FORMULATING FIGURATE NUMBERS

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ABSTRACT

The multiplicative formula for figurate numbers (or binomial coefficients) we use today appeared in Western Europe in verbal form in the late 1500s and in symbolic form in the early 1600s. In this presentation, we first recount the early history of figurate numbers and especially of multiplicative means for computing them. We then focus on the development of multiplicative formulas for figurate numbers in the late sixteenth and early seventeenth centuries by Cardano, Faulhaber, Briggs, and Harriot. Throughout the presentation, we explore what it means to "have a formula for" a mathematical relationship. Indeed, the story of figurate number formulas has implications both for how we teach the history of mathematics and for how we teach mathematics. For example, students (and instructors) may be surprised to learn what historians mean when they report that Cardano had the first multiplicative formula for figurate numbers. While students may imagine a symbolic formula, in fact Cardano's description was verbal and employed a numerical example. In all of our mathematics courses, we should keep in mind, if not discuss outright, how conceptions of what constitutes a mathematical formula have varied over time and also how our and our students' views may differ. Students (and instructors) might argue from their own experience that a few good examples, or even just one well-chosen example, can convey a general formula at least as well as a symbolic formula. On the other hand, symbolic notation seems to aid greatly in further development of ideas. The history of multiplicative formulas for figurate numbers is interesting in its own right and provides rich fodder for a broader discussion of mathematical formulas.

1 Introduction

The figurate numbers, and in particular the triangular numbers and generalized triangular numbers, have a history dating back at least to the ancient Greeks, if not earlier. The triangular numbers 1, 3, 6, 10, 15, 21, … are sums of successive positive integers; the pyramidal numbers 1, 4, 10, 20, 35, 56, … are sums of successive triangular numbers; the triangulo-triangular numbers (as Viete and later Fermat called them[1]) 1, 5, 15, 35, 70, 126, … are sums of successive pyramidal numbers; and so on. See the top third of Figure 1[2] for a table of generalized triangular numbers, as it appears in the manuscripts of Thomas Harriot (1560-1621). The triangular numbers generally are attributed to the Pythagoreans who, legend has it, represented each as an equilaterial triangle composed of pebbles. For instance, the triangular number 10, said to be a particular favorite of the Pythagoreans, was formed using four rows of 1, 2, 3, and 4 pebbles. Nicomachus of Gerasa and Theon of Smyrna (both, *c.* 100 CE) generalized triangular numbers at least as far as pyramidal numbers.[3] Boethius (*c.* 500 CE), following Nicomachus, treated polygonal numbers,

[1] Francois Viete (1540-1603) used the terms "triangulo-triangular numbers" for 1, 5, 15, 35, … and "triangulo-pyramidal numbers" for 1, 6, 21, 56, … in his *Ad Angularium Sectionum Analyticen Theoremata (Universal Theorems on the Analysis of Angular Sections),* published posthumously in 1615 (Witmer, 433, 435), and seems to have been the first mathematician to do so (Edwards, 9). Pierre de Fermat (1601-1665) used the first of these two terms in letters to Mersenne and Roberval in 1636 (Mahoney, 231). Thomas Harriot, however, called the same numbers "triangle piramidall" at British Library Additional MS 6782, folio 38.
[2] BL Add. MS 6782, f. 108.
[3] Edwards, 3.

including triangular, square, pentagonal, hexagonal, and heptagonal numbers, and solid numbers, including pyramidal and cubic numbers.[4] His work would remain in circulation in Europe with little alteration for 1000 years. In the sixteenth century, Stifel and Scheubel in Germany and the famous quarreling mathematicians Tartaglia and Cardano in Italy, among others, published tables of generalized triangular numbers.[5]

The page from which Figure 1 was transcribed is the first page of a manuscript treatise on finite difference interpolation that Harriot wrote in 1618 or later.[6] It contains some of the most beautiful results in all of Harriot's approximately 5000 surviving manuscript sheets; namely, multiplicative formulas for the *n*th triangular number, the *n*th pyramidal number, the *n*th triangulo-triangular number, and so on. Harriot's notation is easy to decipher: at the bottom of Figure 1, for instance, his formula for the *n*th entry of the fourth column – his pyramidal number formula – is

$$\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}.$$

To our modern eyes, these formulas lack a column index, but of course we very much appreciate Harriot's generalization from numbers to symbols. The third page of the same treatise (see Figure 2[7]) also contains formulas for generalized triangular numbers, this time positioned as binomial coefficients. We return to these formulas later in our presentation (see Section 6).

I was intrigued by A.W.F. Edwards' assertions in his splendid 1987 history of Pascal's Triangle, *Pascal's Arithmetical Triangle*, that Cardano, Briggs, and Faulhaber had multiplicative formulas for the generalized triangular numbers in 1570, about 1600, and 1615, respectively,[8] hence before or at about the same time as Harriot, and I wondered exactly what form their "formulas" took. Fermat rediscovered a multiplicative rule for generalized triangular numbers in 1636 and shared it with Mersenne, who almost certainly would have shared it immediately with the larger scientific community, including Pascal.[9] Pascal, of course, stated and proved this rule in his treatise on what we now call Pascal's Triangle, *Traite du triangle arithmetique*, written in 1654 and published in 1665.

2 Multiplicative Rules Before Cardano, Briggs, Faulhaber, and Harriot

A multiplicative rule for the figurate numbers was given by Narayana in India in 1356 in his *Ganita Kaumudi,* the equivalent multiplicative rule for the binomial coefficients was given by al-Kashi in Samarkand in 1429 in his *Key of Arithmetic,* and one or both of these

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[4] Masi, 133-149.

[5] Edwards, 5-7, 43-44, 53-54. Edwards reproduced and discussed tables from Michael Stifel, *Deutsche Arithmetic,* 1545, at pp. 5-6; Johannes Scheubel, *De Numeris,* 1545, pp. 7, 53-54; Niccolo Tartaglia, *General Trattato,* 1556, p. 53; and Girolamo Cardano, *Opus Novum de Proportionibus Numerorum,* 1570, pp. 43-44 (see Figure 3).

[6] For the complete treatise, see BL Add. MS 6782, ff. 107-146v, or Beery and Stedall.

[7] BL Add. MS 6782, f. 110.

[8] Edwards, 10, 11, 14.

[9] Edwards, 14-15, 47. Mersenne published the equivalent combinatorial result in his 1636 *Harmonicorum libri XII* (Edwards, 45-47).

rules possibly were known in China by about the same time.[10] The equivalent combinatorial formula for n objects taken k at a time was given by Mahavira in India in the ninth century, by al-Banna in Morocco before 1321, and by Levi ben Gerson in France in his 1321 *Maasei hoshev* (*Art of the Calculator*), though not using symbols such as n and k.[11]

It is not known if Cardano, Harriot, Briggs, or Faulhaber learned any of these rules from any of these sources, directly or indirectly,[12] or if they devised them on their own. The multiplicative rule for the nth triangular number (on the righthand side of the following equation; the additive formula is on the lefthand side), $1+2+3+\cdots+n=\frac{n(n+1)}{2}$, was known in Europe and almost certainly would have been known to them. It appeared in Francisco Maurolico's *Arithmeticorum Libri Duo* in 1575 (written in 1557), where Harriot read it. Maurolico (1494-1575) stated the formula in words: "The root plus one and half the root are to be multiplied" in one place and "Each natural number multiplied by its successor gives twice the corresponding triangular number" in another.[13] Harriot translated much more complicated verbal statements into algebraic language and would have handled these two easily, rewriting them as $(n+1)\left(\frac{n}{2}\right)$ and $\frac{n(n+1)}{2}$, respectively (but in his vertical arrangement for multiplication).

As for influences between Cardano, Briggs, Faulhaber, and Harriot, to my knowledge, neither Briggs nor Faulhaber credited Cardano with his formula, although Faulhaber did cite Cardano and seven other authors in a discussion of figurate numbers in at least one of his books, the aptly named *Numerus Figuratus,* to be discussed in Section 5. Briggs seemed to claim only Michael Stifel as an influence. However, the influence of Cardano on Harriot is clear: on a page that survives in his manuscripts,[14] Harriot (or an assistant) copied out the relevant proposition from Cardano's book.

3 Girolamo Cardano (1501-1576) and Thomas Harriot (1560-1621)

In Proposition 170 of his *Opus Novum de Proportionibus Numerorum* (1570), Cardano displayed a table of generalized triangular numbers (see Figure 3), each one representing a number of combinations, and showed how to compute by successive multiplications the entries of the eleventh and last southwest-northeast diagonal of the table.[15] A similar table

[10] Edwards, 16-17.
[11] Katz, 149, 177-178, 198-201, respectively.
[12] See, for instance, Katz, 187.
[13] The former is my translation of Maurolico's Latin at his page *a;* the latter, Edwards' more modernized translation of Maurolico's Proposition 7 at page 5 (Edwards, 15). The statement also appears as Proposition 22 at page 9. All of these references are to Book I of the *Arithmeticorum Libri Duo.*
[14] BL Add. MS 6782, f. 44.
[15] Cardano, 185-187. The proposition spans the three pages; the table in Figure 3 is at page 185 and the passages I have translated below from Cardano's Latin are at page 187. The table attributed to Stifel is at page 131. See also Edwards, 44, for a reproduction of Cardano's page 185, and Boyer, 389, for a more modernized translation of the first part of the passage.

had appeared previously in *De Proportionibus*, where Cardano had attributed it to Stifel, who used it to compute powers of binomials and to extract roots. Cardano's instructions follow. For eleven objects, Cardano was concerned initially with the "numbers that come from variation of three" of these objects, but he first needed to compute the number of combinations of two of eleven objects, the number that would occupy the second position:

> First, for the second position, I subtract 1 from 11 to get 10; I divide by 2, the number of the position, obtaining 5; I multiply by 11 to get 55, the number in the second position. Then I subtract 2, which is the number of the difference of the third position from the first, from 11, leaving 9; I divide 9 by 3, the number of the position, obtaining 3; I multiply 3 by 55, the number in the second position, to get 165, the number in the third position. Similarly, [if] I want the number from variation of four objects, I subtract 3, the difference of 4 from the first position, from 11, leaving 8; I divide 8 by 4, the number of the position, obtaining 2; I multiply 2 by 165 to get 330, the number in the fourth position. Similarly, for the fifth, I subtract 4, the difference from the first position, leaving 7; I divide by 5, the number of the position, obtaining $1\frac{2}{5}$; I multiply by 330, the number in the preceding position, to get 462, the number in the fourth position.

Certainly, one could adapt this multiplicative process to any row of Cardano's table to obtain any entry; that is, to obtain the number of combinations of any given number (n) of objects taken any given number (k) at a time.

Harriot experimented with proportionality between generalized triangular numbers in several, now scattered manuscript pages, including British Library Additional Manuscript 6782, folios 38, 163, 237, and 330-334. Folios 33-38 comprise a set of notes titled "Of combinations" and, in folios 39-41, Harriot considered "transpositions" and "combinations" together. (Folios 30, 331, and 332 carry similar headings.) Although Harriot copied Cardano's instructions for counting combinations of eleven objects, together with his table, at folio 44, and labeled it "Cardanus de proportionibus prop. 170," he did not reference it at any of the other folios listed here. (At folio 38, he cited Boetius, Maurolicus, and Jordanus.)

We consider Harriot's work at folio 163, titled "Elements of Triangles" or "Elements of Triangular [Numbers]" ("Elementa triangularium"). Here, he labeled columns of units, counting numbers, triangular numbers, and pyramidal numbers, $\overset{0}{b}$, $\overset{1}{b}$, $\overset{2}{b}$, and $\overset{3}{b}$, respectively, and then noted the relationships $n\overset{0}{b} = 1\overset{1}{b}$, $(n+1)\overset{1}{b} = 2\overset{2}{b}$, $(n+2)\overset{2}{b} = 3\overset{3}{b}$, and $(n+3)\overset{3}{b} = 4\overset{4}{b}$. He solved these equations for $n + 1$, $n + 2$, and $n + 3$, and noted such proportionalities as 3 is to $n + 2$ as $\overset{2}{b}$ is to $\overset{3}{b}$. Finally, Harriot combined his equations to obtain multiplicative formulas very nearly like those at the bottom of Figure 1; namely, $\frac{\overset{0}{b}}{\overset{2}{b}} = \frac{1}{n} \cdot \frac{2}{n+1}$, $\frac{\overset{0}{b}}{\overset{3}{b}} = \frac{1}{n} \cdot \frac{2}{n+1} \cdot \frac{3}{n+2}$, and $\frac{\overset{1}{b}}{\overset{3}{b}} = \frac{2}{n+1} \cdot \frac{3}{n+2}$. We return to Harriot's work in Section 6.

4 Henry Briggs (1561-1631)

Henry Briggs (1561-1631) demonstrated how to calculate generalized triangular numbers (or binomial coefficients) multiplicatively in his *Trigonometria Britannica,* published posthumously in 1633 by his friend and colleague, Henry Gellibrand. This book consisted of extensive tables of sines, tangents, and secants, and of logarithmic sines, tangents, and secants, together with a section by Gellibrand on applications of these tables to trigonometric problems. In his preface to the book, dated November 1632, Gellibrand stated that Briggs had done the work contained therein approximately 30 years before ("ante annos plus minus triginta").[16] Although he provided no evidence for this claim and in fact Briggs did not pay his famous visit to Napier until 1615 (and, as far as we know, did not begin working on logarithms until the publication of Napier's *Descriptio* in 1614) and did not show how to calculate binomial coefficients in his 1624 *Arithmetica Logarithmica* or in any early publication, it is possible that Briggs had worked out his rules for generating binomial coefficients as early as the turn of the century.

In Chapter VIII, Briggs displayed the binomial coefficients in a table he called his "Abacus of Many Uses" (see Figure 4). Note, as did Briggs, that the coefficients of the binomial expansions for positive integer powers appear along the northwest-southeast diagonals. Briggs explained that "any one of these diagonal numbers is in proportion to the next higher in the diagonal, as the vertical of the former is to the marginal of the latter."[17] His initial examples were that 15 is to 20 as 3 is to 4, and that 84 is to 126 as 4 is to 6.[18] He then reversed his order so as to focus on generating the next higher number in the diagonal from the one below it, giving as examples: 2 is to 11 as 12 is to 66; 2 is to 9 as 10 is to 45; 2 is to 22 as 23 is to 253; and 3 is to 21 as 253 is to 1771.[19] In these last two examples, in particular, in which he sought the coefficients of a binomial raised to the 23rd power, Briggs showed how to obtain the third coefficient, 253, from the second, 23, and then the fourth coefficient, 1771, from the third, so that presumably, the reader could continue in this manner to generate all coefficients of this expansion. Of course, the first three examples among these last four showed how to obtain the third coefficient in the expansion from the second – that is, from the integer power – really, the first nontrivial coefficient in the expansion.

How one obtains 253 from 23 is to multiply 23 by ${}^{22}\!/_{2}$, and how one obtains 1771 from 253 is to multiply 253 by ${}^{21}\!/_{3}$. Briggs never wrote a product such as $1771 = 253 \cdot \frac{21}{3}$ or $1771 = 23 \cdot \frac{22}{2} \cdot \frac{21}{3}$, let alone a product containing abstract notation. However, his examples made clear enough how to generate all of the coefficients of a

[16] Briggs, Preface.
[17] Hutton, Tract 21, Logarithms, 392.
[18] Briggs, 20.
[19] Briggs, 22.

binomial expansion for any given positive integer power using multiplication. Hutton emphasized Briggs' having obtained this multiplication formula;[20] however, Briggs himself seemed just as excited to note that, once he had computed 253 and 1771 by multiplication, he could obtain the entries 276 and 2024 by addition: 276 = 253 + 23 and 2024 = 1771 + 253.

5 Johann Faulhaber (1580-1635)

Johann Faulhaber (1580-1635) wrote several books on numerology, arithmetic, algebra, and combinatorics. His first mathematics book, *Arithmetisch Cubiccossischer Lustgarten,* was published in 1604 and his last, *Academia Algebrae,* in 1631. He sometimes is credited with the anonymously authored *Mysterium Arithmeticum* of 1615, on figurate numbers, although Faulhaber scholar Ivo Schneider believes it more likely to have been written by Faulhabers's friend and student, Johann Remmelin.[21] Schneider believes Remmelin also wrote another book generally credited to Faulhaber, *Numerus Figuratus, sive Arithmetica Analytica Arte Mirabili,* a 24-page tract published in 1614,[22] and it is therefore possible that the present section should be titled "Johann Remmelin" rather than "Johann Faulhaber".

At page 11, the author of *Numerus Figuratus* presented tables of triangular, pyramidal, pentagonal, and hexagonal numbers. At page 12, he presented the tables in Figure 5, noting (in words) how proportions may be used to generate each sequence of generalized triangular numbers from the preceding one and even describing some of the required multiplications: "In the second column 6 once with two-thirds sets up 10. In the third 20 once with three-fourths makes 35. In the fourth column 15 once with two-fifths produces 21." [23] Pages 13 and 14 of *Numerus Figuratus* contain a discussion of a number with which Faulhaber and his circle were obsessed, 666. At page 15, before continuing his discussion of figurate numbers, the author cited eight other mathematicians, including Christoff Rudolph, Michael Stifel, Cardano, Simon Stevin, and Peter Roth, in whose works he could have seen tables of figurate numbers. As far as I know, only in Cardano's *De Proportionibus* would Faulhaber or Remmelin have seen a (verbal) multiplicative formula for generalized triangular numbers.

The aforementioned *Mysterium Arithmeticum,* published one year later in 1615, is a 16-page tract (eight sheets) on figurate numbers, which appears to be much more general than *Numerus Figuratus.* It, too, contains a table of generalized triangular numbers with five rows and only four columns and its promised climax is a discussion of the number 666, the "Number of the Beast" ("Numerum Bestiae").[24] This table of triangular numbers is

[20] Hutton, 391-393.
[21] Schneider 2005, 323, or Schneider 1993, 257.
[22] Schneider 1993, 113, 257.
[23] This is my translation of the author's Latin at page 12.
[24] The pages of *Mysterium Arithmeticum* are not numbered. "Numerum Bestiae … 666" appears on the title page, which I have numbered 1. With this numbering, the table of generalized triangular numbers is at page 9.

followed immediately by two two-column charts,[25] the first of which instructs the reader to multiply the "root" by the "root plus 1 divided by 2" to obtain the "square plus 1 root, divided by 2"; then to multiply this expression by the "root plus 2 divided by 3" to obtain the "cube plus 3 squares plus 2 roots, divided by 6"; and so on up to multiplication by the "root plus 5 divided by 6" to obtain the "square-cube plus 15 sursolids plus 85 square-squares plus 22[5] cubes plus 274 squares plus 120 roots, divided by 720. &c." One can check that the author's instructions give, in modern notation for "root" n: $\frac{(n+1)n}{2}=\frac{n^2+n}{2}$,

then $\frac{(n+2)(n+1)n}{3\cdot 2}=\frac{n^3+3n^2+2n}{6}$, and so on, up to

$$\frac{(n+5)(n+4)(n+3)(n+2)(n+1)n}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}=\frac{n^6+15n^5+85n^4+22[5]n^3+274n^2+120n}{720}.$$

(Actually, the multipliers in the lefthand column go up to the "root plus 7 divided by 8" but the products in the righthand column only to the sixth degree expression given.) These are the same formulas that appear in the first two pages of Harriot's manuscript treatise (see Figure 1 and Section 6), but the author of *Mysterium Arithmeticum* described them in words rather than symbols.

The second chart is similar, instructing the reader to multiply the "root, minus 1" by the "root, divided by 2" to obtain the "square minus 1 root, divided by 2"; then to multiply this expression by the "root plus 1 divided by 3" to obtain the "cube plus 0 squares minus 1 root, divided by 6"; and so on up to multiplication by the "root plus 4 divided by 6" to obtain the "square-cube plus 9 sursolids plus 25 square-squares plus 15 cubes minus 26 squares minus 24 roots, divided by 720. &c." One can check that the author's instructions give, in modern notation for "root" n: $\frac{n(n-1)}{2}=\frac{n^2-n}{2}$, then $\frac{(n+1)(n)(n-1)}{3\cdot 2}=\frac{n^3-n}{6}$, and so on, up to $\frac{(n+4)(n+3)(n+2)(n+1)(n)(n-1)}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}=\frac{n^6+9n^5+25n^4+15n^3-26n^2-24n}{720}$. (Actually, the multipliers in the lefthand column go up to the "root plus 5 divided by 7" but the products in the righthand column only to the sixth degree expression given.)

6 Thomas Harriot's symbolic formulas

As noted above, in a manuscript treatise entitled "De Numeris Triangularibus et inde De Progressionibus Arithmeticis: Magisteria Magna,"[26] penned in 1618 or later, Harriot derived symbolic formulas for the generalized triangular numbers and binomial

[25] The charts appear at pages 10 and 11 (my numbering and translation).
[26] BL Add. MS 6782, folios 107-146v, or Beery and Stedall.

coefficients, and also for finite differences and interpolated values based on finite differences. We briefly describe the first four pages of this treatise, which contain tables of and formulas for generalized triangular numbers and binomial coefficients.

As we have seen, at the top of page 1 (see Figure 1[27]) of “De Numeris Triangularibus,” Harriot provided a table of generalized triangular numbers. Note that the third row and column of the table contain the triangular numbers, the fourth row and column the pyramidal numbers, the fifth row and column the triangulo-triangular numbers,[28] and so on. The curved symbols are combined “plus” and “equals” signs, illustrating the additive property of the table. Note, for instance, that, in the lower righthand corner of the table, the “plus/equals” sign helps the reader see that 462 + 462 = 924, or, as Harriot would write this equation, 462 + 462 $=\!\!||\!\!=$ 924. For a better view of Harriot’s “equals” sign, see Figure 2 (page 3 of the treatise). In the table in the middle of the page, Harriot displayed the pattern he would use to write a general symbolic formula for each entry of each column of the table of generalized triangular numbers, noting, for instance, that the entry 462 of the first table may be written as $\frac{7\cdot 8\cdot 9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 5}$. Finally, at the bottom of the page, Harriot wrote a symbolic formula for the nth number in each column, n a positive integer. (He did not index the columns, just the rows.) His formula, for instance, for the nth entry of the fourth column – his pyramidal number formula – is

$$\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}.$$

As far as I know, Harriot was the first mathematician to write symbolic formulas for the generalized triangular numbers. (Such formulas were first published by John Wallis in his 1656 *Arithmetica Infinitorum.*) Harriot’s notations “&c” (“et cetera”) indicated that his tables and formulas could be extended indefinitely downward and/or to the right.

On page 2 of his treatise,[29] Harriot expanded his generalized triangular number formulas from page 1, noting, for instance, that $\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}=\frac{nnn+3nn+2n}{6}$. In an intermediate step, Harriot collected like terms in vertical columns. Note that he did not use exponents, writing, for instance, *nnn* rather than n^3. Notice also that this symbolic equation was described verbally, but quite explicitly, by the author of *Mysterium Arithmeticum.*

At the top of page 3 of his treatise (see Figure 2[30]), Harriot shifted his table of generalized triangular numbers to obtain a table in which the rows contain binomial coefficients. He summed each row of his table of binomial coefficients, almost certainly noticing that the sum of each row is double that of the preceding row, or, more specifically, that the rows sum to successive powers of 2. In the table in the middle of the

[27] Figure 1 is a transcription of BL Add. MS 6782, f. 108.
[28] See footnote 1.
[29] BL Add. MS 6782, f. 109.
[30] Figure 2 was transcribed from BL Add. MS 6782, f. 110.

page, Harriot wrote each entry of the first table as a quotient of products, revealing the pattern he would use to write a general symbolic formula for each entry of each column. Finally, at the bottom of the page, Harriot wrote a general symbolic formula for the *n*th entry in each column. His formula for fourth column entries, for instance, is $\frac{(n-2)(n-1)n}{1\cdot 2\cdot 3}$, where, to obtain the four entries shown in the fourth column of each of the tables above, one should set n = 3, 4, 5, and 6. Again, as far as I know, Harriot was the first mathematician to write symbolic formulas for the binomial coefficients. Note again Harriot's use of "&c" ("et cetera") to indicate that his tables, sums, and formulas could be extended indefinitely. On page 4,[31] Harriot expanded his binomial coefficient formulas from page 3, just as he had done on page 2 for his generalized triangular number formulas from page 1.

7 Conclusion

To our modern mathematical eyes, with their appreciation of concise, symbolic formulas, Harriot's formulas for generalized triangular numbers and binomial coefficients probably seem superior to the others we have considered and may in fact be the only ones we are willing to deem "formulas". Yet, the "formulas" of Cardano, Briggs, and Faulhaber-Remmelin, presented in terms of numerical examples and/or verbally, convey what one needs to know in order to compute any generalized triangular number or binomial coefficient and are therefore general, even if neither as concise nor as symbolic as we would like.

With more time (and languages), one might study the work of the earlier mathematicians mentioned in the introduction to this essay. Looking forward just a few years, the theme of multiplicative formulas for generalized triangular numbers and binomial coefficients would recur in the 1636 works and correspondence of Mersenne and Fermat, in the 1656 *Arithmetica Infinitorum* of John Wallis, and of course in Pascal's 1665 *Traite du triangle arithmetique.*

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[31] BL Add. MS 6782, f. 111.

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