Carl Runge : A Professor of Applied Mathematics at Georg-August Universität, Göttingen

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Abstract

Carl Runge (1856-1927) published major works on numerical methods during the years of 1895, 1901, 1903, 1905, 1908, respectively, on numerical solutions of ordinary differential equations, on the Lagrangian interpolation, on the Fast Fourier Transform, the trigonometric interpolation, and lastly on the numerical solution of partial differential equations by finite differencing. All these achievements had a definite major impact on the European mathematical community. Runge obtained a position at Hanover University in 1886, where he remained for 18 years. Then, in 1904, Felix Klein offered him the first chair of applied mathematics at the prestigious Göttingen University. Felix Klein's objective was to diversify his department. Therefore, in this present paper, we examine the impact of Runge's work on applied mathematics, his intuitionist and constructive philosophy.

I. Introduction: Carl Runge and Felix Klein

Some twenty years ago, Gotfried Richenhagen (1985) published his thesis on Carl Runge (1856-1927), V*on der reinen Mathematik zur Numerik* (From pure Mathematics to Numerics). The book focused on Runge as a young man, and a young professor. Our objective is neither to duplicate this work nor even to try to recreate the atmosphere of mathematical "*Aufklärung*", at Gottingen University, at the beginning of the XX *th* century. We want to situate Runge as a researcher, a professor in applied mathematics who tried to find approximate solutions to difficult mathematical problems, to present his philosophy and the legacy of his work. Already, in 1755, Leonhard Euler, in his famous article of fluid mechanics commented with nostalgia: "*L'analytique n'est pas assez cultivée*". His mathematics were powerless, too limited for analytical solutions, faced with the complexity of fluid equations! The split between pure mathematicians and applied mathematicians or scientists was non-existent during the XVIII *th* century and even during the XIX *th* century. Göttingen University was the prestigious university where Gauss, Dirichlet, Riemann had worked; where Carl Friedrich Gauss also had his magnetic laboratory, and where he found the first mathematical model for the Earth's magnetic field. Gauss was a symbol; his flexibility was a source of inspiration, a model that Felix Klein (1849-1925) was dreaming for Gôttingen. In 1904, Felix Klein, the leading mathematician, created for Runge, the first chair in applied mathematics, a position he held until he retired in 1925. Klein wanted to diversify his department, and we should admire his perspicacity in doing so. Runge's method for numerical solutions of ordinary differential equations was published in Klein's journal in 1895, and must certainly have had attracted Klein's attention. Klein moved to Göttingen in 1886, and his university became a model for mathematical research. This period has been well presented and studied by historians (Struik, 1987**;** Birkoff and Bennett, 1988; Rowe, 1989; Schubring, 1989; Tobies, 1989). At that time, discussions about pure and applied mathematics were extremely rich and intense. Klein's continuous interests for applied science were well known. Also, in 1875, in his earlier days, Klein had accepted a position at the *Technische Hochschule* in Munich, and Carl Runge attended his classes. Let us recall this period around 1900 at Göttingen University. After David Hilbert, in 1902, his friend Minkowski also moved to Göttingen. According to Rowe: "*These appointments* (Minkowski, Runge, Prandtl, Schwarschild, Wiechert) *were part of the policy for establishing disciplinary "Schwerpunkte" within the Prussian university system. Yet even the ministry itself seems to have been taken aback by Göttingen's success."*

Finally, in 1949, his daughter Iris wrote Runge's biography (Runge, I., 1949) and recreated his intense scientific life. She published letters, his life in Berlin, in Hanover or Göttingen, and the period of the First World War. In this present work, we only focus on Runge's scientific contributions in numerical methods.

II. Carl Runge before Göttingen Universität

In 1877, Runge went to Berlin where he did his doctoral dissertation under Weiertrass. From the courses he taught in Berlin, we can see that his interests were not only in pure mathematics, but he already lectured on numerical solutions of equations (1883/1884), numerical calculus (1884/85), and the solutions of partial differential equations, etc. Then, in 1886, Runge obtained a position as a professor of mathematics at the *Technical Institute Hanover*, where he remained for 18 years. From his research at Hanover, we can ask ourselves if we should classify Runge as a physicist or an applied mathematician. In fact, he remained as both! Among numerous papers in experimental physics, Runge published a few important articles in numerical methods: on the numerical solution of differential equations (1895), on the least squares in 1897, on the lagrangian interpolation (1901), the harmonic analysis, and the summation of trigonometric polynomials (1903). Also, Ludwig Prandtl was professor of mechanics at the same *Technical Institute of Hanover*. Did Prandtl influence Runge's research on differential equations? Moreover, he had also moved to Göttingen in 1904. In his 1895 article, Runge observed that the numerical solutions of ordinary differential equations, which do not possess an analytical solution, seemed to have eluded mathematicians. The popular numerical method known as the forward Euler method was first exposed in *Introductio*. Then, Runge proposed a generalisation of the Simpson rule as a more accurate method. Although, other methods of mechanical integration may had induced useful generalisations. The next steps were due to Heun in 1900, who proposed a generalisation of the gaussian quadrature, and Kutta (1867-1944) in 1901 who merged Runge and Heun's ideas. The Runge-Kutta method remained one of the most popular methods in solving ordinary differential equations (Chabert et al., 1993, pp. 425-48; Tournès, 2003). Kutta became himself a professor of applied mathematics.

Then, in 1901, Runge published his major article on the lagrangian interpolation. Most of his articles on applied mathematics were to be published in *Zeitschrift für Mathematik und Physik*, where Runge became a co-editor, and, under Runge's impulsion, this journal became a strong periodical in applied mathematics. For generations, the lagrangian interpolation formula was a corner stone in the theory of interpolation. The formula seemed "perfect". It corresponded to a "global" interpolation process, where all the ordinates are taken into consideration, just like if we tried to represent all the variations of Nature into a single formula. Runge studied the application $f(x) = \frac{1}{1+x^2}; -5 \leq x \leq +5$. He observed that the Lagrange interpolation worked well for $|x| \leq 3.63$, but became worse outside of this limited interval, as the number of equally spaced nodes was increasing. This counter-example was a stimulating blow in the interpolation theory. It illustrated a great danger of the global interpolation polynomial. Runge's explanations were based on the fact that a lagrangian polynomial is a truncated Taylor series, which diverges outside of its radius of convergence. Runge's article is one of the most refereed article in the literature on numerical interpolation. Runge's results were a major blow in the interpolation theory, and his results were immediately presented, commented and acknowledged in the mathematical community. In France, they appeared in Borel's *Leçons sur les fonctions de variables réelles*, and in the 1905 H. Lesbegue's *Leçons sur les séries trigonométriques.* This Runge article was a clear demonstration and therefore legitimazing Klein in that applied mathematicians could help pure mathematicians as a source of new fields or branches in mathematics, and vice-versa.

The next important Runge's contribution in applied mathematics will be on the Euler-Fourier series. Since Euler, Bernoulli, Clairaut, Lagrange, Fourier and Gauss' earlier works, the trigonometric functions basis was incredibly well adapted to the solutions of many problems of mathematical physics, or the study of periodic phenomena or the trigonometric interpolations. These types of problems occurred in Astronomy with Planet motions, or in Geophysics, in the trigonometric interpolation, or the trigonometric least square approximation. In 1904, H. Burkhardt completed his monumental 44-page chapter on the trigonometric interpolation for the *Encyklopädie der Mathematischen Wissenschaften.* Some sixty years later, in response to the 1965 Cooley and Tukey's algorithm for the machine calculation of complex Fourier series, several authors reported that they were using similar techniques. And so they focused on Runge (1903-1905, 1924) as the source of their technique, for a faster calculation of Fourier coefficients. However, in 1908, Charles de la Vallée Poussin explicitly published what he called the Euler-Gauss algorithm (Godard, 2006). Both of them were seeking an easier method, an algorithm for the calculation of the discrete Euler-Fourier coefficients. Euler's memoir was "*Conventui exhibita*", in May 1777. It was published in *Nova acta academiae scientiarum Petropolitanae* (1793) in 1798. In the summary, Euler's method is presented as an easy way for the calculus of A, B, C etc. of a trigonometric series:

"*La sommation se réduit à la détermination de chacun des coefficients A, B, C etc., sommation que l'auteur se facilite par une méthode ingénieuse dont il est facile de donner une idée sans entrer dans des explications du symbolisme particulier dont il est fait usage.* "

III. Carl Runge at Göttingen Universität

From his period at Göttingen, we shall mainly retain Runge's contribution to the numerical solutions of partial differential equations. Carl Runge is generally credited and acknowledged as the initiator for these numerical techniques. We should say more explicitly that Carl Runge seems to be the first scientist in obtaining full numerical results for a Laplace equation in a complex domain. Already, during the XVIIIth century, L. Euler used the following discretisation $f(i+1, j)+f(i, j+1)-f(i-1, j)-f(i, j-1)$ for a function of two variables. However, later, during the period of genesis of the two-dimensional finite difference calculus and grid set-up, we distinguished mainly two streams, one originated in Germany and Austria, and the other one in England (Godard, 2002). In Austria, Ludwig Boltzmann was very conscientious about the difficulty in solving partial differential equations, and as an example he even suggested the finite difference calculus for the solution of the well-known Boltzmann equation in the theory of gases. Then, Hugo Buchholz (1908) published *Das mechanische potential nach vorlesungen von L. Boltzmann bearbeitet*.

The book was published in Leipzig, and it was probably the first textbook treating in detail the numerical solution of elliptic problems. L. Boltzmann died in 1906 and had been Buchholz's professor. Buchholz mentioned Boltzmann's lectures on potential theory given in 1892/93 in Munchen. In page 154, he exposed «*Das Prinzip für das zweifach-dimensionale Gebeit und Der Botzmannsche Beweis*». The computational rectangular domain was divided into elementary rectangular cells and each node had a number (a,b). Figure 1 is the graph of the computational domain:

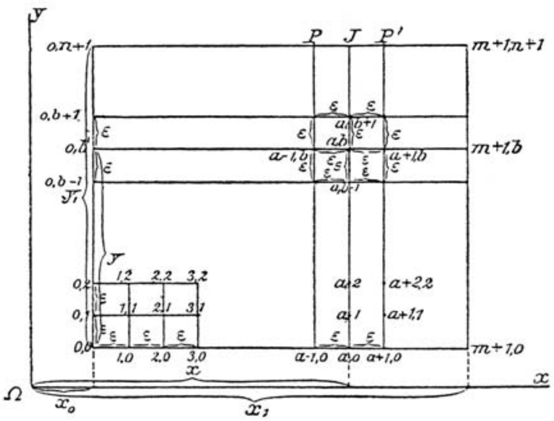

Fig. 52.

Figure 1: Boltzmann's computational domain

The Laplace equation in finite difference form was therefore written as:

$$\frac{u_{a+1,b} - 2u_{a,b} + u_{a-1,b}}{\varepsilon^2} + \frac{u_{a,b+1} - 2u_{a,b} + u_{a,b-1}}{\varepsilon^2} = 0$$

where ε is the grid increment, which must be small, and *u* is the dependant variable (Godard, 2002). At about the same time, Runge (1908) proposed the same discretization of the Laplace equation by finite difference. Runge was interested in the numerical Laplace solution of torsion problems, problems he should have known well when he was a student in Berlin. Figure 2 represents his domain of calculus, which has the shape of a cross.

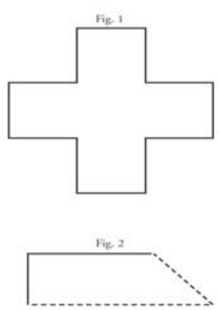

Fig. 2: Runge's domain of interest. The lower figure is his computational domain

Runge was clever enough to use the symmetry of the problem, and the figure, at the bottom, represents his reduced domain of calculus. Runge appears to be the first person to have published numerical results for the solution of the Laplace equation for a complex domain. The Göttingen's *Aufklärung* was represented in Runge and Willers's *Numerische und graphische Quadratur und Integration gewöhnlicher und partieller Differentialgleichungen* (1915), which was published in the *Encyklopädie.* Runge published altogether four articles for the *Encyklopädie,* with the first paper in 1899. In their 1915 article, Runge and Willers explained in detail the method of characteristics for hyperbolic equations, the finite difference schemes for the elliptic equations, the problem of curved boundaries, and the Rayleigh-Ritz method. If one of Felix Klein's objectives for the *Encyklopädie,* was to establish mathematical historical surveys, to re-expose methods lost over time, and to understand the most up-to-date research, in establishing Condorcet's dream about the state of sciences; Runge and Willer's article will remain the source, the bible, the synthesis for partial differential equations. They were, however, very sober for the numerical solutions of the system of linear equations and the truncation errors associated with the numerical approximations, it seems also they had ignored the numerical parabolic equations. We also found some important omissions about Boltzmann or Richardson (1911). Three years later, Liebmann (1918) will publish in Germany, results with the SOR method; and in 1928, Courant, Friedrichs, and Lewy will present stability criteria for hyperbolic and parabolic partial differential equations; i.e., that discretisation in time and discretisation in space are linked together. Curiously, Richard Courant, who married Runge's daughter Nina, had, more than Runge, the mathematical profile Klein was seeking for. Figure 3 represents the sophisticated 1911 Richardson's computational domain for the simulation of a dam.

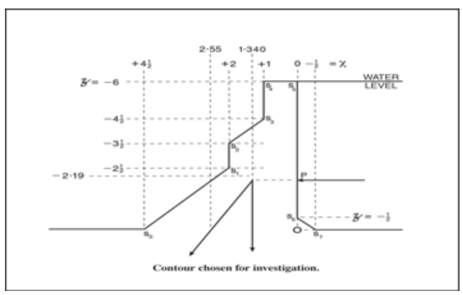

Figure 3: The 1911 Richardson computational domain

IV. The professor

An example of number crunching textbook is Runge's *Praxis der Gleichungen*, which was published in 1901, and a second edition followed in 1921. Even, if the basic techniques such as the numerical solutions of a system of linear equations, zeros of a non-linear equation, systems of non-linear equations, are covered, there was obviously a dramatic gap between applied mathematics and pure mathematics. Applied mathematicians did not always knew how to present, to motivate, to promote applied mathematics, to build a bridge between applied mathematics and pure mathematics, or they assumed that the mathematical background was known and understood. Obviously, if the young Runge was a pioneer in teaching numerical methods, the fields of numerical analysis or, numerical mathematics had to be invented. Another example of Runge's capabilities is *Graphical methods*, published in 1912, by Colombia University Press. The subject was a course given every two years since 1904 at the University of Göttingen. The lectures were combined with practical exercises in laboratory assignments, with the slide rule, logarithms tables with four significant digits and a calculator for computations with greater accuracy. The knowledge of integral and differential calculus was a prerequisite**.** The book was extremely well organized with a large number of figures. The graphical solutions remained at the beginning of the XXth century, an important tool for the solution of ordinary differential equation or hyperbolic problems (Tournès, 2003). As a young professor, Runge designed the construction of a Leibnitz mechanical calculator, and published an article on it in 1905. He succeeded extremely well in his *Vorlesungen über numerisches rechnen.* The book was published in 1924 with Hermann König, his former student, who graduated in 1920. It was the conclusion, the achievement of several German textbooks. The exhibit was very clear and organized. This book is considered as the cornerstone for the Fast Fourier Transform in Fourier analysis, which became so popular around 1965. Runge was also excellent in teaching vector calculus**.** His book on *Vector Analysis* was written in 1919. We realized that Runge should have enjoyed writing his textbook, a very modern one, which will almost be translated into English, in 1923. For Runge, Grassmann's work was quite unnoticed. He indeed acknowledged his colleague from Gôttingen, Woldemar Voigt. The book is full of examples in fluid motion, hydrostatics, or crystal lattice structures.

During his career, Runge had 16 graduate students, and Max Born was one of them! The following selection of thesis subjects illustrated perfectly Runge's interests: Graphical integration in hydrodynamics in 1909, on the Runge-Kutta method for the solution of ordinary differential equations in 1909, on the Helmholtz equation in 1912 and with Woldemar Voigt as the first advisor, on the graphical solution for the Laplace equation in 1914, Isoperimetric functions from the theory of airplanes in 1918 with Ludwig Prandtl as the first advisor. Runge published 123 articles, and his name appeared in seven books as a sole author or co-author. Two of them were translated into English, shortly after their print in German

V. Runge's philosophy

While, the prestigious Göttingen mathematicians such as Felix Klein or David Hilbert participated in the debate about the foundations of mathematics, Carl Runge seemed closer to the intuitionist, constructive viewpoint. He presented his "philosophy" in the introduction of his book *Graphical methods* (1912). A book, which summarized Runge's lectures, while he was on leave at Columbia University for the year 1909-1910. Runge started his introduction as follows:

"A great many if not all of the problems in mathematics may be so formulated that they consist in finding from given data the values of certain unknown quantities subject to certain conditions. We may distinguish different stages in the solution of a problem. The first stage we might say is the proof that the solution sought for really exists, that it is possible to satisfy the given conditions or, as the case may be, the proof that it is impossible…"

Examples of Runge's mathematical problems could have been least square problems, solutions of ordinary or partial differential equations, boundary value problems, where data are the Dirichlet conditions at the border of a domain. We already see that Runge's concepts of mathematics are, in fact, very restrictive. And Runge continues:

"In many other cases, the first stage of a solution may be so easy, that we immediately pass on to the second stage of finding methods to calculate the unknown quantities sought for. Or even if the first stage of the solution is not so easy, it may be expedient to pass on to the second stage. For if we

succeed in finding methods of calculation that determine the unknown quantities, the proof of their existence is included. If on the other hand, we do not succeed, then it will be time enough to return to the first stage."

These sentences may seem strange. It means that if we can't prove the existence and uniqueness theorems, we can still go ahead, and try to find an approximate solution. The example of the history of the Dirichlet principle illustrated the embarrassment of some mathematicians. After, the publication of Riemann's works, Weierstrass objected to Riemann's assumption that the Dirichlet integral necessarily assumes a minimum. Shortly after Dirichlet's death, Weierstrass found a counter-example when the integral does not assume a minimum. The next paragraph shows Runge' sagacity for new and faster methods, and the Göttingen team was the best example:

"So there arises a third stage of the solution of a mathematical problem in which the object is to develop methods for finding the result with as little trouble as possible. I maintain that this third stage is just as much a chapter of mathematics as the first two stages and it will not do to leave it to the astronomer, to the physicist, to the engineer or whoever applies mathematical models…"

While Runge was in Göttingen, Alfred Haar, Walter Ritz and Richard Courant and others, contributed to the numerical Enlightenment.

VI. Conclusion: his heritage in Applied Mathematics and in Numerical Analysis

Runge's legacy is impressive, and we shall give a few examples of his ideas that have stimulated, and contributed to the advancement of mathematics. Charles de la Vallée Poussin (VP, 1908) published a 91 page article: *Sur la convergence des formules d'interpolation entre données équidistantes* (Godard, 2004). His objective was first to reexamine Runge's problem of equally sampled observations. For VP, the purpose of the interpolation was to find "a function", and to write the equation of a curve passing through observations. His interpolatory formula was extremely powerful. Moreover, VP's exercise is one of the first results in numerical mathematics, where not only, de la Vallée Poussin presented a method, but also theorems about its convergence properties. And, in our search of better interpolation methods, the spline interpolation reflected another indirect consequence of Runge's work on the lagrangian interpolation. But, the main Runge's legacy come the Runge-Kutta (R-K) algorithms for the solution of ordinary differential equations, and they represent his most refereed articles. The R-K methods are self-starting, easy to program and accurate in most cases. Only in 1951, Curtiss and J.O. Hirschfelder gave some examples of "*stiff*" problems, and challenged the R-K methods where they may fail. These problems came from the study of chemical reactions, problems of detonations, electric circuit theory and problems of missile guidance, when we have different time constants. Finally, this present work simply outlines some of Runge's contributions to the field of applied mathematics. The next focal point should be to situate the global evolution of the domain of applied mathematics in Germany, its teaching from the period of 1900 to 1925 with its major achievements. It remains to scrutiny in detail all the articles on applied mathematics which appeared in the journals *Mathematische Annalen* and *Zeitschrift für Mathematik und Physik.*

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