AXIOMS IN SEARCH OF A DEFINITION

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ABSTRACT

The fundamental difference between the modern axiomatic method, enunciated by Hilbert, and the ancient, as practiced in Euclid's *Elements*, lies with the role of the basic definitions. In Hilbert's *Grundlagen* the set of axioms constitutes an implicit definition of the, otherwise undefined and philosophically neutral, basic concepts; in the *Elements*, the postulates, as we argue on the basis of the anthyphairetic interpretation of the Platonic Beings, have only an empirical and heuristic role, and are in need and in search of the suitable definitions, whose origin and cause is in the upper realm of Platonic ideas, capable of generating these postulates below. The restored role of the ancient postulates reveals them, not as the finished products presently conceived and widely criticized as antipedagogical, but, on the contrary, as partial empirical constructs, ideal for turning the students into small researchers in search of a definition.

1. The basic definitions in Hilbert's axiomatization are only implicit

For many centuries Euclid's *Elements* was considered as a perfect model of mathematical rigor. Since at least the 16^{th} century the inadequacy and incompleteness of Euclid's postulates to generate the proofs of all propositions in the *Elements*, and the need for additional axioms, was gradually becoming apparent (for congruence (J. Peletier) continuity (Leibniz), betweenness (Gauss)). The proof of the existence of non-Euclidean geometries provided the final impetus for the distancing of modern Mathematics from the ancient *Elements*.

A complete list of axioms for geometry was provided by Hilbert (1899). Hartshorne (2000a, p. 464) succinctly describes Hilbert axiomatization of Geometry as follows: 'Hilbert's axioms for plane geometry postulates a set of points and a set of subsets called lines, a notion of betweenness, and undefined relations of congruence for line segments and for angles. The axioms of *incidence* require that two distinct points lie on a unique line, plus conditions of nontriviality. The axioms of *betweenness* govern the relation that a point B lies between points A and C. The axioms of *congruence* include, among others, that it is possible to lay off a segment congruent to a given segment on a given line; that it is possible to lay off an angle congruent to a given angle at a given point on a line; and the side-angle-side (SAS) criterion for congruence of triangles. These are the basic axioms of a *Hilbert plane*. For Euclidean geometry one also needs the *parallel axiom*, that there is at most one line parallel to a given line through a given point; and the *circle-circle intersection axiom*, that if a circle has a point inside and a point outside a given circle, then the two circles meet in two points.'

These modern developments have led to increasing criticism of the ancient axiomatics. The incompleteness of the Euclidean postulates, well described by Russell (1902), is incontestable and well-founded. However the addition of axioms, as needed, is no more than a corrective step, intended to achieve in an improved manner the ancient ideal and in no way signifies a fundamental change of direction and philosophy for mathematics, in relation to the *Elements*.

There is however, as indicated in Hartshorne's description, a radically new element in Hilbert's axiomatization, in that the basic concepts, such as points, lines (and planes), in opposition to the practice in the *Elements*, are NOT defined in Hilbert's *Grundlagen*. Hilbert expressed this in a rather provocative way by stating that a proper axiomatisation of geometry must be equally applicable to tables, chairs and steins as to points, lines and planes. In this modern approach to axiomatization, which gradually evolved from Grassman (1844) and Pasch (1882) to Hilbert, the fundamental undefined objects and concepts are **implicitly defined** in terms of the properties of these objects and concepts as expressed by the axioms. Later Dieudonné (1964) wrote that 'mathematical objects are to be considered as completely defined by the axioms which are used in the theory of these objects.'

2. The dynamic interaction between definitions and postulates-hypotheses in Aristotle's axiomatization

Kline (1972) thought that Aristotle's approach to the definitions of the basic concepts in an axiomatic system was in agreement with the modern one, and not as in the *Elements*. But this is clearly a misunderstanding on his part, set straight by McKirahan (1992) and Barnes (1993) ('Aristotle explicitly says that we assume the meaning *both* of the dependent terms *and* of the primitives—i.e. in both cases we assume their definitions.'). (cf. Ciolek (2005)). Thus in the matter of definitions of the fundamental concepts there is a genuine difference between the modern and the ancient approach. How are to account for this difference?

Most researchers assume that the ancient definitions play a secondary and dispensable role. Thus Shapiro (1997) states that 'these definitions **play no role** in the subsequent mathematical development' and 'the modern reader may wonder why Euclid included them', and Sklar (1974) calls them '**mathematically useless**'.

But this view does not take into account the interaction that ancient mathematicians and philosophers expected between definitions and hypotheses-postulates. This interaction is clearly expressed by Aristotle in *Analytics Posterior* 76b23-34, according to which the hypotheses and postulates are provable, but we have no proofs of them only because of the imperfect state in which the axiomatized science is in, combined with *Topics* 158b-159a, according to which the unproved hypotheses (and postulates) receive a proof when a good definition is discovered.

Aristotle mentions an instructive historical example of how a good definition can provide proof of a statement hitherto considered as a hypothesis without proof, i.e. as a postulate, as follows:

in the early (quite possibly Pythagorean) era there was no good definition of proportion for magnitudes, and in consequence the statement (referred to, following Fowler (1999), as the *Topics* proposition)

'if a, b, and c are three line segments, then a/b=ac/bc,

could have no rigorous mathematical proof, and was thus accepted as a hypothesis without a proof, namely as a postulate; but when a good definition of proportion was given (presumably by Theodorus and Theaetetus), the statement acquired a simple and natural proof and was turned into a proved proposition. The good definition of proportion (2.2) depended on the

2.1. Definition of anthyphairesis.

For two magnitudes (line segments, areas, volumes) a, b be, with a>b; the **anthyphairesis** of a to b is the following, infinite or finite, sequence of mutual divisions, as defined in Book X of the *Elements* (where e_n is a magnitude and I_n is a natural number for every n):

$a \ = I_0\, b+ e_1$, with $b>e_1$,
$b \ = I_1 e_1 +e_2$, with $e_1>e_2$,
…
$e_{n-1}= I_n\, e_n+ e_{n+1}$, with $e_n >e_{n+1}$,
$e_n \ = I_{n+1}\, e_{n+1}+ e_{n+2}$, with $e_{n+1}>e_{n+1}$,
…

The sequence of successive quotients of the anthyphairesis of a to b is the sequence **Anth(a,b)** = $[I_0,I_1,\ldots,I_n,I_{n+1},\ldots]$.

2.2. The anthyphairetic definition of proportion of magnitudes.

The first rigorous definition of proportion, Aristotle tells us, was the following: if a,b is a pair of homogeneous magnitudes, with a>b, and A, B is another pair of homogeneous magnitudes, with A>B, then we say that the proportion a/b=A/B holds if the sequence Anth(a,b) of successive remainders of the anthyphairesis of a to b is equal to the sequence Anth(A,B0 of successive remainders of the anthyphairesis of A to B.

2.3. The dynamical relation between definitions and postulates in ancient axiomatics.

Thus the role of a good definition, according to the ancient view, lies in its power to turn hypotheses and postulates, hitherto unproved but provable, into proved propositions, and hence the presence of axioms and postulates indicates an incomplete state of the subject under study.

We will be able to better appreciate the example given by Aristotle, if we gradually translate it into a practically equivalent modern one. Towards that purpose let us imagine a modification of Aristotle’s example: instead of employing the historically first rigorous definition of proportion of magnitudes, let us instead employ the final one, namely the one due to Eudoxus and exposed in Book V of the *Elements*. According to the celebrated Eudoxian definition 5 in Book V, the proportion a/b=A/B holds if for all pairs m,n of natural numbers

ma>nb if and only if mA>nB,
ma=nb if and only if mA=nB, and
ma<nb if and only if mA<nB.

Now Aristotle could, equally well, have explained his point on the power of a good definition, by replacing the anthyphairetic definition of proportion by the Eudoxian one (save that it would be an anachronism). The simple proof of the *Topica* proposition, employing the Eudoxian definition, is essentially given in Proposition VI.1 of the *Elements*.

The Eudoxian definition of proportion of magnitudes, as it was well known probably in antiquity but certainly by the Arabs (cf. Vahabzadeh (2002)), is known to be essentially equivalent to the definition of the (positive) real numbers in terms of Dedekind cuts on the set of (positive) rationals (cf. Heath (1926). Thus an equivalence class of a ratio of magnitudes, under the equivalence relation of Eudoxian proportion is into a 1-1 order preserving correspondence with the Dedekinds cuts on the set of positive rationals.

It follows that we can conceive in modern mathematics of an example quite similar to the one given by Aristotle. We can start by considering an axiomatic treatment of the real numbers. This can be achieved, e.g. by the list of properties P1-13, listed in Spivak's text on Calculus (1967), Chapters 1 and 8, on addition, multiplication, order, and order-completeness. However, in Chapter 28 of Spivak's text, the real numbers are defined as Dedekind cuts of rational numbers; with this good (in the sense of Aristotle) definition, all the unproved axioms P1-13, implicitly defining the set of real numbers, become proved propositions.

It is clear that in this case the implicit definition of real numbers in terms of a list of axioms, which is certainly in accordance with the spirit of Hilbert axiomatics, is inferior to the explicit definition of real numbers, in terms of Dedekind cuts (or equivalently in terms of continued fractions, the modern analogue of anthyphairesis). Thus the ancient approach to axiomatics, as described by Aristotle, clearly points to a role of the definition superior to that of the list of axioms.

3. Anthyphairesis in Greek mathematics

We next wish to examine if the same interaction between the definitions and the postulates, as the one described by Aristotle, is at works in Euclid's *Elements*. This examination will turn out to depend crucially on Plato's dialectics, and on the interpretation of Plato's dialectics, in terms of the concept of anthyphairesis, as developed by Negrepontis in (2000), (preprint c). In the section 3, we will briefly review the mathematical concept of anthyphairesis (defined in 2.1 above), developed by the Pythagoreans, Theodorus, and the geometers, principally Theaetetus, in Plato's Academy, and presented, albeit in highly incomplete manner, in Books VII and X of Euclid's *Elements*, while in Section 4 we will give a brief account of the relevance of anthyphairesis to Plato's dialectics

3.1. Definition (Definitions X.1, 2 of the *Elements*).

Let a, b be two magnitudes with a>b; we say that a, b are **commensurable** if there are a magnitude c and numbers n, m, such that a=mc, b=nc, otherwise a, b are **incommensurable**.

The fundamental dichotomy for anthyphairesis is contained in the following

3.2. Proposition (Propositions X.2, 3 of the *Elements*).

Let a, b be two magnitudes, with a>b. Then a,b are incommensurable if and only if the anthyphairesis of a to b is infinite.

An immediate consequence of the anthyphairetic definition of proportion (2.2) is the following

3.3. Proposition ("the logos criterion" for the periodicity of anthyphairesis").

The anthyphairesis of two line segments a, b, with a>b, with notation as in the definition and setting $a=e_{-1}$, $b=e_0$, is **eventually periodic**, with period from step n to step m-1, if there are indices n, m, with n<m, such that $e_n/e_{n+1}=e_m/e_{m+1}$.

3.4. Reconstruction of proof of quadratic incommensurabilities by the Logos.

There are good arguments, not to be given here, that the proofs of incommensurabilities given by Theodorus, reported in Plato's *Theaetetus* 147d3-148b2, of square roots of 3,5,…, up to 17, are anthyphairetic, and employ the Logos Criterion (3.5). Anthyphairetic reconstructions, employing the Logos Criterion, have been proposed by Zeuthen (1910), van der Waerden (1954), Fowler (1999), Kahane (1985), a non-anthyphairetic one by Knorr (1975). We outline, in Table 1 below, a reconstruction of the proof of the incommensurability of the line segments a, b, with $a^2=19b^2$, the first one that Theodorus refrain from giving (abbreviated in the sense that we have omitted the even numbered steps):

Table 1. Anthyphairetic Division and Logos Criterion for $a^2=19b^2$

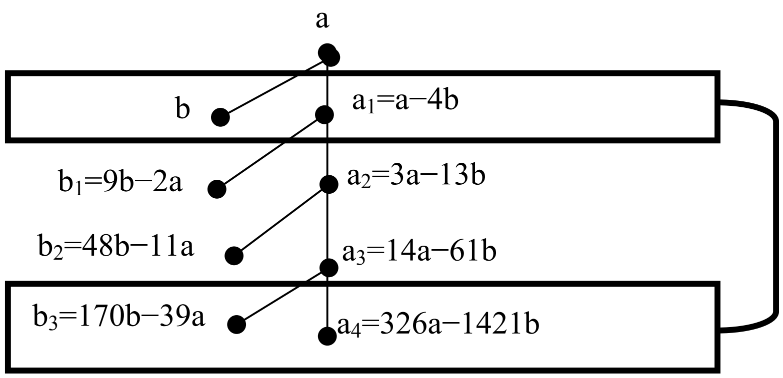


Table 1 is to be understood as follows: we first proceed with the steps of the anthyphairetic **Division** of a by b, employing elementary computations and expressing at the same time the remainders generated in terms of the initial line segments a and b:

$a=4b+a_1$, with $a_1<b$ (hence $a_1=a-4b$), (and $b=2a_1+b_1$, $b_1<a_1$ (hence $b_1=9b-2a$)),

$a_1=b_1+a_2$, $a_2<b_1$ (hence $a_2=3a-13b$), (and $b_1=3a_2+b_2$, $b_2<a_2$ (hence $b_2=48b-11a$)),

$a_2=b_2+a_3$, $a_3<b_2$ (hence $a_3=14a-61b$), (and $b_2=2a_3+b_3$, $b_3<a_3$ (hence $b_3=170b-39a$)),

$a_3=8b_3+a_4$, $a_4<b_3$ (hence $a_4=326a-1421b$); and

we next verify the **Logos Criterion** (indicated in the Table by the coupling of the two expressions in the rectangles), employing the expressions found for the remainders:

$b/a_1=b_3/a_4$.

It follows that, after the initial ratio a/b, the sequence of successive Logoi

b/a_1, a_1/b_1, b_1/a_2, a_2/b_2, b_2/a_3, a_3/b_3,

forms a complete period of Logoi, repeated ad infinitum, and provides **full knowledge** of the initial ratio a/b, i.e. of the quadratic irrational square root of 19, and proving incidentally, the incommensurability of the ratio a/b.

According to the interpretation by Negrepontis (preprint (b)) in the *Theaetetus* 147d3-148b2 passage, the mathematical importance of this condition was realised by Theaetetus who proved the fundamental theorem that for a pair a>b commensurability in power only implies (eventual) periodicity of the anthyphairesis of a to b.

4. The anthyphairetic interpretation of Plato's dialectics

For the purpose of self-containment, we provide a brief account of Plato's dialectics and its anthyphairetic interpretation, given by Negrepontis (2000,2005, preprints (b), (c)).

4.1. A Platonic Being is the mixture of the Infinite and the Finite.

According to the *Philebus* 16c9-10, 23c12-d1, a Platonic Being is the mixture of the two principles of Infinite (=Unlimited) and Finite (=Limited), described in the *Philebus* 23-25?, and in consequence has the nature of intelligible one and many, equivalently, of Division (of the one into many) and Collection (of the many into one), a method described in the *Sophistes* and the *Politicus*.

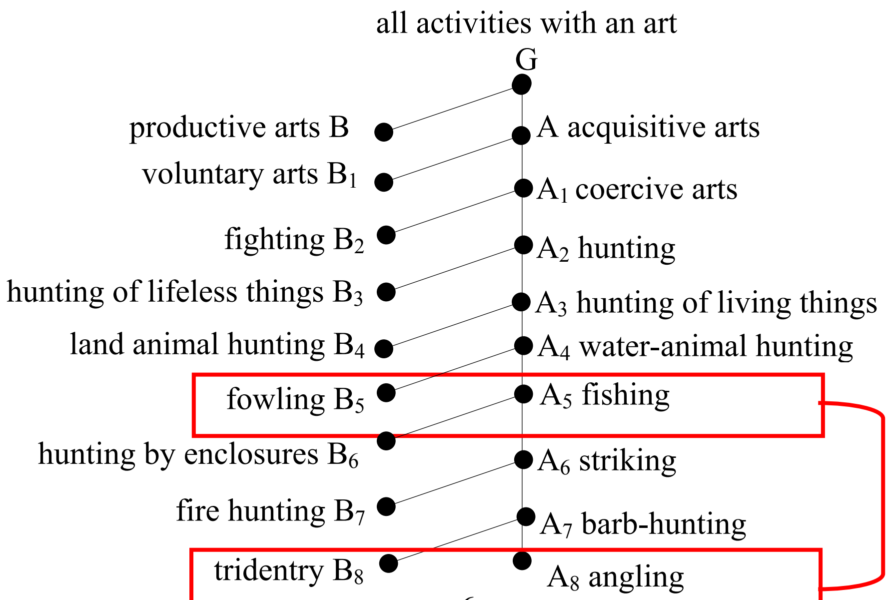
According to the anthyphairetic interpretation, given in Negrepontis (2005), the *Philebus* 23e3-25e3 passage imitates the anthyphairetic dichotomy, and in fact the Philebean principle of the Infinite is a philosophical version of infinite anthyphairesis, while the opposite Phlebean principle of the Finite is a philosophical version of finite anthyphairesis.

As revealed in the Double Measurement passage, *Politicus* 283b6-287b3, a Platonic Being is the mixture of the Infinite and the Finite, in philosophical analogy to the sense in which a pair of lines a >b commensurable in power only is the mixture of incommensurability-infinity of a to b and of commensurability-finiteness of a^2 to b^2.

4.2. Platonic Beings have the power of Division and Collection.

Periodic anthyphairesis and the Logos Criterion has been shown by one of the authors to be at the center of Plato's dialectics (Negrepontis (2000), (2005), preprints (b), (c)). The simplest way to see this is to correlate anthyphairesis with the Platonic Division and Collection, a method, by which Platonic Beings become known to the human soul, described in the Platonic dialogues *Sophistes*, *Politicus*, *Phaedrus*, *Philebus*; and the simplest way to grasp the close connection between Division and Collection and periodic anthyphairesis is to examine the examples of this method provided by Plato in the *Sophistes*. For lack of space, we restrict attention to the Division and Collection of the Angler, given in the *Sophistes* 218b-221c, and summarized in Table 2, below:

Table 2. Division and Collection for the Angler



The Division, thus, starts with the Genus G, and this is divided into two species B and A, of which A is clearly the one containing the Angler. In the next step B remains undivided, but species A is turned into a Genus and is divided again into species B_1 and A_1. After a number of such binary division steps we arrive at the species A_8, the Angler. So far we have only performed Division, obtaining the Name ('Onoma') of the Angler. We maintain that this division process is but a philosophical version of the anthyphairetic division, as in Section 3 and Table 1, for $a^2=19b^2$. There is, additionally, need for the philosophic analogue of the Logos Criterion, what Plato calls Logos or Collection, described in the *Sophistes* 220e3, 221a2, 221b5, 221b7 and summarised as follows:
tridentry B_8/ angling A_8 =
from above downward barb-hunting/from below upwards barb-hunting,
fowling B_5/ fishing A_5 =
from above downward water-animal hunting/from below upwards water-animal hunting,
so that
tridentry B_8/ angling A_8 = fowling B_5/fishing A_5.

In Table 2 the Logos-Collection $B_5/A_5=B_8/A_8$ is indicated by the coupling of the two expressions in the rectangles. We see that the Platonic Logos-Collection is the philosophic version of the Logos Criterion for anthyphairetic periodicity, as in Section 3.

4.3. In conclusion **a Platonic Being** is a **mixture of Infinite and Finite**, namely the philosophic version of incommensurable and commensurable, namely of commensurable in power only ratio, and, hence in analogy to Theaetetus' deep theorem, it exhibits **Division and Collection**, namely a philosophical version of periodic anthyphairesis. It is then easily seen to be a self-similar entity, so that although it is divided into an infinite multitude of parts (Division into many), is nevertheless **partless** and possessing Unity, in the sense that every part is similar to the whole (Collection into one).

5. The interpretation of the basic definitions in Euclid's *Elements* in terms of Plato's dialectics.

We are now in position to obtain an understanding of the first crucial feature that the basic definitions have in the axiomatic foundation of Euclid's *Elements*.

5.1. The interpretation of the definition of the geometric point in the *Elements* as a Platonic Being.

Proclus' comments on the definition 1 of Book I of the *Elements* ('a point is what has no parts ('meros outhen')') occupies the passage 85,1-96,15 of his *Commentary to Euclid*. Proclus makes clear that **the geometric point** possesses the precise character of a **Platonic Being,** as described in Section 4, namely a **mixture of** the two principles **Infinite ('apeiron') and Finite ('peras')** (as stated in 88, 2-7 and 88, 17-22), and possesses (and is made known to the human soul by) **Division and Collection** (as stated in 88, 2-7; 88, 17-22; 89, 10-14; and, 90, 3-6).

It might strike us as artificial, or even downright false, that the geometric point possesses such an intricate structure. However, partlessness, the defining characteristic of the point is also the defining characteristic of a Platonic Being, as explained in 4.3.

According to Aristotle (*Metaphysics* 992a19-22) Plato had rejected the 'geometric point', as the fundamental concept for geometry, considering it simply a doctrine of the geometers, opting instead for the 'indivisible line', a concept that essentially coincides with the One of the second hypothesis in the *Parmenides*. Euclid, although a Platonist, in an evident compromise to the practicing geometers, reintroduced the 'geometric point' but, defining it as 'partless', the emblematic description for a Platonic Being, endowed it, at least according to Proclus, with such a structure.

5.2. The interpretation of the definitions of the straight line and the circle in the *Elements,* as the Division and Collection, respectively, of the Platonic Being geometric point.

The Euclidean definition of a straight line, as 'a line that lies equally with the points on itself' has proved intractable. The older Platonic definition (*Parmenides* 137e) 'straight is whatever has the middle in front ('epiprosthen') of it (i.e. so placed as to obstruct the view of) both its ends', is, according to Heath (1926) 'ingenuous, but implicitly appeals to the sense of sight and involves the postulate that the line of sight is straight'. Heath believes that the Euclidean definition is 'simply an attempt…to express…the same thing as the Platonic definition', but be 'independent of any implied appeal in vision, which, as a physical fact, could not properly find a place in a purely geometric definition' (vol. I, pp. 166, 168); and Proclus' association of the Platonic definition, with the eclipse of the sun (109,25-110,4) has strengthened the belief that the Platonic definition indeed has refers to the sensibles.

However, the thought that Euclid would be more intelligible-minded than Plato himself is preposterous; Proclus at the same passage provides the true intelligible meaning of the Platonic definition of the straight line, as opposed to the circle: 'Perhaps this property of the straight line affords a proof that in the realm of Being, in the process **from** the causes the middle elements become divisive of the existence and of the communication of the [two] extremes to each other, just as in the process of returns [**to** the causes] the elements that have become separate from themselves unite with the initial causes.' (109, 21-110,10). Thus the straight line is the line in which the middle obstructs the 'communication', namely the equalization, of the extremes, something achieved only by the periodic nature of the circle, where the middle unites the extremes.

This suggests that **the straight line** is **the Division**, while **the circle** is **the Collection**, of the Platonic Being 'Geometric Point'. This interpretation of straight line and circle is supported by the fact that the One, in the second hypothesis in the *Parmenides* 145b3-5, participates in the straight and in the cyclic, and by numerous comments in Proclus', *Commentary to Euclid* (104,11-14; 107,11-16; 107,20-108,2; 146,24-147,25; 154,6-24). In particular, the Euclidean definitions of straight line and circle, far from having an empirical content, as erroneously considered by Heath and others, have indeed a philosophic and intelligible one.

Heath is right, however, that the Euclidean definition of straight line is closely connected to the Platonic one. This is made clear in the Philoponus, *Commentary to Categories* 151,1-5 passage, where it is suggested that the relation between the two definitions is the following: the straight line 'lies equally with the points on itself' as in Euclid, and, in consequence, it has 'the middle in front of both its ends', as in Plato.

We note in passing that Euclidean **Arithmetic** (Book VII) is at a state of development more perfect than that if Euclidean Geometry, since no postulates are needed, but only the basic definitions of One (defined as the One that is suitable for Platonic Beings) and of Number. It should be noted that the One of the second hypothesis in the *Parmenides* generates not only Geometry, but (eidetic) numbers (143c1-144e7, 148d5-149d7).

6. The generation of the Euclidean postulates from the basic definitions

We are now in position to obtain an understanding of the second crucial feature that the basic definitions have in the axiomatic foundation of Euclid's *Elements*.

6.1. The generation of the first three Euclidean postulates from the flowing of the point.

Proclus derives the first three postulates from the flowing ('rhusis'), i.e. the motion, of a geometric point. But how is it possible for a partless entity to move? Proclus carefully explains (97,11; 185,25-187,3) that this is an immaterial ('aulos'), non-bodily, intelligbe motion. (The textual 'correction' from 'immaterial' to 'material' ('enulos'), suggested by von Eecke and by Morrow (1992, p. 79, footnote 15), is clearly a misunderstanding). This motion, according to the anthyphairetic interpretation given in Negrepontis (2005, preprint b), is the infinite anthyphairetic Division of the indefinite dyad One, Being, in the second hypothesis in the *Parmenides*.

The first postulate is then derived from the definitions of point and straight line as follows: it is the motion-division process from one point (namely a 'logos' of two successive parts) to another point (namely another 'logos' of two other successive parts); the second postulate follows in the same natural way; the third postulate, the drawing of the circle, follows from the definition of the circle as the Collection of the point (185,8-25).

Thus the basic definitions of the point, straight line and cycle in the *Elements* are by no means useless nor of sensible or physical origin, on the contrary they are intermediate between philosophy and mathematics, namely of intelligible, Platonic origin, and indispensable in producing the first three postulates.

6.2. The derivation of the fourth and the attempted derivation of the fifth postulate.

Lack of space does not allow us to complete the derivation of the remaining two postulates from corresponding definitions.

The fourth postulate is related to the interpretation of the dyad (obtuse vs. acute angle) as an instance of Infinite, while the right angle corresponds to the principle of the Finite. The Platonic Being that represents this particular mixture is none other than the 'logos' diameter to the side of a square, with the infinite dyad determined by the side and diametr numbers, and the right angle expressing the 'logos criterion'. For the the anthyphairetic interpretation we rely on the *Politeia* 510d5-e1, and Proclus' *Commentary* 131,3-134,7 και 191,1-15 (cf. Negrepontis, preprint (a))

The stubborn ancient belief in the provability of the Fifth Postulate (191,16-193,9), a postulate of Pythagorean origin, appeared to rest on the general conviction that all Geometry was under the umbrella of Platonic philosophy. More specifically, Proposition I.30, shows that 'parallelism' is 'similarity of position' (373,5-23) and that the Fifth

Postulate is for parallelism what the Enallax property is for analogy (357,9-16). The attempted proof of the Fifth Postulate by Ptolemy (365,5-368,26) relies on some kind of alternation, reminiscent of the Enallax property, while Proclus' attempted proof, by an appeal to Aristotle's philosophic principle of the finite, proceeds in the same way that the Enallax property for magnitudes is proved in Book V (Proposition V.16), by an appeal to the Eudoxian principle (Definition V. 4 and Proposition V.8).

6.3. The subsumption of all Mathematics under Platonic philosophy.

With the arguments outlined in Sections 5 and 6 the definitions and postulates of Euclidean Geometry, and, in fact, of Arithmetic, are derived from Plato's dialectics. All other Propositions in the *Elements* follow from these postulates by mathematical proof; but, according to Plato's dialectics, every mathematical proof is the Synthesis of a corresponding Analysis, and every Analysis is, according to Negrepontis-Lamprinidis (2007), the Analysis associated with a Division and Collection. Thus all Mathematics are subsumed under Plato's dialectics.

7. The relevance of the ancient approach to axiomatics in modern education

These arguments reveal that an axiomatized system according to the ancient mathematics and philosophy, contrary to the modern notion that regards the axioms as a finished product, is in a defective and empirical state that need be completed, as long as basic definitions, with the power to generate these axioms, are missing.

This fundamentally different role of the modern axioms vs. the ancient postulates has significant relevance to the teaching of Mathematics. Modern educators, such as Freudenthal (1973), Fischbein (1980), have rejected the axiomatic approach, and more generally the structuralist approach, typified by Bourbaki, as a teaching method, precisely because they believe that the student must become a small researcher and find things for him/her self, while the axiomatic method presents only the finished product.

This criticism is valid for Hilbert and the structuralists, who, by rejecting the definitions for the basic concepts, regards the axioms, not as heuristic tools in search a definition, but as the final and perfect product, an end in themselves. Were it true that the Euclidean definitions of the basic concepts were empirical, useless and generally negligible, as widely considered by historians, structuralists (such as Bourbaki (1950), Dieudonne (1964), Thom (1971, 1973)), education reformers and counter-reformers, alike, this criticism would be equally valid for the axiomatics of Euclid; but, as we have shown, these definitions are intelligible and crucial for producing the empirical, imperfect and heuristic only postulates. This relegation of the **axioms** to a lower ontological status makes them ideal to serve as the initial objects **in search of a definition,** thus turning the students into small researchers and possibly discoverers; and we are not thinking of axioms on a grand scale, for, say, Geometry or Set Theory, but of 'local' ones, searching for the definition of the concept of, say, numerical ratio, or real number, or function, or Lebesgue measure, from a list of experimentally observed properties, that we agree should be satisfied.

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