The debate on a “geometric algebra” and methodological implications

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ABSTRACT

A crucial methodological question confronting various approaches regarding the use of history of mathematics for teaching is whether and how original texts can be presented as teaching texts. Given that texts, which are older than, say the nineteenth century, use to be not directly readable and understandable – for several essential reasons (conceptualization, notation, language, epistemology, etc.). For use in teaching, one will try, hence, to “modernize” somewhat the original.

Inevitably, modernization will result in some “distortion” and the question is which degree and kind of distortion can claim to be legitimate or tolerable for the aim of teaching. Particularly sensitive in this regard is the relation between geometry and algebra, viz. the transformability of earlier, largely geometrical texts into - for moderns - readable, algebraized texts. A seminal case study for the legitimacy of distortion will be presented by the debate on the existence of a “geometric algebra” in Greek mathematics, provoked in 1975 by Sabetai Unguru and having famous mathematicians (van der Waerden, Freudenhal, Weil) as reactors. The methodological questions for historiography of mathematics, as implied in this debate, will be shown in their relevance for use of historical texts in teaching.

The use of historical material for teaching

For the use of historical material in the classroom, for teaching, there exist at present two different approaches: the use of original sources and the use of sources which have been adapted in a certain way for the purpose of introducing them to a definite learning group. Even for the first approach, one will have to reflect how original really are the alleged original sources, given that at least the language of the original might be unfamiliar to the intended public so that a translation will be presented to them - and the adequacy of the translation is not *a priori* evident.

Definitely more problematic is, clearly, the case of an adapted source. In how far does it correspond to the content and the methodology of the original? The question is the more pertinent as even professional historians of mathematics are not always aware of the importance and dimension of this question. There are, in fact, prominent editions of historical classics of mathematics in use, which claim to be the true representations of an original but which in reality have transformed the original into another type of mathematics.

This remark concerns even such a classic like Euclid’s *Elements*. The standard edition in use is the English one, edited by Thomas Heath (1861-1940), based on J. L. Heiberg’s text of 1893, the first time in 1908. This edition shows a stark contrast between the main text and the footnotes. While the enunciation of the theorems and of the demonstrations of Euclid’s geometrical books are in fact geometric, the footnotes give an entirely algebraized version (see as example Figure 1). Heath explained his proceeding in the footnotes by a proper section of comment, entitled “Geometrical Algebra”. He points out, however, that the original procedure is geometrical, and that the algebraical method was introduced later on (Heath 1956, 373).

Yet, there are translations of Heiberg’s text into other languages where the difference between the main text and the footnotes disappear. For instance, the standard German version is that edited by Clemens Thaer. There, theorems and proofs are in algebraic formulation – see here the same proposition, Figure 2.

Did a Greek algebra exist?

The practice of transforming a considerable part of Greek geometry into a kind of algebra has for a long time gone unchallenged.

It has for a first time been questioned in a fundamentally methodological manner by Sabetai Unguru in 1975 and has led to a heated debate in which it were essentially mathematicians who contradicted his arguments: Bartelt van der Waerden (1976), Hans Freudenthal (1977) and André Weil (1978). Sabetai had problems to find a journal, which would publish his answer (1979). Eventually, he took up the question in an even more profound manner, in a series of two seminal papers, together with David Rowe, whether the Greeks intended to solve quadratic equations (Unguru/Rowe 1981, 1982). Published in a marginal journal, these studies are still almost unknown. Even Grattan-Guinness who published an account of the discussion and the present state in 1996 does not quote them. It is therefore revealing to present the essentials of the entire debate.

The key issue of this debate had been how to interpret Book II of Euclid’s *Elements*. For van der Waerden, this Book constitutes the kernel of what he understands to be the Greek algebraic geometry. He went even so far to qualify this Book II as “the start of an algebra textbook, dressed up in geometrical form” (van der Waerden 1963, p. 118). According to him, “the line of thought is always algebraic, the formulation geometric” (ibid., p. 119).

Actually, this interpretation was not restricted to a part of Euclid, but it comprised several other Greek mathematicians. Van der Waerden, for instance, affirmed:

> Theaetetus and Apollonius were at bottom algebraists, they thought algebraically even though they put their reasoning in a geometric dress (ibid., p. 265).

Other historians of mathematics have prepared this reinterpretation of Greek geometry, since the end of the 19th century: Paul Tannery, Thomas Heath, Hieronymus Zeuthen, and Otto Neugebauer. Zeuthen, an important Danish historiographer affirmed since 1896 that ‘the Ancients’ knew to treat all forms of equations of the second degree (Zeuthen 1896, p. 50).

I am calling this school of thought, which supposes the existence of a Greek geometric algebra, a “revisionist” school – since all the centuries before, mathematicians had agreed that Greek geometry canonized the synthetic method, emphasizing each particular case and renouncing decidedly any algebraizing attempt at generality.

This revisionism raises several essential methodological issues. A first one has been evoked by Otto Neugebauer’s justifying his transcription of Apollonius’s geometry into algebraic language. According to Neugebauer, this transcription had in no ways affected the content of Apollonius’s mathematics:

> I did not change Apollonius's text except in its exterior form (Neugebauer 1932, p. 250).

Neugebauer pretended hence – and with him the entire revisionist school of a geometric algebra -, that the mathematical content is independent of its form, language and the symbols used to express it. While intended to facilitate understanding old texts, this approach is clearly anhistoric. Fortunately enough, there is now a modern reliable edition of Apollonius, liberated from these anhistoric misinterpretations (Fried/Unguru 2001).

Another methodological issue is addressed by the evident question: Given one would admit that Greek mathematical thinking was basically algebraic – how can one explain that the Greeks disguised this thinking by a geometric language? Hans Freudenthal gave three answers:

> First, historical: it just happened that the Greek end of the torturous path through foundations of mathematics, Eudoxos' theory, was so excellent that the Greeks did not aspire a better one.
>
> Second, philosophical: though in daily use by laymen as well as mathematicians, fractions were taboo in highbrow mathematics, because mathematics forbade the division of the unit.
>
> Third, traditional: Once canonized, the Elements were sacrosanct, liable to additions, but not to change. The mathematical community was small. To be understood within it, you had to quote Euclid and to speak his language (Freudenthal 1977, p. 191).

Regarding the first answer, Unguru followed Wilbur Knorr in refuting that there had taken place the famous, but only pretended crisis of foundations in the pre-Eudoxian period. The second answer contained no pertinent point, according to Unguru. And to the third answer, Unguru put as question:

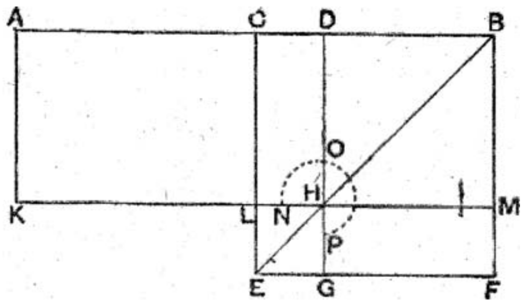
> Fine, but why did Euclid then adopt the very same language? (Unguru 1979, pp. 558-559).

The third controversial point of the debate is of genuinely methodological nature and concerns the key issue in particular, the role of algebra and algebraization. Unguru had emphasized in his discussion of the propositions of Euclid's Book II that neither the propositions nor the proofs do show any equation. Furthermore, there is no use of unknowns and likewise not of symbols and, consequently, no operations on symbols. Unguru had therefore sharply criticized Heath who – although correctly translating Euclid's text of propositions and their proofs – presented these propositions in his extended commentaries as entirely algebraic texts.

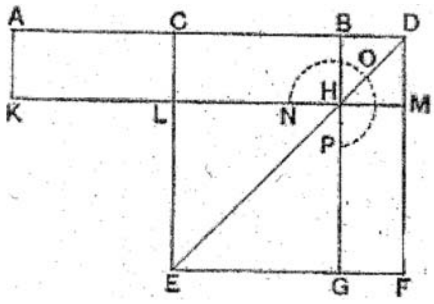
In their replies to Unguru's critique, Freudenthal and van der Waerden claimed that a statement made in words is completely identical to a statement made in symbols. The most radical comment in this regard was published by André Weil, however. Weil, well known for his strong reactions, not only in mathematics but also in history of mathematics, declared any reflection about signs and symbols to be superfluous:

> As everyone knows, words, too, are symbols. The content of a theorem does not change greatly, whether it is expressed in words or in formulas: the choice is, as we all know, mostly a matter of taste and of style (Weil 1978, p. 92).

The necessity of deeper reflection on the relation between statements in words and corresponding statements in symbols, however, was already clear from considerations by van der Waerden of Book II of Euclid's Elements. His algebraic approach to this book led to a contradictory situation concerning propositions II.5 and II.6, which he could explain away. In both propositions, a segment of a straight line is cut into two unequal parts. While II.5 deals with the difference of these tow sub-segments, II.6 deals with their excess:

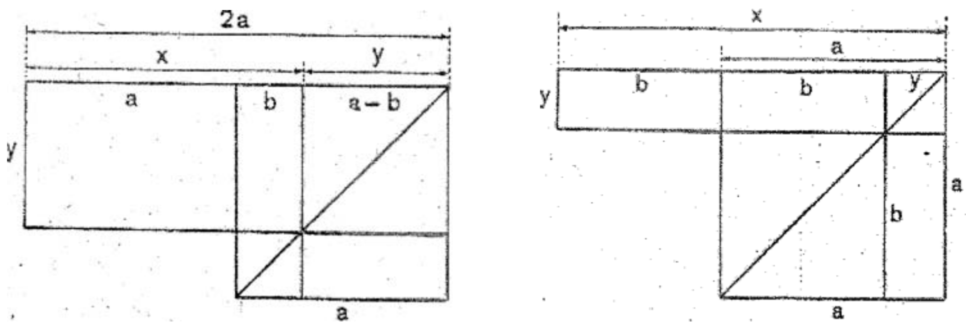


Euclid II.5



Euclid II.6

Due to his zeal to find algebraic formula in Euclid, van der Waerden had introduced designations *a* and *b* on the one hand and *x* and *y* on the other hand and labelled with them the sides in the two figures as shown in Heath, but applied according to his own interpretation:



van der Waerden 1963, 197

In fact, he had succeeded in labelling the sides in both figures in such a manner that he was able to deduce the same formula from the labelled figures:

$$(a+b)(a-b) = a^2 - b^2$$

Perplexed, van der Waerden observed this "strange double form": "Why two propositions for one single formula?" (van der Waerden 1963, 197).

Actually, his perplexity reveals the profound difference between the classical synthetic method and the modern analytic one. Since modern times, one had been conscious of the marked contrast between these two methodologies. While the mathematics of the 'Ancients' treated each case separately and independently, without searching for related cases, which might be regarded as following a common pattern, the analytic method strove for generality. What were for the Moderns just variations of the same property were, for the Greeks, new cases, due to a different position of some lines within a figure.

Van der Waerden's perplexity implies, hence, the failure of the algebraizing approach. Greek geometry was geometry and not algebra. It occurred within another era and another culture that Greek geometric approaches were transposed into algebraic ones.

In fact, it was within the Islamic civilization that a basic principle of the Greek method of comparing geometric entities was bypassed: the principle of homogeneity of the magnitudes to be operated with. Thanks to this rupture with the geometric origins, as initiated by Abu-Kamil (ca. +900), Arab mathematicians were able to develop algebra as a new discipline (Djebbar 2001, p. 7).

Pitfalls

The motivations for the revisionist approach of geometrical algebra have been admitted explicitly by several of its protagonists. Neugebauer lamented about the "entanglement of letters":

> "Offenbar überblickte man das Buchstabengewirr einer Konstruktion mit derselben Selbstverständlichkeit wir wir heute komplizierte Formeln" (Neugebauer 1936, 250).
>
> "Apparently, one was able to understand the entanglement of letters in the same easy manner as we today complicated formulae".

And van der Waerden expressed his uneasiness with the uncommon form of Greek geometry:

> "Reading a proof in Apollonius requires extended and concentrated study. Instead of a concise algebraic formula, one finds a long sentence, in which each line segment is indicated by two letters which have to be located in the figure. To understand the line of thought, one is compelled to transcribe these segments in modern concise formulas" (van der Waerden 1963, 256).

These motivations were clearly dictated from a mathematician's point of view, trying to rediscover mathematics familiar to him.[1] Yet, as Unguru and Rowe have shown, these "transcriptions" lead to distortions of the intended meaning of the texts, even to an entirely different mathematics – in the case of Greek geometric algebra the analysis showed that in this interpretation its ultimate mathematical concept and the object of its operations is **number**, whereas it is clear, taking Greek mathematics seriously, that its underlying concept and the object of its operations is **quantity** ("magnitude").

Similar cases occur when, for instance, earlier entirely rhetorical texts, i.e. without any use of symbols, are "transcribed" into texts with symbolic notations and operations. Such transcriptions occur in particular for didactical reasons and motivations, in order to make these historical texts understandable to readers unfamiliar with texts without symbols. While this constitutes a common practice in projects making use of history of mathematics for teaching mathematics, I am missing a reflection about the differences between the true original and its adapted, transcribed version. One might argue that a

[1] Cf. The methodological discussion of different uses of the history of mathematics in Grattan-Guinness 2004.

certain distortion will be inevitable for teaching purposes, but that should at least be practiced in a “controlled” way. My intention is to instigate such a methodological debate.

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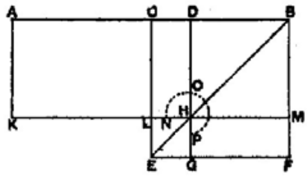
Annex

Proposition 5.

If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

For let a straight line AB be cut into equal segments at C and into unequal segments at D;

I say that the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.



For let the square $CEFB$ be described on CB, [I. 46]
and let BE be joined;
through D let DG be drawn parallel to either CE or BF,
through H again let KM be drawn parallel to either AB or EF,
and again through A let AK be drawn parallel to either CL or BM. [I. 31]

Then, since the complement CH is equal to the complement HF, [I. 43]
let DM be added to each;

therefore the whole CM is equal to the whole DF.

But CM is equal to AL,

since AC is also equal to CB; [I. 36]

therefore AL is also equal to DF.

Let CH be added to each;

therefore the whole AH is equal to the gnomon NOP.

But AH is the rectangle AD, DB, for DH is equal to DB,

therefore the gnomon NOP is also equal to the rectangle AD, DB.

Let LG, which is equal to the square on CD, be added to each;

therefore the gnomon NOP and LG are equal to the rectangle contained by AD, DB and the square on CD.

But the gnomon NOP and LG are the whole square $CEFB$, which is described on CB;

therefore the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.

Therefore etc. Q. E. D.

Geometrical solution of a quadratic equation.

Suppose, in the figure of II. 5, that $AB = a$, $DB = x$;

then
$$ax - x^2 = \text{the rectangle } AH$$
$$= \text{the gnomon } NOP.$$

Thus, if the area of the gnomon is given ($= b^2$, say), and if a is given ($= AB$), the problem of solving the equation
$$ax - x^2 = b^2$$
is, in the language of geometry, *To a given straight line* (a) *to apply a rectangle which shall be equal to a given square* (b^2) *and shall fall short by a square figure*, i.e. to construct the rectangle AH or the gnomon NOP.

Now we are told by Proclus (on I. 44) that "these propositions are ancient and the discoveries of the Muse of the Pythagoreans, the application of areas, their exceeding and their falling-short." We can therefore hardly avoid crediting the Pythagoreans with the geometrical solution, based upon II. 5, 6, of the problems corresponding to the quadratic equations which are directly obtainable from them. It is certain that the Pythagoreans solved the problem in II. 11, which corresponds to the quadratic equation

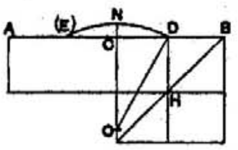
$$a(a-x)=x^2,$$

and Simson has suggested the following easy solution of the equation now in question,

$$ax-x^2=b^2,$$

on exactly similar lines.

Draw CO perpendicular to AB and equal to b; produce OC to N so that $ON=CB$ (or $\frac{1}{2}a$); and with O as centre and radius ON describe a circle cutting CB in D.



Then DB (or x) is found, and therefore the required rectangle AH.

For the rectangle AD, DB together with the square on CD is equal to the square on CB, [II. 5]

i.e. to the square on OD,

i.e. to the squares on OC, CD; [I. 47]

whence the rectangle AD, DB is equal to the square on OC,

or $$ax-x^2=b^2.$$

Figure 1: Heath 1956, 382-384

§ 5 (L. 5).

Teilt man eine Strecke sowohl in gleiche als auch in ungleiche Abschnitte, so ist das Rechteck aus den ungleichen Abschnitten der ganzen Strecke zusammen mit dem Quadrat über der Strecke zwischen den Teilpunkten dem Quadrat über der Hälfte gleich.

Eine Strecke $A\,B$ teile man in gleiche Abschnitte in C und in ungleiche in D. Ich behaupte, daß $A\,D \cdot D\,B + C\,D^2 = C\,B^2$.

Man zeichne über $C\,B$ das Quadrat $C\,E\,F\,B$, ziehe $B\,E$, ferner durch D $D\,G \parallel C\,E$ oder $B\,F$, ebenso durch H $K\,M \parallel A\,B$ oder $E\,F$, und ebenso durch A $A\,K \parallel C\,L$ oder $B\,M$.

Hier ist die Ergänzung $C\,H$ der Ergänzung $H\,F$ gleich (I, 43); man füge daher $D\,M$ beiderseits hinzu; dann ist das ganze Pgm. $C\,M$ dem ganzen $D\,F$ gleich. Andererseits ist Pgm. $C\,M = A\,L$, da $A\,C = C\,B$ (I, 36); also ist auch Pgm. $A\,L = D\,F$. Man füge $C\,H$ beiderseits hinzu; dann ist das ganze Pgm. $A\,H =$ Gnomon $N\,O\,P$ (II, Def. 2). $A\,H$ ist aber $A\,D \cdot D\,B$; denn $D\,H = D\,B$ (II, 4, Zus.); also ist Gnomon $N\,O\,P = A\,D \cdot D\,B$. Man füge $L\,G = C\,D^2$ beiderseits hinzu; dann sind Gnomon $N\,O\,P + L\,G = A\,D \cdot D\,B + C\,D^2$. Gnomon $N\,O\,P$ und $L\,G$ bilden aber zusammen das Quadrat $C\,E\,F\,B$, d. h. $C\,B^2$. Also sind $A\,D \cdot D\,B + C\,D^2 = C\,B^2$ — S.

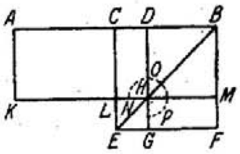


Fig. 52.

Figure 2: Thaer 1991, 36-37