BROUWER'S INTUITIONISM AS A SELF-INTERPRETED MATHEMATICAL THEORY

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ABSTRACT

We introduce the concept of a self-interpreted mathematical theory, construing Brouwer's intuitionistic analysis as an important example of such a theory. Brouwer's aim was to show evidence of all the mathematical properties of the continuum by unfolding its intuitionistic meaning, without using any axioms. Our criticism of Kleene's formalization of intuitionistic analysis derives from this point of view. We give a reconstruction of Brouwer's analysis from the self-interpreted theory point of view, based only on definitions of the fundamental concepts. We argue though, that Brouwer's proof of fan theorem is not intuitionistically acceptable, therefore a different approach on the proof of fan theorem is needed, in order to secure the self-interpreted character of intuitionistic analysis. Finally, we discuss the benefits of incorporating elements of intuitionistic analysis into teaching mathematical analysis, based on some crucial points of our reconstruction of Brouwer's intuitionism.

1 The concept of a self-interpreted mathematical theory

A mathematical theory T is called *self-interpreted* (siT) if it has a unique mental interpretation and this interpretation, not only is part of T, but also generates its syntax. In a siT its formal part (syntax) is a mathematical development of the fundamental intuitions of its interpretation (semantics). A siT contains no axioms, but every proposition derives from the definitions of the related concepts.

The major examples of siT are the following:

Platonic Euclidean geometry (PEG): it consists of the greater part of the Euclidean Elements (except e.g., the axiomatic Book V) and the Platonic philosophical interpretation of the Euclidean geometric concepts. The Euclidean definitions of them are actually of Platonic nature, creating an important link between mental interpretation and mathematical language. Proclus tried to give a 'philosophical proof' of the Euclidean Postulates through the concept 'flowing of the point', using only the Euclidean definitions, in order to secure the siT-character of Euclid's geometry[1].

Brouwer's intuitionistic arithmetic (BA): it is the development of arithmetic from the mental construction which corresponds to a natural number. This construction is founded on Brouwer's primordial intuition of time two-ity[2]. Though Brouwer provided a sketch of his arithmetic (see e.g., [Brouwer 1981] p.90), only Heyting followed his line of presentation in [Heyting 1956] (pp.13-15). A fuller reconstruction of Brouwer's

[1]For the above siT-character of Euclidean geometry see [Farmaki, Negrepontis 2008], who show the platonic nature of the Euclidean definitions and their role in turning postulates into proved propositions.

[2]I.e., the a priori intuition of retaining in memory a *previously* distinguished object and *connecting* it to another object, distinguished *later*, forming that way a new composite object.

arithmetic can be found in [Petrakis 2007, 2008], in which the difference between BA and the formal theory of Heyting's arithmetic is explained.
Brouwer's intuitionistic analysis (BIA): it is the mathematical development of the concept of the intuitionistic continuum, which is described as an appropriate spread, based on a corpus of intuitionistic principles and concepts. Brouwer never used axioms in BIA and he tried to justify all his principles on conceptual grounds.

All the above theories are constructive, i.e., they contain, tacitly or not, some constructive principles which guide the execution of proofs *and* the formation of concepts. In that way, the truth of the fundamental intuitions or concepts is preserved by the defined concepts and proofs. A *siT* is trivially categorical, since there is only one interpretation associated to it, consistent, as long as the fundamental mental interpretations are non-contradictory, and complete, in the sense that a formula of a *siT* is true only if there is some constructively persuasive evidence for it.

In our opinion, it is impossible for a theory to be a *siT* without being constructive. Of course, there are theories with a constructive character, such as various versions of post-Brouwer intuitionistic arithmetic or analysis, which are not self-interpreted. We shall only indicate in section 5 why post-Brouwer intuitionism lost every aspect of the *siT*-character of Brouwer's intuitionism. Post-Brouwer intuitionistic theories, Kleene's system FIM[3] being the best known, became *formal theories* i.e., theories originating from a syntax, which generates a multiplicity of semantics. As any formal theory, post-Brouwer intuitionistic theories had to deal with their consistency, categoricity and completeness. The metamorphosis of the self-interpreted Brouwer's intuitionism to a variety of formal theories brought intuitionism, on the one hand, closer to classical mathematics, but on the other hand, carried it away from the methodological principles of his philosophically motivated inventor. The general view though, is that "it was Brouwer's vision what theory should be attained and Kleene's insight which first found implicit in Brouwer a usable, more conventionally mathematical (axiomatic) road to that theory"[4]. Myhill, for example, insisted that Brouwer's views were not adequately taken into account[5], but he was referring to specific technical issues, rather than on the general foundational scheme. It was Heyting who, commenting on his first intuitionistic logic papers, regretted the fact that these papers '*diverted the attention from the underlying ideas* (of Brouwer) *to the formal system itself*'[6]. There is a remarkable historical analogy between the pairs (Brouwer's intuitionism, post-Brouwer intuitionism) and (Euclidean geometry, axiomatic-Hilbertian geometry). The first elements of the pairs are self-interpreted theories which were translated, in the course of time, to formal (axiomatic) theories. Hilbert's 'Grundlagen' though, are neither constructive nor self-interpreted; actually the syntactical part of it is not interpreted at all. Similarly, the syntax of post-Brouwer intuitionistic analysis is on its own uninterpreted. But Brouwer's view, already from his early period, was that[7]

> [A logical construction of mathematics, independent of the mathematical intuition, is impossible - for by this method no more is obtained than a linguistic structure, which irrevocably remains separated from mathematics

[3] See [Kleene-Vesley 1965].
[4] These are words of Vesley in [Vesley 1980] p.328.
[5] See [Myhill 1968].
[6] See [Heyting 1978] p.15.
[7] See [Brouwer 1907] p.97.

- and more-over it is a contradictio in terminis - because a logical system needs the basic intuition of mathematics as mathematics itself needs it.]

PEG and BIA though, are not fully successful self-interpreted theories: a philosophical proof of the Fifth Postulate has not been given and Brouwer's proof of Fan theorem is not, as we argue in section 5, intuitionistically acceptable. A new kind of proof of Fan theorem is needed in order to be an intuitionistic truth.

In the next section we present the main definitions of BIA. It is the difference between intuitionistic and classical concepts that explains the difference between BIA and classical analysis[8].

2 Basic constructive definitions

Since 1918, Brouwer has described the intuitionistic continuum as a certain spread. A *spread* M is determined through the spread-law Λ_M and (if necessary) the complementary law Γ_M. Λ_M decides which finite sequences of natural numbers[9] are accepted or not. Namely Λ_M:
(i) decides which naturals are admitted as 1-sequences (i.e., as sequences of length 1).
(ii) accepts $(a_1, a_2, ..., a_n)$, if it accepts $(a_1, a_2, ..., a_n, a_{n+1})$.
(iii) decides, for any natural k, whether it accepts $(a_1, a_2, ..., a_n, k)$ or not, if it accepts $(a_1, a_2, ..., a_n)$.
(iv) accepts $(a_1, a_2, ..., a_n, k)$, for some k, if it accepts $(a_1, a_2, ..., a_n)$.
So, Λ_M determines a tree with its branches corresponding to the admissible by Λ_M finite sequences or *nodes* of M[10]. By (iv), all paths of the tree are potentially infinite and they are called *choice sequences of the spread* M[11]. Each a_1 is the *root* $<>$ of the tree.
Γ_M corresponds to any Λ_M-accepted sequence an *already* constructed mathematical object. So, if $(a_1, a_2, ..., a_k, ...)$ is an M-(choice) sequence, then by the following correspondences of Γ_M

$$\begin{gathered} (a_1) \mapsto b_1 \\ (a_1, a_2) \mapsto b_2 \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ (a_1, a_2, ..., a_k) \mapsto b_k \\ \cdots\cdots\cdots \end{gathered}$$

an M-sequence of mathematical objects, other than naturals, is constructed. Each M-sequence is an *infinitely proceeding sequence* (i.p.s), of which we know at any moment only a finite initial segment i.e., an M-sequence is an *on-going mathematical object.*

[8]Feferman, discussing Brouwer's uniform continuity theorem (see section 5), in([Feferman 1997] p.222), remarks: [*...but once it is understood that Brouwer's theorem must be explained differently via the intuitionistic interpretation of the notions involved, an actual contradiction is avoided. Perhaps if different terminology had been used, classical mathematicians would not have found the intuitionistic redevelopment of analysis so off-putting, if not downright puzzling.*]

[9]Brouwer has previously justified the use of natural numbers, founding them on the intuition of time.

[10]The above definition does not specify the nature of Λ_M, only its function. This is not a problem, since BIA uses certain spreads and it is independent from a general theory of spreads.

[11]We will not study here choice sequences outside a spread.

If Λ_M admits any finite sequence of naturals, then M is the *universal spread* ω^ω, which corresponds to the classical Baire space $\mathcal{N}$ of all sequences of naturals[12]. But intuitionistic ω^ω is not a set, but a mechanism of construction of i.p.s. The notion of spread is one of Brouwer's most important conceptual innovations, since it *holds together* all the M-sequences, without *containing* them as a set. The spread concept derives from Brouwer's need to avoid the concept of absolutely infinite set.

A *subspread* K of M, $K \sqsubseteq M$, is a spread such that, if $(a_1, a_2, ..., a_n)$ is Λ_K-accepted, then it is Λ_M-accepted.

The most important spread is the *spread of real numbers* $\Re$. If we define the rational numbers in the classical way, and fix an enumeration $q_1, q_2, ..., q_n, ...$ of them, then $\Lambda_\Re$:
(i) accepts any natural number as a successor of the root $<>$.
(ii) accepts $(a_1, a_2, ..., a_n)$, if it accepts $(a_1, a_2, ..., a_n, a_{n+1})$.
(iii) accepts $(a_1, a_2, ..., a_n, a_{n+1})$ iff it accepts $(a_1, a_2, ..., a_n)$ and

$$\mid q_{a_n} - q_{a_{n+1}} \mid < \frac{1}{2^n}$$

$\Gamma_\Re$ is defined by

$$(a_1, a_2, ..., a_n) \mapsto q_{a_n}$$

$\Lambda_\Re$ guarantees the extension of any $\Lambda_\Re$-admitted sequence $(a_1, a_2, ..., a_n)$, since there always exists rational q such that,

$$q_{a_n} - \frac{1}{2^n} < q < q_{a_n} + \frac{1}{2^n}$$

But q is a q_k, for some k, so $(a_1, a_2, ..., a_n, k)$ is $\Lambda_\Re$-admitted. Of course, this q is not unique, so the extension of $(a_1, a_2, ..., a_n)$ is not absolutely determined by $\Lambda_\Re$. So, Λ_M is, generally, a *non absolutely deterministic law*. Through $\Gamma_\Re$ the sequence $(q_{a_1}, q_{a_2}, ..., q_{a_n}, ...)$ determines an *intuitionistic real number*. Obviously, the intuitionistic continuum is a *holistic* continuum which generates its points, while the classical continuum is an *atomistic* continuum generated by its points as their sum (set). A spread M is called *splitting* iff $(\forall\alpha \in M)(\forall N)(\exists\beta \in M)(\exists M > N)(N_\beta = N_\alpha \;\wedge\; M_\beta \neq M_\alpha)$, i.e., if each finite M-sequence splits at some moment of its evolution into two different sequences.
A *fan* F is a finitely branching spread i.e., each finite Λ_F-admitted sequence can be extended only by finitely many naturals. A *subfan* T of F, $T \sqsubseteq F$, is a subspread of F. If the universal law applies only to 0-1 sequences, then we take the fan 2^ω, which corresponds to the classical Cantor space $\mathcal{C}$ of all 0-1 sequences[13].
An *intuitionistic ω^ω-function*, $\varphi : \omega^\omega \longrightarrow \omega$, is a law[14], such that φ corresponds an ω^ω-sequence α to a unique natural number $\varphi(\alpha)$, based on an initial segment of α of length N, N_α, for some N, or on any extension of it[15]. We call any such node a *critical node*

[12] If we define the metric

$$\rho(\alpha, \beta) = \begin{cases} \frac{1}{\eta(\alpha,\beta)} & , \text{if } \alpha \neq \beta \\ 0 & , \text{if } \alpha = \beta \end{cases}$$

where $\eta(\alpha, \beta)$ is the least index for which $\alpha \neq \beta$, then $\mathcal{N}$ is a metric space, and a spread, classically interpreted, is a closed set of $\mathcal{N}$. Not every closed set corresponds to a spread.

[13] A fan, classically interpreted, corresponds to a compact subset of $\mathcal{N}$.

[14] In Brouwer's words: "...by a function...we understand a law..." ([Brouwer 1927] p.458.

[15] There are many, more or less, equivalent formulations of the same concept.

for φ. So, φ is actually determined by a function φ^* on the finite ω^ω-sequences. This definition is natural, since α is an on-going object and its value must be determined some time in the course of its 'becoming'. If M is an arbitrary spread, an *intuitinistic M-function*, $\varphi : M \to \omega$, is defined likewise.
An *intuitionistic ω^ω-Function*, $\Phi : \omega^\omega \to \omega^\omega$, is a law which corresponds an ω^ω-sequence α to a unique ω^ω-sequence β, based on a law Φ^*, which correlates finite sequences of naturals such that:
(i) if $N \leq M$, then $\Phi^*(a_1, a_2, ..., a_N) \preceq \Phi^*(a_1, a_2, ..., a_M)$, where $\preceq$ means that the sequence $\Phi^*(a_1, a_2, ..., a_N)$ is an initial segment of the sequence $\Phi^*(a_1, a_2, ..., a_M)$.
(ii) Φ^* is not finally constant.
(iii) $\Phi(\alpha) = \sup_N \Phi^*(N_\alpha)$ i.e., $\Phi(\alpha)$ is approximated by the segments $\Phi^*(N_\alpha)$.
This definition is natural, since the image of an on-going object through Φ is another on-going object. If M_1, M_2 are arbitrary spreads an *M_1, M_2-Function* $\Phi : M_1 \to M_2$ is defined in the same way.

A *species* is a constructed property A on pre-determined intuitionistic objects. It is not identified with its extension, despite the fact that we call the objects which satisfy A *elements* of A, it avoids circularity, since it applies on pre-existent objects, and it is not a linguistic expression, since it is constructed. The constructive character of a species is studied in [Petrakis 2008], but it suffices to say here that a species depends on a *common mode of formation* of its elements. So, we study e.g., the species of the M-sequences, for some spread M, or its sub-species of those M-sequences which extend a finite M-segment $(a_1, a_2, ..., a_n)$.

A *well-ordered species* (w.o.s) A is defined inductively as follows:
(i) if A is one-element species, then it is a w.o.s.
(ii) if $A_1, A_2, ..., A_n$ are disjoint w.o.s., then their ordered sum $\bigoplus_{i=1}^n A_i$ is a w.o.s [16].
(iii) if $A_1, A_2, ..., A_n, ...,$ is a constructively given sequence of disjoint w.o.s., then their infinite ordered sum $\bigoplus_{i=1}^\infty A_i$ is a w.o.s.

A *bar* B for a spread M is a species of finite M-sequences such that, each infinite M-sequence α has an initial segment in B or *hits the bar* i.e., $\forall\alpha(\alpha \in M)(\exists n)(n_\alpha \in B)$. A *decidable* bar is one for which there is a method to say in finite time whether a finite M-sequence is in the bar or not. A *thin* bar B contains only the elements necessary to be a bar i.e., it satisfies: $(u \in B) \wedge (v \prec u)$, then $v \notin B$.

3 Programmatic Principles of Brouwer's Foundational Framework $\mathfrak{B}$

(P1) Mathematical objects, besides some fundamental intuitions, are constructions of the mind of a creating subject[17], based on the fundamental ones.
(P2) The fundamental intuitions are the intuition of time two-ity and the intuition of (time) continuum. The latter is formalized in terms of the discrete intuition of two-ity, through the concept of spread $\mathfrak{R}$.
(P3) On-going objects, such as M-sequences, are, for the first time in a constructive theory of the continuum, legitimate objects. It is the ontology of intuitionistic objects,

[16] $\bigoplus_{i=1}^n A_i$ is the union of A_i such that, each A_i preserves its order and $j < k$, then $a \prec b$, for $a \in A_j$ and $b \in A_k$. $\bigoplus_{i=1}^\infty A_i$ is defined analogously.

[17] Our reconstruction of $\mathfrak{B}$ stresses Brouwer's objective, rather than solipsistic, views. E.g., in [Brouwer 1981] p.90, Brouwer says that "the stock of mathematical entities is a real thing, for each person, and for humanity".

such as the M-sequences, which is responsible for the non-acceptance of logical principles, like the Principle of the Excluded Middle (PEM).
(P4) A completed mathematical object exists iff it has been constructed, while an ongoing object exists iff there is a procedure which guarantees its infinite becoming[18].
(P5) Mathematical formulations are mere linguistic structures, if they do not accompany genuine constructive thoughts. Absolute infinity is a linguistic concept completely separated from intuition. Therefore, the only legitimate infinity is the potential one. Generally, language is inferior to genuine thought, which is independent from language.
(P6) A mathematical proposition of BIA is true if persuasive constructive evidence is provided for it, based on the definitions of the concepts found in it.

In the next sections we show that the proof of continuity principle is compatible with these principles, but we argue that Brouwer's proof of fan theorem is not.

4 The Continuity Principle

Continuity Principle (CP) was accepted from Brouwer since 1918, as a self-evident truth. Post-Brouwer formal theories (e.g., FIM) considered it as an axiom[19]. In our reconstruction CP is proved in accordance to P6.

Continuity Principle: If $\varphi : \omega^\omega \to \omega$ is an intuitionistic ω^ω-function, then

$$(*) \qquad \forall\alpha(\alpha \in \omega^\omega)(\exists N)(\forall\beta, N_\beta = N_\alpha \longrightarrow \varphi(\beta) = \varphi(\alpha))$$

Proof[20]: If β is a sequence such that $N_\beta = N_\alpha$, then φ corresponds β to $\varphi(\alpha)$, since φ, by its definition, is activated only by N_α.$\diamond$

$(*)$ expresses the continuity of a function in Baire space $\mathcal{N}$[21]. CP is classically false, since the following function

$$\varphi(\alpha) = \begin{cases} 1 & , \text{if } \alpha \neq \overline{0} \\ 0 & , \text{if } \alpha = \overline{0} \end{cases}$$

where $\overline{0}$ is the constant sequence 0, does not satisfy (*). But the above φ is not an intuitionistic function since, if it were, it would correspond $\overline{0}$ to 0, based on a $N_{\overline{0}}$, for some N. But then, by CP, all sequences with N_0 as initial segment would take the same value 0. So, the following universal version of PEM is false in BIA:

$$(\alpha = \overline{0}) \ \vee \ (\alpha \neq \overline{0})$$

A simple corollary of CP is that ω^ω is not denumerable i.e., there exists no intuitionistic function $\varphi : \omega^\omega \xrightarrow{1-1} \omega$, and the same holds for any splitting spread, of which we know that it is non-empty (like $\Re$)[22]. Therefore, Brouwer reached the same conclusion for

[18] So, an M-sequence is known iff Λ_M is known. Contrary to Brouwer's early views, it is not necessary to know the evolution law of a sequence in advance.

[19] Kleene, in [Kleene 1969], defined a formal system, which generates classical analysis, if PEM is added, and formal intuitionistic analysis, if an expression of CP is added.

[20] $\alpha \in \omega^\omega$ means that α is an element of the species of infinite ω^ω-sequences and not that α belongs to a set. We avoid here though, a different symbolism. Also, $\forall\alpha$ is meant only with the potentially infinite way: "for each given α".

[21] In $\mathcal{N}$, if $\alpha \neq \beta$, then $\rho(\alpha, \beta) < \frac{1}{N} \to N_\alpha = N_\beta$.

[22] This is the intuitionistic version of the generalized Cantor theorem: a non-empty, perfect set of $\mathcal{N}$ has cardinality $2^{\aleph_0}$.

the continuum as Cantor did, but, according to P5, $\Re$ has not cardinality $2^{\aleph_0}$, it is only sequentially inexhaustible.
Since there is a retraction-Function $\Theta : \omega^\omega \to M$, which is the identity on M, CP proves easily the Continuity Principle for a spread M (CP(M))

$$\forall \alpha (\alpha \in M)(\exists N)(\forall \beta, N_\beta = N_\alpha \longrightarrow \varphi(\beta) = \varphi(\alpha))$$

where φ is an intuitionistic M-function[23].

5 An argument against the intuitionistic validity of Brouwer's proof of Fan theorem

Fan theorem is the most important proposition of BIA, since it proves the remarkable, from the classical point of view, following theorem.

Brouwer's uniform continuity theorem: If $\Phi : [\alpha, \beta] \to \Re$ is an intuitionistic Function, then Φ is uniformly continuous[24].

Brouwer's (1924) formulation of FT is essentially the following:

Brouwer's Fan theorem (FT): If T is a fan and $\varphi : T \to \omega$ an intuitionistic function, then there exists a natural N, such that, for each infinite T-sequence α, its value $\varphi(\alpha)$ is determined by its initial segment N. I.e.,

$$(**) \qquad \exists N \forall \alpha (\alpha \in T)(\varphi(\alpha) = \varphi^*(N_\alpha))$$

FT is classically false, since, for example, the following function $\varphi : 2^\omega \to \omega$, based on $\varphi^* : 2^{<\omega} \to \omega$:

$$\begin{gathered} \varphi^*(1,0) = 1 \\ \varphi^*(0,1) = 2 \\ \cdots\cdots\cdots\cdots \\ \varphi^*(\underbrace{0,0,...,0}_{n},1) = n+1 \\ \cdots\cdots\cdots\cdots \\ \varphi(\overline{0}) = 0 \end{gathered}$$

cannot be determined by a global bound N. Although φ is algorithmically defined on $\mathcal{C}$ and non $\mathcal{C}$-continuous at $\overline{0}$, it is not an intuitionistic function, since its value on $\overline{0}$ is not determined by any of its initial segments[25].

Brouwer's proof of FT is based on his famous Bar theorem. Note that, if $\varphi : M \to \omega$ is an intuitionistic M-function, then the species of the critical nodes for φ, B_φ, is a decidable bar. Also, if B is a decidable bar for M, then it contains a unique decidable thin bar B_0; for each $u \in M$ we find its shortest initial segment also in B. A node u is *secured* relative to M iff $(\exists v \preceq u,\ v \in B_0) \vee u \notin M$. An M-node u is B-*securable* iff every infinite M-sequence which extends u hits the bar B.

Bar theorem (BT): if B is a decidable[26] bar for M, then it contains a well-ordered

[23]Though we cannot exhaust here the results of BIA related to CP, it is worth mentioning that CP alone guarantees the continuity of a function $\Phi : \Re \to \Re$ ([Veldman 1982]).

[24]It can be shown that the intuitonstic interval $[\alpha, \beta]$ is a fan. Note that Φ is uniformly continuous just by being defined on $[\alpha, \beta]$.

[25]The essential difference between 2^ω and $\mathcal{C}$ is that $\mathcal{C}$-sequences, like $\overline{0}$, have an existence independent from their generation, while 2^ω-sequences exist only as Λ_{2^ω}-procedures.

[26]Decidability is not explicit in [Brouwer 1927] but it is necessary, as Kleene has shown.

thin bar.

Proof (an outline[27]): Brouwer interpreted the intuitionistic implication $P \Rightarrow Q$ as a constructive method, transforming every possible proof of P to a proof of Q. In BT P is "B is a decidable bar for M" and Q is "B contains a well-ordered thin bar B_0". Brouwer replaced the above P with its equivalent P': "$<>$ is B-securable".

Brouwer proved BT by analyzing the possible proofs of P'. Since there are, in general, infinitely many such proofs, he was forced to adopt a dogma, in order to avoid their infinity.

Brouwer's Dogma (BD)[28]: A proof, R_u, of the "u is B-securable" can be reduced to a *canonical proof* (c.p.), where *only* the following kinds of inference occur:

$$\frac{u \textit{ is secured}}{u \textit{ is securable}} \quad \eta\text{-inference}$$

$$\frac{(a_1,a_2,...,a_n) \textit{ is securable}}{(a_1,a_2,...,a_n,k) \textit{ is securable}} \quad \zeta\text{-inference}$$

$$\frac{(a_1,a_2,...,a_n,1) \textit{ is securable, } (a_1,a_2,...,a_n,2) \textit{ is securable,....}}{(a_1,a_2,...,a_n) \textit{ is securable}} \quad \digamma\text{-inference}$$

A $\digamma$-inference has infinitely many premises, meaning that there is a method generating these subproofs of the proof of securability of $(a_1, a_2, ..., a_n)$. Even if this is not totally clear, in the fan case a $\digamma$-inference becomes:

$$\frac{(a_1,a_2,...,a_n,k_1) \textit{ is securable, } (a_1,a_2,...,a_n,k_2) \textit{ is securable,...,} (a_1,a_2,...,a_n,k_m) \textit{ is securable}}{(a_1,a_2,...,a_n) \textit{ is securable}}$$

where $k_1, k_2, ..., k_m$ are the immediate successors of $(a_1, a_2, ..., a_n)$. Since we mostly care for the proof of FT rather than of BT, the application of BD on a fan is clear.

A securable node u has the well-ordering property (w.o.p.) iff the thin bar B_0^u which bars exactly the M-sequences which extend u is a w.o.s. *A subproof R of R_u has the well-ordering property* iff every node, the securability of which is established in the course of R, has the w.o.p. for nodes. Also, *a subproof R has the preservation property* (p.p.) iff the conclusions in R have the w.o.p. for nodes whenever their premises have the w.o.p. for nodes. Brouwer proves then the following facts:

Proposition 1: R_u has the p.p.

Proof: By BD we prove this inductively. It is trivial for an η-inference. In a ζ-inference, if $B_0^{(a_1,a_2,...,a_n)}$ is a w.o.s., then by an induction on the definition of a w.o.s., $B_0^{(a_1,a_2,...,a_n,k)}$ is a w.o.s. too. In a $\digamma$-inference $B_0^{(a_1,a_2,...,a_n)} = \bigoplus_{k=1}^{\infty} B_0^{(a_1,a_2,...,a_n,k)}$, where $B_0^{(a_1,a_2,...,a_n,k)}$ are appropriately inductively defined.$\diamond$

Proposition 2 R_u has the w.o.p.

Proof: R_u starts necessarily from η-inferences which have one-element well-ordering structure. Since R_u has the p.p., each conclusion of an inference in R_u has the w.o.p. too.$\diamond$

Proposition 3: u has the w.o.p.

Proof: Since u is the last conclusion in R_u and R_u has the w.o.p., then u has the w.o.p. too.$\diamond$

Conclusion of BT: B_0 is a w.o.s.

[27]The details can be found in [Brouwer 1927] or [van Atten 2004].

[28]This is a term of Martino and Giaretta.

Proof: Since P' is the hypothesis of a proof of the securability of $<>$, by BD, there is a c.p. $R_{<>}$, and by Proposition 4 for $u = <>$, $<>$ has the w.o.p., meaning that B_0 is a w.o.s.$\square$

Proof of FT via BT : In case M is a fan, $B_0^{(a_1,a_2,\ldots,a_n)} = \bigoplus_{k=1}^{m} B_0^{(a_1,a_2,\ldots,a_n,m)}$, for some $m \in \omega$. Since there can easily be proved, that an ordered sum generated by one-element species and finite sums only, is finite, then $B_0^{(a_1,a_2,\ldots,a_n)}$ is finite. Especially, the w.o.s. B_0 is finite, so it has a node of maximum length N. In case $B = B_\varphi$, for some function φ, N is the global bound of $(**)$.$\square$

Up till now, the problem with the above proof of BT (or FT) was centered around BD. Brouwer considered BD natural, though "as late as 1952 had to admit that a simpler proof (of FT) had eluded him"[29]. Epple (in [Epple 1997]) rightfully questioned the cognitive transparency of BD, as Brouwer's programmatic epistemology demands (P6). On the other side, van Atten[30] doesn't accept Epple's general tenet, believing that Brouwer's proof doesn't betray the intuitionistic principles, without giving though, some arguments in favor of his opinion. The obscurity of BD justified Kleene's reaction of using the scheme of bar-induction (BI_D)

$$\begin{gathered}(\forall \alpha \in M)\ \exists n(n_\alpha \in B)\ \ \wedge\\ (\forall u \in M)\ (u \in B \vee u \notin B)\ \ \wedge\\ (\forall u \in M)\ (u \in B \rightarrow u \in W)\ \ \wedge\\ (\forall u \in M)\ (\forall k(u \frown k \in W) \rightarrow u \in W)\ \longrightarrow\\ <> \in W\end{gathered}$$

where $u \frown k$ is an immediate successor node of u, as an axiom[31]. In case $u \in W$ means that u has the w.o.p. for nodes, then BT is directly derived. But to prove FT through an axiom violates P6 and the *siT*-character of intuitionism. Brouwer always searched for a conceptual proof of FT, not an axiomatic one. Kleene justified this seemingly evasion of his to postulate an axiom schema by his independence result of bar induction from the other intuitionistic principles[32].

The argument against Brouwer's proof: Someone could assert, like Brouwer, that BD is transparent enough to be accepted, but still, there is a serious problem in Brouwer's proof, from the intuitionistic point of view.

The main idea of the proof is that the proof $R_{<>}$ of the securability of $<>$ starts with the securability of the elements of B_0 (the securability of the, in prnciple, infinite nodes $u \notin M$ is not that essential) and goes down to the securability of $<>$, mainly due to $\digamma$-inferences. But the hypothesis that the proof starts that way and ends in the securability of $<>$, is not constructively innocent. *The real problem in the proof of BT is that B_0 may have infinite elements and in order to consider $R_{<>}$ completed, Brouwer employs the intuitionistically unaccepted concept of absolute infinity (P6).* If $R_{<>}$ is a man-made proof (P1) and we start $R_{<>}$ from η-inferences, we need, in general, absolutely infinite time to complete it.

[29] In [van Stigt 1990] p.379.

[30] In [van Atten 2004] p.40, p.61.

[31] We use the same symbolism $\alpha \in M$ and $u \in M$ for the M-infinite and M-finite sequences respectively.

[32] See [Kleene, Vesley 1965] p.51.

Even if M is a fan and $B = B_\varphi$ is an infinite species, then, at the beginning of $R_{<>}$, we do not know that B_0 is a finite species. $R_{<>}$ starts with inferences

$$\frac{u \in B_0}{u \textit{ is securable}}$$

but we need to know that these inferences are finite, or some other uniform generation of B_0 is needed, in order to start $R_{<>}$. So , *the real problem with Brouwer's proof of FT is that he considers a line of progression in* $R_{<>}$, *without justifying constructively that such a proof is completed in finite time.* Brouwer proves the finiteness of B_0 only on the hypothesis that $R_{<>}$ is completed and proceeds the way he describes. But before staring the proof, Brouwer doesn't know that B_0 is finite and he does not explain how we may go from the η-inferences down to the root $<>$. So, the proof $R_{<>}$ that Brouwer adjusts to hypothesis P' is not an intuitionistic mathematical object.$\Box$

The non-constructive character of Brouwer's argument applies to BI_D too. While in standard induction there is a fixed beginning, in BI_D there is a fixed end without any explication of how the η and $\digamma$-inferences it employs lead to the w.o.p. of $<>$. In our view, the quantification over u has an absolutely infinite character. Moreover, Martino and Giaretta have shown ([Martino, Giaretta 1979]) that BD is equivalent to BI_D, assuming CP.

Our analysis has shown that, in order to secure the siT-character of BIA, we must skip Brouwer's justification of FT[33]. We feel the same towards Kleene's theory FIM and its non-constructive BI_D-axiom. Someone could assert that BD is not more evident than BT itself, therefore it is not necessary. But it is the analysis of Brouwer's use of BD in his proof, which reveals the non-constructive character of BI_D.

6 Intuitionistic analysis in teaching mathematical analysis

Modern textbooks of mathematical analysis or books on the foundations of mathematical analysis are concerned exclusively with classical analysis. A characteristic exception is [Truss 1997]. In his book, Truss studies the classical set of reals together with the Baire space $\mathcal{N}$, the Cantor space $\mathcal{C}$ and the intuitionistic continuum, all in a more or less undergraduate level. We believe that incorporating elements of intuitionism in teaching mathematical analysis will have the following merits, based on our previous reconstruction of intuitionism.

(i) Students will understand the importance of philosophical views in the development of mathematics, since BIA is a philosophically motivated mathematical theory. Becoming familiar with the general philosophical background of intuitionism they will become aware of the non-constructive framework of classical mathematics.
(ii) Despite their familiarity with the axiomatic approach on mathematics, students will have the opportunity to acknowledge a conceptual origin of the continuum. Dedekind cuts is a model of the axioms of reals based on an axiomatic set theory, but the intuitionistic $\Re$ is the mathematical description of an intuition, independent from the concept of set. Also they will discover that using an axiom to prove a theorem is not always the only solution. In a theory like BIA an appropriate definition or a suitable

[33] A different approach to the proof of FT is studied in [Petrakis 2008].

analysis of a definition, is probably the right way to a proof.
(iii) Students will understand that BIA is not in clash with classical analysis, but is a theory based on different principles and definitions. An intuitionistic function $\varphi : \omega^\omega \to \omega$ behaves differently from a classical $f : \mathcal{N} \to \mathbb{N}$, because it has a completely different definition from it.
(iv) Student will find out the significance of the difference between the potential and absolute infinity in the development of a mathematical theory. The absolute infinity is the infinity of classical analysis, while the potential infinity is the infinity of BIA. Brouwer's fundamental concept of spread, for example, is a product of his need to avoid the concept of an absolutely infinite set. Unfortunately, as we have argued in section 5, he didn't manage to avoid it in his famous proof of fan theorem.

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