THE EFFECTS OF STUDYING THE HISTORY OF THE CONCEPT OF FUNCTION ON STUDENT UNDERSTANDING OF THE CONCEPT

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ABSTRACT

This study examines the mathematical learning that occurred when students studied the history of the concept of function. Students experienced an in-depth study of the history of functions during a 5-week unit in the junior-senior level History of Mathematics course. They completed a series of worksheets, readings, and problems.

The research methodology was a teaching experiment and the framework for analysis of data was *APOS* (Action, Process, Object, Schema) Theory. All 17 students enrolled in the course completed an extensive initial questionnaire and 6 were selected to participate in an in-depth interview to reveal their understanding of the function concept. During the unit, each student wrote a series of reflections about his or her understanding. After the unit, students completed a second questionnaire and participated in another in-depth interview to discern the changes in their thinking about the concept.

The findings support the notion that studying the history of a mathematical concept enables a deep reflection of ideas. Four of the six participants notably strengthened their function conceptions. Two moved an entire APOS level. Five of the six exhibited an increased ability to recognize a function in a given scenario. Growth was most profound in the area of graphical representations.

1 Introduction

Professional mathematical societies are deeply concerned about the mathematical education of our teachers and are continuing to search for effective means to deepen students' understanding of fundamental mathematical concepts (Conference Board of Mathematical Sciences, 2001). In the USA, national reports call for better preparation of our mathematics teachers (RAND Mathematics Study Panel, 2003; U.S. Department of Education,2000).

The concept of function takes center stage when it comes to mathematics education. Guershon Harel and Ed Dubinsky argued (1992) that

> The concept of function is the single most important concept from kindergarten to graduate school and is critical throughout the full range of education. Arithmetic in early grades, algebra in middle and high school, and transformational geometry in high school are all coming to be based on the idea of function. (p. vii)

Though some researchers have obtained positive results for student construction of a process conception of function (Breidenbach et al., 1992), others have not (Sfard, 1992) and still others note the continued difficulty students have with the concept (Breidenbach et al., 1992; Carlson, 1998; Even, 1993; Norman, 1992; Sierpinska, 1992; Wilson, 1994).

The purpose of this study was to discern if students learn mathematics by studying its history. In particular, it investigated the changes in pre-service secondary school teachers' thinking about functions resulting from their studying the history of the concept. This study addressed the following questions.

- Does studying the history of the concept of function deepen a student's *understanding* of the concept in any way and if so, in what way? In particular, does studying the history facilitate his or her move from an *action* level understanding to a *process* level understanding as described by APOS theory (***A****ction*, ***P****rocess*, ***O****bject*, and ***S****chema*)?
- In what ways can studying the history of a mathematical concept be used to deepen a student's understanding of the concept?
- Does a student's studying the history of the concept of function facilitate his or her move from a process level understanding to an *object* level understanding?

2 Theoretical Basis for the Study: APOS Theory

The purpose of this section is to explain APOS Theory, a constructivist approach to the learning and understanding of mathematics at the post-secondary level. APOS theory and constructivism are theories that are simultaneously about knowing and coming to know, that is, knowledge and learning.

The developers of APOS theory wanted to use the idea of theoretical cognitive structures from Piaget and relate them to observable behaviors in college-level students (Asiala, Brown, et al., 1997). They created a model for conducting research in mathematics education. APOS is the cognitive aspect of the model. It guides the theoretical analysis of a student's understanding of a mathematical concept and is an attempt to model the epistemology of the concept.

The acronym APOS stands for ***A****ction*, ***P****rocess*, ***O****bject*, and ***S****chema*—mental constructions made by students in their attempts to understand mathematics. *Actions* lead to *processes*, which must come before seeing a concept as an *object.* The origins of this interpretation of mathematical understanding lie in Piaget's work and parallel many concepts in von Glasersfeld's radical constructivism. The developers of APOS see their work as "the result of reconstruction of our understanding of Piaget's theory leading to extension in its applicability to post-secondary mathematics" (Asiala, Brown, et al., 1997, p. 41). To them, the following best describes what it means to learn and know something in mathematics:

> An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situation. (p. 40)

They assumed that what a person knows and is capable of doing is not necessarily available to him or her in any given moment in any situation. As did Piaget, therefore, they see *reflective abstraction* as crucial to a student's construction of mathematical process and objects. Reflection involves paying conscious attention to operations performed. Such reflection is significantly enhanced in a *social context* (Asiala, Brown, et al., 1997). APOS theorists refer to the literature supporting the importance of a student's social interaction while learning and also to a research mathematician's need for interaction with colleagues before, during, and after doing creative work in mathematics.

2.1 The Action Construct

The *Action* construction is similar to Piaget's *action schemes.* A student who has an *action* understanding of functions sees an algebraic expression as a command to calculate. Such a student can carry out a transformation only by reacting to *external cues* (textbook directions, teacher suggestion, etc.) that give exact details on what to do. This conception is like a recipe and they must apply it to some number before it will produce anything. They do not necessarily see the recipe as an object in itself, that is, a result of its own application (Thompson, 1994). Though it is considered the lowest level of abstraction, it is a necessary beginning to the understanding of functions. The reason so many students have trouble understanding piece-wise functions, composition, and inverses of functions and sets of functions, for example, is that the learner is not able to go beyond an *action* understanding of functions (Asiala, Brown, et al., 1997).

Interestingly, Breidenbach, Dubinsky, Hawks, and Nicols (1992) found students who did not display any aspect of a function concept, not even an action conception. The meaning these students gave to the term function was not useful to them as they attempted to deal with activities related to mathematical functions. The researchers categorized such students as having a *pre-function* conception.

2.2 The Process Construct

A *process* is an internal construction that performs the same transformation as the action, but it is internal and hence under the control of the individual. She no longer needs the external stimuli, no longer needs to actually evaluate an expression to think of its result. She can reflect on, describe, or reverse the steps of a transformation without actually performing those steps. A good example is an understanding of the function *cos x.* Since no *explicit recipe* exists for evaluating this function at a given value, one needs to imagine the process of associating a real number with its *cosine.* With this understanding, a student can link one or more processes to construct a composition, or reverse the process to obtain inverses of functions.

When a student moves from an *action* understanding to a *process* one, APOS theorists say the student has *interiorized* an *action* to form a *process.* This *process* relates to von Glasersfeld's *internalization* of a topic, which, according to Battista (1999b, p. 3), is "the process that results in the ability to re-present a sensory item without relevant sensory signals being available." It is at this level, according to von Glasersfeld, that a concept has been formed. To be considered a concept, these constructs must be stable enough to be re-presented without requiring perceptual input (von Glasersfeld, 1995).
Achieving a process conception of function is non-trivial for students (Thompson, 1994) and research has shown that many students do not achieve this level without specific instruction geared specifically to that end (Thompson).

Once students have practice working with processes, groundwork is laid for them to begin thinking about *sets* of inputs in relation to *sets* of corresponding outputs. APOS theorists say students are then ready to begin to reason more formally about functions—they *encapsulate* the process to form an *object.* A *process* understanding of function corresponds to Piaget's "*operations*" whereas the next level—*object* construct—corresponds to Piaget's *objects*.

2.3 The Object Construct

An *object* understanding of a concept sees it as “something to which actions and processes may be applied” (Selden & Selden, 1992, p. 19). One indication that a student is functioning at the *object* level in understanding functions is her ability to reason about operations on sets of functions (Thompson, 1994). It is often necessary, however, to de-encapsulate objects back into processes. For example, one can think about adding or multiplying functions or forming sets of functions, but to actually find these sums or products of sets requires the student to de-encapsulate them back to the processes from which they come. Asiala, Brown, et al. (1997) claimed that reaching this level of abstraction is incredibly difficult for students and has found few pedagogical strategies to be effective in achieving this end. They claimed the reason for this difficulty is the fact that there is very little in our experience that corresponds to performing actions on processes.

2.4 Conclusion

APOS theory is a theory of knowing mathematics that has its roots in constructivism. It was developed as an extension of Piaget’s work so researchers could understand the nature of learning in college mathematics students. It firmly holds to the basic tenets of constructivism: (a) knowledge is not passively received, but built up by the cognizing subject; and (b) the function of cognition is adaptive and serves to organize the experiential world, NOT the discovery of an ontological reality (Von Glasersfeld, 1995). Knowledge is obtained by reflecting on problems and by constructing and reconstructing actions, processes and objects in a social setting. It has specific implications for instructional strategies that are based on inquiry learning.

3 Procedure for the Study

Students enrolled in the History of Mathematics course participated in a five-week unit on the history of the concept of function. These learning materials were nontraditional in that they did not follow a chronological approach of multiple topics typically used in history of mathematics texts. They focused on the development of a single concept and thus did not fit into the curriculum as the course is usually taught. Three of the worksheets were created by the researcher, one was adapted from Usiskin (2003) and five were from Katz & Michalowicz (2005). Since only one section of the course is taught per semester, the researcher was the instructor of the course. The culture of the classroom was inquiry-based, with students working collaboratively.

All students in the class completed questionnaires before and after the instructional program to ascertain their conception of function. Most of the questions on them are adapted from the work of Dubinsky and Harel (1992). Based on student responses to the initial questionnaire, the researcher chose seven students to interview. The initial interviews consisted of questioning students about responses to the initial assessment. In particular, the researcher asked them to explain their thinking as they responded to each question. Each of these interviews was audio taped. Each interview lasted between 60 and 90 minutes. All tapes were transcribed.

All students in the course worked in groups both in class and outside of class to complete the readings and worksheets in the Appendices. After each reading or worksheet, each student wrote a one page summary reflection indicating the following:

- her understanding of the concept of function;
- if and how the worksheet led to new insights concerning the concept of function.

The researcher conducted another individual interview with each of the 6 participants at the conclusion of the unit. These interviews were also between 60 and 90 minutes in length and focused on students' responses on both questionnaires. In particular, the researcher asked the participants again to explain their thinking in detail as they worked through the second questionnaire and asked if they would change any of their answers on the initial questionnaire. These interviews, the completed questionnaires, the student worksheets, and the individual reflections comprised the data for this study.

4 Data Analysis

The theoretical framework for this study was APOS theory as previously described. Analysis closely followed that in the articles by Breidenbach et al. (1992) and by Dubinsky and Harel (1992). Since the questions for the current study are adapted from the Dubinsky and Harel study, their detailed analysis of the situations presented follows.

4.1 Finite Sequences

For a student to identify a function from a sequence, she must have thought in terms of a first term, second term, and so forth, something not given to her in the situation. The authors claimed that if a student can accept a positive integer as the ordinal position of one of the quantities in the sequence, and take that quantity as the output, then that student is using a process conception. However, in these particular questions, the format of the problem strongly suggests that construction (Dubinsky & Harel, 1992). Therefore, a student's success in dealing with these questions might only indicate an action conception. One question involves Boolean values for output, however and it "does provide us with a context for suggesting the possibility that those who were successful were capable of more than an action conception of function" (p. 92).

4.2 Character Strings

The mental constructions required for strings are mathematically equivalent to those required for sequences. They pose more psychologically difficult for students, however, since the outputs are characters rather than numbers, and no suggestion of the construction is evident (Dubinsky & Harel, 1992). All one sees is the result.

4.3 Graphs

The authors believed that graphs have potential to provide indication of one's ability to use a process conception of function, particularly if the graph required use of values on the vertical axis as the domain. One question showed a set of discrete points on the Cartesian plane. It is a good indicator of a process conception since a student with a process conception of function would see the function process even if the domain was very small.

The researchers noted student difficulty with this type of question, reminiscent of Carlson’s (1998) finding that students often believe that a function had to be continuous.

4.4 Sets of Ordered Pairs

Dubinsky and Harel (1992) noted significant student difficulty with this representation of function. Students often confused the process of constructing the set of ordered pairs with the function itself. However, the function can only be constructed if the ordered pairs are already there by identifying the domain as the first element and taking the second element as the result of the function process. The ordered pair representation does not suggest this construction, so it must come from the student himself.

The researchers noted the common student confusion of the uniqueness condition with the notion of one-to-one, results also consistent with Vinner’s (1989) study. Because of this confusion and the necessity for the construction coming from within the student, they claimed that “the set of ordered pairs is a bellweather type of situation for detecting the presence and strength of a process conception of function” (Dubinsky & Harel, 1992, p. 93).

4.5 Tables

Dubinsky and Harel (1992) claimed that, for the purpose of their analysis, tables are similar to ordered pairs. A student who insisted upon a rule relating the first number to the second and/or cannot construct a process as the act of going from an item in the first column to one in the second, is “probably displaying an action conception of function” (p. 93).

4.6 Equations

If a student insisted upon solving an equation in two variables for one in terms of the other, she probably displayed an action conception of function. If she can describe the process without actually doing it, she “probably” exhibited “at least the beginning of” a process conception (Dubinsky & Harel, 1992, p. 93).

With equations involving one or more variables, one can view as input any numerical value(s) and as output a Boolean value, *True* or *False*. Since such a construction must come from the student himself, it is evidence of a process conception of function.

4.7 Statements

These are the most open-ended in Dubinsky and Harel’s (1992) list of situations. They included these simply to observe what type of functions the participants would construct, given a situation with very little structure.

One consistent theme for evidence of a process conception of function is that the subject did the construction himself, that is, the representation does *not* suggest the construction.

5 Findings

This section analyzes the six participants (DB, CW, MJ, BG, MS, and CS) as a group to see what patterns emerged regarding the growth in their understanding of functions as they studied the history of the concept. This section has four subsections. The first part is a glimpse at the participants' growth concerning their definition of function, the second is a look at their increased abilities to recognize functions in situations, and the third summarizes changes in APOS levels. The section ends with comments concerning changes in understanding graphs of functions.

5.1 Definitions of Functions

Recall that Dubinsky and Harel (1992) claimed that an emphasis on equations, numbers, and evaluating expressions is evidence of an action conception of function. On the initial questionnaire, several participants exhibited this tendency. The notion that a function was an equation was strongest in CW and CS. DB and MJ exhibited the tendency to a lesser degree and DB thought that a graph or a formula is the function, rather than just a representation of a function. Interestingly, DB and MJ at times discussed a function as a general process, but reverted to a "function as equation" as they attempted to recognize functions in situations. This tendency suggests that DB and MJ had an emerging, but not strong, process conception initially. MS at times referred to the need for "some operation" but he, like BG, was able to recognize that a function may have more than one representation. BG did not exhibit the "function as equation" understanding.

On the second questionnaire and interview, DB made no mention of function as equation. In her words, "now I feel that a function simply stated is something that takes an input and produces an output." She came to understand the uniqueness criterion, something she overlooked on the initial instruments. Similarly, CW's new definition of function is "a relationship between variables where one input produces one output and the relationship can be shown in a number of ways, not just an analytic expression." MJ and CS also let go of their tendency to look for an equation, with MJ noting, "the mapping can be arbitrary" and CS claiming that a "function is a relationship that provides us with a unique output for a given input." Note that there is no mention of equation in these definitions.

These results suggest that studying the history of functions broadened the participants' definition of function. After working on the worksheet entitled "Definitions of Functions" and the readings associated with it, the class had an extensive discussion about the change in the definition of function over the years and what prompted such change. It is reasonable to conclude that this worksheet and discussion facilitated a move away from the "function as equation" notion in the participants. Table 1 summarizes these results.

Table 1
Participants Holding the Notion of "Function as Formula" Before and After History of Functions Unit

	Before	After
DB	*	
CW	*	
MJ	*	
BG		
MS		
CS	*	

5.2 Functions in Situations

The current study provided other evidence that studying the history of function facilitate a move away from a narrow view of function. Not only did the participants' definitions change, but their ability to find functions in situations improved as well. The following discussion focuses on four tasks on the initial questionnaire and the corresponding tasks on the second questionnaire that may be considered bellwether indicators of a process conception (Dubinsky & Harel, 1992).

Table 2 indicates which participants recognized a function in the given scenario. An asterisk (*) means that participant clearly articulated an appropriate function. The phrase "considered the possibility" indicates that the participant had a vague notion of function in the scenario, but was either unable to articulate it or gave an inappropriate formulation.

Table 2
Appropriate Answers to Specific Tasks Before History of Functions Unit

	String as function	Arbitrary pairing as function (i.e., table or list)	Recognizes a Boolean (true-false output) function	Discrete set of points on graph as function
DB			considered the possibility	considered the possibility
CW				
MJ	*			
BG	*	*	*	*
MS	considered the possibility	*		considered the possibility
CS				

Only two participants considered an arbitrary mapping as a function. The "function as equation" notion was evident in DB ("you apply the function of subtraction"), CW ("if you'd have the previous one plus five, you would have a function"), and MJ ("starting to graph it a little bit and see if there was any kind of relationship"). CS left the task completely blank. On the discrete set of points graph, MS did successfully find a function, but insisted on connecting the dots and trying to find a formula so that he could "find a pattern of how to change it" [the inputs]. Note also that other than BG and MS, the participants had little success with these tasks as a whole.

Table 3 indicates the participants who successfully recognized a function in these scenarios on the second questionnaire or while revisiting the initial questions during the second interview.

Table 3

Correct Answers to Specific Tasks after the Unit on History of Functions

	String as function	Arbitrary pairing as function (i.e., list)	Recognizes a Boolean (true-false output) function	Discrete set of points on graph as function
DB	*	*	*	not discussed
CW	*	*	*	*
MJ	not discussed	*	*	*
BG	*	*	*	*
MS	considered the possibility	*		*
CS	considered the possibility	*		considered the possibility

Other than BG, each participant exhibited an increased ability to recognize a function in the given scenarios. The data shows remarkable progress in the abilities of DB, CW, MJ, and CS. It is worth noting that BG, who recognized functions in each of the initial scenarios, exhibited evidence of an object conception of function in the second interview. Also noteworthy is the fact that MS, who showed the least growth, had put forth the least effort during the functions project, insisting on working alone, and turning in work of mediocre quality. The others, DB and MJ in particular, worked incredibly hard on the project, seeking help when necessary, struggling with the ideas presented. Not surprisingly, they exhibited impressive growth.

Of interest to the researcher is the fact that the class never discussed Boolean functions, nor was topic covered on any of the worksheets. It is possible that participants discussed this scenario in small groups, since they worked together on the function project. It is also likely that the worksheet entitled “Definitions of Functions” and the subsequent class discussion facilitated an ability to recognize this type of function.

5.3 APOS Level changes

The initial APOS conception of each participant is summarized in Table 4.

Table 4

APOS Conception Before History of Functions Unit

	Action	Emerging Process	Process	Object
DB		*		
CW	*			
MJ		*		
BG			*	
MS		*		
CS	*			

Table 5 lists each participant's conception after the unit on the history of functions. A double asterisk (**) indicates a change in level from the initial conception. Note that four participants notably strengthened their function conceptions. Two participants moved an entire level: CS from an action conception to an emerging process one and BG from a process conception to an object. Admittedly, however, the evidence from this study is insufficient to claim that the unit on history of functions enabled BG's move to an object level, since the initial questionnaires did not test for this understanding. All one can claim is that after the unit on the history of functions, BG appeared to hold an object conception. DB and MJ appear to have strengthened their process conception. Recall that though CW did not advance a level in her APOS conception, she appeared to be moving toward a process conception. Recall also that MS was the weakest student among the participants and not surprisingly, showed little growth.

Table 5

APOS Conception After History of Functions Unit

	Action	Emerging Process	Process	Object
DB			**	
CW	*			
MJ			**	
BG				**
MS		*		
CS		**		

5.4 Changes in Understanding Graphical Representations

This section considers the participants' responses to individual graphing tasks and characterizes the APOS level for that task. A summary of the participants' understanding of graphs before the history of functions unit is in Table 6.

Table 6
Interpretation of Graphs Before History of Functions Unit

Task / Participant	1	2c	3a	3d	5
DB	(no answer)	Pre-Function	Pre-Function	Pre-Function	Pre-Function
CW	Process	Process	Pre-Function	Pre-Function	Process
MJ	Action	Object	Object	Pre-Function	Pre-Function
BG	Process	Object	Object	Process	Process
MS	Process	Object	Process	Process	Process
CS	Process	Pre-Function	Object	Pre-Function	Pre-Function

Notable is the low level of graphical understanding for two or more questions in three of the participants, DB, CW, and MJ. Compare these levels with those following the study of the history of functions, in particular, after completion of the worksheet concerning Oresme's techniques. Table 7 lists only those conceptions which changed during the course of study.

Table 7
Interpretation of Graphs After History of Functions Unit

Task / Participant	1	2c	3a	3d	5
DB	Process	Object	Object	Process	Process
CW			Action		
MJ	Process			Process	Process
BG					
MS					
CS				Process	Process

6 Discussion

There appears to be little question that a marked improvement in understanding functions occurred during the course of studying its history for four of these six participants. Less clear is the reason this improvement occurred. This section attempts to answer the question,

- In what ways can studying the history of a mathematical concept be used to deepen a student's understanding of the concept?

In other words, what specific uses of historical material facilitated the change?

Growth was most profound in the area of graphical representations of functions. Interestingly, of all the worksheets, the Oresme worksheet on the history of graphs was the one most dependent upon original sources and provided the most in-depth information about the germination of the concept. It offered a combination of historical reading pinpointing the rationale for a new technique, activities comparing Oresme's techniques to modern ones, and Oresme's proofs with the details omitted. It appears to have cured the "graph as picture" tendency in DB and CW and enabled the understanding of area under the curve for MJ and DB. According to MJ, it helped him understand the "connection between/motivation for integration and area." DB commented that this particular set of exercises clarified her thinking about graphs. One can reasonably conclude that this use of primary sources revealing the germination of an idea along with activities relating original methods to modern-day ones enabled conceptual growth. With the other worksheets it is difficult to say whether the history, the class discussion, or group interaction caused growth, but here, it is evident that the use of these materials facilitated growth. This finding validates the work described by Jahnke (2000) concerning the benefits of using original sources.

A worksheet about the work of Leibniz concerned the first use of the word "function," a geometric interpretation. Perhaps this exercise suggested the idea that a function need not be definable by equations alone. A worksheet about Fibonacci was not historical in the sense that it did not delve into the beginnings of an idea. It was just a problem from history. The learning that occurred as a result of this worksheet supports the claim that history is a good source of problems, however. Both MJ and CW had not considered sequences to be functions until after their work on this assignment. Perhaps it facilitated their new-found ability evidenced on the second questionnaire and in the second interview to see strings as functions as well.

A worksheet on the Definitions of Function showed participants that the definition changed, but did not really go into depth about *why* it changed. One may therefore wonder if the change in participants' understanding was superficial, though their comments in the second interview suggested otherwise.

The above evidence suggests that a wide variety of materials may pull students along in their understanding of a concept: primary source readings about the germination of a concept, problems from history, or simply reading about the changes of a concept over time. The emphasis here, however, is clearly on a thematic approach to history—the germination and development a single mathematical concept. To those who view the history of mathematics as a disjoint collection of anecdotes, facts, and dates, this study offers nothing.

7 Conclusion and Summary

This study has shown that it is possible to create a positive and productive learning environment using historical materials and confirmed what countless mathematicians have conjectured about the value of studying the history of mathematics. History *can* be a vehicle to refresh and strengthen a student's understanding of a mathematical concept. An in-depth study of the birth of an idea and its early development provide the impetus. A thematic approach, studying the development of an idea, reading original sources, and working on problems of former mathematicians did lend insight into the conceptual base. Three students showed significant growth. Five of the six showed improvement in finding functions in situations. One claimed that the unit helped refresh ideas long forgotten.

This study, then, provided evidence that studying the history of mathematics enables a deep reflection of ideas, or as Von Glasersfeld (1995) suggested, a “re-presentation” or reconstruction of ideas. For those that study it for pure enjoyment, this research suggests another good reason to study it, to teach it, to preach it.

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