FROM THE HISTORY OF THE ANGLE TO ITS EPISTEMOLOGICAL NATURE

Contributions to a scholar design

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ABSTRACT

The scholar notion of angle has played an ambiguous role in school. The traditional assumption in schools is that when the concept is defined, characterized, manipulated and its typology exhibited in the mathematics classroom, its use, application and interpretation in other subjects should pose no problem for the students. To the contrary of that assumption, it is in other subjects where the most common conflicts in the handling of this notion are found.

The nature of the concept of angle has been the topic of debate for over 2,000 years and the discussion is not over yet (Matos, 1990). Matos made an historical account of the concept of angle with the aim of understanding how angle was conceived, the properties that were attributed to it, the problems which were resolved and even those that were not, using the concept. This historical review provides important factors to consider when we think of the conflicts the student presents in the classroom when dealing with the concept.

Historically, we find that the angle was used, applied and defined as a quality, a quantity and/or a relation. Considering the definitions and historic uses along with the most common obstacles and conceptions in the student, we have designed a sequence of activities that seek to favor the epistemological nature of the concept.

1 Introduction to the didactic phenomenon

The nature of the concept of angle has been a topic of debate for well over 2,000 years and the discussion is not over yet (Matos, 1990). Perhaps for that reason there is no single definition accepted by the mathematics community and its didactic transposition is in no way a trivial process. Michelmore and White (2000) have called this peculiarity the *multifaceted nature* of the concept of angle, where each facet is made up of the set of related contexts where the angle is located in physical situations. For these authors, the varied textbook definitions are fitted to different formal mathematical structures, but the fact that no definition seems to coincide with all the physical contexts of the angle emphasizes the difficulty of forming a standard general concept.

In the work developed by Casas (2000) some historical elements of the notion of angle were considered as part of the background to the research, centered mainly on articles published by Matos (1990 and 1991). Nevertheless, these elements were considered neither as part of the didactic phenomenon, nor as an explanation for the student's conflicts.

In the same way, an explanation on the cognitive plane can be found in Michelmore and White (1995 and 2000), considering that cognition is situated, related to the didactic plane (the result of didactic transposition), but leaves aside an explanation of the role played by the mathematical knowledge involved, that is, the epistemological nature of the notion of angle.

These two studies report some historic ideas without reflecting on the epistemology of the concept and, in this sense, do not consider that the mathematical object conserves or loses significance on becoming scholar knowledge. The authors' emphasis is on the *cognitive* component of the didactic phenomenon, relating it to the structure of scholar discourse.

We consider it feasible to identify elements on the epistemological nature of the concept of angle from the notes of Matos (1990 and 1991), taking into account not only the historical evolution, but also the stages of construction.

2 History and construction of the notion of angle

Matos made an historical account of the concept of angle with the aim of understanding how angle was conceived, the properties attributed to it and the problems that were resolved, and those that were not, using the concept. From his study we can extract stages in the construction of the concept:

Use

↓

Definition-Classification

↓

Discussion of its Nature

Although on this last, the discussion of its nature, Matos does not report a consensus from the scientific community.

For the purposes of our analysis on the notion of angle that is worked on at secondary level (junior high) of the Mexican education system, we have considered the historical evolution of the concept from the Neolithic culture and up to some formal elements in Peletier (Fig. 1)

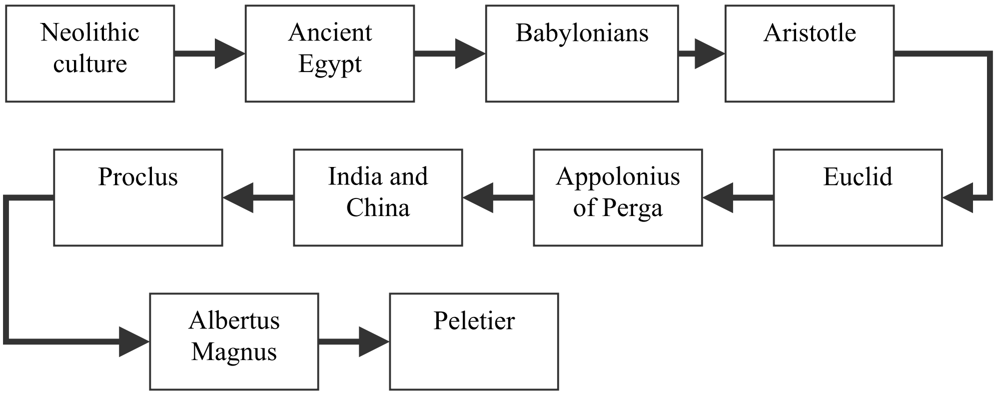


Fig. 1 *Historical evolution of the notion of angle*

2.1 Use of the Notion of Angle

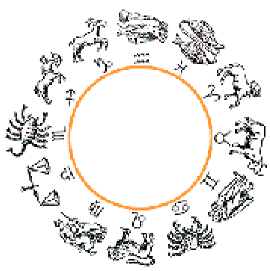
We would like to differentiate between the terms *use* and *application*. For the first we will understand that angle plays an important role in certain activities, but at the level of notion that is still undefined or unidentified as a concept, while application refers to taking the formal concept to the situation that requires it.

In early cultures, the notion of angle did not exist, but the inclination and direction of their constructions were based on the stars.

In this sense, the use of the notion of angle acquires significant importance since, in a school setting, the student can build intuitive conceptions that characterize the angle.

Construction and astronomy were fitting settings for the use of the notion of angle, although underlying in the *inclination* of constructions and the *direction* of celestial bodies, they contributed a significance that would later be represented by geometric objects.

It was the Babylonians who managed the notion of time and space dividing it into 360°



In pre-Columbian times, the Maya engineered constructions to stage celestial phenomena on earth, such as the castle of Chichén Itzá where the descent of the serpent, Kukulkán, can be observed on the vertices of the pyramid during each solstice. The four stairways of the structure have a total of 365 steps, the number of days in a year.

Among the many purposes of astronomy in different cultures, Babylonian, Hindu, Chinese, Maya among others, important progress was made in the measurement of time. It was a natural step to take the change from day to night as the first benchmark for measuring time, and observing the movement of the sun and moon logically followed. Nevertheless, these phenomena would soon prove insufficient for determining the best time for the typical activities of each community (agriculture, fishing, navigation, trade), and so began the observation and recording of the changing positions of the stars and planets, bringing the need for more precise tools and references.

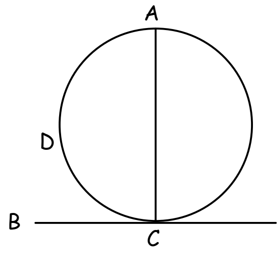
Some of the most noteworthy contributions to the practice of astronomy and geometry-trigonometry are the division of the circle into 360 parts, the arc-chord tables equivalent to today's trigonometric tables and information on the positions of the stars, which would later serve as validation for the first geometric models.

2.2 Definition and discussion on the nature of the concept of angle

The Greeks were the first to have a word for angle and even before Aristotle (384-322 BCE) the distinction had been made between acute, right and obtuse angles. From the Classic Greek Period the angle finds definitions, characterizations and classifications which are ever more precise and broad. What is important to point out is that every discussion is grounded in the philosophical-mathematical rather than in practical *use.*

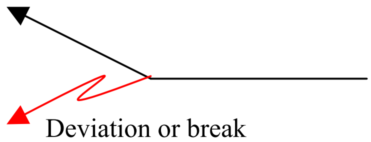
Matos (1990 and 1992) has described with precision and detail the historical evolution of the concept of angle, but for the purposes of this paper we will synthesize the most relevant moments regarding its definition and the discussion of its nature. Our aim is to shed light on the current meanings, and those meanings that have been lost in scholastic mathematical discourse, of rectilinear angles only. From the first Greeks through to the 16th century the discussion was ongoing on how curvilinear or mixed angles, as well as rectilinear angles, were characterized in terms of quality, quantity and relation and it was Peletier (1517-1582) who maintained that:

An angle of contact is not an angle at all… the contact of two circles, is not a quantity… the contact of a straight line with a circle is not a quantity either…[Heath, 1926[1]]



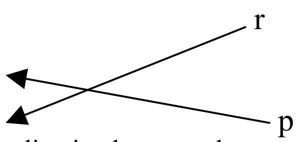
A central problem to Aristotle's discussion (384-322[2]) on geometric objects was determining their nature. Longitude, for example, was a quantity; parallelism was a relation and the triangle a quality.

The straight angle (two right angles) and the circle were figures to him. It is assumed that for Aristotle the angle was a quality, defined in some writings as *the deflection or fracture of a line* (Matos, 1990).

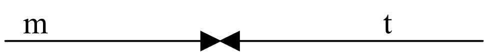


Deviation or break

In addition, definitions 8 and 9 of Book 1 of Euclid's Elements (~300 BCE) establish that *a plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line… and when the lines containing the angle are straight, the angle is called rectilinear*[3].



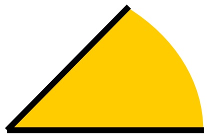
Inclination between the lines "r" and "p"



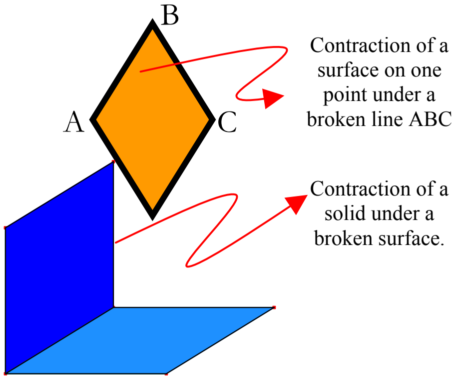
Rectilinear angle

[1] Cited by Matos (1991)
[2] Before the common era (BCE)
[3] Taken from http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html

And in definitions 10 through 12 he makes a categorization of the angles according to their size. From the first two definitions it can be deduced that for Euclid the angle is defined by two particular straight lines, but also suggests angle as a kind of area contained between two straight lines.



Two notable changes can be seen with respect to Aristotle, the angle is defined by two straight lines and not just by one which is fractured, and it can be quantified. Nevertheless, Euclid excludes from being angles the zero, the rectilinear angle and those greater than the rectilinear, indeed he does not consider them as a composition of other angles. Only commentators from the Middle Ages [Tartaglia (1500-1557), Peletier (1517-1582) and Clavius (1538-1612)] considered angles greater than 180° as a single angle (Heath, 1926).
Later, for Euclid the discussion centers on three problems: (1) what is the definition of an angle, (2) where do angles fit into Aristotle's categories of quality, quantity and relation, and (3) what is the nature of curvilinear angles.
In the following table we mention some of the definitions that Matos (1990 and 1991) reports after Euclid, including some that are part of commentaries on the Elements:

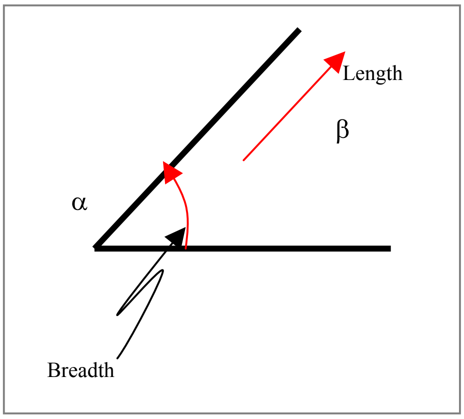
Author	Period[4]	Definition
Apollonios of Perga	262 – 190 BCE	The contraction of a surface on a point under the fractured line or of a solid under a fractured surface. 
Heron	~10 to 75	"An angle is a quantity which a simpler related quantity encloses when it comes to a point".
Plutarch of Athens	~46 to 120	An angle is the first space under a point,

[4] See this site: http://www.ou.edu/cas/hsci/isis/website/thesaurus/IsisCB.Personalnames.A-B.html

		because there must be a first space under the inclination of the lines or planes it contains.
Carpus of Antioch	A precise date could not be found, but he is considered Pythagoric.	An angle is a quantity called distance between lines or surfaces.
Proclus	~410 to 485	An angle is formed by two lines or two surfaces that intersect. The surfaces need not be flat, nor the lines straight.
Avicenna	980 - 1037	The term angle can be used for the quantity in itself, or for the quality of being angular.
Albertus Magnus	1193 - 1280	For a surface, an angle is enclosed by lines, given that it is the medium between a quantity of one dimension and another of two dimensions. Nevertheless, a solid angle ends in surfaces, being the medium between a surface, which has two dimensions, and a solid which has three.
Wallis	1616 - 1703	A point is not a line, but the beginning of a line; a line is not a surface, but the beginning of a surface; thus, an angle is not the distance between two lines, but their initial tendency towards separation.

In the case of Proclus, Matos (1991) emphasized his conclusion on the nature of the angle[5]:

> *an angle was a quantity that provided it with the ability to be divided and compared, it was a quality by virtue of its shape, and it incorporated a relation because it needed the relation of the lines or surfaces that bound it* (taken from Morrow, 1970).



Albertus Magnus considered two dimensions in an angle: α indicates a direction of increasing *breadth* and β a direction of increasing *length*.
Solid angles also have "depth" which apparently would constitute a third dimension.
In his commentaries on the Elements, Albertus Magnus concluded that the angle tells us a *quality* about a certain *quantity:* an angle (angulus) is a quantity, but to be

[5] Taken from Morrow, (1970)

angular (angulatio) is a quality. However, in his comments of Aristotle's Metaphysics he asserts that an angle is a relation because it is a "medium" between a line and a surface.

a. As an angle has breadth it is not a line. Nor is it a body because it might not have depth. Neither is it a surface, because it cannot be divided breadthwise, only lengthwise. An angle is the indivisible contact of two lines.
b. It does not seem to be a quantity because when a particular angle, the right angle, is doubled it is no longer the same kind of continuous quantity.
c. It is a property of a surface or a body so it is not a quantity.
d. It has the ability to divide a figure and this ability is a kind of quality.
e. Angles can be increased and decreased so they seem to be a quality.
f. Acuteness and obtuseness are conditions of quantity.
g. An angle has breadth and length and so is a quantity.

By the arrival of the 19^{th} century the angle had two new contexts of discussion. Non-Euclidean geometry changed the discussion on its nature radically; however, one of the most important extensions was its use to express time intervals between two periodic events. The crucial idea came from the development of functions in trigonometric series, by Fourier (Kline, 1972).

Within Euclidean geometry efforts continued to clarify the notion of rectilinear angles. Veronese (1854-1917) maintained that an angle is an entity in one dimension with respect to the ray and in two dimensions with respect to the points on the plane. His idea was to define angle as the aggregate of the rays issuing from the vertex and comprised in the angular sector (Heath, 1956[6]). This meant that for him angle was the set of all the rays within two given rays.

On the other hand, to Bertrand an angle was the portion of the plane that is common to the two semi-planes limited by the two lines, or the interference of these two semi-planes.

More recent definitions have a more formalist touch using symbolism and terminology of sets, groups, etc., which is only logical given the formalist programs of mathematics itself. However, books of a more historical nature or of the diffusion of the mathematics culture maintain this explicative, illustrative vision of the angle. Such is the case of Maor (1998), who in his book *Trigonometric Delights,* in the chapter on the angle points out:

> *Geometrical entities are of two types: those of a strictly qualitative nature, like the point, the line and the plane and those that can be assigned a numeric value, a* ***measurement.*** *To this last group belong the segments of a straight line, whose measurement is length; a plane region associated with an area; and a* ***rotation,*** *measured by its* ***angle****.*
>
> *There is a certain ambiguity in the concept of angle because it describes the qualitative idea of* ***separation*** *between two lines that intersect and the numeric value of said separation (the measurement of the angle).*

[6] Cited by Matos (1991)

Fortunately, we need not concern ourselves with this ambiguity; trigonometry is only concerned with the quantitative aspects of straight line segments and of angles.

Let us suppose that characterizing the angle as a relationship between two lines is not a necessity for Matos given the context which interests him, trigonometry. That is, given that when discussing trigonometric relations the concept of angle is intimately tied to the triangle, there is no need to explicitly state that an angle needs two lines to be defined or delimited.

Matos' historic review (1990 and 1991) provides important elements to consider when we think of the conflicts the student presents in the classroom on tackling the concept of angle. Matos himself points out that:

…I will assume that a historical investigation on the origins of a mathematical concept is fruitful as a guide for developing a pedagogical perspective…(Matos, 1990; pp. 4)

From the point of view of a mathematics educator, it is interesting to note that these several developments of the concept of angle have their counterparts in contemporary school mathematics. In fact, there are several kinds of angles currently used in schools:

(1) The definitions of Euclid and Hilbert
(2) Angles associated with rotations
(3) Angles as a measure of periodic events

(Matos, 1991; pp. 24)

The task of the mathematics educator is to problematize how it is learned, how it is taught and also what is taught. In this sense, it is important to recover from the historic report that which sheds light on the concepts of angle, its nature, its meanings and the conflicts which lead to its construction, that is, to unravel its epistemology.

3. From the history to the epistemology of the notion of angle

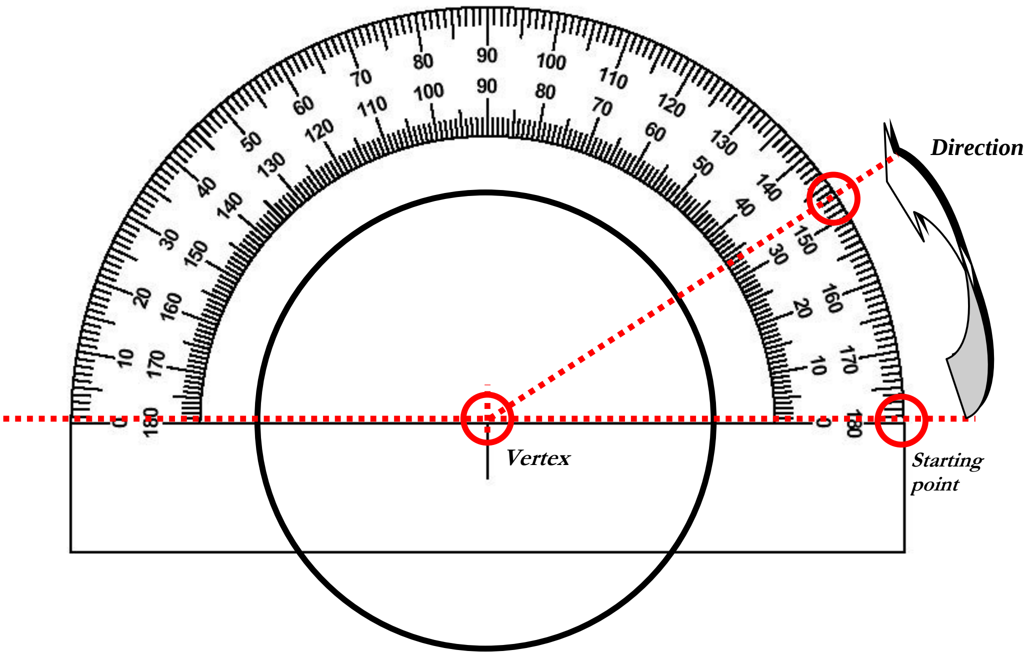
Historically, we find that angle was used and applied considering a quality, a quantity and/or a relation. We will consider, then, that the nature of the concept will be associated with the meanings that the student must construct:

- **Qualitative** for its form.
- **Quantitative** for its size.
- **As a Relation** for the way it is defined.

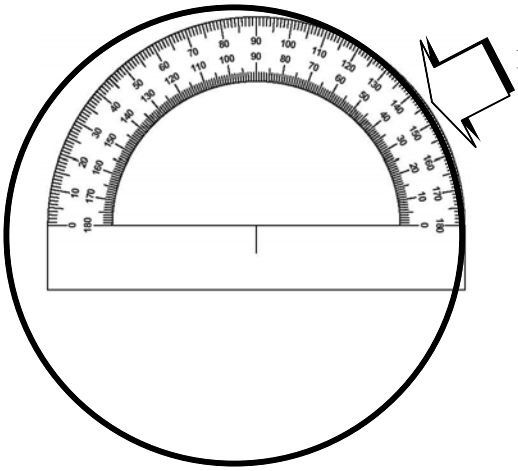
Our concern with the learning of the concept of angle leads us to consider the construction, the task, the distinction and the articulation of these three meanings, that one way or another are found in the different school definitions (from basic education through university), but disjointed and preceding definition and measurement to the handling of forms and the identification of the contexts where the angle has a use or an application.

In the qualitative we can distinguish that the angle is identified as an opening, inclination or even area; all of which are acceptable *forms* for working with the concept, as long as what pertains to the angle is distinguished. On the other hand, depending on the context or mathematical objects to which it relates, the angle is seen as a static object (above all in Geometry and Trigonometry, or in Technical Drawing), in the inclination, opening or fractured line; or a dynamic object (in Analysis or Physics), in turns, rotations or the classic unit circle.

In the quantitative it is important to distinguish the "type of measurement" which should be performed when working with angles. Currently in schools, the angle is preceded by measurements of length where the principal tool of measurement (the ruler) is placed directly over the object to be measured and the measurement is taken. The measurement of the angle requires, in addition to the protractor, the identification of the "starting point" and the "direction" in which the degrees are read.



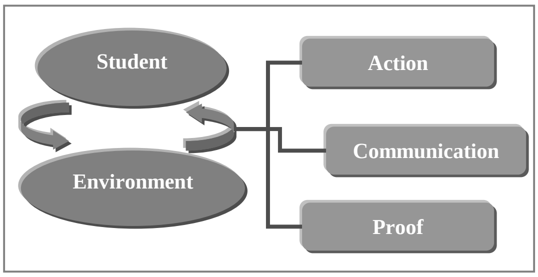
In this sense we can assume that "measuring" an angle is of a different nature to "measuring" a length and the use of the word "measure" may be the trigger for considering the angle as the *length* of an arc.



By characterizing *forms* (their qualitative meaning) we can build the classic subdivision of 360°, without necessarily talking about degrees.
Once the qualitative and quantitative meanings of angle have been worked on, it is possible to identify the elements which comprise it, giving rise to its definition as the *relation* of other objects (lines, for example).
This presupposes the inverse order of that proposed in school, but also considers the diverse situations which give a multifaceted concept sense, meaning and use.
To assume that the angle is simple or trivial may be at the root of why it becomes such a difficulty for the student since its presentation is, in effect, simple and trivial. Nevertheless, as with any mathematical concept, it is the result of broad, complex processes of construction and abstraction.

4. Elements of a didactic theory for a classroom design

The Theory of Didactic Situations focuses its attention on the didactic devices whose aim is for the student to gain certain mathematical knowledge. But to be able to produce such devices, or organize the means through which the didactic activity develops, we must first determine the conditions which produce the student's grasp of the knowledge and the nature and origin of the mathematical concepts in play.
Brousseau's (1987) basic idea is that the process for acquiring a mathematical knowledge consists of diverse facets and is based on specific games, where the actor interacts with an environment on different levels, developing its notions and language. The interaction of an actor with his environment takes place on three levels:

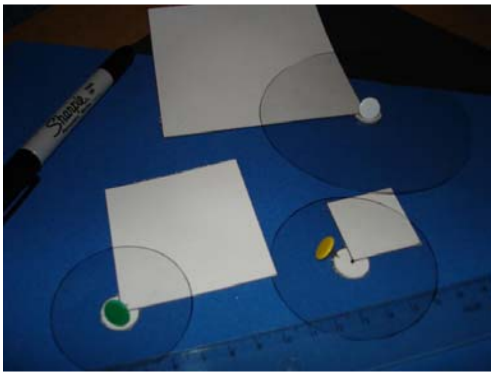


In **action** type interaction the actor fixes a condition of the environment or determines or limits the actions of other actors. **Communication** type interaction consists of modifying the knowledge of another actor through messages carrying information and, lastly, **proof** type interaction tends to the justification or cultural validation of the acts or declarations established either explicitly or implicitly. These interactions cannot take place simultaneously; in fact, they occur in situations with their own characteristics and where the actor plays distinct roles, uses a variety of tools and produces different mechanisms of communication.
In the learning situation of a specific chunk of mathematical knowledge, the student must achieve these interactions in an environment organized by the teacher. He must be capable of acting, talking, thinking and evolving by self-motivation. Nevertheless, even when the student knows that the problem situation presented to him has as its objective that he acquires new knowledge, the teacher must abstain from intervening or suggesting the knowledge that he wants the student to acquire. If we think of mathematics as the product of specific games with dimensions of action, information and veracity, composed of a symbolic system that, depending on the social and personal context where it is used, plays an instrumental role and, in turn, functions as communication, we must cause the student to interact with the environment as an artificial genesis of the target mathematical knowledge.
Thus, then, the theory of situations allows the design and exploration of a set of class sequences conceived by the teacher with the aim of providing an environment for the realization of a certain learning project. To do that the teacher considers epistemological factors (already discussed), didactic factors (how scholastic discourse is structured to present the angle as a static or dynamic object; the different definitions according to grade level and the subject which includes the concept; the classroom explanations and kind of exercises used to work on the concept; and some school applications) and cognitive factors (going back to the cognitive model of Michelmore and White (2000) that suggests the stages of *experience → classification → contexts → abstraction* for the construction of the concept of angle).

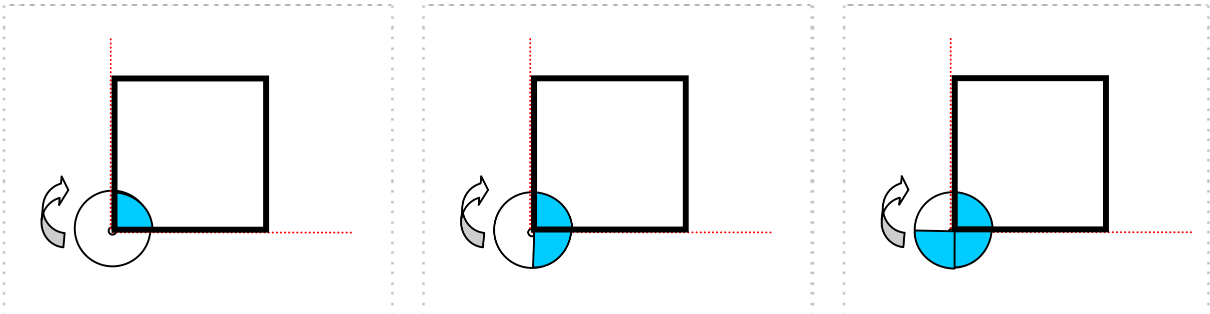
5. Preliminary design of a didactic sequence

The basic principle for the sequence is to allow the student to build the concept through concrete activities with manipulative materials which allow him to find forms, patterns and relationships in the diverse situations that are presented to him.
The design is targeted to students between 12 and 14 years of age and currently in secondary level of the Mexican educational system (junior high). We assume,

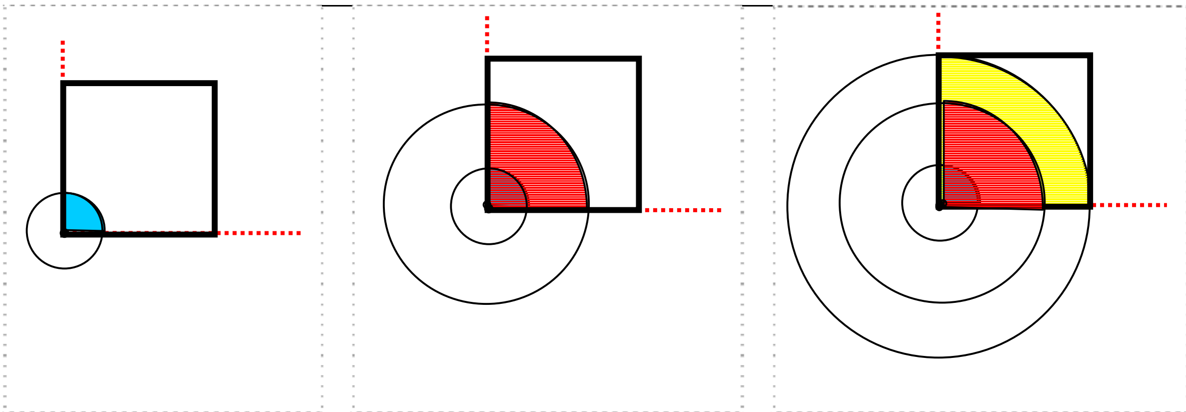
therefore, some basic geometric knowledge (or background) for the resolution of the sequence.
The fundamental idea was to use basic geometric figures over which different sized circles were placed in order to find the "parts" or "portions" which are superimposed.



Working materials

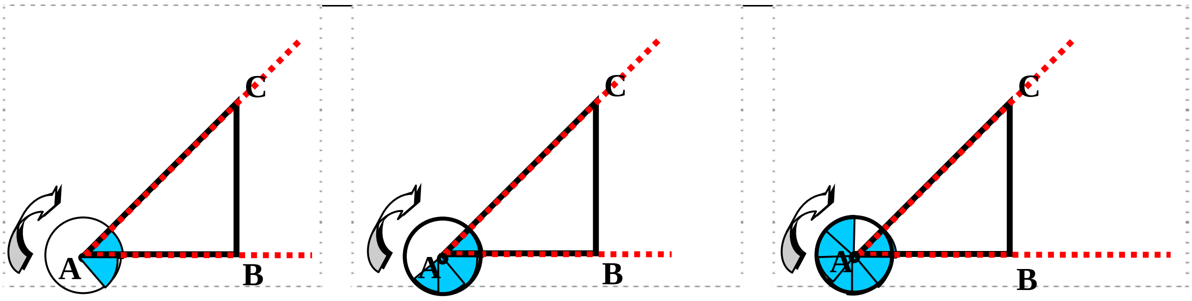


Activity for finding the shaded "parts" of the circle. (Favoring the notion of area, but as a portion of the circle; in addition the transit is made from the static notion to the dynamic notion.)



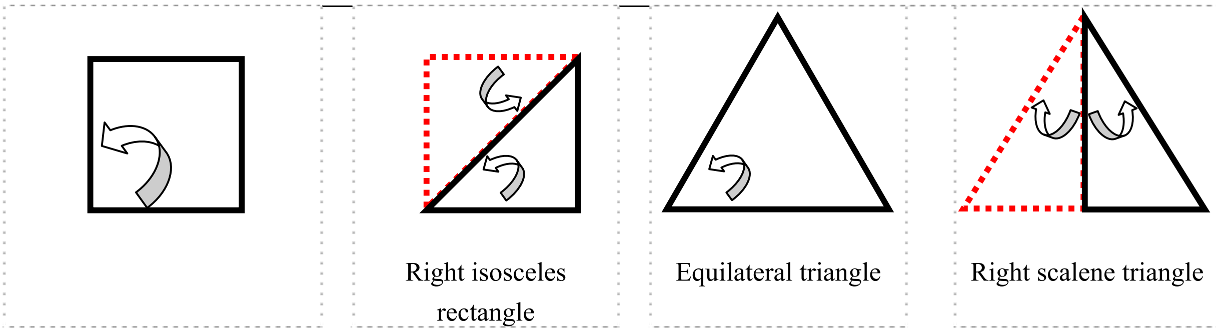
Activity to compare the "shaded parts" (In order to consolidate the idea of the "portions" of the circle regardless of its size)

Recalling the square, an isosceles triangle is constructed and the same type of activity carried out.



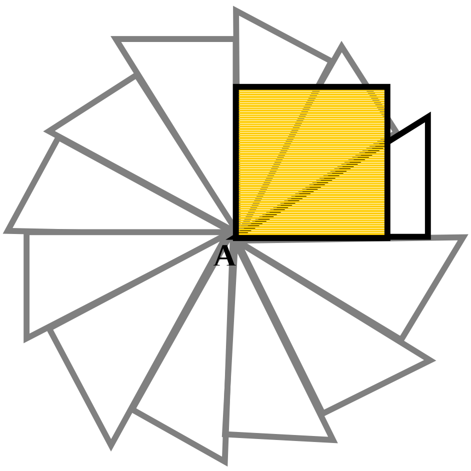
Activity to find the shaded "parts" of the circle but now with half of a square, so that the fraction found is now smaller. (Favoring a notion of area, but with the portion of the circle; in addition the transit is made from the static notion to the dynamic notion.)

The sequence continues with geometric figures used frequently in class such as the equilateral triangle and the right triangle formed with half of the first until the different shaded parts or rotations are identified:

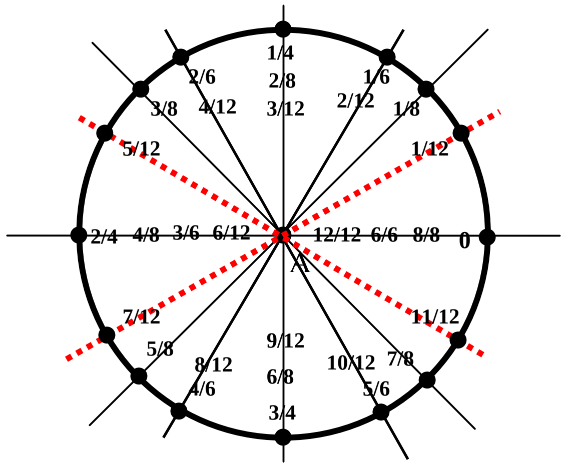


Activity to identify ¼ rotation, 1/8 rotation, 1/6 rotation and 1/12 rotation (Favors a notion of area but as the portion of the circle; in addition the transit of the static notion to the dynamic notion is made by identifying rotations and relating them to figures that are often worked with in mathematics and technical drawing when using geometry set squares.)

The sequence continues with ever more complex figures until arriving at something like this:



The purpose is to construct an instrument of measurement with the student based on *pieces of turns and rotations.*



We are currently in the process of application of the first parts of the sequence intending for the student to move between the stages of *action*, *communication* and *proof*, always working with the notion of angle from the *qualitative* and the *quantitative*. Defining the concept as a *relation* between mathematical objects, that is, giving it a formal status as a school concept will be the task of the teacher, since it constitutes the *Institutionalization* of scholar knowledge.

6. Final reflections

Just like in the historical evolution when we found that ancient cultures **used** the angle in different activities without identifying the notion or concept, students in secondary education use the concept frequently in a variety of school activities without being aware of it.

For this reason we begin the sequence without mentioning the concept of angle as we traditionally do in classes or textbooks. We intend for it to be **used** and little by little identified in school activities and frequently used instruments until an awareness of its characteristics and qualities is acquired through different patterns and it can then be **applied**.

In the same way, the epistemological study gives us valuable information about how the discussion and analysis of the concept began. Of how the different mathematicians involved defined and characterized the angle from its different applications and how, even now, a unique definition remains illusive because of its multifaceted character.

Taking into account that angle can be considered a turn or an opening, it is important that the student handles the static part in the sequence (in the shaded parts of the circle and different geometric figures) and the dynamic part on discovering the possibility of rotation in both directions.

On the other hand, measurement plays a very important role since the angle does not only posses specific qualities but it is also possible to measure and quantify it except that this is under a different scheme to that used in length and area.

The historic review of the concept provides us with important elements when we think of the conflicts the student presents in the classroom on addressing the concept. In this way we find that the angle is used, applied and defined as a quality, a quantity and/or a relation and thus, the activities of the sequence designed seek to favor the epistemological nature of the concept, that is, its **qualitative** nature (manipulation of forms), **quantitative** nature (exercises of measurement) and **as a relation** (to define it).

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