental Theorem of Calculus

$\left(\int_0^x f(t) dt \right)' = f(x)$. Accordingly, term-by-term integration of series ex-

plains Rule III. However, this interpretation does not correspond to the argument's structure regarding the series of arcsine and sine. In calculus, one first expands the series of sine and then gets the expansion of arcsine by the theorem on the inverse function derivative. On the contrary, Newton finds the power series of arcsine first and then the series of sine. The core of this difference is that Newton does not apply the derivative or limit concept.

We present actual techniques applied by Newton, namely Euclidean proportion, indivisibles, 'infinitely close' relation, and formal power series. Regarding the study of arcsine and sine, we present Newton's results in a broader context, ranging from Ptolemy to Euler.

During the workshop, we will walk participants through Newton's infinitesimals and formal power series techniques.

The workshop rests on the first English translation of *De Analysi* [1]. We will provide its electronic version in advance. One can also browse it through the Internet Archive digital library.

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TRANSFORMING INSTRUCTION IN UNDERGRADUATE MATHEMATICS VIA PRIMARY HISTORICAL SOURCES

Looking to the future, with an eye on the past

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The *TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources* (TRIUMPHS) project was a multi-year, multi-institution effort to design and test student projects based on the reading of primary historical source materials, and to research their impact on students and instructors. When HPM met in Montpellier in 2016, TRIUMPHS was just completing its first year of funding from the US National Science Foundation. When that funding ended in 2023, the original group of seven principal investigators had expanded to include nine doctoral students in mathematics and mathematics education, thirteen additional authors, and a sizeable cadre of site testers. This presentation takes a reflective look back at what TRIUMPHS accomplished (or not) with regard to the development and classroom implementation of its collection of “Primary Source Projects” (PSPs), and a realistic look ahead at the prospects for sustaining this effort into the future. Some highlights from TRIUMPHS’s accomplishments, and questions associated with each, that will be considered in this presentation:

- Development of over 100 PSPs for use in university (tertiary) mathematics courses ranging from beginning algebra to topology.
 - What do PSPs look like? Who writes them, and how often?
 - How have PSPs been reviewed for consistency and quality? What aspects of the review process could be continued in the future?
 - Should new PSP development continue? If so, who will write them? What does it take to learn to write one?
- Classroom testing of PSPs by over 120 university mathematics instructors at a wide variety of institutions across the US and Canada.
 - What does classroom implementation of PSPs look like? Who teaches with them, and how often?

- Which PSPs get used the most (or least) frequently? Do these PSPs have common features that might explain these trends?
- How have site testers been recruited and supported? What aspects of site tester support could be continued in the future?

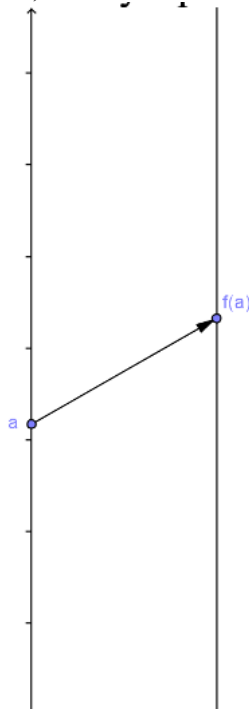
This presentation will be of interest to instructors who wish to bring primary sources into their own teaching, as well as those involved in widespread implementation of instructional innovations and researchers who study the effect of using history to teach mathematics.

TWO EXAMPLES FROM HISTORY: MAPPING DIAGRAM TO VISUALIZE RELATIONS AND FUNCTIONS.

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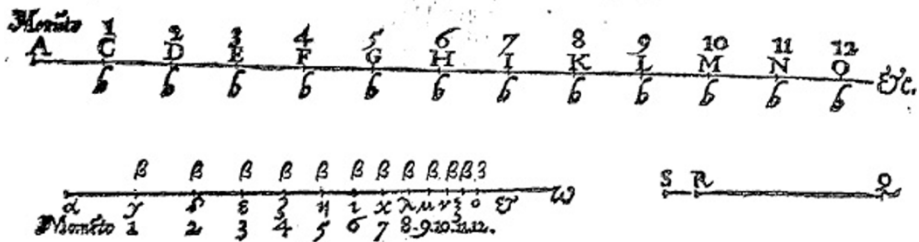
A mapping diagram for a function, f , is a figure consisting of two parallel number lines (or axes) and a set of arrows between these lines. Points on one (source or input) line represent numbers from the domain of f , the source (controlling or independent) variable values. Points on the other (target or output) line represent numbers from the co-domain of f , the target (controlled or dependent) variable values. An arrow in the diagram has its tail on a point on the source (domain), representing a selected number, a . The head of the arrow points to the function value, $f(a)$, for the number a , represented by a point on the target (co- domain) line.



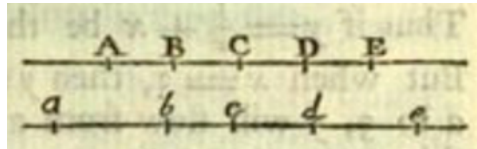
Mapping diagrams are also described as function diagrams, arrow diagrams, dynagraphs, parallel coordinate graphs, or cographs. They visualize functions and relations as an alternative to cartesian graphs.

The history of mapping diagrams predates Descartes' work *La Géométrie* (1637) which introduced numbers and equations to the analysis of geometry. Though originally an appendage to his *Discourse on Method*, this mathematical work eventually became the basis for current coordinate geometry and the graphical representation of functions.

Published in 1614, Napier's work, *Mirifici logarithmorum canonis descriptio* (*A Description of the Wonderful Table of Logarithms*) used mapping diagrams as key visualizations for the introduction to logarithms. His diagrams did not use arrows to indicate the corresponding points. Instead, Napier used alphabetic labels to connect a point moving on one axis with its position increasing arithmetically to a second point moving on a parallel line segment with its position decreasing geometrically with respect to the segment's end-point.



Later in history mapping diagrams were used by Isaac Newton in his *Treatise of the Method of Fluxions* (published posthumously in 1736) to visualize his solution of the problem of finding the relation of two quantities, given an equation involving their fluxions.



The author will explain and connect these two examples (using GeoGebra) to featuring mapping diagrams in contemporary approaches to logarithms and differential equations, illustrating how this history can be integrated into current pedagogy.

SELECTING EPISODES SHEDDING LIGHT ON THE HISTORY OF THE FUNCTION CONCEPT: HISTORICAL AND DIDACTICAL ANALYSES OF A LESSON-STUDY IN GRADE 10

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The design of didactic situations aiming to take students along a cognitive path of amendment and expansion of their concept of “function” has already been studied in an HPM context (Kjeldsen & Petersen, 2014). In 2018, a different session – with roughly the same goals – was designed and implemented in the context of the Shanghai HPM-studio, in the lesson-study format (Liu, Shen & Wang, 2019; Wang & Shen, 2020). The goal of the workshop is to further describe, analyze, and discuss the various choices made by the participants in this lesson-study.

More specifically, we plan to:

- Present a selection of historical episodes – some well-known, some less well-known – which could be used in the design of a session aiming to take students along the expected cognitive path.
- Offer a didactical analysis of the expected benefits of an integration of accounts of these episodes in the session design.
- Share some empirical data collected during the implementation of the session in 2018 so as to discuss the choices made by the stakeholders in the lesson-study, as well as their actual effects on students.

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THE CONSTRUCTIVE AND DESTRUCTIVE ROLES OF CALCULATORS IN MATHEMATICS EDUCATION

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For a long time, calculations themselves and the ability to perform them was an intrinsic part of mathematics education. Until WWII, what we now call “mathematics” was simply called “calculations measuring.” A few decades later, the emergence of pocket calculators brought a new aid, but also a threat to the primary goal of teaching arithmetic. Operating the new device became an important skill, which had to be taught as well, but the spread of the devices led to the belief that we no longer need to learn (and hence also to teach) multiplication tables. In my talk, I will draw attention to how numerical skills were associated with mathematics, how the arrival of the computer and calculating devices shifted the core of mathematics, and how this is reflected in mathematics education.

THE PROOFS OF EUCLID ON GEOGEBRA

A step-by-step visualization

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ABSTRACT

This article presents an example of the use of technology, specifically *GeoGebra*®, to introduce Book I of Euclid's *Elements* (which ends with the Pythagorean Theorem and its converse) into the teaching of mathematics. This dynamic geometry software allows new approaches to Euclid's proofs, enabling students to have access to step-by-step constructed images, rather than just a static final image as seen in a book or a *PDF*. This facilitates a better understanding of the mathematical contents, as well as the argumentations used in each step of the proofs. This is an ongoing work; the ultimate goal of which is to have all the proofs from Euclid's *Elements* Book I accessible on *GeoGebra*® for free use.

1 Introduction

The History of Mathematics is an instrument which can have positive effects on mathematics teaching, as the vast literature produced by the HPM group has shown (Clark et al., 2018 and Clark, 2019). The aim of this work is to use Euclid's *Elements* as an instrument to illustrate the strength and rigour required to achieve mathematical results. Of course, it is also intended that this approach will teach students about this important work in the history of mathematics (i.e., also as a goal), and even *GeoGebra*® itself can be seen as a goal, since the use of dynamic geometry software is compulsory in many countries.

The technology is another instrument that has been discussed over the years, especially the role it should play in the context of mathematics education. In fact, this discussion is now again on the agenda with the release, for example, of *ChatGPT* in November 2022.

Math education has been challenged by calculators, by *Google*, by homework and notes sharing platforms, by *WolframAlpha*, and more. This is another moment for us to think about how we want to adapt our instruction in light of an ever-changing landscape. (Eaton, 2023)

As for the simultaneous use of these two fields (history and technology) in education, studies are very scarce. For example, the article by Isoda and Aoyama (2000) discusses examples of the use of LEGO and dynamic geometry software to approach a mechanical/geometric problem by L'Hospital (1696) in a school context. In 2004, Isoda also demonstrated the potential of using historical tools and computer software by presenting an experiment carried out in secondary education:

The historical tool and software, enabled students to remind developer's reasoning. Because the lesson started from a historical picture of ellipse compass and gave it meaning within a historical tool for reminding developer's perspective, students could understand mathematics as a Human Endeavor. (Isoda, 2004, p. 236)

2 *The Elements of Euclid*

The example presented in this paper has potential in both areas: the technology allows a better understanding of the content of Euclid's *Elements* and, on the other hand, it allows this work to be brought more easily into the school context, exemplifying and helping students to understand the rigour of a mathematical proof and the nature of the discipline itself. It should be noted that many of the propositions in this book are still taught today in elementary geometry, such as: how to construct an equilateral triangle (Elements I,1), SAS Criteria for equality of triangles (I,4), the equality of vertical angles (I,15), the triangular inequality (I,20); the transitivity property of parallel lines (I,30), the sum of the three interior angles of any triangle is equal to two right angles (I,32) and the Pythagorean theorem (I,47). All these results are proved based on some entities that are initially presented and asked to be considered as true: 23 Definitions, 5 Postulates and 5 Common Notions (as presented in Heath's version). For instance, the first common notion is "things which equal the same thing also equal one another", which is easily accepted by students. On the other hand, we must also be careful because some of Euclid's considerations are problematic in terms of today's mathematics, such as the common notion 5 which states that "the whole is greater than the part" (for example, the set of natural numbers has the same cardinality as the set of integers...; what does it mean in mathematics to be bigger?).

The structure of *Elements* shows how the 'mathematical building' is constructed and the origin of the strength of mathematical results, where, starting from a few premises that are assumed to be true, complex results are proved

step by step, such as the Pythagorean Theorem and its converse (the last two propositions of Book I of *The Elements*). On the other hand, once a result has been proven, we can use this knowledge whenever we need it. And, regarding this, the first book is very interesting because it is very sequential in its proofs, since in the proof of a certain proposition, in general, the immediately preceding result is used. For instance, the proposition 1 is used in order to prove proposition 2; proposition 2 is used to show proposition 3; and so on. This structure makes it easy to visualise and comprehend the constructive and recursive nature of mathematical knowledge.

3 *The Elements of Euclid: historical approaches beyond the traditional book*

Euclid's *Elements* was the most widely circulated mathematical book for centuries, even before the introduction of the printing press. When this technology was created by Gutenberg (1430), naturally this work was one of the first math texts to be chosen for publishing. The first edition was published in Venice, in 1482, by Erhardus Ratdolt, but many others have emerged over time (in different places, at different times and in different languages).

One of the most interesting approaches to this work was done in Portugal in the 17th century, at the Jesuit College in Coimbra, which had several tiles in its classrooms with images of Euclid's proofs, reproducing figures from André Tacquet's version (Duarte, 2007). Figure 1 shows an example of one of these tiles, as well as the original image from which it was reproduced. These tiles were larger than usual and were intended to serve as a teaching pedagogical aid for maths lessons, in an example of innovation using Euclid's *Elements*. It should be noted that there were also tiles from other subjects such as astronomy and geography for the same purpose. Today, most of these tiles can still be visited at the Machado de Castro Museum in Coimbra (Portugal).

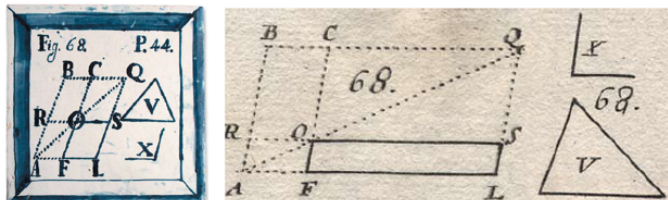


Figure 1. Tile with an image from Euclid's *Elements* edited by Andre Tacquet (I,44: To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle) (Source: Duarte, 2007, pp. 40-41).

Another interesting approach to the work of Euclid was the book published by Oliver Byrne in 1847, where a coloured printout of the images was used to make them easier to understand (the colours black, blue, red and yellow were used). The objects are not defined by letters as usual, but by their colours; with this option, the proof is easier to follow and the whole structure is more visually appealing. Notice also that Byrne's version is slightly different from Heath's version indicated before; for instance, it considers 35 Definitions, 3 Postulates, 12 Axioms (instead of the term Common notions). At the website (Rougeaux, n.d.) it is possible to find a modernisation of Byrne's work. This website is not just a *facsimile* of the original, but an adaptation of this work into several languages (old English, modern English, Greek and Spanish), while maintaining the interesting layout of the original. This is a good example of how technology can help to spread original sources to a wider public, without losing the pedagogical aspects of the original. See figure 2 for an example of a Euclid proof, in Spanish.

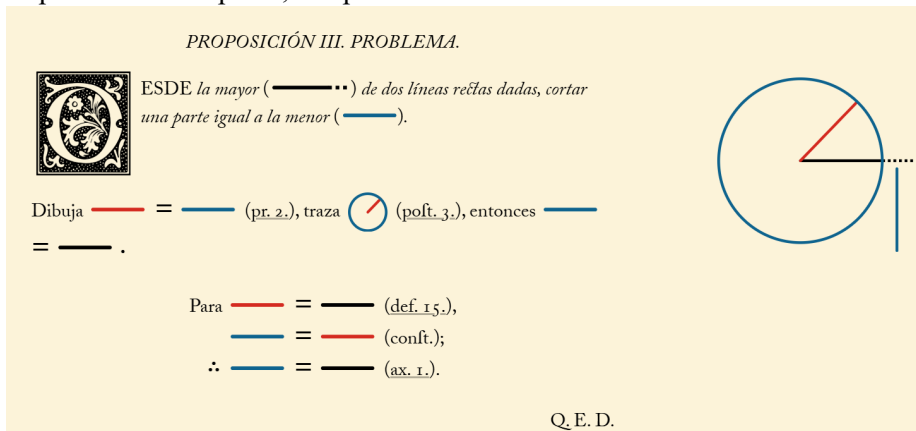


Figure 2. Online adaptation, in Spanish, of Proposition 3 of Byrne's book.
(Source: <https://www.c82.net/euclid/es/book1/#prop3>).

The possibility of creating dynamic geometrical content has grown as computer possibilities have evolved. A well-known website was created by David Joyce (1996-97), where interactive applets of the proofs of Euclid could be found. The idea of this website is that users can move the starting points so that the images in the proofs can be adapted to the chosen data. In fact, the image that appears in a mathematical proof in a book is just a representation of what is being considered in generic terms. That is, for example, the image of the construction of the equilateral triangle on a segment is done

for one specific segment (Elements I,1) but the considerations are valid for any segment that chosen by the user. This website, for each proposition, presents the final image that would appear in a Euclid book, which sometimes contains too many items and can hinder a full understanding of the reasoning used in each step of the proof (see, for instance, figure 3).

Unfortunately, the interactive aspect Joyce's website is not currently functional so that at the moment, the images appear statically, and it is not possible to move the indicated points. This is one of the reasons that the technology needs to be updated to guarantee interactivity in the approach to Euclid's proofs. On the other hand, the intention now is to create step-by-step images that would be built up alongside the used arguments. So, the process is gradual and, at first, only the initial data appears on the image, with everything else only appearing on the image when necessary for the proof.

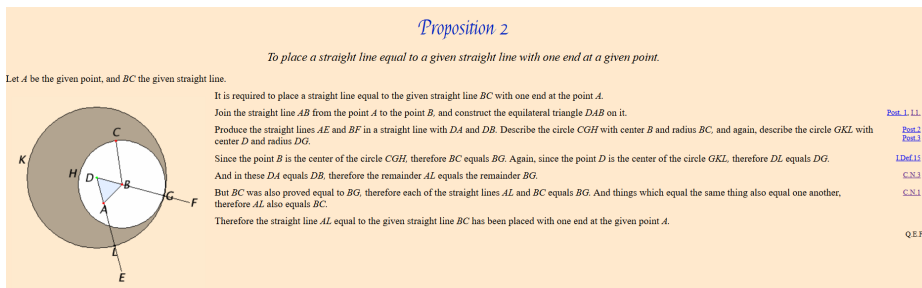


Figure 3. Applet with the proof of Elements I,2 by D. Joyce.

(Source: <http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI2.html>).

4 Using Geogebra® to reach The Elements of Euclid: a proposal

The main idea behind this project is to use Geogebra® software to construct dynamical proofs of the Book I of Euclid's *Elements*. It was decided to follow the set-up of David Joyce (1996-97), with the same exactly math notations. The advantage of this option is being freely accessible online and is based on the well-established version of the *Elements* by Thomas Heath (1956). Notice that this Book I deals with triangles, parallels, and areas, basic contents in the geometry for the first years of school and, as stated earlier, this book ends in Pythagorean Theorem. In fact, plane geometry is the content of the first six books of *Euclid's Elements* (for instance, Book III studies the circles and Book VI presents results about similar figures and proportions).

The selected software was chosen because it is free and because it is widely known among pre-university students and teachers, allowing them to

adapt these contents to their needs. These GeoGebra® applets use a colour scheme to facilitate the understanding of what is being used at each moment of the proof: in black are the initial data, in red is what is being proven at that moment, in green is what is being used to deduce what is in red, and in blue are previous steps of the proof.

As an illustrative example of this work, the majority of the ‘steps’ of the proof of Elements I,1 are reproduced next (Figs. 4-8).

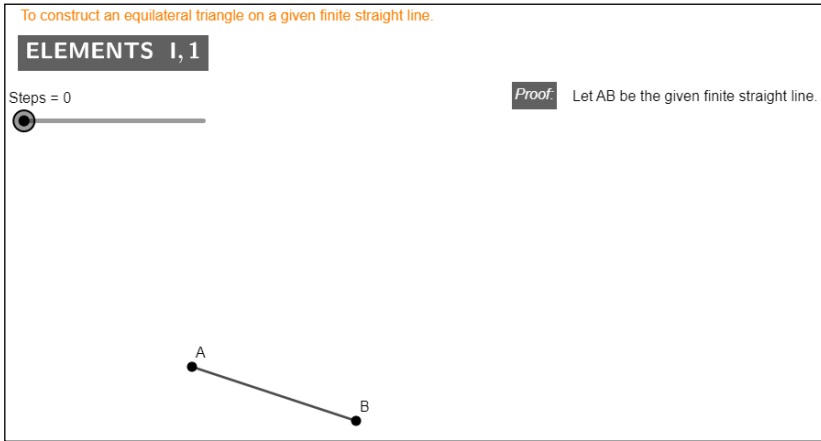


Figure 4. Initial data from Elements I,1. Points A and B can be moved to change the segment. The point ‘Steps’ is movable to proceed with the proof.

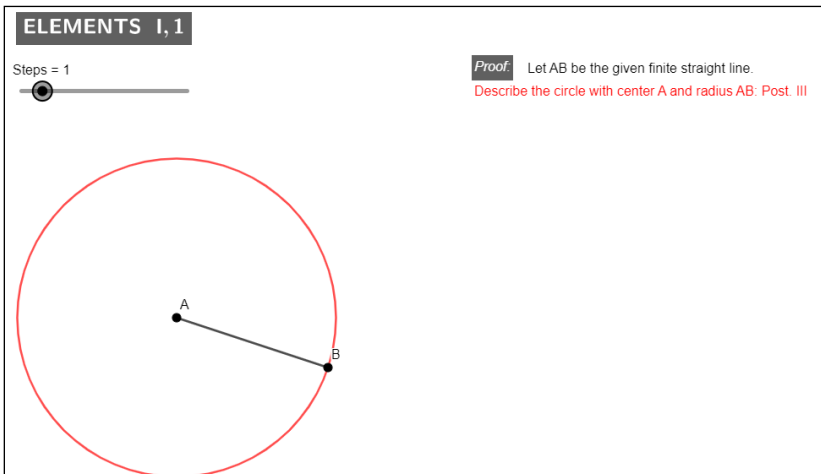


Figure 5. Step 1 of the proof of Elements I,1 and its justification on the right. Notice that is only possible to draw such circumference, given a center and a radius because Postulate III was stated at the beginning of *The Elements*.

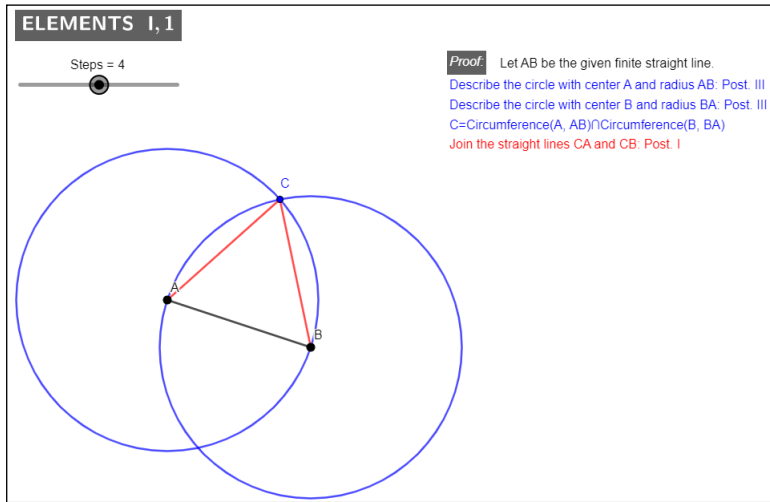


Figure 6. Step 4 of the proof of Elements I,1 and its justification on the right.

Postulate I asserts that, given any two points in the plane, it is always possible to draw a rectilinear segment joining them. In Joyce (1996-97), it is possible to find three historical criticisms of this proof, in particular, one about the question "Why does the point C exist?". This existence is not guaranteed by Euclid's axioms. In the XIX century, Pasch's axiom is stated to address this question.

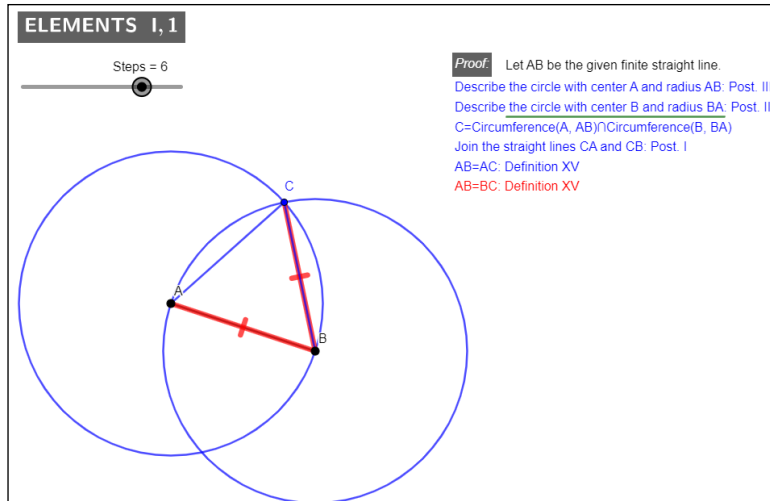


Figure 7. Step 6 of the proof of Elements I,1 and its justification on the right. By the definition of a circumference (XV), all its radii have the same length.

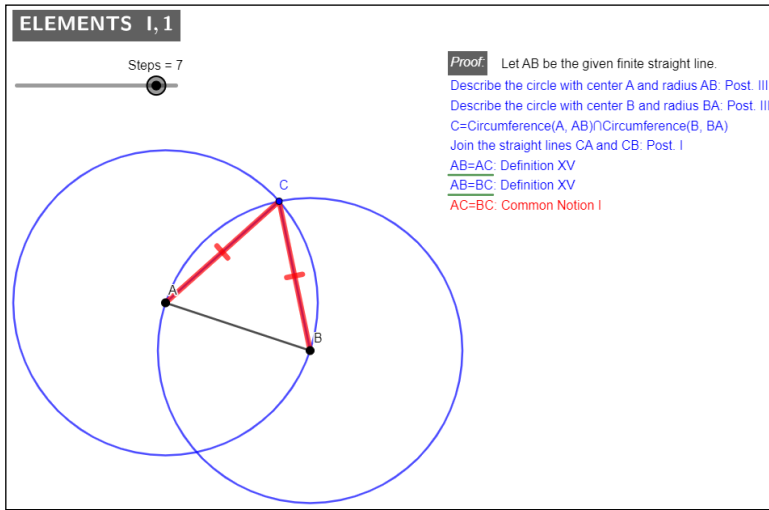


Figure 8. Step 7 of the proof of Elements I,1 and its justification on the right.

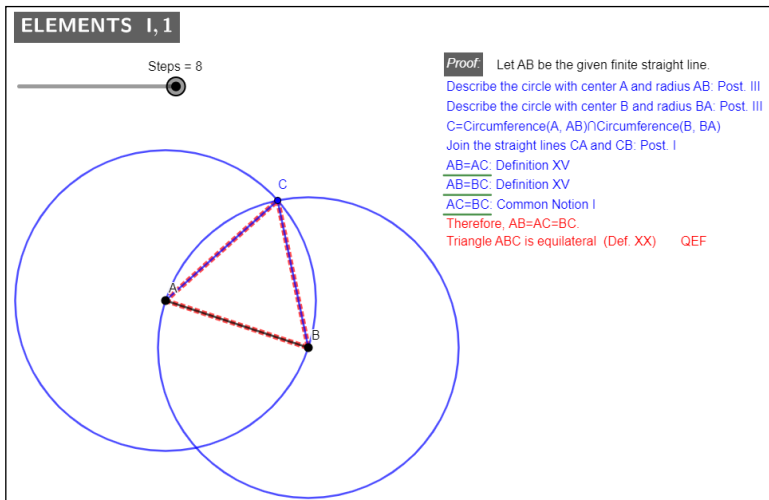


Figure 9. Final step of the proof of Elements I,1. Since the three segments are all equal, the constructed triangle is equilateral (definition XX).

The great advantage of these GeoGebra® applets is that the proof is being built step by step, in front of the students' eyes, and at each moment they can visualise the previous data that are being used. It should also be noted that in many of Euclid's proofs, the final static image shows various points and other auxiliary constructions that complicate the full understanding of the proof

(see, for example, figures 10 and 11 to other propositions where the proofs use various auxiliary constructions to proof the desired result).

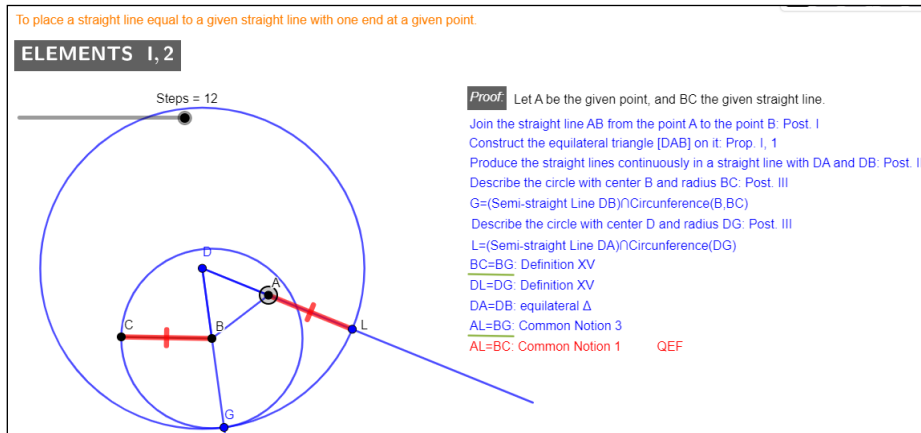


Figure 10. The proof of Elements I,2.

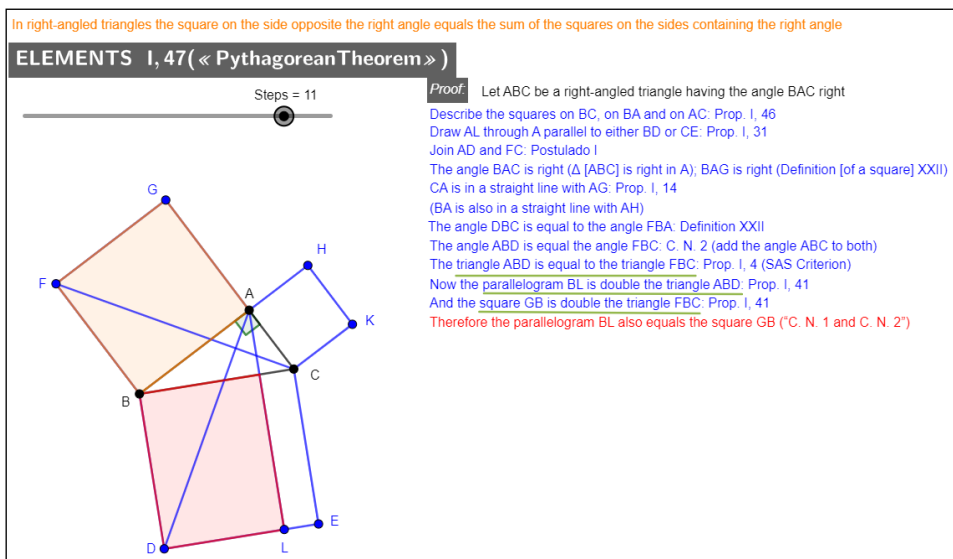


Figure 11. The 'step 11' of the proof of Elements I,47.

5 Final Remarks

Euclid's Elements was a success in part because it presents content that is still essential in elementary/school geometry in the early years. It's still a good example of how the construction of the 'mathematical building' works (what a

mathematical proof is, what the rigour of a proof should be, where each step must be duly justified by previously proved results). In mathematics is still valid the metaphor of dwarfs on the shoulders of giants, with new truths/results based on the previously discovered truths. With today's technology, interactivity and dynamic geometry, it is possible to give a new look and new approaches to dispel some of the difficulty in reading an original text from the past and, hopefully, get a new audience for the History of Mathematics.

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PAST, PRESENT AND FUTURE: THE FRUITFUL INTER-WEAVING OF CULTURAL MATHEMATICS

Plenary Address

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ABSTRACT

Half a century of research in Papua New Guinea has shown that ethnomathematics is strong in PNG cultures and richly influences technologies. Significantly, capabilities for noticing and visuospatial reasoning are particularly strong with pattern and dynamic (including action) imagery extending problem solving. Furthermore, our early childhood project indicated how teachers could build on this strong ethnomathematical beginnings for school mathematics.

1 Why past, present, and future matters

Ethnomathematics research requires a study of both the relationships of culture and of mathematics, noticing them, often through systematic work. Ethnomathematics research will help address the issue presented by Francois, Pinxten and Mesquita (2013. p. 20) who said “we claim that the drop out of school of many groups of children ... has to do with the insufficient learning system at school that fail to fit with the daily background knowledge of the children”. Furthermore, cultural mathematics may provide alternative mathematics in a future world just as indigenous and alternative scientific knowledge has provided mainstream medicines and care for the environment.

1.1 Ways of inquiring

The long distant past (40 000 years ago to first contact with Europeans), recent past (since first contact and during our life times), present, and future matter for mathematical identity of people in Papua New Guinea (PNG). Many past practices are still carried out today while others show modifications such as house building where sawmilled wood might be incorporated into the frames along with bush materials. Over five decades (1973-2020), we

have researched PNG cultural mathematics. We have collated, analysed, and interrogated data obtained from the following sources:

- Reports held at the University of Goroka (UoG) from over 250 mathematics teachers interrogating their own cultural practices;
- A further 50 research reports on cultural practices including theses;
- More than 20 in-depth interviews about measurement and spatial thinking in cultural practices;
- Over 250 responses to an open-ended questionnaire on measurement practices by student teachers and lecturers mainly electronically;
- Lean's (1992) thesis on counting systems based on government, linguistic, and other written records, field trips, and 1000s of responses to a questionnaire on counting systems collated from students at PNG University of Technology (Unitech) and teachers from late 1960s to 1980s;
- Ten team visits in six provinces covering coastal, river valley, and highlands with a member or teacher of that community, observing, discussing and videotaping various practices, reviewing the tapes and checking back with community members;
- Three focus groups of UoG students discussing our findings on measurement and visuospatial reasoning;
- One focus group of professionals living or studying in Sydney on their education in PNG (spanning late 1960s to 2015);
- Research literature in education, history, first contact diaries, linguistic material, art, sociology, mathematics education, and curriculum documents;
- Our own lived experiences and oral histories (Muke is a highlander, Bino is a coastal, Owens lived in PNG for 15 years and has returned 15 times to work with colleagues including Muke and Bino especially between 1998 and 2016. She has visited at least 60 villages in 13 of the 22 Provinces, mostly in Morobe Province).

Most of this data was for our first major project together on measurement and visuospatial reasoning between 2006 and 2008, but also for our projects on number and language from 2000-2018 and for our 2013-2017 design-research project on early childhood education based on cultural mathematical language, thinking and practices. For this we ran 15 workshops in remote areas (except for those in Goroka and Madang) in six provinces.

Our research substantiated cultural space, geometry, measurement and number mathematics, cultural visuospatial reasoning, and the impact of cultural

identity on mathematical identity (Owens, 2015; Owens & Lean with Paraide & Muke, 2018; Paraide, Owens, Muke, Clarkson & Owens, 2023).

1.2 Mathematical identity

Our mathematical identities and mathematics education identities have been affected by our cultural as well as educational experiences. Muke's home language is YuWooi but at primary school, he listened to English but never spoke it. After graduating from school, teachers college, and university, and teaching in PNG primary and secondary schools, he obtained a MEd (2000) from Aetearoa New Zealand on his own counting system. He also taught secondary mathematics in Australia while completing his PhD on quality teaching using the language of the children. He worked at tertiary level in all three countries. Bino's home language is Motu. She taught mathematics at secondary and tertiary level in PNG. Both have advocated for a recognition of home language and cultural practices in schooling PNG students, to overcome disadvantages for village students, and strengthen the resources for learning.

Owens taught mathematics and health education at all levels (early childhood to tertiary) in Australia and PNG. In PNG, while teaching at Unitech, she became aware of ecocultural visuospatial reasoning when colleague Lean with Clements (1981) and Bishop (1979, 1988) began to investigate and write about spatial abilities of Unitech students. She read their collected papers on spatial abilities in PNG. In 1992 during her doctoral studies, several important aspects of problem solving emerged with her PNG sample. Key to problem solving was the responsiveness of students and the sense of ownership. It was clear that visuospatial reasoning as well as other mental and affective variables were involved in influencing the students' responsiveness, sense of achieving mathematically and forming a mathematical identity. In 1997, she noticed architecture students were proud of their cultural heritage and investigated how their cultural mathematical thinking influenced their problem solving (Owens, 1999). When not in PNG, Owens worked with Australian First Nations peoples at Western Sydney and Charles Sturt Universities. Harris (1989) and Yolngu kinship iterative patterns and spatial mathematics (Thornton & Verran-Watson, 1996) influenced her thoughts on an ecocultural perspective of visuospatial reasoning.

After 2000, Owens assisted in setting up the Glen Lean Ethnomathematics Centre at UoG with Lean's work on counting systems (Owens et al., 2018).

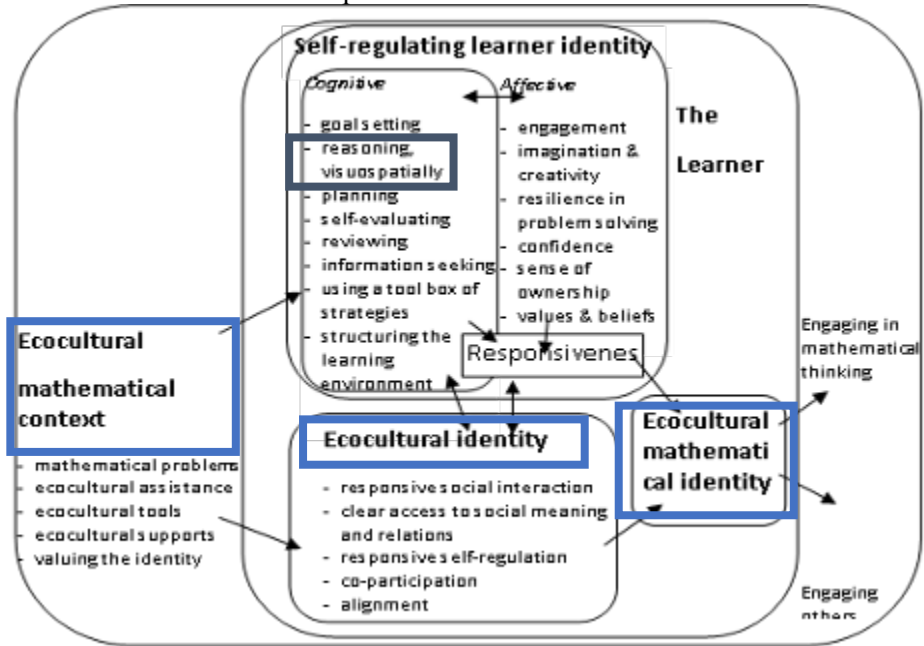
Bino provided the material to UoG students on their intranet. At UoG Owens, Kaleva, Matang, Muke, Bino and Sondo, researched measurement practices including focus group discussions evaluating the results emerging from the field visits (one to Muke's village), responses to the measurement questionnaire, and analysis of the UoG research reports.

From her early years in PNG, Owens observed and discussed practices and technologies which incorporated mathematics and counting systems during village visits and whenever people wanted to share their cultural mathematics. Constantly, she was aware of the strength of their visuospatial reasoning and their pride in their technologies. This was also evident from the UoG students' reports where students expressed pride and gratitude to their elders for sharing their knowledge, often saying "they were doing mathematics but didn't know it". After refining these ideas and interrogating them with other key research foci such as expressed by Gruenewald's (2008) critical place-based learning and Pinxten, van Dooren and Harvey's (1983) Navajo mathematics, Owens developed her 2015 book and the model for developing mathematical identity that could be gained through an ecocultural identity (Figure 1). From her doctoral studies (1993) she identified cognitive and affective processes that emerged and influenced responsiveness in problem solving. This in turn develops mathematical identity as evident with UoG and architecture students. Ecocultural aspects of learning influence these processes and also directly lead to a mathematical identity. Reasoning was often visuospatially. These key aspects are highlighted in blue in Figure 1. As we researched the ethnomathematics of PNG cultures, our own mathematical identities became increasingly aligned with ecocultural mathematics.

PNG cultures are strong but too much is being lost along with language. This loss is the result of many factors including government funding requiring English as the language of instruction for the 70 years of colonisation and in the new independent nation except for a 20 year period where local languages were used in village schools but often without a written language or training of teachers to implement home language literacy and transfer to English. Internal migrations to the cities and the increasing use of Tok Pisin (lingua franca) have added to this loss. There is a sense of urgency to link the past, present and future to establish an ecocultural mathematical identity.

Furthermore, many children need to make links to their cultural experiences to make sense of mathematics. For that to develop, we need to think differ-

ently about mathematics. This includes valuing the mathematical practices of the various cultural groups so students know the connections between their cultural mathematics and school mathematics to be empowered in their mathematical learning. This helps in building a positive mathematical identity as Figure 1 indicates. When school mathematics incorporates culturally relevant examples and practices, math anxiety is reduced and students who might otherwise feel disconnected from the subject are more engaged (Tutuo, Dayal, Averill, & Owens, 2023). Most children in PNG will be reliant on relatives, village, gardens, and their land and not be employed in the mainstream economy mostly because there are few technical and vocational institutions to absorb the growing youth population. A healthy country means healthy culture and a healthy understanding (Shaun Williams, personal communication, 2016) and use of mathematics in each person's circumstances.



Source: Owens (2015).

Figure 1. Mathematical identity through inquiry and ecocultural identity.

Students develop a mathematical identity through noticing their cognitive and affective aspects of thinking leads them to be responsive to their problem situation. That ownership of action and problem solving develops their mathematical identity. Ecological awareness and cultural context is central to the

individual's understanding and relationship with mathematics. This means recognising how mathematical concepts can be applied to ecological systems, environmental issues, and sustainable practices (Owens, 2015).

Well-documented ethnomathematics examples from other countries include Arabic art with geometric patterns and concepts of symmetry and tessellations, Mayan numerical system with a base 20 offering an alternative to base 10 counting systems, African fractals in architecture, art and hair designs with complex patterns that repeat at every scale, recursive algorithms and iterative processes in kinship systems of, for example, Yolngu. Such processes are fundamental concepts in mathematics and computing. Beyond our research, further examples will emerge in PNG if mathematicians and educators take an ecocultural lens in research. It will enable the mathematics teachers to create a more inclusive and engaging learning experience that respects and celebrates the diverse ways in which indigenous people understand and use mathematics (Wildfeuer, 2022).

1.3 The breadth of technology and mathematics

A list of Science, Technology, Engineering and Mathematics (STEM) in Indigenous PNG societies indicates their cultures are rich in the following mathematics (after the dash) (See details, Chs 2 & 3 in Paraide et al., 2023):

- Cultural learning before school in games, helping in the garden and house - A range of ideas develop from basic arithmetic to the probability embedded in games and garden growth, straight edges and slopes for drains, length comparisons and measures, multiplication through rows of mounds, visual area units rather than measured area units;
- Technology for making string from inner bark, weaving bindings, making string bags (bilum), and processing inner bark for tapa (cloth) - topology in practice, designs, patterns, and order in processing science and technology;
- Extraction of minerals and colours from ground; salt, spices, medicines and colours from plants; lime from shells, coral and stone; sago and oil from plants - ratios, relationships of plants, healing, illness, and beliefs;
- Technology, design, and skill for food capture with nets, traps, bows and arrows - physics of the traps, spatial movement, and animal behaviour;
- Bows and arrows, spears, and shields for fighting - strength and pliability of laminated knots, topology, spatial movement, shield and spear designs;

- Engineering, technology and material preparation for house and bridge building – slopes, trigonometry ratios, balance, vertical/horizontal lengths, representations of sine curves, catenaries, ratios for house materials;
- Drying, storing, and cooking food (e.g., in ground, bamboo, and pots) and implements - ratios and rates, volumes;
- Stone, wood, bamboo, shell and bone implements such as axes, adze, clubs, planes, tongs, needles - various shapes and tool usages;
- Colour and painting (for house, shields, canoes, face and body); art, technology and design for bilas (decorations especially for singing and dancing), masks, wigs, mud heads, art and design - ratios, patterns and relationships, lengths, volume and area, shapes and designs;
- Agricultural studies of crop growth and pest control, crop growing itself, spades, drains, trenches, propagating (trenches also used for demarcating land and fighting in some places) - time, weather, seasons, scientific experiment data and decision-making, straight edges, slopes, ratios;
- Longevity and necessity of agriculture indicates practices for 10s of thousands of years - slopes and straight edges, volumes, patterns;
- Longevity of pottery, and pot design including evidence of sailing across the Pacific Ocean - wayfinding and navigation, patterns, three-dimensional shapes and topology, links to sounds for drums, creativity;
- Designs in pots, body decorations (painting and tatoos), shields, canoes, houses, bridges, weaving, bags, in tortoise shell kapa designs for leaders - shapes, position, classifications, and meanings;
- Drums, kundu and garamut, and decoration - shapes, volumes, and sounds;
- Dances often with specific steps, often mimicking birds and activities, and story-telling - time, place, purpose, relationships, and patterns;
- Sound making with long pipes, pan pipes, jaw harp, and bullroarer - experiments, shape, volume and sounds;
- Designing and making canoes for rivers and sea for sailing and racing – often with extensive carving; Model canoes - visuospatial reasoning, trialing, lengths, shapes, physics for movement and balance;
- Trade, trade routes, and sailing (wayfaring across the sea) – accounting, navigation, dynamic mental mapping, all senses for visuospatial reasoning;
- Detailed natural science e.g. sea, marine plants, terrestrial plants, star, moon, and planet movements - time, seasons, noticing differences and behaviours, relationships;

- Negotiations e.g., for payments of compensation and bride price, for land usage, and for feasts, money systems - comparisons of large numbers and sizes, visuospatial reasoning, rates and ratios, group counting;
- Art including tapa cloth, bilum-making, string-figures, and carvings - repeating patterns reflecting the physical geography, and associated with particular clans and cultural practices.

Importantly, cultural practices integrate STEM so STEM and STEAM education benefits from ethnomathematics in schools, beginning at early childhood education. Key to many of these practices is visuospatial reasoning.

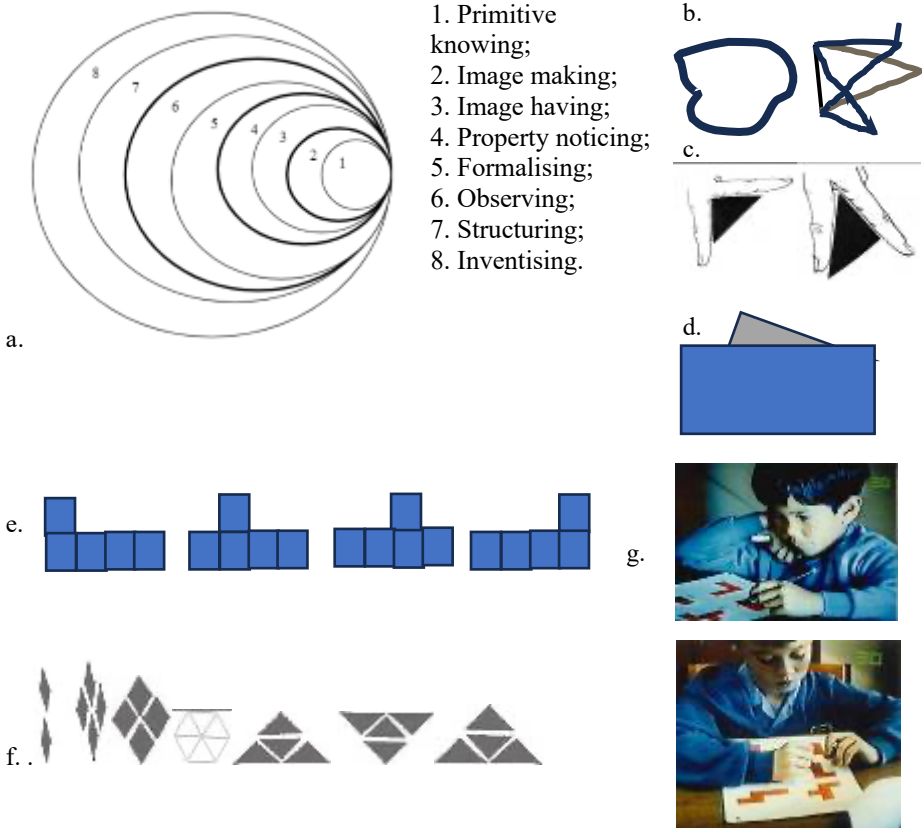
2 Spatial capabilities and visualisation - Visuospatial reasoning

Owens (2015) has studied this area of educational psychology and proposed an ecocultural approach (2020). Two significant theories influenced her research and theory. Pirie and Kieren (1986, see Figure 2a) indicate that primitive knowing and the next two steps have visuospatial aspects that begin from everyday cultural, family-based experiences. If a student cannot easily move forward through the eight steps, they will revert to an earlier stage, usually to image making or primitive knowing. Presmeg (1983) discerned different forms of visuospatial reasoning, namely holistic, pattern, dynamic, and action imagery. Reasoning on holistic imagery begins with noticing aspects. However, students continue to incorporate imagery into their mathematical reasoning through the use of patterns and through dynamic imagery incorporating change and patterns of change. Both types of visuospatial reasoning extend students conceptual understanding and problem solving.

Dynamic imagery may involve action. In Figure 2b, two fingers hold a string taut and then a stick is used to create one or more triangles (indeed some students explain the locus of points for all the triangles). In Figure 2c, students move their fingers apart to form larger angles. In Figure 2e, a student said she could see in her mind moving the square along the row of squares to make new pentomino shapes and then checking if they are reflections. In Figure 2g, students, in their minds, are folding up the shape to form an open cube. Figure 2f shows pattern imagery where students make the larger rhombus with one sized rhombus and repeat the pattern for the other, or make reflections of the pattern of triangles.

3 *Ecocultural visuospatial reasoning*

In the PNG context, the natural physical surroundings are grounds for practical mathematics classes. Traditional knowledge and practices often involve visuospatial reasoning (Owens, 2017) accompanied by physical movements. Fingers are used for counting (fingers are bent down from the little finger), identifying a whole or group, measuring, spatial positioning, bilum design making, and many other learned skills.



a. Pirie & Kieren's mode; b. Holding 2 points on a string and using a stick to make triangles; c. Opening fingers around angles to notice angle size; d. Asking several times what shape may be hidden leads to pattern and dynamic imagery; e. Making new pentominoes; f. Patterning shapes; g. Visualising how to fold up a pentomino shape to see if it makes an open cube. Developed from Pirie & Kieren, Presmeg, Van Hiele, & Owens (Owens, 2015).

Figure 2. Examples of visuospatial reasoning.

Customary land management requires negotiations on size, quality, usage, position, income and support for families, and division of land to different family members. Areas are visually compares, sometimes with taking strides

down the length and breadth. Landmarks are associated with rights to the land or its resources within a tribe. These practices are orally passed down from clan elders or chief, together with sustainable farming, hunting and fishing practices that have been honed over generations. Natural land marks like mountains, rivers, distinct rocks and trees are often used as reference points. The land tenure systems vary from community to community and are tied to kinship, cultural and spiritual beliefs (Hennings, 2021).

Bridge technology requires visuospatial reasoning from selecting materials to putting a design into practice. Designs vary. Log bridges are used to cross narrow rivers while suspension bridges and cantilever bridges (sometimes combined) are made for wide, deep, and fast flowing rivers from bush ropes (strong vines) and bamboos or saplings. These unique bridge building skills and knowledge of complex designs are shared by elders or village chief when the men of the village collectively replace or repair a bridge.



Figure 3. Ecocultural visuospatial reasoning.

Noticing is an important spatial capability for Indigenous communities. They may see a part and know where the rest of the animal or object is. They know the kind of area to find foods like fungus, ferns, nests, shell fish, and animals. They recognise both general and small details. Elders who are familiar with traditional medicines and their uses, select the children who are interested, keen and capable to go and look and bring back medicines, knowing the

environment to find them, finding their way there and back, and distinguishing the medicines from other plants. They also have to observe and listen and remember how to make and mix medicines, and to say the songs that accompany their use (Kopi, 1997).

When traversing the sea (Figure 3a), people recognise their position with all available information by memory of past experiences, look, feel, time, and reason. They recognise particular swell and current patterns by looking and bodily feeling the movement. They recognise landmarks and birds and know their flight patterns and distance out to sea. They know wind patterns and strengths. They notice places in the sea, in the jungle or when traversing bare mountains or swamps.



Figure 4. Pattern visuospatial reasoning in PNG cultures.

Visuospatial reasoning involves movement, not only walking or sailing but arm and hand movements. For example, Vandendriessche (2015) recalled two girls repeating the making of a string figure to remember the next step. Memory is in the hands and muscles as well as in the brain. Many string bag (bilum) makers in PNG, just make the stitches and patterns without much concentration with the number of stitches and changing of colours (Figure 3b). One PNG teacher noted that a trapezium was formed when two friends walked parallel from one path across to another crossing path (Figure 3c).

Patterning occurs in bilum making (Figure 3b) and weaving (Figure 3e, 4) but also in the actions for many technologies, ecoculturally meaningful art (Figure 4 top line), trade, gambling, and adjusting model canoes. Change patterns in card games and betting were socially significant in Goroka (Pickles, 2013). The model canoes were adjusted to suit the winds for better sailing as the model makers adapted their knowledge of sailing to the models (Figure 4, lower line) (Bino, 2022).

3.2 Visuospatial ratio

On numerous occasions in our field research, ratios were noted instantaneously. For example when referring to adding a fourth row of posts for a house on stilts, the comment came from Sondo Sorengke in Malalamai, Madang Province “half as much again” (Figure 3d). He also quickly converted money in current decimal currency to the older shillings (20 shillings for a pound note) for his father. On another occasion, in Muke’s Jiwaka village, the men adjusted the position of the rafter in a model to keep the trigonometry ratios the same for the roof angle (Figure 5a) although they did not know about school trigonometry. Knowing the number of pots of water for a mumu (ground cooking) for a particular number of nuts was also by visuospatial reasoning (Figure 5b). Selecting an area of kunai grass or the correct size sago palm to make sufficient roof coverings for a house was like a ready-reckoner of ratios in the head. (Figure 5c are the morata made from sago leaves folded and sewn over limbom palm sticks.) When one Elder in Kaveve village was describing the building of his house, he immediately discussed the size of the gardens needed to be planted in advance to feed helpers proportional to the house size. Details can be found in Owens (2015), and Paraide et al. (2023). The ratio and rates reasoning was through experientially developed visual memory of events and sizes.



Figure 5. Ecocultural visuospatial ratio reasoning.

Like all the STEM activities, these past activities from before colonisation are still practiced today, albeit not as frequently or as widely or with modifications. Visuospatial reasoning is well developed. Pattern and dynamic imagery, especially with bodily awareness and movement, abound.

3.3 Visuospatial reasoning and number

Number was also considered in terms of size or body position, especially in reference to the whole. The part-whole relationship is strong whether this is for numbers that group to 5 (whole hand, may be 4 or 6 too), 10 (2 hands, may be 8) or to 20 (hands and feet) or as a body-part tally system (various odd numbers like 27) for a whole (Owens et al., 2018). A number of Austronesian Oceanic and some neighbouring non-Austronesian languages express 6, 7, 8 and 9 as so many to make the whole group of 10 while others specify the group of 5 or 10 in connection to the objects in the group. The ecocultural aspects also link with the emphasis on pairs seen in number systems like Motu where 6 and 8 are 3 pairs and 4 pairs. Links to the whole from a worldview are evident in the (2, 5, 20) cycle system of Iqwaye in the crouching positioning for powers of 20 (Mimica, 1988). Their sense of infinity and oneness is embedded in mythology which also links sky-earth and man-woman for one so pairs are needed together. Other areas associated with number and iteration and groups are the naming of alternate generations with the same names, one of many kinship patterns found across the world. Thus their kinship naming can be associated with Abelian and dihedral groups.

Another aspect of number is the association of large quantities with displays. These are associated with ceremonies such as bride price and funeral distributions, and welcomes into community. Thus the visuospatial aspects of numbers are evident. For example, the fathoms of shell money in East New Britain are bundled up into large rings. In the highlands, the pigs are lined up and ceremonially counted, the sweet potatoes put in mounds, and the flowers, leaves and bananas are displayed. Another visuospatial aid is the use of body-parts for numbers. This occurs in the body-part tally systems but also Muke's father pointed to different body-parts to record hundreds to remember exchanges and compensations (Owens et al., 2018).

4 An early childhood project

Above we have noted how ethnomathematics was valuable for primary and high school and tertiary mathematics. Our early childhood training connected cultural mathematics and quality early childhood mathematics education. Teachers would plan details for a week of lessons that the teachers taught to peers or in schools. Discussions involved having more than one answer to open questions and how to say school mathematical terms in language. In this way, they were thinking more deeply about the meaning of the school mathematics and how it related to them. Each week begins with Tuning In with a cultural activity as many contain mathematics, followed by planning to find out, finding out, sorting out, and then going further, making connections, taking action, and sharing, reflecting and summarising new learning (Murdoch, 2019). In these cultural mathematics lessons, teachers explored their cultural setting of their mathematics enhancing their mathematical identity (Owens 2015). However, to be ongoing, each language group requires significant input from Elders, teachers, and people who know their language and mathematics. Teachers in more remote areas are more likely to teach in village ways and with local language to explain to the children (Muke, 2012).

5 Conclusion

Cultural mathematics often arises from practical needs such as agriculture, navigation, trade, and architecture. These applications require a profound understanding of mathematical principles tailored to specific environmental and social contexts. The long past of PNG cultures has fine tuned these STEM areas. Many of the cultural practices occurred in the recent past and even in the present albeit with some modifications and with less involvement of all the people of a particular society. For the future, as part of each student's identity, it is important to know these cultural practices and be proud of their ancestors and elders and their mathematics. Significantly visuospatial mathematical reasoning is a major aspect of these cultural practices. Hence ethnomathematics enriches our understanding of both visuospatial reasoning and number. An ecocultural approach to teaching and learning enhances mathematical identity both through the impact on cognitive processing, especially visuospatial reasoning, and on affective processes through identifying with culture and being emboldened to solve the problem, leading to responsiveness that develops into

a strong ecocultural mathematical identity. Ethnomathematics encourages students to see mathematics as a tool for addressing real-world environmental problems with a cultural lens. This perspective fosters responsibility for future sustainability while applying mathematics in ways that benefit the society including village societies. PNG ethnomathematics demonstrates advanced visuospatial mathematical reasoning and fosters an ecocultural mathematical identity.

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CULTURAL PERSPECTIVES ON MULTIPLICATIVE THINKING IN MATHEMATICS

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ABSTRACT

This paper explores how students understand and express multiplicative relationships and patterns through both conventional representations and culturally grounded pedagogical approaches. The Goompi Model—an Indigenous framework—is examined to show how it supports learners in recognising structure, growth, and generalisation in pattern-based tasks. Drawing on a case of growing patterns, this study shows how students transition from concrete representations to algebraic expressions by engaging with visual and cultural forms. Rather than relying on solely abstract notation, students use spatial logic and culturally embedded reasoning to articulate linear relationships. This approach offers insights into the teaching of algebraic thinking and multiplicative reasoning, bridging formal mathematics with everyday meaning-making.

1 Introduction

Understanding multiplicative relationships is foundational to algebraic thinking and functional generalisation. These relationships support key mathematical ideas such as growth, scaling, and symbolic expression. While concepts like ratio, rate, and function all involve multiplicative reasoning, this paper focuses on how students perceive structure and repeated change—particularly through growing patterns that can be expressed as linear equations of the form:

$$y = mx + c.$$

This paper examines how students transition from concrete representations of change to symbolic generalisation, and how culturally responsive approaches can support this process. Specifically, the Goompi Model, an Indigenous pedagogical framework, is explored to examine whether cultural perspectives, spatial reasoning and symbolic meaning-making can be integrated to support students in constructing mathematical relationships through observation and interpretation.

2 *Theoretical Perspectives Contextualised*

Learning can be understood through different interpretative lenses, each offering unique insights into how understanding develops. The Goompi Model, grounded in Indigenous epistemologies, represents one such lens—highlighting the cultural, symbolic, and experiential dimensions of mathematical learning (Matthews, 2012). While distinct in its foundations, the Goompi approach resonates in important ways with the theories of Bruner (1966) and Vygotsky (1978). Like Bruner’s emphasis on progressive representation (enactive, iconic, symbolic) and Vygotsky’s focus on the social and cultural mediation of learning, the Goompi Model affirms that meaning-making emerges through both internal construction and external cultural contexts. Viewed together, these perspectives enrich our understanding of how students develop multiplicative reasoning through culturally embedded mathematical experiences.

Bruner (1966) proposes that learners construct knowledge by progressing through enactive (action-based), iconic (image-based), and symbolic (abstract) stages. This model informs our interpretation of how students move from visual or spatial patterns to generalised mathematical expressions such as linear equations.

Similarly, students’ engagement in collective pattern creation and discussion, allowing mathematical understanding to emerge from shared cultural experiences directly reflect Vygotsky’s (1978) theory that learning is a socially embedded process shaped by context, tools, and interaction.

In parallel, the Goompi Model offers an epistemological and pedagogical lens rooted in Indigenous perspectives. It conceptualises mathematics as a cyclical process where knowledge emerges through lived experience, is abstracted and expressed through culturally mediated symbols, and is reflected back into real-world contexts. This framework foregrounds cultural identity and symbolic meaning-making as integral to mathematical understanding.

2.1 **The Goompi Model**

The Goompi Model, developed by Matthews (2008), provides a culturally grounded framework for understanding mathematics as a dynamic and cyclical process. As illustrated in Figure 1, the model shows how learners begin with

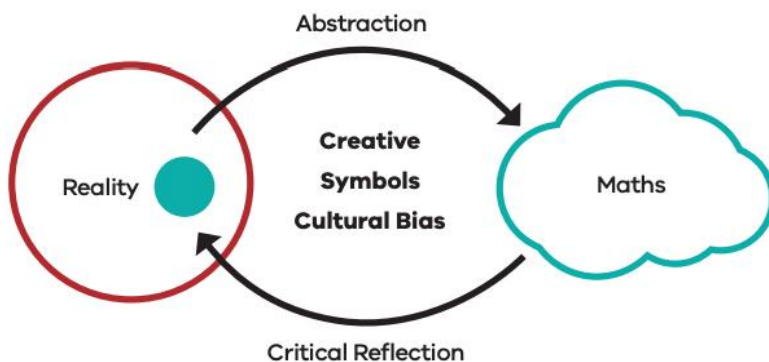


Figure 1. The Goompi Model (Matthews, 2020, p. 5)

reality—lived experiences or phenomena they wish to understand.

Through abstraction, they generate representations such as drawings, language, and symbols, which are shaped by creativity, cultural bias, and symbolic meaning. These become formalised into what we refer to as mathematics.

The cycle continues as students apply their mathematical ideas back to reality through critical reflection, allowing them to test, refine, and contextualise their understanding. This cyclical process foregrounds three key cultural elements: Creativity as self-expression; Symbols as bridges between reality and abstraction; Cultural bias as an inevitable and meaningful lens shaping how knowledge is constructed (Matthews, 2008).

In this paper, the Goompi Model is not used as a prescriptive teaching method but as a conceptual lens (Matthews, 2012) to examine how culturally embedded visual patterns can support students' development of multiplicative reasoning. The model highlights the potential of connecting formal mathematics to students' lived experiences, enriching both understanding and engagement.

2.2 Cultural Pathways to Multiplicative Thinking

The paper draws on cultural and Indigenous perspectives referring to mathematical ways of knowing that emerge from the everyday practices, worldviews, and pedagogies of diverse communities (Matthews, 2008; Ascher, 2018; Lipka et al., 2005). While multiplicative reasoning is found

dational in formal mathematics, it also emerges organically in cultural practices through practical, visual, and spatial tasks.

In Indigenous and artisanal traditions, this form of reasoning is embedded in activities such as time measurement, navigation, weaving, and art (Ascher, 2018). For example, American Indian design practices (Lipka et al., 2005), often generate shapes through diagonal or midpoint cuts along lines of symmetry, ensuring that components are proportionally related. The emphasis on symmetry, congruence, and visual balance—reflected in cultural notions such as “black and white must be in balance; one follows the other” (p. 33), —demonstrates how multiplicative thinking is deeply rooted in spatial and visual design. This alignment reveals a meaningful convergence between community-based knowledge systems (Miller & Armour, 2021) and the formal mathematics taught in schools.

The Goompi Model, developed by Indigenous educator Chris Matthews (2008), exemplifies this integration. It draws on culturally grounded ways of seeing and representing growth. The model’s spiral patterns echo recurring forms in nature (e.g., shells, storms, galaxies) that students may recognise from cultural narratives or lived experience. Its pedagogy is dialogical, participatory, and observation-driven, reflecting Indigenous teaching traditions. In this way, mathematics is not imposed on culture but emerges meaningfully from within it.

3 Research Focus and Question

This paper adopts a practice-oriented lens to analyse how growing patterns can support students’ understanding of multiplicative relationships when presented through culturally situated frameworks. In particular, it draws from documented pedagogical practices, including a linear pattern task rooted in the Indigenous Goompi Model, to explore how mathematical ideas are constructed through cultural tools and representations. Rather than presenting new empirical data, the discussion synthesises insights from existing classroom applications to explore the conceptual significance of such approaches. Accordingly, the research question guiding this paper is: *How can Indigenous pedagogical approaches, such as the Goompi Model, support students’ understanding of multiplicative relationships through growing patterns?*

4 *Conventional Introduction to Linear Relationships: The Growing Pattern Approach*

In conventional instruction, students are often introduced to linear equations through the abstract formula $y = mx + c$, a method that can feel disconnected and cognitively demanding. To ease this transition, teachers frequently use growing patterns, such as the T-block configuration shown in Figure 2 as a visual tool to help students recognise structure and derive symbolic expressions. The growing pattern approach typically involves two steps: first, students identify a number sequence by counting the number of squares in each stage of the pattern. This step is generally intuitive and accessible. However, the next step, which involves formulating a linear equation from that sequence, is often less obvious. It requires students to discern the underlying structure of change and express it algebraically, presenting a cognitive challenge that may not be immediately apparent.

From the sequence, 5, 6, 7 and 8, students are expected to derive a generalised equation, $y = x + 5$, where x represents the number of added blue blocks, and y the total number of blocks. Generating a symbolic expression from the pattern involves a significant conceptual leap. It requires students to distinguish between constant and changing components, and to express that relationship algebraically.

The visual pattern depicts a growing structure, where each figure in the sequence adds one additional unit compared to the previous figure: 5 squares in the first figure, 6 in the second, and so on. This consistent increase by one square per step reflects a linear growth pattern.

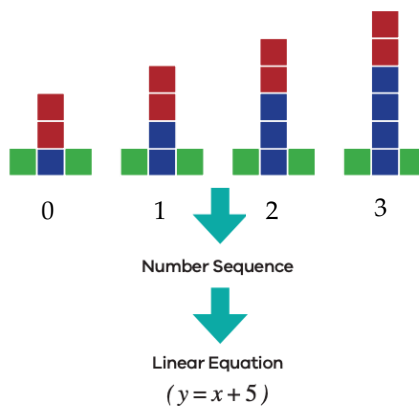


Figure 2. A typical representation of a growing pattern (Matthews, 2020, p. 6)

The equation $y = x + 5$ represents this growth algebraically, where:

- x refers to the step number, starting from 0.
- x can also be interpreted as the number of blue squares added.
- y represents the total number of squares in that step.

So, at Step 0 ($x = 0$): $y = 0 + 5 = 5$ squares

At Step 1 ($x = 1$): $y = 1 + 5 = 6$ squares

At Step 2 ($x = 2$): $y = 2 + 5 = 7$ squares

At Step 3 ($x = 3$): $y = 3 + 5 = 8$ squares.

Although the T-block pattern is designed to help students distinguish between the constant part (the part that stays unchanged) and the part that grows (represented by the multiplicative factor), it can still pose cognitive challenges. While students may grasp the additive aspect, recognising that one block is added at each stage, they often struggle to interpret this repeated addition as a multiplicative relationship. Identifying that the total number of blocks increases by a consistent factor (m) relative to the step number requires abstract thinking. This makes the transition from counting to constructing a generalised algebraic rule, $y = mx + c$, a nontrivial step for many learners.

5 *The growing pattern pedagogy using the Goompi model*

While the conventional growing pattern approach supports students' emerging understanding of linear relationships through structured visuals and colour differentiation, it remains relatively abstracted from personal or cultural meaning. The figures are helpful but arbitrary. In contrast, the Goompi Model (Matthews, 2008) situates mathematical growth within cultural forms that carry meaning—spiral patterns drawn from Indigenous knowledge systems. This model encourages students not only to observe growth but to interpret it through personal perspective, spatial reasoning, and narrative. In doing so, it has the potential to deepen engagement and make the structure of a linear relationship more intuitive, especially for students who benefit from contextual, story-based, or visual-spatial learning. The following section explores how this model supports the same core algebraic concepts—constancy and change—through a culturally grounded representation. Matthews demonstrates how the Goompi model can be applied to enhance the teaching and learning of growing patterns through 3 stages of representation, as shown in Figures 3,4 and 5.

Figure 3 introduces the first stage of Matthew's growing pattern activity

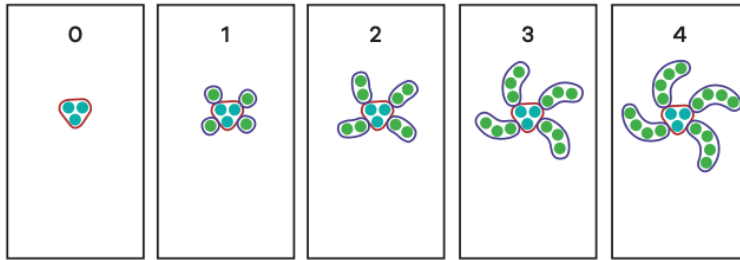


Figure 3. First stage of growing pattern pedagogy using the Goompi Model (Matthews, 2020, p.7)

using the Goompi Model. In this stage, students collaboratively construct a spiral pattern over five days on a large sheet of butcher's paper. The activity is designed to help them connect the visual structure of a growing pattern with the abstract structure of a linear equation.

The pattern begins on Day 0 with a triangular red outline surrounding three blue dots at the centre, representing the constant part of the pattern. Each subsequent day, green dots are added in a regular, outward-spreading spiral. The rule governing this growth is that four green dots are added per day, arranged into a new curved arm that extends from the central core. These arms increase in length and number, forming a swirling spiral shape over time.

The teacher facilitates this process, modelling how to extend the pattern and prompting students to follow the rule: for each new day, add four new green dots, one per arm, so that each arm grows symmetrically. Students may associate these visual forms with familiar natural spirals such as hurricanes, galaxies, or flowers, which helps reinforce the cultural grounding of the activity. To support mathematical understanding, the teacher prompts students to distinguish between the parts that remain constant and those that grow. The constant part (3 blue dots) is marked in red, while the growing green dots are colour-coded to highlight the additive progression.

The use of butcher's paper serves a specific pedagogical purpose: it offers a large, communal workspace that allows the whole class to participate in and reflect on the pattern-building process over multiple days. Its expansive size enables the growing spiral to remain visible and physically accessible throughout the unit, supporting both spatial reasoning and collaborative dialogue.

In the second stage (Figure 4), students begin quantifying the visual pattern. They calculate the total number of dots (S) on each day by summing the constant part (always 3 dots) with the growing part — which increases by 4 dots per day. On Day 1, for example, students observe 3 constant dots + 4 new dots = 7 dots. This progression continues across Days 2 to 4:

$$\text{Day 2: } 3 + 8 = 11$$

$$\text{Day 3: } 3 + 12 = 15$$

$$\text{Day 4: } 3 + 16 = 19$$

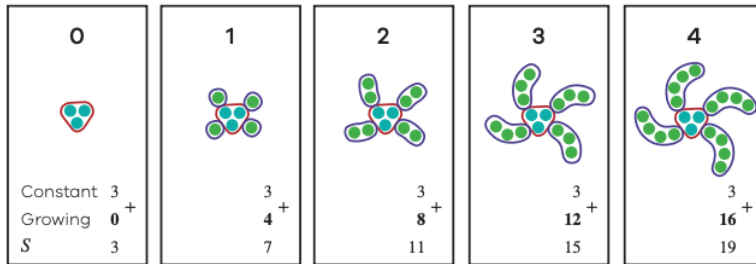


Figure 4. Second stage of growing pattern pedagogy using the Goompi Model (Matthews, 2020, p. 7)

This helps students shift from qualitative pattern recognition to quantitative analysis. They are then asked to make predictions, such as estimating how many dots will appear on Day 365. In doing so, they begin identifying the underlying structure of the pattern.

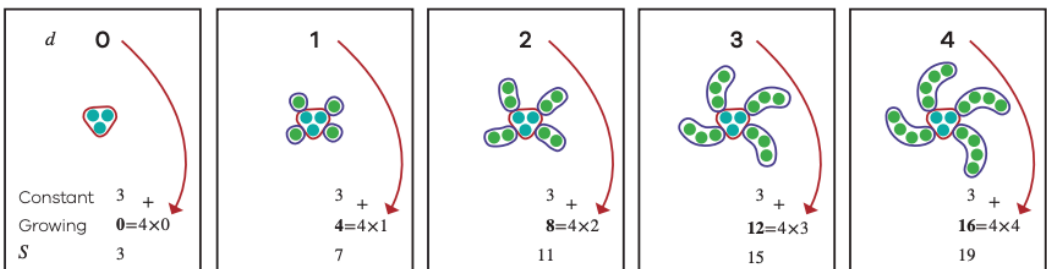


Figure 5. Third stage of growing pattern pedagogy using the Goompi Model (Matthews, 2020, p. 8)

In the final stage, the focus shifts to developing a mathematical relationship between the spiral's size (S) and the number of days (d) by analysing the

constant and growing parts individually. As illustrated in Figure 5, students observe that the total number of dots at each step is composed of a constant core (always 3 dots) and a growing component that increases by 4 dots per day. This leads to the generalisation:

$$S = 4 \times d + 3,$$

where S is the total number of dots and d is the number of days.

This equation captures the structure of the pattern: for every day that passes, 4 additional dots are added to a constant core of 3 dots, resulting in the general form $S = 4d + 3$. Starting the pattern at Day 0 highlights the presence of this existing structure before any growth occurs, making the linear relationship more explicit. It ensures that when $d = 0$, $S = 3$, allowing students to clearly connect the equation to the visual pattern and understand how each part—the constant and the growing—contributes to the overall structure. To reinforce this understanding, the teacher challenges students to interpret alternative rules (e.g., $S = 5d + 2$) and construct new patterns accordingly, encouraging them to explore how different constants and rates of growth affect the structure.

6 *Conceptual Insights and Discussion*

Growing patterns follow a consistent rule, such as adding four dots at each step, which can be expressed using the equation $S = 4d + 3$. This linear form helps students transition from recognising visual patterns to representing them symbolically. Understanding multiplicative relationships builds students' ability to generalise, predict, and model real-world scenarios. Recognising how quantities scale supports critical thinking, especially when students use multipliers to project future values.

The Goompi Model frames linear growth through a culturally responsive lens, embedding mathematical ideas in a pattern within natural forms and familiar designs. By grounding abstract concepts like linearity and multiplicative relationships in visual and experiential learning, students can begin to see mathematics as both universal in structure and diverse in expression.

The relationship represented by the equation $S = 4d + 3$ exemplifies a multiplicative structure, where a consistent rate of increase (4 new elements per day) is added to a fixed starting value. This mirrors the format $y = mx + c$, first introduced through the T-block growing pattern. While the T-blocks offer a

more schematic or algebraic representation, the Goompi approach leads students to construct the same multiplicative structure through observation, spatial reasoning, and cultural narrative.

In the Goompi model, students are not simply applying a given rule; they construct the rule themselves based on observation, spatial reasoning, and collective discussion. This contrast reveals the approach's potential to deepen student's understanding.

7 Conclusion

This paper examined how the Goompi model supports students in constructing mathematical rules through observation and spatial reasoning. It exemplifies how learning can unfold through culturally grounded representations, offering students intuitive access to complex concepts like linear equations. This reflects a constructivist perspective, drawing from Vygotsky's emphasis on social and cultural mediation and Bruner's theory of cognitive representation, where learners move from action-based experiences to abstract symbols. These perspectives affirm that mathematical thinking can evolve through multiple cultural and cognitive pathways.

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HOW CAN THE HISTORY OF THE EXISTENCE OF FOURTH PROPORTIONALS FROM EUDOXOS VIA OMAR KHAYYAM AND NASIR AL-DIN AL-TUSI TO ISAAC NEWTON FOSTER A MODERN MATHEMATICAL NUMBER CONCEPT?

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A concept development towards a modern understanding of real numbers is explored using the historical genetic method. First, we consider the concept of the fourth proportional (given magnitudes a and b of the same kind and another magnitude d , then there is a magnitude c of the kind like d , such that $a : b = c : d$) at the hand of excerpts from book V, attributed to Eudoxos, in Euclid's Elements. Then we discover and discuss situations in contemporary school textbooks in which the existence of fourth proportionals is assumed and used. We look at these textbooks through the eyes of Eudoxos and Omar Khayyam (1048–1131). In Euclid's book V, we study definition 5 of the equality of two ratios and compare it with the “antiphairctic” definition based on the Euclidean algorithm. With Khayyam, for us (positive real) numbers are nothing but proportions of magnitudes that, surprisingly, can be calculated with if at least the calculation laws are proven, which Euclid and Khayyam partly did. The proof of the is in Euclid's book V and later in Nasir al-Din al-Tusi (1201–1274). In the early modern period, the concept of number as a proportion of magnitudes was anew clarified by Isaac Newton (1642-1726) and later attributed to him (cf. Frege, p. 25). The existence of the fourth proportional can serve as “Grundvorstellung” (mental model) for the arithmetic of real numbers, which we try to develop together with the participants along the historical perspectives such that the educational potential becomes visible.

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TEACHING MATHEMATIZING THROUGH STORIES FROM THE HISTORY OF MATHEMATICS: PROMOTING CULTURALLY RESPONSIVE AND SUSTAINABLE MATHEMATICS EDUCATION

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The session will start with a presentation showcasing the use of a children's story as a context for mathematizing (Freudenthal, 1991). We will model the teacher's role in initiating a lesson through storytelling. Participants will then actively engage by responding to questions related to the story and solving embedded mathematical tasks. This will serve as a foundation for discussing key aspects of our methodology (Haydar & Durmaz, 2022) focusing on the integration of mathematics and storytelling (Sims Bishop, 2011), the utilization of contexts for mathematizing, and the incorporation of the history of mathematics in a framework of culturally responsive/sustainable pedagogy (Djebbar et al., 2009).

In the second half, we will work in two groups and analyze two instructional units:

Al Kindi and Cryptography:

The unit begins with notice and wonder routines introducing code-breaking, followed by an online game. A historical storytelling context delves into Julius Caesar's coded messages during wars, leading to mini-lessons on inverse operations and functional rules. The literacy connection includes a reading on the House of Wisdom, featuring the Arab philosopher and mathematician Al Kindi and his work on cryptography. This sets the stage for an investigation guiding students to reconstruct Al Kindi's frequency analysis method. Social studies connections explore the history of Baghdad and the scientific contributions from the House of Wisdom.

Ibn Al Haytham and Visual Proofs:

The unit begins with a children's story about Cubey Cake, using a birthday cake and dice context to introduce a guided proof of adding consecutive whole numbers 1 to 6. A task follows on constructing Ibn Al Haytham's visual proof using Cuisenaire rods investigations. A video story narrating Ibn Al Haytham's biography leads to interdisciplinary connections around his journey and imprisonment in Egypt and his seminal work on Optics. The following mathematical investigation leads to generalizing the formu-

la for the sum of consecutive numbers and culminates with the story of Gauss.

We will conclude with participants' feedback and general discussion.

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INFLUENCES OF A LIBERAL-ART COURSE ABOUT EAST-ASIAN MATHEMATICAL CULTURE ON UNIVERSITY STUDENTS' MATHEMATICS BELIEFS

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This paper examines the impacts of a liberal-art course about East-Asian mathematical cultures on university students' mathematics beliefs. The course, taught in a comprehensive university in Taiwan, explored pre-modern East-Asian mathematics, focusing on traditional Korean culture and mathematics, which have deep links to its Chinese counterparts. This course covered key theories, methods, and problems of pre-modern East-Asian mathematics, highlighting unique reasoning patterns and possible applications in fields such as astronomy and civil engineering. In addition to Chinese treatises, this course exposed students to some Korean scientific artefacts, such as the *Cheomseongdae* ('star-gazing tower'), counting rods, and mathematical texts containing problems related to government functions in pre-modern East Asia. We used a survey research method, with quantitative and qualitative tools. The quantitative tool was a twenty-seven-item Likert-scale questionnaire on mathematics beliefs, which was designed to examine students' beliefs in the dimensions of the 'nature' and 'values' of mathematics; the qualitative tool was students' after-class reflexions. The two tools were analysed with descriptive and inferential statistics, and content analysis, respectively. A total of forty-six students from six different colleges (Science, Engineering, Nuclear Science, Humanities and Social Sciences, Electrical Engineering and Computer Sciences, and Education) of that university took the course, wrote several reflexions, and finished both the pre-test and post-test of the questionnaire. Quantitative results show that, generally speaking, for the dimension of the nature of mathematics, students had a more diversified understanding of the justification of mathematical knowledge; for the dimension of the values of mathematics, students tended to agree more on the links between mathematics and traditional cultures, humanities, and real-world applications. Individual students' changes between the pre-test and post-test have also been compared, and it could be seen that most students were inclined to more diversified and culturally-oriented beliefs about mathematics. Qualitative data were used to triangulate our findings from the quantitative data. In general,

research results show that the liberal-arts course about East-Asian mathematical cultures has a similar impact with earlier studies on university students' mathematics beliefs in terms of the nature of mathematics, in which students, though highly approve of the role of logic in mathematics, also confuse the context of discovery with that of justification in mathematics; results also show that after taking this kind of course students' beliefs in terms of the values of mathematics have a stronger tendency that mathematics can be linked to real-world applications, traditional cultures, and humanities and social sciences.

GLOBALISATION THROUGH THE LENS OF HISTORY OF MATHEMATICS EDUCATION

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ABSTRACT

The emergence of nation states in the 19th century, the institutionalization of school systems and the corresponding teacher training led to the development of national education systems in the following centuries. These systems were shaped by various factors, including economic conditions, military activities, ideological paradigms, and cultural traditions. In the history of education, the internationalization of mathematics education is largely assumed progressive, which requires a critical examination. As we will show, these narratives can be questioned by investigating historical examples and the study of tensions between universal global trends and cultural-historical traditions.

1 Different views on Internationalization and Globalization

The history of mathematics education as a scientific discipline is much younger than the history of mathematics. Nevertheless, it is closely interwoven with the history of mathematics. Topics like the history of institutions, biographical historiography, history of mathematical concepts can also be seen as parts of the history of mathematics and its methodologies can partly be used (Karp & Furinghetti, 2018, p.3). While the history of mathematics often focuses on the study of historical artefacts and sources and historical and cultural contextualizations are rather local and limited to short time intervals, research in the history of mathematics education is often related to long-term processes and their patterns. In particular, the study of recent history like the history of the New Math reforms show, that by analysing processes in the past the history of mathematics education can develop perspectives that allow the present to be seen as a continuation of cultural-historical processes and thus to explore meaningful critical analyses of currently existing social practices (Fleck, 1979, p. 20):

“...whether we like it or not, we can't get rid of the past - with all its errors. It lives further in borrowed terms, in problem formulations, in school teaching, in everyday life, in language and in institutions. There is no *Generatio spontanea* of concepts. They are, so to speak, determined by their ancestors. The past is much more

dangerous - or actually only than dangerous – if the bond with it remains unconscious and unknown.”

This article focuses on phenomena, which are described as global in the educational policies of many countries. Recent examples of such global processes are output - and competence orientation, introduction of educational standards, central tests and global assessments, as well as economization, centralization, digitalization of education systems (e.g. Atweh et al., 2001, Atweh & Clarkson, 2002).

Globalization is often seen as a kind of internationalization of processes that take place locally more or less simultaneously and independently of one another, becoming more international through cooperation and then forming a global process through the large number of participating states. Thereby, a global network is formed and then identified with the process. This narrative enables the participating states to assume that the resulting global network serves as a political tool for solving local problems.

Critical positions on the assumed progressive character of global networks started to develop in the second half of the 20th century, especially with regard to the leadership and control claims made by “Western democracies”. As early as 1961, the OECD demanded (Graupe & Krautz, 2014, p. 2):

“...with regard to developing countries, it would be ‘nothing short of cutting a million people loose from a way of life that has constituted their living environment for hundreds or thousands of years. Everything achieved by these countries’ schools and education until now has served social and religious aims, which have primarily allowed for resignation and spiritual comfort; things that completely go against any economic sense of progress. Changing these century-old approaches may, perhaps, be the most difficult yet also most important task for education to accomplish in developing countries.”

A systematic study of the influence exerted on national educational systems in the context of colonialism and neocolonialism was in the second half of the last century initiated by concepts of critical pedagogy, critical psychology, critical mathematics education and the philosophy of mathematics.

Since the 1980s, representatives of ethnomathematics and critical mathematics education, like Freire, D’Ambrosio and Skovsmose, questioned the leadership and control claims of so-called “Developed Western democracies” over “developing countries”. Their educational approaches take into account the social, cultural and historical conditions of “developing countries” and address the situation of disadvantaged population groups. From this perspective,

though, an unintentional identification of North American, Western European and Asian educational systems takes place under the notion of “Western democracies”. However, the latter are partly very different in terms of their cultural traditions (e.g. Kocka, 1987), the extent of economization, profit orientation and power structures. The educational approaches of socialist Eastern European and Asian states, although very successful in science and mathematics education, are rarely visible from the perspective of the global antagonism “Developed Western countries vs. developing countries”.

Another view, which ascribes enormous importance to the phenomenon of globalization as a key concept for understanding “the transition of human society into the third millennium” but sees globalization without historical contexts and cultural and social common references, was formulated in the 1990s (Waters, 1995, p.1). Here globalization is seen as “a concept which refers both to the compression of the world and the intensification of consciousness of the world as a whole” or as “a social process in which the constraints of geography on social and cultural arrangements recede and in which people become increasingly aware that they are receding”. In mathematics education, such global trends are e.g. convergence of school mathematics and mathematics education curricula around the world, similarity in research questions or epistemological positions such as constructivism. This kind of actions without actors are often associated with “forces that are impersonal and beyond the control and intentions of any individual or groups of individuals” (cf. Atweh et al., 2001, p. 79, Atweh & Clarkson, 2002).

In other words, globalization in this case is not considered to be related to the solution of economic and social issues but to slogans, such as container terms with arbitrarily postulated meaning created by the network itself. For example, “virtual reality, information society, digital natives”, which are related to visions, utopias without restrictions in terms of space and time and therefore without a past. Since the beginning of the millennium, there have been numerous efforts, particularly by critical mathematics education, to systematically study educational and socioeconomic phenomena occurring worldwide (see e.g. Atweh et al., 2001, Atweh et al. 2007).

We will approach the phenomenon of globalization and its conceptualization from the perspective of the history of mathematics education. Depending on the context, internationalization and globalization can be defined differently. However, internationalization processes are traditionally seen as having a

bottom-up dynamic, starting from local, culturally and historically anchored problems. Internationalization here means above all the adaptation of functioning local “best practices” to other local, like national or cultural-historical conditions; in other words, cross-country dissemination based on local needs. In contrast to this, a system-theoretically oriented, structural approach may see internationalization as growing numbers of connections in international networks; according to this understanding, any activity that involves a cross-country collaboration contributes to the internationalization of the activities of the partners (Henry et al., 2013). For example, from a system-theoretic perspective, publications in “international journals” increase the “international visibility” of the author(s); support of collaboration and division of labour are rather not in the focus.

We will follow the first cultural-historical interpretation of internationalization. In this understanding, we see globalization as a contrasting top-down movement whose motives no longer need to be justified locally in cultural-historical or socioeconomic terms. We will use historical examples from the German history of mathematics education to sharpen our understanding of the differences between internationalization and globalization.

2 German History of Mathematics Education in the Context of Internationalization and Globalization

2.1 Germany as a Case Study

One might ask why the history of teaching reforms in Germany can be illuminating for phenomena related to internationalization and globalization. As we will see, Germany's history of mathematics education is of particular interest from several perspectives.

During two centuries, Germany had various socioeconomic systems and thus different views of humanity, educational paradigms and educational systems: an Imperial Empire (German Empire after the Franco-Prussian War), the Weimar Republic, the fascist Nazi Germany, 4 occupation zones, West German capitalist social market economy, East German socialist society and German neoliberal imperialism. German history therefore offers the opportunity to examine internationalization and globalization processes in different social contexts. We will restrict our study to three international reforms; the reform of geometry teaching Newer Geometry in the 19th century, the Meran-

er Reform and the New (or Modern) Math Movement in the 20th century. All three aimed at the modernization of upper secondary mathematics teaching. The restriction to the teaching of geometry does not represent a real restriction here, since the changes in the teaching of school geometry were essential for all three reforms (Weiss, 2022). The consideration of Germany as a case study is also interesting in connection with internationally known protagonists of international associations for the promotion of mathematics education. German mathematicians such as Felix Klein, Walter Lietzmann and later Heinrich Behnke and Georg Steiner played a significant role in the formation, development and management of organizations supporting the foundation of ICMI as well as in the reforms of geometry teaching (Tobies, 2021, Schubring, 2019, Tobies, 2024).

Finally, yet importantly, the development of the German educational system was strongly influenced by supranational movements such as the Enlightenment, the development of dual higher education careers, reform pedagogy and a culture of accompanying instruction and oral examination by the mathematics teacher that is characteristic of the German-speaking educational models and differs from the Anglo-Saxon model.

2.2 German Enlightenment

To this day, the spirit of the Enlightenment permeates the German education system, especially the university law. The idea of man's emergence from his self-incurred immaturity is expressed in fundamentals of the German education system, like trust in the professionalism of the teacher and the orientation towards maturity autonomy, emancipation, responsibility, self-realization, and self-determination from teacher and student. Further ideas of the Enlightenment that were integrated into the German higher education system in the 19th century were and are the unity of research and teaching, as well as the freedom to choose the teaching methods. The symbiosis of autonomy with *Verstaatlichung* (nationalization) and of *Zweckfreiheit* (uncommitted research) with purpose oriented practices (e.g. Tenorth, 2017) was only in this millennium replaced by the dominance of third-party funded research.

Another characteristic of the German education system is the civil servant status of professors and teachers, which entails high social prestige and security on the one hand and loyalty to the state on the other. Since the academiza-

tion at the beginning of the 19th century, teacher education for secondary schools in Germany has been at a high academic level. For career promotion teachers are expected to participate in committees for school development and development of curricula, as well as in the development of textbooks and methodologies and publications in corresponding journals.

2.3 German Dual Education and Reform Pedagogy

Another key factor in the development of mathematics education in Germany was the creation of a higher education path, dual to the classical Humboldtian *Gymnasium*, through the establishment of new school types; the *Oberrealschule* and the *Realgymnasium*. The qualifications at these schools were given the same status as those of classical *Gymnasiums* at the beginning of the 20th century; their curricula were particularly well coordinated with further scientific or engineering studies. Ideas adopted from the Enlightenment and rooted in elementary education such as knowledge does not come from books but from dealing with objects in nature and in the world and the concept of experimental learning (cf. Schubring, 2019) led to methodologies that were oriented towards the cognitive abilities and interests of the child (Fröbel, Pestalozzi, Herbart) and work-based education (Kerschensteiner, Ostrich).

2.4 Guided Instruction and Process Orientation as a German Teaching Tradition

Trust in the teacher was not only shown by the social status of the civil servant and the representation of an office, but also in the complete responsibility for assessing the learning success of the students. The tests and examinations were almost always created by the teacher themselves and were intended to serve more as diagnostics and feedback than as a selection tool, with the focus being on learning progress under accompanying guidance. (Lietzmann, 1926, p. 207):

“That examinations, where examinee and examiner know each other, are the norm in our country seems to me to be an essential advantage of our school system. Where, as in France or in England, both parts are alien to each other or even - if they are limited to written exams - do not even come across each other, the exam itself is more difficult. Still, it doesn't lead to higher performance on average, despite a sometimes unworthy exam drill.”

Access to studying mathematics, technology or natural sciences at university was not via entrance examinations. Examinations were preferably oral

and were taken by the teacher who had also taught the class; this social practices were changed only in this Millennium.

3 *Newer Geometry as a (bottom-up) Internationalization Process*

The development of the teaching reform Newer Geometry, which took place in Germany in the second half of the 19th century, illustrates particularly effectively the importance of cultural-historical circumstances for the development of a reform movement from below. “Newer geometry” meant the disciplines that established themselves in the 19th century as Descriptive geometry (Monge), Geometry of position (Carnot, von Staudt), Geometric constructions (Steiner), Projective properties (Poncelet), Barycentric coordinates (Möbius), Linear algebra, algebra (Graßmann, Plücker), Analytical geometry (Gergonne) as well as the reform of the strict school geometry teaching by Euclid’s Elements. A uniform description of German educational policy is difficult, since educational policy in Germany is a federal matter.

Due to the professionalization and academisation of teacher education at the beginning of the 19th century, mathematics *Gymnasium* teachers were familiar with projective geometry and other non-Euclidean approaches through their university studies. Using descriptive geometry as an example, Barbin et al. (2019) show the spread of these ideas in some European countries and the close connection between the development of descriptive geometry and the development of curricula and teaching methods for the newly founded polytechnic schools and technical universities. In most federal states of Germany at the beginning of the 20th century, important aspects of projective geometry were anchored in the curricula and textbooks of secondary schools. This rapid inclusion of higher mathematics in secondary school curricula was due both to the nature of the subject and to the high qualifications of the teachers, who had partly worked on these non-Euclidean geometries in their dissertations and were themselves involved in the reform of the “strictly by Euclid’s elements” style of geometry teaching (Simon, 1906, p. 5):

“If one takes a look at elementary geometry in the 19th century, it is particularly important to note how the major currents of science also emerge in elementary geometry.”

The inclusion of new methods, such as the use of models, reflections as a change of perspective between spatial and planar geometry, the paradigmatic

reference to conic sections and a task-led, action-oriented concept development is impressively demonstrated in the three-part textbook from Henrici and Treutlein. The great width of the work of Peter Treutlein was characteristic for many protagonists of the Newer geometry and later the Meraner reform: in addition to teaching at school, it included administrative work as director of a *Realgymnasium* and later a *Reformgymnasium*, curriculum development and development of textbooks and methodologies (see also Weiss, 2019).

4 The Meraner Reform as a (bottom-up) Internationalization process

The Meraner or Klein's Reform as an international teaching reform is often equated with international efforts to modernize mathematics teaching and the resulting introduction of differential and integral calculus and analytical geometry in the upper secondary school. However, Felix Klein's reform efforts in Germany can also be seen as an extension of the Newer geometry reform to and as a compromise solution to the contradictions of the dual education system, which became particularly apparent in the anti-mathematical movement.

The anti-mathematical movement developed from concrete problems that had arisen from a strongly theory-oriented teaching of mathematics at technical colleges and universities, which was often measured against the rigour of theoretical mathematics (Hensel et al., 1989, p.53):

“Nevertheless, it was usually not sufficiently taken into account that the contradictions between theory and practice had been amplified because the theoretical-mathematical treatment of some technical science disciplines - measured against their level of development - was one-sidedly overemphasized, while the experimental validation of the theoretical framework was only insufficiently developed.”

On the one hand, Germany's theoretical orientated engineering education enjoyed high recognition worldwide, but on the other hand, the World Exhibition in 1893 had also shown the advantages of pragmatic, mathematically less well-founded approaches like the American. A first result of the broad discussions that followed, which had already begun in the 1870s and were continued broadly and intensively after the World Exhibition, was the introduction of laboratory teaching. This, however, raised the question of which content should be replaced by the new subject. The question was polemicized whether mathematics at technical universities is a *Hilfswissenschaft* (auxiliary science) or a *grundlegende Wissenschaft* (basic science). The answer to this question

was also linked to the place and importance of the basic and special mathematical lectures given by mathematicians with chairs at the technical universities (Hensel, 1987, pp. 253). The introductory courses to differential and integral calculus, as well as to descriptive and analytical geometry, developed and taught at the technical universities and poly-technical schools formed a solid empirical basis for the related discussions about the possibilities to teach these subjects at *Realgymnasien* and *Oberrealschulen*. The introduction of differential and integral calculus and analytical geometry into upper secondary teaching was therefore not an abstract modernisation of mathematics education. It was, rather a result of compromises reached in long discussions to divide the proportions of mathematical education between secondary schools and technical colleges and universities, while maintaining and expanding the desired close relationships between pure and applied mathematics. There was also an interest in ensuring that theoretically inclined engineers had possibilities to receive a well-grounded education in mathematics and science at technical colleges. The strong personal and content-related relationship between the Newer Geometry reform and the Meraner Reform is evident in Felix Klein's efforts to dissolve the anti-mathematical movement, as well as his ideas on functional thinking. In his concept development of a function and his introductions to calculus, geometrical transformations and geometrical approaches and relations are fundamental. The influences of the reform "Newer geometry" on the Meraner Reform are also visible in the methodology of Walter Lietzmann, Klein's close cooperator in the Meraner Reform. Another example that shows the strong personal and content-related connections and continuities between the teaching reform Newer Geometry and Klein's reform is Georg Wolff. Klein sent Wolff to England and France in 1913 and 1914 to study mathematics and physics education in these countries and to prepare reports on behalf of the IMUK (see also Schubring, 2019). As also Walter Lietzmann, Georg Wolf worked as a teacher in an Oberrealschule. In his lecture "Teaching of Geometry in Germany" at the annual meeting of the British Mathematical Association (1937), he classifies the achievements of the reform of geometry teaching and links them to the goals of the Meraner Reform (Wolff, 1937):

"The Merano course of study aimed at promoting those parts of mathematics that have to do with number, and thus by means of the function-concept arithmetic, al-

gebra, analysis, and analytic geometry were unified in a single whole. This unification proved to be very useful and purposeful.”

Wolff describes the development of geometry teaching within the framework of the Meraner Reform as a “quiet” reform, the goals of which are self-concreted through the consistent application of the principle of functional relations. The concept of functional thinking, which is at the heart of the Meraner Reform, succeeds in unifying the various geometric theories that had previously been taught separately.

In the Meraner Reform, the application-oriented approaches, which are closely linked to physical and engineering contexts, as well as the ideas of reform pedagogy are valued, taken up and further developed by the proponents of Klein's ideas.

5 The New Math Reform as a (top-down) Globalization Process

1.1 5.1 West German New Math Reform

The great importance of substantiated mathematical knowledge and conceptual understanding of natural sciences and engineering technology for economic competition among industrial nations remained unchanged after the Second World War. In the post-war years, West Germany benefited from the Marshall plan and the high level of education and training of technician, scientists and engineers in the Weimar Republic.

Adapting the ideas of the Meraner Reform to the new conditions, given by the rapid development of applied mathematics and the greater specialization of engineering sciences, would have been an obvious impetus for reform efforts on the grounds of economic needs. Another social challenge arose from the political supported work immigration of the 50s and thus greater heterogeneity in the classroom. Above all, however, a critical debate should have taken place about the appropriation, ideologization and misuse of mathematics education for racial theories and character training during the Nazi regime (Hamel, 1933, p.307):

“But far more important is the educational value that follows from the connection between mathematics and the Third Reich. The basic attitude of both is the heroic. [...] Both demand service: mathematics the service of truth, sincerity, accuracy. [...] Both are anti-materialistic. [...] Both want order and discipline, both fight chaos and arbitrariness.”

However, a fundamental debate failed to materialize, partly due to the lack of denazification of the West German education system.

At the end of the 1950s, university mathematicians in several European countries called for a modernisation of the ‘backward mathematics education’, demanding an algebraization of geometry teaching and an orientation towards axiomatic set theory, which had already been implemented by Bourbaki in higher mathematics since the 1920s. Although in Germany van der Waerden’s *Modern Algebra* was already in its 5th edition in 1955, nevertheless, modern algebra in the sense of Bourbakism only began to gain acceptance at German universities from the early 1950s. Algebra instruction was, incoherent and out of date in the Bourbaki sense, depending on the lecturer and location. Therefore, a reform aiming at the set theoretical foundation and algebraization of geometry should have started with teacher education (Weiss, 2023).

Central figures of the New Math teaching reform were Heinrich Behnke and his former assistant, Georg Steiner. Heinrich Behnke was a member of the ICMI Executive Committee from 1952-the year the ICMI was restored-until 1966, and served as its president from 1955 to 1958. Since the 1930s, Behnke had continuously campaigned for the close connection between university and higher education. He saw the university’s task as providing teachers with a broad, well-founded mathematical education and maintaining relationships through conferences, training courses and encouraging publications in subject didactics (Hartmann, 2008). Neither Behnke nor any other university mathematician from Germany took part in the meeting, which was supposed to shape the movement for modernisation as an international project: the Roy-aumont seminar in 1959. Heinrich Behnke took part in the ICMI seminar in Aarhus, where well-known mathematicians presented their ideas for a Bourbaki-inspired geometry concept development. In his opening lecture he tried to place the planned reform movement in the tradition of Klein’s ideas, citing Klein (Behnke et al., 1960, p.15):

“My [Klein’s] interest from my time in Bonn ...was to understand the mutual relationship between the opposing mathematical schools and to encompass their contradictions through a unified overall conception. From this perspective, how much he would have won over to Bourbakism today! For it was precisely this unified conception of mathematics, which the French have been striving for with such success for about two decades now, that was always Klein’s main goal.”

And on the next page somehow contradicting, Behnke continues:

“Klein believes that the axiomatic representation necessarily leads to “purism”, the shielding of one part of mathematics from the other. But here he was radically wrong. The Bourbakism of our day is axiomatic, but it connects the individual parts together again. Yes, that is its initial motive. All of mathematics is based on a few parent structures, namely: 1. Algebraic structures, 2. Order structures, 3. Topological structures.”

Although teachers were unprepared for a set-theory-based, axiomatic approach to teaching of geometry, and the axiomatic-deductive method contradicted established reform pedagogical principles, the axiomatic set theory was nevertheless integrated into the curriculum in 1968 through a top-down decision by *Kultusminister Konferenz* (Conference of Federal Ministers of Education), despite significant opposition from teachers (Weiss, 2023). An important change that also took place with the top-down reform, strengthened by trends in international networks, was the new self-determination of mathematics education as a scientific discipline. This involved a division of mathematics educators into school practitioners, dealing with the development of teaching materials and teacher training, and the visionary new mathematics didactics elite, who were increasingly occupied with self-reflection, establishing their own methods and research questions, and distancing themselves from unscientific approaches. The traditional theoretical conceptualization and contextualizations of current developments, as well as traditional activities of mathematics educators like editing and adaptation of teaching materials, were no longer seen as scientific work and research (Weiss, 2024). In the 1980s, the structural and set theoretical approaches of the New Math Reform were completely withdrawn. One of the new global trends that were now followed was Realistic Mathematics Education (RME). However, it was not linked to the application orientation of the Meraner Reform and its close interplay between mathematics and physics education.

5.2 East German New Math Reform

In East Germany, the reform New Math encountered different conditions than in West Germany. The German Democratic Republic (DDR) was not an official member of the International Commission on Mathematical Instruction (ICMI), the organization behind the International Congress on Mathematical Education (ICME). This was primarily because ICMI operated under the International Mathematical Union (IMU), of which the DDR was not a recog-

nized member due to Cold War-era political divisions (Furinghetti & Giacardi, 2023). As a result of the war and the denazification carried out in East Germany, there was a shortage of 40,000 teachers (Bruder, 2024), who were trained as *Neulehrer* (new teachers) in short accompanying courses after World War II. In contrast to the Western Occupation Zones, the unified school was introduced in the Soviet Occupation Zone, consisting of an 8-year, later 10-year polytechnic elementary school and a 4-year-high school (Sträßer et al., 2023). The philosophical basis of the educational principles, the teaching design and the educational policy was historical dialectical materialism. The principles of the Meraner Reform were compatible with these teaching principles; work-based education, a paradigmatic inductive development of concepts and a close integration of mathematics teaching with the natural sciences, especially physics teaching. These guiding principles formed as well a foundation upon which mathematics education in East Germany developed. Due to the initial lack of well-trained teachers and the great heterogeneity of the students in a unified school system, the emphasis was on practicing and understanding examples that were paradigmatic for the subjects. Research into mathematics education was state-controlled and institutionalized and, like teacher training, was strongly oriented towards school practice. In contrast to the constructivist approaches prevalent in West Germany, pedagogical approaches in mathematics education were inspired by Soviet activity theory research (Bruder, 2024). Also, due to the traditional proximity to the ideas of the Meraner Reform, the efforts to modernize geometry teaching concentrated on linking the concept of geometric representation with the concept of function and thus a uniform presentation of the school subjects geometry, algebra and analysis. The initially planned, consistent foundation of geometry on the concept of mapping proved to be impractical and was only implemented as a possible background theory. The concept of geometrical transformations found its way into the curricula primarily through the construction of congruence mappings and their compositions and their use in geometric proofs of triangle geometry. However, a consistent connection between the geometric concept of transformation and the concept of function could not be demonstrated in the students' understanding (Filler, 2016). However, this was partially successful in mathematical and physical contexts. Although the educational

system in East Germany was centrally organized, there were no drastic reorientations or breaks with cultural-historical traditions, due to the New Math movement. Some of the introduced notions, related to set theory, were kept until the reunification, in particular in the context of inequalities, transformations and functions.

6 Conclusions

History of reforms of geometry teaching enables a better understanding of the differences between bottom-up problem-oriented culturally rooted internationalization processes and global networks. The West German top-down New Math Reform was not motivated through practical experiences of schoolteachers. Methods such as axiomatic-deductive concept development and programmed learning had no foundations in the existing social practices of teachers and students. It seems also, that the reformers were more motivated by belonging to global networks than by practical local problems. German history in both West and East Germany also shows that profiling in global networks is often linked to ignorance or the lack of appreciation of one's own cultural-historical traditions.

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WISKUNDE POST, A MATHEMATICAL MAGAZINE FOR STUDENTS SUPPORTING THE “MODERN MATHEMATICS” MOVEMENT IN FLANDERS

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ABSTRACT

In the 1960s, *Wiskunde Post*, a mathematical magazine for young people, played an important role in the promotion and dissemination of “modern mathematics” in Flanders. The initiative came from three enthusiastic mathematicians, Paul Bockstaele, Gaspard Bosteels and Roger Holvoet. The magazine covered topics in the history of mathematics, “traditional mathematics” and “modern mathematics”, but the latter topics in particular were completely new to most readers, not only students but also their mathematics teachers. Many of them were first introduced to the mathematics of sets, relations, algebraic structures and topology through *Wiskunde Post*. Motivation for or applications of the new mathematics were largely lacking, however; it was primarily the mathematics itself that was supposed to captivate the magazine’s readership.

1 Introduction

After WWII, mathematics magazines for young people appeared in many countries. Some notable examples (among many) are *Pythagoras* in the Netherlands, *Mathematical Pie* in the United Kingdom, *Le Facteur X* in France, *Kvant* in the Soviet Union, *La Scienza per I giovani* and *Angolo Acuto* in Italy, *Die Wurzel* in (East) Germany, *Euclid–Appendix of the Bulletin of the Hellenic Mathematical Society* in Greece. The main purpose of these magazines was to stimulate young people’s interest in mathematics, sometimes also to prepare the most talented among them for studies in mathematical science (Aksenteva, 2000). To that end, these magazines presented mathematical puzzles, number games, challenging problems, beautiful or surprising results, elements from the history of mathematics, ... The scope of certain magazines included both mathematical and physical knowledge (e.g., *Le Facteur X*, *Kvant*, *La Scienza per I giovani*). Also, some of these magazines played an important role in the spread of “modern mathematics” in the 1960s (e.g., *Euclid–Appendix of the Bulletin of the Hellenic Mathematical Society*) (Baltzis, 2025). Mathematics journals for young people were rarely the subject of historical analyses; Guichelaar’s (2017) study of *Pythagoras* is one of the rare exceptions. The focus of this paper is *Wiskunde Post* (WP) [Mathematics

Post], a “modern” mathematics magazine for secondary school students, published in Flanders, the Dutch-speaking part of Belgium, in the 1960s and early 1970s (Figure 1). But who were the initiators of *WP*? What content was covered? And, in what ways did *WP* support the “modern mathematics” movement in Flanders?

In the 1960s, the introduction of “modern mathematics” in secondary schools was in full preparation in Belgium. The main initiative to prepare Belgian teachers for the arrival of “modern mathematics” in their classrooms were the “Days of Arlon”, a series of in-service teacher training courses organized annually from 1959–1968 in the city of Arlon in the very south of (French-speaking) Belgium (De Bock & Vanpaemel, 2019). Flemish mathematics teachers also participated in these courses, though to a lesser extent than their French-speaking colleagues. In this context, *WP* was created in the second semester of the 1961–1962 school year. It was intended for secondary school students, but it was also eagerly read by Flemish mathematics teachers; many of whom were introduced to “modern mathematics” by *WP*. On the first page of the first issue, we read about the new magazine’s scope:

We want to find a happy medium. A part of our issue will always be devoted to the history of mathematics, so that you can appreciate the acquisitions. Another part will be devoted to traditional mathematics, as it is taught to you, and finally, we will deal resolutely with the most modern mathematical views. (*WP*, I(1), p. 1)

The initiative to launch a magazine for students was not taken by an association of mathematics teachers, but came from three enthusiastic mathematicians who acted both as editors and main authors: Paul Bockstaele, Gaspard Bosteels and Roger Holvoet. Bockstaele (1920–2009) was a historian of mathematics with particular interest in the work of mathematicians from the Southern Low Countries in the 16th and 17th centuries (Vanpaemel, 2010). Before becoming a professor at the University of Leuven, he worked for some time as a mathematics teacher in a secondary school and as a teacher trainer. Bosteels (1909–2003) was also a mathematics teacher first, then became headmaster of a secondary school (Deelstra, 2023). Both Bockstaele and Bosteels were authors of successful mathematics textbooks. Holvoet (1938–1998) attended secondary school at the school where Bosteels was active and then went on to study mathematics at the *Université libre de Bruxelles* [Free University of Brussels], where he earned his PhD under the supervision of Georges Papy, the figurehead of the modern mathematics movement in Belgium (De Bock & Vanpaemel, 2019). Holvoet became a colleague of Bock-

staele as a professor of algebra at the University of Leuven. Each of these three initiators stood for one of the three tracks in *WP*, respectively, history of mathematics, “traditional mathematics”, and “modern mathematics”, but the latter predominated. Holvoet introduced his readers to the 20th-century mathematics of sets, relations, algebraic structures, linear algebra, topology, and discrete mathematics.

WP appeared until the second semester of the 1973–1974 school year; in total 53 issues were published, spread over twelve volumes: Vol. 1 in 1962 (three issues), Vols. 2–4 in 1962–1965 (six issues per school year); Vols. 5–12 in 1966–1974 (four issues per school year). The number of pages per issue ranged from 8 to 20; a total of 716 pages were published. No editorial explanation was ever given as to the reason(s) why the number of issues was reduced from six to four starting with Vol. 5, as well as why the magazine ceased publication after Vol. 12. *WP* had up to 8000 subscribers per volume, both from Catholic schools and from schools run by the State. It is also worth mentioning that *WP*’s financial profits were spent on scholarships for outstanding students in mathematics (Holvoet, 1989).



Figure 1. Header of Vols. 1–5 (1962–1967) of *WP*, visualizing the path from antiquity to modernity

2 Browsing in *Wiskunde Post*

2.1 The “history of mathematics” track

Bockstaele starts his historical chronicle in ancient Egypt and Babylonia (“4000 year old mathematics books”). In each article, he first provides some general historical information about the ancient civilizations before focusing on mathematical sources. Bockstaele’s articles, always illustrated with photographs of authentic documents, appeal to *WP*’s young readership, and from time to time the author explicitly addresses his readers:

Certainly there are some daring young readers of **Wiskunde Post** who dream of studying mathematics later, but also love languages and take an interest in history of science. Couldn't an Assyriologist-mathematician grow out of one of them? (*WP*, I(2), p. 2, bold in original)

Bockstaele continues his historical chronicle with ancient Greek mathematics, covered successively: “Thales of Miletus”, “Pythagoras and the Pythagoreans”, “Euclides”, and “Archimedes”, the latter supplemented with an article about “The history of the number Π ”. Bockstaele always tries to actively engage his readers in the reasoning of the ancient mathematicians.

In the first issue of Vol. 5 (1966), the researcher Bockstaele steps forward: He describes the contents of *Die manire om te leeren cyffren na die rechte consten Algorismi. Int gheheele ende int ghebroken* [The way to learn to cypher according to the right art of Algorismi, in whole and broken numbers], the oldest known Dutch cyphering book of 1508, printed and perhaps also compiled by the Brussels printer Thomas van der Noot. The book was repeatedly reprinted and also translated into French (Antwerp, 1529) and English (St. Albans, 1537). Bockstaele had previously published about this cyphering book in scholarly journals (Bockstaele, 1959, 1960), but the angle of his article in *WP* is to popularize historical science for young people. Bockstaele explains that ciphering books, aimed at teaching practical calculations and written in popular language, emerged at the end of the 15th century in wealthy commercial cities in northern Italy and southern Germany and, with the discovery of the art of printing, quickly became widespread, including in the Low Countries (see also Meskens, 2013). *Die manire* consists of three parts. The first treats the operations with integers (numeration, addition, subtraction, duplation (doubling), mediation (halving), multiplication and division), the second takes up the same operations with fractions, and the third is a collection of problems on, among others, the “ghulden reghel” [golden rule], i.e., the rule of three. Some of these problems are practical, others more recreational. Bockstaele invites his readers to solve one of the problems in *Die manire*:

A drunkard drinks 1 aem (which is about 130 liters) of beer by himself within 14 days. If his wife drinks with him, they drink that aem together in 10 days. Now the question is: within how much time would his wife drink that aem alone? (*WP*, 5(1), p. 2, free translation)

However, he expresses the hope that his young readers will be more moderate than the “drunkard and his wife...”. Although Bockstaele remains a member of the editorial board of *WP*, only a few of his contributions have been pub-

lished after this article (e.g., on how complex numbers made their appearance in mathematics, Vol. 7).

2.2 The “traditional mathematics” track

Gaspard Bosteels, who signed his contributions with G. Bosteels, G. B. or Gabos¹, showed a broad interest and expertise in various areas of mathematics. He was by far *WP*’s most prolific and versatile author. Vols. 11 and 12 were almost entirely written by him. To list all the topics he has covered in *WP* would lead us too far, we just mention a few: geometric properties that were beyond the scope of the secondary school curriculum (e.g., the nine-point circle), elementary logic (e.g., the link between conjunction and disjunction, on the one hand, and series and parallel circuits in electricity, on the other), physical laws and their translations into differential equations (e.g., Newton’s law of cooling and heating), practical applications of mathematics (e.g., on the height of an access gate to allow a sleeper to enter a building), mathematics and economics (e.g., maximizing the income of a bus company), problems with a historical root (e.g., the trisection of an angle), various computational tricks and games (e.g., about quickly multiplying by certain numbers), subjects that were fashionable in the 1960s (e.g., the binary number system).

Like many “amateur mathematicians”, Bosteels showed particular interest in special types of numbers and their properties: obviously prime numbers, but also Fibonacci numbers, Pythagorean triples, “lucky numbers” (= numbers generated by a certain “sieve” by analogy to the sieve of Eratosthenes), Lagado numbers (named after a fictional city in Gulliver’s Travels), Mersenne numbers, and pentagonal numbers made their entrance in the columns of *WP*. With regard to these special numbers, there are still several unsolved problems that young people can easily understand (e.g., “so far, it remains an open question whether the sequence of Fibonacci numbers contains a finite or an infinite number of prime numbers”, *WP*, 2(1), p. 7). With such topics and related problems, Bosteels was undoubtedly able to spark the interest and curiosity of *WP*’s readership.

From the third issue of Vol. 3, Bosteels started a dictionary of mathematical terms, including their Greek or Latin origin (where applicable), often also

¹ Among Flemish mathematics teachers, *Wiskunde Post* was therefore often called “Gabos magazine” (Deelstra, 2023).

with translations into French, English or German, and explanations of the symbolism used. However, he would not finish this project: the last contribution to “Gabos dictionary” appeared in the second issue of Vol. 11 (with some items beginning with the letter f).

2.3 The “modern mathematics” track

With an opening article on *Verzamelingenleer, een eerste kennismaking* [Set theory, a first introduction] in *WP*, 1(1), Holvoet set the tone:

Often the young student has the impression that mathematics is an (ugly) building with several floors (arithmetic, algebra, geometry, trigonometry, etc.) where it is virtually forbidden to move from one floor to another. In reality, today’s mathematics has a rigorous foundation: **the theory of sets**. All mathematicians of the whole world live **in the universe of sets**. The goal is to explain the elements of this so-important theory of sets. (p. 1, bold in original)

Perhaps for many teachers and students in Flanders, Holvoet’s article was the first introduction to the theory of sets. Successively, Holvoet discusses: the origin of set theory, the concept of a set, the empty set, the equality of sets, and the representation of sets with Venn diagrams. A proof of the theorem “there is only one empty set” is included. In the second and third issues of Vol. 1, Holvoet continues his serial with the concepts of subset and set of subsets, and the operations on sets and their properties. In each case, the new concepts are accompanied by various examples and exercises.

In Vol. 2, Holvoet starts a new serial on the theory of relations “entirely built on set-theoretic foundations” (issue 1, p. 2). The representation with arrow diagrams is introduced and frequently used. The next issues will cover: equivalence relations and partitions, order relations, the Cartesian product, the composition of relations and its properties, functions as relations of a specific type, equipotence of sets, and permutations. Holvoet seems obsessed with the empty set (and the empty relation, actually another name for the same thing); they constantly pop up in examples and exercises (e.g., “the empty relation is the only permutation of the empty set”, *WP*, 3(1), p. 4). To divide his articles into sections, Holvoet uses the binary number system instead of the decimal.

From the second issue of Vol. 3, the readers are introduced to algebraic structures: Holvoet starts a serial on linear algebra, in particular on vector spaces “which play a fundamental role, both in theoretical mathematics and in the applied sciences” (*WP*, 3(2), p. 1). Successively, the following topics are addressed: definition, properties, linear mappings, bases and dimension of a

vector space. The theory is illustrated with many examples and exercises, but the style is purely mathematical: concepts and symbols, theorems and proofs; examples from the “applied sciences” or other (external) motivations for the theory are not given to the magazine’s young readership.

Topology is the subject of Holvoet’s fourth serial on “modern mathematics”; it starts from Vol. 4. As a motivation, the author writes that topological structures give mathematical meaning to intuitive concepts, such as neighborhood, continuity and limit, which initially played an important role in approximation problems. However, he continues, “we will detach ourselves from the real numbers and obtain statements independent of the concept of distance” (*WP*, 4(1), p. 5). Consecutive topics covered are: neighborhoods, open sets, bases of a topology, continuous mappings, homeomorphisms. Concepts are defined and theorems formulated in general topological spaces; “concrete examples” are framed in topologies of the plane, space, the set of real numbers, finite sets, ... Topics that might appeal to young people, e.g., non-orientable surfaces like the Möbius strip or Klein’s bottle, are not included.

Holvoet’s fifth serial, included in Vol. 5, deals with combinatorics, a branch of discrete mathematics. The author first repeats some elements of the theory of relations (particularly mappings) and then uses that theory to address: variations with and without repetition, permutations and combinations. Several theorems are proved on a set-theoretic basis, including that for all natural numbers n , it holds that $n^0 = 1$ (implying $0^0 = 1$; indeed, there exists exactly one mapping from the empty set to the empty set, namely the empty relation). The theory is illustrated with several counting problems, all but one presented within a purely mathematical context. The opportunity to show some practical applications of combinatorics is not taken, probably a deliberate choice in line with Holvoet’s view of mathematics and mathematics education. Interestingly, the final issue of Vol. 5 includes an article proposing an axiomatic approach to probability theory. However, it is not written by Holvoet but by Frank Laforce, a mathematics teacher who would become the first president of the Flemish Association of Teachers of Mathematics in 1975.

A new serial in Holvoet’s crash course on “modern mathematics” is about monoids and groups (Vol. 6, issues 2 and 3). In his introduction, the author refers to the importance of groups (mainly in higher physics):

In the 20th century, group theory is indispensable both to the mathematician, and to the mathematics user. Important applications of groups are found, for example, in *crystallography* (crystal groups), in *nuclear physics*, in *quantum theories* (or-

thogonal groups), in *relativity theory* (Lorentz group), in *commercial sciences* (homology groups). (*WP*, 6(2), p. 1, italics in original)

Unfortunately, Holvoet was unable to illustrate any of these applications at a level suitable for the young readers of *WP*. Holvoet's new serial, entitled "Groups 1", was also shorter than expected: only two articles were published and "Groups 2" would never appear...

In the third issue of Vol. 7, which appeared in the 1968–1969 school year when, with few exceptions, "modern mathematics" was introduced in all first years of general secondary education in Flanders, Holvoet writes again about relations ("Contemporary mathematics is *relational*: we study *sets* connected by *relations*", *WP*, 7(3), p. 1, italics in original). He focusses on the properties of relations and on the correct use of quantifiers to formulate these properties. The article seems to respond primarily to a direct need of teachers who now had to teach "modern mathematics" in their classrooms. More interestingly, in Vol. 8 (issue 2), Holvoet writes about infinite cardinal numbers. He first sketches the historical context: From Georg Cantor's work on infinite cardinal numbers to some (in the 1960s very) recent results (e.g., regarding the continuum hypothesis). In the subsequent sections he provides a modern account of some of Cantor's results.

In Vol. 9, Holvoet announces a follow-up of the serial he had started in Vol. 6 on algebraic structures. It is entitled "From PX (= the set of subsets of the set X) to groupoids, monoids, groups, rings. 1", but unfortunately only one article appears (in the second issue) and rings are not discussed (though they are discussed in the same issue by Bosteels). The emphasis in Holvoet's article is more on introducing modern mathematical jargon than on new content, for example, he distinguishes between a groupoid (a set with an internal law), a semigroup (if that law is also associative), a monoid (if, in addition, there is a neutral element), a commutative monoid, and a group. Holvoet seems to lack inspiration, time or interest; only one more of his articles will appear in subsequent volumes of *WP* (on Fermat's prime numbers, in the second issue of Vol. 12). However, articles on "modern mathematics" were still published in *WP*, but by other authors (Bosteels, Laforce, ...).

In sum, Holvoet offered the readers of *WP* a clear overview of the (naive) theory of sets and relations, as well as an introduction to algebraic and topological structures, topics that were considered of utmost importance in the "modern mathematics" era. Each time, he illustrates the new concepts with

numerous mathematical examples and exercises. However, his articles seem more fodder for teachers than for students, and for teachers Holvoet provides little material to “sell” the “new mathematics” to their students (real-world applications, intriguing problems, historical roots, ...). The motivation had to come mainly from the new mathematics itself. For geometry, a subject on which the reformers of the 1960s were neither very enthusiastic nor in agreement (see, e.g., De Bock & Vanpaemel, 2019), readers were left hungry....

3 *Epilogue*

The “modern mathematics” movement in Belgium was led by Francophones, particularly by Papy. Although Papy’s groundbreaking textbook series *Mathématique Moderne* (Papy, 1963–1967) was translated into Dutch (with the exception of Vol. 6), the translation of the first volume did not appear until 1965 and was not an easy read for a “first introduction” to the new mathematics. In Flanders, this gap was largely filled with *Wiskunde Post*, a magazine that emphasized “modern mathematics” but combined this with articles on “traditional mathematics” and history of mathematics, making it appealing to students and their teachers. Especially in the early years, the initiators succeeded in sparking an interest and love for mathematics in many students in Flanders (Deelstra, 2023). In the early 1970s, however, the enthusiasm of the initiators waned and the magazine died in silence with Vol. 12 (1973–1974). For teachers this gap was soon bridged: already in 1975 the first issue of *Wiskunde & Onderwijs* appeared, the first Flemish journal for mathematics teachers. A new initiative for a youth magazine on mathematics has not been taken in Flanders since. As an alternative, some Flemish secondary school students, encouraged by an enthusiastic mathematics teacher, subscribed to the Dutch magazine *Pythagoras*.

SOURCES

Wiskunde Post, 1962–1974. Vols. 1–5 (1962–1967) were published by *S.M. Ontwikkeling* (Antwerp); Vols. 6–12 (1967–1974) by *De Sikkel n.v.* (Antwerp). Complete collection in authors’ possession.

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EXPECTATIONS REGARDING FRENCH PROSPECTIVE TEACHERS' KNOWLEDGE IN GROUP THEORY

A (tentative) historical survey

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ABSTRACT

History of mathematics and history of mathematics education generate insights for didactical reflection (Artigue, 1991; Chorlay & De Hosson, 2016; Chorlay et al., 2022). This paper aims to present elements of history of mathematics education that will contribute to a PhD dissertation (A. Rolland) on the links between mathematical knowledge acquired in tertiary education and its use by secondary school teachers (a topic known as “Klein's second discontinuity”). The specific topic chosen is that of knowledge about group theory, and the aim of the PhD is to produce a capstone course, that is a programme for pre-service teachers training backed up by general methodological and theoretical reflections on Klein's second discontinuity. One of the challenges is to understand (and to help prospective teachers understand) the future role of abstract theories whose link with the content to be taught is not transparent. Rather than seeking to determine a priori (through mathematical and/or didactic reflection) what this role is in the case of groups, we are seeking, through a historical study of one of the main French competitive examinations for teacher recruitment (*agrégation*) from 1950 to 1990, to identify the expectations and motivations of the institution recruiting teachers with regard to future teachers. The choice of study period allows us to investigate variations in these expectations, showing that they may have been significant even before and after the New Maths period (Gispert, 2023).

1 Introduction

In France, prospective secondary school mathematics teachers typically pursue a degree in mathematics, which almost always includes an introductory course on group theory. This content is often perceived by students as disconnected from what they will be teaching (Zazkis & Leikin, 2010). This results in a (second) “discontinuity”, as defined by Klein (1908/2016), with regard to this theory. He noted that upon completion of their university studies, pre-service teachers seemed unable to “discern any connection between [the task of teaching] and [their] university mathematics”, and thus “[their] university

studies remain only a more or less pleasant memory which has no influence upon [their] teaching.” (Klein, 1908/2016, p. 1)

This phenomenon, first identified in the late 19th century, has since been addressed through the design of adapted teaching resources (Sultan & Artzt, 2018; Usiskin et al., 2003; N. Wasserman, 2018) and by research in mathematics education (Planchon, 2022; Winsløw & Grønþæk, 2014). These studies investigate a range of approaches, including the explicit delineation of the connections between secondary and tertiary mathematics, as well as the identification of teaching practices transferable to secondary education settings. Nevertheless, this remains a topical issue in pre-service teacher education today.

This communication draws upon the ongoing PhD of Agathe Rolland, supervised by Nicolas Grenier-Boley (Mathematics Education) and Frédéric Brechenmacher (History of Mathematics). The aim of the PhD is to understand (and help prospective teachers understand) the future role of abstract theories whose link with the content to be taught is not transparent.

The question the authors seek to address in this paper is as follows: how can history of mathematics education inform reflection on the role of group theory in preservice teacher education?

Firstly, we will describe the type of course design that has been selected in order to address Klein’s second discontinuity regarding group theory. Secondly, we will examine historical sources to clarify the expectations of the institution recruiting teachers, in relation to group theory.

2 Addressing the second discontinuity: capstone courses

One way to address the challenges of Klein’s second discontinuity is by the incorporation of capstone courses for prospective teachers. Winsløw and Grønþæk explain the use of such courses as follows:

(...) the term ‘capstone course’ is used to indicate a study unit which is located towards the end of an academic study program, with the aim of concluding or ‘crowning’ the experience, and to link academic competence and training with the needs of a professional occupation (Winsløw & Grønþæk, 2014, p. 4).

Wasserman et al. (2023) emphasise that the link between tertiary mathematics and the needs of secondary school teachers can be of varying natures, either focused on mathematical content, disciplinary practices, or on didactical practices. Murray and Star (2013) identify two types of capstone courses focused

on mathematical content. The first type, labelled “secondary mathematics from an advanced standpoint”, aims to revisit secondary mathematics content with the depth provided by tertiary mathematics. The second type, labelled “tertiary mathematics with connections”, runs through the curriculum of tertiary mathematics, focusing on the importance of some notions for secondary mathematics.

The main goal of the dissertation is to identify design principles for a capstone course of the first type described above on groups, then design – and possibly implement – such a course. This entails attempting to answer two questions: *what* should pre-service teachers know about groups? And *why* is it useful, necessary or advisable that they do so, for their professional practice as future mathematics teachers?

The PhD will not address two additional questions: the impact of such a capstone course on actual teaching practices and the impact of a “group-informed” or “group-rich” secondary school teaching on student learning. They will be tackled after the PhD.

3 *A heuristic historical detour*

3.1 The French context

In order to take a step aside from our own preconceptions and biases regarding groups, the PhD aims at uncovering the answers given to the “what” and “why” questions by the French Ministry of Education in the 1950-1990 period. The material presented here is still a work in progress, and makes use of historical sources in a heuristic manner.

The specific French context should make it easy to spot explicit answers to the two questions: firstly, the curriculum for secondary education is nationally defined. Secondly, the majority of secondary teachers are public servants. They are hired via a national competitive examination - the *agrégation*¹ - which is designed like a university examination (with a clearly defined official curriculum, published written and oral examination questions and annual reports of the selection board), but with a limited number of places, and which serves as a selective recruitment procedure.

¹ There is another national examination which serves the same purpose, but with a lighter curriculum, the *CAPES*. We will not address it, given that the sources are too incomplete.

A key factor to take into account is whether groups were explicitly taught in secondary school. During the 1950-1990 period, groups were the subject of study in secondary education in an extended “New Maths period”. Regardless of whether or not groups were explicit objects of study in secondary education, some knowledge of groups was expected from future teachers, at least since the 1960s, that is even before the New Maths reform (Gispert, 2023).

3.2 A heuristic use of historical files

Among the sources we studied, two stood out as particularly relevant. In 1956-1957, the APMEP (Association of Secondary School Mathematics Teachers) organized a series of lectures for its members, entitled “Algebraic structures and topological structures²”. The contributors were professional mathematicians advocating a more “modern” teaching of mathematics: H. Cartan, J. Dixmier, R. Godement, C. Pisot, L. Schwartz and J.-P. Serre were members of Bourbaki, and G. Choquet, A. Lichnerowicz and A. Revuz played an active role in the New Maths reform in France. P. Dubreil, P. Lelong and L. Lesieur also contributed to the lecture series.

In the introduction to the lectures series, G. Choquet and G. Walusinski argue (Cartan et al., 1957, p. 3-4):

The rapid expansion of research and the ever-more rapid progress of discovery mean that even those teachers who teach at elementary levels have to renew their theoretical knowledge. (...) Their very teaching must take advantage of new scientific discoveries: today one must teach the mathematics of today. This will better prepare students for the studies and research that await them³.

Thus, it seems that one goal of introducing structures to secondary and elementary levels is to pave the way for the pupils’ future academic careers. The professional mathematicians mostly care about the education of the more mathematically inclined and advanced secondary students, with a view to smoothing out the first Klein discontinuity, i.e. between secondary and ter-

² *Structures algébriques et structures topologiques.*

³ “L’extension rapidement croissante de la recherche, le progrès de plus en plus rapide de la découverte imposent, à ceux mêmes des professeurs qui enseignent à des niveaux élémentaires, de renouveler leurs connaissances théoriques. (...) Leur enseignement même doit profiter des nouvelles acquisitions de la science : il faut enseigner aujourd’hui les Mathématiques d’aujourd’hui. Ainsi mieux préparer les élèves aux études ou aux recherches qui les attendent.” Free trans. Deepl and R. Chorlay.

tiary education. However, it is unclear why pupils who will not pursue further studies in mathematics should also be exposed to these concepts.

In his introductory lecture on “algebraic structures”, H. Cartan formulates cautious but rather undecipherable suggestions. He claims that “This use [of algebra] is by no means new; what is especially new is our awareness of it.” This means that modern mathematics is more algebraic than ever, not so much because of the extension of a specific domain called “algebra”, but because modern mathematicians are now more “conscious” of the pervasiveness of algebra. His opinion is as follows (Cartan et al., 1957, p. 5):

The teaching of mathematics in secondary classes should reflect this development, at least in the final classes; not so much in terms of a change in the syllabus as in the way classical theories are presented, taking into account this new perspective⁴.

Thus, this lecture series sheds light on the fact that, for these professional mathematicians, it was necessary to bring a modern viewpoint on well-known objects to school children. Yet, it is not exactly clear why.

The other important sources we studied are the annual reports of the selection board. These reports help us better understand the scope of the mathematical culture expected from the applicants. We have identified several relevant excerpts, dating from the New Maths period:

The study of geometry seems to be too neglected. However, general theories (which are sometimes derived from it!) have a wide range of practical applications. Any study of geometry is inseparable from the notion of a group operating on a set⁵. [Report, 1975, p. 91]

Here, it is made clear that groups should not be considered solely for their most general aspect, but also for their connection with geometry. Thus, multiple points of view on groups coexist.

Group theory. The candidate should not develop a general theory if it cannot be supported by non-trivial examples⁶. [Report, 1978, p. 75]

⁴ “L’enseignement des Mathématiques dans les classes secondaires doit se ressentir d’une telle évolution, tout au moins dans les classes terminales ; moins sans doute dans le sens d’une modification des programmes que dans la façon de présenter des théories classiques, en tenant compte d’une nouvelle optique.” Free trans. R. Chorlay.

⁵ “L’étude de la géométrie semble trop délaissée. Pourtant les théories générales (qui en sont parfois issues !) y trouvent un large champ d’applications concrètes. Toute étude géométrique est inséparable de la notion de groupe opérant sur un ensemble.” Free trans. R. Chorlay.

⁶ “Théorie des groupes. Le candidat ne doit pas développer de théorie générale s’il ne peut pas l’étayer par des exemples non triviaux” Free trans. R. Chorlay.

The selection board emphasizes the importance of examples, which are often left out by applicants who focus exclusively on general group theory.

In the “paths” of mathematical thought, definitions are very often crossroads; unfortunately, many candidates seem to place them in the middle of the desert because their introduction is so lacking in relevance and their use so wanting in scope. How else can you deal with topics such as (2)⁷, (11), (28), if not by showing how the notion, however elementary, of quotient, dimension or ideal provides the answer to varied and difficult questions, and then - this is the problem of structure - why it makes it possible to solve them, why it makes them their common substance⁸? [Report, 1979, p. 59]

The selection board wanted the applicants to be aware of the fundamental role of structures: to provide answers to a wide range of problems.

This compartmentalisation of knowledge appears to be a general fact; even the best candidates hesitate to bring together concepts which are clearly related but which they have studied at different times, thus depriving their plan of its most fruitful examples and reducing it (...) to an enumeration which reveals little about the structures and lacks the spirit of synthesis⁹. [Report, 1979, p. 59]

The applicants were expected to draw connections between different domains of mathematics.

Before going into detail on the various aspects, let us first stress the two imperatives that must be met in every lesson: to illustrate and to unify. (...) To refuse, for example, on the pretext that we are presenting a lesson on groups, any recourse to linear algebra, analysis, geometry or topology, is a damaging attitude in every respect, a scientific nonsense as well as a pedagogical blunder¹⁰. [Report, 1980, p. 66]

⁷ (2) refers to “Examples of quotient algebraic structures.” (“Exemples de structures algébriques quotients.”)

⁸ “Dans les « chemins » de la pensée mathématique, les définitions sont bien souvent des carrefours ; nombre de candidats semblent malheureusement les placer en plein désert tant leur introduction est dénuée d’à propos et leur usage de portée. Comment traiter des sujets tels que (2), (11), (28), sinon en y faisant ressortir comment la notion, pourtant élémentaire, de quotient, de dimension ou d’idéal fournit la réponse à des questions variées et difficiles, puis - c’est le problème de la structure - pourquoi elle permet de les résoudre, pourquoi elle en fait leur commune substance ?” Free trans. R. Chorlay.

⁹ “Ce cloisonnement du savoir apparaît d’ailleurs comme un fait général ; les meilleurs candidats eux-mêmes hésitent à rapprocher des notions dont la parenté est pourtant claire mais qu’ils ont étudiées à des moments différents, privent ainsi leur plan de ses exemples les plus fructueux et le réduisant (...) à une énumération peu révélatrice des structures et dont l’esprit de synthèse est absent.” Free trans. R. Chorlay.

¹⁰ “Insistons d’abord, avant d’en détailler divers aspects, sur deux impératifs qui s’imposent dans chaque leçon : illustrer et unifier. (...) Se refuser, par exemple, sous prétexte que l’on présente une leçon sur les groupes, tout recours à l’algèbre linéaire, l’analyse, la géométrie ou la topologie, est une attitude dommageable à tous égards, un non-sens scientifique en même temps qu’une maladresse pédagogique.”

During the New Maths period, it seems that with regard to groups, the selection board expected applicants to be able to highlight the generalizing and unifying power of the structural approach, but more besides. They had to be able to flesh out the concept, with applications to domains of mathematics outside of algebra, such as geometry, but also with concrete examples.

The documents we studied show that the “what” question is always only *partially* answered in official documents; the *actual scope* of what is expected from applicants varies in time and needs to be studied as a social construct involving several stakeholders (as we know is the case for school structure, curricula etc.); this scope can be documented by studying the official reports of the selection board, whose purpose is to make its expectations more explicit for future applicants. Moreover, the answer(s) to the “why” question is almost nowhere to be found when it comes to the specific needs of future teachers. While some evidence justify the presence of group theory in the curricula – citing goals such as unifying mathematical knowledge or bringing school mathematics closer to contemporary research - nothing justifies why this content seems relevant specifically for future teachers.

4 Conclusion

The historical study showed that the expectations regarding the content prospective teachers need to know about groups have evolved between 1950 and 1990. Notions about groups were not explicitly taught in secondary school outside of the “New Maths” period. However, some knowledge about groups has been expected from the *agrégation* applicants since 1958. In particular, the role played by groups in geometry seems to have changed over the period: first introduced to give a modern viewpoint on a “classical theory”, the transformations of the Euclidean plane, groups became “inseparable” from the study of geometrical objects during the “New Maths” period. Moreover, the selection board stressed the importance of drawing connections between domains of mathematics. This aspect seems relevant for a capstone course. However, why specific knowledge about groups is expected from secondary school teachers remains unclear. The current situation (2024) presents a typical contrast. Indeed, there is no mention of groups in the secondary syllabus¹¹,

¹¹ One mention, actually: “Sans introduire explicitement les structures algébriques, cet enseignement introduit et étudie certains exemples fondamentaux : corps des nombres

yet some knowledge about groups is required for the written part of the competitive examination. However, in the actual written exams, there were no questions on groups in 2021, 2022, 2023 or 2024. Nothing in official documents (curricula or reports from the selection board) explains this state of affairs.

In order to inform the design of a capstone course in group theory, it thus seems necessary to supplement this heuristic study with other sources. Interviews with mathematicians who are either experts in group theory or who wrote books about group theory aimed at prospective teachers preparing for the *agrégation* should help shed some light on the “what” and “why” questions.

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Secondary sources

complexes, groupes des nombres complexes de module 1 et des racines n -ièmes de l'unité, anneau des entiers relatifs, d'une manière suffisamment approfondie pour préparer à des généralisations. De même, on aborde la notion générale d'équation algébrique, mais pas celle de polynôme formel. Le professeur peut mettre en évidence l'apparition dans divers contextes de notions communes : élément neutre, opposé ou inverse.” (Grade 12, “maths experts”).

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HOW TO LEARN JAPANESE MATHEMATICS “WASAN” IN THE EDO PERIOD (1603-1868)

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ABSTRACT

There were several schools of mathematics in Japan during the Edo period. Each school had its way of writing formulas and expressing mathematics differently. There are few written records of how students learned mathematics during the Edo period, but one small school, the Shisei-Sanka school, has left behind a rare book describing how students learned.

I will introduce the mathematics studied there and the problems published in the *sangaku* (votive mathematical tablet) to show how the Japanese people were familiar with mathematics then. In addition, each school had a license with several levels of teaching mathematics. One of Licenses is also presented.

1 Introduction

There were several schools of mathematics in Japan during the Edo period (1603-1868). The most famous and popular school was the Seki school of the renowned mathematician Seki Takakazu [関孝和](?-1708). In addition to the Seki school, there were various other schools such as Takuma school, Shisei-Sanka school, Saijyo school, Miyagi school, Omura school etc.

I would like to describe how to express mathematics in Seki school, Takuma school and Shisei-Sanka school. And I also introduce how to learn mathematics, a license and a *sangaku* (votive mathematical tablet).

2 How to express formulas in different schools

In this section, I discuss the representation of mathematics in the three schools.

2.1 Seki school's way of writing formula

The Seki school's way of expressing mathematics is inspired by the motif of *Sangi* [算木](arithmetic sticks), which was introduced to Japan from China.

The sticks were the tools used for calculation (fig. 1). The arithmetic sticks were placed vertically for the first, hundred, and ten thousand places and horizontally for the tenth, thousand, and hundred thousand places. The red blocks represent positive numbers, and the black blocks represent negative numbers. For example, 231 and 5089 are positive numbers, so the red arithmetic sticks are used, and -407 and -6720 are negative numbers, so the black arithmetic sticks are used (fig. 2).



Figure 1. Representation of numbers by arithmetic sticks

231			≡	
5089	≡		⊥	≡
-407				≡
-6720	⊥	≡	=	

Figure 2. Example of representation of numbers by arithmetic sticks

When writing in books, negative numbers were represented by diagonal lines to express them in black ink only. In addition, zero is expressed by ○. For example, II 一○ represents 210. Figure 3 shows the pages of the famous book *Katsuyō Sanpō* [括要算法] (Summary Method) (1712), which describes Bernoulli numbers and binomial coefficients. The number of arithmetic sticks were written.

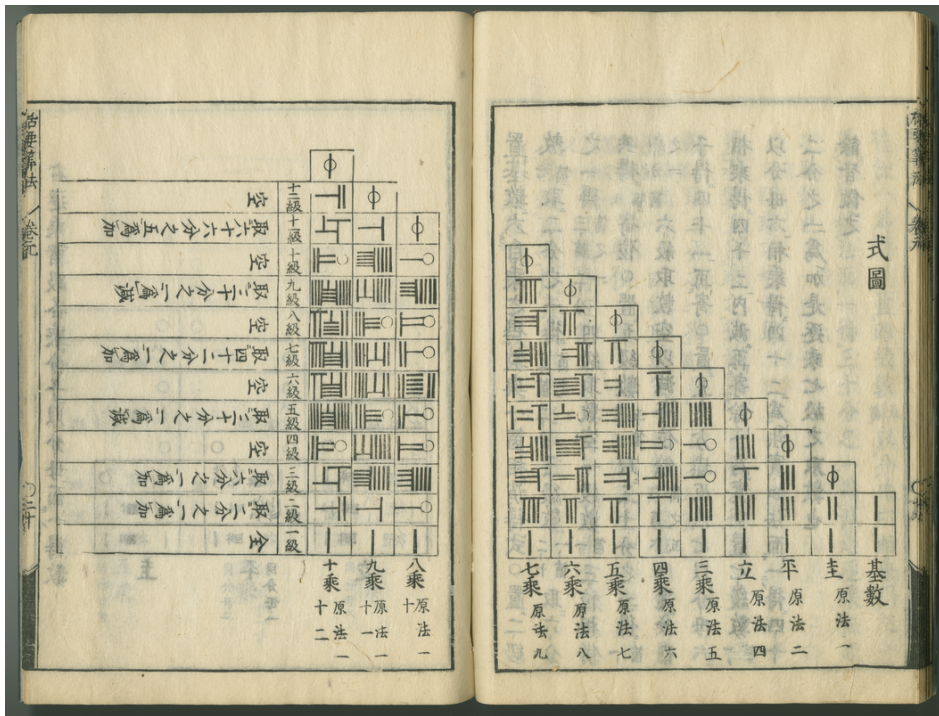


Figure 3. *Katsuyō Sanpō* [括要算法] (1712)

These pages were described Bernoulli numbers and binomial coefficients.

2.2 Takuma school's way of writing formula

Takuma school was a school founded by Takuma Yoshikiyo [宅間能清] in Osaka. Unlike Seki school's way of writing formula, Takuma school does not use numerical expressions of arithmetic sticks. Negative numbers are written as *fu* (フ) in Japanese katakana [カタカナ], and numbers are one(一), two(二), three(三) in Japanese kanji [漢字].

For example, Figure 4 shows

1 上矢(upper arrow) - 1 小円(the diameter of small circle).

ラ is (-1), in other words, it is said to be 1 negative.

Figure 5 is *Myōyoku Shū* [妙矩集] (Exquisite

formulae) No.6 of Takuma School's Collection [起術解路法]. The following is a description in modern notation.

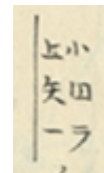


Figure 4. *Myōyoku Shū* [妙矩集] No.6 of Takuma School's Collection [起術解路法].

As shown in the figure (fig. 5), there are two circles C and C' , and C' is inscribed in C . Line L passes through the center of circle C , and 2 lines L and M intersect in circle C . L and M are perpendicular. Also, z is the diameter of circle C' . When C' is tangent to M , answer the following questions for the figure's lengths a , b , x , y , and z .

When 上矢= x , 下矢= y , 小円(the diameter of C')= z , 丑= a , 寅= b , 子= c , the lower right side of Figure 4 is written $1(x-z)y=1b^2$, and the lower left side is written $1yz=1ac$.

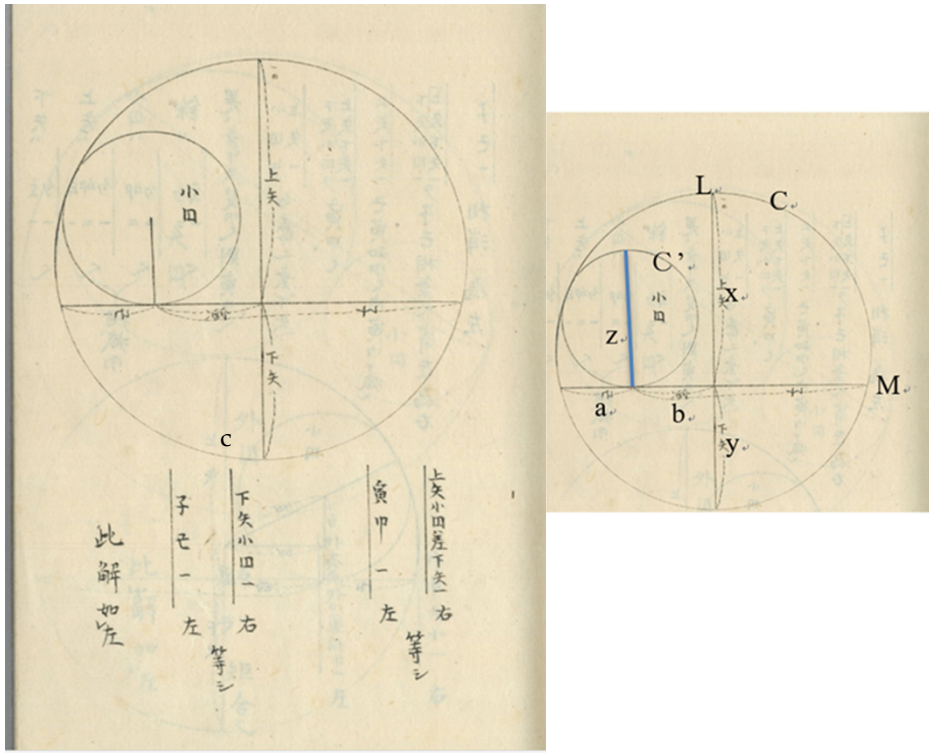


Figure 5. *Myouku Shu* [妙矩集] (Exquisite formulae) No.6 of Takuma School's Collection [起術解路法].

2.3 Shisei-Sanka school's way of writing formula

Shisei-Sanka [至誠賛化] school was a school founded by Hurukawa Uji-kiyo [古川氏清](1758-1820). The meaning of the phrase "Shisei-Sanka [至誠賛化]" is to agree that heaven, earth, and nature create and nurture all things if one is sincere in one's heart.

Fig. 6 is one of the problem of *Kugou-Ruihen* [矩合類編] (a collection of equations found in geometric figures).

(Length of one side of the square) = a ,
 (length of the diagonal) = b ,
 then $\sqrt{2}a - b = 0$

The expression of the formula is almost the same as the expression of Seki school's formula.

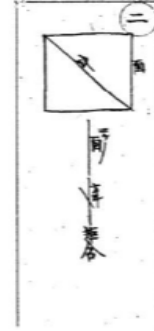


Figure 6. 矩合類編 of Shisei-Sanka School's Collection.

3 How did students learn mathematics during the Edo period?

Few written records exist of how students learned mathematics during the Edo period, but one small school, the Shisei-Sanka [至誠賛化] school, has left behind a rare book *Kiou shu* [淇澳集]. The following is “how to learn mathematics” as written in the *Kiou shu*.

Rule

1. When you hold up a question on a tablet wooden board to a particular student, ask that student by name. When you ask the whole group a question, you should write: 'Ask this to everyone'. Students should have approximately one to two months to solve and answer the question, depending on the complexity of the answer the question requires.
2. If no one can answer a question presented to all students, then all students have lost. Therefore, try your best to solve the math problem and answer first. However, if you cannot find the answer, ask the teacher for the solution.
3. The student should not rely on the help of others, either in the formulation of the problem or in the formulation of the answer. The best way to solve a problem is to use *Tenzan* [天竄]¹, and the second best way to solve a problem is to use *Tengen jyutu* [天元術]². However, since the degree of mathematical training varies from person to person, we do not insist on this limitation.

¹ *Tenzan* [天竄] is a symbolic manipulation technique for solving equations with more than one unknown.

Solve the problem with ingenuity by following the above rules. It is strictly forbidden to discuss our school with those of other schools. Therefore, we do not allow the questions of different schools to be included in the wooden boards of this school. (Jan.1808)

This instruction shows that mathematics was kept secret by each school. Each school had its way of writing formulas and expressing mathematics differently.

The flower arrangement and tea ceremony in Japan today have their schools, and each school has its style of flower arrangement and tea ceremony.

In mathematics in Japan during the Edo period, different schools had different ways of writing mathematical formulas, so it was probably not a good idea to learn from other schools.

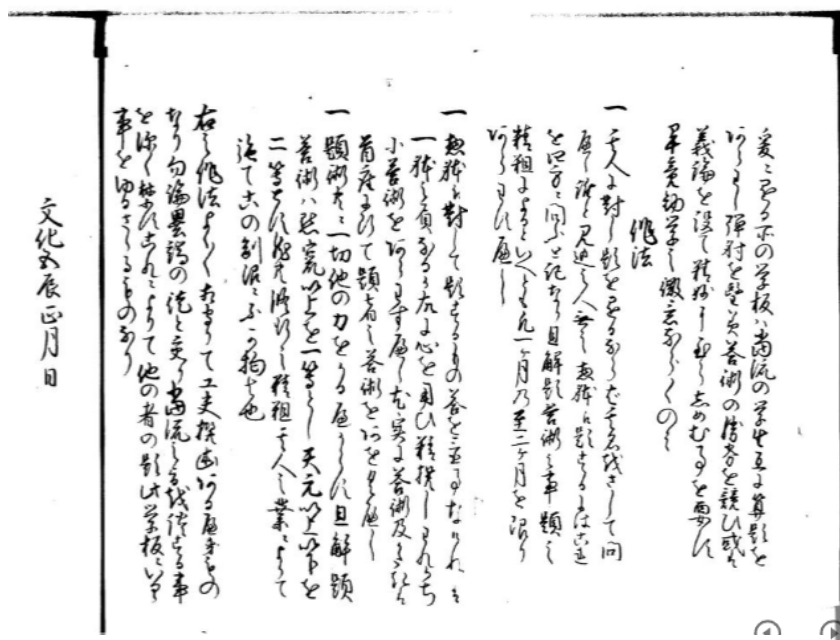


Figure 7. Kiou shu [淇澳集]

The part where the rules for learning are written.

² Tengen jyutu [天元術] is a method of representing and solving equations of higher order using an arithmetic sticks.

4 License

A license was required to teach Japanese mathematics in the Edo period.

Fig.8 is a license for Takuma School of Mathematics (private collection, 1852). A person who is granted a license can teach mathematics. Depending on the type of license, the content of mathematics that could be taught differed. Some licenses allow teaching only basic mathematics, while others enable teaching complicated mathematics. Note that the license is written vertically and reads from right to left. In addition, the license is kept wrapped up.

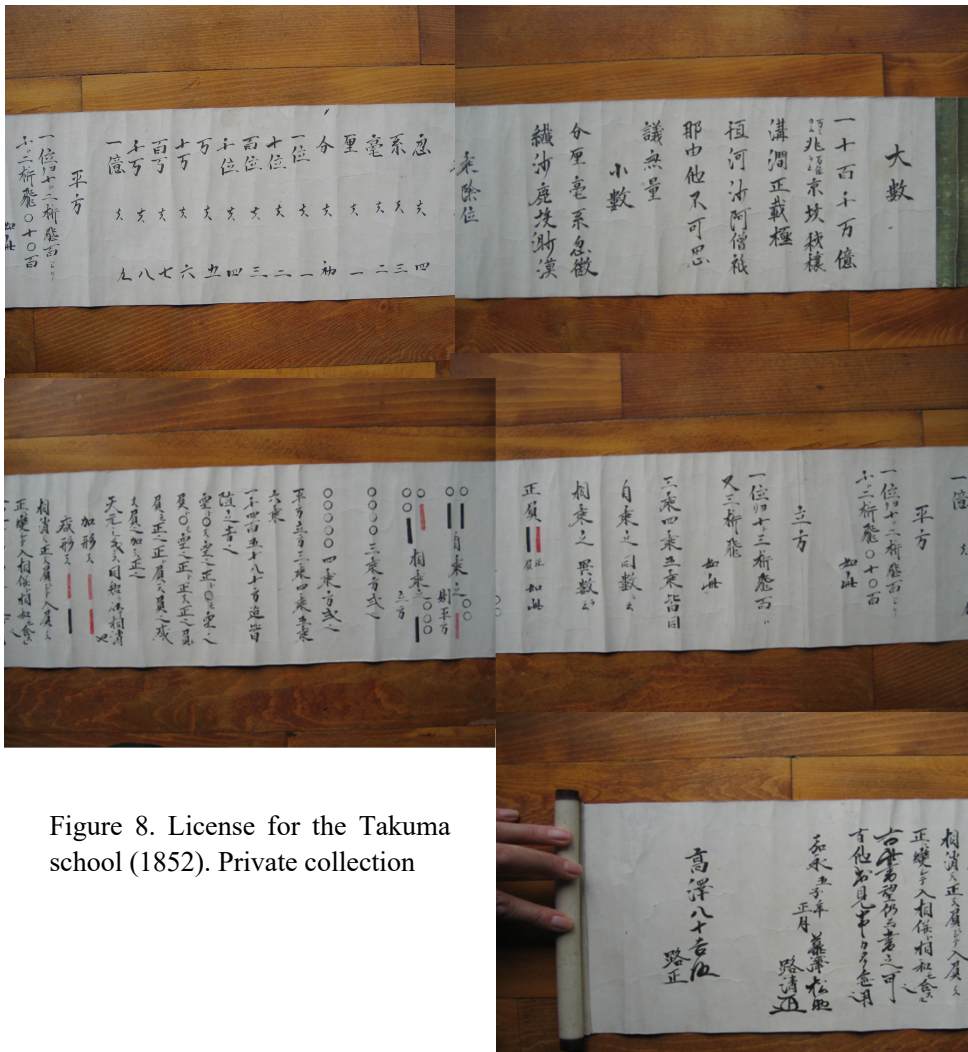


Figure 8. License for the Takuma school (1852). Private collection

5 *Sangaku*

When students solved an excellent mathematical problem, they thanked the gods or Buddha, wrote the problem and the formula for solving it on a wooden panel called *sangaku* (votive mathematical tablets), and dedicated it to a shrine or a temple. In the Edo period, shrines and temples were the most crowded places, not only for festivals, celebrations, and funerals, so by displaying *sangaku* there, students could show many people the problems that could be solved. The names of the students and their schools were written on the dedicated *sangaku*, which would have promoted the schools.

The geometrical figures on the *sangaku* were painted in colorful colors. The *sangaku* are still preserved in shrines and temples in Japan today.

In January 2024, we visited Zenkoji Temple in Nagano Prefecture to see *sangaku* [320cm×133cm] (fig.9). This is relatively large in size. This *sangaku* contains five problems, five geometric figures in colorful colors, with the problem, the answer, and the short explanation for the answer written below each figure. The first large kanji characters on the right side of the *sangaku* are “dedication[奉納],” indicating that the *sangaku* was dedicated to the gods. It was considered beautiful and excellent to use a small number of letters to write on *sangaku*.



Figure 9. Seki school's *sangaku* Zenkoji Temple 1833. Photographed by author

6 From the Edo to early Meiji (1868-1912) period

The many versions of the Edo period mathematics book *Jinkōki* [塵劫記] became a huge bestseller. The people of Edo learned mathematics by reading the *Jinkōki*.

The pace of Western mathematics introduction quickened as the Edo period drew to a close. However, it's important to note that Western mathematics introduced during this period was not in its pure form but rather as a tool for specific applications or practices. The more advanced branches of European mathematics, such as calculus and algebra, were yet to be systematically integrated. From the Meiji period onward, Japan promoted modernization on all fronts, following the lead of the West. A mathematics education policy was instituted based on Western mathematics rather than traditional Japanese mathematics.

This meant that many *Wasanka* [和算家] (Japanese mathematicians) could teach elementary Western mathematics if they could learn it. In other words, many people who knew Japanese mathematics were teaching at elementary schools in the Meiji era. Japanese mathematics has quickly abandoned the more than 200-year tradition of Japanese mathematics. Facing such a situation, I wonder if those involved in Japanese mathematics felt no lingering regret for their Japanese mathematics. Some Japanese mathematician expressed their displeasure with the penetration of Western mathematics.

Despite the discontent expressed by some Japanese mathematicians, the trend towards the adoption of Western mathematics persisted. This trend, rooted in the core support of science and technology, underscored the enduring influence of Western knowledge on Japanese education during the Meiji era.

Japanese mathematicians who also studied Western mathematics from the end of the Edo period to the beginning of the Meiji period were among the first to realize the value of Western mathematics. It did not matter whether the mathematics was Japanese or Western as long as it was useful. Those who promoted the introduction of Western mathematics also studied Japanese mathematics. The shift from Japanese to Western mathematics was an inevitable movement that accompanied a general policy shift in science and technology.

7 *Teaching WASAN in the High School Mathematics Classroom*

The history of mathematics in Japan is not clearly positioned in the curriculum. However, almost all mathematical textbooks provide students with access to the content of traditional Japanese mathematics, "WASAN". Teachers do not deal with it much. I would like to describe a class in which Japanese high school students solve problems by looking at the original text.

I dealt with the problem in Figure 5 (*Myouku Shu* [妙矩集] No.6 of Takuma School's Collection [起術解路法]).

The Math lesson is as follows.

- 1 Target group: 20 interested third-year students of a certain Japanese high school
- 2 Timeframe: 1 Lesson, December 2022
- 3 Contents (Lesson Flow)
 - 1) The teacher handed out copies of the book of the Takuma School and explained about mathematics in the Edo period.
 - 2) While explaining how to read mathematics in the Edo period, the teacher wrote problems on the blackboard in the modern writing style.
 - 3) When the teacher finished writing the problem in the modern writing style, the students thought about the problem.
 - 4) The teacher wrote the answer on the blackboard in the modern writing style in the book.
 - 5) Since the students did not understand the answers, the teacher discussed the solution with the students and they all understood the solution together.
- 4 Student's Impressions
 - 1) This was the first time I saw mathematics written vertically.
 - 2) Just by looking at the problem, it was too difficult to solve.
 - 3) It was difficult to understand the solution written in an Edo period book because it did not describe the way of thinking in the middle of the problem.
 - 4) People in the Edo period may have been amazing in their ability to solve plane geometry problems.
 - 5) I would like to see other problems.

8 Summary

In Japan during the Edo period, there were various schools of mathematics, and different schools had different ways of writing mathematics. In this report, we introduced literature describing “how to learn mathematics” at that time, licenses, and *sangaku* dedicated to shrines and temples. In addition, we discussed how Japanese mathematicians viewed Western mathematics from the Edo to early Meiji period.

Few studies have been conducted on Japanese mathematics in the Edo period from an educational perspective, and I wanted to introduce it internationally this time.

In the future, I would like to deepen my consideration of the spread of the schools of Japanese mathematics during the Edo period and why the Seki school, which was the largest, could not unify all the schools in Japan. I would also like to continue my research to see if there are any other documents written about education at that time.

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THE CONCEPTS OF CURVES AND EQUATIONS IN EARLY AMERICAN & BRITISH TEXTBOOKS ON ANALYTIC GEOMETRY

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ABSTRACT

Curves and equations, also known as the Cartesian Connection, is an important and fundamental concept in plane analytic geometry, but it has been shown that students lack a rigorous understanding. Focusing on the concept of curves and equations, we investigated 84 American and British analytic geometry textbooks published from 1826 to 1963. The study found that three types of non-rigorous definitions were prevalent in early textbooks published before the 20th century. The first type merely recognized a connection between curves and equations but did not express the nature of the strict correspondence. The second type, although intentionally emphasizing that each point needs to be satisfied, examined only one aspect of the correspondence, neglecting the bidirectional nature inherent in the relationship between curves and equations. The third category of non-rigorous definitions confused the concepts of curves with functions. Definitions of curves and equations have become increasingly rigorous in post-20th century textbooks, and these rigorous definitions can be divided into descriptive definitions, definitions based on the concept of set, and definitions based on sufficient and necessary conditions. It is noteworthy that the second category of non-rigorous definitions was not eliminated until the middle of the 20th century. Epistemological barriers in the historical development of the curves and equations concepts can become cognitive barriers for students in the classroom as well. Instructional strategies can be developed to guide students in recognizing non-rigorous definitions, leading them through a historical reconstruction of how concepts related to curves and equations have evolved. This approach facilitates a natural progression from qualitative to rigorous understanding. In addition, early textbooks also provide a variety of methods for verifying the Cartesian Connection, which provide abundant materials for teaching.

1 Introduction

The development of analytic geometry was a gradual process. Before its formal establishment, Apollonius (c. 262 B.C.-190 B.C.) and N. Oresme (1323-1382) used coordinate axes to study curves, while F. Viète (1540-1603) applied algebraic methods to solve geometric problems. In the 17th century, R. Descartes (1596-1650) and P. de Fermat (1601-1665) combined these approaches, establishing the connection between curves and equations within a

coordinate system. This lengthy evolution suggests that students may inevitably encounter difficulties when learning about curves and equations.

Curves and equations, also known as the Cartesian Connection, refers to the principle that “a point is on the graph of the line l if and only if its coordinates satisfy the equation of l .” (Moschkovich et al., 1993). Despite its significance, research on how to effectively teach the Cartesian Connection remains limited. For instance, a study of high school students in Shanghai found that while textbooks introduce curves and equations, teachers rarely require students to verify the “if and only if” conditions for deriving equations, leading to confusion (Ruan et al., 2012). Similar findings by Knuth et al. (2000) and Moon et al. (2013) confirm that curves and equations remain challenging for both students and teachers.

To explore this further, we analyzed 84 American and British analytic geometry textbooks published between 1826 and 1963. The following research questions guided our investigation:

How were curves and equations defined in early textbooks?

How did these definitions evolve over time?

2 Methods

This research is part of a program on studying early American and British textbooks (Figure 1), organized by the HPM Community in Shanghai, China. The program aims to provide resources for teaching from a historical perspective, promote a comprehensive understanding of mathematical concepts among pre-service and in-service teachers, and offer insights for the development of curriculum materials. The program primarily focuses on textbooks published between the 18th and 20th centuries, a period during which modern mathematical knowledge systems gradually evolved from exploratory constructions into standardized forms. Textbooks from this era not only preserve the historical traces of conceptual developments in fields such as algebra, geometry, trigonometry, and analytic geometry, but also reflect the early shaping of modern instructional systems.

For this research, we focused specifically on analytic geometry textbooks, which became established as a distinct genre relatively late in the evolution of mathematics education. Based on a search in the HathiTrust Digital Library using the keywords “Analytic Geometry” and “Coordinate Geometry,” the

earliest suitable American or British textbook we identified was *The Principles of Analytical Geometry*, published in 1826.

A total of 84 analytic geometry textbooks published between 1826 and 1963 were selected. The textbooks were grouped into 20-year intervals, with their publication dates illustrated in Figure 2. In cases where a textbook was republished by the same author, the earliest version was selected unless substantial changes were made in later editions; in such cases, the revised edition was treated as a distinct textbook. Textbooks published after the mid-20th century were generally excluded, as analytic geometry by that time had become a mature and highly standardized subject, rendering later works less informative for the purposes of this historical and conceptual study.

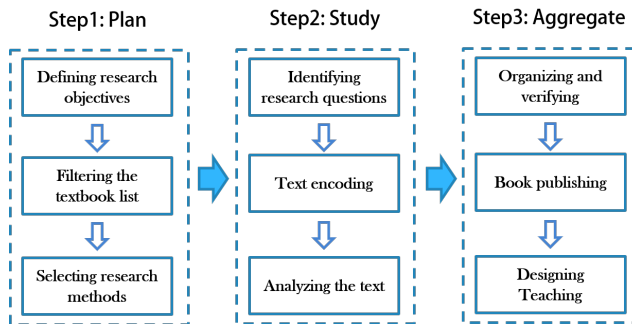


Figure 1. Flowchart of the program

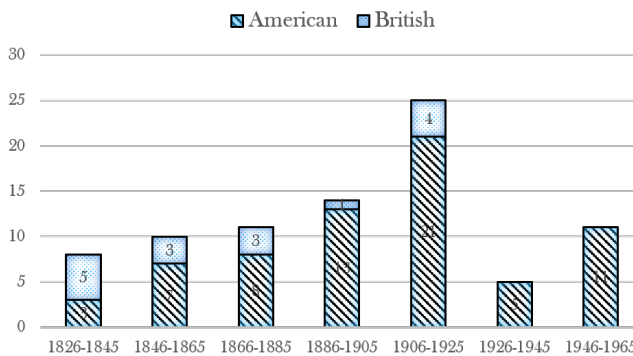


Figure 2. Temporal distribution of 84 early textbooks

3 Findings

Briggs (1881) emphasized the foundational role of this connection in his

textbooks, noting: “This close relationship between curves and equations is the foundation of our discipline and warrants careful study. Once this concept is understood, the subject becomes natural and accessible; however, for anyone who neglects it or has an unclear grasp, analytic geometry will be incomprehensible.” (Briggs, 1881, p. 12).

Although the idea of connecting curves and equations emerged with the works of Descartes and Fermat, their use of coordinates was limited to positive values. It was not until mathematicians such as J. Wallis (1616-1703) began to consciously incorporate negative coordinates that a more complete one-to-one correspondence between plane curves and bivariate equations became possible, as noted by Kline (Kline, 1990, p. 322). The rigorous understanding of this connection, however, took much longer to develop, as evidenced by the definitions found in early textbooks, which transitioned from non-rigorous to rigorous over time.

3.1 Non-rigorous Definitions

Some early textbooks offered definitions of the connection between curves and equations that were not rigorous enough. These can be categorized into three types.

3.1.1 Type 1: Qualitative Definitions

The first category, qualitative definitions, recognized the connection between curves and equations without expressing the strict or formal nature of their correspondence. A total of 15 textbooks adopted this approach. For example, Biot (1840) described the mutual representation of curves and indeterminate equations as follows: “We may regard every line as susceptible of being represented by an equation between two indeterminate variables; and, reciprocally, every equation between two indeterminates may be interpreted geometrically and considered as representing a line, the different points of which it enables us to determine.” (Biot, 1840, p. 27). This explanation addressed the relationship in a general and rhetorical manner, rather than in a symbolic or algebraically rigorous way.

3.1.2 Type 2: Single-direction Definitions

The second category, found in 14 textbooks, fell into the trap of single-direction definitions. Although these definitions emphasized that each point must satisfy

the equation, they focused on only one direction of the correspondence, neglecting mutuality. For instance, Young (1830) stated: “The line which any equation represents, or in which the variable point (x,y) is always found, is called the locus of that equation, or of the point (x,y) .” (Young, 1830, p. 36). This definition focused on the equation determining the curve but overlooked the converse.

A similar asymmetry appears in Hardy’s definition, which stated: “The equation of a locus is the equation which is satisfied by the coordinates of every point on the locus, and by no others” (Hardy, 1897, p. 14). While precise, this description emphasized sufficiency but did not explicitly require that all solutions of the equation must lie on the locus—thus potentially compromising completeness.

3.1.3 Type 3: Definitions Confused Curves with Functions

The third category confused the concepts of curves and functions. For example, Riggs (1911) conflated the terms “equation” and “function” by stating: “In each of the examples to be next studied, some simple locus of points will be considered, and the equation which expresses the dependence of the ordinate of any point of the locus upon the abscissa of the point will be derived. This equation will be known as the equation of the locus.” (Riggs, 1911, p. 41). Similarly, Dowling (1914) argued: “The equation of the locus defines y as a function of x , and the locus itself is the graph of this function.” (Dowling, 1914, p. 53).

The confusion between equations and functions likely arose from historical developments. The function concept, as introduced by J. Bernoulli (1667-1748) and L. Euler (1707-1783), bore a formal similarity to equation expressions in 19th-century algebra textbooks. Early algebra textbooks often defined functions through equations, and F. Klein (1849-1925) later emphasized unifying mathematical content under the concept of functions. This historical entanglement between “equation” and “function” has been well-documented (Liu et al., 2021).

3.2 Rigorous Definitions

Over time, textbooks increasingly provided rigorous definitions of the relationship between curves and equations. These definitions can be grouped into three types.

3.2.1 Type 1: Definitions by Describing

The first type of rigorous definition is derived through direct description. For example, Peck (1876) defined it as follows: “The equation of the locus of points satisfying a given condition is an equation in the variables x and y , representing the coordinates, such that the coordinates of every point on the locus satisfy the equation; conversely, every point whose coordinates satisfy the equation lies on the locus.” (Peck, 1876, p. 42).

3.2.2 Type 2: Definitions Based on Set

A total of 6 textbooks utilized the theory of set to define the relationship between curves and equations. Hamilton (1826) was the first to propose: “Let $f(x,y)$ be an indeterminate equation between x and y ; then, the set of points (x,y) will form a curve, called the locus of the equation $f(x,y) = 0$.” (Hamilton, 1826, p. 52).

Although Hamilton (1826) offered a rigorous definition, his proof that the locus of a linear equation is a straight line fell into circular reasoning, possibly due to the immaturity of set theory at the time (Hamilton, 1826, pp. 55-58). With the later widespread acceptance of Cantor’s (1845-1918) set theory, some textbooks began using set-theoretic notation. For example, Taylor (1962) represented the intersection and union of two curves using set operations (Taylor, 1962, p. 3). However, none of the surveyed textbooks used set notation to define the correspondence between curves and equations explicitly.

3.2.3 Type 3: Definitions Based on Sufficient and Necessary Conditions

Some textbooks provided concise definitions using logical terminology. For instance, after offering a descriptive definition, Taylor (1959) stated: “In other words, for a specific point (x,y) , the ordered pair (x,y) satisfies the equation if and only if the point (x,y) lies on the curve.” (Taylor, 1959, p. 20).

4 The Temporal Distribution of Chapters on Curves and Equations

The analysis above highlights the widespread misconceptions surrounding the concept of curves and equations in early textbooks, as reflected in the prevalence of non-rigorous definitions. In contrast, rigorous definitions demonstrate a variety of approaches converging on the same underlying principles. Figure 3 illustrates the evolution of definitions in early textbooks,

grouped by 20-year intervals.

From the figure, it is evident that prior to the 20th century, more than half of the textbooks provided non-rigorous definitions, with qualitative definitions being the most common. This indicates that many authors of early textbooks approached the relationship between curves and equations primarily from a qualitative perspective. Entering the 20th century, the proportion of textbooks offering rigorous definitions gradually increased, with descriptive definitions remaining the predominant approach.

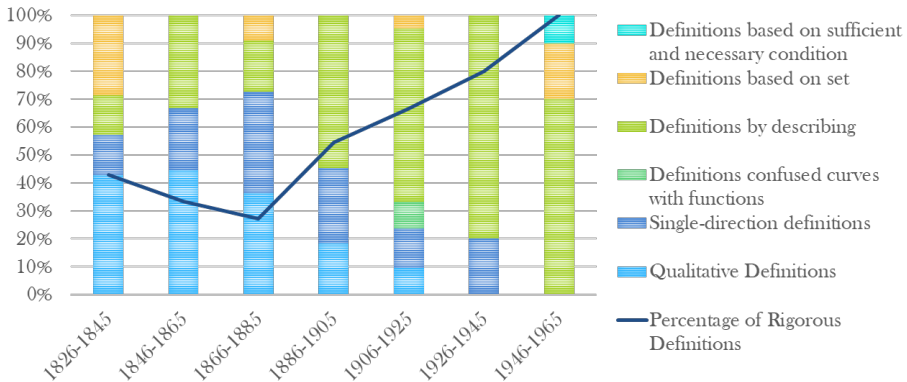


Figure 3. Evolution of the concepts of curves and equations

Notably, while more textbook authors attempted to rationalize the relationship between curves and equations quantitatively, many fell into the trap of focusing on only one direction of the correspondence. This misconception persisted well into the mid-20th century, underscoring the enduring challenges in achieving a comprehensive understanding of this foundational concept.

5 Verification

Peck (1876) emphasized that “The statement of the definition must be demonstrated, and the derived equation of the locus must be verified.” (Peck, 1876, p. 42). The verification methods in early textbooks can be broadly categorized into three types.

5.1 Type 1: Proof by Contradiction

Tanner (1898), after deriving the equation $3y - x - 3 = 0$ for the line passing through the points $P_1(3, 2)$ and $P_2(12, 5)$, provided two types of proof by

contradiction to demonstrate that “any point not on the line does not satisfy the equation.” (Tanner, 1898, pp. 61-63).

Method 1: In the derivation process, he used the property of proportionality between corresponding sides of similar triangles. If a point is not on the line, it is impossible to form a similar triangle, and hence the proportionality $\frac{y-2}{5-2} = \frac{x-3}{12-3}$ does not hold.

Method 2: It was assumed that $P_3(x_3, y_3)$ was not on the line passing through P_1P_2 . A perpendicular was drawn from P_3 to the x-axis, which intersected the line P_1P_2 at $P_4(x_4, y_4)$, where $x_3 = x_4$ but $y_3 \neq y_4$. Substituting into $3y_4 - x_4 - 3 = 0$, it was found that $3y_3 - x_3 - 3 \neq 0$. Therefore, it was evident that no point off the line satisfied the equation $3y - x - 3 = 0$.

5.2 Type 2: Backward Reasoning

Young (1936) used backward reasoning to verify the derivation of the standard equation of an ellipse. Starting from the equation of the locus,

$$PF_1 + PF_2 = 2a \quad (1)$$

he derived the standard equation of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (2)$$

He then verified that points satisfying equation (2) necessarily meet the condition in equation (1). By substituting values from (2) into the derivation steps, he showed:

$$\pm\sqrt{(x-c)^2 + y^2} \pm \sqrt{(x+c)^2 + y^2} = 2a \quad (a^2 - c^2 = b^2),$$

Four possible cases arose for the coordinates (x, y) : Case (a): $++$; Case (b): $-+$; Case (c): $+-$; Case (d): $--$. Young (1936) then demonstrated that only case (a) satisfied the condition $PF_1 + PF_2 = 2a$, as in the other cases, the triangle VPF_1F_2 's two sides differed by more than the third side, $2c$. This confirmed that the standard equation must meet the original condition. (Young, 1936, p. 36).

Cell (1951) further noted: “The verification in the second part can often be simplified by reversing the numerical steps, as this ensures equivalency in the derivation process without repeating each step in detail.” (Cell, 1951, p. 33).

5.3 Type 3: Direct Substitution

Smith (1954) while deriving the equation $2x - y - 1 = 0$ for the locus of points

equidistant from $P_1(3, 0)$ and $P_2(-1, 2)$, set a point $P_0(x_0, 2x_0 - 1)$ satisfying the equation. He substituted P_0 into the distance formula to calculate the distances from P_0 to P_1 and P_2 , verifying their equality and thus confirming the derived equation (Smith, 1954, pp. 20-22).

The above methods exhibit general applications and can be adapted based on specific contexts. Some 20th-century textbooks, such as those by Roberts (1918), acknowledged that exhaustive verification could sometimes be omitted but emphasized its fundamental importance: “This step is so similar in all examples that the student will not be required to give it, unless called for, but he should never lose sight of the fact that this is one of the essential conditions in the determination of the equation of a locus.”(Roberts, 1918, p. 48).

6 Discussion and Implications

As M. Kline (1990) observed, “The polished presentations in the courses fail to show the struggles of the creative process, the frustrations, and the long arduous road mathematicians must travel to attain a sizable structure.”(Kline, 1990, p. xi). In this regard, non-rigorous definitions, rather than being dismissed, can serve as valuable teaching resources to enhance students’ understanding of the connection between curves and equations.

First, qualitative definitions align well with students' initial conceptual understanding and can provide a foundation for deeper exploration. The second type of non-rigorous definition, which focuses on only one direction of the correspondence, highlights a common misconception. This can be utilized as an opportunity to guide students in identifying counterexamples and developing a more complete understanding of the bidirectional relationship. Finally, the third type, which conflates curves with functions, reflects a persistent confusion that many students also experience. This emphasizes the importance of distinguishing between the two concepts in instructional settings, fostering a clearer and more precise understanding.

As Schubring (2011) argues, using “historical errors” rather than exclusively celebrating historical successes can be more effective in helping teachers recognize the origins of common student difficulties. In this sense, the flawed or incomplete definitions in early textbooks are not merely historical artifacts, but potential pedagogical tools. By addressing these non-rigorous definitions systematically, educators can not only help students overcome common cognitive obstacles but also provide historical context that enriches

the learning process. This approach underscores the value of integrating historical perspectives into mathematics instruction, enabling students to appreciate the gradual evolution of mathematical rigor while cultivating critical thinking and problem-solving skills.

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EXPLORING THE TEACHING OF GEOMETRY FOR CHILDREN IN 19TH-CENTURY SPANISH EDUCATIONAL PRESS: THE CASE OF 12-YEAR-OLD CARLITOS TEACHING HIS FRIENDS

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ABSTRACT

Published journals have proven to be a valuable source of information for the history of mathematics and mathematics education. Consequently, this study aims to analyse a geometry section included in a 19th-century Spanish journal targeted at children, *Los niños: revista de educación y recreo*. For this purpose, we carried out a descriptive study using a content analysis technique, which is widely employed in research in the aforementioned field. The results indicate that the author employed an original approach to teach theoretical geometry, incorporating dialogues, materials, and graphical representations.

1 Introduction

In the last few years, different studies have considered the possibilities of analysing journals written in the past to broaden the understanding of how mathematics was disseminated, taught and learnt. For example, Zelbo (2019) analyses the recreational mathematics activities included in two mathematics puzzle columns of nineteenth-century American journals. Madrid et al. (2023) study the contents related to mathematics or mathematics education included in *Semanario de Salamanca*, an 18th-century weekly journal published in the Spanish city of Salamanca. Muñoz-Escolano et al. (2022) examine the mathematics problem section that appeared in the Spanish bulletin *La Escuela. Boletín de la Asociación Provincial del Magisterio de primera enseñanza* from 1913 through 1916.

Gazeta de los Niños, the first journal designed specifically for children, was published in Spain in 1798 (Chivelet, 2010). Since then, various educational journals targeting children emerged in 19th-century Spain and their

study helps broaden our understanding of mathematics education at that time. One of these journals was *Los Niños: revista de educación y recreo* (Children: journal of education and leisure).

In March 1870, Carlos Frontaura founded this children's journal in Madrid; it was modelled on the Parisian journal *Magasin d'Éducation et Récréation*. This publication enjoyed the public favour, and featured carefully created content with contributions from renowned writers who aimed to produce brief, engaging, and reader-appropriate texts (Martínez Arancón, 2005).

Several issues of this journal included mathematical contents, for example in 1874 the journal included a section titled *Problemas* (Problems). Moreover, among these mathematical contributions are 32 articles published between 1871 and 1872 by Eduardo Thuillier, which focus on geometry for children. The aim of our study is to know more about the teaching of geometry at the time through the analysis of these publications.

2 Methodology

To achieve this, we conducted a descriptive and exploratory study of a historical-mathematical nature, employing content analysis techniques for analysing texts from the past. This research employed the instrument developed by Madrid et al. (2022).

For this purpose, each paragraph within the *Geometría de los niños* section was defined as a unit of analysis. These paragraphs were read, analysed, and subsequently categorized. The issues of the journal were accessed digitally through the Virtual Library of Historical Press, provided by the Spanish Ministry of Culture and Sports.

3 Results

The children's geometry section, signed by Eduardo Thuillier, appeared for the first time in issue 15 of May 1871. The first article is titled Carlitos, and it explains the story of a 12-year-old boy named Carlitos, who teaches geometry to his friends in the garden of one of their houses. The author justifies this by explaining that Carlitos' friends have exams and are falling behind in geometry, whereas Carlitos is more advanced. Consequently, they

agree that rather than playing other games, Carlitos will teach lessons on this subject in the afternoon (Thuillier, 1871a). These are the 32 articles included:

ISSUE	TITLE
Volume III Issue 15 - 1871 May	1. Carlitos
Volume III Issue 16 - 1871 June	2. Extension
Volume III Issue 17 - 1871 June	3. Lines
Volume III Issue 18 - 1871 June	4. Two lines
Volume IV Issue 1 - 1871 July	5. Angles
Volume IV Issue 3 - 1871 July	6. Angles continued
Volume IV Issue 4 - 1871 August	7. Two new students
Volume IV Issue 5 - 1871 August	8. Triangles
Volume IV Issue 7 - 1871 September	9. Triangles continued
Volume IV Issue 8 - 1871 September	10. Polygons
Volume IV Issue 9 - 1871 September	11. Quadrilaterals
Volume IV Issue 11 - 1871 October	12. Parallelograms
Volume IV Issue 12 - 1871 October	13. Parallelograms continued
Volume IV Issue 13 - 1871 November	14. Displeasure caused by the geometers
Volume IV Issue 14 - 1871 November	15. Quadrilaterals final
Volume IV Issue 15 - 1871 November	16. Polygons
Volume IV Issue 16 - 1871 December	17. Polygons final
Volume IV Issue 17 - 1871 December	18. Circumference and its lines
Volume IV Issue 18 - 1871 December	19. The previous lesson continues
Volume V Issue 1 - 1872 January	20. Circle
Volume V Issue 2 - 1872 January	21. Finishing the interrupted lesson
Volume V Issue 3 - 1872 January	22. Areas of polygons
Volume V Issue 5 - 1872 February	23. This and that
Volume V Issue 8 - 1872 March	24. Plane and straight line
Volume V Issue 9 - 1872 March	25. Two planes and one plane
Volume V Issue 12 - 1872 April	26. New angles

Volume V Issue 13 - 1872 May	27. New angles continued
Volume V Issue 15 - 1872 May	28. Polyhedra
Volume V Issue 17 - 1872 June	29. Pyramid and prism
Volume VI Issue 1 - 1872 July	30. Round bodies
Volume VI Issue 2 - 1872 July	31. Volume of bodies
Volume VI Issue 3 - 1872 July	32. Results of the lesson

Table 1. The section Children's geometry in the journal.

As shown in Table 1, in each lesson, Carlitos explains various geometrical concepts, both in two-dimensional and three-dimensional space, including: length, lines, straight lines, angles, triangles, polygons, quadrilaterals, parallelograms, circumferences, circles, areas of polygons, planes, polyhedra, prisms, pyramids, cones, cylinders, spheres, or volumes.

Throughout the articles, the different concepts are presented as dialogues among Carlitos and his friends. For example, in article 29, they speak about types of prisms, and to explain oblique prisms they engage in this dialogue:

- Then, cried little Esteban, the lateral faces of the oblique prism will be oblique to the bases.
- Yes, and that is the property that distinguishes them – Carlitos.
- It seems incredible that they don't fall, said Ricardo.
- They would fall, replied Carlos, [...] (Thuillier, 1872).

An interesting aspect is Carlitos' use of tangible materials such as pieces of rope, paper, wood, rods, sticks, or rings to explain concepts. For example, Carlitos uses rods to represent lines as shown in Figure 1.

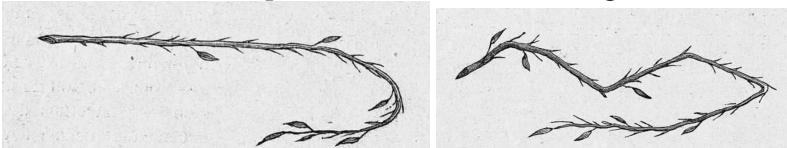


Figure 1. Examples illustrating types of lines (Thuillier, 1871b, p.262).

Additionally, Carlitos provides graphical representations as can be seen for example in Figure 2 when he talks about adjacent and opposite angles.

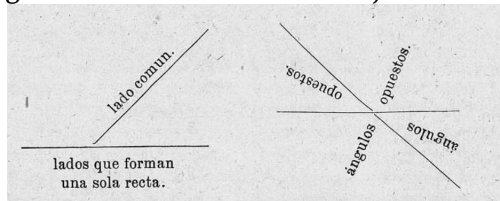


Figure 2. Graphical representation of angles (Thuillier, 1871c, p.43).

4 Conclusions

Published journals allow us to approach the history of mathematics and mathematics education in a non-formal context, showing how it was taught to different target groups, as shown in Madrid et al. (2023) or Muñoz-Escolano et al. (2022). In this case, we focused on the dissemination of geometry in an informal context aimed specifically at children in 19th-century Spain.

The analysis of these 32 articles shows that the author focused on theoretical geometric content, using a variety of materials, and including graphical representations. Future studies will compare this section with similar ones that could be found in the press of the past, thereby expanding our understanding of mathematics dissemination through historical press.

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THE INFINITESIMAL CALCULUS IN TEXTBOOKS PUBLISHED IN SPAIN BEFORE ITS INCORPORATION INTO THE CURRICULUM OF SECONDARY EDUCATION

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The research in the field of learning and teaching Calculus focuses on trends such as cognitive development or task design with particular attention to the concepts of limits, derivatives, and integrals (Bressoud et al., 2016). Studies on Calculus concepts approached through the History of Mathematics Education can help to face this problem since conceptual difficulties with teaching mathematics often correspond with historical periods of conceptual crisis in mathematics (Heeffer, 2006).

This research project aims to analyze the evolution of the contents of calculus and its teaching through teaching textbooks. The period selected is between the introduction of this branch of mathematics in Spain as a scientific discipline, that is, at the beginning of the 18th century, until the implementation of the first study plan that included it among its teachings, the Plan Pidal of 1845.

It is an exploratory, descriptive, and qualitative historical investigation focused on analyzing old textbooks from the perspective of the History of Mathematics Education. It is approached through the historical research method proposed by Ruíz (1976) and the content analysis technique proposed by Maz (2005).

The results of this project may be of interest to improving the training of mathematics teachers. They allow us to identify how, when, and why the difficulties students typically encounter when dealing with calculus-related topics in the classroom arose. It will also allow us to learn about the scientific approaches and teaching strategies used in the past to address these difficulties.

The selected study period coincides with the influence of recognized religious and military and, later, civil institutions dedicated to training engineers and professionals in developing and disseminating calculus from the mid-18th century to the beginning of the 19th century. This period also saw the consolidation of calculus in the rest of Europe. Finally, the calculus textbooks pub-

lished in Spain during this period contain misconceptions, such as the non-consideration of negative algebraic solutions. Therefore, it is relevant to know how and when the advances achieved in Europe were introduced in the teaching of calculus in Spain and whether the same misconceptions continue to be made over the selected period.

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HJELMSLEV'S TEACHING OF HIS GEOMETRY OF REALITY

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During the first half of the 20th century the Danish geometer Johannes Hjelmslev (1873-1950) developed what he called a “geometry of reality”. It was presented as an alternative to the idealized Euclidean paradigm that had recently been completed by Hilbert. Hjelmslev argued that his geometry of reality was superior to the Euclidean geometry both didactically, scientifically and in practice: Didactically, because it was closer to experience and intuition, in practice because it was in accordance with the real geometrical drawing practice of the engineer, and scientifically because it was based on a smaller axiomatic basis than Hilbertian Euclidean geometry but still included the important theorems of ordinary geometry.

It is characteristic of Hjelmslev's geometry of reality that lines have a non-zero width and consequently that two points do not uniquely determine a line: “We have not formulated the requirement that 2 arbitrary points determine a straight line. Indeed, this requirement is in its extreme consequences one of the worst assumptions one has ever introduced in geometry since it is the one that can give rise to the greatest errors”.

Hjelmslev based his own teaching of descriptive geometry at the Polytechnic College in Copenhagen on his controversial new ideas and worked hard (but not entirely successfully) to propagate them to the geometry teaching in primary and secondary schools: For 10 years he taught a course on his geometry of reality to the future mathematics teachers at a teacher training college, and he authored a system of textbooks for primary and secondary school based on his realist ideas.

In the research report, I shall explain the main ideas of Hjelmslev's geometry of reality, their origins and Hjelmslev's attempts to implement them in his own College teaching and in the geometry teaching in primary and secondary schools. I shall also discuss why his ideas had a limited success.

JULES HOUËL (1823-1886): FROM TEACHING GEOMETRY IN HIGH-SCHOOLS TO RESOLVING THE QUESTION OF THE INDEPENDENCE OF EUCLID'S POSTULATE IN FRANCE

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Jules Houël (1823-1886) was a French mathematician and astronomer, from an old Normand Protestant family. After studying at the École Normale, he taught Euclidean geometry at various high schools from 1846 to 1856 (in Bourges, Bordeaux, Pau, Alençon, Caen). On that topic, Houël was a fervent defender of a geometrically pure teaching method and that for this reason he rejected the “hybrid” method developed in the Legendre’s *Éléments de géométrie*. After this period, he taught infinitesimal calculus at the Bordeaux Faculty of Science from 1859 to 1884. Nevertheless, the question of the foundations of classical geometry was a constant subject of reflection to him: as early as 1863, he published several papers/memoirs on the reformulation of Euclid’s postulates, such as *Essai d’une exposition rationnelle des principes fondamentaux de la géométrie élémentaire* or *Essai critique sur les principes fondamentaux de la géométrie élémentaire, ou Commentaire sur les XXXII premières propositions d’Euclide*. Houël’s deep motivations for the foundations of geometry were linked to his experience in high school teaching. He had a point of view that was similar to those of Newton (for the experimental side of axioms and movement), Euler and D’Alembert (for the importance of the assumption of “impenetrability”). He was also interested in the question of the independence of the parallels postulate, which led him to rediscover, translate and publish the work of Lobachevski, Bolyai, Riemann, Beltrami, etc. on non-Euclidean geometries, starting in 1866 in the *Mémoires de la Société des sciences physiques et naturelles de Bordeaux*. The SSPN, a small provincial company until 1866, extended very quickly its branches outside France, as a result of Houël’s work as an archivist and contributor. In fact, in 1866, the SSPN was in an exchange relationship with 20 learned societies, in 1869 with a little over 80 and in 1875 with a little over 170 societies, many of them abroad. Houël also brought the theory of the foundations of geometry onto an epistemological ground; he studied the synthetic nature of geometry while opposing the purely axiomatic vision. Moreover, the conception based on the axioms of elementary geometry and his ideas on the independence of

the postulate of parallels did not vary between 1863 and 1883. The Carton affair at the Paris Academy of Sciences in 1869 precipitated the questioning of the postulate of parallels, and finally, Houël concluded the question in the *Note sur l'impossibilité de démontrer par une construction plane le principe des parallèles*. Our presentation will be based on geometric teaching in high schools in France about 1850, and will outline the epistemological and historical milestones in Houël's reflections on the foundations of classical geometry. Houël's connections with the foundations of geometry and non-Euclidean geometries will also be discussed, in particular with Italian mathematicians (Battaglini, Beltrami, Cremona, etc.). *Nota bene*. Houël's view of the foundations of geometry has nothing to do with Hilbert's *Grundlagen*.

ERRORS DEALING WITH THE NEGATIVE IN SOLVING QUADRATIC EQUATIONS. AN EPISODE IN THE HISTORY OF ALGEBRA AND ITS TEACHING

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The codex Dresden C80 belonged to Johannes Widmann who used it when he lectured on algebra at the University of Leipzig in 1486. Among the documents it contains there is a *Latin Algebra*, Wappler (1887), in which the rule for solving one of the types of second degree equations ($ax^2 + c = bx$, in current terminology) includes an error. The rule is equivalent to

$$\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

what is correct. The error appears in a prompt at the end of the rule: “if you can’t subtract $[c/a]$, you are allowed to add” (Wappler, 1887, p. 14).

This authorization to add instead of subtract also appears in Bombelli’s *L’Algebra*, 1572. After having solved the equation $x^2 + 20 = 8x$ by giving its two imaginary solutions, Bombelli says that “there is another sophistic method, that since 20 cannot be subtracted from 16, it is added”, and he obtains a third solution of the equation, 10, and adds “and this 10 is minus” (Bombelli, 1572, pp. 262-263), without any explanation.

This is not the only error in the rules to solve this type of equation. In a manuscript before 1504, *Die “Algebra” des Initius Algebras* (Curtze, 1902), the rule is stated correctly, but when applied to an example, the order of subtraction is wrongly reversed.

$$\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \pm \frac{b}{2a}$$

And in Adam Ries’ manuscript *Die Coss* (1527), edited by Rüdiger, Rainer & Folkerts (2023), the rule itself is stated with the subtraction erroneously reversed.

Both errors appear in Marco Aurel’s *Arithmetica Algebraica*, published in Spanish in 1552. He uses the erroneous rules in an example and a problem and explains how to obtain the correct result from the erroneous one or how to interpret the wrong result as correct. Its influence on other books written in

Spanish, and especially the success of one of them, Pérez de Moya's *Arithmetica Practica y Especulativa*, published in 1562 and reprinted a lot of times until 1798, made these errors last long in Spain.

In this communication we study the statements of these erroneous rules, their eventual use in examples and problems, the justifications for the way of obtaining the correct result from the incorrect one or interpreting it as correct, and their dissemination.

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**FROM THE MARRIAGE BAR
TO THE HYPATIA SCHOLARSHIP: WOMEN WORKING
IN MATHEMATICS IN AUSTRALIAN UNIVERSITIES**

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The first woman lecturer in the Australian mathematical sciences was appointed in 1921, almost 70 years after the first Australian university was established in 1852. Since then the proportion of women among mathematical sciences students and academic staff has gradually increased, though it has not reached 50% even at the bachelor's completion level. Why so low, and why so slow? What can we do now to create a more inclusive and equitable higher education sector that allows everyone to contribute more fully? What does the past tell us about what works? I'll outline some of the history of women working in the mathematical sciences in Australian universities, with a focus on the national Women in Mathematics Special Interest Group (WIMSIG, founded in 2012) of the Australian Mathematical Society. Drawing on that history, I'll offer insights into our evolving understanding of why women and others have been underrepresented, and the initiatives we've used to address inequity. In addition to describing the Australian historical, societal and legal contexts, I'll consider broader explanatory factors that have acted to inhibit participation of women and other underrepresented groups worldwide. Juicy topics such as the marriage bar and its impact, the female minimum wage, and the Jennifer and John correspondence study will be addressed.

THE DEVELOPMENT OF NEOCOLONIALISM IN PAPUA NEW GUINEA AND ITS IMPACT ON MATHEMATICS EDUCATION

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ABSTRACT

The history of mathematics education in Papua New Guinea began at least 40 000 years ago with evidence of scientific technologies from 10 000 years ago in agriculture, with earlier stone and pottery evidence. However, from the late 1880s, mathematics education was conducted by missionaries and then governments from colonizing countries. The impact of colonialism is summarized especially noting the significant impact of using English language and imported curricula or curricula ideas. Since Independence in 1975, neocolonialism continued to plague mathematics education impacting on social justice and meaningful mathematics education.

1 The significance of this study

Teachers require a knowledge of the history of mathematics education in their own countries in order to assess the merits of the curriculum that might be used in their country. Students from countries that have been (recently) colonised and immigrant students are particularly affected by curricula and education that do not take account of their home cultures and values. In rapidly changing countries, teachers often have a basic teacher education. Curricula are developed through overseas funding and experts and an upwardly mobile teaching profession working under political influences. The resulting neocolonial approaches might lead to a loss of cultural knowledge, values and well-being. Those with monetary wealth or not living in remote areas might be privileged.

2 Background

Papua New Guinea (PNG) had European settlers from the late 1800s with plantations, missions, and governments – in the north Germany and in the south Britain, substantially through its colony Queensland, and later Australia.

Schools in the north were for German and Chinese children in German but the missions also used vernacular languages, not necessarily the local language. The Lutheran Mission in Morobe chose a non-Austronesian language, Kôte, to facilitate learning and worship with mountain people. There are 850 languages in PNG and at least 103 languages in Morobe with more languages in the neighbouring mountains of Eastern Highlands and Madang where Kôte was used. In the south, Lawes, an early long-term missionary, wrote a Motu dictionary in 1886 and developed mathematics plans. After World War 1, the north became a Trust Territory of Australia (Paraide, Owens, Muke, Clarkson & Owens, 2023). Churches still use local languages.

In the first half of the 1900s, government funding was insufficient to run the colony and funds were only available to mission schools (most of the schools) if the schools taught in English and for the number of students who passed English and mathematics examinations set in Queensland (Smith, 1987). Mathematics was and still is valued as an examination subject to progress in school (Paraide et al., 2023).

Pioneering sociologist Mikloucho-Maclay (1975) in Madang, the Royal Cambridge Archaeological Society Expeditions, Malinowski (1917-1920) in the Trobriand Islands, and the Australians who were opening up new areas with administration centres recorded local practices and languages (Owens, Lean, with Paraide & Muke, 2018). Furthermore, there were two long-term Australian administrators, William Macgregor and Sir Hubert Murray, who were understanding in their decision-making (West, 1968). Schooling the ‘natives’ was to overcome exploitation by plantation owners and for pacification as there was not only trading between neighbours along trade routes but conflicts (Paraide et al., 2023).

These first contact references, teachers’ biographies, theses, research, and many government reports, curricula, and education publications, over 300 research reports of cultural practices, responses to questionnaires on counting systems (1000s) and measurement practices (352), in-depth interviews (25), field visits (25), oral histories (17) and our own lived experiences have made it possible to write a history of mathematics education, even before colonisation as many practices continue to today (Owens, 2015; Owens, et al., 2018; Paraide et al., 2023).

3 *Entrenching western mathematics education*

In the 1960s, the Prime Minister, Hasluck (1976) realised that the Papua and New Guinea Territories would be autonomous, and the government of Australia worked towards its independence. There was a hurried move to encourage education into high school and onto tertiary studies. Without university graduates, the country could not self-govern. Australian teachers were trained (many in a 6-month course in PNG) (Freestone, 2011), ready for the mushrooming primary schools with local teachers who watched and learned. Secondary education, staffed initially by Australians and other expatriates, began in towns. Teacher education courses began for Grade 10 graduates. Four National High Schools were set up to bring students from across the provinces together to reach Grade 12 and to learn to recognise and work across diverse cultures. The Universities provided Preliminary Years to recruit more students from Grade 10. Mathematics was taught at both the University of Papua New Guinea (UPNG) and PNG University of Technology (Unitech) providing graduates in all required areas. These Universities had outstanding professors and teaching staff from around the world who loved PNG and their competent students. Unitech soon had National Vice-Chancellors (3rd was Tigali, 5th was Moromoro, and 7th was Baloiloi). The Mathematics Department ensured support for all students to reach the high standard needed and tailored the courses for each degree. Eminent researchers in mathematics education such as Alan Bishop, Gilah Leder, Susan Pirie, Kath Hart, Ken Clements, and Beth Southwell visited its Mathematics Education Centre. The University went on to be an early user of computers for mathematics (Wilkins, 2000).

Before, and soon after Independence, school teachers used whatever textbook or curriculum could be found from overseas sources. Before Independence, the Administrator had even requested Zoltan Dienes (1999) to bring his multi-base arithmetic (MAB) and attribute blocks and introduce group activities for logical learning in mathematics. Teachers who had training found these useful techniques (Kirkby, 2019) but in most schools they were not known or used (Southwell, 1974). It might have seemed a good place to use MAB as the many languages vary in how they traditionally count. While most coastal and neighbouring languages use base 10 (35%), the majority of languages use other systems: 6, or 4, 8 (4%), some use one and two to form higher numbers (at least 10%), often with a body-tally system which might go to quite high numbers (e.g. 35) (16%), but many use cycles of 5 (47%) or (5, 20) (30%) di-

git tally systems (percentages approximate data in Owens et al., 2018). However, in any one place, there would be one (sometimes two) different systems except that in the islands different counting systems are used for different categories of items (Owens, et al., 2018).

4 *Independence and the Education System*

On 16 September, 1973, the country became self-governing. An all-PNG committee under Tololo, a highly respected educator, prepared an Education Plan which valued students' own cultures, encouraged respect, education for girls, and a basic education for all (Department of Education PNG, 1974). However, the expatriate Administrator could not see how it would be implemented and asked an expatriate Dean of Education at UPNG to prepare a plan that was implemented. Children started school at 7; Kindergarten had already been dismissed. The principles of the Tololo report were lost. This included recognition of language and culture. As some compensation, the primary schools were called Community Schools and teachers were encouraged to teach with community in mind, with community resources, and recognising that education needed to cater for village living. There was debate about whether to educate an elite or all children (Smith, 1987).

From a late start, by Independence, government departments were headed by nationals, universities produced graduates, and colleges produced teachers. Curriculum Officers prepared *Maths Our Way*. From 1975-1989, Papua New Guineans with one or two resident expatriates formed curriculum advisory teams for schools and teachers colleges (Paraide et al., 2023).

For school students, their goal is to achieve well in school mathematics and English for paid employment. The School Certificate and Higher School Certificate examinations kept a high international standard, although less relevant to PNG. The system was driven by examinations: Grades 6, 8, 10, 12. At each level there was a 50% push out rate with capable students unable to continue. Some of these continued with 'distance' education, often with a local tutor through a government or mission organisation (Paraide et al., 2023).

However, with external funding and pressure, overseas publishers produced mathematics books. Only Grade One of Community Schools Mathematics considered the local situation (e.g. PNG pictures without words for counting in the vernacular). In practice, mathematics was imported and linked to fur-

ther education. Later changes in curriculum were also accompanied by overseas produced textbooks.

Teachers guides and textbooks disappeared from the schools or never reached them. With curriculum documents, teachers learnt to write down the objective of the lesson, an introduction like “what do you know about ...”, some rote practice, a worked example, and exercises written on the board (usually just using numbers and algebra). Teaching as one was taught, predominately with rote learning, prevailed through generations. The authors were in school or teacher education throughout this period. This was confirmed by other teacher educators such as Kaleva. As part of our research we interviewed a group of PNG people living in Sydney (professionals, students, and residents) who were at school between the start and the end of this period, all noted that they had been forced to use English at school (punished if they did not). They said mathematics was primarily by rote without meaning. Too many times we heard “we have mathematics in school and mathematics out there; they are different and cannot be related”. For teachers and university graduates, there were years of schooling away from village and home language and, with high migration to the cities and a rise in the lingua franca Tok Pisin, there was language and culture loss (Paraide et al., 2023).

From the 1970s to 2020s, we were living through social change in the country, in the schools, and in teacher education. There were eight changes, three major – 1974 Community Schools with International Education Agency schools replacing T and A schools; 1986 with village elementary schools and Provincial and Secondary High Schools going to Year 12 with a Reform curricula, supported in 2001 by Curriculum Reform Implementation Project (CRIP); and 2014 with reversals of the Reform structure.

5 Ending colonisation: Recognition of the Voice¹ too late

Ten years after Independence, in 1986, the Matane report (National Department of Education PNG, 1986) reflected the Tololo report and the Reform and restructure of the school system began upholding the national 8-

¹ In Australia, the Voice became symbolic of giving First Nations peoples a Voice in their own affairs, and for others to listen. Here it is symbolic of Indigenous recognition in the face of neocolonialism and global pressures in mathematics education.

point plan for the nation (King, Lee, & Warakai, 1985). It aimed for universal education so village elementary schools were set up using the vernacular as language of instruction and for literacy. There were some successes (Paraide, 2003). Culture and girls' education were high on the agenda. However, the structural change was very disruptive and administration, already dispersed to the provinces, was difficult. There were elementary schools for Pre-Elementary, Elementary 1 and 2, then Grades 3-8 in Primary School, 9-10 in high school and most high schools offered Grades 11-12 with the best going to National High Schools or a provincial high school (Guy, 2009). Universities were insufficiently funded for 10 years in the 1990s, large lectures began, the support of the mathematics units at Unitech went, with loss of learning and discussion. The money was redirected to the primary or elementary schools but it did not reach them (Paraide et al., 2023).

There was a lack of teacher education for beginning learning which was supposed to be in vernacular in Elementary schools. Teachers often had large classes, were often unpaid, and had no facilities except a chalkboard, sticks and stones (Paraide et al., 2023). In the 1990s schools were issued with large shell picture books in which they could write the story in their own language for class reading. Summer Institute for Linguistics (SIL) provided many courses for teachers in how to select phonics to begin reading in vernacular, ready to transfer to English but this really gained impetus only after 2010.

The 10-year delay to the Matane report in introducing the values of the Tololo report was costly for culture and language and education. It was 15 years later in 2000 there were two large Australian Aid projects: CRIP which introduced Outcomes Based Education (OBE) and Primary and Secondary Teacher Education Project (PASTEP) (Howes et al., 2014), and later UK Volunteer Services Overseas (VSO) gave scripted lessons and phonics - more neocolonialism (Paraide et al., 2023).

Teacher education never achieved a compulsory ethnomathematics subject in teacher education even though its value was recognised in a couple of their mathematics lectures and the University of Goroka (UoG) had the Glen Lean Ethnomathematics Centre and a popular elective course called Mathematics, Language and Culture. PASTEP gave an elective to Teachers Colleges but it was not implemented due to financial and time pressures. For the critical home and culture beginnings of mathematical conceptualisation (Barwell, Clarkson, et al., 2016), the teacher training at the PNG Institute of Education

(PNGIE) did not incorporate ethnomathematics, mostly due to a turnover in staff and not having been introduced themselves to mathematics in that way. The school curriculum and support materials encouraged some ethnomathematics but teachers were struggling to implement the briefly worded OBE.

Our last design-research project (Kravia & Owens, 2014) on elementary school mathematics resulted in supplying the PNGIE with a Self-Instruction Unit (SIU) for Mathematics (Owens, Edmonds-Wathen, & Bino, 2015b) after trials with teachers (Edmonds-Wathen, Owens, Bino & Muke, 2018). We delivered 15 workshops in six coastal and highlands provinces, mostly in remote village areas with no services. Figure 1 presents a summary of the coverage. It introduced quality ethnomathematics teaching for beginning mathematics through inquiry, play, language, and cultural mathematics.

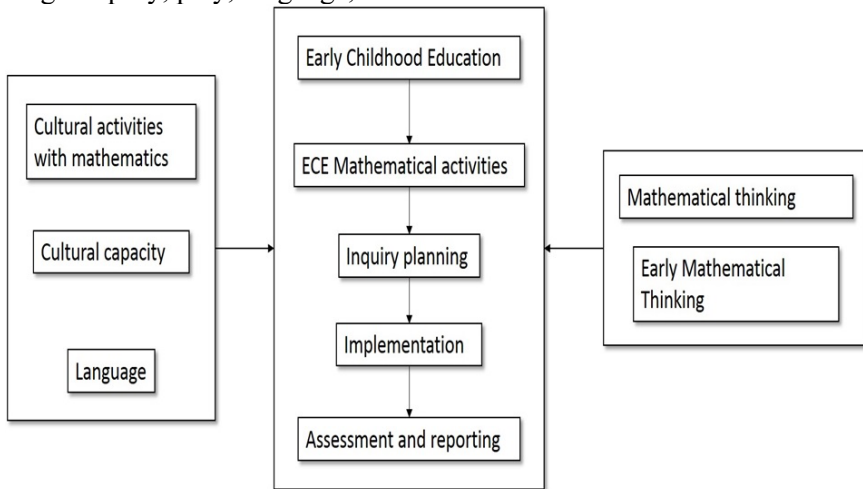


Figure 1. Early Childhood Mathematics.

The computer version was like a webpage and included the SIU with activities for teachers, a list of mathematical terms to be considered in local languages (Edmonds-Wathen, Owens, & Bino, 2019), materials which we used such as games, assessment tasks, early readers on topics like multiplication, area units, and composite unit of 10. It also had videos of cultural practices, teaching and assessment task. The workshop delivery included an introduction to computing so teachers could continue their learning. There were positive evaluations in all sessions and visits to schools indicated that teachers were implementing some of the key ideas such as encouraging more than one answer for an open-ended question, students talking more, and lessons incorpo-

rating cultural activities (Bino, Owens, Tau, Avosa, & Kull, 2014). However, the workshops were too short and needed follow up especially with the Prime Minister's 2014 decree that English was to be the language of instruction.

6 *Response to Reform Failure*

Papua New Guineans saw failure of the reform and blamed it on the use of home languages and OBE but the main reasons were lack of elementary teacher education; no cultural mathematics education; no multilingual or bridging/transition education; lack of funds for work on languages with the need for orthography and discussions with elders around language for mathematical ideas; messy restructure; and neocolonial curriculum designers (Paraide et al., 2023).

The call was to bring back English, and the new global approach was Standards Based Education. Many considered that standards across the nation meant English everywhere from the beginning of schooling even in remote villages where no one spoke English except, maybe, the teachers in the classroom. Some linguists (Temple, Dalsgaard, & Kamene, 2015) have correlated university students' success and age for starting to learn English, claiming that the malleability of the brain at a young age encourages learning of language. However, they barely acknowledge this variable, age for starting to learn English, was connected to socioeconomic status (SES) and lack of rurality. By contrast, a study in Morobe showed that learning sounds of the local languages (Tok Ples) assisted with reading and writing (Labuan, 2013) first in Tok Ples then English supporting Paraide's (2003) study. Furthermore, the literature on learning mathematical and other concepts suggests that students are more likely to learn these early concepts best in their home language and that learning them in another language soon afterwards would strengthen understanding (Barwell, Clarkson, & et al., 2016). Matang and Owens (2006, 2014) showed that students performed better on early mathematical concepts if they were taught in local language in Kôte but in other areas of the country those learning in English did as well but this sample included a school with university-staff children. One notable point is that students learnt less if Tok Pisin was used. This corroborated a study in which students were not learning to read and the worse ones were in schools where teachers were using Tok Ples, English and Tok Pisin. This study indicated the lack of training of teachers,

especially in bilingual education. A crisis was reached, languages were being lost too rapidly and teacher education was none or too little. The solution was to rewrite the curriculum without Tok Ples rather than funding the writing of orthographies, using inquiry learning based on culture or educating in bilingual learning.

Mathematics was supported by Japanese aid including a textbook, simplified and in English, that teachers could buy, avoiding reliance on government distribution. There is a lack of cultural examples, e.g. the symmetry example is tiles, found only in hotels and luxury apartments but many children live in self-help housing. There are many cultural examples that could be used.

There was another accompanying restructure: one year of kindergarten, six years of primary school and six years of secondary school. The majority of the elementary teachers were now out of work and needed primary education training. Another Australian project was set up but the very few PNG advisers keep saying “what about culture?” Enthusiasm, good intentions, and knowledge of the latest trends in mathematics education (not always the best) prevail in yet another neocolonial project. Overall, in education planning there has been no recognition that technology, science, mathematics (STEM) and folk lore are highly integrated in cultural societies, with visuospatial mathematical reasoning (Owens, 2015), understanding of the whole and infinity with traditional counting systems of many different kinds (not just base 10), and other vernacular mathematical language (Paraide et al., 2023).

7 Conclusion

Despite the recognition of the importance of language and culture for PNG identity and success in schools, and education for students in remote areas at Independence (Tololo Education Plan), the education system was built on Australian curricula and this continued for another 10 years before Matane led a further committee for change, reflecting the Tololo report. However, Australian Aid projects dominated the change and the World Bank recommended 3 years in the village schools using Tok Ples but this occurred without adequate teacher education or recognition of the needs of communities for language and literacy development and early bilingual education.

Ethnomathematics research in PNG is extensive: Matang (2006, 2014) and Kaleva (1998) and with teachers at UoG writing reports on their ethnomathe-

matics, Bishop (1988), Lean (1992) on counting systems, Clarkson, Muke, and Paraide on language over decades (see Paraide et al., 2023), and Owens and colleagues (e.g., Kaleva, Matang, Muke, Sondo, Edmonds-Wathen, Paraide, Bino, & Clarkson) between 1997 and 2023 (e.g., Owens, 2015; Owens et al., 2018; Owens, Edmonds-Wathen, & Bino 2015a). From pre-school to high school, it has been shown that ethnomathematics is relevant. Neocolonialism seems to have the stronger voice.

Neocolonialism and a failure of providing adequate teacher education for both bilingual and early childhood mathematics for the elementary schools has created an issue for PNG with cultural loss and disenfranchisement of some, particularly remote, students that may need to be addressed strongly from the grassroots as well as through government and academic policy.

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SEEING THE DEVELOPMENT OF MATHEMATICS EDUCATION IN THE LIGHT OF KHUN'S THEORY OF SCIENTIFIC REVOLUTIONS

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In 1964 Kuhn formulated a theory of scientific revolutions, using the concept of paradigm shifts, where scientific knowledge undergoes radical changes rather than gradual evolution. His ideas about the incommensurability between competing paradigms and the role of normal science have influenced educational research. Even if this theory was originally developed for natural sciences, its frame can be useful for analyzing the development of other phenomena, in particular mathematics education in various countries. During my talk I will formulate the paradigms of mathematics education and present the historical and modern “revolutions” that motivated their shifts. I will discuss suitable examples of the pivoting moments from the past that shaped our Western modern education, for example introduction of compulsory education, introduction of secular education, and public education available to everybody. More modern education reforms that impacted teaching and learning include “New Math”, or “No Child Left Behind Act” replaced later by “Every Student Succeeds Act”. I will explain their motivations, pros and cons in the light of the shifts of the previously defined paradigms.

In the context of Indonesian education, the paradigm shifts went through different routes. Being currently the fourth largest (after China, India, and USA) education system with over 50 million students and 3 million teachers, Indonesian education was nationalized in 1945 and later went through transformations in the sixties and the seventies due to political changes. Regardless of multiple reforms, the schools remain not only under the responsibility of the [*Ministry of Education, Culture, Research, and Technology*](#) but as well under the [*Ministry of Religious Affairs*](#). Facing a variety of geographical, organizational, and ethnic challenges, Indonesian mathematics education emphasizing ethnomathematics and realistic mathematics education was influenced by the ideas of Hans Freudenthal.

A part of the presentation will be devoted to the very recent sudden transformations worldwide that happened during the pandemic. In the context of previously defined paradigms I will discuss their shifts and challenges faced by teachers from Indonesia in the context of journal submissions send by researchers from that country.

THE HPM GROUP: PAST, PRESENT AND FUTURE

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ABSTRACT

This paper presents the HPM group from four perspectives. The first is a personal perspective of the development of the group by one of its initiators. The second considers the changing themes and participants of the HPM over the 50 years of its existence. The third looks at the development of the HPM by way of the HPM Newsletter. Finally, the last perspective is a forward-looking one examining some questions that the HPM will have to confront in the coming years.

Introduction

The International Study Group on Relations between the History and Pedagogy of Mathematics, HPM, was established at the ICME meeting in Karlsruhe in 1976. With nearly fifty years behind it, HPM-11 provided an opportunity to reflect on the past, present, and future of HPM itself, leading to the panel discussion on which this paper is based.

The paper presents the HPM from four different perspectives and is, accordingly, divided into four parts. The first is a brief personal perspective of the development and prehistory of the group by one of its initiators. The second part looks at the HPM from the point of view of its themes and participants, how these have evolved from the organization's early days to the present. The third considers the HPM from the perspective of HPM Newsletter, which began not long after the study group itself was established. Like the second part, one sees in this third part the changing faces of the participants and key figures of the HPM through the list of editors and subscribers of the newsletter as well as changing aspects of interest in the organization.

While the first three parts are mostly retrospective, the fourth part looks towards the future and suggests four domains in which work of the HPM should continue or expand. For the panel, this part was meant to spark discussion among the participants; for the reader, it is meant to stimulate reflection on questions worth pursuing.

1 Beginnings of HPM Group (Leo Rogers)

The British Society for the History of Mathematics (BSHM) was founded in 1971, and has built an international reputation for promoting historical studies of mathematics and the use of the history of mathematics at all levels in mathematics education in order to enhance the teaching of mathematics for the public benefit (Wilson & Flood, 2021). In the first meeting of BSHM on 2 July 1971, there were a talk of Arthur Morley (Nottingham College of Education) on "The history of mathematics in colleges of education" and a talk of Leo Rogers (Digby Stuart College) on the "Relations between the History and Pedagogy of Mathematics".

The International Congress on Mathematical Education – 2 (ICME) was organised in Exeter, England in 1972. My Research supervisor was David Wheeler, who was one of the local organising Committee, and so I proposed a study group on "Relations Between the History and Pedagogy of Mathematics". Since Philip Jones was author of the first article in the 31st Yearbook NCTM entitled *The History of Mathematics as a Teaching Tool*, I asked Philip Jones if he would join me to be co-chair of this group at ICME 2. With the support of these colleagues my proposal was accepted.

At ICME 3 in 1976 at Karlsruhe, the ICMI Executive Committee approved the affiliation of the new Study Group, under the title "International Study Group on Relations between the History and Pedagogy of Mathematics" (IS-GHPM) cooperating with the International Commission on Mathematical Instruction. The list of eight "principal aims" of the HPM Study Group had been given in 1976 (Rogers, 1978, p. 76):

1. To promote international contacts and exchange information concerning:
 - a) courses in history of mathematics in universities, colleges and schools;
 - b) the use and relevance of history of mathematics in mathematics

teaching; c) views on the relation between history of mathematics and mathematical education at all levels.

2. To promote and stimulate interdisciplinary investigation by bringing together all those interested, particularly mathematicians, historians of mathematics, teachers, social scientists and other users of mathematics.
3. To further a deeper understanding of the way mathematics evolves, and the forces which contribute to this evolution.
4. To relate the teaching of mathematics and the history of mathematics teaching to the development of mathematics in ways which assist the improvement of instruction and the development of curricula.
5. To produce materials which can be used by teachers of mathematics to provide perspectives and to further the critical discussion of the teaching of mathematics.
6. To facilitate access to materials in the history of mathematics and related areas.
7. To promote awareness of the relevance of the history of mathematics for mathematics teaching in mathematicians and teachers.
8. To promote awareness of the history of mathematics as a significant part of the development.

2. *The past of HPM Group (Évelyne Barbin)*

In this overview, we indicated the forms of the meetings of the HPM Group and we presented the types of participants in HPM meetings and their evolution. Next, we would like to examine the themes of the HPM Group. We begin by summarizing the aims of HPM Group as they were in 1976 and then we look at the Announcements and the Proceedings of the meetings. We will observe that themes changed, some were introduced, some disappeared or reappeared; at the same time, and perhaps as a result of changing emphases, types of participants also changed.

2.1 HPM sessions and HPM meetings

From 1980 until 1992, the Congress ICME proposed HPM sessions organized by the Chair of HPM. In this period, George Booker also organized the first HPM satellite meeting in Adelaide in 1984. In 1988, Florence Fasanelli

organized the second HPM meeting in Florence, following which ICME-6, held in Budapest, included HPM sessions and a panel on “History and teaching”, organized by Ubiratan d’Ambrosio, Chair of HPM.

There were no HPM sessions in ICME-10 in Copenhagen (1994) but ICME organized that year two Topic Study Groups with historical themes: “The role of the history of mathematics in mathematics education” and “The history of the teaching and the learning of mathematics”. They were organized by two teams of chairs and members, invited by ICME. The first indicated TSG stayed in ICME Congress until 2024, but not the second one.

2.2 Participants in HPM meetings (1980-2020)

In the period 1980-1992, participants of HPM meetings and HPM sessions were mainly historians of mathematics, mathematicians and teacher educators. The next period 1996-2004 saw several changes, from the point of view of the number and type of participants in the HPM meetings. HPM-4 (Braga, 1996) was organized in conjunction with the Second “European Summer University on epistemology and history of mathematics” (ESU). There were more than 550 participants, with many secondary school teachers. HPM-5 (Taipei, 2000) was organized with many secondary teachers also. There were also secondary teachers in HPM-6 (Uppsala, 2004), organized in conjunction with the Fourth ESU. Thus, from 1996, secondary school teachers participated in the HPM meetings, offering workshops as well as oral presentations and plenary lectures.

There were several researchers on mathematics education in HPM-7 (Mexico, 2008), and both HPM-8 (Daejon, 2012) and HPM-9 (Montpellier, 2016) included teachers and researchers on mathematics education. HPM-10 (Macao, 2020) did not take place as planned because of Covid pandemic (an abridged conference was held online). Thus, from 2008, HPM meetings have been open to historians of mathematics, mathematicians, teacher educators, secondary teachers and researchers in mathematics education. The proportions of the different profiles vary from one HPM meeting to another, in particular regarding secondary school teachers. In addition, some participants fall under several profiles.

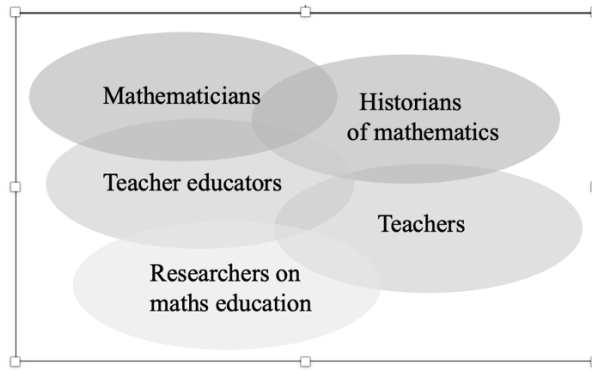


Figure 1. Three successive waves of participants at the HPM meetings

2.3 Themes of HPM meetings (1976-2024)

The list of eight “principal aims” of the HPM Study Group in 1976 can be summarized in eight themes (Fig. 2).

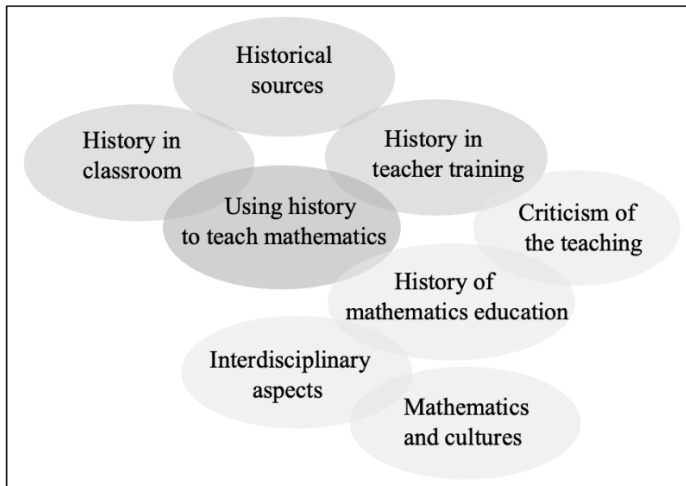


Figure 2. Eight themes for HPM Group (1976)

Four themes, however, predominated during the period 1980-1988: “History in classroom”; “Using history in teaching”; “Epistemological aspects”; “History of mathematics education”. The volume edited after HPM-3 (1992) presents two new themes: “Historical sources” and “Original sources in the classroom” (Calinger, 1996). Papers coming from HPM-4 and ICME-8 (1996) contained new themes also, for example, “General ideas on the use of history in teaching” and “The use of history in teachers’ training” (Katz, 2000). Two new

themes concerned with interdisciplinary and epistemological aspects of history of mathematics in mathematics teaching appeared in HPM-6 (2004). Figure 3 represents all the themes introduced in the period 1980-2004 during which period teacher educators and secondary teachers also began participating in the meetings.

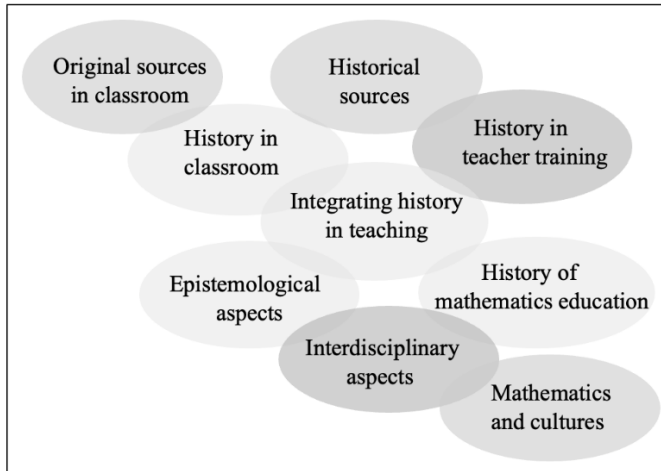


Figure 3. Four waves of themes (1980-2004)

HPM-7 (2008-Mexico) did not include additional themes to those included in HPM-6. But two themes disappeared, “The use of history in teachers training” and “Historical topics”; in this meeting where several participants are researchers on mathematics education. Seven themes were proposed in HPM-8 (2012) which have remained in all the last meetings, HPM-9 (2016), HPM-10 (2020) and HPM-11 (2024). In these last meetings two new themes have appeared, “Theoretical frameworks” and “Teaching materials”, while two other themes have been removed, “In teachers' training”; “Historical topics and sources” (Fig. 4).

These changes corresponded to the arrival of researchers in mathematics education who focused on didactical materials for teachers. Thus, “Integration of history in classroom” was replaced by “Implementation of teaching materials”. Owing to the production of adequate materials by these researchers, the training of teachers in the history of mathematics no longer seemed urgent, nor did studies on historical topics and sources. The theme “criticism of teaching” was present in 1976 but it did not appear as such then. It may have been addressed in some papers or in panels, but it does not seem to have an immediate link with the concern for “teaching and learning in the classroom”.

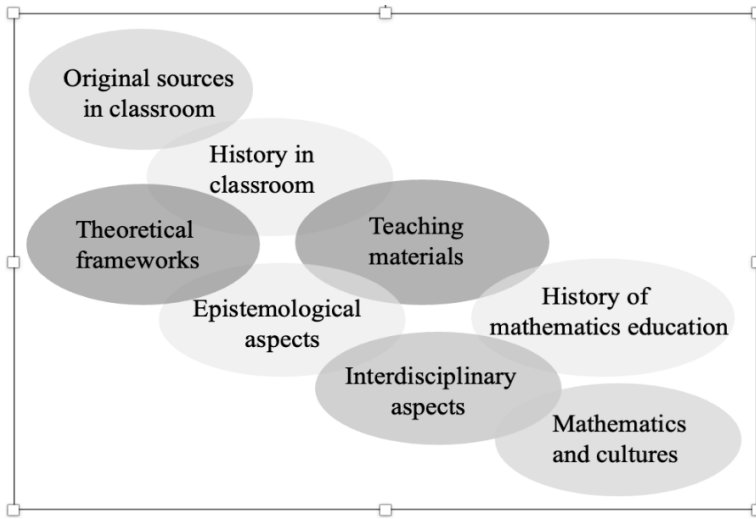


Figure 4. Themes in last HPM meetings (2012-2024)

3. *The history of HPM Group through the Newsletter (Hélder Pinto)*

Many of the changing themes and participants in the HPM meetings, as discussed in the previous section, find a different kind of expression in the HPM Newsletter. The HPM Newsletter is the tool used by the HPM Group and by its Chairs to communicate with all members of the Group and other interested people in this field. It is distributed by e-mail three times a year (March, July and November) and is also published on the Group's institutional website.¹ The newsletter is coordinated by an editorial team and publishes texts by the editors, by the HPM Chair, members of the HPM Group, and others who wish to contribute news, comments, interviews, book reviews, dissemination of congresses, working progress, etc. It should be noted that the Newsletter is not a scientific journal (it does not have blind peer reviews); however, it sometimes publishes short texts on the history of mathematics, for example, *Volume of a sphere in Ancient China and India* (R. C. Gupta, HPM NL, 75, Nov. 2010), *Arthur Cayley and Mathematics Education* (T. Crilly, 68, Jul. 2008) and *Power of Innovation: A Historical View* (W.-S. Horng, 54, Nov. 2003).

¹The website (<https://hpm.sites.uu.nl/newsletter/>) is hosted by Utrecht University. Readers can find there an archive containing almost every issue of the HPM Newsletter.

The Newsletter was created in the 80's (see, for instance, Lawrence, 2005) with the aim of serving as a mean of communication for the Group, and at that time the Newsletter was sent to all its readers by post. Initially, publication had a single editor, but in recent years the option was for a larger team. This can be seen in Table 1, which presents a list of all editors throughout the newsletter's existence (the order of the editors is the same as in the newsletters).

N.º	Dates	Editor(s)
1-4	Missing	Leo Rogers (UK)
5-6	Oct.83-Feb.84	Charles V. Jones (Can) and others (North American Edition)
7-16	Oct.84-May.88 (11 is missing)	Charles V. Jones (Canada)
17-38	Winter89-Jul.96	Victor Katz (USA)
39-44	Nov.96-Aug.98	Gerard J. Buskes (USA)
45 ² -56	Nov.00- Mar.04	Peter Ransom (UK) (nº 46 first to appear online)
57-68	Nov.04-Jul.08	N. Kastanis (Greece), C. Weeks (UK) and B. Smestad (Norway)
69-75	Nov.08-Nov.10	C. Weeks (UK) and B. Smestad (Norway)
76-78	Mar.11-Nov.11	B. Smestad (Norway), S. Lawrence (UK), H. Pinto (Portugal) and K. Clark (USA). In 2011 a <i>Word-Press</i> version appeared in https://groupnpm.wordpress.com/
79-80	Mar.12-Jul.12	B. Smestad (Norway), S. Lawrence (UK), H. Pinto (Portugal), K. Clark (USA) and L. Puig (Spain)
81-103	Nov.12-Mar.20	S. Lawrence (UK), H. Pinto (Portugal), K. Clark (USA) and L. Puig (Spain)
104-113	Jul.20-Jul.23	S. Lawrence (UK), H. Pinto (Portugal), L. Puig (Spain) and G. FitzSimons (Australia)

² Number 45 was wrongly published as number 44. Please, see Editorial from the Newsletter number 46.

114- Present	Nov.23- Present	S. Lawrence (UK), Janet Barnett (USA), L. Puig (Spain) and Antonio Oller-Marcén (Spain)
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Table 1. List of Editors of the HPM Newsletters

At the early years, there were two newsletters, one in the HPM Group and the other in the Americas section, publications; these later merged into the single newsletter that appears today (Fasanelli & Fauvel, p. xiii):

By 1984 the two newsletters had in effect amalgamated and henceforth (from what was called issue no. 7) there was one HPM Newsletter, edited until 1988 by Charles Jones, with occasional special supplements for the Americas Section. It was at the 1983 Michigan NCTM meeting, mentioned below, that Charles Jones agreed to be the editor of the Newsletter. The intention was that the Newsletter would have a calendar of upcoming events, a guest editorial, a 'Have You Read?' column and short reviews and announcement of meetings and activities. (...) By 1988 there were 2500 on the mailing list with readers on every continent (except Antarctica) and in 62 countries.

At present, the Newsletter contains the following sections in almost every issue: «Note from the Chair», «Conference Reports», «Announcements of Events» (from the HPM Group but also from others like the European Society for Research in Mathematics Education and the British Society for the History of Mathematics and even local events such those organized by IREM and others in Europe, North Africa, USA and Asia) and «Have you read these» (in this section are indicated many papers from this field, some special numbers and it's published many Book Reviews; in the last years a strong contribution is done by MAA Convergence, spreading several significant images from the History of Mathematics. Less prevalent nowadays are the sections «Have you been here?», «Work in progress» and «Interview»; notice that in the past this last section was very common and it should be highlighted the Newsletter number 100 (March 2019) that have interviews with the majority of the previous HPM chairs: Roland Stowasser (Germany), Ubiratan D'Ambrosio (Brazil), Christian Houzel (France), Florence Fasanelli (USA), Fulvia Furinghetti (Italy), Constantinos Tzanakis (Greece), Evelynne Barbin (France) and Luis Radford (Canada).

Inevitably in an organization existing for more than 50 years, many obituaries have been published over the years honouring many of the colleagues who have contributed to the growth of the Group; the first one was

the obituary of John Fauvel (UK), in July 2001 who was chair of the Group from 1992 to 1996.

It should also be noted that two numbers of the year 2012 (79 and 80) were translated to Spanish with the help of the Asociación Peruana de Investigación en Educación Matemática (the contact was due the colleague Maria del Carmen Bonilla, who is the distributor of the Newsletter in Peru); this kind of initiative should perhaps be put on the table again so that Newsletter could have more interested readers, since many pre-university teachers and students have difficulty reading in a language other than their native one.

Today, the Newsletter continues to be a valuable organ of the HPM group through the voluntary contributions of its editors and all those who contribute texts, reviews and other news. The goodwill and efforts of the pioneers in creating an effective and comprehensive vehicle of information and for sharing ideas is still strong today but cannot be taken for granted. Notice, as example, the words of the chair of HPM in 2000, when, due to various circumstances, there was a short hiatus in the publication of Newsletter (Van Maanen, 2000):

After 22 issues (Nos. 17-38) of the HPM Newsletter edited by Victor Katz, Gerard Buskes took over in the summer of 1996, and edited six issues, the last one being No. 44 (August 1998). Various problems urged Gerard to step down as the editor in 1999; since that time the editorship is vacant. This is a serious drawback for HPM, since the Newsletter has always been the main uniting force in this informal group, which can work without money since - except for the U.S. - the regional distributors have support from their institutions in order to print and distribute the issues. Active steps are under way in order to restore the Newsletter as a mirror and focus of HPM's activities; a number of possibilities are currently being investigated.

For the future, the Newsletter should encourage more colleagues from other parts of the world who are not currently represented in the HPM Group, such as Mexico, Eastern Europe, Turkey, Middle East, Iran, India, China, North and South Africa, etc. Another suggestion for the Newsletter is to create a «Funding Opportunities» section where suitable open calls/grants/prizes for HPM field would be publicised and help to boost the creation of networks of researchers who can apply for these same calls. The Newsletter should also try to increase the participation of the high school teachers in order to favour the establishment of more teacher-researcher

collaborations, for instance, in the context of teacher in-service training and curricular reforms.

4 Out of the past and into the future of the HPM: Questions for reflection (Michael N. Fried)

This final section will be more pointedly forward looking than the previous sections. We propose four domains to focus our reflections on who we are and where we ought to go as a group wishing to advance the history of mathematics in mathematics education. We do not assert in any way that these are the only domains possible, but they are ones that arise naturally from the history of the HPM. Indeed, some of the questions within these domains have accompanied the HPM from the beginning, though often their meaning, urgency, or even whose character as questions has changed or developed.

4.1 Domain 1: History of Mathematics & Pedagogy of Mathematics

The full name of HPM is, with my emphasis, International Study Group *on Relations between History and Pedagogy of Mathematics*. What is the *nature* of those relations? Are they such that history of mathematics is another approach for advancing the learning of school mathematics? Or are they such that a focus on history of mathematics redefines mathematics education? It is a question, in a way, of part and whole. In other words, is the history of mathematics a part of mathematics education, only a certain aspect of mathematics education? Or is it, in effect, the whole of mathematics education from which ideas to be studied can be derived?

4.2 Domain 2: Teacher-Training & Professional Development

So the question we might ask is whether history in teacher training should be a matter of teaching teachers history *or* teaching teachers to teach history? This, of course, is related to the previous question in perhaps a more radical spirit, for it asks whether history of mathematics should be for the classroom at all but only for the deepening of teachers' knowledge (just as advanced courses in pure mathematics help teaching only indirectly by making teachers' own mathematical knowledge more profound).

4.3 Domain 3: History of Mathematics & Ethnomathematics

The relationship between HPM and ethnomathematics raises questions as to the purview of historical thinking. Is it an overextension of the reach of history of mathematics? Or is it, rather, a *natural* extension of thinking of history of mathematics as the mathematics of cultures of the past? One might well argue that when historical research loses sight of the grounding of history in cultural settings it begins to see mathematical as a historical, that is, unconditioned and independent of time, people, and place. On the other hand, the history of mathematics is clearly not identical with ethnomathematics. So where does one draw the line? And how firmly should a line be drawn, if at all, in the HPM?

4.4 Domain 4: History of Mathematics & Mathematics Education Research

A proposal for an international journal under the auspices of the HPM was prepared by request for the HPM Advisory Board first by Evelyne Barbin & David Pengelley in April 2017 and then revised by Michael N. Fried and Tinne Hoff Kjeldsen in January 2018. The proposal for the international journal recognized the need to define the nature of the papers. This raises general questions about research and HPM. In particular:

- What *is* the nature of research into the relations between history of mathematics and mathematics education?
- Is it really the case that *empirical* research is what is needed?
- What kinds of assumptions do we make when we claim that the questions that concern us in HPM are empirical in character? (Do we, for example, assume a kind of naturalness of studying via history, something like the ontogeny-phylogeny argument?)
- Should HPM research be something different from mainstream mathematics education research?

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