USING HISTORY OF MATHEMATICS TO DEVELOP STUDENT UNDERSTANDING OF NUMBER SYSTEM STRUCTURE

Mala Saraswathy NATARAJ
The University of Auckland
<mala@math.auckland.ac.nz>

Michael O. J. THOMAS
The University of Auckland
<m.thomas@math.auckland.ac.nz>

ABSTRACT

The use of the historical development of mathematical concepts to inform current teaching and learning is a debatable process. In order to investigate the value of this approach in this study we considered the use of a combination of historical development of number systems and modeling with concrete materials as a way of enhancing students' knowledge and understanding of place value. Additionally, we also looked at place value in different number bases and linked multiple representations in an attempt to strengthen understanding of the structure of the number system. The results suggest that this historical and concrete approach helped students to understand positional notation to the extent of generalizing it to other bases, and this may have further implications for the learning of algebra.

1. Background

Understanding the number system and being able to use it is a vital part of the development of numeracy, and is therefore important to all of school mathematics. While the term 'numeracy' has taken on a variety of meanings around the world, the Numeracy Development Project in New Zealand defines it in these terms: "To be numerate is to have the ability and inclination to use mathematics effectively—at home, at work and in the community". Thus numeracy here is the practical application of mathematics, and not just number sense. However, number plays a basic role in all aspects of people's lives and fundamental to the understanding of the number system is the principle of place-value, or positional notation. A real understanding of arithmetic operations rests on a firm grasp of the place value concept. However, evidence (e.g., Thomas, 2004) suggests that this vital and central concept is not well understood by students. One possible reason is that the concept of place value cannot be developed through the teaching of base ten alone and students cannot completely understand the decimal system unless it is seen as a particular case of a more general concept of positional notation. Unfortunately, one cannot just define such a concept into existence for students since "concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples" (Skemp, 1971, p. 32).

Not only is the idea of place value and structure of the number system important to the understanding of operations on numbers, fractions and decimals, but it is also the foundation for algebra, which in turn form the basis for higher mathematics, as supported by the concept map relationships outlined in Schimitau and Vagliardo (2006). Structural understanding of the number system gives a conceptual foundation for the four fundamental operations on whole numbers, and the development of the concept of decimals, fractions, variable, exponent, polynomials and operations on these. For an example of place value application, working with polynomials in algebra is a generalisation of the positional system. Hence positional notation is a significant aspect of algebra preparation. Students' difficulties in algebra have been well documented (e.g., Kieran, 1992; MacGregor & Stacey, 1997) and researchers' views on the various approaches to beginning algebra, such as generalisation, problem solving and function/modeling, are also evident in the literature (e.g., Mason, 1996; Radford, 1996). According to Mason (1996), generalisation is the heartbeat of mathematics and one of the most important sources of generalisation is the domain of number, including detecting and expressing number patterns. Furthermore, general number can be seen as a pre-cursor of variable, the central concept of algebra, a view supported by Radford (1996). Hence a good knowledge of place value, obtained through a generalisation of multiple bases to the notion of a general base *n*, should assist in a smoother transition to algebra.

In this research study we considered the importance of understanding of place value notation and how it might be improved using a combination of concrete materials, linked multiple representations, multiple bases, and historical perspectives. Concrete materials have been recognized and appreciated since Piaget's description of the concrete operational stage of learning, because for these students one must recognize that "in sum, concrete thought remains essentially attached to empirical reality" (Inhelder & Piaget, 1958, p. 250). Multi-base Arithmetic Blocks (MAB) and also known as Dienes' blocks or base ten blocks were also widely used in the eighties. One reason that concrete materials such as these are so useful is that the concept of representational versatility (Thomas & Hong, 2001; Thomas, 2006) lies at the heart of much mathematics thinking. The versatility arises in the ability to translate between representations of the same concept and to interact with these representations in qualitatively different ways. For example, students may interact with a conceptual representation by observing it, for example by noticing properties of the representation itself or of the conceptual processes or object(s) represented, or acting on it.

The third aspect we have employed is the use of history to inform practice. Over the last couple of decades or so there has been a growing interest in the history of mathematics by teachers and educators. Researchers (Katz, 2007; Fauvel & van Maanen, 2000; Gupta, 1995) have highlighted the usefulness of history of mathematics as an excellent resource for motivating students to learn mathematics, and state that one of the greatest benefits is in enhancing the understanding of mathematics itself. However, there are different ways in which historical material may be incorporated in the classroom, either with the direct or indirect integration or addition of the historical elements, and this may be the cause of the opposing views on the value of history in learning. An analytical survey conducted by Tzanakis and Arcavi (2000), regarding how history of mathematics can be incorporated into the classroom, provides for a wide selection of examples for teachers and educators to use. In any case, the inclusion of a historical dimension can bring about a global change in the teacher's approach, characterising the didactic strategy, or the way mathematics is taught. A study of the different number systems and other large contributions of different civilizations, such as the Egyptian, Babylonian, Greek, Mayan, Chinese and Roman, is a specific example of the explicit use of history. The number systems from these cultures presents to students the idea of mathematics as a human endeavour, with twists and turns, false paths, dead ends and triumphs, and helps learners towards a more realistic appreciation of their own attempts. History of mathematics can also be used implicitly; a historical and epistemological analysis (Puig & Rojano, 2004) may help the teacher to understand different stages in learning (Barbin, 2000; Katz, 2007) and why a certain concept is difficult for the student. In turn, this can influence the teaching strategy and development. Katz (2007) believes that there are four conceptual stages in the history of algebraic development (apart from the *rhetorical*, *syncopated* and *symbolic* stages propounded by Nesselmann in 1842) and Barton (2007) in his commentary in the same paper has said that this historical view sheds light on contemporary issues and raises questions about relevant algebra curricula.

Since the Indian system, with its distinct numerals for the numbers from zero to nine, and the place-value value principle within a decimal base has been universally adopted (Datta & Singh, 2001; Sen, 1971; Joseph, 2000) it is reasonable to suppose that the historical development of this system may hold valuable lessons for today. Other Indian achievements include fundamental operations in arithmetic, fractions, surds and irrational numbers, along with general methods of solutions for linear, quadratic, simultaneous and indeterminate equations number in algebra. Hence a deeper study of the history of Indian mathematics, particularly the development of the decimal number system was considered worthwhile, and this research study sought to use such a historical analysis, along with a theory of representational interaction and concrete materials for enhancing students' understanding of the structure of the number system.

2. Research methodology

The research study comprised a historical analysis of the different stages in the development of the number system followed by its application to a case study of a class of 27 Year 9 students in a New Zealand secondary school. Each of these two aspects is considered below.

2.1 Historical analysis

An overview of the history of the decimal place value system with zero in India indicates that it was the result of the confluence of several things: a consideration of very large numbers; the emergence of writing; number symbolism; denominations known as 'places'; and the development of the concept of zero. It must be noted that this happened over very many centuries, and the different stages in the development of the decimal number system in India can be broadly classified as below.

2.1.1 Large numbers

For a very long time prior to the production of the present place value system, the ancient Indians (as with pre-Columbian Maya) seem to have had a particular fascination for very large numbers, due to their interest in astronomy, which was important in order to draw up a calendar and to determine the time and place for rituals. Moreover, Indian cosmology gives a time-scale for the Earth and the universe which consists of billions of years and this necessitated some kind of place value numeration system.

2.1.2 Verbal stage in number notation

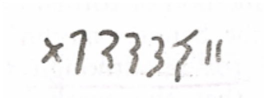
In this stage, numbers (both small and large) were written down in words without the principle of place value, in the same way as we say them, for example:

i) *sahasrani sata dasa* (= one thousand + one hundred + one ten, i.e. 1110).
ii) *sasthim sahasrani panca satani navatim nava* (= sixty thousand + five hundred + nine ten + nine, i.e. 60599) (Bag, 2003, p. 160).
iii) *dvi- navaka* (= twice nine, 2×9, i.e. 18).
iv) *trini satani trisahasrani trimsa ca nava* (= three hundreds +three thousands +three tens + nine, i.e., 3339). (Datta & Singh, 2001, p. 15).
v) *ekonna vimsati* (= one less than twenty, 20 – 1, i.e. 19) (Sen, 1971, p. 142).

2.1.3 Interim stage in number notation

According to Datta and Singh (2001) in the early stages of numerical symbolism, as well as numbers being written out in full in words they assert that symbols were used for the smaller units and words for the bigger units. In this intermediate stage, and before the establishment of the place value principle, numbers were written in numerical symbols with the application of the additive and multiplicative principles. The following are some examples.

a) Given below is the number 274 in Kharoshti numerals inscribed in the stone pillars of Asoka (c. 300 BCE). In Kharoshti script the numerals are written from right to left.



Reading the above numeral from the right : 2×100 +20+20+20+10+4. Both multiplicative and additive principles have been applied here. 4 is the cross at the far left. (Sen, 1971).

b) Figure 1 shows numbers in Brahmi numerals found in a cave in the Nanaghat hills near Poona. Brahmi numerals are written from the left to right. The multiplicative principle is applied here. For example, for 6000 the symbols for 6 and 1000 are conjoined. Therefore 6000 = 1000×6 and not 1000+1000+1000 …six times.

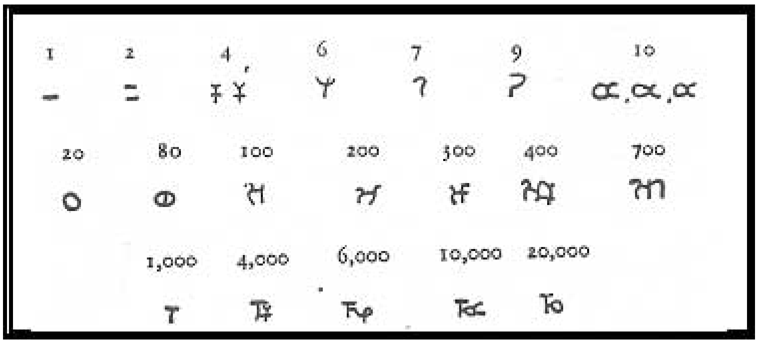


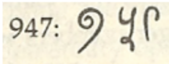
Figure 1. Numbers in Brahmi numerals.

2.1.4 ‘Places’

This is an important stage whereby the denominations of *eka* (1), *dasa* (10), *sata* (100), *sahasra* (1000), etc were known as ‘places’ (the Sanskrit word for ‘place’ is *sthana*). In India, the decimal place value system developed when numbers in successive powers of ten were associated with the values of the ‘places’ of the numbers arranged from left to right or right to left. The following examples are given in Datta and Singh’s (2001) book.

The first use of the world ‘place’ for the denomination is met with in the Jaina canonical work *Anuyogadvara-sutra* (c. 100 BCE). In this work, the total number of human beings in the world is given thus: “a number which when expressed in terms of the denominations, *koti-koti*, etc, occupies 29 ‘places’(*sthana)* or it is beyond the 24th ‘place’ and within the 32nd ‘place’, or it is a number obtained by multiplying the sixth square (of two) by its fifth square, (ie 2^{96}) or it is a number divided by two ninety-six times”. According to Hema Candra (b. 1089 CE) the number *Sirsaprahelika* is so large as to occupy 194 notational ‘places’ or (*anka-sthanehi*).

Aryabhatta I (499) states that the denominations are names of ‘places’. He says: “*Eka* (unit), *dasa*(ten), *sata* (hundred), *sahasra* (thousand*), ayuta* (ten thousand), *niyuta* (hundred thousand*), prayuta* (million), *koti* (ten million), *arbuda* (hundred million), and *vrnda* (thousand million) are respectively from ‘place’ to ‘place ‘each ten times the preceding”. With the advent of denominations as ‘places’, word numerals and numbers were soon written with the place value principle and the following is an example .



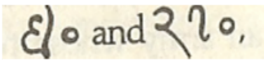
(Joseph, 2000)

2.1.5 The zero concept

The concept of zero, or *sunya,* one of the most important Indian contributions, has a very long history and had varied meanings in the different dimensions of philosophy, language, mathematics and science. The application of *sunya* (zero) in the decimal system of notation was discovered in India sometime in the pre-Christian era (Datta & Singh, 2001). Its symbol was at first a dot as in the Bakshali manuscript and then a small circle.

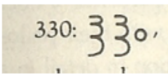
2.1.6 Abstract stage in number notation

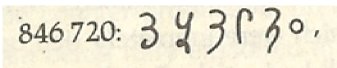
The discovery of zero gave rise to a fully developed place-value system incorporating zero. The following are examples of the final symbolic stage in number notation. In the Gwalior inscription (876 CE) the numbers 50 and 270 are given as



.

In the Bakshali Manuscript the following numbers are found:





(Joseph, 2000)

So far we have briefly looked at the different stages in the historical evolution of the number system and note that it took a very long time before the place value number system was ‘perfected’. Very often students today are confronted with the final product of this number system rather than the different developmental stages during their learning, and we hypothesise that this could be one of the reasons students have difficulty understanding this vital concept. Hence these questions arose: Is it possible to enhance students’ understanding of positional notation through exposing them to large numbers and through the different historical stages that led to full place value structure? What are the implications for the learning of number as well as other topics in mathematics?

2.2 The case study

The second part of this research comprised a case study using concrete materials to model the different stages above in the historical development of the number system, as well as multiple bases and multiple representations for the concept of place value. The research comprised a case study of a Year 9 (13 year-olds) class in a multicultural secondary school in Auckland, New Zealand. The teacher explained to the students what was going to be taught and why it was important to their learning. The teaching module was activity-based and all the class lessons were taught by the first named researcher. The first activity for the students was to create their own number system. Keeping in mind that large numbers were considered for a long time in history and this provided the impetus for the creation of a full number system in both the Indian and Mayan civilizations, the students were given a large number of coloured sticks to assist with their thinking. Students had to decide on the grouping, representations of the numbers, how many symbols were needed, and to show an example of addition of the numbers. The main intent of this task was to get students to think about a need for a number system and how it might have been constructed. Following this, a pre-test (Figure 2) was given to the students comprising questions that addressed their current knowledge of place value.

Section A

1. Write the following in words.

   a) 7905 ______________________________

   b) 100005 ______________________________

2. Write the following in numerals

   a) fifty three thousand eight hundred and ninety two ______________

   b) sixty two thousand and nine______________

3. For the following numbers, what is the actual value for each of the digits?

   a) 35275 b) 6008 c) 7658.32

4. What is the meaning of zero in question 3b?
5. How many symbols do we have in the number system that we use? ________
6. What is the base of the number system that we use?__________
7. How many symbols do we need for a number system with base six? ______
8. How many symbols do we need for a number system with base forty three?_____

Section B

1. Suppose we consider a number system with base **six**. Write the following numbers in words.

   a) 3524 ______________________________

   b) 40035______________________________

   c) 324.15______________________________

2. Suppose we consider a number system with base **seven** where

   1 is written as 1
   2 is written as Γ
   3 is written as Δ
   4 is written as □
   5 is written as ∃
   6 is written as ϖ
   0 is written as 0

   Then write the following numbers in words.

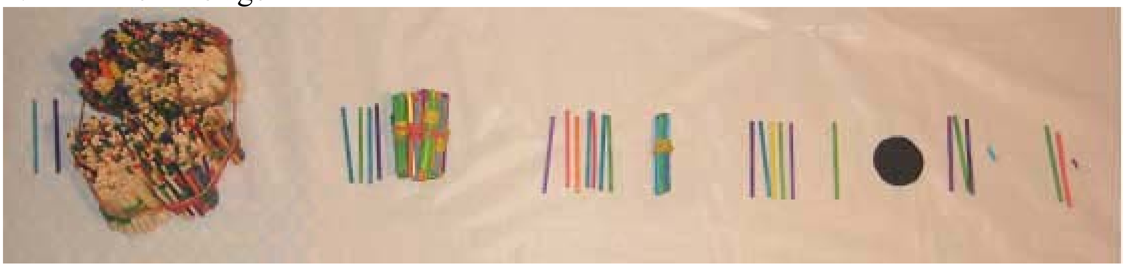
   a) □ϖΔΓ b) Δ0□ϖ c) ΓΔ0.∃□

Figure 2. Some of the pre- and post-test questions.

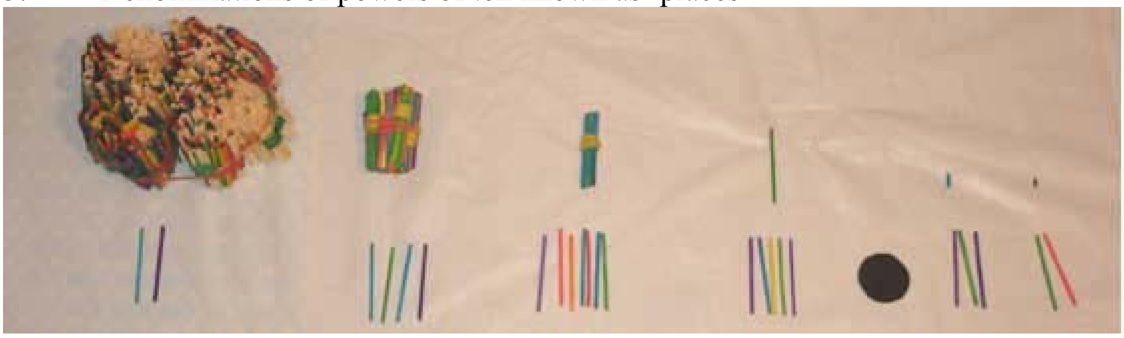
Following the pre-test, the students' second task was to investigate the number systems of past civilizations to see what could be learned from them. These included Primitive, Egyptian, Babylonian, Roman, Greek and Mayan, and finally the present Indian decimal system. The pupils were asked to convert from the different numerals to the present number system and vice versa. The students were given the time to discuss the symbols in the system, along with general features such as place value and zero, the advantages and disadvantages, and to write down their observations on the system. The main purpose of this task was for students to see the way numbers were represented in different cultures and that the numerals that they use everyday are number symbols.

1. Verbal stage

2. Interim stage



3. Denominations of powers of ten known as ‘places’



4. Final abstract stage with place value

Figure 3. Representations paralleling development of the stages of place value.

The third activity was to use concrete materials in the form of large numbers of sticks to understand base 10 numbers. Students were given these sticks and elastic bands to make groups of ten, ten of these to make a hundred, and then a thousand (they even managed one ten thousand!) and these bundles, with part sticks, were used to model numbers such as 12386.52. Keeping in mind that the historical development through the Verbal stage was in place for a long time, the sticks were used to model the numbers as we say the number, that is, one lot of ten thousand, two lots of thousand etc (1 lot of

10×10×10×10 and two lots of 10×10×10). The pictures in Figure 3 shows how the representations used in the activity paralleled what seem to be four main stages in the historical evolution of number in India.

In the next stage, the same number was modeled by applying the multiplicative principle (2×1000 rather than 1000+1000), which is similar to the historical development evidenced in the Kharoshti and Brahmi numerals. In the next stage of the task, only a single bundle of sticks was placed to represent 'place' value. For example, only one bundle of hundred (which was ten groups of ten and not hundred ones) was placed and 3 sticks were placed underneath it to represent 300. Again this paralleled development in history when denominations of powers of ten were known as 'places'. This was done for all the digits and in the final stage, the bundle of sticks representing powers of ten (and 'place') were removed, students had to imagine the value of the 'place' and saw the numbers in their final abstract form.

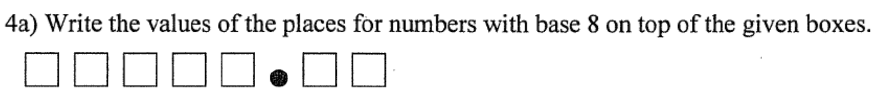
The visual representation of the number was as in 2475.32. (see Figure 3 for the model). The cognitive thinking of these representations is a key step in the construction of the system. There was also discussion on the need for a symbol for zero when we consider a number such as 5006. During this intervention, different numbers were written up on the board with their place values (including their different representations) written on top of each digit leading to a discussion of exponential multiplication and place value. For example, thousand was also written as 10^3, 10×10×10, 10^2×10, 1000 and in words. This was done so that students not only see 1 thousand as thousand ones, but as 10 groups of 10 groups of 10 and also as 10 groups of 100 which has implications for a better grasp of operations on numbers. Figure 4 is an example of what was written up for each one of the positions.

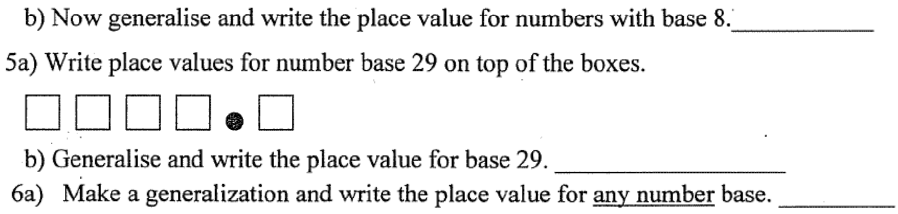
Thousand
1000
$10^2 \times 10$
$10 \times 10 \times 10$
10^3

10^4 10^3 10^2 10^1 10^0 10^{-1}

Figure 4. Use of the place value system.

In the second part of this task exactly the same procedure was followed, but this time the students grouped the sticks in sets of 6's, 36's, 216's etc. and hence different numbers were represented in base six. Again this was written on the board in different representations: in words, exponential and full forms. Eg: 4 lots of 216 (6^3), 5 lots of 36 (6^2). There was discussion on the word base and how many symbols were needed for a particular base. Following these tasks the students were given a post-test, along with extra questions on generalisation (see Figure 5 for some of these questions) involving bases 6,7,8 and 29 as well as base 10.


4a) Write the values of the places for numbers with base 8 on top of the given boxes.


b) Now generalise and write the place value for numbers with base 8.___________

5a) Write place values for number base 29 on top of the boxes.

☐☐☐☐●☐

b) Generalise and write the place value for base 29. __________________

6a) Make a generalization and write the place value for <u>any number</u> base. _________

Figure 5. Some 'extra' post-test questions on generalization.

3. Results

When we accessed the participants' results using a standard New Zealand diagnostic tool available in most schools, the Assessment Tools for Teaching and Learning (asTTle) (Hattie, Brown & Keegan, 2005) we found that place value skills were a problem for many students in the class. In the first task of creating their own number systems, most of the students simply took the base 10 system which they had knowledge of and created their own symbols for it. Only one group used a system of merging two symbols together to create a partial multiplicative arrangement and did not use place value (see Figure 6). This student response is very interesting as it is similar to the Brahmi numerals where the symbols for 4 and 1000 are merged together to give 4000 (see Figure 1). The second task of consideration of historical number systems was interesting to students for varying reasons. Some liked the Egyptian and Roman systems for aesthetic reasons and others liked the Roman system for its ease of use. When asked to represent large numbers students realised they had to repeat symbols many times and also had to create more and more symbols. When asked why they were able to write large numbers with only ten symbols in the present decimal system, students found the question quite challenging and one student said 'it was because of all the zeros'. During the third task with the sticks, there were opportunities for the discussion and construction of other concepts such as the relative sizes of numbers like 10^4 and 10^{-2}. Students' comments about the 'bigness' of 10^{12} and the smallness of 10^{-23} showed that they were thinking about these ideas.

Working with the groups of coloured sticks and looking at the patterns, students came up with 10^0 as 1 and one tenth as 10^{-1}. It was brought to the attention of students that in a number such as 12796.34 the three sticks used represented 3 lots of the tiny bits of sticks, or 10^{-1}. There was discussion on base, how many symbols were needed for a particular system, range of values that the exponent could take, and so on.

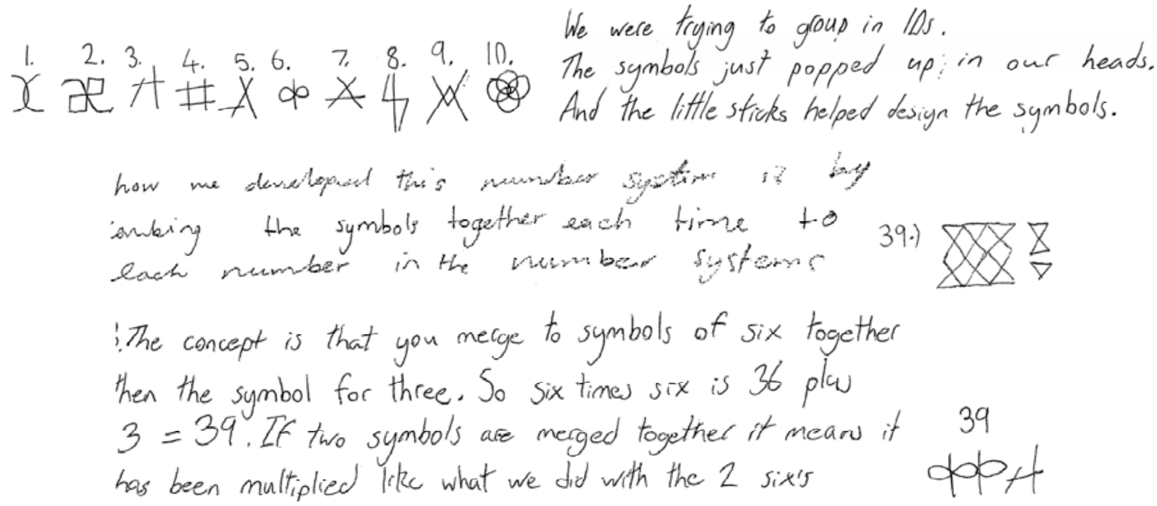


Figure 6. Some students' work on creating their own number system.

3.1 Test results

From the pre-test to the post-test every student except for S13 improved their score, and overall there was a significant improvement in the mean score on the test ($Mean_{pre}$=7.41, $Mean_{post}$=13.63, t=6.22, p<0.0001). There was improvement on every question on the tests (sections A and B), but especially on section A Q's 7 and 8 (from 5 and 3 correct to 23 and 20, respectively), and every question in section B (from 0 on every question to scores from 15 to 17 correct). Questions 7 and 8 asked how many symbols are needed for bases 6 and 43, and this generalisation was clearly better understood after the module of work. Two students, S6 and S19, are attending ESOL classes and were very hindered by language difficulties. Although they only attempted to answer some of the questions they did both improve, from 0 each on the pre-test to 6 and 7 respectively on the post-test. It was pleasing to see that by the end of the module of work 23 of the students could answer Q4a) for base 8 and 19 of these could generalise the place value to 8^x (Q4b)), or equivalent. Similarly 24 students could do the same for base 29 (Q5a)), 21 of these could generalise the place value here too to 29^x, and the same number could even take this to any base and write n^x (see Figure 7).

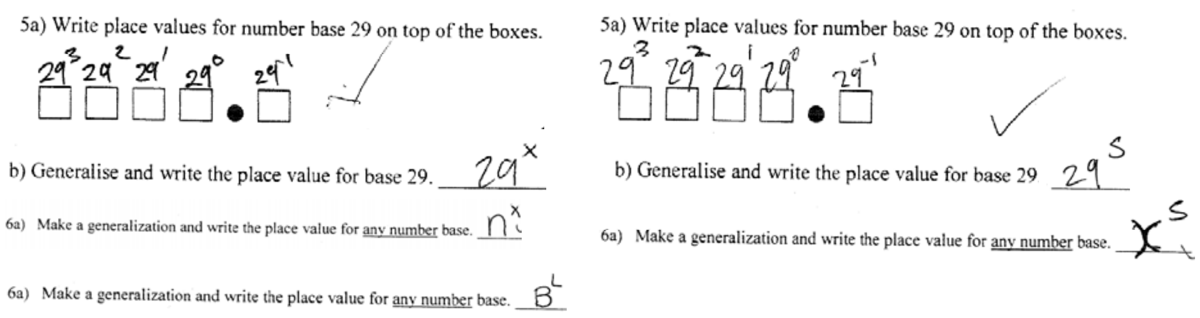


Figure 7. Some of the student answers to the test.

Some of the student comments on the tasks were quite illuminating. For example they mentioned for the task on creating their own number system: "This is lots of fun. Got us thinking about funny names and symbols". "We like working together and bounce ideas off each other but it is hard. It is like making your own language up". "I felt I was designing something for the future". "The group was confused. Different opinions in the group and they all wanted different things/symbols". "quite hard". It seemed that this task was interesting but not easy for the students. Further, on the tasks with the sticks they thought that: "With the sticks it was easier because we saw what we were doing not just hearing it", "When you do it with the sticks it helped because you learn better when you do stuff in person, using your hands". Most of the comments were positive, with the exception of two students who said "Sticks didn't help me much".

4. Conclusion

This study attempted to develop in students a meaningful understanding of place value and a structure of the number system through: considerations of large numbers and exponential multiplication, use of concrete materials, multiple bases, multiple representations and a review of development of historical number systems. The focus was on students' understanding of structure, and recognition that the numerals that they deal with on a daily basis are number symbols forming part of a system. The results show that students achieved a certain measure of success to the extent that most were able to generalise the multiplicative (including exponential) structure of the number system. The study also shows that students respond well when extended beyond what they are responsible for in terms of learning in order to conceptualise what they *have* to learn in the curriculum and this in turn may have implications for mathematics curriculum development. It may be that teaching this crucial concept through ideas structured according to the historical order, employing linked multiple representations of number using concrete materials, and an activity, task-based approach may help students to construct the place value concept.

References

Bag, A. K., 2003, "Need for Zero in the Numerical System in India", in *The Concept of Sunya*, A. K. Bag & S. R. Sarma (eds.), New Delhi: IGNCA, INSA & Aryan Books International, pp. 159-169.

Barbin, E., 2000, "Integrating history: research perspectives", In *History in mathematics education: The ICMI study,* J. Fauvel & J. van Maanen (eds.), Dordrecht: Kluwer Academic Publishers, pp. 63-90.

Barton, B., 2007, "Commentary from a Mathematics Educator on Stages in the history of algebra with implications for teaching", *Educational Studies in Mathematics,* **66**, 185-201.

Datta, B. B., & Singh, A. N., 2001, *History of Hindu mathematics: A source book. Parts I and II.* New Delhi: Bharatiya Kala Prakashan.

Dienes, Z. P., 1960. *Building up mathematics*. London: Hutchinson Educational.

Fauvel, J., & van Maanen, J. (eds.) 2000, *History in mathematics education: The ICMI study*, Dordrecht-Boston-London: Kluwer.

Gupta, R. C., 1995, "Why study history of mathematics?", *Bulletin of the Indian Society for History of Mathematics,* **17**, 1-4, 10-28.

Hattie, J. A., Brown, G. T. L., & Keegan, P., 2005, *asTTle V4 Manual 1.1*, University of Auckland.

Inhelder, B. & Piaget, J., 1958, *The growth of logical thinking*, London: Routledge and Kegan Paul.

Joseph, G. G., 2000, *The crest of the peacock: Non-European roots of mathematics,* Harmondsworth: Penguin.

Katz, V. J., 2007, "Stages in the history of algebra with implications for teaching", *Educational Studies in Mathematics,* **66**, 185-201.

Kieran, C., 1992, "The learning and teaching of school algebra", in *Handbook of Research on Mathematics Teaching and Learning*, D. A. Grouws, (ed.), New York: Macmillan, pp. 390–419.

MacGregor, M., & Stacey, K., 1997, "Students' understanding of algebraic notation: 11 – 16", *Educational Studies in Mathematics,* **33(1)**, 1–19.

Mason, J., 1996, "Expressing generality and roots of algebra", in *Approaches to algebra: Perspectives for research and teaching,* N. Bednarz, C. Kieran & L. Lee (eds.), Dordrecht: Kluwer Academic Publishers, pp. 65-86.

Puig, L., & Rojano, T., 2004, "The history of algebra in mathematics education", in *The Future of the Teaching and Learning of Algebra: The 12th ICMI Study,* K. Stacey, H. Chick & M. Kendal (eds.), Dordrecht: Kluwer Academic Publishers, pp. 189-223.

Radford, L., 1996, "Some reflections on teaching algebra through generalisation", in *Approaches to algebra: Perspectives for research and teaching,* N. Bednarz, C. Kieran & L. Lee (eds.) Dordrecht: Kluwer Academic Publishers, pp. 107-112.

Schmittau, J., & Vagliardo, J. J., 2006, "Using Concept mapping in the development of the concept of positional system" in *Concept Maps: Theory, Methodology, Technology*, A. J. Canas & J. D. Novak (eds.), Proceedings of the Second International conference on Concept Mapping, San Jose, Costa Rica, pp. 590-597.

Sen, S. N., 1971, "Mathematics", in *A Concise History of Science in India,* Bose, D. M., Sen, S., & Subbarayappa, B. V. (eds.), New Delhi: Indian National Science Academy, pp. 136-209.

Skemp, R. R., 1971, *The Psychology of Learning Mathematics*, Middlesex, UK: Penguin.

Thomas, M. O. J., 2006, "Developing versatility in mathematical thinking". in *Retirement as process and object: A festschrift for Eddie Gray and David Tall*, A. Simpson (ed.), Charles University, Prague, Czech Republic, pp. 223-241.

Thomas, M. O. J., & Hong, Y. Y., 2001, "Representations as conceptual tools: Process and structural perspectives", in *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education,* M. van den Heuvel-Panhuizen (ed.), Utrecht, The Netherlands: Utrecht University. vol. **4**, 257–264.

Thomas, N., 2004, "The Development of structure in the number system", in *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, M. J. Hoines & A. B. Fuglestad (eds.), Bergen, Norway, **4**, pp. 305-312.

Tzanakis, C., & Arcavi, A., 2000, "Integrating history of mathematics in the classroom" in *History in mathematics education: The ICMI study,* J. Fauvel & J. van Maanen (eds.), Dordrecht: Kluwer Academic Publishers, pp. 201-240.