# LEARNING DISCRETE MATHEMATICS AND COMPUTER SCIENCE VIA PRIMARY HISTORICAL SOURCES:

## Student projects for the classroom

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**ABSTRACT**

We discuss and present excerpts from classroom project modules based on primary historical sources, being developed by an interdisciplinary faculty team for courses in discrete mathematics, graph theory, combinatorics, logic, and computer science. The goal is to provide motivation, direction, and context for these subjects through student projects based directly on the writings of the pioneers who first developed crucial ideas and worked on seminal problems. Each module is built around primary source material close to or representing the discovery of a key concept. Through guided reading and activities, students explore the mathematics of the original discovery and develop their own understanding of the subject. We describe a dozen projects already available, and give substantial selections from two of them, on Pascal's elucidation of mathematical induction in his treatise on the arithemetical triangle, and Euler's seminal paper in graph theory on the Königsberg Bridge Problem. We also discuss the details of classroom implementation of teaching with historical projects. Preliminary evaluation shows a statistically significant benefit to students' performance in subsequent courses from a course with a historical project. Further evaluation and project development is underway, and two web sites provide expanded materials and information. Ongoing support is provided by the US National Science Foundation.

## 1 Introduction

A discrete mathematics course often teaches about precise logical and algorithmic thought, and methods of proof, to students studying mathematics, computer science, or teacher education. The roots of such methods of thought, and of discrete mathematics itself, are as old as mathematics, with the notion of counting, a discrete operation, usually cited as the first mathematical development in ancient cultures [9]. However, a typical course frequently presents a fast-paced news reel of facts and formulae, often memorized by the students, with the text offering only passing mention of the motivating problems and original work that eventually found resolution in the modern concepts of induction, recursion and algorithm. We will focus on the pedagogy of historical projects for students, always centered on actual excerpts from primary historical sources, that place the material in context, and provide direction to the subject matter.

Our interdisciplinary team of mathematics and computer science faculty has completed a pilot program funded by the US National Science Foundation, in which we have developed and tested over a dozen historical project modules for student work in courses in discrete mathematics, graph theory, combinatorics, logic, and computer science. The projects are slated to appear in print [1], and are presently available through the web resource [3].

Designed to capture the spark of discovery and motivate subsequent lines of inquiry, each module is built around primary source material close to or representing the discovery of a key concept. Through guided reading and activities, students explore the mathematics of the original discovery and develop their own understanding of the subject. To place the source in context, a module also provides biographical information about its author, and historical background about the problems with which the author was concerned. For example, motivated by the problems of computing odds in a game of chance and of finding the summation of powers (with the eventual goal of computing the area under certain curves), Pascal arranged figurate numbers into columns of a table, today called Pascal's Triangle. Having noticed certain patterns in the table which he wished to justify, he formulated verbally what has become mathematical induction. After reading Pascal's original writings in his 1653 Treatise on the Arithmetical Triangle, students are asked to explore the validity of his claims with concrete numerical values, and then grapple with the logic behind induction techniques.

Our team is now beginning a four-year NSF expansion grant through which additional modules based on primary sources will be developed, tested, evaluated, revised, and published. The expansion will support classroom testing by faculty at twenty other institutions, careful evaluation of their effectiveness, and provide training in teaching with these projects to graduate students. The projects being created under the expansion grant are described at our new web resource [2], and we welcome instructors who would like to collaborate in testing or writing projects.

Here we will briefly describe the completed pilot projects, outline the content of two of them, describe how such historical projects can be implemented in the classroom, and discuss preliminary evaluation of their effectiveness.

## 2 Available classroom projects

Each historical project is centered around a publication of mathematical significance, such as Blaise Pascal's "Treatise on the Arithmetical Triangle" [11, vol. 30] from the 1650s or Alan Turing's 1936 paper "On Computable Numbers with an Application to the Entscheidungsproblem" [14]. The projects are designed to introduce or provide supplementary material for topics in the curriculum, such as induction in a discrete mathematics course, or compilers and computability for a computer science course. Each project provides a discussion of the historical exigency of the piece, a few biographical comments about the author, excerpts from the original work, and a sequence of questions to help the student appreciate the source and learn how to do the relevant mathematics. The main pedagogical idea is to teach and learn certain course topics directly from the primary historical source, thus recovering motivation for studying the material.

The following dozen projects (and a few more) are currently available for instructors [1][3]. The introduction to this collection discusses the topic material of each project further, and suggests appropriate courses for each project. Each project also has guiding notes for the instructor on its use in teaching. Listed below are the project titles together with the primary historical author whose work is highlighted in the module.

1. "Are All Infinities Created Equal?" (Georg Cantor, 1845–1918, [5])
2. "Turing Machines, Induction and Recursion," (Alan Turing, 1912–1954, [14])
3. "Turing Machines and Binary Addition," (Alan Turing, 1912–1954, [14])
4. "Binary Arithmetic: From Leibniz to von Neumann" (Gottfried Leibniz, 1646–1716, [8])
5. "Arithmetic Backwards from von Neumann to the Chinese Abacus," (Claude Shannon, 1916–2001, [12])
6. "Treatise on the Arithmetical Triangle," (Blaise Pascal, 1623–1662, [11])
7. "Counting Triangulations of a Polygon," (Gabriel Lamé, 1795–1870, [10])
8. "Two-Way Deterministic Finite Automata," (John Shepherdson [13])
9. "Church's Thesis," (Alonzo Church, 1903–1995, [6])
10. "Euler Circuits and the Königsberg Bridge Problem," (Leonhard Euler, 1707–1783, [4])
11. "Topological Connections from Graph Theory," (Oswald Veblen, 1880–1960, [4])
12. "Hamiltonian Circuits and Icosian Game," (William Hamilton, 1805–1865, [4])

## 3 Two sample historical projects

Here we provide some excerpts from two of the projects listed above. For each project we display selections from the primary historical source in the project, and also from the student assignment questions, to give a flavor of the nature of the project for students.

The project *Treatise on the Arithmetical Triangle* is intended for introductory level discrete mathematics, and presents the concept of mathematical induction from the pioneering work of Blaise Pascal [11, vol. 30] in the 1650s. After arranging the figurate numbers in one table, forming "Pascal's triangle," the French scholar notices several patterns in the table, which he would like to claim continue indefinitely. Exhibiting unusual rigor for his day, Pascal offers a condition for the persistence of a pattern, stated verbally in his Twelfth Consequence, a condition known today as mathematical induction. Moreover, the Twelfth Consequence results in the modern formula for the combination numbers or binomial coefficients. In this project, students will learn first-hand about the issues involved in proofs by iteration, generalizable example, and mathematical induction.

The project *Early Writings on Graph Theory: Euler Circuits and The Königsberg Bridge Problem* is suitable for a beginning-level discrete mathematics course, or for a 'transition to proof' course. In the paper on which the project is based [7], today considered to be the starting point of modern graph theory, Leonhard Euler (1707–1783) undertakes a mathematical formulation of the famous Königsberg Bridge Problem. By introducing modern graph theory terminology alongside Euler's original writing, the project assumes no prior background in graph theory. The first part of the project in which students are required to read and understand Euler's analysis of the 'bridge problem' is well suited for small group discussion. Other questions ask students to compare Euler's treatment of key results to the treatment of these same results in a modern textbook, with the objective of drawing students' attention to current standards regarding formal proof. The project culminates with exercises which require students to 'fill in the gaps' in a modern proof of Euler's main theorem. These questions are ideally suited for individual practice in proof writing, but could also be completed in small groups.

### 3.1 *Treatise on the Arithmetical Triangle*: Blaise Pascal

## TREATISE ON THE ARITHMETICAL TRIANGLE

DEFINITIONS

I call *arithmetical triangle* a figure constructed as follows:

From any point, G, I draw two lines perpendicular to each other, GV, G$\zeta$ in each of which I take as many equal and contiguous parts as I please, beginning with G, which I number 1, 2, 3, 4, etc., and these numbers are the *exponents* of the sections of the lines.

Next I connect the points of the first section in each of the two lines by another line, which is the base of the resulting triangle.

In the same way I connect the two points of the second section by another line, making a second triangle of which it is the base.

And in this way connecting all the points of section with the same exponent, I construct as many triangles and bases as there are exponents.

Through each of the points of section and parallel to the sides I draw lines whose intersections make little squares which I call *cells*.

Cells between two parallels drawn from left to right are called *cells of the same parallel row*, as, for example, cells $G$, $\sigma$, $\pi$, etc., or $\varphi$, $\psi$, $\theta$, etc.

Those between two lines are drawn from top to bottom are called *cells of the same perpendicular row*, as, for example, cells $G, \varphi, A, D$, etc., or $\sigma, \psi, B$, etc.

Those cut diagonally by the same base are called *cells of the same base*, as, for example, $D, B, \theta, \lambda$, or $A, \psi, \pi$. ...

Now the numbers assigned to each cell are found by the following method:

The number of the first cell, which is at the right angle, is arbitrary; but that number having been assigned, all the rest are determined, and for this reason

| Z | 1 | 2 | 3 | 4 | 5 | 6 | 7 | L 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | G 1 | $\sigma$ 1 | $\pi$ 1 | $\lambda$ 1 | $\mu$ 1 | $\delta$ 1 | $\zeta$ 1 | 1 | 1 | 1 |
| 2 | $\varphi$ 1 | $\psi$ 2 | $\theta$ 3 | R 4 | S 5 | N 6 | 7 | 8 | 9 | |
| 3 | A 1 | B 3 | C 6 | $\omega$ 10 | $\xi$ 15 | 21 | 28 | 36 | | |
| 4 | D 1 | E 4 | F 10 | $\rho$ 20 | Y 35 | 56 | 84 | | | |
| 5 | H 1 | M 5 | K 15 | 35 | 70 | 126 | | | | |
| 6 | P 1 | Q 6 | 21 | 56 | 126 | | | | | |
| 7 | V 1 | 7 | 28 | 84 | | | | | | |
| T 8 | 1 | 8 | 36 | | | | | | | |
| 9 | 1 | 9 | | | | | | | | |
| 10 | 1 | | | | | | | | | |

it is called the *generator* of the triangle. Each of the others is specified by a single rule as follows:

The number of each cell is equal to the sum of the numbers of the perpendicular and parallel cells immediately preceding. Thus cell $F$, that is, the number of cell $F$, equals the sum of cell $C$ and cell $E$, and similarly with the rest.

Whence several consequences are drawn. The most important follow, wherein I consider triangles generated by unity, but what is said of them will hold for all others.

## First Consequence

*In every arithmetical triangle all the cells of the first parallel row and of the first perpendicular row are the same as the generating cell.*

For by definition each cell of the triangle is equal to the sum of the immediately preceding perpendicular and parallel cells. But the cells of the first parallel row have no preceding perpendicular cells, and those of the first perpendicular row have no preceding parallel cells; therefore they are all equal to each other and consequently to the generating number.

Thus $\varphi = G + 0$, that is, $\varphi = G$,
$A = \varphi + 0$, that is, $\varphi$,
$\sigma = G + 0$, $\pi = \sigma + 0$,

And similarly of the rest.

## Second Consequence

*In every arithmetical triangle each cell is equal to the sum of all the cells of the preceding parallel row from its own perpendicular row to the first, inclusive.*

Let any cell, $\omega$, be taken. I say that it is equal to $R+\theta+\psi+\varphi$, which are the cells of the next higher parallel row from the perpendicular row of $\omega$ to the first perpendicular row.

This is evident if we simply consider a cell as the sum of its component cells.

For $\omega$ equals

$$R + \underbrace{C}_{\theta \,+\, \underbrace{B}_{\psi \,+\, \underbrace{A}_{\varphi,}}}$$

for $A$ and $\varphi$ are equal to each other by the preceding consequence.

Therefore $\omega = R + \theta + \psi + \varphi$. ...

1. Pascal's Triangle and its numbers

   (a) Let us use the notation $T_{i,j}$ to denote what Pascal calls the number assigned to the cell in *parallel row* $i$ (which we today call just *row* $i$) and *perpendicular row* $j$ (which we today call *column* $j$). We call the $i$ and $j$ by the name *indices* (plural of *index*) in our notation. Using this notation, explain exactly what Pascal's rule is for determining all the numbers in all the cells. Be sure to give full details. This should include explaining for exactly which values of the indices he defines the numbers.

   (b) In terms of our notation $T_{i,j}$, explain his terms *exponent, base, reciprocal, parallel row, perpendicular row,* and *generator.*

   (c) Rewrite Pascal's first two "Consequences" entirely in the $T_{i,j}$ notation.

   (d) Rewrite his proofs of these word for word in our notation also.

   (e) Do you find his proofs entirely satisfactory? Explain why or why not.

   (f) Improve on his proofs using our notation. In other words, make them apply for arbitrary prescribed situations, not just the particular examples he lays out.

... The next consequence is the most important and famous in the whole treatise. Pascal derives a formula for the ratio of consecutive numbers in a base. From this he will obtain an elegant and efficient formula for all the numbers in the triangle.

## Twelfth Consequence

*In every arithmetical triangle, of two contiguous cells in the same base the upper is to the lower as the number of cells from the upper to the top of the base is to the number of cells from the lower to the bottom of the base, inclusive.*

Let any two contiguous cells of the same base, $E$, $C$, be taken. I say that

| $E$ | : | $C$ | :: | 2 | : | 3 |
|---|---|---|---|---|---|---|
| the lower | | the upper | | because there are two cells from $E$ to the bottom, namely $E$, $H$, | | because there are three cells from $C$ to the top, namely $C$, $R$, $\mu$. |

Although this proposition has an infinity of cases, I shall demonstrate it very briefly by supposing two lemmas:

The first, which is self-evident, that this proportion is found in the second base, for it is perfectly obvious that $\varphi : \sigma :: 1 : 1$;

The second, that if this proportion is found in any base, it will necessarily be found in the following base.

Whence it is apparent that it is necessarily in all the bases. For it is in the second base by the first lemma; therefore by the second lemma it is in the third base, therefore in the fourth, and to infinity.

It is only necessary therefore to demonstrate the second lemma as follows: If this proportion is found in any base, as, for example, in the fourth, $D\lambda$, that is, if $D : B :: 1 : 3$, and $B : \theta :: 2 : 2$, and $\theta : \lambda :: 3 : 1$, etc., I say the same proportion will be found in the following base, $H\mu$, and that, for example, $E : C :: 2 : 3$.

For $D : B :: 1 : 3$, by hypothesis.

Therefore $\underbrace{D + B}_{E} : B :: \underbrace{1 + 3}_{4} : 3$

$$E : B :: 4 : 3$$

Similarly $B : \theta :: 2 : 2$, by hypothesis

Therefore $\underbrace{B + \theta}_{C} : B :: \underbrace{2 + 2}_{4} : 2$

$$C : B :: 4 : 2$$

But $\quad B : E :: 3 : 4$

Therefore, by compounding the ratios, $C : E :: 3 : 2$. Q.E.D.

The proof is the same for all other bases, since it requires only that the proportion be found in the preceding base,and that each cell be equal to the cell before it together with the cell above it, which is everywhere the case.

6. Pascal's Twelfth Consequence: the key to our modern factorial formula

   (a) Rewrite Pascal's Twelfth Consequence as a generalized modern formula, entirely in our $T_{i,j}$ terminology. Also verify its correctness in a couple of examples taken from his table in the initial definitions section.

(b) Adapt Pascal's proof by example of his Twelfth Consequence into modern generalized form to prove the formula you obtained above. Use the principle of mathematical induction to create your proof.

Now Pascal is ready to describe a formula for an arbitrary number in the triangle.

PROBLEM

*Given the perpendicular and parallel exponents of a cell, to find its number without making use of the arithmetical triangle. . . .*

## 3.2 *The solution of a problem relating to the geometry of position*: Leonhard Euler

SOLUTIO PROBLEMATIS AD GEOMETRIAM SITUS PERTINENTIS

**1** In addition to that branch of geometry which is concerned with magnitudes, and which has always received the greatest attention, there is another branch, previously almost unknown, which Leibniz first mentioned, calling it the *geometry of position*. This branch is concerned only with the determination of position and its properties; it does not involve measurements, nor calculations made with them. It has not yet been satisfactorily determined what kind of problems are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned, which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position — especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this kind of problem, as an example of the geometry of position.

**2** The problem, which I am told is widely known, is as follows: in Königsberg in Prussia, there is an island $A$, called *the Kneiphof*; the river which surrounds it is divided into two branches, as can be seen in Fig. [1.2], and these branches are crossed by seven bridges, $a$, $b$ , $c$ , $d$ , $e$ , $f$ and $g$. Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he would cross each bridge once and only once. I was told that some people asserted that this was impossible, while others were in doubt: but nobody would actually assert that it could be done. From this, I have formulated the general problem: whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find out whether or not it is possible to cross each bridge exactly once?

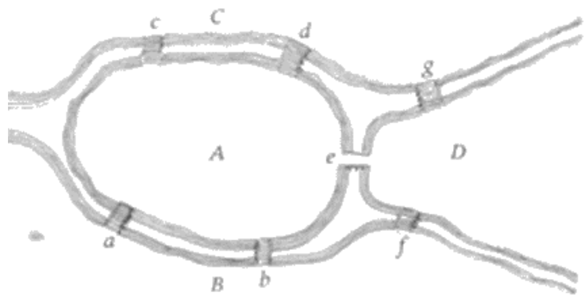
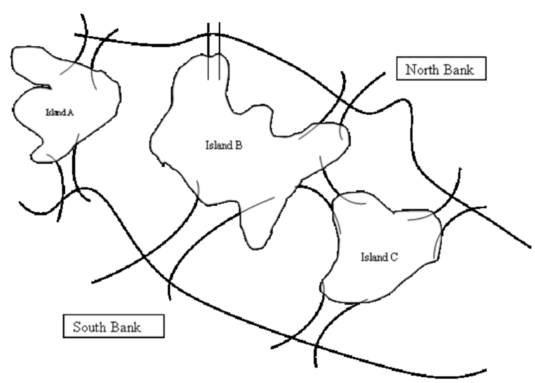


FIG. 1.2

Notice that Euler begins his analysis of the 'bridge crossing' problem by first replacing the map of the city by a simpler diagram showing only the main feature. In modern graph theory, we simplify this diagram even further to include only points (representing land masses) and line segments (representing bridges). These points and line segments are referred to as *vertices* (singular: vertex) and *edges* respectively. The collection of vertices and edges together with the relationships between them is called a *graph.* More precisely, a graph consists of a set of vertices and a set of edges, where each edge may be viewed as an ordered pair of two (usually distinct) vertices. In the case where an edge connects a vertex to itself, we refer to that edge as a *loop.*

1. Sketch the diagram of a graph with 5 vertices and 8 edges to represent the following bridge problem.




. . . . . . . . . . . .

After rejecting the impractical strategy of solving the bridge-crossing problem by making an exhaustive list of all possible routes, Euler again reformulates the problem in terms of sequences of letters (vertices) representing land masses, thereby making the diagram itself unnecessary to the solution of the problem. Today, we say that two vertices joined by an edge in the graph are *adjacent*, and refer to a sequence of adjacent vertices as a *walk.* Technically, a walk is a sequence of alternating (adjacent) vertices and edges $v_0 e_1 v_1 e_1 \dots e_n v_n$ in which both the order of the vertices and the order of the edges used between adjacent vertices are specified. In the case where no edge of the graph is repeated (as required in a bridge-crossing route), the walk is known as a *path*. If the initial and terminal vertex are equal, the path is said to be a *circuit*. If *every* edge of the graph is used *exactly once* (as desired in a bridge-crossing route), the path (circuit) is said to be a *Euler path (circuit)*.

2. For the bridge problem shown in Question 1 above, how many capital letters (representing graph vertices) will be needed to represent an Euler path?

Having reformulated the bridge crossing problem in terms of sequences of letters (vertices) alone, Euler now turns to the question of determining *whether* a given bridge crossing problem admits of a solution. As you read through Euler's development of a procedure for deciding this question in paragraphs 7 - 13 below, pay attention to the style of argument employed, and how this differs from that used in a modern textbook.

. . . . . . . . . . . .

> **13** Since one can start from only one area in any journey, I shall define, corresponding to the number of bridges leading to each area, the number of occurrences of the letter denoting that area to be half the number of bridges plus one, if the number of bridges is odd, and if the number of bridges is even, to be half of it. Then, if the total of all the occurrences is equal to the number of bridges plus one, the required journey will be possible, and will have to start from an area with an odd number of bridges leading to it. If, however, the total number of letters is one less than the number of bridges plus one, then the journey is possible starting from an area with an even number of bridges leading to it, since the number of letters will therefore be increased by one.

Notice that Euler's definition concerning 'the number of occurrences of the letter denoting that area' depends on whether the number of bridges (edges) leading to each area (vertex) is even or odd. In contemporary terminology, the number of edges incident on a vertex $v$ is referred to as the *degree of vertex v*.

4. Let $deg(v)$ denote the degree of vertex $v$ in a graph $G$. Euler's definition of 'the number of occurrences of $v$' can then be re-stated as follows:

   - If $deg(v)$ is even, then $v$ occurs $\frac{1}{2}deg(v)$ times.
   - If $deg(v)$ is odd, then $v$ occurs $\frac{1}{2}[deg(v)+1]$ times.

   Based on Euler's discussion in paragraphs 9 - 12, how convinced are you that this definition gives a correct description of the Königsberg Bridge Problem? How convincing do you find Euler's claim (in paragraph 13) that the required route can be found in the case where 'the total of all the occurrences is equal to the number of bridges plus one'? Comment on how a proof of this claim in a modern textbook might differ from the argument which Euler presents for it in paragraphs 9 - 12.

**14** So, whatever arrangement of water and bridges is given, the following method will determine whether or not it is possible to cross each of the bridges: I first denote by the letters A, B, C, etc. the various areas which are separated from one another by the water. I then take the total number of bridges, add one, and write the result above the working which follows. Thirdly, I write the letters A, B, C, etc. in a column, and write next to each one the number of bridges leading to it. Fourthly, I indicate with an asterisk those letters which have an even number next to them. Fifthly, next to each even one I write half the number, and next to each odd one I write half the number increased by one. Sixthly, I add together these last numbers, and if this sum is one less than, or equal to, the number written above, which is the number of bridges plus one, I conclude that the required journey is possible. It must be remembered that if the sum is one less than the number written above, then the journey must begin from one of the areas marked with an asterisk, and it must begin from an unmarked one if the sum is equal. Thus in the Königsberg problem, I set out the working as follows:

Number of bridges 7, which gives 8

| | *Bridges* | |
|---|---|---|
| A, | 5 | 3 |
| B, | 3 | 2 |
| C, | 3 | 2 |
| D, | 3 | 2 |

Since this gives more than 8, such a journey can never be made.

. . . . . . . . . . . .

5. Apply Euler's procedure to determine whether the graph representing the 'bridge-crossing' question in question 1 above contains an Euler path. If so, find one.

In paragraphs 16 and 17, Euler makes some observations intended to simplify the procedure for determining whether a given bridge-crossing problem has a solution. As you read these paragraphs, consider how to reformulate these observations in terms of degree.

**16** In this way it will be easy, even in the most complicated cases, to determine whether or not a journey can be made crossing each bridge once and once only. I shall, however, describe a much simpler method for determining this which is not difficult to derive from the present method, after I have first made a few preliminary observations. First, I observe that the numbers of bridges written next to the letters A, B, C, etc. together add up to twice the total number of bridges. The reason for this is that, in the calculation where every bridge leading to a given area is counted, each bridge is counted twice, once for each of the two areas which it joins.

**17** It follows that the total of the numbers of bridges leading to each area must be an even number, since half of it is equal to the number of bridges. This is impossible if only one of these numbers is odd, or if three are odd, or five, and so on. Hence if some of the numbers of bridges attached to the letters A, B, C, etc. are odd, then there must be an even number of these. Thus, in the Königsberg problem, there were odd numbers attached to the letters A, B, C and D, as can be seen from Paragraph 14, and in the last example (in Paragraph 15), only two numbers were odd, namely those attached to D and E.

6. The result described in Paragraph 16 is sometimes referred to as 'The Handshake Theorem,' based on the equivalent problem of counting the number of handshakes that occur during a social gathering at which every person present shakes hands with every other person present exactly once. A modern statement of the Handshake Theorem would be: *The sum of the degree of all vertices in a finite graph equals twice the number of edges in the graph.* Locate this theorem in a modern textbook, and comment on how the proof given there compares to Euler's discussion in paragraph 16.

7. The result described in Paragraph 17 can be re-stated as follows: *Every finite graph contains an even number of vertices with odd degree.* Locate this theorem in a modern textbook, and comment on how the proof given there compares to Euler's discussion in paragraph 17.

Euler now uses the above observations to develop simplified rules for determining whether a given bridge-crossing problem has a solution. Again, consider how you might reformulate this argument in modern graph theoretic terms; we will consider a modern proof of the main results below.

. . . . . . . . . . . .

**20** So whatever arrangement may be proposed, one can easily determine whether or not a journey can be made, crossing each bridge once, by the following rules:

*If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.*

*If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these areas.*

*If, finally, there are no areas to which an odd number of bridges leads, then the required journey can be accomplished starting from any area.*

With these rules, the given problem can always be solved.

A complete modern statement of Euler's main result requires one final definition: a graph is said to be *connected* if for every pair of vertices $u, v$ in the graph, there is a walk from $u$ to $v$. Notice that a graph which is not connected will consist of several components, or subgraphs, each of which is connected. With this definition in hand, the main results of Euler's paper can be stated as follow:

**Theorem**: A finite graph $G$ contains an Euler circuit if and only if $G$ is connected and contains no vertices of odd degree.

**Corollary**: A finite graph $G$ contains an Euler path if and only if $G$ is connected and contains at most two vertices of odd degree.

8. Illustrate why the modern statement specifies that $G$ is connected by giving an example of a disconnected graph which has vertices of even degree only and contains no Euler circuit. Explain how you know that your example contains no Euler circuit.

9. Comment on Euler's proof of this theorem and corollary as they appear in paragraphs 16 - 19. How convincing do you find his proof? Where and how does he make use of the assumption that the graph is connected in his proof?

# 4 Implementation

Time spent working on the project is time for explanation, exploration, and discovery, for both the instructor and the student. Instructors are encouraged to adapt each project to their particular course. Add or rephrase some questions, or delete others to reflect what is actually being covered. Be familiar with all details of a project before assignment. The source file for each project together with its bibliographic references can be downloaded and edited from the web resource [3].

For use in the classroom, allow one to several weeks per project with one or two projects per course. Each project should count for a significant portion of the course grade (about 20%) and may take the place of an in-class examination, or be assigned in pieces as homework. For certain course topics, the project can simply replace other course activities for a time, with the main course topics learned directly through the project. Begin early in the course with a discussion of the relevance of the historical piece, its relation to the course curriculum, and how modern textbook techniques owe their development to problems often posed centuries earlier. While a project is assigned, several class activities are possible. Students could be encouraged to work on the project in class, either individually or in small groups, as the instructor monitors and assists their progress and explores meaning in language from time past. A comparison with modern techniques could begin as soon as the students have read the related historical passages. For example, after reading Pascal's verbal description of what today is recognized as induction, the instructor could lead a discussion comparing this to the axiomatic formulation of induction found in the textbook. Finally, the historical source can be used to provide discovery exercises for related course material. For instance, in his 1703 publication "An Explanation of Binary Arithmetic" [8], Gottfried Leibniz introduces the binary system of numeration, states its advantages in terms of efficiency of calculation, and claims that this system allows for the discovery of other properties of numbers, such as patterns in the base two expansion of the perfect squares. An engaging in-class exercise is to examine patterns in a table of perfect squares (base two) and conjecture corresponding divisibility properties of the integers. The pattern of zeroes in the binary equivalent of $n^2$ leads to the conjecture that $8|(n^2-1)$, $n$ odd, where the vertical bar denotes "divides." Construct the table!

## 5 Conclusion

After completion of a course using historical projects, students write the following about the benefits of history: "See how the concepts developed and understand the process." "Learn the roots of what you've come to believe in." "Appropriate question building." "Helps with English-math conversion." "It leads me to my own discoveries." Furthermore, in an initial pilot study of students learning discrete mathematics from primary historical sources, there were 229 cases where students earned course grades above the mean in subsequent courses, compared with 123 cases where students earned course grades below the mean in follow-on courses. The probability that this would occur under the assumption that the historical projects had no positive effect is less than .000001 using a simple binomial sign test. Of course there may be other factors at play, e.g., differing entering preparation for different groups of students in different courses and semesters, but we plan to compensate for these in the much more extensive evaluation now underway during our four-year NSF expansion grant work.

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