Crossing Cultures, Seas, and the Cosmos:
In Search of the Origins of Trigonometry

Glen Van Brummelen (gvb@questu.ca)
Quest University

Abstract: It is a strange paradox that calculus and linear algebra recently have been reacquainted with inspirations in science and industry; yet trigonometry — which owes its very existence to outside needs — remains virtually untouched. It was born in Hipparchus's fusion of astronomical models with Babylonian calculation, forming the first truly exact science. It was rejuvenated with Muslim requirements to determine the direction of Mecca; it flourished with the beginning of European ocean-going navigation. Although trigonometry has a colorful history crossing the boundaries of ancient Greece, medieval India and Islam, and the West, students usually remain unaware both of its cultural richness and of the reasons that these cultures cared. We shall explore examples of these little-known stories accessible to students learning the subject, providing interesting historical motives for some of the more peculiar twists and turns encountered in the classroom.

Multicultural education has been a buzzword in pedagogical circles for some years now. The increasing need for intercultural dialogue in a very quickly developing global monoculture is altering how we teach our classes. In fact, the university I'm helping to build right now uses "International" as one of its three slogans, and we emphasize communication across all sorts of boundaries, disciplinary as well as cultural. Within the mathematics classroom, amazingly fast inroads have been made through ethnomathematics, especially at lower grade levels. Ethnomathematics has helped to increase our awareness of different ways to approach mathematics, which in turn leads us to work more effectively with students from backgrounds different than our own. It might even address the math gender gap. But by the time students reach high school age and take subjects that lead toward calculus, they're learning European mathematics. And they need to, if they're going to be able to function mathematically at the level that society expects of them.

The unfortunate effect of this specialization is that cultural diversity starts to disappear as students begin to scale the peaks of analytic geometry. Efforts to battle the monoculture view of mathematics begin to suffer, in the face of the unity of the content that we must teach. Inevitably, ethnomathematics becomes marginalized as students learn how to factor polynomials, solve logarithmic equations, and apply trigonometric identities. It's enough to make one question: is it even possible to permit different cultural approaches at this level, beyond the superficialities of textbook margins filled with biographies and photographs? Is it really true that culture affects not just elementary mathematics, but secondary as well, in some demonstrably teachable way?

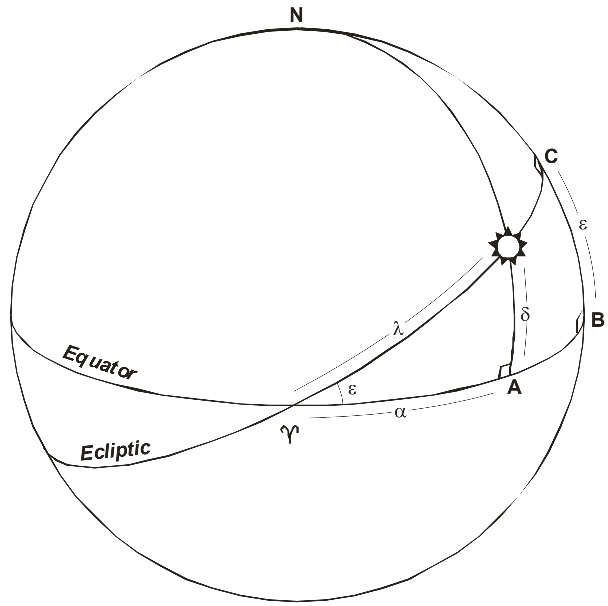


Figure 1: Determining the Sun's declination

It is my contention that the answer to both of these questions is an emphatic "yes". I have just completed a book on the early history of trigonometry, which moves from Greece to India, medieval Islam to the West. One of my most striking revelations during this project was how differently one can view a subject even as advanced in the high school curriculum as trigonometry, depending on who you are. Significant historical issues arise that nicely muddy the interpretive waters, raising questions that get at the heart of what we think mathematics is. What do we mean by the "birth of a mathematical subject"? How is that subject transformed when it transmits into a different society, with different values? Is mathematics a universal discipline, or are our subject boundaries simply a reflection of who we are? Is it even possible to talk about the discovery of a mathematical result? Each of these questions became all too real for me as I wrote my book, and I see no reason why they cannot be asked of high school students as they learn trigonometry. Indeed, for the sake of deeper awareness, it seems to me that they really should.

The Birth of Trigonometry

To bolster this claim, some examples would seem to be in order. Let's begin at the beginning…whenever that is. The word "trigonometry", or "triangle measurement", was coined by Bartholomew Pitiscus in 1595, with his *Trigonometriae*. This was no more than a variant on the phrase in vogue in Renaissance Europe, the "science of triangles", used for instance by Georg Rheticus (Nicolaus Copernicus's student and champion), and the 15th-century astronomer Regiomontanus.

But triangles are a relatively recent development in trigonometry. The transition toward triangles took place several centuries earlier, in the Muslim world around the late 10th and early 11th centuries, when that culture was at the peak of its scientific power. The sea

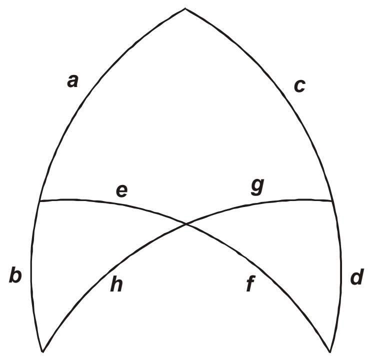


Figure 2: Menelaus's Theorem

change took place within spherical astronomy, which had been the mother of invention for trigonometry since the beginning. Consider the following typical problem (Figure 1). The Sun is in a certain position at a certain time on the celestial sphere; the Earth is an infinitesimal dot at the center of this sphere. The Sun's annual path through this sphere is called the *ecliptic*, and the Sun's position is measured as the arc λ measured from the point ♈ where the ecliptic and equator cross, the *vernal equinox*. These two great circles are inclined to each other at an angle $\varepsilon \approx 23\tfrac{1}{2}°$. Given λ and ε, how far is the Sun removed from the equator — its *declination* δ?

Rather than approaching the obvious triangle in the problem, the Muslims had inherited from the Greeks a solution using Menelaus's Theorem (Figure 2). This apparently cumbersome theorem requires that one find a particular configuration of great circle arcs on one's diagram, and apply one or the other of these two equalities of ratios of sines:

$$\frac{\text{Sin } a}{\text{Sin } b} = \frac{\text{Sin}(c+d)}{\text{Sin } d} \cdot \frac{\text{Sin } g}{\text{Sin } h}$$

$$\frac{\text{Sin}(a+b)}{\text{Sin } a} = \frac{\text{Sin}(g+h)}{\text{Sin } g} \cdot \frac{\text{Sin } f}{\text{Sin}(e+f)}$$

Surprisingly, this configuration is actually easily found in most problems of spherical astronomy, if one is prepared to add some great circles to the diagram. In our case it can be found as *NCBA*♈☼, from which application of the second identity gives rise to the solution $\sin\delta = \sin\lambda \sin\varepsilon$.

But scientists like Abū'l-Wafā' and Abū Naṣr Manṣūr pulled Menelaus out of the foundation of the subject and made it a mere corollary, replacing it with what Abū Naṣr called the "figure that frees" (Figure 3) — what we now call the Rule of Four Quantities

$$\frac{\text{Sin } \overset{\frown}{AD}}{\text{Sin } \overset{\frown}{AE}} = \frac{\text{Sin } \overset{\frown}{BD}}{\text{Sin } \overset{\frown}{CE}},$$

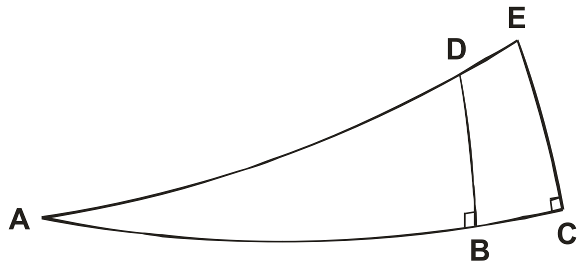


Figure 3: The Rule of Four Quantities

which applied directly to the triangle without the need for extra construction. The difference here is simply one of convenience: the two solutions are qualitatively the same, even though one focuses on triangles and the other doesn't.

So triangles are not *necessarily* the essence of trigonometry; but then, what *is*? Before the "triangular revolution", astronomers in Islam and India were using sines and tangents to solve many problems. But well before that the Greeks were clearly doing trigonometry, and they simply appealed to the chord function. The first extant trigonometric problem, due to Hipparchus of Rhodes, is to determine the eccentricity of the Sun's orbit given the lengths of the seasons (Figure 4). The large circle is the Sun's orbit around the Earth; but it was observed quite early that the seasons are of different lengths. Hipparchus's solution was to displace the Earth E from the center of the Sun's orbit Z; the goal is to determine the length EZ. The data are the lengths of the seasons, which can be transformed easily into the arcs $\widehat{M\Theta}$, $\widehat{\Theta K}$, $\widehat{KL}$, and $\widehat{LM}$.

The first step is to use the known arcs to determine some other arcs, in particular, $\widehat{\Theta Y}$. The key step comes next: Hipparchus uses his chord table to determine the length $YT\Theta$ from $\widehat{\Theta Y}$. Half of this is $T\Theta$, which is equal to XE. Applying a similar technique to $\widehat{QPK}$ determines FK, which is equal to ZX. Finally, the Pythagorean Theorem applied at the middle of the diagram gives EZ.

What makes this trigonometric? The need to convert arbitrary arcs in the diagram to lengths of line segments. This leads to a simple definition of the subject: the *systematic* ability to convert between arcs/angles and lengths.

This ability came about with the fusion of Greek geometric models for celestial motions with the imported Babylonian base-60 number system, roughly in the 2^{nd} century BC. The capacity afforded by efficient numeration transformed their astronomy, from the essentially qualitative model-building science of Eudoxus and Autolycus to the more familiar quantitative one of Hipparchus and Ptolemy. The explanatory models now became capable of *predicting positions*, not merely mimicking behaviour. In a sense, one might say grandly that trigonometry triggered the birth of the first truly exact science.

But what of earlier claimed occurrences of trigonometry? Problems that require triangle measurement occur on the Egyptian Rhind papyrus, 1600 years earlier, involving the

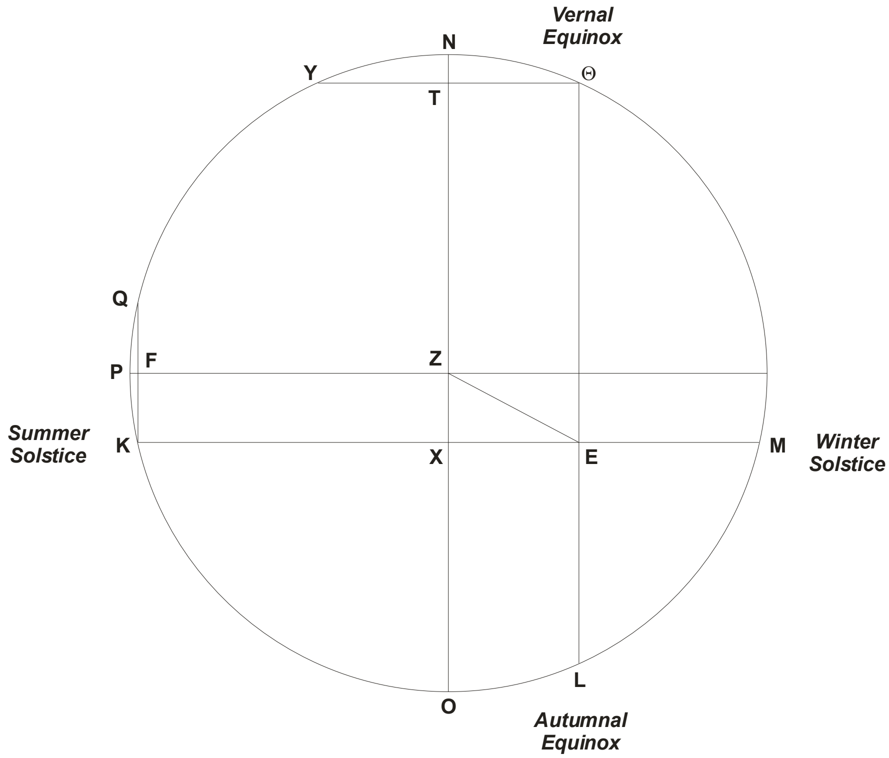


Figure 4: Hipparchus's determination of the Sun's eccentricity

---

measurements of the sides and slopes of pyramids. Another case is the infamous Babylonian tablet Plimpton 322, which contains trios of whole numbers that satisfy the Pythagorean Theorem — the so-called Pythagorean triples. When one considers how the ratios of the numbers in successive rows of this table change with respect to one another, it becomes tempting to think of the numbers as an attempt to measure ratios of sides in right-angled triangles. This interpretation, however, has been scorned by recent historians largely because there simply was no Babylonian conception of arcs or angles as later cultures understood them. So however one might describe what these very ancient people were doing, it shares little with what we, or even the ancient Greeks, were doing.

And here lies a rather meaningful question that is worth addressing, even in the classroom. I've just excluded Egypt and Babylon from consideration as trigonometric forebears. But telling the continuous story from Hipparchus forward, through India and Islam, and into Renaissance Europe and beyond, forces me to include the work (say) of Leonhard Euler on the relation between exponentials, complex numbers, and trigonometric functions. Why should $e^{i\theta} = \cos\theta + i\sin\theta$ be part of the story any more than the slopes of Egyptian pyramids?

Here we run into trouble with the use of labels for the disciplines that we practice, that inevitably change gradually over time. My previous statements notwithstanding, the very business of carving up history with the tool of pre-programmed definitions seems faulty. It leads to the possibility of imposing modern views on our subject (as if that risk wasn't present in almost everything we do in any case), without respecting changes in culture and attitude. It rigidifies the content, forcing us to stay within highway lines that do not follow the river where it wants to go. In short, definitions increase the danger of a Whig approach to history, the royal road to "us" — it's "presentism".

Cultural awareness, then, requires us to respect changing contexts and boundaries as we move from one society to another. This is part of a broader movement in the history of ideas known as "contextualism": the rejection of the notion of an independently existing "perennial idea" altogether, in favour of the recovery of an original author's intention through an appreciation of how language, schools of thought, and other influences shaped a unique instance of intellectual culture, not easily comparable to apparent occurrences of similar ideas elsewhere. The borders of a historical treatment, therefore, should respect narrative continuity more than the subject itself. So, I choose to disconnect Egyptian pyramids from Hipparchus, but *not* disconnect Hipparchus from Euler. $e^{i\theta} = \cos\theta + i\sin\theta$ is a far cry from Hipparchus's solar model, but it is an elaboration 2000 years later of a conversation that began with an examination of the stars.

This changing point of view may make it more of a challenge to use especially ancient history of mathematics in the classroom. The new sourcebook of non-Western mathematics edited by Victor Katz, for example, is strongly influenced by contextualism. It is more difficult to pick out an example of, say, Babylonian multiplication from this book to enhance your mathematical lesson than it was from its predecessors — because, to understand what's presented, your students will need to grasp something of metrological systems in Mesopotamia, their accounting systems, their farming, and so on. It's not a simple plug-in. But there is an incredible opportunity here as well, to break through today's artificial disciplinary boundaries and teach an integrated unit that touches on history, weights and measures, and even agriculture. Treating the subject more honestly, while it causes extra hard work now, cannot help but provide valuable educational rewards.

Transmission and Cultural Dissonance: The Example of Greece versus India

This leads us directly to our most crucial question: how can cultural differences be shown to affect the mathematics *beyond* multiplication? One leading example that I'd like to dwell on for a few minutes is the transmission of trigonometry from Greece to India. Truth be told, we know little about trigonometry in the early stages of either culture. In Greece, the bulk of our knowledge comes from Claudius Ptolemy's *Almagest*, his classic work on geocentric astronomy. At this time trigonometry was not its own subject, but simply the mathematical preliminaries to determining the positions of the planets in the night sky. So, to understand the origins of trigonometry you need to know some ancient astronomy, of which (luckily) we've already seen a taste. However, astronomy was hardly independent of mathematics: Ptolemy's original title for the *Almagest* was the *Mathematical Compilation*.

As "applied geometry", the *Almagest*'s astronomy has a distinctly Euclidean feel. Its trigonometry contains some differences from our own (for instance, the use of the chord function rather than the sine, and a base circle radius of 60 rather than 1), but aside from these trivialities, the math is familiar. To construct a table of chords, Ptolemy begins with

θ	Crd θ
½°	0;31,25
1°	1;2,50
1½°	1;34,15
⋮	⋮
7½°	7;50,54
⋮	⋮
15°	15;39,47
⋮	⋮
30°	31;3,30
⋮	⋮
60°	60;0,0
⋮	⋮
90°	84;51,10
⋮	⋮
120°	103;55,23
⋮	⋮
150°	115;54,40
⋮	⋮
180°	120;0,0

θ	Sin θ
3¾°	7;51
7½°	15;40
⋮	⋮
15°	31;4
⋮	⋮
30°	60;0
⋮	⋮
45°	84;51
⋮	⋮
60°	103;55
⋮	⋮
75°	115;55
⋮	⋮
90°	120;0

Figure 5: Ptolemy's chord table compared with Varāhamihira's sine table

chord values corresponding to the sine values that we start our students off with today: 36°, 60°, 72°, and 90°. Next, he demonstrates chord identities equivalent to the modern sine addition, subtraction, and half-angle laws. With judicious use of these identities in combination with the known chords, it is possible to determine the chords of all arcs that are multiples of 3°.

The remaining chord values, however, are inaccessible by the usual ruler and compass methods; to find the chord of 1°, for instance, would be equivalent to trisecting the angle. Ptolemy and his Muslim successors were forced into various uncomfortable, yet ingenious methods to approximate this elusive value. From it the rest of the chord table could be filled in. But this required a fundamental violation of geometric principles, a replacement of geometric purity with something as crude and unworthy as interpolation. This was so repugnant to at least one Arabic geometer, al-Samaw'al, that he actually restructured the circle to contain 480° rather than the usual 360°.

The transition to India is not a simple story. It seems that Greek astronomy did find its way to India, but *before Ptolemy*. The earliest reliable Indian astronomical texts date

θ	Sin θ
3¾°	225
7½°	449
⋮	⋮
15°	890
⋮	⋮
30°	1719
⋮	⋮
45°	2431
⋮	⋮
60°	2978
⋮	⋮
75°	3321
⋮	⋮
90°	3438

Figure 6: Āryabhaṭa's sine table

---

from around the 5th century AD, and they contain what appear to be elaborations of the Greek epicyclic model of the motions of the planets, the division of the circle into 360°, and some other devices that appear to reflect Greek origins. But none of Ptolemy's inventions — his equant point for planetary longitudes, his planetary latitude model, his handling of the celestial problem children Mercury and the Moon — are to be found. So, many scholars have attempted to find evidence for pre-Ptolemaic Greek astronomy in the earliest Indian documents.

But this is a very difficult thing to do. Indian planetary models, while they share the epicyclic concept with the Greeks, do not employ it the same way. The Greeks did what we would do: assert a physical model that "saves the phenomena", then work from the geometry to the predictions. Indian astronomy is much more concerned with the predictions, and less bothered about the physical reality of the geometry. The model does not attempt to explain the underlying physics; it is there to work more directly with prediction. Thus one finds for each planet a pair of epicycles representing two different phenomena without a geometric link, or even a clear position for the planet on the diagram. There is a transmission here, but the conversation changes qualitatively on a fundamental level. Can the same be said for its supporting mathematics, the trigonometry?

In part, yes. The most obvious change is the transition from the chord function to the sine, but this makes surprisingly little difference, as witnessed by an early sine table by

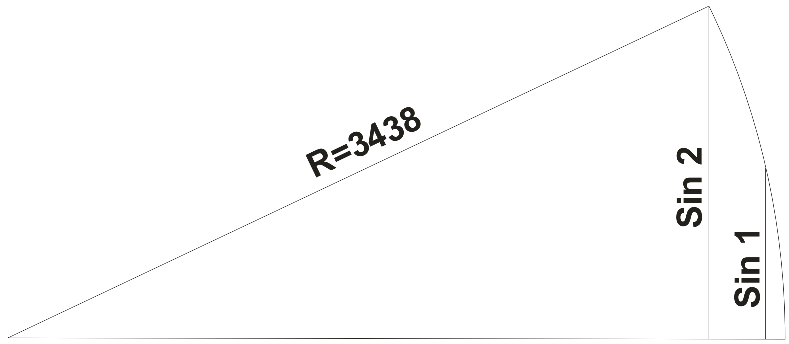


Figure 7: In Indian trigonometry, for small values of θ, $Sin\ \theta \approx \theta$

Varāhamihira (Figure 5). We see the same use of degrees, the same use of sexagesimal numeration, and the same base circle of 60 units. In fact, the table may as well have been rounded from Ptolemy. However, the more typical sine table represented by Āryabhaṭa (Figure 6) is an entirely different matter. The most obvious difference is the base circle radius of 3438. This peculiar choice comes about by dividing the 360° of the circle into minutes of arc, and using each minute as the unit of length. This idea is foreign to the *Almagest* and any further developments, until radian measure was introduced much later in Europe. And, it solved Ptolemy's fundamental problem: determining the geometrically inaccessible $\sin 1^{\circ}$. In India this was easy: sines of small arcs are simply equal to the arcs themselves (Figure 7).

The very idea of measuring inclinations by dividing an enclosing circle into one-minute units of length is foreign to us. But then again, so was the Greek approach: Ptolemy and friends did not use angles, but rather the arcs of circles. Even today, there are alternatives to what seem to us the obvious selection of angles to measure inclinations: Norman Wildberger, in his new book *Divine Proportions: Rational Trigonometry to Universal Geometry*, has replaced angles with what he calls "spread"; in Figure 8, the spread of lines l_1 and l_2 is measured as the ratio of the squares of the lengths of Q and R. This idea would simplify trigonometry considerably, but it will never catch on. For us, it's just too strange.

But the foreignness of India's trigonometry goes well beyond this. Consider the method that Āryabhaṭa used to calculate his sine table, in contrast to Ptolemy's. The only textual evidence we have to reconstruct Āryabhaṭa's reasoning is this passage, which from a modern point of view can only be described charitably as "obscure":

> When the second half [-chord] partitioned is less than the first half-chord, which is [approximately equated to] the [corresponding] arc, by a certain amount, the remaining [sine-differences] are less [than the previous ones] each by that amount of that divided by the first half-chord.

Modern historians have found over 25 possible explanations of this passage; what we shall see here is one outlined by Āryabhaṭa's late successor Nīlakaṇṭha, around AD 1500.

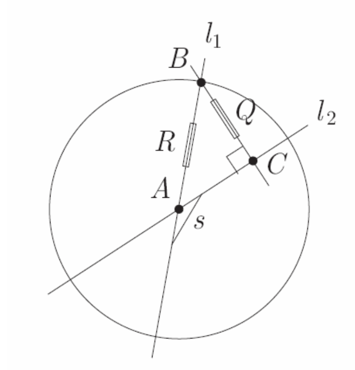


Figure 8: Wildberger's definition of "spread"

Consider the first several values in an Indian-style Sine table (Figure 9); notice that since we're measuring in degrees in this chart, the sines of these arcs are indeed very close to the arcs themselves (60 minutes each). Consider the first differences of these arcs, Delta(i); they change in a manner reminiscent of the cosine function. In fact, this is a reflection of the fact that the derivative of the sine function is the cosine. If we then consider the second differences (i.e., the differences of the differences), they are of course negative; but they grow in a fashion reminiscent of the growth of the original sines (since the second derivative of the sine is the negative sine). If we attempt to verify this by dividing through the second differences by the original sines, we see in the rightmost column of the table that we in fact arrive at a constant value of -0.00030461.

What is this mysterious quantity? Nīlakaṇṭha, 1000 years after Āryabhaṭa, shows that it is equal to $\left[\mathrm{Crd}(1)/R\right]^2$ (where R is the radius of the base circle. By assuming that this column continues to be constant, Āryabhaṭa is able to generate the next value of Sin(i) from the preceding ones, simply by working backward through the columns. Thus he can calculate an entire sine table, one entry at a time, if he knows the first value — which, of course, he does.

The differences in the sensibilities between the Greek trigonometry represented by Ptolemy, and the Indian trigonometry represented by Āryabhaṭa, are remarkable. In India we find a keen awareness of the successive differences between trigonometric table values, which led to this and many other insights — close to an understanding and manipulation of what we might call first- and higher-order derivatives. It took the Greek/Arabic/Western tradition 1400 years, until the work of François Viète, to rediscover it. In India this led eventually to iterative approaches to various problems, including some problems that could have been solved more simply by direct means. Conversely, we find in India a lack of concern for geometric precision and clarity: an absence of respect for the Euclidean dream, and even for direct solutions themselves. Indeed, the very notion of proof is missing from the work of Āryabhaṭa and his colleagues. Thus while the content of the trigonometric conversation is related, the context imposes different goals, and hence different judgments on the texts.

i	Sin(i)	Delta(i)	Delta(i)-Delta(i-1)	…divided by Sin(i)
0	0			
1	59.99695387	59.99695387		
2	119.9756321	59.97867822	-0.018275653	-0.00030461
3	179.9177646	59.94213248	-0.03654574	-0.00030461
4	239.8050924	59.88732779	-0.054804694	-0.00030461
5	299.6193732	59.81428083	-0.073046954	-0.00030461
6	359.3423871	59.72301387	-0.091266964	-0.00030461
7	418.9559417	59.6135547	-0.109459172	-0.00030461
8	478.4418784	59.48593666	-0.127618038	-0.00030461
9	537.782077	59.34019863	-0.145738031	-0.00030461
10	596.958462	59.176385	-0.16381363	-0.00030461

Figure 9: Indian sine table calculations

Crossing Disciplinary Boundaries

Indeed, if we consider the term "context" in a slightly different sense, we find another kind of transmission, one that never did occur in India but does happen with Ptolemy's more direct descendants in Islam. Consider the following remarkable synthesis of spherical astronomy in Nīlakaṇṭha's *Tantrasaṅgraha*, around AD 1500. It refers to the *astronomical triangle*, a figure for which each of its sides and two of its angles are important, named, astronomical quantities. In Figure 10, a is the Sun's altitude above the horizon, δ is the Sun's declination, and φ is the observer's terrestrial latitude; meanwhile, the angle at Z is the Sun's azimuth and the angle at N is the so-called *hour-angle* which tells the time of day. Like their plane counterparts, spherical triangles usually require knowledge of three elements to be solvable. This leads to $\binom{5}{3}=10$ different possible combinations of knowns. Nīlakaṇṭha dispatches each of the 10 cases in turn, systematically, solving for the unknowns given values for the knowns. In fact, his solutions parallel what you might find in a modern textbook on spherical trigonometry.

What Nīlakaṇṭha never does (and nor did any of his compatriots) is to abstract the mathematical theorem from the astronomical context. This raises a curious question: can Nīlakaṇṭha really be said to have solved the spherical triangle, as the textbooks do? This sort of situation happens time and again: for instance, al-Battānī, the 9th-century Muslim astronomer, is often given credit for the spherical Law of Cosines because he solved an astronomical problem in a manner mathematically equivalent to its use. But absent the apparently trivial process of abstraction from context, we must conclude that Nīlakaṇṭha and al-Battānī were not really proving, or even discovering, trigonometric theorems. The implications of their work differ from the obvious mathematical corollaries; successors picked up the crucial astronomical threads, but they did not state the general results and rebuild the mathematical foundations from them as we moderns might expect. In Islam the separation of the mathematics from the astronomy was to occur, but it did not start for another century and took more centuries to be realized in full. And incidentally,

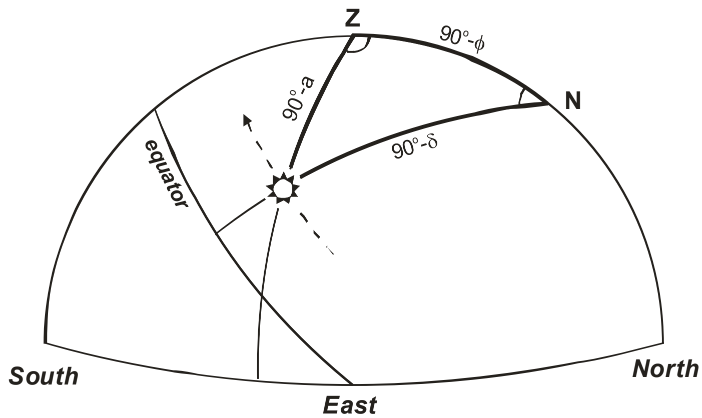


Figure 10: The astronomical triangle

there is a real interpretive danger here: simply by asserting that Nīlakaṇṭha and al-Battānī didn't really demonstrate trigonometric theorems, we run the risk of implicitly devaluing their accomplishments, thereby allowing presentist values to skew our perspective.

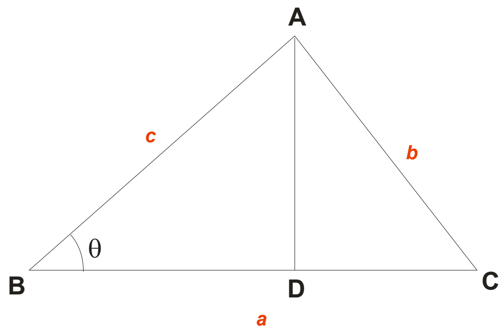
So, there are implied values that can be altered in transmission: both from one culture to another, and from one scientific discipline to another. These are often so subconscious that they can be overlooked, and without due diligence our unspoken points of view on these values can lead us to misjudging the accomplishments of the past.

Chasing Elusive Theorems

We appear to be heading towards a quandary: what, then, does it mean, precisely, to originate a theorem? What would Nīlakaṇṭha have to do before we can say that he solved the general spherical triangle? Clearly the theorem would need to be disentangled from its immediate scientific context, and either stated generally or have been capable of being transposed to different situations. But even this condition can be posed without fear only if Indian mathematics shares with us the same values with respect to clear, abstract mathematical statements, which seems unlikely. With fading hope, one might ask for the earliest proof. But there are countless examples of the nebulous and changing meanings of proof over the centuries. Would the 15th-century Indian derivations of Taylor series expansions of the sine and cosine, playing fast and loose as they do with infinitesimals, qualify? And what of results used heavily in pre-modern China, where the very idea of proof is alien to most mathematical practice?

There is a flip side to this coin. A theorem that appears completely abstracted from physical context might also, by this abstraction, lose its significance. Consider Euclid's *Elements* II.13, which states in no uncertain terms the planar Law of Cosines (Figure 11). But it did so almost two centuries before trigonometry was even conceived, and was used for geometric purposes rather than for mensuration. We appear to be in a bind: in order to determine what we mean by "the discovery of the Law of Cosines", we cannot ask for the result to exist solely within some context, but also we cannot ask for the result to be

Elements II.13



"In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle."

$$\begin{aligned} b^2 &= a^2 + c^2 - 2a \times BD \\ &= a^2 + c^2 - 2ac\cos\theta \end{aligned}$$

Figure 11: The Law of Cosines in Euclid's *Elements*

---

abstracted from its context. It seems to me that we can judge its discovery only by its subsequent use and interaction with other theory. And this leads to the rather strange statement that the Law of Cosines was discovered not when it was first stated, but rather when someone first turned its attention away from pure geometry and toward arc and line measurement. Put starkly, precisely the same text in one context might qualify as a discovery, and in another it might not. The only alternative that I can see to this conclusion is simply to dismiss the issue of the discovery of theorems altogether, a rather drastic solution.

There is yet more to the predicaments one encounters when tracing the history of theorems. Consider the following example, taken from 14th-century Venice, where trigonometric methods were first used for the purpose of seafaring. In order to restore the correct headings for vessels that had been forced off course, a number of navigators were equipped with small tables known as *toleta de marteloio*. Imagine a ship attempting to travel due east in Figure 12, but being forced to travel off course for some time. The ship must then determine a new heading in order to arrive safely at its destination. Among the quantities defined here are the *alargar*, the distance the ship is off course for every 100 miles sailed; and the *ritorno*, the distance the ship needs to arrive at its destination for every 10 miles that it is off course. Each term in the table (Figure 13) corresponds to some multiple of a standard trigonometric function: for instance, the *alargar* is 100 times the sine, and the *ritorno* is 10 times the cosecant.

Of course we have here the same contextual problems I raised before, but I'd like to focus our heading in a different direction. Consider the following typical problem described by Michael of Rhodes (Figure 14). A ship needs to travel 100 miles eastward

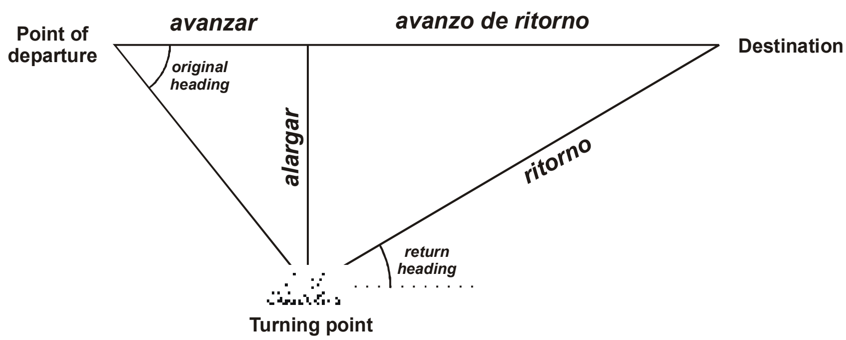


Figure 12: Definitions of terms in the *marteloio*

Quarter	*Alargar* (distance off course)	*Avanzar* (advance)	*Ritorno* (return)	*Avanzar de ritorno* (advance on return)
1	20	98	51	50
2	38	92	26	24
3	55	83	18	15
4	71	71	14	10
5	83	55	12	6 1/2
6	92	38	11	4
7	98	20	10 1/5	2 1/5
8	100	0	10	0
	For every 100 miles		For every 10 miles *alargar*	

Figure 13: A typical *toleta de marteloio*

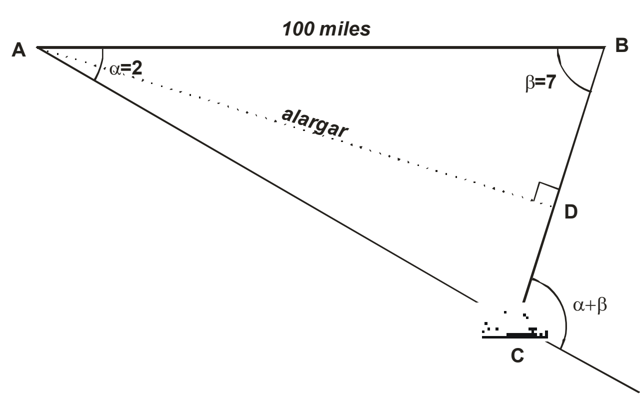


Figure 14: A typical *marteloio* problem

from A to B, but is forced to travel an unknown distance AC in the direction $\alpha = 2$ quarters south of East (2/8 of a right angle). To correct its heading, it must travel in the direction $\beta = 7$ quarters north of East. How far did the ship travel from A to C?

Michael's solution takes two steps. First, he considers ΔABD, and notes that AD is the *alargar* of β. This is a twist on the original meaning of *alargar*, to be sure: an abstraction of the geometric meaning from its original context. Incidentally, this gives us some reason to assert that the navigators really thought of the *alargar* as an independent function rather than solely as a distance off course. Now that AD is known, Michael turns to ΔACD and finds that AC is equal to AD times the *ritorno* of $\alpha + \beta$ (up to a couple of constants). In modern notation,

$$AC = \frac{alargar(\beta)}{10} \cdot ritorno(\alpha + \beta).$$

Now the *alargar* is (more or less) the sine, and the *ritorno* is (more or less) the reciprocal of the sine. So, Michael's solution is effectively the planar Law of Sines, applied to AB with $\angle C$, and AC with angle β.

Or so the historical literature on this topic would have you believe. But this isn't really a fair conclusion. In every similar instance, Michael again and again determines AD, before then proceeding to AC. So there is no direct awareness in Michael's mind, as far as we can tell, about the ratios of sines and angles in the combined triangle ABC. He is capable of solving anything that the Law of Sines can solve, but the actual relationship is disguised by the intermediate step of calculating AD.

Now imagine a plausible sequel to this episode, that Michael and his successors gradually over time turn this calculation into an algorithm. The intermediate calculation of AD gradually disappears into the formula, until finally all that remains is the calculation, identical in every appearance to the Law of Sines. In this imagined process, it seems to me that, if anything, knowledge has been lost: the original geometric content is blurred or forgotten. But what would appear in our hypothetical text might register to a reader as a gain: a rather clear, unambiguous statement of the Law of Sines.

So, what precisely does it mean to know the Law of Sines? Is it the clear geometric statement that we see in modern textbooks? This strikes me as presentist. Is it a pattern of calculation that solves triangles accessible to the Law of Sines? Our hypothetical example seems to cast doubt that such a calculation really would signal sufficient awareness of the theorem. Is it simply a geometric demonstration of the result? This ignores the problem of the "intermediate step" that we've just seen. I don't have an easy answer to this, and I doubt that there is an answer that would satisfy everyone in this room. We're left here with an ambiguity that affects the history of mathematics at least as sharply as it does the history of other disciplines.

Conclusion

I've burdened you with some weighty questions here: to what extent and in what ways does culture affect mathematics, at levels as high in the curriculum as trigonometry? How should one judge and respect the scientific context in which the mathematics emerged? Is there even such a thing as the discovery of a mathematical theorem? Questions such as these, I hope, will cause us to reflect on precisely what mathematics is. Is it a set of universal truths, or a societal construct of the mind, or somewhere in between? Of course, this latter query is another version of the old saw: is mathematics discovered in a Platonic world out there somewhere, or is it invented in an individual or group mind?

Our answers to these questions obviously will impact the approach we take to mathematics education: for instance, should we take a traditional logical approach, or should we adopt social constructivist strategies? How do we rate algebraic performance and ability to achieve correct solutions versus explanatory clarity and group work? But I'd like to suggest a slightly different take on all of this: beyond using our *answers* to inform what we take to our students, we might want to consider taking the *questions* to them as well.

Everyone has their pet peeve about student attitudes in their mathematics classes. Mine is that they see mathematics as an algorithmic game. To them it is a task to accomplish rather than a playground to meander through: where the primary mission is not to take away a lesson that will benefit them in the long term, but to survive with a decent grade in the short term, to climb to the next rung of the academic ladder. But how does this attitude arise? In order to increase examination performance, we strip away as much of the context as we can. You don't need context to get the right answer. Show students how to play the game, and they'll play it in exchange for good grades. Instead, I propose that we give our students the same hard questions that I've been raising with you. Have students debate whether mathematics really is different when practiced the Greek way or the Indian way. Have them struggle with the relation between mathematics and its scientific context. Have them wrestle with what it means to discover a theorem. The sooner that students realize that there are difficult interpretive questions also in mathematics, the sooner that they will begin to recognize that mathematics is an inextricable part of the intellectual development of our and other cultures — not a symbol-pushing game played on the sidelines of history. This sort of understanding leads to broader and more significant cross-disciplinary thinking, and just as importantly, it leads to de-mystification. Use deep questions to make mathematics more ambiguous, and paradoxically, the meaning and significance of mathematics will be made all the more clear.