JACOBI'S LAST THEOREM
The history of Jacobi-Perron algorithm

Fritz SCHWEIGER
IFFB Didaktik und LehrerInnenbildung, Universität Salzburg, Hellbrunnerstr. 34, A 5020 Salzburg
fritz.schweiger@sbg.ac.at

ABSTRACT

J. L. Lagrange proved that the regular continued fraction expansion of a quadratic irrational number becomes periodic. In a paper published after his death C. G. Jacobi proposed a generalization of the regular continued fraction algorithm. Most probably he hoped that for a pair of cubic numbers this algorithm would become periodic. Paul Bachmann stated that according to results of Hermite and Charve on ternary quadratic forms this conjecture should be true. O. Perron developed a profound theory of Jacobi's algorithm and its generalization to higher dimensions. It was here that Perron stated a theorem on positive matrices which later on became a good tool under the names of Perron and Frobenius. L. Bernstein collected a large number of examples for special families of irrational numbers for which he could prove periodicity. Among others this story shows again that number theory is a rich source of problems which are easy to state but hard to solve.

1 Jacobi's Paper

J. L. Lagrange proved that the regular continued fraction expansion of a quadratic irrational number becomes periodic (Lagrange 1770). In his second letter to Jacobi Hermite considered binary quadratic forms. He wonders if the theory of continued fractions can be extended in a suitable way. "Peut-être parviendra-t-on à déduire de là, un système complet de caractères pour chaque espèce de ce genre de quantités, analogue par exemple à ceux que donne la théorie des fractions continues pour les racines des équations du second degré "(Hermite 1850, p. 286).
In the year 1868 E. Heine published the paper "Allgemeine Theorie der kettenbruchähnlichen Algorithmen, in welchen jede Zahl aus drei vorhergehenden gebildet wird" which he found in the legacy of G.G.J. Jacobi (1804-1851). In this paper Jacobi considers three numbers ("unbestimmte Zahlen") a, a_1, a_2 and a sequence of given quantities ("gegebene Grössen") $l, m, l_1, m_1, l_2, m_2, \ldots$. Then he defines

$$\begin{aligned} a_3 &= a + la_1 + ma_2 \\ a_4 &= a_1 + l_1a_2 + m_1a_3 \\ a_5 &= a_2 + l_2a_3 + m_2a_4 \\ \ldots &\ldots \ldots\ldots \end{aligned}$$

Translated into matrix theory this reads as the relation

$$(a_1, a_2, a_3) = (a, a_1, a_2) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & l \\ 0 & 1 & m \end{pmatrix}$$

or generally

$$(a_{i+1}, a_{i+2}, a_{i+3}) = (a_i, a_{i+1}, a_{i+2}) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & l_i \\ 0 & 1 & m_i \end{pmatrix}.$$

From this starting point he derives a lot of formal relations. In §6 of the paper he takes a new approach. Let u_0, v_0, w_0 be three positive numbers and put $l_0 = [\frac{v_0}{u_0}], m_0 = [\frac{w_0}{u_0}]$. Then he sets $n_1 = v_0 - l_0 u_0, v_1 = w_0 - m_0 u_0, w_1 = u_0$. Again, translated into matrix theory we find

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} -l_0 & 1 & 0 \\ -m_0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix}.$$

Clearly, then he proceeds by iteration.

One sees immediately that the matrices involved are only the inverse matrices of the matrices defined above. In §7 he specifies further. The number u_0 be an integer, v_0 and w_0 be numbers of the form

$$\alpha + \beta x + \gamma x^2, \alpha, \beta, \gamma \in \mathbb{Z}, x^3 = n, n \in \mathbb{N}.$$

A wealth of results is presented of which one is remarkable. If $u_i = u_0, v_i = v_0, w_i = w_0$ then the algorithm produces a unit of the cubic number field. Eventually in §14 he comes to the delicate question if the algorithm eventually becomes periodic. He states that in 1839 he found three examples which he communicated to his friends Dirichlet and Borchardt. As help for further investigations he now wants to present the three examples to the public.

i)

$$\begin{aligned} (u_0, v_0, w_0) &= (1, \sqrt[3]{2}, \sqrt[3]{4}) \\ (u_1, v_1, w_1) &= (1, \sqrt[3]{2} + 1, \sqrt[3]{4} + \sqrt[3]{2} + 1) \\ (u_2, v_2, w_2) &= (1, \sqrt[3]{2} + 2, \sqrt[3]{4} + \sqrt[3]{2} + 1) = (u_3, v_3, w_3) \end{aligned}$$

He also gives convergents to $\sqrt[3]{2} - 1$ and $\sqrt[3]{4} - 1$ namely the sequences $\frac{1}{3}, \frac{3}{12}, \frac{12}{46}, \frac{46}{177}, \frac{177}{681}, \ldots$ and $\frac{2}{3}, \frac{7}{12}, \frac{27}{46}, \frac{104}{177}, \frac{400}{681}, \ldots$

ii) $(u_0, v_0, w_0) = (1, \sqrt[3]{3}, \sqrt[3]{9})$
He finds in a similar way $(u_2, v_2, w_2) = (u_4, v_4, w_4)$. Among others he states that according to the proceeding considerations the number $2\sqrt[3]{9} + 3\sqrt[3]{3} + 4$ is a unit (in my copy of the paper someone changed the printed $2\sqrt[3]{9} + 2\sqrt[3]{3} + 4$ to the correct expression).

iii) $(u_0, v_0, w_0) = (1, \sqrt[3]{5}, \sqrt[3]{25})$
After a very long calculation he finds

$$(u_7, v_7, w_7) = (u_{13}, v_{13}, w_{13}).$$

The characteristic equation of the periodicity matrix belonging to $(1, \sqrt[3]{5}, \sqrt[3]{25})$ is given as $\chi(t) = t^3 - 123t^2 + 3t - 1$ with the largest root $t_0 = 41 + 24\sqrt[3]{5} + 14\sqrt[3]{25}$. Unfortunately, Jacobi's paper contains two misprints ($l_6 = l_{10} = 2$ should be corrected to $l_6 = l_{10} = 0$). Jacobi never states the conjecture that the algorithm will eventually become periodic if one performs it on a triple $(1, v_0, w_0)$ where v_0, w_0 belong to a cubic number field. Maybe he was warned off by the long calculations for $(1, \sqrt[3]{5}, \sqrt[3]{25})$ and the obviously never ending calculations on $(1, \sqrt[3]{4}, \sqrt[3]{16})$. In fact, it is still not known if periodicity occurs at all!

Elsner and Hasse 1967 published a list of numerical results on the Jacobi algorithm. It is worth mentioning that they confirm that $(1, \sqrt[3]{2}, \sqrt[3]{4})$ is periodic with preperiod $q = 2$ and period $p = 1$. The triple $(1, \sqrt[3]{4}, \sqrt[3]{2})$ also is periodic but $q = 1$ and $p = 2$. However they do not find any indication of periodicity for the triple $(\sqrt[3]{2}, \sqrt[3]{4}, 1)$ which is equivalent to $(1, \sqrt[3]{2}, \frac{\sqrt[3]{4}}{2})$.

2 Later Developments

The next one to consider the problem of periodicity was P. Bachmann in a paper in Journal für Mathematik (Bachmann 1872); the content has been reproduced with some changes in his "Vorlesungen über die Natur der Irrationalzahlen" (Bachmann 1892)). Paul Bachmann stated that according to results of Hermite and Charve on ternary quadratic forms Jacobi's conjecture should be true. He deduces several equivalent conditions for periodicity. From these conditions he derives the following necessary condition. If the algorithm for $(1, \sqrt[3]{D}, \sqrt[3]{D^2})$ eventually becomes periodic then the algorithm provides the Diophantine approximation

$$\begin{aligned} \left| \sqrt[3]{D} - \frac{A_1^{(s)}}{A_0^{(s)}} \right| &\ll \frac{1}{A_0^{(s)}\sqrt{A_0^{(s)}}} \\ \left| \sqrt[3]{D^2} - \frac{A_2^{(s)}}{A_0^{(s)}} \right| &\ll \frac{1}{A_0^{(s)}\sqrt{A_0^{(s)}}}. \end{aligned}$$

The numbers $A_i^{(s+j)}$ are the entries of the matrix

$$((A_i^{(s+j+1)})) = \beta\begin{pmatrix} l_0 \\ m_0 \end{pmatrix}\beta\begin{pmatrix} l_1 \\ m_1 \end{pmatrix}\dots\beta\begin{pmatrix} l_s \\ m_s \end{pmatrix}, 0 \le i, j \le 2$$

where the matrices

$$\beta\begin{pmatrix} l_k \\ m_k \end{pmatrix}, 0 \le k \le s$$

are now related to the expansion of the triple (u_0, v_0, w_0) by

$$\beta\begin{pmatrix} l_k \\ m_k \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & l_k \\ 0 & 1 & m_k \end{pmatrix}, 0 \le k \le s.$$

The still most important paper on Jacobi algorithm is Perron 1907. He not only generalized the algorithm to any dimension $d \geq 2$ but even for dimension $d = 2$ his paper contains a wealth of interesting results. He states the conditions for admissible sequences of digits, namely for $k \geq 1$ we find

1. $m_k \geq l_k \geq 0$
   $m_k \geq 1$
2. If $l_k = m_k$ then $l_{k+1} \geq 1$.

He proves that if the algorithm stops (if $u_s = 0$) then the triple (u_0, v_0, w_0) is linearly dependent. Strangely enough, only in dimension $d = 1$ (continued fractions) and $d = 2$ the converse is true (this was proved later (Perron 1935, Paley & Ursell 1930)). He gives the first proof for the convergence of the algorithm, i.e.

$$\lim_{s\to\infty} \frac{A_1^{(s)}}{A_0^{(s)}} = \frac{v_0}{u_0}, \lim_{s\to\infty} \frac{A_2^{(s)}}{A_0^{(s)}} = \frac{w_0}{u_0}.$$

He shows that if the algorithm of $(1, x_1, x_2)$ eventually becomes periodic then x_1 and x_2 are rational functions of $\rho_0, x_1 = x_1(\rho_0), x_2 = x_2(\rho_0)$ where ρ_0 is an algebraic number of degree $n \leq 3$. Only later could Perron prove that in fact $n = 3$ and this result is wrong for dimension $d \geq 3$ (see Perron 1908, 1920, but also Paley & Ursell 1930). In this connection Perron detected the theorem (nowadays known as Theorem of Frobenius-Perron) that a matrix with positive entries has a distinguished largest real eigenvalue. Perron did not use matrices and hid his result as a Hilfsatz in §14! For periodic algorithms with period length $p \geq 1$ Perron's results show that

$$\begin{aligned} \left|A_0^{(sp)} x_1 - A_1^{(sp)}\right| &\ll \left|\rho_1\right|^s \\ \left|A_0^{(sp)} x_2 - A_2^{(sp)}\right| &\ll \left|\rho_1\right|^s. \end{aligned}$$

Here ρ_1 is the second root of the "characteristic equation" of the periodic algorithm. For $d = 2$ we always have

$$\left|\rho_2\right| \leq \left|\rho_1\right| < 1 < \rho_0.$$

Therefore the approximation result

$$\left|\frac{A_i^{(s)}}{A_0^{(s)}} - x_i\right| \ll \frac{1}{A_0^{(s)}\sqrt{A_0^{(s)}}} i = 1, 2$$

can only be true if the conjugates ρ_1, ρ_2 of the real root ρ_0 are complex. Clearly this is the case for $x_1 = \sqrt[3]{D}, x_2 = \sqrt[3]{D^2}$. At the end of this important paper Perron notes "Hiernach dürfte es doch zweifelhaft erscheinen, ob der Satz von der Periodizität allgemein richtig ist."

On the other hand Perron was searching for a proof till the end of his life! His very last papers are entitled "Der Jacobische Kettenalgorithmus in einem kubischen Zahlenkörper I und II" (Perron 1971, 1973). In these papers Perron gives a wealth of different formulas. The most important result is the following: Let $\omega > 0$ be the largest root of $x^3 - Ax - B = 0$ with integral coefficients.

$$\begin{aligned} \text{Let } x_1 &= \frac{P_1 + Q_1\omega + R_1\omega^2}{S}, x_1^{(\nu)} = \frac{P_1^{(\nu)} + Q_1^{(\nu)}\omega + R_1^{(\nu)}\omega^2}{S^{(\nu)}} \\ x_2 &= \frac{P_2 + Q_2\omega + R_2\omega^2}{S}, x_2^{(\nu)} = \frac{P_2^{(\nu)} + Q_2^{(\nu)}\omega + R_2^{(\nu)}\omega^2}{S^{(\nu)}} \end{aligned}$$

with integral coefficients P_i, Q_i, R_i, S which are subject to some divisibility conditions.

Let

$$x_1^{(\nu)} = a_\nu + \frac{1}{x_2^{(\nu+1)}}, x_2^{(\nu)} = b_\nu + \frac{x_1^{(\nu+1)}}{x_2^{(\nu+1)}}, \nu \geq 0,$$

then the equation

$$\begin{aligned} x_2^{(\nu)}\left(Q_1^{(\nu)} - R_1^{(\nu)}\omega\right) &+ \frac{x_1^{(\nu)}\left(Q_1^{(\nu-1)} - R_1^{(\nu-1)}\omega\right)}{x_2^{(\nu)}} - \\ &- \frac{Q_2^{(\nu-1)} - R_2^{(\nu-1)}\omega}{x_2^{(\nu)}} = \\ &= \frac{Q_1R_2 - R_1Q_2}{S}(3\omega^2 - A) \end{aligned}$$

is valid, which says that the lefthandside does not depend on the index ν (Perron calls this equation "Hauptrekursion").

But his efforts end with the words "Nun aber muss ich mein Problem jüngeren Kollegen überlassen, die sich wohl dafür interessieren. Mich zwingt meine fortschreitende Erblindung ... jetzt meine Feder aus der Hand zu legen."

Between 1907 and 1970 some other mathematicians worked on Jacobi algorithm. I want to mention again Paley & Ursell 1930 and the efforts of Leon Bernstein (see Bernstein 1971). Further references can be found in Brentjes 1981 and Schweiger 2000. An overview is given in Schweiger 2006. Among others Paley & Ursell proved that

$$\left| \frac{A_i^{(s)}}{A_0^{(s)}} - x_i \right| \ll \frac{1}{A_0^{(s)}}, i = 1, 2$$

and no better result is possible (which was already known to Perron).

Bernstein investigates in a long series of papers the periodicity of pairs (α, α^2). A quite typical result reads as follows. Let $D \geq 2$ be a natural number and let $\alpha = \sqrt[3]{D^3 + 3D}$, then

$$(\alpha, \alpha^2) = \begin{pmatrix} D & D-1 & 0 & 0 & \overline{2D-1} & \overline{D-1} & \overline{0} & \overline{D-1} \\ D^2+1 & D & 1 & D & 3D^2+3 & D & 1 & D \end{pmatrix}.$$

Strangely enough in 1975 it was proved that every cubic number field contains a pair (x_1, x_2) such that the algorithm becomes periodic (Dubois & Paysant-Le Roux 1975).

The idea to consider the triangle with vertices

$$(x_1, x_2), \left(\frac{A_1^{(s+1)}}{A_0^{(s+1)}}, \frac{A_2^{(s+1)}}{A_0^{(s+1)}}\right), \left(\frac{A_1^{(s+2)}}{A_0^{(s+2)}}, \frac{A_2^{(s+2)}}{A_0^{(s+2)}}\right)$$

is due to W. Schmidt: If $A(s)$ denotes the area of this triangle then for infinitely many values of s the inequality

$$A(s) < \frac{1}{2\beta(A_0^{(s+1)})^2 A_0^{(s+2)}}$$

holds, where $\beta = \xi^2 + \frac{2}{\xi}$ and $\xi > 1$ is the largest root of $\xi^3 - \xi^2 - 1$ (Schmidt 1958). Again this result is the best possible (and can be seen as generalization of Hurwitz's famous theorem on continued fractions). The Diophantine inequality

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{\sqrt{5} q_n^2}$$

is true for infinitely many values of n.

3 Probabilistic Considerations

The metric theory of Jacobi algorithm is quite a different story. The aim of this theory is concerned with the statistical behaviour of the sequence of digits as random variables

$$\begin{pmatrix} l_0 & l_1 & \dots & l_s & \dots \\ m_0 & m_1 & \dots & m_s & \dots \end{pmatrix}.$$

It is also related to Diophantine approximation, e.g. there is a $\delta > 0$ such that for $s \geq s(x)$ the inequality

$$\left| \frac{A_i^{(s)}}{A_0^{(s)}} - x_i \right| < \frac{1}{\left(A_0^{(s)}\right)^{1+\delta}}, i = 1, 2$$

is true almost everywhere (see Schweiger 1996 and Broise-Alamichel & Guivarc'h 2002). In the metric theory of Jacobi algorithm one introduces the map $T(x_1, x_2) = (\frac{x_2}{x_1} - l_1, \frac{1}{x_1} - m_1), l_1 = [\frac{x_2}{x_1}], m_1 = [\frac{1}{x_1}]$, on the unit square $0 < x_1 \leq 1, 0 \leq x_2 \leq 1$. If one starts with a triple $(1, v_0, w_0)$ of positive numbers and puts $x_1 = \frac{w_0 - m_0}{v_0 - l_0}, x_2 = \frac{1}{v_0 - l_0}, l_0 = [v_0], m_0 = [m]$ one sees immediately that iteration of the map T leads to the same algorithm.

Since in the metric theory the condition $l_0 = m_0 = 0$ is applied a slightly changed notation is used, namely

$$\beta \begin{pmatrix} l_r \\ m_r \end{pmatrix} = \begin{pmatrix} m_r & 0 & 1 \\ 1 & 0 & 0 \\ l_r & 1 & 0 \end{pmatrix}, r = 1, 2, \dots$$

but the basic recursion

$$A_i^{(s+3)} = A_i^{(s)} + A_i^{(s+1)} l_s + A_i^{(s+2)} m_s, 0 \leq i \leq 2$$

still holds (see Schweiger 2000).

4 Some conclusions

This story shows:

1. Number theory is a rich source of problems which are easy to state but hard to solve.

2. A lot of suggestive data can be collected by elementary methods (the assistance of a computer is clearly welcome).

3. The gap between a wealth of examples and a valid proof is a challenge which is typical for mathematical creativity.

4. The attempt to solve the problem stimulated the theory of multidimensional continued fractions.

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