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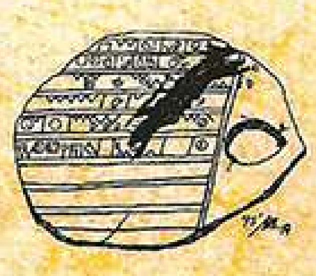
Proceedings of the HPM 2000 Conference

History in Mathematics Education:

Challenges for a new millennium



Vol.I

August 9-14, 2000

Department of Mathematics

National Taiwan Normal University

Taipei, Taiwan

Editors:

Wann-Sheng Horng & Fou-Lai Lin

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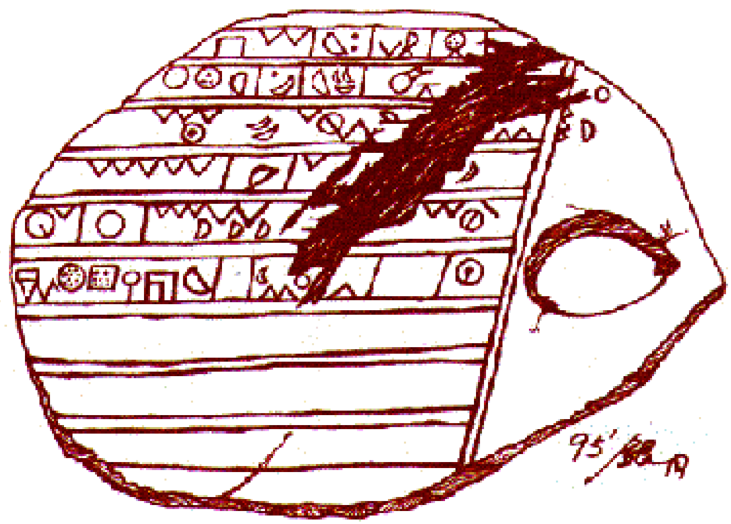

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Two pots and one lid :
The first arithmetic textbooks in the Netherlands before 1600

Marjolein Kool
Hogeschool Domstad, Utrecht

A historical arithmetic problem

Somebody has two silver pots and one silver lid. The price of the lid is 16 guilders. If he places the lid on the first pot, this combination will cost 4 times more than the second pot. If he places the lid on the second pot, this combination will cost three times more than the first pot. What will be the value of each pot?

If you have tried to solve this problem, you probably wrote down two equations: $x + 16 = 4y$ and $y + 16 = 3x$ and had got the answer quite soon. I found this problem in a Dutch arithmetic manuscript of 1568. The author didn't use equations, letters and arithmetical symbols to solve this problem. These mathematical aspects, we are so familiar with, were hardly known in the sixteenth century Netherlands.

Sixteenth century arithmeticians could solve problems like this one, just by using words, numbers and a few lines, as you can see in figure 1. They used the so called 'regula falsi' or 'rule of false position'. This rule is used to find the required unknown number with the help of two arbitrarily chosen numbers.

Let me explain this. The value of the first pot with the lid is 4 times the value of the second pot. Imagine the first pot costs 4 guilders, than the second pot will cost 5 guilders. The second pot with the lid will cost three times the first pot. Three times 4 is 12, but here we have 21, so we have 9 guilders too much. In figure 1 you can see the result of this calculation: Your first try - 4 - is written on the left, the difference - 9 - is written on the right and between them is a long line with a small vertical mark. This mark indicates: 'too much'.

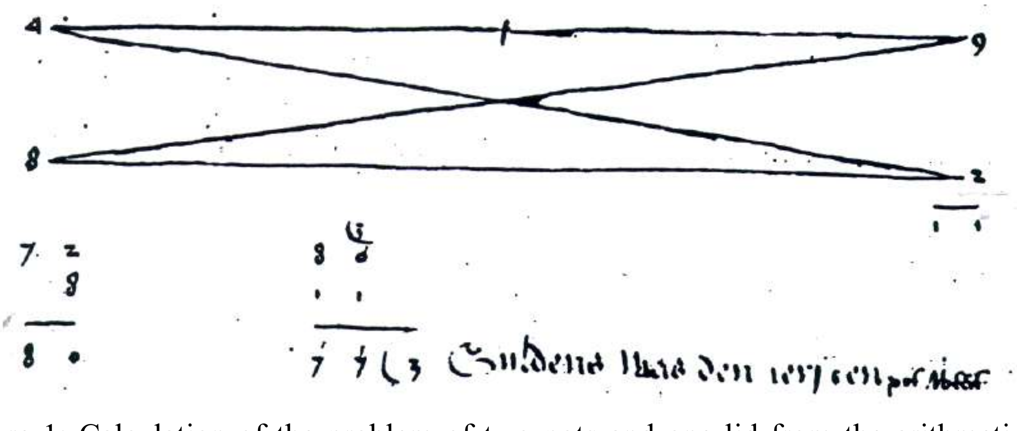

figure 1: Calculation of the problem of two pots and one lid from the arithmetic of Peter van Halle, fol. 145v.

Let us choose a second arbitrarily value of the first pot. Imagine the first pot will cost 8 guilders. In that case you will found 6 guilders for the second pot. The difference is now: 2 guilders too less. In the figure you can see 8 on the left side and the difference - 2 - on the right side, and between them a long line. The following calculations are made:

[(8 x 9) + (4 x 2)] : (9 + 2) = 80 : 11 = 7 3/11. This is the value of the first pot and that means that the second pot will cost 5 9/11.

Pupils will be amazed when they see this solution. On the first sight it seems a kind of magic trick. But if they study on this and perhaps some other examples, they can find out what is happening. If x is the solution of the equation f(x) = 0 and a and b are arbitrarily chosen solutions, you can find x in this way:

$$x = [a \cdot f(b) - b \cdot f(a)] : [f(b) - f(a)]$$

If you have done this problem in your classroom, you can go on by asking children to solve other historical (or modern) problems with the rule of false position and if you have clever pupils you can try to ask them why this rule works. It is always interesting to let pupils compare their own solutionmethod with the historical one.

In working on this problem, your pupils will discover that people in the sixteenth century use arithmetic to solve problems that we would solve with algebra. Why? I am sure they will be very surprised if you tell them that there was a time that people didn't know algebra, even didn't know symbols like 'plus' and 'minus'. In spite of these 'handicaps' they could solve quite complicated problems. Your pupils will be impressed and realize at the same time that mathematics in the sixteenth century was different from their own mathematics. Mathematics has changed during the ages and will change in the future. That will be an eye-opener for many children.

Intermezzo

On the 9th of February 1999 I finished my study on Dutch arithmetic books of the fifteenth and sixteenth centuries. In my thesis I describe the contents of these books, the schools where they were used, the teachers, pupils, the arithmetical vocabulary, etcetera. The reactions on my book were positive and of course I am proud on it, but I feel that my work isn't finished yet. During my research, I saw hundreds, perhaps thousands of arithmetical problems, and I thought many times: 'This could be a good problem to use in our modern classroom.' Unfortunately I had hardly time to collect these suitable problems and experiment with them because the modern mathematical education was no part of my research. Today I will tell you about the results of my research and at the same time I feel free to reveal some of the didactic ideas that came into my mind during the time I was doing this research. Perhaps this will be the start of a new research.

Where did the problem of the two pots come from?

I have found the problem of the two pots and one lid in a huge arithmetical manuscript that was written in 1568 by Peter van Halle in the Dutch language. Nothing more is known about this person than what he has written in a poem at the beginning of his work: He is 18 years old, lives in Mechelen and asks his audience not to blame him for his failures because this is his first book. It contains more than 600 pages and you can find it nowadays in the Royal Library of Brussels (ms. 3552).

Where could this young guy find the inspiration and information to write such a huge book on arithmetic? The problem of the two pots and the lid gives us an idea. The same problem is also found in many other mathematical medieval works in Europe.

The German authors Adam Ries (1533) and Johannes Widman (1508) have variations of the problem. In the work of Widman the story is illustrated by a picture as you can see in figure 2.

Figure 2: The problem of the two pots with one lid from the arithmetic of Johannes Widman (1508)]

The same problem, even more similar, is found in the works of Christoff Rudolff (1525) and Gemma Frisius (1540). They use exactly the same numbers in this problem. Has Peter van Halle used the works of Rudolff and Frisius for composing his own book? I found some indications to suppose so.

Peter van Halle has one surprising chapter in his book: He describes the basic principles of algebra. Although he doesn't use algebra to solve problems, he describes how to do the arithmetical operations with algebraic expressions. That is quite unusual in a Dutch arithmetic book of this time. Where did he find his information concerning algebra? He finished this chapter with the words: 'If you want to know more about the regula coss (= algebra) read the works of Cristophorum Javier and Hieronimium Cardanum.' This makes clear that Peter van Halle probably used the works of Rudolff and Cardano to compose his own work. In another chapter, on the rule of false position, he mentions that Doctor Gemma used a different terminology for this subject. This shows that also the work of Gemma Frisius was one of his sources.

Peter van Halle knew his mathematical predecessors and he used their works. He copied many problems and fragments from several mathematical books, but he never mentioned that he is doing this. That was not unusual in the sixteenth century Netherlands. Authors of textbooks didn't have the intention to be original. They just wanted to write a clear and instructive book and they used the works of their predecessors to reach this. Everybody knew this although the authors didn't mention their sources.

Peter van Halle also used Dutch arithmetic books. I found resemblances with the work of Gielis van den Hoecke (1537) and Peter Heyns (1561). Perhaps he has used even some more sources, although I didn't found them until now.

Peter van Halle and his colleagues: What do they want to reach with their arithmetic books?

I found quite a few Dutch arithmetic books from authors that worked more or less in the same way as Peter van Halle did. To be precise: I found 36 arithmetic books written in the Dutch language: 12 manuscripts and 24 printed editions. The oldest one was written in 1445. The resemblances between these books are impressive. All the authors used at least one or two sources to compose their own book. After much study and research I had only five arithmetic books left of which I couldn't find any source. But this means probably that I didn't look well enough, or perhaps that their sources are getting lost, because these five arithmetic books doesn't look very special compared with the other 31 books.

The way in which the sources are used differs widely. Some authors copy large portions, others revise or translate their sources, using only a few problems or compiling the material at their disposal and supplementing it with their own explanation, comment or examples.

They all have the same aim: They wanted to teach how to do arithmetic with Hindu-Arabic numbers, how to do arithmetic with pen and paper. Two questions will rise at this moment: Why this topic? and Who wanted to learn this?

The old and the new arithmetic

In the arithmetic book of Peter van Halle you can find a few pages on arithmetic with coins. That is to say, on fol. 280r-v he explains with just two examples, how to do multiplication and division with coins. In the introduction to this subject he writes that he doesn't need to describe how to do additions and subtractions with coins because that is already known to everybody.

Calculating with coins was the common and well-known way of doing arithmetic during the Middle Ages. In doing this you need a board with horizontal lines and by placing coins on and between the lines you can express numbers and do arithmetic. This counter-board was a variation of the traditional abacus of the old Greeks and Romans. They used a board with vertical lines. During the Middle Ages this abacus underwent some changes and was turned a quarter like you can see in figure 3.

Figure 3: A medieval counter-board with horizontal lines.

At the beginning of the thirteenth century the Hindu-Arabic numbers came into use in Europe, but that doesn't mean that immediately everybody threw away his coins and started to do arithmetic with Hindu-Arabic numbers. It took a very long time before the new method had replaced the old one. During a very long period both arithmetical methods stayed into use. It seems that many people in the sixteenth century could work with both methods. Adam Ries wrote in his arithmetic book of 1533 that the traditional calculating with coins was a good preparation for the learning of the new method with the pen. The mathematician Petrus Ramus used the new arithmetical method in his 'Arithmeticae libri tres' (1555), but he said, that in private he preferred the traditional way of doing calculations with coins. The Dutch author Adriaen van der Gucht wrote in his arithmetic book of 1569 that he taught the calculating with coins because it could happen that you have to do a calculation but you don't have ink and paper at hand. In that case you will probably always have some money with you, so you can do your calculation with coins.

The main reason why the old method didn't disappear immediately after the new one was introduced was: Many people couldn't write! In the sixteenth century primary schools children first learned to read and thereafter learn to write. Most children left school before they had got writing lessons. Around 1600 about 60% of all men and 40% of all women were able to sign their marriage certificate. That means that during the sixteenth century many people couldn't write and were not able to do the new arithmetic.

Even in 1698 coins were struck in the Southern Netherlands, but as time went by finally more and more people turned to the new method.

The new method with Hindu-Arabic numbers has several advantages compared with the old one: It is much easier to write big numbers, to do extracting roots or calculations with fractions, and finally, you only need one instrument for making your calculation and noting the result.

The target group

This last advantage was important for merchants. During the sixteenth century the job of big merchants became rather complicated. Their travels became longer, they send their travelling salesmen to many different countries, they had to pay salaries, custom rights, costs of transport, assurances of goods, they had to change money. They wrote reports and cash-books. In the beginning these cash-books had roman numerals, but as time went by you see more and more Hindu-Arabic numbers and also calculations with these new numbers.

For merchants it was important to learn the new arithmetical method. It is not clear if Peter van Halle wrote his manuscripts for merchants. He didn't mention his target group, but other authors did. They wrote in their prologue that they were writing for merchants, moneychangers, bookkeepers, bankers, carpenters, bricklayers, gold- and silversmiths etcetera. Other authors wrote that they were an arithmetic teacher at a French schools. At these private schools for future technical, administrative and financial practitioners pupils learn arithmetic, bookkeeping and French. French was the most important business language at that time. These schools were called 'French schools' although the other subjects were taught in the vernacular. It seems that the 'French' arithmetic teachers used their own book to teach arithmetic. And indeed if you turn over the leaves of the arithmetic books you can find many financial and technical arithmetical problems.

The arithmetic of the arithmetic books

Before a pupil can solve these practical problems he first has to learn the arithmetical operations. And before he can learn the operations, he must know how to read and write the Hindu-Arabic numbers. All authors starts with this fundamental topic. Peter van Halle gives in his arithmetic book a long explanation of the new numbers. He compares them with the Roman numerals, like you can see in figure 4.

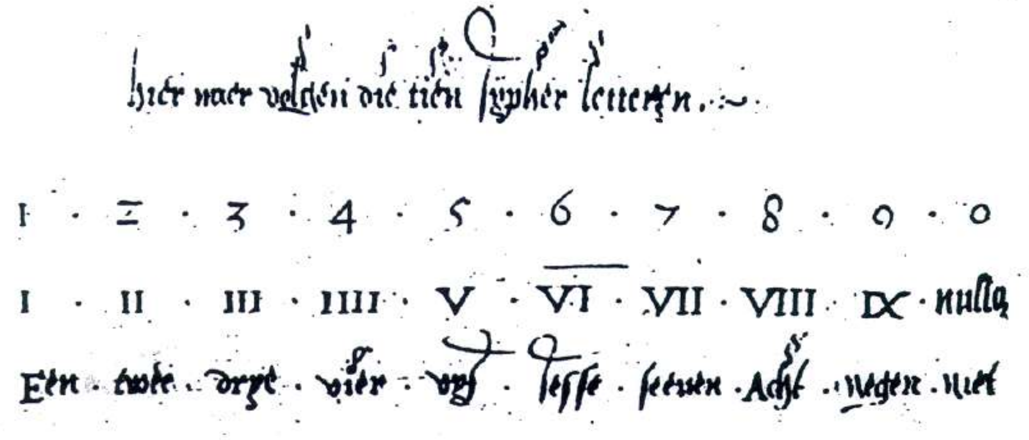

Figure 4: Roman and Hindu-Arabic numerals from the arithmetic of Peter van Halle.

He continues with the following arithmetical operations: addition, subtraction, multiplication and division. Some authors also dealt with mediation and duplication, but Peter van Halle explained that mediation is the same as dividing by 2 and duplication is the same as multiplying with 2.

The biggest part of the arithmetic books is filled with practical problems and different rules to solve them. The secret of a good sixteenth century arithmetic teacher is: to practice, to repeat and to order to learn by heart. The pages are filled with very many rules and problems.

The most important rule is the rule of three. This rule is used to find the fourth number in proportion to three given numbers. Peter van Halle underlined the importance of this rule by presenting it in a beautifully decorated frame, see figure 5. The text says: 'The rule of three, how you can find the fourth number out of three numbers'.

Figure 5: The introduction of the rule of three in the arithmetic manuscript of Peter van Halle, 1568.

With the rule of three you can solve all kinds of problems about buying, selling or exchanging goods, about partnership, changing money, calculating of interest, about insurance, profits, losses, making of alloys, etcetera. If you want to solve a problem with the rule of three, you have to place the three given numbers on a line, multiply the last two numbers together and divide by the first number. Peter van Halle gives many examples, for instance the following one:

'If 9 seamstresses can make 15 shirts within one day. How many shirts 6 seamstresses can make?'

He finds (15 x 6) : 9 = 10 shirts. See also figure 6

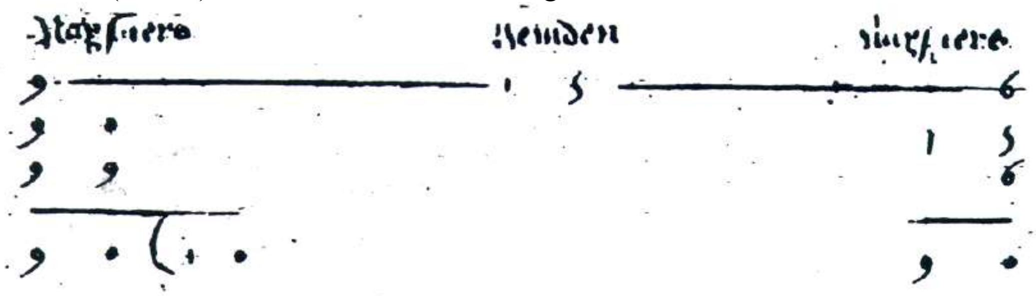

Figure 6: If 9 seamstresses can make 15 shirts in one day. How many shirts 6 seamstresses can make?

This problem is quite simple, but if you go on in the book you will find problems that become more complicated. The situations described in these problems give you an impression of the perils and pleasures of the life of a merchant. Look for instance at the following problem.

Fol. 97r: 'Three merchants are at sea and suddenly they experience a terrible storm. They have to throw overboard a part of their cargo. The value of this part is 100 guilders. They arrive safely at home and then they have to divide the loss. The first person had for 300 guilders cargo in the ship, the second person had for 400 guilders cargo in the ship and the third one had for 500 guilders cargo in the ship. The total cargo had a value of 1200 guilders, 100 guilders of this amount disappeared overboard.' Peter van Halle made three calculations to find the loss for each single person, as you can see in figure 7.

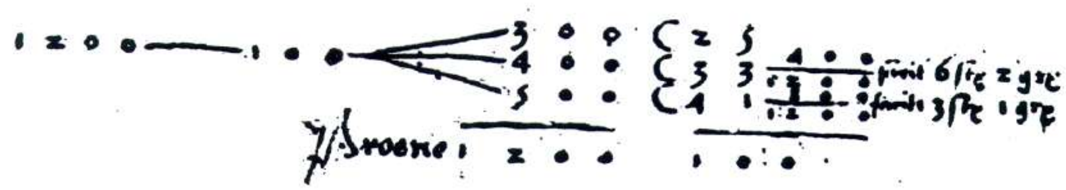

Figure 7: Solution of the problem of the ship in the storm from Peter van Halle.

Merchants had to calculate profit and loss. There are many problems about changing money, partnership and also a few about bartering. The problems really give you an impression of the life of the merchant. Other problems are typically for carpenters, woodworkers, and bricklayers. How many stones do you need to pave a cellar? Or how many tiles will fit on a roof of certain measurements? You can find comparable problems in many other Dutch arithmetic books, and some questions contain features that are typically for Mechelen. I don't mean the many problems about buying and selling linen. Mechelen was famous because of the linen industry, but this was the same in some other Flemish cities. Typically for

Mechelen is the famous Rombouts cathedral, with his 97 meters high tower. They start the building in 1225 and many generations had worked on it. In the arithmetic book of Peter van Halle you can find the following problem:

'If 8 bricklayers can build the Romboutstower within 4½ years, how many years will need 12 bricklayers to do the same job?'

In figure 8 you can see how Peter solved this problem. He used the so-called inverse rule of three. That is to say, he multiplied the first two numbers and divided the product by the last one. The result is 6 half years. In the waving sentence you can read: 'Within 3 years the Romboutstower will be built.'

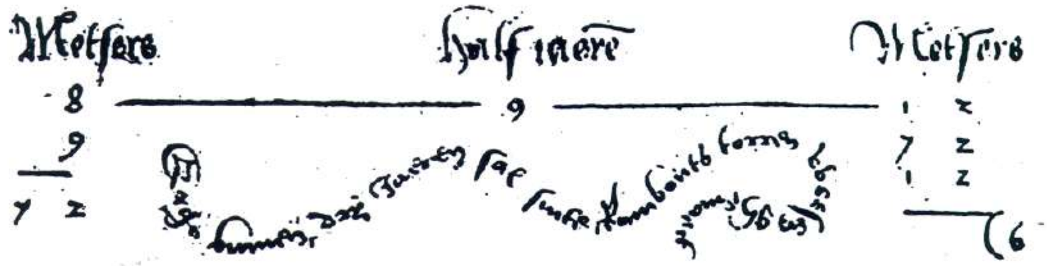

Figure 8: Solution of the problem of the Romboutstower of Mechelen.

Recreational problems

If you want to become a merchant or a technical practitioner and you work through the book of Peter Van Halle I am sure you will learn a lot of useful things. You can practice with hundreds of practical problems, repeat the rule of three and its many variations as much as you want, you will certainly reach your aim.

But among these practical problems you sometimes will find problems that are not realistic at all. These problems contain funny stories that have nothing to do with money or merchandize. Look for instance at the following problem: 'A man can empty a barrel of beer within 20 days. If his wife is drinking with him, they will empty the barrel within 14 days. How many time will need the woman to empty the barrel on her own?' (fol. 146r)

You can find more funny problems like this one. What is the function of this kind of problems in a serious arithmetic book? Peter van Halle doesn't tell us. His colleague Christianus van Varenbraken wrote in his arithmetic book of 1532 above these type of problems: 'Questie uut genouchten', which means: 'Problems for pleasure'. And another colleague, Martin van den Dijcke, had collected all these curious problems in a special chapter at the end of his book. He introduces this collection in this way: 'Here you will find many different beautiful problems to sharpen and enjoy your mind.' It is clear that these problems have an educational and a recreational function. Even in the sixteenth century classroom pupils couldn't be serious always.

A good example of this type of problem is the ring problem:
In a group of people there is a person who has a ring on one of his fingers, around a certain phalanx of that finger. The leader of the game doesn't know where the ring is, but he asks the group to do some calculations and finally when he sees their results he

knows where the ring is. The calculations are: 'Double the number of the person that has the ring. (Each person has a number). Add 5 to this. Multiply the result with 5. Add the number of the finger. Place the number of the phalanx behind the result and subtract 250.' If for instance the third person has the ring on his second finger around his first phalanx, the final result of the calculations will be: 321.

What can we learn from our sixteenth century colleagues?

I believe that the ring problem brought pleasure in the sixteenth century classroom and it can do the same in our modern classroom. You can play the game and you can add extra questions to it. For instance you can ask your pupils what will happen if the ring is on the tenth finger. You can ask them why the trick works and if it will work with all numbers. And perhaps you can even challenge them to create their own hidden ring trick. In doing this you can turn this exciting historical game into a rich and valuable maths problem.

This example shows you one of the ways in which you can use fragments of historical arithmetic books in your own modern maths lesson. You can give your pupils an intriguing problem from the past and they will work on it with their modern mathematical techniques and ideas. I gave several suitable examples of problems in this article that you can use in this way. In the first place the historical arithmetic books can be a rich source of problems to motivate pupils to do mathematics.

But there is more. The problem we start with, about the two pots with the lid, can play a different part. Of course this is also an intriguing problem, that can stimulate children to work on using their own maths. But afterwards it can become very interesting and valuable to show your pupils the historical solution, to let pupils compare their own solution with the rule of false position. They will see that maths wasn't always the same. You can challenge them to think about advantages and perhaps disadvantages of their own method and they will get more insight in their own method. At the same time they will discover that there are different ways of doing mathematics and solving problems and they definitely will realize that mathematics wasn't always the same. Mathematics is a subject that is changing and growing and each generation can add new things to it. If historical problems can teach this to your pupils I think you have reached a historical mile-stone.

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Introduction of Western Sciences in China, Japan and Korea: A Comparative Comment

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Widely recognized, I believe, is the fact that modern West did not develop until the 15th century, and also recognized is that until then the East=West gap in the science and mathematics were at the best negligible, if not there was a favorable tip toward the East. But the Western science was making galloping strides ever since, and finally the strides had made the West far advanced than the East in science-mathematics after several centuries of history ever since.

And the crucial East=West intercourses had started with the coming of the Western missionaries to the East in the 16th century. The intellectual comforts within the basically Confucian tradition were prevailing over the Eastern countries in the period, which were to be gradually threatened with the coming of the West into the East Asia. Of course the first country where the Westerners had arrived was China. After the half-successful adventures by some Western travelers like Marco Polo, China had been known to the Westerners as a place of hope, if not of gold.

1. The Chinese Origins of Modern Sciences: China

The Christian missionaries had begun to reach the Chinese shores from the early 16th century. But their initial ventures were not necessarily successful in quick Christianizing of the Middle Kingdom. Instead their door-kocking had continued throughout the whole 16th century along the several outposts of the southern seashores of China, until Matteo Ricci (1552-1610) had finally succeeded to settle himself down in the nation's capital - Beijing - in 1601.

It is rather well-known thereafter that he had written many books on Christianity and on science-technology and mathematics. The representative of his writings were *T'ien-chu shi-yi* for Christianity, and *Ji-ho yuan-ben* for geometry. With the cooperation from a high official of Chinese government, Hsu Kuang-chi, Ricci could successfully propagate Christianity and some of the fresh knowledge about the world as well as Western sciences. For instance he prepared a world-map and printed it several times with several occasions of revising. So that it had even been introduced very early in 1604 into Korea through the efforts of one Korean ambassador who had visited China that year.

The most distinguished Jesuit missionaries in this early period were Adam Schall von Bell (1591-1666) and Ferdinand Verbiest (1623-1688), as well as Ricci himself. Thanks to the proven superiority of the Western astronomy and calendar-making, those missionary teachers started to teach their method of calculating and predicting the heavenly phenomena, with the time-honoured Office of Astronomy of the Chinese government under their direct command ever since. Particularly notable was the illustrious achievement of Adam Schall at the turnover of the dynasties, from Ming to Qing in 1644. Under his directorship with the assistance of many other missionaries, the new calendar system was formally adopted by the new dynasty, as they were well prepared with the Western astronomical collection, *Xi-yang sin-fa li-xu*.

Only very slowly and gradually, if persistently, this new method of Western astronomy was accepted and adopted in Korea, through many efforts for more than 70 years after the initiation of the new system in China. According to the Korean records of the period, Koreans had successfully begun to use the new method of calendar-calculations from 1708. Still they were not sure about some parts of the new astronomy, thus they had to send more astronomers and mathematicians to China, or sometimes even put formal enquiries to the Office of Astronomy of China. But the Chinese Office and astronomers were reluctant to comply with the Korean enquiries, for the astronomical matters were strictly prohibited to the Koreans for the part of the Chinese. Traditionally the Chinese thought that the Koreans were not allowed to directly conduct astronomy or to separately make calendars. The Chinese thought that the Koreans, as a vassal state under Chinese emperor, were not mandated by Heaven, and for that reason not qualified to do anything about Heaven.

There were a flurry of books published in China, usually written or translated by the Western missionaries. They include, for instance, Sabbatinus de Ursis' book about the Western technology with emphasis on agricultural hydrology. Simple irrigation machines, thus introduced into China through this book, were also a very important source of the new knowledge in Korea. Giulio Aleni had published a book about the contemporary world geography, and this was very well accepted by the Korean literati.

With the successful accomplishment of Ignatius Koegler's massive New Collection of Astronomy in 1742, the more modern knowledge of the Western astronomy, including Kepler's newly discovered laws of orbital movements of the planets, were being introduced into China. Naturally the level of the Western science in China was much higher than the level of it in Japan in the 18th century. Some of the forerunning Chinese scholars were readily imbued with the incoming foreign ideas of science--particularly astronomy and mathematics. Two notable figures in this group were Mei Wen-ting (1633-1721) and Juan Yuan (1764-1849). Mei was a distinguished astronomer-mathematician for himself, whereas Juan was a pioneering historian of science by writing an excellent piece of history of astronomy with his *Biographies of Astronomers and Mathematicians* (1799).

Particularly this *Biographies of Astronomers and Mathematicians* was, as I believe, the best piece of history of science until the end of the eighteenth century. Although this book was supplemented by one of his disciples in the beginning period of the next century, the original by Juan Yuan is composed of 46 volumes to cover the biographies of 243 Chinese (vol. 1-42) and 37 Westerners (vol. 43-46). But here I am not bringing this information to you to emphasize the great achievement of this historian of science, but to show the typically Sinocentric attitude toward the Western science in this early period of the East=West contacts.

In his writing about Matteo Ricci, he ridicules Hsu Kuang-chi's adulation of Ricci as "the Hsi and Ho of the modern days." Juan had very highly evaluated the traditional Chinese astronomy and mathematics, as much higher than that of the West, and continued that "they simply do not know the greatness of the past Chinese astronomy, for they had only absorbed the modern Western knowledge."

From Mei to Juan, and thereafter for a longer period of time, the Chinese scientists and mathematicians had often concluded that all the advanced portions from the Western science and mathematics were originally of the Chinese production in the ancient period. The prevailing attitude of the Chinese scholars in the times was a hypothesis of the Chinese origins of all the Western sciences and mathematics. And this attitude failed to shift to the other side until the end of the nineteenth century.

According to Alexander Wylie's Memorials of Protestant Missionaries to the Chinese, Giving a List of their Publications (1867), the protestant missionaries in China had published 777 kinds of books for more than a half century from 1811 to 1867. Majority of the books were about religion, and some were books about history, geography, languages, education, politics, mass communications, but science-technology were represented with a considerable amount of them--occupying 104 of 777 kinds. And some of them were even republished in Japan.

All in all, China had established an apparent higher plateau, in comparison with Japan and Korea, in the collection of books about the Western sciences and mathematics by the mid-nineteenth century. But there was a serious difference in the collection of Western knowledge in China and Japan, to which we shall turn our attention promptly.

2. Dutch Learning for Modernization: Japan

It was the very next year in 1868, after the compilation of the list of the Missionaries publications by Wylie in 1867, that the epoch-making socio-political reforms were formally started with the Meiji Restoration in Japan. As it is pointed out previously, the Japanese were introducing the Chinese publications of the Western science and mathematics until the mid-19th century. But the borrowing of the Western books from China had suddenly become obsolete for the Japanese, because they themselves had started to translate them directly from the latter half of the nineteenth century.

The first encounter of the Japanese with the Western missionaries was only slightly later than the Chinese in the early 15th century. It is widely recognized in Japanese history that the first meeting of the Westerners had occurred in 1543, when the first Portuguese navigators had landed at an off-island of Kagoshima--the southwestern harbor in Kyushu. For the first time in Japanese history, they had brought to Japan the first Western products--including Western musket and alarm clocks among others. Then came the arrival of the famous Jesuit missionary Francisco Xavier in 1549. It was from this period that the Western influence had really started to infiltrate Japan--Western things from 1543, and Christianity from 1549.

All seems to have been smooth in the earlier years until 1587, when the then strongman Toyotomi Hideyoshi ruled the banning of Christianity in the land, then to the degree of executing 26 Christians in Nagasaki. The Japanese attitude toward Christianity were not necessarily consistent throughout the early period of the Western impacts on Japan. But the next Shogun Tokugawa Ieyasu was not necessarily friendly to Christians, when he announced the prohibition of the Western religion in 1612 and it was followed by the banning of the Christian books in the 1630's. Then in 1640 the sixty-one Christians including many Portuguese, were put to death, and the persecution of the Christians had continued with the execution of 603 Christians in 1658.

It seems that Japan at the turn of the seventeenth century was a dark age for the Westerners and Christianity. But it was not necessarily so dark after all. Before the oppression of Christianity from the period of Toyotomi, who became the actual ruler of the nation from 1585, the Western religion and science had many achievements to hold on as the bases for the future growth. For example, Portuguese missionary Luis Frois was allowed by then actual ruler Oda Nobunaga to live in Kyoto, the capital, in 1569. Then three feudal lords of the Western Japan had even dispatched a special embassy to Vatican, and the Papal emissary Valignano had visited Japan in 1590, and he had a meeting with then ruler Toyotomi in 1591. In 1601 a Christian church was built in Edo, present Tokyo. And a northern lord of Sendai, Date Masamune, had sent his lieutenant Hasekura Tsunenaga to Vatican in September 1613, and he had returned from his European visits in 1620.

When Portuguese trade ships were prohibited to visit Japan any more in 1639, the isolation of Japan seems to be complete. But the fact was that a small but sure window was safely reserved with the moving of the Dutch trade center to Dejima off Nagasaki harbour in 1641. After their initial trade relationship with Japan in 1609, the Dutch became the final victors in the commercial intercourse with Japan. And it was from that early period that Japan had gradually and firmly established their intellectual connection with the West through the Dutch.

Though Western influence was begun to be felt from the early years of Portuguese arrival in the mid-sixteenth century, it was definitely enhanced with the Dutch connection in Nagasaki. There began to witness young Japanese who volunteered to learn the language of Dutch and the trend seemed to have been officially endorsed with the Shogun's order to two of his retainers to learn Dutch in 1740. Under the direct order of Shogun Yoshimune, Aoki Konyo and Noro Genjo had started to learn Dutch, and they had eventually emerged in Japanese history as earlier pioneers of so-called Dutch Learning<rangaku>.

The climax of the development of Dutch Learning in Japan was the translation of a Western book on science-technology by Japanese translaters in 1774 - the translation of a Dutch book on anatomy into <Kaitai shinsho> by Sugita Gempaku and others. The original of this translation was a Dutch book, Ontleedekundige Tafelen, published in 1734, and it was originally a German title Anatomische Tabellen(1732). Significance here is the fact that the science books of the West began to be translated directly by the Japanese and published forty years later than the original in Europe. No similar thing could have occurred in China, not to speak of Korea, in the same period.

With similar time-lag, the Japanese were diligently translating and transforming the Western sciences into Japanese with their knowledge of Dutch. Outlines of Newtonian physics were put into a Japanese book in 1784, and similar books were produced by Japanese Dutch Learning scholars in the modern chemistry, biology, and astronomy throughout the turn of the nineteenth century. An interesting additional information in this connection might be the case of a versatile scholar, Hiraga Gennai (1729-79), who based upon his diligent learning from the Western sciences through Dutch Learning tradition, had reinvented asbestos (1764), a thermometer (1768), and an electric generator(1776) in Japan. One of his electric generator seems to have been smuggled into Seoul by 1830's according to the contemporary records in Korea.

A great assistance had come with the arrival of a German medical doctor, Philip Franz von Siebold (1796-1866) at the Dutch trade center in Nagasaki in 1823. As the resident doctor for the center, he had utilized his well-trained knowledge in the modern medicine and sciences to teach a handful young and intelligent Japanese in biology and medical science, and tried to collect as much materials of East Asia and do his own researches about them. Under his teaching, some able Japanese scientists were trained, and he had left a collection of books about East Asia, among them a book on Korea is included. Incidentally he had never visited Korea in his life, but he did write the book based upon his first hand experiences with some Koreans brought to Nagasaki after the ship-wreck. One surprising thing he did in his efforts to study about Korea is that he became the first scholar of a sort about the Korean language and Korean aklphabets (han-gul).

Throughout the early half of the nineteenth century, many schools for Dutch Learning had emerged in many parts of Japan. And according to one survey, a Western learning school of Osugi Gentaku had trained 94 Dutch learners between 1826 and 1838. One famous Dutch learner in Japanese history is Fukusawa Yukichi, who is widely recognized as the foremost leader in the early Enlightenment Movement about the Meiji Restoration period. In the mid-nineteenth century, the original Dutch learner had realized that English was a more important language than Dutch, and he had switched to learn it and became one of the early English speakers in Meiji Japan. Though we cannot say of him as a scientist by any means, he had even written a book about the Western science in the very year of Meiji Restoration, namely in 1868.

Historians tend to stress only the post-Maiji modernization efforts of the Japanese into limelight. But I would emphasize that all the basics of the modern sciences were already prepared well and sound in Japan in the pre-Meiji Japan through the long period of Dutch learning. For instance, the formal initiation of the modern university system in Meiji Japan was the opening of Tokyo University in 1877. But that university was a development of the Shogun's institute of translating foreign books before it.

Another episode relevant to Korean history deserves our attention in describing the historical development of Tokyo University in its relation with modern science. Edward Morse (1838-1925), an American biologist, had successfully established his name in Japanese history as one of the best "employed"(yatoi) scientists in the Meiji Japan. And his name is recorded in Korean history with his relation to a famous Korean in the period, Yu Giljun (1856-1914), the first Korean student studied in a Western country. Because of some tenuous relationship with some Koreans in Japan, Morse had offered Yu his house as the Korean's place for living for the ten months from the fall of 1883, while he was attending a high school in America.

3. Late Starters via China and Japan: Korea

As shown above, the Western missionaries were the actual teachers of the Western civilization from the mid-16th century in China and Japan, where they had arrived in not too small numbers from then on. But none of the Western missionaries had visited Korea for three hundred years after their first appearances in China and Japan. In Korea some Catholic fathers from France had come to do their evangelical activities in 1835 for the first time, only to be caught and be beheaded immediately the very next year.

For one reason or another, the Westerners did not voluntarily come to Korean peninsula until the early nineteenth century. Then how about the Koreans' contacts with the Westerners, or with the things from the West, during those centuries before their arrivals? There were only two ways of Western connections for the Koreans from the early 16th century to the early nineteenth - one through the occasional opportunities of Koreans meeting the Western missionaries in Beijing at their visits to China, and the ship-wrecked Westerners haphazardly arriving at the Korean shores.

From the latter cases of contacts, Koreans seem to have gained almost nothing, largely because there were very few such occasions, and the Westerners thus arrived at Korea were mostly illiterate sailors, who could hardly give any intellectual awakenings to the Koreans. In Korean historiography only a few cases of them are recorded. One most noted case is a Dutch sailor Hendrik Hamel's record of Korea in the seventeenth century, which he had written after his escape from the land in 1666 after thirteen years' captivity in Korea from 1653. The only merit of the Hamel's adventures in Korea is that he left a very valuable record of Korea during his stay there.

It was natural therefore that the Koreans moderately continuous contacts with the Westerners were possible only through the Korean embassies to Beijing once a year. Each year Korean embassy made of hundreds of Koreans were visiting China to do their tributary duties, as much as for actual trade purposes. But the embassy had much more than these, as their goals, in their visits. And naturally many young scholars were eager to have the opportunities of visiting China at one of these occasions. And only those audacious or intellectually high-motivated among them were courageous enough to visit the Western missionaries stationed in Chinese capital, for their contacts with the foreigners were never encouraged by the Korean court. For instance, Hong Tae-yong (1731-1783) was visiting China as a follower of his uncle in the Korean embassy of 1765. Toward the end of the year they had arrived in Beijing, and left there in the spring of the next year. In the meantime Hong had volunteered to visit Hallerstein and Gogeisl, German missionaries in charge of Chinese office of astronomy at the time, four times at the South Church in Beijing. Though he could not meet them at one of the visits, he had actually met them three times, and had quite long dialogues in writing with them. And we can read part of his conversations from the remaining writings of Hong today. Considering of the magnitude of Western science and mathematics already translated into Chinese until that time, the contents of Hong's conversations with the Westerners are very small and pretty elementary. Until then, Koreans did not fully import the Chinese renditions of the Western science and mathematics introduced into China. It has not been studied fully yet, but we do know that only a fraction of the translated Western materials were introduced into Korea through the efforts of the annual Korean embassies to China. With a meager understanding of the materials from the Western missionaries in China, Hong Tae-yong maintained that the earth rotates once every day, and not the other way around.

Many books about the Western science and mathematics - including the *Elements* of Euclid translated by Matteo Ricci in 1607 - were imported into Korea. But the number of them were rather small, so small that an encyclopedic writings of Yi Ik (1681-1763) seem to show only less than one dozen titles in his writings. Nevertheless this same Yi Ik was confessing that the Western exact science were so superior to the traditional learning of the East. Then he had even said that Confucius, if he were born again, would follow the Western astronomy instead of the Eastern method. So only at the turn of the nineteenth century, some Koreans - like Pak Chega (1750-1805) and Chong Yagyong (1762-1836) expressed their hope of learning the Western science and mathematics through the direct invitation of Westerners into Korea or through the establishment of such an institution for learning of Western things in Korean government.

Ch'oe Han-gi (1803-1877) shows the climax of the Korean-Western intellectual relationship in the pre-opening Korea. He wrote many books about the Western science and mathematics between 1836 and 1867 - all in the forms of digested rewritings from the books published in China in the earlier period. In 1857 he had presented a shortened version of Wei Yuan's book on the world situations, and in 1866 Ch'oe had published another book about medicine, based upon the various books by Banjamin Hobson in China. And his book in 1867--Songgi unhwa-- is again the abridged edition of the Chinese translation of John Herschel's survey of astronomy, whose Chinese edition was published in China in 1857 under the title of Tan-t'ien, translated by Li Shan-lan and Alexander Wylie.

Korea opened its door to the foreigners in 1876, and Ch'oe can be considered as the last scholar who did great contributions to introduce the Western knowledge into Korea before the opening. But as we can tell easily from the contents of his books, which were selected and abridged from the Chinese books, his achievements were very much piecemeal and ineffective in building a modern edifice of science and mathematics in Korea.

It was from the turn of the century that Korea had for the first time become a land for transformation to a modern society. But a brief period of Koreans' own efforts for the transformation was abruptly halted with the colonization of the land by the Japanese. And for the whole period of colonial Korea, from 1910 to 1945, the Imperial government of Japan had only allowed the least amount of science-mathematics education for the Korean people. During the 35-year period of Japanese rule in Korea, only two hundred odd Koreans had received formal college degrees from Japanese colleges in the field of physical sciences and engineering, according to my recent research. It is a shocking contrast against the tens of thousand Japanese who had received the same degrees in the same period of time in Japan. And the Japanese colonial authorities had seldom allowed Korean students to go to Western countries during the period, thus making the Korean college graduates from the West somewhat smaller than those trained in Japan for the same period. Therefore, Koreans' direct and real encounter with the Western civilization had to be postponed until after the Liberation of Korea in 1945, although the first meeting occurred during the latter half of the nineteenth century.

4. From A Mathematical Pioneer to A Patriotic Activist: Yi Sangsol

One additional episode in the history of Korea's absorption of the Western sciences in the post-opening years will serve well for a further illustration of the Korean situation vis-a-vis China and Japan. Yi Sangsol(1870-1917) had started his career as a mathematician in the earlier period of his life, but had to round up as a patriotic activist later. The gloomy socio-political background of the times was to be blamed for the loss of one mathematician at the turn of the century in Korea.

He passed government service examination in 1894, at the age of 24, to be appointed to a high position in Korean royal government. But it was the time for this young Korean to be immensely interested in the modern scholarship inflowing from the outside at the time. And he decided to study science and mathematics, as well as some foreign languages like English and French. As such, he had once become a teacher of Seoul Normal School in 1896 - the first modern high school established by the Korean government at the time.

It was around the end of the nineteenth century that he was asked by the government officials in Education Ministry, to translate a Japanese mathematics text. That book--Ueno Kiyoshi's Modern Mathematics-- was translated with some modifications into Korean and published in 1900 as one of the first modern text about modern mathematics in Korea. He might have been trained himself as the foremost pioneer in Korean history in introducing modern mathematics into Korea.

But his career was to be deflected abruptly and completely at the fall of his fatherland. In 1905 when the Japanese declared Korea as a protectorate of Japan, Yi decided to dedicate himself for the independence of Korea. He had exiled himself in northeastern part of China just across Korean border in 1906. There he started a Korean school, and even there he was still teaching mathematics to his students, obviously using his text book prepared in 1900. But the tumultuous political and nationalistic flurries of the period did not allow him to stay cool to teach mathematics to young Koreans within Korea, or out in China. In the very next year we find him representing the ousted Korean emperor in his declaration to deny the protectorateship of his kingdom at an international peace conference held in the Hagues, the Netherlands. With two other Koreans, he was picked up by the ousted emperor to become the delegates to the meeting, with little to avail.

After the abortive statement delivered to the conference by three Korean representatives in the summer of 1907, Yi continued his anti-Japanese activities in exile. He stayed in America a while, but eventually moved to Russia, just across the Korean border, to stay there until his death in 1917. Today his name is very famous in history books in Korea, as one of the major figures in the independence movement of Korea, but hardly anywhere with the introduction of modern mathematics into the land. In short, Korea was very late in its first encounter with the Western science and mathematics, and the late starters were hampered further in their building of the modern science and mathematics, with the bitter experiences of colonial domination under the Japanese for about half a century.

So Korean science and mathematics in its modern sense had been barely started with the end of Korean War in 1953, with only a handful of scientists and mathematicians, most of them trained largely to the level of college graduation in Japan. Therefore really Korean science and mathematics in its meaningful scope had to be started again with those modern scientists and mathematicians trained and returned to Korea after their studies abroad in the late fifties and the sixties. The real history of Korean science and mathematics can hardly trace back to the nineteenth century, nor to the first half of the twentieth century. It was started only from the late 1950's.

Becoming a mathematician in East and West: some cross-cultural considerations.

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Introduction

It is both a pleasure and an honour to have this opportunity to talk to such a large gathering of mathematics educators and historians and philosophers of mathematics from both East and West.

Today I shall be addressing myself primarily to the educators, in the hope of convincing them that the historians and maybe even the philosophers can have something useful to say to them from time to time. This will involve me, in part, in saying again some things that my historian colleagues have already heard me say more than once, for which I apologise to them. Some of them will recognise my debt to Geoffrey Lloyd in Cambridge so far as Greek matters go, and to Nathan Sivin in Philadephia for some of my statements about China. But of course the blame for any mistakes or confusions is mine alone.

I shall be arguing that if we do not want to mislead ourselves as to what mathematics is about, we need to think about the ways in which mathematics has developed. Since our conceptions of the nature of mathematics affect the ways we set out to teach it, such a historical investigation may turn out to have very practical consequences. Let me say straight away that my remarks on the teaching of mathematics in the Western tradition are not meant as an attempt to give a fair picture of the best modern practice. They are based however on my own personal recollections of being taught mathematics in Britain early in the second half of the twentieth century, and later of seeing the way my students were taught mathematics when I was a physics educator myself rather later than that. No doubt everything is much more enlightened in twenty-first century Taiwan.

But what kind of historical investigation will be most useful to us in reflecting on the nature of the mathematics we may set out to teach? If history is to help at all in the task of seeing our world with clearer eyes, it seems to me essential that it should be comparative history. Once upon a time, different cultures were free to imagine their own histories without the inconvenient distractions of the Other. So it was possible for western Christians to calculate on the basis of the Biblical chronology, as did Bishop James Ussher (1581-1656), that the world was created in 4004 BC, and to assume that all human beings alive were descended from the group who survived the flood in Noah's Ark. Meanwhile at the other end of Eurasia, Ussher's Chinese contemporaries were equally convinced that the origins of human culture could be traced back to Fu Xi and the Yellow Emperor, and that the Flood had been dealt with quite efficiently by Yu without any necessity for taking to the boats. And as for the origin of everything in 4004 BC, an eight century Buddhist astronomer like Yixing had no problems in imagining a cosmos that had been running for nearly 100,000,000 years. When Jesuit missionaries began to arrive in China from about 1600, both the Chinese and the Western sides of the subsequent dialogue found such divergent accounts of the past somewhat disconcerting.

Of course as the centuries have gone by, the study of history in East and West has been immeasurably enriched by the access gained by each side to the other's narratives. Ignoring other people's histories is no longer an option except for the purblind and conservative. As a result, many of the features of our worlds that once seemed to be natural and inevitable have been revealed as historically contingent: they didn't have to be that way, and we can see they didn't have to be, because things turned out so differently elsewhere. That applies to the history of mathematics just as much as to the history of religion, art or politics. In mathematical terms, it turns out that Greece and China provide rather good counterfactual alternatives to one another. I want to suggest that by contemplating these alternatives on a basis of equality, we may liberate ourselves to find new ways of thinking about mathematics, and perhaps new ways of teaching it.

Proof and problematics

Let us start at the Greek end of things. It is well known that ancient Greek texts discussing mathematics frequently contain words such as epideixis, apodeixis, and deiknumi, all with the root meanings 'show' or 'point out', which are used in ways which justify them being translated into modern English as 'proof' or 'prove'. Please take note, if you will, of the way I put that. It seems to me essential in this kind of discussion that we should not just assume at the outset that there exists, outside language, history and culture, any given object of discourse. We can see Greeks constructing sentences with the word "epideixis" in them; and we can see English speakers constructing sentences containing the word "proof". All we can do is to try to map one usage onto another, and try to see whether they overlap to a considerable extent. If they do, then it may not be too misleading to translate one word by another. Fortunately in this case the mapping works quite well, and so it seems safe to start talking comparatively, in English, about the role of proof in ancient Greek thought. But what would we do if there wasn't a word in the Greek mathematical vocabulary that mapped even roughly onto "proof"? We shall have to consider such a problem later on.

To proceed with our discussion, we may note that the general methodological problem of what was to count as a successful proof was often discussed in ancient Greece, most notably in the writing of Aristotle in the 4th century BC. Some scholars would argue that the way Euclid constructed his geometrical proofs in the *Elements of Geometry* around 300 BC was an obvious attempt to put into practice Aristotle's ideal of what a proof should be. It is certainly true that for many later writers in the West Euclid's work was taken as the model to be followed in constructing an argument, and attempts to follow his example were made in many fields other than mathematics. And certainly, for Greeks writers about mathematics it was proving things that mattered most. Typical is the remark of the commentator Proclus in the fifth century AD, discussing Euclid's version of what is known in the west as Pythagoras' theorem:

For my part, while I admire those who first became acquainted with the truth of this theorem, I marvel more at the writer of the *Elements*, ... because he established it by a most lucid demonstration. (Proclus On Euclid, ed. Friedlein 426, 6-14, quoted in Greek Mathematical Works I, p. 185, tr. Ivor Thomas, Loeb Classical Library, Harvard 1998)

Of course the gap of time between the discovery and the proof of this theorem is actually much greater than Proclus though. For him it was a matter of the interval between Pythagoras around 520 BC and Euclid two centuries later. But in fact we have the clearest textual evidence that the theorem was being taught as part of the standard scribal curriculum in Babylon a thousand years before Pythagoras. It seems to me to that we ought to pause for a moment to ask what kind of strange set of priorities is involved here: we have a well-known and very useful fact about geometry, tried and tested in practice for many centuries and never found in error. And yet Proclus is excited because someone has proved that it is always true. To a Babylonian scribe that would have seemed about as useful as proving that water makes things wet. What is more, Proclus thinks that the proof is more important than the original discovery of the fact. There is a rather bizarre mindset here that requires explanation. And yet rather than asking why on earth a small number of Greek thinkers got hung up on this kind of thing, most of the world takes it for granted that this is the way it should be.

Admiration for Euclid continued to influence the Western conception of mathematics for the succeeding one and a half millennia after Proclus. To use the language of Thomas Kuhn, to whom we shall return in a moment, for centuries until comparatively recently the paradigmatic achievement of Western mathematics as represented in schools and colleges remained the *Elements of Geometry*. For many thousands of young Europeans, mathematics beyond the elementary level of doing sums was a matter of following Euclid through his elaborate structure of proofs and corollaries, all the way from his Postulates, Common Notions and Definitions and so on through the propositions of as many Books as could be mastered before one's wit or courage failed. Although my own school no longer used Euclid's actual text, our main introduction to grown-up mathematics certainly came in the form of a careful training in the construction of geometrical proofs, and the style of geometrical argument flavoured much of the rest of our mathematical training. Proper mathematics was about proof, and the ability to solve problems was expected to be a by-product of a training in the construction of proofs. It is clear that the popular image of mathematics today is still influenced by this stereotype: witness the widespread public attention given a few years back to the discovery of the proof of Fermat's theorem, despite the well-known fact that no mathematician had any practical doubt that it was true. The notion that proof is the thing that matters most in mathematics has worked its way deep into the modern mind.

Now please note that I have nothing to say against the elegant construction of Euclid's exposition (even though his proof of Pythagoras seems quite unnecessarily complex), nor do I want to seem ungrateful to my teachers. I really enjoyed being taught to prove things (it makes one feel so nice and secure), and I don't think it does young people any harm to be made to analyse formal arguments with great care. Mathematicians in particular should not stop doing this. However, I am worried by the fact that historians who received the same kind of mathematical training as myself often seem to see real mathematics as mathematics that proves things, with other aspects of mathematics coming a rather poor second. Anyone who looks into correspondence section of the latest issue of the science history periodical *ISIS* will see the kind of problems that can lead to. Two scholars called respectively Chen and Cullen are fighting over whether a certain ancient Chinese text contains a proof of Pythagoras' theorem - or as we had better call it in such a context, the gougu relation. It seems to matter a good deal to the first to be able to show that there is a proof in there somewhere, and he seems to feel that if he does not succeed Chinese mathematics will be thereby lowered in the scale. My point is that while Proclus may have thought so, there is not reason why we have to agree with him today. More to the point in the present company, thinking about mathematics as having the notion of explicit proof at its essential core may lead us to teach in non-optimal ways. Let me explain in more detail why that worries me.

Kuhn's on problems

I shall do so by sharing with you some acute observations made by Thomas Kuhn. His book *The Structure of Scientific Revolutions* sparked a great debate when it was first published in 1962. He was accused, amongst many other things, of reducing the triumphant progress of scientific discovery to a mere succession of changing fashions, each with its own "paradigm" or pattern of how science was to be done, each more or less incommensurable with the others, and none really more objective or truer than the others. So for example there was no way that one could sensibly say that Galileo's theory of motion was better than Aristotle's, since they were quite different ball-games rather than less or more accurate answers to the same question. Kuhn denied many of these interpretations of his work quite vehemently, and in the second edition of his book, published in 1969, he added a Postscript in which he set out to clarify his views. It is worth noting in the present gathering that this Postscript was added at the suggestion of his student and friend Nakayama Shigeru, now one of Japan's senior historians of science.

At one point in this postscript, Kuhn is setting out to explain one of the many senses in which he had used the word "paradigm" in his original writing. One of the most important meanings he wants to illustrate is that of the paradigm as a "shared example" - that is, as one of the examples of problem-solutions that all students of a given science encounter during their training. For Kuhn, problem-solving is crucially important, as the scientific activity which characterises the periods of what he calls "normal science", which take place in the often long intervals between revolutionary paradigm-shifts. Most scientists spend their careers solving problems in the context of normal science rather than overturning the foundations of their speciality. But, as Kuhn tells us, the student's encounter with problems has not normally drawn the attention of those who want to tell us what science is really about, despite its importance as a mode of learning and practising science - and of course, as a mode of learning and practising mathematics too. As he says:

> Philosophers of science have not ordinarily discussed the problems encountered by a student in laboratories or in science texts, for these are thought to supply only practice in the application of what the student already knows. He cannot, it is said, solve problems at all unless he has first learned the theory and some rules for applying it. Scientific knowledge is embedded in theory and rules; problems are supplied to gain a facility in their application. I have tried to argue, however, that this localisation of the cognitive content of science is wrong. After the student has done many problems, he may gain only added facility by solving more. But at the start and for some time after, doing problems is learning consequential things about nature. In the absence of such exemplars, the laws and theories he has previously learned would have little empirical content. (Kuhn, *The Structure of Scientific Revolutions* (second edition) Chicago 1970, 187-8)

With a little translation, these perceptive remarks about science could be made to apply equally well to mathematics. But a few paragraphs later we come even closer to the real meat of the experience of trying to learn something new. Students, Kuhn tells us

> .... regularly report that they have read through a chapter of their text, understood it perfectly, but nonetheless had difficulties solving a number of the problems at the chapter's end. Ordinarily, also, those difficulties dissolve in the same way. The student discovers, with or without the assistance of his instructor, a way to see his problem as like a problem he has already encountered. Having seen the resemblance, grasped the analogy between two or more distinct problems, he can interrelate symbols and attach them to nature it in ways that have proved effective before. [...] The resultant ability to see a variety of situations as like each other [...] is I think the main thing a student acquires by doing exemplary problems, whether with a pencil and paper or in a well designed a laboratory. After he has completed a certain number, which may very widely from one individual to the next, he views the situations that confront him as a scientist in the same gestalt as other members of his specialists' group. For him they are no longer the same situations he had encountered when his training began. He has meanwhile assimilated a time tested and group licensed way of seeing. (Kuhn 1970, p.189)

I think that these observations will be familiar to all of us, whether as teachers or learners. But to speak from my own experience at least, the reality of what Kuhn pointed out has not historically been recognised in the patterns of teaching mathematics in Western cultures. Certainly the best students have quickly learned to recognise the patterns in the problems before them, and they have learned to generalise those patterns in ways that have enabled them to solve problems more and more marginally related to those they have seen solved before. That is what gets you high marks in mathematics exams, and enables you to go on to become a professional mathematician if you want to. But I can remember all too acutely being left to get on with that kind of thing for myself. It was not what my mathematics teachers felt their subject was really about. For them, mathematics was at its best a matter of the development and exposition - in the Euclidean manner - of a beautifully articulated logical structure of propositions, with each part linked to the rest by the most solid and elegant chains of deduction that could be contrived. Typically a lesson consisted of the careful explanation and proof of some proposition in geometry, algebra or calculus, which one was expected to copy into one's notebook, and learn for later reproduction in a test. If you had attended properly to what the teacher was saying, you would understand it all clearly, and would therefore be able to do all the problems at the end of the relevant chapter which were set for your homework assignment. If you couldn't do the problems, you were either unintelligent or had not been listening properly in class.

A sad consequence of this was that those students who could not make the leap into solving problems for themselves, and who did not particularly enjoy the architecture of mathematical argument for its own sake were the first to drop out of doing mathematics. The effects of this spilled over into other subjects, when as an educator later on I often found students reluctant to take physics to more advanced levels because they had been thoroughly alienated by the formal and proof-centred structures of mathematics teaching, and were convinced that they just did not have what it took to use mathematics as a scientific tool. But where in the world might one find a different tradition of doing mathematics, that might be less likely to alienate students such as these? The answer lies in China - of course.

The alternative paradigm - Chinese mathematics

In Greece, as I have already claimed, mathematical discourse ended up by privileging proof. The paradigmatic work of the Greek tradition, Euclid's Elements, set out to use the minimum number of primary propositions as a foundation for proving, if not all possible true propositions at least as many as were known at the time of writing and could be fitted into the author's scheme. Uninformed descriptions of Chinese mathematics have sometimes painted it is a kind of poor relation of Greek mathematics, containing many true facts and some useful methods, but alas all set out with purely practical ends in view, and quite without the essential element of logical demonstration and structure. In fact, just the sort of approach to mathematics condemned by Plato:

> They speak, I gather, in an exceedingly ridiculous and poverty-stricken way. For they fashion all their arguments as though they were engaged in business, and had some practical end in view, speaking of squaring and producing and adding, and so on, whereas in reality, I fancy, the study is pursued wholly for the sake of knowledge ... (Plato, *Republic*, viii, 525A-530D)

But when a large body of intelligent people in some other period and culture seem to be devoting much of their time and energy to doing something rather badly and confusedly, the lesson of history is that they are often engaged in doing rather well and quite effectively something entirely unsuspected by the historian at first glance.

To find out what ancient Chinese mathematicians were really up to, a number of strategies are available to us. One rather obvious one is to listen to them when they are talking about mathematics as an activity, and that is the strategy I propose to adopt here. Of course they did their talking in Chinese, unsurprisingly enough, so the things that are important to them have Chinese labels. Before we start committing ourselves to English language descriptions of their activities, we therefore need to look very carefully at how their explicit use of Chinese words might be mapped onto corresponding English patterns of usage. I have already hinted at one negative result that follows from this: they are certainly not talking explicitly about anything that seems to correspond to the English word "proof". But what do they talk about explicitly? Let us look at some examples.

The mathematics of the *Zhou bi*（周髀）

From the Tang onwards a proud position as the first of the Ten Mathematical Classics was held by the book titled Zhou bi(周髀)"The Gnomon of the Zhou Dynasty", which dates in its final form from around the first century AD , during the Eastern Han dynasty. If however one was writing an old-fashioned history of Chinese mathematics based on the chronology of 'achievements', it would have to be said that the *Zhou bi* does not add a great deal to the credit balance of Han mathematics. The main text of the book assumes that its readers can add, subtract, multiply, divide and find square roots of quite large numbers. It applies simple proportion in some straightforward geometrical situations, and uses Pythagoras' s theorem on just four occasions. And that is that. There is no mathematical principle used in the *Zhou bi* that is not also well exemplified in another much more systematic Han mathematical treatise, the *Jiu zhang suan shu* （九章算術，Nine Chapters on the Mathematical Art）, which is the second of the Ten Classics.

On the other hand, unlike the *Nine Chapters* the *Zhou bi* contains what may be called 'second-order' mathematical material in which a user and teacher of mathematics reflects on his activity. This occurs at the beginning of the second section of the *Zhou bi*, which I have called the 'Book of Chen Zi'. A certain Rong Fang（榮方） opens the dialogue by asking for answers to a number of questions about the dimensions of the cosmos and the movements of celestial bodies. Chen Z（陳子） assures him that mathematics can settle all such problems, but sends him away to find the answers for himself. It is worth listening to their dialogue in detail, not only because it is revealing as to how Chen Zi views the process of learning mathematics, but also because it gives us a great deal of insight into the student teacher relationship that is part of this process. The pattern, it becomes clear, is effectively student : master :: suppliant : benefactor. But as we listen, we shall also hear repeated a Chinese term which seems to be as key to Chen Zi's way of thinking about mathematics as words like epideixis were to a Greek mathematician:

> #B1 [23k] Long ago, Rong Fang asked Chen Zi 'Master, I have recently heard something about your Way. Is it really true that your Way is able to comprehend the height and size of the sun, the [area] illuminated by its radiance, the amount of its daily motion, the figures for its greatest and least distances, the extent of human vision, the limits of the four poles, the lodges into which the stars are ordered, and the length and breadth of heaven and earth?'
>
> #B2 [24e] 'It is true' said Chen Zi.

#B3 [24e] Rong Fang asked 'Although I am not intelligent, Master, I would like you to favour me with an explanation. Can someone like me be taught this Way?'

#B4 [24g] Chen Zi replied 'Yes. All these things can be attained to by mathematics. Your ability in mathematics is sufficient to understand such matters if you sincerely give reiterated thought to them.'

#B5 [24j] At this Rong Fang returned home to think, but after several days he had been unable to understand, and going back to see Chen Zi he asked 'I have thought about it without being able to understand. May I venture to enquire further?'

#B6 [24k] Chen Zi replied 'You thought about it, but not to [the point of] maturity. This means you have not been able to grasp the method of surveying distances and rising to the heights, and so in mathematics you are unable to extend categories (tong lei （通類）- note the first appearance of this important term). This is a case of limited knowledge and insufficient spirit. Now amongst the methods [which are included in] the Way, it is those which are concisely worded but of broad application which are the most illuminating of the categories of understanding. If one asks about one category, and applies [this knowledge] to a myriad affairs, one is said to know the Way. Now what you are studying is mathematical methods, and this requires the use of your understanding. Nevertheless you are in difficulty, which shows that your understanding of the categories is [no more than] elementary. What makes it difficult to understand the methods of the Way is that when one has studied them, one [has to] worry about lack of breadth. Having attained breadth, one [has to] worry about lack of practice. Having attained practice, one [has to] worry about lack of ability to understand. [Note by the way the complete reversal here of the usual sequence of "first theory then problems" criticised by Kuhn] Therefore one studies similar methods in comparison with each other, and one examines similar affairs in comparison with each other. This is what makes the difference between stupid and intelligent scholars, between the worthy and the unworthy. Therefore, it is the ability to distinguish categories in order to unite categories (neng lei yi he lei 能通類以合類) which is the substance of how the worthy one's scholarly patrimony is pure, and of how he applies himself to the practice of understanding. When one studies the same patrimony but cannot enter into the spirit of it, this indicates that the unworthy one lacks wisdom and is unable to apply himself to practice of the patrimony. So if you cannot apply yourself to the practice of mathematics, why should I confuse you with the Way? You must just think the matter out again.' (Zhou bi, section B in Cullen, *Astronomy and Mathematics in Ancient China*, Cambridge 1996)

What we have here is a concise statement of a twofold heuristic strategy, summed up in the words 'distinguish categories in order to unite categories' (lei yi he lei 類以合類). On the one hand the mathematician performs the analytic task of distinguishing different problem types from each other. On the other hand the very act of analysis brings together groups of similar problems which may be treated synthetically. Further, one can then attempt to 'unite categories' at a higher level by finding common structures underlying different problem categories.

Chen Zi's analytic/synthetic approach is in fact not particularly well exemplified in the *Zhou bi* itself. It is however clearly (or so it seems to me) the main rationale of the *Nine Chapters* mentioned earlier. Whereas Euclid was concerned to show how a great number of true propositions could be deduced from a small number of axioms, the anonymous author of the *Nine Chapters* followed a different but no less rational route in the reverse direction. He started from the almost infinite variety of possible problems, and aimed to show that those known to him could all be reduced to nine basic categories solvable by nine basic methods. To a great extent he succeeded, although the contents of some sections still show a degree of diversity. It did not strike him as worthwhile to try to argue explicitly that his methods would always work for the appropriate problem type. In the first place, he already knew they did work on what the examples are before us to this day. Secondly, if it ever turned out that the method failed on a new problem, that would not have been taken as a sign that the method was wrong, but rather that it was necessary to distinguish a new problem category with a new common method for all problems of the new type. "Lei yi he lei" 類以合類, in fact, as Chen Zi says.

In the preface to his brilliant commentary on the *Nine Chapters* written towards the end of the third century AD, Liu Hui（劉徽）makes some of the same points as those I have tried to bring out from the words of Chen Zi:

> When I was young I learned the Nine Chapters and when I grew up I went over them again carefully. I looked into the breaking apart of Yin and Yang, took a comprehensive view of the basis of mathematical methods, and of the suppositions involved in seeking the unknown, and thus attained to realisation of [the work's] meaning. Therefore I have ventured to exert my meagre capacities to the utmost, and to select from what I have seen [in other books?] in order to make a commentary. The categories under which the matters [treated herein fall] extend each other [when compared], so that each benefits [from the comparison]. So even though the branches are separate they come from the same root, and one may know that they each show a separate tip [of the same tree]（事類相推，各有攸歸，故枝條雖分而同本幹知，發其一端而已）. Further, my reason for analysing the underlying principles with verbal explanations (xi li yi ci 析理以辭), and dissecting forms using diagrams, is to enable the reader to get through without confusion, so that merely looking over what I have written will enable your thoughts to go more than half-way [to their goal]. (*Jiu zhang suan shu*, preface, p.177 in the edition of Guo Shuchun（郭書春），Shenyang 1990)

So whereas Euclid aims to go froms a few axioms to many theorems, Liu Hui is telling us that the aim of the Nine Chapters is to go from many problems to a few methods. It seems to me that these two approaches to the practice of mathematics, ancient Chinese and ancient Greek are very different, but at the same time complementary to one another.

Concluding thoughts

Why was it that Greek writings on mathematics are so full of discussions of proofs, and Chinese writings prefer to speak of how one can learn to solve problems? The answer may well be found by looking at the different social processes which generated the two different literatures we are discussing here.

In ancient Greece, most mathematical writing takes place in the context of a wider debate amongst people who start on equal terms. Anyone who claims a right to speak can be challenged to justify everything he says, just as every speaker in the law-courts and political assemblies of ancient Athens could be challenged to prove his point. In this radical democratic atmosphere, no person or organisation can claim authority to lay down the law without having to prove their point against all comers. Even when Socrates was on trial for his life for allegedly attacking the ancient religious ideology of the city in which he lived, the trial took the form of a public debate in front of a randomly selected jury of citizens, none of whom was an official of the state. A citizen was only answerable to his equals. What is more, as Geoffrey Lloyd has pointed out, if you wanted to make a living out of being an intellectual in ancient Greece, you could not expect to get a government job as you might have done in China. You had to win enough of a reputation as a teacher to attract fee-paying students. Winning students, and keeping them, was something you did in open competition with your rivals. To win such competitions, it helped to be original, but it also helped to be able to prove that your rivals were confused and mistaken, and that your own views were absolutely clear and uncontrovertible. It is in the context of this style of debate that Euclid writes, and it is therefore understandable that the issue of justifying what he says is the one he finds most important.

In ancient China, however, the social and political background was completely different. The idea of radical democracy on the Athenian model seems to be quite absent, even in the thinking of the Mohists. The peasant Levellers attacked by Mencius around 320 BC still thought that there ought to be a king, even if he ought to plough the fields together with his people. Political persuasion was a matter of persuading those in power to adopt your advice. What counts as knowledge is not to be argued about openly in the marketplace, but will be revealed to you by your teacher. It seems clear that the same applies to the way that mathematical knowledge was structured and passed on in the example we have been discussing. Chen Zi is speaking to Rong Fang as a teacher to a student who has acknowledged his authority and requested instruction. Rong Fang has already stated his belief in the correctness of what Chen Zi teaches before the process of instruction starts. There can clearly be no question here of the student demanding that the teacher should prove that what he says is correct, and so we it is not surprising that we find no word corresponding to "proof" in Chen Zi's teaching. Now that does not mean that someone like Liu Hui has no interest in showing how his mathematical methods work, and in so doing he may end up doing something like what a Greek might call epideixis. But the teacher's task is to help the student to learn to do things, and that is just what Chen Zi does. His way of doing so is to pass on the underlying method of problem-solving based on the use of the concept of "category" (lei 類), and that is the feature of mathematics that principally and explicitly concerns him. Not just mathematical style, but the substance of mathematical discourse seems to be intimately linked with differences in social circumstances.

But what has all this got to do with the teaching of mathematics today? Something at least, I hope. I have tried to suggest in the short time available here that the proof-centred style of mathematics that for so long was seen in the West as the highest type of mathematical discourse is dependent for its origins on some very special historical circumstances. In another very significant part of the world, people in different socio-political circumstances structured their mathematical expositions differently, but were none the worse for that. There seems no reason at all for saying that Euclid was doing mathematics properly, while Liu Hui was not. Living as we do today in a variety of worlds, none of which resembles ancient Athens any more than it resembles ancient China (and thank goodness for both of those facts), we have in the light of history the wonderful privilege of choosing our own mathematical style to suit ourselves., and to suit our students. So, if our students thrive on proofs and corollaries, they can have them and welcome. But if we find it more expedient, we may equally well teach our students their mathematics in ways that Plato would never have approved of, with the stress laid on the development of effective and reliable problem-solving skills rather than on the inter-relations of theorems. Either way, history liberates and empowers us to do as we choose.

The Use of Technology in Teaching Mathematics with History
- Teaching with modern technology inspired by the history of mathematics -

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Abstract

One of the current issues in mathematics education is curriculum reform incorporating technology. This issue seems far removed from the teaching of mathematics with history, but there are several movements to link history in mathematics teaching with technology. In the first part of the paper, the reasons why we use technology in the teaching of mathematics with history are discussed; in the second part of the paper, some of the movements that aim to link history in mathematics teaching with technology are outlined.

Why we should use technology in teaching mathematics with history

We cannot really taste East Asia dishes without using chopsticks

The teacher who knows little of the history of Mathematics is apt to teach techniques in isolation, unrelated to the problems and ideas which generated them (Ministry of Education in England, 1958, from John Fauvel 1991). Eating without cultural perception is essentially consuming 'fast food'; it is accompanied by a certain attitude towards eating, Similarly, mathematics teaching without cultural perception brings students a certain attitude. (Hans Niels Jahnke, 1994). In order to teach mathematics as a human activity, the history of mathematics involves a model of this activity called mathematization (H. Freudenthal 1973). Research and issues strongly support the humanizing of mathematics education inspired by the history of mathematics. On the other hand, in modern reform movements are observed other humanizing movements supported by modern technology and seem no relation to the humanization inspired by the history of mathematics. However we should say that the later observation is not appropriate[1].

Indeed, there are several researches that focus on using traditional or modern technology for teaching mathematics with history (eg. Jan van Maanen 1991, Maria Bartolini Bussi., David Dennis and Jere Confrey, 1997, Ryosuke Nagaoka and Jan van Maanen et al. 2000) in the last decade of the 20th century. Their opinions need to be heeded today and over the next ten years because we have entered the age of the media revolution and in the near future, everyone will be making much greater use of all kinds of technology in mathematics.

An example to demonstrate the need to use technology for the studying and learning of mathematics with history was given by Descartes (1637). In his Geometry, after showing that geometric construction problems, such as obtaining a segment, could be solved by an algebraic method, Descartes discussed the power of his methods used for curves, about which ancients such as Pappus did not employ. In his discussion, we find that his methods of problem posing were mediated by tools as follows. First, he applied his methods to ask about the curve made by

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[1] Japanese new high school mathematics curriculum from 2003 will enclose mathematics history for humanizing mathematics education in the Basic Mathematics as well as using technology

a linkage mechanics using a triangle (Figure 1). After showing that it is a hyperbola by use of algebraic representation, he changed the condition as follows: 'If CNK be a circle having its center at L, we shall describe the first conchoid of the ancients (figure 2) while if we use a parabola having KB as axis we shall describe the curve which is the first and simplest of the curves required in the problem of Pappus (Figure 5).' Through changing the conditions (or parts of mechanics), he finds other curves using the linkages.

If we were unable to use tools, it would be impossible to picture the curves which Descartes described in this context. Fortunately. we and our students can represent such curves easily using Dynamic Geometry Software (DGS) as in figures 2 to 5. These constructions are made by the same procedure, the only difference being in the means (the selection of parts for mechanics). Prior to figure 1, Descartes wrote as follows: 'If we say that they (ancients) call them mechanical (curves) because some sort of instrument has to be used to describe them, then we must, to be consistent, reject circles and straight lines, since these cannot be described on paper without the use of compasses and a ruler, which may also be termed instruments.' We can say that Descartes had the metaphor of mechanics and he reasoned with algebra to try to overcome the ancients' convention of using a compass and a ruler; this led to the design of his Universal Mathematics (1628).

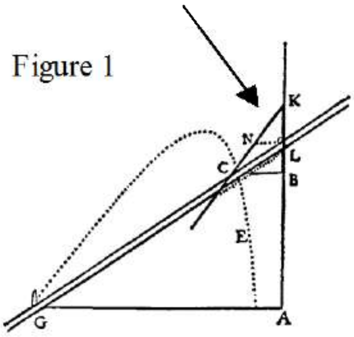

Figure 1

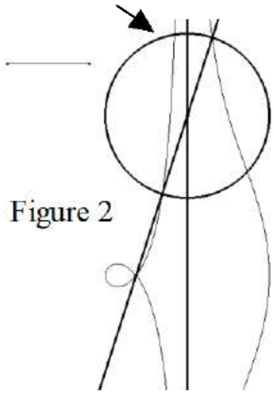
Figure 2

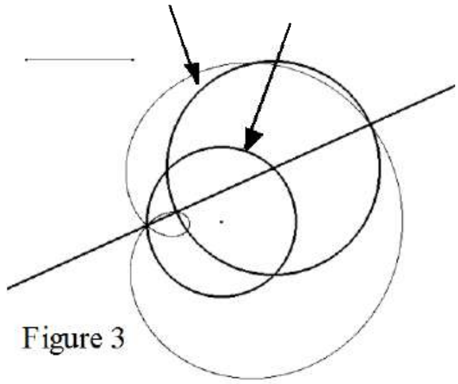
Figure 3

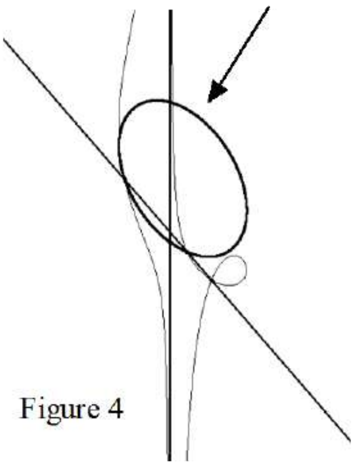
Figure 4

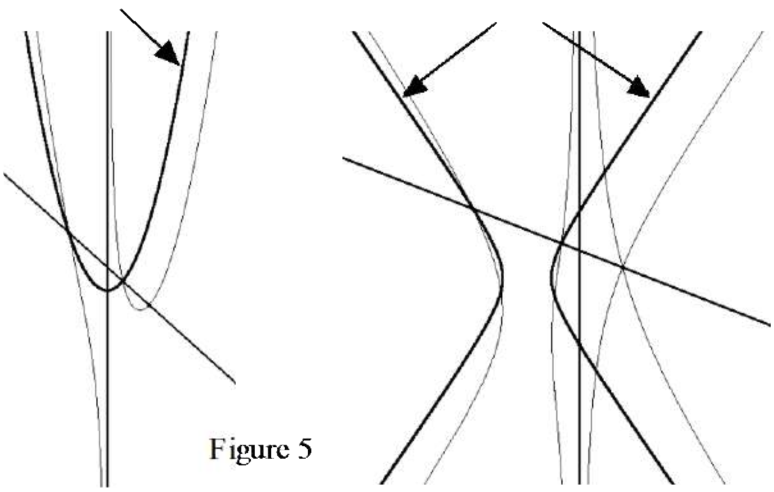
Figure 5

In today's classroom, a graph is defined by an equation (function); teachers and students often find it unusual to see equations leading to curves such as the conchoid (Figure 2) and the limaçon (Figure 3). If students read the texts of the ancients or Descartes with the aid of drawing tools, they can deduce the origins of such curves themselves and represent them algebraically using geometrical observation and reasoning. Such activity is important for geometric feedback as opposed to the customary use of algebraic defintion (Dennis, 1995).

The importance of history using tools lies not only in the power of the tools to enable students to gain feedback but also in providing a cultural perspective such as restriction by mediational means[2] (James Wertsch, 1991) and confrontation originated from the selection of mediational means. Indeed, Descartes was aware of the restriction by the ancients to the use of rulers and compasses, and he lamented the loss of the geometric intuition that they possessed. The necessity of feedback for students itself implies that students' cognition is restricted by algebraic representation, also a tool but a psychological one in the Vygotskian perspective[3], which was developed by Descartes for designing his Universal Mathematics. This shows that the forces that shape mediational means introduce unintended effects into mediated action (Wertsch, 1991). The example of the confrontation can be seen at well known commentary by Pascal used against Descartes in Panse:'79. Descartes - We must say summarily: "This is made by figure and motion," for it is true. But to say what these are, and to compose the machine, is ridiculous. For it is useless, uncertain, and painful. And were it true, we do not think all Philosophy is worth one hour of pain.' (used Japanese translation by Youich Maeda 1978). Against algebraization of geometry by Descartes, Pascal try to keep the spirit of geometry. This confrontation is based on the difference of representations (psychological tools) but related with the definition such that ruler and compass are mechanics or not .

Because tools as mediational means are embedded in human history and culture itself, it is necessary to use traditional tools for teaching mathematics as a human activity.

Why use 'modern' technology? - The history of mathematics is difficult! -

Scientists have always written for a close circle of colleagues and experts. So there is no reason to assume that students can readily access history even if the subject matter looks quite elementary in today's school curriculum (Jahnke, 1994). Original manuscripts and texts may be really important to historians, but their work is not easily accessible to laymen. However, what is new is that everyone can access resources via the world wide web (see Brummelen 2000, eg. Hiroshi Kotera http://www.asahi-net.or.jp/~nj7h-ktr/english.html) easily. The mathematical difficulties with historical concepts, for psychological, philosophical or physical reasons, can be alleviated by virtual tools offered by websites or software (eg. Maria Bartolini Bussi, http://www.museo.unimo.it/theatrum/). Historians know that this kind of experience is just

[2] In mathematics education research, one preferred theory to explain teaching-leaning using technology in the classroom is the Vygotskian or socio-historical-cultural perspective (James Wertsch, 1991). There are other important keywords in the research (eg. Keitel & Ruthven 1991) for using technology such as environment (Colette Laborde), micro-world (Celia Hoyles) and milieu (Nicolas Balacheff). The problem of using tools is closely related to the problem of representation because any tool can be used to represent an idea (David Dennis and Jere Confrey 1997). However, as no social theory of mathematics can be developed without extensive exploration of the interaction between mathematics and technology (Michael Otte), we could recognize that the Vygotskian perspective is useful in explaining the use of technology in mathematics teaching with history.

[3] Vygotsky explains that human higher mental (inter-mental and intra-mental) functioning are mediated by technical tools and psychological tools. Vygotsky noted 'a technical tool alters the process of a natural adaptation by determining the form of labor operations' (1981 in English version, 1987 in Japanese version) and Wertsch described how the forces that shape mediational means such as technical tools typically introduce unintended effects into mediated action.

‘virtual’. On the other hand, only teachers, who know what their students can do, can plan student activity in their classrooms and organize any equipment that should be used. Teachers and students can read original or translated text and behave as historians. Historians knows the importance of activity and the risk of hermeneutic. Thus teachers should read historians works but in many cases the works of historians themselves are not easy for laymen. In Japanese translation of Descartes’s Geometry by Koukich Hara, there are several footnotes written by him using algebraic expressions for readers’ appropriate understanding. Behind of his footnotes, his works are hidden and there was no DGS in his ages. For laymen, his footnotes are usefull but not easy to understand because to imagine curves from algebraic expression is not easy for them and they try to know it in short time without his background works. Thus, if there were no DGS, Descartes’s images mediated by mechanics would not re-emerge even in teachers’ mind. We should reconsider what is preferable and essential mediational means to hermeneutic of the text in the classroom. We can prefer the physical construction of such mechanical devices as the age of Descartes but the physical construction is too complicated for the classroom activity even if the activity is appropriate[4].

The movement to use ‘modern’ technology in the classroom

The above discussion endorses teaching with modern technology inspired by the history of mathematics. Pedagogically, there are several views about the role of history in mathematics education such as the cultural view of mathematics and the epistemological base of mathematical concepts and teaching (eg. Ernest 1994). And there are several views regarding significant ways of using history in mathematics teaching (eg. Phillip Jones 1969, John Fauvel 1991). Given the focus on teaching mathematics with modern technology inspired by history, the following discussion begins with the features of the current movement to use technology and the nature of its use.

The IT revolution in the mathematics classroom with handheld technology

The use of modern technology is a revolutionary change in the mathematics classroom. Figure 6 shows recent software innovations for general users of mathematics. Using functions or macros, some of these packages can be extended to design special tools. Some were developed for research, but the evolution of the interface has made such software more accessible to general users and enabled it to be integrated into handheld technology so that students can use it anywhere. These days, many mathematical software packages incorporate multiple representation features and can also be used on the world wide web via Java.

Several studies have already indicated the power of multiple representation tools for knowledge construction. Through the use of these tools, we can assist students to translate and interpret concepts through various representations (Lesh, Landau & Hamilton 1983, Isoda 1998a) or innovative representations and help students’ inquiry into mathematical ideas. In the following, special features of using modern technology in today’s classroom are discussed and some examples that demonstrate the use of multiple representation tools in classroom inspired by history are shown.

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[4] LEGO project directed by Masami Isoda is an attempt to change this situation. See http://130.158.186.11/mathedu/ForAll/kikou/lego/lego.html

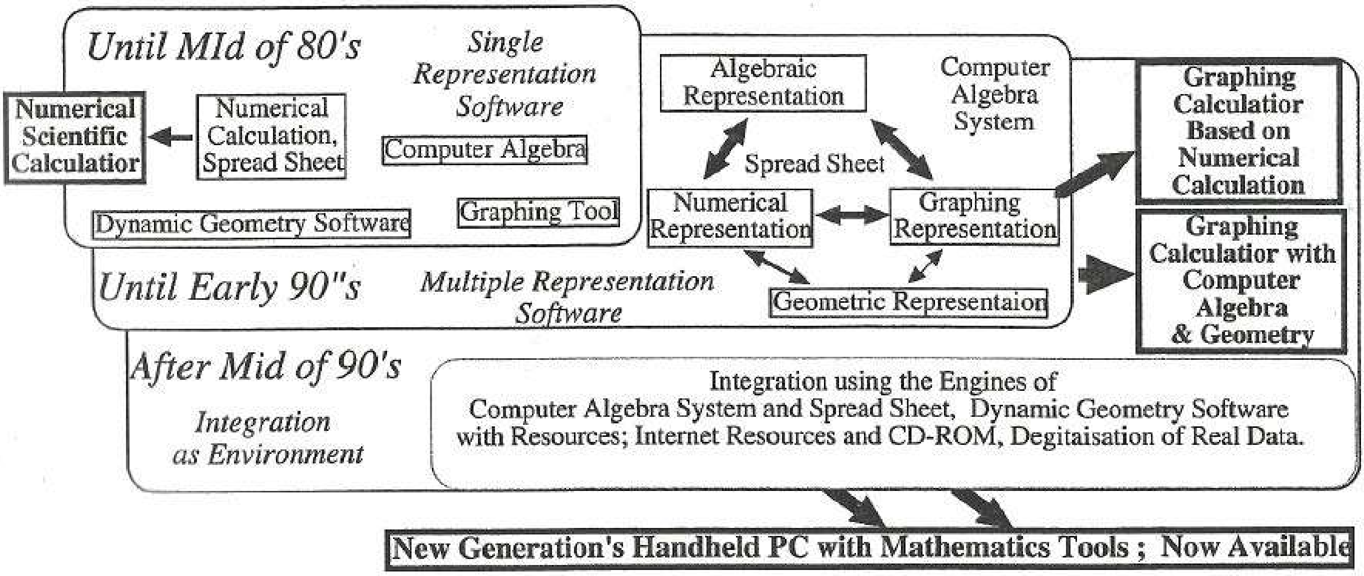

Figure 6. Mathematical Software Innovation and Integration into Handheld Technology

Changing representation as the feature of inquiry using modern technology

The IT revolution in the mathematics classroom has been discussed using keywords such as exploration or inquiry. The laboratory approach becomes a new approach using modern technology (see figure 7, Isoda 1998a). The power of visualization and the manipulation of higher mathematical concepts by modern technology accelerates the use of various representations. Figure 7 explains the inquiry style via the action of changing representation. It is necessary to understand the roles of tools in inquiry and the affective use of tools for mathematics inquiry in the classroom.

As is discussed with the example of Descartes, we can focus on the following roles of tools in inquiry:

a) Determine the subject of the mathematical inquiry.
b) Develop a method for the mathematical inquiry.
c) Reveal the epistemological obstacles inherent in using such tools in the specific context.

We can add the following ideas for using tools in classroom:

1) Change the role of tools depending on the context.
2) Support students' understanding through the changing of tools and representations.
3) Although the generality or viability of mathematical ideas are different depending on the representation, we should give students the opportunity to select, find or create new tools or representations for constructing knowledge.

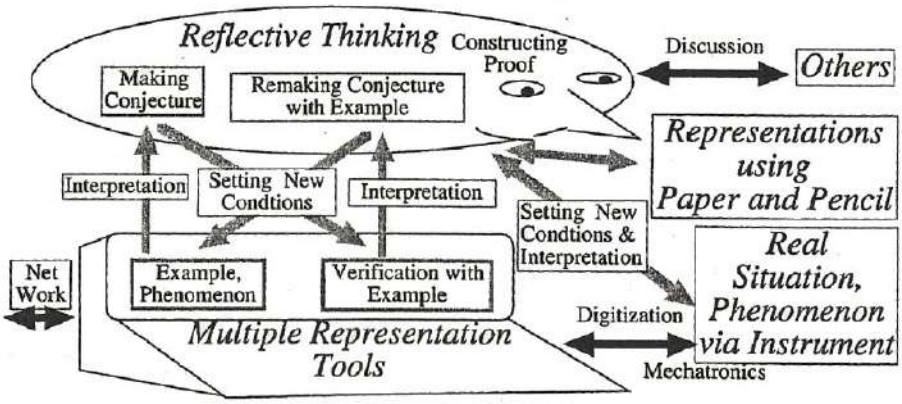

Figure 7. Inquiry based on multiple representation tools

The following are some examples that demonstrate mathematical teaching using modern technology as inspired by history.

The mathematical experience supported by modern tools and activated by its value in history

Jan van Maanen (1991) discussed his classroom teaching activity based on L'Hôpital's Weight Problem (L'Hôpital 1696) with physical instruments: "Let F be a pulley, hanging freely at the end of a rope CF which is fastened at C, and let D be a weight. D is hanging at the end of the rope DFB, which passes behind the pulley F and is suspended at B such that the points C and B are on the ropes that do not have mass; and one asks at what place the weight D or the pulley F will be." (see 44 in figure 8)

Using the problem, L'Hôpital demonstrates the significance of the method of calculus by showing that the result is the same as that obtained by the method of geometry. The problem can be investigated using a concrete model, a Computer Algebra System (CAS) or DGS. Masami Isoda observed undergraduate students' mathematical inquiry: the roles a, b and c and the contexts 1, 2 and 3 were confirmed. Using such tools, the students experienced the visual correspondence between geometric representations of motion (locus) and graphical representations of motion (graph of function) such as figure 8, the emergence of the same equations by differentiation and by geometrical reasoning, the correspondence between data from measurement and the results of mathematics, and so on. These correspondences are not the same as in L'Hôpital's discussion, but students can experience the methodological correspondence between Geometry and Calculus that L'Hôpital wished to highlight.

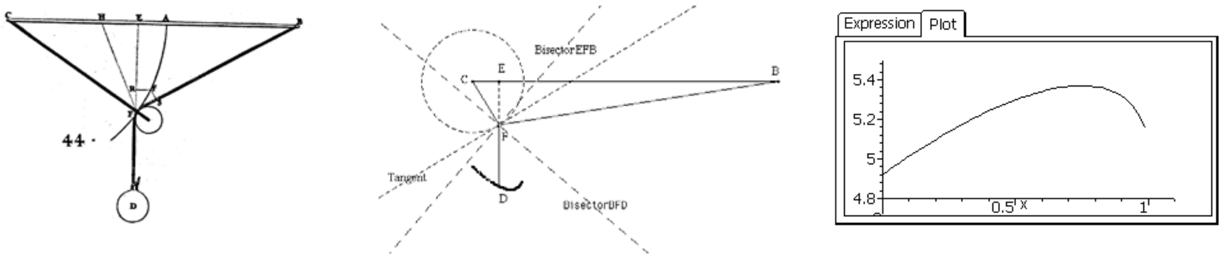

Figure 8. L'Hôpital's Weight Problem

Of course, we can pose the problem without referring to L'Hôpital and students can still comprehend the correspondences between different representations, although they cannot appreciate how he tried to persuade other mathematicians regarding the significance of his new methods using calculus. As this example includes historical value, students can come to appreciate not only the power and beauty of each correspondence but also the mathematician's desire to spread the idea of calculus.

The example shows that simulation by multiple representation software helps us to enhance the understanding of historical problems in their original context. Indeed, in the example, students have to solve cubic equation but Japanese students do not know the formula of it. Thus, if students did not use CAS, they can not appreciate the correspondences. The numerical simulation is also a well known research method used in the history of Japanese Lost Mathematics 'WASAN' when we only know a problem and a numerical result and make a supposition about the solution. If we checked the solution numerically, the supposition is still a conjecture. If the representation is very different from the original one, there is more possibility that the supposition need to reconsider. Thus, knowledge about historically original instruments and modeling them with representation tools (special mediational means) are necessary for following appropriate hermeneutics.

Integrating the approach with tools

There are many research projects that have been designed to examine the integration of mathematics with tools. Some ongoing projects are aimed at curriculum development of mathematics with tools and others at the development of a curriculum which integrates mathematics and history, but each of them adopts history in the classroom and attempts to integrate historical tools and modern tools.

As examples of projects that focus on curriculum development with tools, Jere Confrey and David Dennis (1995, 1997) in the USA, and later Masami Isoda in Japan (1997, 1998a), have designed projects for the integration of geometry, algebra and calculus using drawing instruments, and multiple representation software including DGS. Their physical instruments are made from a set of changeable parts, so new instruments are easy to construct. In order to allow students to construct instruments, Jere Confrey and Masami Isoda (to appear in http://130.158.186.11/mathedu/mathedu/forAll/index.html) began to use LEGO. Their projects used tools for integrating multiple representations and such tools are supported by history.

Two examples of projects that focus on the development of a curriculum which integrates mathematics and history are as follows. The project of Maria G. Bartolini Bussi (http://www.museo.unimo.it/theatrum/) in Italy developed Java tools for her virtual mathematical laboratory and now in her project, many kinds of representation tools, instruments and software are available for the teaching of mathematics and history (see http://www.museo.unimo.it/labmat/). Arzarello's project, also in Italy, originally named EuCart (Euclid & Descartes) is focused on the teaching of proof. The project uses the multiple representation tools of DGS and CAS. In this project, there is a focus on three historical periods represented by Euclid, Descartes and Hilbert, with an introduction of original sources into the classroom, guided by the teacher's introduction. DGS is oriented to developing the semantics of proof whilst CAS is oriented to developing the syntax of proof.

Beyond each tool's restriction

Each instrument looks unique, but it can be made from many kinds of representations such as physical, virtual or mathematical. Thus the mediated action depends on a representation of instrument that includes some specific features implicitly functioning. One of the aims for integrating various tools or representations is to develop the student's competence in selecting and creating appropriate tools or representations. For example, in Algebra, Geometry and Calculus for All project by Isoda (1999), students are asked a problem regarding ellipses with original pictures by van Schooten (Maanen 1992) using various representations. When students used physical pieces of LEGO, they commented on the changing physical resistance when they tried to draw an ellipse. In the case of DGS, they did not. With physical tools, students discussed the difficulty of using them for drawing. With DGS, students could draw some parts of an ellipse, but to draw other parts, they needed additional constructions and this led to some misunderstanding. A student reported that we must first solve equations if we are to represent an ellipse using BASIC. Students began to change parts without teacher intervention because they had experience of changing LEGO parts when they were young. But students did not try to change equation parameters until the teacher suggested it. By using LEGO and DGS, students could find the general equation of an ellipse. When the teacher asked students to make a drawing tool with LEGO for a case with changed parameters, some students first changed parameters of the figure within the DGS and then experienced success. A variety of representations supports students' multiple reasoning and relational understanding.

From this discussion, teaching with modern technology should not be separated from teaching with traditional technology. In the first instance, the necessity of using tools was discussed from the socio-historical-cultural perspective. Modern technology developed to gain some advantages over traditional technology, but such advantages are subject to some restrictions. Thus, we should develop students' ability to select tools depending on what is needed as much as their ability to use each tool in the mathematics classroom. In this context, history of mathematics is didactical means to teach the importance of the selecting tools and appreciate the functions of each tool.

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Theoretical Research Tendency of Mathematics in Pre-Qin Times Viewed from *Suan Shushu*

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1. Introduction

Mathematics in pre-Qin times remains a puzzle to us because no mathematical works have handed down. Most of historians of Mathematics accept Qian Baocong's relevant inferences, describing mathematics in times of the Warring States as showing both practical and theoretical tendency [1], which is undoubtedly correct. It is true to say mathematical concepts of the School of Logicians （名家）and the Mohist School （墨家）show a tendency towards theoretical researches. However, it seems biased to regard the School of Logicians and the Mohist School as the exclusive representative with a theoretical tendency in mathematics. As a matter of fact, we can find the theoretical tendency in mathematical works like *Suanshu shu*（算數書） of pre-Qin times.

As regard to the *Nine Chapters on the Mathematical Art* （九章算術, hereafter, the *Nine Chapters* for brevity）,the most important mathematical classic in ancient China, the author holds

1. that, viewed from the style, internal structure of and prices reflected in the *Nine Chapters*, Liu Hui's inferences on the compiling process of the *Nine Chapters* is perfectly convincing, therefore, the main body of the *Nine Chapters* was finished in pre-Qin times;

2. that, in the main body of the *Nine Chapters*, the compilers adopted a style that questions are dominated by abstract, rigorous and universal rules;

3. that, judging from Liu Hui's annotations and preface to the *Nine Chapters*, methods of demonstration such as the figure-verification, model-verification were invented at the time when the main body of the *Nine Chapters* was compiled;

4.that, the main body of the *Nine Chapters* was compiled according to the *classification*（"類以合類"） principle. [2]

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[1] Qian Baocong. *A History of Chinese Mathematics*. pp. 14—22. Beijing: Science Press, 1964. See also: *The Collected Works on the History of Science of Li Yan and Qian Baocong*, Vol.5, pp. 15 —24. Shengyang: Liaoning Education Press, 1998.

[2] Guo Shuchun. *The Leading Mathematician Liu Hui in Ancient World,* pp.87—106. Jinan: Shandong Science and Technology Press, 1992. See also its revised edition in complex Chinese characters, pp.85—103. Taopei: Mingwen Book Company, 1995. See also: Guo Shuchun: *The Nine Chapters on the Mathematical Art: Translation and Annotation*. pp. 5—24. Shengyang: Liaoning Education Press, 1998.

These reflect one side of theoretical researches of mathematics in pre-Qin times. It is unfair to emphasize the practical side of the *Nine Chapters* but neglect its theoretical contributions. Meanwhile, it is also erroneous to regard the traditional Chinese mathematics as "principally based upon non-logic thinking, i.e., the concepts are formulated and reasoning carried out principally by means of intuition, imagination, analogy, inspiration, etc." [1]

2. The *Suanshu Shu*

The *Suanshu shu* was a batch of mathematical bamboo slips unearthed from Han Tomb M247 in Zhangjiashan, Jiangling county, Hubei province, from December 1983 to January 1984. The reverse side of one of the total 200 slips reads "*Suanshu shu*", which is believed to be the title of the book. Historians dated it before the 2nd year of the Empress Lü's reign (186 B.C.).

The relation of the *Suanshu shu* to the "*Nine Arithmetical Arts*"（九數）and the *Nine Chapters* is an open question. There are more than 60 subtitles in the *Suan-shu shu*, including Rectangular Field (*fangtian*, 方田), Millet and Rice (*sumi*, 粟米), Distribution by Proportion (*cuifen*, 衰分), Short Width (*shaoguang*, 少廣), Construction Consultation (*shanggong*, 商功), Fair Levies (*junshu*, 均輸), Excess and Deficit (*yingbuzu*, 盈不足), which are exactly the names of the "*Nine Arithmetical Arts*" and of the chapters or rules of the *Nine Chapters*. So far two problems in the *Suanshu shu* has been divulged, one of which is as follows:

> Now given one lends 100 coins at a monthly interest of 3 coins. Tell: given one lends 60 coins for 16 days, how much interest?
>
> Answer: 24/25 coin.
>
> Method: Take the product of one month and 100 coins as divisor. Multiply the number of coins lent by the monthly interest of 100 coins, and again by the number of days as dividend. Divide, giving the number of coins. [2]

This problem is not contained in the *Nine Chapters* which has handed down to us, but similar to the last problem in the chapter of Distribution by Proportion, which is more popular in language.

The second problem is as follows:

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[1] Yuan Xiaoming. On the thinking characters of the classic Chinese mathematics. *Studies on the History of Natural Sciences*, 1990, **9**(4).

[2] Chen Junyue & Chen Yanping. The *Suanahu shu* and the *Nine Chapters on the Mathematical Art*. See *Collected papers of the Archaeological Society of Hubei Province* (1), pp.220—222, 1987.

> Short width. The width is $1\frac{1}{2}$ *bu*. Take the unit to be 2, and $\frac{1}{2}$ to be 1. These two being added, 3 is got, which is the divisor. Lay down 240 [square] *bu*, again take the unit to be 2 [and 240 to be 480 as dividend]. Divide, giving 160 *bu*, which is the number of *bu* in length. [1]

Compared with the first problem in the chapter of Short Width in the *Nine Chapters*, this problem lacks the "question", "answer", and the word "method", but its content, data and algorithm are all the same as those of the former, while its language is simpler.

The above comparison shows that either the *Suanshu shu* and the *Nine Chapters* are of the same origin and independent of each other, or the *Suanshu shu* is the prototype or one of the principal sources of the *Nine Chapters*, the latter being more likely.

Any way, the *Suanshu shu* cannot have been compiled as late as the beginning of the Han Dynasty. It must have been finished before the compilation of the main body of the *Nine Chapters*.

Having remained inedited for over 16 years, the entire text of the *Suanshu shu* is still unknown to us today. It is necessary to analyze the problem of "*lending money*" in it. Let's first have a look at the similar problem in the chapter of Distribution by Proportion of the *Nine Chapters* (the answer is not quoted):

> Now given one lends 1000 coins at a monthly interest of 30 coins. Tell: given one lends 750 coins for 9 days, how much interest?
>
> Method: Take one month as 30 days to multiply 1000 coins as divisor. Multiply the interest 30 by the number of coins lent, [and] again by 9 days as dividend. Divide, giving the number of coins. [2]

Here concrete data such as "take one month as 30 days to multiply 1000 coins", "the interest 30" and "9 days" respectively correspond to "the product of one month and 100 coins", "the monthly interest of 100 coins" and "the number of days" in the method of the *Suanshu shu*. Evidently, the method given in the *Suanshu shu* is more abstract, and acts as a universal formula for interest problems; while the one in the *Nine Chapters* is a specific algorithm for the given question.

Another content of the *Suanshu shu* known to us is as follows:

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[1] Li Xueqin. A great discovery about the history of Chinese mathematics. See *The World of Cultural Relics*, 1985, No.1. p. 47.

[2] *The Nine Chapters on the Mathematical Arts*, collated by Guo Shuchun. p. 245. Shengyang: Liaoning Education Press, 1990.

Increasing or decreasing a fraction: to increase a fraction, its numerator must be increased; to decrease a fraction its denominator must be increased. [1]

This is a proposition on the property of a fraction: a fraction increases when its numerator increases and decreases when its denominator increases.

In view of the fact that the Mohist School in times of the Warring States used abstract propositions and definitions on figures, which probably had already existed before and were merely quoted to expound its teachings, we may safely conclude that there must have been propositions and definitions on properties of figures in the *Suanshu shu* and other mathematical works of Pre-Qin times. To discuss the general properties and definitions of numbers and figures, which is the abstract mathematical theory, is an important aspect of mathematical researches in the times of the Warring States.

3. The *Zhoubi Suanjing*

The *Zhoubi suanjing*（周髀算經）, the earliest Chinese classic on mathematical astronomy, whose time is still controversial today, contains Shang Gao's answer to Duke Zhou's questions (11 centuries B. C.) and Chenzi's answer to Rong Fang's questions. The historians Cheng Zhenyi（程貞一）and Xi Zezong（席澤宗）hold that Chenzi lived in the 5^{th} century B. C., [2] i. e., the turn of times of the Chunqiu and the Warring States. While teaching Rong Fang the method of learning mathematics, Chenzi expounded the character of mathematics, ascribing Rong Fang's failure to master this science to his incapability of *perceiving the generality*（通類）.He then proposed that the mathematical "*Daoshu*（道術）is brief in expression but extensive in use, showing a wise *perception of generality*. That *from one kind all others are arrived at*（问一類而以萬事達）is called *knowledge of Dao*（知道）" and "the *classification* is the sage's quality indispensable for him to learn and master knowledge" [3].

The compilation of the main body of the *Nine Chapters* is guided by the ideology identical with that proposed by Chenzi. First, the main body of the *Nine Chapters* was divided into 9 parts---- Rectangular Field (*fangtian*, 方田), Millet and Rice (*sumi*, 粟米), Distribution by Proportion (*chafen*, 差

[1] Chen Junyue,Chen Yanping.The *Suanshu shu* and the *Nine Chapters on the Mathematical Art*. in *Collected papers of the Archaeological Society of Hubei Province* (1), pp.220—222, 1987.
[2] Chen Zhenyi & Xi Zezong. Chenzi's model and the early measurement of the sun. *Studies on the History of Ancient Chinese Science* (the sequel), a research report of the institute of the humanities, Jingdu University.
[3] *Zhoubi suanjing*, pp. 5—6. See *Ten Mathematical Classics* (I), collated by Guo Shuchun & Liu Dun. Shengyang: Liaoning Education Press, 1998.

分), Short Width (*shaoguang*, 少廣), Construction Consultation (*shanggong*, 商功), Fair Levies (*junshu*, 均輸), Excess and Deficit (*Yingbuzu*, 盈不足), Rectangular Array（*fangcheng*, 方程）and Right-angled Triangle（*Pangyao*, 旁要）, according to the nature or use of the rules. That is to say, on the basis of *perception of generality*, through *classification*, the mathematical knowledge at that time falls into nine categories, namely the "Nine Arithmetic Arts". Second, as we have seen, rules given in the main body of the *Nine Chapters*, very succinct, summary and to some extent abstract, but of wide use, are indeed "brief in expression but extensive in use" and show that "*from one kind all others are arrived at*".

It is remarkable that Chenzi, when Rong Fang asked him of the method of measuring the height of the heaven and the distance of the earth, answered that "all these are what arithmetic（*suanshu*, 算術）deals with" [1]. Here "arithmetic" in its broad sense is mathematics, whose essential part is the "*Nine Arithmetic Arts*", or the *Nine Chapters* at that time; in its narrow sense it is probably no other than the *Nine Chapters* at that time, or the main body of the *Nine Chapters* which has handed down to us.

In a word, Chenzi's profound expositions of the character of mathematics and mathematical methods, identical with the compiling practice of the main body of the *Nine Chapters*, reflects an important aspect of the theoretical tendency of mathe-matical researches in times of the Warring States.

4. Conclusion

From above discuss we see the very abstract rules, the derivations and demonstrations which must have actually existed, the expositions of the properties of numbers and figures and the ideology of *classification* in compiling a mathematical wok, etc., make up the major theoretical achievements of mathematics in pre-Qin times, which were the source and foundation of some of thinkers' mathematical propositions. Mathematics in pre-Qin times has the tradition of attaching importance to theoretical researches as well as to practicality. The various schools of thought and their exponents in pre-Qin times argued with each other and inquired into the thinking laws, bringing about an atmosphere of taking abstract thinking seriously in the academic circle. They attached importance to the study of "category"（類）and "classification"（分類）. We are told in *Yijing* that all in the world "falls into its own category" [2], which is exactly the fundamental method of the In-out Complementary Principle. Mo Ti（墨翟） used perception of the class as an important tool to argue with other

[1] *Ibid*.

[2] *Zhouyi*, Book 1. See *Annotations to the Thirteen Classics*. Vol. 1. p.16. Beijing: Zhonghua Book Company, 1980.

schools. “Taking and giving according to the category”（以類取,以類予）[1] was the fundamental logical method of the later Mohists. Scholars’ faculties of abstract thinking and the logical level is considerably high, even much higher than those of the scholars in the Han Dynasty. The thinking modes of Han scholars are principally in images [2]. This is the reason why most of the rules later added by Zhang Cang and Geng Shoucang in the *Nine Chapters* are concrete solutions to concrete problems, the level of abstraction being much lower than that of the part compiled in pre-Qin times.

[1] *Mozi*, p. 286. Shanghai Ancient Book Press, 1986.

[2] Feng Youlan. *A New Book on the History of Chinese Philosophy*. Vol. , p. . Beijing: People Press, 1992

An investigation on Chang Qiujian Suanjing

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ABSTRACT

Chang Qiujian Suanjing(CQS) is one of the ten computational canons in ancient China. It is a work in 3 *juan*(rolls) consisting of 15, 22 and 38 problems respectively. We have the following conclusions:

1. It took 3 for π despite that some more accurate values were known then.
2. It consists problems on squares and circles, also cube and balls with the same measure. Contrast with the geometry construction of Greek, ancient Chinese took the way of convenience.
3. So far as we know, CQS is the first one of the ten canons that uses greatest common divisor (GCD) in solving problems.
4. CQS has an air of algebra.
5. CQS began the multi-solution problem.
6. Contrast with *Sunzi Suanjing*(SS), CQS did not take pure numbers as reality. It gave no *su*(trick, method) to the operations on pure numbers.

Introduction

Chang Qiujian Suanjing(CQS) is one of the ten computational canons in ancient China. *Qian Baocong* asserted that the period of compilation was 466~485 AD. It is a collection of problems, includes 3 *juan*(rolls) consisting of 15, 22 and 38 problems respectively. The content is not well organized. We do not think it done by an author. To simplify the statement, we will use *m-n* to denote the *n*th problem of *m*th *juan.*

The Value of π

The problems of CQS which involved the value of π are 1-20, 2-12, 2-20, 2-21, 3-5, 3-9, 3-24, 3-25, 3-30, 3-31. As the other computational canons, CQS took 3 for π. It is worthy to note that some better approximation was known then. *Liu Shin* took 3.1547 for π while (1~5 AD) he constructed *tong hu*(standard copper container); *Chang Hung*(78~139) took 3.16 for π while he solved the volume of balls. In the well known commentary of *Jiuzhang Suanshu*(JS), *Liu Hui*(end third century) took 3.1416 for π. However, the better approximations are kept in the commentary only. The main text always took 3 for π. It seems that taking 3 for π is the choice with least trouble, also, the authority of JS may form it as a tradition.

Figures with the same measure

The construction of circle with the same area as given square, by compass and straight edge, is a well-known unsolvable problem. Problem 2-20 of CQS asks the circumference of a circle that has the same area as a square with given side; Problem 2-21 asks the reverse. Problem 3-30 asks the diameter of a ball that has the same volume as a cube with given side; problem 3-31 asks the reverse. Essentially, these problems are rather algebraic than geometric. By taking 3 for π, they were satisfied with approximations. As we have mentioned above, this is the way for convenience. It seems that the theoretical interest was not the motivation of these problems. Rigorousness was not concerned.

The use of the greatest common divisor

The ideal of GCD was mentioned in JS and some other canons for simplifying rationals only. However, the *su* for solving problem 1-10 and 1-11 of CQS make the use of GCD other than simplifying rationals. Though the usage was not very properly, we would like to value it highly.

The air of algebraic

Problem 3-37 is a copy of SS. However, the *su*s are quite different. The *su* of CQS is algebraic while that of SS is arithmetic. In fact, many *su*s of CQS are algorithms. They are independent to the numerals given in the problems.

Multi-solution problem

CQS listed three solutions of 3-38. So far as we know, this is the first time that multi-solution was considered. Problem 3-26 of SS is a problem on residue classes, which is easier to see the multi-solution property. However, only one solution was listed. It is worthy to note that the answer (0, 25, 75) is not included. We assert that 0 was not accepted as a number then.

The attitude toward pure numbers

Problems 1-1 through 1-6 of CQS are on the operations of numbers. The answers are given without *su*. It seems that the author(s) of CQS did not accept numbers as reality.

The Notion of Volume in the *Jiu Zhang Suan Shu*, and Japanese Mathematics

JOCHI, Shigeru 城地　茂 [1]

ABSTRACT

In traditional Chinese mathematics, the notion of dimension was vague. In the *Jiu Zhang Suan Shu* 九章算術 (Nine Chapters on the Mathematical Arts), the oldest textbook of the mathematical art in China, the length, the area, and the volume were all expressed in the same unit, that of linear measure. No distinction was made between terms of the second, third, or higher degrees. The linear unit of measure was based on the length of the human body, e.g. "Chi" 尺 (literally one hand length), "Bu" 步 (literally (double) pace).

Liu Hui 劉徽 (3c), one of the greatest mathematician of the 3c A.D., wrote a commentary on this book and differentiated "Ping Fang Chi" 平方尺 (square "Chi") and "Li Fang Chi" 立方尺 (cubic "Chi") from "Chi", yet these terms did not catch on among other mathematicians. However, the unit of fourth "Chi" degree was used later, in the 13c.

On other hand, Japanese scholars in the Edo period used the system of that one cubic “Cun (Sun)”is 1000 cubic “Fen (Bu)”. We can conclude that the influence of the *JZSS* was limited in the Edo period.

1 The Notion of Volume at the *Jiu Zhang Suan Shu*

The numerical unit of Western mathematics is named the new unit by each three (or six in England) places, that is to say, thousand, million, billion and so on. This system must be based on the volume[2]. The side became ten times, the volume would become one thousand times. But Chinese system is the eight places (one hundred million) system[3], it is not related with the notion of the volume.

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[2] Mori Tsuyoshi, 1989: 30.

[3] There were three systems in China; (see Needham, 1954, vol.3: 82-90, however "Middle" must be changed as follows))

		Upper	Middle	Lower	Japan (after 14C)
(Wan	萬)	(10^4)	(10^4)	(10^4)	10^4
Yi	億	10^8	10^8	10^5	10^8
Chao	兆	10^16	10^16	10^6	10^12
Jing	京	10^32	10^24	10^7	10^16
Gai	垓	10^64	10^32	10^8	10^20

This system was described at the *Shu Shu Ji Yi* 數術記遺 (Memoir on Some Traditions of Mathematical Art, Qian Baocong, 1963, vol.2: 540), and ,in Japan, it was described at the *Jiko-ki* 塵劫記 (Permanent Mathematics, Was not the notion of volume in China the same as the Western's one? Thus we will study the notion of volume in China, moreover we will consider the influence of Chinese mathematical notion on Japan.

The notion of dimension must be the basis of geometry in Greece. Euclid's *Elements* was introduced into China in the Ming 明 dynasty (1368 – 1644)[4]. After the Ming dynasty, Chinese mathematics was influenced from Western mathematics, therefore we must consider the ancient Chinese mathematics. The base of Chinese mathematics must be the *Jiu Zhang Suan Shu* (we call it *JZSS* hereafter).

1-1 The *Jiu Zhang Suan Shu*

The *JZSS* was completed in the Eastern Han 東漢 dynasty (25-220)[5] and one of the oldest[6] and most complete mathematical books in China. It is said to be a collection of Chinese mathematics since prehistoric age, so that it is convenient for studying the development of Chinese mathematics in that age.

In the system of Chinese mathematics, there was no definition in as Greek mathematics; therefore we must collect many questions and abstract particular notions from them when we study Chinese mathematics. And *JZSS* is one of the best mathematical books for this kind of work because it described nine fields of mathematics, these are as follows;

Chapter	Main Works
1: Fang Tian 方田 (Field measurement)	plane geometry
2: Su Mi 粟米 (Cereals)	compound proportion
3: Cui Fen 衰分 (Distribution by proportion)	series
4: Shao Guan 少廣 (Width)	square root, cube root
5: Shang Gong 商功 (Construction consultations)	solid geometry
6: Jun Shu 均輸 (Fair Tax)	labor
7: Ying Bu Zu 盈不足 (Excess and deficiency)	1st degree equation
8: Fang Cheng 方程 (Rectangular Arrays)	1st degree equations
9: Gou Gu 句股 (Sides of Triangle)	Pythagorean theorem 2nd degree equation

Shimodaira Kazuo, 1990, vol.1-3: 7). Probably, Japanese mathematicians had used the Japanese system since the Ten books version of the *Iroha Jirui-sho* 伊呂波字類抄 at the Kamakura period (Oya, 1980: 85).

[4] Euclid's *Element*, that is to say, *Jihe Yuanben* 幾何原本, was introduced by Matteo Ricci in 1607, but it was perhaps introduced in the Yuan dynasty (Yan Dunjie, 1943. Zhong Wai Shuxue Jian Shi Bianxie-zu, 1986: 344).

[5] The first evidence about the *JZSS* is the inscription of Cup in 179 A.D. (Ren Jiyu, 1993: vol.1:79).

[6] The oldest material for mathematics in China is the *Suan Shu Shu* 算數書(See Jochi, 1988).

We cannot know even who is the author, but this book became the standard in Chinese schools. Then Liu Hui 劉徽 (3c) commented on this book in 263 A.D.. His works covered all of nine fields, especially two works are very important. One is, he succeed in computing an approximate value of π, and another is, he developed surveying method, which is described in the *Hai Dao Suan Jing* 海島算経 (Sea Island Mathematical Manual).

Then the *JZSS* had probably become a text-book since the Liu Chao age 六朝 (Six dynasties age, 220-581); some famous mathematicians comment about it, Zu Chongzhi 祖沖之 (429-500), his son Zu Gengzhi[7] 祖暅之 (5-6c) and astronomer Li Chunfeng 李淳風 (7c). But the former two versions were lost, only Li Chunfeng version was left the later ages. Li'Chunfeng s version was the official text-book of mathematics department of university in the Tang 唐 dynasty (618-907).

The *JZSS* was introduced into Japan[8], it became the most important textbook in Japan[9].

Li Chunfeng's version was also used for text-book in the Song 宋 dynasty (960-1279), it was republished twice, in the seventh year of Yuan Feng 元豐 period (1084) and Jia Ding 嘉定 period (1208-1224). And five chapters of the latter version is kept in Shanghai Library 上海圖書館 now[10].

The most popular version is Dai Zhen's 戴震 (1724-1777) version in 1774. It

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[7] His name is sometimes recorded Zu Geng 祖暅, Li Chunfeng's comment of *JZSS* is Zu Gengzhi 祖暅之, thus we call him Zu Gengzhi in this paper.

Historical materials of Zu Gengzhi

book	year	name
Wei Shu 魏書 (History of Wei Dynasty)	554	Zu Geng
Qi Gu Suan Jing 緝古算経 (Continuation of Ancient Mathematics)	620	Zu Gengzhi
Liang Shu 遼書 (History of Liang Dynasty)	629	Zu Geng
Chen Shu 陳書 (History of Chen Dynasty)	636	Zu Geng
Sui Shu 隋書 (History of Sui Dynasty)	656	Zu Geng
JZSS (Li Chunfeng's Comments)	7c	Zu Gengzhi
Bei Shi 北史 (History of the Northern Dynasties)	659	Zu Geng
Nan Shi 南史 (History of the Southern Dynasties)	659	Zu Gengzhi and Zu Geng

[8] Liu Hui' s comment, Mr. Xu's 徐 (probably 徐岳) comment and "Zu Zhong"'s 祖中 (or 仲, not 沖) comment of *JZSS* were described on the *Nihon Kenzaisho Mokuroku* 日本見在書目録. "Zu Zhong" is probably Zu Chongzhi (Nihon Gakushi-in, 1954-60, vol.1:148-9).

[9] Jochi, 1987.

[10] It was re-published in 1980 by the Wenwu Chuban-she 文物出版社, Beijing.

was hand-copied from the *Yong Le Da Tian* 永楽大典 (Yong Le Encyclopedia) which was edited in 1407, then this version was published several times[11]. We use Bai Shangshu's comment edition, 1983.

The *JZSS* describes the solid geometry, so we will consider until three-degree dimension.

1-2 Units of Length

Firstly, we consider length, the first degree. The units are not original for mathematics; they are in every day use. The length are defined by human body like as the *Shu Wen Jie Zi* 説文解字 (Dictionary of Characters).

> 尺。･･･周制、寸、尺、咫、尋、常、仞諸度量、皆以人之體爲法 [12]。
>
> The "Chi": The standard in the Zhou 周 dynasty (B.C. 13c-256 B.C.), "Cun", "Chi", "Zhi", "Xun", "Chang" and "Ren", these length are defined by human body.

The standard unit must be "Chi"[13], because most of the other units are defined by multiplying of "Chi", especially "Zhang" 丈 means "Shi" 十 (ten) in the Zhou 周 dynasty (1100 B.C. ?- 256 B.C.).

In the Han dynasty (207 B.C.-220 A.D.), rather than the Qin 秦 dynasty (221-207 B.C.), Chinese empire was founded and Chinese civilization was united. Therefore united standard was established because old definitions of the length had individual variations. It was made from the musical scale. According to chapter Lu Li Zhi 律曆志 (Record of Music and Calendar) of *Han Shu* 漢書 (History of the Western Han dynasty), it was defined as follows;

> 度者、分、寸、尺、丈、引也、所以度長短也。本起黄鐘之長、以子穀秬黍中者、一黍之廣、度之九十分、黄鐘之長。一爲一分、十分爲寸、十寸爲尺、十尺爲丈、十丈爲引、而五度審矣 [14]。
>
> The (units of) length are "Fen", "Cun", "Chi", "Zhang" and "Yin". Are the means for measuring the length? The standard is the length of "Huang Zhong" (literally, yellow bell). Put middle size cereals into this tube, it is one grain thick, ninety "Fen" lengths. The unit is one "Fen", ten "Fen" is a "Cun", ten "Cun" is a "Chi", ten "Chi" is a "Zhang", and ten "Zhang" is a "Yin". And the five units are clarified.

The standard unit was also "Chi" in the Han dynasty. Although the length of "Huang Zhong" 黄鐘 was nine "Cun", but there was the opinion that the length of

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[11] See Ren Jiyu, 1993, vol.1:79-87.

[12] The *Shuo Wen Jie Zi* 説文解字注 (Dictionary of Characters). Section of "Chi" 尺, p.401.

[13] A set phrase of measuring is "Chi Du" 尺度.

[14] vol.4: 966.

“Huang Zhong”must be one "Chi" length[15].

In *JZSS*, units of the decimal system were use more than the others, in table 1, V is used and - is not used in *JZSS* as follows;

Unit	definition	In *JZSS*	note
Cun 寸	1/10 Chi	V	
Chi 尺	standard	V	22.5cm
Zhang 丈	10 Chi	V	
Yin 引	10 Zhang	-	
Zhi 咫	8 Cun	-	18cm,small-Chi
Xun 尋	8 Cun	-	Depth
Chang 常	2 Xun, 16 Cun	-	Depth
Ren 仞	7 Chi	-	height, depth
She 揲	5 Chi	-	
Duan 端	20 Chi	-	Length of cloth
Bi 匹	40 Chi	-	Length of cloth
Kui 跬	3 Chi	-	Length of Field
Bu 歩	6 Chi, 2 Kui	V	
Li 里	300 Bu,1800 Chi	V	

Table 1 Units of Length[16]

V: This unit was used in the *JZSS*.
-: This unit was not used in the *JZSS*.

Then Liu Hui concluded the decimal fraction system in 3rd century[17]. And this system was also used into the length system, thus the units under "Cun" were also defined, these are as follows;

Unit	Definition
Fen 分	1/10
Lin 厘	1/100
Hao 毫	1/1000

[15] See Wu Chengluo, 1937: 20. The typical opinion is in the *Lu Xue Xin Shu* 律學新説 (New Discourse on Music Studies), as follows;

黄鐘長十寸、每寸十分、共合一百分。

The length of “Huang Zhong" is ten “Cun", each “Cun" is ten “Fen", so it is one hundred “Fen".

[16] See Wu Chengluo, 1937: 111 and Needham, 1954, vol.3: 84.

[17] Liu Hui commented the Question12-16, Chapter 4 of *JZSS* (Bai Shangshu, 1983: 103), he said,

微數無名者、以爲分子、其一退以十爲母、其再退以百爲母。

The fractions, which had no name, describe the numerator. The denominator is 10 at the first step, the denominator is 100 at next step.

Table 2　Units of Decimal Fraction[18]

As seen above, these units of length was expanded the decimal system considerably in *JZSS*.

1-3 Units of Area

The units of area are advancing each 100 times because one side became ten times, thus the area would become 100 times. Now, Chinese in the Han dynasty use the system that one "Mu" was 240 (square) "Bu". But it was in 223 B.C. that the definition of one “Mu" was changed to 240 (square) “Bu"[19]. It had used that one "Mu" was 100 (square) "Bu". Moreover, 240 are very close to 225, that is to say, the square of 15. Thus we can conclude that the notion of area in *JZSS* was very close to the Western notion.

Unit	Definition
Bu　歩	1 Bu by 1 Bu
Mu　畝	100　Bu (10 Bu by 10 Bu)
Qiang 頃	100 Mu (10,000 Bu)

Table 3　Units of Area

At first glance, Chinese system distinguished the length and the area. But we must note that the standard unit, 1 "Bu" length by 1 "Bu" length, was also named the same term, "Bu". Even mathematicians confounded two "Bu". For example, question 2 of chapter 1 in *JZSS*.

今有田廣十二歩、從十四歩。問爲田幾何。
答曰。一百六十八歩[20]。
There is a rice field, which is 12 "Bu" wide and 14 "Bu" lengths. How much area is it?
Answer: one hundred sixty-eight "Bu".

We can conclude that the notions of length and area were not clearly classified in *JZSS*.

1-4 Units of Volume

The volume of "Huang Zhong" (Yellow Bell) was the standard unit in daily life, and more units were defined as follows;

Unit	Definition	*JZSS*
Yue 龠 Shao　勺	the capacity of "Huang Zhong", 1,200 grains of middle size cereal　黍	V
He　合	10 Shao	V

[18] See Needham, 1954, vol.3: 84.
[19] Wu Chengluo, 1937: 61.
[20] Bai Shangshu, 1983: 121.

Sheng 升	10 He	V
Dou 斗	10 Sheng	V
Dan 斛、石	10 Dou	V
Dou 豆	4 Dan	-
Qu 區	4 Dou	-
Fu 釜	4 Qu	-
Zhong 鍾	4 Fu	-

TABLE 4 Units of Volume

The "Dou" and other units in table 4 were the quaternary system, but Chinese mathematicians did not use these units, they use the systematic units. And these were a perfect decimal system.

We must note that the system of mathematical unit was the decimal system, not three place-system. The standard unit, "Yue" or "Shao", was defined to the tube of 1 "Chi" length by 1 "Cun" diameter, was not 1 cubic "Chi" or 1 cubic "Chi". This definition will be the decimal system, that is to say, the next larger unit will be the volume of 1 "Zhang" length by 1 "Cun" diameter. The unit of "He" means 10 times of "Yue".

Moreover Chinese mathematicians used the special unit in *JZSS*. The unit of the volume was also used the same term of the length. For example, question 19 of chapter 4 in *JZSS* as follows;

> 今有積一百八十六萬八百六十七尺〔此尺謂立方之尺也。凡物有高深。而言積、曰立方〕。問爲立方幾何。
>
> 答曰。一百二十三尺[21]。
>
> There is (a cube), whose volume are one million and eight hundred sixty thousand and eight hundred sixty-seven "Chi". [Liu Hui's comment: This "Chi" means cubic "Chi". All of objects have the height or depth, "Ji" (product) means the volume] How much volume is it?
>
> Answer: one hundred twenty three "Chi".

Chinese mathematicians used the unit of cubic "Chi" in *JZSS*. They perhaps tried to use more systematic units. The system of volume in daily life was inconvenient for calculation.

But there were no larger unit, e.g. cubic "Zhang". Moreover the term was the same with the unit of length and area, so it was sometimes confused (see 2-1). Although Liu Hui commented that this "Chi" was "cubic Chi", however this clear term has never been used in later periods[22]. Thus we can conclude that the notion of

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21 Bai Shangshu, 1983: 112.

22 Rather "Chi" became the quantity of four degree when any degree higher degree equation was solved in the Song dynasty. For example, it is in the Yuan 元 Dynasty (1279-1367), in the *Suan Xue Qi Meng* 算學啓蒙 (Introduction to Mathematical Studies) in 1299. (vol.3: 27A-B).

今有積一百一十二萬九千四百五十八尺六百二十五分尺之五百一十一。問爲三乘方幾何。

volume was even more vague than that of area.

The term of sphere in *JZSS* is "Li Yuan" 立圓 [23][24] (lit. Standing circle)[25]. That is to say, the volume of *JZSS* was the multiplication of the area by the height.

Next, Chinese mathematicians described the value by the height. The Question 6, Chapter 5 of *JZSS* must be the typical problem, as follows;

> 今有塹、上廣一丈六尺三寸、下廣一丈、深六尺三寸、袤一十三丈二尺一寸。問積幾何。
>
> 答曰。一萬九百四十三尺八寸。〔八寸者、謂穿地、方尺深八寸。此積餘有方寸中二分四厘五毫、棄之、貴欲從易、非其常定也〕・・・[26]
>
> There is a moat, the upper width is one "Zhang" six "Chi" and three "Cun", the lower width is one "Zhang", the depth is six "Chi" and three "Cun", the length is thirteen "Zhang" two "Chi" and one "Cun". How much volume is it?
>
> Answer: ten thousand and nine hundred forty-three "Chi" and eight "Cun". [Liu Hui's comment] Eight "Cun" means that digging the ground one square "Chi" area and eight "Cun" depth. The fraction is 0.245 ("Cun") depth, cut them down to be easier, the value is not correct.

The "Chi" and "Cun" are used into the length and the value. In the case of length, one "Chi" length must be ten10 "Cun" length. But in the notion of the Western mathematics, one square "Chi" must be 1000 square "Cun". And the fraction is 824.5 square "Cun" in the Western system, however it is difficult to indicate it in the Chinese system. Thus Liu Hui must comment that one square "Chi" area and digging 8.245 "Cun" depths. That is to say, "Cun" of the answer is not square "Cun", is 1 square "Chi" area by 1 "Cun" length.

1-5 Confounding among Different Units

Chinese mathematicians calculated among other degree without any doubt. At least, they disregarded about units in *JZSS*. For example, question 7 of chapter 6 is as follows.

> 今有取傭負鹽二斛、行一百里、與錢四十。今負鹽一斛七斗三升少半升、行八十里、問與錢幾何。

> 答曰。三十二尺五分之三。
>
> There are a value of 129458.511/625 “Chi". How much is it to open four-degree root.
>
> Answer: 32.3/5 “Chi".

[23] In day life using, the sphere was called to the "Wan" 丸. (Liu Hui's comment of the Question 23-24 of Chapter 4, Bai Shangshu, 1983: 120.

[24] The word of "Yuan" was also the sphere. Yan Shigu's 顔師古 comment of Shi Huo Zhi 食貨志, chapter 24-2, *Han Shu* 漢書

> 李奇曰、圓即錢也。圓一寸而重九両。
>
> Li Qi said, the "Yuan" is the coin. The sphere (of gold) of one "Cun" diameter is nine "Liang" weight.

[25] The Question 23-24 of Chapter 4.(Bai Shangshu, 1983: 120-1)

[26] Bai Shangshu, 1983: 139.

答曰。二十七錢十五分錢之十一。

術曰。置鹽二斛升數、以一百里乘之爲法。以四十錢乘今負鹽升數、又以八十里乘之、爲實。實如法得一錢。[27]

The cost of a carrier who carries 2 "Dan" of salt is 40 coins per 100 "Li". There is one "Dan" seven "Dou" three and one third "Sheng" salt and carry it 80 "Li". How much does it cost?

Answer: 27.11/15 coins.

Method: Set 2 "Dan" (on the counting board) and change into "Sheng", then multiply 100 "Li", it is the divisor. Multiply 40 coins by the number of salt to carry, multiply it by 80 "Li", it is the dividing number. Divide it by the divisor, the answer is given, the unit is coins.

In this question, Chinese mathematicians calculated among three different units, the length, the volume and even the value.

(40 coins ×173.1/3 "Sheng" ×80 Li)÷ (200 "Sheng" × 100 Li)

＝27.11/15 coins

Of course the unit of capacity means the weight in this question, but they calculated between the unit of length and the unit of capacity, at first glance. At least, they did not have the strict notion of degree.

2-2 Japanese units

In the Nara period (710-794) or a little before, the *JZSS* was transmitted into Japan. The *JZSS* was one of the most important textbooks for the department of mathematics in the Japanese national University. Both of he comment by Li Chunfeng and Zu Chongzhi were transmitted into Japan. But the tradition of ancient mathematics in Japan was destroyed before the Edo 江戸 period (1603-1867).

Japanese mathematics was based on the *Suan Fa Tong Zong* 算法統宗 (Systematic Treatise on Arithmetic)[28]. The “Huang Zhong” and the system of volume was described in the *Suan Fa Tong Zong*[29]. But the description of the *Jinko-ki* 塵劫記 (Permanent Mathematics) is quite simple, there are only names of units in chapter 1, there is no explanation about the units of volume. Did Japanese mathematicians doubt Chinese traditional system?

The scholar of old documents, Kariya Ekisai 狩谷棭𣏐 (1775 - 1835) described new system at the chapter *Honcho Ryo Ko* 本朝量攷 (Studies of Volume in Japan) of *Honcho Ryo Ko Ko* 本朝度量衡攷 (Studies of Units in Japan) in the 18th century,

酒井家に参河以来、用ひ來りしと云ふ量を傳へらる。縦五寸、横四寸九分、深さ二寸

[27] Bai Shangshu, 1983: 200.

[28] In 1675, the *Suan Fa Tong Zong* was re-published in Japan (Nihon Gakushi-in, 1654, vol.1: 400)

[29] It is described in the preface. (Mei Rongzhao et al (ed.), 1990: 54)

六分。 [積六十三寸七百分。今の九合八勺七撮弱を受く。][30]

There is a measuring cup of Lord Sakai since that the family had lived at Mikawa. It is five "Cun (Sun)" length, and four "Cun (Sun)" and nine "Fen (Bu)" width, and two "Cun (Sun)" and six "Fen (Bu)" depth. [Comment] the volume is sixty-three cubic "Cun (Sun)" and seven hundred cubic "Fen (Bu)", that is to say, about nine "He (Go)" and eight "Shao (Shaku)" and seven "She (Satsu)" now.

This "Fen (Bu)" means cubic "Fen (Bu)". The notion of Japanese scholar in the Edo period was not the same of the *JZSS*.

Conclusion

The notion of dimension was not the same as in China and Japan. Of course, it had the difference for 1500 years between the *JZSS* and the *Honcho Doryo Ko*, therefore we cannot compare two books. But the most basic notion was not the same clearly. Thus we can conclude that the influence of the *JZSS* was very limited. Perhaps, Japanese mathematicians and scholars in the Edo period could not read the *JZSS*.

The *JZSS* in that time, that is to say, in the Qing 清 (1644-1911) dynasty in China, was re-found by Tai Zhen in 1782. Before 1782 even Chinese mathematicians could not read the *JZSS*. Japanese mathematicians could read the *JZSS* after the Tai Zhen's edition was transmitted into Japan. But Seki Takakazu and other Japanese mathematicians already established the Wasan (Japanese mathematics in the Edo period). Therefore the influence of *JZSS* was quite little in Japanese mathematics in Wasan period.

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A History of Calculus Education in Japan

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Abstract

In former times, calculus was a branch of higher mathematics and was taught only in post-secondary level. The idea of differential and integral calculus was introduced into school mathematics in Japan by a drastic change of the curriculum in 1942. However, this curriculum was not carried out completely due to the World War II. After the War, elements of calculus were introduced into upper secondary mathematics systematically, and, as a result of the spread of upper secondary education, "popularization of the calculus" has been made. However, there remain problems concerning calculus education, so calculus education should be improved.

1. Establishment of the system of mathematics education in Japan.

Modern educational system was introduced into Japan in 1872. Since then, Western mathematics has been taught, with some consideration of the traditional way of calculation, the use of *soroban*, in elementary arithmetic. Before that, Japanese people learned traditional Japanese mathematics, *Wasan*, which had been developed in Japan since the seventeenth century based on Chinese mathematics that had been introduced into Japan several centuries before; they studied elementary mathematics as a useful knowledge for their daily life and for their occupations. There were also Japanese mathematicians of those days who studied mathematics itself as an art. Their main concern was often merely skillful solving of complicated problems, a great many of which were numerical computations of quantities related to geometric figures by solving algebraic equations of higher degrees numerically. However, there were researches closely related to calculus: for instance, researches on the arc length of a circle. Only few people studied Western mathematics up to the sixties of the nineteenth century. Many of them were either military officers or engineers in the sixties of the nineteenth century and they studied mathematics as a preliminary subject for learning about their special fields: for example, navigation and gunnery for naval officers. Some of them studied calculus. Thus, in Japan, mathematics had been regarded as a skill or a tool.

The educational system was established firmly by the end of the nineteenth century. As to mathematics, arithmetic was taught in elementary schools, and arithmetic, algebra, Euclidean geometry and trigonometry were taught in secondary education. Calculus was a subject of higher mathematics, and was taught only in post-secondary education. However, secondary education was not popular at that time, and post-secondary education was only for the elite.

In 1895, Rikitaro Fujisawa, a professor of mathematics at the Imperial University of Tokyo and a leading mathematician at that time, wrote a book on education of arithmetic ([4]). It was the first book in Japan considering mathematics education seriously. He regarded the main objectives of mathematics education of secondary level as students' acquisition of mathematical knowledge necessary for their further studies (as to arithmetic, also for their daily life) and a mental discipline. He attached importance to the latter. Mathematics was regarded as a subject for mental discipline. His view of mathematics education had great influence on the national curriculum of mathematics in Japan until the early thirties of the twentieth century.

In 1902, the syllabus for the subjects of "middle schools" (literal translation of *chugakko* (in the former system), which were boys' five-year secondary schools for general education), was announced officially. Mathematics was divided into four subjects: Arithmetic, Algebra, Geometry and Trigonometry, and the syllabus indicated that these four subjects should be taught separately and rigorously. In the subject Geometry, the study of which began from the third year, Euclidean geometry was treated rigorously. Geometry was regarded as a subject suitable for mental discipline, so that it should be placed in the curriculum as late as possible and it should be taught rigorously and deductively. Intuitive geometry before studying formal Euclidean geometry was rejected.

Emphasis was laid on problem solving. Emphasis on problem solving, which was a tradition in *Wasan*, was also regarded as a mental discipline as well as preparation for the entrance examinations to advanced schools. On the other hand, utility and applications of mathematics were de-emphasized.

The syllabus was revised in 1911. However, revision of the syllabus of mathematics was rather small, and there was no influence of the movement for reform of mathematics education advocated by John Perry and others since the beginning of the twentieth century. There was neither intuitive geometry nor the concept of functions except the words "trigonometric functions" in the syllabus.

As mentioned above, calculus was a subject of higher mathematics, and was taught only in post-secondary education. However, an outline of the contents of a course of calculus as an introduction of higher mathematics and a preparatory subject for further studies was established by the end of the nineteenth century. The textbooks of calculus which were widely used at that time were the books written by Todhunter ([20], [21]). However, the textbooks of calculus were changed to the ones written in Japanese by Japanese mathematicians by the end of the first quarter of the twentieth century except calculus in universities.

At this point we mention briefly about the mathematics education at the Imperial College of Engineering (*Kobu Daigakko* in Japanese), the predecessor of the School of Engineering of the University of Tokyo. This college was established in 1873 and the principal was Henry Dyer. At this college, engineering education was carried out by integrating theory, application and practical training ([2], [19]). It was a great experiment, and it succeeded. John Perry came to Japan in 1875 as a teacher at this college and taught civil engineering and mechanical engineering at this college, made researches with William Edward Ayrton, a professor of electricity at this college, and stayed in Japan until 1879. Ayrton and Perry used squared papers for mathematics and mathematical treatment of various problems in engineering ([1], [14]). Perry's teaching experience at this college and at other institutions in England resulted in his idea of practical mathematics and advocacy for the reform of mathematics education. The Imperial College of Engineering developed to the School of Engineering of the Imperial University in 1886.

2. Influences of the movement for reform of mathematics education.

The movement for reform of mathematics education early in the twentieth century, advocated by John Perry, Eliakim Hastings Moore, Felix Klein and others, was intended to reform school mathematics placing the concept of functions as the central idea of school mathematics. This idea for reform was introduced into Japan, and some mathematicians and mathematics educators stressed the reform of school mathematics in Japan following this idea. However, the curriculum of mathematics was not revised until the thirties.

Meanwhile, the scale of secondary and post-secondary education grew gradually, and, to promote higher education and researches, the new regulation for universities and that for higher schools were announced officially in 1918. Since then, the scale of post-secondary education grew steadily. The Mathematical Association of Japan for Secondary Education, the predecessor of the present Japan Society of Mathematical Education, was established in 1919 to improve mathematical education in Japan. In 1924, Kinnosuke Ogura and Ryoitiro Sato stressed reform of mathematics education in middle schools in Japan by placing the concept of functions as the central idea in school mathematics, and they also stressed that the introduction of the idea of differential and integral calculus into middle school mathematics as a final step of teaching the concept of functions ([11], [16]). Ogura argued the necessity of radical reform of mathematics education by getting rid of Euclid; according to him, "the essence of mathematics education is development of scientific mind." Ogura's book [11] had influence on teachers and educators of mathematics, especially those of elementary schools, and it resulted in the sweeping revision of the textbooks of arithmetic in elementary schools in the thirties. Sato explained in his book of 1929 ([17]) mathematics education in middle schools more in detail by giving teaching plans that he had practiced already at the Middle School attached to Tokyo Higher Normal School. He laid emphasis on pupils' understanding of the concept of limits and the idea of differentiation and integration. These teaching plans were practicable at that time if a teacher wanted to do so.

The curriculum of middle schools was revised in 1931 to cope with the spread of secondary education and to meet the demand of the time. The new syllabus of mathematics was very simple. It indicated only main contents of each grade without dividing mathematics into subjects such as arithmetic, algebra, geometry. Teaching mathematics whether by dividing into subjects such as algebra, geometry and so on or not was left to each school. As to geometry, intuitive geometry before studying formal geometry was introduced in the first year. The syllabus also indicated that, in teaching mathematics, cultivation of the concept of functions should always be kept in mind. In this way, the concept of functions was introduced into school mathematics in Japan. The new syllabus made it possible to teach mathematics in various ways. However, mathematics was taught, in many cases, still as before. Treatment of functions and intuitive geometry was unsatisfactory as a whole.

The government-designated textbooks of arithmetic for elementary schools were renewed entirely in the thirties. New textbooks, "*Jinjo Shogaku Sanjutsu*" (Arithmetic for Elementary Schools), which came into use since the first grade children of the year 1935, were edited to develop mathematical thinking of school-children through their various activities. Materials related to the concept of functions and graphs as well as intuitive geometry were treated extensively, even a material related to the concept of limits was treated in the textbook of the sixth grade.

3. Calculus in post-secondary education.

We mention here briefly calculus in post-secondary education in those days. A higher school (*Koto-gakko* in the former system), more exactly, upper division of a higher school (*Koto-gakko Koto-ka*), was a three-year post-secondary boys' school for general education. The graduates enter universities. Each school has two courses: Literature and Science. As to mathematics, in litarature course, the syllabus said to teach an outline of trigonometry, analytic geometry and differential and integral calculus of one variable; however, actual treatment of the topics were diverse. In science course, contents of mathematics were solid geometry, analytic geometry, algebra, differential and integral calculus, and mechanics. As to calculus, calculus of one variable and that of two or three variables were treated. Elements of differential equation were taught in many cases, though differential equations were not indicated in the syllabus. There were two methods of teaching calculus at higher schools:

(1) Differential calculus first, then integral calculus.
As many textbooks of calculus were divided into two volumes "differential calculus" and "integral calculus" (e.g., Todhunter's books), calculus was taught in this way in former times.

(2) Calculus of one variable first, then calculus of two or three variables.
Some textbooks were written in this way, and this way of teaching became popular since the thirties.

The situation is similar at other higher educational institutions: for example, mathematics at higher technical school is similar to that of science course of higher schools mentioned above, but more application-oriented.

4. Introduction of the idea of calculus into school mathematics --- a drastic change.

Since the latter half of the thirties, the Ministry of Education had been planning a drastic change of school system and curriculum to meet the national demand of that time, and it was an urgent need and a matter of the highest priority. In this way, "elementary schools (*shogakko*)" were changed into "national schools (*kokumin-gakko*)", which was literal translation of "Volksschule" in German, in April 1941, and the curriculum of elementary education was revised entirely, and, two years after, in 1943, sweeping revision of secondary education was made.

Meanwhile, in Japan, the "New Structure Movement" started in 1940. This was a big political-social movement in Japan at that time supported by the government. The main aim of this movement was to break down old order --- old system, old sense of values and so on --- and to establish a new order in Japan that would be of help to carry out the national policy of that time. This movement had soon spread all over the country.

Improvement of the education of mathematics and natural science was also the pressing need of that time, and a drastic change was required so as to cultivate students' scientific mind. In this way, the syllabi of mathematics and natural science for middle schools and also those for girls' schools were revised entirely and drastically in March 1942. The new syllabus of mathematics was intended to develop mathematical thinking of pupils and their creativity through their various activities and to meet the national demand of the time. Utility and applications of mathematics were emphasized. Mathematical contents of the new syllabus were greatly influenced by the movement of reform of mathematics education since the beginning of this century. New topics were introduced in school mathematics: elements of analytic geometry, nomography, descriptive geometry, elements of probability and statistics, and the idea of differential and integral calculus. We mention here briefly about geometry. Though main theorems of Euclidean geometry were treated in the syllabus, the ways of treatment were quite different from those in a traditional course of Euclidean geometry.

Concerning the idea of calculus, the syllabus says:

The Fourth Year:	Natural numbers and series,
	Observations and treatment of sequences,
	Observations and treatment of continuous change.
The Fifth Year:	Variation of values of functions,
	Observations on forces and motion.

In this way, the idea of differential and integral calculus was introduced into school mathematics. As to the calculus in secondary education, the commentary of the syllabus says as follows ([9], p. 119):

> "Introduction of calculus into secondary mathematics without due reflection is very dangerous. It may make secondary mathematics too formal. To be introduced into secondary mathematics is not the technique of calculus but the methods of observation and treatment based on the concept of limits."

It was a drastic change in mathematics education in secondary schools in Japan, in both contents and methods of teaching. Before that, mathematics in middle schools had been taught in a traditional way: classical elementary algebra and Euclidean geometry had been taught mainly in lecture-and-exercise style. The new syllabus, however, laid emphasis on doing mathematics and heuristic methods: to let students discover mathematical facts through their activities such as observations, experiments, and considerations, and to let them synthesize and systematize these mathematical facts, here the word "systematize" was used in much wider sense than to systematize these facts from a mathematical point of view and to construct a mathematical system (in the usual sense).

In spite of such a drastic change of the curriculum, textbooks conformed with the new syllabus were not published in the year 1942, due to the shortage of time for preparing new textbooks; new textbooks for the first to the third year were published in 1943, and those for the fourth and the fifth year were published in July 1944. The style of new textbooks was entirely different from that of former ones. The new textbooks were just like workbooks or problem books. Problems and works to be done by students were indicated in the textbooks, and doing these works with appropriate introductory teaching and advice by their teachers would be of help to them for their discoveries of mathematical facts by themselves and their systematization of these mathematical facts. Definitions, theorems and proofs were not stated systematically in the textbooks, sometimes these were "given" as problems such as:

> "Sum up what you have found from the considerations made above, and write down these in the following blanks."

Perhaps a great many of the students were not able to understand mathematics by merely reading the textbooks. Doing was required. To acquire mathematical knowledge and to understand mathematics, they had to do various activities first, and then to make deep consideration. It was very hard.

As it was a drastic change, it was necessary to make thoroughgoing preparation for carrying out the new syllabus. Good reference books for students and those for teachers, mathematical laboratories, especially, teachers of ability were needed. The syllabus, however, was carried into effect without any preparations. It was a reckless attempt to carry out the new syllabus in 1942, during the World War II, as Ogura and Nabeshima wrote in their book ([13], pp. 412 - 413). It was quite unfortunate for the syllabus and also for teachers and students at that time, as the syllabus was the one in peacetime and not the one in wartime. Moreover, due to the World War II, it was impossible to carry out secondary education normally when all the textbooks were published, so this syllabus was not carried out completely.

The author considers that, though the syllabus was influenced by the national policy of that time, some of the ideas for the improvement of mathematics education in the syllabus are still worth consideration.

5. After the War --- calculus in upper secondary mathematics.

After the World War II, the educational system in Japan was reformed entirely, and the new system came into operation in 1947.

In upper secondary schools, which were started in 1948, mathematics was composed of three subjects at first: Analysis I, Analysis II, and Geometry. In the subject Analysis II, the elements of differential and integral calculus of one variable were treated systematically and extensively. Topics such as differential and integral calculus of trigonometric, exponential, logarithmic and inverse-trigonometric functions, integration by substitution, integration by parts, the meaning of differential equations, and, though very briefly and without mentioning convergence, Maclaurin expansions of simple elementary functions were treated in the textbook of Analysis II published in 1947. This textbook was written in the "ordinary" style of a textbook of mathematics.

The intended curriculum of mathematics was overloaded, however; so the contents of each subject were reduced from the beginning: for instance, calculus of inverse-trigonometric functions was deleted.

Since then, the curriculum for upper secondary schools was revised several times, about every ten years.

The revision of the curriculum in 1960 (effective from April 1963) was intended to develop students' basic knowledge and skills to cope with the rapid development of science and technology as well as the spread of upper secondary education. As to mathematics, contents were enriched and stress was laid on students' acquisition of skills. The revised curriculum of mathematics was influenced by the movement for modernization of school mathematics to some extent.

Since the revision in 1960, calculus in upper secondary schools has been taught in two steps, namely, calculus for simple polynomial functions first, and then that for simple elementary functions. As a result, many upper secondary students learn elements of calculus. By this, in addition to the spread of upper secondary education, "popularization of calculus" has been made.

However, there arises new problems. Stress on students' acquisition of skills has changed school mathematics to routine problem-solving by applying formulas. Moreover, we can differentiate and integrate polynomials formally, without any concept of limits, as we see in the standard textbooks of algebra (e.g., [22], pp. 84 - 86). As a result, though many upper secondary students have manipulative skills in differentiation and integration of simple elementary functions, their understanding of the concepts of limits, differentiation and integration is poor. This is partly due to the order of learning of limits of sequences, limits of values of functions of one real variable, definite integrals and indefinite integrals in the curriculum. We should improve the situation.

6. Calculus in universities.

Now we consider calculus in universities after the World War II. Roughly speaking, in universities, mathematics is taught in two different ways, that is, mathematics as a science worth studying in itself and mathematics as a language and a tool for science and technology. In the former case, mathematics is taught rigorously, and many professors of mathematics intend to teach their students methods of modern mathematics. Applications of mathematics are treated lightly because of limited school hours. In the latter case, mathematics is taught as a preliminary subject for students' studies in their majoring fields. Some theorems are stated without proof.

As science or engineering students have already studied elements of calculus of one variable at upper secondary schools, calculus for them is often taught rigorously so as to let them deepen their understanding of the concepts of functions, limits, differentiation and integration as well as to let them study the method of modern mathematics and prepare for their further studies. Facts which are admitted as intuitively obvious in school mathematics such as the intermediate value theorem are proved based on the axiom of the completeness of the reals (existence of upper bounds, etc.). However, the statement of the axiom of the completeness of the reals is difficult for many students, and they do not understand the necessity of such rigour, and have difficulty in such a course, --- for them, school mathematics and university mathematics are of quite different nature. Recently, excessive rigour has been improved in most cases. In the latter case, though transition from school mathematics to university mathematics may be smooth, what students learn are some mathematical facts and techniques to apply these facts to practical problems, or, in some cases, techniques of manipulating a mathematical software.

Another problem in calculus in universities is the transition from calculus of one variable to that of several variables. Many students have difficulty with calculus of several variables. This is partly due to their poor knowledge of geometry and lack of spatial intuition.

Therefore, improvement of mathematics education in both universities and upper secondary schools is needed. Especially, smooth transition from school mathematics to university mathematics is needed. For this, calculus plays an important role as a bridge between school mathematics and university mathematics.

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The Brief Sketch of Mathematics Education for Immediately Follow the World War II in Japan

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The purpose of this paper is to outline and state the feature of mathematics education for immediately follow the World War II in Japan. A historical survey of GHQ/SCAP Records were conducted, which resulted in the following three observations. 1. Japanese mathematics education for immediately follow the war age was established while there were two comparing positions: accepting American progressive education and keeping Japanese pre-war education. 2. Under the leadership of C.I.E., Japanese ministry officers were forced to lower the level of mathematics teaching and develop experimental units for subject matter. 3. Japanese ministry officers tried to make sure that mathematical significance was not lost in these experimental units.

Introduction

After World War II, Japan was under the occupation of the G.H.Q. (General Headquarters of the Allied Powers). Under the occupation Japan carried out drastic reforms in many fields, including education. In the case of educational system, the most emphasized point was the elimination of militarism. This emphasis required the changing form training based on uniformity in the war age to child-centered education.

In curriculum study and design, teaching of subject matter was reexamined, and children's problem solving in their lives was stressed. History, Geography and Civics were integrated into a new subject, Social Studies and its contents were based on the core-curriculum of the state of Virginia in United States. The progressive education such as the Virginia core-curriculum was a model of child-centered education in Japan at that time. In mathematics education, the children's interests, the relationships between mathematics and everyday life and the relationships with other subjects especially Social Studies were considered, too. In addition, a new teaching approach, which was called 'teaching by unit'[1] was proposed and practiced.

The period of new education for immediately follow the war age

In Japanese mathematics education history, the 'teaching by unit' has often described the changing period of mathematics education for immediately follow the war age. The beginning of it was April 1947 when new school system started. The typical textbooks were published in 1949. 'Teaching by unit' was included in Tentative Course of Study published in 1951.

Most of the studies and reviews that discussed this period have a tendency that just showed resources such as statements of the course of studies and textbooks. Many of them have taken critical stance against 'teaching by unit'. According to those studies the 'teaching by unit' neglected mathematics and was forced upon by the occupation forces (Ogura & Kuroda. 1978, Matsuda. 1981). Such interpretations have not correctly considered Japanese ministry officers' intentions, which proposed 'teaching by unit'. Moreover there had not been historical resources, which verify them.

Inagaki (1982,1984,1985a,1985b,1988) points out those studies that are critical at 'teaching by unit' look at limited aspects. He suggests necessity of considering the role of problem solving in 'teaching of unit'. Oku (1989) also points out that the historical studies of the period have been based on indirect information but the GHQ/SCAP Records will provide more direct responses. In these records, there are reports of conferences between Civil Information and Education Section (C.I.E.)[2] and Ministry of Education.

This paper will reexamine new mathematics education for immediately follow the war period using the GHQ/SCAP Records, and to describe the features of the 'teaching by unit'.

The mathematics education for immediately follow the war age

The beginning of the new school system for immediately follow the war. In April 1947, the school system for immediately follow the war period started. Under the system, six years of elementary school and three years of lower secondary school were established as compulsory education. Mathematics

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1 'Unit' means a single complete activity of leaning. But in this case, it was not like a chapter in textbook. It suggests a problem solving activity by children and through the activity children learn many things and solve their own problems based on their interests.

2 Japanese Ministry of Education was under the leadership of C.I.E in G.H.Q. The ministry officers had many meetings with C.I.E. officers and were required their approval for any decision.

was one of the subjects of these schools, and its aims, contents, grade placements, teaching methods and evaluations were stated in the course of study. "Tentative Course of Study for Mathematics" was published in May 1947.

In this course of study, there was no statement about the 'teaching by unit'. As Mr. Wada who was an officer of mathematics in Ministry of Education explained in the panel discussion of 33rd Annual Meeting of the Mathematical Educational Society of Japan, the 'teaching by unit' was not included intentionally in the mathematics course of study, although there were statements in other subjects (Kanehara et al., 1951). It suggests that the Japanese ministry officers in the beginning, thought they could continue with the existing mathematics education (Wada 1988).

The model textbook for the 'teaching by unit'. After the tentative course of study was published, C.I.E. suggested the ministry officers to revise it in June 1947[3]. By this suggestion the officers were forced to restart mathematics curriculum development for mathematics education immediately follow the war. In this process, Ministry of Education compiled two kinds of textbook series.

The first volume of a series for each grade level had already been in use, but the second volume was not published yet. Especially compiling for lower secondary school textbooks had been delayed. Because of this delay, in the process of the compiling the second volume, the officers were forced to make changes in their textbooks, though the first volume was already published based on the tentative course of study.

In 3 October 1947, Mr. Wada had a meeting with C.I.E. about the textbook of the second volume of "Secondary Mathematics" for 2nd grade in lower secondary school (8th grade). In this meeting following three points were suggested by C.I.E.[4]

1. The section dealing with trigonometric ratio should be deleted from the 8th grade text, possibly moved up to the 9th grade.
2. The section of the 9th grade textbook containing a unit on "Family Accounts" should be moved down to the 8th grade.
3. It should contain an experimental unit of the new type, which has been discussed recently with Mr. Wada, on "Banking", "Insurance",

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[3] GHQ/SCAP Records Conference Report 4 June 1947(in National Diet Library in Japan)

[4] GHQ/SCAP Records Conference Report 3 October 1947(in National Diet Library in Japan) (See Appendix 1. In this report suggestions form C.I.E were four, but one of them "The text should be reorganized" was omitted.)

"Taxation", etc.

From these items, we can see two things. First, Trigonometric ratio was content of 2nd grade of junior high in Tentative Course of Study, but the officers were suggested to change this grade placement. Second, we can see the term 'Unit'. It means that the development of materials using 'Unit' had started at that time, and the development of 'experimental unit' as a new type of 'Unit' was suggested. But Mr. Wada was opposed to any change at first. C.I.E. didn't accept his opposition and withheld any approval.

As a result Mr. Wada was forced to accept these suggestions, then the 'teaching by unit' was implemented in the new mathematics education for immediately follow the war.[5] In September 1948, Ministry of Education revised a grade placement of some of mathematics contents.

The second textbooks series were published in 1949 with all suggested changes as an acceptance for these changing. Especially in "Mathematics for Lower Secondary School Student", there were materials, which developed as the 'teaching by unit'.[6] This textbook was quite different from older one that contained explanation of rules of calculations and repetitions of drill works. The new textbook included problems based on social issue and students' lives, and by solving problems, students were expected to learn.

The 'teaching by unit' in Course of Study in 1951. The revised course of study of 1947 was published in 1951. There was a chapter about the 'teaching by unit' in "Tentative Course of Study for Mathematics in Lower and Higher Secondary School".

In this chapter 'teaching by unit' was explained as one of a teaching methods for mathematics. It aimed at promoting the development of following four characteristics.

1. Improving of their own life by using mathematics
2. Independent learning
3. Problem solving with their own ability and cooperation
4. Problem posing by using knowledge and skills already learned

The course of study emphasized was not only children's everyday life but also

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[5] GHQ/SCAP Records Conference Report 6 October 1947(in National Diet Library in Japan)

[6] "Mathematics for Junior High School Student" was published only for first grade in junior high school. In 1949, the school textbook selection system was changed form government-ready to authorization. This textbook was made as a model of new authorized textbook and Ministry of Education went out from textbook compiling after this textbook.

problem solving activities by using mathematics and how mathematics works for them. These problem solving contexts were not limited in everyday life and social issue.

The feature of the 'teaching by unit'

"Mathematics for Lower secondary school Student" was a model textbook of the 'teaching by unit'. The first unit in the textbook was "Our Home". In this unit, it was presented that there were many different shapes of house in the world. There were questions about the relationships between the shapes of the houses and functions of house in human life. Some of questions touched upon solid geometry such as a cone and prism etc.[7]

One feature of progressive education was that mathematical concept such a cone, pyramid or prism in this case was learned incidentally, thus, the approach in the new textbook appears to be the same. In the development of the issue, we can see the intension into mathematical significance.

After identifying the shapes from those houses, there were questions about the structures and properties of the solids so that students can build their own models. Then the shapes were abstracted to geometric solid. Finally, after understanding of mathematical conditions for these solids, the definitions were given. The definitions satisfy these conditions but they don't use mathematical terms. For example, the definition of corn solid, "The shape we can see at while umbrella is called 'corn.'" Implicit in this definition are the facts that the frame of umbrella had same length and the edge of umbrella was on a circle. More technically, the generating lines of a cone have same length and the base of cone is circle. This is a necessary and sufficient condition for cone. Other solids and the plane figures were treated the same manner.

As this example shows, mathematics contents were not picked up accident. In the background of this unit, there was an intension that children understand mathematical significance. It was intended that students learn about geometrical properties of solids and understand concept of the solids through paying attention to the shapes of house.

In this case, mathematical idea wasn't valuable for solving problem of life, but in the later part of this unit there were problems that are more mathematical in nature. In that part, problems related housing shortage are presented and the statistic was used to solve these problems. In that section, understanding and mastering mathematical contents were in the center of activities.

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[7] See Appendix 2.

The Japanese ministry officers had to use experimental unit to satisfy the recommendations by C.I.E., even though that was not their original intention. However, we can see that the Japanese ministry office tried to maintain the emphasis on mathematical significance in these experimental units.

This paper discussed how a new mathematics course of study and textbook were developed immediately after the World War II. How this course of study and textbook were implemented in actual classroom is in another question, but will have to be discussed in a separate paper.

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Appendix

1. GHQ/SCAP Records Conference Report 3 October 1947

R-E-S-T-R-I-C-T-E-D

3 October, 1947
Education Division,
CI&E Section

[illegible] L. OSBORNE /mm
Secondary Schools Officer

Mr. Wada, Mathematics Textbook Compiler, Bureau of Textbooks, Ministry of Education

Education

Miss Hollingshead & Mr. Osborne, Secondary Schools [illegible] Officers, Education Division, CI&E.
SUBJECT: Mathematics Text for Grade VIII.

During the past two weeks Section II of the textbook to be used for the General Mathematics course in Grade VIII has been reviewed by two of the Secondary Schools Officers. Much of the material was considered too difficult for the 8th grade level; the book was badly organized; and the problems listed bore only the remotest relationship to real life. The Secondary Schools Officers made several recommendations:

1. The section dealing with trigonometric ratio should be deleted from the 8th grade text, possibly moved up to the 9th grade.

2. The section of the 9th grade textbook containing a unit on "Family Accounts" should be moved down to the 8th grade.

3. The textbook should be reorganized.

4. It should contain an experimental unit of the new type which has been discussed recently with Mr. Wada, on "Banking", "Insurance", "Taxation", etc.

Mr. Wada was opposed to any change of any sort. The Secondary Education Officers withheld approval of the textbook until some changes were made.

Subsequently, at Mr. [illegible]' request, it was decided that the experimental unit might be published separately in pamphlet form. This would be a unit worked out in accordance with the modern concept of mathematics. Teachers would be asked to experiment with this unit in school, work out their own units on many subjects during the balance of this school year and a part of the next, and send the results to the Ministry for use in the new 1949 textbooks.

R-E-S-T-R-I-C-T-E-D

2. Monbusho(Ministry of Education) (1949). Mathematics for Lower Secondary School Student(textbook)


カナダのインデヤンの家


エスキモーの氷の家


南米のインデヤンの家


フィリッピンのその族の水上の家

B. 未開人の住居

外国の書物や，外国の地理風俗について書いた本を見ると，私たちは，上にあげたような未開人の住居の写真を，しばしば見うける。

問 1. 上にあげた住居を，わが国の大昔の住居と比べて見よ。どんなところが似ているか。また，どんなところがちがってい

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るか。

今まで調べた住居は，私たちの住居に比べると，ずいぶんそまつなものに見える。しかし，住んでいる人たちにとっては，なくてはならないたいせつなものである。どんな役目を果しているだろうか。そのおもなものとして，次のようなことがらが考えられる。

(1) 雨や風，また，夜の冷たい空気などから，自分や家族の者をかばう。

(2) おそろしい動物が近よるのを防ぐ。

(3) 家族の楽しい場所や，休養の場所を作る。

(4) 必要な物を保存しておく。

これらは，人類が住居を必要とした根本的な理由であったといえるだろうか。

問 2. 今までに調べた住居を，上にあげた四つの点から見て特に，くふうしてあると見られる点をあげよ。

問 3. たて穴住居・平地住居・高床住居を建てる時に，柱を立てるのに，どんな苦心をしたかを考えよ。

問 4. これらの住居の住みごこちはどうだろう。次の点から考えてみよ。

(a) 日光をじゅうぶんうけられるか。

(b) 冬の寒さが防げるか。夏の暑さが防げるか。

(c) 風通しはよいか。

(d) 湿気が多くて，じめじめしないか。

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C. 家 の 形

今までに調べた住居の模型を作ってみよう。

(1) 未開人の家

問 5. 14 ページの写真にあるカナダにいるインデヤンの小屋の形をかいてみよ。このような形をした小屋を見たことはないか。

問 6. 今かいた小屋の形は，からかさをひろげていく時にもできる。各自に作ってみよ。

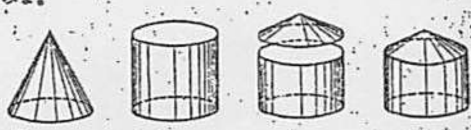
問 7. カナダにいるインデヤンは，どんなにして，住居の骨組みを作るのだろう。

(a) 住居のまわりの丸太を，その端がどんな線の上にあるようにして，立てかけたらよいか。

(b) 丸太を地面につきさした所から，丸太を結んだ所までの長さを，比べたとする。この長さの間に，だいたいどんな関係があるか。

からかさをひろげていく時にできる形を円すいという。

問 8. わりばしと紙を使って，円すい形の小屋の模型を作ってみよ。



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アメリカ北部のインデヤンの家

問 9. エスキモーの住居の屋根の形をいえ。また，その住居を横から見た形をいえ。

問 10. 前のページの下にあげた図形を，各自にかいてみよ。

問 11. 厚紙を使って，エスキモーの家の模型を作れ。

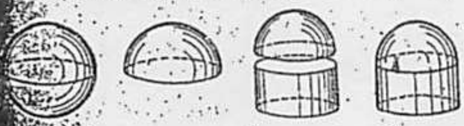
問 12. 上の図は，アメリカ北部にいるインデヤンの住居を示したもので，今までに作ったものに似ている。

この住居の形は，円すいと，どんな所が似ているか。また，どんな所がちがっているか。屋根の骨組みを見て考えよ。

ボールの形を球という。

球を，まっぷたつに切ってできる二つの図形のおのおのを，半球という。

問 13. 次の図形を各自にかいてみよ。



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Mathematics Textbooks and Terminology: The Impact of Calvin Winson Mateer's Work on the Transformation of Traditional Chinese Mathematics Education into a Modern One

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China began to experience the arduous transition from an arrogant, feudal, traditional China into a humble, half-feudal and half-colonial, and relatively modern one as a direct result of the humiliating defeats of the two Opium Wars (1839-42, 1856-58). In late 19^{th} century, education was one of the most important results of this transition. Missionaries, particularly American Protestants, played an important role in the process of educational modernization.

Under the treaty of Tianjin, which signed in 1858, and went into effect in 1860, American missionaries gained the right to establish missionary schools in China. Among the pioneer missionaries who devoted themselves to educate Chinese, Calvin Winson Mateer was a prominent. In July of 1863, Mateer, together with his wife Julia Brown, were sent by the Presbyterian Board of Foreign Missions to Shandong, China. Mateer would spend the rest of his life in China. Over the next forty years, Mateer would successfully establish a primary school at Dengzhou, Shandong, which later developed into the first college run by missionaries in China.

This paper will examine the mathematics textbooks compiled by Mateer for his school and for the Educational Association of China, as well as his work on English-Chinese mathematical terms. It will focus on their impact on the modernization of Chinese mathematical education, and on the unification of Chinese mathematical terminology under China's Department of Education in 1909. After establishing the general background of the mathematical textbooks available to the Chinese, this paper goes on to analyze Mateer's five textbooks: Bi Suan (*Arithmetic*, published in 1878); Xing Xue (*New Geometry in Chinese*, published in 1885); Dai Shu Bei Zhi (*The Outline of Algebra*, published in 1887); Dai Xing He Can (*Analytical Geometry*, published in 1887); Wei Ji Feng (*The Integral and Differential Calculus*, published in 1890). These texts differed from the traditional Chinese ones, as well as those compiled by other missionaries, in the following ways:

1. Using Arab-Indian numerals and Western conventional symbols, instead of traditional Chinese ones;
2. Adopting the Western format of books, namely left to right;
3. Using the colloquial style rather than the formal literal one;

4. Creating textbooks rather than direct translation; and
5. Including exercises.

Mateer's textbooks were widely used not only in missionary schools, but also in many official Chinese schools, e.g. Tong Wen Guan (Beijing Imperial College). Based on the circulation of Mateer's textbooks and the fact that in 1905 they became the official textbooks designated by the Education Department of China, this paper shows the positive influence of Mateer's texts on the transformation of mathematical textbooks in China.

The second part of the paper will analyze Mateer's work on usable Chinese mathematical terms under the auspices of the Educational Association of China. Using examples of terms such as arithmetic, geometry, logarithm, and prime number, I will discuss Mateer's deep understanding of traditional Chinese mathematics, and his contributions to the standardization of Chinese mathematical terms.

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Teaching Fortification as part of Practical Geometry : A Jesuit case.

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Fortification, practical geometry, Jesuits : what a strange collection ! But what can have their relationship been ? Practical geometry deals with two different questions : firstly, construction problems, such as inscribing regular figures into a circle or describing quadrilaterals with given sides/angles/area (as we already find in Euclid's *Elements*) ; secondly, problems of measurement of lines (above all, distant and unreachable lines) areas (including the quadrature of the circle) and capacities. It is the euclidian tradition meeting with the practical one (roman "agrimensores", egyptian "arpedonaptai", arab inventors and users of the astrolabe, navigators, etc.) Famous examples are given by J. Stoffler's *Tractatus de astrolabio* (Tübingen, 1512), Oronce Fine's *Protomathesis* (Paris, 1532, containing one book on practical geometry, often republished), Gemma Frisius' *Usus annuli astronomici* (Paris, 1557), Cosimo Bartoli's *Delle modo di misurare le distantie, le superficie i corpi* (Venice, 1582) and, last but not least, Jean Errard's *Géométrie et Practique générale d'icelle* (Paris, 1594).

Jean Errard (called "Errard de Bar-le-Duc") is a special case : he wrote the first French mathematical treatise on fortification[1], only a few years after Stevin's one (1594) and after his own *Practique de Géométrie*. The book is mathematical in the sense that every assertion is proved (by Euclid's *Elements*.) In France a lot of treatises on military architecture appeared in the 17th century, but, even among the most famous ones, there is a lot of differences. Errard de Bar-le Duc was a renowned "Ingénieur du Roy", he had built or modified many strongholds (Amiens, Sedan…) and had made war (hero of the siege of Sedan) : he wrote for skilled engineers ; Blaise-François Pagan, who never built any defence work but lost an eye in the battle, was yet recognised as "the father of fortification *à la françoise*". At the end of his life, he wrote on all kinds of sciences, of course including fortification ; we can see his state of mind in the preface of its treatise[2] *If the Fortification science was purely Geometrical, its Rules would be perfectly demonstrated : but since its object is Matter & its main foundation is Experience, its most essential maxims do only depend on conjecture*. We can see that even in those matters, there was a fight between the believers of theory and the supporters of practice. Note that Vauban himself, in spite of a serious scientific education, claimed there was no theory, that everyone had to adapt to the reality of the battlefield.

Jesuits and Fortification

The question of education is of great significance : in France, in the seventeenth century, most schoolboys entered Jesuit Colleges (famous or not), that's why the introduction of mathematics as a worthwile and independent subject in the syllabus is mostly due to the Jesuit teachers and administrators. It was not pure mathematics, but general science, mathematical physics or applied mathematics first. For instance, Jacques de Billy's lecture

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[1] La *fortification réduicte en art et démonstrée par Jean Errard de Bar-le-Duc, Ingénieur du très chrestien Roy de France et de Navarre*, Paris, 1600.
[2] *Les Fortifications de Monsieur le Comte de Pagan,…,*3rd edition, Paris, Cardin Besongne, 1669.

(Godrans College, Dijon) near 1670 contained arithmetic, study of the sphere, astronomy, optics, perspective, theory of sight, fortifications and military art, universal and practical geometry, and so on. According to Steven J. Harris[3], "the Company of Jesus, with its professorships of mathematics, was one of the most powerful agents of legitimation of mixed mathematics". It is clear that fortification was one of the most famous of these emerging sciences, because of the war in Europe (only two years of total peace all along the seventeenth century.) Young people in the Colleges had to know the "Officers' science". The "cursus mathematicus" flourished (Schott, Pardies, Rabuel, Milliet de Chales, Castel, …) and particular treatises on fortification, too. We have noticed several of them : manuscripts by M. Cuvelier (1635), J. Bertet (ca 1650), I.G. Pardies (1673), B. Durand (1700), J. M. Aubert (ca 1710), Yves André (1758), A. Fijan (ca 1740) ; printed books by G. Fournier (1649), P. Bourdin (1655), J. du Breuil (1665), C.F. Milliet de Chales (1677), P. Ango (1679), this list is not exhaustive.

Silvère de Bitainvieu (pen name, anagram of Jean du Breuil Jesuit) is of great interest for us. He is the author of two reference works on perpective[4] and on military art[5], the latter especially written for young men (maybe in the Dijon' College where Silvère died in 1670.) Who were those persons, and why would they have to learn fortification ? Silvère himself answers in the preface : his goal is to *[contribute] something to make [the nobility] able to better serve the King.* He writes to the young noblemen : *When exercising will have made you more learned in the fortifying Art, you will be esteemed by Sovereigns & State ; & in the controversies that may happen, you will be made the judge of the ones involved in them.* He also confesses that he wants to allow a good quality in conversation…War was everywhere, even in the salons ! This is confirmed by Rohault, in his treatise on fortifications (posthumously published in 1682), speaking of the different manners of fortifying in France : *they have become so famous that one can't do without speaking of them unless being taken for an ignoramus.*

Pierre Ango[6] sheds another light : *All the glory I want to take is only to obey the orders of the greatest & the wisest of the Monarchs, who wanted the Art of fortifying to be taught, by means of books & public instruction in Schools [...] We aren't tired of saying how fine this Art is, & we even discover its most secret mysteries, for those who understand a little geometry, & who have at least read the six first books of Euclid's Elements*

Now, to try to know what the necessary mathematics were, and how they were brought into play, the best thing may be to read original texts : we will first see the capacities required from the students, and then study a particular manuscript.

What was the syllabus ?

It's been very difficult for us to find official instructions, except in the *Ratio Studiorum* of 1599. Every College might have followed its own curriculum, or every teacher his own philosophy. Nevertheless, the major part of the jesuit printed works on fortification are cast in

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[3] Les chaires de mathématiques, in Luce Giard (ed.), *Les Jésuites à la Renaissance*, Paris, PUF, 1995(in French)

[4] *La perspective pratique, nécessaire à tous peintres, graveurs, sculpteurs*, Paris, 1651. Augustus de Morgan mentioned it in a letter in 1863 as the *jesuit perspective*, reminding that it is *better known than that of Pozzo*, he asserts it *had more influence on common methods than any other*.

[5] *L'Art universel des fortifications françoise, hollandoise, espagnole, italienne…*, Paris, 1st edition, 1665.

[6] *Pratique générale des fortifications pour les tracer sur le papier & sur le terrein, sans avoir égard à aucune méthode particulière*, Moulins, 1679.

the same mould : a lot of precise definitions, practices of geometry on paper and on the field, construction of bastions and outworks on polygons, and so on. Is there any difference between the books and the lectures ? In other words, what was really taught ? What were the pupils supposed to be capable of doing ?

We can have good indication of what was required at Godrans College[7], by considering a little printed booklet[8] published in Dijon in 1762, just one year before the Jesuit' final throwing out of the city. This is an announcement for public examination concerning three students in the College. The oral test focused on military art, especially fortification (and some questions on fortresses attack and defence), involving arithmetic and geometry.(see illustration below)


EXERCICE
DE MATHÉMATIQUE
SUR
L'ART MILITAIRE
Dans la Fortification, l'Attaque & la Défense des Places, & sur un Traité préliminaire d'Arithmétique & de Géométrie.

RÉPONDRONT,
MESSIEURS,
JACQ. FRAN. ADELON DE CHAUDENAY, de Dijon,
LOUIS BEGUIN, de Baigneux,
JACQUES MAGNIEN, de Dijon.

Dans la Classe de Mathématique du College de la Compagnie de JESUS, le Vendredi 2. Avril 1762. à deux heures après midi.

A DIJON, de l'Imprimerie de la Veuve de PIERRE DE SAINT, seul Imprimeur du Roi, de Monseigneur l'Evêque & du College.

That kind of publication was delightful for the Jesuits and the students' families, the former because of the publicity given by the latter to their teaching, and for the public proof of their efficacity and relevance. The publication of "theses", or examination programs, was common, not only in Jesuit Colleges but also among other institutions[9].

The chapters are all built on the same pattern : definitions, theorems or properties, problems, showing that students were expected to master every notion in their conceptual aspect as well as their practical applications. It begins with questions on arithmetic, especially calculation (four operations) with simple or "complex"[10] numbers and the rule of three. The following section deals with geometry of lines, angles, polygons (including special chapters on triangles and quadrilaterals), surfaces and solids. The second part treats of military art, particularly the art of fortifying regular or irregular towns (old systems, then Vauban's"three systems"), different ways of representing them, wash painting the maps, and the third part is about attack and defence.

Thanks to the first part of the booklet, we can see that, actually, fortification was a matter of geometry : it takes in fact the major part of the "preliminary treatise", when arithmetic matters are only elementary ones. Let us see in more details what we could call theoretical and practical geometry.

1) <u>Lines</u> : first, questions about different types of lines (straight or not), their relative positions, circles and their radius, diameters and tangents. Then, problems, like these :

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[7] The biggest college of the jesuit region of Champaign. Founded in 1581, thanks to a legacy of Odinet Godran. Close the first time in 1595 (Henri IV), reopened in 1603.The fathers wanted a class of mathematics and obtained it in 1666, through a grant accorded by Pierre Odebert. The Jesuits were throwed out in 1763 (one year after the King's decision). The College had been in perfect harmony with the good society in Dijon.

[8] *Exercice de Mathématique sur l'Art Militaire, ...*. Manuscript n° 3357 (7), given by Louis Beguin's family to the public library in 1999.

[9] Beaune public library owns Monge's examination program, in the city Oratorian college.

[10] *i e* numbers expressing time, money, distance or any old measures, before metric system (and if you belong to those who still use what I call "old units", please accept my apologies and congratulations too.)

I. Draw a straight line from one point to another one, on paper as well as on the field.
V. Divide a given straight line into as many equal parts as you will.
VII. Find the centre of any given circumference or arc.
IX. Draw a perpendicular on a given line from any point on this line.
X. Draw a perpendicular at the end of a straight line without producing it.

2) Angles : definitions and questions of measurement ; equal arcs ; opposite, corresponding, alternate angles. Theorems corresponding to Euclid I-13, and I-29 (the students surely had to be able to prove it) and four problems on measurement (on paper and on the field), transferring and division in two.

3) Triangles : classification, cases of similarity, theorems corresponding to Euclid I-32 (with corollaries) and Euclid I-5 & 6, problems of construction, some caracteristics being given (a side and two angles adjacent to it, two sides and the angle between them, two sides and the opposite angle, three sides, three equal sides.)

4) Quadrilaterals : trapezium, parallelogram, square, rectangle, lozenge ; theorems similar to Euclid I-34, construction problems (to make a parallelogram on given sides and a given angle, to make a square on a given straight line.)

5) Other polygons : definitions (of n-gons, $5<n<12$), radius,angles ; theorems (the sum of the angles equals -in right angles- twice the number of sides minus four ; the sum of the angle of the center and the angle of the polygon equals two right angles) ; problems about finding the value of angles of the polygons, the ratio of the radius to the side, inscription of any polygon into a circle, drawing any polygon on the field, on the basis of its map or not.

6) Drawing up maps : including calculating lengths using a scale, reducing and enlargement.

7) Measurement of surfaces and solids : Euclid I-36 & 37, then problems about the area of each plane figure. The same with solids.

Well, it becomes clear this is not a complete and abstract examination on geometry. Propositions out of Euclid's *Elements* are not present for themselves, but in a pragmatic aim : one can't study fortification without being a geometer (ideally) or at least knowing the matters listed above (in reality). The second chapter (*Upon military art, in fortification, attack and defence of places*) shows that it is important not only to be able to build or draw fortified towns but, first of all, to know what we are talking about : the first page is entirely devoted to definitions of lines, angles, masonry works, structures, their use and interest !

For instance, the title of this first chapter is *upon the principal parts of fortifications*, and it contains more than sixty questions, such as : what is the rampart and what are its width and height ? What is the scarp and where does it begin, whether the rampart is surfaced or not ? What is called magistral line ? What is a bastion ? What are the faces ? the flanks ? the demigorges ? What is called curtain wall ? flanked angle ? angle of the flank ? What is the moat and what is its use ? and so on… I will stop here, but there was a whole page on the subject !

The following chapters get to the heart of the matter, without forgetting definitions. After fortification maxims (the defence line should be as long as the range of the musket, the interior of the fortress must command the outworks, …) and before questions on attack, government and defence of towns, the questions turn on building (=drawing) different parts of the ramparts, bastions, ravelins, hornworks, crownworks, etc., especially after Mr Vauban's systems. We will now leave the *Exercice de Mathematique* to take an interest in another manuscript, which will give us details on the methods.

Bernard Durand's *Architecture militaire* of 1700

Durand was a professor of physics in Godrans College from 1698 to 1701 ; his treatise[11] could be a transcription of his course, written up by some student (the words *B. scripsit* appear at the end of the manuscript), but most probably a revised and corrected version of it ready to be published or come down to posterity.(the last words are : *le temps presse*, which means "time is getting short" ; Bernard Durand died in 1701)

Title page (partly printed) of Durand's manuscript

The first page of the treatise announces it is divided into three books : "military architecture teaches us to fortify places, that is to say, to arrange their parts in such a manner that a few persons can easily resist the efforts of numerous ennemies" ; according to Durand, this is the reason why fortification is a part of architecture. The second one is a book on civilian architecture "to build houses inside the walls", and the third one is about perspective, which "allows us to see both from different points of view." The real order of the treatise is 1) military architecture, 2) perspective, 3) civilian architecture, with another handwriting.

The mathematical part is at the very begining of the first part : "To proceed with method in the knowledge of this Art, we must know some principles of geometry without which we couldn't understand it in its depth." What are those principles ? They are "the terms of point, line, , parallel lines, occult, perpendicular, circle, diameter, angle, polygon.

The definitions are both in Euclid's manner and with a practical touch, reminding us of land surveying handbooks. For example : "line is a length without width[12] ; it is made by slipping the feather along a ruler". Durand immediately joins the essential practices, like drawing parallel lines (the second one being the common tangent of two half circles centered on the first one and with the same radius) or perpendicular lines. Here is a difficulty : the practice depends on the situation of the point of intersection. If it is in the "middle" of the line, we use the median of any segment [AE] of the line, the point of intersection being equidistant

[11] *Traité d'Architecture civile et militaire donné par le R.P. Durand de la Compagnie de Jésus. L'An de notre Seigneur 1700. Le 6^e du Mois d'Avril. A Dijon.* Ms 469, Dijon public library.

[12] Strictly euclidian definition, like the one of the point as "having no part".

between A and E. But if it's "at the end" of the line, it is more difficult (don't forget you can only use ruler and compasses, or a rope on the field) :

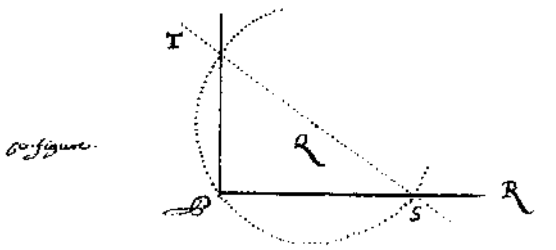

figure 1 : construction of a perpendicular

"for example, if, from the point P, 6th figure, you want to drive a perpendicular take any point on the line P·R·, as Q· and from Q· draw a circle which touches the point P· and which cuts the line P·R· as in S· then draw a line from S· through the point Q· to T· and T·P· will be the perpendicular."

"You can also proceed this way[13] on the line I·K· 7th figure put a leg of the compasses at the point I· and with the other one draw a portion of a circle L·M· as long as you want Then put the compasses with the same degree of opening at the point M· and with the other leg cut the aforesaid portion of circle at the point N· put again the same opening of the compasses to the point N· and make the little arc P·Q· then drive a line from M· to N· which should cut P·Q· from the intersection of P·Q· let a line down on the point I· : it will be perpendicular."

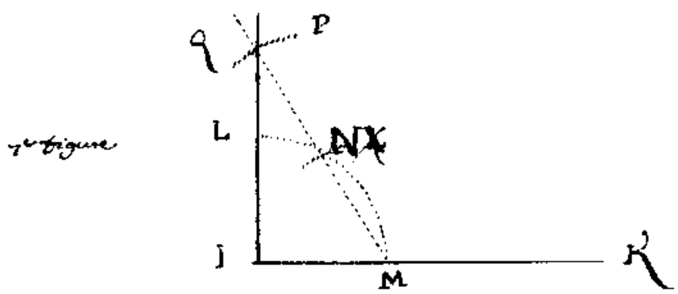

figure 2 : alternate construction (note the deletion)

These methods were well-known ; in France, we do not teach them anymore[14], who would need to draw anything "at the end" of a line ? On the other hand, we still are interested in dividing any line into equal parts (now an application of "Thales theorem") and inscribing regular polygons into the circle ; the method for pentagon is still the same, but do you how to inscribe an enneagon into the circle ?

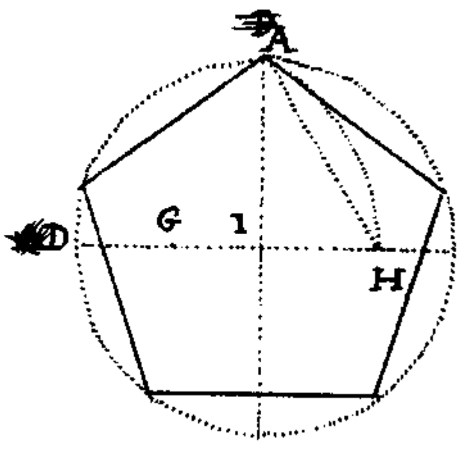

figure 3 : classical pentagon.

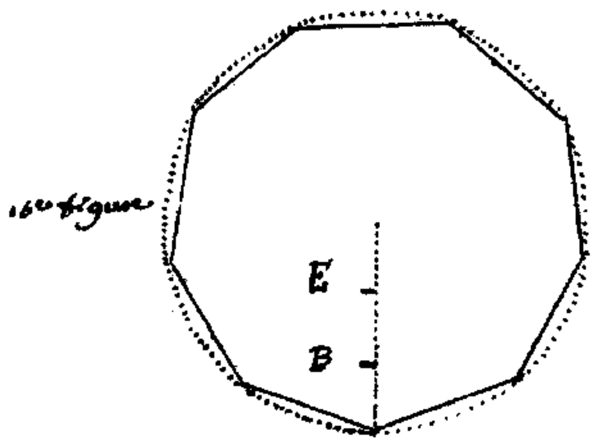

figure 4 : is it the real enneagon ?

To make the pentagon : [AI] and [DI] are two orthogonal diameters, G the middle of [DI], H is defined by GH=GA, AH is the side of the required polygon.

[13] The lack of punctuation comes from the original.

[14] Anyway, in our mathematics common-core syllabus ; it may not be the same for pupils in technical classes.

And what about the enneagon ? : "To make the enneagon, that is to say nine angles, just take two thirds of a half-diameter as E·B for its side." (quoted from Durand) That's all ! If you want to know whether it is true or approximate, compare the values of $\frac{2}{3}$ and $2\sin(\pi/9)$…

This part on geometry ends with questions on measurement of angles and the use of the protractor, contruction of a scale for the maps, and we can read then : "First method to draw fortifications outside using but the ruler and common compasses ". The first practice is about the square, it will be followed by other ones, on polygons from five to ten sides ; after that, Durand will examine "fortification inside", and different types of outworks, such as hornworks and crownworks. Different methods will be explained and finally the conception of a fortification with the help of tables or compasses of proportion. Let us see in detail how to build bastions on a square fortress, following the simplest of the methods. (Durand, page 14)

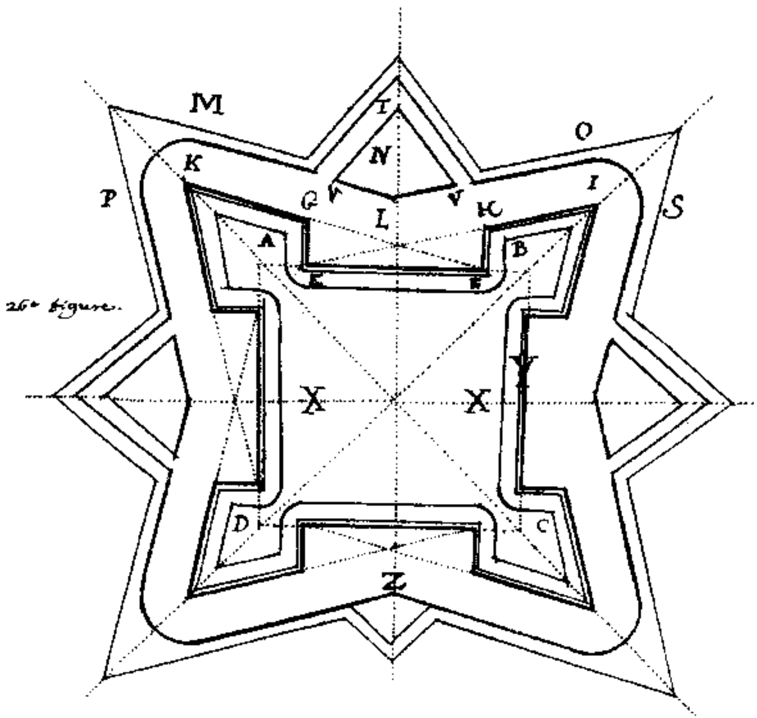

figure 4 : how to fortify a square ?

"You will take 1°) the square : let then be proposed A·B·C·D Draw 1st the diagonals A·C D·B that you will produce outside of the square as long as you want. These lines being produced outside of the square are called capital lines. Every side of the square such as A·B has to be divided into six equal parts, two of them such as A·E··F·B will be for the demigorge and E·F for the points from where the flanks raise up to A·B (which is the extended curtain wall) and those flanks will have the length of one of the six parts ; it means that you will make E·G equal to E·A and F·H equal to F·B and also from the point B you draw a line up to I passing through H This line which is the grazing defence will form the pan of the bastion H·F and will finish the flank F·H ; do the same from the point F passing through G and you will get the other half bastion E·G·K and the complete side A·B fortified ; do the same with the other ones. This method making the gorge quite reasonable gives the pans of the bastions shorter than by any other method and consequently less expensive and more advantageous, leaving a wider moat. Supposing the side A·B is 100 toises[15] long, each one of the six parts dividing this side will be 16 toises and 4 feet long. So will the demigorge. The curtain wall E·F will be 66 toises and 4 feet and the defence line F·K a little more than 100 toises."

Here we have the bastions, and the square is fortified. But it is not over with the defence of the square, because those basical works are not enough. You have to make the covered ways, the outworks, especially ravelins, and later (if you read the entire manuscript, this is a

[15] about 650 french feet, or 198 meters ; otherwise, about 213 yards.

next lesson) other types of outworks, in the style of Vauban. In concrete terms, it means new parallel lines on the diagram. Let us go back to Durand's text.

"For the moat, you have to take the flank E.G. with the compasses and to bring the space to the flanked angle K. and with the other make a portion of circle M.P. ; do the same at every tip of the bastions and with the same degree of opening setting the compasses on L. meeting point of the grazing lines make a portion of circle again L.N. that you will bring in front of every curtain, then applying the ruler on M.N. in such a way that it grazes the portions of circle M.O. ; draw lines which meet in N. where the capital line has been produced.

If you want ravelins, which are outworks with four or five angles, placed in front of the curtains to defend them, please note how they are formed. Produce the capital line from the middle of the curtain wall, through the angle of the counterscarp .N. and give to N.T. on the aforesaid capital line two thirds of the face of the bastion. Then from the point T. draw lines to the angles of the flanks of the bastions A.B. These lines cutting the counterscarp will give the sides of the ravelins T.V.T.V.

For the rampart, which is a path upon the walls inside, take half of the flank E.G. and apply this half parallel to the curtainwalls inside, such as X.X. to turn all around, even inside the bastions, rounding the lines near the reflex angles of the flanks with the same degree of opening for the compasses.

Add, near the walls inside, the line of the parapet, to which you will give the fifth part of the rampart. The parapet Y includes a little degree near the wall, from which the gunmen are under cover.

Give the covered way turning all around the place on the ditch, half the width of the rampart and to the moat of the ravelin half the biggest moat. It is called covered way because it is a path upon the moat, covered with a six or seven feet high parapet to cover the ones who watch it.

If you can't make any ravelin in front of a curtain wall, such as in Z, you must make there a housing or a hideout with salient angles inside and outside."

Conclusion

These construction algorithms would be suitable for learning geometrical design, and even better for using construction softwares ! Have you noticed that there was no proof ? Bernard Durand seems not to be interested in calculating the values of lines or angles. Nevertheless, he gives tables at the end of his treatise, but doesn't give any method. Didn't he live in the times of the "calculating geometry of Mr Descartes" ? We can imagine that the aim of his course was not building fortresses at all : we could hardly find real concern with the building on the field. Are you convinced it was a matter of geometry on paper ?

Now, if we compare with nowadays situations, do you believe that our taught mathematics are justified by future applications or by necessity ? Are they really useful for our students' future life (everyday life or not) ? Ten years ago, we heard about "mathematics for the citizen" (anniversary of the French Revolution…), isn't it still valid ? Indeed, Jesuit teachers spoke to noblemen (it is not entirely true, Odinet Godran's will imposed free public lectures) who would become Officers (fighting Officers), and we speak (in the desert) for the whole generations of kids. Indeed, little by little, computers are changing our ways of life and work and thinking. But what about good old geometry ?

An Episode in the Development of Mathematics Teachers' Knowledge? the Case of Quadratic Equations

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> "When asked what it was like to set about proving something, the mathematician likened proving a theorem to seeing the peak of a mountain and trying to climb to the top. One establishes a base camp and begins scaling the mountain's sheer face, encountering obstacles at every turn, often retracing one's steps and struggling every foot of the journey. Finally, when the top is reached, one stands examining the peak, taking in the view of the surrounding countryside - and then noting the automobile road up the other side!"
>
> Robert J. Kleinhenz in H. Eves' *Return to Mathematical Circles*

In the framework of a Mathematics Teaching Methods course for secondary school pre-service mathematics, the students were analyzing the different teaching sequences adopted in Israeli schools when teaching the general solution of a quadratic equation: a) linear functions-linear equations-quadratic functions-quadratic equations, or b) linear equations - linear functions - quadratic equations - quadratic functions. Following the first approach, a "classic" algebraic proof by completing the square was presented. Following the second approach, a proof relying on the line symmetry of the graph of the quadratic functions was revealed, this one entirely unknown to the students. Innocently, they were asked: "How old is the "classic" proof?" and all of them were sure that it was known *as such* already to the ancient Greeks. I decided to design an activity for the following two meetings of this group. The purpose of this paper is to describe the activity and the main questions raised.

According to the curricula of Oranim, the students don't take any courses connected to the History of Mathematics so, the Teaching Methods course provides suitable opportunities to discuss the development of some of the main concepts and ideas they are supposed to teach in school and these discussions may be considered important for many purposes:

a) To strengthen and to deepen the students' content knowledge;

b) To help students appreciate the power and elegance of the mathematical tools and methods used today;

c) To have students compare solving techniques and *think about* the Mathematics they will soon teach;

d) To improve the students' inventory of teaching techniques;

e) To draw the students' attention to cultural, social and emotional aspects involved in the creation and learning of Mathematics, to expose the students to the social and intellectual context in which the Mathematics was developed.

The topic of the activity was Quadratic Equations and it was decided to discuss it through historical material since I believe that "[Humanistic Mathematics] puts the discussion of quadratics into the human-mathematical context that gives the mathematical topic its sense and its beauty." (Tymoczko, 1993, p.12). Moreover, I considered it important for my students' education as future mathematics teachers to emphasize that "Standard approaches to the quadratic formula embed the quadratic formula in purely utilitarian mathematics. They suppress the aesthetical, the historical and the purely mathematical aspects of this mathematical problem in favor of touting the practical significance of answering various canned word problems. Students spend half a year mastering a variety of techniques leading up to a general solution which eliminates the need for their mastering of those techniques. But they were never told why anyone would think a general solution was intrinsically interesting for its own sake" (ibid., p.13). Moreover, the presentation of other proofs of the quadratic formula may enhance the students' appreciation of the classic proof, making them aware - following Kleinhenz's metaphor - that they've reached the peak of the mountain by using automobile route and not by climbing the difficult way as our ancestors did.

THE ACTIVITY

Two different ways to utilize the history of mathematics in order to teach it were adopted cooperative learning and the collective reading of an original mathematical text. Some of the students were Hebrew speakers and the others were Arabic speakers so, to make the experience relevant and meaningful for all of them, the materials chosen dealt with mathematics originally written in Hebrew and with mathematics originally written in Arabic.

The first task dealt with Al-Khwarizmi's algebra and it was implemented in a cooperative learning setting. Three different working cards were designed in order to present Al-Khwarizmi's method of solving three different types of quadratic equations $x^2 + ax = b$, $x^2 = ax + b$, $x^2 + b = ax$ for $a > 0, b > 0$. Each card also asked for possible generalizations of the described method and for a discussion of its limitations. After a short presentation of Al-Khwarizmi and his treatise *Hisab al jabr wa-l-muqabala* (The Condensed Book on the Calculation of al Jabr and al-Muqabala) the students were organized into pairs and each pair was given a different card. They worked for almost twenty minutes on their cards and at the end of this part of the activity, the three methods were presented by the students and their answers to the general questions were given and discussed.

For the second task implemented during the following meeting of the group a week later, three pages from Bar Hiyya's *Hibbur ha-Meshihah ve ha-Tishboret* (Treatise on

Measurement and Calculation) were selected and read together with the students. These pages written in the XIIth century by a Spanish Jewish mathematician in medieval Hebrew, also presented rhetorical solutions to three different types of quadratic equations. The first case solved by Bar Hiyya was the quadratic equation $x^2 - 4x = 21$, the second equation was $x^2 + 4x = 77$ and the third one was $4x - x^2 = 3$. In each case, the algorithm is followed by a geometric proof. After a short introduction about Bar Hiyya, his time and his work, a student read the text aloud and stopped after each sentence to interpret its meaning to construct the corresponding drawing and to connect it with the "known" method of solving quadratic equations. At the end of the meeting, we compared between this task and Al-Khwarizmi's methods of solving quadratic equations.

THE REACTIONS

The first task stimulated students to ask interesting mathematical questions, so the questions I thought should be asked to summarize the task were in fact raised by the participants. I present the questions as they were originally formulated by the students followed by a summary of the answers provided for each one of them.

1. Why did Al-Khwarizmi use geometry to solve an algebraic problem?

Al-Khwarizmi 's geometric proofs demonstrate both his Greek and Babylonian heritages. The Babylonians did not explain how their algorithm to solve quadratic equation was derived the just presented a numerical solution, without any connection to geometrical meanings (Katz, 1997, p.28). The Islamic mathematicians looked at the methods developed by the Babylonians, combined them with classical Greek geometry and proved their own results. The only justification accepted during these time was geometric proof, even the justification of algebraic results or number theory theorems (Katz, 1993, p.228). That seems to be the reason why Al-Khwarizmi used Euclidean tools to provide a geometric solution for the quadratic equations.

2. Why did Al-Khwarizmi have so many different cases of quadratic equations instead of a general one like ours?

Al-Khwarizmi did not work on *the* quadratic equation in general he identified different types of quadratic equations, involving three kinds of quantities? the square (of the unknown), the root of the square (the unknown itself) and the absolute numbers (the constants of the equation):

1. squares equal to roots $ax^2 = bx$;
2. squares equal to numbers $ax^2 = c$;
3. squares and roots equal to numbers $ax^2 + bx = c$;
4. squares equal to roots and numbers $ax^2 = bx + c$;
5. squares and numbers equal to roots $ax^2 + c = bx$.

Since negative numbers were not accepted in those early stages of mathematical development, several different algorithms were needed to handle the various cases resulting from the requirements of positive coefficients.

This question also led to a discussion of the difficulties involved in the adoption of suitable mathematical symbolism and the contribution of the use of letters for different classes of numbers only later, in Descartes' work. Only modern symbolism has enabled the mathematicians to abandon both the rhetorical style of presentation and algorithms and also to leave the presentation of examples. As Friedelmeyer wrote: Used as we are today to using letters in calculations, we find it difficult to appreciate the major change that took place in that development and even in the concept of mathematics itself. The use of literal symbols was not just a simple matter of presentation, like the use of a more efficient typeface. The use of a letter set the mathematician free from his attachment to the concrete." (Friedelmeyer, 1997, p.317). This kind of activity may foster the students understanding and appreciation of the huge impact that symbols had and still have in the development algebraic thinking.

3. **Does Al-Khwarizmi's work constitute a proof that his methods are correct?**

This question was also a fruitful one, since it led us to discuss the notions of *proof* and *rigor* and their relativity (Kleiner, 1991; Barbin, 1994). The issue of *proof* was present in the frame of this Teaching Methods course but this task fostered another perspective Concerning the learning of proof, one should not ask "How do you prove this result" but, "How did you obtain this result?". This should be the first step followed by the perfecting of methods of proof arising from confrontations with those produced by the pupils themselves. Then, step by step, one could arrive at a more subtle method of proof which is that of deductive reasoning (Barbin, 1994, p.51). In that sense, the geometrical method used by Al-Khwarizmi may be considered not only a proof of the quadratic equation algorithm but mainly an enlightening answer to the frequently asked question "How did he obtain this result?"

4. **Why isn't Al-Khwarizmi's method as accurate as ours? Why does it miss solutions?**

Since Islamic mathematicians - unlike the Hindus - did not deal with negative numbers nor with the number 0, only the positive roots were admitted.

The exploration of how many positive solutions (if at all) has an equation of each type was a significant exercise since the students engaged their knowledge of Viete's relationships of the roots to the coefficients of a polynomial equation.

The second task also provoked questions but questions of a different nature?

1. **Did Bar Hiyya write other books?**

Aside from the book mentioned before - *Hibbur ha-Meshihah ve-ha-Tishboret* (Treatise on Measurement and Calculation), translated into Latin as Liber Embadorum, Bar Hiyya wrote the book *Yesod ha-Tebunah u-Migdal ha-Emunah* (The Foundations of Understanding and the Tower of Faith) which is the first encyclopedia in Hebrew. He also wrote books on Astronomy - *Zurath ha-Haretz ve-Tavnith ha-Shamaim* and *Heshbon Mahalkhoth ha-Kokhavim* - and calendar calculations - *Sefer ha Hibbur* (Book of Intercalation). He can be considered the father of Hebrew mathematics.

2. **Is Bar Hiyya an exception, or were there more examples of Jewish mathematicians in Medieval Europe?**

During this time, Jewish mathematicians flourished mainly in Spain and Provence. Two important names may be mentioned: Abraham ben Meir ibn Ezra and Levi ben Gershon (Gersonides).

The Spanish Jew Abraham ibn Ezra (1090-1167) one of the greatest Biblical commentators of the Middle Ages, was one of the first to translate Arabic writings into Hebrew. For example, he translated Al-Biruni's commentaries on Al-Khwarizmi's tables. He worked on combinatorics, discussing the number of possible conjunctions of the seven planets known then, since it was believed that these conjunctions might have an influence on human life and on the decimal system in his treatises *Sepher Ha-Echad* (Book of the Unit) and *Sepher Ha-Mispar* (Book of the Number).

The French Jew Levi ben Gershon (1288-1344) wrote philosophical texts Biblical commentaries and mathematical books such as *Maasei Hoshev* (The Art of the Calculator) dealing with combinatorics and induction but without introducing the new numerals and the decimal system, and *On Sines Chords and Arcs* dealing with trigonometry.

CONCLUSION

I would like to end this paper with a quotation used in the activity to summarize part of the message I tried to transmit to the students through this introduction to the history of mathematics?

> *"The power and beauty of algebra lie in its generality. Its generality is possible only because of its extensive use of symbols... Mathematical symbols permit calculation otherwise beyond our powers and quantitative expression that otherwise would require an infinity of time. It is no exaggeration to say that mathematical symbols are second only to the alphabet as an instrument of human progress. Light moves fast and space is large, but symbols can shoot past them at a walk"* (Boyer, 1945, p. 139).

I believe that the use of the history of mathematics is a *necessity* in Mathematics Teaching Methods and I hope that this paper has illustrated that it is possible and that it allows high level discussions to develop among pre-service teachers.

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Teachers' Teaching Beliefs and Their Knowledge about the History of Negative Numbers

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ABSTRACT

Currently in Taiwan, it is the practice for secondary school teachers to teach their mathematics students only that information, which will be needed for the entrance examinations. From my experiences of contacting with thousands of secondary school mathematics teachers, I feel that most of them neither deeply consider the value of mathematics, nor thoroughly think about why they teach such mathematics in such ways. Their beliefs about mathematics and its teaching come mainly from their secondary school learning experiences, and it is hard to broaden or change their beliefs. For this sake, the present study was emerged to investigate whether the knowledge of mathematics history can cause changes to secondary school teachers' beliefs about mathematics and its teaching. Moreover, the feasibility for secondary school teachers to learn the history of mathematics was also investigated. The negative number was chosen to be the mathematics topic for this research. The research was a qualitative inquiry that implemented in a natural classroom setting with 24 in-service secondary school mathematics teachers in the class.

Some of the research results were described below. Before learning the history of negative numbers, many teachers' beliefs about 'what the negative number is' included the element of 'real life phenomena.' The 'reason for the negatives to exist' and the 'values of negative numbers' were also elements in the teachers' belief systems. However, the teachers not only seldom perceived negative numbers from the perspectives of history, but also rarely viewed why their students needed to learn negative numbers from the historical aspects. During the learning section, through teaching discourse as well as group discussion, the teachers of this study showed their surprises and struggles about the development of negative numbers. After the learning section, the teachers' beliefs apparently changed. The historical viewpoints emerged when the teachers described what the negative number was and also why their students need to learn negative numbers.

INTRODUCTION

When teaching a poem, we do not just introduce the meaning of the poem. We also introduce who wrote the poem, what motives prompted the poet to write poems, and what backgrounds of the society originated the poets' motives. However, these are not seen in mathematics classes. Currently in Taiwan, most of the secondary school mathematics teachers teach only for the sake of entrance examinations, so they try to clearly state the definitions or the meanings of new concepts and give a lot of problems to the students to practice. It is never made clear to the students why they need to learn the concept and what the value is for them to learn. Mathematics is not automatically inherited interesting. Therefore, it will be less available for students, who lack intrinsic motivations about mathematics or those who can learn only when the information make sense to them, to accept new mathematics concepts.

It is because, perhaps teachers' teaching has strong relationship with their teaching beliefs, lots of researchers conducted researches to study teachers' beliefs (Grant, et al, 1986; Munby, 1984; Raymond, 1993; Raymond, 1997; Schifter, 1998; Shulman, 1986; Wilson & Goldenberg, 1998). Despite the popularity of teachers' beliefs for researchers, Thompson (1992) mentioned that, '…the concept of belief has not bee dealt with in a substantial way in the educational research literature.' She then described some distinctive features of beliefs, such as 'they can be held with varying degrees of conviction;' 'they are not consensual.' Green (1971) identified three dimensions of belief systems, which were primary vs. derivative, central vs. peripheral, and clustered. Lerman (1999) reviewed research perspectives on mathematics teachers and discussed the issues about the relationship between teaching beliefs and teaching itself. He declared that the research results were not consistent in this issue.

Despite the hard interests of researchers, few were conducted in order to provide a big conflict to teachers' cognition for them to change their beliefs. The phenomena of lacking stress on mathematics value or its learning reasons might come from the fact that teachers believe mathematics is an absolutely true knowledge and is not considered as a progressive knowledge in the world. These beliefs about mathematics are generally emerged from the secondary school learning experiences of teachers, which contain merely the introduction of current and static mathematical definitions and meanings. It is very hard to broaden or to change these fixed, stabled, and long-tern beliefs.

The history of mathematics describes the construction of mathematics and contains the origins of mathematics. Besides, it comprises the stories or anecdotes of mathematicians' investigation, struggling, and even failure. These might be good resources for the teachers to 'reknow' mathematics concepts in a new way and then help them to rethink the meaning of mathematics in a new view. As a result, the history of mathematics might give a support to teachers to make sense of the meaning of their mathematics learning as well as that of their students'. It has been few researches conducted in this area, if any, while Jardine (1997) is one who tried to use mathematics history in his calculus class and reported positive results on learning motivation, but he did not examine the beliefs of his students.

In Taiwan, at the junior-high-school level, the first concept all students learn is the concept of negative numbers, which is also the first concept that is hard for students to make sense from their daily lives. In the textbooks used at all schools, the connection of 'different signs of numbers' and 'different directions of moving' is given. Also, 'earn' and 'lose' are used to give metaphors for 'positive' and 'negative' numbers respectively. Teachers usually teach as what the textbook says, and make no change except giving more or alternative metaphors. It is not clear why students need to learn negative numbers and why teachers teach in the way that follows the textbook without making adaptation contingent upon thorough consideration. One possible reason is that, to mathematics teachers, mathematics is absolute right truth and does not worth to be doubted. However, as educators, we all know that, if one can not doubt, one can not invent or think deeper. There must be some ways to help teachers doubt and then to clarify. One possible way is to increase teachers' knowledge about the history of negative numbers, which is unfamiliar to them, in order that the teachers have some foundation knowledge to doubt and clarify. Although, there are some papers relate to the negative numbers, none of them are related to teachers' beliefs (Streefland, 1996; Thomaidis, 1993).

As a result, the purpose of this study was to investigate secondary school mathematics teachers' beliefs of the negative number and its teaching, also to explore how the history of negative numbers influenced the teachers' teaching beliefs. Moreover, the feasibility for the teachers to learn the history of negative numbers was investigated.

METHOD

Design

Since one purpose of this study was to investigate the feasibility for teachers to learn the history of negative numbers, the more nature and realistic the research setting was, the more unsuspected the research results were. Also, it is more adequate to provide deeper descriptions about teachers' beliefs than just give quantitative statistics. Therefore, the study was designed to be a qualitative inquiry conducted in a natural classroom setting.

Because the spirit of adaptability and creativity in designing qualitative research is aimed at being responsive to real-world conditions, the teaching discourse as well as group discussion was used for the learning activities in the class. Both qualitative and quantitative data were collected. The qualitative data included data from questionnaires, field notes, and informal interviews. Then a content analysis was performed to the data and data triangulation was conducted to emerge labels and patterns (Patton, 1990, p.194). The quantitative data was used not to evaluate, but to understand how much history knowledge teachers could retain at a period time after learning activities.

Instrument

This research adopted the spirit of qualitative study. Therefore, I myself was an instrument of the study (Patton, 1990, p.14). One examination and one inquiry questionnaire, entitled 'History Knowledge of Negative Number' and 'Teaching Belief of Negative Number' respectively, were developed to collect information of teachers' beliefs and historical knowledge about negative numbers. I describe them below.

The examination problems were constructed based on the following frame of the history of negative numbers:

1. Information Proposed to Influence Teaching Belief (General)
   - The concept of negatives now conventionally been accepted
   - It took long time and encountered lots of difficulties to reach the current state of negative numbers, especially in the western society
   - In ancient China, the relation between negatives and solving systems of equations
   - The way ancient China mathematicians operated negative numbers and the differences of the ways between now and then
   - The differences between the development of the number concept and the operations for negative numbers
   - The century that all mathematicians accepted negative numbers
   - The various metaphors used to illustrate the negative number's concept and operations
   - The comparison of text flows in textbooks and historical development over years for negative numbers
2. Information Proposed to Increase Personal Competence (Specific)

- In ancient China, when did ‘negative’ appeared first in documents
- Excluding China, which country was the earliest one that had mathematicians accepted the use of negative numbers
- Why negative operations were necessary in order to achieve the processes of solving systems of equations
- What were the similarities and dissimilarities among ‘the number concept of negatives,’ ‘the operations of negatives,’ and ‘the negative magnitudes’
- What daily-life contexts were negatives used at the earliest stages both in China and western world
- What were the different stages of the developmental changes for negative numbers
- Why did a lot of famous mathematicians rejected negative numbers

Three experts in mathematics education and six secondary school teachers established the above frame. According to the frame, one examination was developed with fifteen yes-no questions, four multiply choice questions, and two short essay questions.

The ‘Teaching Belief of Negative Number’ questionnaire contained two versions. One version, I call it pre-questionnaire when mentioned, was used before the learning activities of negative numbers, and the other version was used after the learning activities, which I call it post-questionnaire. The pre-questionnaire consisted of four questions as shown below:

1. What do you think the negative number is?
2. What do you think why secondary students need to learn negative numbers?
3. What do you think how the negative number and its operations should be taught?
4. Has there been any change made with your teaching methods or contents since you started to teach negative numbers? If there is any, please describe it.

The above questions 1, 2, and 3 were also included in the post-questionnaire and were renumbered as 2, 3, and 4. Another question was added to the post-questionnaire as question 1:

Q: Are there any new appreciation or perception of you about negative numbers after the group discussion? If there are any, please describe them.

<u>Subject</u>

Because one purpose of this study was to investigate the feasibility for in-service teachers to learn mathematics history, the more nature and realistic the research setting was, the more unsuspected the research results were. Accordingly, the subjects were 24 in-service mathematics teachers who were pursuing their specialist master degrees on mathematics teaching. Out of them, 13 were teaching in senior high schools and 11 were teaching in junior high schools. Only three of them have never taught in junior high schools before. Their teaching years altered from 3 to 21 and the mean was 8.7 years.

All subjects attended the class ‘Introduction to Mathematics Education’, which was offered by me. They were very active in the class, and were free to ask questions, share experiences, as well as express opinions in the class. Before we started to discuss about the history of negative numbers, only one of the subjects showed knowing some history of negatives, but not much. The rest of them either have never touched any history of negatives or could not retrieve any events of it. (Data came from oral interviews and field notes recorded during teaching discourse and group discussion.)

Procedure

At the beginning of the academic year, the subjects were divided, according to their willingness, into four groups with six people each. During one class section, the subjects were told that the class would do some activities relating to the history of negative numbers. Then the pre-questionnaire was distributed to the student. It took about thirty minutes for all subjects to finish their questionnaires. The time they submitted back the questionnaires were only differed in two minutes.

After that, handouts of the history of negative numbers were distributed to all students. I spent about a half-hour to discuss some documented development of negatives in ancient China. Then I asked the subjects to discuss the material on the handout in-group for two hours, which was the rest of the class time. One and half-hours after the group discussion started, I passed the post-questionnaire to each group. Group members were asked to finish the questionnaire in cooperation.

All groups could not stop to discuss when the class time was over, and so I allowed the subjects to continue their discussion longer without time constraint. Four groups spent 2 hours and 15 minutes to 2 hours and 40 minutes variedly with group discussion and submitted back the post-questionnaires afterwards.

Three class sections, eleven days, after the negatives' learning activities, without announced in advance, the subjects were given an examination using the 'History Knowledge of Negative Number' examination problems. At the beginning of the examination, I declared to the class that the score of the examination would not be counted except for the students that were on the borderline to pass the course.

RESULT AND DISCUSSION

The subjects' beliefs, which were collected before the learning activities of the mathematics history about negative numbers, were first labeled according to the views emerged from the subjects' reflections. Then the labels were analyzed and grouped into patterns. Data from pre-questionnaires, field notes, and informal interviews were triangulated to interpret emerged labels and patterns.

Before Learning Activity

The beliefs of what are negative numbers

Before the learning activities of the negatives' history proceeded, the subjects' beliefs about 'what is negative numbers' and 'Why secondary students need to learn negative numbers' will be described below.

There were four kinds of views mentioned more than the other kinds were when the subjects tried to explain what they thought negative numbers were. The four kinds were listed below. (1) Using like mathematical definitions: Nine subjects (38%) mentioned that 'negative numbers are the opposites of positive numbers.' (2) Attaching physical meaning: Eight (33%) subjects thought that negatives were numbers that represent the opposite directions of daily life phenomena. (3) Describing the opposite characteristics: Five (21%) subjects tried to describe negative numbers as something that represent opposite concepts, principles, and situations. (4) Using Symbols: Five (21%) subjects declared that the negative number was a kind of symbols or signs. (5) Comparing with zero: Four (17%) subjects compared negative numbers with zero and stated that negative numbers were the kind of numbers that are smaller than zero.

After analyzing the data, I found that there was some phenomena worth to be mentioned and I will discuss them below.

(1) When Describing what negative numbers were, only about 42% subjects mentioned something relate to mathematics definitions or principles, which were classified into three kinds: the opposite numbers of positive numbers, the numbers smaller than zero, and the results of smaller numbers minus larger numbers. It raised a situation that conflicts somewhat with the common sense about mathematics teachers. When a mathematics teacher want to explain a mathematics concept, he/she usually gives mathematics definitions or at least gives some illustration that closely related to mathematics definitions. However, more than a half of the subjects in this study, who were all mathematics teachers, abandoned the mathematics ways.

(2) When the subjects described what a negative numbers was, 15 of them (62.5%) mentioned the words ' real life' or 'physical world.' Some said negative numbers existed in the physical word while some others said negative numbers were used to express world objects or phenomena without mentioned the existence of negatives. However, some of the subjects held opposite opinions, that is, negative numbers did not exist or had no real meaning in the world.

(3) Nine (37.5%) subjects included some sentences that tried to give a reason for the existence of negative numbers. Six of them said that negative numbers were created by human being in order to represent 'not enough' or 'opposite' situations. Two others mentioned that the negative number was an inescapable production of a smaller number minus a larger number.

(4) From the view of history, one important reason for the negative number to be developed is that it helps the process of solving equations and provides consistent forms of solutions for some equations with different coefficients and constants. However, none of the subjects tried to touch this point.

<u>The beliefs of why secondary students need to learn</u>

The subjects' beliefs about why secondary students needed to learn negative numbers, before the learning activities of negatives' history proceeded, is described in the following paragraphs.

Almost every subject listed a bunch of items that they thought were the values of negative numbers for the secondary students to learn. The subjects mentioned the following seven patterns listed the most among all the reasons they provided. (1) To build or learn the number system (46%). (2) To be the foundation knowledge for later learning (42%). (3) Negative numbers exit in the world, so students should learn (33%). (4) To improve mathematics ability (29%). (5) To use to recognize, record, and represent daily life problems (29%). (6) To learn opposite concepts (17%). (7) For examinations (13%). The rest reasons could not be classified into the same category with more than 10% subjects in it.

In general, most of the subjects (71%) believed that students needed to learn negative numbers for the sake of gaining mathematics knowledge. If we broaden the concept of mathematics knowledge to include mathematics abilities, then there will be 88% of the subjects thought to gain more mathematics knowledge is one of the reasons to learn negatives for secondary school students. Comparing this percentage (88%) with the percentage (42%) of mentioning mathematics definitions when the subjects described negative numbers, it became strange that mathematics knowledge stood in different positions. Exactly a half of the subjects believed that students had to learn the negative number because it relates to our life. They either thought the negative number was an element of the world or could be used to manage daily life problems.

During and After Learning Activity

During the group discussion in the class, the subjects expressed surprising and struggling about the development of negative numbers.

After the group discussion, subjects showed some new appreciation and perception about negative numbers. I will extract some of their words in the following section.

<u>New appreciation or perception</u>

Group 1

'The feeling of negative numbers are related to personal views of philosophy. Throughout the history, the eras mathematician lived are not the same as the era our students live. Now the use of negative numbers is very widespread. Students have encountered "negative numbers" often since they were little kids. So, our students may doubt the reasonableness of the negative number but they are willing to accept it …'

'For those students who do not want to accept negative concept, we can give suitable examples, But the problem is that [the suitable examples] are special cases, which can not be used for all situations. When students want to transfer or analogy the concepts [which they learned from the examples], they will still encounter difficulties.'

Group 2

'From the development of negative numbers in the history, we know that the concept of negatives is different from the concept images we had before. (We thought: [negative numbers are] in everyday life, [which is] made out from nothing.)'

'The negative number was developed according to the need of solving equations.'

'[The negative number] allowed equations with the same forms to have the same forms of solutions.'

'[The negative number] made it convenient to represent quantities.'

'Chinese way of using sticks to operate negatives are one way [well] explains "the plus and minus operations of negative numbers."'

The forms [both in concepts and ways to use] of negative numbers now are formulated through a lot of debates by mathematicians. We ought to recognize that it will be hard for students to accept negative numbers.'

Group 3

'We have a deeper understanding of the development of negative numbers.'

‘We would never think that “the operations of negative numbers” are certainly reasonable.’

‘Now we are more able to realize the confusions of students in learning [negative numbers.]’

‘When giving examples [about negative numbers to students], we will be a little bit lacking in confidence.’

Group 4

‘We suddenly know that the development of negative numbers took long time to reach the current stage and did not purely come from the opposite concept of positive numbers.’

‘We found some new perspectives about the operations of negative numbers, but in teaching, we should still guide students to think from the concept of the opposites of positive numbers.’

The beliefs of what are negative numbers

I will not report this issue as I did to the same issue which described the subjects’ beliefs before the learning activities. In stead, I will compare the subjects’ beliefs before the learning activities with those after the activities; in the meanwhile, the comparison will be based on groups’ labels or patterns of their beliefs.

For group 1, the original beliefs of group members were classified into 7 different labels. After the group discussion, the group members only agreed two labels, which were ‘the negative number is the opposite of the positive number’ and ‘the negative number represents opposite quantities.’ These two labels were appeared in their beliefs before the learning activity and no other new beliefs emerged.

For group 2, the original beliefs of group members were classified into 11 different labels. After the group discussion, there was only one label repeated, which was ‘the negative number is the opposite concept of the positive number.’ The label ‘the negative number represents opposite quantities’ appeared and which differed from the previous one with the word ‘is’ switched to ‘represents.’ Another new belief emerged, which is related to the history of negative numbers. The belief was that the negative number was developed in order to allow equations with the same forms to have the same forms of solutions.

For group 3, the original beliefs of group members were classified into 12 different labels. After the group discussion, there was only one label repeated, which was ‘the negative number is the opposite concept of the positive number.’ Another new belief emerged, which was that the negative number was a creation for solving equations.

For group 4, the original beliefs of group members were classified into 13 different labels. After the group discussion, there was only one label repeated, which was ‘the negative number is a kind of symbols that could be attached with some principles and operations.’ Another new belief emerged, which was that the negative number was developed to enlarge the number system of positive numbers in order to allow some equations or operations to operate.

In summary, after the group discussion, the subjects’ beliefs about what the negative number was were contracted. All the beliefs they mentioned could be classified into four thoughts: the opposites of positive numbers, the numbers representing opposite quantities, a kind of symbols, and a production of historical affairs about mathematics.

The beliefs of why secondary students need to learn

Although the subjects made big changes with their beliefs about what negative numbers were the labels and patterns of the Beliefs about why secondary students need to learn negative numbers were not departed from the original ones. Something worth to be mentioned was that about the issue of the history point of view. About 42% of the subjects, prior to the learning activities, thought to learn negative numbers was for the sake of enriching the foundation knowledge for later learning, however, among these subjects, only one subject (in group 3) regarded solving equations as one of so called 'later learning.' The rest subjects either did not come up any mathematics concepts or declared that it was for the learning of the plane coordinate system and the vectors to learn negatives. After the group discussion, all four groups thought that to be the foundation for later learning of mathematics was one of the reason for the students to learn negatives, meanwhile, three of the four groups mentioned solving equations as one of the later learning subject.

The members of group 4 came up one interesting point, which was 'for the students to know the arduous progress of negative numbers, and so as to develop students' confidence about learning. "To doubt" in the place "doubtless" is the spirit of studying knowledge.'

Knowledge of negative number

As I mentioned above, the purpose of the examination was to investigate the feasibility of learning negatives' history and not to give grades to the students. So the result of the 'Knowledge of Negative Number' examination would only be used to give a broad view of the remained concepts of the subjects after a period time of the learning activities. As a result, I will report this portion by general descriptions.

For the 15 items in the yes-no questions, the percentage of correct responses was 85.42%. The percentage of correct responses for each individual question was varied from 37.5% to 95.8%. The most difficult one was the question: 'It is well documented that the operations of negative numbers on the number line helped the development of the negative concept a lot.' For the multiple-choice questions, the percentage of correct answers was 75%. For each individual question, the lowest correctness was 54.2% and the highest one was 100%. For the first essay problem, all of the subjects, except one, could provide suitable examples to explain why some mathematicians could not accept the concept of negatives. For the second essay problem, seven subjects could not successfully give some illustrations on the developing orders between 'the number concept' and 'the operations' of negative numbers. The mean score of this examination for the whole class was 82 points (total 100).

SUMMARY

From the discussion above, we can conclude that learning the history of negative numbers would change secondary school mathematics teachers' beliefs of the negative number and its teaching. Moreover, it is feasibility for the teachers to learn and retain the history of negative numbers when the learning environment provides a change of group discussion.

The study was definitely not a complete one. There are a lot of issues can be studied further. I provide two below. In this study, more than half of the subjects did not think about the necessity to give mathematical explanations. The question popped now is that, when asking teachers about another mathematics concept such as linear equations, how many teachers will still not give mathematical explanation?

When teachers described the values of negative numbers for secondary students, did it somewhat reflect that the underline beliefs of teachers were not only for secondary students, but also for all level of students, even for themselves as teachers?

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What Are Teachers' Views of Mathematics？

- An Investigation of How They Evaluate Formulas in Mathematics.

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Abstract

Apparently, mathematics formulas are usually expressed in symbols, but there should be more meanings than only the outlook symbols of formulas. Mathematics formulas somewhat described the ultimate conditions of the development of mathematics rules and the generalized outcomes of the world phenomena. It is interesting to know if we dig deep into people's views of mathematics formulas, will it give us more understanding of what people's specific thoughts about mathematics? Therefore, the purpose of this study was to investigate teachers' views of mathematics through their evaluations of mathematics formulas from the following three dimensions: the beauty of these formulas, the usefulness of these formulas, and the needs toward the history of these formulas.

The subjects of this study consisted of 419 pre-service and in-service teachers. The questionnaire contained 5 questions about the views of mathematics formulas and provided 38 alternatives of mathematics formulas for the subjects to choice. All of the subjects were asked to fill out the questionnaire and 9 of them were interviewed afterwards to get deeper information of their views. The results of this study showed that teachers' views differed in different groups; however, some of the formulas were popular for every group of teachers. Some characteristics were the keys to judge the beauties of formulas for the teachers, such as "symmetry and regularity," "domain-overlapping," "conciseness," and so on. One prevailed view of teachers to judge the usefulness of formulas was "the function of the formulas toward solving mathematical problems." Some hidden views of mathematics of the teachers were also found in this study.

Introduction

Although the student-centered teaching activities on mathematics are more and more emphasized now, the mainstream of mathematics teaching in Taiwan is still to lecture by teachers presenting mathematical knowledge in class and students absorbing it. Therefore, "what teachers teach" and "how they teach" determine "what students learn" and "how they learn". Besides, the latter are usually influenced by teachers' beliefs and values of mathematics and mathematics education. This is the reason why to understand the mathematics teachers' concepts of " what mathematics is " and "why people should learn math" becomes so important. The foreign and native researchers have made great efforts to study mathematics teachers' views of mathematics and how the change of the views affects teachers' teaching. These researches are expected to improve the current situation of mathematics education, which is also the main purpose of this study.

What has to be mentioned first is that it is very difficult to give "the views of mathematics" a precise definition. Furthermore, the subjects in the study were in-service teachers and pre-service teachers for secondary schools. Thus, the term "the views of mathematics" here refers to the general ideas of teachers for questions such as "What is math?" "Why should people learn math?" and so on. We neither seriously regard the term as "belief" or "value" nor think over the nature of mathematics from the perspective of mathematics philosophy, like ontology and epistemology. We adopt the more ordinary and the more general ideas for mathematics.

For the study of the views of mathematics, one convenient way is to get the answers by questionnaires or interviews. However, Munby (1982) considered that this simple approach is unsuited for some reasons. He described, "*...individually we may not be the best people to clearly enunciate our beliefs and perspectives since some of these may lurk beyond ready articulation.*"(p.217) In our on-the-spot survey by questionnaires and interviews, when teachers were asked the questions, like "What do you think mathematics is?" or "Why do students need to learn math?" two kinds of reactions usually came out:

1. The interviewees were not able to completely describe their opinions by words. They responded: "The question is too huge and too hard to answer"; "I've never thought about it" even "This is a philosophical question and I know nothing about it." They might just give unorganized and hollow descriptions. All these responses revealed that teachers themselves seldom seriously considered these questions.
2. The other kind of interviewees, most of whom were in-service teachers with teaching experience for several years, answered the questions in a variety of simplified viewpoints. In addition, it was not easy to tell whether their views of mathematics were only direct imitations from books or someone else, for examples, to regard mathematics as "the ability to deal with numbers and figures," "the science to train the ability of logical thinking," "a tool for calculation," "a process of discovery, thinking and problem -solving" "a philosophy" and etc.

We believed it did not mean that the first kind of interviewees did not have their own views of mathematics; likewise, it did not mean that the second kind of interviewees did really have their personal views and meanwhile could completely demonstrate them.

Moreover, many researchers went into the classroom to study if the teachers' beliefs accord with their teaching behaviors. This method could not only further study the correlation between the teachers' beliefs and their teaching behaviors but also provide the evidences about whether the questionnaire or interview inquiry could really reveal the teachers' views of mathematics. However, some researches showed that teachers' beliefs were consistent with their teaching behaviors, while others showed inconsistent results. For instance, Thompson (1984) discovered an inconsistency between the teachers' beliefs and teaching behaviors. When being asked to explain the reason for the discrepancies, these teachers gave two explanations: 1. They may contain unexamined clusters of conflicting beliefs; 2. there were various sources of influence on their instructional practice, causing them to subordinate their beliefs (Thompson, 1992). For the inconsistencies might exist, it was unreliable to study the views of mathematics by their actual teaching behaviors. Therefore, we implemented another way to inquire the teachers' views of mathematics—the evaluation for mathematics formulas.

Mathematics Formulas vs. Mathematics

There are some reasons to focus the discussion of the view of mathematics on mathematics formulas. First, the mathematics formulas are peculiar to mathematics. It is not like other mathematics entities, such as logic or reasoning, which are intertwined with other fields. Second, mathematics formulas usually described the ultimate conditions of the development of mathematics rules and the generalization outcomes of the world phenomena. For example, Pythagorean theorem: $a^2 + b^2 = c^2$ represents not only an equation expression but also the process and the result of mathematics development. Third, mathematics formulas are succinct and universal. For example, Euler's polyhedron theorem, v-e+f=2, can demonstrate the certain relation among the points, the lines and the surfaces of all the cubes. Finally, and the most important of all, we can get more concrete and delicate answers easier by focusing on mathematics formulas than on the broad and abstract concept "mathematics". It is reasonable to believed that the specialty of mathematics formulas might remind the teachers of the viewpoints they neglect previously and make the teachers easier to express their opinions.

Besides, for teachers and students at school, using “math” usually means using “mathematics formulas.” They use them to assist calculation as well as to solve mathematics problems and even they regard the ability to operate mathematics formulas as the most important assignment in mathematics course. Consequently, the mathematics formula becomes an effective indicator to represent mathematics when analyzing the views of mathematics.

Some Perspectives of Mathematics Formulas

1. *Beauty*

The beauty has the objectivity (the original characters and nature of an object) and subjectivity (the personal emotion toward the object); therefore, the beauty is very difficult to be separated from math. Hardy (1998) asserted, “Beauty is the first test: there is no permanent place in the world for ugly mathematics”. Russell (1942) claimed: “Mathematics, rightly viewed, possesses not only truth, but supreme beauty---a beauty cold and austere. Aristotle would not divorce mathematics from aesthetics, for order and symmetry were to him important elements of beauty, and these he found in mathematics. (Kline, 1972) Evidently, in order to understand one’s views of mathematics, it is absolutely helpful to explore his viewpoints of the beauty of mathematics, and the mathematics formulas are exactly one of the easiest way to exhibit the beauty of mathematics.

2. *Usefulness*

Why should we learn mathematics? What can mathematics do? These kinds of questions are the basic indicators to study the views of mathematics. They reflect the value for mathematics on mathematics teachers’ minds and also affect their motivations and attitudes toward learning and teaching mathematics. Compared with “beauty” of mathematics formulas, the application of mathematics in other fields added the subjectivity to the dimension of ”usefulness.” We had to admit that without mathematics many subjects become obstructed. Besides, using the usefulness of mathematics formulas, we could reveal the interaction and connection of different branches of mathematics (we will call it **domain-overlapping** since now). For instance, “de Moivre’s formula” can combine the complex number and trigonometry.

Mathematics teachers are in the forefront of mathematics education. Teachers decide the contents of teaching and teach what they feel importance, which often comes form their own learning experiences. As the mathematics experiences of teachers differ form those of students, what teachers emphasized may not correspond to what students’ need and still teachers does not pay attention to the discrepancy. Therefore, it becomes important to explore the viewpoints of both teachers and students and to compare them.

3. *Needs for history of mathematics formulas*

No one would doubt the importance of teachers’ knowledge and its impact (Fennema, 1992). In Taiwan, most of the mathematics teachers of secondary schools hardly have difficulties about mathematical knowledge in the secondary school’s mathematics-curriculum. The mathematics knowledge or ability has been fully emphasized in teacher-training programs of the normal universities. However, during the whole process of the training program, the knowledge about history of mathematics has not been emphasized. It results in a lack of the sensitivity and the humanistic concern of math. Many teachers feel that they learn mathematics because they have to, but they do not understand the background of it. Consequently, teachers are not able to demonstrate the history of mathematics freely in class, which reduced the opportunity to “soften” the class. As a result, it becomes important to know what mathematics history teachers like to learn the most in order to supplement their needs and to use in their teaching.

The Purpose of the Study

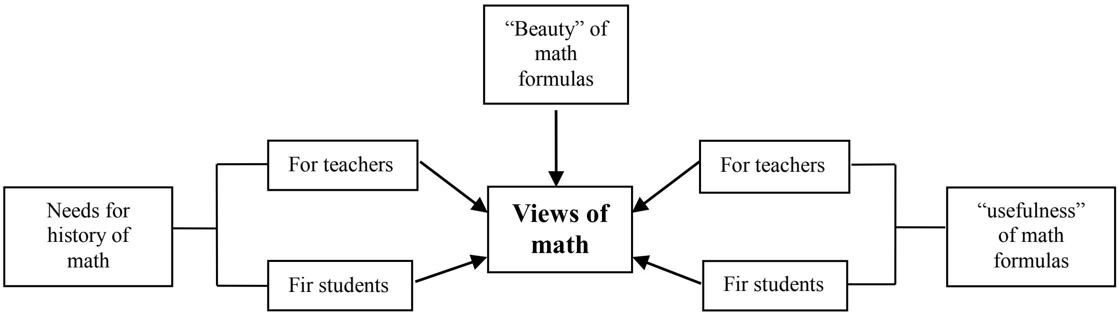
In order to understand the issues we discussed above, we formulated the present study. The purpose of this study was to investigate teachers' views of mathematics through their evaluations of mathematics formulas from the following three dimensions: the beauty of these formulas, the usefulness of these formulas, and the needs toward the history of these formulas.

The method

In the study, we adopted both the quantitative questionnaire method and the qualitative interview method. The term "teachers" below includes pre-service teachers and in-service teachers.

Frame

According to the issues we discussed above, we started from three chosen dimensions: "beauty" of mathematics formulas, "usefulness" of mathematics formulas, and "needs for history of mathematics". The frame of study was shown below:



Subjects

The subjects included 338 undergraduates, 35 graduates, 22 practice teachers and 24 in-service teachers. All the undergraduates and graduates were studying at the mathematics department of National Taiwan Normal University (NTNU). For the undergraduates, there were 84 freshmen, 81 sophomores, 96juniors, and 77 seniors. Almost all of them would be mathematics teachers, and all of the seniors had teaching practice for four weeks and most of them have taken the course of history of mathematics. All of graduate had teaching experience for at least one year.

The practice teachers were all graduated form NTNU and they all had taken the course of history of mathematics. The in-service teachers were all studying mathematics education in graduate school of mathematics department at NTNU. Out of these 24 teachers, 13 were senior high school teachers and 11 were junior high school teachers. Their teaching experiences were from 4 years to 21 years, and the average was 9.7 years.

To contrast the students' viewpoints with the teachers', 139 grade twelfth students in Taipei County were chosen to participate in our study. Among them, there were 58 in the science curriculum section and 81 in the liberal arts curriculum section.

Table 1. The Statistics of the people taking the questionnaire

Classification	Group		Amount	Taken the course of History of mathematics	Total	Footnote
Pre-service teacher	Freshman		84	None	338	Students of National Taiwan Normal University
	Sophomore		81	None		
	Junior		96	Few		
	Senior		77	Most		
	Graduate school (Master)		35	24 among 35 persons	35	25 have teaching experience
In-service teacher	Practice teacher		22	All	46	All over Taiwan
	Students in graduate school of	Senior high school teacher	13	Taking the course right now		

	math education	Junior high school teacher	11			
Students of the twelfth grade	Natural science section		58	None	139	Taipei County
	Social science section		81	None		

In order to further understand the teachers' thoughts and investigate the causes more deeply, we picked nine teachers from different groups as interviewees. They were two in-service teachers (T1 and T2), two practice teachers (PT1 and PT2), three seniors (S41, S42 and S43) and two juniors in NTNU (S31, S32).

Table 3. Data of the interviewees' backgrounds

Code	Sex	Backgrounds	Year of teaching	Place of teaching	The course of history of mathematics	Footnote
T1	F	Junior high school teacher	5	Kaoshiung City	Taking	The graduate school student of math education
T2	M	Junior high school teacher	6	Taipei City	Taking	The graduate school student of math education
PT1	M	Practice teacher in junior high	1	Taipei City	Taken	
PT2	F	Practice teacher in senior high	1	Taipei County	Taken	
S41	F	The senior in university	4 weeks	Taipei City	Taken	
S42	M	The senior in university	4 weeks	Taipei City	Taken	
S43	M	The senior in university	4 weeks	Taipei City	Taken	
S31	M	The junior in university	None	None	No	
S32	M	The junior in university	None	None	No	

Instrument

A questionnaire was conducted to investigate teachers' views of mathematics formulas. As we mentioned before, it is hard for teachers to express in words the hidden views of their minds only by asking broad questions. We felt that it was necessary to set an approach to make teachers exceed their obstacles in perception as well as expression and then exhibit the hidden views; therefore, in the questionnaire we asked them to vote mathematics formulas. We notice that when people make choices, they would take their own points of views into consideration either consciously or unconsciously. For example, in "the Beauty Contest for Miss China", the esthetic senses and views of social expectations of the judges will be taken into consideration. By taking the vote on mathematics formulas, it might be helpful to exceed the interviewees' "limitation of expression and perception" and directly find out the viewpoints of mathematics that are beyond words or unperceived.

According to our research frame, we devised 5 questions in the questionnaire as shown below:

(1) The questions

① In your opinion, what are the most beautiful mathematics formulas?
② In your opinion, what are the most useful mathematics formulas?
③ Which mathematics formulae's history do you want to know the most?
④ In your opinion, what are the most useful mathematics formulas to the secondary school students?
⑤ In your opinion, which history of the mathematics formulae you would like to introduce to secondary school students the most?

(2) The alternatives

The questionnaire offered alternatives in order to avoid the trouble in statistics and people would be more willing to answer. However, by this approach, the choice of alternatives must be more carefully. We followed several principles as described below.

① Covering wide ranges of domain in mathematics and being not rarely seen.
② Being representative and not having too many homogeneous qualities.
③ Being restricted in the required courses in secondary schools or universities.
④ Appearing in the forms of symbols as long as possible.

⑤ Having simply forms instead of verbose formulas.

To be cautious, we had a pilot study. We adopted 15 alternatives from the book "*The Most Beautiful of Mathematics Formulas"* (Salem, Testard, & Salem, 1992) and other 14 alternatives from the textbooks of secondary school. The total was 29. We also kept the blanks for the people to avert cases the provided alternatives were not sufficient enough. People who took part in the pilot study were about 300 students in mathematics department at NTNU.

We revised the alternatives according to the results of the pilot study and after discussing with the professors of mathematics department, then we made a draft plan. The graduate school students from different majors confirmed the form of the formulas in the draft plan. Then, we had a final and formal version of the questionnaire (see Appendix I) with thirty-eight alternatives distributed to six classifications and the blank kept in reserve. The alternatives from the first to the twenty-fourth were the materials of secondary school and the others were the contents of courses in university.

Table 2. The classification of alternatives (A: alternative)

Classification	Unit	A	Classification	Unit	A	Classification	Unit	A
一. Number theory	Decimal system	1	3. Algebra	Exponentiation	13	4. Calculus	Series	31
	Fundamental theorem of arithmetic	2		Logarithm	14		Derivation & differential	32
	Fibonacci sequence	25		Series	15 16		Taylor series	35
	Prime number theorem	26		Product representation	17		Integral calculus	33
2. Geometry	Triangular sides and angles	4. 5		Inequality	18		Fundamental theorem of calculus	34
	Measurement of area	6. 7 8. 9		Trigonometric formula	19 20 21	5. Complex analysis	$e^{i\pi} = -1$	36
	Analytic geometry	10		Equation	23		Euler formulae	37
	Conic	11		Complex number	24		Cauchy integral formula	38
	Solid geometry	12		Fundamental theorem of algebraic	25	6. Others	Set	3
	Topology	27					Combination	22
	Differential geometry	28					Probability	30

Procedures

We conducted the survey of the questionnaire three months later after the pilot study to avoid the possible disturbances of formal questionnaire. The survey was made in class and the time for the survey was no more than ten minutes. When answering question①, ② and ③, they could pick up at most seven alternatives from all the provided alternatives. The answers to question④ and ⑤ were limited in the curriculum of secondary school (alternative 1-24) and people could pick up at most five alternatives. Because we made the quantitative survey in this part, they were not asked to arrange the picked alternatives in ranks.

The main purpose of the interview was "understanding the thoughts of the interviewees at the time of answering the questionnaire and interviewing" and "what caused these thoughts" in order to make sure if we did the proper interpretations and to refine upon our interpretations. At the beginning of every question, the interviewees were asked to recall why they chose those formulas.

In the part of "beauty," the interviewees were asked to predict which alternatives would not change with time and to describe the common qualities of these alternatives. Afterward, they would be asked to compare the beauty of three pairs of formulas: (1) $e^{i\pi} = -1$ and $e^{i\pi} + 1 = 0$, (2) $e^{i\pi} = -1$ and $e^{ix} = \cos x + i \sin x$, and (3) Fiboncci sequence general form "$F_n = F_{n-1} + F_{n-2}$" and number form "1.1.2.3.5…." By this way, the more refined and detailed viewpoints were obtained. Then, they were asked to give their opinions for what the beauty of mathematics was, and compare the beauty of mathematics with mathematics formulas. Finally, the interviewees were asked to see the statistical results and give comments.

In the part of "usefulness", after the interviewees stated the reason for their alternatives, the conversation was focused on the differences between question② and question ④, and where to use them. At last, they were asked how to respond the students' questions, "Why should I learn math?"

In the part of "the needs for history of mathematics", the focus was on the differences between question③and question⑤, and why they need them. They were asked to answer the questions that if they would use history of mathematics in class, how to use it and what could be for the students' own good.

The Results

The Results of the Questionnaires

The statistical results of the questionnaires were separated in nine groups and presented the top 7.

1. *The most beautiful mathematics formulas*

"Pythagorean theorem" was the top one of all the nine groups and got 51% of all the votes. It revealed that this theorem with long history was still full of glamour. The formula " v-e+f=2" rose to the top 7 of the groups from the sophomore and the higher in the groups. (Why?) Both "Cauchy inequality" and "Heron's formula" were popular in this dimension. The favorite of mathematicians, "$e^{i\pi} = -1$", didn't appear until we went over the list to the senior's. The top 2 of the masters, Euler series $\frac{\pi^2}{6}$, did not stand high in other groups' favor. Among all of the alternatives, the Gauss-Bonnet formula was the least favored (4%).

Table 4. The rank and the votes of the most beautiful mathematics formulas

Group (total) / TOP	1		2		3		4		5		6		7	
Freshman (84)	Pythagorean Thm.	42	Cauchy ineqn.	35	Taylor series	26	F.T. of calculus	25	Sine rule	23	De Moivre's Thm.	22	$F_n = F_{n-1} + F_{n-2}$	21
Sophomore (81)	Pythagorean Thm.	37	e^{ix} =conx +isinx	26	Sine rule	24	Heron's Formula	23	Cauchy ineqn.	23	$\cos^2\alpha + \sin^2\alpha = 1$	22	v-e+f=2	21
Junior (96)	Pythagorean Thm.	49	Heron's Formula	33	Cauchy ineqn.	30	F.T. of calculus	30	v-e+f=2	29	$F_n = F_{n-1} + F_{n-2}$	28	Taylor series	27
Senior (77)	Pythagorean Thm.	38	v-e+f=2	32	Cauchy ineqn.	22	$e^{i\pi} = -1$	22	$F_n = F_{n-1} + F_{n-2}$	21	F.T. of Calculus	21	Taylor series	21
University (338)	Pythagorean Thm.	166	Cauchy ineqn.	110	v-e+f=2	98	F.T. of calculus	96	Taylor series	92	Heron's formula	90	$F_n = F_{n-1} + F_{n-2}$	90
Master (35)	Pythagorean Thm.	19	Euler series $\frac{\pi^2}{6}$	15	v-e+f=2	14	Heron's formula	12	Cauchy ineqn.	10	F.T. of Algebra	10	$e^{i\pi} = -1$	10
Teachers (47)	Pythagorean Thm.	28	v-e+f=2	20	Heron's formula	17	Cauchy ineqn.	16	$e^{i\pi} = -1$	15	$F_n = F_{n-1} + F_{n-2}$	15	De Moivre's Thm.	14
12th grader/ social science (81)	Pythagorean Thm.	51	Heron's formula	33	$\cos^2\alpha + \sin^2\alpha = 1$	32	S.A.T=180°	26	Product representation	24	Sine rule	24	Circle= πr^2	22
12th grader/ natural science (58)	Pythagorean Thm.	24	Heron's formula	21	Cauchy ineqn.	21	Circle = πr^2	18	Distance from a point to a line	15	Sine rule	15	S.A.T=180°	14

(Note. Thm.= theorem; Se. =Sequence; F.T.= Fundamental Theorem; S.A.T.=the Sum of Angles of a Triangle; Q.E.= quadratic equation; ineqn.= inequality)

2. *The most useful mathematics formulas*

"Pythagorean theorem" took the first seat again and gained more votes (71%). The "Fundamental Theorem of calculus" and the "Cauchy inequality" were still favored by all teachers. "Distance from a point to a line" and "Discriminant of Q.E." were on the list of teachers and high school students while "Heron's formula" and "v-e+f=2" were out off the list. It was interesting that the Circle= πr^2 considered beautiful by high school students did not appear on their list of "usefulness." On the contrary, it appeared on the list of the teachers'. It revealed that "beauty" did not necessarily come along with "usefulness."

"L'hospital rule" was highly grade on the freshman and the sophomore students while the junior were in favor of "C_n^m," which is obviously affected by the courses, which they are taking now. This also shows us that the views of mathematics were likely to be influenced by the present experiences. The least favored is Prime number theorem; only two people put it on their list.

Table 5. The rank and the votes of the most useful mathematics formulas

TOP Group (total)	1		2		3		4		5		6		7	
Freshman (84)	Pythagorean Thm.	58	Distance from a point to a line	32	Circle= πr^2	31	L'Hospital rule	29	Cauchy ineqn.	26	F.T. of calculus	25	Discriminant of Q.E.	24
Sophomore (81)	Pythagorean Thm.	59	Cauchy ineqn.	28	L'Hospital rule	24	Discriminant of Q.E.	23	S.A.T=180°	21	F.T. of calculus	20	Distance from a point to a line	18
Junior (96)	Pythagorean Thm.	71	C^m_n	31	Distance from a point to a line	28	Cauchy ineqn.	26	Discriminant of Q.E.	26	Circle= πr^2	25	F.T. of calculus	24
Senior (77)	Pythagorean Thm.	55	F.T. of calculus	22	Cauchy ineqn.	20	Decimal system	19	Arithmetic Series	19	C^m_n	17	Circle= πr^2 v	16
University (338)	Pythagorean Thm.	234	Cauchy ineqn.	100	F.T. of calculus	91	Distance from a point to a line	89	Circle= πr^2	86	Discriminant of Q.E.	86	Decimal system	79
Master (35)	Pythagorean Thm.	22	F.T. of calculus	17	L'Hospital rule	13	Discriminant of Q.E.	12	$\int udv = uv - \int vdu$	11	Cauchy ineqn.	10	Taylor series	10
Teachers (47)	Pythagorean Thm.	36	Cauchy ineqn.	19	F.T. of calculus	13	Discriminant of Q.E.	13	Taylor series	12	Cosine rule	10	Circle= πr^2	10
12th grader/ social science (81)	Pythagorean Thm.	64	Distance from a point to a line	48	Arithmetic Series	42	Discriminant of Q.E.	42	$\frac{1}{2}ab\sin\theta$	36	$(a+b)^2 = a^2 + 2ab + b^2$	32	S.A.T=180°	27
12th grader / natural science (58)	Pythagorean Thm.	41	Discriminant of Q.E.	29	Distance from a point to a line	22	$\frac{1}{2}ab\sin\theta$	20	Cosine rule	20	Arithmetic Series	19	$(a+b)^2 = a^2 + 2ab + b^2$	18

(Note. Thm.= theorem; Se. =Sequence; F.T.= Fundamental Theorem; S.A.T.=the Sum of Angles of a Triangle; Q.E.= quadratic equation; ineqn.= inequality)

3. *The mathematics formulas whose history was most wanted*

In this dimension, the chosen alternatives were widespread among six classifications and different groups had their own favorites. Generally, teachers answered this question for two reasons: "the history of mathematics can be used in real mathematics curriculums" and "their thirst of knowledge can be satisfied." The formulas " v-e+f=2" "Cauchy inequality" " Heron's formula" "$F_n = F_{n-1} + F_{n-2}$ " and "Pythagorean theorem" were chosen because of the former although the last two formulas did not appear on the list of high school students; "Prime number theorem" "$e^{i\pi} = -1$" "L'hospital rule" were chosen because of the latter.

Table 6. The rank and the votes of the mathematics formulas whose history is most wanted

TOP Group (total)	1		2		3		4		5		6		7	
Freshman (84)	v-e+f=2	38	Cauchy ineqn.	28	$F_n = F_{n-1} + F_{n-2}$	25	F.T. of calculus	25	Pythagorean Thm.	24	L'Hospital rule	22	Taylor series	21
Sophomore (81)	Cauchy ineqn.	33	Pythagorean Thm.	29	v-e+f=2	29	$F_n = F_{n-1} + F_{n-2}$	28	Heron's formula	17	$e^{i\pi} = -1$	17	Cauchy's integral formula	16
Junior (96)	Cauchy ineqn.	29	v-e+f=2	26	Heron's formula	25	Pythagorean Thm.	24	C^m_n	21	$F_n = F_{n-1} + F_{n-2}$	21	L'Hospital rule	21
Senior (77)	F.T. of calculus	24	v-e+f=2	20	$e^{i\pi} = -1$	17	Cauchy's integral formula	17	Cauchy ineqn.	16	Pythagorean Thm.	15	F.T. of Algebra	15
University (338)	v-e+f=2	113	Cauchy ineqn.	106	Pythagorean Thm.	92	$F_n = F_{n-1} + F_{n-2}$	86	F.T. of calculus	82	Heron's formula	73	Cauchy's integral formula	68
Master (35)	Prime number theorem	14	v-e+f=2	12	Cauchy's integral formula	11	$F_n = F_{n-1} + F_{n-2}$	10	F.T. of Algebra	9	Heron's formula	8	$e^{i\pi} = -1$	8
Teachers (47)	v-e+f=2	14	Taylor series	14	L'Hospital rule	13	Heron's formula	13	Cauchy ineqn.	13	F.T. of calculus	13	Volume of a spheroid	12
12th grader/ social science (81)	Cauchy ineqn.	36	Heron's formula	29	C^m_n	28	Discriminant of Q.E.	26	Distance from a point to a line	23	Logarithm rule	21	De Moivre's Thm.	18
12th grader / natural science 58)	Heron's formula	21	De Moivre's Thm.	16	Cauchy ineqn.	15	Volume of a spheroid	13	v-e+f=2	13	Discriminant of Q.E.	12	F.T. of calculus	12

(Note. Thm.= theorem; Se. =Sequence; F.T.= Fundamental Theorem; S.A.T.=the Sum of Angles of a Triangle; Q.E.= quadratic equation; ineqn.= inequality)

4. *The most useful mathematics formulas to secondary school students*

Compared with other results, the results here were of high homogeneity. The alternatives on the list of every group were almost the same, only with different orders.

For both students and teachers, including pre-service teachers, "Pythagorean theorem" "Discriminant of Q.E." and "$\frac{1}{2}ab\sin\theta$" were the most useful mathematics formulas to high school students. "the Sum of Angles of a Triangle=180°" and "Circle=πr^2", which appeared on the list of teachers, were not found on the list of students. "Cosine rule" and "$(a+b)^2 = a^2 + 2ab + b^2$" are ignored by the pre-service teachers. In the part, "Sets" gets the least votes (5%).

Table 7. The rank and the votes of the most useful mathematics formulas to secondary school students

TOP Group (total)	1		2		3		4		5		6		7	
Freshman (84)	Pythagorean Thm.	53	Distance from a point to a line	29	$\frac{1}{2}ab\sin\theta$	27	Cauchy ineqn.	27	Arithmetic Series	26	Discriminant of Q.E.	26	S.A.T=180	23
Sophomore (81)	Pythagorean Thm.	62	S.A.T.=180°	28	Circle= πr^2	26	the circumference of a circle	23	$\frac{1}{2}ab\sin\theta$	23	Distance from a point to a line	23	Arithmetic Series	21
Junior (96)	Pythagorean Thm.	77	Distance from a point to a line	38	Discriminant of Q.E.	33	S.A.T=180°	32	$\frac{1}{2}ab\sin\theta$	26	Arithmetic Series	24	Circle= πr^2	23
Senior (77)	Pythagorean Thm.	64	S.A.T=180	26	Circle= πr^2	24	Discriminant of Q.E.	24	Distance from a point to a line	23	Arithmetic Series	19	Decimal system	18
University (338)	Pythagorean Thm.	256	Distance from a point to a line	113	S.A.T=180°	109	Discriminant of Q.E.	100	$\frac{1}{2}ab\sin\theta$	92	Arithmetic Series	90	Circle= πr^2	89
Master (35)	Pythagorean Thm.	22	Discriminant of Q.E.	13	S.A.T.=180°	10	Cosine rule	10	$(a+b)^2 = a^2 + 2ab + b^2$	9	Cauchy ineqn.	9	the circumference of a circle	8
Teachers (47)	Pythagorean Thm.	38	S.A.T =180°	16	Discriminant of Q.E	15	Cosine rule	14	$(a+b)^2 = a^2 + 2ab + b^2$	12	$\frac{1}{2}ab\sin\theta$	11	Cauchy ineqn.	11
12th grader/ social science (81)	Pythagorean Thm.	59	Distance from a point to a line	44	Arithmetic Series	35	$\frac{1}{2}ab\sin\theta$	28	Discriminant of Q.E.	28	Cosine rule	19	$(a+b)^2 = a^2 + 2ab + b^2$	18
12th grader / natural science (58)	Pythagorean Thm.	35	$(a+b)^2 = a^2 + 2ab + b^2$	23	Arithmetic Series	20	Discriminant of Q.E.	20	$\frac{1}{2}ab\sin\theta$	17	Distance from a point to a line	17	Cosine rule	16

(Note. Thm.= theorem; Se. =Sequence; F.T.= Fundamental Theorem; S.A.T.=the Sum of Angles of a Triangle; Q.E.= quadratic equation; ineqn.= inequality)

5. *The mathematics formulas whose history should be taught to secondary school students the most*

Table 8. The rank and the votes of the formulas whose history should be taught the most

TOP Group (total)	1		2		3		4		5		6		7	
Freshman (84)	Pythagorean Thm.	46	Cauchy ineqn.	38	Heron's formula	21	Discriminant of Q.E.	18	Distance from a point to a line	16	S.A.T.=180	15	Circle= πr^2	14
Sophomore (81)	Pythagorean Thm.	54	Cauchy ineqn.	26	Circle = πr^2	25	the circumference of a circle	19	Heron's formula	18	Volume of a spheroid	18	C^m_n	16
Junior (96)	Pythagorean Thm.	63	Cauchy ineqn.	31	S.A.T.=180°	24	Discriminant of Q.E.	22	Circle= πr^2	20	Distance from a point to a line	18	Arithmetic Series	18
Senior (77)	Pythagorean Thm.	58	Circle= πr^2	20	Cauchy ineqn.	20	Volume of a spheroid	18	S.A.T.=180	17	the circumference of a circle	16	Decimal system	12
University (338)	Pythagorean Thm.	221	Cauchy ineqn.	115	Circle = πr^2	79	S.A.T.=180	69	Discriminant of Q.E.	65	Heron's formula	64	Volume of a spheroid	64
Master (35)	Pythagorean Thm.	22	Circle= πr^2	13	S.A.T.=180	11	Arithmetic Series	11	the circumference of a circle	9	C^m_n	9	Decimal system	8
Teachers (47)	Pythagorean Thm.	34	Cauchy ineqn.	21	Circle = πr^2	12	Logarithm rule	11	S.A.T.=180	10	Cosine rule	9	Heron's formula	8
12th grader/ social science (81)	Cauchy ineqn.	29	Pythagorean Thm.	27	Distance from a point to a line	25	Heron's formula	24	Discriminant of Q.E.	23	Arithmetic Series	22	C^m_n	20
12th grader / natural science (58)	Cauchy ineqn.	22	Heron's formula	18	Discriminant of Q.E.	17	Distance from a point to a line	16	Pythagorean Thm.	11	Circle= πr^2	11	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	11

(Note. Thm.= theorem; Se. =Sequence; F.T.= Fundamental Theorem; S.A.T.=the Sum of Angles of a Triangle; Q.E.= quadratic equation; ineqn.= inequality)

It seemed that high school students did not want to learn the history of "Circle $= \pi r^2$", while most teachers thought it should be taught to students. "Cauchy inequality" stood high in teachers' and students' favors but it did not show up on the list of the last dimension. Besides, when checking the data of question third, "Cauchy inequality" was found to be one of the mathematics formulas whose history teachers wanted to know the most. The result was very interesting. Teachers thought students should learn the history of the formula, which the teachers were not familiar with.

The Results of Interviews

We discovered that when the interviewees answered the questionnaires, they did not have enough time to think completely, so they picked out the formulas intuitively. Their primitive thoughts were reserved. But, in the interviews, they got enough time to think, and therefore some different opinions came out. The impacts forced them to reflect their mind further and more latent viewpoints were brought out of their mind.

In the part of “beauty”, the interviewees’ ideas had something in common.

(1) “**Amazing**” was probably the most used adjective. However, its definition differed a lot. For example, by **S41**’s definition, it meant the amazement of the process of discovering the formula. In **S42**’s opinion, the formulas magically connected different branches of math. **S43** thought the formulas were amazing because only knowing some of the requirements could lead in the whole results. **S32** felt amazed about the coincidence. **PT1** said, “ The magic of mathematics formulas is beyond words.”

(2) “**Symmetry and regularity**”: This was really compatible with Aristotle’s views of “beauty.” Mathematics formulas were the most remarkable form of symmetry and regularity. **S31** mentioned that symmetry was helpful to memorize them. **S32** thought the beauty of symmetry not only had visual effects but also provided a way to predict if the formula was right. For PT1, symmetry looked comfortable. And **S41** had a direct comment: “ Is there anything more beautiful than regularity?”

(3) “**Domain-Overleaping**” was also popular in the interviews. Alternative 24 and alternative 37 were relative to complex number and trigonometric; alternative 34 connected integration and differential. Alternative 31 presented the transcendental number in the form of rational number. Alternative 27 combined the points, the lines and the surfaces of solid. Alternative 36 integrated several fundamental elements in one formula. The interviewee, **S32**, even sensed that the theorem of prime numbers was the connection between continuous and discrete.

(4) “**Conciseness**”: As mathematics is the essence of rationality, mathematics formulas are to the essence of mathematics. **T1** said, “ mathematics is like a miniskirt, brief and beautiful.” **PT2** said, “mathematics is succinct and powerful; the more simple it is, the more beautiful it will be.” **S41**and **S31** indicated that beauty is necessary to be concise.”

(5) When making a comparison between “$e^{i\pi}=-1$” and “$e^{i\pi}+1=0$,” favors of the nine interviewees were different. Those who chose the former considered it was more succinct and more balancing and that it was more difficult for human to develop “negative number” in history of mathematics. The others chose the latter for the formula contained many fundamental elements of mathematics, such as e, i,π ,and the basic operational symbols (+, =). In the comparison between $e^{i\pi}=-1$ and $e^{ix}=\cos x+i\sin x$, the general form, was obviously popular.

For many people, “usefulness” was also a factor to determine if a mathematical formula was beautiful. They believed that a mathematical formula could never only go with visual effects. Without understanding the formula, people would hardly appreciate its beauty. And to use formula was the good way to know them.

In question two and question four, interviewees answered the questions from their own experiences or from the perspectives of teaching. However, almost all of them focused on mathematics learning. For example, they said these formulas were useful for students to solve mathematical problems, to get answers more quickly, to avoid making errors and so on. We believed that these responses were more likely to consist with the reality of mathematics education in Taiwan than ones like “to develop the ability to adapt to the environment” “to train people’s thinking”.

In the comparison between question two and question four, **PT2, S42** and **S43** made their choices for the similar reasons. **T1, T2**and **PT1** took the current teaching experiences into consideration for question four, but **S41**, **S31** and **S32** could only remind of their learning experiences in secondary school because of the lack of teaching experiences. At last, when answering the students the question “Why should I learn math?” they confessed that the question came upon them many times but they seldom could provide the answer completely. During the interview, we could also feel their confusion and uncertainty for this question. Some of them focused on the daily use or the career life (**S41, S43, S31**); some of them go back to the knowledge and the ability (**T1, PT1, PT2, S43, S32**); the others responded in an authoritative way (**T2, S42**), and answered: “Just to learn it without asking, you will know later”. Only **S42** and **S32** referred to the problems solving and the exams.

In question three, the interviewees’ reasons most came form the lack (they did not have chance to learn them) and the thirst, which comes from the curiosity (**T1, PT1, S41, S42** and **S31**) and the need for teaching (**T1, T2** and **S41**). They thought that a mathematics teacher should have some fundamental knowledge of the history of mathematics. It is helpful both for students to learn mathematics and for they themselves to teach. However, we found that teachers were very passive in finding this kind of information. Most of them said they have neither time nor access, and wished someone else could provide this information. This phenomenon should be taken seriously and worried by us.

As **for** the part of the formulas whose history should be taught the most, “used frequently or not” was the most important factor (**T1, PT2, S42, S31** and **S32**). The interviewees believed that the history of mathematics could contact the deeper thoughts of students. **T2** took the interest of students into consideration. **PT1** and **S43** would like to share with students because they felt interested. There is an exception that **S41** think it was not necessary to teach the history of mathematics in mathematics classes in order not to cause extra work for students and the time for the class was limitary. She thought that the teachers that should understand the history of mathematics themselves and put it into the teaching material beforehand and naturally.

What is **the** advantage of using history of mathematics in teaching and learning? The first one was to stimulate the motivation and to make class more interesting. Besides, some teachers believe that the use of history of mathematics could add more humane concern into mathematics education, make the teaching more systematized, make the learning more complete and make the predecessors good examples for the present.

Discussion and Suggestion

Comparison of Related Survey

David (1988) conducted a similar research on the readers of *The Mathematical Intelligencer* (vol.10 no.4 p.30). Wells also provided the readers twenty-four alternatives in advance and asked them to score the alternatives from 0 to 10 points. Because the readers were mostly professional mathematicians, his questionnaire contained the alternatives of a broader and deeper range, including the statements on the theorems and the properties. The collected questionnaires, Seventy-six in total, were mostly from mathematicians. The following table was their top-ten of the beauty. The result and that of the graduate school students in our study were much alike. Moreover, in Wells’ report on the further discussion of the ten themes of the beauty, many comments that the mathematicians presented also were seen in our interviews.

Table 10. The most beautiful ten mathematics formulas

Formula	Average score	Formula	Average score
1. $e^{i\pi} = -1$	7.7	6. A continuous mapping of the closed unit disk into itself has a fixed point	6.8
2. Euler’s formula for a polyhedron: V+F = E+2	7.5	7. There is no rational number whose square is 2.	
3. The number of primes is infinite.	7.5	8. π is transcendental.	6.7
4. There are 5 regular polyhedron.	7.0	9. Every plane map can be colored with 4 color.	6.5
5. $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$	7.0	10. Every prime number of the form 4n+1 is the sum of two integral squares in exactly one way.	6.2 6.0

(Quoted from David, 1990)

Discussion and Suggestion

In the study, we did not expect that all the views of mathematics of mathematics teachers could be discovered completely at this stroke. On the contrary, we tried from a special perspective to observe the roles of mathematics in teachers' minds. We had to admit that from our research design , the results of research could not completely depicted the views of teachers. However, we believed that we have explored out some views of mathematics in this study. The voting activity also provided the teachers a chance to re-think about and reflect on mathematics.

Mathematics formulas cannot represent mathematics completely. Many interviewees said: "without mathematics formulas, mathematics is still mathematics, but taking away mathematics formulas from it, the beauty and the usefulness would definitely reduce." According to this viewpoint, we found that mathematics teachers tended to **consider mathematics formulas a symbol representation of mathematical knowledge**. The existence of the mathematics formulas does not influence the nature of mathematics, but by the formulas mathematics can be revealed more clearly and succinctly. The characteristic of simplification is teachers' most general view of mathematics.

Teachers could not catch the beauty of mathematics formulas without mathematical knowledge and could not perceive the beauty only by sight. The judgment of beauty of mathematics was related to mental activities, and this kind of judgment became harder be analyze. For example, to the teachers in the study, the beauty of mathematics formulas had to be succinct, orderly, useful, domain-overlapping and so no. These judgments were specific for mathematics, but probably, not for literature, music or paintings.

When teachers describe the beauty of formulas, they showed highly surprising at "how the formulas were discovered", which made them decide to choice the formula as beautiful ones. However, they did not care about "who" developed the formulas. This phenomena was some what like "out of personality" for the mathematics. We will illustrate this further. As the teachers might not be familiar with Pythagoras or Cauchy at all, they still appreciated "Pythagorean Theorem" and "Cauchy inequality". Could we really appreciate Li-bei's poems, Beethoven's music or da Vinci's paintings without paying attention to the creator, their personalities, and their passions at the moment of creation? The front mentioned figures seem to be part of the life of the creation, but the life of mathematics seem to merely come from truth.

"Domain-overlapping" may not be specific to mathematics, for instance, electromagnetism is a large branch in physics combines electricity and magnetics. We cannot deny that the connection of different branches is helpful to develop a discipline. In this study, "Domain-overlapping" also influenced the judgment of teachers' views of beauty. The concepts seemed irrelevant or even contradictory could be connected together. Also many problems could usually be solved in various ways. We believed the teachers had a hidden view that **there is great capacity within every branch in mathematics**.

As I mentioned above, almost all of the teachers said that mathematics formulas were useful for students to solve mathematics problems. Their views of the usefulness of mathematics formulas were limited completely within mathematics. When answering the question "why should people learn math?" the nine interviewees gave ideal responses, like "to develop the ability to adapt to the environment" "to train people's thinking" and etc. But when further inquired, they could not give any real example for all mathematics concepts, except logic. Therefore, we could infer that these ideal responses are not truly innate within themselves. "The usefulness of mathematics is to solve mathematical problems" was the main views of the teachers. Consequently, we thought that in teachers' mind **mathematics was closed.**

Where does the closure viewpoint come from? Hsieh's study might give us a hint (Hsieh, 2000). In her survey, the factor that influenced the teachers' teaching most was "The Examinations", such as the National Entrance Examination and the quizzes. The examinations not only restricted the classroom teaching but also affected the views of mathematics of teachers. To fulfill their students' need for high scores on the exams teachers concentrate on the proficiency of the problem solving in mathematics . the things like those ideal education goals , which have no obvious relation to exam, are neglected usually. It might result in that mathematics are used only to solve mathematics problems.

The results of inquiring teachers' need for the knowledge of history of mathematics revealed least two things: first, in teachers' views, it was possible for mathematics teaching to include a sense of perceptual or sensitivity, which could be enhanced by using history of mathematics; second, the mathematics teaching in classroom was too inhumane so adding the history of mathematics would be good to the atmosphere. Most of the teachers expected to use the history of mathematics to improve their teaching context. That meant they thought their teaching in textbooks was out of context.

What mathematics formulas will you teach their history to your students? In this study, teachers had different favorites. In the interviews, most teachers would like to teach the history of those mathematics formulas that were used the most frequently. in this case, the function of the history of mathematics formulas was to help students learn the formulas. This gave us some worries. Under the environment of emphasizing exams, the ability of teachers to actively search for teaching resources had been reduced. If the situation of over-emphasizing exams is not improved, probability, teachers will give more exams on the history of mathematics they taught. Then all the effort to introduce the history of mathematics in class will become a disaster.

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Appendix I : The alternatives of mathematics formulas in questionnaires

1. Decimal system：ex: $23 = 2\times 10+3\times 1$
2. Fundamental theorem of arithmetic
3. $(A\cup B)\cap C = (A\cap C)\cup(B\cap C)$
4. the Sum of Angles of a Triangle $\alpha+\beta+\gamma=180°$
5. Pythagorean theorem： $a^2+b^2=c^2$
6. The circumference of Circle = $2\pi r$
7. The Area of a Circle = πr^2
8. The Area of a Triangle = $\frac{1}{2}ab\sin\theta$
9. Heron's formula : A = $\sqrt{s(s-a)(s-b)(s-c)}$
10 Distance from a point to a line = $\dfrac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$
11 Ellipse's equation： $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$
12. Volume of a spheroid = $\dfrac{4}{3}\pi r^3$
13. $a^m\times a^n = a^{m+n}$
14. log(ab)=loga + logb
15. 1+2+3+…+$n = \dfrac{n(n+1)}{2}$
16. $1+x+x^2+x^3+... = \dfrac{1}{1-x}$ (|x|<1)
17. $(a+b)^2 = a^2+2ab+b^2$
18. Cauchy inequality：

$$(a_1b_1+a_2b_2+...)^2 \le (a_1^{\,2}+a_2^{\,2}+...)(b_1^{\,2}+b_2^{\,2}+...)$$

19. $\cos^2\alpha+\sin^2\alpha=1$
20. $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$
21. $a^2=b^2+c^2-2bc\cos A$
22. $C_n^m = \dfrac{m!}{n!(m-n)!}$
23. Discriminant of quadratic equation: $\Delta = b^2-4ac$.
24. $[r(\cos\theta+i\sin\theta)]^n = r^n(\cos n\theta+i\sin n\theta)]$
25. Fibonacci sequence： $F_n = F_{n-1}+F_{n-2}$
26. The prime theorem： $\lim_{x\to\infty}\dfrac{\pi(x)}{x/\log x}=1$
27. Euler's formula for a polyhedron：v-e+f=2
28. Gauss-Bonnet theorem：

$$\oint_c k_g\,ds+\iint_D K\,d\sigma = 2\pi-\sum_{i=1}^{n}\alpha_i$$

29. Fundamental theorem of algebraic: Every polynomial of positive degree has a root that is a complex number.
30. Bernoulli's theorem： $P_n(k)=C_k^nP^k(1-P)^{n-k}$
31. $\dfrac{\pi^2}{6}=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\dfrac{1}{5^2}.....$
32. L'Hospital rule：

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}\quad(\tfrac{0}{0},\tfrac{\infty}{\infty}\text{型})$$

33. Partial integration ： $\int udv = uv-\int vdu$
34. Fundamental theorem of calculus：

$$\int_a^b f(x)dx = F(b)-F(a)$$

35. Taylor series: $f(x)=f(a)+\dfrac{f'(a)}{1!}(x-a)+\dfrac{f''(a)}{2!}(x-a)^2+...+\dfrac{f^{(n)}(a)}{n!}(x-a)^n+R_n(x)$
36. $e^{i\pi}=-1$
37. $e^{ix}=\cos x+i\sin x$
38. Cauchy integral formula：

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta-z}d\zeta = f(z)$$

39. (else others) ____________________

Developing of the area of calculus in history and its connection to development of ways of thinking

Ada Sherer (Israel)

In the history of mathematics we may make a distinction between voyage of the process of inventing by single human being and voyage in the process of invention when human beings continue each other's ideas through time.

In the first instance described, researches of processes of invention and creating solutions to problems can be used to follow thought paths in the past. Hypothesizing a direction, which is parallel to homogeneous characteristics of pattern of invention, data gathered from highly selected samples, does this. We might use it by looking in original text for certain common features in invention process, or by empirical observation, and elaborate an assumption we made.

The following three researches or descriptions that try to pull out repeated features in processes of high creation. They point on typical features in processes. (I) and (III) from observations or introspections (II) in the second instance, (III)parallelism between first and second instances. (I) Not specified to mathematics, (II) and (III) are such. The three descriptions show possibility to characterize successive stages in process of high creative thinking. In all of them verification occurs, and always preceded by well characterized other stages. Thus high-creative mathematicians can some times work correctly with mental objects before precise definition or proof and come to correct results before the doing of verification[1]. But it is not to be considered mathematically till rigor proof is established. In those cases in which rigor has eventually been accomplished, to my humble opinion, planning curriculum and context along the lines of thinking of high creative mathematicians might be strengthening the natural way of thinking of high school pupils. Might be will encourage them to rereveal. Along the beginnings of paths might be proof not rigor. In fact this is what we do in several themes before university. In some explanations it is pedagogical to do so, it is conditioned we teachers keep in mind that there is a formal axiomatization to prove it's correct when it is the case. In the area of calculus Harnik, [Harnik 1986], says "The most important thing in Robinson's reveal [*logic of non-standard analysis*] for high school level is that it enables teachers to talk about infinitesimals while feeling that behind this intuitive concept stands (at last!) mathematical validity". One thing is that we can at last do so. Second thing is why think it is the nature of thinking in mathematics and there fore decide to do so. Lets go back to the empirical studies. In the case of high creative process, which is not in mathematics, there were also researches, which concluded differently than the conclusion in (I). In mathematics: one description is thought supported by historical-mathematical examples,[Harnik 1986], about the formation of a mathematical concept evolving in time in general in mathematics, and the other one observation through solving a special problem, and the resemblance in process and images' chain through 9 years in history, [---- 1999, 2000].

The empirical studies:

(I) Based on introspections of known creators Wallas [Wallas1926] proposed a four-stage sequence as the <u>necessary</u> pattern for inventions: The first stage is a period of "preparation"; this is followed by "incubation"; this is followed by "illumination" or insight to a solution; and the last stage is "verification".

An example: Neuton's and Leibniz's calculations. [1]

But later, there were studies, also not specifically to mathematics, which concluded a different pattern. Ghiselin's, [Ghiselin 1952], report on the creative processes of 39 world known personalities gave some descriptive support to these stages, but did not generalize about the invariant order of their sequence. Eidhoven and Vinacke [Eidhoven & Vinacke 1952], studying stages of the creative process of artists versus nonartists, using experimenter's observations and subjects' introspection, concluded that: "Thus, the 'stages' are not stages at all, but processes which occur during creation. They blend together and go along concurrently" (p. 168).

(II) Harnik, [Harnik 1986], describing what he calls the genesis of a new mathematical concept, says one can distinguish three stages "a) The preliminary stage. A new concept is born out of necessity. At the beginning it is often vague, and even the inventors of the concept may feel uncomfortable with it"... "b) The familiarization stage. The new concept is used again and again with increasing confidence until it is fully crystallized and understood. This stage may precede considerably the axiomatic treatment (which is the essence of the third stage). The most convincing example is the case of the natural numbers. These numbers were not only discovered but fully understood in the distant past while their axiomatic treatment (by Peano) was accomplished only at the end of the 19^{th} century!". . ."c) The axiomatzation stage which comprises two sub-stages. c)1. Material axiomatization. Certain basic properties of the concept are singled out as axioms. All the other known properties are then deduced from the axioms. The material axiomatization of the real numbers was achieved in the late 19^{th} century".. . "c) 2.Formal axiomatization. This is a modern stage, the need for which became apparent only at the late 19^{th} century".. . "The modern mathematician looks at a list of axioms describing a certain concept and asks how he can be sure they are not contradictory. After all, it is conceivable that our intuition misleads us and there is no structure that satisfies all the axioms on the list. This question is usually answered by building a model for the axioms in question, that is, a system of objects that satisfy all the axioms.

(III) Observing Kakeya problem solving verbal and successive images process led by mathematicians (lecturers and assistants giving exercise lessons in the department of mathematics in the Technion), Wallas's stages were occurring among other processes. Research conditions: neither being known to me when investigating nor any other characteristics of high creative solving process in mathematics so I then did not expect a pattern of thinking. At that time reading literature in the first or second instance was a far task for me. Hence parallelism, repeated patterns are occurring by experiment. By simply following the chain of images parallelism is seen. The historical evolution shows the same first three of four successive stages pattern.

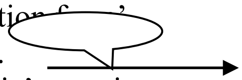
The two descriptions concerned mathematical developments of ways of thinking are also evident in the second instance. The repeated characteristics of developing processes of revealing in mathematics supports the following: Today researches about brain, cognition, anthropology, babies (at the age of one month psychologists can do that), animals, prove existence of "mathematical module" programmed in genes. Circles in brain are responsible for specific capacities, mathematics amongst,[Liron 2000]. As far as I found out about it (by hearing from a pupil a quotation of a lecture held by Haim Shapira of the department of socio-biology in Tel-Aviv University) the common opinion today in the genetic field science is that human beings develop by moving forward in roads designed by their potentials (in a metaphoric way of description). There is the line along the center of the road and to it's left and right. One can choose to go in the middle line or in the margins but cannot move outside his

or hers road to another road. This does not mean that a pupil who does not have the mathematical module in his brain can't study the highest 5 points score (this is the notion in Israel) in high school mathematics regular classes, but he or her will not grow to be a mathematician. Maybe the following is a self-selection example to support this opinion: Stendhal said that knowing how to do (-)*(-)=(+) is not to understand. His way question "why?" could not accept that a model which obeys rules and has no inner contradictions is an answer. He chose literature road though he was thought to be quite good in mathematics. A pupil might be gifted with other specific capacities than mathematics to which we have different groups of high creative processes performing. This may explain why the researches and descriptions mentioned above in (I) showed different patterns at highly selective in creativity samples.
The existence of a mathematical module might in a way explain why in different civilizations there were, without connections, some similar processes of evolution in mathematics, as well as occurrence, in the same country same mathematical department but group of mathematicians educated in different countries, of homogeneous judgement of level in two categories by high-creative mathematicians [-----1984] contradicted to low-creative mathematicians' judgement. And why Neuton and Leibniz were developing the same revealing and thought each one of them by his own, at the time doing so not aware of the others work. But in another sheet of questions in the same group of mathematicians an answer to one of 4 simple computational questions divided the group not to high and low creative but to analysis & descriptive geometry people and (as many as I happened to ask) because answered contradictory algebra people. So might be its a few genes which have a clutching in the same chromosome not independent from one another and there is crossing overs among chromotides. No matter how it goes (I don't know genetics), the phenotype expression (expression of a quality as a result of activity of genotype in given surroundings' condition) of this gene or genes' clutcher is no wonder possible to be repeated by different people who have those genes. Examples of same phenotype expressions are blended in the different absorbed surroundings. In order to distinguish them there are several ways, for example sorts of questionnaires. Differences in surroundings might be: at infants' period there are different images babies absorb, and at childhood and youth different emphasizes teachers make causing anthropological differences, different countries or places have different scales of what's important in different times (example significance of practicing lists of theorems in Euclidean geometry leaving deductive thought behind. In physics from cause to result in comparison to studying by heart situation of mechanical vectors).Though there are differences in physical surroundings and atmospheres, the Wallas' pattern might suit to cultivate mathematical ability to reveal or rereveal in the same way common to mathematical oriented pupils brought up in different surroundings' conditions. So we can look at any mathematical oriented pupil as a whole without considering the possible differences in attitudes when making plan of teaching. While we decide how to apply connection between history and development of ways of thinking to calculus. While teaching- be sensitive. The changes relative to a pattern that already occurred in history (which seems following natural development of cognitive growth) suggested in this article are made as I conjecture that they would have progressed to a verification stage (non-standard analysis). In this case, the fact that it did not happen straightforward, is not indicating that to teach it (up to a certain level) straight forward be a non-natural successive step. Apparently, the inventors in the 17th century (development in the first instance) were well aware of the distinction between the two

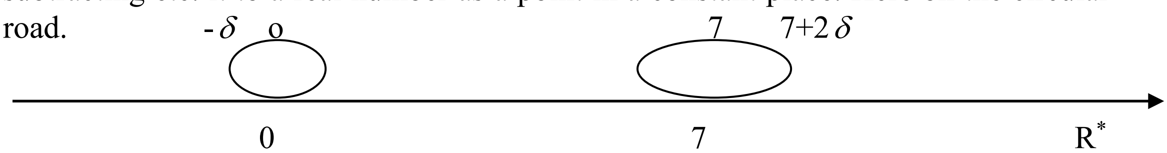
kinds of equality (thus to be in the direction of establishing verification) but they never distinguished them notationally [Harnik 1986].In the Hebrew version adds "the idea to use different notions, which seems so plain, was too dairy in their time". This supports to apply historically postponed verification ("jump" of 285 years) and after it continue along what happened earlier- lines of historical developments through 18 th – middle 20 th century. Thus the argument for this certain change in parallelism, in calculus is the naturality of successive stage. A connected argument seems the following: Rogers, [Rogers 1997] says "the error of claiming some kind of parallelism between the difficulties that students might have today and the apparent difficulties of people conceptualizing mathematical ideas in the past".

In fact the criticism like Berkley's(see [Harnik 1986])originated from difficulties at that time. Another apparent reason for such a gap in time for verification of the intuitive starting way is that a rigor new way not based on infinitesimals was found so mathematicians were not motivated any more to seek for verification. After a rigor definition (Weierstrass) was conveyed mathematicians had no conflicts whether their argument for a proof or negation a definition is correct. Thus the second typical stage in (II), familiarization, had not been performed. Seems for reasons of conservativity this continues to be the situation today. Chronologically the idea of infinitesimal treated as constant segments (none running end point of segments) which is simple for beginners was abandoned. Instead, mathematicians were happy to pass to limits and that how we all studied it. This was and still is strengthened by Hilbert's establishment of regours' demands (end of 19th century). This situation was lasting for one and a half century. The grasping of concept of continuity, derivative by beginners conveyed by ifinitesimals and principle of continuity of Leibniz, was not possible. Till forty years ago. Conclusion seems to be: precede hyperreal numbers model to limits while start of calculus. For pupils intuitively, explain scheme of illustration and pass to abstractization based on extract of formal axioms which seems sensible from the scheme of illustration. For teachers, in a rigorous way. Harnik, [Harnik 1986], conveys an argument of simplicity to grasp meaning (as well as following development in history argument): "Let us add [to an example of proof in nonstandard tools] that the nonstandard definitions have a simpler logical structure than their standard counter-parts. For example, the nonstandard definition that limit of f(x) when x moves towards a is *l* : if, whenever the value of x is *near* a but different from a, the value of f(x) is *near l* . Thus, the nonstandard definition of limit notion simply states that *for all x* a certain simple, *quantifier-free* condition is fulfilled (the expression "for all" is called a universal quantifier and "there is" is called an existential quantifier). In contrast to this, the standard definition involves *alternations of quantifiers,* such as: *for all* epsilon*, there is* a delta, s.t. *for all* x. . . ". The more alternations (i.e., changes from universal to existential or vice versa) of quantifiers a statement has, the more complex and hard to grasp is its meaning". An empiric one case study support to this claim (and if we make a step forward to why for pure mathematical oriented high school pupils it is more natural to advance in their roads beginning calculus with hyperreal numbers)is a paragraph in the autobiography of the Hungarian mathematician Halmos: "notions and symbols did not bother me. I could play with them without difficulty. . .[but] I was completely embarrassed by the infinitesimal subtlety of the epsilonic analysis. I could read analytic proofs, recall them after some efforts, and recapitulate them in this way or another. But I did not know what really was going on there. . .And one day as I arrived to room number. . . all at a sudden I understood what is epsilon what is a limit". To the group of pure mathematicians, we already mentioned Halmos, Harnik, claiming hardness to grasp Weierstrass' meaning

(at least at the beginning) we add Keisler’s pointed remark: “The nonstandard definition has lower quantifier complexity than its standard counterpart” [Harnik 1986]. Arguments of simplicity are coherent with the linked both to history and to development of thought repeatedness in pattern. This might be an answer to why do it when by recent revealings we can. How?-start with intuitive concept of infinitesimals (original texts by Leibniz, yet still did not find [Bonnefoy&Keller 1999]), and then as a hyperreal numbers. Then follow development in history: limit-the intuitive explanation of the concept by Cauchy. To this devote few lessons and move to definition by Weierstrass. Teaching In a spiral way according to Brunerian approach[Bruner 1959]. Next inner deeper loop teaching the definition of limit with ε, δ and proofs of lemmas and theorems part of them been familiar as already proved before. Following 2 examples to how to build teaching in high school calculus in a spiral attitude. Before the examples now follow schemes of illustration. Read first possibility to scheme of illustration in [Harnik 1986]. I’ll present transmission to abstractization from a variation of scheme of illustration to the one in Keisler, [Keisler 1976]. The variation is adding a metaphor for seeing differently with a microscope and without it. Its up to teachers to decide Both schemes provide a way to work with rules such as the continuity principle of Leibniz. (“In simple terms, this principle stated that *infinitesimals* are indeed extraordinarily small, but otherwise *behave just like the usual real numbers”.. .* “While the principle of continuity is vague, what counts is that Leibniz” arrive to correct results- like the curve produced by a string sagging under its own weight). Perhaps my addition to thought about the scheme is more convincing for a smaller group of pupils-those who have also tendency to literature. Convincing as a metaphor why some results effect some disappear, why the whole picture disappears though we are looking at the same location when transferring from hyperreal numbers to real numbers and vice versa. The connected to literature: In a narrow angle of view we see details but loose aspects of the whole picture the sense of the whole situation, in a wide angle details are not seen but we see the whole picture. Non-standard definition of an infinitesimal is that $x \approx 0. x \neq 0$ or x=0. x is near 0. In the microcosms (when we look in a narrow angle of view) we see it and this in a constant place. Where is this place? What do we see? The necessity to see when formulating mathematics is, (from quationaires and interview methods at least in small samples), a phenotype expression of mathematicians when formulating mathematics, might be in most surroundings’ conditions. It brings us to look for visual scheme of illustration. The one proposed by Keisler is to see an image of fragment of segment in related to a point drawn inside an eliptic ‘explanati[illegible]’
And a fragment of a graph related to a point in the graph in the same way.



Happened that while I presented Keislers’ illustration with the Leibniz’ notations brought in [Harnik 1986], before a group of teachers from and around the Jordan valley region, through discussion it was suggested another illustration to hyperreal axis: see numbers on circular roads. Then the illustration of hyperreal axis is the real to which circular roads emerging from any real points are added. And we know the circular roads does not exist when we move our thought to wide angle of view. In both schemes of illustration it’s essential we see an infinitesimal and result of adding, subtracting etc. it to a real number as a point in a constant place. Here on the circular road.



a scheme of illustration for hyperreal number axis

set of points presenting the hyperreal set of numbers

When we look in macrocosms we don't see it (i.e. when we adjust looking through wide angle it does not exist, but there is certain kind of results of calculation – direction, angles- which does remain). Let x be seen in a narrow angle of view. In this angle we of course see the point on real number axis from which circular road emerged, denote it by a. When writing st(x) = a this is what we mean. The standard part of x is the focus of narrow angle of view or the microscope which is a. In case a is the point 0 on the real axis (we also find its location in "middle" of circular road) then x is an infinitesimal. st(dx)=0 Whereas Leibniz did not consider 0 as an infinitesimal according to any list of axioms in nonstandard analysis 0 is an infinitesimal. In circular road emerging from which real is 0 located? from 0 thus st(0)=0 and 0 is according to definition an infinitesimal. Another way of putting it: near which only real is the hyperreal 0 ? near 0. So it is an infinitesimal. It is the single infinitesimal which is real. Another way of writing the microscopic equality st(x)=0 is: when we look at a in a wide angle of view we see it as part of an axis. We are now in macrocosm's world knowing that near a in narrow angle of view we will reveal numbers "near" it for instance x. Writing the second possibility of macroscopic equality:

$$x \approx a$$

f(x) is continuous at point a if : $\forall x\ ((x \approx a) \Rightarrow f(x) \approx f(a))$

or:

$$\forall x((st(x) = a) \Rightarrow st(f(x)) = f(a))$$

As long as a list of axioms does not raise contradictory conclusions, and we are sure about it, because there exists a group of objects in which the whole list of axioms are fulfilled. One of the axioms – every hyperreal x is near a real standard a, which, scheme of illustration helps us to accept. It is in a circular road emerging somewhere on the real number axis. Of course x cannot be near two different standard numbers. In formal writing, this is expressed by for example: $7 + 2\delta \approx 7$. or $7 = st(7 + 2\delta)$.

The mental conceptualization of infinitesimals and hyperreal numbers must be preceded by examples of getting results influencing macro from things we don't see and there for we adopt a model seen in micro. We try for instance in quantum mechanics to elaborate models.

Might be that due to two reasons: spiral approach and going along the lines of development in history and ways of thought, the meaning of epsilontic proofs will be easier than described by Halmos, or what many teachers of mathematics experienced in their own studies. The teaching in spiral or along the line in history coincide according to Haeckel's, [Haeckel ~1900],biological law: 'ontogenesis recapitulates phylogenesis'. But one have to be aware of regressions in history. The same as the situation child's development was suppressed. At these points leave chronological path stick to development of thought pattern. Another reason for teaching calculus in spiral following history: thus education does not break the link between history and mathematics. Radelet[Radelet-de Grave 1999] says: "if education breaks. . . , the result is poverty, scantiness". The precise logic conveyed by Keisler, [Keisler 1986], to my humble opinion might be made after Cauchy and Weierstrass. Only after it return to it for mature in mathematics high school pupils or in university as an area. Axioms listed in [Harnik 1986]and computation rules seem at the beginning of calculus to be enough. Purpose is starting with intuitive concept (As we said before teachers who teach this way know it is valid, and study how).Teach successive pattern : preliminary stage and familiarization stage. Continue in a spiral attitude to bring pupils to validity, to rigor but by Weierstrass' way. The formal axiomatization after it. Perfect rigor was adjusted to infinitesimals in the middle of 20th century in the Hebrew university. Seems to be an evolution as in the Hebrew University those years there were lecturers who studied in Gettingen, where Hilbert established the formal axiomatization. The suggested features are to build a voyage in Leibniz's thought and define intuitive invisible infinitesimal by Robinson's simple two macroscopic equalities and computational rules. Continuing in a spiral way- again teach the pupils the concept continuity and derivative as was mentioned based on limits. Heuristically same as Cauchy who envisaged infinitesimals as variable quantities that tend to zero. This made limit a primary concept. He arrived to the idea of intuitive limit. What we'll do is :from constant infinitesimal seen only in an illustration scheme we move to seen but not constant segment on the real axis.

$a + \Delta x$

runs toward a. Correspondingly point

$(a + \Delta x, f(a + \Delta x))$

runs on seen graph (what we called before our macro world) to point (a, f(a)). In any of the intermediate situation we compute mean slope which is the slope of a chord equals tangents of changing angle. Running point tends to overlap (a, f(a)). and changing chords to have one point in common with graph, angle changes towards angle between tangent to graph and x axis. In the constant infinitesimal approach seen in micro the illustration of circular road, the road connects a point to itself. After based on intuitive limit the based on rigorous definition of limit. Now to see what definition claims might be easier. See the transfer in explanations in the examples. maybe as we follow the road of mathematicians' potential now it be grasped easier and well understood. Then rigor formal axiomatization.sticking more or less (beside the notations) Kiesler's book.

When we teach beginners in calculus I suggest maybe precede to it examples of transmission of thought from seeing in a narrow angle (microscopic world) to seeing in wide angle (macroscopic world) and vice-versa. Teachers might start calculus with examples taken from physics, mathematics, biology, literature. Thus conceptualizing infinitesimals and then hyperreal numbers might be done by: 1) Preceding to calculus examples of how we get results in macro from things we don't see, we get it from a scheme of things we adopt a scheme seen in micro. Like warmth or coolness of a glass of water (temperature degree) is felt in macro- kinetic energy of molecules (do we see them? what is the connection ?) the computation in micro gives the result felt in macro. *In physics we are used to deal with things we don't see*. Adjust a model and deal with as simply being seen. In quantum mechanics we are used to deal with things we don't see but we look for scheme of illustration and treat it as seen. Thus get used to doing intermediate steps and correct consequences transmissions from macro to micro so to get a consequence in macro. Seeing examples of how to conclude by transmission between angles of view. 2) As Harnik,[Harnik 1986],

"reminding students that the concept of number has been extended several times in school by the introduction of fractions, negetive numbers, and irrational numbers (it may be useful to mention that the latter are necessary for measuring the ratio of any two line segmants) the teacher can describe one more kind of numbers – the (non-zero) infinitesimals which are different from zero but smaller in absolute value than any positive real. they can be illustrated in a very simple and palpable way by using a device of Keisler's". Teachers are acquainted with another definition of a field of numbers according to given rules, to which a scheme of illustration was built – the complex numbers illustrated in Gaus plain. 3) While calculus- teach history. how in second instance one idea came from the other. Acknowledge teachers with the difficulties in the past: on one hand dx small to infinity but not zero, on the other hand based on this pseudo definition we get to results which lead to dx=0. Reason why present teachers with these difficulties, though might be not faced at class, is in order to increase their sensitivity to pupils' problems conceptualizing the idea if happens difficulty been aroused in class [Arcavi 1991].

Time axis of process of development of calculus along history

Archimedes- technique of going smaller and smaller parts of line, of areas, -300
of layers computing centre of gravity.
Explanation of Abraham bar Hiyya Ha-Nasi to compute area of full circle 1150
“those going around the area of circle will become straight lines becoming
smaller and smaller till they turn to be a single point” When the shape of steps,
the narrow trapezes becomes a triangle we think of the heights of the trapezes
as of infinitesimal amounts.
Descartes – a right-angled system of axis. Presenting connection between 1616
values being expressed by a formula by sample of points joining to a
geometrical figure – graphs.
Neuton trying to prove that celestial bodies in solar system are always going 1665
to move in elliptic orbits, had to answer the question what is a momentary
velocity? Same question considering mathematical basis what should be
momentary rhythm of transferring warmth to surrounding. Leibniz while 1675
examining from every point of view Pascal triangle built Leibniz triangle
and adding sums of series in triangle arrived to asking what is rhythm
of local change in graph in a certain point in domain that does not taking place?
Both of them applied numbers-concepts they understood intuitively-
infinitesimals. These are numbers small to infinity that are not 0, and are steady
not keeping changing. And arrived to define the derivative. Leibniz used the
notion $\frac{dy}{dx}$. At the same time proved rule for differetiating a compound
function. Neuton uses a point above the letter which represents the function.
And it’s still used to day for differentiating with respect to time. In order to
compute with the infinitesimals Leibniz formulated his continuity principle. At
the same time of “defining” the derivative Leibniz continues to area and
integral. Neuton arrives to integral from needs in physics.
First book of analysis by L’Hospital using infinitesimal language. 1696
The”Royal society” accuses Leibniz of plagiarism. 1711
Sounds of criticism against infinitesimals are raised. Berkely formulates 1734
proofs to his opinion that the definition is not valid mathematically. But
Lagrange, D’Alembert , Bolzano and Cauchy are enthusiastic to find a way
that the using infinitesimals will be mathematically based. While this, grasping
the infinitesimal to be a changing quantity moving to zero (i.e. having limit
zero) Caucy arrives to place the concept of limit in a central place. His
definition is intuitive. Based on this concept Cauchy does the building of
mathematical analysis. Not intending to do so, his work did not provide basis
to infinitesimals but on the contrary opened the door to use them not. Because
if there is something which does not arouse inner contradictions (so they
thought using infinitesimals does)- why use it?
Another book of analysis –Euler 1748

Notion f ‘(x) to derivative- Lagrange. Cauchy mades a conjecture and tried to prove it on base of infinitesimals. Claims he did so. 1821

Abel builds a contradictory example to Cauchy’s claim that he thinks that proved on basis of infinitesimals. An unusual situation was created In the mathematical community that mathematicians cannot transfer one another what they see in their mental eyes. It is taken for sure that no hope to find mathematical basis for the infinitesimals. 1826

Cauchy did not give up and argues all the rest of his life that his theorem is correct, and Abel had not understand his intention. Writes clarifications to his proof. 1853

Bolzano started and Weierstrass completed, defining rigorously the concept of limit “for every amount epsilon>0 there exist amount deltha>0 so that. . .” and based on it continuity and differentiability. Their definition is not based at all on the concept of the infinitesimals. The mathematical community seized to use the concept. 1872

Hilbert in the university of Gettingen started his axiomatic period. Unified material axioms revealed before and added axioms to base real numbers’ meaning. Continued to formal axiomatization. 1898

Robinson in the Hebrew University developed a mathematical basis based on formal axiomatization he built to using the infinitesimals and to do prove rigorously. One might say, the fact that after three hundred years verification stage to the intuitive start is looked for and is accomplished, qualifies that mathematical intuitive start of calculus this way is a natural way of thinking for this area. 1960

Still the debate whether Cauchy’s theorem is correct or his intention was not well understood continues. 1978 and forward

Why we should apply connection between history and development of ways of thinking to calculus and to be able after Robinson’s revealing to do so, is one thing. Knowing how- is a different thing altogether.

For example, when doing it we must put in front of us a question: how to transfer from illustrating scheme passed to abstraction, to a regular routine in such a way that it would not be mechanic.

Example 1

Find a rule for the derivative of a compound function f[g(x)]. Assume g ‘(x) and f ‘(g) exist.

Stage based on intuition aware of it’s being based on rigor (non-standard analysis): One way of putting it :

In micro the mean slope in segment dx is: we divide amounts: $\frac{d[f(g(x))]}{dx} = \frac{df}{dx}$ $= \frac{df}{dg} * \frac{dg}{dx}$ (when dg is not the infinitesimal 0). We consider dx, df as quantities. They are stable quantities

dx is the length of part of the circular road emerging from a real point x in one of the schemes of illustration or <u>the</u> hyperreal length of straight line segment seen in narrow angle of view focused to x if we stick to Keisler’s scheme.. df amount added (can be negative) to amount of the function at the real focus point x after substitution of g(x+dx) in f(g), seen in micro.

$\frac{df}{dg} \approx$ f ‘ (g) (is near to the real number f ‘ (g). Definition of derivative –the single **near** value in macro to amount computed in micro of mean slope. We use Lagrange's notion for the derivative), $\frac{dg}{dx} \approx$ g ‘ (x). Saying the same thing in microscopic equality: $\frac{df}{dg}$ = f ‘ (g) + δ , $\frac{dg}{dx}$ =g ‘ (x) + r .

where δ , r are infinitesimals.

(f ‘ (g) + δ)*(g ‘(x) +r)= f ‘(g)*g'(x)+ f ‘(g)*r +δ *g ‘ (x) +δ *r= according to computational rules, multiplying infinitesimal with real number (which is commutative, real number is a finite hyperreal) gives infinitesimal- point located in the same circular road emerging from 0 (or in the same straight line seen in a drawn picture through a mental microscop)as the original infinitesimal was. Multiplying an infinitesimal with another infinitesimal gives infinitesimal. Sums of infinitesimals is an infinitesimal =f ‘(g)*g ‘(x)+infinitesimal.

Hence is near product $\approx$f ‘(g) *g ‘ (x) of two real numbers which is real, and is the single real near.

[f(g(x))]'= f ‘ (g)*g ‘ (x)

In case dg=0 for dx not zero, we get in micro f(g+dg) =f(g). Hence df=0. The mean slope df/dx=0 in micro. 0 is the only real infinitesimal. Being as all infinitesimals in the neighborhood of the single real number 0 is near 0.When we turn to macro [f(g(x)]'=0, the same value we would have got from the right side of equality [f(g(x))]'=f '(g)*g '(x) which is therefore correct also in this case.

Or:

The derivative i.e. focus of narrow angle of view is the standard part of df:dx .

(*) st $\frac{df}{dx}$ = st [$\frac{df}{dg} * \frac{dg}{dx}$] =

We need to prove multiplication of standards equals standard of multiplication. For let's b and c be finite hyperreals. In narrow angle of view(=microscopic equality):

$b = st(b) + \delta; c = st(c) + r$

$st(b) = b - \delta; st(c) = c - r$

$st(b) * st(c) = (b - \delta) * (c - r) = b * c - b * r - \delta * c + \delta * r$

Since st(b)*st(c) is real hypperreal so is

$b * c - b * r - \delta * c - \delta * r$

so its equals its standard, which is st(b*c).(Either from scheme of illustration or by standard of sums equals sum of standards)

(*) equals (st $\frac{df}{dg}$)*(st $\frac{dg}{dx}$) (each of this standard parts exists since its given those derivatives exist) and **is** the derivative

[f(g(x)] ‘ = f ‘ (g))*g ‘ (x)

In which writing routine pupils understand better what they answer? Or better suggest the writing and saying of thoughts while solving and developing becomes a regular part of doing the mathematics in either of these ways. Besides being then not mechanic, the research literature on the topic the “talking aloud” technique suggests that talking aloud while solving a difficult problem might actually improve performance (for instance [Gagne&Smith 1962]). Another research shows that high-creative students with comparison to low- creative have a lower degree in criterion of silence when solving problems task was asked to be performed in “talking aloud “ technique. By “method of interview” =”talking aloud” technique, especially in mathematics indicates high-creative mathematicians have good communication with themselves (research I once did). Thus the routine of writing ones thoughts while finding and proving in both routines would not “allow” mechanic performance. Seems fitting study of calculus by method beginning with infinitesimals.

Adopting spiral method of Brunner,[Brunner 1959], i.e.proving the same again in the Weierstrass’ way. We go forward to rigor.

We move from seen only in micro steady amounts dx (or announced infinitesimals noted by other letters) to seen on real axis

Δx

moving in domain

$(0,\delta)$

and will develop derivative.

The price we pay when we earn seeing on ‘simple’ real number axis might be evaluated by counting number of changes from universal to existential quantifiers or vice versa. What is simpler to understand, as a beginning, is answered not homogeneously by high-creative pure mathematicians. And might be we’ll arrive to “theko”. Still this question is not the point, but how to follow the based on history epistemology for classes good in mathematics for arguments some of them were mentioned.

What is [f(g(x)]’ ?

It is the limit of

$$\frac{f(g(x+\Delta x))-f(g(x))}{\Delta x}$$

when

$\Delta x \to 0$

or

$x+\Delta x$

moves towards x, or x towards x_0. Correspondingly the point on graph moves continuously to (x, f(x)).”almost overlap point on point”. Towards which number runs mean slope of changing chord? Is it a single number or depends on our choice of presenting descending distance from x?

Either we can use what we are acquainted with when developing derivatives based on infinitesimals, or this time it is easy to guess since

$$\frac{f(g(x+\Delta x))-f(g(x))}{\Delta x}=\frac{\Delta f}{\Delta x}=\frac{\Delta f}{\Delta g}*\frac{\Delta g}{\Delta x}$$

that derivative might be f ‘ (g)*g ‘(x). So we have to prove this to be the limit. Rigor proof is when we show that: for every arbitrary value

$\varepsilon > 0$

as small as wished, there exists a value

$\delta > 0$

such that for every delta x

$$0<|\Delta x|<\delta \quad \Rightarrow \quad \left|\frac{\Delta f}{\Delta x}-f'(g)g'(x)\right|<\varepsilon$$

As long as
$|\Delta x|$
runs within the domain from 0 (not included) to
$\delta(\varepsilon)$
the chord's slope (changing mean slope of function in changing segment) is restricted to be a value in domain
$(f'(g)g'(x)-\varepsilon, f'(g)g'(x)+\varepsilon)$

Thus the pupils do arrive to rigor but not without preparation stage conceptualizing intuitively what is a derivative. Like a journey in the inventors' of the concept thought.
f '(g) exists. According to definition of f '(g), for every positive
ε' δ'
there exists a positive value
so that for every
Δg $0<|\Delta g|<\delta' \Rightarrow \left|\frac{\Delta f}{\Delta g}-f'(g)\right|<\varepsilon'$

Per definition of g '(x) for every
$\varepsilon''>0$
there exists
$\delta''>0$
so that for each

$$0<|\Delta x|<\delta''$$

$\left|\frac{\Delta g}{\Delta x}-g'(x)\right|<\varepsilon''$
Since g(x) is continuous in the neighborhood of x, for each
$\delta'>0$

there exists a
$\delta'''>0$

so that for each
$0\le|\Delta x|<\delta'''\Rightarrow 0\le|\Delta g|<\delta'$

We choose
$\delta=\min(\delta'',\delta''')$

Thus for every

$$0<\left|\Delta x\right|<\delta \Rightarrow f'(g)-\varepsilon'<\frac{\Delta f}{\Delta g}<f'(g)+\varepsilon'$$

and

$$g'(x)-\varepsilon''<\frac{\Delta g}{\Delta x}<g'(x)+\varepsilon''$$

For simplicity sake lets assume all sides of inequalities are positive, multiplying and opening brackets we get

$$f'(g)g'(x)-f'(g)\varepsilon''-g'(x)\varepsilon'+\varepsilon'\varepsilon''<\frac{\Delta f}{\Delta g}*\frac{\Delta g}{\Delta x}<f'(g)g'(x)+f'(g)\varepsilon''+g'(x)\varepsilon'+\varepsilon'\varepsilon''$$

$$-f'(g)\varepsilon''-g'(x)\varepsilon'+\varepsilon'\varepsilon''<\frac{\Delta f}{\Delta x}-f'(g)g'(x)<f'(g)\varepsilon''+g'(x)\varepsilon'+\varepsilon'\varepsilon''$$

$$\left|-f'(g)\varepsilon''-g'(x)\varepsilon'+\varepsilon'\varepsilon''\right|<f'(g)\varepsilon''+g'(x)\varepsilon'+\varepsilon'\varepsilon''$$

For every

$\varepsilon>0$

we can find solutions to the inequality

$0<f'(g)\varepsilon''+g'(x)\varepsilon'-\varepsilon'\varepsilon''\leq\varepsilon$

First we chose

$\varepsilon',\varepsilon''<\min(f'(g),g'(x))$

$\varepsilon'>0,\varepsilon''>0$

then

$f'(g)\varepsilon''+g'(x)\varepsilon'-\varepsilon'\varepsilon''>0$

and put for instance that also

$$\varepsilon''<\frac{\varepsilon}{f'(g)*2}$$

$$\varepsilon'<\frac{\varepsilon}{g'(x)*2}$$

substituting we get

$$f'(g)\varepsilon''+g'(x)\varepsilon'-\varepsilon'\varepsilon''<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}-\varepsilon'\varepsilon''<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

As we proved before for every

$\varepsilon',\varepsilon''$

and these amongst them there exists a

$\delta>0$

so that for every

$0<\left|\Delta x\right|<\delta$

$$\left|\frac{\Delta f}{\Delta x}-f'(g)g'(x)\right|<\varepsilon$$

Example 2

Find and prove your steps in a spiral approach ~ along the line thought advanced in history.

$$\frac{dx}{\sin dx} = \frac{\delta}{\sin \delta} \approx ?; st\frac{\delta}{\sin \delta} = ?$$

$$dx \approx 0; \delta \approx 0; st\delta = 0$$

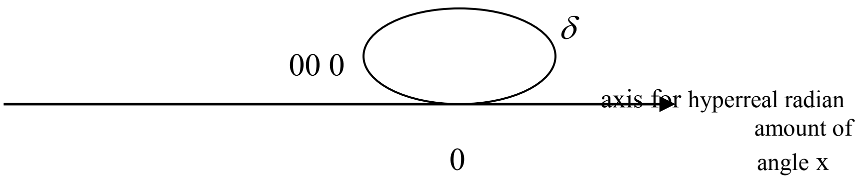
st(dx)=0,

$dx \neq 0$

Solution :

Leibniz's-Robinson's way: preparation intuitive understanding, teachers aware of the fact it is based on rigor.

Plot the unit trigonometric circle. Situation central angle 0.We see In micro an infinitesimal angle not zero its amount dx radian. Its amount on circular road emerging from 0.Lets denote it by delta. 0



Remark: real radian is a pure real number. We deal as usual knowing this is seen in the microscopic world only. Build triangles and sector of circle in the trigonometric unit circle. In the microscopic world we deal as usually with amounts of area. Seeing which shape is completely located to be part of another one we come to inequalities

$$\frac{\cos\delta \sin\delta}{2} < \frac{\delta * 1}{2} < \frac{\tan\delta}{2}$$

$$\delta \approx 0; \delta \neq 0; \sin\delta \neq 0$$

if

$\delta > 0$

$$\cos\delta < \frac{\sin\delta}{\delta} < \frac{1}{\cos\delta}$$

$\cos\delta$

hyperreal

near

1

smaller

$\frac{1}{\cos\delta}$

hyperreal

near

1

larger

hyperreal order axioms the same as for real numbers (in a non-rigor way Leibniz's continuity principle). Both quantities are located in the same circular road. Such a road is continuous therefor must be in it, among the two locations the location of amount

$$\frac{\delta}{\sin\delta}$$

there for this quotient is in the same circular road. Cannot be in a second circular road. As well as the two quantities

$$\cos\delta; \frac{1}{\cos\delta}$$

also

$$\frac{\delta}{\sin\delta} \approx 1$$

or

$$st(\frac{\delta}{\sin\delta}) = 1$$

When

$\delta < 0$

same arguments.

When we raise the mental microscope we see no more that the angle 0 has around it vicinity of angles, what we see is an angle that equalis 0 and its sine is 0. but the result we arrived to, this effect of consequences got to in the microscopic world remains. Quotient of an angle near 0 (macroscopic equality to 0) to its sine is a number near the seen number 1.(equals macroscopically to 1). Or in micro its standard part must be 1.

*Cauchy way**- we move from constant dx which we see in micro as none-zero amount to changing amount***

Δx

running to 0 from a sensible point.

$$\lim \frac{\Delta x}{\sin(\Delta x)} = ?$$

as

$\Delta x \to 0$

More known to us in the notation

$$\lim \frac{x}{\sin x}$$

$x \to 0$

We make same comparison of areas in the unit trigonometric circle. Allowing the angle to run towards 0. By 'sandwich rule' get 1. The real numbers' quotient approaches the standard part of the hyperreal numbers' quotient. Or, to what the quotient has macroscopic equality relation is the limit when x runs towards 0.

Rigor- Weierstrass' way: now we see

Δx

$0 < \Delta x < \delta$
on the real number axis running from
δ
towards 0. Knowing it is rigor but proving in a not completely rigor way that the limit is 1 we now put what we have to prove:
$\forall \varepsilon > 0$
small as we wish
$\exists \delta > 0$
such that
$\Uparrow$

$$\forall \Delta x((0 < |\Delta x| < \delta) \Rightarrow \left|\frac{\Delta x}{\sin(\Delta x)} - 1\right| < \varepsilon)$$

$\Uparrow$

We compare areas in macroscopic world and arrive to

$$\cos(\Delta x) < \frac{\Delta x}{\sin(\Delta x)} < \frac{1}{\cos(\Delta x)}$$

$$\frac{1}{\cos(\Delta x)} - \cos(\Delta x) > \frac{1}{\cos(\Delta x)} - \frac{\Delta x}{\sin(\Delta x)}$$

$$\frac{1}{\cos(\Delta x)} > 1$$

$$\frac{1}{\cos(\Delta x)} - \frac{\Delta x}{\sin(\Delta x)} > 1 - \frac{\Delta x}{\sin(\Delta x)}$$

If we find a delta such that

$$\forall \Delta x((0 < |\Delta x| < \delta) \Rightarrow \varepsilon > \left|\frac{1}{\cos(\Delta x)} - \cos(\Delta x)\right|)$$

then by transitivity its also valid to demanded per definition of limit inequality.

$$\left|\frac{1 - \cos\text{^}2(\Delta x)}{\cos(\Delta x)}\right| < \varepsilon$$

$$\left|\frac{(\sin \Delta x)\text{^}2}{\cos \Delta x}\right| < \varepsilon$$

$\Uparrow$

$|\sin \Delta x| < |\Delta x|$

$$\left|\frac{\Delta x\text{^}2}{\cos \Delta x}\right| < \varepsilon$$

$\Uparrow$

Choose

$\Uparrow$

$1 > \cos \Delta x > \frac{1}{2}$

$\Uparrow$

$\frac{\Delta x\text{^}2}{\cos \Delta x} < 2\Delta x\text{^}2 < \varepsilon$

$\Uparrow$

$\Delta x\text{^}2 < \varepsilon / 2$

$\Uparrow$ $\qquad\qquad |\Delta x| < \frac{\pi}{3}$

$0 < |\Delta x| < \sqrt{\frac{\varepsilon}{2}}$

$\Uparrow$

$\delta(\varepsilon) = \min(\sqrt{\frac{\varepsilon}{2}}, \frac{\pi}{3})$

Acquaintance with a difficulty in historical development of calculus

The most famous and outspoken opponent of infinitesimals was Bishop Berkeley. . .in a work. . .in 1743. His objections were in the following vein:

(One) There can be no nonzero infinitesimals because any number whose absolute value is smaller than any positive real must be zero.

(Two) if f(x) = x^2 then f '(x)= 2x while

$$\frac{f(x+dx)-f(x)}{dx} = 2x + dx$$

Since these two expressions ***must be equal***, we have 2x+dx = 2x. Hence dx =0, contrary to the assumption that dx is a nonzero infinitesimal.

Suggestion: an open discussion in classroom.

SUMMARY

Purpose in more or less keeping an invariant order of successive stages in development of high creative ideas, and try to do it simple, is: to bring pupils to reinvent (a key word for learning of mathematics coined by Freudenthal) what others have invented before him or her, and thus understand better because arriving up to a certain degree by themselves to develop a theme, a solution. The phenomenon that evidently one can find repeating features in pattern of invention process, together with how Freudenthal saw learning of mathematics go along concurrently to provide a kind of empirical support to one of the arguments why try follow the path that mankind has traveled in history. Argument based on Haeckel's biogenetic law [Haeckel ~1900). We can look at a narrower aspect of any period of solving a mathematical problem pupils go through. In this short instance he goes through a pattern in which some features are repetitions of what others before him went through, a process which originates from his or hers own cognitive growth. Following historical path means for instance to teach

$\int xdx = x^2/2$

as part of intuitive and familiarization stages as voyage in the process of invention of Leibniz's thought,[http://www-groups.dcs.st-and.ac.uk/~history/Mathrmaticians/Leibniz.html] this two stages for pupils working on calculus are abandoned. Yet in following the path in history we are restricted in planning number of lessons, thus if something is very simple then don't devote long periods to understand preceding methods. In calculus we can look at the question of why to teach by following lines in history also from its subject matter.

I find that the starting original development, to use an intuitive concept of an infinitesimal, was always in the back of the minds of certain pupils, natural to mentally visualize it and understandable. Those infinitely small segment on the horizontal axis, which is not 0 to which, fits infinitely small fragment on graph, which is not one point. But, we, educators in mathematics, pure mathematicians, lecturers and high school teachers, teach pupils the way which was taught from end of 19th century strengthened by demands of rigor founded by Hilbert from that same time and we arrive somehow to Weierstrass. This because behind us a long period and deep tradition and because rigorous concept was revealed, and from then on no difficulty for mathematicians to convey to each other their mental pictures of limits, continuity and derivative. The intuitive start was abandoned (completely from 1872) most of us never heard it existed. Its hard to figure out by dialogues what would had had be our attitude towards infinitesimals at beginning of studying calculus (before we know how to treat its concepts with other mental pictures) if there would have not been a pause of 128 years. The advancement in thought starting with an intuitive stage rather than with the rigor formal one seems to be the case in the voyage of inventing by single human being as well as it **was** the way evolving along time. There fore seems to my humble opinion developing calculus by connecting it to its original start in history and at the same time to ways of thought, might encourage classes of gifted in mathematics pupils to rereveal in mathematics and inside this frame they'll develop fluent performance of rigor mathematics. In calculus it was not possible till Robinson's reveal. The reasons this is not done:

1) Pure mathematicians cannot go along with the situation they cannot convey each other what they see in their mental eyes (Cauchy and Abel).[Harnik 1986].

2) At the moment a rigorous concept was established the problem of what is a limit was solved, so there was no motivation to find rigorous theory supporting the way how the derivative was first thought of by Leibniz or Neuton.

3) Starting with Hilbert at the end of the 19th century the common way that today pure mathematicians do mathematics is to base on formal axiomatization- everything have to be rigor. Start with Weierstrass is rigor.

4) The great deal of work on researching the use of history in mathematics education have been started only recently [Fauvel/van Maanen 1997]. In calculus [Bonnefoy Keller 1999], p.72 abstracts of IIIrd European Summer University, Lauvain. Earlier [Harnik 1986],not because of this trend.

Today we know that to start Calculus with an approach which is near to Leibniz's **is** provided with the required rigor justification. According to, [Harnik 1986], Leibniz and L'Hospital were aware of the possibility to move from the intuitive ideas to a rigor expression. Knowing it is based on rigor we can teach calculus to pupils at high school in a connected both to history and to ways of thinking method. At the beginning not going with the pupils through too high level deep logic.

As for the question how, some of the concepts of nonstandard analysis we convey to the pupils. Like distinguishing between

$\approx$

to =, and use of some concepts like the standard part. We can explain pupils who just start calculus there exist objects which fulfil the axioms and this makes it valid, and the computational rules of hyperreals. The list of rules and explanations we should bring as basis to infer to are extracted in [Harnik 1986]. As we are going to deal with numbers we see in micro and don't see in macro it's suggested to start with examples from physics, biology, literature. While we deal with a model in micro (whether it is really so or not) our attitude is the same as to seen images. Thus also to parts of curves in micro just as we see in the real world. The transmission to real world seen consequences is now valid. Also I tried to make it easy to grasp how we move from dx infinitesimal we see in the narrow angle of view as a constant amount to

Δx

we see without having to pass from one angle of view to another but then it's a changing amount. And might be (I don't know about a research concentrating on this question) that high school pupils will understand better the limit's, continuity's and derivative's meaning in the alternating quantifiers' definition or find it is simple.

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THE ROYAL OBSERVATORY IN GREENWICH; ETHNOMATHEMATICS IN TEACHER TRAINING.

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Introduction

The history of mathematics is illustrated all around us as shown by writers in Africa, South America and other parts of the world. When students are training to become teacher of mathematics for primary or secondary schools, they need to learn to extract the mathematics from the place they are, to place it in a time frame or chronological time line in order to understand what they see. That experience will enable them as future teachers to show their own students that mathematics has a history and to challenge the myth that mathematics is an unchanging, objective body of truth. The declared aim of our unit 'The History and Nature of Mathematics' was to change our students' attitudes to mathematics. The results of our research were written up for the HPM in Braga 1996.

At Greenwich University – the Greenwich Royal Observatory provides rich pickings for historians of mathematics together with the opportunity for teacher training students to extract and decode the mathematics embodied in its artifacts. The Observatory and Greenwich Maritime museum are set in Greenwich Park, which is only 4 miles from Avery Hill campus where the students study. It is almost in their back yard. Many of the students will teach in schools in this area and so have the opportunity to use the Observatory itself with their own pupils. Others are expected to use the visit as a model for planning their own visits to mathematical sites in other areas and for some, in other countries.

There are two main groups of students who have visited the Observatory with me each year from 1992 to 1999. One group was studying to be general primary teachers but specialising in mathematics, and the other group was studying to be secondary mathematics teachers. Each group numbered between 10 and 25 students and took a compulsory unit entitled 'The History and Nature of Mathematics'. The approaching Millennium made the visit especially topical and gave an undercurrent of excitement about being at the centre of the celebrations.

Unfreezing frozen mathematics

To those of you unfamiliar with the work of Paulos Gerdes and his colleagues in Maputo, Mozambique, I used an article published in June 1986 in the journal: For the Learning of Mathematics, entitled "How to Recognise Hidden Geometrical Thinking". In this initial article, the mathematics hidden within basket weaving is unfrozen from its frozen state in the finished basket. By actually weaving using coloured card strips, the mathematics becomes clear in the action of the weaving: angles of 60°, parallel lines, equilateral triangles, hexagons and rhombuses. The mathematics is necessary to meet the design requirements of the basket or hat to make it useful, decorative and rigid. This process of unfreezing or extracting the mathematics is intrinsic in the making of an object does not just apply to baskets, but also to many other artifacts. The craft of making is often regarded as a merely mechanical skill passed on from one generation to the next, and not requiring any mental skill. Actually, much mathematics is used both in the design and in the strategies for making the object. Intending teachers need to be able to unfreeze the mathematics around them to help their own students link their school mathematics to the world around them. At the Observatory, the students' task is a kind of unfreezing of the objects on display into their mathematical parts and then relating the objects and their mathematics to their place and time.

History for students of mathematics

From the beginning of their course, the students are introduced to the study of history so they can more effectively research for themselves. I find as many books called 'The History of Mathematics' as I can and we analyse the balance and subject matter of their contents. In our library at Avery Hill in Greenwich at least, nearly every book is by a man, and most volumes are written in the United States. A few are from Britain, again by men only. We are expecting our students to act as amateur historians who need to be aware of the bias and interests of the books, articles and encyclopedias that they are consulting. All writers, historians included, can only write limited by their own cultural perspective and experience. Certainly for the Royal Observatory visit, all the books and pamphlets available, contain information which is at best second hand as the writers themselves may have consulted the original manuscripts if they exist.

Another important idea for the students to keep in their minds, is that of evidence. The history books are based on some sorts of evidence. We spent some time in college and at the British Museum looking at what original or primary evidence looks like. They saw tally bones, clay tablet calculations, hieroglyphic inscriptions and sets of weights. This is what histories of the origins of mathematics are based on, together with written interpretations from centuries, if not thousands of years later. With the notions of primary and secondary evidence firmly in their minds, our students could look at the objects on display in the Royal Observatory and make their own histories from mainly primary evidence. Of course, they also read anything else written about their topic, but were encouraged to be critical readers.

Astronomy, Navigation and Time

Throughout the history of mathematics, there have been several key themes which have motivated mathematical developments. Intending teachers need to have some knowledge and background in these key areas to inform their teaching. The student-teachers' own experiences of mathematics are still likely to fall into discrete blocks of learning: school mathematics from lessons, exam or textbook; 'Do It Yourself" mathematics as necessary for everyday life, done in your head or on a piece of paper; and work mathematics which is whatever specialist techniques you need for your work. Linking different areas of mathematics through historical study can help students make sense of what they already know. Astronomy, navigation and time measurement are key themes in the history of mathematics and have spurred on many mathematicians to significant discoveries which have benefited people beyond the immediate problem solution. They are found from ancient times across the world, in all societies struggling to control their environment and please their gods. Examples are found across the world: In northern Indian cultures, sacrificial brick altars demanded high levels of mathematical skill to construct, Egyptian pyramids are aligned with the stars; The title of the oldest source of Chinese mathematics, the Chou Pei Suan Ching is translated as; The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven. This illustrates that in most ancient societies, using astronomy was a major force behind mathematical development.

In today's school mathematical curriculum, no astronomical or navigational mathematics is specifically taught. Even the chapters that used to teach the angles of latitude and longitude have been deleted from modern texts and are only found in geography lessons. This means that school mathematics no longer offers any non-Euclidean geometry; all geometrical surfaces are planes and all geometry is plane geometry. School students will find it increasingly difficult to relate to their classroom mathematics, since many results and techniques, like trigonometry, were developed to solve astronomical problems but are now presented in isolation. This gap at school level is concurrent with satellites giving us amazing pictures of earth from outer space and their use in the Global Positioning System (GPS) which can tell us our position on earth by pressing buttons on a hand set. The displays of objects in the Royal Greenwich Observatory help make the link between the key themes in mathematics of astronomy, navigation and time measurement in the past and present day.

Historical background to the Royal Greenwich Observatory

In recognition of a national need for an almanac of the movements of the moon and common stars, Charles II of England, Scotland and Wales founded the Royal Observatory on 22 June 1675, 'in order to the finding out of the longitude of places for perfecting navigation and astronomy'. A site was chosen in Greenwich Park, which since it was on high ground and on the outskirts of London, gave good views of both the heavens and London city. The first Astronomer Royal was appointed in the person of John Flamsteed whose job it was to catalogue the stars, with the help of one assistant and £100 per year. The aim of the first Greenwich astronomers was to produce an almanac with the moon and star positions at all times during the year. Such an almanac would make possible the idea of a lunar-distance method of finding longitude as suggested by the French mistress of the King.

But why would King and Parliament go to such expense employing Christopher Wren himself to design and build the new observatory? After all, Wren had already made his name for his architecture after the Great Fire of London in 1666, designing the new St. Paul's Cathedral and numerous other churches and important buildings. However, Wren was an astronomer before he turned to architecture, so his design was practical enough to house the long telescopes and large quadrants that were the instruments of the day. He said, "We built indeed an Observatory at Greenwich….it was for the Observators habitation and a little for pompe…"

The driving force behind the founding of the new observatory was the Longitude problem, which was not solved until a hundred years after the Observatory's founding. Hundreds of seamen and many ships full of valuable cargo were lost at sea each year since although a ship's latitude could be found if measurements were accurately taken, there was no method of surely finding its longitude. Just inside the first room of the Royal Observatory's exhibition, is a map, which tells this story in harrowing detail. South America is the subject, and 1741 is the date, with a line drawn showing the route of Commodore George Anson's fleet sailing from the Caribbean, round Cape Horn and on northwards roughly up the west coast of South America. The route is obviously plotted from details given in his own ship, the Centurion's log, which is also on display. The fleet of 6 boats reached Patagonia (southern Argentina) together but lost 5 boats around Cape Horn. The horror now of seeing how the remaining ship meandered north and south, east and west along the coast trying to reach Juan Fernández Islands to replenish supplies, is quite striking. Half the 500 seamen on board the Centurion, died of scurvy and related diseases as they could not tell how far east or west they were. This map sets a context for the rest of the Royal Observatory's exhibits as a museum showing the quest for the Longitude and accurate time measurement so sea travel could be as safe as land journeys.

The state of measuring instruments in 17th century Europe was such that telescopes, timekeepers, angle measurers, direction finders, and charts all existed but none were accurate enough to use at sea. While ships could follow coastlines or sail along parallels (line of latitude) then the problems of navigation were minimised. However, for English and Dutch ships trying to get into the English Channel to get to their home ports, an error of measurement or calculation of position could land the ships on the rocks of the Scilly Isles. This happened to Admiral Cloudisley Shovell's forces in 1707, losing 2000 lives. The founding of the Observatory pushed up the demand for more accurate instruments both for making observations on land and for instruments specifically designed for use at sea. The solution to the Longitude Problem in England was solved mainly due to the invention by John Harrison, of a sea-worthy time-keeper which could tell a ship at sea the exact time in London (or Amsterdam, or Lisbon, depending on the home port). The situation at sea only improved gradually from the 1780's as chronometers became affordable and available for all ocean ships.

The students' task

Each pair of students (occasionally one on their own, or even three together) chose one particular aspect to look at on the day. I spent part of the previous session in college introducing the Observatory and telling the story of its founding, setting the scene. Museum objects on their own are quite dry but once they are seen as props to an important story, then each individual person can start to relate to the quest for a practical, foolproof method of measuring longitude at sea, and of improving the accuracy of all measurements made at sea. In the light of this introduction, students were often inspired to choose something straight away before the visit, because they wanted to start reading and finding out more in a focused way. My own books were often borrowed that day despite pressing assignments for other course units. Their task was to look at everything at the Observatory, make notes towards their topic, research afterwards in the college library and finally to present their findings to the whole group. They not only had to unfreeze the mathematics, but also they had to place it carefully in its historical context to show how important it was at the time. They also had to make their own personal sense of the mathematics, so it was understood well enough that it could be explained clearly to the whole group. After all they are aspiring to be good teachers.

The Observatory trail is organised chronologically in the main. As you enter the first room, you see sketches, pictures and a computer-animated construction of the observatory's beginnings together with Anson's map. Next is a room of the artifacts showing the state of knowledge at the time, across the known world, followed by several living rooms remade in the style of the late 17th century. Then the story of Longitude Problem unfolds with a whole room dedicated to John Harrison and his clocks H1, H2, H3 and H4, the winner of the Longitude Prize. Downstairs, the story of time is brought up to date. Over in the main building, the telescopes, sextants and quadrants are displayed as they were in use until the 1950s, with other examples for comparison.

The students and I met by General Wolfe's statue, where we are able to see the whole of Greenwich and the river Thames. On a clear day, we can see St. Paul's Cathedral and the city of London to the west, and down river to the east towards the Millennium Dome and the Thames Barrier. We went into the Observatory to buy student-group tickets then out again into the courtyard to stand with one foot in each hemisphere across the Zero Meridian line. Before going into the main exhibition, we spend ten minutes in a darkened room looking at the lighted round table. Here a Camera Obscura projects down the same view of the river as we saw outside, panning back and forth to show moving boats and cars on the road.

On one of our visits, as we reached the end of the signed trail, the last exhibit which is a 30-foot telescope in its own dome structure was being demonstrated by one of the museum's astronomer-scientists. He not only talked about the telescope's history but also showed how the telescope could be moved round to any part of the circle and could be tilted at any angle all by turning a handle to operate a ratchet mechanism.

Topics chosen by the students

Tom chose celestial navigation, which he described as finding your way by the stars using a sextant (or other angle measure), Greenwich Mean Time, (or other time measurer), maps and an almanac. He took all of us through the sequence of making an educated guess as to your position, called your Assumed Position, identifying and taking readings from two stars, checking the readings with an almanac then adjusting the Assumed Position in the light of your knowledge. This process could be repeated several times until the error triangle was so small that the Assumed Position readings closely matched the readings of the Geographic Position given in the almanac. During his presentation, Tom was able to use transparencies and sketches to illustrate the mathematics. He had found a site on the Internet to download that helped him explain clearly. Angles of declination, right-angle triangle geometry, table reading, time measuring and calculation were some of the mathematics he identified.

Beth researched telescopes, which are on display in the Royal Observatory, and the Camera Obscura. She explained how each of the main types of telescope worked by drawing diagrams to show how the light was reflected or refracted by the mirrors and lenses. Her imaginative presentation included layers of transparencies to show how the colours in light can be split and joined. Ingeniously she also used the Overhead Projector light to shine through her model pinhole camera and project a cutout L shape in her box.

Trevor had visited the Observatory on his own after we had made our group visit so he did not get much on the background. Instead, he did the work himself and told us his version of the story of longitude again backed up with his own chronology. Most usefully, he made some tables of equivalent measures between degrees latitude and longitude, distance in miles on the ground or at sea, and time in hours and minutes.

Dan made a simple card quadrant, which he demonstrated, for measuring angle above the horizon. He then showed how a Sextant superseded the quadrant and why the smaller 60º segment was a good base for a whole circle of measures. He showed us how you could see a star and the horizon at the same time using the combined glass and mirror-viewing device then read the angle off the scale. Since horizons could be hazy or not visible, Dan explained how an imaginary horizon could be added to a sextant to make it even more versatile.

Jeremiah had done a lot of work to understand the notion of latitude. He drew diagrams of different places on the earth's surface like the equator, New York, Greenwich and the Poles. He, then showed how their angle of latitude was determined in theory using the centre of the earth, and then in practice from aboard a ship.

All the material on looking at the moon and stars inspired Andy to look at the effect of the moon's gravitational pull on the earth. He explained that the earth, moon and satellites all either exert a gravitational pull themselves or are being held by the pull of some other body. Andy put the facts together to show how the gravitational pulls of the earth and moon act on each other to produce a different level of tide on the earth's oceans and seas. Every two weeks, we get high, high tides and low, low tides since both the sun and moon pull the waters on earth. At full moon, the earth has the maximum pull from the sun and the moon. When it is a new moon, the moon is in between the earth and sun, so the two bodies also give a combined pull on the earth, which gives a high, high tide or Spring tide. A graph of spring and neap tides helped show how the oceans rise and fall during the moon's orbit around earth.

Vicky made a model astrolabe, which is itself a model of the heavens. There are many different Western and Islamic types of astrolabe to see as the museum's collection of astrolabes is the second largest in the world. Vicky explained how the instrument could be used by an imaginary observer to show the sun, moon and stars at a particular time and latitude. Then observations could be taken using the scales round the edges. There are several different types; the Planispheric astrolabe used for astronomy, the Mariner's type used at sea, and a Universal type often used for astrology or fortune-telling. From one of the museum's reference books, Vicky was able to label the parts and demonstrate the functions of this multi-faceted measuring instrument.

These students were a small group training to be secondary teachers, so they had worked extra hard to do individual, instead of paired, presentations. This benefited all of them since they share their findings both through speaking and often also making copies for their fellow students.

Feedback

No presentation would be complete without feedback to help the students improve their future presentations and classroom performance. First, feedback from their peers, as their fellow students were supportive yet critical adding other points from their own research. The variety of materials, models, overhead transparencies, diagrams, maps and illustrations were all good ideas for our future teachers, which they could use in their own classrooms to clearly illustrate the mathematics. Second, from me as their tutor and lecturer, I fed back privately with each student. I made notes both on their presentation skills and on their explanations of the mathematics. The most common faults in presentation were speaking very fast due to nervousness, facing the writing board instead of the audience or failing to maintain eye contact. The mathematical explanations often assumed too much prior knowledge, some diagrams were too rough to help and one could not write legibly on the board. These are sensitive issues that can be addressed in college but are much more difficult to change once in a school classroom. Also, some of the students' work was displayed on the walls together with maps, diagrams, portraits, writing and photographs. These helped students on other courses see some mathematical contexts for their own studies.

At the end of each History and Nature of Mathematics unit, I asked the students to fill in a questionnaire to see how their attitudes towards the subject had changed. 90% had broadened their view to see mathematics as part of history and over half were enthusiastic about introducing some history into their classrooms to liven up their lessons. I had a telephone call from one secondary mathematics student, two years after he had finished his course. He wanted to discuss how he could bring his class of 13 year-olds to the Observatory to do mathematics. Visits live on in both teachers' and students' memories and demonstrate classroom theories with real history and mathematics.

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History of mathematics: why, and is that enough?

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History of mathematics—we all have some notions of what it is, but should it be included in school mathematics. Rather than be concerned with empirical studies on the effectiveness of history in the teaching of mathematics, this paper considers theoretical justifications for the inclusion from a constructivist perspective, and by considering the implications of enactivism and other ways of thinking. While one can justify teaching the history of mathematics, I think that we may be asking the wrong questions and may need to consider broader issues such as teaching in a more integrated way.

Introduction

Over the last few years there has been debate in a number of school subjects about having historical components included within the curriculum. While empirical studies could well show that such topics assist in teaching, within the context of school mathematics I have two concerns. First we need to know why we might want the history of mathematics education included, and that raises issues such as whose history is to be included and how this might be achieved. The second point that arises if we decide the history of mathematics should be taught is, will that be enough? or, on the basis of justifying the inclusion, is there a need to consider more drastic curriculum reforms?

Justification

An initial justification for including the history of mathematics in school programmes is likely to be based on notions such as that one should start with the interests of the students which may include history. Alternatively one may think that there is a need to put a human face onto mathematics and this could possibly be achieved with some historical overview. Another notion is that our students need to see how the subject evolved and how mathematicians worked and continue to work, thus gaining a broad view of the subject, rather than merely considering the outcomes of the work of mathematicians.

Such justifications usually lead us to consider whether history of mathematics topics and/or activities are intended as ways into traditional mathematics, as another topic within mathematics, as enrichment material, or as redefining an appropriate mathematics course where historical activities are integrated with traditional material.

As well as considering the place of history in the mathematics curriculum, we also need to decide what we mean by history, or as in the first point above, whose history are we talking about. This will involve questions such as, what mathematics are we talking about when we are thinking of historical topics? Is it academic or everyday mathematics? Is it Western mathematics or mathematics from all cultures? Is it history or herstory? And what should be done in countries when mathematics is not seen as a separate subject? Indeed we are aware that mathematics is part of a Western partitioning of knowledge that does not occur in the same way in other cultures. At this stage we may be drawn into the ethnomathematics debates and need to consider the six activities that Bishop (1988) sees as pre-conditions for mathematics—counting, measuring, locating, designing, playing, and explaining, and use these to introduce mathematics into elementary schools. However, the history of these may be inadequate for more than starting points if the goal in high schools is Western mathematics.

If we want to be inclusive of historical topics from different societies and from different cultures then we may need to add a fifth option to introduction, separate topic, enrichment topic, or integrated with all mathematical topics. That is the option of a holistic curriculum that is not divided into subjects.

Making connections

Within current mathematics curriculum debates the term 'making connections' is often used (National Council of Teachers of Mathematics, 1989). For me this term means making connections within subjects, with other subjects, with everyday life and with future possible work. It also means linking with the prior knowledge of students and the familiar social and cultural contexts in which they find themselves. With these notions in mind the inclusion of history of mathematics seems axiomatic.

Constructivist justification

During the last two decades constructivism has become a dominant theory in education. Within constructivism the notion of making connections includes the construction of knowledge 'schemas' (concept maps, mental maps, graphs) where links are made between 'bits' of knowledge and the relationships between bits are as important as the bits themselves.

From this perspective one may see a justification of history of mathematics based on starting where the students are and starting with the interests of the students but although some students may be studying history one can not assume that it is a real interest of the majority. At the same time, giving the subject a more human face and making connections through interesting historical accounts will add an additional dimension to the subject and thus be theoretically justified. The notions of 'making connections' with prior knowledge, with the world of the students, and with the people of that world, warrant the inclusion of history if the topics chosen reflect the mathematical activities they see in their everyday world. This view, which includes making connections with other subjects, can also be used as a justification for a holistic curriculum. However, having a justification based on theory does not remove the question—Whose history of mathematics should be taught, and, does including the history of mathematics go far enough?

Connected knowledge

"Making connections" also resonates for me with what Gilligan (1982) and Belenky, Clinchy, Goldberger and Tarule (1986) have called connected knowledge as distinct from separated knowledge. In broad terms they see separated knowing as how males think, and connected knowing as how women think. There is, in my opinion, considerable evidence for their conclusion within 'white' society (in Anglo-American and European cultures), but there is also evidence of connected thinking within numerous non-Western cultures.

With the idea of 'connected knowledge' in mind, the notions associated with "making connections" in mathematics by including aspects of the history of the subject may be further justified. This is in terms of making the subject more female-inclusive, and, if my hypothesis is true, more non-Western inclusive. However, if the history remains a white male version then this is unlikely to be achieved—Wertheim's (1997) book provides many insights into how and why the history of mathematics is so male oriented. For me however, two questions remain unanswered—Which mathematics are we thinking about when we look for associated history? Does the inclusion of the history of mathematics go far enough or should we still consider the option of a holistic curriculum?

Enactivist justification

Enactivism (Davis, 1996; Sumara & Davis, 1997) is an emerging learning theory. It has been influenced by interrelated notions from:

- phenomenology (Merleau-Ponty, 1962),
- ideas about Cartesian dichotomies (Damasio, 1994),
- consideration of non-cognitive knowing (Maturana & Varela, 1987), which includes 'mindful awareness' from Theravadin Buddhism (Nhat Hanh, 1987) and from Zen Buddhism (Batchelor,1999), and from other Eastern thinkers (Krishnamurti, 1956),
- the neural biological work which emphasises evolutionary or Darwinian notions (Edelman, 1987; Plotkin, 1998; Sacks, 1995; Varela, Thompson & Rosch, 1991),
- systems theory (Bertlanffy, 1968), and
- the idea of embodiment (Varela, Thompson & Rosch, 1991; Lakoff & Johnson, 1999).

One aspect in enactivism that may appear strange to some is the inclusion of Buddhist and Eastern ways of thinking. We from the West have been conditioned to undervalue the Eastern contribution, but this has started to be recognized more generally in other social sciences, for example in psychoanalysis (Suzuki, Fromm, & De Martino,1960; Molino, 1998), and in sociology (Bell, 1979). These Eastern ideas fit well with the arguments against the Cartesian dichotomies where we separate body from mind, self from others, and so on.

I can only give a hint of what enactivism means for me and will do this by providing some ideas from Dawson (1999), but I suggest that we need to familiarise ourselves with this theory. Enactivism involves becoming aware of what you are doing without judging it, and it moves *from a culture based on judgement to one based on possibility.* It requires us to see knowledge not as independent of individuals and their environments, or as something that can be tested/matched against external standards, but rather as embodied action with all of us being responsible for our actions. Enactivism and embodied action mean we maintain the *yin-yang* of self and world, and of ideas 'out there and 'in here'. From this perspective the aim in teaching is not to link learners' experiences to some external curriculum*, but to view the curriculum as being occasioned by the learners' experiences in their school environment*. Such ecological perspectives are located within a complex web of relations with all *decisions and actions being both constrained by and influencing all nodes of the web.* The enactivist classroom could be construed as a dynamic system with *the teacher listening not to check or model the students*, but *to participate* with them (Kieren, 1995).

For me, enactivism provides a different perspective from that provided by constructivism. It rejects the dichotomies in our thinking and suggests that all aspects of a person and their environment are interrelated or connected. Enactivism emphasizes "being connected" which is stronger than "making connections". It implies for me that school subjects, everyday life, and the culture and history of us all is connected. Assuming this connectedness highlights the need to include the histories (and herstories) of all mathematics (academic, everyday, etc), and address the issue of the Western partitioning of knowledge.

Subsequent challenges

From these different perspectives three challenges emerge for me. The first is a mathematical issue; the second is related to education in general, not only to mathematics education; and the third is to do with immediate concerns. The three are:

1. What elements of knowledge from a culture might be regarded as mathematical so that we can include their associated history?
2. Should we separate knowledge into different subjects, and if not, how can we more appropriately organize the curriculum?
3 What could we be doing to get started with histories within mathematics in the meantime?

Mathematics

Is it Western mathematics or multicultural mathematics, and is it academic mathematics or all aspects of mathematical activity we want to consider in terms of history? The answers to these questions are related to the theoretical assumptions we make about knowing and learning; to what we intend to teach in schools; and to how we are going to use historical topics (introduction, separate topic, enrichment, integrate into mathematics, or an integrated curriculum). If we are clear about our assumptions and know what we intend to teach then the problem is straightforward although it may be laborious. If we are not certain then we have some important issues to consider, and this consideration may need to involve people from the community. I do not believe that the answer should be determined only by those of us who are steeped in the traditions of male Western mathematics. No doubt all involved in the determination will have strong connections in their minds and may not always provide a balanced viewpoint, but Western male views could well dominate as they have in the past. That is not to say that we do not all have a role to play. We can encourage consideration of the questions and affirm their importance. We can support the decisions that are made. We can ensure that the dominant curriculum documents allow space for the inclusion of topics such as historical ones. We can legitimize the claims of our communities and help by working alongside the people who use mathematics and find out where the modes of working have come from. Having said this, I would repeat, it is important that Western viewpoints do not dominate—in my experience it is very easy for them to do so. Westerners find it very easy to make assumptions that are bound up with our own expertise, education, ways of thinking, history, and language; and undervalue other profound issues.

Education

The suggestion of a holistic curriculum for many, in particular secondary teachers, implies a radical review of education. No doubt as mathematics educators we would be be willing to participate in such a review, and because of our thinking in terms of history (and ethnomathematics and contexts) we could provide valuable input. While I believe we need to facilitate this review, it is important that we do not dominate it nor expect too much too quickly. For too long many cultures in the East and of indigenous people have been denigrated by the West, thought of as inferior, and not respected because they were different and not understood. Our Western colonization has been of little service to these countries, teaching them Western values of consumerism and capitalism has sometimes caused problems for not only these people but also for the world. A review of education that considers cultural diversity and all its implications in depth, gives us an opportunity to reassess the societal trends of the West, the society we would want for our children, and the sort of education that might help us achieve this. Of course such a rethink is much broader than education but much can be done if we take the viewpoint of Postman and Weingartner (1971) and regard "teaching as a subversive activity".

Such a review would need to look at the aims of education (and of society). Traditionally our aims have been academically or practically oriented. Recently they have included more social and personal aims, and very recently some aims linked with economic growth have been included. A review needs to examine all of these, be prepared to put a different emphasis on them, and include others that may arise such as those to do with culture. After the general aims have been considered we would need to think about how they might be operationalised for subjects such as mathematics in terms of learning, teaching and lecturing, assessing, and curriculum and resource development.

At the same time, if we regard "teaching as a subversive activity" (Postman & Weingartner, 1971) then we can do much in schools, especially in elementary schools with students before subject specialisation occurs and restructure our sets of subject curricula into a holistic one.

Including the history of mathematics in school mathematics?

If we want to include the history of mathematics, however we define it, in our school programmes, then there is much to be done. Most new initiatives require at least three interrelated aspects of development—curriculum, professional, and resource—to be considered. At the start it is usually a small group of interested people that get an project moving. As the initiative slowly gathers momentum more people join in but in my experience it takes a considerable time period before regional projects are initiated. Thus we need grass roots networks with interested people developing some draft resources (and doing their professional development as part of this resource development). If the topics are to be taught as introductory material or as enrichment then no changes need to be sought with the curriculum. If it is to move past this then space will need to be negotiated for some optional material because all the teachers in a region are initially not likely to feel the same confidence to teach the history of mathematics.

Interim solutions

When one has a grand vision it is sometimes easy to think that what one does in the meantime is of little consequence. I would suggest that nothing could be further from the truth. The maxim "think globally, act locally" provides us with a guide to action. We know that the process of change is slow, it often takes more than one generation. New ideas need to be legitimated by being considered over a considerable time span and by comparison with alternatives, this means that many options need to be offered. We need to challenge the status quo by suggesting other ways forward and to show that these at least work to some extent. Thus, while we keep the big picture in our minds, we need to rephrase the three challenges so that we can consider regional, local and personal ways forward.

If we take up this notion then the challenges might become:

1. What elements of mathematics related to the backgrounds of the students in my classes and to the topics we need to cover can I research in terms of history? Will the material I find be for introducing topics or as enrichment? How can I use such material in my teaching?
2. In what ways can I integrate mathematics with history and other topics either in my classroom or with a colleague from the history department? Can I teach some, part, or all of my present curriculum (mathematics and other subjects) in a way that better integrates the knowledge and helps my students make more connections?
3 Can I contribute to a major reform of curriculum by joining a network (perhaps electronic) where such material is shared in a way that each teacher can modify

material they receive to suit their situation but do not have to start at the beginning each time?

As teachers we can all move in this direction to some extent. As researchers we can legitimate such action from others by researching the processes that they work through, document it for others, and share the findings so that a broader view of these issues arises. As teacher educators who have people from different backgrounds in our classrooms, we can become more aware of the different cultures and encourage students to talk to people from different nationalities and different vocations about the changes that have occurred in the mathematics they use. As scholars we can read whatever we can on such matters and share what we find. As members of teams involved in curriculum we can ensure that there is at least space for and encouragement of such innovative changes.

Is this enough?

I expect that many people think that these challenges are too difficult and that my suggestions indicate that I am out of touch with the real world. I would suggest that if we who care do not make a stand then the present problems with mathematics will continue. There is no doubt that over the last one hundred years many changes have occurred in school mathematics and most of these started with a concerted effort by a few keen people. My belief is that the existence of this organization has already started shifting people's views about history of mathematics, the next stage is to develop practical resources to help people who wish to teach some history of mathematics.

At the same time, this is not enough. I hope that people will also keep the big picture in mind and do what they can whenever curriculum is discussed to push to more holistic perspectives.

Conclusion

Many of us are interested in the history of mathematics, but how far are we willing to go? Are we content to dabble and to produce our academic papers? Or are we committed to change and ready to work alongside and encourage the teachers who want to build the history of our subject into their teaching? I believe that our actions and decisions can influence the direction of mathematics and mathematics education. Are we ready to stand up and be counted?

Kia Ora Katoa.

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Lifelong Learning and the History of Mathematics: An Australian Perspective

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Governments around the world have espoused the rhetoric of lifelong learning for social and economic reasons, and it might be imagined that the History of Mathematics would play a prominent role in the mathematics education of the whole person across the span of a lifetime. However, in Australia this does not appear to be the case for school children or adult and vocational learners, according to course documentation. This paper will draw briefly upon the work of Basil Bernstein to analyse the discrepancies between the rhetoric and the reality. It will suggest a reconstructive role for the history of mathematics in re-appropriating the discourse of lifelong learning for broader social and economic purposes than are apparent under neoliberalist regimes.

Introduction

This paper will briefly outline my positioning as a vocational education and training (VET) teacher and researcher. It will continue the conversation begun at the 1996 meeting in Braga (FitzSimons, 1996) concerning the impact of economic rationalism on the teaching of the history of mathematics in the Australian VET sector. It will examine the conditions of possibility for the inclusion of history of mathematics under neoliberal government agendas for lifelong learning which see the increasing convergence of general and vocational education. In countries such as Australia there has been a trend over the last two decades towards ministerialisation of educational bureaucracies, concomitant with the de-institutionalisation of post-compulsory education through mechanisms of flexible learning and workplace-based learning. This has been accompanied by the demise of disciplinary knowledge in the academy but not in schools — notwithstanding the promotion of a set of generic Key Competencies described as being essential for effective participation work and in other social settings. Mathematics continues to be portrayed as an essentialised object, dehumanised, located in vacuums of space and time, inserted as/when/where necessary, to be unproblematically transferred from (potential) worker's 'tool box.' Historical and cultural aspects, when mentioned at all in the rhetoric, become marginal in the recontextualised texts.

As a teacher and researcher my work is on the borderlands of three subfields of educational research: *mathematics* education, *adult* education, and *vocational* education — connecting all three. Perhaps it is due to these borderland crossings that my research remains at the margins of each of the research communities to which I subscribe. I am also both insider and outsider to the subfield of history of mathematics — I have no post-graduate qualifications but an ongoing interest in how it can inform and promote understandings of mathematics as part of peoples' cultural heritages.

A positioning such as my own can, however, be turned to advantage in that it compels the researcher to draw on a broad range of literatures which inform the various subfields of education; each of these discourses (mathematics, adult, and vocational) has broadened my own horizons; and of course they are not mutually exclusive. The growing trend in educational research towards interdisciplinarity has encouraged the cross-fertilisation of ideas, the apprehension of new shades of meaning and insights.

Lifelong Learning

In 1996 I made the observation that in Australia "current VET curricula fail to include ethnomathematics, historical research projects, aesthetics, and so on. That is, non-vocational uses of mathematics are ignored, and the teaching of critical mathematics is implicitly discouraged" (FitzSimons, 1996, p. 133). The situation has, if anything, deteriorated. This is somewhat surprising since neoliberal governments such as those in Australia have been pursuing agendas for lifelong learning which promote the increasing convergence of general and vocational education (e.g., Australian National Training Authority [ANTA], 1999; Department for Education and Employment [DfEE], 1999). The United Nations Scientific, Cultural and Scientific Organization [UNESCO] made the following statement (Delors, 1996):

> There is a need to rethink and broaden the notion of lifelong education. Not only must it adapt to changes in the nature of work, but it must also constitute a continuous process of forming whole human beings — their knowledge and aptitudes, as well as the critical faculty and the ability to act. It should enable people to develop awareness of themselves and their environment and encourage them to play their social role in work and in the community. (p. 21)

It might be supposed a statement such as this would imply an extended role for mathematics in compulsory and post-compulsory education, to invite contributions from ethnomathematics and the history of mathematics, not to mention critical mathematics education. In fact the discourse of lifelong learning has been colonised by neoliberal governments (among others) in the quest for production of self-managing subjects, achieved partly through inculcation of the aspirations of citizenship and partly through encouraging people to invest in themselves through education (ANTA, 1999; Butler, 1998; Marginson, 1997). Ministerialisation of educational bureaucracies has enabled education to be used as a tool of social and economic reform (see Gjone, 1998, for discussion of the effects on school mathematics curricula). The era of 'frank and fearless' advice from experienced, long-serving educators has been replaced by 'can-do' mentalities of short-term political appointees as Department Heads in the form of professional bureaucrats selected for their ability to pursue the corporate management strategies of reformist governments.

The De-institutionalisation of Education

Under the pretext of making lifelong learning more accessible, *flexible learning* and *workplace-based learning* are being promoted as important strategies, supposedly encouraging learning when, where, and how the learner wants it (ANTA, 1998, n.d.). As old boundaries between theory/practice, disciplines, learner/worker, working/learning become blurred, and therefore contested, McIntyre and Solomon

(1998) suggest that the state has actively been seeking to reshape institutional forms of vocational education (in both university and VET) away from what they term the *socio-democratic settlement* in education through the mechanism of work-based learning. In essence, they claim, this amounts to externally mounted challenges to traditional discipline-based knowledge production — these call into question the institutional power of the universities by means of a politics of curriculum, attacking the accepted order or codification of academic knowledge. Curriculum in workplace-based learning is almost completely defined by work activities. While there is contestation between the workplace and the academy, as mentioned above another layer of complexity is added with the worker/learner confronting their own issues of identity and subjectivity, as well as taking responsibility for their own learning.
Flexible learning implies the modularisation and customisation of courses, and delivery outside of the regular institutional sites and times — possibly face-to-face but increasingly through print and electronic media; almost always self-paced. Lindsay (1999) suggests that the concept of *socially distributed knowledge* is at the heart of flexible learning; in the neoliberal strategy of conflation of learning and work, the implementation of flexible learning is reduced to a technical problem. In other words, the system of flexible learning structures and associated credentials are being harnessed ostensibly for the good of the nation and individuals as members of society, but it would appear that they are actually serving as a means of control over worker/students and their teachers. The theorisations of Gibbons et al.'s (1994) 'new modes' of knowledge production help to explain the shift from culturally concentrated to socially distributed knowledge.

Gibbons et al. (1994) argue that the adequacy of traditional knowledge producing institutions is being called into question with the emergence of a new mode of knowledge production, *Mode 2*, characterised by transdisciplinarity and heterogeneity, heterarchy and transience. In contrast to the academically constrained problems of *Mode 1*, those of Mode 2 are set in the context of application and are more socially accountable and reflexive. It would appear that the mathematics used in the workplace is aligned with Mode 2, emanating as it does from a broad range of considerations. Is there, then, a role for history of mathematics in the new order of learning and work?

Even though the culturally concentrated knowledge of the academy is challenged by the instrumentality of workplace-based learning, according to McIntyre and Solomon (1998) workplace-based learning still requires disciplinary knowledge in order to constitute itself. They conclude that workplace-based learning is a particular commodification of knowledge (see also Marginson, 1997). As Habermas (1963/1974) observes, under an ideology of technical control, subjective value qualities are filtered off and respective claims cannot be evaluated. Authorities wishing to secure socially binding commitments, posits Habermas, complement positivism by mythology — see, for example, ANTA (1999), DfEE, 1999, or Kemp (2000).

Reflections on the Current Situation Regarding the History of Mathematics

Drawing on the work of Basil Bernstein (1996), the de-institutionalisation of post-compulsory education may be described as the weakening of classification between sites of knowledge production — the academy and the workplace. This is reflected in the demise within the Australian academy of disciplines such as history and mathematics, even meta-disciplines such as economic history, concurrent with the rise in transdisciplinary courses such as business studies and information technology. What might this mean for history of mathematics? The field of education has not been immune from funding cutbacks in Australia, placed as it is in an ambiguous position within the academy. According to Green and Lee (1999), historically it has been held in lower regard than other professional courses; it has also been marginalised through its emphasis on process rather than disciplinary content. Tensions experienced in its normative theoretical orientation towards the traditional Ph.D. have been overcome to some extent by the evolution of the practice-oriented professional doctorate (Ed.D.). Education faculties are responding to demographic changes through reinvention in their provision of qualifications in education and training for workplaces other than schools (including human resource development). In the move from culturally concentrated knowledge to socially distributed knowledges, traditional mathematics education is likely to be replaced workplace numeracy (however defined).

Within what Bernstein (1996) terms the *recontextualising field*, knowledge produced elsewhere is distributed according to a set of rules which determine who has access to what, then transformed into pedagogic curriculum content by educators (or entrepreneurs in the case of the Australian VET sector) under the surveillance of the state in the form of industry advisory bodies or curriculum boards. Curricula must conform to fundamental overarching requirements (e.g., competency-based training, external examinations). These are further recontextualised by curriculum guides, textbooks, and so forth. Finally pedagogic practice is regulated at the classroom level, in the form of selection of content, forms of transmission, and distribution to different groups in different contexts. The history of mathematics has not had a strong presence in sites of academic knowledge production in Australia and, following from the above, this is likely to weaken in the move from culturally concentrated to socially distributed knowledge.

In Australia, as in other English-speaking and European countries, a generic set of competencies (Mayer, 1992) or core skills has been promoted to inform senior school and vocational education curricula, underpinning the new convergence of learning and work. They are as follows: (a) collecting, analysing and organising information, (b) communicating ideas and information, (c) planning and organising activities, (d) working with others and in teams, (e) using mathematical ideas and techniques, (f) solving problems, and (g) using technology. An eighth competency, (using) cultural understandings, was proposed but not officially endorsed on the grounds that it was included in the other seven. Their existence is further evidence of the convergence of learning and work — a situation reminiscent of technological decision-making as described by Habermas (1963/1974). Consequently the cultural understandings competency has virtually remained invisible, not least in the *Using Mathematical Ideas and Techniques* competency which, more often than not, becomes reduced to 'basic skills' of number and measurement, and quickly checked off. Industry-driven vocational curricula, not surprisingly, fail to appreciate fully the conditions of possibility of the mathematics competency (FitzSimons, in press) and are most unlikely to seriously address cultural understandings — certainly not in the context of the ethnomathematical and historical interests of this meeting.

The Adult, Community and Further Education Board [ACFEB] (1996) programme of general education for adults addresses the cultural understandings in their general curriculum stream, and claims that adults should be taught numeracy skills "in a way that recognises the cultural and historical origins of mathematics" (p. 175). However, in the following 80 pages of learning outcomes, assessment criteria, and sample activities covering four modules, this exhortation receives scant attention with the notable exception of two books listed under resource material — in contrast to the possibilities outlined in FitzSimons (1997) and the Kluwer ICMI-HPM publication (Fauvel & van Maanen, 2000), for example.

For school children the National Statement on Mathematics for Australian Schools (Australian Education Council, 1990) was to set the foundation for further development by individual states and territories. This publication explicitly addresses the need for students to experience the processes through which mathematics has developed, indicating a definitive role for the history of mathematics — such as exploration of the erroneous concepts and errors of reasoning, the motives for pursuing the subject, the origins of particular ideas and techniques, and the importance of particular social and cultural contexts, both historically and in the present, to meet utilitarian and aesthetic needs. Activities are suggested to enable students to recognise that geometries are invented systems, to investigate various attempts to find areas and volumes, and to explore various forms of information collection and recording over time, and so forth. Perusal of a selection of recent Victorian school mathematics texts will reveal that these recommendations have virtually been ignored and that historical anecdotes, if they appear at all, are generally confined to the margins. Thus, although historical and cultural aspects of mathematics may be identified as of importance in official guides, the chances of their manifestation in classrooms or other sites of learning are minimal. This is a grim prognosis, indeed. So, what might be the conditions of possibility for the history of mathematics?

Towards a Renaissance for the History of Mathematics?

I believe that it is possible to re-appropriate the discourse of lifelong learning, colonised by neoliberal (and other) governments and peak bodies, in the interests of the individual citizen and the social and economic wellbeing of their community. As I have argued elsewhere (FitzSimons, in preparation), an understanding of the discipline of mathematics, beyond the superficial descriptions of numeracy as basically confined to elementary arithmetic and measurement, indicates its imbrication in each of the Australian Key Competencies (Mayer, 1992). Further, a mapping may be made between these 'core skills' and a theoretically well-founded model of broad occupational competence (Onstenk, 1998) which includes:

> Broad professional skill . . . is defined as a multi-dimensional, structured and internally connected set of occupational technical, methodical, organisational, strategic, co-operative and socio-communicative competencies, geared to an adequate approach to the core problems of the occupation. (p. 126).

Onstenk (1999) provides a further elaboration of this model. In my experience mathematics education texts for teachers and students, school and vocational, rarely address any but the technical competencies associated with production problems (e.g., FitzSimons, 1996); organisational and sociocultural competencies are all but ignored in practice — even though they may occasionally be given lip-service in the rhetoric — and the tenets of lifelong learning completely ignored (FitzSimons, in press).

In order to meet the criteria of educating the whole person the most pressing need for lifelong learning at all ages is to turn around the near-universal distancing from the discipline of mathematics particularly, but not only, among less privileged groups in terms of gender, class, race, age, and so forth. This is highlighted by the continual exhortation in official documentation (not always realised by any means!) to engender confidence and practical competence in the learner, as illustrated by the following set of goals for Australian school students (AEC, 1990):

> As a result of learning mathematics in school all students should:
>
> - realise that mathematics is relevant to them personally and to their community;
> - gain pleasure from mathematics and appreciate its fascination and power;
> - realise that mathematics is an activity requiring the observation, representation and application of patterns;
> - acquire the mathematical knowledge, ways of thinking and confidence to use mathematics to:
>
> conduct everyday affairs such as monetary exchanges, planning and organising events, and measuring;
> make individual and collaborative decisions at the personal, civic and vocational levels;
> engage in mathematical study needed for further education and employment;
>
> - develop skills in presenting and interpreting mathematical arguments;
> - possess sufficient command of mathematical expressions, representations and technology to:
>
> interpret information (from a court case, or media report) in which mathematics is used;
> continue to learn mathematics independently and collaboratively;
> communicate mathematically to a range of audiences;
>
> - appreciate:
>
> that mathematics is a dynamic field with its roots in many cultures;
> its relationship to social and technological change.
>
> (p. 15)

This rather functional approach may be contrasted with that of Ernest (1998) which offers more scope for constructing a teaching programme for learners of all ages based upon the notion of capability. He promotes an appreciation of mathematics as:

1. Having a qualitative understanding of some of the big ideas of mathematics such as infinity, symmetry, structure, recursion, proof, chaos, randomness, etc.;
2. Being able to understand the main branches and concepts of mathematics and having a sense of their interconnections, interdependencies, and the overall unity of mathematics;
3. Understanding that there are multiple views of the nature of mathematics and that there is controversy over its philosophical foundations.

4. Being aware of how and the extent to which mathematical thinking permeates everyday and shopfloor life and current affairs, even if it is not called mathematics;
5. Critically understanding the uses of mathematics in society: to identify, interpret, evaluate and critique the mathematics embedded in social and political systems and claims, from advertisements to government and interest-group pronouncements.
6. Being aware of the historical development of mathematics, the social contexts of the origins of mathematical concepts, symbolism, theories and problems.
7. Having a sense of mathematics as a central element of culture, art and life, present and past, which permeates and underpins science, technology and all aspects of human culture. (p. 50)

To overcome the affective dissonance, let alone the cognitive, would be a huge achievement. And it is here that the history of mathematics, in concert with sensitively portrayed ethnomathematical and critical mathematics education, could play a critical role in re-humanising that which is seen to be cold, hard, and impersonal; meaningful only to a small élite, harnessed by the powerful to control the powerless, potentially destructive to society and the environment. Mathematical themes addressed could be powerfully underscored by human-centred historical accounts of struggle, success and failure, construction and reconstruction. Then students, young and old, successful or otherwise, would begin to see the study of mathematics as an evolving journey, a human endeavour in which they are but a fragmentary part, but not as something mysterious, closed, supremely rational, absolute, discovered and recorded immaculately elsewhere in time and space. As well, adults (if not children also) need to develop a critical consciousness of the pedagogies they have experienced in the past, recognising the hegemonic history of mathematics education itself, so that they might redefine their own identities and sense of agency in more positive terms. Then the espoused goals of lifelong learning in a technological society might have a chance to benefit citizens, local communities, industry, and society at large.

How might this be achieved? The first step is to establish a theoretical framework, including an epistemology and a methodology, and to accumulate a body of research to counter the inevitable economic rationalist arguments against the proposition. Another is to overcome the barriers of technological decision-making (Habermas, 1963/1974), through networking — starting from this meeting. Perhaps we can continue the conversation . . . .

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The quarrel between Descartes and Fermat to introduce the notion of tangent in high school

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In 1637, René Descartes published in the Nederlands, *Le Discours de la Méthode* and his appendix *La Géométrie*, in which, among many results, he described a general method to obtain the tangents to a curve. Descartes was very proud of his method, the first general one to be given. The Minime friar Marin Mersenne, who served through a steady correspondence with many scientists and philosophers all throughout Europe as a clearing house, asked Pierre de Fermat, who was living in Toulouse in the South of France his feelings about Descartes'method. Fermat answered by sending to Mersenne a short essay, *Methodus ad disquirendam maximam et minimam*, in which he exposed another method for determining the tangents to a curve, that he claimed to have invented already a few years earlier and to be much easier than Descartes. Mersenne sent his essay right away in Holland to Descartes, who became very upset by Fermat's criticizing, and attacked Fermat's method as incorrect : he pretended to apply Fermat's method to certain curves and to obtain false results. In reality Descartes didn't understand Fermat's method well, and, being quite sarcastic, he even proposed to correct Fermat's errors, which led him to imagine a new method for determining tangents, using lines instead of circles. Descartes'misunderstandings and the different arguments in the harsh exchange of letters between both mathematicians during the year 1638 brings to light many different aspects of the concept of tangent, which is quite difficult to understand by high school students. Following step by step the quarrel until Descartes'second method, combined whith infinitesimal ideas connected to Fermat's theory regarding the search of extremum can lead to the modern definition taught in high school.

I intend to explain both Descartes and Fermat's method, to study different arguments of their quarrel, to follow mostly Descartes' difficulties in understanding and his reinterpretation of Fermat's method ; I will expose the various activities this study caused me to propose to my students in order to help them to discover and learn the concept of tangent to a curve and the derivability of a function.

At the beginning of the XVIIth century the definition of a tangent was still the Greek one : "A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle" (Euclid's Elements III, definition 2)[1]. They also had in mind proposition III, 16 and applied it to other curves than the circle : "The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed"[2](...). - Our students also entertain such intuitive ideas of tangent when they begin the first initiation to calculus : they have only met the

[1] Euclid , *The thirteen books of the Elements*, translated by Sir Thomas Heath, Dover pub. New York, 1956, vol. 2, p. 1

[2] ibid. p.36

tangent to a circle.- With this type of characterisation Apollonius determined in the first book of *The Conics* the tangent to the ellipse, parabola and hyperbola. The fifth book, in which he characterizes a normal to a conic by an extremum property was not yet known at that time. There was no general method known ; Descartes was very concerned by this question, because he was, among other things, trying to define the anaclastic curve in order to solve optical problems and to build a glass lense having special reflection properties. This is why he wrote in the second book of *La Géométrie* : "I shall have given here a sufficient introduction to the study of curves when I shall have given a general method of drawing a straight line making right angles with a curve at an arbitrary chosen point upon it. And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know."[3]

Descartes' method is the following : the tangent to the curve at the point is determined by the position of the normal MP, P being taken on the axis of the curve or axis of coordinates. P is to be defined as the center of a circle passing through M and intersecting the curve in two coincident points in M. The method is purely algebraic, grounded on expressing that a certain equation has two equal roots. The equation of the curve has to be equivalent to a polynomial one, and calculations can be complicated in case of high degree equations. The tangent to the curve is exactly the tangent to the circle in M, which is very easy to accept by a beginner.

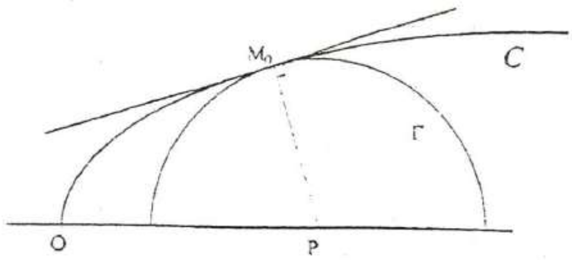

fig.1

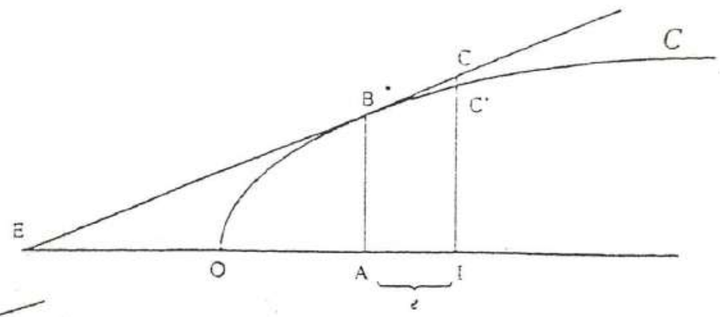

fig.2

Fermat, asked by Mersenne to give his impressions about *La Géométrie*, answered by the end of 1637 that he considered Descartes' tangent method rather complicate and leading most generally to lengthy calculations. He explained that he had had since 1629 a simplier method, grounded on searching the extremum of a certain quantity, exposed in the short essay *Method for finding maxima and minima* [4], that he was sending along. Fermat considered that if f(a) realises the extremum of a quantity f (the notation is

[3] *The Geometry of Rene Descartes* translated by D. E. Smith and M. L. Latham, Dover Publ. New York, 1954, Book II, p.95

[4] *Methodus ad disquirendam maximam et minimam*, in *Oeuvres* de Fermat edited by Paul Tannery and Charles Henry Paris, Gauthier-Villars, 1841, t.I, 133-139 ; french translation, t-III, 121-126

modern, not found in Fermat's writings, nor the concept of function), then f(a) is "allmost equal" to f(a+e) ; he divided f(a+e)-f(a) by e, and then by taking e=0, he obtained an equation whose solution is the value of a. He applied this rule, sometimes called "pseudo-equality rule", for searching the tangent passing through E to a curve ***C***, but did not explain what quantity would be maximal or minimal if line (EB) was the tangent to ***C*** at B (cf figure 2). In fact IC - IC' is that quantity : it is strictly positive as long as C' is on the curve ***C***, and C on the tangent (EB) which is supposed to be entirely exterior to the curve, and becomes minimal (in that particular case. equal to zero) if and only if C' is in B. Fermat's procedure for determining a tangent consists in postulating IC' as "allmost equal" to IC. This type of explanation is crucially missing in the essay and even in the following letters that Fermat would write during the year 1638, and that is the reason why the method was impossible to be understood and accepted by Descartes. Let us note that we can see premisses of infinitesimal methods in the search of extremum, but Fermat never spoke of e being infinitesimal nor approaching zero.

Descartes, very upset by the critics against the method he was so proud of, answered very bitterly in January 1638, that he was surprised by Fermat trying to rival with "such bad arms". Even if Fermat did by chance get an exact result, Descartes would show that his method was incorrect : following the idea of looking for an extremum, he interpreted Fermat's procedure as the search of "the longer line one can draw from the point E to the curve", applied the "pseudo-equality rule" by positing EB to be "almost equal" to EC. The result was not the expected one, so he concluded that Fermat's rule was manifestly false, and that "no other rule could be found as good and as general" as the one he gave in *La Géométrie* [5]. There were again a few letters exchanged in February and March 1638, where the two scientists were not able to understand each other : Descartes applying incorrectly Fermat's rule and Fermat bringing to light the errors of his correspondent, but unable to clearly explain the base of his own method.

The most interesting letter for my purpose is Descartes'letter[6] dated May 3d 1638. The reader understands the fundamental position of Descartes, the importance he was lending to establishing a general method, with an explicit and clear foundation : Fermat's method was not acceptable because it had been imagined gropingly, without clearly conceiving its principles. Still thinking that that the principle of the method was to look for the longer line exterior to the curve that could be drawn from the point E to the curve, but forgetting in his calculations to express this condition of exteriority, he showed that he could not obtain a solution.

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5 Descartes à Mersenne, 18 janvier 1638, in Descartes, *Oeuvres*, edited by C. Adam and P.Tannery, Paris, Vrin, 1964-1974, vol. I , 487, 1.

6 Descartes à Mersenne, 3 mai 1638, in Descartes, *Oeuvres*, edited by C. Adam and P.Tannery, Paris, Vrin, 1964-1974, vol. II, 129, 1.

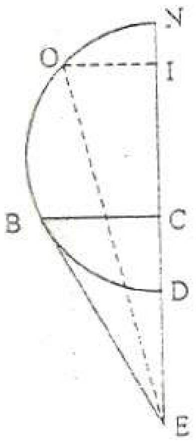

fig. 3

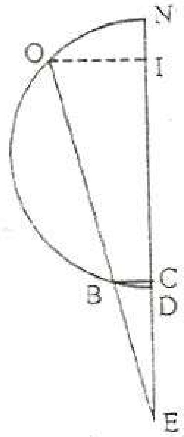

fig. 4

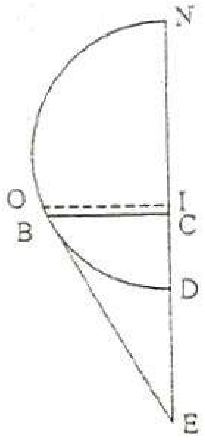

fig. 5

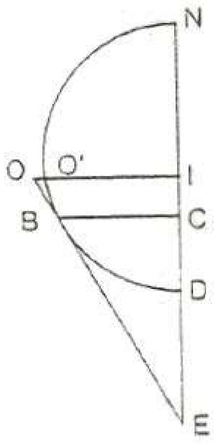

fig.6

Then he offerred to correct Fermat's "erroneous" procedure : he pretended that Fermat was using the "pseudo-equality rule" by assigning OI to be "almost equal" to CB as in figure 3, reproduced from the letter, as are figures 4 and 5 (Fermat's well understood procedure was to assign OI to be "almost equal" to OI' as in figure 6, which is not to be found in Descartes'letter). Instead, he suggested a new method, to use the "pseudo-equality rule" by assigning OI to be "almost equal" to CB as in figure 4 or 5 ; the "pseudo-equality rule" is not used here by Descartes in relation with the notion of extremum, but is a short way to express two equal roots of an equation (O and B are the two intersecting points between the curve and the line, they coincide when EB is tangent to the curve at B). Although once again, Descartes did not consider that e was infinitely small nor approaching zero, the procedure of "pseudo-equality rule" applied as explained by Descartes was equivalent to take the limit position of the secant (OI) to the curve when O is approaching B. This way the modern definition of the tangent is coming out, and even becomes self-evident to the students.

In the summer of 1638, Fermat answered[7] Descartes latest objections, and was attempting a short explanation of the link between the determination of a tangent and the problem of extremum : one should not look for the longer line that can be drawn from a point O to the curve, but on the contrary for the shorter line, which is then the normal from O to the curve. This does'nt explain really why, as in figure 6, he posits IO' to be "almost equal" to IO. Descartes answered Fermat directly for the first time on July 27th, recognizing finally that Fermat's method was "very good". He was even extremely gracious with Fermat , telling him that his letter gave him as much joy as one coming from a mistress[8]. Unfortunately he was not sincere, since the same day he sent a letter to Mersenne[9] where he explained that he had never been convinced by Fermat's method, which had finally turned out to be exact and efficient thanks to Descartes' latest corrections.

[7] Fermat à Mersenne, juin-juillet 1638, in *Oeuvres* de Fermat edited by Paul Tannery and Charles Henry Paris, Gauthier-Villars, 1841, t.II, 154-162

[8] Descartes à Fermat, 27 juillet 1638, in Descartes, *Oeuvres*, edited by C. Adam and P.Tannery, Paris, Vrin, 1964-1974, vol. II, 280, 1.

[9] Descartes à Mersenne, 27 juillet 1638, in Descartes, *Oeuvres*, edited by C. Adam and P.Tannery, Paris, Vrin, 1964-1974, vol. II, 272, 1.

Work in the classroom

I will expose the work done with and by rather slow students aged approximately 16. They are familiar with some elementary algebra (linear and quadratic equations and inequations), have very little ability in analytical geometry (they know how to write and use the equation of a straight line, to recognize orthogonality of lines on their slopes, and to express the distance between two points), have begun to recognize the equation of some classical curves (parabolas mostly), they know what the graph of a function is and have learnt to compute some limits.

I first teach them to write and recognize the equation of a circle. I ask them to write, in the simplest case, the equation of a tangent T (at the point A(0,1) to the circle C centered in the origin). Their only knowledge of the word "tangent" is a perpendicular to the radius of the circle. Then, we consider all the right lines passing through A and given by their slope m, I make them aware of the fact that the line is completely determined when slope m is known (it is difficult for them to work with this parameter m) ; they draw some of them, count the number of intersecting points with the circle and are asked to verify by solving the equation, that tangent T is the only straight line in the family of lines, enabling the two intersecting points to coincide. Then they work again the same way with a second family of straight lines passing through a second point B of the circle. The tangent in a point M has now become the only line having a double intersecting point M with the circle. A complementary approach is followed by using graphic calculators ; in this type of classgroup, with students who don't intend to study math after high school, they are only few such calculators in the classroom. They draw the upper half of the circle ($y > 0$), two or three lines passing through A, zoom around point A , do the same with the tangent at A and discover that, at some scale, they cannot distinguish the circle and the tangent, but can still do it with another straight line.

To introduce the notion of tangent to a more general curve, I first talk about Descartes, his optical concerns, the importance he lent to the determination of tangents and we read some excerpts of *La Géométrie*, for example the following : "...observe that if the if the point P fulfills the required condition, the circle about P as a center and passing through the point C will touch but not cut the curve ; but if this point P be ever so little nearer to or to farther from A than it should be, this circle must cut the curve not only at C but also in another point. Now if this circle cuts CE, the equation involving x and y as unknown quantities (...) must have two inequal roots. (...) The nearer together the points C and E are taken however, the less difference there is between the roots ; and when the points coincide, the roots are exactly equal, that is to say, the circle through C will touch the curve CE at the point C without cutting it."[10]
To help them to understand better, I ask them to draw half of the parabola $y=x^2$ ($x>0$) and few circles centered in P on the ordinate axis, passing through the point A(1,1) and to look for the position of P so that the circle ***C*** "touches" the parabola in two coincident points. They discover that it happens when P coordinates are : (0,1.5). They verify it by solving the intersection equation. We define the tangent to the parabola at A as the tangent at A to the circle

At that point I tell the story of the quarrel between the two mathematicians, without explaining in detail Fermat's method. I just insist on what Descartes understood : to draw a tangent passing through N to a curve, one must look for the longer line which

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10 *The Geometry of Rene Descartes* translated by D. E. Smith and M. L. Latham, Dover Publ. New York, 1954, Book II, p.101-104

can be taken between N and the curve. With some sketches on the blackboard, they soon contradict this characterisation. We read some lines from Descartes' letter dated May 3d and closely look at the figures in it (fig. 3, 4, 5). The students formulate themselves the second Descartes'method, to express that the tangent cuts the curve in two coincident points, recognizing that they have already done this type of calculations in the case of the circle, and they consider the family of straight lines passing through A (each one of them determined by its slope), to look for the line cutting the parabola at two coincident points. They are convinced that the algebraic procedure is easier than the previous one with circles. Using graphic calculators we verify that the tangent will, at some scale, not be distinguishable from the parabola, on the contrary to another line passing through A.

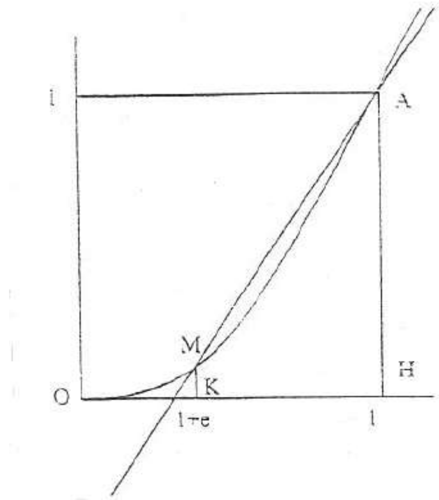

fig.7

I ask then the question of how they will express the condition of two equal roots of the intersection equation if the equation of the curve is $y=x^3$ or of a higher degree. Their first answer is naturally to use the "nullity of the discriminant" as they did with the quadratic equation. Then I explain how to use Fermat's "pseudo-equality rule" in the manner Descartes suggested, in the particular case of the tangent of the parabola at A : he proposed to posit HA to be "almost equal " to KM, M being defined as the point on the parabola with first coordinate 1+e ; this is equivalent to posit HA - KM "almost equal" to 0 ; he divided by e ; we follow and obtain 2+e ; then he took e=0 and we obtain 2, in which my students recognize the slope of the tangent at A they have got by the previous method. They observe that they can follow the same procedure in order to look for the tangent at A to the curve $y=x^3$; some of them verify graphically with the graphic calculators that the obtained line has the same undistinguishable position relative to the curve. Then I ask them to recognize what the graphic interpretation of the operation : "(HA - KM) divided by e" is (translated into $(y_M - y_A) / (x_M - x_A)$ to help them). They are disturbed by taking e=0 after having divided by e, but finally accept to recognize the procedure of taking a limit when e approaches zero. They are quite ready to accept the modern definition of the slope of a tangent to a curve. But before I make them read an excerpt from D'Alembert[11] explaining that the tangent at A to the curve is the limit position ot the secants (AB) when B approaches A ; and that

11 D'Alembert, *Eclaircissements sur les Eléments de philosophie* in *Mélanges de littérature, d'histoire et de philosophie*, vol.V, 1767

all the straight lines passing through A are defined by their slopes. So it only necessary to look for the limit of the slope of (AB) when B approaches A. I derive the definition of the derivability of a function in a, from the existence of a tangent having a (finite) slope.

A complementary exercice later, when the students know how to use the derivative, is to go back to the first misinterpretation of Descartes, and to look for the smaller line one can draw from a given point to a curve.

I hope to have shown how historical mistakes and misunderstanding can also be quite fruitfull to our contemporary teaching. You have certainly noticed that these activities are mostly based on tentative and sometimes erroneous ideas and not on the accomplished formulations of the concepts.

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Using Mathematics History and PCDC Instruction Model to Activate Underachievement Students' Mathematics Learning

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Abstract

The vocational high school students were mostly underachieved in mathematics, therefore causing their vigor toward mathematics learning to die away, especially for those students in evening class. These students' attitudes and values toward mathematics learning need to be promoted. This study reports an action research of a teacher who cooperated with researchers and used both mathematics history and a Problem-Centered Double Cycle (PCDC) instruction model to improve students' mathematics learning in an evening class of a vocational high school. The PCDC instruction model was compatible with the social constructivist view of the genesis of mathematics knowledge, and recommendations of the Standards advocated by NCTM. The teacher gave out tasks for the students to accomplish by individual or small group, and share their experiences and ideas. The tasks included accessing the Internet to gather data about history of mathematics and group discussion in the classroom on the mathematics history and mathematics problems. The findings of this study were that these students gradually regained their vigor for learning mathematics; they came to agree on this new way of learning and appreciate mathematics as a feature of humanism. Students' views of mathematics, attitudes and values have changed positively.

Key Words: mathematics history, PCDC, underachievement, views of mathematics, action research.

1. Introduction

The reform of mathematics education by using historical perspective of mathematics has been growing in the past twenty years. There are no agreement upon 「Why」 and 「How」 to set up the relationship between Mathematics History and Mathematics instruction and these opinions are still widely divided (ICMI Study Book, vol. 6; HPM 2000 Taipei Net Work). The aim of setting up the international HPM is to develop the application of mathematics history to mathematics education. In the development of HPM 2000 Taipei, there are more and more teachers in Taiwan trying to improve mathematics teaching by applying history of mathematics.

Most of these teachers were proceeding under the traditional teaching rather than the constructivist teaching. The purpose of this action research is to report a mathematics teacher of vocational high school who applied both the constructivist teaching and history of mathematics history to improve students'mathematics learning.

Lillian (an assumed name) is a mathematics teacher of vocational high school at the north of Taiwan. Seven years ago, she went back to the university and learned the constructivism and the Problem-Centered instruction. She was successful in carrying out the Problem-Centered instruction for three years and was confident in the method of the Problem-Centered teaching (Hsiao, 1999).

Recently Lillian changed her teaching class from day school to evening school. Lillian hadn't been successful in her teaching. She felt so upset. Evening school students rejected the new way of learning. Lillian didn't understand why the desire of learning mathematics is so low. She was so proud of the Problem-Centered instruction in day school but it almost couldn't be implemented in evening school at all. Students were scared of mathematics and didn't believe they had the ability to learn mathematics, they felt that mathematics is no use except for examination.

As soon as she observed the inert attitude of students in the study of mathematics she couldn't help but feel helplessness. Lillian started to question about constructivist teaching and doubted if it was suitable for the low achievers. In order to clear these suspicions, Lillian went back to university to get consultation from professor Ching-Kuch Chang. After the professor had understood the teaching problems he shared with Lillian what he had learned about mathematics education recently.

The Problem-Centered double cycle instruction model basically is almost the same as Problem-Centered teaching but is more sophisticated. But how to teach the value teaching Lillian had no idea about that. She had a further discussion with Chang, and decided to adopt the history of mathematics to enhance the mathematics values and inspire the interests of learning mathematics.

Lillian was puzzled about how to introduce mathematics history to her class, because she had learned little about mathematics history in the past and heard a great deal of failure ending about employing mathematics history to mathematics classroom. Chang pointed that the common problems that caused a failure ending was that these teachers used the way of traditional teaching. That means they employed mathematics history to mathematics classroom under the traditional teaching method.

Therefore Chang suggested Lillian use the mathematics history in new teaching style in the mathematics classroom, and to think over guiding students to access the Internet in searching for mathematics history and mathematics materials related to the class topics. These students were familiar with the computer network. So Lillian decided to encourage the students to access the Internet to search for the material about the mathematics history. Lillian and Chang started an action research by employing mathematics history to Problem-Centered double cycle instruction （PCDC） to improve mathematics teaching in a vocational evening high school. The students went through exam twice in three months.

The preliminary findings of this study were that these students graduallyregained their vigor for learning mathematics, their attitudes and values for mathematics have been improved. Surprisingly, the students' records were also greatly improved.

2.The Nature of Mathematics and Problem-Centered Double Cycles Instruction （PCDC）

In recent years , the nature of mathematics has been changed from tradition absolutism to social constructivism （Von Glasfeld, 1991）. PCDC instruction was designed based on social constructivism （Chang,1995）.

Basically the core of PCDC instruction is Problem-Centered learning model. There are two cycles existed in PCDC , the inner cycles is learning while the out cycle is teaching cycle. The learning cycle is to guide the learning activities of individuals so the subjective knowledge of individuals and objective knowledge of the social identification can be mutually created and circled. Outside the learning cycle there is the teaching cycle. The goal of teaching cycle is to facilitate learning cycle to work（Fig. 1）.

In the Problem-Centered instruction model the teacher assigns problems as the chief task, allowed the learning model of students using group cooperation to solve problems and to share the expression to reach a common consensus. The teaching cycle reminds the teacher to give the suitable well-designed 〞task〞and let the students to accomplish the task.

It is necessary for the teacher to provide a good learning 〞environment〞and sometimes to offer hints for students to fulfill the task. The teacher should improve step by step by continuing to 〞analyze〞 the teaching.

From assigning the problem, making the planning, analyzing the effect and reconsidering the problems, this continuously improving process is the basic principle of the PCDC teaching model, it is also a basic principle of the action research.

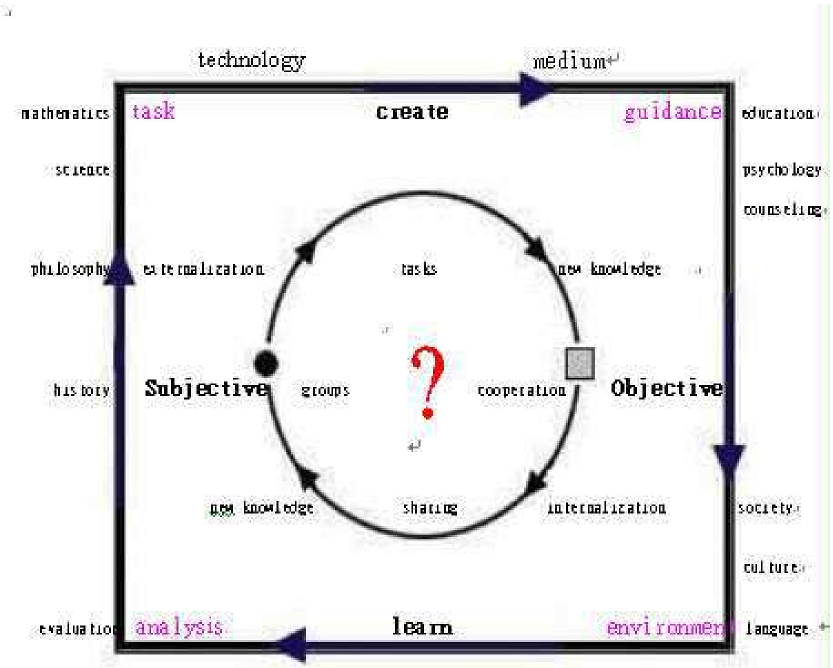


Fig.1. PCDC instruction model (Chang，1995)

3.Teaching Activities Illustration

Fifty students from one of the three classes of the first-year data processing class in an evening vocational high school were chosen for the study. The teaching experiment started from March and finished at the end of May 2000. The main topics covered in teaching are trigonometric function and trigonometrical survey. At first, Lillian guided the student to access the Internet to " kimo website", and key in "mathematics history" "trigonometric function", then from the net page, where there are plenty of information, the students can pick up materials they want and print them out. These materials became to class for discussion information.

There are six students in each group cooperating together to achieve the tasks. The chief tasks were sharing with mathematics history and problem solving in each group cooperation；the next step students were shared ideas with each other. The former stage usually continued for fifteen minutes to twenty-five minutes, the latter stage continued for fifteen minutes to twenty-five minutes.

Besides sharing the story about mathematics, Lillian also put those mathematics problems in a mathematics history in teaching. For example, to teach cosine law the "Samos Tunnel" story was dissolved in the teaching. This teaching materials of "Samos Tunnel" story was excepted from high industrial mathematics textbook （chang,2000）

Lillian entered the classroom where the students had arranged the desks in-groups order and Lillian gave loose-leaf materials to students:

The Story of Samos Tunnel

In Greece there was an island called Samos, which was the hometown of Pythagorem . About the B.C.540, Samos was the capital of Polycrates. As the population was increasing the water for drinking is not enough. On the other side of the Castro Mountain there was a fountainhead. Hence there came a need to dig a tunnel and induce the water to the other side.

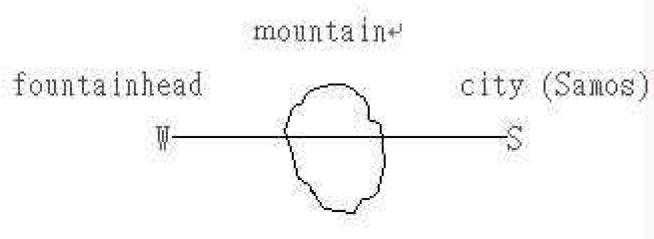
Eupalinus was an engineer who wanted to accelerate the progress by working from two sides of the mountain simultaneously. The tunnel should connect at midpoint. The workers dug from two sides of the mountains but couldn't connect precisely at the midpoint. The only choice was to find another way to connect. How to connect the tunnel on both sides precisely?

Group discussion:

How did Euplinus decide the right direction to get through?

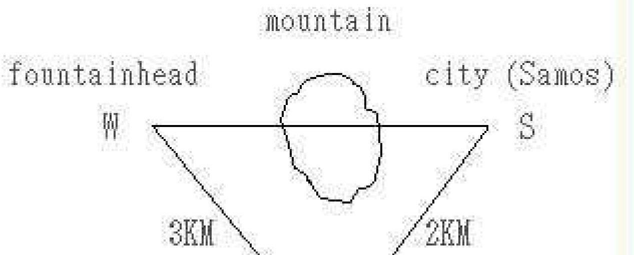
fountainhead mountain city (Samos)

Lillian talked to students : "Everyone gets the loose-leaf material should read it attentively ! Let us come to appreciate the famous Samian tunnel story of Greek. How did the Greeks solve the problems in their life ? How to set through and find out the most efficient way ? First, group discussion for 15 minutes and share with your views for 20 minutes, lastly, the teacher has 10 minutes for discussion. "



Eupalinus thought, if there wasn't the mountain, just to pull a long line connected with S and W, along the line set up a water pipe, and then the problem was solved. Although the mountain was there , if we can find the line direction between S and W , then the dig will be no problems. Now how to decide the direction of WS ? How to estimate the length of the water pipe we need to establish from fountainhead W to the city S ?

To find a reference point O, from O we can see W and S simultaneously. We can get the distance OW of 3 kilometer by measure, and the distance OS of 2 kilometer, and measured an included angle WOS is 54°.



O

The problem of Samian tunnel

The problem of Samian tunnel had already transformed to the question: We know an included angle of a triangle and both sides length of WO and OS, and then to solve the two other angles and the length of the third side. We can get the third side length using cosine law and sine law to get the other two angles. By using the obtained two angles, one from S and the other form W, the mountain can be penetrated precisely from both sides and in a line. Then the difficult construction problem can be solved.

COSINE LAW: to explain the relation of cosine law and the length of the sides

$$a^2=b^2 + c^2 - 2bc\cos A$$

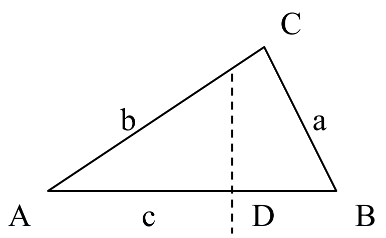
$$b^2 =a^2 + c^2 - 2ac\cos B$$

$$c^2 =a^2 + b^2 - 2ab\cos C$$

In group discussion of Samian tunnel at first, Lillian didn't remind students to find reference point O but give more open questions to inspire student's imagination, creation, and critical thinking which Lillian believed that were more important than the solution itself.

Lillian didn't let students concentrate on solving the correct direction of both sides ,but to reflect the tunnel construction problem to solve the problem of triangular— two sides with one angle. This means to imply the cosine law.

The principle of hinting is " late and less", as long as the students have the ability to do the problem then the teacher should not demonstrate in order to avoid depriving of student's creation and learning. By discussing tunnel problems one implied the cosine theorem. After proving the cosine theorem then one came back to solve the tunnel question.



By using the auxiliary line CD, the cosine theorem can be proved. Assuming A is acute angle, then

$$a = CD + DB$$
$$= (b \sin A) + (c - b \cos A)$$

$$= b \ \sin A + c \ - 2bc \cos A + b \ \cos A$$

$$= b \ + c \ - 2bc \cos A$$

4. Data Collection and Analysis

Gathering data included classroom observation, teaching protocol, teacher interview, mathematics history materials and trigonometric function, and student interviews. Quantitative data were monthly exams reasult. Qualitative data were the analysis interviews, observation notes, and triangulation.

5.Preliminary findings

(1) Enhanced student learning interests

From viewpoints of the students, the teacher thought to apply mathematics and let the students feel as if they were in the real situation using mathematics to solve problems and then to affirm the effect of mathematics teaching. The students considered this model not only allowed them to better understand the mathematics history but also made the course more practical and beneficial for their daily life.

Applying teaching material （chang,2000） to PCDC teaching caused positive reactions for students. There are forty-two students who considered that using mathematics history to mathematics teaching was vivid and interesting. It raised interests in studying mathematics. They believed that using mathematics one can solve daily life problems and they changed their views of mathematics.

There are three students who considered mathematics history might influence their learning, and wondered using mathematics history in mathematics would influence scheduled teaching.

After doing PCDC instruction there are five students who felt that discussion could help mathematics learning to progress. However, they are still not used to the PCDC teaching model; they hoped teacher could teach mathematics more. In general, the response of students' reaction was positive. Merging mathematics history into mathematics really enhances mathematics teaching.

（2）Attitudes become better

Concerning students' attitudes about mathematics, transforming from the thought that mathematics was useless into that it could solve problems; it also transformed from that mathematics was isolated from our life into the comprehension that it played a crucial role in our daily life. They felt, therefore, they must study hard in mathematics.

Student understood that mathematics was very abstruse, but it was very practical. They thought it could save their time and labor whether in their daily life or applied in the way science is applied. Students would never forget by applying this because most of the answers were derived from discussions; it is not easy to get but it is also not easy to forget.

Students thought that it was really difficult to accustom the PCDC instruction; by this new model, they had changed their attitude of studying mathematics mechanically. This made mathematics courses more interesting, living, exciting and so on.

Students positively affirmed PCDC instruction; through group discussion and sharing they had general concepts about those hard-dissolvable problems. If students comprehend by impressing themselves with a story, they would understand the formula, and calculated questions more carefully and attentively, they would say, ″Mathematic is so easy.″

（3）Monthly exam achievement to move great advance

Students thought that the results of monthly exams would not be better than before because the teacher taught little. Besides learning mathematics history was useless for monthly exam, and it seemed nothing was gained. However, the following monthly exam showed there were 35 students whose advancements were too much to be believed.

From Table 1, the first monthly average exam score was 50.12,and the second monthly average exam score was 60.32. The second monthly exam average was advanced 10.2, a remarkable achievement. There were three classes in the first grade of the department of data processing in the vocational high school.

Next, classes comparison were as follow：

The first monthly exam average score; the experiment class was 50.12,the other two classes were 49.3 and 46.1. After first monthly exam experiment class had the mathematics history and PCDC instruction and the other two classes had the tradition instruction.

In the second monthly exam average score, the experiment class was 60.32; the other two classes were 45.1 and 41.4. The experiment class was above the first monthly exam by 10.2, and the other two classes were lower than the first monthly exam by 4.2 and 5.

It was clearly the influence of mathematics history applied into PCDC instruction.

Table 1、Mathematics score of three classes in two monthly exams

mean and standard deviation

Class / Monthly Exam		Class A experiment class	Class B Tradition teaching	Class C Tradition teaching
exam Ⅰ	M (SD)	50.12 (23.67)	49.3（24.1）	46.1（22.4）
exam Ⅱ	M (SD)	60.32 (23.05)	45.1（24.4）	41.1（24.9）

Table 2. The stem-and-leaf plot of experimental class monthly score

	Monthly exam Ⅰ	Monthly exam Ⅱ
	Stem leaf 9 256 8 5 7 069 6 0011233355566666 5 004488 4 124589 3 0559 2 013 1 245568 0 09	Stem leaf 10 0 9 266 8 00004888 7 126666 6 00444588888 5 2226666 4 044 3 266666 2 4 1 2266

Comparing two monthly exams scores from the stem-and-leaf plot:

high score group above 80 scores from 4 students increased to 12 students；above 70 scores from 7 students increased to 18 students . low score group under 20 scores from 11 students decreased to 5 students；under 40 scores decreased from 21 decrease to 14 students.

To show the individual progress、regress，there were 35(70%) progress among the 50 students . 1 (2%) of students without progress and regression, and 18 (36%) students having regressed. The most progressed student was number 2, with progress from 0 to 60, next was number 54 from 15 to 68 progress 53 points, as well as number 1 who progressed from 65 to 96，and student number 8 who progressed from 70 to 92 . There were 14 students who regressed. The most regressed student was number 26 regressed from 54 to 12, regressed 42 points, and next was student number 39 regressed 15 points, number 15 regressed 15 points, and the other 11 students regressed less than 10.

According to the findings of the study above, using mathematics history with PCDC mathematics instruction activated students learning interests and motivation.

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On the effectiveness of history in teaching probability and statistics

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1. Introduction

Over the past twenty years there has been growing interest in using history of mathematics in mathematics education (Freudenthal 1983, Sierpinska 1994). Teachers and researchers provide us with various arguments in favour of including historical dimension in the teaching of mathematics (Sierpinska, Kilpatrick 1998).

In my presentation I would like to show how the knowledge on historical development of probabilistic and statistical concepts can influence the process of probability and statistics – stochastics – learning in such a way that it regards student's cognitive development. Results of my research point out that using history of mathematics in mathematics education can be effective not only in creating a didactical approach to teaching which takes account of the student's actual abilities, but also in recognising student's progress in mathematics learning.

On the basis of various cases of student's actual arguments and solutions, the effectiveness of using history in probability and statistics education will be carefully discussed.

2. The development of probabilistic and statistical concepts – from historical viewpoint

Analysis of old authentic or reconstructed reasonings – which were made not only by the best scientists but also by common people living in various historical periods – allows us recognise many interesting regularities and pose hypotheses concerning historical development of probabilistic and statistical concepts (Lakoma 1992, 1999). Here we will focus on these historical hypotheses, which seem to be of great importance from didactical point of view.

2.1. Ian Hacking (1975) argues that *the concept of probability* in its historical development *has a dual nature*. He distinguishes two aspects of this concept. One of them – *epistemological* – is implied by the general state of our knowledge concerning a phenomenon under consideration, and is related to the degree of our belief, conviction or confidence, which arise in connection to an argument – related to this phenomenon – and are supported by this argument. The other aspect – *aleatory* – is related with the physical structure of the random mechanism and with the admission of its tendency to produce stable relative frequencies of events. The first aspect gives the basis for the „chance calculus" and the other – for the „frequency calculus". The analysis of old probabilistic reasonings shows that both these aspects became inseparable and pierced each other starting from time of Pascal (about 1660). Before that time they were developed independently. Conscious sticking them together, subtle contrasting and verbalising it – which was made mainly thanks to Blaise Pascal and Pierre Fermat – has caused an act of illumination in the way of probabilistic thinking. From that moment the concept of probability acquired a dual, mathematically mature, character (Lakoma 1992, 1999a). Thus, the history points that in order to acquire the probability concept it is necessary to make conscious its dual nature.

2.2. People in the past were faced with some concrete problems, which they tried to solve by means of mental tools being actually to their disposal. When analysing their works, it is possible to distinguish four main steps of a process of problem solving: *discovering and formulating of a problem; constructing of a model of „real" phenomenon; analysing of a model; confronting results obtained from a model with „real" situation.* This way arises directly from common sense thinking while considering simple problems as well as more advanced ones. When we look at history, we find this methodology in works of Isaac Newton (1678). Mathematical models used by people in the past were usually as simple as possible, had a local character and a strong *explanatory value*. These are *the local models*.

2.3. Historical process of forming the dual probability concept was accompanied by crystallising the notion of *expectation* (*expected value*). These two processes have been interlaced and supported each other already in the old pre-pascal time. The notion of expectation in a more sophisticated form has become present only about 1660 (i.e. in reasonings of Blaise Pascal or Christiaan Huygens). Both concepts: probability and expectation were usually confronted and distinguished each other. It is evident especially in definition given by Huygens: "*Expectatio – the chance of a profit* – is worth for somebody as much as he is able to play as if he buys this chance in the fair and honest game." It seems that involving the concept of expectatio and careful distinguishing it from the probability made the probability calculus more understandable and clear for many people in the past and let the theory develop more intensively.

2.4. All historical problems, which appeared in pre-pascal time, could be characterised by finite probability spaces. The first problem, which was solved, using *an infinite probability space*, was presented by Christiaan Huygens in his work "De ratiociniis in aleae ludo" (1657). It seems to be very symptomatic to find that people became able to solve this class of problems just at the same time when both fundamental concepts: dual probability and expectation were crystallised.

2.5. Historical evolution of probabilistic concepts exposes another important regularity – *developing probability together with statistics*. Both these processes were initiated in ancient times and always were interrelated very closely, supported and influenced each other. This connection was indicated especially trough the specific duality of probability. Long observation was a source of many problems appearing in the past. On the other hand it became a necessary aid in the process of problem solving. Simultaneously, people realised that conclusions made only on the basis of empirical data were not sufficient, and that it would be necessary to support them by some theoretical consideration.

2.6. Observing historical development of the probability concept indicates another important hypothesis: a clear *evolution of styles of leading old probabilistic reasonings*. The oldest arguments were closely connected with concrete problems, which appeared in relevance to random phenomena. Models, which were created to solve these problems, fit only to situations under consideration. This style of matching a model to every particular situation separately was common in pre-pascal time. In time of Pascal (1660), an essential change of a style of reasonings was initiated – on the basis of a concrete probabilistic problem people tried to find a solution in a way of thinking, which could be adequate to wider class of problems. From that time we can observe a stable tendency of people to generalise their probabilistic arguments.

Hypotheses presented above lead to the conclusion that this natural way of evolution of probabilistic reasonings, observed in history, should be respected also in the process of probability and statistics learning in today's classroom.

3. Influence of historical perspective on the process of stochastics teaching

Hypotheses arisen from analysis of historical reasonings suggest that the history – showing considerable diversity of ways of developing probabilistic concepts – provides us with some hints to construct a didactical approach, which will regard student's cognitive development. Let us express the main didactical hypotheses basing on the history (Lakoma 1990, 1999).

3.1. I believe that in the process of probability teaching – from the very beginning of education – it is necessary to create such conditions that it will be possible *to form* in students' mind the *dual probability concept*.

3.2. It seems that the main aim of probabilistic teaching would be to create didactical situations, in which students will have an opportunity *to solve probabilistic problems – using local models*. Historical facts suggest also to let students create such a models which directly arise from the natural context of considered situation, expose mechanism of random phenomenon under consideration in a clear way, and use the concept of symmetry not only within models but also in leading of arguments. These models – adequate to actual students' potentialities – have for them the greatest *explanatory value*. What is essential is the general way of reasoning and acting which can be developed in unique way on every level of learning.

3.3. It seems also to be very important *to develop* – from the very beginning of education – both fundamental concepts: *the expected value together with the probability*, remember however to distinguish and to confront each other. Simultaneous considering probabilistic problems not only by perspective of the question "How often?" but also of the question: "Is it worth-while?" seems to stimulate developing in student's mind the fundamental probabilistic concepts.

3.4. From the point of view of reaching mature probabilistic competence, it seems to be important for students to *consider problems leading to infinite probability spaces* and to create conditions to solve them by means of tools which would be available for students.

3.5. The history clearly suggests that in the process of stochastics teaching we should *introduce elements of probability together with elements of statistics*.

3.6. Observing the natural evolution in styles of reasonings in the past suggests let students being able to solve probabilistic problems, according to actual stage of developing their own ways of probabilistic reasoning. What is important for students is *to experience the evolution of their own style of thinking.*

The history serves also as a source of examples, which are fundamental from the point of view of stochastics learning. These *paradigmatic examples* can be scaffolding for this process. Among many such examples I would like to remind the well known problems (Hacking 1975, Lakoma 1990, 1999): the problem of stakes, the problem of a throw of dice until the first success – solved by Huygens – and the problem of shooting to the target (Lakoma 1999a).

In the next chapter – on the basis of students' solutions of these problems – I will show how the history can help a teacher in investigating the process of developing student's probabilistic concepts and reasonings.

4. History as a tool of diagnosing students' progress in stochastics learning

The history can help not only to construct and organise a process of stochastics learning but also to understand students' natural ways of thinking. In recognising symptoms of understanding in student's probabilistic arguments, or mental objects, which student creates and by means of which he forms in his mind dual probability concept in a proper way, it is useful to compare his reasonings with old historical ones.

Duality of the probability concept – duality – serves as a tool for diagnosis and evaluation of a degree of maturity of student's probabilistic knowledge and understanding. During analysis of students' works it would be profitable to distinguish two aspects of probability, using the methodology of Leibniz (Hacking 1975). Leibniz identifies epistemological aspect of probability with arguments *de dicto*, which means – with arguments referred to this, what we know about the phenomenon and what we would express. On the other hand, an aleatory aspect is identified by arguments *de re*, that means by arguments concerning physical characteristics of a random phenomenon. Thus, such mental objects like *experimental frequencies* express an *aleatory* side of probability, whereas *facilities, theoretical frequencies, chances* – based on analysis of a model – express *epistemological* side of probability. Careful supporting and piercing each of these sides are symptoms of understanding of the *dual* probability concept.

In order to show an example how we can recognise different aspects of probability in students' reasonings, let us consider the simplest random phenomenon – throw of a coin. Why we argue that probability of throwing a head is ½? When we pose this question to our students, we usually obtain various arguments like following: 1) *Probability is ½ because there are two possible outcomes.* 2) *A coin has two sides – both are equal (the same), so probability of obtaining a head is one half.* 3) *When we throw a coin many, many times, we will obtain one half of heads.* There are also some doubts: 4) *There is also another outcome – when a coin stands up on its edge.*

Students often give all these arguments. It is easy to notice that argument 1) is not sufficient to answer the question correctly. In this case it is necessary to add that we consider a coin as symmetric one. Only this conviction justifies the probability given in the question. But it is very symptomatic that argument 2) is also not sufficient for many of students. They feel a need to find a support of this *epistemological* consideration – they analyse throw of a coin from *aleatory* point of view, by observing experimental frequencies or by trying to predict their values in a long observation of this random mechanism. When students use both arguments: 2) and 3) they usually are satisfied with their reasonings and they decide to formulate final conclusions. If argument 4) appears in student's reasoning, it seems to be a symptom of *lack of balance* in both sides of arguments. We can observe an excessive attachment to considerations *de re* – concerning physical structure of a random mechanism. This disturbs correct thinking.

This example also shows the natural way of presenting and exploring a random mechanism – students in their arguments usually describe it by specification of possible outcomes and predicting chances of their appearing. We know this way of thinking from the history very well – i.e. reasonings of Galileo or Cardano (Hacking 1975, Lakoma 1999).

When we – mathematics teachers – use the mature probability we certainly do not distinguish and do not expose explicite two aspects of this concept in our reasonings. Our students do not have the dual concept of probability at their disposal yet. Thus we are able easily to recognise in their reasonings both probability aspects which appear separately and independently each other. The way, in which students use these aspects, shows us what is a degree of advance of their probabilistic thinking and what is their stage of forming the dual probability. Reaching a balance in using both arguments *de re* and *de dicto* usually shows us that their probability concept is developing properly and that they are able to gain a relational understanding of probabilistic concepts such as probability, or frequency.

Let us consider some examples of probabilistic activities of students.

4.1. *The Problem of Stakes* – formulated by Chevalier de Mere (1654):

A play of two persons consists of several games, in which each player has the same chance of winning. The winner is the player who wins 6 games as the first. The play was stopped when the player A has won 4 games, and B – 3 games. How to share the stake between them?

In solutions of students we can usually distinguish two ways of thinking. One of them relays on sharing a stake according to results of the game already obtained (advantage of arguments *de re*). The other way – usually much rare – takes into account various possibilities of continuation of the game (arguments *de dicto*). Analysing results, which can happen in future, leads to the right solution. Students often try to find an appropriate graphic representation of situation when continuing the game and draw a tree. Many of them are able to do it without any trouble. But making right conclusions is usually not easy. In example given in Fig.1 we can analyse two solutions of this problem made by students at tertiary level. Both of them represented a situation under consideration by a tree. The solution 1a turned out to be wrong. Student A tried to conclude on the basis of symmetry of the tree – four possible outcomes. His reasoning is in *a style of d'Alembert* (Hacking 1975, Lakoma 1992). It is evident that the concept of *the model of even chances* became an obstacle for him, and caused that his solution is wrong. The other solution presented on Fig.1b is right. Student B counted all *possibilities* of continuation of the game and distinguished those, which indicated a winning of the player A and the player B respectively. However he wrote fractions expressing chances of winning on the every possible stage of continued game, but he did not expressed chances to win in the whole game by means of fractions, describing them very clearly in his own way. In his reasoning he uses *symmetry* – treating all possible events (look at the tree on the right) as equally probable. His reasoning is similar to those led by Galileo or Cardano (Hacking 1975, Lakoma 1992). He uses *possibility* – expressing it as *chances* – as a mental object, which seems to be developed in future towards the concept of probability in a proper way. Teacher – working with student A – should carefully observe student's ways of thinking and pose him in such didactical situations that it would be possible for him to overcome the obstacle of excessive attaching to the model of even chances.

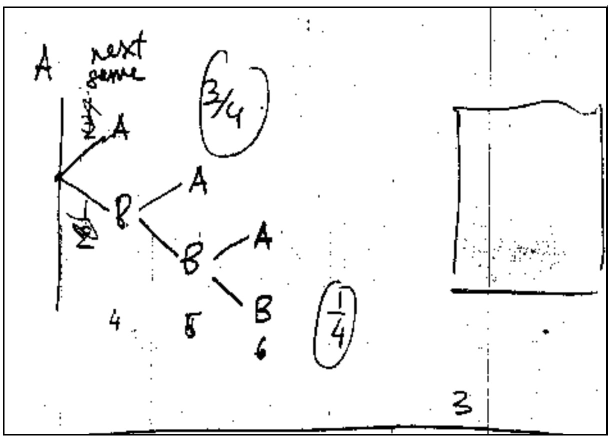


Fig 1a.

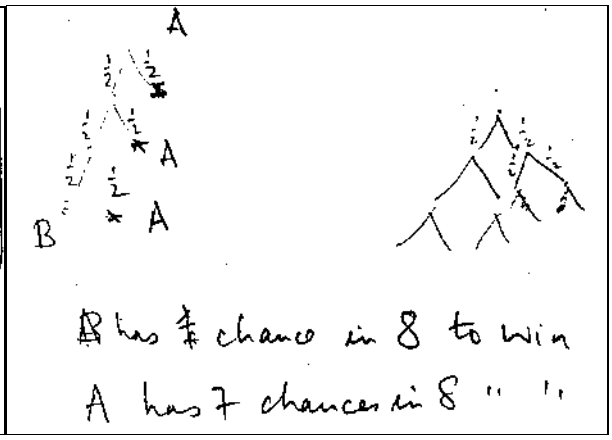


Fig.1b

4.2. *The Problem of Dice* formulated by Christiaan Huygens – the Proposition XIV in his work "De ratiociniis in aleae ludo" – I posed to students in easier form of the 'First Hit Problem':

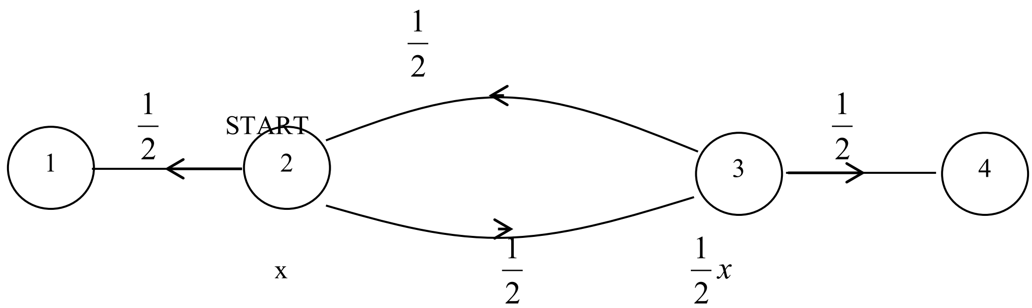
Two boys: Alan and Bob try to score at a basketball target. They both have the same frequencies of success: 50%. They decided to play a game: each will throw the ball until he fails to score. When one fails, the other takes over the scoring. The first who scores a hit is a winner. Alan always starts first.

Students usually try to understand rules of this game just by making a Monte Carlo simulation (throws of a ball to the basket were replaced by throws of a coin), and counting experimental frequencies of outcomes (arguments *de re*). Then they try – on the base of symmetry – to consider the random mechanism theoretically (arguments *de dicto*). Many of them realise that this situation there is not symmetrical, so the game is not fair. Then they try to draw a tree of the game. This model is easily transformed to a graph shown in the Fig.2. In order to calculate chances of winning for each of the players, students try to discover some regularities of this model. Student 13 years old expressed it in a very original way:

> *The tree can continue theoretically to the infinity. Here Bob tries, he fails – with a chance one half, Alan tries again .... and this is as if the game begins again! One half of one half of a chance of the second boy – this is as if he starts to play again.*

He was able to grasp the fact of infinite number of possible outcomes in recursive way. This expression lets him to compose an equation and to solve it nearly immediately. What is interesting, after obtaining the result he confronted it with empirical frequencies obtained during simulation of the game (*de dicto* versus *de re*). His way of thinking which led to the model in a form of equation was very close to reasoning, presented by Huygens in the solution of his problem (Hacking 1975, Lakoma 1999).

A teacher – analysing student's reasonings – can suppose that his arguments base on a proper image of the concept of probability.



$$x = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}x\right)$$

$$\frac{3}{4}x = \frac{1}{2}$$

$$x = \frac{2}{3}$$

Fig.2

5. Final remarks

Giving an historical perspective to the teaching of probability and statistics has become helpful for inspiring the design of teaching approach, which takes into account the student's actual abilities – so called Local Models' Approach. In this didactical conception taking into account the epistemological structure of mathematics, in the matter of students' cognitive development, lets students learn mathematics in an active way. Students are posed in situations in which they are involved to discover and formulate problems, to search for solutions according to their individual potentialities. They are able to construct, step by step, own mathematical knowledge, and to develop both aspects of probability keeping them in balance (Lakoma 1999, 1996-2000).

From the point of view of a teacher, the history can help him to understand student's reasonings and to diagnose his progress in the process of learning. Thanks to historical knowledge, a teacher is able to predict some "typical" situations and difficulties in a classroom practice, and to react on actual didactical situations in such way that it would possible to help his students in their conceptual development.

Qualitative analysis of using history of mathematics in mathematics education, rather than quantitative one, seems to be useful to recognise an effectiveness of the process teaching. Using history of mathematics seems to be effective when we try *to recognise general mathematical competencies – performance – of students rather than particular skills*. It is possible to evaluate this kind of effectiveness after some years of learning mathematics in active style – by observing students' progress and acting in real situations when they use their knowledge. We can recognise whether they dispose of instrumental or relational understanding of mathematical concepts (Skemp 1976).

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The Use of History of Mathematics To Increase Students' Understanding of the Nature of Mathematics

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Abstract

For much of the past century, the philosophy of mathematics has mainly focused on establishing the solid foundation of mathematical knowledge. These foundationist views have been challenged in many ways and a call for pursuing alternative views about the nature of mathematics has increasingly received much attention among mathematics philosophers and mathematics education researchers during past decades. It was found that teachers' and students' beliefs about mathematics has profound influence in their teaching and learning behavior. It is therefore argued that the view-shifting about the nature of mathematics has to be reflected in our teaching and learning of mathematics. A cross-culture observational study was conducted by the author attempting to investigate the Taiwan and United States college students' views about mathematics and mathematicians. Both sites showed inappropriate views about mathematics and mathematicians. The article makes some suggestions of how to use the history of mathematics to increase students' understanding of the nature of mathematical knowledge and mathematical thinking.

A Brief Review of Foundationists

Mathematics has long been held by many (either mathematicians or non-mathematicians) as a body of infallible, highly valid knowledge. For much of the past century, the philosophy of mathematics has mainly focused on mathematical knowledge as a product but has eschewed the process aspect of its epistemology (Ernest, 1992). Once guided by a product-oriented view, the certainty and validity of these scientific products inevitably receive much concern. The pursuing of the foundation of mathematics, therefore, had constituted the chief task of the interested scholars.

Traditionally, four major schools, Platonism, logicism, intuitionism, and formalism, made the effort to establish an infallible foundation of mathematics. Platonists are convinced that they are dealing with an absolutely objective truth associated to reality. However, the view was undermined by the emergence of non-Euclidean geometries, which shows that there is more than one geometric system with intrinsic consistency. The establishment of non-Euclidean geometry not only freed mathematics from the responsibility of adhering to the external world, but demonstrated the ability of the human mind to create new mathematical systems without underlying upon the perceived reality. The situation, however, is intolerable because geometry had served the supreme exemplar of the certainty of human knowledge (Hersh, 1986). Logicists, such as Russell, declared that the mathematical system creates knowledge according to deductions and logical rules. Yet, logicism failed to resolve the Paradoxes in its own system. As Frege claimed, just as the building was completed, the foundation collapsed. Intuitionism, represented by Brouwer and Kronecker, was mostly concerned with the appearance of paradoxes in set theory and argued that mathematical concepts are only valid by constructing from a first principle. This position makes the use of proof by contradiction become not possible, different from classical mathematics, and thus failed to attract a large number of proponents. To save mathematics from the crisis of its foundation, the formalist view of Hilbert contended that mathematics is the meaningful manipulation of meaningless symbols. The effort is aimed to ground mathematical ideas in terms of formal axiomatic systems. The attempt nevertheless is unattainable owing to the establishment of Godel's incompleteness theorem showing that it is impossible to maintain the consistency in an axiomatic system. The above schools all shared the common position that mathematics must be provided with an absolutely reliable foundation and mathematical truth should possess absolute certainty, which are categorized as "founationism" by Lakatos. The anxiety of the collapse of the foundation of mathematics can be seen from Hilbert's statement that "If mathematical thinking is defective, where are we to find truth and certitude?" (Hilbert, 1983, P. 191). Yet, as seen above, these foundationist views have been challenged in many ways and a call for pursuing alternative views about the nature of mathematics has increasingly received much attention among mathematics philosophers and mathematics education researchers during past decades.

Toward Alternative Views of the Nature of Mathematics

Despite the debate among the foundationists, professional mathematicians are still working on their vocation of looking for and proving mathematical truth. Few mathematicians are concerned about the crises of mathematical foundations or the nature of their working subject, just like few philosophers are concerned about how mathematicians execute their tasks. Mathematicians seem to distance themselves from the work of the philosophy of mathematics because, in their thinking, philosophy of mathematics is almost a branch of mathematical logic or foundational studies which leaves little room for mathematicians. Revealing the nature of mathematics is one of the chief tasks of the study of the philosophy of mathematics. Yet the attempt of pursuing the foundation and absolute certainty of mathematical knowledge is "wrong headed" (Ernest, 1992). Traditional mathematical philosophers not only ignored the fact that mathematics is one kind of human endeavor in our society but overlooked "the working philosophy of the professional mathematician" (Hersh, 1986), even though some of them are also practicing mathematicians. What is need now, as Hersh suggested, is a new beginning, but not a continuation of the various schools of foundationism.

If Hersh's argument could be accepted, then where is our new beginning? First of all, the anxiety about the collapse of the foundation of mathematics should be set aside and go back to the intrinsic nature of mathematics in which mathematics is a discipline dealing with ideas but not infallible knowledge, at least not at the beginning of its development. Historically, the construction of mathematical knowledge rooted in an empirical basis, as Lakatos (1976) showed. The main driving force of the progress of mathematics is heuristic intuition and reasoning but not formally rigorous proof. According to Polya (1945), heuristic means "serving to discover" and heuristic reasoning is a reasoning that is provisional and plausible only. Mathematical ideas are empirically tested in mathematicians' minds (either logical type or intuitive type as Poincare distinguished) and then written in a mathematical way. During this individual construction stage, the development of a mathematical idea mostly, if not all, is empirical. Even after a mathematical idea was presented in a mathematical way, it may not be accepted as a mathematical truth immediately. A social agreement process of scrutinizing this mathematical idea is indispensable before being admitted as mathematical knowledge. Working under criteria commonly accepted by a certain community at the time, community peers would decide whether a new-born mathematical idea is allowed to be regarded as a mathematical truth. During this social construction stage, the development of mathematical knowledge is subject to the standards set up by certain professional communities (e.g., Cantor's paper showing there are strictly more real numbers than algebraic numbers by producing an infinite counting argument was rejected to be published by Kronecker, who was an intuitionist and unable to accept the actual infinity of real numbers) at the given time (e.g., the conclusion that the equation $x + 1 = 0$ is solvable was definitely unacceptable before negative numbers were admitted by mathematics community). The key issue here is that the certainty of mathematical knowledge changes over the course of time. As Ernest (1992) pointed out, the "social constructivist account of mathematics is the assumption that the concepts, structures, methods, results and rules that make up mathematics are the inventions of humankind" (p. 93).

Once realizing the above empirical and social construction features of mathematical knowledge, we then have a position to explore the humanistic aspect of mathematics. Traditional views of mathematical philosophers placed little emphases on the human side of mathematics. Though they did care about the significant contribution made by some great figures, their main concern is after all the completeness of the whole mathematical structure, but not human activities. As seen in the preceding paragraph, both empirical and social construction features of mathematics are parts of human activities; thus, revealing the humanistic feature of mathematics is far more important and valuable than exploring the completeness of mathematical structure and pursuing the solid foundation of mathematics when pondering what mathematics is all about. Tymoczko (1993) argued that "mathematics requires a perspective, and a human perspective is the only perspective we can get" (p. 11). In his point of view, pursuing the answer of the question "What is important in mathematics?" is less valuable than asking the question "What is important in mathematics to human being?". It should be appropriate to say that the final purpose of constructing or discovering knowledge of any kind is "concerning with the human". Too often mathematical knowledge was presented in a polished format in which human struggle is diminished. Not only does mathematics posses technical aspects but precious human aspects. Yet, the technical aspects of mathematics has been overwhelmingly emphasized in our education. The recognition of the humanistic feature of mathematics allows us to admit that mathematics has ambiguity, paradox, and mystery (Davis, 1993), which are intolerable by traditional mathematics philosophers.

The Profound Influence of the Nature of Mathematics in Mathematics Education

A teacher's sense of mathematical enterprise may determine the nature of the classroom environment that the teacher creates (Schoenfeld, 1992). "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics." (Thom, 1973, p.204). The assumption underling the argument is that one's conception of mathematics would have a potential to affect one's conception of how it should be presented. If the assumption could be accepted, then, as Hersh (1986) further indicated, "The issue, then, is not, What is the best way to teach? But, What is mathematics really all about?" (p.13).

The above philosophical arguments have been to some extent supported by some empirical studies in mathematics education. Raymond (1997) reported a case study of a beginning mathematics elementary teacher's instruction revealing that the participating teacher maintained a controlled, discipline atmosphere where students were quiet and on task, which was more likely to be consistent with her traditional beliefs about mathematics (mathematics is fixed, predictable, absolute, and certain), but not with her non-traditional professed beliefs about teaching and learning mathematics (students primarily learn mathematics through problem solving tasks; the teacher's job is to facilitate and guide students' learning with little lecture). In addition, the three case studies of experienced mathematics teachers' consistency between beliefs about mathematics and teaching practice conducted by Thompson (1984) also suggested that teachers' instructional behavior were correlated to their professed beliefs about mathematics. For instance, two participating teachers possessing a view of mathematics which is a consistent, certain and an exact discipline showed a traditional approach to teaching mathematics in which mathematics was demonstrated as a precise, prescriptive procedure, and their students were asked to follow the standard procedures. On the contrary, another participating teacher claimed that mathematics is a challenging discipline, a body of knowledge continually expanding its content and undergoing changes to accommodate new developments. In practical teaching she employed a variety of problem solving approaches to stimulate students' interest. A similar finding can also be implicitly seen from Cooney's (1985) study though he did not plan to explore the issue.

The studies cited above, of course, may not allow us to conclude that teachers' conceptions of the nature of mathematics do place a direct impact on their classroom behaviors. Still some educational methodology issues should be cleared up. Further, in the area of science education research, there is a considerable body of literature showing that the connection between teachers' beliefs about the nature of science and their instructional behavior is neither simple nor direct (Bell, Lederman, & Abd-El-Khalick, 1998), thus challenging the above finding in the area of mathematics education. Yet uncritically neglecting the effect of teachers' conception of mathematics is also unwise. Therefore, Thompson's (1992) review of research on teachers' mathematics beliefs indicated that research should more closely examine links between conception of mathematics and instructional practice.

On the other hand, how students' conceptions about mathematics may place an impact on their learning behavior has received much attention (Franke & Carey, 1997; Higgins, 1997; Kloosterman & Stage, 1992; Schoehfeld, 1983, 1985, 1989). As noted above, teachers' beliefs about the nature of mathematics play a profound role in their beliefs about teaching mathematics and instructional behavior. It is, therefore, a plausible assumption that students' beliefs about the nature of mathematics could be related to their beliefs about the learning of mathematic and learning behavior in some ways. Rodd (1993) explored the relationship between high school students' beliefs about the nature of mathematics and beliefs about the learning of mathematics. The working hypothesis in Rodd's study was that if one's beliefs about mathematics could be described as fallibilism then he or she is more likely to hold an investigative style of learning in which students are encouraged to be active learners. On the other hand, once someone holds an absolutist view toward mathematics, he or she would be more comfortable with a didactic style of learning in which students are passive knowledge receivers. Based on the interview data collected from nine students, the finding partly supported the working hypothesis. Ruthven and Coe (1994) also conducted a study by administering a 28-item Likert type questionnaire to high school students to investigate students' epistemological views on mathematical knowledge, activity and learning. According to structural analysis, the relationship between epistemological dispositions and pedagogical preference was moderate, though not linear.

A cross-culture observational study was conducted by the author attempting to investigate Taiwan and the United States college students' views about mathematics and mathematicians. The Taiwan students were 50 engineering majors enrolled in a technical college and the 20 students of the United States were diverse majors in a state university. Some commonalties and differences of students' views were found between the two counterparts. When asked, in their point of view, "What is mathematics?" both the Taiwan students (22 out of 50) and the U.S. students (15 out of 20) showed a narrow view of mathematics by saying that mathematics is a subject containing rules or operations for dealing with the problems of numbers. None of them regarded mathematics as a discipline dealing with ideas. It, therefore, can be seen that students of both sites failed to comprehend the intrinsic nature of mathematics. In addition, some Taiwan students (n = 15) were more likely to associate mathematics to calculation, however, this was not found in the U.S. students' responses. The difference may be explained by the fact that the U.S. students are allowed to use calculators when solving mathematics problems but these Taiwan students are not. Thus, an image that the ability to calculate is very important in learning mathematics was held by Taiwan students and this image surfaced when responding to what they think mathematics is about. Only four Taiwan students and one of the U.S. students mentioned that mathematics is a subject involving thinking.

Reflecting their calculation-oriented image of mathematics, Taiwan students (11 out of 50) tended to view the work of mathematicians as exploring and resolving calculation problems. On the other hand, the U.S. students (12 out of 20) imagined that the work of mathematicians is to solve mathematics problems, a narrow and vague response. Few students at both sites were aware of mathematicians' hard-work and struggle while they are devoting themselves in pursuing mathematical facts. Because mathematicians usually hide their struggles from students when they teach and that mathematicians' works are not well known by the public, the students' answers, though inappropriate, are not too surprising.

The students were also asked to think about whether there are any differences between mathematicians' personality and laypersons'. On this question, the both sites showed more similarities than differences. They concurrently indicated that the major difference between mathematicians' personality and laypersons' is about "thinking". Mathematicians' thinking was seen by these students to be more organized, structured, analytical, logical, creative, and abstract. The finding is noteworthy in two ways. First, from students' answers, it is plausible to reason that the students were likely to view that these factors are critical in doing mathematics and probably they considered themselves as being deficient in these necessary features. If this implication is valid, it thus reminds all mathematics educators that fostering students' mathematical thinking in these respects should be given more attention in our school mathematics teaching. Second, despite their recognition of the above important factors in doing mathematics, the students naively viewed mathematics as a subject of relating to numbers and calculation, as reported earlier. Schoenfeld (1989) reported a finding that students considered mathematics is best learned by memorization but simultaneously agreed that doing mathematics involves creativity. The finding in this article is quite similar to Schoenfeld's in nature. According to Schoenfeld, students seem to hold two distinct views about mathematics. One is the view about school mathematics that they have experienced for years and the other is the view about abstract mathematics that they have heard about but never experienced. It should be appropriate to say that there is only one mathematics in existence, i.e., the mathematics constructed by mathematicians in our human history. Immersing students in the experience of this empirical, humanistic and social-constructed mathematics not only is philosophically correct but should be regarded as educationally necessary.

Using the History of Mathematics to Increase Students' Understanding of the Nature of Mathematics

Mathematics as taught in school has the reputation of being a dull drill subject and the polished product of mathematics has been mistakenly seen as what mathematics is all about. A new direction of stressing the pristine process of mathematics should not merely stay at the level of philosophical discussion but ought to be brought into our practical school mathematics teaching and learning. That is, the alternative views about the nature of mathematics described above need to be reflected in our school education. It is argued here that the term "the nature of mathematics" is composed of two major distinct but intertwined components which are "the nature of mathematical knowledge", a macroscopic view regarding the historical development of mathematical knowledge and "the nature of mathematical thinking", a microscopic view of investigating the heuristic approaches to the solution of a mathematics problem.

It was found that, as reported earlier, neither Taiwan nor the U.S. college students observed by the author regarded mathematics as a part of human endeavor. A misconception is frequently held by students that mathematics can only be figured out by certain great and smart people who create mathematical knowledge without laborious working and little struggle is involved. There is no doubt that history of mathematics may adequately serve the purpose of correcting this mistaken view since, when students are taught the history of mathematical processes, they will see that mathematics "were not handed down from a mountain of inspiration but are the fruits of laborious thinking in the long and tedious journey up from barbarism to civilization and were developed for the most part as necessary demand" (Hassler, 1929, p.171). A most recent and appropriate example is Andrew Wiles' resolution of Fermat's Last Theorem. During its over 300 years long history, Fermat's Last Theorem attracted numerous professional and amateur mathematicians engaging in revealing its mysterious veil, which fully reflects the fact that the establishment of mathematical knowledge is a human endeavor. Wiles' personal perseverance in proving the theorem and the subsequent peer-review process exactly demonstrate the empirical and social construction aspects of mathematical knowledge.

The doctrine that mathematical knowledge is indubitable and timeless certainty has long been rooted in the public minds. Yet, once recognizing the empirical, social-construction, and human facets of mathematical knowledge, the above stereotype is inevitably arguable. As Kline (1982) pointed out, owing to the limitation of the human mind and the illogical development of mathematical knowledge, "the hope of finding objective, infallible laws and standards had faded" (p. 7). The certainty of mathematics is lost and, according to Kline, the concept of a universally accepted, infallible body of reasoning is merely a grand illusion. Based on the development of complex number, Glas (1998) suggested the use of a fallibilist model of trials and tests in mathematics teaching to show students that mathematical knowledge actually is the products of a series of thought-experiment activities. In addition, the mistakes made by great mathematicians are also served as appropriate examples of showing the possible fallibility of mathematical knowledge. For instance, despite the great achievement he had made, Euler mistakenly established the identity of

$$1+\frac{1}{x}+x+\frac{1}{x^2}+x^2+\frac{1}{x^3}+\ldots=0$$

by adding the following two identities of

$$x+x^2+x^3+\ldots+x^n+\ldots=\frac{x}{1-x}$$

and

$$1+\frac{1}{x}+\frac{1}{x^2}+\ldots+\frac{1}{x^n}+\ldots=\frac{x}{x-1}$$

which ignored the fact that either the series

$$x+x^2+x^3+\ldots+x^n+\ldots$$

or the series

$$1+\frac{1}{x}+\frac{1}{x^2}+\ldots+\frac{1}{x^n}+\ldots$$

fails to have a sum.

Another famous mistake made by Leibniz is showing the divergent series

$1-1+1-1+1-1+\ldots$ equals to $\frac{1}{2}$ by letting

$$1-1+1-1+1-1+\ldots=S$$

thus,

$$1-(1-1+1-1+1-1+\ldots)=S$$

$$1-S=S$$

therefore he got

$$S=\frac{1}{2}.$$

Yet, it should be noted that the fallibilist view of mathematics should not be implied that no absolute mathematical truth could exist at all. Instead, it means that mathematical knowledge is revisable and improvable over the course of time (Glas, 1998).

Studying the history of mathematics may also benefit students' thinking in mathematics. According to Polya, mathematical thinking is not only a formal process concerning axioms, deduction, and logical reasoning, but an informal approach involving guessing, induction, and heuristic reasoning. In Polya's mind, unlike most mathematicians, Euler is a great author as well as a great mathematician who "tries to impress his readers only by such things as have genuinely impressed himself" (Polya, 1954, p. 90). Siu (1995a, b) proposed several examples of Euler's heuristic reasoning and mathematical way of thinking, which may serve as a model of what mathematical thinking is about. On the other hand, based on the empirical and social-construction perspectives of mathematics, Schoenfeld (1994) created an environment that was functioning like a mathematics community in his problem-solving class. Two beliefs held by Schoenfeld are (a) mathematics is a living, breathing, and exciting discipline of sense-making, and (b) students will come to see it that way if and only if they experience it that way in their classrooms. Thus, it is argued here that a history-oriented and problem-center approach should be a rather appropriate way, if not the best, in developing students' mathematical thinking. The use of historical problems in mathematics classroom has been advocated by many scholars (Avital, 1995; Rickey, 1995; Swetz, 1995). Swetz suggested that using problems from the history of mathematics in mathematics instruction may provide teachers an alternative way to teach problem-solving. For instance, some ancient mathematics texts only provided a specific and tricky solution for a certain mathematics problem but not a general approach. Students can be encouraged to find out the rationale hidden behind the specific solution leading to "discovery situations." Historical material may also help teachers correct students' misconceptions about mathematical thinking. It has been reported that students usually hold a view that there is only one correct way to solve any mathematics problem (Schoenfeld, 1992). To break down this misconception, teachers may provide historical sequences of problems from different time periods and cultures for students to solve and compare (e.g., the "Pythagorean Theorem", Swetz, 1995). Further, in modern textbooks, the divergence of harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ is normally presented as a theorem for students to learn and memorize. Yet, in this manner, students' opportunity of discovery is deprived. To avoid the deficiency, students had better be first asked to evaluate the sum of this series on their own and then follow by showing them the alternative solution proposed by past mathematicians such as Pietro Mengoli, Johann Bernoulli, and Nicole Oresme. The merits of this approach are not only that the students' joy of discovery can be retained but the view of the uniqueness of mathematical solution can be dispelled.

Conclusion

Any fundamental change in the intellectual outlook of human society should necessarily be followed by an educational revolution (Whitehead, 1967). The current study of the philosophy of mathematics has been gradually set free from the cage of pursuing a solid foundation of mathematical knowledge and turned to the "working philosophy of the mathematician" as Hersh (1986) proposed. School mathematics education has been suffered by the static and rigid views of mathematical knowledge in which students are asked to follow pre-existed procedures to solve mathematics problems. Schoenfeld (1983) suggested that, in addition to the required facts and procedures, students' beliefs about mathematics also play a profound role in influencing their problem-solving performance. Therefore, bringing a dynamic and empirical view of mathematics into mathematics classroom has its own merit. History may provide necessary and fruitful nutrition for reaching the goal.

However, one should be reminded that the intuitively appealing assumption that a historical approach course would necessarily increase students' understanding of the nature of mathematics may not be valid. Teaching the history of mathematics itself may not suffice to improve students' views of mathematics without students' integration and reflection on what they have learned. Thus, inquiry-base instruction in which adequate time is allowed for students to discuss and compare the multiple facets of the nature of mathematics is necessary. In addition, the historical problems used in the classroom should be bonded to practical school curriculum. History of mathematics ought to be treated as a part of the curriculum but not be separated from it. Using history only for history's sake by no means is the position taken by this article.

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Epistemological complexity of multiplication and division from the view of dimensional analysis

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Abstract

This article deals with an obstacle that students encounter in learning multiplication and division. In view of comprehension of this obstacle, the author tries here to confirm the historical process of it and propose a theoretical framework by using ideas taken from dimensional analysis and mental model theory (Johnson-Laird, P.N.1983) . According to this framework, we analyze a process of solving two multiplicative problems posed to two secondary school students. In addition, implications for teaching would be derived as a consequence.

1. Introduction

Multiplication and division are fundamental conceptions in arithmetic and mathematics. Multiplication and division problems that have dimensional complexity are more difficult than addition and subtraction problems that have only unidimensional one. This epistemological complexity of the former problems has been revealed behind the mathematical simplicity. Many researchers have been trying to tackle this dimensional complexity of multiplication and division (e.g.,Schwartz,1988,Tompson,1994).To investigate these problems, an analysis of student's activities is needed, as well as an analysis of historical processes. Such analysis may allow an explanation of some difficulties peculiar to multiplication and division.

The purpose of this paper is two fold: 1) to discuss a basic framework for comprehending multiplication and division from the view of dimensional analysis, and 2) to point out controversial points in mathematics education. Section1 deals with an obstacle in comprehension of multiplication and division, as pointed out the difference of multiplicative structures (Vergnaud,1983). Section 2 deals with Euclidean and Cartesian's view about multiplication and division in the history of mathematics and proposes two types of mental model as a framework to draw some aspects of shifting from scalar operation to functional operation. Finally, Section 3 reports a case study.

2. Epistemological complexity of multiplication and division

In many cases, as many research studies point out, when students are requested to interpret multiplication and division, they emphasize not merely numerical aspects but also physical ones (e.g.,Schwartz,1988). Students usually deal with quantities that are not pure numbers but magnitudes of various kinds. Although number is resulted from a process of abstraction through which it wrenches themselves from any magnitude, it cannot exist without reference. Thus we can not comprehend multiplication and division in view of the numerical framework alone. If we are to analyze relationship in which students take into account, it requires dimensional framework that is familiar to physicists but it is usually disregard by mathematicians. This framework would be led from the product and quotient of homogeneous, inhomogeneous magnitudes.

According to Vergnaud (1983), we could identify three different multiplicative structures 1) ; (a) isomorphism of measure, (b) product of measures, and (c) multiple proportion. The isomorphism of measures is a structure that consists of simple direct proportion between two measure-spaces M1 and M2. The product of measure is a structure that consists of the Cartesian composition of two measure-spaces, M1 and M2, into a third, M3. What has to be noticed is that the product of measure is often less natural than isomorphism of measure 2), because the only way for analyzing and managing the latter structure seems to be by means of dimensional analysis. So it is difficult for students to master well this structure, and fail to understand multiplication and division when it is introduced as Cartesian product (Anghileri,1989). Naturally, other researchers have mentioned the distinction between scalar and functional aspects (Freudenthal,1978, Lamon,S.J.1994,etc). It is easy for us to image the difficulties that students would meet in extending the meaning of multiplication from "Isomorphism of measures"(scalar aspect) as a primitive conception to "Product of measures" (functional aspect).

3. The significance of multiplication and division : a historical view.

Above difficulties can be illuminated to consider historical parallel. Development of multiplicative reasoning would recapitulate in brief the whole history if mathematics could not be dissociated from its history. It is well known that Euclid had expressed the view in book V that the Greeks would only compare homogeneous magnitudes by forming their quotient (Boyer,C.B.1968). Greek mathematics of Euclid would not have been able to introduce conceptual product P×L, and conceptual quotient P÷L for two magnitudes P and L in general when P is one kind of magnitude (in one measure-space) and L is another kind of magnitude (in another measure-space). So Greek mathematician would envisage the proportion P1 : P2 = L1 : L2 if L1 and L2 are two values of the same magnitude such as lengths and P1 and P2 are two values of any other magnitude such as weights, they would not convert the proportion into an equality P1 : L1 = P2 : L2, or P1×L1 = P2×L2. While they had a prenotion of it surely, but there was an obstacle in the metaphysical background of their reasoning which kept Greek mathematics from conceptualizing it in respect of dimensional analysis. Essentially, it is due to the lack of conception of real numbers, but at the same time it is well known that Greek mathematics develop directly a mathematical theory of general physical quantities as the fifth book of Euclid. So we would like to focus attention on metaphysical background of their reasoning. They could not philosophize about extension the product and quotient to other magnitudes because they could not make sense these extensions in their geometrical sense 3). It is the points to be specially considered that one of intellectual failures of their mathematics was representational (geometrical) rather than operational (analytical). These issues made Greek mathematics unsuited to promote dimensional analysis in our sense. Thus the mathematical conceptualization-cum-symbolization of physical dimension is retarded by two thousand years.

Mathematician divided magnitude by inhomogeneous one in the 17th century. It was expressly conceptualized during and after the Renaissance since Descartes in his geometry acts as if he spread the acute awareness of the role of symbolization by mathematically controlled reasoning and represents magnitude on the natural ordering of the one-dimensional linear continuum (Boyer,C.B. 1956) 4). He would try to emergent the general coordinates that represent physical magnitude in the natural linear ordering, and introduce this system into Euclidean space. When we venture to compare Greek mathematics with after Renaissance mathematics, we could give the following account. Greek mathematics intended to construct for a large class of magnitudes, and after Renaissance mathematics intended to construct the real number and envisages the isomorphism between the real number and the magnitudes of the large class. On the whole, Bochner (1966) has proposed plausible explanation as for the given above. After pointing out that the Greek horror of multiplication, he goes on to mention "Greek mathematics was both made and unmade by the efforts of the Greeks to conceptualize simple scalar magnitudes like length, area, and volume".

As shown above, the Cartesian product, as a product of measure, is historically difficult to be mastered. We should keep in mind that the mathematical conceptualization-cum-symbolization of a product like P×L and quotient P/L is one of merkmal for development of multiplicative reasoning. Student constructs notion that historically took centuries to evolve. Although we spend a lot of time coming to this phase, still we hope students to be able to coordinate of distance, time, speed, acceleration etc, and to emergent new magnitude according to each situation.

It follows from what has been said thus far that it is better for us to introduce the frame of "mental model" as it is used in mental model theory (Johnson-Laird, P.N.1983). Although an analysis of multiplicative structures provides for us only a framework for research, they do not provide any description of students reasoning. As concerns mental model theory, it is possible for us to feature students reasoning. Mental models are the backbone of our reasoning process. They constrain activities in various ways. They could be a source of constraints on reasoning and

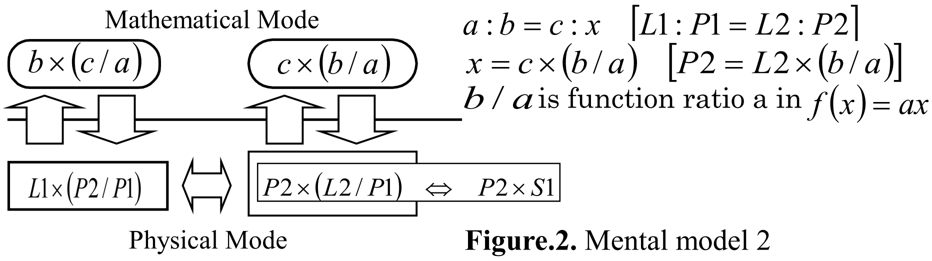


Figure.2. Mental model 2

problem solving process. Here, I venture to propose two types of mental model as follows :

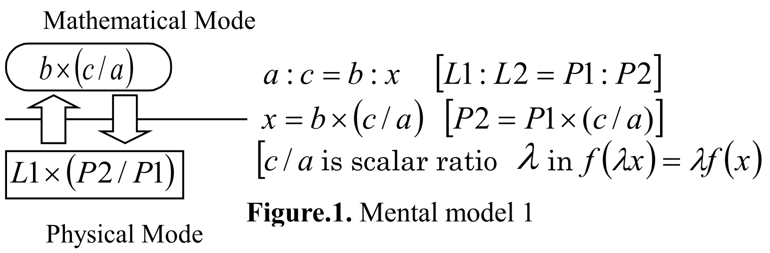
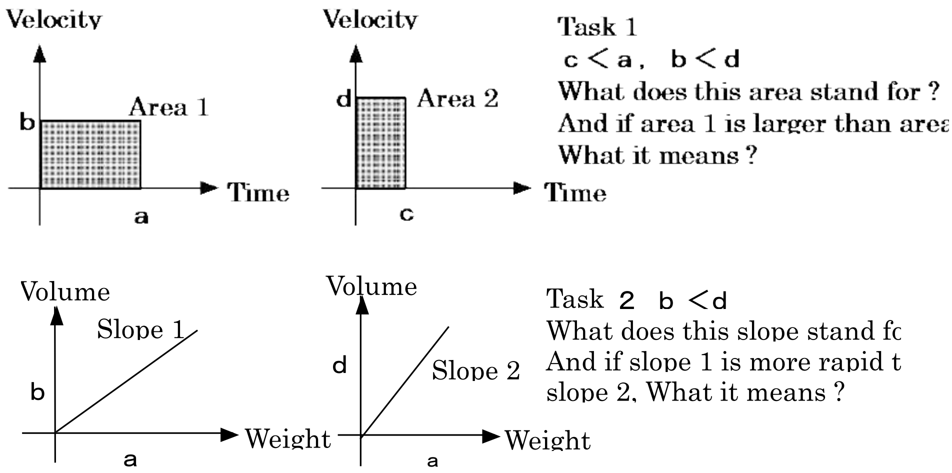


Figure.1. Mental model 1

The students who have mental model 1 regard multiplication as a scalar operation modeling the increase of a quantity. The students who have mental model 2 regard multiplication as a functional operation modeling the combination of two quantities. In figure 1 and 2, a,b,c,x are real numbers, and P1, P2 and L1, L2 and S3 belong to different measure spaces. In "mathematical mode", each student deals with pure number that has unidimension, and in "physical mode", student deals with directed number that has dimensional complexity. Of course, each model is selected according to the student's aims, so it does not merely mean the extension of multiplication and division. The product P・L, and the quotient P/L, are not only multiplies and divide the numerical values of P and L but it also forms a new magnitude out of the two magnitudes P and L. Thus, the shift between two mental models has profound implications for comprehending multiplication and division. To establish the link between these models is an important issue to develop multiplicative reasoning.

4. Case study.

The previous discussion does not attempt to make it clear "what is dimensional analysis in student's sense" or even "how to overcome obstacles for multiplication and division". Rather, the purpose is to provide a framework for discussing student's multiplicative reasoning. Here,to actualize their mental models, and to illustrate how do students talk on the above issue, the problem solving activities given by Yumiko and Tadashi who are in secondary school students (14 year old) will be presented. They are asked to talk out loud while doing the following tasks. These tasks are modified from Schwarz's(1988) study.



4.1. Protocol of Tadashi. [Case 1.]

Session I (for task 1).

Tadashi wrote the words "distance" on the word "velocity", and number on the paper near the graph. He then points at the area 1and says: "I do not know it (time-velocity plane)...hmm...what is it ? " In session 1, there is no solution of continuity in time. (I: Interviewer, T: Tadashi)

I: Excuse me why are you in trouble ?
T: Um... I saw the left one is bigger than the right one, and....so...
I: Yeah, How did you get that ?
T: well, for instance, if the left one (he points at left plane) shows...2km per hour, and 3hour, and the right shows 5km per hour and 2hour, then.....um...area is length multiply by width, so... oops...it is strange.
I: What is strange ?
T: ....(pause)
I: Okay, well, what does this area stand for ?
T: ... I could not understand it. I have never seen it.

Session 2 (for task 2).

Tadashi read the task2, then says " I know"

T: This one is... too steep than the left one (then he divides b by a by writing b/a near the left graph).
I: How did you get that ?
T: Well, if the horizontal line is x , and the vertical line is y , rate of changes is... divided y by x, so b by a, on graph, if the same amount of weight is changed, the volume of the right one is larger

than the left one. .. for instance, if a is 1 kg , the volume of right one would be larger than the right one.

I: Yeah, what does this slope 1 and 2 stand for ?

T: (pause)... rate of changes (in values of function)

4.2. Protocol of Yumiko. [Case 2.]

Session1 (for task 1).

Yumiko draws some numbers near a, b, c, d on the paper, and shows a bit of confusion and immediately change her approach. Then, she says "..um...so... so that... " (Y: Yumiko)

Y: So, if we multiply time by velocity, the area shows the distance. Since time and velocity are so related, then we can multiply them.

I: What? What do you mean by "so related" ? please write down on the paper if you need.

Y: (She wrote down the word expression (velocity) = (distance) / (time)). So, concretely…. (she wrote down "3km/h×4h =12km"), and I can reduce this fraction (she cut out "h").

I: Indeed, so, please tell me what $a \times b < c \times d$ means ?

Y: ... well ... the right one might move more, the distance is longer than the left one, ... may be.

Session2 (for task 2).

Yumiko changes the sheet of paper and writes down some number near the graphs and says:

Y: This slope means.… That is.… rate of changes of function...

I: Okay, so what does this slope 1 and 2 stand for ?

Y: Slope...it is weight per volume, so...here (she wrote down the word expression (density) = (weight) / (volume)) , and...might be.. density... No !... (pause)...what do I call it ?

I: How did you get that ?

Y: Well..... dividing volume by weight to get one that.... look like density ...

4.3. Discussion

Although task 1 and 2 are actually constitutive ideas of the differential and integral calculus respectively, as the purpose of this paper is concerned, we would concentrate on student's dimensional analysis. Task 1 and 2 were given to chart the presence of dimensional analysis in their reasoning. As a matter of fact, Yumiko and Tadashi had different meaning from an epistemological point of view. Session 1 for them reveals the gap between their reasonings. In Session 1 , unlike the Yumiko, Tadashi did not have a feeling of obviousness about time-velocity plane. For all that he knew the coordination of time, distance, and velocity. It is inferred from his trouble "what is area 1 and 2 in session1 that he avoid the epistemological obstacle (Sierpinska,1988) caused by dimensional analysis. It was hard for Tadashi to conceive area as distance. In session 2, while Tadashi was able to successfully interpret task 2 and applied his reasoning to two graphs, he would not mention the reciprocal magnitude for "density" that is coordinated from volume and weight when he compared the slope 1 with 2 . One interpretation of this things is that mental model 1 only allowed Tadashi to divide b by a depending on the isomorphic properties of the linear function so as to operate scalar ratio. His method of analysis would be constrained unidimensionally.

On the other hand, in session 1 for Yumiko, she gave an account of area 1 and 2 writing the word expression (velocity) = (distance) / (time), in consequence "product of measure". Thus, she would regard multiplication as an operation to coordinate(generate) new magnitude in order to justify her conjecture, she analyzed dimension of "$a \times b$" and "$c \times d$" in session 1. And in session 2 , she interpreted the "b/a" in physical mode, and tried to look for the new word which express new magnitude as a tool to compare two graphs in session 2. So it is reflected in her ability to coordinate another magnitude, and the process of "trying to name" is typically involved mental model 2 (to output S1), and crucial in the process of dimensional analysis. Hence, Yumiko displayed an ability to deal with both dimensional and the numerical aspect during session 1 and 2. We can not disregard the fact that Yumiko do dimensional analysis in her reasoning which required at mental model 2. Here the gap between Tadashi who have mental model 1 and Yumiko who have mental model 2 is quite evident. To compare their reasonings brings us to the idea that it is hopeless to try to propose better conditions for students to make sure their mental model, if we do not make the effort to get them to interpret graphical representations of relationships between the three quantities. From this point of view, to encourage the shift between two mental models, it is necessary to take into account : i) the activation of dimensional analysis, and ii) the shift through verbalization (symbolization) of new magnitude. Not to mention, we have to discuss both epistemological and didactical nature that are two main causes for beginning of obstacles. But they are too complicated to be examined in here.

5. Concluding remarks

When we have students to comprehend multiplication and division, it is essential for us to take into account the dimensional analysis as well as the analyze of number systems. I would say that it is wise to be firmly aware of two kinds of consideration. First point, it is sometimes better to dare to be actualized this obstacle that are not easy for them to overcome it. Second point, we should support to advance student's reasoning by verbalizing (symbolizing) that generates new magnitude. Student's reasoning advance only if they encounter situations that they fail to assimilate, and then teachers would be able to help students to accommodate their views. A way to actualize it and get student to shift between mental models might be to use graphical representations for which need dimensional analysis. While it is right in a way to deal only with numerical aspects eventually in mathematics, it is better for us to consider the dimensional analysis as a key to conjecture students reasoning.

Notes

1) The third structure "multiple proportion " ,that is not canonical way of choosing unit as f(1.1) =1 has similar relationship to "Product of measure"(Vergnaud,1983), so we do not deal with this third structure in this paper.

2) We are able to see"product of measures" as a double isomorphism or double, and reciprocally "isomorphism of measures" is able to be explained as a product such as volume × density = mass, in short it depends on our interpretation.

3). There were two exceptional cases for P and L. First, P and L are both the lengths, and their product is an area. Second, P is the length and L is the area, and their product is a volume.

4) In strictly, this method was traditional one since Merton school, and Fourier,J.(1822) had been given systematic method to coordinate physical dimension.

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Method of Indivisibles in Calculus Instruction

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1. Introduction

Assessing students' understanding of limits, integration, and the definite integral, Orton [8], states that among 19 calculus items addressed by him in a clinical study, a core of four items, all concerned with understanding the integral as the limit of a sum, constituted a real stumbling block. The majority of students could not grasp these important steps. In his experiment, where he wished to test for understanding rather than memorization, Orton found that those students that understood the concepts did not use the usual method, rather "the Wallis method" – which we shall describe shortly. While discussing the students' errors and misconceptions, Orton points out that the concept: "limit of the sequence equals area under the curve" - a concept which partakes in the standard approach - was correctly understood only by 10 students out of 69, while the concept: "limit from sequence of fractions from the general term", a concept that incorporates elements of the Wallis method was correctly comprehended by 62 out of 79 students.

In a paper to be published in the College Mathematics Journal, Czarnocha, Dubinsky, Loch, Prabhu and Vidakovic [5] discuss the re-discovery of a rather unexpected intuition, pertaining to the area under a curve using the method of Riemann sums, demonstrated by some students of calculus. In addition to the standard idea, held by some students, of approximating the area by sums of rectangles inscribed under the given curve, other students in the experimental group saw the area as a sum of line-segments from the abscissa to the curve. More precisely, instead of seeing the area as a limit of the process of taking partial Riemann sums, they saw it as a sum of the ordinates of the function in question.

Orton's article points us in the direction of a historical approach adopted by John Wallis in the seventeenth century. Our students' views motivated us to look into history to check how this topic was dealt historically. Both approaches bring us very close to the same general method used for computing and/or visualizing the definite integral. In fact, a historical study of the concept of the area under a curve reveals that the image held by students corresponds in its general outline to the viewpoint presented by Archimedes in the Method [3], Cavalieri in the Geometria Indivisibilibus [4] and John Wallis in Arithmetica Infinitorium [10]. We will refer to this as the method of indivisibles. The difficulties that students of calculus have may now be attributed, at least partially, to a mismatch between the students' spontaneous intuition and that cultivated by the standard instruction of that topic. The standard instructional approach to the definite integral is via the method of Riemann sums based on the concept of the area of inscribed (or circumscribed) rectangles. However, our research points out that the intuition of a sizable number of students [5] proceeds (is constructed) via the sum of the ordinates of lines. A possibility thus arises that one might increase the students' understanding of integration if the instruction is more closely correlated with the students' spontaneous intuition, The intuition of indivisibles. Thus the practical concerns about students understanding of the concept of definite integral had led us toward the point of view which integrates the results of investigations into mathematics education with the historical development of the concept of an area under an irregular curve.

In this presentation we will discuss briefly our students' background and our findings. We will compare the standard Riemann approach to the Wallis approach for the problem of finding the area under a curve. We will then introduce a technique modified from the Cavalieri ratio and Wallis' techniques to illustrate the arithmetic side of the rather visual views displayed by our students. Finally we will present an instructional sequence which integrates both, Riemann with Cavalieri-Wallis approaches to the problem of finding the area under some irregular curves.

2. Historical and Educational Background

A brief historical account of the mathematical development of the notion of indivisibles will help to situate the conceptual content of the problem being considered and students' views obtained from the clinical interviews. The image of the area under an irregular curve as represented by the vertical parallel lines enclosed in that region appears for the first time in Archimedes' treatise, On the Method re-discovered in 1906 [1]. There, the area under a parabola is computed with the help of Archimedes' lever principle, which by a very ingenious argument transforms the region under the parabola into the region of a properly situated triangle whose area is known to be 4/3. The lever principle is used to show that every vertical line under the parabola is in balance with a unique line in the triangle. In this work Archimedes thought about "the areas of the parabolic segment and of the triangle as the totality of a set of lines" contained within these figures.

It is a fascinating accident of history that Cavalieri, who lived about 1800 years after Archimedes, approached the problem of the area of an irregular region in essentially the same way, without, however, knowing about Archimedes' work on that subject. For Cavalieri, too, the area under the curve was "all the [parallel] lines" contained in the figure and perpendicular to the base, just like a solid was seen as "all the planes" parallel to it and contained in the solid. These images of area and volume were the underlying notions of the Cavalieri Principle and Cavalieri's Theorem, and find their analogs in contemporary mathematics research.

Cavalieri's Theorem [2]: If two solids have equal altitudes, and if sections made by planes parallel to the bases and at equal distances from them are always in a given ratio, then the volumes are also in this ratio.

The Cavalieri Principle [6](1) If two planar pieces are included between a pair of parallel lines, and if the two segments cut by them on any line parallel to the including lines are equal in length, then the areas of the planar pieces are equal. (2) If two solids are included between a pair of parallel planes, and if the two sections cut by them on any plane parallel to the including planes are equal in area, then the volumes of the solids are equal.

Both Archimedes' and Cavalieri's approaches, despite their powerful results, had important conceptual drawbacks centered on the issue of atomicity of lines and planes [7]. Because of that, Archimedes provided a separate proof of his results via the exhaustion principle, while the followers of Cavalieri such as Toricelli, Roberval and Wallis created an alternative approach exploiting the same basic intuition. For Wallis, each of the “all the lines” of Cavalieri, i.e. every indivisible, is seen as the “limit” of an inscribed rectangle whose width goes to 0 as the number of rectangles under the curve increases. This view is of extreme importance from the point of view of a balanced, integrated instruction, because it integrates the intuition of indivisibles with the intuition of inscribed rectangles on which the standard Riemann construction is based.

The excerpt below from a clinical interview with a student of calculus is an illustration of Wallis-like approach and the naturalness of the transition from Riemann to Wallis approach:

Excerpt

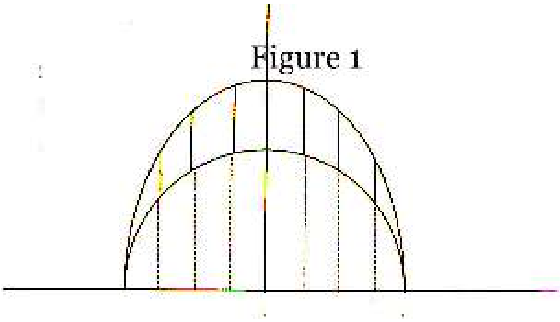
[Student C:] Right. Um. Well, the Riemannn Sum breaks this up into n, an infinite number of rectangles. And it's difficult to use the theory behind it. It's difficult for me. Um... basically the Riemannn Sum was just summation of the area of a number of rectangles where you would always have a number of some error, because they wouldn't fit directly to each point.
[Interviewer:] ***How would you get the closest...***
[Student C:] Um...
[Interviewer:] ***...possible area***?
[Student C:] Closest possible area would be by taking the length of a line segment from the x axis to the function itself. And that would give you an infinitely many..and many areas to add up. ***And that's what the definite integral gives you. It just allows you, you know, to be able to work with basically a rectangle with no width, just height. So, you calculate, you know, the length of a line segment. So, you add the lengths of all the individual line segments and, um, get an area. (Pause)***. You can't, I am trying to recall the theory behind how the definite integral gives you summation of rectangles. I don't remember that. Hmm...

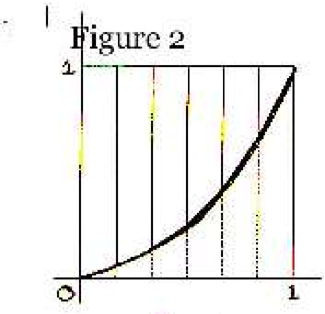
3. Cavalieri and Wallis constructions

The two examples below encapsulate the basic ideas on which the instruction of the Cavalieri principle and of the Wallis technique in our project will be based.

In Figure 1, we consider an ellipse with a semi-minor axis b, semi-major axis a, and and an inscribed circle of radius b. Given that the area of the circular segment is $\frac{\pi b^2}{2}$, we can find the area of the elliptic segment using Cavalieri’s theorem. The ratio of the ordinate in the elliptic segment to the ordinate of the circular segment can be computed to be $\frac{a}{b}$. Therefore, by Cavalieri theorem, the ratio $\frac{\sum all\ lines\ in\ the\ ellipse}{\sum all\ lines\ in\ the\,circle} = \frac{a}{b}$, and since the area of the circular segment =

$\frac{1}{2}\sum all\ lines\ in\ the\ circle = \frac{\pi b^2}{2}$, we get that the area of the elliptical segment = $\frac{1}{2}\sum all\ lines\ in\ ellipse = \frac{\pi b^2}{2} \times \frac{a}{b} = \frac{\pi ab}{2}$





Our instructional strategies will extend this technique to other figures and solids in order to give the students a strong sense of the concept of corresponding lines and of the area as "all the lines" in a figure (or volume as "all the planes" in a solid).

A similar ratio, referred to as the Cavalieri-Wallis ratio, occurs in Wallis' approach to the computation of the area under a parabola. Here the ordinates of the parabola are compared with the corresponding unit line segments in a circumscribed unit square.

In figure 2 we consider the parabola $y = x^2$ in the first quadrant on the interval [0,1]. The following sequence of ratios of the sum of ordinates of the parabola to the sum of corresponding unit lines computed for every partition into n equal intervals is constructed by Wallis and shown to converge to 1/3 which is the sought-after area:

$$\frac{0+1}{1+1} = \frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{0+1+4}{4+4+4} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$$

$$\frac{0+1+4+9}{9+9+9+9} = \frac{14}{36} = \frac{1}{3} + \frac{1}{18}$$

....

$$\frac{0^2 + 1^2 + 2^2 + \ldots + n^2}{n^2 + n^2 + n^2 + \ldots + n^2} = \frac{1}{3} + \frac{1}{6n}$$

Note the explicit use of the technique to decompose every the term of the series into the sum of the limit and extraneous term. It is this aspect of Wallis' approach, which was observed by Orton to be so successful in students' understanding of a limit. We will use this computational technique in our instructional sequence for both, Riemann and Wallis approaches as the common element between the two. Both approaches can be seen then as resulting in two different series with the same limit - the area under the curve.

4.The instruction

The instructional sequence is based on two important ideas. The global idea is that of determining the area under the parabola, which can be done using the Riemann construction (Task B) and the Cavalieri-Wallis technique (Task C). To develop this idea, we begin both sets of activities with a similar sequence that underlines the contrast between the two methods, one dealing with the sum of areas of inscribed rectangles and the other with the sum of ordinates of the function. The common theme for both methods is the process of sequence formation, where each term is decomposed in the limit as 1/3 + another term. (It is this way of seeing the limit of the sequence which, according to Orton [8], was so successful in his study.) Students will be led to observe that we are dealing with two different sequences with the same limit 1/3, which is equal to the area under the curve. The sequence obtained by the Riemann construction has the general term $\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}$ whereas the sequence obtained via the Cavalieri-Wallis technique has the general term $\frac{1}{3}+\frac{1}{6n}$.

The second, more specific new idea introduced by the instruction is that of the ratio $\frac{\sum lines, the\ unknown\ figure}{\sum lines, known\ figure}$ as the central concept in the Cavalieri-Wallis technique.

The excerpt below, taken from another clinical interview demonstrates how the concept of an area as "all the lines" can be described by a student of calculus:

Excerpt

[Student B:] Um, well the integral is, is um basically the sum of f(x)I for I in this case in [-3,-1], ***not necessarily integers but every number, every single point between -3 and -1 is going to have an f(x) value***.

[Interviewer:] Okay

[Student B:] ***And if you add all those together***

[Interviewer:] Um-hum

[Student B:] ***You should get an area under the curve***.

[....]

[Student B:] Now, if you took on this one you'd have to take like an integral to find the base of this thing, and that you have that you'd want to integrate like this, like vertical for well, however you have your plane, but ***let's say if you cut out like a, a plane, like a bunch of horizontal planes..***

[Interviewer:] uh-huh

[Student B:] ***that are parallel, then you gonna want to add all those planes together***.

In general, in the case of the Cavalieri Principle, each sum contains "all the lines" and is seen as equal to the area under the curve, whereas in the case of the Cavalieri-Wallis technique the sum contains a finite number, viz., n+1 lines. The ratio used by this technique, which we call the Cavalieri-Wallis ratio, was seen historically as approaching the Cavalieri ratio as $n\rightarrow\infty$.

The instruction begins with (a) sequences, (b) limits of sequences, proceed to (c) the method of Riemann sums, and end with (d) the Cavalieri-Wallis method. The instructional activities related to developing sequences and limits begin with an analysis of patterns of numbers, analyzing their behavior, and drawing conclusions about their end-behavior based on the observed pattern. Once students can make predictions about the end-behavior, the limit of a sequence, they would then be motivated to analyze activities that would help them realize the general term of the sequence and the limit of the sequence from its general term. We present below some sample instructional activities:

TASK A: [Adapted from Orton]

Sketch the graph of $y = x^2$. The area under the graph for x = 0 to x = 1 is to be determined.

In order to obtain the area under the curve, we will divide [0,1] into six equal intervals, and draw rectangles on five of the six intervals, (do not draw a rectangle on the interval [0,1/6]. The height of each rectangle is the y-value for the function at the left-hand end of each interval.

(i) What is the width of the base of each rectangle?
(ii) List the heights of the five rectangles.
(iii) How would you obtain the areas of the rectangles?

In the next task, students are asked to analyze the sequence obtained by summing the areas of two rectangles, three rectangles, etc., and to find a formula for the general term in the sequence of fractions obtained.

<u>**TASK B** Riemann sum construction</u>

We have so far obtained the areas for six rectangles and for five rectangles. Now look at the answers we have obtained and study where the numbers have come from.

(i) If we took four rectangles in the same way, what would be the area?
(ii) If we took three rectangles what would be the area?
(iii) If we took two rectangles what would be the area?
(iv) If we took just one rectangle what would be the area?
(v) List the answers to all of the total area questions, starting with the area for one rectangle and finishing with the area for six rectangles.
(vi) This sequence could be continued indefinitely by taking more and more rectangles. Can we ever obtain an exact answer to the area under the curve $y = x^2$ for x = 0 to x = a from this sequence?
(vii) If your answer to (vi) is 'Yes', what is the area under the curve $y = x^2$ for x = 0 to x = a?
(viii) Write two paragraphs explaining your reasoning in (vi) and (vii) <u>(generalized narrative writing task</u>)
(ix) Rewrite your six fractions so that each one is 1/3 + another fraction. Call this other fraction the extraneous term.

(x) Can you find a formula for the general term in the above sequence of fractions? Use n as the variable.

(xi) What is the area under the curve $y = x^2$ for $x = 0$ to $x = a$ on the basis of what we have just been doing?

(xii) Write a paragraph explaining your reasons in (x)) (generalized narrative writing task)

(xiii) Compare your answer in (xii) with your answer in (viii) (analogic writing task)

The Cavalieri-Wallis method:

Before we list the instructional activities, we indicate briefly the background that we will develop before beginning this segment of the instruction. Building on the students' high-school familiarity with the notion of similarity of figures, and with the dependence of the ratio of their areas on the ratio of their linear dimensions; we will demonstrate that it is possible to obtain the area of an unknown irregular figure by considering a properly chosen known regular figure, determining the ratios of corresponding lines in the two figures, and determining, as a result, the ratios of their areas. Next we develop the concept of a bijection between the lines (or, in the case of volume, the planes) of different figures (solids). Students are thus be familiar with the notion of the Cavalieri Principle as a means of determining the areas (volumes, measures in general) of unknown regions from those of known regions using the Cavalieri ratio

$$\frac{\sum \textit{all lines in the unknown figure}}{\sum \textit{all lines in the known figure}}.$$

The concept of the Cavalieri-Wallis ratio is developed. The Cavalieri-Wallis ratio is defined to be the ratio of the sum of a finite number of ordinates of the given function to the sum of the same number of heights of the circumscribed figure.

Sample instructional activities to develop the Cavalieri-Wallis method:

TASK C

Sketch the graph of the function $f(x) = x^2$ on the interval [0,1]. Circumscribe the graph with the unit square.

1. Divide [0,1] into four equal parts. This gives five points of subdivision of [0,1]. Label the six points and find the ordinates of these five points of subdivision. Find the Cavalieri-Wallis ratio for this subdivision.
2. Find the Cavalieri-Wallis ratio for four points of subdivision.
3. Find the Cavalieri-Wallis ratio for three points of subdivision.
4. Find the Cavalieri-Wallis ratio for two points of subdivision.
5. List the Cavalieri-Wallis ratios for two, three, four and five points of subdivision.
6. This sequence could be continued indefinitely by taking more and more points of subdivision. Can we ever obtain an exact answer to the area under the curve $y = x^2$ for $x = 0$ and $x = 1$ from this sequence?
7. If your answer to 6 is 'yes', what is the area under the curve $y = x^2$ for $x = 0$ to $x = 1$?
8. Write two paragraphs explaining your reasoning in 6 and 7. (generalized narrative writing task)

9. Rewrite your ratios as 1/3 + another term. Recall, we will call the second term 'extraneous term'.
10. Can you find a formula for the general term in the above sequence of ratios? Use n as the variable.
11. What is the area under the curve $f(x) = x^2$ for $x = 0$ to $x = 1$ on the basis of what we have just been doing?
12. Write a paragraph explaining your reasons in 10.
13. Compare your answer in 12 with your answer in 8. (analogic writing task)

At the end of the activities of type C above, students are carefully led to deduce the formula $\int_0^1 x^n\, dx = x^{n+1} /(n+1)$.

TASK D

In Task B , we formed a partition of the interval [0,1] and found the area under the curve by summing the areas of the rectangles. In Task C, however, we did not form rectangles, but rather considered the Cavalieri-Wallis ratios.

(i) Refer to the solutions of the previous tasks and determine the sequence of extraneous terms for both methods.
(ii) Determine the area obtained by both methods.
(iii) Write two paragraphs explaining the similarities and differences you observe in the two methods. (analogic writing task)

The next step of the instruction applies activities of the above type to the task of determining areas of regions bounded by two functions, volumes of solids of revolutions, arc lengths, etc., using both the Riemann and the Cavalieri-Wallis approaches.

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Using History to Enliven the Study of Infinite Series

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The impetus for this paper was an obvious lack of enthusiasm for infinite series in my upper division mathematics students. Their disinterest, together with their weakness in knowledge and skills related to infinite series, was all the more dismaying, since they had recently passed third term calculus, which includes the study of infinite series. Investigation of what was wrong led to the work of Otto Toeplitz, whose **Calculus, a Genetic Approach** was written with the goal of avoiding the symptoms my students presented.

Biologists used to teach that "ontogeny recapitulates phylogeny," meaning that the development of each organism retraces the evolution of the species. Modern biologists are skeptical of that simple phrase, but it may offer insight into learning mathematics. The van Hieles have shown that geometry is learned in stages of increasing abstraction; these stages correspond roughly to the historical development of geometry. Toeplitz, without the benefit of formal research on learning, arrived at similar insight, which led him to develop what he called the "genetic" method. This method exploits the genesis (key episodes in history) of mathematics to make central issues come to life again and thus provide a "gradually though gently ascending" path to higher levels of understanding. "The genetic method is the safest guide to this gentle ascent, which otherwise is not always easy to find."[1]

With this in mind, I reconsidered the chapter on infinite series in the textbook used by my students, Thomas and Finney's **Calculus, 9th edition**.[2] Chapter 8, "Infinite Series," has the following sections:

8.1 Limits of Sequences of Numbers
8.2 Theorems for Calculating the Limits of Sequences
8.3 Infinite Series
8.4 The Integral Test for Series of Nonnegative Terms
8.5 Comparison Tests for Series of Nonnegative Terms
8.6 The Ratio and Root Tests for Series of Nonnegative Terms
8.7 Alternating Series, Absolute and Conditional Convergence
8.8 Power Series
8.9 Taylor and Maclaurin Series
8.10 Convergence of Taylor Series; Error Estimates
8.11 Applications of Power Series

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[1]From Louise Lange's translation of G. Köthe's preface to Toeplitz' **Calculus, a Genetic Approach**.

[2]Other leading calculus texts offer similar treatments of infinite series. Those of Stewart and Grossman are described in the appendix. The texts used in high schools for Advanced Placement Calculus courses are quite different. Typically, they avoid the pitfalls of the logical but boring preoccupation with sequences and convergence, and they rely heavily on graphing calculators. However, they confine themselves strictly to the list of topics to be covered on the Advanced Placement Calculus test and therefore do not get far on their "gentle ascent."

The chapter is logically organized, but, as is common in mathematics exposition, the logical organization is based on hindsight, reflecting later efforts to clarify issues which had arisen in bursts of creativity by enthusiasts. In this case the price of logical clarity is delayed gratification. Before getting to the 12 pages of applications in section 8.11, students must go through 75 pages dealing largely with questions they probably would not have asked on their own at this stage. Can the material be taught in a way that will engage the students better? What would a genetic approach be like in this situation? There is no chapter on infinite series in Toeplitz' book; he had worked on such material, but it was too fragmentary to include. Fortunately, suitable material is readily at hand. Below are some ideas that might be helpful in creating a genetic approach to infinite series.

Begin with Geometric Series

The series most often encountered in early work are geometric series. A problem on the Rhind papyrus asks for the sum 7+49+343+2,401+16,807, and geometric series have continued to capture the imaginations of thinkers through the ages. Comparison between arithmetic and geometric series was the key to the invention of logarithms and was later the heart of Malthus' gloomy models of growth for food supply and population. Zeno's paradoxes show confusion about convergent geometric series, but Archimedes used one successfully in his quadrature of the parabola, even as he avoided taking an infinite sum.

Geometric series played a central role in the development of mathematics in the 17th and 18th centuries. They remain important today, both in calculus and more generally, but in the United States their role in the curriculum has been diminished. Once a standard topic in intermediate algebra, geometric series are now covered in "precalculus" (a name which seems to accord the subject matter no intrinsic value), lumped with the binomial theorem toward the back of the book, where it comes too late in the year to be taken seriously by high school seniors.

One implication of this is that calculus instructors cannot expect their students to bring a solid knowledge of geometric series into their calculus classes. This can be glossed over to some extent until attention turns to the study of infinite series themselves. Then a working knowledge of geometric series is indispensable. Some minor modifications of the basic calculus curriculum could ensure adequate familiarity with geometric series by the time students reach the study of infinite series.

1. **Use geometric series to differentiate x^n**

To calculate the derivative of x^n, it is common practice today to write the difference quotient as $\dfrac{f(x+h)-f(x)}{h} = \dfrac{(x+h)^n - x^n}{h}$, and then apply the binomial theorem. Instead, one could write the difference quotient as $\dfrac{x^n - a^n}{x-a}$, rewrite it as $x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1}$, and then take the limit by simply letting $x = a$.

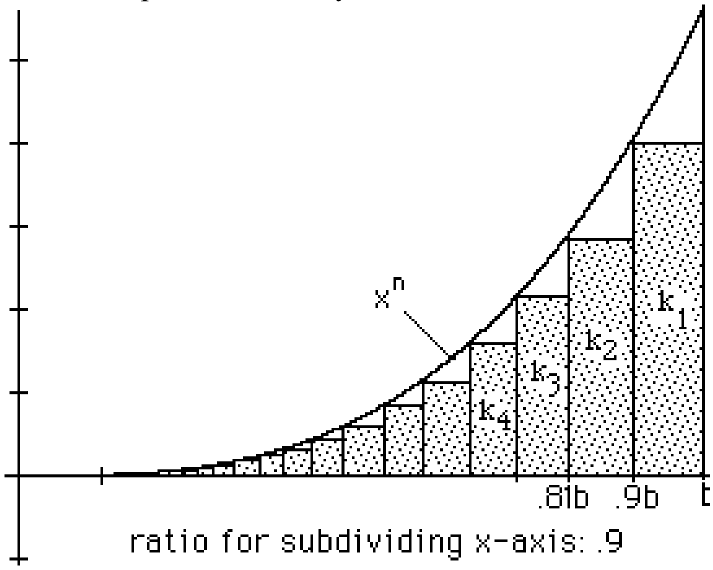
This generalizes easily to negative and fractional exponents. For the latter, $\frac{x^{\frac{p}{q}}-a^{\frac{p}{q}}}{x-a}=\frac{\left(x^{\frac{1}{q}}\right)^p-\left(a^{\frac{1}{q}}\right)^p}{\left(x^{\frac{1}{q}}\right)^q-\left(a^{\frac{1}{q}}\right)^q}$. With $z=x^{\frac{1}{q}}$ and $b=a^{\frac{1}{q}}$, this is just $\frac{\frac{z^p-b^p}{z-b}}{\frac{z^q-b^q}{z-b}}$, which, in the limit, is a quotient of limits already found. A fringe benefit of this approach is that differentiation of $x^{\frac{p}{q}}$ need not be delayed until after the differentiation of inverse functions.

2. **Use Fermat's elegant proof to integrate x^n**

I do that as an exercise in my own classes.

Exercise 1. Fermat's Beautiful Proof That $\int_0^b x^n dx = \frac{b^{n+1}}{n+1}$

Divide the interval [0, b] not into equal subintervals but into subintervals determined by the points b, rb, r^2b, r^3b, r^4b . . ., where r, the ratio of this geometric progression, is between 0 and 1. For example, for r=.9 they would be b, .9b, .81b, .729b, ... as shown.



If A_r is the area of all the rectangles below the curve, let A=$\lim_{r\to 1^-} A_r = \int_0^b x^n dx$ Area of rectangle k_1, whose base runs from rb to b, is:

Area of rectangle k_2, whose base runs from r^2b to rb, is:

Area of rectangle k_3, whose base runs from r^3b to r^2b, is:

Area of rectangle k_4, whose base runs from r^4b to r^3b, is:

$$A_r = \sum_{1}^{\infty} \text{Area of } k_i = (1-r)b\left[(rb)^n + r(r^2b)^n + r^2(r^3b)^n + r^3(r^4b)^n + \cdots\right]$$

$$= (1-r)b^{n+1}[\qquad\qquad\qquad\qquad\qquad\qquad]$$

$$= (1-r)r^n b^{n+1}[\qquad\qquad\qquad\qquad\qquad\qquad]$$

Sum the geometric progression in the brackets, and fill in the blanks:

$$A_r = (1-r)r^n b^{n+1}\left(\frac{\qquad\qquad}{\qquad\qquad}\right) = b^{n+1}r^n\left(\frac{\qquad\qquad}{\frac{\qquad\qquad}{1-r}}\right).$$

Replace the denominator with an equal **finite** geometric progression, and let $r \to 1$ to complete the determination of A.

3. Give Additional Historical Examples Using Power Series

The work with geometric series outlined above is an inherently interesting way to approach topics that must be covered anyhow in basic calculus, and it builds a foundation for treatment of other series. The following classroom exercises involve binomial series.

Exercise 2. Newton and the Area under the Hyperbola

Isaac Newton read Wallis' **Arithmetica Infinitorum** in 1661 at the age of 19, and he decided to apply Wallis' methods to the problem of finding the area under a rectangular hyperbola. Fermat had established the formula (written in our modern notation) $\int_a^b x^n dx = \frac{b^{n+1} - a^{n+1}}{n+1}$ for all integers n except -1, provided a and b are both greater than 0.

The case n=-1 is clearly quite different. It involves the area under the rectangular hyperbola xy=1. Newton wanted, in effect, a general formula for $\int_1^x t^n dt$ that would enable him to find the case n=-1 by Wallis' method of interpolation. Note that $\int_1^x t^n dt = \int_0^{x-1} (t+1)^n$.

1. Evaluate each of these integrals, either by expanding the integrand or by using u = t +1. Do not reduce the fractions in your answers.

a. $\int_0^x (t+1)^0 dt$ b. $\int_0^x (t+1)^1 dt$ c. $\int_0^x (t+1)^2 dt$ d. $\int_0^x (t+1)^3 dt$

e. $\int_0^x (t+1)^4 dt$ f. $\int_0^x (t+1)^5 dt$

2. a. In each of the above integrals, you may have noticed that each power of x goes with a denominator equal to the exponent. Following Newton, tabulate the **numerators** of the coefficients of each power of x with n≥0 (For now, skip the leftmost column of the table, where n=-1).

factor in numerator of coefficient of	value of n in integral of $(t+1)^n$							
	-1	0	1	2	3	4	5	6
$\frac{x}{1}$								
$\frac{x^2}{2}$								
$\frac{x^3}{3}$								
$\frac{x^4}{4}$								
$\frac{x^5}{5}$								
$\frac{x^6}{6}$								
$\frac{x^7}{7}$								

b. Find the pattern in the table from part a. It should be familiar!

3. Newton extended the table from question 2 to n=-1, using formulas in the rows similar to those Wallis used, namely

$$n, \quad \frac{n(n-1)}{1\cdot 2}, \quad \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, \quad \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$

4. Use your results from question 3 to write down the first several terms of a series for area Newton sought, $\int_0^x (t+1)^{-1}\,dt$.

5. Soon after he did this, Newton learned of the work of Gregory of Saint Vincent (1584-1667), which implied that the area under the hyperbola xy=1 to the right of x=1, if viewed as a function of its upper limit, has the property of a logarithmic function, that f(ab)=f(a)+f(b). As it turned out, Newton had found the series for ln(1+x). He and others used this and related series to calculate natural logarithms.

Exercise 3. Newton and the Binomial Theorem

1. Newton extended his table from exercise 2 beyond -1. Do so yourself, letting formulas and patterns be your guides.

term	-4	-3	-2	-1	0	1	2	3	4	5	6
$\frac{x}{1}$											
$\frac{x^2}{2}$											
$\frac{x^3}{3}$											
$\frac{x^4}{4}$											
$\frac{x^5}{5}$											
$\frac{x^6}{6}$											
$\frac{x^7}{7}$											

2. The above table led Newton to extend the binomial theorem to negative exponents.
a. Use this idea to write down a series (at least the first six terms) for $(1+x)^{-1}$ based on the binomial theorem.
b. Is the answer from part a one which would be familiar from other contexts? Comment.

3. a. Use the binomial theorem to write down a series (at least the first six terms) for $(1+x)^{-2}$ based on the binomial theorem.
b. Is your answer to part a reasonable? Find and carry our a way to check your answer based on the derivative of the series from question 2.

4. a. Use the binomial theorem to write down a series (at least the first six terms) for $(1+x)^{-3}$ based on the binomial theorem.
b. Is your answer to part a reasonable? Find and carry our a way to check your answer based on the derivative of the series from question 3.

Exercise 4. Newton and the Area of a Circle[3]

Newton revisited the question of the area of a circle, taking as his starting point indefinite integrals of the form $\int_0^x (1-t^2)^n\,dt$.
1. Calculate the above indefinite integral for n= 0, 1, 2, 3, 4, and 5.
2. Enter coefficients of your answers to question 1 in the following table.

[3]This exercise is motivated by the treatment in Dunham's **Journey Through Genius**.

term	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\frac{x}{1}$											
$-\frac{x^3}{3}$											
$\frac{x^5}{5}$											
$-\frac{x^7}{7}$											
$\frac{x^9}{9}$											
$-\frac{x^{11}}{11}$											
$\frac{x^{13}}{13}$											

3. Extend the table above to the case n = -1 by extending patterns in the rows.

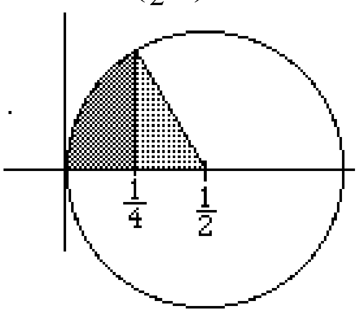
4. Interpolate to extend the table from questions 2 and 3 to the half-integer values of n shown.

5. The column for n = 1/2 yields an infinite series for the integral $\int_0^x \sqrt{1-t^2}\,dt$.
a. Write out the first seven terms of this series.
b. Put x = 1 in your answer to part a. To what number would you expect the resulting series to converge? (Hint: What is the area represented by the integral?)
c. Check your answer to part c by adding up the first seven terms of the series when x = 1. Is the result reasonable? Comment on this.

Problems 3.61-3.69 in Polya's **Mathematical Discovery**,on Newton's testing his binomial as theorem for fraction exponents, are appropriate here, .

Exercise 5. Newton calculates π

Newton found another approach to the calculation of π based on infinite series. For this he used the circle of diameter 1 centered at $\left(\frac{1}{2},0\right)$.



1. Write the equation for the upper half of this curve, expressing y as a function of x.

2. Factor out $\sqrt{x}$, and expand $\sqrt{1-x}$ by Newton's binomial theorem. Carry the expansion to 9 terms.

3. Multiply your each term of your expansion from question 2 by $x^{1/2}$ to get the first nine terms of the series expansion for the equation of the upper half of the circle.

4. Integrate your series term by term.

5. Use the result of question 4 to get a numerical series for $\int_0^{1/4}\sqrt{x-x^2}\,dx$, and use the series to estimate this integral to eleven decimal places.

6. Calculate the area of the entire shaded sector (both light and dark) in terms of π.

7. Use the area of the lightly shaded triangle (easy to calculate) and the numerical estimate found in question 5 to estimate π to several decimal places.

The Next Steps: Increased Generality

To continue "gently but gradually ascending," the "genetic" approach to infinite series must deal with the general question of local power series representation -i.e. Taylor's theorem, the series for e^x and Euler's discovery that $\cos\vartheta + i\sin\vartheta = e^{i\vartheta}$. This last is the subject of the next exercise.

Exercise 6. Euler Links Exponential and Trigonometric Functions[4]

Having identifed the unique non-zero function which is its own derivative as $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\cdots$, Euler audaciously defined e^z for complex numbers z as $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!}\cdots$. Accept this in a purely mechanical sense, as Euler did, and consider the following manipulations, where i, the "imaginary unit" is defined by the property that $i^2 = -1$, so that $i^3 = -i$, $i^4 = 1$, and so on. Then if x is a real number, $e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!}\cdots = e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$ and $e^{-ix} = 1 + -ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \frac{(-ix)^5}{5!}\cdots = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \cdots$, so that, adding and dividing by 2, $\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots$. Euler recognized this as the series for cos x.

1. Show, in a similar way, that $\frac{e^{ix} - e^{-ix}}{2i} = \sin x$.

[4]Roger Cotes had made the same connection in 1714, decades before Euler, but his discovery attracted remarkably little attention.

2. Taking sin x=$\frac{e^{ix}-e^{-ix}}{2i}$ and cos x=$\frac{e^{ix}+e^{-ix}}{2}$, show that $\sin^2 x+\cos^2 x=1$.

3. Taking sin x=$\frac{e^{ix}-e^{-ix}}{2i}$ and cos x=$\frac{e^{ix}+e^{-ix}}{2}$, show that $\frac{d}{dx}\sin x=\cos x$

4. Taking sin x=$\frac{e^{ix}-e^{-ix}}{2i}$ and cos x=$\frac{e^{ix}+e^{-ix}}{2}$, show that $\frac{d}{dx}\cos x=-\sin x$

5. Taking sin x=$\frac{e^{ix}-e^{-ix}}{2i}$ and cos x=$\frac{e^{ix}+e^{-ix}}{2}$, show that $\cos x+i\sin x=e^{ix}$

6. Let $x=\pi$ in the result of question 5 to deduce Euler's identity $e^{i\pi}+1=0$, which elegantly relates the five most important mathematical constants.[5]

Students who have gone this far will surely see the importance of infinite series and have some facility with them. Now, at last, they are ready to consider points that were glossed over or ignored initially. To motivate such reconsideration, one could begin with Dunham's chapter on "The Bernoullis and the Harmonic Series"[6] and then present some of the non-intuitive results that came along later. A particularly appealing example is presented in the appendix of Lakatos' **Proofs and Refutations**.

Conclusion

There is abundant material available on the history of the study of infinite series. Use of historically based examples and exercises can make the study of infinite series come alive for today's calculus students. "Follow the genetic course, which is the way man has gone in his understanding of mathematics, and you will see that humanity did ascend gradually from the simple to the complex. . . . Didactical methods can thus benefit immeasurably from the study of history."[7]

Appendix. Infinite Series in a Sample of US Calculus Texts

a. University Texts

James Stewart's **Calculus, 4th edition**
Chapter 12, "Infinite Sequences and Series"
- 12.1 Sequences
- 12.2 Series
- 12.3 The Integral Test and Estimates of Sums

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[5]After deriving this identity, Professor Benjamin Pierce turned to his Harvard class and said, "Gentlemen, that is surely true, it is absolutely paradoxical, we can't understand it, and we don't know what it means but we have proved it, and therefore we know it must be the truth." Raymond Clare Archibald, **Benjamin Pierce, 1809-1890**, **Biographical Sketch and Bibliography**, Oberlin, Ohio. Mathematical Association of America, 1925, page 6.

[6]Dunham, William. **Journey Through Genius**, John Wiley and Sons, NY, 1990.

[7]Toeplitz, Otto. op cit

12.4 The Comparison Tests
12.5 Alternating Series
12.6 Absolute Convergence and the Ratio and Root Tests
12.7 Strategy for Testing Series
12.8 Power Series
12.9 Representations of Functions as Power Series
12.10 Taylor and Maclaurin Series
12.11 The Binomial Series
12.12 Applications of Taylor Polynomials

In length, depth, and overall organization, Stewart is very similar to Thomas/Finney. Stewart uses termwise differentiation and integration a bit more than Thomas and Finney, and he actually shows that the binomial and Taylor series for $(1+x)^k$ in powers of x are the same.

Grossman, **Calculus of One Variable, 2nd ed.**

Chapter 14, "Sequences and Series."
14.1 Sequences of Real Numbers
14.2 Bounded Monotonic Sequences
14.3 Geometric Series
14.4 Infinite Series
14.5 Series with Nonnegative Terms I: Two Comparison Tests and the Integral Test
14.6 Series with Nonnegative Terms II: The Ratio and Root Tests
14.7 Absolute and Conditional Convergence: Alternating Series
14.8 Differentiation and Integration of Power Series
14.9 Taylor and Maclaurin Series
Review Exercises for Chapter 14

In 68 large pages, Grossman scarcely mentions binomial series, but he includes a brief mention of analytic functions.

b. High School Advanced Placement Calculus Texts

Paul Foerster. **Calculus, Concepts and Applications**.

Chapter 12, "The Calculus of Functions Defined by Power Series:"
12-1 Introduction to Power Series
12-2 Geometric Sequences and Series as Mathematical Models
12-3 Power Series for an Exponential Function
12-4 Power Series for Other Elementary Functions
12-5 Taylor and Maclaurin Series, and Operations on These Series
12-6 Interval of Convergence for a Series-The Ratio Technique
12-7 Convergence of Series at the Ends of the Convergence Interval
12-8 Error Analysis fo Series
12-9 Chapter Review and Test
12-10 Cumulative Reviews

The chapter is 50 pages long with less on each page than a typical university level book. Coverage is limited to topics on the Advanced Placement Calculus tests. Foerster states results informally (rarely using the word "theorem") and gives concise but informal sketches of their proofs. Foerster uses the graphing calculator throughout but never mentions binomial series or the root test *per se.*

Finney, Demana, Waits, and Kennedy, **Calculus, 2nd edition**.
Chapter 9, "Infinite Series"

- 9.1 Power Series
- 9.2 Taylor Series
- 9.3 Taylor's Theorem
- 9.4 Radius of Convergence
- 9.5 Testing Convergence at Endpoints
  - Key Terms
  - Review Exercises

The chapter is 54 full-sized pages long and uses graphing calculators both in the exercises and in the profuse illustrations. Many topics, such as differentiation and integration of series and binomial series, are covered but not mentioned in the titles of the chapter sections. This text has interesting problems and applications throughout the chapter from the very beginning.

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Using mathematical text in classroom: the case of probability teaching

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If we wish to foresee the future of mathematics, our proper course is to study the history and present condition of the science.

Poincaré (1854-1912)

Ⅰ.Introduction

In mathematics teaching, incorporating history of mathematics into classroom can help teacher improve their mathematics teaching. With this in mind, the author has always tried to include history of mathematics into many topics and hopes to broaden students' horizon. As to how history of mathematics can be applied to mathematics teaching, I think it can be divided into five categories:

1.Cite primary mathematics questions as teaching material and pass it on (to students) directly.
2.Tell mathematicians' stories to inspire students.
3. Re-construct mathematics activity, to compare different ideas of mathematicians, and to cultivate recognition in each direction.
4.A cross-reference between the myth of mathematics and students' mistake.
5.Present mathematics in the context of different cultures.

Thus, in this article I would like to show how I use history of mathematics in classroom and to explain how students responded. So, first, I will present the unit that I chose. Then, I will demonstrate the materials I collected and the questions I made. At last, we can pay attention to the various answers of students and find something significant in it.

Ⅱ. The contents of the official curriculum in probability

In the mathematics textbook for Taiwan high school students, the contents of Book Four (Basic Mathematics) are greatly different from those of previous Books. For instance, the chapters of the previous books are number, sequence, polynomial, trigonometry, vector, conic section, and so on. But Book Four discusses probability and statistics. Some students who are unable to learn a lot in the previous three books can surprisingly make great progress in Book Four. According to my observation, it is because of the less prior knowledge for Book Four. Besides, the contents show stronger connection with our everyday life, which in turn would be more interesting to students. Perhaps this may explain the reason why students show better performance in Book Four than in the previous books.

The Book Four covers three chapters. They are permutation & combination, probability, and descriptive statistics in order. Thus, Chapter Two on probability serves to pave ways for later chapters. Indeed, it plays a very important role in this book. The goals of editing the textbook are as follows:

2-1 Event and set
1.To be able to understand the basic concept of set
2.To be able to understand the concept of sample space and events

2-2 The property of probability
1.To be able to understand the definition and calculation of classical probability
2.To be able to understand the basic property of probability

2-3 Conditional probability and Bayes' theorem
1.To be able to understand the meaning of conditional probability
2.To be able to prove Bayes' theorem by means of the concept of conditional probability and disjoint events
3.To be able to decide the conditional probability of some events by means of Bayes' theorem
4.To be able to find out the probability of some synchronic events by means of conditional probability

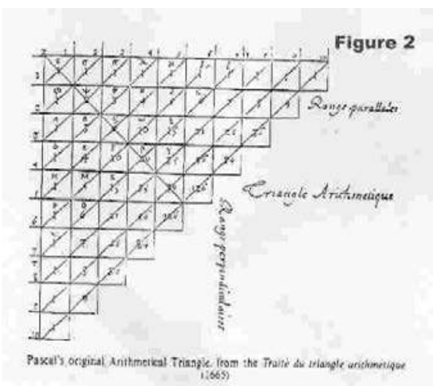
2-4 Independent event
1.To be able to understand the meaning of the dependent and the independent events
2.To be able to understand the necessary and sufficient conditions of independent events
3.To be able to find out the probability of independent events and dependent events

2-5 Mathematics expectation
1.To be able to understand how the expectation is defined
2.To be able to understand the meaning and the property of expectation

From the above citation, we can find the intention of the units design is to help students be acquainted with the Laplace classical probability. The textbook editors introduce Pascal and Fermat's contribution to probability in the beginning of the chapter. But I think it is not enough. Since the Book Four can bring better and obvious improvement in the performance of students who had trouble in learning at their previous stages, I try to say more about the history of the theories on probability in order to inspire and motivate students in learning the subject. In the next section, I will demonstrate some materials that I provided for students.

Ⅲ. Text

1. Jia Xian and Yang Hui's Triangle: explains its origin and applications (Figure 1)
2. Pascal's Triangle: explains its origin and applications (Figure 2)
3. Pascal's portrait: briefly presents his life (Figure 3)
4. Origins of probability: alternative argument
5. Cardano's approach to probability

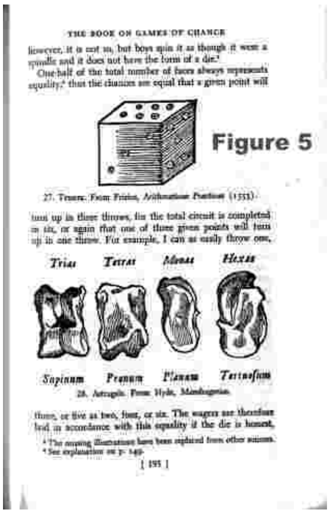


Pascal's original Arithmetical Triangle, from the *Traité du triangle arithmétique* (1665)

For probability, Cardano gave a rough definition as a ratio of equally likely events. A significant passage in a chapter entitled " On the Cast of One Die" reads: "…I can as easily throw one, three, or five as two, four, or six". For the first time, we found a transition from empiricism to the theoretical concept of a fair dice. In making it , Cardano probably became the real father of modern probability theory. (Figure 4 & 5)

Figure 3

Figure 4


THE BOOK ON GAMES OF CHANCE

however, it is not so, but boys spin it as though it were a spindle and it does not have the form of a die.

One-half of the total number of faces always represents equality; thus the chances are equal that a given point will

27. Tessera. From Frisius, *Arithmeticae Practicae* (1553).

turn up in three throws, for the total circuit is completed in six, or again that one of three given points will turn up in one throw. For example, I can as easily throw one,

Trias *Tetras* *Monas* *Hexas*

Supinum *Pronum* *Planum* *Tortuosum*

28. Astragals. From Hyde, Mandragorias.

three, or five as two, four, or six. The wagers are therefore laid in accordance with this equality if the die is honest.

The missing illustrations have been replaced from other sources.
See explanation on p. 149.

[195]

Figure 5

6. Tartaglia's depression(Figure 6)

Tartaglia gave his answer to the Problem of Points. However, he concluded as follows:

"Therefore I say that the resolution of such a question is judicial rather than mathematical, so that in whatever way the division is made there will be cause for litigation."

Judging from the above- mentioned remarks; we would be able to easily imagine his depression.

Figure 6

7. Problem of Points: Pascal, Fermat, Tartaglia

8. Pascal's wager

Pascal began by applying the theory to gambling and ended by applying it to God. He stood at a turning point in history, at the time when the new science had begun to challenge vigorously the old faith. Being religiously enthusiastic and famous as a person who made great contribution to science and mathematics, what he felt would be undoubtedly much stronger than what other common people did at that time. At last, Pascal thought of his early writing on probability and the questions he had solved about gamble. Perhaps the probability theory would show some implications to the issue of religious belief. The answer he has got is exactly the famous Pascal's wager. Perhaps, through this passage, we can find the sign of mathematics expectation.

Pascal's wager

- This is what I see that troubles me. I look on all sides and I find everywhere nothing but obscurity. Nature offers nothing which is not a subject of doubt and disquietude; if I saw nowhere any sign of a Diety I should decide in the negative; if I saw everywhere the signs of a Creator, I should rest in peace in my faith; but, seeing too much to deny and too little confidently to affirm, I am in a pitiable state, and I have longed a hundred times that, if a God sustained nature, nature should show it without ambiguity, or that, if the signs of a God are fallacious, nature should suppress them altogether: Let her say the whole truth or nothing, so that I may see what side I ought to take.
- The value of a ticket in a lottery is the product of the probability of winning and the prize at stake. Even though the probability may be small, if the prize is very great, the value of the ticket is great. So, reasoned Pascal, though the probability that God exists and that the Christian faith be true is indeed small, the reward for belief is an eternity of bliss. The value of this ticket to heaven is, then, indeed great. On the other hand, if the Christian doctrine is false, the value lost by adherence is at most the enjoyment of a brief life. Let us then wager on the existence of God.

[Cited from Kline(1954),pp.374-375]

Ⅳ. Teaching approach

In teaching Chapter 2, I begin with the briefing about the origins of probability theory by presenting the solution of wagering problem connected with games of chance, and the processing of statistical data for such matters as insurance rates and mortality tables. Next, some more words are said to Pascal, who played an important role in the development of probability theory. In addition, I present something about Pascal's Triangle once more. In the meantime, a comparison is given between the Jia Xian and Yang Hui's Triangle. Finally, by introducing the mathematics expectation, I also provide the famous Pascal's wager. In concluding the learning of the chapter, I spend one-hour helping students review the whole story of the development of probability theory and asking them to answer the questionnaire.

Ⅴ. Evaluation

1.Questionnaire

To evaluate the teaching, I give each student a questionnaire about " applying history of mathematics to mathematics teaching ", with the following questions:

1.What was your feeling before mathematics class?

____

2.What do you think about probability before mathematics class?

____

3.What do you say to have mathematics class this way?

____

4.Which part of presentation impressed you most?

____

5.Do you think it helps your learning to have the related background of probability introduced? Why if it does?

____

6.Do you feel the same way about mathematics after having gone through such learning activity and program? Why?

____

7.Do you feel it helps your mathematics learning with such approach?

____

8.Do you feel it helps to improve your personality with such approach?

____

2.Analysis

With the forty-seven students' answers (to the above mentioned questions) put together, the findings of learning evaluation are as follows. Answers to Questions 2 & 3 are not listed due to the fact that nothing in them deserve to be reported here.

Question one
Most students' answers are as follows:

- Mathematics is dull.
- Mathematics is boring.
- Mathematics is difficult.
- Mathematics is a tool of exams.
- Mathematics has nothing to do with daily life.

Another six students' answers are as follows:

- Mathematics has fun.
- Mathematics provides a sense of achievement that comes after the successful solution to questions.
- Mathematics is interesting .
- Mathematics is less disgusting.

Question four
The answers are as follows:

- Pascal's wager.
- The Chinese mathematics is quite advanced, too.

Question five
There are twenty-eight students who consider the introduction to the closely related background of

probability would be helpful to learning. The reasons they have are:

- Mathematics class can be more graphic and not boring.
- It provides realistic things rather than something abstract.
- It can be more informative.
- It tells us the primary source instead of having us memorize formulas.
- Concepts become clearer than ever.
- It brings fun to learning mathematics.

Question six

Eighteen students' answers are as follows:

- There is still something interesting about mathematics.
- Mathematics seems to be finally helpful to daily life.
- Mathematics has history, too.
- New learning material can get me in faster.
- Formulas are no dead stuff. Instead, they are the result of certain changes.
- Mathematics is not just another subject of science. Instead, it is the accumulation of experience and wisdom through long human history.
- It makes me feel more confident.
- We will never feel that mathematics class is nothing more than endless calculation any more and we will never get bored again.

Question seven

Thirty-four students think that it helps in learning mathematics. Students who hold negative attitude towards such approach mostly think:

- This is not important for students who must take mathematics exams.
- The approach and presentation are both interesting and very inviting to students but what students don't understand would still remain what they don't understand after class."

Question eight

A few students' answers are as follows:

- Such approach would free the presentation from the previous deadlock, and would be able to create new view of recognition.
- Also, when confronted with something which I don't quite understand, I would be encouraged to try actively to find out a solution.
- It would be something global and complete.
- After receiving such presentation, I will tend to learn the religious belief of Pascal and change my own way of thinking.

From the above citation, readers can find most students hold a positive attitude towards this way and think that it helps in learning mathematics. This would be a tremendously great encouragement to me. And I am sure that mathematics teaching can be supported, enriched and improved through integrating the history of mathematics into the educational process. The reason why the most impressive to students is Pascal's wager may be because it's new to students and also it is easily comprehensible to them. In addition, students can apply this idea to solving the problems of mathematics expectation. As for few students who hold negative attitude towards such an approach, perhaps it is because of my teaching skill and the pressure of entrance examination for college. Obviously, under the main impact of the social value that school diploma counts in Taiwan, it is a great challenge for a mathematics teacher who intended to use history of mathematics in classroom more effectively and properly.

Ⅵ. Concluding Remarks

What have been stated above are the experiences of the author only. From this experience, I realize I had better improve the questions further. In doing so, I will understand much more about students' thinking process and their responses. Therefore, I can improve my teaching performance and enable history of mathematics to play a more effective role in mathematics classroom.

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