# FROM THE ANALYSIS OF THE ARTICULATION OF THE TRIGONOMETRIC FUNCTIONS TO THE CORPUS OF EULERIAN ANALYSIS TO THE INTERPRETATION OF THE CONCEPTUAL BREAKS PRESENT IN ITS SCHOLAR STRUCTURE

**Gustavo MARTÍNEZ SIERRA**

Programa de Matemática Educativa,
CICATA-IPN, Unidad Legaria,
Calzada Legaria #694 Col. Irrigación
Del. Miguel Hidalgo, C.P.11500, México, D.F.

gamartinezsierra@gmail.com, gmartinezs@ipn.mx

**ABSTRACT**

This article presents the results of an investigation on the construction of knowledge from the *socioepistemological approach.* We are particularly interested in the study of the processes present in the articulation of conceptual mathematics systems to what we have called *processes of mathematical convention and articulation* (Martínez-Sierra, 2003, 2005). More specifically, the aim here is to present our advances in the quest to identify the present processes of mathematics convention of the articulation of the trigonometric functions (TF) to the corpus of Eulerian analysis. We will also present the interpretations that said analysis has allowed us to make in order to become aware of the conceptual breaks in the scholastic construction of the trigonometric functions.

**Key words**: Socioepistemology, knowledge production, mathematical convention, function, trigonometric functions.

**RESUMEN**

En el presente artículo se ofrecen resultados de una investigación sobre construcción del conocimiento desde la aproximación socioepistemológica. En particular estamos interesados en el estudio de los procesos presentes en la articulación de los sistemas conceptuales matemáticos a los que hemos llamado procesos de convención y articulación matemática (Martínez-Sierra, 2003, 2005). De manera más específica lo aquí escrito tiene por objetivo presentar los avances en nuestra búsqueda por identificar los procesos de convención matemática presentes de la articulación de las funciones trigonométricas (FT) al corpus del análisis euleriano y presentar las interpretaciones que tal análisis nos ha permitido para dar cuenta de las rupturas conceptuales presentes en su construcción escolar de las funciones trigonométricas.

**Palabras clave**: Socioepistemología, construcción de conocimiento, convención matemática, función, funciones trigonométricas.

## 1. Introduction

One of the theses used to develop part of the investigations from the *socioepistemologic* perspective in Mathematics Education in Mexico (Cantoral and Farfán, 2003, 2004, Buendía and Cordero, 2005) is that which argues that mathematics teaching and learning processes are specific to the concept or conceptual mathematics system they are dealing with. To this can be added the consideration that chunks of mathematical knowledge were not constructed to be the objects of teaching; “school mathematics” is qualitatively distinct from “mathematics”. Aided by the above considerations investigations have been developed which offer explanations on the particularities, as far as their conceptual construction, of transcendental logarithmic (Ferrari and Farfán, 2004), exponential (Lezama, 2005; Martínez-Sierra, 2002, 2003) and the trigonometric functions (Buendía and Cordero 2005; Montiel, 2005).

In addition, in previous works (Martinez-Sierra, 2005) we have developed some theoric notions which have been useful, on one hand, in the explanation of some didactic phenomena and, on the other, in the interpretation of knowledge production processes. In particular, on the knowledge production plane we have provided evidence that certain pieces of knowledge, which we have called *mathematics conventions*, can be understood as the product of a process of mathematical articulation or process of knowledge integration. In this same way, on the plane of explanation of didactic phenomena, we have realized that some of the conceptual breaks at school have their origins in the disarticulation of a certain part of the corpus of scholastic mathematics (Martínez-Sierra, 2005).

More specifically, the aim here is to present our advances in the quest to identify the present processes of mathematics convention of the articulation of the trigonometric functions (TF) to the corpus of Eulerian analysis. We will also present the interpretations that said analysis has allowed us to make in order to become aware of the conceptual breaks in the scholar construction of the trigonometric functions.

## 2. Socioepistemological approach in mathematics education

Socioepistemology is a systemic approach which enables the phenomena of knowledge production and diffusion to be dealt with from a multiple perspective by incorporating the study of interactions between the epistemology of knowledge, its sociocultural dimension, the associated cognitive processes and the mechanisms of institutionalization through teaching[1] (Cantoral and Farfán, 2004). More accurately, within the socioepistemological theory in mathematics education at least four large interdependent dimensions are thought to condition/determine the construction and diffusion of mathematical knowledge: the cognitive, didactic, epistemological and social dimensions. The latter, in turn, conditions/determines the first three. The *didactic dimension* attends to those circumstances typical of the functioning of different didactic systems and of teaching. The *cognitive dimension* concerns the circumstances relating to our mental functioning and activity. The *epistemological dimension* deals with those circumstances inherent in the nature and meaning of mathematical knowledge. The *social dimension* addresses the circumstances shaped by the social standards and evaluations of the knowledge and the way in which these influence the other dimensions. In this sense, the practices of the craftsman, engineer, physician or, more broadly, of an epoch or a culture, are considered inseparable constituents of scholar knowledge.

[1] "La socioépistémologie procède d'une approche systémique qui permet d'aborder les phénomènes de production et diffusion de la connaissance dans une perspective multiple, qui intègre l'étude des interactions entre l'épistémologie du savoir, sa dimension socioculturelle, les procédés cognitifs asocies et les mécanismes de l'institutionnalisation *via* l'enseignement" (Cantoral and Farfán 2004, p. 139).

## 3. The process of mathematical convention in the construction of theTF

A process of *mathematical convention* may be understood as a consensus-seeking process within the community that works to give unity and coherence to a set of knowledge. The production of consensus is possible because the *practice of systemic integration of knowledge* exists in this community. This means that there is a *standard of activity to relate diverse pieces of knowledge and articulate them into a coherent and interrelated whole.* By nature this practice is found on the plane of mathematical theorization, understanding by this the elaboration of interrelated concepts which try to describe, to explain an object of study which is, in this case, the system of accepted knowledge. This process of synthesis brings about the appearance of emergent properties unforeseen by earlier knowledge. Mathematics conventions would be a part of these emergent properties (Martínez-Sierra, 2003, 2005).

In the previous sense, then, a mathematical convention can be understood as an agreement by the community which works to give unity and coherence to a knowledge set. Two examples relating to exponents, taken from the history of mathematical ideas, will serve to illustrate our "convenience principle" on which rests our characterization of mathematical agreement (first example) and its character in relation to the reference knowledge set (second example).

***First example.*** Towards the end of the XVI century it was known that the curves $y = kx^n$ ($n$ = 1, 2, 3, 4,…), called by index *n,* had a property called "characteristic ratio". This knowledge was a general part of the fundamental problem of the time of the mechanical and algebraic calculation of areas defined by distinct curves (Bos, 1975), and to the meaning that the areas preserve in terms of variation[2]. Taking as an example the curve $y = x^2$ it was said that the characteristic ratio was equal to 1/3; since, if we take an arbitrary point on the curve, **C** (Figure 3) the area of **AECBA** defines a proportion of 1:3 with respect to the area of the rectangle **ABCD**, in the same way as the proportion between the area of **AECBA** and the area of **AECDA** is 1:2. In general, it was known that the characteristic ratio of the index curve $n$ is $1/(n+1)$ for all positive whole numbers $n$ [3].

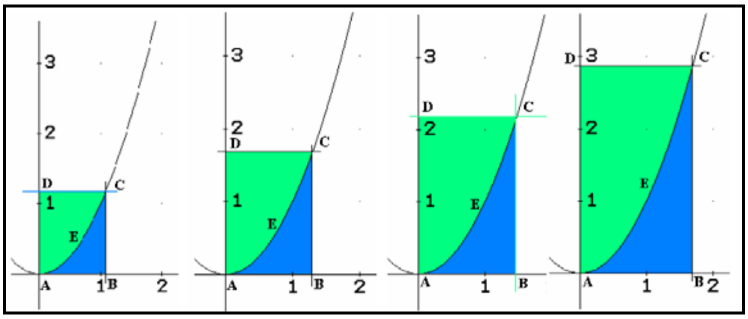



*Figure 1.* Characteristic ratio of the curve $y = x^2$

[2] For example, it is well known that Galileo established his law of falling objects by grasping that the area defined by a speed-time graph was the distance traveled by the body.

[3] In modern terms the notion of characteristic ratio is helped in that $(a > 0)\ \left(\int_0^a x^n dx\right) : a^{n-1} = 1 : (n+1)$

In his investigations on the quadrature of curves, John Wallis (Struik, 1986) used the above to make the following reasoning, which is basically a way of agreeing that the index of $y = \sqrt[2]{x}$ must be equal to ½ in order to unify the notion of characteristic ratio with the notion of index (a paraphrase of the reasoning is given here):

> *"As the curve $y = x^2$ has a characteristic ratio of 1/3, the curve $y = \sqrt[2]{x}$ should also have a characteristic ratio and must be equal to 2/3 (it can be observed that the areas under both curves complement each other to make a rectangle). Also, as the curve of index 2 has a characteristic ratio it can be supposed that that a curve which has a characteristic ratio also has an index, so, what index should the curve $y = \sqrt[2]{x}$ have? As 2/3=1/(1+1/2) the index must be 1/2"*

**Second example.** Wallis also interpreted negative numbers as indices[4]. He defines the index $1/x$ as -1, the index $1/x^2$ as -2, etc. He goes on to try to give coherence to these indices and to the notion of characteristic ratio (Confrey and Dennis, 1996). In the case of curve $y = 1/x$ the characteristic ratio must be $\frac{1}{-1+1} = \frac{1}{0} = \infty$[5]. Wallis accepted this quotient as reasonable since the area under the curve $1/x$ diverges; which, it seems, was a known fact at that time. The above can be interpreted as the proportion between the area **ABCEFA** (Figure 4) and the area of the rectangle **ABCD** being 1:0. When the curve is $y = 1/x^2$ the characteristic ratio must be 1/(-2+1)=1/-1. Here, Wallis's conception on the ratio differs from modern arithmetic of negative numbers. He doesn't use the equivalent 1/-1 = -1, but instead constructs a coherence between diverse representations; that is the essence of a mathematical convention. Due to the shaded area under curve $y = 1/x^2$ being larger than the area under curve $1/x$, he concludes that the ratio 1/-1 is greater than infinity (*ratio plusquam infinita).* He goes on to conclude that 1/-2 is even larger. This explains the plural in his title.

*Arithmetica Infinitorum*, of which the most accurate translation would be "The Arithmetic of Infinities".

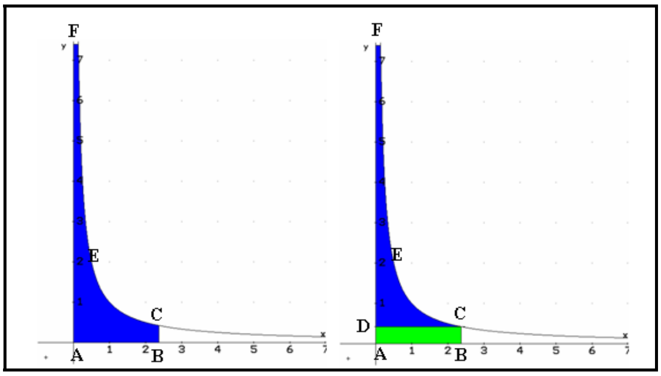



*Figure 2.* Characteristic ratio of curve $y = 1/x$

The above leads us to center our attention on the processes of systemic integration of a knowledge set. Theoretically, from the beginning, this search for integration, which is

---

[4] We would like to clarify that through the literature consulted it was not possible to clearly determine the motives that Wallis had for realizing said definitions; but it is supposed that they were taken from the agreements of exponents already in use at that time in the algebraic context (Martinez-Sierra, 2003).

[5] What we understand today as fractions were conceived in Wallis's time as proportionality so that 1 is to 0 (nothing) as ∞ is to 1.

a search for relationships, could take two paths: 1) *rupture* caused by leaving aside one meaning in favor of another which is eventually built for the task of integration; that is, changing the focus of the meaning, and 2) *continuity* by conserving the meaning in the integration task. The mathematical convention, then, as a product, can be interpreted as an emergent property to establish a relationship of continuity or rupture of meanings.

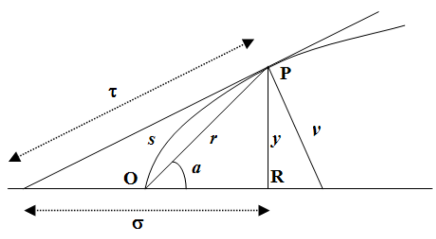
In our examples of Wallis's formulations, the search for coherence between the notion of index and the characteristic ratio (where the ratio/proportion has specific meanings which are different from considering it as a number) bring up two conventionalisms: the index of $y = \sqrt[2]{x}$ as 1/2 and diverse kinds of infinity represented by 1/0, 1/-1, 1/-2, etc. This indicates the *convenience and relative* character of the mathematical agreement with respect to the integration of the notions of index and characteristic ratio and the algebraic and graphic representations.

## 4. Articulation of the trigonometric functions to the corpus of Eulerian analysis

### 4.1. Trigonometric quantities as geometric quantities

According to Bos (1975) the main object of study in end of the 17th century mathematics was the curve. A curve in a system of reference (independently of the conceptions of this: as the sketch of a rule of construction, as a point in motion, or as a polygon with infinitely small sides, etc.) involves the relations between distinct variable geometric quantities defined with respect to a variable point on a curve. Such variable geometric quantities are, for example, (see Figure 3): ordinate, abscissa, arc length, radius, polar arc, subtangent, normal, tangent, area between curve and axis, circumscribed rectangle, solid of revolution, etc.

> "The relation between variables [ordinate, abscissa, radius, subtangent, among others] were expressed where possible by equations. This was not always possible, since just before the end of the 17th century there were no formulas for transcendent relations and these were expressed by means of explicative prose that basically expressed the geometric method for the construction of the curve" (Bos, 1975).




*x*: abscissa, *y*: ordinate, *s*: arc length, *r*: radius, *a*: polar arc, σ: subtangent, τ: tangent, *v*: normal, OPR: area between the curve and the abscissa

*Figure 3*. Variable geometric quantities defined in relation to a variable point on a curve

The search for relations between geometric quantities triggered the search for diverse methods for its achievement. This produced, among other aspects, different infinite series which established the relations between the quantities. For example Gregory, (Malet, 1994-1994) establishes, around 1670, that in the cycloid MOPA (Figure

4) that is generated by point A and called DA=$r$ and Dβ=$b$, the ordinate PQ of any point on the cycloid can be expressed in terms of the abscissa $a$ = OQ by

$$\frac{ra}{b}-\frac{r^2a^2}{2b^3}+\frac{r^3a^3}{2b^5}-\frac{ra^3}{6b^3}+\frac{7r^2a^4}{24b^5}-\frac{5r^4a^4}{8b^7}+\frac{7r^5a^5}{8b^9}-\frac{r^3a^5}{2b^7}+\frac{ra^5}{120b^5}.$$

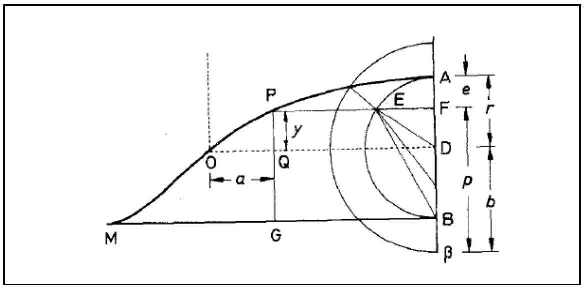
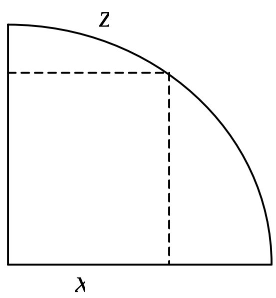



*Figure 4*. Geometric quantities present in a Cycloid according to Gregory (Malet, 1994-1994)

In the same sense Babini (1978, p. 121), affirms that among the series sent by Gregory to Collins, in correspondence dated in 1671, are the following equations to find the arc, given the tangent, and the tangent given the arc (where $r$ is the radius of a circle, $a$ the arc and $t$ the tangent):

$$a=t-\frac{t^3}{3r^2}+\frac{t^5}{5r^4}-\frac{t^7}{7r^6}+\frac{t^9}{9r^8}-etc.$$
$$t=a+\frac{a^3}{3r^2}+\frac{2a^5}{15r^4}+\frac{17a^7}{5r^6}+\frac{62a^9}{2835r^8}+etc.$$

Around the same time Newton (1669), through his method of fluxions, found the relations between an arc ($z$) and its corresponding sine ($x$) in a circle of radius 1 (Figure 5).




$$x=z-\frac{z^3}{6}+\frac{z^4}{24}-\frac{z^6}{720}+\frac{z^8}{40320}-\frac{z^{10}}{3628800}+\ldots$$

$$z=x+\frac{x^3}{6}+\frac{3x^5}{40}+\frac{5x^7}{112}+\ldots$$

*Figure 5*. Series that relates an arc (z) and its corresponding sine (z) (Newton, 1669)

## 4.2. Trigonometric quantities as analytical variables

Montiel (2005) mentions that it was, perhaps, the new uses of the trigonometric functions that took away their geometric character, described above, since they went from being considered lines of circle to quantities that described certain phenomena, particularly periodic movements. As soon as the study of motion was underway and adequate mathematical instruments became available, and insofar as his laws began to be introduced as the foundation of physics, it became apparent that it was not possible to continue considering the determined number or its geometric equivalents (point, straight line, circle, etc.) as the only object of the investigations (Loi, 199l cited in Montiel, 2005). In other words, the mathematical entity was no longer the number: the law of variation, the function, became the center around which science was organized. According to Katz (1987):

> "…no textbook until 1748 dealt with the calculus of these functions. That is, in none of the dozen or so calculus texts written in England and the continent during the first half of the 18th century was there a treatment of the derivative and integral of the sine or cosine or any discussion of the periodicity or addition properties of these functions. This contrasts sharply with what occurred in the case of the exponential and logarithmic functions. We attempt here to explain why the trigonometric functions did not enter calculus until about 1739. In that year, however, Leonhard Euler invented this calculus. He was led to this invention by the need for the trigonometric functions as solutions of linear differential equations. In addition, his discovery of a general method for solving linear differential equations with constant coefficients was influenced by his knowledge that these functions must provide part of that solution." (Katz, 1987, p. 311)

In this way, Euler, in his book *Introductio in analysin infinitorum* (Euler, 1738/1845) provides a treatment of what can be called the precalculus of the trigonometric functions. He defines them numerically, discusses several of their properties including formulas of addition and the development of his series of powers, with which he gave them the status of function[6]. In the first volume, Chapter VIII, *Des Quantités transcendantes qui naissent du cercle* (Figure 6), he defines the trigonometric functions as transcendent quantities that are born from the circle and points out that π is the semicircumference of a circle (of radius 1) and in consequence is the length of the arc of 180° and then he establishes sin 0π =0, cos 0π = 1, sin 2π =0 and cos 2π = 1.

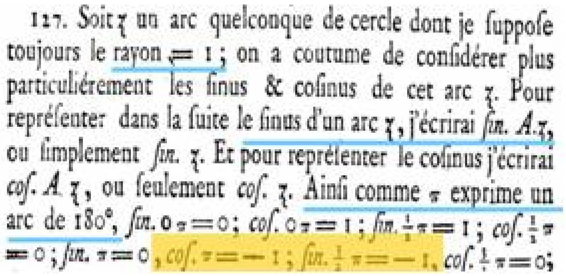

117. Soit ʒ un arc quelconque de cercle dont je ſuppoſe toujours le rayon = 1; on a coutume de conſidérer plus particuliérement les ſinus & coſinus de cet arc ʒ. Pour repréſenter dans la ſuite le ſinus d'un arc ʒ, j'écrirai *ſin. A.ʒ*, ou ſimplement *ſin.* ʒ. Et pour repréſenter le coſinus j'écrirai *coſ. A.* ʒ, ou ſeulement *coſ.* ʒ. Ainſi comme π exprime un arc de 180°, *ſin.* 0 π = 0; *coſ.* 0 π = 1; *ſin.* $\frac{1}{2}$ π = 1; *coſ.* $\frac{1}{2}$ π = 0; *ſin.* π = 0, *coſ.* π = − 1; *ſin.* $\frac{3}{2}$ π = − 1, *coſ.* $\frac{3}{2}$ π = 0;

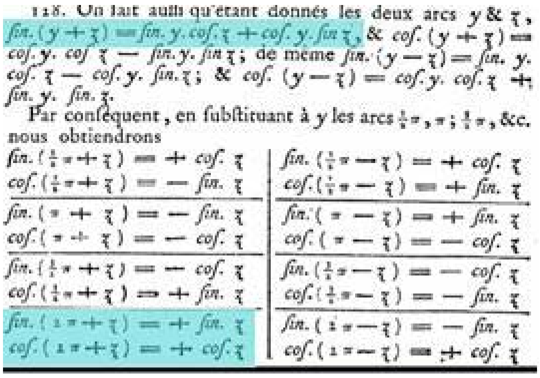
*Figure 6*. Transcendent quantities that are born from the circle (Euler, 1738/1845. p. 93)

[6] We remember that for Euler *"A function of a variable quantity is an analytic expression formed arbitrarily with this variable and with numbers or constant quantities"* Euler (1738/1845, p. 3) and that when he establishes: *"...analytic expression formed arbitrarily...*" he is accepting the use of the usual algebraic operations such as addition, multiplication, differences, quotients and transcendent operations like exponential, logarithmic and trigonometric. He also admits the extension of these to the infinite and the solution of algebraic equations – including to the infinite – where the constants can be included in complex numbers.

Euler does not mention, why, for example, $\cos\pi = -1$. The information we have up to now only allows us to speculate that perhaps Euler used the sine formula of the sum of two arcs: & $\cos(A+B) = \cos A\cos B - \sin A\sin B$ to build the convention $\cos\pi = -1$. One possible reasoning is the following:

> Supposing we want to assign a meaning to the symbol $\cos\pi$. What meaning will it take? If we take the formula
>
> $$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
>
> as our knowledge base, which we want to preserve, it must follow that
>
> $$\cos\pi = \cos\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \cos\frac{\pi}{2}\times\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\times\sin\frac{\pi}{2} = 0-1 = -1$$
>
> for which we must agree that $\cos\pi = -1$.

The previous conjecture is supported when we note that in a following article Euler makes a series of calculations based on the sine and cosine formulas of the sum of two arcs (Figure 7).



118. On sait aussi qu'étant donnés les deux arcs $y$ & $z$, $\text{ſin.}(y+z) = \text{ſin.}\,y.\,\text{coſ.}\,z + \text{coſ.}\,y.\,\text{ſin.}\,z$, & $\text{coſ.}(y+z) = \text{coſ.}\,y.\,\text{coſ.}\,z - \text{ſin.}\,y.\,\text{ſin.}\,z$; de même $\text{ſin.}(y-z) = \text{ſin.}\,y.\,\text{coſ.}\,z - \text{coſ.}\,y.\,\text{ſin.}\,z$; & $\text{coſ.}(y-z) = \text{coſ.}\,y.\,\text{coſ.}\,z + \text{ſin.}\,y.\,\text{ſin.}\,z$.

Par conſéquent, en ſubſtituant à $y$ les arcs $\frac{1}{2}\pi$, $\pi$; $\frac{3}{2}\pi$, &c. nous obtiendrons

| | |
|---|---|
| $\text{ſin.}(\frac{1}{2}\pi + z) = +\text{coſ.}\,z$ | $\text{ſin.}(\frac{1}{2}\pi - z) = +\text{coſ.}\,z$ |
| $\text{coſ.}(\frac{1}{2}\pi + z) = -\text{ſin.}\,z$ | $\text{coſ.}(\frac{1}{2}\pi - z) = +\text{ſin.}\,z$ |
| $\text{ſin.}(\pi + z) = -\text{ſin.}\,z$ | $\text{ſin.}(\pi - z) = +\text{ſin.}\,z$ |
| $\text{coſ.}(\pi + z) = -\text{coſ.}\,z$ | $\text{coſ.}(\pi - z) = -\text{coſ.}\,z$ |
| $\text{ſin.}(\frac{3}{2}\pi + z) = -\text{coſ.}\,z$ | $\text{ſin.}(\frac{3}{2}\pi - z) = -\text{coſ.}\,z$ |
| $\text{coſ.}(\frac{3}{2}\pi + z) = +\text{ſin.}\,z$ | $\text{coſ.}(\frac{3}{2}\pi - z) = -\text{ſin.}\,z$ |
| $\text{ſin.}(2\pi + z) = +\text{ſin.}\,z$ | $\text{ſin.}(2\pi - z) = -\text{ſin.}\,z$ |
| $\text{coſ.}(2\pi + z) = +\text{coſ.}\,z$ | $\text{coſ.}(2\pi - z) = +\text{coſ.}\,z$ |

*Figure 7.* Use of the sine and cosine formulas of the sum of two arcs (Euler, 1738/1845, p. 93)

In the same sense we have been able to interpret that the articulation of the trigonometric functions to Eulerian analysis was possible through the "analitization" of the *quantities that are born from the circle* through the following relations:

(A) $\sin^2 z + \cos^2 z = 1$
(B) $\sin(y+z) = \sin y\cos z + \cos y\sin z$
(C) $\cos(y+z) = \cos y\cos z - \sin y\sin z$

Using the above relations Euler breaks down (A) into

(D) $(\cos z + i\sin z)(\cos z - i\sin z) = 1$

And using (B), (C) and (D) he finds the relation

(E) $(\cos z + i\sin z)(\cos y + i\sin y) = \cos(y+z) + i\sin(y+z)$

Using (E) he finds that $\cos nz = \dfrac{(\cos z + i\sin z)^n + (\cos z - i\sin z)^n}{2}$

Finally (Figure 8) developing the above powers and using the notions of "infinitely large whole number" and "infinitely small quantity" he finds that

$$\cos y = 1 - \frac{y^2}{1\cdot 2} + \frac{y^4}{1\cdot 2\cdot 3\cdot 4} - \frac{y^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \ldots .$$

So that the cosine emerges as a function in Euler's sense.

134. Soit $z$ un arc infiniment petit, alors *fin.* $z = z$, & *cof.* $z = 1$; foit en même temps $n$ un nombre infiniment grand, pour que l'arc $nz$ foit d'une grandeur finie, pour que $nz$, par exemple, $= v$; à caufe de *fin.* $z = z = \frac{v}{n}$, on aura

$$cof.\ v = 1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3.4.5.6} + \&c. \ \&$$

EULER, *Introduction à l'Anal. infin.* Tome I. N

*Figure 8*. Use of sine and cosine formulas of the sum of two arcs (Euler, 1738/1845, p. 97)

Insofar as the process through which the trigonometric functions were made part of differential and integral calculus, this was grounded in the development of powers and the relations of sine and cosine of the sum of two arcs (Euler, 1755). Let us take an extract from *Institutiones Calculi Differentialis* (Euler, 1755) where the differential of the sine (Figure 9) is calculated using (B) the series of powers of sine and the consideration that sine ($dx$) = $dx$ if $dx$ is infinitely small.

140 CAPUT VI.

& fin $x$ denotet eius finum, cuius differentiale inveftigemus. Ponamus $y =$ fin $x$, ac pofito $x+dx$ loco $x$, quia $y$ abit in $y+dy$, erit $y+dy =$ fin $(x+dx)$, & $dy =$ fin $(x+dx) -$ fin $x$. Eft autem fin $(x+dx) =$ fin $x$ . cof $dx$ + cof $x$ . fin $dx$. atque cum fit, uti in introductione oftendimus

$$\text{fin } z = \frac{z}{1} - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \&c.$$

$$\text{cof } z = 1 - \frac{z^2}{1.2.} + \frac{z^4}{1.2.3.4} - \&c.$$

erit reiectis terminis evanefcentibus cof $dx = 1$ & fin $dx = dx$, unde fit fin $(x+dx) =$ fin $x + dx$ cof $x$. Quare, pofito $y =$ fin $x$, erit $dy = dx$ cof $x$. *Differentiale ergo finus arcus cuiufvis aequatur differentiali arcus per cofinum multiplicato.* Si igitur fuerit $p$ functio quaecunque ipfius $x$, erit fimili modo $d$ . fin $p = dp$ cof $p$.

*Figure 9*. Use of sine formulas of the sum of two arcs for the calculation of the sine differential (Euler, 1755. p. 140)

## 5. Conceptual breaks in the scholar construction of the TF

From the point of view of the articulation of scholar mathematics we have been able to affirm that there are different conventions in the scholar structure of the trigonometric functions that can be interpreted, in turn, as carriers of conceptual breaks: 1) the transit from degree-radian-real for the trigonometric functions and 2) negative angles and angles greater than 360º. In this respect we have been able to interpret that in scholar construction of the trigonometric functions in the Mexican education system, the definition of negative angles and angles greater than 360º and the transit to e use of radians are the product of a "*scholar mathematics convention*" for the definition of the trigonometric functions as real variable functions.

This last aspect, in relation to the scholar use of radians as a step prior to the definition of the real domain of the trigonometric functions, brings about a series of conceptual breaks that are the origin of a varying number of didactic phenomena in relation to the status of the trigonometric functions in the framework of Differential and Integral Calculus. In general terms, we consider the didactic phenomena mentioned to be subsidiaries of at least two *social practices*[7] that are reproduced in school settings. The first consists of considering radians as another system of measurement of angles that fulfills the same functions as the sexagesimal or any other measurement system. This practice is easily detected in textbooks in those sections dedicated to practicing the rules of transformation of units from one system to another. The second practice we have detected consists of the *dethematization* (that is, considering them as an object of study from the conceptual point of view) of the transit of radians to real numbers as an argument of the trigonometric functions.

The two practices above determine starkly different conceptions that students and teachers have at Mexican middle school level (students between 12 and 15 years) in relation to the trigonometric functions. An example of such conceptions is that which causes the belief that the domain of the trigonometric functions is dimensional with the unit in degrees or radians. This conception makes it impossible, on mixing the values of $x$ with real numbers and quantities in degrees, to properly interpret diverse frequently used expressions in Differential and Integral Calculus, for example: $f(x) = x + \sin x$, $f(x) = x^2 + \sin x$, $\int_0^1 (\sin x)dx = -\cos x\Big|_0^1$ o $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

The methodology we have followed to identify and classify the phenomena mentioned has been the realization of different analyses such as textbook analysis and the analysis of interviews with teachers and students in Mexican middle schools. Below we show some evidence to support the foregoing affirmations[8].

### 5.1 Breaks present in middle school textbooks.

After analyzing different textbooks used in Mexican middle school (MMS) education we can identify the presence of a common pattern in the construction of the TF which consists of following the transitions degrees →radians →real in the domain of the TF (See Figure 10).

[7] Socioepistemology shares the inclusion of a social and cultural vision in the discipline and specifically contributes to the search for "that" (which we call social practice) which being present in culture and thought is not part of scholar knowledge; it does, however, make possible its construction and diffusion.

[8] See (Méndez, 2008) for more details.

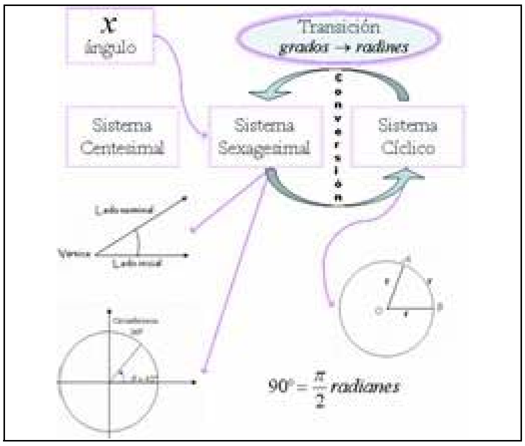



*Figure 10*. Pattern in the construction of the
in textbooks

There are two important points to bring out from the textbook analysis. The first is the observation that the reason for the sudden appearance of radians as a measurement of angle is never made explicit. The second consists in observing the *dethematization* (that is, considering them as an object of study from the conceptual point of view) of the transit from radians to real numbers as an argument of the trigonometric functions (see Figure 11). This can be perceived in the phrases presented in the textbooks, for example: "*it is commonplace to omit the word radians", "when the value of an angle is used in radians, the units are not normally given*", "*for convenience and simplicity we will omit the word radians"*.

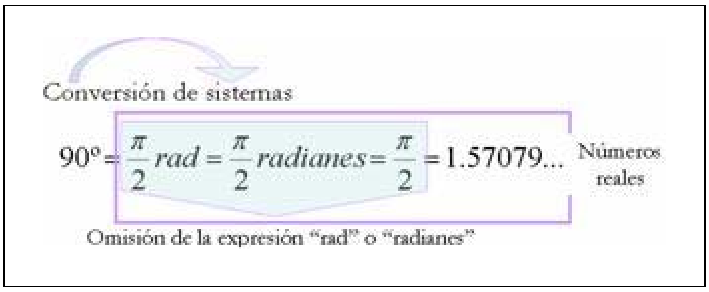



*Figure 11*. Textbook pattern in the *dethematization* of the
transit from radians to real numbers

## 5.2 Breaks present in MMS teachers

Based on the findings from the textbook analysis an interview was designed with four teachers from different MMS institutions. The interview had two phases. The first phase consisted of five activities aimed at detecting the teachers' conception of the domain (the value of *x* in its different possibilities as degrees, radians, or real numbers) and images of the trigonometric functions. The second phase of three activities was aimed at detecting the teachers' conceptions of the significance of operations between the trigonometric functions and algebraic functions.

The main sign of a conceptual break was found when teachers were faced with deciding in what moment to utilize the sexagesimal, the cyclical or real number system to measure angles. For example, in one of the activities a teacher assigns real values to *x*, whereas in the *x* of the sin *x* he assigns values in degrees in spite of both expressions constituting a single expression, that being an addition or a reason. Towards the end of the activity he realizes that he has given values to $x$ in one system and in another and thought it impossible to do so. Nevertheless, he makes no correction to the activity given that he was still not convinced which angular measurement he should use at what time (see Figure 12). Similarly, the confusion over which angular measurement to use in certain activities was such that in several the teachers simply did not answer arguing that these topics are not dealt with in MMS and mentioning that they are complex themes which, due to lack of time, can only be seen as theorems or characteristics of the TF.

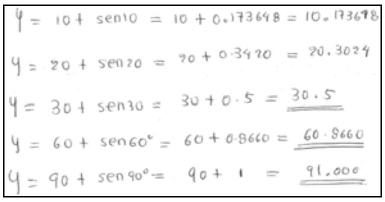
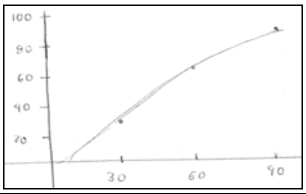
<table>
<tr><td colspan="2">Activity 2.1 How would you explain to a student the construction of the graph of the function $y = f(x) = x + \sin(x)$?</td></tr>
<tr><td>At this point the interviewee tries to clarify that the values of $x$ are in degrees, so he adds the symbol ° (for degree) to the last two expressions, nevertheless he does not do this in the expression of $x$ but only in the expression $\sin(x)$</td><td>He gives values to $x$ without determining if they are degrees or radians and he begins to substitute the values of $x$ in $y = x + \sin(x)$<br><br></td></tr>
<tr><td>Finally, trying to explain his use of degrees he realizes that he would be adding a real number to an angular measurement expressed in the sexagesimal system. He then says that this can't be done and that the values of $x$ were, from the start, real numbers.</td><td>He draws the expression in $y = x + \sin(x)$ in the following way.<br><br></td></tr>
</table>

*Figure 12.* Assignation of values to the variable $x$

## 6. In conclusion

In the framework of the study of the processes present in the articulation of conceptual mathematics systems which we have called *processes of mathematics convention and articulation* (Martínez-Sierra, 2003, 2005), we have presented the advances in our quest to identify the processes on mathematics convention present in the articulation of the trigonometric functions to the corpus of Eulerian analysis and presented the interpretations which this analysis supports to illustrate the conceptual breaks present in scholar construction of the trigonometric functions.

Insofar as the studies on the articulation of the trigonometric functions to the corpus of Eulerian analysis we have demonstrated the presence of mathematics conventions that enable the achievement of such articulation. In particular, through analysis of Euler's work (1738, 1755) we have been able to interpret that the articulation mentioned was possible through the "analitization" of the *quantities that are born from a circle* through the following relations: 1) $\sin^2 z + \cos^2 z = 1$, 2) $\sin(y+z) = \sin y \cos z + \cos y \sin z$ and 3) $\cos(y+z) = \cos y \cos z - \sin y \sin z$. Based on this, Euler (1738) could develop a series of powers to the functions *sin* and *cos* and so elevate them to the quality of functions (in the Eulerian sense) and also, for the first time, (Katz, 1987) the trigonometric functions became part of differential and integral calculus in (Euler, 1755) based on the development of powers and the above relations.

In the same vein, but from the point of view of the articulation of scholar mathematics, we have been able to interpret that in the scholar construction of the trigonometric functions in the Mexican education system, the definition of negative angles and angles greater than 360º and the transit to the use of radians are the product of a "*scholar mathematics convention*" for the definition of the trigonometric functions as real variable functions.

This last aspect, in relation to the use of radians in school as a step prior to the definition of the real domain of the trigonometric functions, brings about a series of conceptual breaks which are the origin of a varied number of didactic phenomena in relation to the status of the trigonometric functions in the framework of Differential and Integral Calculus. In general terms, we consider the didactic phenomena mentioned to be subsidiaries to at least two *social practices* that are reproduced in school settings. The first consists of considering radians as another system of angular measurement that fulfills the same functions as the sexagesimal or any other measurement system. This practice is easily detected in textbooks in those sections dedicated to practicing the rules of transformation of units from one system to another. The second practice we have detected consists of the *dethematization* (that is, considering them as an object of study from the conceptual point of view) of the transit of radians to real numbers as an argument of the trigonometric functions.

**BIBLIOGRAFIA**


- Babini, J. ,1978, *El cálculo diferencial. Origen y polémica*. México: Balsal Editores S.A.
- Bos, H.J.M. ,1975, "Differentials, Higher-Order Differentials and the derivative in the Leibnizian Calculus". *Archive for History of Exact Science* **14**, 1-90.
- Buendía, G. y Cordero, F. ,2005, "Prediction and the periodical aspect as generators of knowledge in a social practice framework. A socioepistemological study. *Educational Studies in Mathematics* **59(2)** 299 - 333.
- Cantoral, R. & Farfán, R. M. ,2003, "Mathematics Education: A vision of its evolution". *Educational Studies in Mathematics* **53**, 255–270.
- Cantoral, R. & Farfán, R. M. ,2004,. "La sensibilité à la contradiction: logarithmes de nombres négatifs et origine de la variable complexe". *Recherches en Didactique des Mathématiques* 24(2.3), 137 - 168.
- Euler, L. ,1738/1845, *Introduction a l'analyse infinitésimale*. París, Francia: L'Ecole Polytechnique.
- Euler, L. ,1755/1787, *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum* (Vol. 1). TICINI. Tiphographeo Petri Galeati.
- Farfán, R.M. y Martínez, G. ,2002, "Explicación de algunos fenómenos didácticos ligados a las convenciones matemáticas de los exponentes". En C. R. Crespo (Ed.) *Acta Latinoamericana de matemática Educativa*. Vol. 15 Tomo I. (pp. 225-231). México: Grupo Editorial Iberoamérica.
- Ferrari, M. y Farfán, R. M. ,2004, "La covariación de progresiones en la resignificación de funciones". En L. Díaz (Ed*.). Acta Latinoamericana de Matemática Educativa* Vol. XVII. (pp. 145 -149).
- Katz, V. ,1987,. "The Calculus of the Trigonometric Functions". *Historia Mathematica* **14**, 311-324
- Lezama, J. ,2003, *Estudio de la reproducibilidad*. Cinvestav Doctoral Thesis - IPN. México.
- Lezama, J. ,2005, "Una mirada socioepistemológica al fenómeno de reproducibilidad**".** *Revista Latinoamericana de Investigación en Matemática Educativa* **8(3)**, 287-318
- Loi, M. ,1999, *Pensar la Matemática*. España: Tusquets Editores.
- Newton, I. ,1669,. "De Analysi per equationes infinitas (june 1669)". En D.T. Whiteside (1967) (Ed.), *The mathematical papers of Isaac Newton*. Vol. II (1667-1700) (pp. 206-247). Gran Bretaña: Cambridge University Press.
- Malet, A. ,1993-94, "James Gregorie on Tangents and the 'Taylor' Rule for Series Expansions". *Archive for History of Exact Science* **46**, 97-137.
- Martínez-Sierra, G. ,2002, "Explicación sistémica de fenómenos didácticos ligados a las convenciones

de los exponentes". *Revista Latinoamericana de Investigación en Matemática Educativa* **5(1)**, 45-78.
- Martínez-Sierra, G.,2003, *Caracterización de la convención matemática como mecanismo de construcción de conocimiento. El caso de de su funcionamiento en los exponents*. CICATA Doctoral Thesis - IPN. México.
- Martínez-Sierra, G.,2005, "Los procesos de convención matemática como generadores de conocimiento". *Revista Latinoamericana de Investigación en Matemática Educativa* **8 (2)**, 195-218.
- Méndez, C. ,2008, *Sobre la construcción escolar de las Funciones Trigonométricas: La transición grados → radianes → reales en el Nivel Medio Superior*. Masters Disertation. University of Guerrero-México
- Montiel G. ,2005, *Estudio socioepistemológico de la función trigonométrica*. CICATA Doctoral Thesis-IPN, México.
- Struik, D. J. ,1986, *A source book in mathematics 1200-1800*. EEUU: Princeton University Press.