EULER'S CONTRIBUTIONS TO MATHEMATICAL CARTOGRAPHY

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ABSTRACT

Euler's appointment as a full professor to the Academy was not the most exciting news in St. Petersburg in 1730. Possibly that news was carried by Vitus Bering, returning from five years exploring the Siberian far east. It fell to the Academy geographer Nicolas Deslisle and his young colleague, Leonhard Euler, to organize the mass of data that Bering brought back.

More than 40 years laters, Euler published a series of three articles about mathematical cartography:

We examine Euler's interest in cartography in the context of the developing science of cartography, the developing Russian nation-state, and the internal politics of the St. Petersburg Academy.

1 Introduction

Leonhard Euler's principal contribution to mathematical cartography was a series of three articles, published in 1777 in *Acta Academiae Scientarum Imperialis Petropolitinae.*

- On the representation of the spherical surface on a plane[1],[2]
- On the geographic projection of a sherical surface[3]
- On Delisle's Geographic Projection and its Use in the General Map of the Russian Empire[4]

In the present work, we do not discuss in detail the contents of these articles. Rather, our intention is to put them in context of the political, social, and scholarly setting of the eighteenth century.

The general trend is from abstract principles in the first article to a general application in the second, to a very specific application in the third. In the interest of improved clarity, we shall discuss the articles in reverse order, beginning with the third.

2 On Delisle's Geographic Projection and its Use in the General Map of the Russian Empire

In 1716, Gottfried Wilhelm von Leibniz wrote one of many memoirs addressed to Tsar Peter of Russia. After proposing the foundation of a central library and a scientific museum ("collection of curiosities"), he went on to say[5]

[1] All English translations of Euler's text are the author's.

[2] De repraesentatione superficiei sphaericae super plano

[3] De projectione geographica superficiei sphaericae

[4] De projecitione geographica De-Lisliana in mappa generalii imperii russici usitata

[5] Leibniz, 1716

> So much for the equipment for sciences and arts. Let us now consider the ways and means of bringing them to the people. These include schools for the children, universities and academies for the young people, and finally scientific and learned societies and other [associations] for those who are advanced in their studies and are concerned with the improvement [of knowledge].
>
> Lastly we must consider the institutions for new discoveries by which sciences are advanced; here the extensive lands of the Russian Empire, with so many possessions in Europe and others adjoining Asia, offer excellent opportunities; for Russia is almost virgin soil and is still insufficiently explored; thus it should yield many plants, animals, minerals, and other natural objects that have not yet been described.
>
> At Your Tsarist Majesty's command it could be found out whether Asia can be circumnavigated on the north, or whether the edge of the ice cap is attached to America, which is something that the English and the Dutch have tried in vain to discover during their dangerous sea explorations.

Peter partially implemented Leibniz' ideas about schools, founding the Imperial Petersburg Academy of Science in 1724. But he enthusiastically embraced Leibniz' final suggestion. Two successive "Great Northern Expeditions", headed by the Danish explorer Vitus Bering, were sent across Siberia to explore and claim new lands for the Tsar. True to Leibniz' vision, geography was one of the principal roles of the Academy in its early years; its geographic projects received the heaviest funding.[6]

The French astronomer Joseph Nicolas Delisle was appointed head of the Academy's Department of Geography, and was assigned to produce an atlas of the Russian Empire. Leonhard Euler, as a member of the Geography Department, was to assist Delisle. Although Delisle was undoubtedly skilled and experienced, under his direction the project suffered repeated delays and setbacks. On at least two occasions, Delisle refused orders to meet with Euler and seek his assistance, further announcing that he would not head the Geography Department as long as Euler worked there.[7] Finally, in 1740, when Delisle was away on an expedition to Siberia, Euler took over the project and began publication a few years later.

Data from the Bering expeditions was considered a state secret by Russia. The Senate decreed in 1733 that

> "...observations, maps, and other materials submitted by the Kamchatka expedition and transmitted to the Academy should be guarded carefully so as to prevent covert or overt knowledge about them and in order that they should not become known in foreign parts earlier than in this country."[8]

Much later, it was discovered that Delisle had in fact sending copies to France of almost everything he received at the Academy. Bagrow (1945) suggests that Delisle might have been secretly working for France, and deliberately putting delays into the Russia geography project. Perhaps this also explains his reluctance to meet with Euler.

The Russian land mass, spanning nearly 180 degrees of longitude, presented new challenges to cartographers. These are discussed in Euler's paper "On de Lisle's Geographic Projection". The Geography Department at the Academy chose to use a

[6]Calinger, 1996, p. 146.
[7]Bagrow, 1945, p.180.
[8]Bagrow, 1945, p.179.

"conic" map projection (see Figure 1), in which the meridians are drawn as segments of straight lines meeting at common point, and the circles of latitude ("parallels") are drawn as circular arcs which meet each meridian in a right angle. Two parallels can be chosen along which there is no scale distortion; this choice determines the angle at which the meridians meet. The Academy used an "equidistant conic" projection, with equally spaced arcs representing all parallels; thus the choice of the two "standard parallels" essentially determined the projection.

Although Euler did not criticize DeLisle directly in his article (DeLisle had in fact died in 1768), perhaps we can detect the echo of a past quarrel in Paragraph 6:

> Delisle, the most celebrated Astronomer and Geographer of the time, to whom the care of such a map was first entrusted, in trying to fulfill these conditions, made the relationship between latitude and longitude exact at two specific Parallels. He was of the opinion that if the named circles of Parallel were at the same distance from the middle Parallel of the map as from its outermost edges, the deviation could nowhere be significant. Now the question is asked, which two circles of paralell ought to be chosen, so that the maximum error over the entire map be minimized.

Euler started with Delisle's assumptions and computed the angle of the meridians, errors at the north and south edges and in the middle of the map, and the deviation between the image of a great circle and a straight line. He never directly contradicted Delisle, but showed that the maximum distortion does not lie exactly in the middle of the map, which casts doubt on the original assumption. Euler concluded (Paragraph 25) with a carefully worded judgement about the Delisle projection:

> In this projection is obtained the extraordinary advantage, that straight lines, which go from any point to any other point, correspond rather exactly to great circles and therefore the distances between any places on the map can be measured by using a compass without considerable error. Because of these important characteristics the projection discussed was preferred before all others for a general map of the Russain Empire, even though, under rigorous examination, it differs not a little from the truth.

Today Delisle is credited with the discovery of the equidistant conic projection with two standard parallels.[9] Euler's work inspired a number of published articles by cartographers over two centuries, describing different ways to minimize distortions with this kind of map. Most notable was the work of Kavraisky in 1934,[10] which seems to have been one of the standard map projections used during the Soviet era.

3 On the placement of a spherical area on a map

The stereographic projection, or planisphere (see Figure 2), may have been known in ancient Egypt[11]. Synesius of Cyrene attributed its discovery to Hipparchus[12]. Ptolemy's

[9] This is ironic, since a form of the projection was developed by Claudius Ptolemy in the second century C.E.; also since, for technical reasons, Delisle's map was not in a true conic projection. See Snyder, 1987, p. 111.

[10] Snyder, 1978

[11] Keuning, 1955

[12] Heath 1981, p. 293

Planisphaerum described its use, but in astronomy, not geography; in fact this projection seems to have been used exclusively for star charts and astrolabes until the sixteenth century. Rumold Mercator [13] published, in 1595, an atlas with maps based on the equatorial stereographic projection[14]. After this it became very popular among cartographers, and its popularity continued during the time of Euler (see Figure 3.)

One reason for the popularity of this projection (which also accounts for its use in medieval astrolabes) was probably its ease of construction. All circles on the globe (not just great circles) are projected onto circles on the plane (or exceptionally, onto straight lines). This property was known to the ancients. In fact, a proposition[15] at the beginning of Apollonius' *Conics* seems almost designed with the planisphere in mind. Since all meridians and all circles of latitude are circles, only three points are needed on each to construct it, and the entire graticule can be constructed with only a compass.

Euler's second paper treated the stereographic projection in great detail. He found the point locations on the projected plane of the north and south poles, and of the center and radii of the circles which are images of the equator, the meridians, and the small circles of latitude. He concluded (Paragraph 20):

> Moreover, let it be remarked, that this method of projection is extraordinarily appropriate for the practical application required by geography, for it does not strongly distort any region of the earth. Nevertheless, the most important is that with this projection, not only all meridians and circles of parallel are represented as circles or even as straight lines, but also all great circles on the sphere pass into circular arcs or straight lines...

Euler gave no proof of his assertion that the stereographic projection "does not strongly distort any region of the earth"[16]. In fact, this is not literally true; distortion increases without limit in the hemisphere opposite to the center of the projection. However, perhaps Euler was thinking of the maps, common in his century, which represented the eastern and western hemispheres by separate equatorial stereographic projections. Certainly these had less distortion than the world chart of Mercator, which we discuss next.

4 On the mapping of Spherical Surfaces onto the Plane

In response to inquiries from Portuguese ship captains, Pedro Nuñes in 1537 discovered that the course followed by a ship which kept to a single compass bearing (a "rhumb line", or "loxodrome") is not in general a great circle. Unless the bearing is one of the four cardinal directions, the rhumb line is a logarithmic spiral approaching one of the poles of the earth. In 1569, Gerhardus Mercator (father of Rumold Mercator mentioned in the previous section) published *A new and enlargedj description of the earth, with improvements for use in navigation*[17]. This is the familiar map that depicts meridians as vertical lines, parallels of latitude as horizontal lines (it is a *cylindrical* projection) and it greatly exaggerates the scale at polar latitudes, so that, e.g., Greenland appears about the size of South America. However, minimal distortion was not

[13] son of Gerhardus Mercator, whom we discuss in the next section
[14] Snyder, 1978
[15] Proposition 6 of Book I
[16] veram figuram regionum terrestrim non admodum detorquet
[17] Nova et aucta orbis terrae descriptio ad usum navigatorum emendatae accomodata

the point. As its title indicates, the map was an instrument for use in navigation; and in fact, the Mercator projection has the unique property that all loxodromes appear a straight line on the map.

By the early seventeenth century, the principles of computing the Mercator "sea chart" were known. (Mercator himself did not explain them, and probably used a purely graphical method.) Since the circle of latitude ϕ had to be stretched to equal the length of the equator, the horizontal distortion at a point with latitude ϕ was equal to $\sec\phi$. To compensate, the vertical distortion was also set equal to $\sec\phi$. Since the distortion was the same in two orthogonal directions, it was the same in all directions. The mapping preserves angles, or, in modern terms, is *conformal*.

Edward Wright[18] presented this reasoning in a colorful metaphor (and also explained why the Mercator projection is called "cylindrical"):

> Suppose a sphaerical superficies with Meridians, Parallels, Rumbes, and the whole hydrographicall description drawn thereupon, to be inscribed into a concave cylinder, their axes agreeing in one.
>
> Let this Sphaerical superficies swel like a bladder, (whiles it is in blowing) aequally alwayes in every part thereof (that is, as much in longitude as in latitude) till it apply, and joyn itself (round about, and all alongst also towards either pole) unto the concave superficies increasing successively from the *Aequinoctial*[19] towards either pole, until it come to be of equal diameter with the cylinder, and consequently the *Meridians* still widening themseleves, till they come to be so far distant every where each from other as they are at the *Aequinoctial.* Thus it may most easily bee understood , how a sphaerical superficies may (by extension) be made a cylindrical, and consequently a plain Parallelogram superficies; because the superficies of a cylinder is nothing else but a plain paralleleogram wound about two equal equidstant circles that have one common axtree perpendicular upon the centers of them both, and the peripheries of each of them equal to the length of the parallelogram as the distance betwixt those circles, or height of the cylinder is equal to the breadth thereof. So as the nauticall planisphaere may be defined to be nothing else but a parallelogram made of the sphaerical superficies of an Hydrographical Globe inscribed into a concave cylyinder, both their axes concurring in one; and the Sphaerical superficies swelling in every part equally in longitude and latitude, till every one of the Parallels thereupon be inscribed into the cylinder (each parallel growing as great as the *Aequinoctial:* or till the whole sphaerical superficies, touch and apply it selfe every where to the concavity of the cylinder.

Mercator's projection is easily understood to be conformal, since horizontal and vertical scaling are explicitly constructed to be the same. Surprisingly, the stereographic projection also turned out to be conformal. Edmund Halley (1656–1742) was the first to publish a proof, although Thomas Harriot proved the result in a manuscript from about 1614.[20]

As of the mid-eighteenth century, two methods of mapping from a sphere to a plane were known to possess the pleasing property of conformality.[21] The open question was,

[18] Wright, 1657

[19] equator

[20] See Pepper, 1968. The short, intuitive proof in Hilbert and Cohn-Vossen, 1952, is similar to Harriot's original.

[21] The term *projectio conformalis* was not actually used until 1793.

were there any others? J.H. Lambert[22] was the first to answer this question, by presenting a family of conic projections that lay, by a clever parametrization, in a continuum between the Mercator and the polar form of the stereographic projection. Lambert also discovered another family of conformal projections, the "transverse Mercator", constructed in the same way as Mercator's, but starting from a designated meridian rather than from the equator.

Lambert's treatise was of immense importance to practical cartographers (indeed, Lambert's projections are still in common use today), but did not present a complete theory. Lambert did not claim to categorize all possible conformal projections, nor to give a complete catalog of useful or interesting properties other than conformality. Perhaps this was Euler's motivation for his own work on the subject. *On the representation of a spherical surface on the plane* does work to build a more complete theoretical foundation. Early in the paper, Euler gives a purely analytic proof that no mapping can be "perfect", i.e., accurately preserve both distance and angles. This was in some sense superfluous, since everyone recognized this already (as Lambert remarked), from the obvious fact that the angles of a spherical triangle sum to more than 180 degrees. However, perhaps Euler was trying to show the power of the analytic method, or perhaps was trying to show that the entire theory could be founded on the methods of differential analysis.

In any case, Euler's work was an impressive tour de force. After showing that a "perfect" mapping was impossible, he proposed three different sets of conditions that it might be desirable for a map projection to satisfy:

I. Preserve angles (conformal)

II. Preserve areas (equal-area)

III. All meridians and all paralells of latitude are straight lines, and each meridian meets each parallel of latitude in a right angle. (cylindrical).

He characterized the mappings satisfying each condition as the solutions to a pair of differential equations in two variables, and gave a general form for the family of solutions. Euler's paper on the stereographic projection, discussed in the previous section, showed that all stereographic projections satisfy the general form for (I). Thus Euler gave a new proof that the stereographic projeection is conformal.

Unlike Lambert (and unlike another contemporary work by Lagrange), Euler dealt exclusively with mappings from a sphere to a plane. He ignored the fact that the earth itself is not a pefect sphere. This was well known by the time of his writing, having been first suggested by Newton in *Principia*, and confirmed by the French Academy's expeditions to Lapland and to South America in the 1730s. Once again, perhaps Euler was more interested in building a theory than in applications to practical cartography. And this ambition was ultimately realized in the nineteenth century when Gauss, Schwartz, and Riemann built on Euler's work as a foundation for the theory of conformal mappings.

[22] Lambert, 1772

5 Conclusion

Euler's work on cartography is not well known to mathematicians. The topics presented in the three Euler papers might provide interesting, and entertaining, material suitable for second semester calculus students.

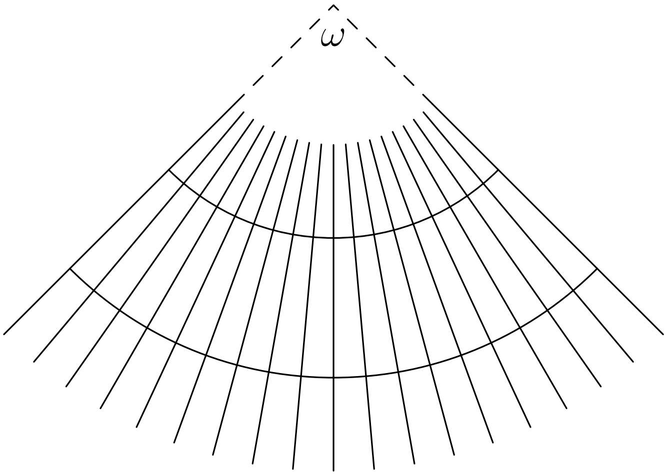

Figure 1: Basic graticule for a conic projection. At most two parallels of latitude can be chosen along which the scale is constant. This choice determines the angle ω.

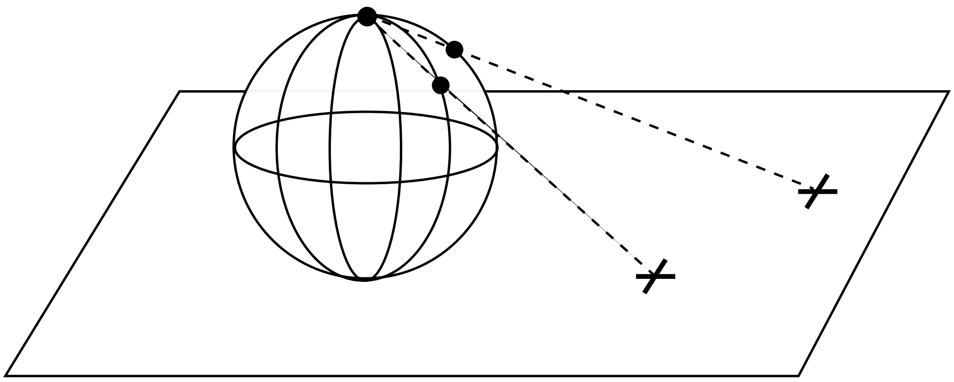

Figure 2: Construction of the stereographic projection

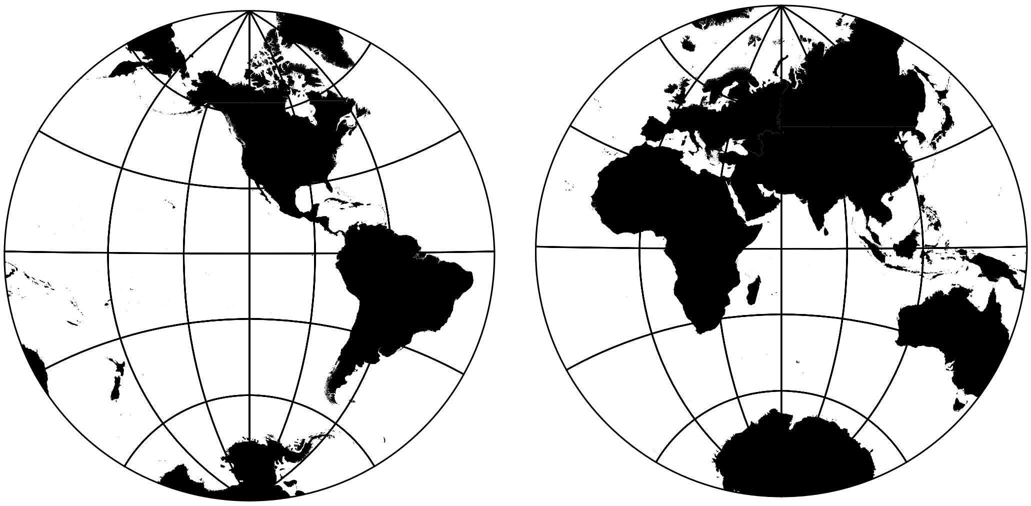

Figure 3: World as two hemispheres, mapped in the equatorial aspect of the stereographic projection. All latitude and longitude lines are projected as straight lines or circular arcs. This was a popular representation of the world in Euler's time.

Figure 4: Mercator's projection or "nautical chart", on which all compass courses are straight lines, at the expense of great distortion in the polar regions.

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