The multiplicity of viewpoints in elementary function theory : historical and didactical perspectives

Renaud CHORLAY, Anne MICHEL-PAJUS
IREM de l'Université Paris 7, 175-179 rue du Chevaleret, 75013 Paris, France
renaud-chorlay@noos.fr annie.pajus@club-internet.fr

ABSTRACT

The so-called rigorization of Analysis in the 19th century is a standard topic in the history of mathematics, and has indeed provided material for didactical research works centred either on notions (e.g. limit, continuity) or on shifts in levels of abstraction (Advanced Mathematical Thinking). For four years, members of the "history of mathematics" group of the French IREM network endeavoured to establish new connections between historical and didactical questions. For the Paris group, the starting point was the identification of four viewpoints on functions : point-wise, infinitesimal, local and global. We gathered historical material – sometimes standard, sometimes less well-known – showing typical interactions between these viewpoints at different stages of the rigorization process. We also tried to identify the contexts in which these viewpoints first emerged, then were explicitly differentiated one from the other. We eventually devised two epistemological models – the "world of quantity" and the "world of sets" – in order to describe two distinct forms of "functional thinking". These high-level descriptive tools helped us gain new insights into didactical questions relevant for the teaching of Analysis at elementary or advanced level. After a short case-study, we will present the main features of the epistemological models. We shall eventually consider more general teaching perspectives.

Introduction

From 2002 to 2006, the "history of mathematics" group of Paris 7 IREM[1] contributed to a research project funded by the *Institut National de la Recherche Pédagogique* (INRP). We chose to work on the multiplicity of viewpoints on functions. In spite of the fact that some didactical and some historical research work was available on this topic, we felt the relevant connections still needed to be pointed to and explored. We also made use of fresh historical research work, namely R. Chorlay's doctoral dissertation on the emergence of the concepts of "local" and "global" in mathematics [Chorlay 2007(b)].

We borrowed from didactical works the notion of viewpoint (as opposed to theoretical frame and semiotic register [DIDIREM 2002]) and the distinction between four viewpoints in mathematical Analysis : point-wise, infinitesimal, local and global. Didactical work focused either on issues of cognitive flexibility (versatility) – the ability to change viewpoints, frames or semiotic registers in problem-solving – and its growing importance in higher education (Advances Mathematical Thinking), or on curricular discontinuities : a point-wise / global dialectic when the concept of function is first encountered, then the infinitesimal and local viewpoints come into play with calculus, then an all-encompassing abstract theoretical frame in higher education. We focused our more historical investigation on a series of *hot spots* in which the four viewpoints interact and, eventually, were made explicit in the 19th century : proofs of the mean value theorem ; proofs of the link between the sign of *f'* and the variations of *f* ; differentiation of the three notions of maximum, local maximum and upper bound ; emergence of the domain

[1] IREM : Institut de Recherche sur l'Enseignement des Mathématiques. The e-mail address for the history of mathematics group is iremmath@yahoo.fr . The members of the Paris group were : Philippe Brin, Renaud Chorlay and Anne Michel-Pajus.

concept. From an curricular viewpoint, these topics cover material that students study from the beginning of high-school (maximum) all the way to undergraduate college Analysis.

In this talk, we shall present some of the results of this investigation. After a general and a-historical introduction of the four viewpoints we will briefly present a short case-study. We will then introduce a more general explanatory framework, distinguishing between a "world of quantity" and a "world of sets". We will eventually outline didactical perspectives.

1. The interplay between four viewpoints

Let us give a few examples in order to illustrate the intricacy of the interplay between the four viewpoints in rather simple statements in function theory. Let's first consider "function f is positive in 2", the property it states is clearly point-wise. Things get more tricky with "function f is differentiable in 2" : here stated an infinitesimal property, which is also a local property of function f ; in addition, from a purely syntactical viewpoint, the statement looks perfectly point wise ("when x = 2"). Now "function f has a local (or relative) maximum when $x = 2$" is of a local nature, yet stated in a point-wise fashion, and often (but not necessarily, since the property if perfectly well defined for non-differentiable functions) related to the infinitesimal behaviour of the function. If one is to give an example of a global property, "function f is bounded on [0,1] " can do : the fact that the property if global is reflected in the fact that a domain relative to which the property holds appears explicitly in the statement (though is could be left implicit if a domain had been set once and for all). This domain-component of the statement helps differentiate global statements from the three other kinds of statements, in which the domain is usually left implicit. It should be noted straight away that, in this context, the term "viewpoint" doesn't primarily denote a subjective property : in spite of the fact that the word "viewpoint" places the emphasis on cognitive processes, our four viewpoints refer to mathematical properties of mathematical statements and objects.

The basic element is the fact that, to say if roughly, functions are objects of a relational nature, they "do something somewhere". The syntactic structures which reflects this relational nature is

Function f is [*property*] on [*domain*],

a syntactic structure which (French) students are required to use systematically as from their first encounter with the function concept (at age 15). One could consider characterising the four viewpoints by a classifying domain-types : the property is point-wise if it can be defined on a single-point domain, it is local if its proper definition requires neighbourhoods of a point. It is hard to go much further in this direction, for two essential reasons. The first one is specific to the infinitesimal level : it can not be characterised through a specific domain type, unless one introduces tools of unreasonable sophistication such as tangent spaces (if only the first order is concerned) or "infinitesimal neighbourhoods" of the kind differential geometry or modern algebraic geometry consider. The other reason is specific to the dialectic between local and global. A property is not global because it deals with a certain *type* of domain, it is global because the domain involved plays a specific *role* in the statement of the property. This was explicitly remarked by the first (at least to our knowledge) mathematician who tried to explain the meaning of "local" and "global" in a mathematical treatise, in 1901. In the article on the theory of functions of complex variables which he write for Felix Klein's Encyclopaedia of mathematics, American mathematician Willian Fogg Osgood gave the following criterion (we paraphrase the German original) : in function theory, the behaviour of a function is local if it refers to the neighbourhood of a point (or a subset), it is global if it refers to a domain which had been set right from the start, as opposed to domains whose extent is determined afterwards to fit the requirements of the problem [Osgood 1901 p.12]. The wording may, at first, seem a little obscure, but it stresses the fact that what matters is the role that domains play in the syntactic structure of the statement. What Osgood had in mind was the difference between a local and a global inversion theorem : in the global theorem, the conclusion holds for the very domain which had been set at the start; in the local case, the conclusion holds for some domain which is usually a sub-domain of the domain we started with. A more elementary example can be found in the distinction between a global maximum (a maximum over the whole domain you started with) and a local maximum (for which some unspecified yet specifiable domain is referred to). This distinction between "given domains" and "specifiable domains", or, to put it differently, between "domains given right from the start" and "domains determined afterwards" is reflected in the order of quantifiers in the formal statement. Unfortunately, the illuminating Osgood distinction doesn't say it all, at least in the local case. The global nature of a property may be completely characterised by the role of the domain, but it is not so for local properties : a neighbourhood is also a *type* of domain ; the local / global dialectics may be captured on the syntactic level, but the local part has deeper, non syntactic, topological roots.

It seems there is no easy way out, yet our job is to design ways in for students, whether at high-school or university level. Of course, one could argue that these intricacies are to be avoided completely. We think, and there is empirical evidence to support this claim, that complete avoidance of these problems has a high cost in the long term in terms of cognitive flexibility (in problem solving) and ability to adapt to evolving theoretical frameworks ; we think some degree of awareness of this interplay is necessary for students to be able to do more than routine calculation in Analysis, and that teachers and those who train them cannot shun the topic altogether.

Before reflecting on two historical case-studies, let us try to show how reasonable classroom work could help raise awareness. We will consider three couples of statements ; in each case, students could be asked if they are true or false ; in the latter case, they could be required to exhibit (graphically, for instance) some counter-example.

> Statement 1 : *if f is continuous in 2 and positive in 2, then f is positive in the neighborhood of 2.*
>
> Statement 2 : *if f is continuous in 2 and positive in 2, then f is positive in 2,00001.*

A graphical counter-example to statement 2 should help point to the fact that no given size can be assigned to the domain over which the conclusion is valid, in spite of the fact that there is such a domain. According to the teacher's goal for the discussion, statement 1 could either be considered to be the right one (as opposed to false statement 2), or too vague to be *the* right one. The first statement could then be amended by introducing "*on a neighbourhood of 2*" (in which sentence "neighbourhood" is a definable domain type and not only a metaphor from daily life) or quantified statements such as "*one can find a positive number A such that f is positive of*]2-*A* , 2+*A*[".

Statement 3 : *if f(2) = g(2) then f'(2) = g'(2)*
Statement 4 : *if f'(2) = g'(2) then f(2) = g(2)*

Both statements are false and are classical mistakes, but they point to two different kinds of errors. If one tries to characterise these errors, the first one comes from an improper understanding of the relative scopes of the hypothesis (which is point-wise) and the conclusion (which is infinitesimal, thus local) ; the second one can be ascribed to the intrinsic loss of information that derivation entails, a loss of information that student often fail to see in spite of the fact that they learn that a primitive (over an interval) of a continuous function is given up to an additive constant. Whether the explicit wording of the reasons for which the statements are false should be a goal for classroom work is left to the teacher's choice. For instance, it could be considered irrelevant in high-school but highly relevant at the beginning of university Calculus or in a teacher training session. In any case, high school students should be able to identify these statements as false, and draw counter-examples.

Statement 5 : *a continuous function is bounded.*
Statement 6 : *an everywhere locally increasing function is an increasing function*.

Both statement deal with passage from local hypotheses (at every point though) to global conclusions. They should help point to the fact that, in order for global conclusions to hold, some knowledge about the nature of the domain is required. For instance, the first statement is valid if the domain is a closed and bounded (i.e. compact) interval, and student should be able to come up with counter-examples, even on bounded intervals (consider for example the tangent function *tan* on]-π/2 , π/2[). As for statement 6, it is valid for intervals (for reasons of connectedness) but not for disconnected domains, as $-1/x$ on **R**\{0} clearly shows. Both can help stress the importance of the domain part in the "function *f* is [*property*] on [*domain*]" structure, a part whose importance usually fails to strike students at any level of the educational system.

2. A case-study: sign of *f'* and variations of *f* according to Cauchy.

We conducted a few case-studies, some of which we will leave aside here – for instance the study of the history of the implicit function theorem [Chorlay 2003], or that on the evolution of the meaning of "maximum" [Chorlay 2007(b)] We will only present some elements from one case-study, on the theorem linking the sign of the derivative and the variations of the primitive.

This theorem is one of the first ones that students encounter in Calculus and, in (French) high-schools, it tends to become the main application of the notion of derivative (approximation aspects play a lesser role in the current curriculum). Heuristic arguments for this theorem are given at high-school level, but no proof ; the proof that has become standard since the late 19th century (which depends on the mean value theorem, whose proof, in turn, depends on a maximum arguments which depends on topological properties of **R**) is usually given at the very beginning of College Calculus, and is one of the first occasions to experience the wealth of links between the four viewpoints. As far as History is concerned, we studied the historical emergence of this proof-scheme in the works of

Bonnet, Serret, Dini and Jordan, to name a few. We also studied a former generation of proofs, namely those of Ampère and Lagrange.

Yet another proof-scheme can be found in Cauchy's 1823 treatise on differential calculus. Here is a (crude) translation from the French :

> "Problem. Assuming that function $y = f(x)$ is continuous relative to x in the neighbourhood of specific value $x = x_0$, one asks whether the function increases or decreases as from this value, as the variable itself is made to increase or decrease.
> Solution. Let Δx, Δy denote the infinitely small and simultaneous increments of variables x and y. The $\Delta y/\Delta x$ ratio has limit $dy/dx = y'$. It has to be inferred that, for very small numerical values of Δx and for a specific value x_0 of variable x, ratio $\Delta y/\Delta x$ is positive if the corresponding value of y' is positive and finite. […]
> This being settled, let's assume function $y = f(x)$ remains continuous between two given limits $x = x_0$ and $x = X$. If variable x is made to increase by imperceptible degrees from the first limit to the second one, function f shall increase every time its derivative, while being finite, has a positive value." [Cauchy p.37]

The proof architecture is quite clear : the behaviour of the function is studied in the neighbourhood of a given point (and Cauchy's inference is correct), then the conclusion is extended to intervals of continuity. A few things are worth stressing.

First, what Cauchy proves in the first part of the proof is not that the function is locally increasing, as counter-example $x + 10x^2 \sin\frac{1}{x}$ shows when $x = 0$: the derivative in $x = 0$ is positive, yet the function is monotonous in no interval around 0. Of course Cauchy never claimed he had proved such a thing, but this way of reading it is naturally induced by our definition of increasing and decreasing functions.

By the way, and this is the second point that is worth stressing, in no part of Cauchy's treatises does one find a definition for increasing functions, in sharp contrast to current curricula which (in France) include a definition of what it is for a function to increase : a function, defined on a domain D, is an increasing function if, whenever a and b are any two elements of D such that $a \leq b$, then $f(a) \leq f(b)$; increasing functions are "order preserving" functions, a notion in which only the point-wise and the global viewpoints are involved, but which requires arbitrary pairs of points to be considered. The latter definition is mathematically elementary, but proves for most students difficult. More often than not, this *definition* is not included in the *concept image* of increasing functions, a concept image which is sufficient to tell increasing functions from decreasing functions when a graph or a table is exhibited. In his proof, Cauchy doesn't rely on this (or any) definition, but rather on the following *cognitive root* : a function is increasing for a specific value $x = x_0$ of the variable if sufficiently small increments Δy and Δx have same signs; a cognitive root whose *embodied* nature is striking. The formal definition which one can draw from this cognitive root may be less easy to handle than the order-preservation one, but the link between the two notions may be worth eliciting. Field-work on this topic is currently being conducted with high-school students (age 16).

The third point that should be stressed is the role of the domain. In the current curriculum, the statement of the theorem goes : "if function f is defined and differentiable on an interval I, and if its derivative is positive on I, then f is an increasing function on I"; the nature of the domain is an essential part of the statement (for the theorem is false for non-connected domains) but usually overlooked by students. Things are quite different in

Cauchy. First, the pre-definition cognitive root on which he relies implicitly entails connectedness of the domains over which the final conclusion can hold. Second, the fact that the conclusion holds for intervals is not entirely captured by the conjuring up of "limits" x_0 and X. One must recall that Cauchy's approach of continuity is not entirely ours. For instance, we teach students that the reciprocal function $1/x$ is defined and continuous on $\mathbf{R}^*$; saying that it is discontinuous in 0 is just a common mistake : this function has no property at all in 0, since it lies outside the definition domain. In Cauchy's treatises the notion of domain of definition never appears, and the behaviour of the reciprocal function in 0 is exactly what he calls "being discontinuous". In the proof above, the connectedness of the domain on which the conclusion holds is partly hidden (for us) behind the continuity hypotheses " let's assume function $y = f(x)$ remains continuous between two given limits $x = x_0$ and $x = X$".

3 Ways of world-making : the case of functions

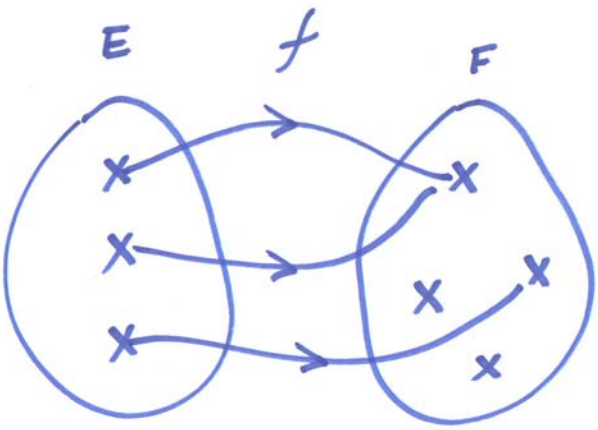
The case-studies, of which only one is sketched above, helped us characterise different ways of "working with and speaking about" functions, different ways of (function) world-making. We shall sketch the outline of two such ways of world-making, the "world of quantity"(WOQ) and the "world of sets" (WOS). One of the goals is to get a *positive* grasp of the pre-Weierstrass framework. By "positive" we do not mean a comparison of the respective values of both frameworks ; we mean to provide a description of what it does and how it is structured which doesn't rely entirely on negative descriptive elements : lists of shortcomings, of implicit assumptions, forms of syntactic vagueness etc. In short, the "world of quantity" is not only something that fails to be the "world of sets".

A first feature of the world of quantity (or "world of magnitude") is what we call the *universally local approach* : everything is known about a function when its *behaviour* is known *everywhere* (that is, for each and every specific value of the variable). In the WOS, the point-wise viewpoint is the fundamental one, the starting point : for a specific value of the variable, the function has a *value*. In the WOQ a function has a behaviour and not only a value. A striking example is that in the 19th century functions do note take on value 0, they *vanish*. The fundamental viewpoint is not the point-wise one, it is a mixture of local and infinitesimal viewpoints ; what is described is not only a value but an event : functions do something / something happens to them. In the WOQ, mathematical statements and proofs are written in what we called the *narrative style*, as opposed to the *static,* explicitly quantified Weierstrassian style.

Three things should be emphasised in this regard. First, this notion of behaviour has a formal equivalent in the WOS, if one uses the (somewhat) sophisticated notion of function germ ; the notion of germs-ring and of quotient rings can even enable us to state infinitesimal properties in a clean-cut, set-theoretic fashion. Second, in the universally local approach, local and infinitesimal properties are not distinguished. This has deep historical reasons, which one of us tried to study in some detail [Chorlay 2007(b) chapter 6]. Third, in the universally local approach, some global aspects elude grasp since the reference to *all* points or *all* values is of a distributive nature : in this context, an important class of global properties, namely uniform properties (such as uniform continuity for a function or uniform convergence for a sequence of functions) seem to be out of reach. In this respect, the advent of the WOS in the work of Weierstrass has (at least) three related components : (1) considering the purely point-wise viewpoint to be the most fundamental one (since it is the only one fitted for “arbitrary” functions), (2) distinguishing between point-wise and uniform properties (3) distinguishing between “infinitely near points” (infinitesimal neighbourhood) and “sufficiently near points” (topological neighbourhood). In this respect, the systematic use of explicit quantifiers appears as a *tool*, an essential tool indeed, but a tool nonetheless.

In defence of the WOQ, it can be argued that in expresses global properties rather efficiently in terms of *systems of singularities*, an epistemic scheme which proved highly seminal in the hands of Riemann and Poincaré ; it lead to (algebraic) topology and to the qualitative theory of differential equations. In this respect, Riemann is the perfect example of the *versatile* thinker, switching between WOQ and proto-WOS in his works in complex or real function theory (respectively).

To contrast sharply the WOQ against the WOS, one can use the purely set-theoretic description of a function as a map between sets :



In this formal scheme, two types of “objects” come into play : sets on the one hand – the domain of definition E and the target set F –, whose explicit statement is mandatory ; another object, the map, which is of a second-order, relational nature : the map is the set of arrows (i.e. a part of the cartesian product E×F). At this level of generality, the only relevant viewpoints are point-wise and global. This system of arrows may be arbitrary up to a certain point, and sets E and F play asymmetrical roles : every element of E has to have one (universality) and only one (univocity) counterpart in F, while some elements of F may have no counterpart (i.e. the image-set may not coincide with the target set) or more than one counterpart in E. In the WOS, a function is just a map between number sets. Many of these elementary but structural features of the WOS are in sharp contrast to the WOQ : in the WOQ no statement of sets is necessary and the roles of what could be considered as definition and target sets are completely symmetrical, in particular since many-valued functions are the rule. If today’s functions are just special maps, the multi-valued function of the 19th century can be seen, in retrospect, as general relations whose maximal domain is to be determined.

4 From historical research to didactic engineering and research in didactics

We mentioned in the first part of this paper various statements (1-6) which can provide starting points for the designing of classroom sessions, whether in secondary or higher education : the exhibition of counter-examples may be adequate at high-school level ; in higher education, discussion of those statements could trigger the search for precise definitions and proof-ideas. They could also help show students how to manage proof-tasks when no formula is given for the functions which are being studied : this new type of task is quite specific to higher education (at least in France) and proves quite unsettling for most students. From a research viewpoint, we touch on questions which have been studied from a purely didactical perspective, by Aline Robert for instance [Baron & Robert 1993] (see also [Praslon 1994]). Adjective such as "local" or "global" belong to what A. Robert calls the *meta*-level : they're instances of a form of mathematical knowledge that say something *about* mathematics ; they help sort mathematical statements and definitions in high-level categories ; they enable you to "find your way around" in an ever-growing, ever more complex mathematical environment. This knowledge about mathematics helps you spot potential difficulties (e.g. "in this problem we are to go from local to global, I know this is usually quite tricky"), identify the right theoretical tools (e.g. to choose between Taylor-Lagrange and Taylor-Mac-Laurin formulae, that is between a global and an infinitesimal formula) or conjure up problems of a similar kind whose solution you remember. Whether this meta-level knowledge is to be made explicit for the students, or even taught as such, is a debated issue.

We mentioned in part 2 of the essay how we found some classical notions from the psychology of mathematical learning – such as that of cognitive root, or the difference between concept image and concept definition – be helpful to link historical work and teaching issues. As far as the notion of functional variation is concerned we decided to venture in the world of didactic engineering and field-work is under way.

The distinction between two formal models, WOQ and WOS, helped us point to a general problem in the current introduction of the function concept. In France, students encounter from the very beginning (age 15) the abstract notion of "arbitrary function on a given domain of definition". This notion is a mixture of elements which, as far as History is concerned, emerged in rather distinct contexts. On the one hand, the notion of arbitrary function emerged in the debate on the foundations of real Analysis ; no *natural* domain can be ascribed to an arbitrary function ; domain restrictions and extensions are completely trivial in this context. On the other hand, questions about domains emerged in the case of highly non-arbitrary functions ; for instance, to some convergent power-series (usually in a complex variable), one can assign two natural domains : its region of convergence, and the domain of holomorphy of the unique holomorphic functions which is represents (domains of meromorphy can also be considered). More generally, questions of domain extension become relevant only when functions with specific properties are concerned. In fact, empirical work would certainly show that this "arbitrary function on a given domain of definition" is usually not included in the function concept-image in high-school, partly for ecological reasons but maybe also for the reason we point to here.

Finally, the historical perspective may help to guide didactical choices, in particular by telling apart more clearly (mathematical) necessity from convention. "Elementary" and seemingly "natural" – mathematically natural, not psychologically natural – notions such as those of function (as arbitrary, one valued correspondence between sets), domain of definition, maximum etc. are partly *conventional* ; which by no means entails that they are arbitrary ! They appear to us as "elementary" and "natural" because mathematicians left us a body of mathematics in which massive conceptual restructuring had taken place, especially in the 19th century. It is not mathematically necessary to assume that a function is one-valued ; for rigorous mathematics to be written, it is not necessary that domains of definitions be stated from the outset : choosing between either conventions will change the priority between notions, the syntactic rules for writing rigorous mathematics, alter the meaning of a theorem. By their very nature, such conventional parts of our function-world cannot be expected to emerge from classroom work as the best solution to a well-designed problem ; there lies an intrinsic limit for didactic engineering. Conventions are neither true or false ; the shift of conventions is a slow and high-level process in which mathematicians react to the global state of mathematics and not to a specific problem.

Conclusion

By way of conclusion, we would like to stress the interest of a macro-historical approach which aims at identifying ideal and coherent "worlds" such as the world of quantity and the world of sets. This is neither historical work in the strict sense – in terms of the standards of the community of historians of mathematics – nor directly didactical work, but we think it helps build bridges without resorting to dubious "ontogeny parallels phylogeny" arguments. In particular, the classical Bachelard-Piaget notion of *obstacle* can be interpreted in a less teleological way ("as mathematics developed, mathematicians overcame obstacles which students are, in turn, faced with …"). The issue of the multiplicity of coherent worlds – none of them being *the* right one (of which all others are superseded archaic forms) – helps us to see obstacles as translation problems that are bound to come up when changing worlds. Such translations are difficult for at least two different reasons. First, in such a translation task both semiotic *and* conceptual aspects are deeply interconnected ; second, because of the large scale coherence of such "worlds", there is no such thing as a strictly autonomous, purely local move. This type of history-based investigation should throw light on both versatility issues and long term curricular discontinuity problems.

REFERENCES

- ARTIGUE, M., 1992, "Functions from an Algebraic and Graphic Point of View: Cognitive Difficulties and Teaching Practices", in *The Concept of Function: Aspects of Epistemology and Pedagogy,* E. Dubinsky & G. Harel (eds.), Washington DC: The Mathematical Association of America, pp. 109-132.
- BARON, M., ROBERT, A. (eds.), 1993, *Métaconnaissances en IA, en EIAO et en Didactique des Mathématiques*, Cahier DIDIREM spécial. Paris : IREM Paris 7
- CAUCHY, A.-L., *Œuvres Complètes* tome 16. Paris : Gauthiers-Villars, 1882-1974.
- CHORLAY, R., 2003, " Fonctions Implicites : de la Notion au Théorème", *Mnémosyne* n° 18, p.15-58 .
- CHORLAY, R., 2007(a), "La multiplicité des points de vue en Analyse élémentaire comme construit historique", in E. Barbin & D. Bénard (eds.) *Histoire et enseignement des mathématiques : rigueurs, erreurs, raisonnements*. Lyon : INRP, p.203-227.
- CHORLAY, R., 2007(b), *L'émergence du couple local / global dans es théories géométriques, de Bernhard Riemann à la théorie des faisceaux (1851-1953)*, Thèse de l'Université Paris 7.
- DIDIREM, 2002, *Actes de la Journée en Hommage à Régine Douady* (14 juin 2001). Paris : IREM Paris 7.
- EPPLE, M., 2003, "The End of the Science of Quantity : Foundations of Analysis, 1860-1910", in H-N. Jahnke (ed.) *A History of Analysis*. New-York : AMS, 2003, p.291-324.
- JAHNKE, H.-N. (ed.), 2003, *A History of Analysis*. New-York : AMS.
- OSGOOD, W. F., 1901, "Analysis der komplexen Grössen", in *Encyclopädie der Mathematischen Wissenschaften* II.2, Leipzig : Teubner, 1901-1921, p.1-114.
- PRASLON, F., 1994, *Analyse de l'aspect Meta dans un enseignement de Deug A concernant le concept de dérivée. Etude des effets sur l'apprentissage* (Mémoire de D.E.A.). Paris : Université Paris 7.
- TALL, D. (ed.), 1991, *Advanced Mathematical Thinking.* Boston : Kluwer, 1991.
- TALL, D., VINNER, S., 1981, "Concept Image and Concept Definition in mathematics, with particular

Reference to Limits and Continuity”, *Educational Studies in Mathematics*, 1981, vol.12(2), p. 151-169.
- VOLKERT Klaus, 1987, “Die Geschichte der pathologischen Funktionen”, *Archive for History of Exact Sciences* **37**(1), p.193-232.