TEACHING WITH PRIMARY HISTORICAL SOURCES: SHOULD IT GO MAINSTREAM? CAN IT?

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ABSTRACT

Many are now teaching mathematics directly with primary historical sources, in a variety of courses and levels. How far should this be taken? Should we adapt or redesign standard courses to a completely historical approach, chiefly from primary sources? If so, what are the obstacles to achieving this? Materials? Instructor attitudes? What should and can we do about such things?

1 Introduction

I am truly honored to be asked to speak on integrating the history of mathematics in mathematics education. Advocating the teaching of mathematics using history is presumably not very controversial at this conference, more like "preaching to the choir", as one says in English. But I wish to be somewhat provocative, perhaps even controversial, by suggesting a dream I have had for some time, that all students should learn the principal content of their mathematics directly from studying primary sources, as is done in the humanities, where students read the great original literature, not just about the great literature. In other words, I propose that we rebuild the entire mathematics curriculum at all levels around translated primary sources studied directly by our students. If you think this is extreme, then at least I am fulfilling the role of being a provocative speaker.

My belief that we should and can aim for a mathematics curriculum that is rich throughout in primary sources has developed only very slowly from my own experiences in the past twenty years. First I would like to describe this personal evolution, because it reflects very clearly some of the important challenges involved in implementing my dream.

2 A personal odyssey as an illustration of issues

First I codeveloped two one-semester courses for beginning and advanced undergraduate university students, based entirely on primary historical sources. Somewhat ironically, I was motivated by William Dunham's description of a great theorem enrichment course for teachers in which he rewrote the original source material in his own words, but I and my collaborator Reinhard Laubenbacher decided to skip the rewriting step and toss the original sources at our students, partly because it seemed like too much work to rewrite things; of course in retrospect this became my chief pedagogical goal, to have students read original sources themselves, the only compromise being translation into English. These courses each follow several great mathematical themes and problems through millennia via primary sources. The courses have been continually successful now for two decades, and have led to two books [4, 6] each with multiple chapters built entirely around primary sources, in which different chapters can be taught in different incarnations of the course [10]. They fully embody my vision of courses and books in which primary sources are the principal objects of study and learning.

However, while they are solid mathematics courses, they are not focused on syllabi in the standard curriculum, i.e., they do not fall into the category of a course in calculus, or discrete mathematics, or real analysis, or abstract algebra, or the many other compartmentalized topics in the typical institutional curriculum. They were instead designed around historical development of great ideas viewed through primary sources, not just a purely modern vision of mathematics; and they are flexible, with different subjects covered in different semesters. In other words, they were designed and implemented in total freedom, rather than under the constraint of an existing course syllabus. So while these courses are taken by many university mathematics students as an elective, and by students studying other disciplines, they are not in the "mainstream" of the curriculum. Moreover, students and colleagues alike tend to consider them as "history of mathematics" courses, simply because no other mathematics courses have any meaningful history in them. Together these features leave the two courses somewhat outside the main path of a standard undergraduate mathematics student's degree coursework, and hinder the adoption of the courses elsewhere.

At about the same time, I also became heavily involved in collaborative developing, teaching, and publishing of student projects for calculus courses [5]. While these were not historical in nature, this began my very slow process of moving away from lecturing in regular teaching, towards a more student-centered, problem-driven classroom, which I personally find prerequisite to engaging primary historical sources as the principal objects of study. Only by combining student project activity with an active classroom and historical materials have I ultimately managed to even begin building a standard curriculum around primary sources. I have recently written on my current thoughts [8] about creating a classroom dynamic in which students are engaged in high-level active work, rather than listening to lecture.

Then about thirteen years ago, I cocreated a graduate level mathematics course on the role of history in teaching mathematics, in which each graduate student develops a written teaching module based on historical material. While this is a successful mathematics education graduate course, it is a course on mathematics education, not a mathematics course based on historical material.

Around this time I also made my first attempt to inject a primary historical source in a substantial way into a regular course, namely Arthur Cayley's first paper on group theory in an abstract algebra course when students first encounter groups [9]. I began to realize that my students could benefit tremendously from having their very first encounters with the notion of an abstract group be through the wonderful mathematics emerging in the nineteenth century that motivated Cayley to define and develop the abstract idea and first steps towards a theory of groups. This was my first indication that one can very profitably simply start using primary sources as key documents of study in a regular course, without dramatically changing the "content" of the course.

This idea has expanded greatly during the past six years into an increasing collaboration with an expanding circle of colleagues. Some of us had experience both teaching with primary sources and with designing substantial student projects, which engage students in large multi-step assignments and written reports on their investigation, and which may last from one to several weeks at a time. We decided to combine these pedagogical approaches, and to focus on regular course content on the discrete side of mathematics, very broadly conceived. So with support from the National Science Foundation, I am now part of a team of seven or more faculty with collaborating writers and testers at numerous institutions, who are developing, testing, and evaluating student projects based on primary historical sources, for teaching regular course syllabi in discrete mathematics, abstract algebra, graph theory, combinatorics, logic, and computer science (e.g., courses on algorithms or automata theory). We hope also that a useful statistical evaluation of the effects of our historical projects will emerge from the nature and scope of this endeavor. Details of our 20–30 student projects based on primary sources, some completed, tested, and published, some yet to be written, are available in a resource book [1] and at our web sites [2, 3]. Below I will use one of these projects to illustrate how I believe primary sources can be central to the curriculum. And a conference workshop will allow attendees to experience several of these historical projects firsthand.

Most recently, in teaching an upper undergraduate level geometry course, I realized that I could have students learn most of the course content on the hyperbolic non-Euclidean plane from the original sources by Euclid, Legendre, Lobachevsky, and Poincaré presented in the geometry chapter of my first book [6] of annotated primary sources, so these pre-prepared primary sources fit well and easily into the course.

I finally feel I am on a route towards a standard course curriculum in which primary historical sources play a core role, but you can see that it has been a long, slow road, that I am benefiting from working in collaboration with others, and that so far I still only have a part of some courses based this way. Nonetheless, I now actually see that this could grow into courses built entirely around primary sources. Of course I realize that I am only one of many around the world who are working to incorporate primary sources in key ways into the regular curriculum, and I would like to acknowledge and applaud everyone else's efforts as well; this gives me the inspiration of community to continue, and I hope that together we may have an impact. My intent in this section has simply been to show by example what some of the challenges are in basing courses on primary sources, but that it may be possible.

3 Motivations: why or why not?

Why should we use primary sources at the foundations of our teaching? Or why not?

The reasons for doing this have already been enunciated by many others over the years, but I will merely mention here motivation and deep connections along time, understanding essence, origin, and discovery, mathematics as a humanistic endeavor; participating in the process of doing mathematics through experiment, conjecture, proof, generalization, publication and discussion; more profound technical comprehension from initial simplicity; also dépaysement (disorientation, cognitive dissonance, multiple points of view), a question-based curriculum that knows where it came from and where it might be going. Questions before answers, not answers to questions that have not been asked.

On the other hand, one can think of reasons why teaching with primary sources at the core might not be good to aim for. In his intentionally "devil's advocate" article [11], Man-Keung Siu lists possible unfavourable factors, some of which could apply to primary sources. There are pedagogical ones:

- How can you set questions on it on a test?
- It can't improve the student's grade.
- Students don't like it.
- Students regard it as just as boring as the subject mathematics itself!
- Students do not have enough general knowledge of culture to appreciate it!
- Progress in mathematics is to make difficult problems routine, so why bother to look back?
- What really happened can be rather tortuous. Telling it as it was can confuse rather than enlighten!
- Does it really help to read original texts, which is a very difficult task?
- Is it liable to breed cultural chauvinism and parochial nationalism?
- Is there any empirical evidence that students learn better when history of mathematics is made use of in the classroom?

I now have enough experience actually teaching with primary sources to say that I personally have found all these concerns to be either untrue or irrelevant with my students and my chosen primary sources. I could elaborate and explain, but here I will only affirm my clear experience that with carefully selected and prepared primary source material, and the right pedagogical method in the classroom, these objections or concerns can and should be rejected. Thank you for raising them, Man-Keung!

Man-Keung also listed concerns that are logistical in nature, and I will address these next.

4 Logistical obstacles

Man-Keung Siu's logistical concerns about using history certainly can apply to teaching with primary sources:

- There is a lack of resource material on it!
- There is a lack of teacher training in it!
- I have no time for it in class!

The good news is that the first two concerns are the things we can all work on, and doing so will influence the third!

4.1 Is there a lack of resource material?

Yes, but the availability of published primary sources and translations in all aspects of mathematics has been growing at great speed in the past few decades, thanks to the work of many wonderful people, and this work we should all continue. It would be wonderful to have a continually updated central listing of these sources. Providing such a central resource online is something incredibly useful that HPM could sponsor, and I believe it is necessary to widespread adoption of teaching with primary sources. I will address the important issue of convenient packaging of primary sources for teaching below.

4.2 Instructor training, motivation?

Is there a lack of teacher training in using primary sources? Yes, of course, and this challenge can and should be relieved by more formal training opportunities, which is another task for us. But the question, I think, really hints at a deeper issue. How do we interest other instructors in teaching with history, and in particular in using primary sources? It will only be through instructors' desire to teach with history that it will happen, not by coercion, since mathematics instructors like to make their own pedagogical decisions. Since I believe that enticement is the only way, we should entice with wonderful source material packaged to make instructors salivate at the idea of learning and teaching with them. Some teachers want or need prepared guiding materials in the form of textbooks or projects, while others like to create their own. So I believe that the solution lies in providing a variety of packaging for primary source materials, and flexibility in how they can be used, along with our own leading by example in our teaching. Let us discuss packaging.

4.3 Packaging into textbooks and projects versus time and pedagogical style

In many parts of the world, textbooks are the driving force behind curriculum and pedagogy, whether we like it or not. So I believe that success in attracting others to teaching mainly with primary sources will require us to create textbooks that have this as their theme. But student projects are also playing a more substantial role in teaching these days, so projects based on primary sources can complement or even supplant portions of a standard textbook, and thus play an intermediate role as stepping stones in the direction I am advocating. This is the approach I am currently working on, as mentioned earlier [1, 2, 3]. In fact, a course could be built entirely on a sequence of projects based on primary sources, and I am thinking in that direction.

These issues cannot be divorced from the remaining question of whether there is time in class for teaching with historical sources. My personal experience is a resounding yes, there is time, and I constantly become stronger in that conviction, by changing my teaching in two respects.

First, one can move entirely away from lecturing, whether using historical sources or not, by having students first do advance reading of all new material at home, and working and writing profitably about it, entirely before initial class contact with the material. This makes lecture totally unnecessary and unfruitful, and means that class time is spent on student work and interaction with the instructor and each other, and some whole class discussion, that already starts at a higher level; the time thus saved and redirected from lecture is enormous. I have written about the details of how I implement this [8].

Second, one can implement learning from primary sources through projects in such a way that it literally takes over core topics from the textbook, i.e., one can find and develop primary sources to teach core material of the course. Then the textbook becomes at most ancillary, perhaps a source of modern notation, extra exercises, and an alternative, more modern point of view. The time otherwise spent with the textbook will instead be spent learning the same material from primary sources, and the textbook becomes a supplement, not the other way around. I would like to elaborate on one example of this.

5 A sample project: Pascal on induction and combinatorics

To see a detailed example of how core syllabus material can be taught directly from a historical project, and its effect on students, consider an introductory discrete mathematics course intended to have students start learning to make proofs in mathematics, in which some key content is to learn mathematical induction as a proof technique, and to become comfortable with binomial coefficients, combination numbers, factorials, and some elementary number theory. I have combined all these core topics in a three week class project [1] centered on Blaise Pascal's *Treatise on the Arithmetical Triangle* [7], from which I present here some selected excerpts and discussion. Pascal's treatise expounds the principle of mathematical induction, and his triangle leads into combinatorics.

After a good bit of historical background and context, students begin reading Pascal:

TREATISE ON THE ARITHMETICAL TRIANGLE

Definitions

I call *arithmetical triangle* a figure constructed as follows:

From any point, G, I draw two lines perpendicular to each other, GV, Gζ in each of which I take as many equal and contiguous parts as I please, beginning with G, which I number 1, 2, 3, 4, etc., and these numbers are the *exponents* of the sections of the lines.

Next I connect the points of the first section in each of the two lines by another line, which is the base of the resulting triangle.

In the same way I connect the two points of the second section by another line, making a second triangle of which it is the base.

And in this way connecting all the points of section with the same exponent, I construct as many triangles and bases as there are exponents.

Through each of the points of section and parallel to the sides I draw lines whose intersections make little squares which I call *cells*.

Cells between two parallels drawn from left to right are called *cells of the same parallel row*, as, for example, cells G, σ, π, etc., or φ, ψ, θ, etc.

Those between two lines are drawn from top to bottom are called *cells of the same perpendicular row*, as, for example, cells G, φ, A, D, etc., or σ, ψ, B, etc.

Z	1	2	3	4	5	6	7	L 8	9	10
1	G 1	σ 1	π 1	λ 1	μ 1	δ 1	ζ 1	1	1	1
2	φ 1	ψ 2	θ 3	R 4	S 5	N 6	7	8	9	
3	A 1	B 3	C 6	ω 10	ξ 15	21	28	36		
4	D 1	E 4	F 10	ρ 20	Y 35	56	84			
5	H 1	M 5	K 15	35	70	126				
6	P 1	Q 6	21	56	126					
7	V 1	7	28	84						
T 8	1	8	36							
9	1	9								
10	1									

Those cut diagonally by the same base are called *cells of the same base*, as, for example, D, B, θ, λ, or A, ψ, π.

Cells of the same base equidistant from its extremities are called *reciprocals*, as, for example, E, R and B, θ, because the parallel exponent of one is the same as the perpendicular exponent of the other, as is apparent in the above example, where E is in the second perpendicular row and in the fourth parallel row and its reciprocal, R, is in the second parallel row and in the fourth perpendicular row, reciprocally. It is very easy to demonstrate that cells with exponents reciprocally the same are in the same base and are equidistant from its extremities.

It is also very easy to demonstrate that the perpendicular exponent of any cell when added to is parallel exponent exceeds by unity the exponent of its base.

For example, cell F is in the third perpendicular row and in the fourth parallel row and in the sixth base, and the exponents of rows 3 and 4, added together, exceed by unity the exponent of base 6, a property which follows from the fact that the two sides of the triangle have the same number of parts; but this is understood rather than demonstrated.

Of the same kind is the observation that each base has one more cell than the preceding base, and that each has as many cells as its exponent has units; thus the second base, $\varphi\sigma$, has two cells, the third, $A\psi\pi$, has three, etc.

Now the numbers assigned to each cell are found by the following method:

The number of the first cell, which is at the right angle, is arbitrary; but that number having been assigned, all the rest are determined, and for this reason it is called the *generator* of the triangle. Each of the others is specified by a single rule as follows:

The number of each cell is equal to the sum of the numbers of the perpendicular and parallel cells immediately preceding. Thus cell F, that is, the number of cell F, equals the sum of cell C and cell E, and similarly with the rest.

Whence several consequences are drawn. The most important follow, wherein I consider triangles generated by unity, but what is said of them will hold for all others.

First Consequence

In every arithmetical triangle all the cells of the first parallel row and of the first perpendicular row are the same as the generating cell.

For by definition each cell of the triangle is equal to the sum of the immediately preceding perpendicular and parallel cells. But the cells of the first parallel row have no preceding perpendicular cells, and those of the first perpendicular row have no preceding parallel cells; therefore they are all equal to each other and consequently to the generating number.

Thus $\varphi = G + 0$, that is, $\varphi = G$,
$A = \varphi + 0$, that is, φ,
$\sigma = G + 0$, $\pi = \sigma + 0$,

And similarly of the rest.

Second Consequence

In every arithmetical triangle each cell is equal to the sum of all the cells of the preceding parallel row from its own perpendicular row to the first, inclusive.

Let any cell, ω, be taken. I say that it is equal to $R + \theta + \psi + \varphi$, which are the cells of the next higher parallel row from the perpendicular row of ω to the first perpendicular row.

This is evident if we simply consider a cell as the sum of its component cells.

For ω equals $R + \underbrace{C}_{\theta + \underbrace{B}_{\psi + \underbrace{A}_{\varphi,}}}$

for A and φ are equal to each other by the preceding consequence.

Therefore $\omega = R + \theta + \psi + \varphi$.

...

Then come some exercises for students, connecting to modern notation, indexing, summation notation, terminology, and the adequacy of Pascal's proofs by example or initial iteration:

"1. Pascal's Triangle and its numbers

   (a) Let us use the notation $T_{i,j}$ to denote what Pascal calls the number assigned to the cell in *parallel row* i (which we today call just *row* i) and *perpendicular row* j (which we today call *column* j). We call the i and j by the name *indices* (plural of *index*) in our notation. Using this notation, explain exactly what Pascal's rule is for determining all the numbers in all the cells. Be sure to give full details. This should include explaining for exactly which values of the indices he defines the numbers.
   (b) In terms of our notation $T_{i,j}$, explain his terms *exponent, base, reciprocal, parallel row, perpendicular row,* and *generator.*
   (c) Rewrite Pascal's first two "Consequences" entirely in the $T_{i,j}$ notation.
   (d) Rewrite his proofs of these word for word in our notation also.
   (e) Do you find his proofs entirely satisfactory? Explain why or why not.
   (f) Improve on his proofs using our notation. In other words, make them apply for arbitrary prescribed situations, not just the particular examples he lays out.

"2. Modern mathematical notation

   Read in a modern textbook about index, summation, and product notations, and recurrence relations. Do some exercises."

Then Pascal, and students, begin to ease into the concept of proof by mathematical induction:

Fifth Consequence

In every arithmetical triangle each cell is equal to its reciprocal.

For in the second base, $\varphi\sigma$, it is evident that the two reciprocal cells, φ, σ, are equal to each other and to G.

In the third base, A, ψ, π, it is also obvious that the reciprocals, π, A, are equal to each other and to G.

In the fourth base it is obvious that the extremes, D, λ, are again equal to each other and to G.

And those between, B, θ, are obviously equal since $B = A + \psi$ and $\theta = \pi + \psi$. But $\pi + \psi = A + \psi$ by what has just been shown. Therefore, etc.

Similarly it can be shown for all the other bases that reciprocals are equal, because the extremes are always equal to G and the rest can always be considered as the sum of cells in the preceding base which are themselves reciprocals.

“3. Symmetry in the triangle: first contact with mathematical induction

Write the Fifth Consequence using our index notation. Use index notation and the ideas in Pascal’s proof to prove the Consequence in full generality, not just for the example he gives. Explain the conceptual ideas behind the general proof.

“4. Mathematical induction: gaining more familiarity

(a) Read in a modern textbook about mathematical induction.

(b) Prove Pascal’s First Consequence by mathematical induction. (Hint: for a proof by mathematical induction, always first state very clearly exactly what the n-th mathematical statement $P(n)$ says. Then state and prove the base step. Then state the inductive step very clearly before you prove it.)

(c) Write the general form of Pascal’s Second Consequence, and give a general proof using summation notation, but following his approach.

(d) Now prove the Second Consequence by mathematical induction, i.e., a different proof.

(e) **Optional**: More patterns.

i. Write the Fourth Consequence using summation notation. Hint: You can write it using a sum of sums. Try writing Pascal’s proof in full generality, using summation notation to help. If you don’t complete it his way, explain why it is difficult.

ii. Prove the Fourth Consequence by mathematical induction.”

...

“The next consequence is the most important and famous in the whole treatise. Pascal derives a formula for the ratio of consecutive numbers in a base. From this he will obtain an elegant and efficient formula for all the numbers in the triangle.”

Twelfth Consequence

In every arithmetical triangle, of two contiguous cells in the same base the upper is to the lower as the number of cells from the upper to the top of the base is to the number of cells from the lower to the bottom of the base, inclusive.

Let any two contiguous cells of the same base, E, C, be taken. I say that

E	:	C	::	2	:	3
the lower		the upper		because there are two cells from E to the bottom, namely E, H,		because there are three cells from C to the top, namely C, R, μ.

Although this proposition has an infinity of cases, I shall demonstrate it very briefly by supposing two lemmas:

The first, which is self-evident, that this proportion is found in the second base, for it is perfectly obvious that $\varphi : \sigma :: 1 : 1$;

The second, that if this proportion is found in any base, it will necessarily be found in the following base.

Whence it is apparent that it is necessarily in all the bases. For it is in the second base by the first lemma; therefore by the second lemma it is in the third base, therefore in the fourth, and to infinity.

It is only necessary therefore to demonstrate the second lemma as follows: If this proportion is found in any base, as, for example, in the fourth, $D\lambda$, that is, if $D : B :: 1 : 3$, and $B : \theta :: 2 : 2$, and $\theta : \lambda :: 3 : 1$, etc., I say the same proportion will be found in the following base, $H\mu$, and that, for example, $E : C :: 2 : 3$.

For $D : B :: 1 : 3$, by hypothesis.

Therefore $\underbrace{D + B}_{E} : B :: \underbrace{1 + 3}_{4} : 3$

$$E : B :: 4 : 3$$

Similarly $B : \theta :: 2 : 2$, by hypothesis

Therefore $\underbrace{B + \theta}_{C} : B :: \underbrace{2 + 2}_{4} : 2$

$$C : B :: 4 : 2$$

But $B : E :: 3 : 4$

Therefore, by compounding the ratios, $C : E :: 3 : 2$. Q.E.D.

The proof is the same for all other bases, since it requires only that the proportion be found in the preceding base,and that each cell be equal to the cell before it together with the cell above it, which is everywhere the case.

"6. Pascal's Twelfth Consequence: the key to our modern factorial formula

(a) Rewrite Pascal's Twelfth Consequence as a generalized modern formula, entirely in our $T_{i,j}$ terminology. Also verify its correctness in a couple of examples taken from his table in the initial definitions section.

(b) Adapt Pascal's proof by example of his Twelfth Consequence into modern generalized form to prove the formula you obtained above. Use the principle of mathematical induction to create your proof.

Now Pascal is ready to describe a formula for an arbitrary number in the triangle."

Problem

Given the perpendicular and parallel exponents of a cell, to find its number without making use of the arithmetical triangle.

Let it be proposed, for example, to find the number of cell ξ of the fifth perpendicular and of the third parallel row.

All the numbers which precede the perpendicular exponent, 5, having been taken, namely 1, 2, 3, 4, let there be taken the same number of natural numbers, beginning with the parallel exponent, 3, namely 3, 4, 5, 6.

Let the first numbers be multiplied together and let the product be 24. Let the second numbers be multiplied together and let the product be 360, which, divided by the first product, 24, gives as quotient 15, which is the number sought.

For ξ is to the first cell of its base, V, in the ratio compounded of all the ratios of the cells between, that is to say, $\xi : V$

in the ratio compounded of	$\xi : \rho,$	$\rho : K,$	$K : Q,$	$Q : V$
or by the twelfth consequence	$3 : 4$	$4 : 3$	$5 : 2$	$6 : 1$

Therefore $\xi : V :: 3 \cdot 4 \cdot 5 \cdot 6 : 4 \cdot 3 \cdot 2 \cdot 1$.

But V is unity; therefore ξ is the quotient of the division of the product of $3 \cdot 4 \cdot 5 \cdot 6$ by the product of $4 \cdot 3 \cdot 2 \cdot 1$.

N.B. If the generator were not unity, we should have had to multiply the quotient by the generator.

"7. Pascal's formula for the numbers in the Arithmetical Triangle

(a) Write down the general formula Pascal claims in solving his "Problem." Your formula should read $T_{i,j} =$ "some formula in terms of i and j." Also write your formula entirely in terms of factorials.

(b) Look at the reason Pascal indicates for his formula for a cell, and use it to make a general proof for your formula above for an arbitrary $T_{i,j}$. You may try to make your proof just like Pascal is indicating, or you may prove it by mathematical induction."

And now Pascal is ready for applications, in particular to combinatorics:

VARIOUS USES OF THE ARITHMETICAL TRIANGLE WHOSE GENERATOR IS UNITY

Having given the proportions obtaining between the cells and the rows of arithmetical triangles, I turn in the following treatises to various uses of those triangles whose generator is unity. But I leave out many more than I include; it is extraordinary how fertile in properties this triangle is. Everyone can try his hand. I only call your attention here to the fact that in everything that follows I am speaking exclusively of arithmetical triangles whose generator is unity.

...

USE OF THE ARITHMETICAL TRIANGLE FOR COMBINATIONS

The word *combination* has been used in several different senses, so that to avoid ambiguity I am obliged to say how I understand it.

When of many things we may choose a certain number, all the ways of taking as many as we are allowed out of all those offered to our choice are here called the *different combinations*.

For example, if of four things expressed by the four letters, A, B, C, D, we are permitted to take, say any two, all the different ways of taking two out of the four put before us are called *combinations*.

Thus we shall find by experience that there are six different ways of choosing two out of four; for we can take A and B, or A and C, or A and D, or B and C, or B and D, or C and D.

I do not count A and A as one of the ways of taking two; for they are not different things, they are only one thing repeated.

Nor do I count A and B and B and A as two different ways; for in both ways we take only the same two things but in a different order, and I am not concerned with the order; so that I could make myself understood at once by those who are used to considering combinations, simply by saying that I speak only of combinations made without changing the order.

We shall also find by experience that there are four ways of taking three things out of four; for we can take ABC or ABD or ACD or BCD.

Finally we shall find that we can take four out of four in one way only, $ABCD$.

I shall speak therefore in the following terms:

1 in 4 can be combined 4 times.
2 in 4 can be combined 6 times.
3 in 4 can be combined 4 times.
4 in 4 can be combined 1 time.

Or:

the number of combinations of 1 in 4 is 4.
the number of combinations of 2 in 4 is 6.
the number of combinations of 3 in 4 is 4.
the number of combinations of 4 in 4 is 1.

But the sum of all the combinations in general that can be made in 4 is 15, because the number of combinations of 1 in 4, of 2 in 4, of 3 in 4, of 4 in 4, when joined together, is 15.

After this explanation I shall give the following consequences in the form of lemmas:

Lemma 1.

There are no combinations of a number in a smaller number; for example, 4 cannot be combined in 2.

...

Proposition 2

The number of any cell is equal to the number of combinations of a number less by unity than its parallel exponent in a number less by unity than the exponent of its base.

Let any cell be taken, say F in the fourth parallel row and in the sixth base. I say that is is equal to the number of combinations of 3 in 5, less by unity than 4 and 6, for it is equal to the cells $A + B + C$. Therefore by the preceding proposition, etc.

"1. Combinations according to Pascal

(a) Explain in your own words what Pascal says about how many combinations there are for choosing two things out of four things.

(b) Write Pascal's Proposition 2 using our $T_{i,j}$ notation for numbers in the triangle. In other words, fill in a sentence saying "$T_{i,j}$ is the number of combinations of choosing ____ things from ____ things." Pascal's justification for his Proposition 2 is based on his Lemma 4 and Proposition 1, which are not included in this project. However, the reader is encouraged to study and understand them, to wit:

(c) **Optional:** From Pascal's treatise [7, vol. 30], rewrite his statements and explanations of his Lemma 4 and Proposition 1 in your own words. State and prove Lemma 4 in the general case; that is, show that the number of combinations of k in n is the sum of the combinations of $k-1$ in $n-1$ and the combinations of k in $n-1$. Also explain why Proposition 2 follows from Proposition 1.

"2. Combinations and Pascal's recursion formula

(a) The modern symbol $\binom{n}{r}$ means the number of ways ("combinations") of choosing r things from amongst n things. Explain how this is related to what we have been learning about the Arithmetical Triangle from reading Pascal. In particular, explain how the numbers $T_{i,j}$ are related to the numbers $\binom{n}{r}$. Do this by writing an equation expressing $T_{i,j}$ in $\binom{n}{r}$ notation, and also writing an equation expressing $\binom{n}{r}$ in $T_{i,j}$ notation. Now use the formula we learned earlier, from Pascal's solution to his *Problem,*[1] to write a formula for the combination number $\binom{n}{r}$, and manipulate it to express it entirely in terms of factorials.

[1] *Given the perpendicular and parallel exponents of a cell, to find its number without making use of the arithmetical triangle.*

(b) Now read in a modern textbook about the multiplication rule for counting possibilities, about permutations, and about combinations. Explain how a combination is different from a permutation.

(c) Read in a modern textbook about the algebra of combinations, Pascal's recursion formula, and how the text presents Pascal's Triangle. How is it different from Pascal's presentation?"

And one can even connect all Pascal's combinatorics and proof by induction to Fermat's Theorem in number theory, and cryptography.

"Now we can put together all of what we have learned from Pascal to prove an extremely important result in number theory, called *Fermat's Little Theorem*, which is at the heart of today's encryption methods in digital communications. The ingredients will be the binomial theorem, proof by mathematical induction as learned from Pascal, Pascal's formula for the numbers in his triangle (solved in his *Problem*), and uniqueness of prime factorization.

"1. The binomial theorem and combinations

Read in a modern textbook about the binomial theorem. Write an explanation of the proof of the binomial theorem using the idea of counting combinations.

"2. Discovering Fermat's "Little" Theorem: prime numbers and congruence remainders

(a) Make a table of the remainders of a^n upon division by n for positive integer values of both a and n ranging up to 14. To do this you should learn about congruence arithmetic, and figure out how to do these calculations quickly and easily without a calculator.

(b) Based on your table, make a conjecture of the form $a^p \equiv ?$ (mod p) for p a prime number and a any integer. This is called Fermat's "Little" Theorem; it is one of the most important phenomena in number theory. Also make some other interesting conjectures from patterns in your table, and try to prove them, perhaps using the binomial theorem.

(c) Write up the details of proving Fermat's Theorem by mathematical induction on a, with p held fixed. Use the binomial theorem, our knowledge of Pascal's "factorial" formula for binomial coefficients, and the Fundamental Theorem of Arithmetic (uniqueness of prime factorization) to analyze divisibility of the binomial coefficients by a prime p.

(d) **Optional:** Read about what Fermat was trying to do when he discovered his Theorem [6, p. 159ff]. Describe what you find in your own words.

"3. **Optional:** The RSA cryptosystem

Read and study the RSA cryptosystem and its applications to digital security, including how it works, which follows from Fermat's Theorem. Write up the details in your own words, with some example calculations."

I make one final comment on the efficacy of a core project like this. On part of a final exam I gave my students a choice between a proof by induction of a standard homework-like summation formula from their textbook or digesting, explaining, and adapting a modern proof by induction from a Consequence in Pascal's treatise that they had never seen before. Half the students chose to do new interpretation and modern proof work from the Pascal treatise!

6 Finale

I must end with an exhortation of one more reason to teach core material from primary sources: It is inspiring, fun, lively, rewarding and enriching for instructors as well as students. It will keep you happy, excited, and alive.

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