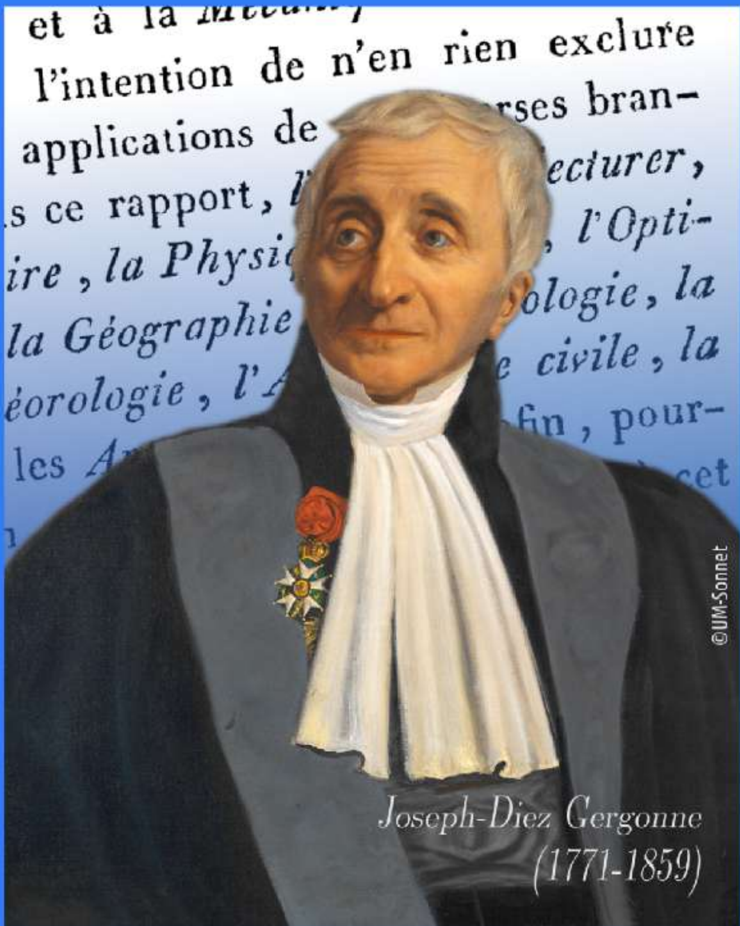
International Study Group on the Relations Between the

History and Pedagogy of Mathematics

Proceedings of the 2016 ICME Satellite Meeting



HPM 2016

Montpellier, July 18-22, 2016

Edited by:
Luis Radford
Fulvia Furinghetti
Thomas Hausberger

International Study Group on the Relations between the

History and Pedagogy of Mathematics

Proceedings of the 2016 ICME Satellite Meeting

July 18-22, 2016
Montpellier, France

Editors:
Luis Radford (Université Laurentienne, Canada)
Fulvia Furinghetti (Università degli Studi di Genova, Italy)
Thomas Hausberger (Université de Montpellier, France)

Organized by: HPM & IREM de Montpellier

Editors:	Luis Radford (Université Laurentienne, Canada) Fulvia Furinghetti (Università degli Studi di Genova, Italy) Thomas Hausberger (Université de Montpellier, France)
Scientific Committee:	Evelyne Barbin (France), Renaud Chorlay (France), Viviane Durand-Guerrier (France), Abdellah El Idrissi (Morocco), Gail FitzSimons (Australia), Fulvia Furinghetti (Italy), Thomas Hausberger (France), Masami Isoda (Japan), Luis Puig (Spain), Anjing Qu (China), Luis Radford (Canada), Man Keung Siu (Hong Kong SAR, China), Bjørn Smestad (Norway), Constantinos Tzanakis (Greece)
Publisher:	IREM de Montpellier
Place:	Montpellier, France
Year:	2016
ISBN:	2-909916-51-0
Copyrights:	© 2016 left to the authors
Cite as:	Radford, L., Furinghetti, F., & Hausberger, T. (Eds.) (2016). Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics. Montpellier, France: IREM de Montpellier.

Sponsors:

HPM 2016 was organised by HPM and the IREM de Montpellier (Institut de Recherche sur l'Enseignement des Mathématiques), with the financial support of the Université de Montpellier, the CNRS (Centre National de la Recherche Scientifique), the Languedoc Roussillon region, the CFEM (Commission Française pour l'Enseignement des Mathématiques). It was hosted by the Faculté d'Education of the University of Montpellier.

Table of contents

INTRODUCTION

Luis RADFORD, Fulvia FURINGHETTI, Thomas HAUSBERGER
Université Laurentienne, Canada
Università degli Studi di Genova, Italy
Université de Montpellier, France

Welcome to HPM 2016. HPM 2016 is the ninth quadrennial ICME satellite meeting of the *International Study Group on the Relations between the History and Pedagogy of Mathematics* (the HPM group).

HPM is one of the affiliate groups of ICMI — the International Commission on Mathematical Instruction. HPM 2016 is a satellite meeting of the corresponding ICME International Congress on Mathematical Education (ICME) organized by ICMI.

The quadrennial satellite meetings are a major activity of HPM. They bring together individuals with a keen interest in the relationship between the history of mathematics and mathematics education. They include: (1) researchers in mathematics education who are interested in the history of mathematics and mathematical thinking; (2) mathematics teachers at all levels who are eager to gain insights into how the history of mathematics can be integrated into teaching and how they can help students to learn mathematics; (3) historians of mathematics who wish to talk about their research; (4) mathematicians who want to learn about new possibilities to teach their discipline; and (5) all those with an interest in the history of mathematics and pedagogy.

The general theme of HPM 2016 is "Mathematics in the Mediterranean." The program and activities are structured around the following topics:

1. Theoretical and/or conceptual frameworks for integrating history in mathematics education.
2. History and epistemology in students' and teachers' mathematics education: Classroom experiments and teaching materials.
3. Original sources in the classroom and their educational effects.
4. Mathematics and its relation to science, technology, and the arts: Historical issues and interdisciplinary teaching and learning.
5. Cultures and mathematics.
6. Topics in the history of mathematics education.
7. Mathematics in Mediterranean countries.

HPM 2016 convenes this year in the Mediterranean city of Montpellier, France, from July 18 to July 22, 2016. HPM 2016 is the result of a strong collaboration between the HPM group and the IREM de Montpellier (a well world-wide known research institute on the teaching of mathematics). The papers presented at HPM 2016 are grouped in the Proceedings by type of presentation: plenary sessions, discussion groups, panels, oral presentations, workshops, posters, and exhibitions.

We are grateful for the support received from the IREM of Montpellier, the Université de Montpellier, the city of Montpellier, and the various sponsors and volunteers that have helped us bring HPM 2016 to fruition. The Local Organizing Committee has done tremendous work to ensure that the participants feel at home in one of the most beautiful French towns. Montpellier is, indeed, very famous for its culture and history, and has one of the oldest universities in the world, where Joseph D. Gergonne published in the early 19th century one of the oldest mathematical journals. Gergonne's portrait appears in the logo and the website banner of the Conference.

1. Plenary sessions

HISTORICAL SOURCES IN THE CLASSROOM AND THEIR EDUCATIONAL EFFECTS

Renaud CHORLAY
ESPE de Paris, 10 rue Molitor, 75016 Paris, France
IREM de Paris, Université Paris Diderot
renaud.chorlay@espe-paris.fr

ABSTRACT

Let History into the Mathematics Classroom is an upcoming book by the French *Commission inter-IREM Histoire et Épistémologie*. Starting from an example from this book, we will first endeavour to identify specific features of the output of this community of practice. On this basis, we will formulate a list of research questions that are not specific to the HPM community, so as to foster collaboration with mathematics education researchers working in other fields. In the second part of the paper, we will report on a recent experiment in primary school which sheds light on the extent to which meta-tasks – a notion introduced in the first part – can be successfully entrusted to young students.

1 *Let History into the Mathematics Classroom*, and reflections thereupon

Since the 1970s, in France, the *Commission inter-IREM Histoire et Épistémologie* (CiIHE) has been producing resources for teaching and teacher-training, through publications, conferences and in-service teacher-training sessions. A new book will soon be available – under the title *Let History into the Mathematics Classroom* – which reports in details on ten experiments of using historical sources in the classroom, at all levels of secondary education (Barbin, forthcoming)[1]. After presenting one of them, we will use this new book to endeavour to identify specific features of what the CiIHE[2] does, and mention some of the things it does not usually do. This will enable us to formulate a list of research questions that we feel are central for the HPM community, yet not completely specific to it, hopefully paving the way for more intense collaboration with researchers in other fields of mathematics education.

1.1 An example: When Leibniz plays dice

In the early 2000s, I came across a paper in *Historia Mathematica* on hitherto unpublished work of Leibniz bearing on games of chance (Mora-Charles, 1992). I was struck by the combination of rich conceptual content and low technical demands (hardly any prerequisites, elementary calculations). I decided to use this text for an introductory session on probability theory, for students of age 15-16.

At that time, in France, students of this age had no prior knowledge of probability theory, but were familiar with basic notions in descriptive statistics: frequencies, relative frequencies, the arithmetic mean and some of its basic properties (linearity; the fact that it can be worked out using relative frequencies only). Before the session, students were asked to

[1] Since the book is not available yet (this paper being written in April 2016), the references will not include page numbers.

[2] This endeavour is descriptive and not normative. Needless to say, many of these features are common to several groups in the HPM community.

look up a few words in a dictionary: heuristic, *a priori*, *a posterior*, empirical, "*aléatoire*" (random), and "*hasard*" (chance). For "aléatoire" and "hasard" they had to look up the etymological origin of the words (namely the word "*dé*"/die, in Latin and Arabic, respectively).

The format I chose for the session is rather unusual: 1½ hours devoted to explaining the content of a mathematical text; guidance was not provided by a worksheet; the students were given the original source, and the teacher engaged in a dialogue with the whole class about the text, along a list a carefully chosen questions and requests.

The text is taken from a rather long letter, written in French, from which I selected the following extract:

> But in order to make this matter more intelligible, I first of all say that appearance [probability] can be estimated, and even that it can be sold or bought.
>
> (…) Let us take an example. Two people are playing at dice[3]: one will win if he scores eight points again, the other if he gets five. It is a question of knowing which of the two it would be best to bet on. I say that it should be the one who needs eight points, and even that his advantage compared with the hope that the other must have, is three to two. That is to say that I could bet three écus to two for the one who needs eight points against the other without doing myself any harm. And if I bet one against one, I have a great advantage. It is true that notwithstanding the chance that I might lose; especially since the chance of losing is like two and that of winning is like three. But as time goes by, observing these rules of chance, and playing or betting often, it is constant that at the end, I will have won rather than lost.
>
> But to show that there is a greater probability for the player needing eight points, here is a demonstration. I suppose that they are playing with two dice, and that the two dice are well made, without any cheating. This being the case, it is clear that there are only two ways to reach five points; one is 1 and 4, the other 2 and 3. However there are three ways to score eight points, i.e. 2 and 6, 3 and 5 and also 4 and 4. Now each of these ways has in itself as much probability as the other as, for example, there is no reason why it cannot be said that there is more probability of getting 1 and 4 than 3 and 5. Consequently, there are as many probabilities (equal amongst themselves) as there are of ways. So if five points can only be made in two ways, but eight points can be made in three ways, it is clear that there are two chances of getting five and three chances of getting eight.
>
> (…) That being the case, it is obvious that the estimate I have just made is the one to follow. That is to say that this fundamental maxim will be the case:
>
> The chance or probability of outcome A keeps the same proportion to the chance or probability of outcome B as the number of all the ways capable of producing outcome A has proportionally to all the ways of producing outcome B, supposing all these ways are equally doable. (Chorlay, forthcoming)[4]

[3] With two dice.
[4] Many thanks to Peter Ransom for the translation.

A detailed account of the session is available in (Chorlay, forthcoming). Let us just summarize the main points:

- What does the following ratio mean: the advantage of the one who bets on a sum of eight "compared with the hope that the other must have, is three to two"?
    - If the two players bet on the results, this ratio enables them to decide whether the wages are fair or not: if you win 1 écu three times out of five (relative frequency 60%), and lose 1 écu two times out of five (relative frequency 40%), the mean win is 1 x 0.6 + (-1) x 0.4 = 0.2 écus; hence the game is structurally favourable to the player who bets on a sum of 8. Another bet would be fair for the two players: If you win 2 écus for a sum of 8 and lose 3 écus on a sum of 5, the mean win is 2 x 0.6 + (-3) x 0.4 = 0 écus.
    - More generally, this ratio enables the players to make rational decisions (here: to engage in a game of chance on terms which are not structurally detrimental to them). The ratio has no dimension, but it can be used to determine values (here in écus). Since the 17th century, the rational pricing of random events has become fundamental in the banking and insurance business: Leibniz was quite right when he said that "appearance (...) can be sold or bought"!
- How does Leibniz determine the ratio 3/2 (in favour of a sum of eight), or the ratio 60% / 40% ? He explains his method in the paragraph starting with "But to show that (...)".
    - He proceeds by pure reasoning and not by observation; hence, strictly speaking, the values 60% and 40% are not relative frequencies of the kind used in descriptive statistics: descriptive statistics relies on surveys (empirical data) and works out parameters *a posteriori*. Leibniz relies on pure reasoning, the values are determined prior to any observations, that is *a priori*. In this context, these values are called probabilities and not relative frequencies; in this context, the mean is called the expected value.
    - To determine the specific values, he works out "all the ways" of getting a sum of 8 (3 ways) and a sum of 5 (2 ways). When dealing with a random experiment, we call these "ways" outcomes. Leibniz states very clearly that this reasoning makes sense only if the dice are fair, so that "each of these ways has in itself as much probability as the other"; we shall say that the outcomes are equally likely.
- Back to the first question, with a twist: "what do these values 60%, and 40% mean? Do they enable us to say what will happen at the next throw of two fair dice?" Leibniz deals with this quite clearly "It is true that notwithstanding the chance that I might lose; especially since the chance of losing is like two and that of winning is like three. But as time goes by, observing these rules of chance, and playing or betting often, it is constant that at the end, I will have won rather than lost." In this passage, Leibniz connects the probabilities (which are *a priori*, theoretical values) to relative frequencies: on a large sample of the same random experiment, the empirical relative frequencies should provide reasonable approximations of the probabilities; for these

approximations to become exact, one would have to repeat the experiment an infinite number of times (betting "often" until the "end of time"!). This statement is an informal version of an important mathematical theorem, called the law of large numbers.

- "Does the law of large number provide a means to check that the values which Leibniz determined by pure reasoning give a correct quantitative description of the random experiment?" Sure, we can use a spreadsheet or write a simulation program (taking in to accounts sums of 8 or 5 only).
    - 10 simulations of 100 repeats each show a large dispersion of the relative frequencies of sum 8. It's hard to say anything.
    - Working out a dispersion parameter (the spread, the interquartile range, or the standard deviation) with 10 simulations of 1000 repeats, then 10 of 10 000 repeats, then 10 of 100 000 repeats seems to confirm our intuitive understanding of the law of large numbers: dispersion becomes smaller as the number of repeats tends to infinity; hence the empirical relative frequencies should provide ever more accurate values of the probabilities.
    - However, something seems to be wrong: according to Leibniz, the probability of sum 8 is 0.6; but in the simulations, the estimated probability stabilizes between 0.55 and 0.56. Maybe Leibniz's reasoning is wrong, or the informal version of the law of large numbers is deceiving ...
    - There is a way out of this predicament, which is compatible with both the "fundamental maxim" (which provides a means to work out probabilities by "counting ways" when the outcomes are equally likely) and the law of large numbers. If the two dice were of different colours, or if one die was thrown twice, we would be able to distinguish between nine different and equally likely outcomes. Hence the following table is incorrect:

Case	$2+6=8$	$3+5=8$	$4+4=8$	$1+4=5$	$2+3=5$
Probability	0.2	0.2	0.2	0.2	0.2

It should be replaced by one of the two tables:

Total of 8	**Total of 5**
2 then 6	1 then 4
3 then 5	2 then 3
4 then 4	3 then 2
5 then 3	4 then 1
6 then 2	
5 ways	***4 ways***
Probability 5/9	***Probability 4/9***

Case	$2+6=8$	$3+5=8$	$4+4=8$	$1+4=5$	$2+3=5$
Probability	2/9	2/9	1/9	2/9	2/9

The same random experiment has two models which account for the empirical relative frequencies. In the first model the 9 outcomes are equally likely, hence the probabilities can be worked out by the easy formula $\frac{number\ of\ succesful\ outcomes}{number\ of\ possible\ outcomes}$. For instance, in this random experiment, the probably of getting at least one "4" is 3/9 (three out of nine). In the second model, there are 5 unequally likely outcomes. Probabilities can be worked out using another (easy) reasoning; for instance, the probability of getting at least one "4" is 1/9 + 2/9. Both methods lead to the same values; the first one is a special case of the second one.

The teaching goals for this session are quite clear: the curriculum requires that students this age become acquainted withe the notions of "probability", "probability distribution", "expected value", "equally likely outcomes", "fair bet"; it requires they be able to determine the probability distribution on finite sample spaces, in simple cases; that they be able to work out probabilities on the basis of a probability distribution, whether the outcomes are equally likely or not. The curriculum also suggests that more epistemological aspects be tackled, such as the connection between descriptive statistics and probability theory, using an informal statement of the law of large numbers; also that, on some occasions, students come accross a multiplicy of random models for the same real-life experiment. Moreover, it is required that students use ICT in all parts of the curriculum where it can help them make sense of the maths.

From an HPM viewpoint, this session is typically on the "HM as a tool" side, as opposed to "HM as a goal". A *history* of maths course on the emergence of probability theory would probably dwell on the works of Galieo, Fermat, Pascal, Huygens; would discuss the problem of points, Pascal's wager on the expected benefits of a christian life etc. Of course it is not historically uninteresting to see that Leibniz wrote several texts on probability theory, but investigating this fact is interesting only for professional historians (of mathematics, or of philosophy). Moreover, this short passage does not say much about Leibniz's interest in probability theory; and as far as the history of probability theory is concerned, it is hardly worth a footnote. To put it in a nutshell: its potential didactical value is not directly correlated to its historical import[5].

1.2 An example of what?

On the whole, many features of this lesson plan are common to those presented in the upcoming book *Let History into the Mathematics Classroom*, which reflects the fact that the French *Commission inter-IREM Histoire et Epistémologie* (CiIHE) is not only a network but also a community of practice. Let us list some of these features:

[5] De Vittori's analysis of standard non-standard teaching sessions, namely those relying of historical sources, fully applies here: « *l'usage de l'histoire des mathématiques dans le contexte scolaire de l'exercice justifie l'extraction d'une partie épistémologique de l'histoire non dans le cadre d'une pratique historienne, mais afin d'élaborer une nouvelle forme de pédagogie* . » (de Vittori, 2015, p.19). Meaning: The use of history of mathematics in the context of a school exercise justifies the extraction of some epistemological part of history; here, the framework is not that of the practice of historians, but the elaboration of a new form of pedagogy.

1. The starting point for the design of a lesson plan is usually the recognition of the fact that some local teaching need is somewhat echoed in a specific historical source. The teaching is often a syllabus requirement: modelling basic random experients in high-school, studying angles in middle-school (Guichard, forthcoming), solving problems where proportionnality holds (Morice-Singh, forthcoming) etc.
2. The background knowledge of the designer is usually historical, and the background know-how usually comes from teaching experience. Results or concepts from the didactics of mathematics are seldom mentioned.
3. Designing such a lesson plan does not require a comprehensive knowledge of the history of mathematics.
4. The chapters are explicit when it comes to background knowledge (mathematical, historical, epistemological) and motivation (including intended educational effects). They are sometimes quite explicit as to the tasks entrusted to the students. As a rule, very little is said about actual student activity, and observable educational effects.
5. The range of task is wide and has some rather specific features. Let us distinguish between two types of task:
    a. Text-reading tasks.
    b. Meta-tasks (M-tasks). I am not in a position to provide a clearcut definition for this class of tasks. To delineate it, suffice it to say that (1) they differ from directly transformative tasks (draw ..., work out ..., factorize ..., solve equation ...); transformative tasks for which the didactical system usually provides techniques and a technology – to use ATD terminology (Chevallard, 2007, 133); (2) they include: justifying, comparing, assessing, criticizing, summarizing; proving and reformulating/translating/rewriting can be listed among them, in cases when no standard background techniques are provided by the didactical system. The book has many examples of these task requests: "In each case there were issues around the understanding of the texts and checking the proofs, even of completing or adding to them." (Guyot, on inscribing a square in a triangle); "Summarize the solution and explain the method. (…) Is it true or false? (…) The translator made a mistake in the problem. What is the incorrect word and what word should replace it? (...) Transcribe the problem in French everyone can understand (take care with the wording). Represent the situation with a simple diagram" (Métin, on false position methods).
6. As far as timing is concerned, we're talking medium range: beyond 10-min exercises, yet not complete chapter designs. Several chapters of the book deal with chapter design (Guichard on angles for 11-12 year-olds), or provide series of exercises on one topic to be used as a guiding thread in a given chapter (divisions of triangles in basic plane geometry (Moyon), proportionnality (Morice-Singh), graphical methods for differential equations (Tournès, chap. 7)). Even in these cases, the activities based on history are fully compatible with curriculum requirements, and interwoven with standard texbook exercises. *Enrichment* rather than *reconstruction*; from a cognitive

viewpoint, cognitive flexibility is aimed for, with no hidden background recapitulationist model of the "ontogeny recapitulates phylogeny" type.

7. The tasks that are entrusted to the students are usually rather demanding, even difficult, which means two things: first, students say they are taxing. Second, the session designer is fully aware of this fact, and is ready to live with it (which means he/she is comfortable occasionally asking a lot from students, and running the risk of complete failure). Occasionally he/she embraces it: "I realized that the mathematical content was both accessible for 16 year-old students and sufficiently 'hidden' to require interpretation" (Métin). In this respect, the Leibniz session is not fully representative, at least as far as the strictly mathematical content is concerned.
8. The chapters are meant to be used as ressources for other teachers. Consequently, one has to distinguish between two intended audiences: on the one hand, the intended audience of the teaching sessions is that of secondary school students; on the other hand, the intended audience of the book, and more generally of the output of the CiIHE, is an audience of teachers and teacher-trainers.

1.3 Possible research questions

Reflecting on this list of features enables us to to elicit a range of research questions, only one of which will be touched upon in the second part of the paper.

A first series of questions concerns the first intended audience, that of students, and the (still tentative) notion of M-task:

- Can (and should) a more clearcut caracterization of this class of tasks be given ? Some features can be mentioned, to complement the first elements brought up above (5b).We chose to call them meta-tasks, since students are required to do something *with reference to*, or *about* a piece of mathematics; something which is not limited to acting *on* or *within* the mathematics involved (as is the case in "factorize (...) then solve equation (...)"). To successfully perform the task requires the identification of the relevant mathematical background knowledge, and this knowledge may involve pretty diverse and distant elements (e.g. write a computer simulation to test Leibniz' forecast as to a game of dice); beyond background mathematical knowledge, other abilities are required, in particular when it comes to reading and writing mathematical – and, more generally, argumentative – texts. Specifying this class of tasks would help to identify:
  - The expected educational effects. In particular, is the main expected effect the passage from the "mobilizable knowledge" to "available knowledge"[6]? Another possibility was investigated by Tinne Kjeldsen, that of fostering "commognitive" conflicts (Kjeldsen, 2012).

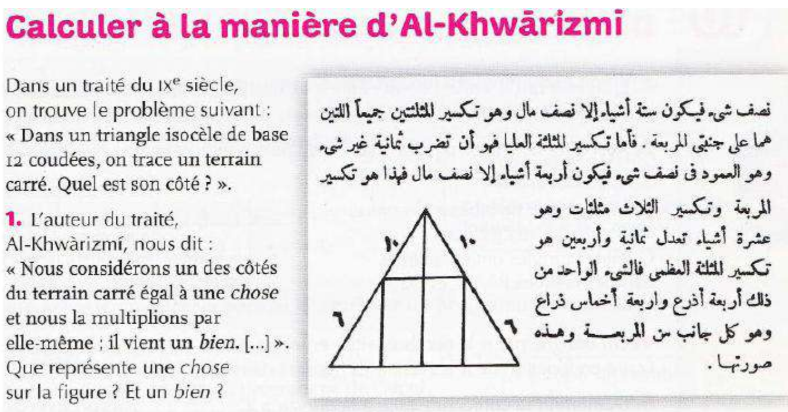
[6] This distinction has become classical in the French didactical community, on the basis of Aline Robert's work (Robert, 1998). Some knowledge (or know-how) is mobilizable if students can apply it reasonably successfully upon request (e.g. "Use Pythagoras' rule to work out the length of side AC"); it has become available when students are able to identify it as the relevant tool even when no indications are given.

- The conditions for such tasks to actually trigger the expected student activity.

- Many of the characteristics of M-tasks apply to “proving”, and the relation between this class and this specific task raises several questions: is the task “proving” always an M-taks, or does it depend on conditions that need to be identified? If mastering “proving” in some mathematical contexts is a major goal of mathematics education, does the inclusion of “proving” in the larger class of M-tasks help to reach this goal? Can M-tasks other than “proving” be conducive to “proving”, and in what way?
- Are M-tasks specific to the “historical sources in the classroom” context? My contention is that they are not[7], but that using a historical document as teaching (raw-)material on the one hand, and entrusting students with M-tasks on the other hand, enjoy a special connection. It would be interesting to investigate the connections between M-tasks and other protocols, such as: “True/False. Justify”, and “True/False/No way to know. Justify” questionnaires; scientific debates (Legrand, 1996); open problems for the classroom (in French: *problèmes ouverts*).
- In M-tasks, the epistemological position of students with respect to the mathematical document is not standard: students are to *investigate* an artefact (usually a mathematical text) that is a non-standard object in the maths-class and is somewhat hard to access; they are to reformulate, and assess some mathematical content on the basis of the maths they have learnt, thus acting as *experts* endowed with background knowledge. Does this imply that a specific didactical contract should be listed among the conditions for success?

A second series of questions concerns the second intended audience, that of teachers and teacher-trainers.

The most common teaching aids are textbooks, and several studies have been carried out on the way history of mathematics appears – or not – in them. On this basis, it is worth mentioning a rather unsual example. A textbook for students in the third year of middle-school featured the following exercise (fig.1), in its chapter on algebraic calculation:



Calculer à la manière d’Al-Khwārizmi

Dans un traité du IX^{e} siècle, on trouve le problème suivant : « Dans un triangle isocèle de base 12 coudées, on trace un terrain carré. Quel est son côté ? ».

1. L’auteur du traité, Al-Khwàrizmí, nous dit : « Nous considérons un des côtés du terrain carré égal à une *chose* et nous la multiplions par elle-même ; il vient un *bien*. [...] ». Que représente une *chose* sur la figure ? Et un *bien* ?

نصف شيء فيكون ستة أشياء إلا نصف مال وهو تكسير المثلثين جميعاً اللتين
هما على جنبتي المربعة . فأما تكسير المثلثة العليا فهو أن تضرب ثمانية غير شيء
وهو العمود في نصف شيء فيكون أربعة أشياء إلا نصف مال فهذا هو تكسير
المربعة وتكسير الثلاث مثلثات وهو
عشرة أشياء تعدل ثمانية وأربعين هو
تكسير المثلثة العظمى فالشيء الواحد من
ذلك أربعة أذرع واربعة أخماس ذراع
وهو كل جانب من المربعة وهذه
صورتها .

[7] In this respect, we do not share de Vittori’s view (de Vittori, 2015).

2. Al-Khwârizmí nous donne ensuite le calcul suivant :
« Quant aux deux triangles qui sont sur les flancs [...] leur aire est que tu multiplies une *chose* par six moins un demi d'une *chose*, il vient six *choses* moins la moitié d'un *bien*. »
Expliquer ce calcul.

3. Avec nos notations actuelles, si on note ℓ une *chose*, comment s'écrit un *bien* ?
Écrire le calcul précédent avec ces notations.

EXPOSÉ Qui était Al-Khwârizmí ? Pourquoi son nom est-il important en mathématiques ?

Figure 1. Calculating in the manner of Al-Khwārizmi[8]

A few pages down, a second exercise was based on the same document; students were asked to solve the problem and "check Al-Khwārizmi's solution". This problem from the book of *al-jabr* is well-known in the CiIHE, and bears witness to the active role of Professor Ahmed Djebbar (Djebbar, 2005). For instance, an account of its use with students in vocational high-school can be found in the *Let History in the Mathematics Classroom* book (Guyot). It is however quite unusual to find such exercises in textbooks, in which parts of the original text are given, and typical text-related M-tasks are entrusted to students.

Beyond the question of tasks[9], one could investigate whether or not teachers actually use this exercise, and (if so) how? To use a metaphor from economics, is there any demand for this supply? I don't have a clue as to the "how" part of the question; however, in 2014, two in-service teachers investigated the reception of this exercise by maths-teachers, as part of a project which I supervised[10]. This project is on a much smaller scale than Professor Siu's (Siu, 2006), with a sample of 30 middle-school maths teachers; but it was focused on the reactions of the teachers to this specific exercise, on the background of their general stated view and practice of history of maths in teaching[11].

In this survey, 90% of the sample say they use the history of mathematics in their teaching: occasionally (80%) or on a regular basis (10%). In most cases (83%) this historical perspective boils down to mentioning names of famous mathematicians or well-known anecdotes. Still, 49% say they sometimes introduce new mathematical concepts or methods using history, and 41% say they occasionally ask their students to solve historical problem.

Studying their reactions to the specific exercise (fig.1) helps us to go beyond this pretty standard distribution of statements. Only 12% of the sample say they occasionnaly use original sources. 50% say they would use this exercise with their students. For those who say

8 (Chesné, & Le Yaouanq, 2011, p.85).
9 This exercise is commented upon in (de Vittori, 2015).
10 Banakas P, Kerboul C. (2015) *Deux problèmes d'arpentage et de transaction commerciale en algèbre. Quelle appropriation par les enseignants ? Quels usages en classe ?* (unpublished Master project). This project was carried out in the context of the course on « history of maths and the sciences in teaching and teacher-training », in the Master of didactics of Paris-Diderot University. From a methodological point of view, this project does not meet the requirements of a scientific work; in particular, no information is given on sample selection and possible biases.
11 Only statements were studied, using questionnaires. Studying actual practice would be a pretty challenging endeavour.

they would not, lack of sufficient historical background information is generally not the main argument – although it is mentioned by 30 % of them. They find the exercise too difficult and time consuming; it involves too much reading and writing. Of course, it could sound encouraging to see that half of the sample say they would use this exercise; in this case, they were also asked to choose a protocol (they could tick several boxes). The results were: 2 teachers answered "in the classroom, students working individually", 11 answered "in the classroom, in groupwork", and 7 answered "as a homework project" (*devoir-maison*).

Although the sample is too small (and probably biased) for any conclusions to be based upon this survey, it suggests at least two avenues for research:

- If we think that this type of exercises is educationally valuable – and it seems the CiIHE does, since it is very close to what they've been supplying for years – how to make it more attractive to teachers? My interpretation of the data is that a majority of them would avoid using it, since I interpret the "as a homework-project" answer as a mere show of good-will[12]. The CiIHE books and brochures usually provide background historical information, but it may not be what teachers feel they need most. Our hypothesis is that teachers do identify M-tasks, which is why they are reluctant to use this exercise: either because they do not see the expected educational benefits; or because they would not feel comfortable/competent surpervising and assessing student work, in particular because what constitutes a "correct" answer is somewhat less easy to identify than in more standard cases.
- The fact that M-tasks are not specific to sessions based on historical sources suggests another lead. Our hypothesis is that some teachers *are* willing to occasionally engage in classroom work which *is* of an unusual nature, *does* take time, proves really *demanding* for students, and has mainly long-term expected educational effects; and some are not. To test the existence of this *venturesome-teacher profile*, one could investigate the correlation between the involvement in several teaching protocols which are M-task-rich, in particular open-ended problems (*problèmes ouverts*); maybe modelling problems as well.

In the second part of this paper, we will report on an experiment recently carried out at the primary level. The emphasis will be on a feature which is often left implicit in the publications of the CiIHE: the actual activity of students. Besides, it should contribute to establishing that, under conditions which remain to be investigated, genuine M-tasks can be entrusted to students even at the primary level.

[12] This interpretation needs grounding, of course.

2 Students' activity in an M-task-oriented teaching sequence in primary school.

2.1 Algorithmic tasks in primary school

In the fall of 2015, an experiment was carried out in four classes in the final year (or final two years) of primary school, in France (Chorlay, Masselin, & Mailloux, submitted). The original motivation for the focus on algorithm, however, lies outside of the primary school context. It lies – first – in current historical work on ancient mathematics carried out, in particular, in the SPHERE[13] research group; second, in what we feel was a fruitful introduction, that of a focus on algorithmic thinking in the current French high-school maths curriculum[14]. I must say that, as a historian and a teacher-trainer, my own view of algorithms and algorithmic thinking was significantly enriched by this unexpected partial overlap. In particular, it occurred to me that meaningful reflexive tasks can be carried out in an algorithmic context, *beyond* the two classical justificatory tasks: to prove that an algorithm is correct (relative to an *a priori* set goal); to prove that it terminates (when termination depends on some condition). In particular, two challenges can lead to M-tasks that are not directly of a justificatory nature. First, since a given algorithm has to be able to process a large class of input values, its formulation requires that some indeterminate objects be denoted and handled. Second, some pragmatic properties of an algorithm can be studied: surveyability, user-friendliness, time-and-labour cost etc.; the investigation of these properties comes naturally to mind when two algorithms performing the same function are to be compared by a user.

For primary school children we[15] selected a context which is pretty familiar to them, that of multiplication of two whole numbers; and an algorithm that is not, that of the *per gelosia* technique. Multiplication, however, was not our object of study; rather, we wanted to know to what extent reflexive tasks bearing on algorithms could be entrusted to 8-to-10-year-old students. Three 1-hour classroom sessions were designed and implemented: in the first session, the two tasks were: to perform/emulate an algorithm, and to formulate hypotheses as to its function; in the second session, the task was to write an algorithmic texts; in third session, the task was to compare the *per gelosia* technique with the one that is familiar to French students.

Before we describe the three sessions in more details, and provide elements as to what students managed to do (or did not), let us say that the whole project was of a rather exploratory nature. Since the current curriculum sets no goals as to reflexive tasks on algorithms in primary education, complete failure would have been harmless to the students – but not to our egos! Hence we felt comfortable putting them in really challenging situations,

[13] The History and Philosophy of mathematics and the Sciences research group, affiliated with the CNRS and Paris-Diderot University (UMR 7219).

[14] A third context must be mentioned, namely the in-service 3-day teacher training programme that the Paris IREM history-of-maths group designed and implemented in 2014-2015 and 2015-2016.

[15] The experiment was designed and implemented by Renaud Chorlay, Blandine Masselin and François Mailloux. In this section, the « we » denotes us three.

providing as little mathematical support as we could. The general design of the three sessions was: once the initial document was given and the task(s) spelled out, the students worked in groups of 4 for 20 to 30 minutes; the groups were eventually asked to present their work to the class under the supervision of the teacher. Our emphasis on autonomy accounts for the fact that we did not choose to work on the justification of the technique. We feel it would be very interesting at this level (and beyond), but probably with a lot of help from the teacher.

2.2 Session #1

The first session was the only one involving historical documents. The children were told that a chest had been found, which held four documents (fig. 2). The first question was: “Can you guess what these are, and why they were together?”

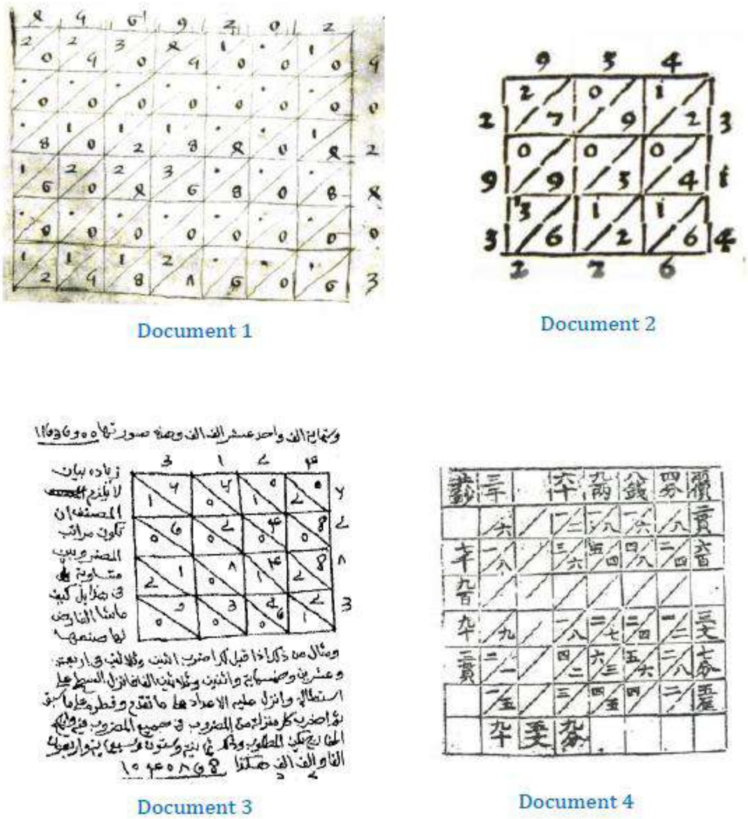


Figure 2. The documents found in the chest[16].

The students turned out to be keen observers when it came to spotting similarities and dissimilarities among the four documents. However, none surmised that they had anything to

[16] Documents 1, 2, and 4 are taken from (Chabert, 1999, p.30-32); Document 3 from (Abdeljaouad, 2005, p.61). Document 1: Latin manuscript *Tractatus de minutis philosophicis et vulgaribus* (Oxford, Bodleian Library, circa 1300). Document 2: *Treviso Arithmetic* (1478). Document 3: Fac-simile of the $Sharh\ al-talkh\bar{\imath}s$ d’$Al-Qalas\bar{a}d\bar{\imath}$ (manuscrit du 15$^{\text{ème}}$ siècle, Bibliothèque de Gotha (Allemagne)). Document 4: *Jiuzhang suanfa bilei daquan*, China, 1450.

do with multiplication, or even calculation. The most common hypothesis was that these were some sort of number games; Sudoku-style. This is what we had anticipated: without the chronology of operations, and the distinction it displays between input values, intermediate steps, and output value, no calculation technique is readily identified. This is why the chest held another document – of a less historical nature however! One of us had been taped while working out the product 93x52 on a blackboard by the *gelosia* method. The movie was silent. After showing it twice (to a mesmerized audience) and freezing on the initial picture (with numbers 93 and 52 written horizontally and vertically, respectively), we asked students to reproduce on paper what they had seen; then to explain what it was all about.

The first task created a double challenge: to redraw the geometric diagram, then to recover the numbers written in it. The first challenge turned out to be really difficult, as shown in figure 3.

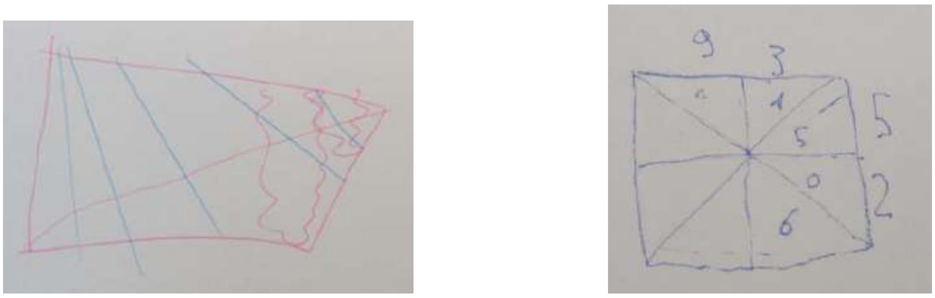


Figure 3. Incorrect diagrams.

The second challenge was quite demanding too. Some students tried to remember the numbers they had seen in the video, but it was clearly impossible to remember them all. Working in groups of four helped, since some remembered some of the numbers, and some were willing to try out ways of finding them through calculation. Eventually, most groups could complete the grid through multiplications of 1-digit numbers followed by additions. However, none spotted the structural similarity with the written technique for multiplication to which they are accustomed. To find out “what it was all about”, many groups tried adding, then subtracting, then multiplying 93 and 52. The fact that 93x52 is, indeed, equal to 4 826 convinced everyone that this was a multiplication technique. Not one student asked for any kind of justification; indeed, the scenario had not been designed to trigger the need for justification.

The teacher acknowledged the fact that this is a multiplication technique, introduced the words “input values”, “output value”, and reminded the students of the words “factors”, “product”, “units digit”, and “tens digit”. At the end of the first session, the students were asked to carry out a few multiplications using this new technique: 23x17, 32x25, 625x8, 625x32. The last two products created a new challenge: until then, the diagrams had been 2x2 “square” grids. A new feature came into play when the initial technique had to be adapted to numbers of various lengths. It usually took students several attempts to adjust the grid, and many used a “square it up” tactic (fig. 4).

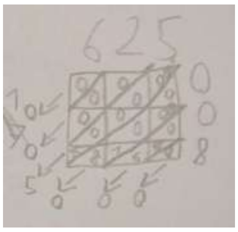


Figure 4. Square it up!

2.3 Session #2

During the next two weeks, a regular basis, students were asked to carry out multiplications either by the familiar or the by the *gelosia* technique. The second session began with a deceptively harmless request: "On paper, explain the *gelosia* method for students with no access to the video. They should be able to apply your method starting from any two whole numbers." In our analysis of the outcomes, we focused on (1) the overall correctness of the algorithm (for instance: no steps missing), (2) the semiotic means used to express the algorithm (intended for an audience with no prior knowledge of this technique), and (3) if and how these semiotic means were used to face the generality challenge ("for *any* two whole numbers"[17]).

We will not go into (1) in any details: we expected very satisfactory results, and very satisfactory they were. As to (2), the most common form was that of the "comic strip" displaying the successive states of a grid while multiplication was performed on an example assuming a generic role; in addition, arrows associated relevant parts of the successive diagrams with short textual instructions. This form has many advantages: it does capture the chronological aspect of the algorithm; it expresses the diagrammatic steps with a diagram, and the numerical operations through a written explanation of a general instruction ("multiply (...)") applied to a generic example ("(...) this by that"). For the students, it is probably reminiscent of what they are regularly exposed to, either in textbooks or when the teachers show a new calculation technique. However, relying on a single – if generic – example fails to meet all the requirements, since it gives no clue as to how to adjust the grid to the length of the input numbers. At the other end of the spectrum, two groups provided a purely textual explanation, using no diagrams and no generic examples. These are fully general description of the goal and the main steps, but the objects to which the instructions apply are not captured specifically enough for anyone to be able to perform the algorithm on two given numbers using only these guidelines.

Several groups faced the generality challenge, more or less directly, and with a variety of semiotic means. In order to denote the fact that their explanation had to apply to any numbers, some drew specific grids (displaying the variability of size) but left them blank, or wrote question marks to denote the location of indeterminate input digits (fig.5).

[17] Although this paper is no place to dwell on this topic, it is worth mentioning that the role of generality as an epistemological value, and the ways of expressing/capturing the general has been a topic of historical research in the SPHere team for over a decade. A long-term project recently led to the publication of (Chemla, Chorlay, & Rabouin, 2016).

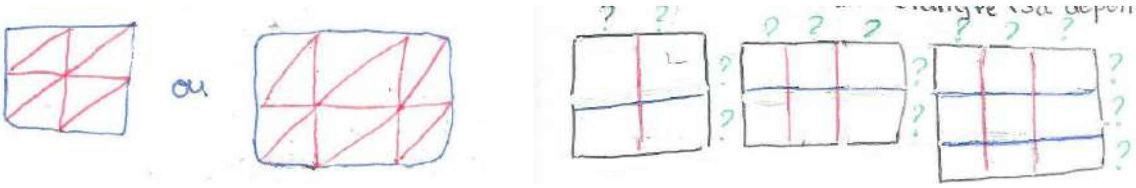


Figure 5. Grids suitable for "any" numbers

One group (fig.6) opted for a textual approach (as opposed to the comic-strip approach) but inserted two (non-generic) examples to illustrate the fact that "one draws a table which depends on the numbers".

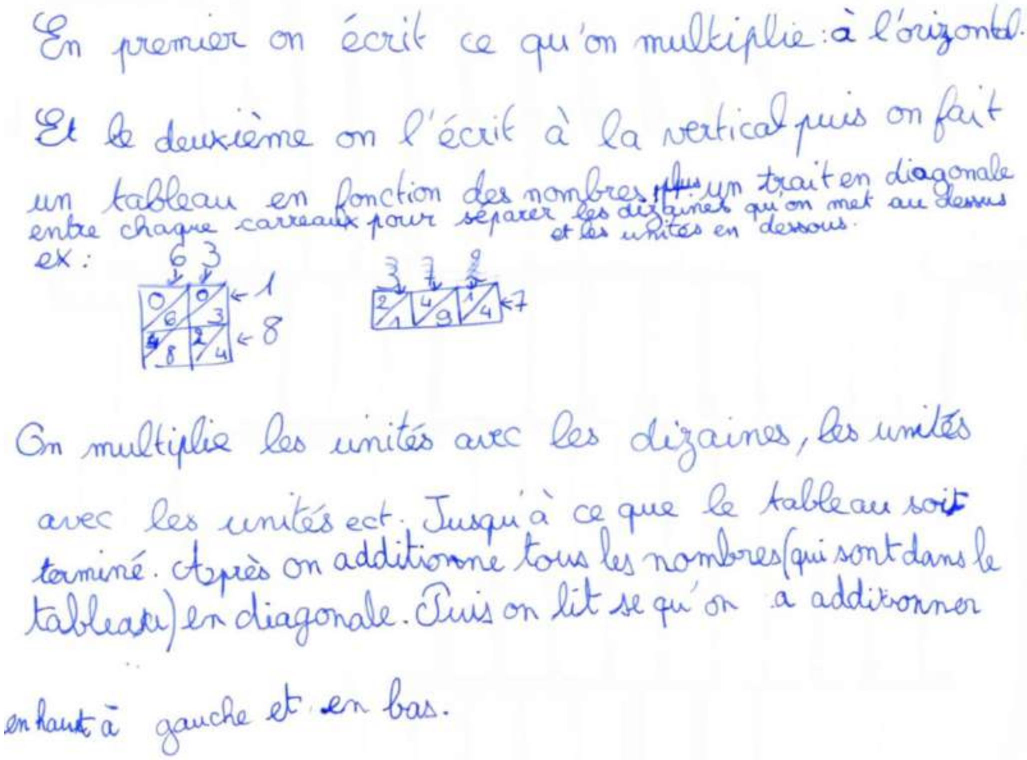


Figure 6. Moving away from generic examples.

In other groups, the step where the dimension of the grid is chosen is either hinted at ("it depends on the factors") or expressed as a semi-formal rule ("the more numbers, the more squares in the grid"). During the final part of session 2, the teacher asked more specifically for a general rule to choose the dimension of the grid. In one class, one student mentioned "the number of digits"; everyone seemed to agree.

2.4 Session #3

The third session was devoted to the comparison between the *gelosia* method and the one that has been familiar to the students for about two years: "If you were to recommend one of the two techniques for another class, which one would you choose, and why?" Again, we chose to entrust students with a set of quite complex tasks; in particular, we chose not to provide or hint at criteria for comparison, so students had to identify criteria themselves before engaging in comparison.

All groups of students came up with objective and explicit criteria – i.e. went beyond comments such as “we recommend ... because we like it better than the other”. As a matter of fact, all the students but one (in four classes of about 25 students each) preferred the *gelosia* method; the fact that they had a point to make was probably a necessary condition for a real involvement in the task. Autonomous work led to the following arguments: (1) the *gelosia* method is more surveyable (once it’s been performed you can easily check all the steps in the grid); (2) in the *gelosia* method, carries may appear only in the additive phase, whereas in the usual method carries may appear in the multiplicative phase, which is a standard cause for mistakes; (3) in the usual method, when the second factor is not a 1-digit number, one has to shift calculation lines to the left (usually by writing extra zeroes), which is a common cause of mistake when the shift is forgotten or faulty. It is striking that all these criteria had long ago been identified by researchers in the didactics of mathematics, in particular in the comparative study that Guy Brousseau carried out since the 1970s (Brousseau, 2007).

All these criteria are of a pragmatic (or ergonomic, as Brousseau put it) nature; they all dealt with the comparative user-friendliness of the techniques. We also wanted to see if students would come up with criteria that would be reminiscent of algorithmic complexity; that is, if the number of steps (of a similar nature) would be worth comparing, for the same two input values. No groups tackled this issue in the written sheets. However, some hinted orally at the fact that they found one of the two methods faster than the other. For these groups, the teacher prompted further investigation into this aspect: “Can you think of ways to measure which is the faster?” In various groups, two different strategies were selected. Some used watches to actually measure the time it took to work out the same product by either method. Others began counting the number of intermediate steps, which turns out to be both tricky (the steps are of different types) and tedious. However, the written outcomes sometimes display an unexpected semiotic feature (fig.7)

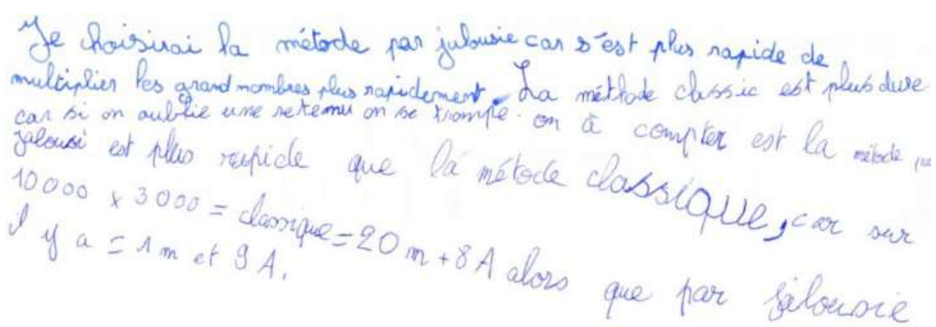


Figure 7. Comparing the two techniques in terms of number of steps.

Here the students summarized their finding using a letter coding: “10 000 x 3 000 = classical = 20m + 8A” *vs* “by gelosia there are = 1m + 9A”. Of course this is not symbolic algebra: the expressions are not calculated upon; numbers are not to be substituted for letters. It would be well worth investigating where this shorthand comes from[18].

Conclusion

It goes without saying that HPM – as a field of practice and research – *has* specific features, in particular when we take into account the connection with history of mathematics as a field of knowledge (Fried, 2001) (Chorlay & Hosson, 2016). However, in this paper we chose to focus on what we feel is *not* specific to the use of historical sources in the classroom, in spite of the fact that all the lesson plans or teaching sequences we presented make explicit use of such sources. Rather, we focused on tasks, intended student activity, and actual student activity. In the first part of the paper, among other things, we attempted to delineate the class of Meta-tasks (M-tasks), which is central to the HPM approach – at least as far as it is practiced in the CiIHE – yet not specific to it.

In the second part of the paper, we reported on an experiment carried out recently on a medium scale (four classes), in the final two years of primary education. Although multiplication of whole numbers was the underlying mathematical content, this is not what was at stake for us in the experiment. Rather, we wanted to investigate the extent to which typical M-tasks such as "formulating a *general* method" and "comparing two methods" could be entrusted to young students, in conditions where very little support was provided by the teachers. It turned out that the session which proved the most difficult for students was generally the first one, in which the tasks were more standard (to draw a complex diagram from memory, and identify/recover numerical operations on an example). When given sufficient time to think things through, students did remarkably well in sessions #2 and #3. A more thorough report should appear soon in (Chorlay, Masselin, & Mailloux, submitted).

We would like to conclude assuming the position of the devil's advocate, in order to point to open questions and suggest further comparative work. The experiment in primary school raises at least three interconnected questions: What was our teaching goal? Why did the 3-session-sequence work? What were its actual educational effects? To the first question, we could argue that we had research questions and not teaching goals. But that wouldn't say it all, because we probably also believe that entrusting M-tasks to students does have beneficial educational effects; work has to be done in order to identify these precisely. As to "why it worked" (if it did), the issue can be investigated by comparing with other sessions which did not work so well. In this respect, we are looking forward to the completion of Charlotte de Varent's doctoral work[19] on the use of Ancient mathematics as a means to make rather advanced students (age 15-16) question seemingly elementary mathematics (namely: the formula for the area of a rectangle, and the role of units of lengths and areas). As to the study of the educational effects, we feel we could benefit from a comparison with recent work in the

[18] Two usual suspects would be: (1) expressing magnitudes with several units (e.g. a length of 1km 300 m, a weight of 1kg 850 g), (2) expressing whole numbers in terms of units, tens, hundreds etc. (e.g. to carry out an addition: 7t + 3u + 5t + 3u = 12t + 6u = 10t + 2t + 6u = 1h + 2t + 6u = 126).

[19] *Ancient mathematics in secondary schools: issues, current practices, and perspectives*. Dissertation supervised by Christine Proust (SPHere. UMR 7219-Paris Diderot) and Nicolas Decamp (LDAR. Paris Diderot).

didactics of physics, where the effect of teaching sequences based on history – and occasionally of original sources – is studied in terms of conceptual change (Merle, 2002), (Décamp & Hosson, 2011).

REFERENCES

Abdeljaouad, M. (2005). *Les arithmétiques arabes (9e-15e siècles).* Tunis : Ibn Zeidoun.

Barbin (forthcoming). *Let History into the Mathematics Classroom*. New-York: Springer.

Brousseau, G. (2007). Le calcul « à la plume » des multiplications et des divisions élémentaires. Communication à l'ARDM 14-01-2007. Available from http://www.ardm.eu/contenu/guy-brousseau-le-calcul-%C2%AB-%C3%A0-la-plume-%C2%BB-des-multiplications-et-des-divisions-%C3%A9l%C3%A9mentaires (accessed 3/2016).

Bühler, M. (forthcoming). The congruence machine of the Carissan brothers. In: E. Barbin (Ed.) *Let History into the Mathematics Classroom*. New-York: Springer.

Chabert, J.-L. (Ed) (1999). *A history of algorithms: from the pebble to the microchip*. New-York: Springer.

Chemla, K., Chorlay, R., & Rabouin, D. (2016). *The owford handbook of generality in mathematics and the sciences.* Oxford: Oxford University Press.

Chesné, J.-F., & Le Yaouanq, M.-H. *et al.* (2011). *Horizon 4ème*. Paris : Didier.

Chevallard, Y. (2007). Readjusting didactics to a changing epistemology. *European Educational Research Journal*, *6/2,* 131-134.

Chorlay, R., & de Hosson, C. (2016). History of science, epistemology, and mathematics education research. In B. Hodgson, A. Kuzniak, & J.-B. Lagrange *The didactics of mathematics: Approaches and issues. A Homage to Michèle Artigue.* (pp.155-189). Springer International Publishing Switzerland.

Chorlay, R. (forthcoming). When Leibniz plays dice. In: E. Barbin (Ed.) *Let History into the Mathematics Classroom*. New-York: Springer.

Chorlay, R. Masselin, B., & Mailloux, F. (submitted for publication). Tâches spécifiquement algorithmiques en cycle 3: trios séances sur la multiplication *per gelosia*.

Decamp, N, & de Hosson, C. (2011). Quelques éléments historiques et didactiques sur l'expérience d'Eratosthène. *Bulletin de l'union des professeurs de physique et de chimie*, *105*, 1065-1082.

Djebbar, A. (2005). *L'algèbre arabe: genèse d'un art.* Paris: Adapt-Vuibert.

De Vittori, T. (2015). Les tâches des élèves dans une acticité mathématique à dimension historique. *Petit x, 97,* 5-25.

Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education, 10,* 391-408.

Guichard, J.-P. (forthcoming). Angles in secondary school: surveying and navigation. In: E. Barbin (Ed.) *Let History into the Mathematics Classroom*. New-York: Springer.

Guyot, P. (forthcoming). A square in a triangle. In: E. Barbin (Ed.) *Let History into the Mathematics Classroom*. New-York: Springer.

Hamon, G. (forthcoming). The probability of causes according to Condorcet. In: E. Barbin (Ed.) *Let History into the Mathematics Classroom*. New-York: Springer.

Kjeldsen, T. (2012). Uses of history for the learning of and about mathematics – Towards a general framework for integrating history of mathematics in mathematics education. *Proceedings of the 12th conference on History and Pedagogy of Mathematics*, *1,* 1-21.

Legrand, M. (1993). Débat scientifique en cours de mathématiques et spécificité de l'analyse. *Repères-IREM, 10,* 123-158.

Métin, F. (forthcoming). The arithmetic of Juan de Ortega: equations without algebra. In: E. Barbin (Ed.) *Let History into the Mathematics Classroom*. New-York: Springer.

Mora-Charles, M.S. (1992). Quelques jeux de hazard selon Leibniz. *Historia mathematica, 19*, 125-157.

Moyon M. (forthcoming). Dividing a triangle in the Middle Ages: An example from Latin works on practical geometry. In: E. Barbin (Ed.) *Let History into the Mathematics Classroom*. New-York: Springer.

Morice-Singh, C. (forthcoming). Indian calculation: The Rule of Three – quite a story … . In: E. Barbin (Ed.) *Let History into the Mathematics Classroom*. New-York: Springer.

Robert, A. (1998). Outils d'analyse des contenus mathématiques à enseigner au lycée et à l'université. *Recherches en didactique des mathématiques*, *18/2*, 139-190.

Siu, M,K. (2006). No, I don’t use history of mathematics in my class. Why? *Proceedings of HPM2004 and ESU4*. 268-277.

Tournès, D. (forthcoming #1). A graphic approach to Euler’s method. In: E. Barbin (Ed.) *Let History into the Mathematics Classroom*. New-York: Springer.

Tournès, D. (forthcoming #2). Calculating with hyperbolas and parabolas. In: E. Barbin (Ed.) *Let History into the Mathematics Classroom*. New-York: Springer.

LES MATHÉMATIQUES DANS L'ESPACE MÉDITERRANÉEN: L'EXEMPLE D'AL-ANDALUS ET DU MAGHREB

Ahmed DJEBBAR
Université Lille 1, Cité Scientifique, 59650 Villeneuve-d'Ascq Cedex, France
ahmed.djebbar@wanadoo.fr

RESUME

Dans l'Histoire des activités mathématiques autour de la Méditerranée, l'espace constitué par le Maghreb et l'Andalus a joué un triple rôle : celui d'un conservatoire d'une partie des savoirs mathématiques préislamiques (particulièrement ceux de la Grèce et de l'Inde), celui d'un lieu de production de savoirs mathématiques nouveaux mais aussi d'enseignement et de publications d'ouvrages ; celui, enfin, d'un intermédiaire important dans la diffusion partielle, vers l'Europe médiévale et l'Afrique subsaharienne, des corpus mathématiques grecs, indiens et arabes.

L'article présentera les éléments essentiels illustrant, à travers ce triple rôle, les activités mathématiques observées, entre le IX^e^ et le XIV^e^ siècle, dans les foyers scientifiques d'une partie de la rive sud de la Méditerranée, avec leurs prolongements vers la rive nord.

1 Introduction

Entre le IXe et le XVe siècle, et en relation étroite avec les foyers scientifiques de l'Orient musulman, deux régions méditerranéennes ont connu une activité mathématique relativement importante et multiforme. La plus ancienne est celle du Maghreb. Elle correspond, aujourd'hui, au nord de l'Afrique (sauf l'Egypte) avec ses extensions sahariennes. La seconde est celle d'al-Andalus, nom donné par les historiens arabes au territoire de la Péninsule ibérique qui a été gouvernée, entre 711 et 1492, par des pouvoirs musulmans[1].

Avant le VIIIe siècle, le patrimoine mathématique de ces deux régions était relativement modeste, comparé bien sûr à ce qui avait été produit au cours de la dernière phase de la tradition scientifique grecque, celle de la période hellénistique[2]. Au Maghreb, aucun témoignage ne permet d'affirmer qu'à l'arrivée des premiers conquérants musulmans (vers 670), il y avait une tradition mathématique écrite dans la région. Mais les données ethnomathématiques connues laissent à penser qu'il y avait un ensemble de savoir-faire intervenant dans les activités quotidiennes qui utilisaient des constructions géométriques (comme la décoration murale ou celle des poteries et des tapisseries), des numérations et des procédures de calcul (dans le cadre des transactions commerciales). Ces savoir-faire ne vont pas disparaître du jour au lendemain. Une partie sera intégrée au nouveau savoir venu

---

[1] Il n'y a pas unanimité sur l'origine du terme « *al-Andalus* ». L'hypothèse communément admise est celle qui le rattache aux Vandales (*Wandāl* en arabe), un peuple européen du Nord qui a migré vers la Péninsule ibérique autour de 409 avant de s'installer au Maghreb et d'y rester jusqu'en 533.

[2] Période que l'on situe, généralement, entre la mort d'Alexandre le Grand (en 323 av. J.C.) et celle de Cléopâtre (en 30 av. J.C.).

d'Orient et l'autre partie continuera à être pratiquée et transmise dans le cadre des activités économiques et artistiques locales.

En Andalus, la situation était différente. Avant la conquête musulmane, les élites de la région avaient accès à certains écrits mathématiques, comme l'*Institution arithmétique* de Boèce (m. 524)[3] et certains chapitres des *Etymologies* d'Isidore de Séville (m. 636)[4]. Mais le contenu de ces ouvrages était bien en deçà de la production grecque des siècles précédents. En tout les cas il n'a pas été à l'origine de la réactivation des activités mathématiques dans cette région au cours de la période wisigothique (508-710). Et, comme pour le Maghreb, il faudra attendre l'arrivée des premières traductions du grec, du sanskrit et du persan à l'arabe, toutes réalisées au Proche Orient, pour qu'une nouvelle tradition commence à émerger avec un premier corps d'enseignants et d'auteurs de manuels. Comme on le verra par la suite, cette tradition, qui s'est exprimée exclusivement en arabe[5], a connu une longue période de maturation avant de s'affirmer, comme un ensemble d'activités d'enseignement, de publication et de recherche. La production qui en a découlé a concerné, à la fois, les domaines mathématiques anciens, comme la géométrie, la théorie des nombres et le calcul (avec leurs aspects théoriques et appliqués) mais, également, les nouveaux chapitres apparus d'abord en Orient, comme l'algèbre et la trigonométrie.

2 Les premiers pas des mathématiques arabes au Maghreb et en Andalus

C'est à Kairouan, dans le Maghreb oriental, qu'apparaissent les premières activités mathématiques en arabe. Puis, ce sera le tour de Tahert, une ville concurrente sur le plan idéologique et politique et qui avait des relations privilégiées avec les premiers foyers scientifiques d'Orient[6]. A ses débuts, la production dans ce domaine était essentiellement utilitaire dans la mesure où elle devait répondre à des besoins précis : le calcul transactionnel et la répartition des héritages. Le premier manuel connu, publié dans cette région, est le « *Livre sur le calcul indien* » d'Abū Sahl al-Qayrawānī (IX^e s.)[7]. Il s'agit, comme le titre l'indique clairement, d'une reprise du système décimal positionnel indien avec les algorithmes arithmétiques qui l'accompagnent, c'est à dire la matière du livre d'al-Khwārizmī (m. 850) qui porte d'ailleurs le même titre et qui a été publié à Bagdad avant 833.

---

3 Boèce (1995). *Institutions arithmétiques*. J.-Y. Guillaumin (Ed. & Trad.). Paris: Les Belles Lettres.

4 Isidore de Séville (1983-2010). *Etymologiae*. Paris: Les Belles Lettres.

5 Nous entendons par « *mathématiques arabes* » toutes les activités de traduction, d'enseignement, de publication et de recherche qui ont eu lieu en pays d'Islam et qui ont été exprimées essentiellement en arabe entre la fin du VIII^e siècle et le milieu du XI^e par des scientifiques de langue, de culture et de confessions différentes. Après cette date, d'autres langues pratiquées dans l'empire musulman, comme le persan, l'hébreu, le turc et, dans une moindre mesure, le berbère, ont exprimé les sciences. Mais leur production s'est inscrite, sans aucune rupture, dans la tradition mathématique arabe.

6 Djebbar, A. (1988). Les activités mathématiques dans les villes du Maghreb Central (IX^e-XVI^e s.). In *Actes du 3^e Colloque maghrébin sur l'Histoire des mathématiques arabes* (pp. 73-115). Alger: Office des Presse Universitaires.

7 Djebbar, A. (1990). Quelques éléments nouveaux sur l'activité mathématique arabe dans le Maghreb oriental (IX^e -XVI^e s.). In *Actes du 2^e Colloque Maghrébin sur l'Histoire des Mathématiques Arabes* (pp. 53-60). Tunis: Publications de l'Université de Tunis.

Un des domaines d'application des mathématiques, dans le domaine savant, a été l'astronomie. Certains de ses chapitres ont concerné des pratiques cultuelles de la nouvelle religion : les prières quotidiennes, l'orientation des mosquées et la réalisation du calendrier lunaire. Il n'est donc pas étonnant que la ville de Kairouan ait été aussi le premier foyer scientifique ayant connu une production substantielle dans ce domaine. Et, jusqu'au X^{e} siècle, ses publications astronomiques, en particulier celles de Dūnash Ibn Tamīm (ca. 956), étaient une référence, pour les spécialistes de la région[8]. C'est d'ailleurs ce que confirme Ḥasdāy Ibn Shaprūt (m. vers 970), un célèbre médecin de Cordoue devenu un grand diplomate au service du calife omeyyade ^{c}Abd al-Raḥmān III (912-961) qui écrivait à ses correspondants à Kairouan pour leur demander de lui procurer des ouvrages produits par des scientifiques de la ville[9].

En Andalus, il faut attendre le début de ce qui est appelé, communément, la « *période des émirats* » (756-900) pour observer les premières signes d'une activité scientifique, à travers les noms de quelques pionniers, comme Ibn Fuṭays et Ibn Nāṣiḥ pour les mathématiques, Yaḥyā Ibn ^{c}Ajlān et Ḥabāb al-Faraḍī pour la science des héritages, enfin Ibn Shamīr à qui on attribue la première table astronomique et Ibn Ḥabīb, auteur d'une « *Epître sur la connaissance des étoiles* »[10].

3 La période des califats d'Occident (900-1008)

Au Maghreb, cette période correspond à la phase fatimide dont l'idéologie et la politique était en totale rupture avec celles des autres pouvoirs musulmans et, en particulier, celle des Omeyyades d'al-Andalus qui ont, eux aussi et à la même époque, revendiqué le titre de calife. Mais, sur le plan scientifique, cet antagonisme s'est transformé en une émulation féconde qui a assuré la continuité de la dynamique déjà amorcée et qui a permis une avancée significative dans les deux régions.

Même si aucun écrit mathématique produit au Maghreb au cours de cette période ne nous est parvenu, nous savons, par les sources historiques que les califes de cette dynastie ont encouragé les activités scientifiques et l'ont même parfois financée, comme l'a fait, par exemple, le calife al-Mu^{c}izz (953-975), lui-même un passionné de sciences et plus particulièrement d'astronomie. Pour la phase maghrébine du califat fatimide (909-969), les bibliographes nous fournissent quelques noms de personnes qui ont pratiqué l'astronomie, la science du calcul ou la géométrie, comme al-^{c}Utaqī (m. 955), Ibn Killīs (m. 990) et al-Huwarī (m. 1023), mais ils ne nous renseignent pas sur le contenu de leur production dans ces différents domaines. On est par contre mieux informé sur les résultats de cette politique de

---

[8] Sezgin, F. (1979). *Geschichte des Arabischen Schrifttums*, *Band VII*. Leiden: Brill.

[9] Urvoy, D. (1990). *Pensers d'al-Andalus*. Paris: Editions du CNRS-Toulouse, Presses Universitaires du Mirail.

[10] Balty-Guesdon, M.-G. (1992). *Médecins et hommes de sciences en Espagne musulmane (II^{e}/VIII^{e}-V^{e}/XI^{e}s.)*. Thèse de Doctorat (Vol. III, 599-600). Paris: Université de la Sorbonne Nouvelle – Paris III.

mécénat en Egypte lorsque ces mêmes Fatimides s'y installeront à partir de 969 et jusqu'à 1171[11].

En Andalus, la période allant du dernier tiers du IX^{e} siècle à la fin du X^{e}, est considérée comme celle du véritable « *décollage* » scientifique. Ce phénomène a d'abord bénéficié des initiatives prises au cours du siècle précédent au niveau de l'enseignement et, surtout, au niveau du transfert vers l'Andalus des savoirs produits en Orient, en particulier de l'achat de dizaines d'ouvrages mathématiques traduits du grec à l'arabe ou produits à Bagdad à l'époque d'al-Ma'mūn (813-833) et de ses successeurs du IX^{e} siècle. Parmi les mécènes qui ont financé cette opération de transfert dans le but d'enrichir leurs propres bibliothèques, il y eut deux califes importants, ^{c}Abd ar-Raḥmān III que nous avons déjà évoqué et son fils al-Ḥakam II (961-976).

C'est au cours de cette période que l'on observe une augmentation nette du nombre de praticiens des mathématiques, en particulier dans les domaines du calcul, de la géométrie et des chapitres des héritages qui utilisent l'arithmétique des fractions. C'est également à cette période que sont rattachés les premier professeurs de haut niveau, comme al-Magrīṭī (m. 1007) qui s'est occupé essentiellement d'astronomie mathématique, az-Zahrāwī (m. 1009) à qui est attribué un « *Livre des principes des transactions par la méthode de la preuve* », Aḥmad Ibn Naṣr, auteur d'un « *Livre sur le mesurage* » et, surtout, Ibn as-Samḥ (m. 1035), un auteur prolifique dont deux ouvrages ont retenu l'attention des mathématiciens postérieurs: le « *Livre sur les fruits des nombres* » et, surtout, son « *Grand livre de géométrie* »[12]. Malheureusement, en dehors de ces titres, et à l'exception d'un fragment du dernier ouvrage cité, rien ne nous est parvenu de leur production qui a pourtant circulé dans la région et même au Maghreb jusqu'à la fin du XII^{e} siècle. Pour prendre l'exemple de l'algèbre, nous sommes sûr aujourd'hui que des copies du « *Livre abrégé sur le calcul par la restauration et la comparaison* » d'al-Khwārizmī et du « *Livre complet en algèbre* » d'Abū Kāmil (m. 930) ont circulé en Andalus relativement tôt, soit au cours de cette période soit quelques décennies plus tard[13]. On sait aussi que des éléments importants de l'algèbre, telle qu'elle était pratiquée en Orient avant l'avènement de l'Islam, se trouvent dans « *L'épître sur le mesurage* » d'Ibn ^{c}Abdūn (m. après 976), un mathématicien de Cordoue qui fut également médecin[14].

La science du calcul était pratiquée plus que d'autres disciplines à cause des besoins exprimés dans le domaine des transactions et de la répartition des héritages. C'est ce que confirme le nombre d'enseignants ou d'auteurs considérés par les bibliographes comme étant

---

[11] Djebbar, A. (1990). *Quelques éléments nouveaux sur l'activité mathématique arabe dans le Maghreb oriental*, op. cit., 61-63.

[12] Rosenfeld, B. A., & Ihsanoglu, E. (2003). *Mathematicians, astronomers and other scholars of Islamic civilization and their works (7th-19th c.)* (p. 106, p. 117, p. 121). Istanbul: IRCICA.

[13] Djebbar, A. (2002). La circulation des mathématiques entre l'Orient et l'Occident musulmans : Interrogations anciennes et éléments nouveaux. In *Actes du Colloque International "From China to Paris : 2000 Years Transmission of Mathematical Ideas"* (pp. 213-236). Stuttgart: Steiner Verlag.

[14] Djebbar, A. (2005). L'épître sur le mesurage d'Ibn ^{c}Abdūn, un témoin des pratiques antérieures à la tradition algébrique arabe. *Suhayl* 5, partie arabe, 7-68.

« *versés* » ou « *connaisseurs* » ou même « *savants* » dans ces deux matières[15]. Mais, jusqu'à maintenant, nous n'avons pas d'informations précises sur le contenu des écrits qu'ils auraient consacrés aux différents procédés de calcul en usage à cette époque (indien, digital ou alphabétique) et à leurs différents domaines d'application.

On est relativement mieux informé sur la géométrie et sur certains de ses chapitres classiques. Une grande partie des ouvrages grecs, qui avaient été traduits à Bagdad à partir de la fin du VIIIe siècle, est disponible à Cordoue au plus tard dans la seconde moitié du Xe siècle. C'est le cas, en particulier, des « *Eléments* » d'Euclide (IIIe s. av. J.C.), des « *Coniques* » d'Apollonius (IIIe s. av. J.C.) et de « *L'épître sur la mesure du cercle* » d'Archimède (m. 212 av. J.C.)[16]. A ce corpus de base, il faut ajouter les premiers écrits arabes produits aux IXe-Xe siècles, dans le prolongement de la tradition grecque, comme ceux des frères Banū Mūsā, de Thābit Ibn Qurra (m. 901) et d'an-Nayrīzī[17].

4 La période des principautés (XIe s.)

Au Maghreb, l'activité mathématique et astronomique de cette période reste encore cantonnée, en grande partie, en Ifriqya. En plus de quelques auteurs dont les sources connues n'ont retenu que les noms[18], deux scientifiques émergent du lot. Le premier est Ibn Abī ar-Rijāl (m. 1034) qui a publié des ouvrages en géométrie et en astronomie qui ne nous sont pas parvenus. Mais il a acquis une certaine renommée, en particulier en Europe, pour son grand traité d'astrologie, intitulé le « *Livre brillant sur les jugements des étoiles* »[19]. Le profil du second mathématicien est représentatif d'une communauté de scientifiques dont le nombre ne va cesser d'augmenter à partir de la fin du XIe siècle, à la suite des évènements politiques qui ont perturbé les équilibres régionaux. Il s'agit d'Ibn Abī aṣ-Ṣalt (m. 1134). Originaire de Dénia, en Andalus, où il a acquis sa première formation, ce mathématicien-astronome, doublé d'un homme de culture, quitte sa ville natale vers l'âge de vingt ans pour l'Egypte où il séjourne une vingtaine d'année et où il fait de la prison sur ordre du pouvoir local. Il passe les vingt dernières années de sa vie au Maghreb, dans la ville de Mahdia[20].

En Andalus, malgré l'instabilité politique qui a caractérisé cette période, suite à l'éclatement du califat et à l'émergence d'une vingtaine de petites principautés rivales, la dynamique scientifique ne s'est pas ralentie et, pour certains domaines, comme les mathématiques, elle a même connu un développement encore plus important, avec l'émergence de scientifiques de haut niveau et la publication de travaux originaux. Parmi les nombreux savants de cette époque, rares sont ceux dont des écrits nous sont parvenus. Ceux

---

15 Balty-Guesdon, M.-G. (1992). *Médecins et hommes de sciences…*, op. cit., 614-632, 637, 641.

16 Steinschneider, M. (1956). Die Europäischen Übersetzungen aus dem Arabischen bis Mitte des 17 Jahrhunderts. *Graz: Akademische Druck-U. Verlagsanstalt, 21*, 24-26.

17 Lévy, T. (1995). Fragment d'Ibn as-Samḥ sur le cylindre et sur ses sections planes conservé dans une version hébraïque. In R. Rashed (Ed.), *Les mathématiques infinitésimales du IXe au XIe siècle* (vol. 1, pp. 929-973). Londres: Al-Furqān.

18 Comme ceux d'al-Kindī et d'al-Kalaᶜī qui étaient des spécialistes en géométrie.

19 Sezgin, F. (1979). Geschichte des Arabischen Schrifttums*, Band VII.,* op. cit., 186-188.

20 Suter, H. (1900). *Die Mathematiker und Astronomen der Araber und ihre Werke*. Leipzig: Teubner.

que nous allons présenter sont représentatifs du dynamisme de leur communauté, à la fois d'après le témoignage du grand biobibliographe Ṣā^{c}id al-Andalusī (m. vers 1078) qui était leur contemporain [21] et, surtout, d'après les informations précises concernant leurs contributions scientifiques.

Le premier d'entre eux est Ibn Mu^{c}ādh (m. vers 1079) qui était originaire de la ville de Jaén. Il fait partie de ces scientifiques de l'Occident musulmans qui ne se sont pas contentés des ouvrages venus d'Orient, pour compléter leur formation ou leur information, mais qui se sont déplacés et qui ont fréquenté certains de leurs collègues des cités du Proche Orient. En plus de sa formation d'astronome, Ibn Mu'adh était très versé en mathématique. En géométrie, il s'est intéressé à la théorie des rapports du Livre V des « *Eléments* » d'Euclide à laquelle il a consacré une épître qui nous est parvenue[22]. Mais son ouvrage le plus important est le « *Livre sur les arcs inconnus de la sphère* » qui est la première publication occidentale connue consacrée essentiellement aux objets et aux outils de la trigonométrie[23]. Cette contribution s'inscrit dans le prolongement des travaux des astronomes d'Asie centrale, en particulier ceux d'al-Bīrūnī (m. vers 1050) sans être une simple copie de leurs contenus. On y trouve en effet, comme éléments nouveaux, la définition de la tangente comme rapport du sinus et du cosinus, la résolution des triangles sphériques, sans le passage par les triangles rectangles, et la discussion des cas impossibles[24].

Le premier savant est al-Mu'taman (m. 1085). Avant de régner sur le petit royaume de Saragosse, il a eu une excellente formation en mathématique, en logique et en philosophie. Il a en outre bénéficié de la richesse de la bibliothèque qu'avait constituée son père al-Muqtadir (1049-1081), pendant son long règne. En attendant d'accéder au trône, al-Mu'taman s'est engagé dans un projet ambitieux, celui de réaliser une sorte de synthèse de ce qu'il a jugé comme indispensable dans les connaissances mathématiques grecques et arabes utilisées à son époque. Il avait conçu son ouvrage, intitulé « *Le livre du perfectionnement* », en deux volumes, mais seul le premier a pu être reconstitué. Il est divisé en cinq grands chapitres regroupant plus de 400 propositions et traitant des thèmes les plus importants du corpus mathématique grec et de certains problèmes résolus en Orient entre le IX^{e} siècle et le début du XI^{e}.

Pour la tradition grecque, il y a d'abord les « *Eléments* » d'Euclide (théorie des nombres des Livres VII, VIII et IX, théorie des rapports du Livre V, grandeurs incommensurables du Livre X, géométrie des figures planes constructibles exposée dans les Livres restants). A un niveau supérieur, il y a les « *Coniques* » d'Apollonius et la « *Sphère et le cylindre* » d'Archimède. Il y a enfin, à l'usage des astronomes, les *Sphériques* de Théodose (m. vers

---

[21] Ṣā^{c}id al-Andalusī (1985). Livre des catégories des nations. In H. Bu^{c}alwān (Ed.), Beyrouth: Dār aṭ-ṭalī^{c}a, pp. 180-181.

[22] Plooij, E. B. (1950). *Euclid's conception of ratio*. Leiden: Rijksuniversiteit.

[23] Villuendas, M. V. (1980). *La trigonometria europea en el siglo XI, Estudio de la obra de Ibn Mu^{c}ādh El Kitāb Mayhūlāt*. Barcelone: Instituto de Historia de la Ciencia de la Real Academia de Buenas Letras.

[24] Samso, J. (1980). Notas sobre la trigonometria esferica de ibn Mu^{c}ādh. *Awrāq, 3*, 60-67.

100), les « *Sphériques* » de Ménélaüs (m. vers 140) et certaines propositions de l'« *Almageste* » de Ptolémée (II^{e} s.).

De la tradition arabe, al-Mu'taman a retenu les contributions suivantes : « *L'épître sur les nombres amiables* » de Thābit Ibn Qurra[25], le calcul du volume de la sphère par les frères Banū Mūsā et la détermination de l'aire d'une portion de parabole par la méthode d'Ibrāhīm Ibn Sinān (m. 946). A ces thèmes purement mathématiques, il faut ajouter la résolution, longue et difficile, du célèbre problème tiré du « *Livre de l'optique* » d'Ibn al-Haytham (m. après 1040), celui de la détermination du ou des points de réflexion, sur un miroir plan, sphérique, conique ou cylindrique (convexe ou concave), d'un rayon lumineux partant d'une source donnée et aboutissant à un point également donné, extérieur au miroir[26].

L'analyse du contenu de ce premier volume révèle une démarche constante : celle qui a consisté à modifier la structure interne des différents chapitres empruntés à la tradition grecque en introduisant un nouvel agencement des propositions anciennes avec, parfois, de nouvelles preuves, et en ajoutant des propositions établies par ses prédécesseurs de la tradition arabe mais formulées ou démontrées différemment[27]. Il ne semble pas qu'al-Mu'taman ait eu le temps de finaliser et de publier le second volume de son ouvrage. Mais la découverte de sa table des matières nous permet d'en savoir plus sur le projet global. Il s'agissait de présenter, dans un même moule, les aspects théoriques et appliqués des sciences mathématiques. Ainsi, dans les chapitres de la seconde partie, l'auteur avait prévu de montrer comment intervenaient les résultats et les outils théoriques en astronomie, en optique, en mécanique et en musique[28].

Ibn Sayyid est le troisième grand mathématicien de ce XI^{e} siècle, peut-être le plus novateur de sa génération. Mais, son projet le plus important est également resté inachevé. Et c'est son étudiant, le célèbre philosophe Ibn Bājja (m. 1138) qui nous en parle et qui évoque ses contributions originales[29]. Un premier projet a consisté en une nouvelle rédaction des « *Coniques* » d'Apollonius dans laquelle Ibn Sayyid aurait introduit des notions et des définitions nouvelles qui lui ont permis de réduire le nombre de proposition de l'ouvrage et de modifier leur agencement. Le second projet, plus ambitieux, a concerné la définition et l'étude des propriétés de nouvelles courbes planes qui feront partie plus tard, en Europe, des courbes de degré supérieur à 2. Son étudiant nous dit que la première catégorie de ces courbes a été obtenue par projection de certaines courbes gauches résultant de l'intersection de solides dont

---

[25] Deux nombres *a* et *b* sont dits « *amiables* » si la somme des diviseurs de l'un est égale à l'autre. Exemple : *a* = 220 et *b* = 284. Djebbar, A. (1999). Les livres arithmétiques des *Eléments* d'Euclide dans une rédaction du XI^{e} siècle : le *Kitāb al-istikmāl* d'al-Mu'taman (m. 1085). *Lull, 22*(45), 589-653.

[26] Hogendijk, J. P. (1991). The geometrical part of the *Istikmāl* of Yūsuf al-Mu'taman ibn Hūd (11th century), An analytical table of contents. *Archives Internationales d'Histoire des sciences*, *127*(41), 207-281.

[27] Guergour, Y. (2006). *La géométrie euclidienne en Occident musulman à travers le Kitāb al-istikmāl d'al-Mu'taman (XI^{e} s.)*. Thèse de Doctorat. Annaba : Faculté de mathématiques.

[28] Djebbar, A. (1997). La rédaction de l'Istikmāl d'al-Mu'taman (XI^{e} s.) par Ibn Sartāq un mathématicien des XIII^{e}-XIV^{e} siècles. *Historia Mathematica, 24*, 185-192.

[29] Djebbar, A. (1993). *Deux mathématiciens peu connus de l'Espagne du XI^{e} siècle : Al-Mu'taman et Ibn Sayyid*. In M. Folkerts & J.-P. Hogendijk (Eds.), *Vestigia Mathematica, Studies in medieval and early modern mathematics in honour of H.L.L. Busard* (pp. 79-91). Amsterdam-Atlanta: GA.

les bases sont des sections coniques. Puis, à partir des courbes planes obtenues, et par itération du procédé, il a obtenu de nouvelles courbes de degré supérieur. Puis, grâce à ces courbes, Ibn Sayyid aurait établi des résultats nouveaux concernant la multisection d'un angle et la détermination d'un nombre quelconque de moyennes entre deux grandeurs données. Il faut préciser que ces derniers résultats généralisaient à la fois ceux de la tradition grecque et ceux des mathématiciens de l'Orient musulman[30].

5 La période impériale (XII^e-XIII^e s.)

Au cours de cette période, l'Occident musulman a été, pour la première fois de son histoire, « unifié » sous la bannière d'un même pouvoir, celui des Almoravides (1040-1147) puis celui des Almohades (1147-1269). Cette unification a renforcé les relations entre les deux régions de l'empire mais, compte tenu du contexte régional, c'est le Maghreb qui en a le plus profité, en particulier dans le domaine scientifique. En mathématique, la recherche est restée vivante dans certains secteurs et les aspects théoriques de la discipline continuent à être cultivés mais avec moins de vigueur que pendant la période précédente. Ce sont surtout les domaines appliqués qui sont les plus représentés dans les informations données par les bibliographes. Quant au contenu des mathématiques pratiquées, nous en avons une connaissance partielle à cause du nombre réduit d'ouvrages qui nous sont parvenus. Mais, comme leurs auteurs étaient importants à leur époque, on peut considérer que leurs contenus reflètent l'état des disciplines concernées. Les quatre auteurs dont nous allons présenter la production qui leur est attribuée illustrent parfaitement la situation nouvelle que connaît la communauté des scientifiques dans la mesure où ils ont tous eu, à un moment ou un autre de leur vie professionnelle, des activités en Andalus et au Maghreb.

Abū l-Qāsim al-Qurashī (m. 1184) est peut-être le plus ancien. Il est originaire de Séville où il a acquis toute sa formation mais il a longuement séjourné à Bejaïa, dans le Maghreb central et il y a enseigné l'algèbre et la science des héritages[31]. Dans la première discipline il a publié un commentaire au « *Livre complet en algèbre* » d'Abū Kāmil qui était considéré par le grand historiens Ibn Khaldūn (m. 1406), comme l'un des meilleurs ayant été produit dans ce domaine[32]. D'après les informations fournies par des mathématiciens du XIV^e siècle, l'ouvrage contient quelques aspects nouveaux, dans l'agencement de l'exposé général, dans la classification des six équations canoniques et dans les justifications de l'existence des solutions des équations du second degré[33]. Cet auteur était également apprécié pour sa méthode de résolution des problèmes d'héritage, basée sur la décomposition des nombres en facteurs premiers. Malgré la grande réticence des juristes (dont les compétences

---

[30] Djebbar, A. (1998). Abū Bakr Ibn Bājja et les Mathématiques de son temps. In *Etudes Philosophiques et Sociologiques dédiées à Jamal ad-Dīn Alaoui* (n° spécial 14, pp. 5-26). Fez: Publications de l'Université de Fez.

[31] Zerrouki, M. (1995). Abū l-Qāsim al- Qurashī : Sa vie et ses écrits mathématiques. *Cahier du Séminaire Ibn al-Haytham*, *5*, 10-19.

[32] Ibn Khaldūn (2002). *Le Livre des exemples*. A. Cheddadi, (Trad.). Paris: Gallimard, p. 951.

[33] Djebbar, A. (1980). *Enseignement et Recherche mathématiques dans le Maghreb des XIII^e-XIV^e siècles*. Paris: Publications Mathématiques d'Orsay, *81*(2), 8-10 ; Djebbar, A. (2005). *L'algèbre arabe, genèse d'un art*. Paris: Adapt-Vuibert.

mathématiques étaient souvent très limitées), cette méthode a eu beaucoup de succès grâce au soutien des mathématiciens, comme al-ᶜUqbānī (m. 1408) et al-Qalaṣādī (m. 1486), qui l'ont enseignée et qui lui ont consacré des commentaires[34].

Abū Bakr al-Ḥaṣṣār (XIIe s.) est le deuxième spécialiste dont il nous est parvenu une partie des écrits mathématiques. Comme son contemporain al-Qurashī, il est originaire de Séville qu'il aurait quittée pour s'installer à Marrakech sous le règne d'Abd al-Mu'min, le deuxième calife almohade (1147-1163)[35]. Il est connu comme auteur de deux écrits sur la science du calcul. Le « *Livre de la démonstration et du rappel* » un manuel qui a eu beaucoup de succès auprès des enseignants du Maghreb tout au long des XIIIe-XVe siècles. Il y traite de la numération indienne, des opérations arithmétiques sur les entiers et les fractions, du calcul exact ou approché de la racine carré d'un nombre entier ou fractionnaire et de la détermination des sommes de certaines suites d'entiers. Au delà de son contenu classique, cet ouvrage a une double importance historique : d'abord parce qu'il est, pour le moment, le plus ancien manuel qui nous soit parvenu et qui nous renseigne sur le contenu de l'enseignement mathématique de base au Maghreb. Ensuite, il est le plus ancien témoin de l'utilisation d'un symbolisme arithmétique permettant d'exprimer toutes les formes de fractions utilisées dans l'enseignement du calcul et dans ses applications à la science des héritages[36]. Son second ouvrage est le « *Livre complet sur l'art du nombre* » dont seul le premier volume et la table des matières du second ont été retrouvés dans une copie anonyme qui a pu être identifiée grâce à des références à son contenu découvertes dans des ouvrages postérieurs. L'auteur y développe les chapitres déjà traités dans son manuel et ajoute d'autres thèmes, comme la décomposition d'un nombre en facteurs premiers, la recherche des diviseurs communs, l'extraction de la racine cubique exacte d'un nombre entier et le calcul des nombres amiables[37].

Le troisième mathématicien est d'origine maghrébine mais il a également séjourné un certain temps à Séville. Il s'agit d'Ibn al-Yāsamīn (m. 1204) qui signifie « *fils de fleur de jasmin* », en référence à sa mère qui était noire et à qui on a probablement attribué ce prénom par dérision. Quant à son père il était originaire d'une tribu berbère du Maghreb extrême[38]. Il s'est rendu célèbre grâce à ses poèmes mathématiques et plus particulièrement à celui qu'il a consacré à l'algèbre et qui est une sorte de « fiche technique » versifiée, facile à mémoriser

[34] Harbili, A. (1996). L'enseignement des mathématiques à Tlemcen au XIVe siècle à travers le commentaire d'al-ᶜUqbānī au Talkhīs. *Cahier du Séminaire Ibn al-Haytham*, Alger: E.N.S., *7*, 6-22 ; Laabid, E. (2011). Ibn Ṣafwān al-Mālaqī (m. 1362) et sa contribution dans la tradition mathématique des héritages. In *Actes du 10e Colloque maghrébin sur l'Histoire des mathématiques arabes* (pp. 198-210). Tunis: Publications de l'A.T.S.M.

[35] Lamrabet, D. (2014). *Introduction à l'Histoire des mathématiques maghrébines*. London: Lulu.

[36] Suter, H. (1901). Das Rechenbuch des Abû Zakarîyâ el-Ḥaṣṣâr. *Bibliotheca Mathematica. Série 3, 2*, 12-40.

[37] Aballagh, M., & Djebbar, A. (1987). Découverte d'un écrit mathématique d'al-al-Ḥaṣṣār (XIIe s.) : Le Livre I du Kāmil. *Historia Mathematica, 14*, 147-158.

[38] Ibn Saᶜīd (1945). *Les branches mûres sur les mérites des poètes du septième siècle*. I. Al-Ibyarī (Ed.). Le Caire: Dār al-maᶜārif.

par l'enseignant[39]. Son ouvrage le plus important, écrit en prose et intitulé le « *Livre sur la fécondation des esprits avec les symboles des chiffres de poussière* ». Il y traite de la science du calcul, de l'algèbre et de la géométrie du mesurage, en combinant des éléments et des démarches caractéristiques aux trois traditions arabes de l'époque : celles d'Orient, d'al-Andalus et du Maghreb[40]. Mais, malgré sa richesse, cet ouvrage n'a pas eu le même succès que le poème algébrique puisqu'il a été peu copié. De plus, à notre connaissance, il n'a bénéficié d'aucun commentaire alors que, dans le même temps, le poème a été enseigné pendant plusieurs siècles et qu'il a été commenté par une quinzaine de mathématiciens du Maghreb et de l'Orient musulman[41].

Comparé à ses trois prédécesseurs qui viennent d'être présentés, le quatrième scientifique de cette période est incontestablement le plus important pour le nombre de ses écrits et, surtout, pour sa contribution originale dans le développement d'un chapitre nouveau en mathématique. Il s'agit d'Ibn Mun^{c}im (m. 1228), un andalou de Dénia qui a été attiré, relativement jeune, par Marrakech, la capitale de l'empire almohade, et qui s'y est installé définitivement. Il y a mené plusieurs activités en parallèle puisque son biographe dit de lui qu'il a pratiqué la médecine, enseigné les mathématiques, publié des ouvrages et fait des recherches originales. Les ouvrages qui lui sont attribués portaient sur la géométrie, la science du calcul et la théorie des nombres[42]. Mais un seul nous est parvenu, le « *Livre de la science du calcul* », dans lequel on trouve des chapitres classiques sur le « *calcul indien* » et d'autres plus originaux comme celui sur les « *nombres figurés* », celui sur les « *nombres amiables* » et, surtout, celui consacré entièrement à un problème combinatoire et qui est intitulé « *Chapitre sur le dénombrement des mots qui sont tels que l'être humain ne peut s'exprimer que par l'un d'eux* ». Il s'agit là d'investigations nouvelles à partir d'un problème ancien posé par les linguistes arabes du VIII^{e} siècle, celui du dénombrement de tous les mots d'une langue en vue de réaliser son lexique. Parmi les résultats obtenus par Ibn Mun^{c}im, soit à l'aide de la preuve par induction (récurrence faible) soit par la technique des tableaux, il y a le dénombrement des permutations des lettres d'un mot, avec ou sans répétition, celui des combinaisons des lettres d'un alphabet, avec permutation et répétition, et celui de toutes les lectures possibles d'un mot, compte tenu des voyelles qu'on peut associer à chaque phonème[43].

6 La periode des quatre royaumes (xiv^{e}-xv^{e} s.)

---

39 Shawqi, J. (1987). *Les poèmes d'Ibn al-Yāsamīn sur les procédés de l'algèbre et du calcul*. Koweit: Mu'assasat al-Kuwayt li t-taqaddum al-^{c}ilmī.

40 Zemouli, T. (1993). *Les écrits mathématiques d'Ibn al-Yāsamīn (m. 1204)*, *Magister d'Histoire des Mathématiques*. Alger: E.N.S.

41 Op. cit., 322-334 ; Abdeljaouad, M. (Ed.) (2003). *Ibn al-Ha'im, Commentaire du poème d'Ibn al-Yasamin*. Tunis: Publications de l'A.T.S.M. ; Lamrabet, D. (2014). *Introduction à l'Histoire des mathématiques maghrébines*, op. cit., pp. 148-152.

42 Ibn ^{c}Abd al-Malik (1973). *L'appendice et le complément aux deux livres al-Mawṣūl wa ṣ-Ṣila*. In I. ^{c}Abbās, & M. Benshrifa (Eds.) (vol. VI, pp. 59-60). Beyrouth: Dār ath-thaqāfa.

43 Djebbar, A. (1985). *L'analyse combinatoire au Maghreb : L'exemple d'Ibn Mun^{c}im (XII^{e}-XIII^{e} siècles)*. Paris: Publications Mathématiques d'Orsay, n° 85-01.

Au cours de cette période, on observe un ralentissement sérieux des activités scientifiques en Andalus avec un dynamisme plus soutenu au Maghreb même si l'analyse du contenu des ouvrages produits durant ces deux siècles révèlent un arrêt de la recherche, dans tous les foyers scientifiques connus, et un rétrécissement général des programmes d'enseignement au détriment des thèmes théoriques[44]. La région d'al-Andalus encore sous le contrôle des musulmans est désormais réduite au petit royaume de Grenade dont dépendaient aussi Malaga, Almeria et Guadix. Dans ces quatre cités s'est maintenue une activité mathématique et astronomique, d'enseignement et de publication, tournée vers les domaines appliqués. C'est ce que montrent clairement les ouvrages qui nous sont parvenus et qui ont été analysés, comme les manuels de calcul d'Ibn Zakariyā (m. 1404) et d'al-Qalaṣādī[45] ou le poème sur le mesurage d'Ibn Luyūn (m. 1346)[46]. Le contenu de cette dernière publication est encore plus révélateur du phénomène de réduction de la matière d'enseignement et de son niveau. En effet, ce qui est exposé (qui ne concerne pourtant que des problèmes de mesurage et d'arpentage liés à la vie de tous les jours) est très en deçà de ce qu'on découvre dans des manuels du XIII^e^ siècle récemment découverts et analysés, comme le « *Livre qui facilite le mesurage et le découpage* » d'al-Mursī[47] et le « *Livre qui vulgarise et facilite pour faire profiter le débutant dans l'art du mesurage* » d'Ibn al-Jayyāb[48].

Au Maghreb, la production mathématique est quantitativement importante. Mais son contenu est, essentiellement, une reprise sous différentes formes (commentaires, résumés, poèmes), d'une partie de la production des siècles précédents. Il y a bien encore quelques apports nouveaux mais ils sont rares. On les trouve chez Ibn al-Bannā (m. 1321), un mathématicien de Marrakech qui a publié des ouvrages sur la science du calcul, la géométrie du mesurage et l'algèbre. Il a également eu une contribution originale en analyse combinatoire qui s'inscrit dans le prolongement des travaux d'Ibn Munᶜim que nous avons déjà évoqués. Il est en effet le premier à avoir établi la formule donnant les combinaisons de n objets p à p, évitant ainsi le recours à la technique des tableaux qu'avait abondamment utilisé son prédécesseur[49]. Il a également résolu certains problèmes en faisant intervenir des démarches purement combinatoires[50]. En algèbre, il a proposé, pour la résolution des

---

44 Djebbar, A. (1980). *Enseignement et recherche mathématiques dans le Maghreb des XIIIᵉ-XIVᵉ siècles*, op. cit., 1-5.

45 Lamrabet, D. (2014). *Introduction à l'Histoire des mathématiques maghrébines*, op. cit., 108-109, 208-211.

46 Moyon, M. (2016). Ibn Luyūn at-Tujībī (1282-1349) : Un témoin de la science du mesurage en Occident musulman. In A. Bouzari (Ed.), *Actes du XIᵉ Colloque maghrébin sur l'Histoire des mathématiques* (pp. 333-352). Alger: Publications du L.E.H.M.

47 Djebbar, A. (2007). La géométrie du mesurage et du découpage dans les mathématiques d'Al-Andalus (Xᵉ-XIIIᵉ s.). In P. Radelet de Grave (Ed.), *Liber Amicorum Jean Dhombres* (pp. 113-147). Turnhout: Editions Brepols.

48 Djebbar, A. (2016). Les techniques de découpage dans un ouvrage géométrique d'al-Andalus. In *Actes du XIᵉ Colloque maghrébin sur l'Histoire des mathématiques arabes* (pp. 109-138). Alger: Dar al-Khalduniya.

49 Djebbar, A. (1980). *Enseignement et recherche mathématiques dans le Maghreb des XIIIᵉ-XIVᵉ siècles*, op. cit., 76-98.

50 Djebbar, A. (2003). Mathématiques et société à travers un écrit maghrébin du XIVᵉ siècle. In *Actes du colloque international « De la Chine à l'Occitanie, chemins entre arithmétique et algèbre »* (pp. 29-54).

équations de second degré des preuves pour l'existence des solutions positives qui ne font intervenir que des identités remarquables, c'est à dire sans recours explicite au discours et aux outils de la géométrie[51].

Mais ce ne sont pas ses contributions originales que retiendront ses successeurs. C'est un petit manuel, sans aucune nouveauté dans son contenu, qui le fera connaître un peu partout et pendant plusieurs siècles. Il s'agit de « *L'abrégé sur les opérations du calcul* » qui expose, d'une manière très concise, et sans aucune justification ni aucun exemple, les éléments de base du calcul indien et les algorithmes arithmétiques (méthode de fausse position, règle de trois) et algébriques permettant de résoudre les problèmes qui aboutissent à des équations du premier ou du second degré. Ce manuel a bénéficié d'une quinzaine de commentaires d'auteurs originaires d'al-Andalus[52], du Maghreb[53] et même de l'Orient musulman[54]. La seule particularité à signaler, à propos de la plupart des commentaires réalisés en Occident musulman, c'est la présence d'un symbolisme algébrique tout à fait performant dont on ne connaît pas l'initiateur.

L'analyse comparative des contenus de ces différents commentaires, qui ont été produits entre la fin du XIII[e] siècle et le milieu du XV[e], confirme la tendance à la réduction de la matière mathématique enseignée : des thèmes, comme l'extraction de la racine cubique approchée d'un nombre ou le calcul de nouveaux couples de nombres amiables, qui étaient présent dans les ouvrages des X[e]-XIII[e] siècles, ont tout simplement disparu. A cela, il faut ajouter l'absence de nouveauté dans les domaines qui ont continué à être enseignés ou qui étaient la matière des ouvrages publiés.

7 Le rôle d'al-Andalus et du Maghreb dans la circulation des mathematiques arabes en Europe

Pour conclure cette brève présentation des sept siècles d'activités mathématiques en Occident musulman, il nous reste à évoquer le rôle joué par l'Andalus et le Maghreb dans le phénomène de diffusion des savoirs et des savoir-faire mathématiques de la rive sud de la Méditerranée vers l'Europe. Très discret à ses débuts, ce phénomène ne cessera de se développer au moment même où l'activité mathématique en pays d'Islam entrait dans une phase de ralentissement. A partir de la fin du X[e] siècle, des initiatives multiformes vont permettre la diffusion d'une partie de la production mathématique grecque, indienne et arabe

Toulouse: Editions du C.I.H.S.O.; Taïbi, N. (2015). *Mathématiques et société au Maghreb : L'exemple du Tanbīh al-albāb d'Ibn al-Bannā*. Mémoire de Magister: Alger, E.N.S.

51 Aballagh, M. (1988). *Le Raf' al-ḥijāb d'Ibn al-Bannā*. Thèse de Doctorat. Paris: Université Paris I-Panthéon-Sorbonne.

52 Al-Qalaḥādī (1999). Commentaire de l'Abrégé sur les opérations du calcul. In F. Bentaleb (Ed.), Beyrouth: Dār al-Gharb al-islāmī.

53 En particulier al-Misrātī (XIV[e] s.), al-Muwāhidī (XIV[e] s.), Ibn Haydūr (m. 1413) et d'Ibn Ghāzī (m. 1514) du Maghreb Extrême, al-[c]Uqbānī (m. 1408), al-Ḥabbāk (m. 1463), al-Ghurbī (XIV[e] s.) et Ibn Qunfudh (m. 1406) du Maghreb central.

54 Djebbar, A., & Aballagh, M. (2001). *La vie et l'œuvre d'Ibn al-Bannā*. Rabat: Publications de la Faculté des Lettres et Sciences Humaines.

vers les espaces culturels latins et hébraïques de l'Europe du sud. Cela a commencé, semble-t-il, par la circulation de la numération positionnelles indienne et de certains instruments astronomiques produits en Andalus. Puis ont été réalisées les premières traductions latines de textes arabes exposant les principes de construction de ces instruments et de leur mode d'emploi. A partir de la fin du XI^e siècle, ce sont des scientifiques chrétiens ou juifs d'al-Andalus, parfaitement arabisés, qui vont assurer le relai en publiant, directement en hébreu ou en latin, des manuels mathématiques dont le contenu était puisé dans les écrits arabes de l'époque. C'est ce qu'a fait Abraham Bar Ḥiyya (m. 1145) avec son « *Livre de la surface et de la mesure* »[55] suivi, quelques décennies plus tard, par Abraham Ibn ᶜEzra (m. 1167)[56] puis par un anonyme du XII^e siècle, auteur d'un ouvrage beaucoup plus riche que ceux de ses prédécesseurs dont le titre, « *Liber Mahamelet* » [Livre des transactions] évoque son origine arabe et andalouse parce qu'il se rattache à toute une tradition locale dite « *calcul des transactions* »[57]. Il faut enfin signaler le rôle important joué par Fibonacci (m. après 1240) dans l'appropriation d'une partie de la matière mathématique produite en Occident musulman, en particulier celle qui correspond à la tradition indienne du calcul. On la trouve exposée dans son « *Liber Abaci* » avec le symbolisme arithmétique qui était en usage au Maghreb depuis le XII^e siècle[58].

Mais la forme la plus importante qu'a connue la circulation des mathématiques de l'espace musulman vers l'espace latin a été celle des traductions de l'arabe au latin, à l'hébreu puis à certaines langues vernaculaires, comme le catalan ou le toscan. Pour les mathématiques et l'astronomie, ces traductions ont été réalisées, essentiellement, en Espagne, en Sicile, en Italie et dans le midi de la France[59]. Elles ont concerné bien sûr des ouvrages grecs parvenus en Andalus dans leur version arabe, mais également des livres écrits en arabe et dont la production s'est développée entre le début du IX^e siècle et la fin du XIII^e. Dans la seconde catégorie, on trouve des ouvrages de géométrie de calcul et de trigonométrie qui ont été produits dans les deux régions de l'Occident musulman qui concernent cette étude. Certains d'entre eux ont circulé en Europe dans une version latine, comme les livres de géométrie d'Ibn as-Samḥ (m. 1035), d'Abū Bakr (X^e s.et d'Ibn Muᶜādh et le traité de trigonométrie sphérique de Jābir Ibn Aflaḥ (m. 1145). D'autres seront accessibles à travers des versions hébraïques, comme le « *Livre de la démonstration et du rappel* » d'al-Ḥaṣṣār et « *L'abrégé des opérations du calcul* » d'Ibn al-Bannā[60].

---

[55] Bar Hiia, A. (1931). *Llibre de geometria*. M. M. Guttmann, & J. M. Vallicrosa (Eds. & Trad.). Barcelone: Editorial Alpha.

[56] Lévy, T. (1996). Hebrew Mathematics in the Middle Ages : An Assessment. In F. J. Ragep, & S. P. Ragep (Eds.), *Tradition, Transmission, Transformation. Proceedings of two Conferences on pre-modern science held at the University of Oklahoma* (pp. 71-88). Leiden: Brill.

[57] Sesiano, J. (2014). *Liber Mahameleth*. New York: Springer.

[58] Sigler, L. E. (2002). *Fibonacci's Liber Abaci*. New York: Springer.

[59] Steinschneider, M. (1904). *Die Hebräischen Ubersetzungen des Mittelaters und die Juden als Dolmetscher. Berlin* : Bibliographisches Bureau; Steinschneider, M. (1956). *Die Europäischen Übersetzungen*, op. cit.

[60] Levy, T. (2003). L'algèbre arabe dans les textes hébraïques. I : Un ouvrage inédit d'Isaac Ben Salomon al-Aḥdab (XIV^e siècles). *Arabic Science and Philosophy, 13*, 269-301.

THE MATHEMATICAL CULTURES OF MEDIEVAL EUROPE

Victor J. KATZ
University of the District of Columbia (Emeritus)
Washington, DC, USA
vkatz@udc.edu

ABSTRACT

When one thinks of medieval mathematics in Europe, the first ideas that come to mind are the introduction of the Hindu-Arabic number system with its algorithms as well as the first beginnings of algebra based on Latin translations from the Arabic. But there was far more mathematics developed and discussed in the European Middle Ages, not only in Latin but also in Arabic and Hebrew. In particular, there were three different mathematical cultures in medieval Europe, the dominant Latin Catholic culture, the Hebrew culture found mostly in Spain, southern France, and parts of Italy, and the Islamic culture that was dominant in Spain through the thirteenth century. We will compare and contrast these three mathematical cultures and consider how they interacted with each other in the pre-modern period, laying the groundwork for the explosion of mathematical knowledge in Europe beginning in the Renaissance.

1 Mathematics in medieval Europe

Mathematics in medieval Europe was not just the purview of scholars who wrote in Latin, although certainly the most familiar of the mathematicians of that period did write in that language, including Leonardo of Pisa, Thomas Bradwardine, and Nicole Oresme. These authors – and many others – were part of the Latin Catholic culture that was dominant in Western Europe during the middle ages. Yet there were two other cultures that produced mathematics in that time period, the Hebrew culture found mostly in Spain, southern France, and parts of Italy, and the Islamic culture that predominated in Spain through the thirteenth century and, in a smaller geographic area, until its ultimate demise at the end of the fifteenth century. These two cultures had many relationships with the dominant Latin Catholic culture, but also had numerous distinct features. In fact, in many areas of mathematics, Hebrew and Arabic speaking mathematicians outshone their Latin counterparts. In what follows, we will consider several mathematicians from each of these three mathematical cultures and consider how the culture in which each lived influenced the mathematics they studied.

We must begin by clarifying the words "medieval Europe", because the dates for the activities of these three cultures vary considerably. Western Europe, from the time of Charlemagne up until the mid-twelfth century, had very little mathematical activity, in large measure because most of the heritage of ancient Greece had been lost. True, there was some education in mathematics in the monasteries and associated schools – as Charlemagne himself had insisted – but the mathematical level was very low, consisting mainly of arithmetic and very elementary geometry. Even Euclid's *Elements* were essentially unknown. About the only mathematics that was carried out was that necessary for the computation of the date of Easter.

Recall that Spain had been conquered by Islamic forces starting in 711, with their northward push being halted in southern France in 732. Beginning in 750, Spain (or al-

Andalus) was ruled by an offshoot of the Umayyad Dynasty from Damascus. The most famous ruler of this transplanted Umayyad Dynasty, with its capital in Cordova, was ‘Abd al- Raḥmān III, who proclaimed himself Caliph early in the tenth century, cutting off all governmental ties with Islamic governments in North Africa. He ruled for a half century, from 912 to 961 and his reign was known as “the golden age” of al-Andalus. His son, and successor, al-Ḥakam II, who reigned from 961-977, was also a firm supporter of the sciences who brought to Spain the best scientific works from Baghdad, Egypt, and other eastern countries. And it is from this time that we first have mathematical works written in Spain that are still extant.

Al-Ḥakam’s son, Hishām, was very young when he inherited the throne on the death of his father, and he was deposed by a coup led by his chamberlain. This man instituted a reign of intellectual terror that lasted until the end of the Umayyad Caliphate in 1031. At that point, al-Andalus broke up into a number of small Islamic kingdoms, several of which actively encouraged the study of sciences. In fact, Sā‘id al-Andalusī, writing in 1068, noted that “The present state, thanks to Allah, the Highest, is better than what al-Andalus has experienced in the past; there is freedom for acquiring and cultivating the ancient sciences and all past restrictions have been removed” (Sā‘id, 1991, p. 62).

Meanwhile, of course, the Catholic “Reconquista” was well underway, with a critical date being the reconquest of Toledo in 1085. Toledo had been one of the richest of the Islamic kingdoms, but was conquered in that year by Alfonso VI of Castile. Fortunately, Alfonso was happy to leave intact the intellectual riches that had accumulated in the city, and so in the following century, Toledo became the center of the massive transfer of intellectual property undertaken by the translators of Arabic material, including previously translated Greek material, into Latin. In fact, Archbishop Raymond of Toledo strongly encouraged this effort. It was only after this translation activity took place, that Latin Christendom began to develop its own scientific and mathematical capabilities.

But what of the Jews? There was a Jewish presence in Spain from antiquity, but certainly during the time of the Umayyad Caliphate, there was a strong Jewish community living in al-Andalus. During the eleventh century, however, with the breakup of al-Andalus and the return of Catholic rule in parts of the peninsula, Jews were often forced to make choices of where to live. Some of the small Islamic kingdoms welcomed Jews, while others were not so friendly. And once the Berber dynasties of the Almoravids (1086-1145) and the Almohads (1147-1238) took over al-Andalus, there were frequent times when Jews were forced to leave parts of Muslim Spain. On the other hand, the Catholic monarchs at the time often welcomed them, because they provided a literate and numerate class – fluent in Arabic – who could help the emerging Spanish kingdoms prosper. By the middle of the twelfth century, most Jews in Spain lived under Catholic rule. However, once the Catholic kingdoms were well-established, the Jews were often persecuted, so that in the thirteenth century, Jews started to leave Spain, often moving to Provence. There, the Popes, in residence at Avignon, protected them. And, of course, by the end of the fifteenth century, all the Jews were forced to convert or leave Spain.

It was in Provence, and later in Italy, that Jews began to fully develop their interest in science and mathematics. They also began to write in Hebrew rather than in Arabic, their intellectual language back in Muslim Spain.

2 The Mathematics of the Muslims

As noted above, it was the rulers of the individual Islamic states in al-Andalus who decided whether or not to support mathematics and other sciences. So why would a ruler support a mathematician? Generally, it was because he felt that the mathematicians could contribute to the wealth and, perhaps, the prestige of the kingdom. And a mathematician definitely needed support. Certainly, he could have a non-mathematical position that earned him a living, but it was better for scientific work if he was given the funds so he could spend sufficient time on mathematics. There were no institutional structures in Islamic Spain, or indeed in the Islamic world in general, that would allow a mathematician to flourish. There were no universities and the madrasas, in general, provided instruction in the religious sciences, but not the secular ones.

So we are left with looking at the relationship between a ruler and a mathematician. We will consider four examples. The first is Abū ʻAbdullah Muḥammad ibn ʻAbdūn (923-976) a mathematician who was born and taught mathematics in Cordova, the capital of the Umayyad caliphate. He became a physician as a result of his studies in the East, and then returned to Cordova as the physician of the caliph, al-Ḥakam II. His only known mathematical work is *On measurement*, of which only one copy survives. Many of the methods in this treatise can be found in texts written in ancient Babylon. In fact, ibn ʻAbdūn's treatise marks the extension of a pre-algebraic tradition of measuring surfaces from the eastern Islamic lands to al-Andalus and, then, to the Maghreb. This treatise is basically a practical manual, and not a theoretical one. Thus it is not surprising that the author of such a treatise would be supported. This was mathematics that could be used.

At the beginning of the manuscript of this treatise ibn ʻAbdūn is referred to as *muhandis* and *faraḍī*. The first denotes someone involved with measuring (theoretical or practical, e.g. surveying), and the second denotes a specialist in the arithmetical procedures necessary to calculate the legal heirs' shares of an inheritance according to Islamic law. The treatise is basically a collection of problems in which the author presents algorithms for finding areas or lengths. He begins with rectangles, squares, triangles, and parallelograms, then moves to circles, where he uses the standard approximation of 22/7 for π in his calculations. But he also shows knowledge of the old methods of solving what we would call quadratic equations when he asks the reader to find certain lengths given information about areas or diagonals.

In these problems, he does not use the al-Khwārizmīan terminology of "thing" for the unknown and *māl* (treasure) for square. He simply converts all his measurements to numbers and gives an algorithm for finding the answers. The algorithms are similar to those from ancient Mesopotamia and are ultimately based on manipulation of geometric figures. But ibn ʻAbdūn leaves out any justification at all, as in the following examples:

> If you are told, "We add the sides and the area and it is one hundred forty, what are the sides?" The calculation is that you add up the number of the sides, which is four, and take its half, two. Multiply it by itself, it is four. Add it to one hundred forty, which is one hundred forty-four. Then take the root of that, twelve, and take away from it half of the four and the remainder is equal to each of its sides.

> If you are told that the diagonal is ten and one side exceeds the other by two, what are its two sides? The way to calculate this is that you multiply the diameter by itself, which is one hundred, and you multiply the two by itself, which is four, and you subtract it from one hundred. The remainder after that is ninety-six. You take half of that, which is forty-eight, which is the area. Now it is as if you are told, "A rectangle, whose length exceeds its width by two, and its area is forty-eight. What is each of the sides?" So you work as we described to you [earlier], and you will hit the mark, Allah willing (Katz *et al*, 2016, p. 452).

A century after ibn ʻAbdūn, we find another mathematician involved in a very practical subject, spherical trigonometry, the key to the understanding of astronomy. This was ibn Muʻādh al-Jayyānī, whose work, the *Book of Unknowns of Arcs of the Sphere*, written probably in the middle of the eleventh century, is the earliest extant work on pure trigonometry, not as an introduction to a work on astronomy. The last part of his name implies that he was from Jaén in Andalusia. He is known to have been a *qādī* (religious judge) and in fact came from a family whose members included a number of such learned officials. Thus, given that he was active after the end of the Umayyad caliphate, he was probably supported by the ruler of one of the small Islamic kingdoms in the south of Spain. What we do not know is how ibn Muʻādh learned his trigonometry. His work is similar to material that had been widely discussed in eastern Islam, but nothing of his book points to any particular known eastern source.

Despite this work being a purely mathematical one, ibn Muʻādh obviously intended it to help in the study of astronomy. But it was not an elementary work. As he wrote in the preface,

> In this book we want to find the magnitudes of arcs falling on the surface of the sphere and the angles of great arcs occurring on it as exactly as possible, in order to derive from it the greatest benefit towards understanding the science of celestial motions and towards the calculation of the phenomena in the cosmos resulting from the varying positions of celestial bodies. […] So we present something whose value and usefulness in regard to understanding this [subject] are great. As for premises that were derived by scholars who preceded us, we give just the statements, without proof, so that we may arrive at acknowledgement of their proof. […] We have written our book for those who are already advanced in geometry, rather than for beginners (Katz *et al*, 2016, p. 503).

There are many starting points for the basic results of spherical trigonometry. Ibn Muʻādh chose as his starting point the transversal theorem, a theorem well-known from Greek times, although written in terms of chords rather than Sines. This theorem shows the relationship of certain ratios of Sines of arc segments in a figure consisting of four intersecting great circle arcs. Given this result and various similar ones, ibn Mu'adh then sets out his goal for the book:

> We say that there are two kinds of things found in a triangle, sides and angles. There are three sides and three angles, but there is no way to know the triangle completely, i.e. [all] its sides and its angles, by knowing only two of the six. Rather, from knowing only two things, be they two sides or two angles or one side and one angle, it [the triangle] is unspecified. For it is possible that there are a

> number of triangles, each of which has those [same] two known things, and so one must know three things connected with it [the triangle] to obtain knowledge of the rest. Thus it is impossible to attain all of it knowing less than three members: three sides, three angles, two sides and an angle or two angles and a side (Katz *et al*, 2016, p. 504).

In other words, his goal is to solve spherical triangles, given the knowledge of three of the six "things". On the way to doing this, he proves various important results. For instance, he shows that if the ratio of the Sines of two arcs is known as well as their difference (or their sum), then the arcs are determined. He also demonstrates "a theorem of great usefulness and abundant benefit in general."

> In any triangle whose sides are arcs of great circles, the ratio of the Sine of each of its sides to the Sine of the opposite angle is a single ratio (Katz *et al*, 2016, p. 512).

Ibn Muʿādh has a long discussion of the properties of right spherical triangles, including results involving Cosines as well as Sines. Finally, he systematically shows how to solve triangles, when any set of three "things" is known, often by dropping perpendiculars and then using the properties of right triangles. Probably the most difficult of the solutions to accomplish is when all three angles are known, obviously a result that has no parallel in plane trigonometry.

The two works mentioned above were reasonably practical. After all, measurement was necessary in all sorts of contexts, and spherical astronomy was important for astronomy, which was in turn necessary for calculating the direction and times of prayer. In fact, ibn Muʿādh described how to find the *qibla*, the direction of prayer. On the other hand, he also wrote a very theoretical treatise on ratios, a work explaining in detail Euclid's definition of ratio in Book V of the *Elements*. Other mathematicians too worked on quite theoretical material.

For example, consider ibn al-Samḥ (984-1035), who lived in Cordova toward the end of the Umayyad caliphate, when that government was in turmoil. He was a student of the famous astronomer, Maslama al-Majrīṭī and wrote on astronomy, astrology and mathematics. Evidently, however, he earned his living as a practicing physician. Here, we look at his geometrical text, *The Plane Sections of a Cylinder and the Determination of their Areas*, which today only survives in a Hebrew translation by Qalonymos ben Qalonymos of Provence.

Ibn al-Samḥ's treatise is in two parts. In the first part, he introduces a figure constructed by what he calls a "triangle of movement" and then considers an oblique section of a right circular cylinder, which he knows is an ellipse. The "triangle of movement" is constructed by fixing one side of a triangle and moving the intersection of the other two sides in such a way that their sum is always equal, although the lengths of each will vary as their intersection moves. He then shows that this figure and the section of the cylinder share the same properties and therefore are the same figures. In the second part, Ibn al-Samḥ finds the area of the ellipse by relating its area to that of its inscribed and circumscribed circles. In order to do this, however, he determines various ratios among the ellipse, its inscribed and circumscribed circles, and the major and minor axis. For example, he proves that the ratio of the inscribed circle to the ellipse is the same as the ratio of the minor to the major axis. Also, the ratio of the inscribed circle to the ellipse is the same as the ratio of the ellipse to the circumscribed circle.

Finally, the proposition giving the area of an ellipse is phrased in a way that echoes Proposition 1 of Archimedes' *Measurement of the Circle*, each expressing the area of a curved figure (an ellipse in the one case, a circle in the other) in terms of a certain right triangle. Further, he actually calculates the area:

> Every ellipse is equal to the right triangle of which one of the sides containing the right angle is equal to the circumference of the inscribed circle and of which the second side is equal to half of the greatest diameter. […] It results from what we have established that if we take five sevenths and one half of one seventh of the smallest diameter, and multiply this by the greatest diameter, we obtain the area of the ellipse (Katz *et al,* 2016, p. 467).

Another important mathematician from Spain at this time is Al-Mu'taman Ibn Hūd (d. 1085). Until recently his works were thought to have been lost, but in the late 1980s Professors Ahmed Djebbar and Jan Hogendijk discovered manuscripts of his extensive survey of the mathematics of his time, his *Kitāb al-Istikmāl* (*Book of Perfection*). Ibn Hūd had planned for the book to have two "genera" but he had only finished the first when he became King of Saragossa, one of the small Islamic kingdoms on the peninsula, in 1081 and evidently had no time to write the second before he died four years later. Ibn Hūd had an elaborate division of his "genera" into species, subspecies, and sections. The work, definitely not intended for beginners, sheds unexpected light on the mathematics of Ibn Hūd's time and is a fascinating blend of mathematics from Greek and Arabic sources, as well as what appear to be some original contributions of ibn Hūd himself. Obviously, given his position as a member of the dynasty that ruled Saragossa from 1038 to 1110, he was free to study whatever mathematics he wished. He clearly had the means to immerse himself in translated Greek mathematics and then to work on problems coming from these Greek sources. Consider these samples from the *Book of Perfection*:

> [Heron's Theorem:] [For] each triangle the ratio of the surface that is made of half the sum of its sides by the excess of that half over one of the sides to the surface of the triangle is as the ratio of the surface of the triangle to the surface that is made from the excess of half the sum of the sides over one of the two remaining sides by [the excess over] the other (Katz *et al*. 2016, p. 479).

Ibn Hūd's proof is different from the one given by Heron. It makes central use of the incircle of the triangle, the triangle whose center is the intersection of the angle bisectors of the triangle and which is tangent to all three sides.

Ibn Hūd also stated and proved a theorem thought to have been originated by the Italian geometer Giovanni Ceva in 1678.

> [Ceva's Theorem:] In every triangle in which from each of its angles a line issues to intersect the opposite side, such that the three lines meet inside the triangle at one point, the ratio of one of the parts of a side of the triangle to the other [part], doubled with the ratio of the part [of the side] adjacent to the second term [of the first ratio] to the other part of that side is as the ratio of the two parts of the

> remaining side of the triangle, if [this last] ratio is inverted, and conversely (Katz *et al*. 2016, p. 483).

Probably the greatest accomplishment of ibn Hūd was his study of a famous problem in geometrical optics generally referred to as “Alhazen’s Problem”, which concerns reflection in mirrors whose surfaces are curved. (Alhazen is the Latin version of the name of ibn al-Haytham.) Suppose one is given a spherical or conical mirror, concave or convex, and an object (thought of as being a point) visible in the mirror to an observer (represented by another point). The question is: At what point on the mirror will the observer see the object? As part of his solution to this problem, ibn al-Haytham gave six difficult geometrical lemmas, which were adapted by ibn Hūd in his *Istikmāl*. In some cases ibn Hūd followed ibn al-Haytham’s ideas, but in a number of cases he introduced new techniques, which simplify and shorten ibn al-Haytham’s proofs.

It is clear that the men we have discussed were quite able mathematicians. How did a mathematician operate in Muslim Spain? In general, it appears either that they had another career to provide support, such as medicine or as a religious functionary, or else, they were supported by – or in the case of ibn Hūd, actually were – rulers of the state in which they lived. There was no structure in this society that could support a steady flow of intellectual development, such as a university. If one wanted to study a particular field, one had to find an expert with whom to study. Since Spain in this time period was in the far reaches of the Islamic domains, someone who wanted to study some advanced mathematical topic had to go to the east – to Egypt or Persia or Baghdad. But there certainly were people who were able to produce interesting mathematics in Muslim Spain. However, they restricted themselves to certain topics, in particular, geometry and trigonometry. Obviously, both of these were based on Euclid’s *Elements*, which had been translated many times into Arabic, beginning in the ninth century. But Muslim mathematicians had also read Archimedes, Apollonius, and Ptolemy, among other Greek authors. They certainly absorbed the Greek notion of mathematical proof, and we see this demonstrated in treatises written in Spain. Sā‘id names many other mathematicians active in al-Andalus up to the mid-eleventh century besides the ones mentioned above, but in virtually all cases, their fields of interest were geometry and astronomy.

Although Muslim authors in the East were developing algebra during the period of Islamic rule in Spain, there is little evidence that any algebraic work more advanced than that of al-Khwārizmī was studied in Spain. In fact, the algebra that does appear in Spain is more closely related to the older geometric strain of the subject than to the more modern use of unknowns. Furthermore, even though Averroes (1126-1198) translated and commented extensively on the work of Aristotle, and his translations were quite influential later in Catholic Europe, Muslim mathematicians did not attempt to develop any of the mathematics implied by some of Aristotle’s physical ideas.

There is little evidence in Spain that there were any religious restrictions to the practice of mathematics. So the reasons why one topic was studied or another was not had to do with practical reasons, such as the availability of teachers, or, more simply, with the inclinations of a particular mathematician.

To complete the story, we recall that after the Battle of Navas de Tolosa, in 1212, in which Catholic armies defeated the Almohads, the Muslims rapidly lost control of most of

Spain. In fact, Cadiz and Cordoba were conquered by the Catholics in 1236 and Seville in 1248. Muslim Spain was then just reduced to Granada, a province in which little mathematics was done in the next two hundred years. That is not to say that Muslims stopped doing mathematics. There was certainly significant mathematics done in the 12th and 13th centuries in North Africa, including important work in combinatorics, but that is not part of our story of mathematics in medieval Europe.

3 The Mathematics of the Jews

There was a significant Jewish community in Spain under Muslim rule and, in many times and places, Jews were able to integrate into the Muslim society. They often served the rulers in administrative or financial capacities. The Jews became fluent in Arabic and used this language in their intellectual pursuits. However, we are not aware of any mathematical work by Jews until the late 11th century, by which time Toledo and many other parts of Spain were already in Christian hands. We should emphasize that when Jews were living closed off from their neighbors, their creativity was mainly displayed in interpretations of the bible. But once Jews were able to participate in the general society, as in this time period in Spain, they started to display creativity in other fields, such as mathematics.

One of the first Jewish mathematicians of whom we are aware is Abraham bar Ḥiyya of Barcelona (1065-1145), where he was a community leader as well as a scholar. His Jewish title was *nasi* (honorary leader), and Arabic title *ṣāḥib ash-shurṭa* (head of the guard, transliterated in Latin as "Savasorda"), a title he probably received from the ibn Hūd dynasty in Saragossa, where he spent time before that dynasty was overthrown. He wrote on mathematics, astronomy, astrology and philosophy, and is distinguished as the first Jewish scholar in the Arabic speaking world to write on science in Hebrew. This choice of language was, at least in part, due to the lack of access of Jews in Provence, where he visited, to the Arab language. His work includes translations from Arabic to Hebrew, and he collaborated with Plato of Tivoli on translations into Latin as well.

Bar Ḥiyya's most important mathematical work was *The Treatise of Measuring Areas and Volumes.* This book was partially translated into Latin in 1145 by Plato of Tivoli, perhaps with Abraham's help, and made an impact on European scholarship. The treatise opens with a motivational introduction, stating explicitly that Abraham wrote the book to teach the appropriate geometry necessary for both secular and holy affairs. After presenting versions of the early books of Euclid's *Elements*, Abraham proceeds to deal with measurements of squares, rectangles and rhomboids (deriving their areas, sides, diagonals, etc. from each other), and includes a geometric treatment of quadratic problems. He continues on to triangles, general quadrilaterals, and circles and then studies measurement of polygons by triangulation as well as giving some practical suggestions for measuring sloping and curved lands. There is then a section on division of plane areas, perhaps based on Euclid's own no longer extant book on the same topic.

This work is not a full scholarly geometry, but a compromise between an introduction to abstract geometry and a measurement manual. It provides a good intuitive introduction to geometrical reasoning and has some problems similar to those of ibn 'Abdūn. As we will see later, Fibonacci seems to have used this book, probably in the

Latin translation, as one of the sources of his *Practical Geometry*. But also, it seems clear that in this time period, Jews generally did not study abstract subjects. If they were interested in mathematics at all, they tended to concentrate on practical subjects.

We begin with Abraham's motivations for studying geometry at all:

> [The scriptures say] "I the Lord am your God, instructing you for your own benefit, guiding you in the way you should go", that is, instructing you in whatever is useful for you, and guiding you on the way you follow, the way of the *torah*. From which you learn that any craft and branch of wisdom that benefit man in worldly and holy matters are worthy of being studied and practiced.
>
> I have seen that arithmetic and geometry are such branches of wisdom, and are useful for many tasks involved in the laws and commandments of the *Torah*. We found many scriptures that require them, such as "In buying from your neighbor, you shall deduct only for the number of years since the jubilee", and "the more such years, the higher the price you pay; the fewer such years, the lower the price", followed by: "Do not wrong one another, but fear your God". But no man can calculate precisely without falsification unless he learn arithmetic. […] Moreover, the *Torah* requires geometry in measuring and dividing land, in Sabbath enclosures and other commandments. […] But he who has no knowledge and practice in geometry cannot measure and divide land truly and justly without falsification. […] It suffices to note that the blessed God prides himself in this wisdom, as is written: "He stood, and measured the earth" and "Who measured the waters with the hollow of His hand, and gauged the skies with a span". So you see from these writings that the blessed God created his world in well founded and weighed out measurement and proportion. And a man must be like his creator with all his might to win praise, as all scholars agree, so from all this you see the dignity of these branches of wisdom. He who practices them does not practice something vain, but something useful for worldly and holy matters.
>
> Arithmetic, which is useful for worldly matters and crafts as well as for the practice of many commandments, is not difficult to understand, and most people understand it somewhat and practice it, so one does not need to write about it in the holy tongue. Geometry is also as useful for as many matters as arithmetic in worldly matters and commandments from the *Torah*, but is difficult to understand, and is puzzling to most people, so one has to study and interpret it for land measurement and division between heirs and partners, so much so that no one can measure and divide land rightfully and truthfully unless they depend on this wisdom.
>
> I have seen that most contemporary scholars in Spain and Provence are not skillful in measuring land and do not divide it cleverly. They severely belittle these matters, and divide land between heirs and partners by estimate and exaggeration, and are thus guilty of sin. […] Their calculation might mete out a quarter to the owner of a third, and a third to the owner of a quarter, and there is no greater theft and falsification (Katz *et al*, 2016, pp. 297-298).

Although there are many occasions in the Talmud where approximations are used, Abraham insists that

> Our fathers did not allow us to dismiss calculations, nor steal from heirs, nor give any of them more or less than their fair share. […] They warned us and gave us strict orders against stealing and falsifying in measuring land (Katz *et al*, 2016, pp. 298-299).

Thus, Abraham concludes, one needs to study the principles of measurement carefully, so that one calculates shares of heirs correctly. So, unlike ibn ʻAbdūn, Abraham presents careful proofs of his rules for calculation, generally based on the *Elements*. Consider the following examples:

> A square quadrilateral that you take away from the number of its area the number of its four sides, and are left with 21 cubits of its area: what is the area and what is the number of each side of the square? Answer: Divide the number of the sides, which is four, into two. Multiply the two by itself, which is 4. Add this number to the given number that's left over from the square, and the total is 25. Find the root of 25, which is 5. Add half the sides, which is 2, so the total is 7. This is the side of the square, and its area is 49. He who posed the question subtracted from the area, which is 49, the number of the four sides, each of which is 7 and all four 28, leaving from the square 21, as he told you (Katz *et al*, 2016, pp. 300-301).

After presenting this algorithm, bar Ḥiyya draws a diagram of the square *ABCD*, then subtracts off rectangle *BECG* with *BE* of length 4 (i.e., the four sides), leaving a rectangle *EADG* of area 21 (Figure 1). He then divides *BE* in half at *H* and quotes *Elements* II-6 to conclude that the square on *HA* is the sum of rectangle *EADG* and the square on *HE*, that is 25. Therefore *HA* itself equals 5 and *AB* equals 7, as desired.

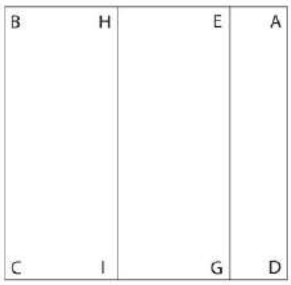


Figure 1. Bar Ḥiyya's justification of his solution

As an example of dividing fields, Abraham begins with a region bounded by an arc of a circle and two straight lines, neither of which are radii (Figure 2). His goal is to find a straight line dividing the region in half. Here *E* is the midpoint of line *AC* and *EG* is perpendicular to *AC*. Line *BG* intersects *AC* at *I*; then *HE* is drawn parallel to *BG*. Then *GH* divides the region in half. To prove the result, note that *BE* divides triangle *ABC* in half, while *EG* divides segment *AECG* in half. But triangles *GHE* and *BHE* are equal. It follows that region *AHG* is equal to the sum of triangle *ABE* and region *AEG*, so is half of the entire region *ABCG*.

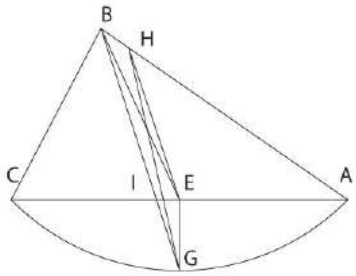


Figure 2. Division of field
bounded by a circular arc

Abraham ibn Ezra (1090-1167) was a younger contemporary of Abraham bar Hiyya. He was born in Tudela, when it was part of the kingdom of Saragossa, but then traveled widely during his adult life. Among his numerous works were books on arithmetic and numerology, as well as a work dealing with astrology which had some interesting combinatorial aspects. As he wrote,

> Only when one knows the natural sciences and their proofs, learns the categories that are the 'guardians of the walls' taught by the science of logic, masters the science of astronomy with its absolute proofs based on mathematical knowledge, and comprehends the science of geometry and the science of proportions, can one ascend to the great level of knowing the secret of the soul, the secret of the supernal angels, and the concept of the world to come in the Torah, the Prophets, and the sages of the Talmud (Ibn Ezra, 1995, chap. 1).

In other words, the reason for studying mathematics was ultimately to get closer to God. Thus, in general, that mathematics could be studied that was useful toward that end.

In his *Sefer ha-Mispar* (*Book of Number*), ibn Ezra expounded on "the science of proportions", as he showed how to use the rule of three to solve problems. He began with methods of calculation, in which he explains the Hindu-Arabic number system, although with Hebrew characters for the digits. But he then shows how to solve numerous commercial problems, such as

> Reuven hired Simon to carry on his beast of burden 13 measures of wheat over 17 miles for a payment of 19 *pashuts*. He carried seven measures over 11 miles. How much shall be paid? (Katz *et al*. 2016, p. 231)

Another of the subjects that ibn Ezra thought was useful was astrology. He wrote a series of books on the subject. In particular, in *Sefer Haʿolam* (*Book of the World*), ibn Ezra discusses the meaning of celestial conjunctions and aspects. It opens by counting all possible conjunctions of the seven known planets, demonstrating some systematic combinatorial reasoning. In order to calculate the number of different sets of *n* elements out of 7 planets, a recursive method is used, taking partial sums of the sequence 1,2,...,7, then taking partial sums of the sequence of these partial sums, etc.

> There are 120 conjunctions [of the seven planets]. You can calculate their number in the following manner: it is known that you can calculate the number that is the sum [of all the whole numbers] from one to any other number you wish by multiplying this number by [the sum of] half its value plus one-half. As an illustration, [suppose] we want to find the sum [of all the whole numbers] from 1

> to 20. We multiply 20 by [the sum of] half its value, which is 10, plus one-half, and this yields the number 210. We begin by finding the number of double conjunctions, meaning the combinations of only two planets. It is known that there are seven planets. Thus Saturn has 6 [double] conjunctions with the other planets. [Jupiter has 5 double conjunctions with its lower planets, Mars has 4, and so on. So we need to add the numbers from 1 to 6]. Hence we multiply 6 by [the sum of] half its value plus one-half, and the result is 21, and this is the number of double conjunctions (Katz *et al*, 2016, p. 272).

Ibn Ezra next finds the triple conjunctions, effectively showing that $C_{7,3} = C_{6,2} + C_{5,2} + C_{4,2} + C_{3,2} + C_{2,2}$, where each $C_{k,2}$ is shown to be the sum of integers up to $k-1$. He continues in this manner with quadruple conjunctions and so on until he has found the total of 120 indicated above.

In a further work, the *Book of Measure*, bin Ezra gives without proof numerous procedures for determining areas of geometrical figures. Many of these are similar to material found in bar Hiyya's work, such as the following problems:

> We have added the sides and the area; this gives so much. How much is the side? Take the square of half the number of all the sides [= 4] and add it to the sum [of the area plus the four sides]; subtract from the root of this result half the number of the sides [= 2].
>
> Or, for the circle: If one is dealing with a semicircle, its area is like that of half a circle. If it is smaller or larger [than a semicircle], you must know the diameter of the circle from which the circular segment has been cut, and the length of the chord of the arc and of the sagitta. When you know two of these [three] elements, you can determine the third. Problem: The chord is 8, the diameter, 10. How much is the sagitta? Subtract from the square of half the diameter the square of half the chord; take the root of the remainder, and subtract it from half the diameter; you will find the sagitta [= 2] (Katz *et al*, 2016, p. 289, 291).

Ibn Ezra further presents a table of sines and later displays the standard medieval method of using an astrolabe to calculate heights and distances. If one knows the distance to the tall object whose height is to be measured, one uses the astrolabe to measure the proportion of height to distance, from which the height can be calculated. If one does not know the distance, one takes two measurements with the astrolabe from different places and then uses a formula known in China and elsewhere for centuries to calculate the height.

Although ibn Ezra had stated the reasons one could study the sciences, he was not the only one. Bahya ibn Paquda, a Jewish philosopher from Saragossa in the mid-eleventh century, wrote the following in his *Duties of the hearts*:

> All departments of science, according to their respective subjects, are gates which the Creator has opened to rational beings, so that they may attain to a comprehension of revealed religion and of the world. But while some sciences satisfy primarily the needs of religion, others are more requisite for the benefit of the world. The sciences specially required for the affairs of the world are the lowest division – namely the science that deals with the natures and accidental

> properties of physical substances – and the intermediate division – namely the science of mathematics. These two branches of knowledge afford instruction concerning the secrets of the physical world and the uses and benefits to be derived from it, as well as concerning arts and artifices needed for physical and material well-being. But the science that is needed primarily for revealed religion is the highest science, namely the divine science, which we are under obligation to study in order to understand our revealed religion and to reach up to it. To study it, however, for the sake of worldly advantages is forbidden to us (Freudenthal, 1995, p. 34).

It was Maimonides (1135-1204), however, whose work was much more important in permitting Jews to study science. Recall that Maimonides was born in Spain. His family left Spain for North Africa during the reign of the Almohads, but simply settled in another part of the Almohad empire. Eventually, he traveled to Palestine and then spent the rest of his life in Egypt as a physician to the sultan as well as the most important philosopher in Jewish history. For Maimonides, the study of science and philosophy was actually a religious obligation:

> It is certainly necessary for whoever wishes to achieve human perfection to train himself first in the art of logic, then in the mathematical sciences according to the proper order, then in the natural sciences, and after that in the divine science (Freudenthal, 1995, p. 32).

And Maimonides emphasized that it was only truth that counted, and that it did not matter who discovered it. On the other hand, since it was the "divine science" of metaphysics that was the ultimate goal, Maimonides emphasized that science was legitimate and desirable only in so far as it contributed to the divine science. Thus Medieval Jews were to study mathematics either because they regarded it as essential for metaphysics, preparing the intellect to apprehend abstract truths, or because they needed it since it was a prerequisite for the study of mathematical astronomy, important for calculating the calendar.

It seemed clear that the study of Euclid's *Elements* was legitimate, and indeed it was widely and continuously studied. And trigonometry, which the Jews learned from the Muslims but to which they made contributions, was also a valued study. But somehow, at least in the 11^{th} and 12^{th} centuries, it was argued that the study of algebra was pointless, indeed harmful. Medieval algebra was construed as a mere technique, allowing one to solve equations, and as such it had no philosophical value; nor was it apparently of practical use. Abraham ibn Daud of Toledo (1110-1180) writes:

> Among those who spend their time on vanities, thereby depriving their soul of afterlife is he who consumes his time with number and with strange stories like the following: A man wanted to boil fifteen quarters of new wine so that it be reduced to a third. He boiled it until a quarter thereof departed, whereupon two quarters of the remaining wine were spilled; he again boiled it until a quarter vanished in the fire, whereupon two quarters of the rest were spilled. What is the proportion between the quantity obtained and the quantity sought? (Freudenthal, 1995, p. 37)

Maimonides himself wrote that the books on conics and on devices (i.e. algebra), and on the science of weights are instances of inquiries that must not be pursued as ends in themselves. They are only worth studying if they help to "sharpen the intellect" to help man achieve knowledge of God. Interestingly, Maimonides himself drew on the demonstrated existence of asymptotes to show that imaginability is not a criterion of existence. "Hear what the mathematical sciences have taught us and how capital are the premises we have obtained from them" (Freudenthal, 1995, p. 37).

The foremost medieval Jewish mathematician, Levi ben Gershon (1288-1344), certainly read Maimonides' works. Yet he interpreted Maimonides differently from most others. Namely, he felt there should be no restriction on what he could write about in science or in mathematics. Because all knowledge of God's works has religious significance, the acquisition of scientific knowledge about the world is a legitimate end in itself. Thus Levi explored many different aspects of science and mathematics.

His earliest mathematics work was the *Maasei Hoshev* (*The Art of the Calculator*), a book using Euclidean methodology and, in essence, mathematical induction, to prove numerous results in number theory and combinatorics. The first part of the book is very abstract. The earlier results deal with such topics as summing integers or squares and sometimes with rather unusual problems such as the following:

> To find three numbers such that the sum of the first and third contains the second as a factor as many times as a given number and such that the sum of the second and third contains the first as a factor as many times as a second given number (Katz *et al*, 2016, p. 259).

Presumably, this result was included because, in the problem section of the text, Levi wanted to include a numerical version of this challenge and because this is an abstract version of a problem that had appeared earlier in Latin mathematics as the problem of men finding a purse. Levi probably wanted to present combinatorial results because Jews had for centuries been interested in the question of how many words could be formed with the letters of the Hebrew alphabet. Now Levi does not answer such a question, but just presents results about combinations and permutations of sets of objects. For example,

> When you are given a number of terms and the number of permutations of a second given number from these terms is a third given number, then the number of permutations of the number following the second given number from these terms is the product of the given third number by the excess of the first given number over the second number (Katz *et al*, 2016, p. 274)

In modern terms, this results says that $P_{m,n+1} = (m - n)P_{m,n}$. This result is the inductive step for proving that $P_{m,k} = m(m-1)(m-2) \dots (m-k+1)$, a theorem Levi states next.

Levi wrote several other mathematical works, two of which were quite theoretical. Namely, he wrote a commentary on Euclid's *Elements*, in which he spent quite a bit of time giving a proof of Euclid's parallel postulate. His argument was quite rigorous, but he began with a different postulate:

> The straight line which is inclined [to another straight line] approaches [the second line] on the side where an acute angle is formed [with a line crossing both of these that is a perpendicular from the first line to the second] (Katz *et al*, p. 328).

He also wrote a number theory work, at the request of a French music theorist. Here he gave a very clever proof of the theorem that a power of 2 must differ from a power of 3 by at least 2, except in the cases 1,2; 2,3; 3,4; and 8,9. Presumably this result was of use in music theory, but it is not clear that this would meet Maimonides' criteria for what could be studied.

There were a few other Jewish mathematicians in Spain and France who also ignored Maimonides' strictures. For example, consider the work of Abner of Burgos (1270-1348), who lived in Castile. He was originally a Jew, but converted to Christianity and was then known as Alfonso di Valladolid. His most important mathematical work is the *Sefer Meyasher 'aqov* (*Book of the Rectifying of the Curved*), whose aim is to enquire whether there possibly exists a rectilinear area equal to a circular area truly and not by way of approximation. Unfortunately, in the only manuscript we have, the final chapter, where the aim was to be accomplished, is missing. But it is the third chapter in which Alfonso considers many interesting geometrical questions related to curves and solids. In particular, Alfonso defines and uses the conchoid of Nicomedes. It is usually accepted that interest in Nicomedes' work – and his original treatise is lost – was only revived in the late sixteenth century, where it is mentioned and used by Viéte and then later by Descartes and Newton, among others. But, in fact, this curve is discussed by Alfonso, with some important applications. So what is the conchoid?

Given a straight line (the "ruler" or "canon" *AB*), a point outside it (the "pole" *P*) and a distance (d), the conchoid of Nicomedes is the locus of all points lying at the given distance d from the ruler *AB* along the segment that connects them to the pole *P* (Figure 3). If *P* is the origin, and *AB* is the line $y = a$, then the curve is defined by the polar equation $r = a/\sin\theta + d$ (Figure 3). The curve has two branches on opposite sides of the ruler, to which both are asymptotes. The branch passing on the side of the pole has three different distinct forms, depending on the ratio between a and d: If $a < d$, it has a loop (as is in the diagram); if $a = d$, then *P* is a cusp point; and if $a > d$, the curve is smooth. The other branch does not change topologically.

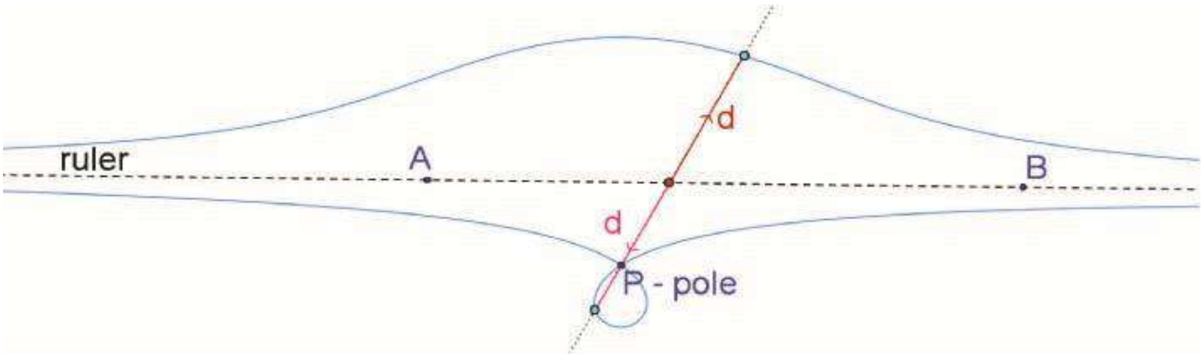


Figure 3. The conchoid of Nicomedes

The importance of the curve to Nicomedes, and later to European mathematicians, was that its use allowed the trisection of an angle, the construction of two mean proportionals between two line segments, and the doubling of the cube. Alfonso, in fact, demonstrates each of these. His angle trisection is similar, but not identical, to that attributed to Nicomedes, but his construction of two mean proportionals is not found in any of the Greek or Arabic literature, and his use of this to construct the doubled cube is unique. In fact, he constructs a generalization of the Delian problem: To construct a polyhedron which is equal in volume to a given polyhedron and which is similar to a second given polyhedron. To get the doubled cube, simply assume the first given polyhedron is any parallelepiped of volume 2, while the second one is a cube of volume 1.

Somewhat later, we find Isaac ibn al-Ahdab (1350-1430) and Simon Motot (mid 15th c.) actually studying and writing about algebra. The former was born in Castile, but ended up in Sicily after leaving Spain. He studied the algebra of the Maghrebian mathematician Ahmad ibn al-Banna' and wrote a detailed commentary on it. The latter lived in Italy and probably learned his algebra in the Italian abacus tradition of his time. His treatise was the first original Hebrew work giving a detailed treatment of the al-Khwārizmīan form of algebra.

Still Levi was, without doubt, the most accomplished Jewish mathematician of the Middle Ages. Even though he went beyond the standard interpretation of Maimonides in deciding that he could study and write on any topic he thought interesting, there were few followers. There was a conflict within the Jewish community regarding what subjects could legitimately be studied, with a significant proportion of "traditionalists" insisting that only the Torah and Talmud were worthy of study. A further issue was that there was no institutional infrastructure for new students to learn the works of their predecessors. One could always arrange to study privately with an individual, and certainly there were "study groups" established by various people, including Levi himself. But there were no Jewish universities – just as there were no Muslim universities.

Leon Joseph of Carcassonne, who lived around the turn of the 14th – 15th centuries in the south of France, writes about this very issue:

> Many years ago I directed my attention toward the study of and research into the profane sciences, which are several in number and nature. […] In my eyes, the merits of these sciences were above all praise. […] I therefore followed in the footsteps of the learned men of our own times, […] so that they should illuminate my way with the light of their intelligence and understanding. […] But I realized that the lack of knowledge that they, and some of my people at this time, found themselves submerged in was great and immense. […] I perceived that said lack of knowledge on the part of one sector of our nation was by no means strange. Its cause was not unknown and I was not unaware of the Talmudic law which referred to it… Then I heard a voice telling me that there was not one single cause, but many, for the lack and absence of this knowledge among some of our scholars. Sciences defeated them because their subject matter is more rational than in the bosom of our people, and they are as far from them as east is from west, and all the more so from the fundamentals of the Torah and of religious faith. [Those few who did study the sciences] had no right to propound [their knowledge] in the squares and streets, or to discuss it, to show themselves to be favorable toward it, nor to conduct public debates with the aim of reading the complete truth, for knowledge of the truth can only be attained by means of the contrary. […] On seeing the obstacle that these causes represented and aware that the aforementioned sciences were known among the Christians, I said to myself: I shall study their language a little. I shall attend their schools and houses of study. I shall follow their footsteps so that I am able to make use of whatever I might learn from their words. […] I found great benefits in this, because in general their discussions on these sciences do not stray from the subject matter; they leave out

> nothing when it is a question of debating the truth or falsehood of a proposition; they are very rigorous concerning the questions and answers of a debate, which are linked together in such a way as eventually to bring out the truth by means of an analysis of opposing points of view (Garcia-Ballester *et al*, 1990, pp. 106-110).

4 Mathematics in Catholic Europe

It is, in fact, the existence of universities in Europe, beginning in the twelfth century, along with the concurrent flood of translations from the Arabic, that provided the impetus for the study and practice of mathematics (and other sciences) in Europe from that time on. However, the first important mathematician in Catholic Europe was Leonardo of Pisa (Fibonacci) (1170-1240). He introduced parts of Islamic mathematics to Europe, because he had accompanied his merchant father on trips to North Africa and elsewhere in the Mediterranean, where he studied with Muslim mathematicians. He mastered the Hindu-Arabic number system as well as the elements of algebra, geometry, and trigonometry. So in his first book, the *Liber Abbaci* of 1202, he spent many chapters describing computational methods and then another several chapters showing how to solve numerous types of problems. His methods of solution were varied, including the well-established method of the rule of three. But since he had learned some algebraic methods as well, he sometimes included these. One of the standard types of problems solved by the rule of three was the "tree problem," a problem to which he later reduced other types of problems:

> There is a tree 1/4 + 1/3 of which lies underground, and it is 21 palms. It is sought what is the length of the tree (Katz *et al*, 2016, p. 80).

Another standard problem that we have already seen in the work of Levi ben Gershon, is the problem of men finding a purse:

> Two men who had *denari* found a purse with *denari* in it; thus found, the first man said to the second, If I take these *denari* of the purse, then with the *denari* I have, I shall have three times as many as you have. Alternately, the other man responded, And if I shall have the *denari* of the purse with my *denari*, then I shall have four times as many as you have (Katz *et al*, 2016, p. 81).

The problem asks, of course, how much each men had and how much was in the purse. Interestingly, Leonardo does not mention that the problem is indeterminate; he just shows how to find one solution.

Besides these recreational problems, Leonardo devotes many pages to very practical problems such as calculation of profits, currency conversions, alloying of money, barter, determining values of merchandise, and so on. Given that his father was a merchant and that he lived in an Italy where commerce was quickly developing, it is not surprising that these kinds of problems would be of great interest to his readers. Although many of the problems are solved by seemingly ad hoc methods, Leonardo devotes a chapter to explaining the method of false position.

Leonardo credits the method to the Arabs, and, of course, this method is found in Arabic texts written in North Africa, such as the work of ibn al-Banna'. This kind of problem is also found both in Hebrew and Arabic works. But Leonardo also devotes the final chapter of *Liber abbaci* to the Muslim method of solving quadratic equations, basically the work due to al-Khwārizmī. He then presents about 100 quadratic problems, taken from the works of such authors as al-Khwārizmī, Abū Kāmil, and al-Karajī.

In his *Practical Geometry*, he solves problems similar to those solved by ibn ʻAbdūn, ibn Ezra, and Abraham bar Ḥiyya. These are generally problems in measurement – of triangles, rectangles, squares, parallelograms, trapezoids, and parts of circles. Just like the earlier authors, sometimes he needs to solve quadratic equations to complete the solutions. But it should be noted that Fibonacci did not merely "copy" problems from earlier authors. He may well have read these authors, but he used his own genius to expand on their methods and often to figure out ingenious solutions. Thus he presented a long series of problems on dividing a region into two equal parts. Some of these methods presumably come from the no-longer extant work of Euclid, and some of the methods are found in the work of bar Ḥiyya. But he very carefully explains his procedures and gives careful proofs.

But Fibonacci also displays a talent for abstract mathematics, demonstrated in his *Book of Squares*, initially prompted by a question from Master John of Palermo to "find a square number from which when five is added or subtracted always arises a square number." He solved this problem and various associated problems through a series of 24 theorems, all given careful and detailed proofs. So it is clear that Leonardo felt that there was a readership for non-practical problems.

Now Leonardo was not connected to a university, unlike most of the mathematicians of medieval Europe that followed him. So a few words about the universities are in order here. Well before the end of the twelfth century the Masters at the School of Saint Victor, together with the Masters at the Schools of St Geneviève and Notre-Dame de Paris, would construct the cradle of the University of Paris. Oxford arose from dissatisfied mostly English Masters and students who left Paris for their homeland; similarly Cambridge was founded from Oxford. The origins of universities in other countries have their own histories, such as the earlier University of Bologna, formed by the students who hired the Masters. And the University of Montpellier, among several others, was founded in the thirteenth century. Some schools followed the English model with Masters in charge. Others followed the Italian model with students in charge. Regardless, if there be universities, there must be students, Masters, and a curriculum. The new curriculum was the gift of the translators, operating mostly in Spain. The curriculum in arts at all of the universities was based on the ancient trivium of logic, grammar, and rhetoric and the quadrivium of arithmetic, geometry, music, and astronomy. This study in the faculty of arts provided the student with preparation for the higher faculties of law, medicine, or theology. The centerpiece of the arts curriculum was the study of logic, and the primary texts for this were the logical works of Aristotle, all of which had recently been translated into Latin. The masters felt that logic was the appropriate first area of study since it taught the methods for all philosophic and scientific inquiry. Gradually, other works of Aristotle were also added to the curriculum. For several centuries, the great philosopher's works were the prime focus of the entire arts curriculum. Other authors were studied insofar as they allowed one better to understand this most prolific of the Greek philosophers. In particular, mathematics was studied in the universities primarily as it related to the work of Aristotle in logic or the physical sciences. (Algebra, on the other hand, was a non-university subject.) The mathematical curriculum itself – the quadrivium – usually consisted of arithmetic, taken from such works as Boethius's adaptation of Nicomachus or a medieval text on rules for calculation, geometry, taken from Euclid and one of the practical geometries, music, taken also from a work of Boethius, and astronomy, taken from Ptolemy's *Almagest* and some more recent Latin translations of Islamic astronomical works.

What is important to realize is that, because the universities were corporate bodies generally operating under a royal charter, they were independent of church control. Now, Aristotle's philosophy did pose problems for Catholic theologians. From Aristotle's point of view, the world was eternal – it had always existed and would continue to exist. But for Catholics, as for Muslims and Jews, the world had been created by God out of nothing. In fact, in 1277, the Bishop of Paris drew up a list of 219 "errors" in which he alleged that "some scholars of arts at Paris" were transgressing the limits of their own faculty. In particular, he wrote that it was an error to doubt God's omnipotence, that, in fact, God had absolute power to do whatever he wills, including creating the world out of nothing. That is, he condemned those ideas that could not be maintained in light of the revealed truth of the Catholic religion.

But this condemnation was too little and too late. The religious elite who dominated intellectual thought had already come to the conclusion that rational thought and an empirical methodology were the tools for understanding the world. In fact, a new canon law had been developed in the 12^{th} century stating that "anyone (and not just priests) ought to learn profane knowledge not just for pleasure but for instruction, in order that what is found therein may be turned to the use of sacred learning." (Huff, 1993, p. 195) In essence, the study of the natural sciences and the pursuit of philosophical truth had become institutionalized in the universities and nothing would disturb this state of affairs. Scientists in Catholic Europe, including mathematicians, were free to study what they wished.

One group of mathematicians who worked at a university were the so-called Oxford calculators, associated with Merton College, Oxford during the fourteenth century. Because they were involved in university teaching, they had to figure out how to explain difficult concepts to students, with the basic method of teaching being disputations with participation from both masters and students. Thus they concentrated on logical argument, based on Aristotle's principles, and then used the argument to try to determine what Aristotle meant in his discussions of physical problems. One of the first of the Mertonians was Thomas Bradwardine (1290-1349).

In his *On the Continuum* he mentions five different opinions presented by scholars of his time and earlier:

> One must know that the old and modern philosophers have five famous opinions about the composition of the continuum. Some of them, such as Aristotle, Averroes, Algazel [al-Ghazalī] and most of the moderns, argue that the continuum is not composed of atoms, but of parts that can be divided without end. Others say that it is composed of two kinds of indivisibles, because Democritus had assumed that the continuum consists of indivisible bodies. Others say that it consists of points, and this [assumption is divided] into two parts: Pythagoras (the father of this position), Plato, and our contemporary Walter [Chatton] assume that it is composed of a finite number of indivisibles, but others say that [it is composed] of

an infinite number. This group, too, is divided into two parts. Some such as our contemporary Henry of Harclay say that it is composed of an infinite number of indivisibles that are directly joined. But others such as Lincoln [Robert Grosseteste], say [that it is composed] of an infinite number [of indivisibles] that are indirectly joined to one another. Therefore the conclusion is this: “If one continuum is composed of indivisibles in some way” (the “way” includes any of the precedent ways), it then follows that “any continuum is composed of indivisibles according to a similar way” (Katz *et al*, 2016, pp. 178-179).

Bradwardine then gives arguments to reject most of these possibilities. For example, to reject the assumption that the continuum is composed of a finite number of points, he proves:

> If this [is true], then the circumference of a circle is double of its diameter. This is: half of the circumference is equal to its diameter. From the different points of the diameter, [assuming that] they are 10, ten perpendiculars are drawn directly to different points on half the circumference. It follows that there are 10 points on half the circumference, because only one point on half the circumference corresponds [to] a perpendicular. Therefore equally, there are the same number of points on half the circumference as are on the diameter. Therefore according to the second conclusion, half the circumference equals the diameter (Katz *et al*, 2016, p. 179).

Bradwardine realizes that this is impossible. Later, he rejects the hypothesis that a continuum is composed of an infinite number of indivisibles:

> If this is true, a terminated surface can exceed another surface equal to it by any finite proportion. Let *AB* and *CD* be parallel lines. Atop base *CE* a right-angled parallelogram *AFCE* is constituted, and atop the same base another parallelogram *CGHE* is constituted with sides that are as much longer as you want than the sides of the parallelogram *AFCE* (Figure 4). Then all lines of *CGHE* which are drawn from all points of *CE* to the opposite points of *GH* are equal in number to those points, and consequently to all perpendiculars of *AFCE* which are drawn from the same points to the opposite points. But they are longer than those [lines]. Therefore, *CGHE* is larger than *AFCE*. But according to I 36 of Euclid’s *Elements*, the parallelograms are equal (Katz *et al*, 2016, p. 180).

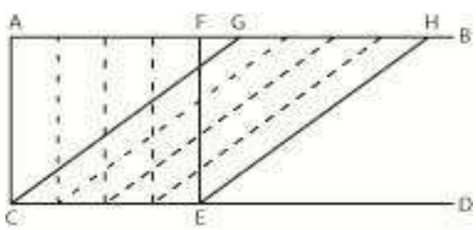


Figure 4. Bradwardine’s proof

Bradwardine finally announced his true view of the composition of continua:

> No continuum is made up of atoms. From here follows and elicits: Every continuum is composed of an infinite number of continua of the same species as it, […] that is, every line is composed of an infinite number of lines, every surface

composed of an infinite number of surfaces, and so on concerning other continua (Katz *et al*, 2016, p. 180).

Similarly, Bradwardine investigated four differing theories regarding relationships among speed (V), force (F), and resistance (R) in his *Treatise on Proportions*. First he explained and demolished the thinking of Aristotle in *On the Heavens and Earth*: The proportion between the speeds with which motions take place varies as the difference whereby the power of the mover exceeds the resistance offered by the thing moved. Then he explained and rejected Averroes' *Comment 36* on Aristotle's *Physics Book VI*: The proportion of the speeds of motions varies in accordance with the proportion of the excesses whereby the moving powers exceed the resisting powers. Next he destroyed a generalization built on remarks in Aristotle's *Physics* and *On the Heavens* and from *On Weights*: With the moving power remaining constant, the proportion of the speeds of motions varies in accordance with the proportion of resistances, and with the resistance remaining constant, it varies in accordance with the proportion of moving powers. Finally he took apart *Comment 79* on Aristotle's *Physics VIII* by Averroes: There is neither any proportion nor any relation of excess between motive and resistive powers. Then he began his own contribution.

> Now that these fogs of ignorance, these winds of demonstration, have been put to flight, it remains for the light of knowledge and of truth to shine forth. For true knowledge proposes a fifth theory, which states that the proportion of the speeds of motions varies in accordance with the proportion of the power of the mover to the power of the thing moved. […] Furthermore, there does not seem to be any theory whereby the proportion of the speeds of motions may be rationally defended, unless it is one of those already mentioned. Since, however, the first four have been discredited; therefore the fifth must be the true one. We therefore arrive at the following theorem:
>
> Theorem I. The proportion of the speeds of motions follows the proportion of the force of the mover to that of the moved, and conversely. Or, to put it another way, which means the same thing: The proportion of motive to resistive power is equal to the proportion of their respective speeds of motion, and conversely. This is to be understood in the sense of geometric proportionality (Katz *et al*, 2016, pp. 189-190).

Symbolically then the first theorem can be expressed as $V = \log_n (F/R)$ or as $n^V = F/R$. That is to say, doubling the velocity squares the ratio of motive power to resistance, tripling the one cubes the other, and so on. For the formula to be correct universally, n is necessarily a constant equal to F/R when $V = 1$. Although the result is not our modern relationship, Bradwardine was able to use mathematical principles to prove various theorems dependent on his result.

An obvious question here is why, since Aristotle was important both to Muslim scientists, in the translation and adaptation of Averroes, and to Jewish mathematicians, in the Hebrew translations of Averroes, these mathematicians never considered the mathematical problems connected with kinematics. Perhaps in both cases these ideas would not be considered important enough religiously to be studied. But more certainly, the ideas of Aristotle were never discussed in a setting in which one could debate these questions. It seems clear from Bradwardine's style, that it was through disputations that he was able to demolish certain arguments and therefore prove the correct one.

Another prominent member of the Mertonian school was William Heytesbury (1313-1373). Continuing further the discussion of velocity, he was one of the first to state the Mean Speed Theorem: A body that moves with uniformly accelerating speed traverses in a given time the same distance as a body that in the same time moves with a constant speed equal to the accelerating body's speed at the middle instant. Heytesbury gave a demonstration of this result by an argument from symmetry, and then proved the easy corollary, that under uniformly accelerated motion from rest, a body in the first half of a given interval will traverse one-third of the distance it covers in the second half of the interval.

Heytesbury's slightly younger contemporary Nicolas Oresme (1320-1382), connected with the University of Paris, made some further advances by using a graphing technique to visualize continuous quantity. As he put it,

> Every intensity which can be acquired successively ought to be imagined by a straight line perpendicularly erected on some point of the space or subject of the intensible thing, e.g., a quality. For whatever ratio is found to exist between intensity and intensity, in relating intensities of the same kind, a similar ratio is found to exist between line and line, and vice versa (Katz *et al*, 2016, p. 197).

Oresme applied his result to all sorts of "intensities" and drew figures to represent uniform difformity as well as "difform difformity". He could then give a simple geometric proof of the Mean Speed theorem. Oresme, however was a bit puzzled when he tried to apply his methodology to the idea of curvature, because he could not figure out how to compare them.

> For curvature, like the other qualities, has both extension and intensity, and one kind of curvature is uniform while another is difform. But still it is not manifest, in regard to the ratio of the intensity of curvatures, whether one is double another or exists in another ratio to it, or whether or not curvatures are unrelatable one to the other by ratio (Clagett, 1968, p. 215).

Oresme wanted to define increase in curvature as a function of "its departure from straightness," but could not actually measure this. He could certainly tell if one curve was more "curved" than another by looking at whether one could be included in the space between the second and a straight line – but this now involved the whole notion of the angle of contingence, that is, the angle between a circle and its tangent and it was reasonably clear, even to Oresme, that measurement of these "quantities" was not possible. On the other hand, Oresme noted that "every circular curvature is uniform and vice versa, and every other curvature is difform." And he could measure circular curvature: "its intensity is measured by the quantity of the radius of the circle whose curve is… the circumference, so that by the amount the radius is less, so proportionally the curvature will be greater." (Clagett, 1968, p. 221)

This was quite an insight for the fourteenth century. However, Oresme was less successful when he tried a counting argument for figuring out the total number of combinations of simple six types of quality figures. Although Hebrew writers before

Oresme, including both ibn Ezra and Levi ben Gershon, had shown how to calculate such values, Oresme somehow made an error. In fact, combinatorial reasoning was not to be fully developed in Europe for another two hundred years.

While mathematics at the universities was clearly important, with the practitioners delving into philosophical questions coming out of Aristotle, the growth of commerce in Italy beginning in the thirteenth century spawned a different kind of mathematics. The Italian merchants of the Middle Ages generally were what today we might call venture capitalists. They traveled themselves to distant places in the East, bought goods which were wanted back home, then returned to Italy to sell them in the hope of making a profit. These traveling merchants needed very little mathematics other than the ability to determine their costs and revenues for each voyage. By the early fourteenth century, a commercial revolution spurred originally by the demands of the Crusades had begun to change this system greatly. New technologies in shipbuilding and greater safety on the shipping lanes helped to replace the traveling merchants of the Middle Ages with the sedentary merchants of the Renaissance. These "new men" were able to remain at home in Italy and hire others to travel to the various ports, make the deals, act as agents, and arrange for shipping. Thus, international trading companies began to develop in the major Italian cities, companies that had a need for more sophisticated mathematics than did their predecessors. These new companies had to deal with letters of credit, bills of exchange, promissory notes, and interest calculations. Business was no longer composed of single ventures but of a continuous flow of goods consisting of many shipments from many different ports en route simultaneously. The medieval economy, based in large part on barter, was gradually being replaced by a money economy.

The Italian merchants needed a new facility in mathematics to be able to deal with the new economic circumstances, but the mathematics they needed was not the mathematics of the quadrivium, the mathematics studied in the universities. They needed new tools for calculating and problem solving. To meet this need, a new class of "professional" mathematicians, the *maestri d'abbaco* or abacists, appeared in early fourteenth century Italy. These professionals wrote the texts from which they taught the necessary mathematics to the sons of the merchants in new schools created for this purpose.

In addition to the algorithms of the Hindu-Arabic number system, the abacists taught their students methods of problem solving using the tools of both arithmetic and Islamic algebra. The texts written by the abacists, of which several hundred different ones still exist, are generally large compilations of problems along with their solutions. These include not only genuine business problems of the type the students would have to solve when they joined their fathers' companies, but also plenty of recreational problems typical of the kind found in modern elementary algebra texts. There were also sometimes geometrical problems as well as problems dealing with elementary number theory, the calendar, and astrology. The solutions in the texts were written in great detail with every step fully described, but, in general, no reasons were given for the various steps. Perhaps the teachers did not want to disclose their methods in written form, fearing that then there would no longer be any reason to hire them. In any case, it seems clear that these abacus texts were designed not only for classroom use, but also to serve as reference manuals for the merchants themselves. A merchant could easily find and readily follow the solution of a particular type of problem without the necessity of understanding the theory.

Among the many extant abbacist texts, we consider two examples written in Montpellier early in the fourteenth century. Of course, Montpellier was not only a university town, but also a center for trade in the south of France. Although the texts were written by abbacists from Italy, it is not surprising that they travelled to Montpellier. The earlier text was the *Tractatus algorismi*, written by Jacobo da Firenze in 1307. This work was a model of an abbacist text, containing problems on such topics as the arithmetic of fractions, the rule of there, partnership, alloying, and even some practical geometry. There is some scholarly controversy over whether it in fact originally included a chapter on quadratic equations, because only one of the three extant manuscripts contains such a chapter.

On the other hand, it is clear that Paolo Girardi, also from Florence, did include the basic al-Khwārizmī rules for quadratic equations in his own abbacist manual of 1327. He tried to appeal to his students by writing the problems as practical ones:

> A man loaned 20 *lire* to another for two years at compound interest. When the end of 2 years came he gave me 30 *lire*. I ask you at what rate was the *lire* loaned per month?
>
> There is a man who went on 2 voyages. On the first voyage he earned 12 *denarii*. On the second voyage he earned at the same rate that he made on the first voyage, and at the end he found [he had] 100 *denarii*. I ask you with how many *denarii* did he leave? (Katz *et al*, 2016, pp. 211-212).

Interestingly, Paolo did not write his problems so they would have simple whole number answers. The answer to the first problem was that the *lire* were loaned at the rate of the root of 600 minus 20 *denari* per month. The answer to the second problem was that the merchant began with the root of 1300 plus 38 *denari*.

5 Conclusions

There were clearly mathematical geniuses in all three of these medieval mathematical cultures, most of whom shared a common mathematical background of the Hindu-Arabic number system, the works of Aristotle and Euclid's *Elements*. Anyone with an interest in mathematics had certainly studied the *Elements*, and quite possibly knew other works of Euclid. Also, he was familiar with many texts of the great philosopher and believed that any philosophical work had to contend with Aristotle's thoughts, either by attacking or defending them. Finally, it was the Muslims who brought the Hindu-Arabic system to Europe from the East; the Jews learned it from them; and the Catholics eventually mastered it as well, learning both from translations from the Arabic and from the material Fibonacci brought back from his travels to Muslim lands.

Yet starting with the same basic information, the mathematicians from the three cultures were interested in different mathematics. Algebra, of course, had been developed in eastern Islam, but it seems that the only algebra work available was that of al-Khwārizmī from the ninth century. The more advanced Muslim algebraic work was not available in Spain, although Fibonacci discovered some of it in his travels. In any case, the Spanish Muslims were not apparently interested in algebra. But as we have seen, there was definite interest in geometry, both practical geometry (which was also of interest to the other cultures) and also quite theoretical geometry. Muslim geometers had mastered the basic Greek techniques of proof and did not hesitate to prove all sorts of interesting results. And along with geometry, there was also trigonometry, important for astronomy, which in turn was necessary for religious purposes.

The Jews too were not interested in algebra, at least until the late fourteenth century. And even then, the Hebrew work in algebra was basically limited to material found in al-Khwārizmī. Of course, just as in Islam, quadratic equations were solved earlier in the context of measuring areas and lengths, but the methodology was the older one of manipulation of geometric figures rather than the newer methodology of "things". On the other hand, Jewish authors seemed to be very interested in geometry. There were quite a few authors who investigated advanced geometric topics, being careful to give strict Euclidean proofs. And there were also several investigations of topics in combinatorics, both intuitively and, in the case of Levi ben Gerson, with careful proofs. Levi and others also investigated some pure number theoretic problems. And, of course, trigonometry was studied, since, as for the Muslims, this was necessary for astronomy and therefore for calendrical questions.

Catholic Europe was interested in mathematics different from the kinds studied by the Jews and Muslims. First of all, there was more interest in developing algebra beyond al- Khwārizmī. Even Fibonacci had problems from later authors, and certainly Jordanus de Nemore developed additional material. Interestingly, however, many of the algebraic techniques developed in eastern Islam did not reach Europe during the medieval period. On the other hand, there was little interest in advanced geometry. Euclid was mastered, and there was some interest in the works of Archimedes, but there was nothing in Catholic Europe like the advanced geometry developed in Muslim Spain. In addition, even though astronomy was part of the medieval university curriculum, there was little development of trigonometry beyond what was already known in Greece. It was not until the work of Regiomontanus in the mid-fifteenth century that Europe had a trigonometric work comparable to the works written in Muslim Spain centuries earlier. And probably one of the reasons for this was that astronomy was not nearly so important for calculations involving the Julian calendar as it was for both the Muslim and Jewish calendar. Similarly, the subject of combinatorics, of interest to the Jews and also to the Muslims of North Africa, was barely mentioned by Catholic mathematicians, although Jordanus de Nemore did display the Pascal triangle as part of a discussion of ratios. The most important mathematical topic studied in Catholic Europe – and not in Muslim or Jewish Europe – was the set of developments coming out of the study of Aristotle's physical theories. In particular, as we have noted, mathematicians in Oxford and Paris were very interested in the ideas of motion, and it was the study of kinematics as well as mechanics that was crucial the work of Galileo and others during the Renaissance.

Although mathematical geniuses existed in each of the three religious groups we have considered, men who could successfully attack any interesting problem, it seems clear that the culture in which they lived was crucial in their actual choice of problems to consider. We can see in this study of mathematics in medieval Europe, as in other times and places, that mathematics is not, and indeed cannot be, a culture-free subject.

REFERENCES

Clagett, M. (1968). *Nicole Oresme and the medieval geometry of qualities and motions.* Madison: University of Wisconsin Press.

Freudenthal, G. (1995). Science in the medieval Jewish culture of southern France. *History of Science* 33, 23-58

Garcia-Ballester, L., Ferre, L., & Feliu, E. (1990). Jewish appreciation of fourteenth-century scholastic medicine. *Osiris* 6, 85-117.

Huff, T. E. (1993). *The rise of early modern science: Islam, China, and the West*. Cambridge: Cambridge University Press.

Ibn Ezra, Abraham (1995). *The Secret of the Torah: A translation of Abraham Ibn Ezra's Sefer yesod mora ve-sod ha-Torah*. Translated by H. N. Strickman. New York: Jason Aronson, Inc.

Katz, V. J., Folkerts, M., Hughes, B., Wagner, R., & Berggren, J. L., eds. (2016). *Sourcebook in the mathematics of medieval Europe and North Africa.* Princeton: Princeton University Press.

Sā'id al-Andalusī (1991). *Science in the medieval world: Book of the categories of nations.* Translated and edited by Semar'an I. Salem and Alok Kumar. Austin: University of Texas Press.

FORMATIVE YEARS: HANS FREUDENTHAL IN PREWAR AMSTERDAM

Harm Jan SMID
Delft University of Technology (retired), Lammenschansweg 51, 2313 DJ, Leiden, The Netherlands
harmjansmid45@gmail.com

ABSTRACT

Hans Freudenthal started his public career in mathematics education after the war. During the war, when he was expelled by the Germans from the university, he made a thorough study of the didactics of teaching arithmetic. Freudenthal himself made on several places remarks about the influences he underwent before the war, he even suggested that his "framework" on mathematics education was already formed then. Although he was in those year active as a mathematician in the first place, there can be no doubt that he was then already seriously interested in the problems of the teaching of mathematics. He had inspiring discussions with other mathematicians with unorthodox ideas about the teaching of mathematics and read publications that helped him to develop his own ideas. The seeds that were sown in those years would bear fruit long afterwards.

1 Introduction

On the morning of the 16th of November in 1930, a young man 25 years of age arrived with the night train from Berlin in Amsterdam. It was a mathematician who just had finished his PhD with a thesis on topology. He came to Amsterdam on invitation of L.E.J. Brouwer, professor in Amsterdam and a mathematician who not only was quite famous, but also highly disputed. The name of this young mathematician was Hans Freudenthal, and he came to Amsterdam as a next step in his career as a mathematician, a career that of course only just had started. It was a successful career move; since during the years that followed he produced a number of papers that established his reputation as a first class mathematician. When he came to Holland, he will not have foreseen that he would spend the rest of his long life in that country, or that he would become perhaps even more famous as a mathematics educator than as mathematician.

That career in mathematics education, what could be called a second career, started more than twenty years later, and went well under way in the sixties and seventies of the last century. In those years he was president of the ICMI, he organized the first ICME conference in 1969 in Lyon and he founded in the same year the journal *Educational Studies in Mathematics*. The ICMI honored him by creating a "Hans Freudenthal award" for "a "major cumulative program of research" in the field mathematics education, of course a clear sign of his importance.

Although Freudenthal started his public activities in mathematics education after the war, it is well known that he began to work seriously in this field during the war. Freudenthal was from a Jewish family, and although his marriage to a non-Jewish Dutch girl gave him some protection against immediate deportation, he lost his job at the University of Amsterdam. He was arrested, set free again, and later on send to a labour camp from which he escaped. On the whole he had, unplanned and no doubt unwelcome, a lot of spare time. His wife, Susanna Lutter, was highly interested in education, especially for the young children they had together, and she stimulated her husband in teaching arithmetic to their children. Freudenthal took an interest in observing how his children learned arithmetic and he began to read all he could lay his hands on about the didactics of teaching arithmetic. He filled 300 pages of a notebook with critical comments on what he had read and he composed a manuscript of 103 pages about the didactics of teaching arithmetic. The manuscript was never finished or published, but can be found in Freudenthal's personal archives that are kept in Haarlem, together with the notebook.

Figure 1. Hans Freudenthal 1905-1990.
Noord-Hollands Archief, Archief Freudenthal, inv.nr. 1914.

These activities are usually considered as his start in mathematics education. His recent biographer goes even a step further when she says about his post-war interest in mathematics education: "The seeds for this interest were, of course, already be sown during the occupation". (Bastide-van Gemert, 2015, p. 52) The use of the word "seeds" supposedly sown during the war suggests that before the war such an interest did not exist. Now that is certainly not the case. Freudenthal himself, in his autobiographical sketches, says that he was interested in educational matters already at a young age. He says: "The framework of my activities concerning mathematics education was already been formed in 1942, *or maybe even ten years before"*. (Freudenthal 1987a, p. 344, italics by the author of this paper) In a letter to Geoffrey Howson, he says that his remark in the preface of his *Mathematics as an Educational Task*, that "the first suggestions to occupy myself theoretically with education came from my wife", is most likely not correct. He wrote to Howson that "I should perhaps have said that she reinforced my interest in mathematics education". (Freudenthal, 1983). Apart from Freudenthal's own remarks, what can be said more about the formation of this framework for his interest concerning mathematics education? What are the indications that the seeds for his later didactical activities were in fact already sown in the pre-war years? Were there in those years people in the Netherlands who might have influenced him? I think there were and my aim in this presentation is to tell you something about them.

2 Some remarkable personalities

In the preface of his *Mathematics as an Educational Task,* Freudenthal remarks about L.E.J. Brouwer the following: "My educational interpretation of mathematics betrays the influence of L.E.J. Brouwer's view on mathematics (though not on education)". (Freudenthal, 1973) Freudenthal was Brouwer's assistant and we might suppose that his influence was exerted in the years before the war. So, our first question is: what was Brouwer's view on mathematics and how should we interpret this remark by Freudenthal?

There are others, not mentioned in the preface of his *Mathematics as an Educational Task*, but elsewhere, who certainly influenced him in the pre-war years. One of them was David van Dantzig, born in 1900, so five years older than Freudenthal. When the latter arrived in Amsterdam, Van Dantzig had already finished his mathematical studies there. But although he had a job elsewhere, he still frequently visited the Amsterdam mathematicians and he knew Freudenthal personally. Van Dantzig had highly original ideas about mathematics education and had published some articles about it. (Smid, 2000) He became after the war professor of mathematics in Amsterdam and when he died unexpectedly in 1959, Freudenthal held the memorial speech for the Dutch Mathematical Society. (Freudenthal, 1960) In this speech Freudenthal fully recognizes the influence that David van Dantzig had on him in those early years in Amsterdam. So we will have to discuss the ideas of Van Dantzig, and when we speak about Van Dantzig, we cannot but speak also about Gerrit Mannoury, a man who on his turn influenced Van Dantzig deeply and who was the colleague of Brouwer in Amsterdam.

The last person I want to speak about is usually connected with Freudenthal's activities after the war. It is Tatyana Ehrenfest-Afanassjewa, who was the heart and soul of the Dutch Mathematical Working Group, of which Freudenthal became a member after the war. But in an article by Pierre van Hiele, on occasion of Freudenthal's 70th birthday, Van Hiele recalls how enthusiastic Freudenthal in the early thirties was about one of the publications of Tatyana Ehrenfest. Van Hiele was then one of Freudenthal's students, and he describes how Freudenthal urged his students to read a small booklet, the *Uebensammlung.* (Van Hiele, 1975) Certainly in this case, it seems most appropriate to speak about a seed that was sown in Freudenthal's mind in those years, a seed that would bear fruit much later.

3 Luitzen Egbertus Jan Brouwer

L.E.J. Brouwer – Bertus for his intimi – was a mathematical genius. This reputation is based in the first place on his fundamental contributions in topology. In the years 1909-1914 he published a series of articles which, as his biographer wrote, contains not only spectacular results, such as the theory of dimension and the fix-point theorem, but also furnished new tools to breathe new life into the research of topology, which was more or less in a dead end. (Van Dalen, 2013)

Figure 2. L.E.J. Brouwer
Picture made available by the Brouwer-archives

The other element that contributed to his fame is his work on the foundations of mathematics. Brouwer was not only a mathematician, but also a philosopher with a strong mystical accent. He completely rejected the two then current theories on the foundations of mathematics; the logicism by Russell, and the formalism by Hilbert. For Brouwer, mathematics was rooted in, and created by the human mind. Crucial is the observation of the permanency of objects during consecutive points of time. That fundamental intuitive sensation of time gives, according to Brouwer, birth to the concept of counting and the series of natural numbers. All mathematics is built on this series. The continuum on the other hand, cannot it be constructed from natural numbers, nor does the continuum consist of all numbers. The continuum is in fact the separation between two different numbers, and you can construct new numbers on this continuum, but that does not mean that the continuum **is** merely a collection of numbers. As a consequence, Brouwer accepted the existence of mathematical objects only when they could be constructed in a finite number of steps. This principle gave birth to his intuitionistic mathematics, in which parts of the traditional mathematics were rejected, or at least were questioned.

Brouwer had the essence of his philosophical ideas already exposed in his PhD of 1907, but during his work on topology this part of his work remained in the shadows. It was not before the twenties that he returned with new energy to the foundations of mathematics. Brouwer was then, with Hilbert as editor in chief, one of the editors of the *Mathematische Annalen*, and so far both man respected each other. Hilbert interpreted Brouwer's new publications on intuitionism however as an almost personal attack on his ideas about the foundations of mathematics, and moreover, he feared that the future of mathematics was seriously endangered if Brouwer's ideas would prevail.

The differences between Hilbert and Brouwer resulted in a clash that deeply divided the German mathematical community. In 1927 Brouwer gave s series of lectures on intuitionism in Berlin, where he was greeted with enthusiasm. There was a certain rivalry between Berlin and Göttingen, the kingdom of Hilbert, so someone who had the guts to stand up against Hilbert received admiration in Berlin, certainly from the Berlin students. Freudenthal, being a student then in Berlin, made himself acquainted with Brouwer's ideas, followed a seminar on intuitionism, attended Brouwer's lectures and posed – in writing - some intelligent questions. (Freudenthal, 1987b p.10) The two men met, and in the following years Freudenthal sent his results in topology to Brouwer. In the summer of 1930, Brouwer invited him to come to Amsterdam to work there as his assistant. Freudenthal worked in Amsterdam on topology, and although he of course was well aware of Brouwer's intuitionism, that theory doesn't play an important role in his work. So what does Freudenthal mean by saying that "his educational interpretation of mathematics betrays the influence of L.E.J. Brouwer's view on mathematics"?

That can be explained best by a quotation of some words of Brouwer himself. In 1946, in a speech to his former colleague Gerrit Mannoury, Brouwer spoke about his early years as mathematics student in Amsterdam, when he could see mathematics only as, I quote, "a collection of truths, fascinating by their immovability, but horrifying by their lifelines". But Mannoury, said Brouwer, had shown to him that the work of a mathematician was something else then collecting such lifeless truths. The undertone of Mannoury's exposure of mathematics had been – I quote again – as follows:

> Look what I have built for you out of the structural elements of our thinking. – These are the harmonies I desired to realize. This is the scheme of construction which guided me – Behold the vision which the completed edifice suggests to us, whose realization may perhaps be attained by you or me on one day" (Van Dalen, 2013, pp. 43-44).

The tone may sound a bit swollen and exaggerated, but the meaning is clear: mathematics is made by living people; it has to be constructed by mind. Mannoury, like Brouwer himself, did not believe in eternal mathematical truths. Mathematics is not a Platonic world lying outside just waiting to be discovered, nor is it just a formal game without any bond with reality. The quotations from Brouwer can be found in a recent biography of Brouwer that has as a subtitle: *How mathematics is rooted in life*. (Van Dalen 2013) That subtitle is also a quotation, in this case from a letter of Brouwer to his PhD supervisor Korteweg, explaining to him what he wanted to do in his thesis: to show how mathematics is rooted in life. The same basic idea is summarized in a famous formulation by Hermann Weyl, in the early years one of Brouwer supporters: "Mathematics is more a way of doing then a theory".

Brouwer himself was not interested in education, only in high level mathematics and in philosophy. But Freudenthal of course really was an educator, and in this capacity one of his adages was: "mathematics as a human activity". Brouwer could have said the same, although he would not apply that to mathematics education for children, as Freudenthal did. There can be no doubt that Freudenthal was right: without being an intuitionist himself, his educational interpretation of mathematics *does* betray the influence of L.E.J. Brouwer's view on mathematics.

4 Gerrit Mannoury and David van Dantzig

In his memorial speech for David van Dantzig in 1960 for the Dutch Mathematical Society, Freudenthal told that even before he came to the Netherlands, he had read two articles written by Van Dantzig about the didactics of mathematics.(Freudenthal, 1960, p. 61) That may seem a bit unlikely at first sight, since the articles are in Dutch, but maybe Freudenthal wanted in the period from August 1930, when he received the invitation by Brouwer to come to Amsterdam, until his depart in November to learn some Dutch, and he could very well have done so by reading Dutch articles. Freudenthal had a talent for learning languages, and German and Dutch are relatively cognate; so with a grammar and glossary at hand, he could certainly do such a thing. He said in his speech that he had to admit that he was very impressed by these articles. Who was Van Dantzig and what was the content of these articles?

Figure 3. David van Dantzig in 1934

David van Dantzig – not to be confused with the much more famous American mathematician George Dantzig, the so called father of linear programming – was born in 1900 in Amsterdam. He first started studying chemistry, but financial reasons made an end to this study. Gerrit Mannoury, whom we just met as a colleague of Brouwer, was then a lecturer at the university teaching mathematics to the chemistry students, and Van Dantzig, in fact much more interested in mathematics than in chemistry, wrote a letter with some questions on mathematics to Mannoury. When Van Dantzig stopped with his study in chemistry, both men stayed in contact and when van Dantzig financial circumstances improved, Mannoury persuaded him to study mathematics. Van Dantzig remained close friends with Mannoury all his life and was influenced deeply by him.

Mannoury, who as we have seen also influenced Brouwer, was a most remarkable man. He was professor of mathematics, but did not hold a university degree. He had been a schoolteacher, but in mathematics he was almost completely an autodidact. Mannoury had not only highly original ideas about mathematics, but also about the teaching of mathematics. The way mathematics was taught and the content of school mathematics was in his view worthless. There was no reason at all to bore the children with all that rubbish. Mathematics teaching was in those days always defended for its supposed "transfer" or "formative value", as a kind of gymnastic for the mind. Learning mathematics disciplines the mind and helps you to think logically. Nonsense, said Mannoury. When you learn mathematics, you just learn mathematics and nothing more, he declared, learning mathematics is like learning to play chess: you learn just that and no more. It will come to no surprise that Mannoury's ideas were not very popular or influential in the world of the Dutch mathematics teachers.

Figure 4. Gerrit Mannoury in 1917

Van Dantzig was in complete agreement with Mannoury's views and in one of the articles Freudenthal mentioned, he discusses the problem of transfer. Its title is (translated) *On the social value of teaching of mathematics.* (Van Dantzig, 1927) The traditional teaching of mathematics teaching has, according to Van Dantzig's opinion, simply no social value at all, and it should be better to teach the great majority of the children only some simple, essential techniques that they can use in daily life, and nothing more. Only for a small minority which is going to work in professions where they made use of mathematics, teaching more mathematics is useful. The teaching of mathematics could have only some *social* value if it was taught completely different: in a much more "living form", with a strong linguistic accent. Van Dantzig was not very clear how this could be done, and I think we do him no injustice if we suppose he did at that time not have a clear vision how that should be done himself.

Freudenthal remarks in his autobiographical sketches that he rejected the idea of transfer as long as he could remember. (Freudenthal, 1987, p. 359) Some pages earlier he is a bit more specific, where he says that he is convinced that he rejected the idea that mathematics was instrumental in “learning to think” already in 1932, when he gave a didactical seminar at the university in Amsterdam. (Freudenthal, 1987, p. 338) It is impossible to say if the reading of Van Dantzig article from 1927 gave him this conviction, or that its reading just corroborated an already existing, perhaps vague opinion with Freudenthal, but there can be little doubt that Van Dantzig influenced the young Freudenthal in this aspect.

The other article by Van Dantzig that Freudenthal referred to is about a problem that was heavily disputed in The Netherlands in those years: the way how mechanics should be taught on secondary education. The article had lost most of its relevance for long, but it contains some remarks that sound familiar for everyone who is acquainted with Freudenthal’s ideas. Discussing the way mathematics should be taught, Van Dantzig remarks that “ready-made mathematics does not arouse anybody’s interest“. It is essential, he writes, that also in schoolbooks the process of *mathematization* is actively carried into effect; otherwise mathematics remains a dead object. (Van Dantzig, 1929, p. 97) Almost fifty years later Freudenthal would, in his *Mathematics as an Educational Task*, write a chapter with the title *Organization of a field by mathematizing*. In that chapter he says: “There is no mathematics without mathematizing. (…) This means teaching or even learning mathematics as mathematization”. (Freudenthal, 1973, p. 134)

In his autobiographical sketches Freudenthal wrote that Van Dantzig belonged to the group of colleagues and students with whom he in the early thirties discussed the situation of Dutch mathematics teaching. It may be clear that both young men shared many ideas.

5 Van Dantzig and his ICMI-report

Van Dantzig wrote no more articles on mathematics education for more than twenty years, but in 1955 he published *The function of mathematics in modern society and its consequence for the teaching of mathematics*. It was a report for the ICMI conference of 1954 in Amsterdam. In a way, it contains the answer by Van Dantzig to the problem he posed more than two decades earlier: what is the social value of mathematics? Then, while rejecting the idea of transfer, he could not really find an answer, but now he could. (Van Dantzig, 1955) Mathematics had become an indispensable tool in modern society, and the use of mathematical models could be found everywhere. As a consequence, Van Dantzig pleads for a thorough reform of the traditional school mathematics into what he called a “consumers-mathematics”, in which concepts as mathematical models, testability, and what he calls good working knowledge of graphs, algebraic computing and elementary calculus are taught. His rejection of the long outdated school mathematics is an echo of his opinion of more than twenty years earlier, but his idea for a “consumers-mathematics” is new.

Van Dantzig’s report was much cited in those years in The Netherlands, but Freudenthal, while expressing his admiration for Van Dantzig didactical publications from 1927 and 1929, did not mention it at all in his memorial speech of 1960. That could be coincidental, but I think also that Van Dantzig’s report contained elements that Freudenthal did not appreciate. One of these elements will have been Van Dantzig’s plea to teach this consumers-mathematics as cost-effectively as possible, and to treat the student results as a mass product, to which we should apply all methods for a satisfactory quality control. That would include the extensive use of statistical methods, something Freudenthal abhorred in education. Clearly, in Van Dantzig's approach not the individual student stands in the center, but the benefit for the society as a whole. Freudenthal essentially wanted to educate individuals, and teaching mathematics was an educational task for him. For Van Dantzig, it was much more an social or economical task, to be performed for the benefit of the modern society as a whole. Van Dantzig's ideas fitted perfectly in the first decades after the war, when the reconstruction of the Western societies after the war, and the *Cold War* rivalry with the communist countries were central issues. Freudenthal's ideas on mathematics education were much more congenial with the educational climate of the seventies, when the optimal development of the individual stood in the centre.

A second reason why Freundertghal did not mention Van Dantzig's publication of 1955 might have been that Freudenthal had been very active in those years in working on a new mathematics curriculum for the Dutch schools. That new curriculum was certainly a step forward, but compared with the ideas exposed by Van Dantzig in his report, one has to admit that it was still rather old fashioned. Freudenthal however was very happy with it and saw no reason for a new curriculum at that moment. One might suspect that he did not share on that time already Van Dantzig's ideas about such a consumers-mathematics.

Some thirty years later however, such a consumers-mathematics was introduced at last. It was intended for those students who had to use mathematics as a tool, but who did not receive much instruction in mathematics in their tertiary education. Many elements described by Van Dantzig in his report of 1955 can be traced in this new program. The institute founded by Freudenthal, then still called the I.O.W.O., was the main driving force behind its introduction, and no doubt Freudenthal, then still very active, strongly supported it. But of course, there were differences. Van Dantzig focused on the content, not on the didactical approach as Freudenthal and his followers did, and Van Danztig's ideas about "cost-effectively" mathematics teaching and "quality control" of the teaching results were completely put aside.

6 Tatyana Ehrenfest-Afanassjewa

Freudenthal was not the only foreigner who deeply influenced Dutch mathematics teaching in the 20th century. The other one was a Russian lady, Tatyana Afanassjewa, born in Kiev 1876. At a young age she moved to St. Petersburg, where she attended secondary education, including the local gymnasium for girls. She followed courses on pedagogy and studied physics and mathematics at the St. Petersburg Woman University. She was no doubt a talented girl, and after her studies in Petersburg she went to Göttingen where she studied with Klein and Hilbert. There she met the physicist Paul Ehrenfest, with whom she married. The young couple returned to St. Petersburg where Ehrenfest tried in vain to find a job. Tatyana however could work there as a mathematics teacher and was involved in several experiments to modernize the teaching of mathematics.

In 1912, Paul Ehrenfest was at last appointed in Leiden as professor in theoretical physics, as successor of Lorentz. Tatyana followed her husband to Leiden, and although she felt never really at home in The Netherlands, she stayed there until her death in 1964.[1]

Figure 5. Tatyana Ehrenfest – Afanassjewa around 1910

Before World War II she returned several times for months during the summer to the Soviet Union to assist in the training of mathematics teachers there. In The Netherlands she taught only for one year mathematics on a school for secondary education, but her unofficial role and influence were much more important. Soon after her arrival she published a Dutch translation of one of her Russian lectures on the teaching of geometry, to "introduce herself to the Dutch mathematics teachers", as she formulated it. She succeeded in forming a discussion group of interested mathematics teachers, a group that in 1936 was transformed into the *Mathematics Working Group,* belonging to the Dutch branch of the still existing *New Education Fellowship* .[2]

In 1924, she published a brochure that would have important consequences. Its title (translated from the Dutch) was "What could and should the teaching of geometry offer to a non-mathematician". (Ehrenfest-Afanassjewa, 1924). The general opinion then in The Netherlands, but certainly not only there, was that the teaching of geometry in a deductive way, more or less in a Euclidean manner, was profitable for everybody: it made you think better and more logically. Ms. Ehrenfest had participated in 1920 and 1921 in a series of meetings at the department for pedagogy of the University of Amsterdam, in these meetings this question of transfer was discussed. (Van Dantzig, 1927) One of the other participants was Gerrit Mannoury, as we have seen a strong opponent of the idea of transfer. It seems likely that this brochure was a result of these meetings. Tatyana Ehrenfest was in favor of the idea of transfer, but certainly not in a naïve or unconditional way. The way geometry was taught in those days in The Netherlands: in a strict deductive way, right from the start, was in her opinion only detrimental for the possibility of transfer. She pleaded for a "propaedeutic"

[1] More biographical details about her life can be found in the preface by Bruno Ernst in her *Didactische Opstellen Wiskunde* (Ehrenfest-Afanassjewa 1960)

[2] Now the *World Education Fellowship*

introductory course in geometry for young children, without the proving of theorems, in which they could develop their spatial ability. Only afterwards a more systematics course, in which geometry is taught in a more traditional way, was appropriate.[3]

In his autobiographical sketches Freudenthal discusses her article on the teaching of geometry of 1924, which he, as he says, read again for the occasion. He does not say when he read the article for the first time, but it seems unlikely that he did not read already much earlier. Freudenthal wrote to Geoffrey Howson in the letter that we have already cited before, that had met Tatyana Ehrenfest already before the war, when he attended once or twice a seminar on physics at her house (Freudenthal, 1983). Freudenthal added that he did not attend her didactical seminars on mathematics in those years. His participation in the Mathematical Working Group started only after the war, in 1947.

In the same letter to Howson, Freudenthal wrote that he partly developed his ideas "on the teaching of mathematics by opposing hers", certainly referring also to their continuing discussion about the problem of transfer. They published together in 1951 a brochure with the title "Can the teaching of mathematics contribute to educate the ability of thinking?" in which they discuss their different opinions about these problems. As Pierre van Hiele however has pointed out later, these differences are not as big as they seem at first sight, and are partly due to a vague formulation of the question. (Van Hiele, 1975).

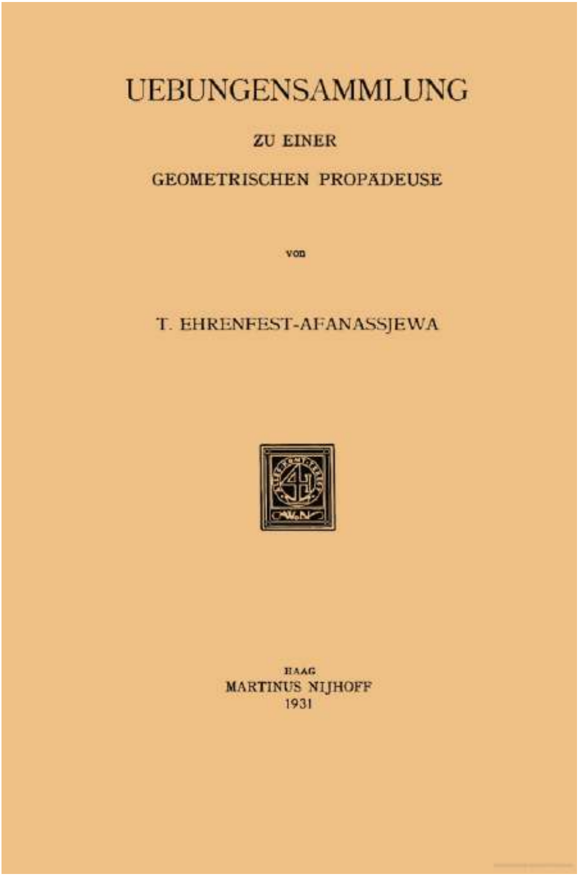
7 The 'Uebensammlung'

While we cannot be completely sure if Freudenthal read Ms. Ehrenfest's publications of 1915 and 1924 already before the war, he certainly read her publication of 1931 soon after it had appeared. Van Hiele described Freudenthal's enthusiasm about this booklet, her *Uebensammlung zu einer geometrischen Propädeuse* . (T. Ehrenfest-Afanassjewa 1931). Pierre van Hiele was in those years one of Freudenthal's students in Amsterdam and he attended a didactical colloquium that was organized by Freudenthal. Each of the students had to present some piece of mathematics and, as van Hiele adds: "Freudenthal very cunningly choose those parts that were rather badly taught to us, so we could profit from it in two ways". The presentations of the students gave in a natural way rise to didactical discussions, and Freudenthal drew their attention to the recent published work of Ms. Ehrenfest. He lent the booklet for a few days to study at home to all the students who attended the didactical colloquium. (Van Hiele 1975).

The *Uebensammlung* is a collection of problems that should be posed within the framework of the propaedeutic geometry course that she already had proposed in 1924. She had experimented with such a propaedeutic course both in Russia before she came to Leiden as well as in The Netherlands, and she claims that when children followed this propaedeutic course, they performed better in the systematic course afterwards. The *Uebensammlung* consists of two parts: an introduction and the collection of problems. In the introduction she

[3] Her brochure elicited a firm reply from E.J. Dijksterhuis, who defended the deductive approach right from the start. Their discussion gave the impulse to the birth in 1925 of the magazine *Euclides*, now the still existing magazine of the Dutch Mathematics Teachers Association.

repeats some aspects of her 1924 brochure and explains how the collection of problems should be used. She remarks that many of the assignments should be connected with other school activities, such as drawing, making models, using toys, tools, clothes, or visiting factories and making outdoor excursions. The whole booklet breathes an atmosphere that is completely the opposite of the traditional geometry lessons of those days.



UEBUNGENSAMMLUNG

ZU EINER

GEOMETRISCHEN PROPÄDEUSE

von

T. EHRENFEST-AFANASSJEWA

HAAG

MARTINUS NIJHOFF

1931

As an example the assignments 30 and 31 are shown below. (translated from the German original)

> 30. Someone walks along the edge of a quadrangular square, departing from the middle of one of the sides. Which angle did he pass through when he arrives at his starting point? The same question for a triangular, a pentagonal and a round square. The same question when he describes a shape in the form of the number 8.
>
> 31. The pupil should, on his way from home to school, on a piece of cardboard draw all angles he passes through on each crossing point he passes, by which he must determine which angle the front of his house makes with the front of the school. Let him control his result by means of a map when the streets are not straight ones, and draw his attention to possible mistakes.

E.W.A. de Moor has pointed to the connection between Ms. Ehrenfest and the Russian mathematics educator Semen Il'ich Sjochor'-Trotskij, whose courses she followed in St.

Petersburg (De Moor 1999, pp. 271-274). There is no doubt some affinity between the ideas of Sjochor'-Trotskij and Ms. Ehrenfest, but De Moor also underlines that the problems in the *Uebensammlung* are not only much more realistic, but also much more creative and original that those found with Sjocher'-Trotskij.

Freudenthal calls the *Uebensammlung* a "masterpiece", on condition that you do not use it as merely a propaedeutic step on the way to a systematics course. (Freudenthal 1987) He adds that he only slowly began to see the extra-value of these problems; it was in fact when he began to see the outlines of his *realistic mathematics education.* In the seventies, the former I.O.W.O., now the Freudenthal Institute developed a new curriculum for teaching geometry in primary schools. This curriculum had remarkably much in common with Ms. Ehrenfest's view on the intuitive introduction to geometry and its content had also a striking resemblance to several activities described in her *Uebensammlung.* (De Moor, 1999, p. 686).

In his letter to Howson of 1983, Freudenthal writes that he regrets that he mentioned Tatyana Ehrenfest as late as on page 405 of his *Mathematics as an Educational Task,* where he tells about her *Uebensammlung.* He should have mentioned her, he wrote, already in the preface, among those who influenced him. Now this is a curious mistake by Freudenthal, since he mentions her not as late as on page 405, but already on the pages 119-120. What he remarks there, makes clear that her influence was even more fundamental. One of the elements that Freudenthal underlines again and again is that the teaching of mathematics should be done by the method of what he called *guided re-invention.* In *Mathematics as an Educational Task* a complete chapter is devoted to that method. In page 120 Freudenthal writes the following illuminating phrase: "It [that is the method of re-invention] dawned upon me when I was studying the work of T. Ehrenfest and her disciples, both in their classrooms and in discussions with them". Visiting their classrooms and having discussions with them will have taken place after the war, when Freudenthal participated in the Mathematics Working Group. But "studying the work of T. Ehrenfest" refers certainly also to the reading of her *Uebensammlung*, since he mentioned that work on the page before as a "beautiful older example" of an analysis of active geometry . When Freudenthal read the *Uebensammlung* in the early thirties, he certainly did not fully realize its importance, as he writes himself. But, to paraphrase his biography, the seeds were sown.

8 Conclusion

When Freudenthal came from Berlin to Amsterdam, he experienced what we would call now a culture shock. The mathematics department in Amsterdam was small and teaching was old fashioned. Brouwer was a man not easy to work with, as "almost as approachable as a minefield", as somebody wrote. The mathematical scientific climate was not in any way comparable with that in Germany. According to Freudenthal, it was only his cooperation with Brouwer's other assistant, Witold Hurewicz, that helped him to survive scientifically. (Freudenthal 1987a, p. 117)

But it is also possible to draw a more positive picture. The teaching of mathematics in the schools and universities of the Netherlands in those years might have been rather backward, things started to change. Brouwer's ideas about the foundations of mathematics helped Freudenthal to form his own educational interpretation of mathematics. He met people like Gerrit Mannoury and David van Dantzig, men with unorthodox ideas about the teaching of mathematics with whom he could have inspiring discussions. He became acquainted with the work of Ms. Ehrenfest, who had not only unorthodox ideas, but also showed him a way how these ideas could be put into practice. He could organize didactical colloquia, that were attended by students as Pierre van Hiele and Dina Geldof, who married later on and became after the war his first PhD students in mathematics education. Pierre van Hieles theoretical work, known as the Van Hiele level theory, and Dina Geldofs practical work influenced Freudenthal deeply. Their contacts, already laid before the war proved to be very fruitful after the war.

The thirties may have been difficult years for the young Freudenthal, but they were certainly not barren or fruitless. Concerning mathematics, he established his reputation, concerning mathematics education; he could build slowly on his framework. The seeds sown in those years would bear fruit later, in his second career as a mathematics educator.

REFERENCES

Bastide-van Gemert, S. la. (2015). *All positive action starts with criticism. Hans Freudenthal and the didactics of mathematics.* Dordrecht: Springer.

Dalen, D. van (2001). *L.E.J. Brouwer 1881-1966 Een biografie.* Amsterdam: Bert Bakker.

Dalen, D. van (2013). *L.E.J. Brouwer topologist, intuitionist, philosopher. How mathematics is rooted in life.* Dordrecht: Springer.

Dantzig, D. van (1927). De maatschappelijke waarde van onderwijs in wiskunde. *Euclides, 3*, 187-197.

Dantzig, D. van (1929). Woord en Werktuig. De strijd om het mechanika- onderwijs. *Euclides, 5,* 86-102.

Dantzig, D. van (1955). The function of mathematics in modern society and its consequences for the teaching of mathematics. *Euclides, 3,* 88-102. Also published in *l'Enseignement Mathématique,* tome I, 1/3, 1955.

Ehrenfest-Afanassjewa, T. (1924). *Wat kan en moet het meetkundeonderwijs aan een niet-wiskundige geven.* Groningen: J. B. Wolters.

Ehrenfest-Afanassjewa, T. (1931). *Uebensammlung zu einer Geoemtrischen Propädeuse.* Den Haag: Martinus Nijhoff.

Ehrenfest-Afanassjewa, T. (1960). *Didactische Opstellen Wiskunde.* Zutphen: W. J. Thieme & Cie.

Freudenthal, H. (1960). In memoriam David van Dantzig. *Nieuw Archief voor Wiskunde, VIII*(3), 57-73.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht: Reidel.

Freudenthal, H. (1983). Letter to A.G. Howson. Regionaal Archief Noord-Holland, Archief Freudenthal, 615, inv. nr. 38.

Freudenthal, H. (1987a). *Schrijf dat op, Hans. Knipsels uit een leven.* Amsterdam: Meulenhoff.

Freudenthal, H. (1987b). *Berlin 1923-1930. Studienerinnerungen von Hans Freudenthal.* Berlin: de Gruyter.

Hiele, P. van. (1975). Freudenthal en de didaktiek der wiskunde. *Euclides, 51,* 8-10.

Moor, E. W. A. de. (1999) *Van vormleer naar realistische meetkunde.* Utrecht: Freudenthal Institute.

Smid, H. J. (2000). De betekenis van David van Dantzig voor het onderwijs in de wiskunde. In: G. Alberts en H. Blauwendraat (Eds.), *Uitbeelden in wiskunde* (pp. 39-57). Amsterdam: CWI.

MATHEMATICS & PHYSICS: AN INNERMOST RELATIONSHIP

Didactical implications for their teaching & learning

Constantinos TZANAKIS

Department of Education, University of Crete, 74100 Rethymnon, Greece

tzanakis@edc.uoc.gr

ABSTRACT

The interplay and mutual influence between mathematics and physics all along their history and their deep epistemological affinity are explored and summarized in three *theses* with a basic didactical moral: In teaching and learning either of them, neither history should be ignored, nor the close interrelation of the two disciplines should be circumvented or bypassed. Moreover, the *key issues of the integration of history* in mathematics or physics education are addressed and a common framework is outlined based on work done in the past. These ideas are illustrated by means of *three examples*: (a) measuring the distance of inaccessible objects; (b) rotations, space-time and special relativity; (c) differential equations, functional analysis and quantum mechanics.

1 Introduction

At all levels of education, teaching and learning mathematics is usually kept separated from physics, and vice versa, corresponding to a distinction between the two sciences at the research level: mathematicians are supposed to stay in a universe of ideal logical rigor, while physicists are simply users of (possibly very sophisticated) mathematics (Tzanakis 2000, §2). This is reflected in physics education (PE), where mathematics is merely a tool to describe and calculate, whereas, in mathematics education (ME), physics is only a possible context for applying mathematics previously conceived abstractly (Tzanakis & Thomaidis 2000, p.49). Overcoming this dichotomy (which creates significant learning problems for the students of both disciplines) demands systematic research in different fields, especially when aiming at informing educational practices by reflecting on historical, philosophical and sociological aspects of scientific knowledge (Karam 2015).

However, the historical development of the two sciences, does not verify this separation. On the contrary, it is a relatively recent fact, not older than 100 years (or, even less; Arnold 1998, p.229), characterizing their development up to the 1960-70s at the peak of the effort to extremely formalize mathematics (reflected in ME in the "New Math" reform). No such clear-cut separation existed before, whereas, in the last decades there is a strong tendency to overcome it, reflected in the often deeply interwoven research in both sciences[1]. I believe this historical fact is due to the deep epistemological affinity of the two sciences, concisely expressed by numerous famous mathematicians and physicists; Galileo's phrase that "*this grand book, the Universe, … is written in the language of Mathematics*"[2]; Wigner's (1960) view of the "*unreasonable effectiveness of mathematics in the natural sciences*"; or Hilbert's

[1] E.g. Casacuberta & Castellet 1992, where 5 of the 7 (Fields medallists) authors think that developments in mathematics will be in areas related to problems in physics.

[2] *The Assayer* (1623); in Drake 1957.

6th problem in his famous 1900 list of problems[3].

On the other hand, over the last 40 years, there has been an increasing awareness of the educational community that integrating history of mathematics in ME could be a promising possible way to teach and learn mathematics because, in principle, it provides the opportunity to appreciate the evolutionary nature of mathematical knowledge, hence, to go beyond its conventional understanding as a corpus of finished deductively structured intellectual products. Especially in the context of the international HPM Study Group, this has evolved into a worldwide intensively studied area of new pedagogical practices and specific research activities (Fasanelli & Fauvel 2006 for an account up to 2000; Furinghetti 2012, Barbin 2013, Barbin & Tzanakis 2014 for later developments). However, the educational value of history outlined above is not restricted to mathematics. It is also valid for physics, because physics - just like mathematics - is not only a deductively structured corpus of knowledge (additionally constrained by its compatibility with experiment), but has been ever-changing and evolving.

These comments will be further detailed in section 2 as **three** main theses on the **ontological status** of mathematics and physics, their **historical interrelation**, and their **epistemological affinity** as scientific disciplines, constituting a historical-epistemological framework with a basic didactical moral: *In teaching and learning mathematics or physics, neither history can be ignored, nor their close interrelation can be circumvented or bypassed*. In section 3, this general point of view is elaborated by addressing the main issues related to a *history-pedagogy-mathematics/physics* (*HPM/Ph*) perspective (§2.1): *Which* history for didactical purposes, with *which role(s)* and in *which way(s)* to be realized in practice? Here much work has been done in recent years, to be briefly reviewed mainly in the light of and aiming to support an interactive interplay in the teaching and learning of mathematics **with/for** physics, and **vice versa**. In section 4 the framework of sections 2, 3 is illustrated by three examples of increasing sophistication - from junior high school to advanced undergraduate or graduate level - each one pinpointing equally well the main issues raised, presented and advocated in sections 2 & 3. The paper's main points are summarized in section 5.

2 Historical-epistemological framework

2.1 What is mathematics? What is physics?

It is customary to think of mathematics as a deductively structured, logically self-consistent corpus of knowledge; a point of view deeply influencing education and getting stronger at its higher levels. It is adopted by many mathematicians and mathematics teachers either because it leads to the quickest way of presentation (lectures, books etc), or/and because the **logical** clarity thus obtained, is often thought **identical** with the complete **understanding** of the subject. Similar comments hold for physics, additionally requiring this knowledge to be consistent with known empirical facts.

But because mathematics and physics are intellectual enterprises with a long history and a

[3] "*Mathematical treatment of the axioms of physics: To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics*" (Hilbert 1902, p.454).

vivid present, knowledge gained in their context is determined not only by the circumstances in which this knowledge becomes a deductively structured theory and/or supported by experiments, but also by the procedures that originally led or may lead to it. Thus learning mathematics or physics includes not only the "polished products" of the associated intellectual activity, but also the understanding of implicit motivations, the sense-making actions and the reflective processes of scientists, which aim to the construction of meaning. Although the "polished products" of both disciplines form the part of this knowledge that is communicated and criticized, and forms the basis for new work, didactically the **processes** producing this knowledge are equally important. Perceiving mathematics or physics **both** as a *logically structured collection of intellectual products* **and** as *knowledge-producing endeavours*[4] should be the core of their teaching and central to their image communicated to the outside world. Along these lines, putting emphasis on integrating historical and epistemological issues in the teaching and learning of mathematics and physics constitutes a possible natural way for exposing them in the making that may lead to a better understanding of their specific parts, and to a deeper awareness of what they are as disciplines. This is very important from an educational viewpoint: It helps to realize that they have been undergoing changes over time as a result of contributions from many different cultures, in uninterrupted dialogue with other scientific disciplines, philosophy, the arts and technology; a constant force for stimulating and supporting scientific, technical, artistic and social development.

The last paragraph frames the *HPM/Ph perspective* (Clark *et al.* 2016, §1.1), summarized in the following thesis on the **ontological** status of both disciplines and its educational implications (cf. Tzanakis & Thomaidis 2000, p.45).

Thesis A: Mathematics and physics should be conceived (hence, taught and learnt) both as the *result* of intellectual enterprises and as the *procedures* leading to these results. Knowledge gained in their context has an evolutionary character; by its very nature, *historicity*[5] is a deeply-rooted characteristic.

2.2 The interrelated historical development of mathematics and physics

Numerous important historical examples attest the following

Thesis B: Throughout their historical development from antiquity to the present, mathematics and physics have been evolving in a close, continuous, uninterrupted, bidirectional, multifaceted and fruitful way.

It goes from Hero's geometrical proof of the *law of reflection*, Eratosthenes' estimation of the *earth's circumference*, and Archimedes' "mechanical arguments" to compute areas and volumes in his *Method*, to Poincaré's derivation of the *Lorentz transformations* in special relativity using group-theoretic arguments, Hilbert's deduction of *General Relativity's field equations* from a variational principle[6], von Neumann's rigorous formulation of quantum

[4] Kjeldsen & Lützen 2015, p.552.
[5] Barbin & Bénard 2007; Barbin 2010.
[6] Both Poincaré and Hilbert got their results before Einstein's **physically-oriented** work, by following **mathematically-oriented** and totally different approaches from his. Though curious at first sight, this is due to the fact that these great mathematicians followed closely every development in physics of their time, which they knew deeply (Tzanakis 1999a, pp.114-115; Mehra 1973); just one striking example supporting *Theses* B & C of this paper.

mechanics, to more recent examples, like Penrose's *singularity theorems* in general relativity, Feynman's *path-integrals* in quantum mechanics and *functional integration,* Thom's *Catastrophe Theory*, or Connes' *Non-commutative Geometry* and its relation to quantum field theory (Tzanakis 1999b, §§4, 5; 2002, §3).

A simplified scheme of 3 scenarios highlights this relation (Tzanakis 2000, §2; 2002 §1.2):

(S_1) *Parallel development*: The physical problems asking for solution and the formulation of appropriate mathematics (concepts, methods, or theories) evolve in parallel.

Typical examples: Infinitesimal calculus and classical mechanics in the 17th century; the interrelated development of vector analysis, electromagnetic theory, and fluid mechanics in the 19th century (Crowe 1967); the multifaceted development in the 19th century of statistical concepts and methods, error theory in celestial mechanics, kinetic theory, and social statistics (Kourkoulos & Tzanakis 2010); Hamilton's unified treatment of geometrical optics and classical mechanics and the solution of 1st order partial differential equations (§ 4.3 below).

(S_2) *Mathematical concepts, methods or theories precede their integration into physics*: The corresponding physical problems naturally stress the need for the appropriate mathematics.

Typical examples: Riemannian geometry and tensor calculus as the indispensable framework for Einstein's formulation of General Relativity (Tzanakis 1999b, §4.3); matrix algebra and Heisenberg's experimentally-induced matrix mechanics (§4.3 below).

(S_3) *Physical problems precede the formulation of mathematics appropriate to tackle them*: Partially intuitive, formal or experimentally-induced models, and logically incomplete or ill-defined concepts, motivate and/or guide the development of new mathematics.

Typical examples: Langevin's equation modelling Brownian motion in statistical physics and the development of stochastic differential equations (Tzanakis 1999b, §5.3); Dirac's introduction and use of his δ-function in quantum mechanics as a key initial step for the development of distribution theory (Lützen 1982); Feynman's path-integral approach to quantum mechanics and the development of functional integration (Gelfand & Yaglom 1960).

A **word of caution**: This **simplified** picture aims just to emphasize the historically rich and bidirectional relation of the two disciplines. In fact, none of the three "pure" scenarios appears isolated. If one appears as the original step in the historical development, it is usually followed by fruitful feedback in the context of the others. This is supported by the examples in section 4.

2.3 The epistemological affinity of mathematics and physics

The conventional view of the epistemological relation between mathematics and physics, shared by many mathematicians, physicists and teachers of these disciplines, is that mathematics is simply the **language** of physics, and physics is an **exterior** to mathematics **source of problems** to be solved mathematically and/or a **domain to apply** already **available** mathematical **tools**. At least indirectly, this means that mathematics is simply a toolkit for handling physical problems, but otherwise is **extrinsic** to it, whereas physics is a huge, but **external** reservoir of a priori non-mathematical problems, which are nonetheless capable of mathematical formulation. If this were so, it would be difficult to understand the interrelation of the two sciences (§2.2). In fact, this conventional viewpoint has been challenged by distinguished mathematicians and physicists.

Maxwell stated his "*… opinion as to the necessity of mathematics for the study of Natural Philosophy…*" as "*Natural philosophy is, and ought to be, Mathematics… the greatest advances in mathematics have been due to enquirers into physical laws*" (in Harman 1990).

Hilbert wrote that *"... while the creative power of pure reason is at work, the outer world ... comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus ... advance most successfully the old theories. ... the numerous and surprising analogies and that* ***apparently prearranged harmony*** *which the mathematician so often perceives in the questions, methods and ideas of the various branches of his science,* ***have their origin*** *in this* ***ever-recurring interplay between thought and experience***" (Hilbert 1902, p.440; my emphasis).

Weyl declared that "*...gaz*[ing] *up… towards the stars is… to strengthen faith in reason, to realise the "harmonia mundi" that transfuses all phenomena... it was my wish to present this great subject* [the general theory of relativity] *as an illustration of the intermingling of philosophical, mathematical, and physical thought...*" (Weyl 1952, pp.x, ix) and that "*We shall see more and more clearly… that Geometry, Mechanics, and Physics form an* ***inseparable theoretical whole***" (*ibid*, p.67, my emphasis; cf. Pyenson 1982).

Einstein was convinced that "*…pure mathematical construction enables us to discover the concepts and the laws ... which give us the key to the understanding of the phenomena of Nature. Experience can… guide us in our choice of serviceable mathematical concepts…* [and] *remains the sole criterion of the serviceablility of a mathematical construction for physics, but the truly creative principle resides in mathematics*" (Einstein 1934) and stressed the criterion of "*naturalness*" (or "*inner perfection*") for trusting a physical theory (Einstein 1969).

Wigner argued that the unreasonable effectiveness of mathematics in the natural sciences "*…shows that it is in a* ***very real sense****, the correct language…*" and its predictions, often in amazing agreement with experimental data, indicate that "[s]*urely… we 'got something out' of the equations that we* ***did not put in***" (Wigner 1960, pp.8, 9; my emphasis)[7], whereas, Dirac (referring to General Relativity) argued that "*Anyone who appreciates the fundamental* ***harmony*** *connecting the way* ***nature*** *runs,* ***and*** *general* ***mathematical*** *principles, must feel that a theory with… beauty and elegance…* ***has to be*** *substantially correct*" (Dirac 1979, my emphasis).

These quotations suggest that mathematics is the language of physics in the deepest sense, often determining the content and meaning of physical concepts and theories (Tzanakis 1999b, §3), or even instigating revolutions in physics (Brush 2015, pp.495, 511). Conversely, physics furnishes and provides to mathematics, not only problems, but also ideas, methods

[7] In Tzanakis 1997 §5, this is called "*naturality*", meaning "*… that the explanation of known unintelligible facts, or the prediction of new ones, is not originally in the intentions of the founders of the theory, whose starting point may be quite independent of these facts*" (*ibid*, p.2101).

and concepts, crucial for many mathematical innovations (Tzanakis 1999b, §3; Kragh 2015, particularly §8). These comments are summarized in the first part of the following

***Thesis C*(a):** Mathematics and physics have always been closely interwoven in the sense of a bidirectional[8] process:

***From* mathematics *to* physics**: Mathematics is the language of physics, not only as a tool for expressing, handling and developing logically physical concepts, methods and theories, but also as an indispensable, formative characteristic that shapes them, by deepening, sharpening, and extending their meaning, or even endowing them with meaning.

***From* physics *to* mathematics**: Physics constitutes a (or maybe, *the*) natural framework for testing, applying and elaborating mathematical theories, methods and concepts, or even motivating, stimulating, instigating and creating all kinds of mathematical innovations.[9]

This historically-evidenced mutual process of dialectical interplay between mathematics and physics helps to moderate Wigner's puzzlement (Kjeldsen & Lützen 2015, p.544) and seems to be based on a deeper epistemological affinity of the two disciplines. Arnold (1998, p.231) has claimed that "…*the scheme of construction of a mathematical theory is exactly the same as that in any other natural science*". Detailed analysis suggests that both disciplines use the same procedures as invention/discovery patterns, or for (partial, in general) justification of results: logical reasoning (by deduction, induction, or analogy), algorithmic procedures, and experimental investigations (Tzanakis & Kourkoulos 2000, §2; Tzanakis 1998, §1). In fact, the supposedly key difference between the two disciplines, namely, what constitutes a "true" statement (logical self-consistency and consistency with available empirical data, respectively) is less sharp than conventionally claimed (Tzanakis & Kourkoulos 2000, §3; Tzanakis & Thomaidis 2000, §2), a point of view (at least indirectly) adopted by important physicists and mathematicians. Dirac claimed that "*A theory with mathematical beauty is more likely to be correct than an ugly one that fits some experimental data*" (Dirac 1970; see also Brush 2015, §9; Kragh 1990) and Weyl declared that "[his] *work always tried to unite the true with the beautiful; but when* [he] *had to choose one or the other* [he] *usually chose the beautiful*" (quoted in Chandrasekhar 1987). Arnold went further arguing that "*Mathematics is a part of physics… the part of physics where experiments are cheap*" (Arnold 1998, p.229). Thus, we are led to the second part of

***Thesis C*(b):** Mathematics and physics as embodiments of general attitudes in regard to the description, exploration, and understanding of empirically and/or mentally conceived objects, are so closely interwoven, that any distinction between them is related more to the point of view adopted while studying particular aspects of an object, than to the object itself.[10].

Theses **A-C** will be further illustrated and supported by examples in section 4. A basic educational conclusion to be drawn from this analysis is summarized as follows:

By *Thesis* **A**, history cannot be ignored in the teaching and learning either of mathematics, or

---

[8] Kragh has called it "*dialectical, or loop-like*" reflecting the historical interaction between the two disciplines which "*… is multifarious and cannot be encapsulated by a single formula*" (Kragh 2015, p.525).

[9] In Tzanakis 1999b, these two aspects are called "*mathematical physics*" & "*physical mathematics*", respectively.

[10] Tzanakis & Thomaidis 2000, §4. This is close to Weyl's view of an "*inseparable, theoretical whole*" above.

physics; By *Theses* **B** & **C** the teaching and learning one of the two should take into account, be supported, or include aspects of the other[11]; *Thesis* **C** gives general orientation to motivate, stimulate, support, deepen and widen the teaching and learning of either discipline specialized for particular examples into precise guidelines with the aid and/or in the light of Thesis B.

For the actual implementation of this conclusion, a sufficiently clear framework for the *HPM/Ph* perspective is needed, which is the subject of the following section.

3 The *HPM/Ph* issues and framework

To proceed further having adopted *Theses* **A-C**, some fundamental issues should be faced related to the integration of history in ME and/or PE. Although much of what follows originated in the context of mathematics, it is **equally valid** for physics, which is reasonable in view of the close interrelation of the two disciplines. The presentation is based on Clark *et al* 2016, §2.3.

3.1 *Which history is suitable, pertinent, and relevant for didactical purposes?*

This has been a permanent issue of debate among historians and educators with an interest in the HPM/Ph perspective. Implicit to some objections against the introduction of history in ME or PE is the idea that the term "history" is used in the same sense by historians, mathematicians, physicists, or teachers. That this is not so was stressed early by Grattan-Guinness, in relation to the history of mathematics (Grattan-Guinness 1973)[12]. On the other hand, it is undeniable that often history followed a complicated zig-zag path, led to dead ends, included notions, methods and problems no longer used in mathematics or physics today etc. Thus, its integration in education is nontrivial, posing the question why it must be done at all, since in this way history may be forced "…*to serve aims not only foreign to its own but even antithetical to them*" (Fried 2011). In other words, there is real danger of either unacceptably simplify or/and distort history to serve education as still another of its tools by adopting a "Whig" (approach to) history, in which "…*the present is the measure of the past. Hence, what one considers significant in history is precisely what leads to something deemed significant today*" (Fried 2001).

An important step to clarify existing conflicts and tensions between a mathematician's and a historian's approach to mathematical knowledge, with due attention to the relevance of history to ME, was Grattan-Guiness' distinction between *History* and *Heritage* (Grattan-Guiness 2004a, b): The ***History*** of a particular mathematical subject *N* refers to

> "… *the development of N during a particular period: its launch and early forms, its impact* [in the following years and decades], *and applications in and/or outside mathematics. It addresses the question 'What happened in the past?' by offering descriptions. Maybe some kinds of explanation will also be attempted to answer the companion question 'Why did it happen?'"*... "[It] *should also address the dual questions 'what did not happen in the past?' and 'why not?'; false starts, missed*

[11] Arnold argues that "[a]*ttempts to create 'pure' deductive-axiomatic mathematics have led to the rejection of the scheme used in physics (observation, model, investigation of the model, conclusions, testing by observations) and its replacement by the scheme definition, theorem, proof*", which "*can do nothing but harm to the teaching and practical work*" (Arnold 1998, pp.233).

[12] See Kjeldsen 2011a; 2011b; Kjeldsen & Blomhøj 2012, for a different approach.

opportunities…, sleepers, and repeats are noted and maybe explained… ***differences*** *between N and seemingly similar more modern notions are likely to be emphasized*" (Grattan-Guiness, 2004b, p.1; 2004a, p.164).

The ***Heritage*** of a particular mathematical subject *N* refers

"*…. to the impact of N upon later work, both at the time and afterward, especially the forms which it may take, or be embodied, in later contexts. Some modern form of N is usually the main focus, with attention paid to the course of its development. ... the mathematical relationships will be noted, but historical ones… will hold much less interest.* [It] *addresses the question 'how did we get here?' ... The modern notions are inserted into N when appropriate, and thereby N is unveiled...* ***similarities*** *between N and its more modern notions are likely to be emphasized; the present is* ***photocopied*** *onto the past*" (Grattan-Guiness, 2004a, p.165).

Though "***both kinds of activity are quite legitimate***, and… *important in their own right*...", they are **incompatible** in the sense "…*that both history and heritage are legitimate ways of handling the mathematics of the past; but muddling the two together, or asserting that one is subordinate to the other, is not*." (Grattan-Guinness 2004a, p.165; 2004b, p.1).

This distinction - clearly valid for physics as well - is close to similar ones between pairs of methodological approaches: for mathematics Tzanakis, Arcavi *et al* 2000, pp.209-210; for physics Tzanakis 1998, Tzanakis & Coutsomitros 1988. Hence, this distinction is potentially of great relevance to education (Rogers 2009; Tzanakis & Thomaidis 2012), contributing to answering the recurrent question "*Why and which history is appropriate to be used for educational purposes?*" (Barbin 1997).

3.2 *Which role can history of mathematics and/or physics play in their teaching and learning?*

This question has been extensively discussed from various angles, especially in relation to the appropriateness and pertinence of original historical sources in ME. In this context, history can play three **mutually complementary roles** (Barbin 1997; Jahnke *et al* 2000; Furinghetti *et al* 2006; Furinghetti 2012, §5; Jankvist 2013) valid for physics as well:

(I) *Replacement*: Replacing mathematical and/or physical knowledge as usually understood (a corpus of knowledge consisting of final results; an externally given set of techniques to solve problems; school units useful for exams etc) by something different (not only final results, but also mental processes leading to them; hence perception of this knowledge, not only as a collection of well-defined deductively organized results, but also as a vivid intellectual activity).

(II) *Reorientation*: Changing what is (supposed to be) familiar, to something unfamiliar; thus modifying the learners and teachers' conventional perception of mathematical and/or physical knowledge as something that has always been existing in its current established form, into a deeper awareness that this knowledge was an invention based on a dialectical interplay between human mind's creativity and careful intelligent experimentation; an evolving intellectual activity.

(III) A *cultural* role: Making possible the awareness that mathematics and physics develop in a specific scientific, technological or societal context at a given time and place; thus appreciating knowledge gained in the context of these disciplines as an integral part of human intellectual history in the development of society; hence, perceiving mathematics and/or physics from perspectives beyond their currently established boundaries as (separate) disciplines.

Considered from the viewpoint of the *objective* of integrating history in ME and/or PE, there are five main areas where this could be valuable (Tzanakis, Arcavi *et al*, 2000, §7.2; Tzanakis & Thomaidis 2012, §3):

1. *Learning specific pieces of mathematics and/or physics*: Historical development vs. polished final results; history as a resource; history as a bridge between different domains and disciplines; history's general educational value in the development of personal growth and skills.

2. *Views on the nature of mathematics, physics and the associated activities*: About their content (to get insights into concepts, conjectures & proofs by looking from a different viewpoint; to appreciate "failure" as part of mathematics and physics in the making; to make visible the evolutionary nature of meta-concepts); about their form (to compare old and modern; to motivate learning by stressing clarity, conciseness and logical completeness).

3. *The didactical background of teachers and their pedagogical repertoire*: Identifying motivations; becoming aware of difficulties & obstacles; getting involved and/or becoming aware of the creative process of "doing mathematics and/or physics"; enriching the didactical repertoire; deciphering & understanding idiosyncratic or non-conventional approaches.

4. *The affective predisposition towards mathematics and physics*: Understanding mathematics and physics as human endeavours; persisting with ideas, attempting lines of inquiry, posing questions; not getting discouraged by failures, mistakes, uncertainties, misunderstandings.

5. *The appreciation of mathematics and physics as a cultural-human endeavour*: They form part of local cultures; they evolve under the influence of factors intrinsic and/or extrinsic to them.

From the point of view of the *way* history is accommodated into education, Jankvist's (2009a) distinction in relation to mathematics is pertinent:

(i) *History as a tool*: That is "… *the use of history as an assisting means, or an* ***aid****, in the learning* [or teaching] *of mathematics* [or physics]... *in this sense, history may be an aid*…" "*as a motivational or affective tool, and… as a cognitive tool*…" (Jankvist 2009b, p.69; 2009c, p.8).

(ii) *History as a goal*: That is, history is "*…an* ***aim*** *in itself... posing and suggesting answers to questions about the evolution and development of mathematics,* [or physics]... *about the inner and outer driving forces of this evolution, or the cultural and societal aspects of mathematics* [or physics] *and its history*" (Jankvist 2009b, p.69).

3.3 *In which way(s) history's role can be realized in educational practice?*

There are three broad ways to integrate history in ME and PE, each one emphasizing a different **aim**. They **complement** each other in the sense that each one, if taken alone, is insufficient to exhaust the multifarious constructive influence history can have on education (Tzanakis, Arcavi *et al* 2000; §7.3; Tzanakis & Thomaidis 2000, §3):

(I) *To provide direct historical information*, aiming to learn history;

(II) *To implement a teaching approach inspired by history* (explicitly or implicitly), aiming to learn mathematics and/or physics;

(III) *To focus on mathematics and/or physics as disciplines and the cultural & social context in which they have been evolving*, aiming to develop deeper awareness of their evolutionary character, epistemological characteristics, and relation to other disciplines, and the influence exerted by factors both intrinsic and extrinsic to them.

From a **methodological** point of view, Jankvist (2009a, §6) classified the teaching & learning approaches in three categories:

1. *Illumination approaches*: teaching and learning is supplemented by historical information of varying size and emphasis.

2. *Module approaches*: instructional units devoted to history, often based on specific cases.

3. *History-based approaches*: history shapes the order and the way of presentation, often without history appearing explicitly, but rather being integrated into teaching.

Approaches may vary in size and scope, according to the specific didactical aim, the subject matter, the level and orientation of the learners, the available didactical time, and external constraints (curriculum regulations, number of learners in a classroom etc).

4 Examples

In sections 2 & 3 a general framework for integrating history in ME and PE was presented, based on the epistemological similarities between the two disciplines and their interrelated historical evolution, and on a unified approach to the basic issues involved in any such integration. Below, this is illustrated by three examples, explicitly indicating their placement within the general framework.

4.1 Measuring the distance of inaccessible objects: *Mathematics & Physics in their wider cultural context*

The general theme is the determination of the size or distance of completely inaccessible objects like the celestial bodies,[13] some important cases being:

(a) *Eratosthenes' measurement of the Earth's circumference*.
(b) *Aristarchus' method of measuring the earth-sun-moon relative distances*.
(c) *Copernicus' method of measuring inner planets' relative distances from the sun*.
(d) *Trigonometric parallax* for measuring: (i) The *earth-sun distance* using inner planets' transits across the sun's disk; (ii) *Star distances* using *stellar parallax*.

This is a **rich** example that can be extended in several different directions, depending on the course orientation and the available didactical time. It could constitute a sufficiently self-contained *teaching module* (§3.3.2) aiming to reveal the wider strong cultural significance of mathematical thinking and its applications to important real problems in the course of history, seen and evaluated from our modern viewpoint (that is, adopting a *heritage-like* perspective;

[13] Along the lines of this example, see Siu *et al* 2000, for the distances of objects on earth.

§3.1)[14]. It reveals the fruitful, far-reaching connections among elementary Euclidean geometry, modelling of physical situations, astronomical observations, the significance of technically accurate instrumentation, and the crucial role of approximate computations. Or, parts of such a module could be used with the same objectives, as *illuminating* examples (§3.3.1) in high-school or university courses on Euclidean geometry, trigonometry, geography, or history of science and mathematics.

From a **mathematical** point of view, examples (a)-(d) are **elementary**. However, the **emphasis** is on how **elementary geometrical ideas** and **reasoning** led historically to **astronomically** and **physically non-trivial** consequences with **far-reaching cultural implications** of the highest importance that can be posed didactically (§3.3.III). Therefore, in this example, *history* appears mainly as a *goal* (§3.2(ii)), playing a *cultural role* (§3.2.III) by helping the learners to appreciate the significance of mathematics and natural sciences in the development of human civilization. That is, the focus is on the highly innovative and revolutionary boldness of **non**-mathematical hypotheses **implicit** to this example (tacitly taken for granted as self-evident from our **modern** perspective), which, **once** formulated and **accepted**, lead to **consequences** of **far-reaching** cultural importance by means of **elementary** high-school mathematics! Besides bridging mathematics with other subjects (history, physics, astronomy, geography, philosophy) as a vast resource of specific information and meaningful problems (§3.2.1), this example greatly enriches and widens teachers' didactical repertoire (§3.2.3), and develops students' awareness that mathematics and the natural sciences may evolve in a constant dialogue with societal needs and philosophical queries that has a strong mutual influence on both the sciences and the society as a whole (§3.2.5). Lack of space allows only a few comments on each case. The *italicized text* points to possible stimulating elaborations.

4.1.1 ***Eratosthenes' measurement of the earth's circumference***: This is a well know example; see figure 1 (for history e.g. van Helden 1985, pp.4-6; original texts in Thomas 1941, §XVIII(d); Heath 1991, pp.109-112; for a didactical implementation, de Hosson 2015). The elementary geometry involved lies on three non-mathematical assumptions: (i) *Earth is spherical*; (ii) Alexandria and Syene *lie on the same meridian;* (iii) The *sun is so far away* that its light rays are *practically* parallel.

Assertions (i), (iii) are **bold** hypotheses of far-reaching implications. Considerable didactical elaboration can be designed by raising culturally and physically important questions e.g.:

(1) *How do we know that the earth is spherical?* This could be approached in various ways, including *Aristotle's reasoning* in *De Cœlo* (Book II, 14, 297b, 25ff) based on the earth's circular shadow during lunar eclipses, and the appearance of new constellations while moving southward (Berry 1961, §29; Crowe 1990, ch.2, p.27).

(2) *How do we know that two places lie on the same meridian?* A problem central during the geographical expeditions in Renaissance until the 18th century, for the determination of *geographical longitude in the sea*, closely related to *the accurate measurement of time* (Berry 1961, §§127, 226; Whitrow 1988; Dörrie 1965, problem 78); and the much easier: *How do we*

[14] A *history-like perspective* is also possible by putting emphasis on original sources and their proper contextualization.

determine the geographical latitude of a place? (Dörrie 1965).

(3) *How can we check that the sun is really so far away*? (cf. 4.1.2) etc.

Any discussion of these questions will reveal their interrelations (evidence for one can be used, or has been used to illuminate the others), their importance in the emergence and establishment of our modern view of the Cosmos, their appearance and treatment within different cultures etc.

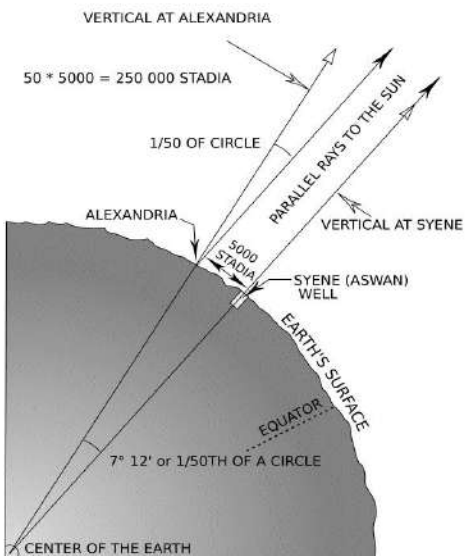


Figure 1: Eratosthenes' measurement of the earth's circumference[15]

4.1.2 *Aristarchus' "lunar dichotomy" to measure* the *earth-sun-moon relative distances*[16]: Inherent to drawing figure 2 are *two hypotheses* about the moon: That it is (i) *spherical*; and (ii) *illuminated by the sun*, so that when the three bodies form a right-angled triangle, half of the moon disc is seen from the earth. By measuring φ, the relative distances of the sun and moon are obtained: $\cos\varphi = a_M/a$.

Remarks: (1) *How do we know that (i), (ii) hold?* Assertion (ii) is explicitly hypothesized by Aristarchus and as suggested by Pappus, the existence of *lunar eclipses strongly supports it* (Heath 1991, p.102).

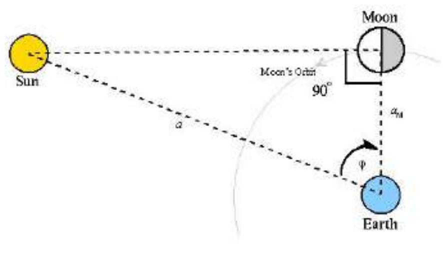


Figure 2: Aristarchus' *lunar dichotomy* method for the earth-sun-moon distances

(2) Aristarchus measured φ^o as 29/30 of a right angle (87^o), deducing that 18< a/a_M <20. The modern value $a/a_M \cong 390$ amounts to $\varphi^o \cong 89^o52'$; hence, Aristarchus' was a great underestimation. This can be discussed in relation to (i) the *instruments* available in antiquity (*dioptra, quadrant, astrolabe* etc), their limited accuracy, the subsequent development of more

[15] https://en.wikipedia.org/wiki/Eratosthenes
[16] Original in Thomas 1941, §XVI(b); see also Heath 1981, chs. II.II, II.III; van Helden pp.6-7.

accurate ones (*telescope, sextant, theodolite*) etc (King 1979) - cf. §4.1.4; (ii) the *sensitive dependence of computations on the data used*: for many problems 2^o-3^o is a negligible error, but not for secφ near $\varphi = 90^o$ (to be discussed in various contexts; trigonometry, calculus etc).

(3) *Aristarchus' work* (and its further elaboration by Hipparchus) on *measuring the solar and lunar radii* using, the accidental empirical fact that the solar and lunar apparent diameters almost coincide (about 32′ on the average), *solar eclipses*, and a_M/a, provides a more complicated – though still elementary – geometrical problem in the context of this example (see e.g. van Helden 1985, pp.7-14; Crowe 1990 pp.27-30).

4.1.3 *Copernicus' method for measuring inner planets' relative distances from the sun*: At greatest angular elongation from the sun (*existing only for the inner planets*; Mercury and Venus) the planet's distance $a_P = a \sin\theta$ (a being the earth-sun distance). Already by drawing figure 3, *it is **assumed** that the inner planets **revolve** around the sun* in ***circular*** orbits (for a more accurate description of Copernicus model see van Helden 1985, pp.42-44).

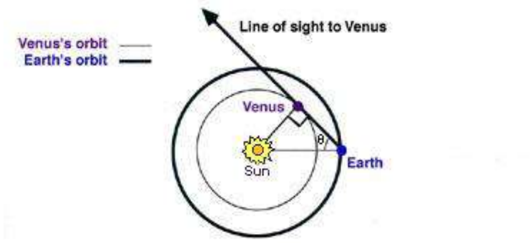


Figure 3: Copernicus' measurement of the inner planets' distances at their greatest elongation

Remarks: (1) Asking for *Copernicus' motivation of such a bold, counter-intuitive assumption* opens a fruitful discussion both on whether the earth moves and whether the planets' orbits are circular, with far-reaching implications on historical and philosophical issues (Kuhn 1973, ch.5; for a recent outline see Brush 2015, §3).

(2) The ***observed*** *greatest elongation of the inner planets*, fitted *naturally* in Copernicus' system (but not in Ptolemy's where extra hypotheses were needed), hence constituting a *strong supporting argument* (Crowe 1990, p.92).

(3) *Tycho Brahe's semi-heliocentric system*[17](and the crucial role of *his* careful astronomical *observations*) can be further explored as an intermediate step towards *Kepler's heliocentric system of elliptic orbits* (Berry 1961, §105; Crowe 1990, pp.140-143; King 1979, pp.16-23).

(4) A similar, though more elaborated *method was invented by Copernicus* for *the distances of the outer planets* (Crowe 1990, pp.95-96) that could be discussed in this context.

4.1.4 *Trigonometric parallax*: (i) The *earth-sun distance using Mercury or Venus' transits across the sun's disk.*

In this method, originally suggested by Halley (van Helden 1985, p.144; Berry 1961, §§202, 227), the planet's projections *A′*, *B′* onto the solar disk are seen by observers *A*, *B* at two distant places on earth and their angular separation is measured simultaneously (figure 4).

[17] The planets revolve around the sun but the latter revolves around the immovable earth.

Remarks: (1) *It is **assumed*** that the *planet is very far away*, so that PA, $PB \gg AB$ the *linear distance* of A, B (cf. example 4.1.1)

(2) $\varphi^{o} < 0^{o}.5$ (sun's angular diameter), hence by (1), *approximately* $\sin\varphi \cong \varphi \cong AB/PA$ (φ expressed in radians; $\sin 0^{o}.5 \cong 2\pi/720$ to 6 decimals).

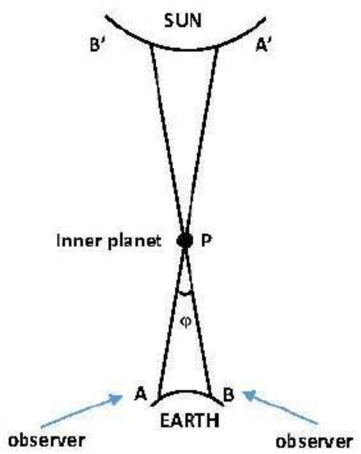


Figure 4: The earth-sun distance using inner planets' transitions across the sun

(3) $PA \cong PB$ is known either from example 4.1.3, or – more accurately – from *Kepler's 3rd law of planetary motions* (e.g. for Venus $PA = 0.72AA'$).

(4) The *earth-sun distance* gives *meaning* to all celestial *relative* distances obtained by *other methods* (examples 4.1.2, 4.1.3, 4.1.4(ii)).

(5) A pair of *Venus' transits* with an 8-year time difference occurs *rarely*, every 110 years at least. *Mercury's transits* are more frequent (every few years) but provide a *less favorable method*, because Mercury is considerably fainter than Venus.

(6) Today we use *radar methods* to accurately get AA' directly (*conceptually simple*, but *technically sophisticated*).

(ii) *Stellar (annual) parallax*

This is a method to measure the *distance of "nearby" stars*, applicable to distances less than about 100 *light years*[18] (*l.y.*; 1 light year = 9.46×10^{12}km) by observing them from two diametrically opposite places on the earth's orbit; that is, in a time span of 6 months (figure 5; AU ≡ astronomical unit, the semi-major axis of the earth's orbit $a = 1.49\times10^{8}$km). Even for the nearest star, $p < 1''$ ($0''.7687$), hence $a \cong d\sin p$ and $\sin p \cong \tan p \cong p$ to 11 decimals. Given that $p = (2\pi/360\times3600)p'' = p''/206{,}265$, one gets $d = (206{,}265/p'')$AU, where p'' is p expressed in seconds of arc[19].

Remarks: (1) There are two *crucial assumptions* inherent in this method: (i) *the earth revolves around the sun*; (ii) *the faint stars are (statistically) very far away* compared to the star whose parallax is sought, so that they constitute a sufficiently *immovable background* on which the star's two projections are formed and their angular separation is measured.

(2) The *conceptually* simple, but *technically* sophisticated idea of *parallax* was used by Copernicus' and Galileo's *Aristotelian opponents* and critics (including Tycho Brahe) *against* the *earth's motion*, given that no parallax had been observed at that time (Berry 1961, §129, Crowe 1990, pp.99-100, 141)[20].

[18] Earth-based measurments. This limit has been greatly extended by using modern sophisticated photographic techniques in space telescopes.

[19] Or, $d = (1/p'')$ parsec; one parsec being the distance of an object of parallax $p = 1''$, i.e. 206,265AU=3.25*l.y.*

[20] Galileo argued that this was due to the smallness of *p*, because of the huge stellar distances (Galilei 1967).

(3) *Technically possible measurement of parallax* required the telescope and became possible as late as 1838 (originally by Bessel), thus giving a *final definite experimental test of the earth's revolution* (Berry 1961, §§278-279). *Other such tests can be discussed*.

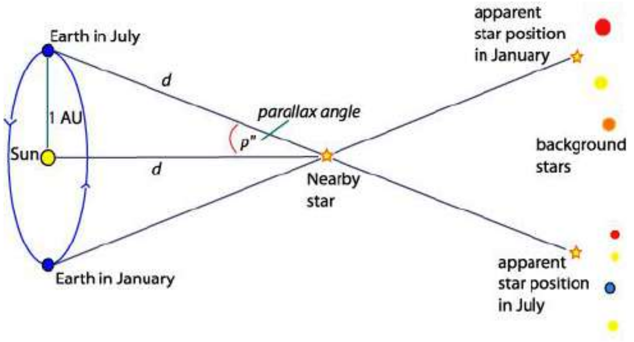


Figure 5: Trigonometric parallax for measuring stellar distances[21].

4.2 Rotations, Space-Time and the Special Theory of Relativity: *How did we get here?*

The *Theory of Special Relativity* (SR) is a standard subject for physics (but not mathematics) undergraduates. On the other hand, *matrices* and *linear algebra* are introduced early in undergraduate mathematics and physics curricula, or even last-year high-school math (as 3D analytic geometry and matrix algebra). Since the power of algebra lies in the unified-through-abstraction approach to otherwise distinct concrete problems, early undergraduates or last-year high-school students meet grave difficulties in the study of abstract algebraic concepts (vector spaces, matrices, linear transformations, groups etc) because of their limited mathematical maturity. Therefore, if such concepts are taught at this level, this ought to be done by giving as many concrete examples as possible.

A nice historical example, illustrating the framework of sections 2 & 3, is provided by an elementary**,** but fairly complete account of the foundations of the theory of *SR* in the spirit of Minkowski's original ideas about space-time and using simple matrix algebra. Below, an outline of a possible *illumination approach* (§3.3.1) is given, strongly *inspired by and based on history* (§3.3(II)), with history playing a *re-orientation role* (§3.2(II)) and serving mainly *as a tool* (§3.2(i)) for *learning mathematics & physics* (§3.2.1; by unfolding the interplay between these disciplines) and for *enriching teachers' didactical repertoire* (§3.2.3). For a detailed presentation, see Tzanakis 1999a, §3; 2000, §3.2. The didactical steps do not necessarily respect historical order; the approach is mainly *heritage-oriented* (§3.1) aiming to provide insights into the development and establishment of basic modern mathematical and physical results, and - whenever necessary - using notions and methods not available or used at the time[22]. This helps to link basic innovations in physics and their mathematical formulation, to their modern counterparts, thus illuminating better "*how did we get here?*"

Key historical elements[23]

(a) *SR* is based on *Lorentz transformations* (LT), the coordinate transformations between

[21] http://www.atnf.csiro.au/outreach/education/senior/astrophysics/astrometry1.html

[22] Vector methods and matrix algebra; a posterior custom in physics, after quantum mechanics in the 1920s.

[23] For details see Tzanakis 1999a, pp.114-115.

inertial systems (IS), i.e. systems moving with constant velocity relative to each other. Einstein derived LT in 1905 introducing and using the basic principles of *SR*: the *Special Relativity Principle* (*SRP*: the invariance of physical laws under IS coordinate transformations); and the *Principle of the constancy of the light speed* (the vacuum light speed c is constant in all IS, whether or not the source is moving). His derivation is based on the epistemological analysis of "simultaneity" (what does it mean that "two events are simultaneous"?), is mathematically elementary and appeals much on physical intuition and some "common-sense" assumptions about the homogeneity of space (Sommerfeld 1952, paper III).

(b) Others before him (especially Lorentz in 1899 and 1904) had obtained the *LT* while searching for the coordinate transformations that leave unaltered Maxwell's equations, without however attributing to them any **general** physical significance. They were simply the transformations between IS moving relative to the *ether* (by its definition in *absolute rest*), necessary to explain the null result of experiments that aimed to detect motion relative to the ether (especially the famous Michelson-Morley experiment).

(c) Poincaré in 1904 was the first to realize the deep general significance of the *SRP* and to derive the *LT* by a mathematically-oriented approach: He explicitly used the **group structure** of the sought transformations implied by the *SRP* and got their general form and fundamental consequences of *SR* (e.g. the relativistic velocity addition law); see Pais 1982, pp.128-130.

(d) In a seminal lecture in 1908 (Sommerfeld 1952, paper V), Minkowski introduced the *space-time* concept, unfolding the rich geometrical content implicit to Einstein's 1905 paper[24].

physically-oriented approach		mathematically-oriented approach	
Lorentz (1904) **Einstein** (1905)	transformations leaving invariant *Maxwell's equations*	**Poincaré** (1904) →	*group structure* of transformations between *inertial systems*
Einstein (1905) →	- analysis of *simultaneity* - *Special Relativity Principle* - Constancy *of the light speed*	**Minkowski** (1908) →	- the *space-time* concept - transformations leaving invariant the *light cone*: $x^2 + y^2 + z^2 - c^2 t^2 = 0$

Figure 6: The initial key steps in the historical development of the *Theory of Special Relativity*

Brief sketch of a possible didactical implementation

These basic historical facts are schematically shown in figure 6. An outline of their didactical reconstruction follows:

(1) The LT in 2D (one spatial x, one temporal t) result by following Minkowski's key ideas: (i) The introduction of *space-time* as a natural concept inherent to Einstein's 1905 analysis of the relative character of the *simultaneity* of events; and (ii) The constancy of c trivially implies that the sought transformations leave invariant the *light cone* (i.e. the surface

---

[24] Though *SR* was conceived **without** the *space-time* concept, Einstein's formulation of *General Relativity* ten years later would have been impossible without it (Tzanakis 1999b, §§4.3, 4.4).

on which light signals lay). This derivation uses elementary *matrix algebra* and proceeds in **close analogy** with the determination of **plane rotations** in 2D-analytic geometry: Rotations in the *xy*-plane conserve the Euclidean distance x^2+y^2; LT in the *xt*-plane conserve the Minkowski (pseudo)distance $x^2-c^2t^2$ that vanishes on the light cone (Tzanakis 1999a, §3). Proceeding along these lines, Euclidean rotations are represented by orthogonal transformations with matrix parametrized by the rotation angle φ,

$$\mathrm{R}_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

whereas the LT are represented by pseudo-rotations with matrix parametrized by φ, with $\tanh\varphi = v/c$; v being the relative speed of the two IS

$$\mathrm{L}_\varphi = \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}$$

(2) Strictly speaking, conservation of the *light cone* implies only that the sought transformations are *conformal*, a fact whose significance was appreciated first by Weyl in 1918 (Sommerfeld 1952, paper XI). It is more advanced - especially in 4D - to show that in the context of *SR* these conformal transformations are indeed isometries of the Minkowski (pseudo)-distance if it is assumed that they map straight lines to straight lines, as a consequence of *Newton's law of inertia*. This derivation may constitute a student project along the lines of Poincaré, using explicitly the **group structure** of the sought transformations. Fundamental results of *SR* are obtained in this way (velocity addition, length contraction etc), at the same time illustrating important abstract concepts, like *group*, *pseudo-Euclidean structure*, *conformal transformations* etc. For instance, the geometrically clear fact that successive plane rotations by angles φ and φ' coincide with a single rotation by an angle $\varphi+\varphi'$, is equivalent to the orthogonal matrices R_φ forming a group under multiplication, $\mathrm{R}_\varphi\mathrm{R}_{\varphi'}= \mathrm{R}_{\varphi+\varphi'}$; a fact verified by direct calculation and using standard trigonometric identities. **Similarly**, by direct calculation and using the corresponding identities for the hyperbolic functions, the LT, hence the matrices L_φ, form a multiplication group, $\mathrm{L}_\varphi\mathrm{L}_{\varphi'}= \mathrm{L}_{\varphi+\varphi'}$. But now, in view of $\tanh\varphi = v/c$, the identity

$$\tanh(\phi+\phi') = \frac{\tanh\phi + \tanh\phi'}{1+\tanh\phi\tanh\phi'}$$

becomes the important, **highly nontrivial** and counter-intuitive **relativistic law of velocity addition** for three inertial systems with parameter φ, φ', $\varphi''=\varphi+\varphi'$, moving respectively with velocities relative to each other v, v', v''; i.e. $v'' = \dfrac{v+v'}{1+\frac{1}{c^2}vv'}$ Since $|v/c|<1$[25], this composition law if seen algebraically, is a new "addition" $\bullet$ in (-1,1], such that this interval with the operation $x \bullet x' = \dfrac{x+x'}{1+xx'}$ becomes a commutative group (isomorphic to the 2D Lorentz group),

[25] Negative values of v in one dimension express motion to the left of the axis.

with $x \bullet 1 = 1$ for any x in (-1,1][26] (Tzanakis 1999a for details and original references).

(3) Conformal transformations in the special case of *similarities* can be introduced naturally by looking for the *symmetry group* of *Maxwell's equations*. It is a nice example – dual to (1) above - to consider this problem for both the 2D *Laplace* and the *wave equation* and to arrive at the **orthogonal** and the **Lorentz** group of transformations, respectively (cf. Heras 2016). In fact, this is the idea behind the pre-relativistic derivations of the *LT* by Larmor and Lorentz (Schaffner 1972; Sommerfeld 1952, paper II; Whittaker 1951).

(4) At a more advanced level, one can pursue further Weyl's original idea that the two basic principles of *SR* alone, imply **only** that the transformations between IS are *conformal*[27], hence that the basic geometrical structure of space-time as a manifold is **not** its (pseudo)-metric, but vector *parallelism*. Actually, Weyl was trying to formulate a unified theory of gravitation **and** electromagnetism. In this way one has a natural path to introduce the first example of what - from today's perspective - was the first *gauge field theory* (Weyl 1952, §35, 36; Sommerfeld 1952, paper XI; see also Tzanakis 2002, §3.1).

4.3 Differential Equations, (Functional) Analysis and Quantum Mechanics: *What did (or did not happen) in the past and why (or why not)?*

For most mathematics and physics undergraduates (at least), the following subjects, if ever taught, are taught separately so as heterogeneous; thus learnt as unrelated, and finally conceived as completely alien to each other:

- *Jacobi's general method* to solve 1st order partial differential equations (PDEs) by solving an equivalent system of 1st order ordinary differential equations (ODEs).
- The *canonical (Hamiltonian) formulation of classical mechanics* and its associated *Hamilton-Jacobi theory* for solving mechanical problems.
- The formal analogy as variational principles of Fermat's *Principle of Least Time* in Geometrical Optics (GO) and Maupertuis' *Principle of Least Action* in Classical Mechanics (CM).
- *Schrödinger's equation* as the cornerstone of Quantum Mechanics (QM).
- Heisenberg's *Matrix Mechanics* and infinite dimensional matrices.
- Infinite dimensional linear spaces; in particular (separable) *Hilbert spaces*.
- Observable quantities in QM as (*hermitian/self adjoint*) operators in linear spaces and their non-commutative algebraic structure.
- Fourier analysis, Lebesgue integration and square-integrable functions.

This is related to the fact that key issues are introduced unmotivated e.g. (i) *Schrödinger's equation* is introduced ad hoc as a basic axiom, whose significance is to be evaluated **a posteriori** by its compatibility with experiment (see Tzanakis & Coutsomitros 1988, §3); (ii) *Separable Hilbert spaces* are introduced rather mysteriously as linear spaces with a countable, dense subset (e.g. Richtmyer 1978).

However, historically these subjects have strong interconnections that motivated,

[26] Physically this means that in any IS a light signal travels at speed c, no matter what the speed of this IS is.
[27] They become isometries only if they map straight lines to straight lines (see (2) above); an extra assumption abandoned after the formulation of general relativity in 1915.

stimulated, and guided the course of development to their current form, which can be beneficial for their teaching and learning today. This example has been presented from a variety of perspectives elsewhere[28]. Here, a possible didactical illustration is given within the framework of sections 2 & 3: Briefly outlining a possible *history-based approach* (§3.3.3) *inspired by history* (§3.3(II)), in which history has a *replacement role* (§3.2(I)), serving mainly *as a tool* (§3.2(i)) for *learning mathematics & physics* (§3.2.1; contrasting the historical development vs. its polished final results, and unfolding the strong bonds between mathematics and physics), *enriching teachers' didactical repertoire* (§3.2.3; furnishing ample insight to the motivation for introducing new concepts and theories) and possibly amending students' *affective predisposition* towards learning abstract and difficult concepts (§3.2.4; by stressing the constructive role of posing questions and of attempting lines of inquiry despite any existing vagueness and uncertainties). The emphasis is on a *history-oriented* approach (§3.1) by enlightening "*what and why did/did not happen?*", but a *heritage-oriented* approach (§3.2) is also possible if the didactical sequence is structured differently to fit better dealing with the question "*how did we get here?*"

Key historical elements

(a) In the 18th century the formulation of Maupertuis' *Principle of Least Action* in CM was motivated by and in analogy with Fermat's *Principle of Least Time* in GO (Dugas 1988, part III, §§V.4-V.7, V.11, V.12). For a mechanical system with finite degrees of freedom and constant total energy, Maupertuis' principle as formulated by Hamilton (in the mid 1830s) was a main motivation for Hamilton's **mathematically unified** development of the two theories, which (together with Jacobi's important work since 1837; Nakane & Fraser 2002) provided a general method for solving 1st order PDEs and constituted another formulation of CM that became central to the solution of mechanical problems as well, the *Hamilton-Jacobi method* (Dugas 1988, part IV, §§VI.2-VI.5; Klein 1979, pp.179-203; Lanczos 1970; Yourgrau & Maldestam 1968; Goldstein 1980, ch.10; Tzanakis 2000, §3.3).

(b) Hamilton's ideas stimulated de Broglie to see this similarity as an indication of a deep relation between mechanical and optical phenomena in his 1924 doctoral thesis[29]. By relativistic arguments, he extended to matter the Planck-Einstein quantization relations for radiation, thus predicting the wave nature of atomic particles (Tzanakis 1999b, §4.2). In 1926, elaborating on this idea, Schrödinger arrived at the formulation of *Wave Mechanics*[30].

(c) In the 1920s, atomic physics was a complicated mixture of CM, electrodynamics, semi-empirical rules, and heuristic arguments. Physicists were trying hard to develop models of atomic phenomena in a rather vague and confusing landscape. Heisenberg's breakthrough in 1925 (van der Waerden 1967, paper 12) was the development of an algebraic manipulation of

[28] Tzanakis, 1998, 1999b, 2000, 2002, Tzanakis & Coutsomitros 1988; Tzanakis & Thomaidis 2000.

[29] "*Guidé par l'idée d'une identité profonde du principe de moindre action et de celui de Fermat, j'ai été conduit… à* ***admettre*** [que] *les trajectoires dynamiquement possible de l'un coïncidaient avec les rayons de l'autre*" (de Broglie 1925).

[30] "*…our classical mechanics is the* ***complete*** *analogy of geometrical optics…Then it becomes a question of searching for an undulatory mechanics…* [by] *working out… the Hamiltonian analogy on the lines of undulatory optics*" (Schrödinger 1982, paper II, p.18); see also Dugas 1988, p.401; Goldstein 1980, §10-8.

atomic quantities, **in analogy** with *Fourier series operations*, with the crucial difference that the Fourier-like frequencies and coefficients were **doubly** indexed as a consequence of *Ritz' combination principle* in atomic spectroscopy (see figure 8). The novel fact was the non-commutative multiplication of atomic quantities, originally unintelligible to Heisenberg. However, it was immediately realized by the mathematically educated Born and Jordan, that Heisenberg's calculus was just the algebra of (generally, infinite-dimensional) matrices (Born 1969, §V.3; Heisenberg 1949, Appendix §1). This led to *Matrix Mechanics*, the first formulation of QM (van der Waerden 1967, paper 15; Mehra & Rechenberg 1982).

(d) After the formulation of *Wave Mechanics,* physicists were puzzled by the existence of two **conceptually** and **mathematically** totally **different** theories of atomic phenomena (*Matrix Mechanics* and *Wave Mechanics*), which nevertheless gave **identical results** compatible with the empirical data. Schrödinger (in 1926) and von Neumann (from 1927) showed that the two theories were mathematically equivalent. However they proceeded in different ways:

Schrödinger provided a **formal** proof: Functions in his *Wave Mechanics* were elements of the linear space $L^2(\boldsymbol{R})$ of Lebesgue quadratically integrable complex-valued functions, equipped with the scalar product $< f, g >= \int_{-\infty}^{\infty} f^* g d\mu$, f^* being the complex conjugate of *f*. The matrices in *Matrix Mechanics* could be seen as acting on the infinite-dimensional linear space l^2 of complex sequences $\alpha=(\alpha_1, \alpha_2, ...)$, such that $\sum_k |\alpha_k|^2<+\infty$; a straightforward generalization of the familiar *n*-dimensional Euclidean space. By expanding the wave functions in an orthonormal (ON) basis of $L^2(\boldsymbol{R})$, solving his PDE becomes the solution of the eigenvalue problem for the Hamiltonian matrix $\hat{H}$ of *Matrix Mechanics* and vice versa; i.e. Schrödinger's equation reduces to a system of linear equations, whose matrix was $\hat{H}$ (Schrödinger 1982, paper 4).

von Neumann's approach was **mathematically-oriented** (von Neumann 1947, ch.I). He identified the basic properties of the objects with which the two theories were dealing, emphasizing the **linear structure** of the **function spaces** (that is $L^2(\boldsymbol{R})$ and l^2) underlying them. Thus he was led to introduce axiomatically what became known as a *separable Hilbert space* (von Neumann 1947, ch.II). Then he resolved the puzzle by proving that all these (normed) spaces are isometric: The two **conceptually different** theories were **different representations** of the **same abstract mathematical structure** that underlies the formalism of QM (von Neumann 1947, ch.II, theorem 9).

Outline of a possible didactical sequence

In view of these historical facts, an approach to this vast and rich subject is outlined below:

(1) The *Least Action Principle* and the *Principle of Least Time* constitute **important** examples of variational principles, leading to mathematically interesting equations that are central in CM (the *Hamilton-Jacobi equation*) and (Geometrical) Optics (the *eikonal equation*)[31]; Pauli 1973;

[31] In Schrödinger's concise wording: "*The **inner** connection between Hamilton's theory and the process of wave propagation is anything but a new idea... Hamilton's variational principle… correspond*[s] *to Fermat's **Principle** for a wave propagation in configuration space… and the Hamilton-Jacobi equation expresses Huygen's **Principle** for the wave propagation… we must regard* [this] *analogy as one between mechanics and **geometrical** optics and not… **undulatory** optics*" (Schrödinger 1982, pp.13, 17; see also next paragraph).

Courant & Hilbert 1962 §II.9.2. Both are **generic** examples, making possible the establishment of a general result in the theory of PDEs: the solution of a large class of 1st order PDEs (the Hamilton-Jacobi equation being an important special case) is *equivalent* to the solution of a system of 1st order ODEs; the associated *canonical* (or *Hamilton*'s) equations (cf. Nakane & Fraser 2002, pp.49, 56). This result is of central importance both in the theory of differential equations and the calculus of variations (e.g. Courant & Hilbert 1962 §§II.8-II.10; Sneddon 1957; Gelfand & Fomin 1963) and in CM (e.g. Goldstein 1980, §§10.1, 10.3; Lanczos 1970). In fact, one can proceed close to Hamilton's and Jacobi's approaches to illuminate the subject from two different, but equally important angles (Dugas 1988, part IV, ch.VI; Klein 1979, pp.182-196).

(2) Schrödinger's elaboration of Hamilton's mathematically unified treatment of CM and GO, was based on **arguing by analogy**: If *CM* is mathematically similar to *GO*, and since the latter is **only** an approximation to *Wave Optics*, *CM* may be **only** an **approximation** to a *Wave Mechanics*, which is similar to Wave Optics in the same way that CM is similar to GO. Thus, entirely **within** the conceptual and mathematical framework of **CM** an equation results as the **mechanical equivalent** of the *wave equation*, which is **formally** identical with *Schrödinger's* up to an undetermined constant σ. This is schematically shown in figure 7.

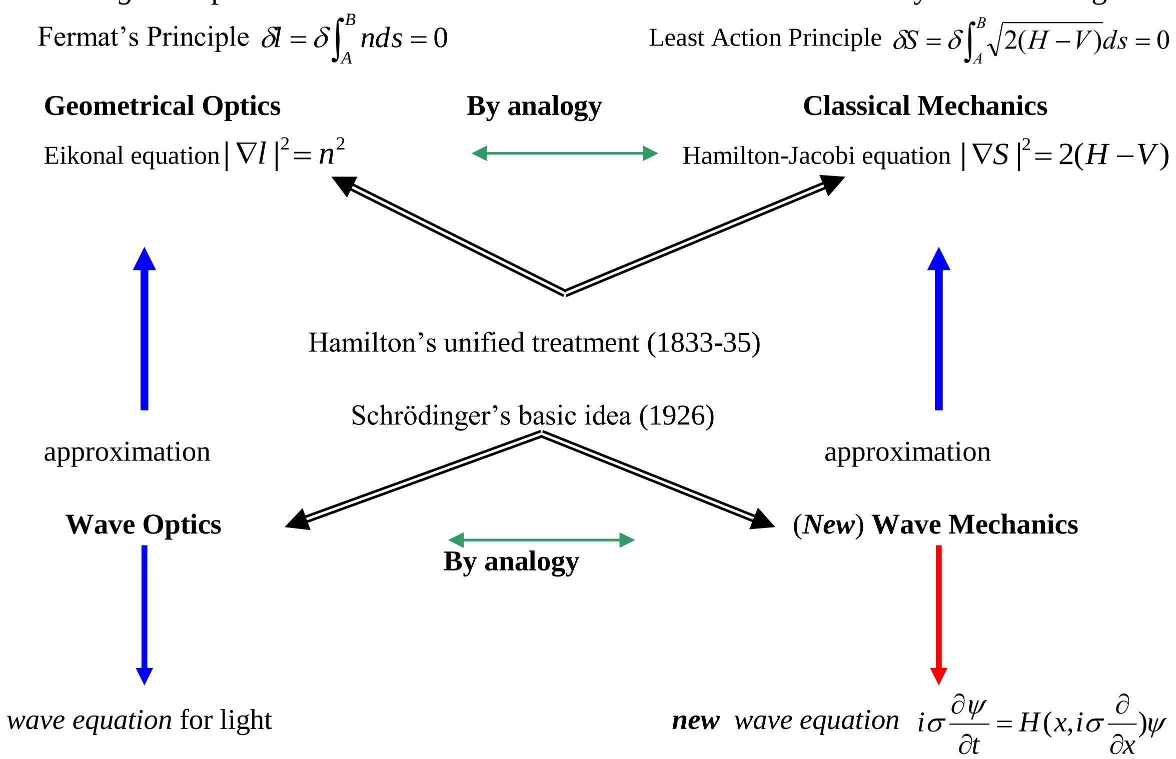


Figure 7: A schematic representation of Schrödinger's reasoning by analogy

Here *l*, *n* are the *optical length* and the *index of refraction* $n=c/v$ (*c*, *v* the speed of light in vacuum and in the optical medium); *S* the *action* of a mechanical system with potential energy *V* and total (mechanical) energy $H(x, p)$, *x* and *p* the coordinates in configuration space and the corresponding generalized momenta, and *ds* the element of arc length in real and configuration space respectively (Tzanakis 1998, §2; cf. Schrödinger 1982, pp.160-163, 189-192).

But then, why Hamilton did **not** formulate *Wave Mechanics*? To this end an extra condition was needed, supplied by de Broglie, who - based on *SR* theory - postulated the wave nature of **matter** by "symmetrising" the Planck-Einstein conception of the **corpuscular** nature of **radiation** with the aid of the same formal relations: as for the *light quanta* (photons), the energy E and the momentum $\boldsymbol{p}$ of a particle are proportional to the frequency ν and the wave number $\boldsymbol{k}$ of the associated wave; $E=h\nu$, $\boldsymbol{p}=h\boldsymbol{k}$, h being Planck's universal constant (Tzanakis 1998). In this case, σ necessarily coincides with h. This was the crucial **physical** idea that led from Hamilton's **mathematically** unified treatment of the **conceptually different** theories of *GO* and *CM*, to a deep, fruitful **physical** theory of atomic phenomena. Lack of this crucial physical idea prevented Hamilton from inventing wave mechanics.[32]

(3) Heisenberg formulated *matrix mechanics* (see (c) above) reasoning by analogy, as schematically shown in figure 8 (n, l, m, k being integers and $\boldsymbol{I}$ the identity matrix).

Classical positions & momenta $\boldsymbol{q}, \boldsymbol{p}$		**Quantum** positions & momenta $\boldsymbol{q}, \boldsymbol{p}$
Fourier representation ***Singly*** indexed frequencies $\nu_k = k\omega$		(looking for a) ***new*** representation (because of) ***Doubly*** indexed spectral frequencies ν_{nm} obeying ***Ritz principle***
$\nu_k + \nu_l = \nu_{k+l}$		$\nu_{nl} + \nu_{lm} = \nu_{nm}$
$q(t) = \sum q_k \exp(i\, \nu_k t)$	**by analogy** $\longleftrightarrow$	$q(t) \sim q_{nm} \exp(i\, \nu_{nm} t)$
	Operations **by analogy**	
$p+q \longrightarrow (p_k+q_k) \exp(i\, \nu_k t)$	$\longleftrightarrow$	$p+q \longrightarrow (p_{nm}+q_{nm}) \exp(i\, \nu_{nm} t)$
$pq \longrightarrow (\sum_l p_l q_{k-l}) \exp(i\, \nu_k t)$	$\longleftrightarrow$ **by analogy + Ritz Principle**	$pq \longrightarrow (\sum_l p_{nl} q_{lm}) \exp(i\, \nu_{nm} t)$
		$qp - pq = \frac{h}{2\pi i} I$

(hence, $pq \neq qp$ leading to Heisenberg's uncertainty relations)

Figure 8: Schematic representation of Heisenberg's formulation of *Matrix Mechanics*

It is important to note that (i) the non-commutativity of quantum quantities was **imposed** by the doubly-indexed spectral frequencies so that the **experimentally obtained** *Ritz principle* is satisfied; (ii) the famous *Heisenberg uncertainty relations* are a necessary consequence of this non-commutativity (Heisenberg 1949, §II.1; Born 1969, Appendices XII, XXVI; for the

[32] In Hamilton's time, there was no **physical** reason to consider the **formal** similarity between *GO* and *CM* nothing more than a **mathematical** analogy. Hamilton never attributed to it deeper physical meaning and it never attracted enough attention after Hamilton, despite Klein's use of this optical analogy to develop the Hamilton-Jacobi theory as a kind of optics in a multidimensional configuration space (Tzanakis 1998, p.73; Goldstein 1980, pp.491-492; Jammer 1966, pp.237-238). At that time neither the wave, nor the corpuscular theory of light were generally accepted. Hamilton's motivation was the desire to formulate GO in a way capable of interpretation in the context of either of the two theories (Dugas 1988, pp.390-391; cf. Goldstein 1980, p.489).

history see Jammer 1966, §7.1).

(4) One can proceed along these lines to motivate the unforced introduction of important concepts and to prove basic results in *Functional Analysis* and *Fourier Analysis*. For instance:

(i) To present in a simple form the basic **mathematical problem** of the Heisenber-Born-Jordan *Matrix Mechanics* and Schrödinger's *Wave Mechanics*: the diagonalization of the hamiltonian matrix $\widehat{H}$ in l^2, and the solution of Schrödinger's equation in $L^2(\boldsymbol{R})$, respectively.

(ii) Given that these physically and mathematically a priori different theories yield identical, experimentally correct predictions, the question of their relation naturally arises. Hence Schrödinger's heuristic and non-rigorous arguments can be used to show formally that Schrödinger's equation reduces to a matrix eigenvalue problem, once an ON basis of $L^2(\boldsymbol{R})$ has been chosen (Schrödinger 1982, paper 4).

(iii) Motivated by this approach, a rigorous proof that l^2 and $L^2(\boldsymbol{R})$ are isometric Hilbert spaces can be given, a fact already included in the works of Riesz and Fisher in 1907 (Bourbaki 1974; Dorier 1996).

(iv) In view of the heuristic approach in (ii), it is reasonable to reverse the argument and to consider linear spaces with a scalar product spanned by a countable ON basis, thus arriving at their isomorphism and their various equivalent characterizations (spaces having either an ON sequence spanning the space, or an ON sequence not perpendicular to any element, or obeying a generalized Parseval identity with respect to an ON sequence, or having a countable dense subset). This is essentially von Neumann's approach (1947, chs.I, II; see also Dieudonné 1981).

(5) Continuing along these lines, to introduce many other important concepts and results of functional analysis: *bounded* vs *unbounded operators* and the associated concept of a *closed operator*; the distinction between *hermitian* and *self-adjoint operators*; the *extension of an operator* etc; all of which are both basic in functional analysis and indispensable to QM (and modern theoretical physics, in general); see Tzanakis 2000, §3.4.

5 Final comments

In this paper the innermost relationship of mathematics and physics considered both from the point of view of their epistemological characteristics and their historical development has been explored and arguments have been presented to support the three main *theses* formulated in section 2. From an educational point of view, they imply that in mathematics and physics education this relationship should be taken into account explicitly. The main issues to be faced in any such attempt have been addressed in section 3 and a framework for integrating history into teaching and learning these disciplines has been outlined on a common ground. The general ideas presented in sections 2, 3 have been illustrated by analyzing three examples of quite different content and orientation. Hopefully, enough evidence has been presented to support that (a) it is impossible to deeply understand either mathematics or physics without being sufficiently aware of their interconnections and mutual influence; (b) on the contrary, taking into account their interrelation is beneficial for teaching and learning either discipline.

REFERENCES

Arnold, V. I. (1998). On teaching mathematics. *Russian Mathematical Surveys*, **53**(1), 229–236.

Barbin, E. (1997). Histoire et enseignement des mathématiques. Pourquoi? Comment? *Bulletin AMQ*, Montréal, **37** (1), 20-25.

Barbin, É. (2010). Epistémologie et histoire dans la formation mathématique. *Repères-IREM*, **80**, 74-86; p.79.

Barbin, É. (2013). History and pedagogy of mathematics in France. In: Hansen, V. L. & Gray, J. (Eds.), *History of Mathematics*, in *Encyclopedia of Life Support Systems* (e-book). Oxford, UK: EOLSS Publishers.

Barbin, É., & Bénard, D. (Eds.) (2007). *Histoire et enseignement des mathématiques: Rigueurs, erreurs, raisonnements*. Lyon: Institut National de Recherche Pédagogique; p.7.

Barbin, É., & Tzanakis, C. (2014). History of mathematics and education. In: Lerman, S. (Ed.), *Encyclopedia of Mathematics Education* (pp.255-260). New York: Springer.

Berry, A. (1961). *A short history of astronomy: From earliest times through the nineteenth century*. New York: Dover.

Born, M. (1969). *Atomic physics.* New York: Dover.

Bourbaki, N. (1974). *Éléments d'histoire des mathématiques*. Paris: Hermann; p.267.

Brush, S.G. (2015). Mathematics as an instigator of scientific revolutions. *Science & Education*, **24** (5-6), 495-513.

Casacuberta, C., & Castellet**,** M. (Eds.) (1992). *Mathematical research today and tomorrow*. New York: Springer.

Chandrasekhar, S. (1987). *Truth and beauty: Aesthetics and motivation in science*. Chicago: The University of Chicago Press; p.65.

Clark, K., Kjeldsen, T.H., Schorcht, S., Tzanakis, C., & Wang, X. (2016). *History of mathematics in mathematics education: Recent developments*. http://www.clab.edc.uoc.gr/HPM/HPMinME-TopicalStudy-18-2-16-NewsletterVersion.pdf

Courant, R., & Hilbert, D. (1962). *Methods of mathematical physics*, vol.II. New York: J. Wiley.

Crowe, M.J. (1967). *A history of vector analysis*. New York: Dover; chs.4-7.

Crowe, M.J. (1990). *Theories of the world from antiquity to the Copernican revolution*. New York: Dover.

de Broglie, L. (1925). Recherches sur la théorie de quanta. *Annales de Physique*, **3**(1), 22-128; p.45.

de Hosson, C. (2015). Promoting an interdisciplinary teach through the use of elements of Greek and Chinese early cosmologies. In: Barbin, É., Jankvist, U., & Kjeldsen, T.H. (Eds.), *History and Epistemology in Mathematics Education: Proc. of the 7th ESU*. (pp.571-583). Copenhagen: Danish School of Education, Aarhus University.

Dieudonné, J. (1981). *History of functional analysis*. Amsterdam: North Holland; p.172.

Dirac, P.A.M. (1970). Can equations of motion be used in high energy physics? *Physics Today*, April 1970, 29-31.

Dirac, P.A.M. (1979). The test of time. *UNESCO Courier*, **32**, 17-23; p.21.

Dorier, J-L. (1996). First vector spaces of functions. In: Lagarto, M.J., Vieira, A., & Veloso E. (Eds.), *Proc. of the 2nd European Summer University on History and Epistemology in Mathematics Education*, vol.II (pp.238-245), Braga: Universidad do Minho; §4.

Dörrie, H. (1965). 100 great problems of elementary mathematics: Their history and solution. New York: Dover.

Drake, S. (1957). *Discoveries and opinions of Galileo.* New York: Anchor Books, Doubleday; pp.237-238.

Dugas, R. (1988). *A history of mechanics*. New York: Dover.

Einstein, A. (1934). On the method of theoretical physics. *Philosophy of Science*, **1**(2), 163-169; p.167.

Einstein, A. (1969). Autobiographical notes. In: Schilpp, A. (Ed.), *Albert Einstein: Philosopher-Scientist.* Vol.I (pp.1-94). La Salle (IL): Open Court; pp.22-23.

Fasanelli, F., & Fauvel, J. (2006). History of the international study group on the relations between the history and pedagogy of mathematics: The first 25 years 1976–2000. In: Furinghetti, F., Kaisjer, S., & Tzanakis, C. (Eds.), *Proc. of HPM 2004 & ESU 4* (pp.x-xxviii). Iraklion: University of Crete.

Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, **10**, 391–408; p.395.

Fried, M. (2011). History of mathematics in mathematics education: Problems and prospects. In: Barbin, E., Kronfellner, M., & Tzanakis, C. (Eds.), *Proc. of the 6th European Summer University on the History and Epistemology in Mathematics Education* (pp.13-26). Wien: Holzhausen Verlag; p.13.

Furinghetti, F. (2012). History and epistemology in mathematics education. In: Hansen, V. L., & Gray, J. (Eds.), *History of Mathematics*, in *Encyclopedia of Life Support Systems* (e-book). Oxford, UK: EOLSS Publishers.

Furinghetti, F., Jahnke, H.N., & van Maanen, J. (Eds.) (2006), *Report No22/2006 on the Mini-Workshop on studying original sources in mathematics education*. Oberwolfach: MFO; pp.1286-1287.

Galilei, G. (1967). *Dialogues concerning the two chief world systems*. Berkeley: University of California Press; pp.364-365.

Gelfand. I.M., & Yaglom, A.M. (1960): Integration in functional spaces and its applications in quantum mechanics. *J. Math. Physics*, **1**(1), 48-69.

Gelfand. I.M., & Fomin, S.V. (1963). *Calculus of variations*. Englewood Cliffs (NJ): Prentice Hall; §23.
Goldstein, H. (1980). *Classical mechanics*, 2nd edition. Reading (MA): Addison Wesley.
Grattan-Guiness, I. (1973). Not from nowhere. History and philosophy behind mathematical education. *Int. J. Math. Educ. Sci. and Tech.*, **4**, 421-453.
Grattan-Guiness, I. (2004a). The mathematics of the past: Distinguishing its history form our heritage. *Historia Mathematica*, **31**, 163-185.
Grattan-Guiness, I. (2004b). History or Heritage? An important distinction in mathematics for mathematics education. *Am. Math. Monthly*, **111**(1), 1-12.
Harman, P.M. (1990) (Ed.). *The scientific letters and papers of James Clerk Maxwell*, Vol.I. Cambridge: Cambridge University Press; pp.429, 430.
Heath, T.L. (1981). *Aristarchus of Samos: The ancient Copernicus*. New York: Dover.
Heath, T.L. (1991). *Greek astronomy*. New York: Dover.
Heisenberg, W. (1949). *The physical principles of the quantum theory*. New York: Dover.
Heras, R. (2016). Lorentz transformations and the wave equation. *Eur. J. Physics*, **37**(2), 025603 (4pp).
Hilbert, D. (1902). Mathematical problems. *Bulletin of American Math. Society*, **8**, 437-479.
Jahnke, H. N. *et al*. (2000). The use of original sources in the mathematics classroom. In: Fauvel, J. & van Maanen, J. (Eds.), *History in Mathematics Education: The ICMI Study* (pp.291-328). Dordrecht: Kluwer; §9.1.
Jammer, M. (1966). *The conceptual development of quantum mechanics*. New York: McGraw-Hill.
Jankvist, U.T. (2009a). A categorization of the 'whys' and 'hows' of using history in mathematics education. *Educ. Stud. Math*, **71** (3), 235-261.
Jankvist, U.T. (2009b). On empirical research in the field of using history in mathematics education. *Revista Latinoamericana de Investigatión en Matemática Educativa*, **12**(1), 67-101.
Jankvist, U.T. (2009c). History of modern applied mathematics in mathematics education. *For the Learning of Mathematics*, **29**(1), 8-13.
Jankvist, U.T. (2013). History, applications, and philosophy in mathematics education: HAPh—A use of primary sources. *Science & Education*, **22**, 635-656; §7.
Karam, R. (2015). Introduction to the thematic issue on the interplay of physics and mathematics. *Science & Education*, **24** (5-6), 487-494; p.487.
King, H.C. (1979). *The history of the telescope*. New York: Dover.
Kjeldsen, T.H. (2011a). Uses of history in mathematics education: development of learning strategies and historical awareness. In: Pytlet, M., Rowland, T., & Swoboda, E. (Eds.), *Proc. of CERME 7* (pp.1700-1709). Rzeszow, Poland: University of Rzeszow; pp.1700-1701.
Kjeldsen, T.H. (2011b). History in a competence-based of the practice of mathematics in a competence based mathematics education: A means for the learning of differential equations. In: Katz, V. & Tzanakis, C. (Eds.), *Recent Developments on Introducing a Historical Dimension in Mathematics Education*. MAA Notes #78 (pp.165-173). Washington: MAA; pp.166-167.
Kjeldsen T.H., & Blomhøj, M. (2012). Beyond motivation: history as a method for learning meta-discursive rules in mathematics. *Educ. Stud. Math.*, **80**(3), 327–349; §3.
Kjeldsen T.H., & Lützen, J. (2015). Interactions between mathematics and physics: The history of the concpet of function – teaching with and about nature of mathematics. *Science & Education*, **24** (5-6), 543-559.
Klein, F. (1979). *Development of mathematics in the 19th century*. Brookline (MA): Math Science Press.
Kourkoulos, M., & Tzanakis, C. (2010). History and students understanding of the variance in statistics. *BSHM Bulletin*, **25** (3), 168-178; pp.170-175.
Kragh, H. (1990). *Dirac: A scientific biography*. Cambridge: Cambridge University Press; ch.14.
Kragh, H. (2015). Mathematics and Physics: The idea of a pre-established harmony. *Science & Education*, **24** (5-6), 515-527.
Kuhn, T.S. (1959). *The Copernican revolution*. New York: Vintage Books.
Lanczos, C. (1970). *The variational principles of mechanics*. Toronto: University of Toronto Press; ch. VIII.
Lützen, J. (1982). *The prehistory of the theory of distributions*. New York: Springer; ch.4.
Mehra J., (1973). Einstein, Hilbert and the theory of gravitation. In: Mehra, J. (Ed.), *The physicist concept of nature*. Dordrecht: D. Reidel.
Mehra, J., & Rechenberg H. (1982). *The historical development of quantum theory*, vol. 3. New York: Springer; chs. II, III.
Nakane, M., & Fraser, C.G. (2002). The early history of Hamilton-Jacobi dynamics 1834-1837. *Centaurus*, **44**, 161-227.
Pais, A. (1982). *Subtle is the Lord…The science and the life of Albert Einstein*. Oxford: Oxford University Press.
Pauli, W. (1973). *Optics and the theory of electrons*, Cambridge (MA): MIT Press; ch.I.

Pyenson, L. (1982). Relativity in late Wilhelmian Germany: The appeal to a pre-established harmony between mathematics and physics. *Arch. Hist. Exact Sci.*, **27**, 137-155; pp.153-154.
Richtmyer, R. D. (1978). *Principles of advanced mathematical physics*, Vol. 1. New York: Springer; §1.5.
Rogers, L. (2009). History, heritage and the UK mathematics classroom. In: Durand-Guerrier, V., Soury-Lavergne, S., & Arzarello, F. (Eds.), *Proc. of CERME 6* (pp.2781-2790). Lyon: INRP.
Schaffner, K.F. (1972) *Nineteenth century aether theories*. Oxford: Pergamon Press; §§2.9, 2.11.
Schrödinger, E. (1982). *Wave mechanics*. New York: Chelsea.
Siu, M.K. *et al* (2000). Historical support for particular subjects. In: Fauvel, J. & van Maanen, J. (Eds.), *History in Mathematics Education: The ICMI study* (pp.241-290). Dordrecht: Kluwer.
Sneddon, I. (1957). *Elements of partial differential equations*. Reading (MA): Addison Wesley; §§2.13-2.14.
Sommerfeld, A. (Ed.) (1952). *The principle of relativity*. New York: Dover.
Thomas, I. (1941). *Greek Mathematics*, vol.II. Cambridge (MA): Harvard University Press.
Tzanakis, C. (1997). The quest of beauty in research and teaching of mathematics and physics: An historical approach. *Non-linear Analysis, Theory, Methods and Applications*, **30**, 2097-2105.
Tzanakis, C. (1998). Discovering by analogy: The case of Schrödinger's equation. *Eur. J. Physics*, **19**, 69-75.
Tzanakis, C. (1999a). Unfolding interrelations between mathematics and physics motivated by history: Two examples", *Int. J. Math. Educ. Sci. and Tech.*, **30**(1), 103-118.
Tzanakis, C. (1999b). Mathematical Physics and Physical Mathematics: A historical approach to didactical aspects of their relation. In: Radelet-de Grave, P., & Janssens, D. (Eds.), *Proc. of the 3rd European Summer University on the History and Epistemology in Mathematics Education*, vol.I (pp.65-80) Leuven: Université Catholique de Louvain.
Tzanakis, C. (2000). Presenting the relation between mathematics and physics on the basis of their history: A genetic approach. In: Katz, V. (Ed.), *Using history to teach Mathematics: An international perspective*, MAA Notes #51(pp.111-120). Washington DC: MAA.
Tzanakis, C. (2002). On the relation between mathematics and physics in undergraduate teaching. In: Vakalis, I., Hughes Hallett, D., Kourouniotis, C., Quinney, D., & Tzanakis, C. (Eds.), *Proc. of the 2nd International Conference on the Teaching and Learning of Mathematics (at the undergraduate level).* New York: J. Wiley (e-book).
Tzanakis, C., & Coutsomitros, C. (1988). A genetic approach to the presentation of physics: The case of quantum theory. *Eur. J. Physics,* **9,** 276-282.
Tzanakis, C., Arcavi, A., *et al* (2000). Integrating history of mathematics in the classroom: an analytic survey. In: Fauvel, J., & van Maanen, J. (Eds.), *History in Mathematics Education: The ICMI study* (pp.201-240). Dordrecht: Kluwer.
Tzanakis, C., & Kourkoulos, M. (2000). Justification in mathematics and procedures on which it is based: A historical approach for didactical purposes. In: Horng, W-S., & Lin, F-L. (Eds.), *Proc. of the HPM Conference "History in Mathematics Education: Challenges for a new millennium"* vol II (pp.31-51). Taipei: National Taiwan Normal University.
Tzanakis, C., & Thomaidis, Y. (2000). Integrating the close historical development of mathematics and physics in mathematics education: Some methodological and epistemological remarks. *For the Learning of Mathematics*, **20**(1), 44-55.
Tzanakis, C., & Thomaidis, Y. (2012). Classifying the arguments & methodological schemes for integrating history in mathematics education. In: Sriraman, B. (Ed.), *Crossroads in the History of Mathematics and Mathematics Education*. The Montana Mathematics Enthusiast Monographs in Mathematics Education, vol. **12** (pp.247-294). Charlotte NC: Information Age publishing Inc.
van Helden, A. (1985). *Measuring the universe: Cosmic dimensions from Aristarchus to Halley*. Chicago: The University of Chicago Press.
van der Waerden, B.L. (Ed.) (1967). *Sources of quantum mechanics*. New York: Dover.
von Neumann, J. (1947). *Les fondements mathématiques de la mécanique quantique*. Paris: Presses Universitaires de France. Originally published as *Mathematische Grundlagen der Quantenmechanik*, Berlin: Springer, 1932.
Weyl, H. (1952). *Space, Time, Matter*. New York: Dover.
Whitrow, G.J. (1988). *Time in history*. Oxford: Oxford University Press; ch.9.
Whittaker, E.T. (1951). *A history of the theories of aether and electricity*, vol.II. New York: Philosophical Library; pp.31-33.
Wigner, E. (1960). The unreasonable effectiveness of mathematics in the natural sciences. *Communications on Pure and Applied Mathematics*, **13**(1), 1–14.
Yourgrau, W., & Maldestam, S. (1968). *Variational principles in dynamics and quantum theory*. Philadelphia PA: Saunders; §6.

L'INTÉGRATION DE L'HISTOIRE DES MATHÉMATIQUES DANS L'ENSEIGNEMENT DES MATHÉMATIQUES: QUELQUES EXPERIENCES EN CHINE

Wang XIAOQIN
East China Normal University, Shanghai, China
xqwang@math.ecnu.edu.cn

NO TEXT AVAILABLE IN THESE PROCEEDINGS

JOSEPH-DIEZ GERGONNE (1771-1859), PROFESSEUR ET RECTEUR D'ACADÉMIE À MONTPELLIER

Éditeur du premier grand journal de l'histoire des mathématiques et de leur enseignement : les *Annales de Gergonne* (1810-1831)

Christian GERINI

Université de Toulon & laboratoire GHDSO (Université Paris 11 – Orsay), France
gerini@univ-tln.fr

ABSTRACT

Joseph-Diez Gergonne est surtout connu pour ses *Annales de mathématiques pures et appliquées*, journal qui a marqué l'histoire de la communication entre les mathématiciens de toutes catégories : enseignants et élèves du secondaire et des universités, académiciens, anciens polytechniciens, militaires, etc. Ce mensuel, édité pendant près de vingt-deux années, a en effet inauguré la forme actuelle de la communication entre chercheurs et/ou pédagogues de cette science, au plan national comme international, et a participé à l'entrée des mathématiques dans leur champ de spécialité séparé des disciplines dont elles dépendaient jusqu'alors, la philosophie au premier rang d'entre elles. Mais le personnage est intéressant sur de nombreux autres aspects du fait de son implication pédagogique (professeur dans les établissements secondaires puis universitaires), académiques et enfin institutionnelles puisque sa carrière de recteur d'académie à Montpellier est un témoignage de premier plan sur le fonctionnement des institutions de l'époque (la Monarchie de Juillet) et sur la difficulté du métier de recteur.

1 Préambule

Les organisateurs du congrès HPM 2016 m'ont fait l'honneur de m'inviter, et je les en remercie vivement, à donner une conférence publique dont le sujet concerne un personnage important de l'histoire des mathématiques, de leur enseignement, de leur diffusion, et de la ville de Montpellier où se tient le congrès et dans laquelle ce mathématicien-philosophe Joseph-Diez Gergonne occupa une place de tout premier plan en sa qualité de professeur à l'université, d'éditeur de ses *Annales de mathématiques pures et appliquées*, et de recteur d'académie sous la Monarchie de Juillet.

Il est difficile de montrer en un temps limité les multiples facettes du personnage et de son œuvre, ni de résumer en un seul texte le fruit d'un travail de plus de vingt années sur Gergonne, sa carrière, son journal de mathématiques, ses correspondances innombrables avec les différents ministres dont il dépendait, les auteurs d'articles de ses *Annales de mathématiques pures et appliquées*[1] (1810-1831), etc. Ce qui suit est donc constitué en grande partie d'extraits de textes publiés depuis une quinzaine d'années et actualisés, complétés, corrigés quand les découvertes au cours des recherches successives faisaient apparaître des lacunes, voire des erreurs dans ce qui avait été écrit auparavant. La bibliographie en fin de texte propose une liste de certaines de ces publications, et nous prions le lecteur de n'y voir aucune prétention : elle est là pour témoigner seulement de cette réflexion menée sur le long terme sur les multiples facettes du personnage et de son œuvre et par souci d'honnêteté concernant les emprunts qui y ont été faits et qui ont alimenté le texte qui suit.

[1] Que nous noterons AMPA dans les références à suivre.

Les découvertes récentes dans les fonds anciens de la bibliothèque universitaire de Montpellier de nombreux manuscrits de Gergonne et de ses correspondants ouvrent en outre une voie complémentaire par rapport à ce qui va suivre et permettront certainement à nouveau des compléments et autres corrections, même si les intitulés de ces manuscrits suite à leur inventaire récent semblent plutôt confirmer les résultats antérieurs sur le personnage et son œuvre.

Nous allons dans ce qui suit :

- Tout d'abord présenter le Gergonne, son travail et ses fonctions et activités successives.
- Donner ensuite un aperçu plus précis sur ce que nous considérons être les trois périodes qui ont jalonné sa vie : la période nîmoise jusqu'en 1816 (il fut alors professeur d'école centrale puis de lycée, membre très actif de l'académie de Nîmes, et y lança ses *Annales* en 1810), la première période montpelliéraine de 1816 à 1830 (où il devint professeur de faculté tout en continuant à assurer la publication de son journal), et la seconde période montpelliéraine de 1830 à la fin de sa vie durant laquelle, toujours professeur à la faculté, il occupa le poste de recteur de l'académie de Montpellier et dut de ce fait cesser sa fonction d'éditeur de ce que l'on peut considérer aujourd'hui comme le premier grand journal de l'histoire des mathématiques.
- Montrer enfin rapidement l'importance de ses *Annales*, à la fois sur un plan éditorial, mathématique, philosophique et didactique.

2 Joseph-Diez Gergonne (1771-1859)

J.-D. Gergonne est né à Nancy le 19 juin 1771, et mort à Montpellier le 4 avril 1859. Fils du peintre Dominique Gergonne et de Thérèse Masson, il fit d'abord sur les désirs de son père un peu de dessin et de peinture dans l'atelier du paysagiste Claudot, un parent. Il suivit ensuite les cours du collège de Nancy, alors dirigé par des chanoines, où il s'intéressa surtout aux mathématiques et aux sciences physiques.

À 17 ans, il était apte à donner des leçons dans ces matières : ayant perdu son père à l'âge de 12 ans, il fut mis dans l'obligation de subvenir, cinq ans plus tard, aux besoins de sa mère et de sa sœur. En 1792, il s'enrôla comme volontaire dans les armées de la République et prit part à la bataille de Valmy, puis durant un an devint secrétaire d'un oncle ex-constituant à Paris. Réquisitionné, il fut alors envoyé à l'armée de Moselle et affecté au secrétariat de l'état-major. Il se présenta en 1794 au concours d'entrée de l'Ecole d'artillerie de Châlons où il entra premier le 21 février. D'un niveau nettement supérieur aux autres élèves, il obtint son brevet un mois plus tard, provoquant l'admiration de son examinateur, S.F. Lacroix ; il en sortit en mars comme lieutenant et fut envoyé à l'armée des Pyrénées Orientales où il participa à la prise de plusieurs villes sur les espagnols. En juillet 1795, la paix ayant été conclue avec l'Espagne, son régiment fut dirigé sur Nîmes. Une chaire de mathématiques à l'Ecole Centrale de cette ville ayant été mise au concours, il se présenta et l'obtint (mars 1796).

Après son mariage en 1802 avec Eugénie Leclair (qui lui donnera quatre filles), il fut nommé, par décret impérial, professeur de mathématiques transcendantes au lycée de Nîmes en 1804.

Pendant cette première décennie du XIX° siècle, outre son travail d'enseignant, Gergonne occupa diverses fonctions à l'Académie de Nîmes; mais surtout, grand admirateur des savants éclairés de la Révolution (Monge, Bailly, Laplace...), il tenta de les convaincre de fonder un journal scientifique. Les seules publications de ce type étaient alors les *Mémoires de l'académie des sciences*, qui paraissaient épisodiquement, et le *Journal de l'Ecole polytechnique*, consacré uniquement aux travaux de celle-ci.

On peut se demander comment Gergonne pouvait entretenir des relations avec les milieux scientifiques parisiens alors qu'il n'était qu'un simple professeur de province. En fait, aussi bien lors de sa carrière militaire que pendant l'année 1792 passée à Paris, il avait su s'attirer la considération de personnalités militaires, politiques et scientifiques. Son bref passage à l'Ecole d'artillerie de Châlons avait, comme nous l'avons dit plus haut, considérablement impressionné Lacroix (1765-1843), mathématicien, ancien élève de Monge, et qui joua un rôle très important dans la diffusion et l'enseignement de sa discipline[2]. Des mathématiciens d'importance de cette époque avaient aussi été élèves ou collaborateurs de Lacroix à Châlons puis à Paris, ou de Monge à Polytechnique, et Gergonne en avait côtoyé certains et les avait même étonnés par ses capacités: on retrouvera leurs signatures au bas des articles des *Annales de mathématiques pures et appliquées* que Gergonne lancera finalement en 1810, comme sur de nombreux traités publiés alors, et Gergonne avait entretenu avec eux le lien qui lui permit de mener à bien le projet que ses plus illustres correspondants refusèrent d'entreprendre[3].

C'est donc après avoir tissé ce réseau de correspondants et d'abonnés potentiels que Gergonne se « résignera » (le mot est de lui) à publier ce journal qui à ses yeux faisait défaut dans le monde éclairé des « géomètres » de son époque. Aidé pendant les deux premières années (1810-1812) de Thomas-Lavernède, lui aussi professeur au lycée de Nîmes, il poursuivit seul ce travail de rédaction, d'édition, correction, réponses, correspondance, gestion (des abonnements, des frais d'édition...), dessin (Gergonne gravait lui-même les planches de géométrie insérées dans les Annales), *etc*., et cela pendant les vingt années qui suivront. Nous allons voir que ce travail considérable était mené en parallèle avec d'autres activités d'importance.

En effet, fort de la renommée rapide de ses Annales et de l'estime de tous les mathématiciens européens, il se vit confier en 1812, outre son cours de mathématiques, un poste de suppléant de philosophie et le titre de secrétaire de la Faculté des lettres de Nîmes. En 1816, il fut nommé professeur de mathématiques spéciales au collège royal de Montpellier, puis professeur d'astronomie à la faculté de la même ville où il enseignera ensuite la physique sur proposition de l'un de ses anciens élèves, François Guizot, entré en politique après avoir enseigné l'histoire à la Sorbonne ; Guizot deviendra en 1832 ministre de l'Instruction publique et restera un fervent

---

2 Lacroix, ancien élève de Monge, fut le protégé de celui-ci et de Condorcet : on peut lire à ce sujet le fort complet article de René Taton : « Condorcet et Sylvestre-François Lacroix », in: *Revue d'Histoire des Sciences*, Paris, 1959, N°2 pp. 127-158, N°3 pp. 243-262. On a retenu de Lacroix essentiellement son *Traité du calcul différentiel et du calcul intégral* (1797-1799) qui fut réédité pendant près de cinquante années et influença l'enseignement de ce calcul en France et à l'étranger.

3 Michel Chasles, dans son *Aperçu historique sur l'origine et le développement des méthodes en géométrie* (1837 ; réédition Jacques Gabay, Paris, 1989) écrit à propos de Monge et de sa géométrie descriptive : « L'école de Monge est très-redevable aussi à M. Gergonne, qui l'a servie utilement par ses propres travaux, toujours empreints de vues philosophiques profondes, par l'accueil qu'il a fait dans ses Annales de mathématiques, aux productions des anciens élèves de l'école polytechnique ».

défenseur de Gergonne, comptant même sur lui pour l'aider dans sa propre carrière[4] ; son successeur au ministère, Salvandy, fera même attribuer à Gergonne le titre d'officier de la Légion d'Honneur en 1839, le félicitant en ces termes : « *J'ai remarqué votre zèle, votre droiture, vos lumières ; quel prix avait pour moi votre infatigable correspondance et à quel point je me sentais d'accord avec vous, dans votre amour de nos communs devoirs et dans la manière de les remplir.* »[5]. Auparavant, suite aux bouleversements de la révolution de 1830, et à l'avènement de Louis-Philippe (2 août), il devint (le 23 août), sur proposition du conservateur le duc de Broglie, Recteur de l'Académie de Montpellier, – dépendant de l'Université, nom encore donné à l'époque au ministère de l'instruction publique après que les lois napoléoniennes de 1808 l'aient appelé Université impériale –, tout en conservant à son expresse demande son cours à la faculté[6]. Il prit sa retraite en 1844.

La période couverte par la vie de Gergonne fut agitée de tels soubresauts et de tels bouleversements qu'on appréciera l'obstination du personnage dans son ambition d'imposer, de sa province nîmoise, une œuvre à tout ce que l'Europe comptait d'esprits scientifiques, influençant aussi bien les mathématiques de son temps et leur enseignement que l'organisation politique et institutionnelle de son pays.

Travail, rigueur, intégrité, exigence, esprit critique, souci pédagogique et sévérité sont autant de traits qui caractérisent le personnage tout au long de sa vie, et en particulier de sa carrière pédagogique et administrative dans l'enseignement. Le travail effectué par Gergonne révèle à lui seul le premier aspect fondamental de sa personnalité : sa capacité de travail était impressionnante et pouvait le conduire dans de nombreuses voies à la fois avec la même ardeur. Sans cette qualité, nous pensons qu'il n'aurait jamais pu mener à bien et sur une aussi longue durée l'œuvre des *Annales*.

Si nous le considérons sur la période allant de son arrivée à Nîmes (en 1796, comme professeur à l'Ecole centrale) jusqu'au son départ pour Montpellier (1816), nous le voyons devenir professeur de mathématiques transcendantes au lycée (1804), tout en assumant à l'Académie du Gard les fonctions de vice-président (1807), trésorier (1810 à 1816), et de secrétaire perpétuel (1815). A en juger par le nombre de rapports, notices, et autres publications qu'il effectua pour cette dernière institution, on constate qu'il prenait ces fonctions très à cœur : il fut le rapporteur, lors des séances de cette académie, de nombreux travaux qui n'avaient pas forcément de lien avec les mathématiques, et il participait sans compter à sa vie et à son organisation financière et matérielle[7]. Il était en outre membre correspondant d'autres académies françaises, ainsi que d'académies étrangères telles celles de Turin et Berlin. Il trouvait même le temps d'envoyer des mémoires lors de concours ouverts par les sociétés savantes : c'est ainsi qu'il devint membre correspondant de l'Académie de Stanislas (Nancy) en 1811, et qu'il reçut de la société savante de Bordeaux, en 1813,

---

[4] Il lui écrit en 1832: « S*i je ne me trompe, nous nous sommes toujours entendus... aidez-moi à laisser quelques traces de mon passage dans l'instruction publique*. » (In : Alceste en rectorie: Joseph-Diez Gergonne, recteur à Montpellier (1830-1844), *Annales du Midi*, Montpellier, N°193, Jan-Mars 1991, p. 67). En fait, Louis Philippe appellera Guizot à la tête du gouvernement en 1840 et l'entraînera dans sa chute en 1848.

[5] *Ibid*., p.67.

[6] Ce qui explique en partie la fin de la publication des *Annales* en 1831.

[7] L'Académie du Gard venait de rouvrir ses portes, après la tourmente de la Révolution, comme le signalent ses propres notices (Bibliothèque municipale, Nîmes) : « *Société d'agriculture, des sciences, lettres et arts, établie à Nismes** *en 1801, sur les débris de l'ancienne académie royale de cette ville, qui avait été fondée en 1682, associée à l'académie française en 1692, et supprimée par décret de la Convention nationale en 1793.* »

le premier prix sur une « dissertation » dont le sujet imposé concernait l'analyse et la synthèse dans les mathématiques.

Nous ne pouvons pas détailler toutes les activités parallèles de Gergonne durant sa longue carrière. Nous en venons donc à présent rapidement aux trois périodes annoncées en introduction.

3 La période nîmoise

Considérons tout d'abord son parcours[8]:

-1801 : l'Académie du Gard rouvre ses portes ; Gergonne, alors professeur à l'Ecole centrale de Nîmes, y entre comme membre résidant.

-1805 : il y est membre non résidant.

-1807 : il y est vice-président.

-1810 ; 1811 ; 1813 : il y est trésorier.

-1815 : il y est trésorier et secrétaire perpétuel.

-1816 : il y est trésorier.

- Novembre 1816: Gergonne démissionne (il part pour Montpellier : lettre de démission datée du 27 novembre 1816) ; il est remplacé par Thomas-Lavernède.

Nous ne détaillons pas ici la liste de ses communications (au nombre de trente-quatre) à l'Académie de Nîmes, bien qu'elle reflète bien l'étendue de ses compétences et le sérieux de son investissement dans cette institution: elles sont consultables dans les registres de l'Académie du Gard à la bibliothèque municipale de la ville de Nîmes[9].

Parallèlement à ces activités, si on ne considère par exemple que les années scolaires 1812-1814, on le voit alors rédiger le tome III et le tome IV des *Annales* : il signe de sa main dans ces deux volumes quelques 28 articles représentant un total de 227 pages et relevant de 13 domaines mathématiques différents et de la rubrique *philosophie mathématique*[10]. Parallèlement à cette production, il adresse à l'académie de Bordeaux son important mémoire sur l'analyse et la synthèse dans les mathématiques, et à celle de Nancy deux manuscrits, le premier sur l'optique, le deuxième sur le tracé des voûtes, ce qui montre là encore son éclectisme. Il assure aussi bien sûr ses fonctions à l'Académie de Nîmes, son enseignement de mathématiques au lycée, de suppléant de philosophie et de secrétaire à la faculté de lettres. La publication des *Annales* représente à elle seule un travail considérable : il doit lire, comprendre, corriger et retranscrire tous les articles émanant de ses correspondants, les annoter, dessiner (et graver de sa main) les planches de géométrie, surveiller et corriger les épreuves chez l'imprimeur, envoyer les fascicules aux abonnés, lire, comme le montrent les nombreuses références de sa main, les publications nouvelles, et en particulier tous les recueils, publiés ou non, de ses correspondants, *etc*.

8 Sources : Archives de l'Académie du Gard et notices de la même académie (*cf.* note suivante)

9 Notices de l'Académie du Gard, Nîmes, Carré d'Art (cote : 12 924).

10 Nous n'aurons pas la place ici de détailler l'œuvre philosophique de Gergonne, ni ses croisements avec les mathématiques. À titre d'indication, et ceci est représentatif de la période de transition et de spécialisation correspondant à sa vie, signalons par exemple le fait que dans ses *Annales*, des articles de mathématiques pouvaient être classés à la fois dans une rubrique spécialisée (par exemple : « Analyse transcendante ») et dans la rubrique « Philosophie mathématique ». Quant à la pensée essentiellement anti-condillacienne de Gergonne, nous en avons donné un aperçu dans les chapitres référencés (Gerini, 2013) et (Gerini, 2015) à la fin de cet écrit.

Une partie des manuscrits du fonds Gergonne récemment inventoriés par la bibliothèque universitaire de Montpellier, et qui font l'objet de l'exposition qui accompagne ce congrès, témoigne de cette importante activité à Nîmes, y compris avant le lancement des *Annales* en 1810.

4 La première période à Montpellier

La première période montpelliéraine dure de 1816 jusqu'à la fin de la publication des *Annales*, qui suivit d'à peine un an et demi sa nomination en qualité de Recteur en 1830.

Il y poursuivit, bien sûr, les activités liées à la publication de son journal décrites ci-dessus. Mais, malgré ce travail et la charge que représentaient ses cours d'astronomie et de mathématiques, il inaugura à la faculté des sciences de Montpellier ce qui peut être considéré comme l'un des premiers cours d'épistémologie, qu'il intitula « Philosophie des sciences », et dont nous avons pour traces les mémoires du philosophe anglais John Stuart Mill qui assista en 1820, à l'âge de quatorze ans, à ses premiers cours. Le journal de ce dernier[11] permet en effet de reconstruire le plan de cet enseignement de Gergonne (Carol de Saint Victor, 1990) :

Introduction : la nécessité de la logique dans les sciences.
Les sensations et les idées.
Mécanisme des sensations.
Classification des idées : - distinctes et confuses
- complètes et incomplètes
- simples et complexes.
Art d'abstraire: - idées générales
- idées de genre, d'espèces, etc.
Idées abstraites: - idée d'une essence universelle
- le corps comme cause des modifications de notre âme.
Division.
Définition.
Noms, langues, langages : - langues hiéroglyphiques et alphabétiques
- langues de la musique, des mathématiques, etc.
- les défauts des langues actuelles
- manière de faire une langue philosophique.
Idées innées ou acquises.
Jugement et proposition : - propositions simples et compliquées
- termes
- genres des propositions
- notation: l'étendue dans les propositions de quantité et de qualité
- l'opposition
- la conversion.

Parallèlement à ses cours à la faculté il continua à rédiger d'importants articles dans son journal et donc à approfondir à la fois son travail mathématique et ses recherches didactiques et philosophiques. Certains d'entre eux sont d'ailleurs ambitieux et auraient pu s'imposer parmi les grands textes de son temps (c'est le cas cependant pour ses articles sur le principe de dualité et sur

[11] "*John Mill's boyhood visit to France: being a journal and notebook written by John Stuart Mill in France, 1820-1821*", éd. Anna Jean Mill, Toronto 1960, pp.77-96.

le fameux « point de Gergonne » dans le triangle) s'il n'avait dû en interrompre la production et l'élargissement en raison de sa promotion au poste de recteur. On peut citer parmi eux:

<u>Tome VII (1817; pp. 189-228)</u> : Essai *de dialectique rationnelle*, véritable essai de logique mathématique qui reprend les théories classiques des syllogismes, les situe dans une perspective moderne de classes et d'étendues, et entreprend de ranger sous la forme de formules symétriques les propositions et les classes, en introduisant un symbolisme neuf.

<u>Tome VII (1817; pp. 345-473)</u> : De *l'analise*[12] *et de la synthèse dans les sciences mathématiques*, reprise de son mémoire couronné en 1813 par l'Académie de Bordeaux.

<u>Tome IX (1818; pp. 1-35)</u> : *Essai sur la théorie des définitions*, tentative de fixer définitivement, au nom d'un nominalisme pascalien, les règles nécessaires à la rigueur d'exposition seyant aux sciences exactes.

<u>Tome XII (1822; pp. 322-359)</u> : *Dissertation sur la langue des sciences en général, et en particulier sur la langue des mathématiques*, véritable manifeste de près de quarante pages dans lequel il remet en question ici, et approuve là, le système général des signes utilisés jusqu'alors, montre l'aspect réducteur et relatif de tout symbolisme, entre dans le détail du mode (actif : opérations, fonctions ; passif : nombres, variables) des données d'une écriture mathématique, et ouvre donc la voie à une ébauche de métamathématique.

<u>Tome XIII (1822; pp. 1-94)</u> : *Essai sur la recherche des maxima et minima, dans les formules intégrales indéterminées*. Reprenant dans ce travail conséquent les travaux de Lagrange sur le « calcul des variations », il les simplifie, en renforce la rigueur, tout en rendant hommage à son illustre prédécesseur dont il vantera souvent par ailleurs les qualités.

<u>Tome XVI (1826; pp. 209-231)</u> : *Considérations philosophiques sur les éléments** *de la science de l'étendue*. Dans ce long article, que l'on pourrait qualifier d'historique, il montre, en mettant en parallèle vingt-huit théorèmes de géométrie plane, et autant de géométrie spatiale non métrique (donc de géométrie dites *de position*), la dualité entre les théorèmes, la similitude entre leurs démonstrations, où il suffit parfois d'échanger le nom des termes sans changer le raisonnement lui-même, l'aspect « réversible » de certaines propriétés. On peut citer ici François Bouisson : « *La dualité en géométrie, comme on l'entrevoit d'après l'expression elle-même, indique un double aspect dans une proposition. Ainsi l'on enseigne que, sur une droite donnée, on peut concevoir une infinité de points; Gergonne retourne la proposition, et trouve que sur un point donné, on peut concevoir une infinité de droites. La science admet que deux droites données déterminent un point; l'auteur du principe de dualité prend le contrepied et établit avec non moins de réalité, que deux points donnés déterminent une droite*. » (Bouisson, 1859, p. 10) C'est certainement sur le développement ultérieur de telles idées que l'on en est venu à construire par exemple des théories valables pour n'importe quelle dimension, en déduisant immédiatement, aux cas particuliers de la droite, du plan, de l'espace, des polygones, polyèdres, etc. les propriétés qui n'en devenaient que des cas particuliers ne nécessitant plus de démonstration. Gergonne le déplore d'avance dans son article: « *Nous craignons bien toutefois que ce que nous venons d'écrire passe sans être aperçu ou que du moins après un examen superficiel, beaucoup n'y voient qu'un de ces rapprochements forcés qui n'ont de consistance que dans l'esprit de ceux qui les imaginent* » (p. 229). Il est important de noter à propos de cet exemple une des caractéristique de l'œuvre

[12] Nous avons respecté l'orthographe du texte original. Nous indiquerons par un * les mots orthographiés différemment à l'époque, comme nous l'avons déjà fait avec Nismes au lieu de Nîmes dans la note 7.

mathématico-philosophique de Gergonne, à savoir sa volonté de subsumer sans cesse les extensions du champ mathématique en un ensemble de principes universels. Ce travail sur la dualité lui coûtera une polémique en paternité avec Jean-Victor Poncelet (1788 – 1867) (Gérini, 2011-b).

Tome XX (1830 ; pp. 213-284) : *Exposition élémentaire des principes du calcul différentiel*, article de 72 pages qui n'est élémentaire qu'aux yeux de son auteur et en raison de sa (fausse?) modestie. Nous nous proposons dans un proche avenir de comparer cet essai aux divers manuscrits sur le même sujet, datés de 1806 à 1816, que la bibliothèque universitaire de Montpellier vient de retrouver.

Tome XXI (1831 ; pp. 305-326) : *Préliminaires d'un cours de Mathématiques pures*, article de 21 pages en appelant d'autres (qui ne viendront jamais : on est dans l'avant-dernier volume des *Annales*) puisqu'il est publié sous le titre général de *Premières leçons d'un cours*. On y trouve donc les deux premières leçons de ce projet ambitieux, situé à la frontière des mathématiques et de leur philosophie: 1/ *Première leçon: Objet des sciences mathématiques* 2/ *Deuxième leçon: Nous ne connaissons que des rapports*. Gergonne, qui venait donc d'être nommé Recteur de l'Académie de Montpellier, tout en conservant sa chaire de mathématiques, reconnaît que cet essai a été « *écrit fort à la hâte* », mais pense « *n'avoir rien dit d'inutile* ».

Tome XXI (1831 ; pp. 329-361) : *Première leçon sur la numération*, premier article (32 pages) d'un projet qui se voulait lui aussi très ambitieux, mais qui s'arrêtera là: « *Ce que je désirerais que l'on fît ici pour le système entier des connaissances humaines, je n'ai jamais négligé de la faire, dans mon enseignement, par rapport à la langue des nombres. Je vais présenter ici une première leçon sur cet intéressant sujet, telle que je désirerais qu'elle fût faite dans nos écoles.* » (p. 330).

Parallèlement à ces tâches, Gergonne lisait, ses notes critiques dans son journal comme l'inventaire de sa bibliothèque que détiennent les archives départementales de l'Hérault en témoignent, une grande part des ouvrages nouveaux, aussi bien mathématiques que philosophiques. Allant même plus loin, il se proposait d'en traduire certains, et non des moindres, comme l'indique ce passage: « *Dans une traduction des* ***Disquisitiones arithmeticae*** *de M. Gauss, que nous avions entreprise, et que la publication de celle de M. Delille nous a fait discontinuer...* » (AMPA, T. XII, juin 1822; p. 337)

On le voit, Gergonne devait consacrer à son travail une grande part de son temps : cet acharnement se doublait de convictions et d'un caractère qui transparaissent dans tous les domaines et, fort de ses appuis politiques importants à Paris (des savants aux ministres, comme nous l'avons vu plus haut), il se permit, avant et après sa nomination au poste de Recteur, de prendre des positions tranchées vis à vis non seulement des réformes du système éducatif, mais aussi de l'organisation politique du pays.

Ses courriers au ministère furent nombreux comme l'indique A. Lafon[13] et il inséra même dans les *Annales* un essai tentant d'introduire l'arithmétique dans la politique afin d'en corriger des règles jugées par lui injustes : *Quelques remarques sur les élections, les assemblées délibérantes et*

13 « *Il expédiait si promptement les affaires que dans l'espace de quinze années, il a écrit lui-même 44 000 lettres* » (A. Lafon, *Gergonne, sa vie et ses travaux. Discours de réception à l'Académie de Nancy.* Nancy, 1864, p.14) : le nombre affiché nous paraît invraisemblable mais a été repris ailleurs.

le système représentatif, (AMPA, 1815-1816, T.VI, pp. 1-11)[14]. Citons justement A. Lafon parlant de Gergonne et de son article[15]:

> « Il fait voir, en effet, par un raisonnement des plus simples, combien était vicieux le mode d'élection alors employé pour les députés, et à ce sujet, il ajoute « c'est une excellente chose que la liberté, mais la justice est une chose plus excellente encore; et c'est précisément parce que cette liberté est un bien précieux, qu'il ne faut pas que la jouissance en soit réservée à une classe quelconque de citoyens. » A la même époque, il envoya à M. le baron Pasquier, alors Ministre de la Justice, un mémoire intitulé: **Examen critique de quelques dispositions de notre code d'instruction criminelle**. On assure que ce mémoire ne fut pas sans influence sur les modifications qu'on apporta à ce code en 1830. »

5 La deuxième période à Montpellier

5.1 Un recteur exigeant et contesté

« Voué depuis trente-cinq ans, par un goût des plus prononcés, aux fonctions de l'enseignement, pour lesquelles j'ai même abandonné de très bonne heure, la carrière militaire qui s'offrait à moi sous un point de vue assez brillant ; j'ai vu jusqu'ici, sans envie et sans regrets, les fonctions rectorales dévolues à des hommes qui me semblaient avoir beaucoup moins de titres que moi pour les obtenir ; content de pouvoir me livrer, sans distraction aucune, à des travaux scientifiques qui avaient pour moi le plus grand charme. Quelque sensible que je sois donc, Monsieur le Duc, à la haute confiance dont vous voulez bien m'honorer, je n'aurais pas hésité, à toute autre époque, à vous supplier de me laisser à mes paisibles et modestes occupations, et de faire tomber votre choix sur quelque autre fonctionnaire de l'Université. Mais, les circonstances sont difficiles ; le gouvernement a besoin qu'on lui prête assistance, pour organiser et consolider l'ordre de choses qu'il a dessein d'établir ; et dès lors je ne dois pas balancer à me dévouer et à concourir à ses vues ; et je le fais d'autant plus volontiers que le changement qui doit en résulter dans ma situation ne m'apporte aucun avantage pécuniaire, si même il ne m'est pas onéreux. Heureux si, dans le nouveau poste où je suis appelé, je puis faire quelque bien, et surtout prévenir l'autorité, lorsqu'elle daignera me faire l'honneur de me consulter, contre d'imprudents bouleversements auxquels des gens intéressés au désordre pourraient tenter de l'entraîner ».

C'est en ces termes que Gergonne rédige son acceptation de la fonction de recteur[16]. La liberté de ton, l'adhésion au nouveau régime, la critique implicite du peu de moyens dont dispose un recteur, l'engagement à servir l'autorité jusqu'à assumer un rôle de police, sont révélateurs à la fois de la position de force dans laquelle se sent l'auteur et de son tempérament affirmé. S'il parle (avec fausse modestie) de ses « paisibles et modestes occupations », il sait très bien que ses *Annales de mathématiques*, sa qualité de correspondant de l'Institut et ses soutiens politiques à Paris lui permettent de traiter quasiment d'égal à égal avec les ministres, ce dont il ne se privera pas par la suite.

La fonction de recteur, encore récente en 1830, est toujours mal acceptée par les cadres des autres corps de l'état (qui y voient une concurrence sur le terrain de leurs attributions), comme par les élus locaux (l'État s'est déjà déchargé sur les communes, par exemple, de la gestion financière des établissements secondaires, gestion et établissements surveillés par les recteurs) et le clergé (qui est dessaisi par la loi de son influence sur l'enseignement).

---

14 Nous mentionnons l'intérêt de ce texte dans le chapitre d'ouvrage référencé (Gerini, 2011-a) en fin d'article.

15 A. Lafon, *Ibid.* p. 4.

16 Archives Nationales (AN), F[17] 20829, lettre de J.D. Gergonne au Duc de Broglie, 28 août 1830.

L'organisation de l'instruction primaire est à cette époque très insatisfaisante : la scolarisation des enfants est faible, leur éducation confiée à des maîtres souvent sans formation : « Dans la plupart des villages, le maître est sous l'autorité directe du curé, qui lui demande de sonner la cloche, d'enseigner le catéchisme, et seulement de s'occuper d'apprendre un peu à lire : l'instruction n'apparaît pas comme nécessaire » (Nique, 1999, p. 41). L'enseignement secondaire, placé pourtant depuis la loi de 1808 sous l'autorité de l'Etat, ne s'est amélioré que dans les lycées : les collèges communaux, trop liés aux gestionnaires locaux dont ils dépendent, malgré le contrôle de la puissance publique via les recteurs, « sont de niveau très inégal » (*ibid*. p. 72). Et la question de la liberté de l'enseignement continue à miner les rapports entre le clergé, les recteurs et les maires : c'est ce qu'on appellera la crise du Monopole. Le nouveau recteur va chercher, à l'image de son ancien élève devenu ministre François Guizot, à la fois à lutter contre les ignorances et le laxisme des pouvoirs locaux, à élever le niveau d'exigence aussi bien des élèves que de leurs maîtres, tout en adhérant au principe d'une liberté de l'enseignement limitée et sous contrôle. Tout cela est motivé, en arrière-plan, ou plus tard ouvertement, par ses convictions politiques, proches elles aussi de celles de Guizot et du Parti de la Résistance que Louis Philippe installe au pouvoir en mars 1831. Sachant imminente sa mise à la retraire à la fin de 1843, il avouera sans retenue cette parenté dans une lettre adressée à son ministre où il revient sur les évènements de juillet 1830 et sur sa nomination :

> « Ce fut dans ces circonstances critiques que M. le Duc de Broglie me fit l'honneur de m'appeler à l'administration de l'académie, fonctions que je n'avais ni sollicitées ni même désirées ; et que je n'aurais probablement pas acceptées dans un tems[*] plus tranquille ; parce qu'elles me sortaient tout à fait de mes goûts et de mes habitudes ; mais je me sentais de force à faire tête à l'orage, et je pensai qu'un bon soldat ne devait pas reculer devant le danger. Les avanies de tous genres ne m'ont point faute (SIC), en effet, pendant les six premières années. Je me constituai tout d'abord le Casimir Perrier de l'académie. »[17]

5.2 Le conseil académique, les notables locaux

Dès son installation, Gergonne, en fonctionnaire zélé comme en défenseur acharné des prérogatives que la loi lui attribue, réorganise la gestion de son académie, et par là même commence à s'attirer quelques inimitiés. Cette situation est vécue par la plupart des recteurs mis en place depuis 1808.[18]

Chargé par exemple de nommer les membres du conseil académique, il tente d'en évincer le premier président du tribunal (« je jugeai M. le Premier Président trop favorable à l'opinion vaincue à la fin de Juillet ») et le procureur général (« trop favorable à celle qui a été vaincue à la fin de décembre »[19]), en stricte application de la loi du17 mars 1808 (art. 85) qui stipule que « les conseils académiques ne doivent être composés que de dix membres, tous

17 Archives nationales : AN F[17] 20829, Lettre de Gergonne au ministre de l'instruction publique datée du 30 novembre 1843

18 On pourra consulter à propos de la création des académies et de la fonction rectorale en 1808 les deux ouvrages collectifs parus à l'occasion du bicentenaire de cet évènement :

- Jean-François Condette & Henri Legohérel (Dir). *Le recteur d'académie. Deux cents ans d'histoire*. Paris : Editions CUJAS, 2008.
- Jean-François Condette (Dir.). *Les Recteurs. Deux siècles d'engagement pour l'École (1808-2008).* Presses Universitaires de Rennes, 2009.

19 Archives nationales : AN, F[17] 20829, lettre de J.D. Gergonne au Procureur Général, 12 janvier 1831

pris dans le corps enseignant.»[20] Et d'ajouter : « Ce n'est, en conséquence, qu'en infraction à la règle que, depuis environ dix ans, on a introduit dans ces conseils un nombre illimité de fonctionnaires étrangers à l'Université. Me figurant qu'on voulait rentrer dans la légalité, je proposai au Ministre, dans le courant de novembre, un Conseil académique conforme au décret ».

Il défend en fait son propre point de vue politique (« Désirant donc un conseil d'hommes modérés, tels que je me suis constamment fait gloire de l'être depuis plus de quarante ans »), mêlant sans cesse gestion administrative et convictions personnelles. Il renvoie donc aux extrêmes les légitimistes d'avant juillet et les républicains contrés en décembre, et feint de s'en tenir, dans la constitution du conseil, à la stricte application de la loi. Le ministre de l'instruction publique, plus pragmatique, lui imposera finalement la nomination du procureur général au conseil[21].

Gergonne se pose en fervent défenseur de l'Université, et en fonctionnaire zélé voulant appliquer à la lettre les règles du monopole de l'Université, offensant là encore les fonctionnaires des autres corps de l'état et les représentants du clergé. Luttant par exemple en 1832 contre les écoles clandestines, il saisit le substitut du procureur du roi de Sainte Affrique pour dénoncer certains instituteurs « clandestins », traitant l'un d'entre eux de « contrebandier ». Ses propos et son jugement sur le substitut lui-même parviendront jusqu'au Garde des Sceaux qui transmettra le dossier au Ministre de l'Instruction Publique, accompagné de ces mots : « avec une ironie offensante il révoque en doute la véracité de Mr Pouget et se plaint amèrement de son refus de poursuivre un Sieur Vernière qu'il traite de contrebandier. En général les plaintes de Mr. le Recteur de l'académie de Montpellier sur la tolérance des Magistrats pour les écoles clandestines sont loin d'être fondées. »[22]

Cette rigueur sans concessions s'exerce en effet tout d'abord dans la gestion des affaires strictes de l'instruction publique au sein de son académie : problèmes de locaux des facultés, questions de dérogations pour les candidats au baccalauréat, affaires de police lors de divers troubles liés au contexte politique de l'époque, mouvements d'étudiants, etc.

Les termes de la loi instaurant la fonction de recteur sont sans ambiguïté quant au rôle de censeur et d'administrateur : les recteurs « se feront rendre compte par les doyens des facultés, les proviseurs des lycées et les principaux des collèges, de l'état de ces

---

20 Ibid.

21 Gergonne conclura sa lettre au procureur général, qu'il vient donc d'accuser d'être trop favorable aux républicains, de façon plus diplomatique : « Je désire et j'espère, Monsieur le procureur général, que les relations qui résulteront entre nous de votre admission dans le conseil académique, me prouvent chaque jour davantage que mes préventions ont été tout à fait injustes.»

22 Lettre du Garde des Sceaux au ministre de l'instruction publique du 24 mars 1832. Archives nationales : A.N., F^{17} 20889. Cette lettre est aussi révélatrice du malaise qui régnait sur la question du monopole, et qui engendrera les crises que l'on connaît tout au long de la Monarchie de Juillet. Le garde des Sceaux ajoute, suite à la plainte du procureur : « une circulaire a été adressée à ce sujet à tous ses substituts pour activer leur surveillance et qu'après comme avant cette circulaire des poursuites ont été intentées contre plusieurs Etablissements dénoncés. Le Ministère public ne cesse de veiller à l'exécution des lois concernant l'Université, sauf la prudence et les ménagements que peuvent commander le besoin des localités, la qualité des personnes et l'intérêt même de l'Instruction publique. »

établissement ; et ils en dirigeront l'administration, sur-tout* sous le rapport de la sévérité dans la discipline, et de l'économie dans les dépenses.»[23]

5.3 La gestion administrative et l'« économie dans les dépenses »

Gergonne remplit son rôle en fonctionnaire zélé et intègre. Il défend systématiquement et de pied ferme les budgets des établissements de son académie, se plaignant souvent de leur insuffisance, aussi bien au niveau des collèges royaux que des facultés.

Il écrit par exemple au ministre, à propos de la demande par un professeur du collège royal de Montpellier d'un rattrapage de salaire pour des fonctions exercées avant son arrivée dans l'académie :

> « Dans le cas où vous jugeriez juste de lui accorder un traitement plus élevé, il ne le serait peut-être pas que ce fût aux dépens de la caisse du Collège royal de Montpellier, qui est déjà fort chargée, et qui ne doit payer, ce me semble, que les services rendus à cet établissement. »[24]

On le voit aussi dans une lettre au Doyen de la faculté des sciences (non datée), rendre compte à ce dernier des courriers qu'il a adressés au ministère à propos de la volonté de la ville de Montpellier de déplacer la faculté : non seulement il demande au ministre de s'opposer à ce déménagement, mais il ajoute que, « si la ville déplace la Faculté, elle doit supporter les frais qui résulteraient de sa translation »[25].

Malgré son souci d'excellence et son attachement à la loi, il sera parfois amené à composer avec la réalité et les lacunes dont il hérite dans son académie lors de sa nomination. Ainsi tente-t-il en 1833 de régler une question épineuse qui met en jeu un ecclésiastique (l'abbé Bous, autorisé un an plus tôt, sous réserve de passer son baccalauréat par la suite, à ouvrir un pensionnat à Prades), un règlement d'examen (l'abbé demande à le passer sous son ancienne forme, car « il ne savait ni mathématiques, ni physique »), et un ancien ministre lui ayant recommandé cet abbé (le duc de Montalivet, ministre de l'instruction publique et des cultes de mars 1831 à avril 1832, *cf.* Yvert, 1990, pp. 160-161). Il écrit prudemment au ministre, toujours dans la même lettre et en lui donnant d'autres exemples afin d'orienter son choix dans un sens favorable à l'abbé :

> « ... vous pouvez, tout comme il vous plaira, ou délivrer purement et simplement le diplôme à l'abbé Bous, ou lui accorder un délai plus ou moins long pour se préparer à subir un nouvel examen, ou enfin lui faire immédiatement fermer son pensionnat. Mais avant de prendre une détermination, veuillez bien considérer, je vous prie, qu'à Céret, dans le voisinage de Prades, un homme, contre mon avis formel, a été élevé au rang de Principal de collège, sans avoir et sans être tenu de prendre le grade de Bachelier, ce qui n'a pas empêché deux de ses régents d'être dernièrement destitués pour ne pas s'en être pourvus dans le délai qui leur avait été assigné. Veuillez bien considérer aussi qu'à S^{t} Affrique (Aveyron) un sieur Vernière, pour prix de cinq mois d'audace durant lesquels il a exercé la profession de maître de pension sans aucune autorisation et sans payer aucune redevance à l'université, a obtenu délai sur délai pour se pourvoir du

---

23 Décret portant organisation de l'Université, 17 mars 1808, Titre XII, article 97. Bulletin des lois de l'Empire français, N° 185, pp. 145 & s.

24 Archives nationales : AN F^{17} 1769, lettre de Gergonne au ministre de l'instruction publique datée du 26 janvier 1832. Cette requête du professeur demandera près d'un an avant d'être réglée, et fera l'objet d'une abondante correspondance (même référence aux AN) entre l'intéressé, le recteur, le ministre, le maire de Montpellier, l'académie d'origine, etc.

25 Archives municipales de Montpellier, cote R12-152.

> grade de Bachelier ès lettres qu'il n'a pas encore, et qu'il n'aura peut-être jamais, et qu'à tout prendre, mieux vaut encore être Bachelier suivant l'ancienne mode que de ne l'être pas du tout ».

On mesure dans ce texte l'état de l'instruction en France à cette époque, et les problèmes auxquels se trouvent confrontés depuis déjà plus de vingt ans les recteurs dans la mise en place de cette Université, de ses règles, et de ses contrôles. Il est difficile alors de ménager toutes les susceptibilités, et de résister aux pressions, ce que le recteur Gergonne tente de faire avec acharnement. L'exception qu'il demande ici pour le baccalauréat de l'abbé est exceptionnelle : il la refusera par ailleurs quasi systématiquement à tous les candidats.

On pourrait multiplier les exemples : les registres des correspondances du recteur Gergonne sont tenus par lui-même avec rigueur et nous montrent l'étonnante dimension du travail accompli. Ces registres permettent aussi de constater qu'un recteur est sollicité sur des domaines extrêmement variés et peut avoir à gérer aussi bien les questions financières de tous les établissements secondaires et supérieurs de son académie que les affaires de police et de surveillance politique en leur sein, les demandes de dérogations aux examens, d'exonération de droits d'inscription, les mouvements des personnels, les validations des diplômes, etc. Ainsi par exemple peut-on voir dans le seul registre d'avril à octobre 1839, et sur la seule période du 6 au 8 mai, l'inscription des lettres reçues portant les numéros 35369 à 35406 : rapports hebdomadaires des collèges et des facultés, instructions ministérielles, candidatures d'enseignants, demande d'affectation de personnels par les ecclésiastiques, transmissions de diplômes obtenus dans d'autres académies, etc.[26] Les réponses de Gergonne sont elles aussi répertoriées par ordre chronologique et portent quasi systématiquement référence du numéro de la lettre reçue[27].

5.4 La « sévérité dans la discipline »

Gergonne exerce malgré tout avec délectation son rôle et sa sévérité aussi bien à l'encontre des fonctionnaires que des étudiants. On le voit souvent se poser en juge, menant de véritables interrogatoires, et prenant autoritairement des décisions qu'il impose alors aux instances sous sa responsabilité, avec compte-rendu systématique au ministre.

Ainsi par exemple en 1837, à propos de la validation du titre de bachelier pour un étudiant dénommé Carbonnel : Gergonne soupçonne ce dernier de s'être fait représenter l'année précédente au baccalauréat, et « retient » le diplôme de l'étudiant, attendant que celui-ci vienne le lui réclamer, ce qu'il fera le 7 juillet 1838, soit un an après l'examen, afin de pouvoir poursuivre ses études à la faculté de médecine. Le rapport au ministre que rédige

---

26 A titre d'exemple : le ministre « consulte sur une dispense d'âge pour le baccalauréat » (n° 35371), « transmet l'acceptation d'engagement du Sieur Séguret » (n° 35372), « transmet le titre de pension de M. Villiers » (n° 35379), « nomme les membres de la commission administrative du jardin » (n° 35389), etc.; le « comité de Carcassonne sollicite autorisation pour une institutrice » (n° 35381); « le docteur Dumas écrit qu'il postule une chaire à Rennes » (n° 35 376) ; « le recteur de Lyon transmet un diplôme de bachelier ès sciences » (n° 35 385) ; « l'abbé Valade me dénonce l'instituteur Séguret de Garrigues » (n° 35391) et « demande un instituteur pour Berlou » (n° 35392) ; « le proviseur de Rodez accuse réception de la décision sur les troubles » (n° 35 394) ; « le principal de Clermont écrit sur les retenues » (n° 35 374) ; etc.

27 Par exemple : « Suivant la lettre 30550 du Ministre, annoncé au Principal de St Genis qu'un congé est accordé à M. Combescure jusqu'à la fin de l'année scolaire. Demandé ses vues sur la suppléance. » (9 avril 1839) ; «Suivant la lettre 30467 du Recteur de Nîmes et un rappel de la lettre du 12 mars, demandé M. Troussard au Ministre pour censeur à Rodez » (1er avril 1839).

Gergonne sur cette entrevue est pour le moins édifiant sur sa « méthode »[28]. Faisant mine de rechercher un document dans un dossier « en rendant à dessein cette recherche un peu longue afin de me donner tout le temps d'examiner sa taille et ses traits », le recteur mène ensuite un véritable interrogatoire qu'il relate au ministre à la façon d'un compte-rendu de procès. Le conseil royal de l'instruction publique se basera sur le seul rapport de Gergonne pour suivre ce dernier dans ses décisions (Archives nationales : AN F^{17} 4391, Arrêté du conseil royal de l'instruction publique en date du 20 juillet 1838), suivi peu après par le conseil de la faculté de médecine de Montpellier.

Nous avons choisi cet exemple (non isolé[29]) sur le chapitre de la sévérité car il est révélateur de la politique inquisitoire et autoritaire du recteur qui le rendra si impopulaire auprès des étudiants comme de la population toute entière. Mais il met aussi par ailleurs en avant sa méfiance à l'égard des élus et de leur improbable impartialité dans les attestations qu'ils délivrent (dites « certificats d'études ») pour l'autorisation des élèves à passer l'examen du baccalauréat. Adressant à son ministre les déclarations de candidature de neuf élèves à cet examen en 1841, il écrit par exemple :

> « Vous m'avez fait l'honneur de m'écrire en novembre dernier, Monsieur le Ministre, au sujet d'un pareil envoi, que j'aurais dû y joindre mon opinion personnelle, sur la sincérité des déclarations ; mais le mot <u>immédiatement</u>, inséré dans l'arrêté, s'y oppose évidemment ; et alors même que j'aurais des agents secrets, des espions, dans toutes les localités où ces candidats prétendent avoir étudié, je ne me flatterais pas de recueillir leurs renseignements en moins de trois mois. […]. Mais à qui s'adresser, pour avoir des renseignements ? Aux maires dira-t-on peut-être. Mais d'abord, il n'est pas certain que tous me répondraient et me répondraient dans le délai que je leur aurais indiqué. Car les maires, <u>fonctionnaires élus</u>, ne se croyent[*] pas tenus de déférer aux demandes des recteurs, <u>fonctionnaires salariés</u>. Et d'ailleurs, un maire qui a mis au bas d'un certificat d'études, que ce certificat contient vérité, ira-t-il le trahir lui-même, s'accuser de faux, en disant au recteur le contraire de ce qu'il a signé ? J'ajouterai que la mesure relative aux certificats d'études étant frappée d'une réprobation générale, peu de gens se feraient scrupule de faire des faux pour l'étude ; et tel est l'effet inévitable des mauvaises lois. »[30]

Gergonne fait référence ici (et critique) l'arrêté du 28 août 1838 portant obligation aux candidats de fournir un « certificat d'études » délivré par les maires, pour pouvoir se présenter au baccalauréat.

Mais sa sévérité s'exerce aussi à l'encontre du corps enseignant. Dans un courrier de 1837 adressé à son ministre suite à des évènements qui défrayèrent la chronique, Gergonne

---

28 Archives nationales : AN F^{17} 4391, Lettre du recteur Gergonne au ministre de l'instruction publique datée du 7 juillet 1838.

29 Gergonne avait pour principe de rejeter systématiquement les demandes de dérogations en tous genres. On le voit par exemple en 1832 s'opposer à une requête d'un dénommé Baubil, étudiant en médecine qui, ayant arrêté ses études pendant huit ans, et n'ayant donc pas dans ses bagages les disciplines nouvelles exigées au baccalauréat depuis lors (grec, mathématiques et physique), demande à passer l'examen sous son ancienne forme. Le fait que la demande soit appuyée par un professeur de médecine, l'étudiant arguant d'un problème de santé, n'influence en aucune sorte le recteur, bien au contraire : « Quand on a mal de tête, on doit se faire soigner et suspendre ses études jusqu'à complète guérison ». Et d'ajouter, imaginant qu'on accepte une complaisance en pareil cas, que : « bientôt, la foule s'y précipiterait, le mal de tête gagnerait de proche en proche ; et il ne manquerait pas de médecins officieux pour en attester l'existence ». Archives nationales : AN F^{17} 1769, Ref. 832, lettre de Gergonne au ministre de l'instruction publique, 17 avril 1832.

30 Archives nationales : AN F^{17} 4391. Lettre de Gergonne au datée du 20 avril 1841.

revient sur une affaire remontant à 1833, et nous renseigne sur la nature détestable de ses rapports avec le corps enseignant des facultés :

> « Lorsqu'en 1833 je me mis sur la brèche pour faire cesser, au péril de ma vie, les outrages auxquels M. le professeur Rech était journellement en butte, c'était évidemment la cause de la faculté de médecine que je défendais ; car de tous ses professeurs, M. Rech m'était le moins connu. Cependant, loin de trouver de l'appui dans ses collègues, lorsqu'on leur parlait de mes efforts pour le rétablissement de l'ordre, ses amis mêmes, ceux qui avaient le plus efficacement contribué à sa nomination, répondaient fraîchement : mais aussi, pourquoi le recteur va-t-il se fourrer là-dedans ? que ne laisse-t-il Rech de débattre avec les étudiants comme il l'entendra ?
>
> Il est donc prouvé qu'on ne peut se promettre le concours efficace de l'ascendant de MM. les professeurs de la faculté de médecine sur leurs élèves pour le rétablissement et le maintien de l'ordre qu'autant qu'un intérêt puissant viendra les faire sortir de leur apathie et de leur rôle de spectateurs passifs ».[31]

Son jugement sur les maîtres de pensions et les enseignants en général n'est pas davantage complaisant, et il se permet des descriptions d'entre eux qui ne laissent pas de doute sur ces autres raisons de son impopularité : « charlatan », d'une « complète nullité », « contrebandier », *etc*.[32]

5.5 Les émeutes et le procès de 1837

Il faut reconnaître que les temps sont difficiles. Les révoltes étudiantes sont nombreuses sur tout le territoire, attisées par des mouvements politiques occultes dirigés depuis Paris par les royalistes, le clergé ou les républicains.

Un exemple paroxysmique de cet état permanent de conflit implique pendant une année entière, en 1837, les notables locaux, les ministres, et la justice elle-même.

> « Ce fonctionnaire d'un caractère extrêmement difficile et d'une raideur inflexible, a plusieurs fois par des paroles peu mesurées, excité des désordres et des improbations fâcheuses de la part de nombreux élèves qui suivent ses cours de l'université, et compromis ainsi la tranquillité publique.
>
> Pour empêcher que l'exaltation des élèves ne les porte à des excès qu'on ne pourrait réprimer que par des mesures de rigueur, M. le Procureur général, M. Le Préfet de l'Hérault et M. le Général Colbert pensent qu'il conviendrait de faire passer M. Gergonne à un autre rectorat sans toutefois porter autrement préjudice à un homme supérieur comme savant, et intéressant comme père de famille, mais dont la présence prolongée à Montpellier peut avoir des conséquences fâcheuses ».

Ainsi s'adresse au ministre de l'Instruction Publique, dans une lettre datée du 11 mai 1837, le « Ministre, secrétaire d'état de la guerre »[33]. Ses propos résument bien les inimitiés des acteurs des divers corps de la fonction publique à l'encontre du recteur Gergonne. Il dénonce aussi la responsabilité de ce dernier, en son rôle de professeur, dans les troubles qui agitent le milieu estudiantin de Montpellier et qui débordent de ce cadre pour embraser la ville entière. On retrouve ici le problème de la confusion des rôles et de la difficulté d'être à la

---

31 Archives nationales : AN F^{17} 20829, Lettre de Gergonne au ministre de l'instruction publique datée du 30 avril 1837.

32 Exemples empruntés à Louis Secondy, *Alceste en Rectorie. Joseph-Diez Gergonne, recteur à Montpellier*, op. cité, pp. 68-69.

33 Archives nationales : AN, F^{17} 20829.

fois un professeur, avec ses exigences et sa sévérité, et un recteur, avec son devoir de gérer les affaires locales de son académie et d'y être aussi bien le représentant de l'autorité que le modérateur. Gergonne reconnaît lui-même cette ambiguïté, mais il en fait porter la responsabilité aux politiques :

> « Je reconnais comme vous, Monsieur le Préfet, que ma double qualité de Recteur et de professeur ajoute à la difficulté de la situation ; mais telle est la règle constitutive de l'Université, dont le maintien a été plus d'une fois réclamé par nos chambres législatives, qui voulaient même que tous les proviseurs et principaux de collèges enseignassent. On avait pensé dans l'origine, avec beaucoup de raison, ce me semble, qu'un recteur serait plus considéré quand on verrait qu'il était quelque chose de plus qu'un homme de bureaux. Ce n'a été que par exception que plus tard, soit que les académies n'eussent pas toutes des Facultés, soit que, sous la restauration, le gouvernement ait voulu mettre à la tête des académies des hommes à sa main, on a vu des recteurs ne professant pas. »[34]

Mais le recteur se garde bien de détailler les raisons de son impopularité, causes réelles des troubles qui vont déborder largement le cadre de son cours, et dont nous avons déjà vu quelques exemples.

L'affaire débute le 4 avril 1837, durant son cours de physique à la faculté des sciences. Son impopularité est telle que, prévenues des risques de troubles, la mairie de Montpellier, la police et l'armée (vingt-cinq hommes du 61ème régiment) se tiennent prêtes à réagir.[35] Les étudiants (et parmi eux des instituteurs) s'en prennent effectivement au recteur au début de son cours, le sifflant et le huant copieusement :

> « M. Gergonne fut obligé de cesser son cours, après quelques façons d'épigrammes lancées par lui contre les élèves et les professeurs de la Faculté de médecine. Les cris et les huées l'accompagnèrent dans les rues jusqu'au Collège royal ; et ici ce ne furent pas les étudiants seuls, mais bien aussi des gens de la ville qui donnèrent à M. Gergonne des témoignages assez positifs de leur peu de sympathie pour lui. »[36]

Cette affaire va impliquer durant deux mois trois ministres, les élus locaux, les officiers de police, le préfet, le procureur du roi, etc. Elle permet de mesurer à la fois les inimitiés que s'était attirées le recteur au plan local et les protections dont il bénéficiait à Paris. Elle nous renseigne aussi sur la réalité de l'enseignement supérieur de l'époque. Une autre lettre du préfet de l'Hérault au ministre de l'instruction publique nous permettra de conclure provisoirement sur cette très inconfortable position d'un recteur de la première moitié du 19ème siècle.

Le récit que fait Lombard des incidents et du procès est confirmé par de nombreux commentaires que les différents acteurs ne peuvent s'empêcher d'ajouter à leurs propres

---

34 Archives départementales de l'Hérault, 1M918, lettre du recteur au préfet datée du 2 juin 1837.

35 Archives départementales de l'Hérault, 1M 918. Rapport du Commissaire Bernarde, auxiliaire du procureur du roi.

36 Ainsi témoignera, dans un compte-rendu très personnel du procès qui s'ensuivit, L. Lombard, un instituteur traduit devant la justice en compagnie de neuf étudiants après enquête de police (L. Lombard, Compte rendu du procès intenté à un docteur et à neuf étudiants de la faculté de médecine de Montpellier ; Bibliothèque de l'Agglomération de Montpellier, p.1). Ce procès fit grand bruit, et fut suivi d'un autre en appel du même Lombard, condamné en première instance ainsi que trois des autres prévenus. Le procureur du roi « a condamné savoir Lombard en 20 jours d'emprisonnement et 100 F d'amende, Veilhan et Bergougnoux en 10 jours d'emprisonnement et 25F d'amende, et Gairard en 8 jours d'emprisonnement et 16F d'amende, le tout solidairement et par corps. » (Archives Départementales de l'Hérault, dossier 1M 918, lettre du Procureur du roi au procureur général datée du 27 avril 1837).

comptes rendus et communications officielles. Ainsi le procureur du roi précise-t-il dans son envoi au procureur général :

> « Après avoir rempli mon devoir en poursuivant M. les étudiants et en requérant des peines contre eux, je crois le remplir encore en déclarant que leur conduite aux débats, ainsi que celle de tous les étudiants en médecine qui encombraient la salle d'audience, a été toute de décence et toute de respect pour la justice, qu'il serait fâcheux qu'à la punition prononcée vînt se joindre des peines académiques qui détruiraient l'avenir de ces jeunes gens. J'ai lieu d'espérer que de leurs chefs ils ne se rendront plus coupables d'un pareil délit. Je dois ajouter que l'âge de plusieurs d'entre eux les rapproche encore de l'adolescence, et sous ce rapport inspirent (SIC) quelques intérêts. Enfin, Monsieur le Procureur général, je dois ajouter et c'est le plus pénible de mon devoir, que si malgré mes prévisions, M. le Recteur était encore outragé, ce serait peut-être aux manières dures, au langage sec et aux formes acerbes et quelque fois peu honnêtes de M. Gergonne, que l'on devrait imputer le renouvellement de scènes affligeantes. Quoiqu'il en soit je serai toujours à poursuivre et à faire réprimer un manque de respect et un outrage envers un fonctionnaire public, quel que soit le caractère du personnage de ce fonctionnaire, parce qu'il faut toujours assurer le maintien de l'ordre et le respect de la loi. »[37]

Dans son propre compte-rendu des évènements à son ministre, Gergonne écrit par exemple :

« Je considérai ensuite qu'excepté un petit nombre d'élèves avides d'instruction, les étudiants de 2ème, 3ème et 4ème année, parmi lesquels se trouvent précisément les perturbateurs, n'avaient qu'un médiocre intérêt à ce que ce cours eût lieu, puisqu'ils ne sont point eux, comme ceux de 1ère année, assujettis à se pourvoir du grade de bachelier, et que conséquemment ils seraient toujours disposés à le troubler et à le faire interrompre. »[38].

Entre temps, les courriers du préfet de l'Hérault à son ministre de tutelle et au ministre de l'instruction publique en appellent à Paris pour soulager l'académie de Montpellier de ce recteur si impopulaire. Là aussi, l'ambiguïté de la double fonction de Gergonne est mise en avant, même si l'on dénonce son caractère par la même occasion :

> « A l'occasion des désordres qui ont troublé les cours de Mr Gergonne, recteur de l'académie et professeur de la faculté des sciences de Montpellier, permettez-moi de vous signaler un grave inconvénient du cumul des fonctions d'un recteur d'une académie et professeur d'une faculté.
>
> Le principal devoir d'un recteur est le maintien des règlements universitaires. Il est donc souvent chargé de l'application de certaines mesures qui contrarient les intérêts ou les désirs des élèves de faculté.
>
> Si ces élèves n'ont que des rapports individuels avec le recteur dans son cabinet, leur mauvaise humeur fut elle poussée jusqu'à une vive irritation ne saurai avoir de fâcheuses conséquences. Le recteur ferait mettre à la porte ou punir quiconque oserait lui manquer de respect et tout serait fini là.
>
> Mais, si le lendemain de la publication d'un arrêté qui les blesse ou les irrite, tous les élèves de faculté retrouvent dans un professeur en chaire le recteur qui a pu provoquer ou qui exécute avec plus ou moins de prudence ou d'adresse, la mesure qui les indisposes, il est difficile que ces jeunes gens ne succombent à l'occasion de manifester les mauvais sentiments qui les animent. Individuellement ils seraient incapables d'insulter le recteur ou ils ne l'oseraient pas ; réunis, ils le feront, entraînés par l'exaltation que produit leur réunion, encouragés par l'espoir qu'a chacun de n'être pas remarqué dans la foule.

---

37 Ibid.

38 Archives nationales : AN F17 20829, Lettre de Gergonne au ministre de l'instruction publique datée du 30 avril 1837.

Entre les inconvénients du cumul des fonctions administratives et enseignantes, je ne signale que le danger qui me touche le plus comme administrateur. Mais il est certain que le recteur s'occupe activement de son administration il ne peut pas lui rester assez de temps pour se tenir au courant de la science qu'il est chargé d'enseigner.

Il est certain que professeur d'université, il ne se montrera pas toujours impartial dans ses fonctions de recteur, entre la faculté à laquelle il appartient et les autre facultés placées sous sa direction ; celles-ci du moins, seront portées à le soupçonner de partialité et son action sur elles sera moins utilement exercée.

Veuillez, Monsieur le ministre, excuser ces observations que le seul intérêt du service en espère et les prendre en considération, dans le cas où vous jugerez utile de donner une autre destination à Mr Gergonne. »[39]

Dans une seconde lettre datée du même jour, le préfet est plus direct dans sa requête :

« J'ai l'honneur de vous prier de vouloir bien examiner s'il ne serait pas convenable de donner une autre destination à Mr Gergonne. Il ne m'appartient pas de réclamer cette mesure dans l'intérêt de l'enseignement universitaire, mais je la réclame dans l'intérêt de la tranquillité publique, dans l'intérêt politique du gouvernement du Roi qui doit chercher à être représenté par des fonctionnaires qui le fassent aimer. »[40]

Cette requête sera appuyée par pas moins de deux ministres[41]. Et Gergonne, fort de ses convictions et de l'appui sans faille sur lequel il semble compter à Paris, donnera dans plusieurs lettres à Salvandy[42] non seulement sa version des faits, mais des consignes pour faire approuver en conseil royal les mesures qu'il préconise :

« C'est, Monsieur le Ministre, par toutes ces considérations, et préoccupé de l'idée qu'il faut faire souvent beaucoup de peur pour éviter beaucoup de mal, que j'aurais désiré, comme simple mesure préventive, comme simple épouvantail, un arrêté du conseil royal conçu à peu près en ces termes :

Le conseil royal de l'instruction publique,

Vu le rapport du recteur de l'académie de Montpellier en date du 3 courant,

Considérant que le cours de physique de la faculté des sciences, le seul cours de physique qui ait lieu dans la ville de Montpellier, en même tems* qu'il est éminemment utile à tous les étudiants en médecine est surtout indispensable aux étudiants de première année, assujettis au baccalauréat ès sciences.

---

39 Archives nationales : AN F^{17} 20829, deuxième lettre du préfet de l'Hérault au datée du 29 avril 1837

40 Archives nationales : AN F^{17} 20829, première lettre du préfet de l'Hérault au datée du 29 avril 1837.

41 Le 7 mai, le ministre de l'intérieur écrit à Salvandy, ministre de l'instruction publique alors en place : « Distingué comme mathématicien, honorable comme homme privé, digne d'intérêt comme père de famille, M. Gergonne, selon M. Le préfet, porte dans l'exercice de ses fonctions administratives un esprit satirique et mordant, une dureté de forme et des habitudes désobligeantes qui l'ont mis dans une fausse position, non seulement avec les élèves, mais même avec la plupart des professeurs placés sous sa direction, et même avec plusieurs des fonctionnaires de l'ordre administratif, qui ont des rapports avec lui. (…) J'ai cru devoir, Monsieur et cher collègue, porter à votre connaissance les faits dont me rend compte le préfet de l'Hérault ainsi que ses observations et la demande dont il les accompagne. » (Archives nationales : AN, F17 20829). Nous avons vu plus haut les termes de la lettre adressée à Salvandy le 11 mai par le « Ministre, secrétaire d'état de la guerre ».

42 Nous pourrions détailler plus avant les contenus des courriers de Gergonne à son ministre, et montrer en quoi s'y imbriquent des jugements personnels, des questions politiques, des dénonciations, voire des complots. Il faudrait y consacrer trop de place tant ces textes sont révélateurs du contexte, et en appellent à des lectures suivant des éclairages fort nombreux : histoire et philosophie politique, contexte social, liberté d'expression, obsession du complot, police et contrôle des populations, rivalités entre les divers corps de l'Etat, etc.

Considérant que le cours n'a été interrompu que par l'effet de graves désordres et de scènes scandaleuses dont on ne saurait impunément tolérer le retour,

Arrête ce qui suit :

Art. 1er : le cours de physique de la faculté des sciences de Montpellier sera repris au jour qui sera ultérieurement fixé par le recteur ;

II Dans le cas où la reprise de ce cours deviendrait le prétexte de nouveaux désordres, soit dans l'intérieur de la faculté soit au dehors, le Recteur est autorisé, de l'avis du conseil académique, à ordonner immédiatement la clôture des deux facultés, jusqu'au renouvellement de l'année classique ; dans ce même cas, l'inscription d'avril serait perdue pour tous les étudiants en médecine de Montpellier, sans distinction ; et, jusqu'au 1er novembre ils ne pourraient prendre aucune inscription, soit dans les deux autres facultés, soit dans une école secondaire.

Voici, Monsieur le Ministre, l'usage que j'aurais fait de cet arrêté.

Je l'aurais fait imprimer en placard et affiché partout ; j'en aurais adressé un exemplaire à chacun de MM. les professeurs des deux facultés ainsi qu'à chacun de MM. les agrégés. J'aurais fixé la reprise du cours à dix ou quinze jours après les publications, afin de donner à MM. les professeurs et agrégés, aux étudiants paisibles et aux personnels intéressés de la ville, le tems* d'user de leur influence sur les étudiants perturbateurs ; et à l'époque fixée, j'aurais été paisiblement reprendre mon cours, bien sûr de n'en être aucunement troublé.

Inutile de vous dire d'ailleurs, Monsieur le Ministre, que je n'aurais pas réputé renouvellement des désordres quelques coups de sifflets qui, sans m'empêcher de poursuivre mes leçons, se seraient échappés çà et là des bancs les plus élevés de l'amphithéâtre. »[43]

On peut mesurer là encore, derrière l'audace et la liberté de ton, la position de force dans laquelle se sent Gergonne dans son rapport à Paris. Le ministre Salvandy rejettera bien les « conseils » de son recteur, mais aussi les demandes dirigées contre lui.

A Gergonne, il répond, lui laissant le choix de la date de reprise de son cours à la faculté :

« Je vous demande, Monsieur le Recteur, de me tenir exactement et promptement informé de tout ce qui intéresse la discipline. Vous pouvez compter sur l'appui (…) du gouvernement, comme j'aime à compter d'après ce que je sais de votre caractère et de vos lumières, sur votre modération autant que sur votre fermeté. »[44]

Au ministre de l'intérieur, il répond, très diplomatiquement et en mettant ainsi un terme au débat sur la mutation de Gergonne comme sur les évènements et leurs suites :

« J'ai fait connaître à ce magistrat [le préfet] qu'à part les égards dus aux services, au caractère et à la renommée du Recteur, il ne me paraissait pas convenable de délibérer sur une intervention conforme au vœu qu'il a exprimé, tant que l'ordre n'aura pas été rétabli et consolidé ; que seulement alors il pourra y avoir lieu à examiner définitivement ses propositions ; mais qu'en attendant le cours de physique doit être rouvert sans perturbation et à cet effet j'ai prié Monsieur le Préfet de vouloir bien accorder son appui à Monsieur le Recteur dont la destination ne peut être changée actuellement ; des recommandations ont été

43 Archives nationales : AN F17 20829, lettre de Gergonne au ministre de l'instruction publique datée du 30 avril 1837.

44 Archives nationales : AN F17 20829, lettre du ministre à Gergonne datée du 8 mai1837

adressées à ce fonctionnaire pour l'engager à éviter ce qui pourrait occasionner de nouveaux désordres et à choisir le moment pour la réouverture de son cours, en ayant soin de faire afficher les articles des ordonnances qui infligent des peines disciplinaires aux élèves dont la conduite serait répréhensible soit au dedans, soit au dehors des écoles. »[45]

Les couteaux seront provisoirement rangés[46], mais Gergonne attendra l'injonction de son ministère en novembre 1837 pour enfin reprendre ses cours à la faculté.

6 Les *Annales* de Gergonne

Les mathématiques ont bénéficié, dans l'Europe du XIX° siècle, d'une nouvelle forme de diffusion qui changea radicalement la communication et les échanges entre les mathématiciens de tous horizons : les périodiques qui leur ont été spécifiquement dédiés[47].

Le premier journal d'importance édité sur le continent le fut donc à partir de 1810 par Gergonne (sous le titre : *Annales de mathématiques pures et appliquées*, publiées mensuellement jusqu'en 1831, et que l'on nomme aujourd'hui *Annales de Gergonne*. Pour preuve du changement radical que les *Annales* provoquèrent dans le paysage éditorial des mathématiques européennes, il n'est qu'à regarder les initiatives qu'elles suscitèrent en France et dans l'ensemble de l'Europe, imposant un mode de communication et de diffusion entre mathématiciens qui perdure encore de nos jours :

- Le *Journal für die reine und angewandte Mathematik* (*Journal de mathématiques pures et appliquées*), aujourd'hui connu sous le nom de *Journal de Crelle*, édité à Berlin à partir de 1826 par Léopold Crelle (1780-1855).

- La *Correspondance mathématique et physique* que Jean-Guillaume Garnier (1766-1840) et Adolphe Quételet (1796-1874) publièrent en Belgique de 1825 à 1835.

- Le *Journal de Mathématiques Pures et Appliquées*, connu sous le nom de *Journal de Liouville*, publié par Joseph Liouville (1809-1882) à partir de 1836.

- Les *Nouvelles Annales, journal des candidats aux écoles Polytechnique et Normale*, nommé à présent *Nouvelles Annales*, que lancèrent Orly Terquem (1782-1862) et Camille Gerono (1799-1891) en 1842.

---

45 Archives nationales : AN F^{17} 20829, lettre du ministre de l'instruction publique au ministre de l'intérieur datée du 23 mai1837

46 Dans une lettre au préfet datée du 2 juin, Gergonne écrit : « Monsieur le préfet, l'assistance que vous voulez bien me promettre par votre lettre du 30 du mois dernier, pour le cas où je croirais opportun de reprendre mon cours, prouve que vous ne pensez pas plus que moi qu'à défaut de précautions convenables ce cours puisse être repris sans risques de nouvelles perturbations ; et, parmi les personnes que leurs positions met le plus en contact avec les étudiants, et que j'interroge journellement avec beaucoup de soin, je n'en rencontre aucune qui veuille m'en donner la certitude. (…) On a vu à toutes les époques et dans tous les pays la jeunesse des écoles se livrer à des actes plus ou moins répréhensibles par l'effet d'une légèreté naturelle à son âge et par là même fort excusable ; mais alors le désordre n'existait que dans les actes, tandis que présentement il a fait invasion dans les idées et n'en est pour ainsi dire que la conséquence logique, ce qui rend le mal beaucoup plus grave. Dans un tel état de choses, et vu surtout l'époque avancée où nous sommes déjà parvenus, je pense, Monsieur le Préfet, qu'il y aurait peut-être prudence, sinon a abandonner formellement, du moins à laisser dans un vague indéfini et la reprise du cours de physique et l'application des peines disciplinaires. » Archives départementales de l'Hérault, cote 1M918.

47 On pourra consulter à ce propos les deux ouvrages dont font partie les chapitres référencés (Gerini, 2014) et (Dhombres, J. & Otero, M., 1993) en fin de texte.

Certes, des tentatives avaient déjà vu le jour au XVIII° siècle, mais elles ne durèrent que peu de temps et ne permirent pas une pérennité sur le long terme dans les échanges entre les savants qui s'occupaient de mathématiques. Cette production éditoriale antérieure aux *Annales de Gergonne* peut se décomposer en deux périodes : les journaux publiés essentiellement hors de France jusqu'en 1794, puis les deux premiers journaux français laissant une place non négligeable aux mathématiques à partir de 1794.

6.1 Avant les *Annales*

La question que nous devrions poser *a priori* avant de nous intéresser aux journaux de mathématiques est celle de la définition du champ mathématique lui-même (et donc du mot « mathématiques ») avant l'édition du premier numéro des *Annales de Gergonne* comme à l'époque de celles-ci. Cette question serait trop longue à développer ici mais un document de référence, et concomitant à l'apparition des *Annales* (puisque publié en 1810, bien que remis à Napoléon 1er en 1808) permet de situer ce que l'on entendait officiellement par « mathématiques » depuis 1789 : il s'agit du *Rapport à l'Empereur sur le progrès des sciences, des lettres et des arts depuis 1789*, de Jean-Baptiste Delambre (1749-1822), et plus particulièrement de sa section I (mathématiques). Jean Dhombres a publié une édition critique de ce rapport en y adjoignant une comparaison de la classification des sciences mathématiques faites à l'Institut en 1808 avec celle établie par Delambre lui-même (Dhombres, 1989). Nous lui empruntons la classification suivante, qui donne un aperçu de ce qui constituait à l'époque (et depuis au moins 1789) les mathématiques : Géométrie, Géodésie et tables, Algèbre, Mécanique analytique, Astronomie, Géographie et voyages, Physique mathématique, Mécanique, Manufactures et arts.

On peut noter parmi les premiers journaux dédiés *a priori* aux mathématiques et publiés en Europe antérieurement aux *Annales de Gergonne* et avant 1794 :

- Les *Beyträge zur Aufnahme der theoretischen Mathematik* publié en Allemagne de 1758 à 1761 par W. J. G. Karsten.

- Les cahiers du *Leipziger Magazin für reine und angewandte Mathematik* publiés trimestriellement de 1786 à 1788 par Jean Bernoulli et Carl Friedrich Hindenburg.

- Les onze cahiers de l'*Archiv der reinen und angewandten Mathematik* publiés semestriellement par C. F. Hindenburg seul de 1795 à 1800. On peut voir dans ce journal les prémisses de ce que Gergonne parviendra à imposer dix ans plus tard avec ses *Annales*. Il est vrai qu'on y trouve une périodicité respectée sur la courte durée de publication : Hindenburg, qui souhaitait à la base un journal paraissant quatre fois par an, parvint à maintenir une publication semestrielle. La population d'auteurs s'élargit aussi, même si elle resta géographiquement située dans le nord de l'Allemagne. Elle était représentative d'une nouvelle génération de mathématiciens et pas seulement de ceux qui, jusque-là, partageaient avec Hindenburg la passion de l'analyse combinatoire et avaient rédigé avec lui la majorité des articles du *Leipziger Magazin*.

- En Angleterre, à la fin du XVIIIe siècle, le *Leybourn's Mathematical Repository*, publié par Thomas Leybourn, professeur au Royal Military College de Great Marlow, fut aussi l'un des premiers journaux à installer durablement le principe de la publication périodique en mathématiques. Mais ses auteurs étaient essentiellement Anglais, du moins sur la période qui précéda le lancement des *Annales de Gergonne*, et peu nombreux. En témoigne la liste des auteurs d'articles originaux du volume III publié en 1814 : Gough, Knight, Cunliffe, Barlow, J. F. W. Herschel (le découvreur de la planète Uranus), Bransby, White et le baron Maseres. Ce ne fut donc pas un journal largement diffusé et ouvert à l'ensemble de la communauté mathématique européenne. En outre, seulement 14 numéros parurent dans la première série (1795-1804) et 24 dans la seconde (1804-1835) (Albry & Brown, 2009), ce qui représente en moyenne environ un numéro par an : on est donc loin d'une périodicité mensuelle comme celle des *Annales de Gergonne*.

Nous ne mentionnerons pas ici les journaux qui, bien que publiant dans leurs colonnes des textes de mathématiques - ou plutôt plus souvent des textes *sur* les mathématiques, étaient très majoritairement consacrés à d'autres informations. Citons seulement pour mémoire le *Journal des savants*, publié en France depuis 1665[48], le *Ladies diary* publié en Angleterre de 1704 à 1841 (Costa, 2002), ou les périodiques généralistes ou populaires allemands qui ne publiaient que très peu d'informations sur les mathématiques (en général des comptes rendus de lecture) et étaient diffusés dans des cercles restreints : par exemple les *Göttingische Anzeigen von gelehrten Sachen* (Göttingen, 1753-1801), l'*Allgemeine deutsche Bibliothek*, publiée à Berlin par Friedrich Nicolaï de 1765 à 1796, la *Neue Bibliothek der Schönen Wissenschaften*, (Leipzig, 1765-1806) ou l'*Allgemeine Literatur-Zeitung* de Bertuch publiée à Yéna de 1785 à 1803.

Les journaux des académies ou des universités avaient quant à eux une périodicité aléatoire (mais souvent seulement annuelle) et ne traitaient, loin s'en faut, pas que de mathématiques, comme par exemple les *Acta Eruditorum* de Leipzig, et les *Mémoires de l'Académie de Berlin*.

A partir de 1794, la création en France de l'École polytechnique, accompagnée de son *Journal* (que nous noterons par la suite JEP, pour « Journal de l'École polytechnique »), modifia le paysage éditorial mathématique. Jusque-là, le seul *Journal des savans*[*] n'abordait que de manière très mineure la discipline et il ne peut donc pas être considéré comme un organe de diffusion de cette science. Le JEP fut rapidement suivi par une autre initiative éditoriale, la *Correspondance sur l'École Polytechnique* de Jean Nicolas Pierre Hachette (1679-1834). Mais le constat que fit Gergonne en 1810 de l'absence de véritables journaux dédiés aux mathématiques, dans son « Prospectus » d'introduction au premier numéro des *Annales*[49], était pertinent, comme nous l'avons démontré en détail ailleurs (Gérini, 2014). Il y déplorait le fait que : « les Sciences exactes, cultivées aujourd'hui si universellement et avec tant de succès, ne comptent pas encore un seul recueil périodique qui leur soit spécialement consacré ». Et il ajoutait :

> On ne saurait, en effet, considérer comme tels, le Journal de l'école Polytechnique, non plus que la Correspondance que rédige M. Hachette : recueils très précieux sans doute, mais qui, outre qu'ils ne paraissaient qu'à des époques peu rapprochées, sont consacrés presque uniquement aux travaux d'un seul établissement.[50]

---

[48] Successivement orthographie « sçavans », « savans » et « savants ».

[49] Joseph-Diez Gergonne (1810), Prospectus, in (*AMPA*, 1, pp. i–iv).

[50] *Ibid.* Le « Prospectus » dont nous citons souvent comme ici des extraits fut en fait l'éditorial dans lequel Gergonne et son collaborateur Thomas Lavernède annonçaient leurs intentions, le champ couvert par le journal, et en fixaient en quelque sorte la ligne éditoriale.

On peut donc avancer l'hypothèse que, historiquement, les *Annales* de *Gergonne* furent le premier journal consacré uniquement aux mathématiques et qui posséda les qualités à la fois de stabilité et de périodicité rapprochée (pas d'interruption dans la publication, périodicité mensuelle), de durée (vingt et une années de parution) et d'envergure : ce sont donc ces *Annales* qui inaugurèrent véritablement le principe de journal international de spécialité dans ce champ disciplinaire. Elles participèrent ainsi de façon essentielle à l'entrée des mathématiques dans la spécialisation et dans la modernité même si elles s'inscrivaient dans une période de transition qui voyait les mathématiques se constituer peu à peu en science autonome et structurée et se détacher progressivement de la philosophie.

6.2 Auteurs et lecteurs des *Annales*

Gergonne ne parvint pas au départ à intéresser les élites parisiennes à son projet : son éloignement de la capitale et la concentration dans celle-ci de ces élites qui fréquentaient les mêmes cercles (Ecole polytechnique, Institut, Faculté des sciences de Paris) et se contentaient donc des organes de diffusion qu'elles partageaient, à savoir les deux publications que nous avons citées et les comptes rendus des académies nationales européennes, expliquent le peu d'attention que suscita au départ son projet.

S'il s'obstina, c'est justement en raison de l'isolement où il se sentait et que partageaient avec lui tous les mathématiciens, professeurs, officiers, ingénieurs, - anciens élèves ou pas de l'École polytechnique - qui, dans leurs établissements de province ou dans leurs cantonnements, aspiraient comme lui à échanger leurs savoirs et leurs avancées.

Il prit de ce fait la précaution, toujours dans son « Prospectus », d'en appeler à toutes les bonnes volontés :

> Un recueil qui permette aux Géomètres d'établir entre eux un commerce ou, pour mieux dire, une sorte de communauté de vues et d'idées ; un recueil qui leur épargne les recherches dans lesquelles ils ne s'engagent que trop souvent en pure perte, faute de savoir que déjà elles ont été entreprises ; un recueil qui garantisse à chacun la priorité des résultats nouveaux auxquels il parvient; un recueil enfin qui assure aux travaux de tous une publicité non moins honorable pour eux qu'utile au progrès de la sciences.[51]

En sa qualité de professeur, et soucieux d'intéresser les enseignants, il mit en avant dans ce même éditorial le souci pédagogique : « Ces Annales seront principalement consacrées aux Mathématiques pures, et surtout aux recherches qui auront pour objet d'en perfectionner et d'en simplifier l'enseignement.»[52]. La référence à l'enseignement, et donc l'appel aux professeurs de toutes catégories, est explicite, mais Gergonne laissa largement la porte ouverte aux avancées théoriques et pas seulement utiles à la pédagogie. Le terme « surtout » de la citation précédente est en effet trompeur : il publia de fait en majorité des articles purement mathématiques souvent novateurs et importants pour la circulation des idées et concepts nouveaux, mais dont les retombées sur l'enseignement de la discipline étaient loin d'être évidentes à l'époque.

L'entreprise de Gergonne connut rapidement le succès et toucha effectivement d'abord le monde enseignant, comme le montre la liste des auteurs de l'année 1811-1812 : dix-neuf auteurs dont neuf professeurs du secondaire, deux principaux de collèges, deux élèves (du

---

51 AMPA, 1, 1810, pp. i-iv.

52 Ibid.

même lycée d'Angers que l'un des professeurs), quatre professeurs d'universités ou académies, deux professeurs d'écoles militaires (dont un ancien polytechnicien).

Cela se confirme sur l'ensemble des *Annales* et on voit bien là une première lacune comblée : les auteurs des provinces françaises représentent environ 37% de la population totale d'auteurs et ils écrivent environ 63% de la totalité des articles. Ils sont majoritairement professeurs dans des collèges et lycées, plus rarement dans des facultés, et signent au total 676 articles (Otero, 1997, p. 25) sur les quelques 900 que comptent les *Annales*[53]. Ce rapide impact auprès d'une population qui n'avait jusqu'alors que peu d'occasions de faire connaître ses travaux eut une première conséquence non négligeable. Alors qu'il était extrêmement difficile de se faire lire et publier, donc reconnaître, par le canal très réservé du JEP et celui très fermé et sélectif de l'Académie des sciences, les *Annales* offrirent à des mathématiciens de qualité de faire connaître des travaux qui n'auraient peut-être pas trouvé sans elles l'écho qu'ils méritaient (certains de ces auteurs devinrent plus tard académiciens : Sturm, Poncelet, Chasles, Lamé, Liouville). Notons que Gergonne lui-même est comptabilisé dans ces statistiques, et qu'il écrivit pas moins de 180 articles dans son journal, sans compter les innombrables notes de bas de pages et commentaires qu'il ajoutait aux textes de ses auteurs et les articles anonymes dont on peut penser qu'ils étaient de sa plume (ce qui nous a été en grande partie confirmé par ses propres annotations en marges d'une collection complète des *Annales* détenue par la bibliothèque municipale de Nancy).

On considère donc souvent les *Annales* de Gergonne comme « le journal d'un homme seul au profit d'une communauté enseignante » (Dhombres & Otero, 1993), ce qu'elles furent à leurs débuts avant d'atteindre une notoriété en France et à l'étranger qui attira dans leurs pages des personnalités de premier rang. La participation massive d'anciens élèves de Polytechnique (souvent militaires non enseignants) montra rapidement que leur frustration perçue par Gergonne face à la « fermeture » du JEP fut aussi corrigée par son journal.

La répartition statistique des articles selon l'origine géographique des auteurs montre bien aussi la réussite de l'ambition de Gergonne de donner la parole à une large communauté de mathématiciens qui, on l'a vu, ne pouvaient prétendre à être publiés ailleurs ou n'avait que peu d'espoir de l'être. Gergonne souhaitait en outre provoquer une émulation au sein de cette communauté mathématique isolée et privée de moyen de communication. Il l'exprime clairement dans son « Prospectus » :

> Chaque numéro des *Annales* offrira un ou plusieurs *Théorèmes* à *démontrer*, un ou plusieurs *problèmes* à *résoudre*. Les Rédacteurs, dans le choix de ces théorèmes et problèmes, donneront la préférence aux énoncés qui pourront leur être indiqués par leurs correspondans*; et ils consigneront, dans leur recueil, les démonstrations et solutions qui leur seront parvenues ; ils espèrent ainsi provoquer chez les jeunes géomètres une utile et louable émulation. Personne n'ignore d'ailleurs combien ces sortes de défis ont ajouté de perfectionnement à l'analise*, au commencement du

---

53 Si l'on n'a malheureusement pas retrouvé d'archives personnelles de Gergonne permettant de chiffrer le nombre d'abonnés à son journal (ni d'archives de son éditeur Bachelier à Paris), ces chiffres, additionnés aux effectifs et nombre d'articles d'élèves, de professeurs des institutions parisiennes, de militaires et d'étrangers, laissent deviner une diffusion rapide et importante des *Annales* sur le territoire français puis en dehors de ses frontières.

> dernier siècle ; et il n'est point déraisonnable de penser qu'en les renouvelant, on peut, peut-être, lui préparer encore de nouveaux progrès.[54]

Il instaura donc le principe des « problèmes à résoudre », « questions posées » et « théorèmes à démontrer » que reprendront la quasi-totalité des rédacteurs des journaux qui paraîtront ensuite. Mais l'émulation et les controverses s'exercèrent bien au-delà des simples défis que représentaient les théorèmes non démontrés ou les problèmes à résoudre. De simples professeurs de lycée, polémiquant parfois sur des questions de paternité de démonstrations de théorèmes ou d'avancées théoriques, firent de la sorte progresser les mathématiques.

Les figures majeures des mathématiques de l'époque, ou les mathématiciens qui allaient atteindre rapidement une notoriété que leurs publications dans les *Annales* facilita, ne s'y trompèrent pas puisqu'ils finirent par alimenter eux aussi à partir de 1820 le journal de Gergonne en articles et essais, sachant que l'impact et la diffusion de leurs écrits, en France comme dans le reste de l'Europe, seraient ainsi nettement supérieurs à ceux obtenus jusqu'alors par leurs communications auprès de différentes académies ou dans l'éphémère et multidisciplinaire *Bulletin des sciences mathématiques, astronomiques et chimiques* (1824-1831) (Taton, 1947) ou bien encore le *Bulletin de la société philomathique*. Citons parmi cette population d'auteurs : Ampère, Cauchy, Dupin, Lacroix, Francoeur, Poncelet, Poisson, Chasles, Poncelet[55].

Les auteurs étrangers commencèrent plus tôt à alimenter les *Annales* en articles, preuve de l'influence et de la connaissance de celles-ci hors des frontières françaises. On trouve par exemple dès les tomes 1, 2 et 3 des *Annales* des articles de Simon Lhuillier, alors professeur à l'Académie de Genève[56]. En revanche, on ne voit apparaître qu'à partir de 1820[57] des mathématiciens tels que Schmidten, Querret, Quételet, Plucker, Libri, et bien évidemment le jeune Niels Henrik Abel (1806-1829), recommandé par Crelle en 1826[58].

L'internationalisation des *Annales* participe aussi à notre thèse qui soutient que le journal de Gergonne est le premier périodique de l'histoire des mathématiques au sens où l'on entend aujourd'hui l'expression « périodique scientifique spécialisé » et qu'il contribua, par cet élargissement au-delà des frontières de la seule France, à l'émergence de ce que l'on appelle la « modernité » dans les mathématiques. Jean Dhombres et Mario Otero (1993, p. 39) avaient déjà souligné ce fait : « Avec vingt-huit auteurs et cent vingt-deux articles, les étrangers sont assez bien représentés aux *Annales*, et ceci constitue une réussite car une *communauté internationale* n'existait pas encore ».

C'est donc un « mathématicien-philosophe-enseignant » qui lança en Europe le premier journal d'envergure en mathématiques. Comme nombre de ses auteurs, Gergonne était un

54 AMPA, 1, 1810, pp. i-iv.

55 On sait par exemple que la controverse entre Poncelet et Gergonne sur le principe de dualité dont nous avons parlé plus haut servit la reconnaissance académique du premier, reconnaissance qui lui avait été refusée dans un premier temps. Cette reconnaissance passa donc en partie par l'échange très vif entre les deux hommes dans les *Annales*.

56 Mais il est vrai que les registres de l'Académie de Nîmes de 1806 à 1809 ont permis de constater que Gergonne entretenait avant le lancement de ses *Annales* une correspondance régulière avec le mathématicien genevois.

57 *AMPA*, 11.

58 Niels Abel (1826), *Recherche de la quantité qui sert à la fois à deux équations algébriques données*, AMPA, 17, 1826-1827, pp. 204-213

personnage de transition entre un XVIII° siècle où les mathématiques étaient encore largement imprégnées de philosophie et un XIX° siècle où elles se constituèrent en véritable champ scientifique spécialisé. Ses *Annales*, bien qu'encore souvent teintées de philosophie, furent aussi le lieu d'un découpage de la science mathématique en de très nombreuses subdivisions : il tentait d'organiser son champ disciplinaire en suivant une classification à ses yeux conforme à sa vision générale et philosophique du savoir scientifique détaillée dans les essais que nous avons mentionnés plus haut. On voit donc au fil des années de parution se défaire le lien entre mathématiques et philosophie et se constituer une science mathématique plus structurée et plus théorique, en particulier en raison de l'élargissement de la population d'auteurs aux élites françaises et étrangères.

7 Conclusion

Il est difficile de faire ici une conclusion après ce qui précède. Si le tour d'horizon que nous venons de proposer sur l'apport de Gergonne à la fois aux mathématiques, à la philosophie, à l'enseignement, à la diffusion de la discipline via son journal, et enfin à l'histoire des institutions de l'instruction publique, est, nous l'espérons, convainquant pour estimer l'importance du personnage dans l'histoire des sciences en général et des mathématiques en particulier, il n'en est pas moins réducteur au regard de la somme de données et analyses que Gergonne et son œuvre ont engendrées depuis vingt ans que nous nous y intéressons. En outre, il faut entrer dans le détail des contenus des textes pour l'estimer à sa juste valeur, ce que nous n'avions pas la place de faire ici. Et il faut de plus s'intéresser à son réseau d'auteurs, à leurs biographies et bibliographies, comme à l'histoire des institutions, pour mettre en contexte cet apport fondamental à nos yeux. Enfin, ce travail nécessite le croisement de nombreuses disciplines : mathématiques bien sûr, mais aussi philosophie, didactique, histoire au sens large et histoire des institutions en particuliers, aspects locaux comme nationaux et internationaux.

Les découvertes récentes à Montpellier d'un fonds d'archives personnelles de Gergonne ajoute évidemment à la difficulté de conclure puisqu'elles ouvrent au contraire de nouvelles perspectives, donc sont plus de l'ordre d'une introduction que d'une conclusion.

Nous préférons donc terminer notre texte par des considérations d'un autre ordre. À l'époque où nous n'avions pas encore conduit le programme de numérisation avec NUMDAM des *Annales* de Gergonne, nous avons été sollicités pour fournir des textes dont certains chercheurs sur des sujets très actuels avaient l'utilité mais qui ne leur étaient pas accessibles. Nous pouvons en citer deux exemples. Le premier sur des recherches en combinatoire d'un chercheur Vénézuélien qui avait besoin d'articles de Gergonne lui-même pour ses propres travaux. Le second, plus étonnant, de chercheurs travaillant sur le programme Galileo de l'union européenne qui développe un système concurrent et indépendant du GPS américain : là aussi c'est un texte de Gergonne (sur le problème des trois cercles) qui leur était nécessaire.

Nous voyons donc ici l'importance de l'histoire des sciences, des mathématiques en l'occurrence, non seulement pour la mémoire de notre patrimoine, bien sûr, mais aussi pour les sciences actuelles qui ont parfois besoin de revenir aux sources et qui n'en ont pas les moyens sans le concours des historiens des disciplines concernées. Les programmes sur les autres journaux du 19ème siècle ont produit les mêmes effets. Un chercheur en mathématiques des systèmes dynamiques, Jean-Marc Ginoux, a résolu une question ancienne sur un « attracteur étrange » en balayant les méthodes mathématiques du 20ème siècle qui avaient échoué dans cette tâche et en revenant à des fondamentaux de géométrie différentielle exposés dans les *Nouvelles Annales* de Terquem et Gerono (premiers éditeurs de ce journal, paru de 1842 à 1927 et numérisé lui aussi grâce à un programme ANR et au CNRS via leur programme NUMDAM).

Nous pourrions faire le même constat sur l'histoire de la philosophie : la lecture de textes de Gergonne nous montre un positivisme repris parfois quasiment au mot près trente ans plus tard par Auguste Comte, considéré pourtant comme le père de ce courant épistémologique.

Et enfin, les conflits sur des questions d'enseignement (sur les méthodes, sur le rôle de l'église, sur le débat public-privé, *etc.*) dans lesquels Gergonne, en sa qualité de recteur comme d'enseignant, prit part et produisit un courrier très abondant avec ses ministres, nous éclairent sur notre histoire récente et sur des thématiques finalement récurrentes qui animent nos sociétés et les choix politiques et idéologiques en la matière. C'est là aussi un apport fondamental pour comprendre les débats d'aujourd'hui en découvrant ceux d'hier.

C'est finalement peut-être la meilleure conclusion que l'on puisse proposer après notre exposé : Gergonne, ses écrits, son journal, ses fonctions, nous éclairent finalement aussi bien sur l'hier qui était le sien que sur le présent et le futur qui sont les nôtres, et cela sur de nombreux registres dont ceux qui sont l'objet du congrès HPM2016 : les mathématiques, leur histoire et leur didactique.

RÉFÉRENCES

Albry, J. & Brown, S.H. (2009). A valuable monument of mathematical genius: The Ladies' Diary (1704-1840). *Historia Mathematica, 36*, 10-47.

Bouisson, F. (1859). *Notice biographique sur J.-D. Gergonne, Ancien Recteur de l'Académie de Montpellier, Montpellier*. Boehm, imprimeur de l'Académie.

Costa, S. (2002). The Ladies Diary: Gender, mathematics, and civil society in early-eighteenth-century England. *The History of Science Society, 17*, 49-73.

Dhombres, J. (1989). *Rapport à l'Empereur sur le progrès des sciences, des lettres et des arts depuis 1789 et leur état actuel, Vol. 1 : Sciences mathématiques*, édition critique du texte de 1810 de Jean-Baptiste Delambre, préface de Denis Woronoff, présentation et notes de Jean Dhombres. Paris : Belin.

Dhombres, J. & Otero, M. (1993). Les Annales de mathématiques pures et appliquées : le journal d'un homme seul au profit d'une communauté enseignante. In : Ausejo, Elena & Mariano Hormigon (Eds), *Messengers of Mathematics: European Mathematical Journals (1800-1946)* (pp.3-70). Madrid : Siglo XXI de España Editores.

Gerini, C. (2008).Un recteur de la Monarchie de Juillet face à la crise du monopole. Joseph-Diez Gergonne, recteur à Montpellier de 1830 à 1844. In : Jean-François Condette & Henri Legohérel (Dirs.), *Le recteur d'académie. Deux cents ans d'histoire* (pp. 77-86). Paris : Editions CUJAS.

Gerini, C. (2009). Le recteur de la Monarchie de Juillet et la culture des élites. Joseph-Diez Gergonne (1771-1859) : le zèle d'un fonctionnaire et l'esprit critique d'un libre penseur. In : Jean-François Condette (Dir.), *Les Recteurs. Deux siècles d'engagement pour l'École (1808-2008)* (pp. 77-86). Presses Universitaires de Rennes.

Gerini, C. (2011-a). Joseph-Diez Gergonne (1771-1859) et ses Annales de mathématiques : un acteur engagé dans l'évolution des mathématiques et de leur enseignement au début du XIXe siècle. In : R. D'Enfert & V. Fonteneau (Dirs), *Espaces de l'enseignement scientifique et technique. Acteurs, savoirs, institutions, XVIIe – XXe siècles* (pp. 61-74). Paris : Hermann.

Gerini, C. (2011-b). Pour un bicentenaire : polémiques et émulation dans les Annales de mathématiques pures et appliquées (1810-1832) de Gergonne, premier grand journal de l'histoire des mathématiques. In : Commission inter-IREM (Ed), *Circulation, transmission, héritage. Actes du 18ème colloque inter-IREM, histoire et épistémologie des mathématiques* (pp. 241-254). Éditions de l'IREM de Basse Normandie, Université de Caen.

Gerini, C. (2012). Les Annales de mathématiques pures et appliquées de Gergonne et l'émergence des journaux de mathématiques dans l'Europe du XIXe siècle : un bicentenaire. In : Daniel Ulbrich (Ed.), *Jahrbuch für Europäische Wissenscaftskultur / Yearbook for European Culture of Science, Vol. 7* (pp. 345-376). Stuttgart : Franz Steiner Verlag.

Gerini, C. (2013). Approche transdisciplinaire d'un document polymorphe. Les Annales de Gergonne, premier grand journal de l'histoire des mathématiques. In A.-L. Rey (Dir), *Méthode et histoire. Quelle histoire font les historiens des sciences et des techniques ?* (pp. 397-417). Paris : Classiques Garnier.

Gerini, C. (2014). Les Annales de mathématiques pures et appliquées de Gergonne et l'émergence des journaux de mathématiques dans l'Europe du XIXème siècle. In : C. Gerini & N. Verdier, N., *L'émergence de la presse mathématique en Europe au 19ème siècle. Formes éditoriales et études de cas (France, Espagne, Italie et Portugal), série « Cahiers de logique et d'épistémologie » 19* (pp. 5-36). Oxford : College Publications.

Gerini, C. (2015). Joseph-Diez Gergonne sur l'expérience : une critique du sensualisme de Condillac et de la « méthode des inventeurs ». In : E. Barbin & J.-P. Clero (Dirs.), *Les mathématiques et l'expérience. Ce qu'en ont dit les philosophes et les mathématiciens* (pp. 173-194). Paris : Hermann.

Nique, C. (1999). *François Guizot. L'école au service du gouvernement des esprits*. Paris : Hachette éducation.

Otero, M. (1997). *Joseph-Diez Gergonne, 1771-1859, Histoire et philosophie des sciences*. Sciences et techniques en perspective, première série de la revue du centre François Viète, Vol. 37. Université de Nantes.

Saint Victor ((de), C. (1990). La fâcheuse lacune: John Stuart Mill à Montpellier. In: *Bulletin historique de la ville de Montpellier, 1990-2, n°14* (pp. 18-24).

Taton, R. (1947). Les mathématiques dans le Bulletin de Férussac. *Archives internationales d'histoire des sciences, 26*, 100-125.

Verhoef, N. C., & Broekman, H. G. B. (2005). A process of abstraction by representations of concepts. In : H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, 4*, (pp. 273-279).

Yvert, B. (1990). *Dictionnaire des ministres de 1789 à 1989*. Paris : Perrin.

HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION

Recent developments

Kathleen CLARK, Tinne Hoff KJELDSEN, Sebastian SCHORCHT, Constantinos TZANAKIS, Xiaoqin WANG

School of Teacher Education, Florida State University, Tallahassee, FL 32306-4459, USA

kclark@fsu.edu

Department of Mathematical Sciences, University of Copenhagen, Denmark

thk@math.ku.dk

Justus Liebig University Giessen, Germany

s.schorcht@gmx.de

Department of Education, University of Crete, Rethymnon 74100, Greece

tzanakis@edc.uoc.gr

Department of Mathematics, East China Normal University, China

xqwang@math.ecnu.edu.cn

ABSTRACT

This is a survey on the recent developments (since 2000) concerning research on the relations between History and Pedagogy of Mathematics (the *HPM domain*). Section 1 explains the rationale of the study and formulates the key issues. Section 2 gives a brief historical account of the development of the *HPM domain* with focus on the main activities in its context and their outcomes. Section 3 provides a sufficiently comprehensive bibliographical survey of the work done in this area since 2000. Finally, section 4 summarizes the main points of this study.

1 Introduction

This is a survey describing the state-of-the-art on the themes of the ICME 13 *Topic Study Group* (TSG) 25: ***The Role of History of Mathematics in Mathematics Education***. It gives a brief account of the developments since 2000[1] on the relations between History and Pedagogy of Mathematics, in order to illuminate and provide insights on the following general questions:

- *Which history is suitable, pertinent, and relevant to Mathematics Education (ME)?*
- *Which role can History of Mathematics (HM) play in ME?*
- *To what extent has HM been integrated in ME (curricula, textbooks, educational aids/resource material, teacher education)?*
- *How can this role be evaluated and assessed and to what extent does it contribute to the teaching and learning of mathematics?*

These are the key issues explicitly addressed in and/or implicitly underlying what we call the ***HPM perspective*** as detailed below.

[1] The year of publication of *History in Mathematics Education: The ICMI Study*, edited by J. Fauvel & J. van Maanen (2000); a highly collective work that has been a landmark in the area (see section 2, p. 4).

1.1 The HPM Perspective

Mathematics is a human intellectual enterprise with a long history and a vivid present. Thus, mathematical knowledge is determined not only by the circumstances in which it becomes a deductively structured theory, but also by the procedures that originally led or may lead to it. Learning mathematics includes not only the "polished products" of mathematical activity, but also the understanding of implicit motivations, the sense-making actions and the reflective processes of mathematicians, which aim to the construction of meaning. Teaching mathematics should give students the opportunity to "do mathematics." In other words, although the "polished products" of mathematics form the part of mathematical knowledge that is communicated, criticized (in order to be finally accepted or rejected), and serve as the basis for new work, the process of producing mathematical knowledge is equally important, especially from a didactical point of view. Perceiving mathematics both as a logically structured collection of intellectual products and as processes of knowledge production should be the core of the teaching of mathematics. At the same time, it should be also central to the image of mathematics communicated to the outside world.

Along these lines, putting emphasis on integrating historical and epistemological issues in mathematics teaching and learning constitutes a possible natural way for exposing mathematics in the making that may lead to a better understanding of specific parts of mathematics and to a deeper awareness of what mathematics as a discipline is. This is important for ME, helping to realize that mathematics:

- is the result of contributions from many different cultures;
- has been in constant dialogue with other scientific disciplines, philosophy, the arts and technology;
- has undergone changes over time; there have been shifting views of what mathematics is;
- has constituted a constant force for stimulating and supporting scientific, technical, artistic and social development.

This helps to improve ME at all levels and to realize that although mathematics is central to our modern society and a mathematically literate citizenry is essential to a country's vitality, historical and epistemological issues of mathematics are equally important. The harmony of mathematics with other intellectual and cultural pursuits also makes the subject interesting, meaningful, and worthwhile. In this wider context, history and epistemology of mathematics have an additional important role to play in providing a fuller education of the community: not being a natural science, but a formal science closer to logic – hence to philosophy – mathematics has the ability inherent in itself to connect the humanities with the sciences. Now that societies value and want young people educated in the sciences, but have a hard time finding out how to get people to "move" from humanities to the sciences, integrating history and epistemology in ME can make this connection visible to students. This is most important, especially today when there is much concern about the level of mathematics that students are learning and about their decreasing interest in mathematics, at a time when the need for both technical skills and a broader education is rising.

1.2 Summary of the content of the present survey

The rationale underlying this perspective has formed the core and main concern of the approaches adopted towards integrating History and Epistemology of Mathematics in ME (the ***HPM domain***), especially in the context of the ICMI affiliated *International Study Group on the Relations Between the History and Pedagogy of Mathematics* (the ***HPM Group***) since its formation in 1972.

What follows consists of four sections:

Section 2 gives a brief historical account of the development of the *HPM domain* with focus on the main activities in its context and their outcomes since 2000 (§2.1); a short presentation of journals and newsletters (§2.2); and an outline with comments on the key issues mentioned in section 1 and references to the literature for details (§2.3).

Section 3 constitutes the major part of the survey. It provides a sufficiently comprehensive bibliographical survey of the work done since 2000: Collective works in §3.1 (collective volumes, special ME journal issues, conference proceedings, resource material); individual works in §3.2 (books & doctoral dissertations, papers in scientific journals, collective volumes and conference proceedings). Though the emphasis is on research results of an as broad as possible international interest, due attention is paid to nationally-oriented implementation of the *HPM perspective* as well.

Finally, section 4 summarizes the main points of this study and section 5 contains all references given in section 2.

Remarks: (a) In section 3, next to each reference the TSG 25 themes to which it is related are indicated, numbered as in the Appendix below.

(b) Collective works exclusively on the *HPM perspective* (i.e. those in §3.1) contain several important contributions. However, in order to keep this survey to a reasonable size, these contributions are not included as separate items in §3.2, though some of them are quoted in section 2 (hence, they appear in section 5). Instead, these collective works are annotated briefly. Also note that several abbreviations are used for the titles of journals, conferences etc., are explained at the beginning of section 3, and many lengthy URL have been integrated into the title of the reference to avoid making the text difficult to follow and non-appealing. These URLs are not displayed in a printout of this document, so the reader is advised to use its electronic version available at http://www.clab.edc.uoc.gr/HPM/HPMinME-TopicalStudy-18-2-16-NewsletterVersion.pdf.

(c) For several references (especially those in the annotated bibliography in section 3.1) hyperlinks are provided, where one can find online additional information in the form of an abstract, review, outline of contents etc.

APPENDIX: *Main Themes of TSG 25*

T1: Theoretical and/or conceptual frameworks for integrating history in ME.

T2: History and epistemology implemented in ME, considered from either the cognitive or/and affective points of view:

a. Classroom experiments at school, the university and teacher pre- & in-service education;
b. Teaching material: textbooks, resource material of any kind.

T3: Surveys on:
a. Research on the HM in ME;
b. The HM as it appears in curriculum and/or textbooks.

T4: Original sources in the classroom and their educational effects.

T5: History and epistemology as a tool for an interdisciplinary approach in the teaching and learning of mathematics and the sciences by unfolding their fruitful interrelations.

T6: Cultures and mathematics fruitfully interwoven.

2 An outline of the historical development of the HPM domain

Integrating HM in ME has been advocated since the second half of the 19^{th} century, when mathematicians like De Morgan, Poincaré, Klein and others explicitly supported this path and historians like Tannery and later Loria showed an active interest on the role HM can play in ME. At the beginning of the 20^{th} century, this interest was revived as a consequence of the debates on the foundations of mathematics. Later on, history became a resource for various epistemological approaches; Bachelard's historical epistemology, Piaget's genetic epistemology and Freudenthal's phenomenological epistemology, at the same time stimulating the formulation of specific ideas and conclusions on the learning process (Barbin & Tzanakis, 2014, p. 256 and references therein; Fauvel & van Maanen, 2000, p. 202).

This interest became stronger and more competitive in the period 1960-1980 in response to the *New Math* reform, when its proponents were strongly against "a historical conception of ME," whereas for its critics, HM appeared like a "therapy against dogmatism," conceiving mathematics not only as a language, but also as a human activity. In 1969, the NCTM in USA devoted its 31^{st} Yearbook to the HM as a teaching tool (NCTM 1969) and in the 1970s a widespread international movement began to take shape, greatly stimulated and supported by the establishment of the *HPM Group* at ICME 2 in 1972 and its scope in 1978 (HPM Group 1978).

Thus, during the last 40 years, integrating HM in ME has evolved into a worldwide intensively studied area of new pedagogical practices and specific research activities and a gradually increasing awareness has emerged of what was described in §1.1 as the *HPM perspective* (Fasanelli & Fauvel (2006) for a historical account and references prior to 2000; Barbin (2013), Barbin & Tzanakis (2014), and Furinghetti (2012), for a concise outline of later developments, and references therein).

The rising international interest in the *HPM perspective* and the various activities of the HPM Group worldwide, led to the approval by ICMI in 1996 of launching a 4-year *ICMI Study* on the relations between HM and ME. After a *Discussion Document* written by the *Study* co-chairs (Fauvel & van Maanen, 1997) and a *Study Conference* in 1998, at Luminy, France, the Study culminated in the publication of a 437-page volume written by 62 contributors working together in 11 groups (Fauvel & van Maanen, 2000). This was a landmark in establishing and making more widely visible the *HPM perspective* as a research domain in the context of ME and greatly stimulated and enhanced the international interest of the educational community in this area.

Below we give an account of the main regular activities and their outcomes concerning educational research and its implementation in educational practice, relevant to the *HPM domain* and mainly realized in the context of the *HPM Group*.

2.1 Meetings and related collective volumes

2.1.1 *ICME Satellite Meetings of the HPM Group*

These quadrennial meetings are a major activity that bring together individuals with a keen interest in the relationship between the HM and ME; researchers in ME interested in the HM in relation to mathematical thinking, mathematics teachers at all levels eager to gain insights into the *HPM perspective*, historians of mathematics wishing to talk about their research, mathematicians wanting to learn about new possibilities to teach their discipline, and all those with an interest in the *HPM domain*.

They are organized just after, or before the ICME:
1984 Adelaide; *ICME 5*, Adelaide
1988 Florence; *ICME 6*, Budapest
1992 Toronto; *ICME 7*, Quebec
1996 Braga, *HEM Braga 96* conjointly with the *2nd ESU*; *ICME 8*, Seville
2000 Taipei, *HPM* 2000; *ICME 9*, Tokyo-Makuhari
2004 Uppsala, *HPM 2004*, conjointly with the *4th ESU*; *ICME 10*, Copenhagen
2008 Mexico City, *HPM* 2008; *ICME 11*, Monterrey
2012 Daejeon, *HPM 2012*; *ICME 12*, Seoul
2016 Montpellier, *HPM 2016*; *ICME 13*, Hamburg

The books published as a result of these *HPM* meetings are listed below using the abbreviations introduced in section 3:

Swetz et al.,1995 (after ICME-6); Calinger, 1996 (after HPM 1992); Lagarto et al., 1996 (during HPM 1996); Katz, 2000 (after HPM 1996); Horng & Lin, 2000 (at HPM 2000); Bekken & Mosvold, 2003 (before ICME 10 & HPM 2004); Horng et al., 2004 (before HPM 2004); Furinghetti et al., 2004 (at HPM 2004; revised edition Furinghetti et al., 2006; see §3.1.3); Cantoral et al. (at HPM 2008); Barbin et al., 2012 (at HPM 2012; revised edition in progress).

2.1.2 The *European Summer University on the History and Epistemology in Mathematics Education* (ESU)

The initiative of organizing a *Summer University* (SU) on the *History and Epistemology in Mathematics Education* belongs to the French ME community in the early 1980s. The French IREMs organized the first interdisciplinary meeting in 1984, in Le Mans, France, followed by another three in France. The next one was organized in 1993 on a European scale; the 1st *European Summer University on the History and Epistemology in Mathematics Education*, (a name coined since then, abbreviated as ESU since 2004), though many participants come from outside Europe. Since 2010, ESU is organized every four years to avoid coincidence with the HPM meetings.

Since its original conception, ESU has been developed and established into one of the major activities in the HPM domain. It mainly aims to: provide a school for working on a historical, epistemological, and cultural approach to mathematics and its teaching, with emphasis on actual implementation; give the opportunity to mathematics teachers, educators, and researchers to share their teaching ideas and classroom experience related to a historical perspective in teaching; and motivate further collaboration along these lines among teachers of mathematics and researchers on the HM and ME in Europe and beyond, attempting to reveal and strengthen the *HPM perspective*. Below is a list of the ESUs:

1993, *ESU* 1 Montpellier
1996, *ESU* 2 Braga (conjointly with *HEM Braga 96*)
1999, *ESU* 3 Leuven & Louvain-la-Neuve
2004, *ESU* 4 Uppsala (conjointly with *HPM 2004*)
2007, *ESU* 5 Prague
2010, *ESU* 6 Vienna
2014, *ESU* 7 Copenhagen

The following works were published after the *ESU*s: Lalande et al., 1995; Lagarto et al., 1996; Radelet-de-Grave & Brichard, 2001; Furinghetti et al., 2004; Barbin et al., 2008; Barbin et al., 2011a; Barbin et al., 2015.

2.1.3 The *HPM domain* at ICMEs

Activities related to the *HPM perspective* have always been present in the ICMEs; Fasanelli & Fauvel 2006 for ICMEs before 2000. Since 2000, such activities have formed an established part of the ICMEs' scientific program:

(a) *ICME 9, Tokyo,* Japan, 2000

WG for Action 13: *History and Culture in Mathematics Education* coordinated by J. van Maanen, W.S. Horng.

Summary in H. Fujita, Y. Hashimoto, B. R. Hodgson, P. Y. Lee, S. Lerman & T. Sawada (Eds.) (2004). *Proc. of the 9th ICME* (pp. 287-291). Dordrecht: Kluwer Academic Publishers, in CD (accessed 18/2/16)

(b) *ICME-10*, *Copenhagen*, Denmark, 2004

TSG 17: *The role of the history of mathematics in mathematics education*, organized by M. K. Siu, C. Tzanakis, A. El Idrissi, S. Kaisjer, L. Radford.

Summary in M. Niss (Ed) (2008). *Proc. of the 10th ICME* (pp. 363-367). IMFUFA, Roskilde University, in CD (accessed 18/2/16)

This TSG led to a post-conference publication (Siu & Tzanakis, 2004)

(c) *ICME 11 Monterrey,* Mexico, 2008

TSG 23: *The role of the history of mathematics in mathematics education*, organized by A. El Idrissi, A. Miguel, F. Furinghetti, A. Garciadiego, E. Barbin (accessed 18/2/16).

(d) *ICME 12 Seoul,* Korea, 2012

TSG 20: *The role of history of mathematics in mathematics education*, organized by R. Chorlay, W-S. Horng, M. Kronfellner, K. Clark, A. El Idrissi, H. Chang (accessed 18/2/16).

Summary in S. J. Cho (Ed.) (2015). *Proc. of the 12th ICME: Intellectual and attitudinal challenges*. New York: Springer (pp. 485-487) http://www.icme12.org/sub/sub02_05.asp
12 papers and 13 posters available at
http://www.icme12.org/sub/tsg/tsg_last_view.asp?tsg_param=20 (accessed 18/2/16).

(e) *ICME 13 Hamburg,* Germany 2016

TSG 25: *The role of history of mathematics in mathematics education*, organized by C. Tzanakis, X. Wang, K. Clark, T. H. Kjeldsen, S. Schorcht (accessed 4/4/2016)

2.1.4 The *HPM domain* at CERME

CERME is a regular activity of the *European Society for Research in Mathematics Education* (ERME), organized every two years in the form of presentations, discussions, and debates within thematic working group (WG). Though relatively new, the *HPM perspective* has exhibited great potential at CERME and is expected to play a central role in the future:

(a) *CERME 6*, *Lyon*, France, 2009

WG 15: *Theory and research on the role of history in mathematics education*, organized by F. Furinghetti, J-L. Dorier, U. T. Jankvist, J. van Maanen & C. Tzanakis.

A new WG structured along 7 themes; 13 papers and 1 poster accepted and included in the proceedings (see §3.1.3).

(b) *CERME 7*, Rzeszów, Poland, 2011

WG 12: *History in Mathematics Education* organized by U. T. Jankvist, S. Lawrence, C. Tzanakis & J. van Maanen.

Structured along 9 themes; 13 papers and 1 poster accepted and included in the proceedings (see §3.1.3).

(c) *CERME 8*, Antalya, Turkey, 2013

WG 12: *History in Mathematics Education* organized by U. T. Jankvist, K. Clark, S. Lawrence & J. van Maanen.

Structured along 9 themes (the same as CERME 7); 12 papers and 3 posters accepted and included in the proceedings (see §3.1.3).

(d) *CERME 9*, *Prague*, Czech Republic, 2015

Thematic WG 12: *History in Mathematics Education*, organized by R. Chorlay, U. T. Jankvist, K. Clark & J. van Maanen (accessed 4/4/2016).

The WG themes were modified considerably, becoming more specific. This reflects further deepening of research in this area, with emphasis both on empirical work and its assessment on sharpening theoretical ideas, and developing conceptual frameworks adequate for describing and understanding phenomena relevant to the *HPM perspective* (see §3.1.3)

2.2 Journals and Newsletters

2.2.1 *Convergence: Where Mathematics, History, and Teaching Interact*

Since 2004, the MAA has published *Convergence: Where Mathematics, History and Teaching Interact,* a free online journal in HM and its use in teaching.

Aimed at teachers of mathematics at both the secondary and collegiate levels, *Convergence* includes topics from grades 8-16 mathematics, with special emphasis on grades 8-14. Its resources for using the HM in mathematics teaching include informative articles about the HM, translations of original sources, classroom activities, projects and modules, teaching tools such as its *Mathematical Treasures*, reviews of new and old books, websites, *Problems from Another Time*, and other teaching aids that focus on utility in the classroom.

2.2.2 The *Bulletin of the British Society for the History of Mathematics (BSHM Bulletin)*

The *BSHM Bulletin* aims to promote research into the HM and to encourage its use at all levels of ME. Articles on local HM and the use of HM in ME are particularly encouraged. It was originally published as a Newsletter, until 2004 when its 50th issue became Bulletin 1. Under the influence of the late J. Fauvel, president of BSHM (1992–94), editor of its Newsletter (1995–2001) and chair of the HPM Group (1992-96) and his successor, the late J. Stedall, the Newsletter changed from providing information to members into a scientific journal with a regular Education Section directly related to issues relevant to the HPM perspective since 2002 (issue No 46).

2.2.3 The *HPM Newsletter*

The Newsletter appears three times per year since 1980. Originally it was available by contacting the regional distributors; however, for the last 13 years it is also available online from the HPM Group website and its *Newsletter* webpage

It includes a calendar of upcoming events, a guest editorial, a '*Have You Read?*' section, short reviews and announcements of meetings and activities. Furthermore, for the last 13 years it has also included short articles, reports on research projects and PhD theses, book reviews, lists of relevant websites, and particular themes that are suggested for further research.

2.3 Comments on some key issues

From what has been presented so far and will become clearer from the bibliographical survey in Section 3, the last two decades have generated considerable research activity related to the *HPM perspective* of great variety: doing empirical research based on actual classroom implementations; designing specific teaching units; developing various kinds of teaching aids; exploring and understanding students' response to the introduction of the HM in teaching (including teacher education); designing, applying, and evaluating interdisciplinary teaching; drawing and/or criticizing parallels between the historical development and learning in a modern classroom;[2] mutually profiting from theoretical constructs and conceptual frameworks developed in the context of other disciplines, especially philosophy, epistemology and cognitive science; and evaluating the effectiveness of all this in practice.

The key issues mentioned in section 1 permeate all these activities as recurring themes that form the *leitmotif* of the HPM domain. Though impossible to present all the work done and the not always mutually compatible opinions and results, below a few general ideas are outlined

---

[2] The old but still discussed issue of "historical parallelism" – if and to what extent "ontogenesis recapitulates (aspects of) phylogenesis"; e.g., Radford et al., 2000; Schubring, 2006, 2011; Thomaidis & Tzanakis, 2007.

with reference to the literature for details.

Whether HM is appropriate, or even relevant at all to the teaching and/or learning of mathematics, is an issue that, despite the extended research and the many insightful and sophisticated applications in the last few decades, has not reached universal acceptance even today. In fact, a number of objections against the *HPM perspective* have been raised (Furinghetti 2012, §7; Siu 2006; Tzanakis, Arcavi et al. 2000, p.203; Tzanakis & Thomaidis 2012, §3.4):

A *Objections of an epistemological and methodological nature*

(a) *On the nature of mathematics*

1. This is not mathematics! Teach the subject first; then its history.
2. Progress in mathematics is to make difficult problems routine, so why bother to look back?
3. What really happened can be rather tortuous. Telling it as it was can confuse rather than enlighten!

(b) *On the difficulties inherent to this approach*

1. Does it really help to read original texts, which is a very difficult and time-consuming task?
2. Is it liable to breed cultural chauvinism and parochial nationalism?
3. Students may have an erratic historical sense of the past that makes historical contextualization of mathematics impossible without having a broader education in general history.

B *Objections of a practical and didactical nature*

(a) *The background and attitude of teachers*

1. Lack of didactical time: no time for it in class!
2. Teachers should be well educated in history: "I am not a professional historian of mathematics. How can I be sure of the exposition's accuracy?"
3. Lack of teacher training.
4. Lack of appropriate didactical and resource material.

(b) *The background and attitude of the students*

1. They regard it as history and they dislike history class!
2. They regard it just as boring as mathematics itself.
3. They do not have enough general knowledge of culture to appreciate it.

(c) *Assessment issues*

1. How can you set questions on it in a test or exam?
2. Is there any empirical evidence that students learn better when HM is made use of in the classroom?

Each of these objections addresses one or more of the four key issues mentioned in Section 1. Below we comment briefly on them in the light of these objections.

2.3.1 *Which history is suitable, pertinent and relevant to Mathematics Education?*

This has been a permanent issue of debate among historians and mathematics educators with an interest in the HPM perspective. As early as 1984 at ICME 5, d'Ambrosio stressed the need to develop three separate histories of mathematics: history as taught in schools, history as developed through the creation of mathematics, and the history of that mathematics which is used in the street and the workplace. To deal with these differences he introduced the concept of *ethnomathematics* as compared to *learned mathematics* (Booker, 1986).

In fact, implicit to the objections A(a1), A(a2), A(b1) is the idea that the term "history" is the same, whether used by historians, mathematicians, or teachers and mathematics educators. That this is not so lies at the heart of Grattan-Guinness' early refutation of some of these arguments (Grattan-Guinness, 1973; see also Kjeldsen, 2011a, pp. 1700-1701; 2011b, pp. 166-167; Kjeldsen & Blomhøj, 2012, §3, and references therein for a different recent approach). On the other hand, it is undeniable that quite often the historical development was complicated, followed a zig-zag path, led to dead ends, included notions, methods and problems that are no longer used in mathematics as we know and work with today, etc. (A(a2), A(b3)). Thus, its integration in ME, on the one hand is nontrivial, and on the other hand poses the question why it must be done at all. Therefore, integrating HM in the teaching and learning of mathematics, may force history "*…to serve aims not only foreign to its own but even antithetical to them*" (Fried, 2011, p. 13). In other words, the danger of either unacceptably simplifying or/and distorting history to serve education as still another of its tools is real by adopting what has been called a "Whig" (approach to) history, in which "*…the present is the measure of the past. Hence, what one considers significant in history is precisely what leads to something deemed significant today*" (Fried, 2001, p. 395).

In this connection, an important step was Grattan-Guiness' distinction between what he called *History* and *Heritage* trying to clarify existing conflicts and tensions between a mathematician's and a historian's approach to mathematical knowledge, and paying due attention to the relevance of HM to ME (Grattan-Guiness, 2004a, b). In the context of the *HPM perspective*, this is a distinction close to similar ones between pairs of methodological approaches; *explicit* & *implicit* use of history, *direct* & *indirect* genetic approach, *forward* & *backward* heuristics (Tzanakis, Arcavi et al., 2000, pp. 209-210). Hence, this distinction is potentially of great relevance to ME (Rogers, 2009, 2011; Tzanakis & Thomaidis, 2012), serving, among other things, to contribute towards answering the recurrent question "*Why and which history is appropriate to be used for educational purposes?*" (Barbin, 1997).

2.3.2 *Which role can History of Mathematics play in Mathematics Education?*

Perhaps, this is the question that has been discussed and analyzed most on the basis of both a priori theoretical and epistemological arguments and of empirical educational research. At least implicitly, such analyses try to refute some of the above objections, especially those concerning the barriers posed by the complexity of the historical development (A(a2), A(a3), A(b1)) and/or by students' predisposition to and general knowledge of both mathematics and history as taught subjects (objections (B(b1), B(b2) and B(b3), A(b2), respectively).

This question has been extensively discussed from several points of view, especially in relation to the appropriateness and pertinence of original historical sources in ME. In this context, HM can play three mutually complementary and supplementary roles (Barbin, 1997; Furinghetti, 2012, §5; Furinghetti et al., 2006, pp. 1286-1287; Jahnke et al., 2000, §9.1; Jankvist, 2013, §7):

A *replacement* role: Replacing mathematics as usually understood (a corpus of knowledge consisting of final results/finished and polished intellectual products; a set of techniques for solving problems given from outside; school units useful for exams etc.) by something different (to emphasize not only final results, but also mental processes that may lead to them; hence to perceive mathematics not only as a collection of well-defined and deductively organized results, but also as a vivid intellectual activity).

A *reorientation* role: Changing what is (supposed to be) familiar, to something unfamiliar; thus challenging the learner's and teacher's conventional perception of mathematical knowledge as something that has always been existing in the form we know it, into the deeper awareness that mathematical knowledge was an invention, an evolving human intellectual activity.

A *cultural* role: Making possible to appreciate that the development of mathematics takes place in a specific scientific, technological or societal context at a given time and place; thus becoming aware of the place of mathematical knowledge as an integral part of human intellectual history in the development of society; hence, seeing mathematics from perspectives that lie outside its nowadays established boundaries as a discipline.

Considered from the point of view of the *objective* of integrating HM in ME, there are 5 main areas in which the *HPM perspective* could be valuable:

The learning of mathematics;
The development of views on the nature of mathematics and mathematical activity;
The didactical background of teachers and their pedagogical repertoire;
The affective predisposition towards mathematics; and
The appreciation of mathematics as a cultural-human endeavor.

These are analyzed in detail into more specific arguments in Tzanakis, Arcavi et al., 2000, §7.2, describing in this way the role of history in the educational process.

From the point of view of the *way* HM is accommodated into this perspective, a distinction was made by Jankvist (2009); namely, history serving as a *tool* for assisting the actual learning and teaching of mathematics, and history serving as a *goal* in itself for the teaching and learning of the historical development of mathematics (see also Jankvist & Kjeldsen, 2011). A similar distinction between *history for reflecting on the nature of mathematics as a socio-cultural process* and *history for constructing mathematical objects* was made by Furinghetti (2004; 2012, §5).

In this way, a finer and more insightful categorization of the possible roles of HM in ME resulted, reflecting the variety of their possible implementations in practice.

A small selection appears below (many more are given in section 3).

- Fauvel & van Maanen (2000): Chapters 7, 8 provide a variety of examples of possible classroom implementations, for several mathematical subjects; chapter 9 gives examples of using original sources in the classroom and specific didactical strategies to do so.
- Katz & Tzanakis (2011), Chapters 9, 10, 13, 14, 16, 19, and Sriraman (2012), Chapters 2, 7, 14 provide particular examples, most of them emphasizing empirical results of

actual implementations.

- Katz et al. (2014): Rich on recent work in the *HPM domain*, including a sufficiently comprehensive old and recent bibliography in the editors' introduction and in its 12 papers. The papers concern theoretical issues on the history, philosophy and epistemology of mathematics, and on empirical investigations both in school and teacher education.
- Doctoral dissertations with considerable work on both the theoretical issues of the *HPM perspective* and on empirical investigation and evaluation of actual implementations: e.g., Clark (2006); Glaubitz (2010); Jankvist (2009a); Su (2005); van Amerom (2002).

2.3.3 *To what extent the History of Mathematics has been integrated in Mathematics Education (curricula, textbooks, educational aids/resource material, teacher education)?*

Considerable work has been done over the last 15 years on understanding better and formulating more sharply the methodological issues raised by the integration of HM in ME, on producing appropriate educational aids of various types (B(a4)), and on designing and implementing teaching approaches to specific subjects and instructional levels in this context, with special emphasis on teacher education (B(a2), B(a3)).

According to the classification of the various approaches to integrate HM in teaching and learning mathematics given in Tzanakis, Arcavi et al. (2000), there are three broad ways that may be combined (thus complementing each other), each one emphasizing a different aim:

To provide direct historical information, aiming to learn history;
To implement a teaching approach inspired by history (explicitly or implicitly), aiming to learn mathematics;
To focus on mathematics as a discipline and the cultural and social context in which it has been evolving, aiming to develop a deeper awareness of its evolutionary character, its epistemological characteristics, its relation to other disciplines and the influence exerted by factors both intrinsic and extrinsic to it.

From a methodological point of view, Jankvist (2009b) classified the teaching & learning approaches in three categories:

Illumination approaches, in which teaching and learning is supplemented by historical information;
Module approaches, in the form of instructional units devoted to history, often based on specific cases;
History-based approaches, in which history shapes the sequence and the way of presentation, often without history appearing explicitly, but rather being integrated into teaching.

Approaches may vary in size and scope, according to the specific didactical aim, the mathematical subject matter, the level and orientation of the learners (A(b1), B(b3)), the available didactical time (B(a1)), and external constraints (curriculum regulations, number of learners in a classroom etc.).

The crucial role of teachers' training for effectively following the *HPM perspective* has been stressed repeatedly (e.g., Alpaslan et al., 2014, pp.160-162; Barbin et al., 2000, p. 70;

Barbin et al., 2011b; Furinghetti, 2004, p.4). Similarly, the need for appropriate didactical resources is equally crucial (e.g., Pengelley, 2011, pp. 3-4; Tzanakis, Arcavi et al., 2000, pp. 212-213).

Though accommodating the *HPM perspective* in an essential way into the official national curricula does not seem to have attained wide applicability,[3] intensive efforts have been made to train teachers and explore changes in their attitude and/or teaching, and to design, produce and make available didactically appropriate resources. Some indicative examples (more are given in section 3):

Teacher training**:** Arcavi & Isoda, 2007; Bruckheimer & Arcavi, 2000; Clark, 2011; Liu, 2003; Mosvold et al., 2014; Povey, 2014; Smestad, 2011; Waldegg, 2004.

Resource material & educational aids**:** The need for didactical resources along the lines of the *HPM perspective* has been satisfied to a considerable extent in the last 15 years, so that such material is available nowadays in a variety of forms. Some examples:

- A wide spectrum of resource material can be found in *Convergence*; e.g., (i) *HPM Newsletter*, No90/2015, pp. 10-12 for a recent sample; (ii) Clark, 2009 (detailed description of a teaching module).
- Katz & Michalowicz (2004): didactical source material within 11 mathematical modules.
- Siu (2007): a useful survey of the literature and available resources.
- Pengelley et al. (2009): Didactical material for discrete mathematics based on original sources.
- Pengelley & Laubenbacher (2014): A website with many references to published work and material available online.
- Barnett et al. (2014): Extensive information on teaching with historical sources and bibliography on its theoretical framework and available resource material.
- Books with material that can be used directly and/or inspire teaching; e.g., Barbin, 2015 (review in *HPM Newsletter* No 89/2015, pp. 13-14); Stein, 2010 (review in *HPM Newsletter* No77/2011, pp. 8-9).

2.3.4 *How can this role be evaluated and assessed and to what extent it contributes to amend the teaching and learning of mathematics?*

Evaluating the effectiveness of the *HPM perspective* on improving ME from the point of view of both teaching and learning mathematics is an issue clearly stressed in objections B(c1), B(c2). Those who oppose, or are reserved about the role of HM in ME rightly ask for sufficient empirical evidence about its effectiveness. Quite early it has become clear that this is a key issue (e.g., Siu & Tzanakis, 2004, p. 3; Jankvist, 2007), and that any such evaluation is a complex process relying more on qualitative, than quantitative methodologies: to consider changes induced in teachers' own perception of mathematics; to examine how this may influence the way they teach mathematics; and to explore if and in which ways this affects

[3] One exception is Denmark (see Jankvist, 2013, §3; Kjeldsen, 2011b, §15.2; Niss & Højgaard, 2011, ch.4). For a recent discussion and survey see Boyé et al., 2011.

students' perception and understanding of mathematics (Barbin et al., 2000, particularly §§3.1, 3.2).

Additionally, any such evaluation goes together with actual classroom implementations, in school teaching and teacher pre- and in-service education. Therefore, many, if not all, works referring to such implementations necessarily address evaluation issues about the effectiveness of the approach considered in each case (e.g., those listed in §§2.3.2, 2.3.3).

This is an area of currently active research with no established results of universal acceptance because of several reasons:

(a) Such a complex process is not expected to lead to spectacular changes in a short time interval. Preconceptions, misconceptions, predispositions either of the teachers or the students are too stable to be easily and/or quickly modified. Therefore, one should expect to see such changes after a considerable time exposure to an approach adopting the *HPM perspective*; often this time is not available.

(b) There is strong dependence on the instructional level (primary, secondary, tertiary) and orientation of the students, teacher-students included (science or humanities; elementary or secondary school teachers etc.), as well as, on their entire previous educational path, which has determined their knowledge of, attitude towards, and preconceptions about mathematics.

(c) There is influence by external "technical" factors that may favor, impede, or even prevent the implementation of an approach based on the *HPM perspective*: the curriculum and the corresponding regulations; the number of students in the class (e.g., a small number facilitates group work and teacher's effective supervision); the structure of the educational system (e.g., in a centralized system, teachers have less freedom, hence fewer possibilities to apply an innovative teaching approach not necessarily falling into the official curriculum regulations).

(d) Not all mathematical subjects are equally accessible or appropriate to be taught and/or learned in a historically motivated/driven context.

All of this constitutes a complex network of factors interfering with each other, so that empirical findings of different research works are not easily comparable. Therefore, despite many thoughtfully designed and carefully applied empirical investigations, much work is still needed to evaluate the effectiveness of the role of HM in ME in an undisputable way.

3 A bibliographical survey in the HPM domain since 2000

This section provides a comprehensive bibliographical survey of work related to the *HPM perspective* since 2000, indicating the TSG 25 themes to which each item is related and relevant, except those collective works that practically touch upon all themes (cf. remark (a) and Appendix in §1.2). Within each subsection, items appear by publication year and for each year by authors' alphabetical order. For those works included in Section 5 only author names and a note "see section 5" are given.

Throughout this topical study, names of journals, proceedings, and conferences are abbreviated as follows:

ABBREVIATIONS

Journals

BSHM: BSHM Bulletin: Journal of the British Society for the History of Mathematics
ESM: Educational Studies in Mathematics
FLM: For the Learning of Mathematics
Int. J. ME Sci. Tech.: International Journal of Mathematical Education in Science and Technology
Int. J. Math. Teach. Learn: International Journal for Mathematics Teaching and Learning
Int. J. Sci. ME: International Journal of Science and Mathematics Education
JMB: Journal of Mathematical Behavior
JRME: Journal for Research in Mathematics Education
MER: Mathematics Education Review
MT: The Mathematics Teacher
MTL: Mathematical Thinking and Learning
MedJRME: Mediterranean Journal for Research in Mathematics Education
NOMAD: Nordic Studies in Mathematics Education
PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies
R-IREM: Repères-IREM
RME: Research in Mathematics Education
S&E: Science & Education
TJME: Taiwan Journal of Mathematics Education
TME: The Mathematics Educator
ZDM: ZDM - The International Journal of Mathematics Education

In Chinese:
MB: Mathematical Bulletin 数学通报
JME: Journal of Mathematics Education 数学教育学报
CTMM: Curriculum, Teaching Material and Method 课程·教材·教法
MTMS: Mathematics Teaching in Middle Schools (Junior High School)/ (Senior High School) 中学数学教学参考
ERR: Educational Researches and Reviews (High School Education and Instruction) 教育研究与评论
HSMM: The High School Mathematics Monthly 中学数学月刊
CER: Comparative Education Review 比较教育研究

Conferences

CERME: Congress of European Research in Mathematics Education
ESU: European Summer University on the History and Epistemology in Mathematics Education.
HPM XXXX: The HPM Group Satellite meeting of ICME-Y[4]
ICME: International Congress on Mathematical Education
ICTM: International Conference on the Teaching of Mathematics (at the undergraduate level)

[4] XXXX being the year that ICME-Y took place.

Books

Hand Int RME: Handbook of International Research in Mathematics Education
ICMI Study Volume: J. Fauvel & J. van Maanen (Eds.), *History in Mathematics Educatio*n, *The ICMI Study*. Dordrecht: Kluwer, 2000.
Int. Hand. Res. Hi. Phil. Sci. Teach.: International Handbook of Research in History, Philosophy and Science Teaching
TMMEM: The Montana Mathematics Enthusiast Monographs

Other

ERME: European Society for Research in Mathematics Education
HPM Group: International Study Group on the Relations between the History and the Pedagogy of Mathematics
ICMI: International Commission on Mathematical Instruction
MAA: The Mathematical Association of America
NCTM: National Council of Teachers of Mathematics (USA)
Proc.: Proceedings
TSG: Topic Study Group
WG: Working Group

3.1 Collective works

3.1.1 *Collective volumes* in this area, with research papers, reviews of work, etc.

2000

Dorier, J.-L. (Ed.). *On the teaching of linear algebra*. Dordrecht: Kluwer. T2, 5

This volume consists of two parts: On an epistemological analysis of the historical development of vector space theory; and on didactical issues addressed and actual implementations at the undergraduate level, to which this analysis is related.

Fauvel & van Maanen (see section 5).

Vol. 6 of the *New ICMI Study Series*: The outcome of a 4-year collectively realized international study of 62 scholars in 11 groups, each one authoring one chapter under the coordination of a convenor. A landmark in establishing the HPM domain.

Katz (see section 5).

26 chapters based on papers presented in the 1996 HPM Meeting: 7 on general issues of HM in ME; 5 on the teaching of a particular subject using the HM; 3 on teacher education; 11 on the HM.

2001

Barbin, É., Duval, R., Giorgiutti, I., Houdebine, J., & Laborde C. (Eds.). *Produire et lire des textes de démonstrations*. Paris: Ellipses.

This work addresses all problems encountered in learning proof, by following a variety of approaches: mathematical, historical, epistemological, didactical, linguistic, cognitive. Revealing the diversity of the points of view on proof and its teaching is the strong point of this book.

2002

Boniface, J. (Ed.). *Les constructions des nombres réels dans le mouvement de l'arithmétisation* *de l'analyse. Comprendre les mathématiques par les textes historiques*. Paris: Ellipses.

Cerquetti, F., & Rodriguez A. (Eds.). *Faire des mathématiques avec des images et des manuscrits historiques*. Créteil: CRDP de Créteil. T2a, 4

An empirical study with high school students on rediscovering geometry using various educational aids; among others, excerpts from original sources (16-18th centuries)

2003

Bekken & Mosvold (see section 5). T1, 4, 2a, 5

A continuation of Swetz et al. (1995). Of its 27 chapters, 9 are related to the HPM perspective; the others concern either the HM or the history of ME.

2005

Shell-Gellasch, A., & Jardine, D. (Eds.). *From calculus to computers: Using the last 200 years of mathematics history in the classroom*. MAA Notes 68. Washington, DC: MAA. T2, 1, 5

22 contributions focusing on 19th and 20th century mathematics, emphasizing recent history in the teaching of mathematics, computer science, and related disciplines; for details see http://www.maa.org/publications/books/from-calculus-to-computers (accessed 18/2/2016)

2006

Thomaidis, Y., Kastanis, N., & Tzanakis, C. (Eds.). *History and mathematics education*. Thessaloniki: Ziti Publications (in Greek). T4, 5, 1

Collective volume with 5 papers on the HPM perspective; 6 on the history of ME and 3 on historiographic approaches to ancient Greek mathematics (more in *HPM Newsletter*, No63/2006, pp. 7-9).

2007

Barbin, É., & Bénard, D. (Eds.). *Histoire et enseignement des mathématiques: Rigueurs, erreurs, raisonnements*. Lyon: Institut National de Recherche Pédagogique. T5, 4, 1

12 chapters on rigor, experiment and proof in geometry, reasoning between geometry and algebra, and multiplicity of points of view in analysis. Contributions are based on original texts and anchor on epistemological and didactical reflections.

Boero, P. (Ed.). *Theorems in school. From history, epistemology and cognition to classroom practice*. Rotterdam: Sense Publishers. T5, 1

15 chapters in 4 sections exploring: the historical-epistemological nature of proof; the way national mathematics curricula and the underlying epistemological beliefs might influence students' perceptions of proof; the similarities and differences between argumentation and proof; the teaching of proof.

2008

Shell-Gellasch, A. (Ed.). *Hands on history: A resource for teaching mathematics*. MAA Notes 72. Washington, DC: MAA. T2b, 5, 6

16 chapters on historical examples appropriate for incorporating hands-on learning: From simple devices, to elaborate models of descriptive geometry, and detailed descriptions on how to build and use historical models in the high school or collegiate classroom; for details see http://www.maa.org/press/books/hands-on-history-a-resource-for-teaching-mathematics (accessed 18/2/2016)

Robson, E., & Stedall, J. (Eds.). *The Oxford handbook of the history of mathematics.* Oxford, UK: Oxford University Press. T6, 5

This is not just a 'history of mathematics'; it is unusual in that it is organized in 3 major themes: geographies and cultures; people and practices; interactions and interpretations that are sensitive to modern historiography and include social and cultural backgrounds and epistemologies.

2009

Greek Society of the Didactics of Mathematics (Eds.). *The value of the history of mathematics in mathematics education*. Thessaloniki: Ziti Publications (in Greek). T2, 4, 1

9 contributions on one or more of the following issues: Whether and how crucial historical steps can be integrated into ME; whether and to what extent historical knowledge allows the prediction of students' difficulties; whether the integration of HM in ME is possible at all, given the constraints imposed by the official curriculum and the teachers' and students' established preconceptions.

2010

Barbin, É. (Ed.). *Des grands défis mathématiques d'Euclide à Condorcet*. Paris: Vuibert. T2, 4, 5, 1

9 examples of introducing a historical perspective in ME, having as a starting point specific historical problems and organized in 4 parts: measuring magnitudes, representing magnitudes, calculating the probable, approaching a curve; thus illustrating the great domains of today's taught math: analysis, algebra, probability and geometry (for details see http://culturemath.ens.fr/node/2582 (accessed 18/2/2016) or *HPM Newsletter* No75/2010, pp. 1-2).

Hanna, G., Jahnke, N., & Pulte, H. (Eds.). *Explanation and proof in mathematics: philosophical and educational perspectives*. New York, NY: Springer. T1, 5

The 17 chapters assemble perspectives from ME, its history and philosophy to strengthen mutual awareness and share recent findings and advances in these interrelated fields. By a variety of examples, the authors explore the role of refutation in generating proofs, the varied links between experiment and deduction, the use of diagrammatic thinking in addition to pure logic, and the uses of proof in ME.

2011

Katz & Tzanakis (see section 5).

An all-embracing outcome of activities within the HPM Group during 2007-2009; to present an overview of the state of the art in this area after the appearance of the ICMI Study volume. 7 chapters on theoretical aspects of HM in ME; 10 with concrete implementations; 4 with particular focus on teacher education; 3 invited papers on the HM (indicative examples in section 5).

Shell-Gellasch, A., & Jardine, D. (Eds.). *Mathematical time capsules: Historical modules for the mathematics classroom*. MAA Notes 77. Washington, DC: MAA. T2b, 5

35 chapters offering teachers historical modules for immediate use in the math classroom (undergraduate or secondary math curricula). Each capsule presents at least one topic or a historical thread that can be used throughout a course and further references and resources on the chapter subject; for detail see http://www.maa.org/press/ebooks/mathematical-time-capsules (accessed 18/2/2016).

2012

Barbin, É. (Ed.). *Les mathématiques éclairées par l'histoire: des arpenteurs aux ingénieurs*. Paris: Vuibert. T2, 4, 6

The 9 chapters provide examples of the influence or use of HM in the math classroom. Authors work with students from upper secondary school to tertiary, including teacher training. They show how their own reading and reflection has led to direct use of historical material in the classroom, either through the use of original texts or by devising tasks based on the methods or examples of their subjects (*HPM Newsletter* No80/2012, pp.12-13).

Chemla, K. (Ed.). *The history of mathematical proof in ancient traditions*. Cambridge: Cambridge University Press. T2b, 6

A number of important contributions from international scholars on the historiography, history and epistemology of mathematical proof, relevant for today's mathematics classroom.

Sriraman, B. (Ed.). *Crossroads in the history of mathematics and mathematics education*, TMMEM vol.12. Charlotte, NC: Information Age Publishing. T1, 2b, 5

15 chapters: 7 in the history and didactics of calculus and analysis, 4 in the history and didactics of geometry and number and 4 on HM in ME.

2013

Barbin, É., & Moyon, M. (Eds.). *Les ouvrages de mathématiques dans l'histoire. Entre recherche, enseignement et culture*. Limoges: PULIM. T3b

For contents & summary, see http://www.pulim.unilim.fr/index.php/notre-catalogue/fiche-detaillee?task=view&id=750 (accessed 18/2/2016).

2014

Matthews, M. (Ed.). *Int. Hand. Res. Hi. Phil. Sci. Teach*. Dordrecht: Springer. T1, 4, 5, 3b

Of the 76 chapters, 7 are devoted to ME, 4 of which concern the HPM perspective.

2015

Barbin, É., & Cléro, J.-P. (Eds.). *Les mathématiques et l'expérience: ce qu'en ont dit les philosophes et les mathématiciens*. Paris: Hermann. T5, 6

17 contributions on philosophical issues, pertinent to mathematics and its epistemological characteristics, on the basis of texts of various philosophers through the ages; for details see http://les-livres-de-philosophie.blogspot.gr/2015/10/barbin-evelyne-clero-jean-pierre-dir.html (accessed 18/2/2016).

Barbin, É. (Ed.). *Les constructions mathématiques avec des instruments et de gestes (Mathematical constructions: making and doing)*. Paris: Ellipses-Editions. T2b

Any study of geometry involves both thinking and doing and it is through the doing (drawing, measuring, copying) that we develop a sense of what geometry is. School mathematics has today lost much of its geometry. It is in an attempt to recall this loss, and to report on the history of geometrical constructions, that provides the spur for this collection (see *HPM Newsletter* No 89/2015, pp.13-14).

Rowe, D.E., & Horng, W.S. (Eds.). *A delicate balance: Global perspectives on innovation and tradition in the history of mathematics: A festschrift in honor of Joseph W. Dauben*. Berlin: Birkhäuser. T6

A collection of contributions on various historical issues, from the perspective and/or in the context of different cultures; for content & summary see http://www.springer.com/gb/book/9783319120294 (accessed 18/2/2016).

To appear

Jardine, D., & Shell-Gellasch, A. (Eds.). *The courses of history: Ideas for developing history of mathematics courses*, Washington, DC: MAA. T1, 3b, 4, 5, 6

3.1.2 *Special issues* of international journals of ME

2004

Siu & Tzanakis (see section 5). T1, 2, 3a, 5

A special double issue with 10 papers originally presented at ICME 10, TSG 17 "The role of the History of Mathematics in Mathematics Education" on epistemological issues; teacher education; didactical material; particular examples.

2007

Furinghetti, F., Radford, L., & Katz, V. (Eds.). The history of mathematics in mathematics education: Theory and practice. *ESM, 66*(2), 107-271. T1, 2a, 3b, 4

A special issue with 10 papers, seeking to deepen the understanding of the pedagogical role HM may play in contemporary ME. Some provide examples of the use of the HM in school practice and teacher education; others address theoretical questions that have become crucial to understanding the profound intertwining of past and present, conceptual developments on spreading new epistemologies and theories of learning.

2010

Stedall, J. (Ed.). Special Issue: The history of mathematics in the classroom. *BSHM, 25*(3), 131-179. T2a

4 articles on the HM in classroom practice, from teaching basic arithmetic in primary school to teaching statistics to young adults.

2014

Clark, K., & Thoo, J.B. (Eds.) The use of history of mathematics to enhance undergraduate mathematics instruction. *PRIMUS, 24*(8), 663-773. T2, 4, 6

8 papers focused on undergraduate math teaching and learning using HM as a vehicle: 4 on some aspect of including HM within coursework required of undergraduate students; 4 on authors' experiences with developing and using primary source material with undergraduate math students. Results from: classroom experiments; teaching HM courses for math, ME, or philosophy majors; ways of integrating original sources in the undergraduate classroom.

Katz et al. (see section 5). T1, 2a, 5

Special issue with an introduction accompanied by an extensive bibliography and 12 papers directly related to the HPM perspective, divided into 4 sections: theoretical issues in the use of the HM in teaching; direct uses of the HM in the classroom; HM in teacher education; relations between the philosophy, the epistemology, the teaching and the sociology of mathematics.

Kourkoulos, M., & Tzanakis, C. (Eds.). History of mathematics and mathematics education. *Education Sciences*. Special Issue for 2014, 5-198. T1, 2a, 3a

Bilingual issue with 9 papers related to the *HPM perspective* (3 in English, 6 in Greek): 3 concern general ideas, conceptual frameworks and methodological schemes and 5 refer to specific issues with focus on classroom implementations (from elementary school to the university).

2015

Karam, R. (Ed.). Thematic issue: The interplay of physics and mathematics: Historical, philosophical and pedagogical considerations, *S&E, 24*(5-6), 487-805. T1, 5, 6, 2a

11 papers challenging the typical situation found in educational contexts: In physics education, math is a mere tool to describe and calculate; in ME, physics is only a possible context for applying mathematical concepts previously defined abstractly. Overcoming this dichotomy (which creates significant learning problems for the students) demands a systematic research effort in different fields, especially when aiming at informing educational practices by reflecting on historical, philosophical and sociological aspects of scientific knowledge.

3.1.3 *Proceedings of conferences and meetings* (with reference to their accessibility via the Internet, wherever possible)

2000

Horng & Lin (see section 5).

5 plenary lectures, 5 round tables, 4 panels, 35 oral presentations, 1 workshop with special emphasis on the effectiveness of the HPM perspective, cultural aspects and history of ME.

2004

Horng et al. (see section 5).

Proceedings of a meeting preceding HPM 2004; see HPM Newsletter No55/2004, pp. 12-14; No56/2004, p. 16.

2006

Furinghetti et al. (see section 5). T4, 1, 2a

Workshop on original sources in ME. Research questions were identified which evolved from work in the past and are helpful in orienting future work. They reflect central issues related to the integration of original sources from the HM into ME, both in learning and teaching mathematics (see *HPM Newsletter*, No62/2006, pp. 7-10).

Furinghetti, F., Kaisjer, S., & Tzanakis, C. (Eds.) *Proc. of HPM 2004 & ESU 4*, Iraklion: University of Crete, Greece (accessed 18/2/2016).

Revised edition of the *HPM 2004 of ICME 10* & the *4th ESU* proceedings (Furinghetti et al. 2004). 78 papers divided into 6 sections, corresponding to the 6 main themes of this joint meeting.

2008

Barbin et al. (see section 5).

78 full papers & 42 abstracts, in 6 sections corresponding to the ESU 5 six main themes: 6 plenary lectures, 2 panel discussions, 19 workshops based on didactical and pedagogical material, 25 workshops based on historical & epistemological material, 44 oral presentations and 26 short communications.

Cantoral et al. (see section 5).

70 papers on the 6 themes of the meeting, stemming from 4 plenary lectures, 5 workshops, and 61 oral presentations.

2010

Furinghetti, F., Dorier, J.-L., Jankvist, U. T., van Maanen, J., & Tzanakis, C. (Eds.). WG 15: Theory and research on the role of history in mathematics education. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proc. of CERME 6* (pp. 2677-2810). Lyon: Institut National De Recherche Pédagogique. T2a, 4, 3b, 5 (accessed 18/2/16)

Group work structured along 8 main themes, leading to 14 papers included in this proceedings.

2011

Barbin et al., 2011a (see section 5).

55 texts and 35 abstracts divided into 6 sections corresponding to the ESU 6 six main themes (see *HPM Newsletter No79/2012*, pp. 9-11).

Jankvist, U. T., Lawrence, S., Tzanakis, C., & van Maanen, J. (Eds.). WG 12: History in mathematics education. In M. Pytlet, T. Rowland, & E. Swoboda (Eds.), *Proc. of CERME 7* (pp. 1636-1779). Rzeszow, Poland: University of Rzeszow. T1, 2a, 4 (accessed 18/2/2016).

Group work structured along 4 general topics, leading to 15 papers included in this proceedings. Topics: Research questions and relevance of research; use of HPM theory and ME theory; methods, data and analysis; validity, reliability and generality of research results.

2012

Barbin et al. (see section 5).

78 contributions on the 7 themes of the meeting, stemming from 7 plenary lectures, 7 workshops, 57 oral presentations, 5 posters & 2 exhibitions. Revised version in progress.

2013

Jankvist, U. T., Clark, K., Lawrence, S., & van Maanen, J. (Eds.). WG 12: History in mathematics education. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proc. of CERME 8* (pp. 1945-2067). Ankara: Middle East Technical University. T2a, 4, 5, 6 (accessed 18/2/2016).

Group work structured along 5 general themes leading to 13 papers and 3 posters included in this proceedings. Themes: interdisciplinarity; theoretical frameworks in history of ME; history in pre high school ME; history in high school ME; HM in teacher education and design.

2015

Barbin et al. (see section 5). T1, 2a, 2b, 4, 6

53 texts and 34 abstracts corresponding to all types of activities during ESU 7, divided into 8 sections corresponding to its 7 themes and a poster session.

Chorlay, R., Jankvist, U. T., Clark, K., & van Maanen, J. (Eds.) WG 12: History in Mathematics Education. In K. Krainer & N. Vondrová (Eds.), *Proc. of CERME 9* (pp. 1778-1884). Prague: Charles University in Prague, Faculty of Education & ERME (accessed 4/4/2016).

Group work structured along 9 themes leading to 14 papers and 2 posters included in this proceedings, dealing with four areas of questions on history in ME-the student perspective; history in ME-the teacher perspective; history of ME-the mathematical education landscape; methodological reflections on history in/of ME.

3.1.4 *Resource material*, collectively produced, especially material available on the web.

2002

Bazin, M., Tamez, M., & Exploratorium Teacher Institute. *Mathematics and science across cultures. Activities and investigations from the Exploratorium*. New York, NY: The New Press. T2b, 6

Addresses teachers of grades 4-12 who wish to develop hands-on activities in classroom, suggested by artifacts or instruments produced by different current and past civilizations.

2004

Barnett, J., Bezhanishvili, G., Leung, H., Lodder, J., Pengelley, D., & Ranjan, D. *Teaching discrete mathematics via primary historical sources* (accessed 18/2/2016). T2a, 2b, 4

Resource material on teaching projects via original sources, its rationale and extensive bibliography.

Katz & Michalowicz (see section 5); T2b

An electronically available resource based on work done by a larger group of people, with material on 11 mathematical modules (secondary education), useful in the classroom and/or for teachers to adapt it according to their needs and those of their classroom.

2005

IREM Basse-Normandie. *L'espérance du Hollandais ou le premier traité du calcul du hasard.* Paris: Ellipses. T2b

Annotated presentation of Huygens' book "De rationiciis in ludo aleae" (the 1st treatise on probability calculus 1657) and the commentaries on it by its contemporaries and successors.

2006

Demattè, A., & Furinghetti, F. *Fare matematica con i documenti storici: Una raccolta per la scuola secondaria di primo e secondo grado*. Trento: Editore Provincia Autonoma di Trento – IPRASE del Trentino. T2b

Includes selection of excerpts from original sources and aims to provide secondary school teachers with activities for their integration in the classroom; thus promoting alternative ways of teaching through text-based activities and exercises in order to consolidate or/and introduce mathematical skills (one book for the teacher and one for the student; see *HPM Newsletter* No64/2007, pp. 6-8).

Groupe d'Histoire & d'Epistémologie des Mathématiques de l'IREM de Lyon. *Textes fondateurs du calcul infinitésimal.* Paris: Ellipses. T2b

Annotated texts of Newton and Leibniz on infinitesimal calculus; for details see http://www.decitre.fr/livres/si-le-nombre-m-etait-conte-9782729868222.html (accessed 18/2/2016).

Swetz, F. J. *Shu Xue You Ji*, (*The history of mathematics: A study guide*) Taipei: Transoxania Ltd (in Chinese). T2b

Based on an earlier teaching course on HM.

2007

Katz, V. (Ed.). *The mathematics of Egypt, Mesopotamia, China, India, and Islam: A sourcebook*. Princeton, NJ: Princeton University Press. T2b, 6.

English translations of key mathematical texts from the 5 most important ancient and medieval non-Western mathematical cultures, putting them into full historical and mathematical context. A firsthand understanding and appreciation of these cultures' important contributions to world mathematics. An essential resource and indispensable guide for math teachers who want to use non-western mathematical ideas in the classroom.

Siu, M.K. *Some useful references for course MATH2001 (Development of mathematical ideas) Department of Mathematics, University of Hong Kong* (accessed 18/2/201). T2b

An extensive list of papers, collective volumes, individual books, special journal issues, websites on the HM and its relations to ME. References either in English or Chinese. A useful resource for those interested in the HPM perspective.

2009

Clark (see section 5). T2b, 4

Introduces undergraduate students preparing to teach mathematics to Stevin's pamphlet on decimal fractions and to encourage prospective math teachers to think about connections with pupils' initial learning of decimals' multiplication.

Pengelley et al. (see section 5). T2b, 4

An extended presentation focusing on the pedagogy of historical projects, which offer excerpts from original sources, place the material in context, and provide direction to the subject matter.

Rogers, L. *Resources from Leo Rogers history and the mathematics curriculum* (accessed 18/2/2016). T2b

A website with a lot of information; useful websites; bibliography; suggestions/advices for HM in ME etc.

2012

Swetz, F. J. *Mathematical expeditions: Exploring word problems across the ages.* Baltimore, MD: Johns Hopkins University Press. T2b, 4, 6

Review at http://www.maa.org/press/periodicals/convergence/review-of-emmathematical-expeditions-exploring-word-problems-across-the-agesem (accessed 18/2/2016)

2013

Barnett, J., Bezhanishvili, G., Leung, H., Lodder, J., Pengelley, D., Pivkina, I., Ranjan, D., & Zack, M. Primary historical sources in the classroom: discrete mathematics and computer science. *Convergence*, *10*. doi:10.4169/loci003984 T2b

Presentation of 16 separate curricular modules, each a project for students based on excerpts from primary historical sources, to provide context, motivation and direction for selected topics in discrete mathematics and computer science as an alternative form of instruction.

Siu, M. K. A reading list for teachers interested in history of mathematics.
See http://hkumath.hku.hk/~mks/Whynot%20Histh_MKS_XZYS.pdf and http://hkumath.hku.hk/~mks/HistMathTeachingReadingListChinese.pdf (accessed 18/2/2016). T2b

An interesting list of books, collective volumes, and papers on the HM and on history in ME.

2014

Friedelmeyer, J-P., & Lubet, J-P. (Eds) *L'analyse algébrique – un épisode clé de l'histoire des mathématiques.* Paris: Ellipses. T2b

Original texts by Euler, Lagrange, Brisson, Servois and others, contextualized by appropriate comments and introductions.

Pengelley & Laubenbacher (see section 5). T2a, 2b, 4

A website with information and materials on using original historical sources in math teaching, including the authors' own experiences and materials, and those of others.

2015

Commission interIREM, Commission IREM de Basse-Normandie, Barbin, É., & Legoff, J-P. *Si le nombre m'était compté*. Paris: Ellipses. T2b

A useful resource for the generalization of the number concept beyond natural numbers

3.2 Individual books and papers

3.2.1 *Books and Doctoral Dissertations*

2001

Marshall, G. L. *Using history of mathematics to improve secondary students' attitudes towards mathematics*. Dissertation. Illinois State University, Illinois. T2a

2002

Pizzamiglio, P. *Matematica e storia. Per una didattica interdisciplinare*. Brescia: Ed. La Scuola. T5

van Amerom, B.A. (see section 5). T2, T1, T3a

2003

Scimone, A. *Pupils' conceptions about a historical open question: Goldbach's conjecture. The improvement of mathematical education from a historical viewpoint*. Dissertation. University of Bratislava, Bratislava. T1, 2a

2004

da Silva Souza, E. A prática social do cálculo escrito na formação de professores: A história como possibilidade de pensar questões do presente. Dissertation. UNICAMP, Campinas, Brazil. T2a

2005

Su (see section 5). T2a

van Gulik-Gulikers, I. *Een studie naar de waarde en de toepassing van de geschiedenis van de meetkunde in het wiskundeonderwijs*. Dissertation. Rijksuniversiteit Groningen, The Netherlands. T2a, 3a, 5

2006

Bagni, G.T. *Linguaggio, storia e didattica della matematica*. Bologna: Pitagora Editrice.

Clark (see section 5). T2a, 3a

Thomaidis, Y., Petrakis, Y., Touloumis, K., & Stafylidou M. *Language, history and Euclidean geometry*. Thessaloniki: University of Macedonia (in Greek). T2a, 2b, 4, 5

2007

Goodwin, D. M. *Exploring the relationship between high school teachers' mathematics history knowledge and their images of mathematics*. Dissertation. University of Massachussets, Lowell MA (accessed 18/2/2016). T3a, 2a

Knoebel, A., Laubenbacher, R., Lodder, J., & Pengelley, D. *Mathematical masterpieces – further chronicles by the explorers*. New York, NY: Springer. T2b

2008

de Jesus Brito, A. *A geometria de Euclides a Lobatschweski. Um esstudo histórico-pedagógico*.

Natal, RN, Brazil: Editora da UFRN. T2b

Filloy, E., Rojano, T., & Puig, L. *Educational algebra. A theoretical and empirical approach*. New York, NY: Springer. T1, 5, 2a

Haile, T. *A study on the use of history in middle school mathematics: The case of connected mathematics curriculum*. Dissertation. Faculty of the Graduate School of the University of Texas, Austin (accessed 18/2/2016). T3a,b, T1

Stedall, J. *Mathematics emerging: A sourcebook 1540 – 1900*. Oxford, UK: Oxford UP. T2b, 4

2009

Jankvist (2009a) (see section 5). T1, 2a

2010

Stein (see section 5). T2b

Wardhaugh, B. *How to read historical mathematics*. Princeton, NJ: Princeton University Press. T4

Glaubitz (see section 5). T2a, T4

2015

Shell-Gellasch, A., & Thoo, J. *Algebra in context: Introductory algebra from origins to applications*. Baltimore, MD: Johns Hopkins University Press. T2b

3.2.2 *Individual papers in Scientific Journals*

2000

Arcavi, A., & Bruckheimer, M. Didactical uses of primary sources from the history of mathematics. *Themes in Education*, *1*(1), 55-74. T4, T2

Barbin, É. Que faut-il enseigner? Pour qui? Pourquoi? Des réponses dans l'histoire des mathématiques, in *R-IREM*, *38*, 43-51. T2b

Barbin, É. Construire la géométrie élémentaire. *R-IREM*, *40*, 5-9. T1, 5

Barbin, É. Pourquoi démontrer? Les mathématiques grecques, *Cahiers de Science & Vie*, *55*, 42-49. T1

Bkouche, R. Sur la notion de perspective historique dans l'enseignement d'une science. *R-IREM*, *39*, 35-59. T1

Furinghetti, F. The history of mathematics as a coupling link between secondary and university teaching. *Int. J. ME Sci. Tech.*, *31*(1), 43-51. T1, 2

Horng, W-S. Nam Byung Gil's comments on the "Tian Yuan Shu" versus "Jie Gen Fang": An HPM perspective. *Chinese Journal of Science Education*, *8*(3), 215-224 (in Chinese). T2b

Li, B.C. A survey on mathematics teachers' knowledge about the history of mathematics. *MB*, *39*(3), 39-40. T3a

Matos, J. The historical development of the concept of angle (1). *TME*, *10*(2), 49-56. T2

Testa, G. L'enseignement des coniques à travers une approche historique: comment saisir un texte? *R-IREM*, *41*, 105-119. T4, T2

Tzanakis, C., & Thomaidis, Y. Integrating the close historical development of mathematics and physics in mathematics education: Some methodological end epistemological remarks. *FLM*, *20*(1), 44-55. T1, 5

2001

Barbin, É. Qu'est-ce que faire de la géométrie? *R-IREM*, *43*, 59-83. T1, T5, T4, T6

D'Ambrosio, U. What is ethnomathematics, and how can it help children in schools? *Teaching Children Mathematics*, *7*(6), 308. T6, T1

Filep, L. The development, and the developing of, the concept of a fraction, *International Journal for Mathematics Teaching and Learning*. Available from http://www.cimt.plymouth.ac.uk/journal/lffract.pdf (accessed 18/2/201). T2b

Fried (see section 5). T1

Friedelmeyer, J.-P. Grandeurs et nombres: l'histoire édifiante d'un couple fécond. *R-IREM*, *44*, 5-31. T2a

Furinghetti, F., & Somaglia, A. The method of analysis as a common thread in the history of algebra: reflections for teaching. *Themes in Education, 2*(1), 3-14. T2a, 4, 1

Jahnke, H.N. Cantor's cardinal and ordinal infinities: An epistemological and didactic view. *ESM*, *48*, 175-197. T1, 2

Lit, C-K., Siu, M. K., & Wong, N. Y. The use of history in the teaching of mathematics: Theory, practice, and evaluation of effectiveness. *Educational Journal*, *29*(1), 17–31. T2, 1

Mendez, E. P. A history of mathematics dialogue in textbooks and classrooms, *MT, 94*(3), 170-173. T2, 3b

Percival, I. An artefactual approach to ancient arithmetic. *FLM*, *21*(3), 16-21. T2a

Povey, H., Elliott, S., & Lingard, D. The study of the history of mathematics and the development of an inclusive mathematics: Connections explored. *MER*, *14,* 8-18. T2a, 1

van Gulik-Gulikers, I. & Blom, K. 'A historical angle', a survey of recent literature on the use and value of history in geometrical education. *ESM*, *47*(2), 223-258. T3

2002

Boyé, A., & Comairas, M.-C. Moyenne, médiane, écart-type, quelques regards sur l'histoire pour éclairer l'enseignement des statistiques au lycée. *R-IREM 48*, 27-40. T2b, 5

Nooney, K. A. Critical question: Why can't mathematics education and history of mathematics coexist? *TME*, *12*(1), 1-5. T1

Quinton, P. Activités mathématiques à propos de la mesure de la Terre. *R-IREM*, *49*, 73-92. T2b, a, 5

2003

Friedelmeyer, J.-P. Euclide peut-il encore apprendre quelque chose au professeur de mathématiques d'aujourd'hui ? *R-IREM*, *53*, 23-42. T2b, 4

Furinghetti, F., & Paola, D. (2003). History as a crossroads of mathematical culture and educational needs in the classroom. *Mathematics in School, 37*(1), 37-41. T2b,a, 1

Glaubitz, M., & Jahnke, H. N. Die Bestimmung des Umfangs der Erde als Thema einer mathematikhistorischen Unterrichtsreihe, *Journal für Mathematik-Didaktik*, *24*(2), 71-95. T4

Glaubitz, M., & Jahnke, H. N. Texte lesen und verstehen. Eine Erwiderung. *Journal für Mathematik-Didaktik*, *24* (3/4), 252-260. T4

Guichard, J.-P. D'un problème de Diophante aux identités remarquables. *R-IREM*, *53*, 5-19. T2b, 4, 6

Jahnke, H. N. Numeri absurdi infra nihil. Die negativen Zahlen. *Mathematik lehren*, *121*, Dez. 2003, 21-22; 36-40. T2b

Liu (see section 5). T1

Ransom, P. Drawing instruments – their history and classroom use. *BSHM*, *47*, 53-56 & *48*, 45-48. T2b

Streefland, L. Learning from history for teaching in the future. *ESM*, *54*, 37–62. T1, 5, 2a

Tournès, D. Du compas aux intégraphes: Les instruments du calcul graphique. *R-IREM*, *50*, 63-84. T2b

van Amerom, B. A. Focusing on informal strategies when linking arithmetic to early algebra. *ESM*, *54*, 63–75. T2a, T4

van Gulik-Gulikers, I. The seventeenth-century surveyor in class. *BSHM*, *47*, 56-63. T2a, 4

Wang, X.Q., & Ouyang, Y. Historical notice on HPM. *JME*, *12*(3), 24-27. T1

2004

Barbin, É. L'outil technique comme théorème en acte. *Sciences et avenir*, *140***,** 26-27. T1

Grattan-Guiness 2004a (see section 5). T1

Grattan-Guiness 2004b (see section 5). T1

Keiser, J. M. Struggles with developing the concept of angle: Comparing sixth-grade students' discourse to the history of the angle. *MTL*, *6*(3), 285-307. T2a, T1

Li, H. T. The new horizon of the curriculum reform: The history of mathematics entering the new mathematics curriculum. *CTMM*, *25*(9), 51-64. T1

Lopez-Real, F. Using the history of mathematics as a starting point for investigations: some examples on approximations. *Teaching Mathematics Applications*, *23*(3), 133-147. T2b

Mercier J.-P. Le problème des cinq carrés ou comment montrer l'intérêt des identités remarquables. *R-IREM*, *57*, 47-67. T2b, 4, 6

Merker, C. La méthode des indivisibles racontée lors d'un stage. *R-IREM*, *54*, 57-76. T2b, 4

Métin, F. Avez-vous lu Euclide. *R-IREM*, *55*,101-103. T2b

Michalowicz, K., & McGee, R. Using historical problems in the middle school. *Convergence*, *1* (accessed 18/2/2016). T2b, 4

Swetz, F. J. Using problems from the history of mathematics, *Convergence*, *1* (accessed 18/2/2016). T2b, T6, T4

2005

Burn, B. The vice: Some historically inspired and proof-generated steps to limits of sequences. *ESM*, *60*, 269-295. T2b

Carroget, L., & Gairin C. Les Mathématiques sans ignorer nos anciens. *R-IREM*, *58*, 5-15. T2a, 1

Dubinsky, E., Weller, K., McDonald, M. A., & Brown, A. Some historical issues and paradoxes regarding the concept of infinity: An APOS-based analysis: Part 1. *ESM*, *58*(3), 335-359. T1

Durand-Guerrier, V., & Arsac, G. An epistemological and didactic study of a specific calculus reasoning rule. *ESM*, *60*(2), 149 – 172. T2a, 5, 1

Su, Y. W. Designing HPM work sheets. *Taiwan Journal of Mathematics Teachers*, *2*, 2-13 (in Chinese). T2b

Wang, X. Q. The Historical-genetic principle viewed from a test on the imaginary numbers and divergent series. *JME*, *14*(3), 30-33. T3b

2006

Bakker A., & Gravemeijer, K. P. E. An historical phenomenology of mean and median. *ESM*, *62*(2), 149-168. T1, 2a

Barbin, É. Apports de l'histoire des mathématiques et de l'histoire des sciences dans l'enseignement. *Tréma*, *26*, 20-28. T1, 5

Liu, P-H. & Niess, M. L. An exploratory study of college students' views of mathematical thinking in a historical approach calculus course. *MTL*, *8*(4), 373-432. T2a

Martínez-Torregrosa, J., López-Gay, R., & Gras-Martí, A. Mathematics in physics education: Scanning historical evolution of the differential to find a more appropriate model for teaching differential calculus in physics. *S&E*, *15*(5), 447-462. T5

Savizi, B. Applicable problems in history of mathematics; practical examples for the classroom. *Philosophy of Mathematics Education Journal, 19* (accessed 18/2/2016). Also in *Teaching Mathematics and its Applications*, *26* (2007), 51-54. T1, 2b

Thiénard, J.-C. Les transformations en géométrie, introduction à une approche historique. *R-IREM*, *63*, 27-52. T2b, 4

Tillema, E. Chinese algebra: Using historical problems to think about current curricula. *MT*, *99*(4), 238-245. T2, 6

Wang, X. Q., & Zhang, X. M. Researches on HPM: Contents, methods and examples. *JME*, *15*(1), 16-18. T1

2007

Barbin, É. L'arithmétisation des grandeurs. *R-IREM*, *68*, 5-20. T2b, 5

Barbin, É. Les récréations: des mathématiques à la marge. *Pour la science*, *30*, 22-25. T3

Barbin, É. Les avatars de la rigueur mathématique. *Pour la science*, *356*, 10-13. T1

Carson, R. N., & Rowlands, S. Teaching the conceptual revolutions in geometry. *S&E*, *16*(9-

10), 921-954. T2b, 1

Demattè, A. (2007). Primary sources from the history of mathematics for secondary school students. *Acta Didactica Universitatis Comenianae Mathematics, 7*, 47-66. T2b, 4, 1

Farmaki V., & Paschos, Th. Employing genetic 'moments' in the history of mathematics in classroom activities. *ESM*, *66*(1), 83-106. T2a, 5

Gérini ,C., & Verdier, N. Les Annales de Gergonne (1810-1832) et le Journal de Liouville (1836-1874): Une mine de textes numérisés à exploiter dans notre enseignement. *R-IREM*, *67*, 55-68. T2b

Jankvist (see section 5). T3a

Li, T. A., & Song, N. Q. Teaching of analytic geometry from the historical perspective. *JME*, *16*(2), 90-94. T1

Meavilla, V., & Flores, A. History of mathematics and problem solving: A teaching suggestion. *Int. J. ME Sci. Tech.*, *38*(2), 253-259. T2

Ren, M. J., & Wang, X. Q. Senior high school students' understanding about the mathematical function. *JME*, *16*(4), 84-87. T3b

Thienard, J.-C. Redonner du sens aux mathématiques enseignées. *R-IREM*, *66*, 62-72. T1, 5

Thienard, J.-C. Introduction aux lois de probabilités continues: Problèmes épistémologiques. *R-IREM*, *67*, 31-42. T2b, 5, 4

Yang, Y. H., Wei, J., & Song, N. Q. An analysis of historical materials in primary mathematics textbooks. *JME*, *16*(4), 80-83. T3b

2008

Bagni, G. T. A theorem and its different proofs: History, mathematics education and the semioticcultural perspective. *Canadian Journal of Science, Mathematics, and Technology Education*, *8*(3), 217-232. T5, 1

Bühler, M., & Michel-Pajus, A. Les démonstrations en arithmétique: à propos de quelques preuves historiques du petit théorème de Fermat. *R-IREM*, *71*, 23-39. T2b, 4

Clark, K. M., & Robson, E. Ancient accounting in the modern mathematics classroom. *BSHM, 23*(3), 129-142. T2

Doorman, M., & van Maanen, J. A historical perspective on teaching and learning calculus. *Australian Senior Mathematics Journal*, *22*(2), 4-14. T2

Fried, M. N. History of mathematics in mathematics education: A saussurean perspective. *TMMEM*, *5*(2), 185-198. T1, 2

Gérini, C. Variations pédagogiques sur un article de géométrie analytique d'Haton de la Goupillière, paru en 1872. *R-IREM*, *71*, 65-80. T2b,a, 4

Jahnke, H. N., & Richter, K. Geschichte der Mathematik. Vielfalt der Lebenswelten – Mut zu divergentem Denken. *Mathematik lehren*, *151*, Dez. 2008, 4-11. T1

Jamm, F. Le rêve de Ptolémée réalisé. *R-IREM*, *73*, 5-19. T5, 2a, b, 4

Jankvist, U. T. A teaching module on the history of public-key cryptography and RSA. *BSHM*, *23*(3), 157-168. T2

Jankvist, U. T. Den matematikhistoriske dimension i undervisning—gymnasialt set (The dimension of the history of mathematics in teaching—the case of upper secondary level). *MONA*, *4*(1), 24-45 (in Danish). T2, 1, 4

Zhao, Y. Y., & Zhang, X. M. Discussions about the Historical Parallelism: an Overview. *JME*, *17*(4), 53-56. T1

2009

Charalambus, C., Panaoura, A., & Philippou, G. Using the history of mathematics to induce changes in preservice teachers' beliefs and attitudes: insights from evaluating a teacher education program. *ESM*, *71*(2), 161-180. T2

Henry, M. Emergence de la probabilité et enseignement: Définition classique, approche fréquentiste et modélisation. *R-IREM*, *74,* 76-89. T2b, 5, 4

Jankvist 2009b (see section 5). T1

Jankvist, U. T. On empirical research in the field of using history in mathematics education. *Revista Latinoamericana de Investigatión en Matemática Educativa*, *12*(1), 67-101. T1, 2a

Jankvist, U. T. History of modern applied mathematics in mathematics education. *FLM*, *29*(1), 8-13. T2

Kang, S. G., & Hu, G. H. Researches on the history of mathematics and mathematics education in primary and secondary schools in China: Analysis and reflections. *JME*, *18*(5), 65-68. T3b

Kjeldsen, T. H., & Blomhøj, M. Integrating history and philosophy in mathematics education at university level through problem-oriented project work. *ZDM*, *41*, 87–103. T1,2a,5

Lin, M. S. &, Su, Y. W. Letting mathematics being interesting by history of mathematics. *The Educator Monthly*, *509*, 81-83 (in Chinese). T2a

Liu, P-H. History as a platform for developing college students' epistemological beliefs of mathematics. *Int. J. Sci. ME, 7*(3), 473-499. T2, 1

Mayfield, B., & Kimberly, T. A Locally Compact REU in the History of Mathematics: Involving Undergraduates in Research. *Convergence*, *6*. doi: 10.4169/loci003263. T2a

Shen, J. L., & Su, Y. W. Integrating history of mathematics into elementary teaching. *New Horizon Bimonthly for Teachers in Taipei*, *163*, 70-77 (in Chinese). T2a

Zhang, X. M., & Wang X. Q. Integrating the history of mathematics into mathematics teaching in senior high schools: An action research*.* *JME*, *18*(4), 89-92. T2b

2010

Barbin, É. Epistémologie et histoire dans la formation mathématique. *R-IREM*, *80*, 74-86. T1, 5

Bartolini Bussi, M. G., Taimina D., & Isoda, I. Concrete models and dynamic instruments as early technology tools in classrooms at the dawn of ICMI: From Felix Klein to present applications in mathematics classrooms in different parts of the world. *ZDM*, *42*(1), 19-31. T2

Bressoud, D. M. Historical reflections on teaching trigonometry. *MT*, *104*(2), 106-112. T2b

Clark, K. M. Connecting local history, ancient history, and mathematics: The Eustis Elementary School pilot project. *BSHM*, *25*(3), 132-143. T2

da Conceição Ferreira Reis Fonseca, M. Adult education and ethnomathematics: appropriating results, methods, and principles. *ZDM*, *42*(3), 361-369. T2, 6, 1

Jankvist, U. T. An empirical study of using history as a 'goal'. *ESM, 74*, 53-74. T2a

Lih, K. W. A remarkable Euler square before Euler. *Mathematics Magazine, 83*, 163-167. T3b

Lin, F. M., & Horng, W. S. *The housekeeper and the professor* by Yoko Ogawa. *Mathematical Intelligencer*, *32*(2), 75-76. T6

Papadopoulos, I. "Reinventing" techniques for the estimation of the area of irregular plane figures: From the eighteenth century to the modern classroom. *Int. J. Sci. ME*, *8*(5), 869-890. T2

Rowlands, S. A pilot study of a cultural-historical approach to teaching geometry. *S&E*, *19*(1), 55-73. T2a

Tan, X. Z. Spiritual heritage and educational value of Greek mathematical culture. *JME*, 19(1): 27-29. T3a

Tatar, E., Tevfik I., Yasin S., & Levent A. Teaching square root numbers: from the perspective of Ibrahim Hakki of Erzurum. *Procedia – Social and Behavioural Sciences*, *2*(2), 1102-1106. T2b

Xu, Z. T., Wang, X. Q., & Mei Q. X. The genetic approach to teaching and learning mathematics: From the history to the classroom. *JME*, *19*(1), 10-12. T1

Zhu, F. Q., & Xu, B. H. The modes of integrating the history of mathematics into teaching: an international perspective. *JME*, *19*(3), 22-25. T1

2011

Heeffer, A. Historical objections against the number line. *S&E*, *20*(9), 863-880. T1

Jankvist, U. T. Anchoring students' metaperspective discussion of history in mathematics. *JRME*, *42*(4), 346-385. T1, 2a

Jankvist & Kjeldsen (see section 5). T1, 2a, 5

Klyve, D., Dtemkoski, L., & Tou, E. Teaching and research using original sources from the Euler archive, *Convergence*, *8*. doi:10.4169/loci003672. T2b

Panagiotou, E. Using history to teach mathematics: The case of logarithms. *S&E*, *20*, 1-35. T2a

Pu, S. P., & Wang, X. Q. Hans Freudenthal's thought on HPM and its implications. *JME*, *20*(6), 20-24. T1

Shih, Y. K., & Su, Y. W. Integrating digital materials of history of mathematics into elementary teaching. *National Education*, *52*(3), 65-71. T2a

Schubring (see section 5). T1

Su, Y. W. An initial investigation into the design of materials of history of mathematics for elementary school education. *Research and Development in Science Education Quarterly*, *62*, 75-96 (in Chinese). T2b

Wang, X. Q., Wang, M., & Zou, J. C. Teaching mathematics from the HPM perspective: The case of ellipse. *JME*, *20*(5), 20-23. T2b

2012

Chan, Y. C., & Siu, M. K. Facing the change and meeting the challenge: Mathematics curriculum of Tongwen Guan in China in the second half of the nineteenth century. *ZDM*, *44*(4), 461-472. T6

Chevalarias, N., & Minet, N. Des séances “Maths-Histoire” en classe de seconde. *R-IREM*, *86*, 5-25. T2a, b 4

Clark, K. M. History of mathematics: illuminating understanding of school mathematics concepts for prospective mathematics teachers. *ESM*, *81*(1), 67-84. T2

Fernandez, J. M. L., & Rodriguez, O. A. H. Teaching the fundamental theorem of calculus: A historical reflection. *Convergence*, *9*. doi:10.4169/loci003803. T2b

Kjeldsen & Blomhøj (see section 5). T1, 2a, 5

Li, G.Q., & Xu, L.H. Levels of mathematics teachers’ literacy in the history of mathematics based on the theory of SOLO. *JME*, **20**(1), 34-37. T1

Liu, Z. D., Wang, Q. J., & Shao, R. Students’ motivation to learn mathematics: the historical perspective. *JME*, *20*(1), 30-33. T3b

Pu, S. P., & Wang, X. Q. How to integrate the history of mathematics into mathematics textbooks: the cases of Chinese and French mathematics textbooks. *CTMM*, *32*(8), 63-68. T3b

Pu, S. P., & Wang, X. Q. Junior high school students’ understanding about the letters: An empirical study of the historical parallelism. *JME*, *20*(3), 38-42. T3b

Shih, Y. K., & Su, Y. W. Integrating digital materials of history of mathematics into elementary teaching. *National Education*, *523*(3), 65-71 (in Chinese). T2a

Swetz, F. J. Similarity versus the “out-in complementary principle”: An example of a cultural faux pas in mathematical understanding. *Mathematics Magazine*, *85*(1), 3-11. T6, 4

Wang, X. Q. The history of mathematics in French junior high school mathematics textbooks. *MB*, *51*(3), 16-20. T3b

2013

Bernard, A. Résoudre un problème par l’algèbre sans en perdre le sens: sur les traces de Diophante d’Alexandrie. *R-IREM*, *92*, 59-74. T2b, 4

Branson, W. B. Solving the cubic with Cardano. *Convergence*, *10* (accessed 18/2/2016). T2a, 2b, 4

Diamantopoulos, J., & Woodburn, C. Maya geometry in the classroom. *Convergence*, *10.* (accessed 18/2/2016). T2b, 6

Frejd, P. Old algebra textbooks: a resource for modern teaching. *BSHM*, *28*(1), 25-36. T3b

He, B. T., & Wang, X. Q. Senior high school students’ misconceptions of the tangent lines. *JME*, *21*(6), 45-48. T3b

Jahnke, H. N., & Wambach, R. Understanding what a proof is: a classroom-based approach. *ZDM*, *45*(3), 469-482. T2

Jankvist (see section 5). T4, 5, 2

Kjeldsen, T. H., & Blomhøj, M. Developing students' reflections about the function and status of mathematical modelling in different scientific practices: History as a provider of cases. *S&E*, *22*(9), 2157-2171. T1, 2a, 5

Liu, Z., & Yang, G. W. A comparative study on historical materials in junior high school textbooks of China and Japan. *HSMM*, *10*, 28-30. T3b

Moyon, M., & ERR. Diviser en multipliant les approches... Quand les mathématiques remontent aux sources, *R-IREM*, *93*, 47-77. T2a, b 4, 6

Wu, J., & Huang, Q. Y. The understanding of mean, median and mode based on the history of mathematics. *MB*, *52*(11), 16-21. T3b

Xie, Y. M. Historical materials in senior mathematics textbooks in Hong Kong: Characteristics and implications. *JME*, *21*(2), 67-70. T3b

Wu, J., & Wang, X.Q. Integrating the history of mathematics into mathematics teaching: Some foreign experience. *CER*, *8*, 78-82. T1

Zhang, X. G., & Zhang, X. A review on using history in mathematics teaching and learning abroad. *JME*, *21*(4), 43-46. T3b

2014

Barnett et al. (see section 5). T4, 1, 2

Clark, K. M., & Thoo, J. B. Introduction to the special issue on the use of history of mathematics to enhance undergraduate mathematics instruction. *PRIMUS*, *24*(8), 663-668. T1

Fenaroli, G., Furinghetti, F., & Somaglia, A. Rethinking mathematical concepts with the lens of the history of mathematics: an experiment with prospective secondary teachers. *S&E, 23*, 185-203. T2a, 1

Goodwin, D., Bowman, R., Wease, K., Keys, J., Fullwood, J., & Mowery, K. Exploring the relationship between teachers' images of mathematics and their mathematics history knowledge. *Philosophy of Mathematics Education Journal*, *28*, 1-15. T3a, 2a

Jankvist, U. T. A historical teaching module on "the unreasonable effectiveness of mathematics": Boolean algebra and Shannon circuits. *BSHM*, *29*(2), 120-133. T2

Jankvist, U. T. The use of original sources and its potential relation to the recruitment problem. *FLM*, *34*(3), 8-13. T4, 2a

Kjeldsen, T. H., & Petersen, P. H. Bridging history of the concept of a function with learning of mathematics: Students' meta-discursive rules, concept formation and historical awareness. *S&E*, *23*(1), 29-45. T2a, 1, 4

Smestad, B., Jankvist, U. T., & Clark, K. Teachers' mathematical knowledge for teaching in relation to the inclusion of history of mathematics in teaching. *NOMAD*, *19*(3-4), 169-183. T2a

Su, J. H., & Ying, J. M. Enhancing in-service mathematics teachers' professional knowledge with an HPM approach as observed in teachers' reflections. *TJME*, *1*(1), 79-97. T2a

Su, Y. W., Huang, J. W., Chen, C. H., & Lin, M. Y. Enhancing teachers' professional development through HPM script design (in Chinese). *TJME*, *1*(2), 25-52. T3a

Wu, J., & Wang, X. Q. A review on the practice of integrating the history of mathematics into mathematics teaching. *MB*, *53*(2), 13-16. T2b

Wu, J., & Zhao, R. A survey on teachers' statistics knowledge for teaching from the perspective of HPM. *MB*, *53*(5), 15-18. T3a

Ying, J. M. Mathematical canons in practice: The case of a nineteenth-century Korean scholar Nam Pyŏng-Gil and his evaluation of two major algebraic methods used in East Asia. *East Asian Science, Technology and Society: An International Journal, 8*(3), 347-362. T3b

2015

Fried, M. N., & Jahnke, H. N. Otto Toeplitz's 1927 paper on the genetic method in the teaching of mathematics. *Science in Context*, *28,* 285-295. T1

Fried, M. N., & Jahnke, H. N. (transl.) Otto Toeplitz, The problem of university courses on infinitesimal calculus and their demarcation from infinitesimal calculus in high schools. *Science in Context*, *28*, 297-310. T1

Jankvist, U. T. Changing students' images of "mathematics as a discipline". *JMB*, *38*, 41-56. T2a

Jankvist, U. T., Mosvold, R., Fauskanger, J., & Jacobsen, A. Analysing the use of history of mathematics through MKT. *Int. J. ME Sci. Tech.*, *46*(4), 495-507. T1, 2

Kjeldsen T. H., & Lützen J. Interactions between mathematics and physics: The history of the concept of function – teaching with and about nature of mathematics. *S&E*, *24*(5-6), 543-559. T1, 2a, 5

Lim, S. Y., & Chapman, E. Effects of using history as a tool to teach mathematics on students' attitudes, anxiety, motivation and achievement in grade 11 classrooms. *ESM*, *90*(2), 189-212. T2a

Lombard, P. L'invention du zéro ou la revanche des bergers. *R-IREM*, *101*, 33-44. T6

Pont, J.-C. À propos de l'introduction des nombres négatifs à l'école secondaire. *R-IREM 101*, 69-86. T2

Petrocchi, A. A new theoretical approach to sample problems and deductive reasoning in Sanskrit mathematical texts. *BSHM*, *30*(1), 2-19. T6

Pu, S. P., & Wang, X. Q. Mathematics teachers' professional development promoted by HPM: researches and implications. *JME*, *23*(3), 76-80. T4

Swetz, F. J. Pantas' Cabinet of mathematical wonders: Images and the history of mathematics, *Convergence*, *12* (accessed 18/2/2016). T2b, 4

Xu, N. N., Kong, F. Z., & Liu, P. F. A study on the presentation of history of mathematics in senior high school mathematics textbooks. *JME*, *23*(2), 61-65. T3b

Ying, J. M., Huang, J. W., & Su, Y. W. An exploratory study on influences of a mathematical culture course on university students' mathematics beliefs – the case in a medical university. *TJME*, *2*(2), 1-24. T2a

Zhong, P., & Wang, X. Q. The teaching of the concept of logarithm from the HPM perspective. *HSMM*, *5*, 50-53. T2a

2016

Barnett, J., Lodder, J., & Pengelley, D. Teaching and learning mathematics from primary historical sources. *PRIMUS*, *26*(1), 1-18.T4, 1, 2

To Appear

Barnett, J., Bezhanishvili, G., Lodder, J., & Pengelley, D. Teaching discrete mathematics entirely from primary historical sources. *PRIMUS*. T4, 2

3.2.3 *Individual papers in Collective Volumes*

2000

Dennis, D. The role of historical studies in mathematics and science educational research. In D. Lesh & A. Kelly (Eds.). *Research design in mathematics and science education* (pp. 799-819). Mahwah, NJ: Lawrence Erlbaum (accessed 18/2/2016). T1, T3a, T2

2001

Radford, L. The historical origins of algebraic thinking. In R. Sutherland et al. (Eds.), *Perspectives on school algebra* (pp.13–36). Dordrecht: Kluwer. T1

Testa, G. Per un avvio alla ricerca "storica" in campo scientifico: studential lavoro. *L' insegnamento della matematica e delle scienze integrate*, *24B*, 39-68. T2a

2002

Furinghetti, F., & Radford, L. Historical conceptual developments and classroom learning in mathematics: from phylogenesis and ontogenesis theory to classroom practice. In L. English (Ed.), *Hand Int RME* (pp. 631–654). Mahwah, NJ: Lawrence Erlbaum Associates. T1, 3a

van Maanen, J. Research on history in mathematics education in the Netherlands: The 'Reinvention Studies'. In F. L. Lin (Ed.), *Common sense in mathematics education* (pp. 191–201). National Taiwan Normal University, Taipei. T3

2003

Boyé, A. La musique au carrefour des mathématiques, des sciences et des arts. In É. Barbin (Ed.), *La pluridisciplinarité dans les enseignements scientifiques, Histoire des sciences*. Caen: CRDP Basse-Normandie. T5

Jahnke, H. N. Texte von Studierenden zur Geschichte der Mathematik. In L. Hefendehl-Hebeker & S. Hußmann (Eds.), *Mathematikdidaktik zwischen Fachorientierung und Empirie. Festschrift für Norbert Knoche* (pp. 105-116). Hildesheim: Franzbecker. T4

2004

Bagni, G., Furinghetti, F., & Spagnolo, F. History and epistemology in mathematics education:

2000-2003. In L. Cannizzato, A. Pesci, & O. Robutti (Eds.), *Italian research in mathematics education: 2000-2003* (pp. 207-221). Milano: Ghisetti & Corvi (accessed 18/2/2016). T3a

Puig, L., & Rojano, T. The history of algebra in mathematics education. In K. Stacey et al. (Eds.), *The future of the teaching and learning of algebra. The 12th ICMI study* (pp. 189–223). Dordrecht: Kluwer. T1

2005

Grattan–Guinness I. History or heritage? An important distinction in mathematics and for mathematics education. In G. van Brummelen & M. Kinyon (Eds.), *Mathematics and the historian's craft* (pp. 7–21). New York, NY: Canadian Mathematical Society & Springer. T1 (identical to Grattan–Guinness 2004b).

2006

Fischer, W. L. Historical Topics as indicators for the existence of fundamentals in educational mathematics. In F. K. S. Leung, K.-D. Graf, & F. J. Lopez-Real (Eds.), *Mathematics education in different cultural traditions. A comparative study of East Asia and the West. The 13th ICMI study* (pp. 95-110), New York, NY: Springer. T6

2007

D'Ambrosio, U. Mathematical marginalisation and meritocracy: Inequity in an English classroom. In S. Barath (Ed.), *Peace, social justice and ethnomathematics*, TMMEM vol.1 (pp. 25-34). Charlotte, NC: Information Age Publishing. T6

Heeffer, A. Learning concepts through the history of mathematics, In K. François & J. P. van Bendegem (Eds.). *Philosophical dimensions in mathematics education* (pp. 83-103). New York, NY: Springer (accessed 18/2/2016). T5, 1

Pinxten, R., & François, K. Ethnomathematics in practice, In K. François & J. P. van Bendegem (Eds.). *Philosophical dimensions in mathematics education* (pp. 213-227). New York, NY: Springer (accessed 18/2/2016). T6, 1

2008

Clark, K. M. Heeding the call: History of mathematics and the preparation of secondary mathematics teachers. In F. Arbaugh & P. M. Taylor (Eds.), *Inquiry into mathematics teacher education: Association of Mathematics Teacher Educators (AMTE)* (pp. 85-95). San Diego, CA: Association of Mathematics Teacher Educators. T2

Furinghetti, F., & Radford, L. Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics. In L. English (Ed.), *Hand Int RME*, 2nd Edition (pp. 626–655). New York, NY: Routledge, Taylor and Francis (accessed 18/2/2016). T1, T3a

2009

Jankvist, U. T. A teaching module on the early history of error correcting codes. In I. Witzke (Ed.), *18th Novembertagung on the history, philosophy and gidactics of mathematics* (pp. 153–163). Berlin: Logos Verlag. T2

Swetz, F. J. Culture and the development of mathematics: An historical perspective. In B. Greer et al. (Eds.), *Culturally responsive mathematics education* (pp. 11-41). New York, NY: Routledge. T6

Swetz, F. J. Word problems: Footprints from the history of mathematics. In L.Verschaffel et al. (Eds.), *Words and worlds: Modelling verbal descriptions of situations* (pp. 73-91). Rotterdam: Sense Publishers. T2b, 4

2010

Costabile, F., & Serpe, A. Archimedes in secondary schools: A teaching proposal for the math curriculum. In S. A. Paipetis & M. Ceccarelli (Eds.). *The genius of Archimedes: 23 centuries of influence on mathematics, science and engineering*, History of Mechanism and Machine Science, vol. 11 (pp. 479–491). Netherlands: Springer. T2

2011

Jahnke, H. N. The conjoint origin of proof and theoretical physics. In M. Pitici (Ed.), *The best writing on mathematics 2011* (pp. 236-256). Princeton, NJ: Princeton University Press. T2

2012

Furinghetti (see section 5). T3a, T1

Siu, M. K. Proof in the western and eastern traditions: Implications for mathematics education. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education: The 19th ICMI study* (pp.431-440). New York, NY: Springer. T1, 6, 5

2013

Barbin (see section 5). T3a, b, 1, 4

Wang, X. Q. Research on HPM: An overview. In J. S. Bao & B. Y. Xu (Eds.), *Guide to the research to mathematical education II* (pp. 403-422). China: Jiangsu Education Publishing House. T1, 2, 3, 4

2014

Barbin & Tzanakis (see section 5). T3a

Clark, K. M. History of mathematics in mathematics teacher education. In M. R. Matthews (Ed.), *Int. Hand. Res. Hi. Phil. Sci. Teach.*, Volume 1, (pp. 755-791). Dordrecht: Springer. T1, 2

Fried, M. N. History of mathematics and mathematics education. In M. R. Matthews (Ed.) *Int. Hand. Res. Hi. Phil. Sci. Teach.*, Volume 1, (pp. 669-703). Dordrecht: Springer. T1

Fried, M. N. Mathematicians, historians of mathematics, mathematics teachers, and mathematics education researchers: The tense but ineluctable relations of four communities. In M. N. Fried & T. Dreyfus (Eds.). *Mathematics & mathematics education: Searching for common ground* (pp. 94-98). Dordrecht: Springer. T1

Jahnke, H. N. History in mathematics education. A hermeneutic approach. In M. N. Fried & T. Dreyfus (Eds.), *Mathematics & mathematics education: Searching for common ground* (pp. 75-88). Dordrecht: Springer. T1, 4

Jankvist, U. T. On the use of primary sources in the teaching and learning of mathematics. In M. R. Matthews (Ed.), *Int. Hand. Res. Hi. Phil. Sci. Teach.*, vol. 1, (pp. 873-908). Dordrecht: Springer. T4

Kjeldsen T. H. & Carter, J. The role of history and philosophy in university mathematics education. In M. R. Matthews (Ed.), *Int. Hand. Res. Hi. Phil. Sci. Teach.*, vol. 1, (pp. 837-871). Dordrecht: Springer. T1

Radford, L. et al. History of mathematics and mathematics education. In M. N. Fried & T. Dreyfus (Eds.), *Mathematics & mathematics education: Searching for common ground* (pp. 89-109). New York, NY: Springer. T1, 2

Rogers, L. History of mathematics in and for the curriculum. In H. Mendick & D. Leslie (Eds.), *Debates in mathematics education* (pp. 106-122). London: Routledge. T3

2015

Siu, M. K. "*Zhi yì xíng nán* (knowing is easy and doing is difficult)" or vice versa? – A Chinese mathematician's observation on HPM (History and Pedagogy of Mathematics) activities. In B. Sriraman, J. F. Cai, K. Lee, L. Fan, Y. Shimuzu, C. Lim, K. Subramanian (Eds.), *The first sourcebook on Asian research in mathematics education* (pp. 27-48). Charlotte, NC: Information Age Publishing. T3a, 6

3.2.4 *Individual papers in Proceedings of Conferences*

2000

Fried, M. N. Some difficulties in incorporating history of mathematics in teaching-training. In A. Ahmed, J. M. Kraemer, & H. Williams (Eds.), *Cultural diversity in mathematics (education): CIEAEM 51* (pp. 73-78). Chichester, UK: Horwood Publishing. T1

2002

Kronfellner, M. A genetic approach to axiomatics. In I. Vakalis, D. Hughes-Hallett, Ch. Kourouniotis, D. Quinney, & C. Tzanakis (Eds.), *Proc. of the 2nd ICTM* (e-book). New York, NY: J. Wiley & Sons. T2b

Safuanov, I. S. Design of genetic teaching of algebra at universities. In I. Vakalis, D. Hughes-Hallett, Ch. Kourouniotis, D. Quinney, & C. Tzanakis (Eds.), *Proc. of the 2nd ICTM* (e-book). New York, NY: J. Wiley & Sons. T2b

2004

Jahnke, H. N. Historical sources in the mathematics classroom: ideas and experiences. In H. Fujita, Y. Hashimoto, B. R. Hodgson, P. Y. Lee, S. Lerman, & T. Sawada (Eds.). *Proc. of the 9th ICME* (pp. 136-138). Dordrecht: Kluwer Academic Publishers. T4

2005

Mason, R., & Janzen Roth, E. Thinking like Archimedes: An instructional design experiment. In *Proc. of the 8th international congress of the history, philosophy, sociology and science teaching*, Leeds, UK. (accessed 10/10/ 2015); abstract in http://www.mern.ca/abstract.asp?id=23 (accessed 18/2/ 2016)

2006

El Idrissi, A. (2006). L'histoire des mathématiques dans les manuels scolaires. In N. Bednarz & C. Mary (Eds.), *Actes du Colloque EMF2006* (in CD-Rom). Sherbrooke: Faculté d'Éducation. T3b

Fried, M. N. & Bernard, A. History of mathematics in the curriculum and the possibility of a new Paideia. In F. Spagnolo & B. Di Paola (Eds.), *Proc. of CIEAEM 57: Changes in society: A challenge for mathematics education,* (pp. 217-220). Piazza Armerina, Sicily. T1, 3

Heeffer, A. The methodological relevance of the history of mathematics for mathematics education. In G. Dhompongsa, F. M. Bhatti, & Q. C. Kitson (Eds.) *Proc. of the international conference on 21st century information technology in mathematics education* (pp. 267-276). Chang Mai, Thailand. (accessed 18/2/2016) T1

Rowlands, S., & Carson, R. Proof, reason, abstraction and leaps: A cultural-historical approach to teaching geometry In D. Hewitt (Ed.), *Proc. of the British Society for Research into Learning Mathematics, 26*(1), 71-76 (accessed 18/2/2016). T1, 2b

Schubring (see section 5). T1

2009

Montelle, C., & Clark, K. M. Using history to enrich undergraduate mathematics: Beyond the anecdote. In D. Wessels (Ed.), *Seventh southern right Delta Conference on the teaching and learning of undergraduate mathematics and statistics* (pp. 185-195). Stellenbosch, South Africa: DELTA. T1, 2

2010

Kjeldsen, T. H. History in mathematics education – why bother? Interdisciplinarity, mathematical competence and the learning of mathematics. In B. Sriraman & V. Freiman (Eds.), *Interdisciplinarity for the 21st century: Proc. of the 3rd international symposium on mathematics and its connections to arts and sciences* (pp. 17-48). *TMMEM*, vol. 11. Charlotte, NC: Information Age Publishing. T1, 2a, 5

2011

Boyé, A., & Moussard, G. L'enseignement des vecteurs au XX^{e} siècle: Diversité des héritages mathématiques et circulation entre disciplines. In IREM Basse-Normandie, (Eds.). *Circulation, Transmission, Héritage* (pp. 219-238). Caen: IREM de Basse-Normandie. T2b

2012

Rogers, L. (2012). Practical mathematics in 16th century England: Social-Economic contexts and emerging ideologies in the new Common Wealth. In K. Bjarnadσttir, F. Furinghetti, J. Matos, & G. Schubring (Eds.), *"Dig where you stand" 2. Proc. of the conference on the History of Mathematics Education* (pp. 421-444). Universidade Nova: Lisbon. T6

2013

Barbin, É. L'histoire des mathématiques dans la formation: une perspective historique (1975-2010). In J.-L. Dorier (Ed.), *Actes du Colloque Espace Mathématique Francophone* (pp. 546-554). University of Genève: Genève. T2b

2015

Smestad, B. Uses of history of mathematics in school (pupils aged 6–13). In S. J. Cho (Ed.) *The proceedings of the 12th international congress on mathematical education: Intellectual and attitudinal challenges* (pp. 601-603). New York, NY: Springer. T2a

4 Concluding remarks

The *HPM perspective* described in §1.1 emerged gradually over the last decades as a perception of mathematics worth exploring, thanks to research and teaching work done worldwide, thus establishing the *HPM domain* as a valuable research area in the context of ME. Launching the ICMI Study volume in 2000 was a decisive step in this direction. This highly collective work motivated, stimulated, oriented, encouraged and supported research in this area, to a large extent realized in the context of the *HPM Group* and the main activities related to it. At that time central issues were (and still are):

- To put emphasis on *pre- and in-service teacher education* as a necessary prerequisite for the *HPM perspective* to be possible at all.
- To design, produce, make available and disseminate a variety of *didactical source material* in the form of anthologies of original sources, annotated bibliography, description of teaching sequences/modules to serve as a source of inspiration and/or as generic examples for classroom implementation, educational aids of various types, appropriate websites, etc.
- To perform systematically, carefully designed and applied *empirical research* in order to examine in detail and *evaluate* convincingly the *effectiveness* of the *HPM perspective* on improving the teaching and learning of mathematics, as well as students and teachers' awareness of mathematics as a discipline and their disposition towards it.
- To acquire a deeper understanding of theoretical ideas put forward in the HPM domain and to carefully develop them into coherent *theoretical frameworks and methodological schemes* that will serve as a foundation for further research and applications in this area.

In the last 10 to 15 years much work has been done on these issues and more is still in progress. In this survey:

- An attempt was made to provide enough evidence – mainly based on the literature – that HM is relevant to ME in several ways and may have a multifaceted influence on improving the teaching and learning of and about mathematics. More specifically:
- An outline of the development of the HPM domain has been given;
- The key issues in this domain have been formulated and briefly discussed; and
- A sufficiently comprehensive survey of the existing literature has been included.

We hope that the present survey will serve both as a working document and as a motivation for all those who desire information about the *HPM perspective* and to explore further the possibilities offered for supporting and improving ME.

Acknowledgments: We much appreciate contributions to section 3 from É. Barbin, F. Furinghetti, Y. Hong, W.-S. Horng, N. Jahnke, U.T. Jankvist, M. Moyon, D. Pengelley, L. Rogers, M.-K. Siu, Y.-W. Su, F. Swetz, Y. Thomaidis. They provided detailed references of relevant publications, though lack of space did not allow including all of them here.

REFERENCES

Alpaslan, M., Işıksal, M., & Haser, Ç. (2014). Pre-service mathematics teachers' knowledge of history of mathematics and their attitudes and beliefs towards using history of mathematics in mathematics education. *Science & Education*, *23*(1), 159-183.

Arcavi, A., & Isoda, M. (2007). Learning to listen: From historical sources to classroom practice. *Educational Studies in Mathematics*, *66*(2), 111-129.

Barbin, É. (1997). Histoire des Mathématiques : Pourquoi? Comment? *Bulletin AMQ*, *37*(1), 20-25.

Barbin, É. (2013). History and pedagogy of mathematics in France. In V. L. Hansen, & J. Gray (Eds.), *History of mathematics*, in *Encyclopedia of Life Support Systems* (e-book). Oxford, UK: EOLSS Publishers.

Barbin, É. (Ed.) (2015). *Les constructions mathématiques dans l'histoire : Avec des instruments et de gestes (Mathematical constructions: Making and doing)*. Paris: Ellipses-Editions.

Barbin, E. *et al* (2000). Integrating history: Research perspectives. In J. Fauvel, & J. van Maanen (Eds.), *ICMI study volume* (pp. 63-90). Dordrecht: Kluwer.

Barbin, É., Stehlikova, N., & Tzanakis, C. (Eds.) (2008). *History and epistemology in mathematics education: Proc. of the 5th ESU*. Prague: Vydavatelsky servis, Plzeň.

Barbin, E., Kronfellner, M., & Tzanakis, C. (Eds.) (2011a). *History and epistemology in mathematics education: Proc. of the 6th ESU*. Wien: Holzhausen Verlag.

Barbin, É., Furinghetti, F., Lawrence, S., & Smestad, B. (2011b). The role of the history and epistemology of mathematics in pre-service teachers training. In É. Barbin, M. Kronfellner, & C. Tzanakis (Eds.), *Proc. of the 6th ESU* (pp. 27-46). Wien: Holzhausen Verlag.

Barbin, É., Hwang, S., & Tzanakis, C. (Eds.) (2012). *Proc. of HPM 2012 – The HPM satellite meeting of ICME-12*. Daejeon, South Korea: Korean Society of Mathematical Education & Korean Society for History of Mathematics.

Barbin, É., & Tzanakis, C. (2014). History of mathematics and education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 255-260). New York, NY: Springer.

Barbin, É., Jankvist, U., & Kjeldsen, T. H. (Eds.) (2015). *History and epistemology in mathematics education: Proc. of the 7th ESU*. Copenhagen: Danish School of Education, Aarhus University.

Barnett, J. H., Lodder, J., & Pengelley, D. (2014). The pedagogy of primary historical sources in mathematics: Classroom practice meets theoretical frameworks. *Science & Education*, *23*(1), 7-27.

Bekken, O., & Mosvold, R. (Eds.) (2003). *Study the masters*. Nationellt Centrum för Matematikutbildning NCM, Göteborg: Göteborgs Univesitet.

Booker, G. (1986). Topic area: Relationship between the history and pedagogy of mathematics. In M. Carss (Ed.), *Proc. of the 5th ICME* (pp. 256-260). Boston, MA: Birkhäuser.

Boyé, A., Demattè, A., Lakoma, E., & Tzanakis, C. (2011). The history of mathematics in school textbooks. In É. Barbin, M. Kronfellner, & C. Tzanakis (Eds.), *Proc. of the 6th ESU* (pp. 153-163). Wien: Holzhausen Verlag.

Bruckheimer, M., & Arcavi, A. (2000). Mathematics and its history: An educational partnership. In V. Katz (Ed.), *Using history to teach mathematics. An international perspective*, MAA Notes 51 (pp. 135-146). Washington, DC: MAA.

Calinger, R. (Ed.) (1996). *Vita mathematica,* MAA Notes 40, Washington, DC: MAA.

Cantoral, R., Fasanelli, F., Garciadiego, A., Stein, R., & Tzanakis, C. (Eds.) *Proc. of HPM 2008: The HPM satellite meeting of ICME 11*. Mexico City: Centro de Investigación y de Estudios Avanzados del IPN.

Clark, K. (2006). *Investigating teachers' experiences with the history of logarithms: A collection of five case studies*. Dissertation. College Park, USA: Faculty of the Graduate School of the University of Maryland.

Clark, K. (2009). 'In these numbers we use no fractions': A classroom module on Stevin's decimal fractions. *Convergence 6*.

Clark, K. (2011). Reflections and revision: Evolving conceptions of a *Using History* course. In V. Katz, & C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education* (pp. 211-220). MAA Notes 78. Washington, DC: MAA.

Fasanelli, F., & Fauvel, J. (2006). History of the international study group on the relations between the history and pedagogy of mathematics: The first 25 years 1976–2000. In F. Furinghetti, S. Kaisjer, & C. Tzanakis (Eds.), *Proc. of HPM 2004 & ESU 4* (pp. x-xxviii). Iraklion: University of Crete, Greece.

Fauvel, J., & van Maanen, J. (1997). The role of the history of mathematics in the teaching and learning of mathematics: Discussion document for an ICMI Study (1997-2000). *Educational Studies in Mathematics*, *34*, 255-259.

Fauvel, J. G., & van Maanen, J. (Eds.) (2000). *History in mathematics education: The ICMI study*, New ICMI Study Series, vol. 6. Dordrecht: Kluwer.[5]

Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, *10*, 391-408.

Fried, M. (2011). History of mathematics in mathematics education: Problems and prospects. In É. Barbin, M. Kronfellner, & C. Tzanakis. (Eds.), *Proc. of the 6th ESU* (pp. 13-26). Wien: Holzhausen Verlag.

Furinghetti, F. (2004). History and mathematics education: A look around the world with particular reference to Italy. *MedJRME*, *3*(1-2), 1-20.

Furinghetti, F. (2012). History and epistemology in mathematics education. In V. L. Hansen & J. Gray (Eds.), *History of mathematics*, in *Encyclopedia of Life Support Systems* (e-book). Oxford, UK: EOLSS Publishers.

Furinghetti, F., Kaisjer S., & Vretblad, A. (Eds.) (2004). *Proc. of the HPM satellite meeting of ICME 10 & the 4th ESU*. Uppsala: Uppsala University.

Furinghetti, F., Jahnke, H. N., & van Maanen, J. (Eds.) (2006). *Report No22/2006 on the mini-workshop on studying original sources in mathematics education*. Oberwolfach: Mathematisches Forschunsinstitut Oberwolfach.

Glaubitz, M. R. (2010). *Mathematikgeschichte lesen und verstehen: eine theoretische und empirische Vergleichstudie*, Disssertation. Duisburg, Essen: Universität Duisburg-Essen.

Grattan-Guiness, I. (1973). Not from nowhere. History and philosophy behind mathematical education. *Int. J. ME Sci. Tech.*, *4*, 421-453.

Grattan-Guiness, I. (2004a). The mathematics of the past: distinguishing its history from our heritage. *Historia Mathematica*, *31*, 163-185.

Grattan-Guiness, I. (2004b). History or heritage? An important distinction in mathematics for mathematics education. *American Mathematical Monthly*, *111*(1), 1-12.

Horng, W.-S., & Lin, F.-L. (Eds.) (2000). *Proc. of the HPM conference history in mathematics education: Challenges for a new millennium - A satellite meeting of ICME-9*. Taipei: National Taiwan Normal University.

Horng, W.-H., Lin, Y.-C., Ning, T.-C., & Tso, T.-Y. (Eds.) (2004). *Proceedings of Asia-Pacific HPM 2004 Conference: History, culture and mathematics education in the new technology era*. Taichung, Taiwan: National Taichung Teachers College, Department of Mathematics Education.

International Study Group on the Relations between the History and the Pedagogy of Mathematics (1978). *Historia Mathematica*, *5*(1), 76.

Jankvist, U. T. (2007). Empirical research in the field of using history in mathematics education: Review of empirical studies in HPM2004 & ESU4. *NOMAD, 12*(3), 83–105.

Jankvist, U. T. (2009a). *Using history as a "goal" in mathematics education*. Dissertation. Denmark: Department of Science, Systems and Models, IMFUFA, Roskilde University.

Jankvist, U. T. (2009b). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.

Jankvist, U. T. (2013). History, applications, and philosophy in mathematics education: HAPh—A use of primary sources. *Science & Education*, *22*, 635-656.

Jankvist, U. T., & Kjeldsen, T. H. (2011). New avenues for history in mathematics education: Mathematical competencies and anchoring. *Science & Education*, *20*, 831-862.

Jankvist, U. T., Lawrence, S., Tzanakis, C., & van Maanen, J. (Eds.) (2011). In M. Pytlet, T. Rowland, & E. Swoboda (Eds.), *Proc. of CERME 7* (pp. 1636-1779). Rzeszow, Poland: University of Rzeszow.

Jahnke, H. N. *et al.* (2000). The use of original sources in the mathematics classroom. In J. Fauvel, & J. van Maanen (Eds.), *ICMI Study Volume* (pp. 291-328). Dordrecht: Kluwer Academic Publishers.

Katz, V. (Ed.) (2000). *Using history to teach mathematics. An international perspective*. MAA Notes 51.

[5] Abbreviated as *ICMI Study Volume* throughout this study.

Washington, DC: MAA.
Katz, V., & Michalowics, K. D. (Eds.) (2004). *Historical modules for the teaching and learning of mathematics* (CD). Washington, DC: MAA.
Katz, V., & Tzanakis, C. (Eds.) (2011). *Recent developments on introducing a historical dimension in mathematics education*. MAA Notes 78. Washington, DC: MAA.
Katz, V., Jankvist, U. T., Fried, M. N., & Rowlands, S. (Eds.) (2014). Thematic issue: History, philosophy and mathematics education. *Science & Education*, *23*(1), 1-250.
Kjeldsen, T. H. (2011a). Uses of history in mathematics education: Development of learning strategies and historical awareness. In M. Pytlet, T. Rowland, & E. Swoboda (Eds.), *Proc. of CERME 7* (pp. 1700-1709). Rzeszow, Poland: University of Rzeszow.
Kjeldsen, T. H. (2011b). History in a competence-based of the practice of mathematics in a competence based mathematics education: A means for the learning of differential equations. In V. Katz, & C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education* (pp. 165-173). MAA notes 78. Washington, DC: MAA.
Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation: History as a method for learning meta-discursive rules in mathematics. *Educational Studies in Mathematics*, *80*(3), 327–349.
Lagarto, M. J., Vieira, A., & Veloso, E. (Eds.) (1996). *História e Educação Matemática: Proc. of Deuxième Université d'Été Européenne sur l'Histoire et Épistémologie dans l' Éducation Mathématique and the ICME-8 Satellite Meeting of HPM*. Braga: Portuguese Association of the Teachers of Mathematics & Department of Mathematics, University of Minho. Available online Vol. I, Vol. II.
Lalande F., Jaboeuf F., & Nouazé, Y. (Eds.) (1995). *Actes de la première Université d'Été Européenne sur l'Histoire et Épistémologie dans l'Éducation Mathématique*. Montpellier : IREM de Montpellier, Université Montpellier II.
Liu, P.-H. (2003). Do teachers need to incorporate the history of mathematics in their teaching? *MT*, *96*(6), 416-421.
Mosvold, R., Jakobsen, A., & Jankvist, U. T. (2014). How mathematical knowledge for teaching may profit from the study of history of mathematics. Science *& Education*, *23*(1), 47-60.
National Council of Teachers of Mathematics. (1969/1989). *Historical topics for the mathematics classroom* (31st Yearbook). Reston, VA: Author.
Niss, M., & Højgaard, T. (Eds.) (2011). *Competencies and mathematical learning-ideas and inspiration for the development of mathematics teaching and learning in Denmark*. Roskilde: Roskilde University.
Pengelley, D. (2011). Teaching with primary historical sources: Should it go mainstream? Can it? In V. Katz & C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education* (pp. 1-8). MAA Notes 78. Washington, DC: MAA.
Pengelley, D., Lodder, J., Barnett, J., Bezhanishvili, G., Ranjan, D., & Leung, H. (2009). Historical projects in discrete mathematics and computer science. In B. Hopkins (Ed.), *Resources for teaching discrete mathematics* (pp. 163-276). MAA Notes 74. Washington, DC: MAA.
Pengelley, D., & Laubenbacher, R. (2016) *Teaching with original historical sources in mathematics, a resource web site*.
Povey, H. (2014). 'Walking in a foreign and unknown landscape': Studying the history of mathematics in initial teacher education. *Science & Education*, *23*(1), 143-157.
Radelet de Grave, P., & Brichard, R. (Eds.) (2001). *Proc. of the 3rd ESU*. Leuven & Louvain-la-Neuve : Université Catholique de Louvain.
Radford, L. *et al.* (2000). Historical formation and student understanding of mathematics. In J. Fauvel, & J. van Maanen (Eds.), *ICMI study volume* (pp. 143-170). Dordrecht: Kluwer Academic Publishers.
Rogers, L. (2009). History, heritage and the UK mathematics classroom. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello, (Eds.). *Proc. of CERME 6* (pp. 2781-2790). Lyon: Institut National De Recherche Pédagogique.
Rogers, L. (2011). Mapping our heritage to the curriculum: Historical and pedagogical strategies for the professional development of teachers. In V. Katz, & C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education*. MAA Notes 78 (pp. 221-230). Washington, DC: MAA.
Schubring, G. (2006). Ontogeny and phylogeny. Categories for cognitive development. In F. Furinghetti, S. Kaijser, & C. Tzanakis (Eds.), *Proceedings of HPM 2004 & ESU 4* (pp. 329-339). Iraklion: University of Crete.
Schubring, G. (2011). Conceptions for relating the evolution of mathematical concepts to mathematics learning –

Epistemology, history, and semiotics interactive. *Educational Studies in Mathematics*, *77*(1), 79-104.
Siu, M.-K. (2006). No, I don't use history of mathematics in my classroom. Why? In F. Furinghetti, S. Kaijser, & C. Tzanakis (Eds.), *Proceedings of HPM 2004 & ESU 4* (pp. 268-277). Iraklion: University of Crete, Greece.
Siu, M.-K., & Tzanakis, C. (Eds.) (2004). The role of the history of mathematics in mathematics education. *Med JRME*, *3*(1-2), 1-166.
Smestad, B. (2011). History of mathematics for primary school teacher education, or: Can you do something, even if you can't do much? In V. Katz, & C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education* (pp. 201-210). MAA Notes 78. Washington, DC: MAA.
Stein, R. (2010). *Math for teachers: An exploratory approach*. Dubuque, IA: Kendall Hunt.
Su, Y.-W. (2005). *Mathematics teachers' professional development: Integrating history of mathematics into teaching*. Dissertation. Department of Mathematics. Taipei: National Taiwan Normal University (in Chinese).
Swetz, F., Fauvel, J., Bekken, O., Johansson, B., & Katz, V. (Eds.) (1995). *Learn from the masters!* Washington, DC: MAA.
Thomaidis, Y., & Tzanakis, C. (2007). The notion of historical "parallelism" revisited: Historical evolution and students' conception of the order relation on the number line. *Educational Studies in Mathematics, 66*, 165-183.
Tzanakis, C., Arcavi, A. *et al*. (2000). Integrating history of mathematics in the classroom: An analytic survey. In J. Fauvel, & J. van Maanen (Eds.), *ICMI study volume* (pp. 201-240). Dordrecht: Kluwer.
Tzanakis, C., & Thomaidis, Y. (2012). Classifying the arguments & methodological schemes for integrating history in mathematics education. In B. Sriraman (Ed.), *Crossroads in the history of mathematics and mathematics education* (pp. 247-294). Charlotte NC: Information Age Publishing.
van Amerom, B. A. (2002). *Reinvention of early algebra. Developmental research on the transition from arithmetic to algebra*. Dissertation. Utrecht: CD-ß Press, Centre for Science and Mathematics Education. Freudenthal Institute, University of Utrecht.
Waldegg, G. (2004). Problem solving, collaborative learning and history of mathematics: Experiences in training in-service teachers. *Med JRME*, *3*(1-2), 63-71.

BREAKING NEWS, NOVEMBER 1813 – ACCUSED 'NUMBERS' CLAIM GEOMETRIC ALIBI:

A Dramatic Presentation Celebrating Joseph Gergonne

Gavin HITCHCOCK
South African Centre for Epidemiological Modelling and Analysis (SACEMA),
University of Stellenbosch, Stellenbosch, South Africa
aghitchcock@gmail.com

ABSTRACT

Joseph-Diez Gergonne provides an early model for the importance of the role of journal editor in filtering novel and valuable ideas, encouraging correspondence, mediating disputes, influencing research priorities, shaping disciplinary discourse, and negotiating disciplinary boundaries. A paper of J.-F. Français appeared in Gergonne's journal, *Annales de mathématiques pures et appliquées*, in 1813, on the idea of representing imaginary numbers geometrically. Gergonne placed the article and responses by himself and others in the '*Philosophie Mathématique*' section of his journal. We present, in the form of a dialogue between the participants, a paraphrase of the lively interchanges in the *Annales*, including Gergonne's copious editorial footnotes. The play is introduced, and a closing commentary given, by a historian of mathematics and a modern mathematics teacher in dialogue, providing historical and pedagogical context. Cameo appearances are made by historian Gert Schubring, John Wallis, Abraham De Moivre, Caspar Wessel, H. Valentiner, A.-M. Legendre, and the modern mathematician Barry Mazur.

1 Context and characters

JOHN WALLIS (1616 –1703) became Savilian Professor of Geometry at the University of Oxford in 1649. The University had been purged by Parliament of all Royalists, and Wallis may have won the Professorship as a reward for his work for the other side in the English Civil War – decrypting a Royalist letter. ABRAHAM DE MOIVRE (1667–1754) was raised a Protestant in France, in a period of religious tension. In his late teens he was imprisoned for over two years for his religious beliefs, and subsequently went to live in England. There he taught himself mathematics, published his own work, and was elected to the Royal Society when only 30. He made a living as tutor and risk consultant, solving problems of chance for financial speculators and gamblers.

Two prominent players in the early development of the geometric or graphical representation of complex numbers were not perceived as mathematicians at all. CASPAR WESSEL (1745–1818) was a Danish-Norwegian surveyer, who made his contribution in a talk to the Royal Danish Academy of the Sciences which appeared in their *Mémoires* in 1799, in Danish; a French edition appeared only in 1897, and a full English edition another century later. It has been commonly supposed until recently that the ARGAND of our story, who produced the *Essai sur un maniére de représenter les quantités imaginaires dans les constructiones géométriques* in 1806, is one Jean-Robert Argand (1768–1822), a bookseller in Paris. However a good case has been made by Gert Schubring that our Argand was a different person, and did not publish his work in any official sense, merely distributed a few copies of his manuscript to individuals. These two works, by Wessel and Argand, went almost unnoticed at first. An article of JACQUES-FRÉDERIC FRANÇAIS (1775–1833), a mathematician from Alsace, published in Gergonne's *Annales*, was instrumental in bringing Argand to communicate with Gergonne, and lay claim to having given his essay to the well-known mathematician ADRIEN-MARIE LEGENDRE (1752–1833) in 1806. Legendre, though impressed, did not pursue the ideas, but wrote about them to the older brother of Français, who found it among his deceased brother's effects; the letter has recently been discovered by Schubring. It was through Gergonne's *Annales* that Argand's work became known to many French mathematicians. Hermann Hankel's work on complex analysis in 1867 drew attention once more to Argand's contribution, provoking a French translation in 1874, and ensuring that the name of Argand would be forever attached to the complex plane. Two other relative outsiders broke the news of the geometric representation of imaginaries to the English. Adrien-Quentin Buée was a French priest who escaped to England in 1792 rather than take the oath of allegiance to the Revolutionary Constitution. He mostly wrote religious and political tracts, but published his 'Mémoire sur des quantités imaginaires' in the *Philosophical Transactions of the Royal Society* in 1806. Strangely his work was also largely unnoticed, and English mathematicians generally awoke to 'Mr Warren's planar representation of imaginaries' when John Warren published his booklet: *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*, in 1828.

JOSEPH-DIEZ GERGONNE (1771–1859) spent some years as a Captain in the French army in the late 18th century, before taking the chair of 'transcendental mathematics' at the new École Centrale in Paris, and then moving to the University of Montpellier in 1816. He made some significant contributions to mathematics, including discovering the principle of duality in projective geometry, but he was probably most influential as the editor of the *Annales des mathématiques pures et appliquées* (*Annals of Pure and Applied Mathematics*). This journal was founded by him in 1810 when he realised how important and how hard it was to get work published, and it was sometimes called simply the *Annales de Gergonne*. Between 1810 and 1830, when he was appointed Rector of the University, he published in his journal about 200 of his own articles, as well as articles by some of the leading mathematicians of his time, including Poncelet, Servois, Bobillier, Steiner, Plücker, Chasles, Brianchon, Dupin, Lamé, and Galois.

FRANÇOIS JOSEPH SERVOIS (1767–1847) was ordained a priest shortly after the French Revolution broke out, but before long he left the priesthood to join the army, doing mathematics in his leisure moments. He was assigned to his first academic position in 1801, recommended by Legendre, and went on to teach mathematics at a number of artillery schools. In 1804 he published a text on practical geometrical constructions for use by military officers in action, called *Little-known Solutions to Various Problems in Practical Geometry*. His most striking mathematical contribution came in 1814 when his work on the foundations of calculus came to full bloom. Following on from the work of Lagrange and of L. F. A Arbogast (1759–1803), he produced a remarkable paper 'Essai sur un nouveau mode d'exposition des principes du calcul différential', in which he defined and studied the differential in the form of what we would now call a 'linear operator'. Arbogast and Servois also initiated the study of what we now call 'functional equations', and in considering the algebraic rules needful for manipulating the symbols in the resulting 'calculus of operations', and 'calculus of functions', they made the first explicit extension of algebraic operations to entities which were neither numbers nor geometrical magnitudes. It is Servois to whom we owe the words 'distributive' and 'commutative' – standard terms in algebra today.

2 THE PLAY

MARIA, a historian of mathematics, and EMMY, a modern mathematics teacher, converse frontstage, while cameo appearances are made by GERT SCHUBRING, JOHN WALLIS, ABRAHAM DE MOIVRE, BARRY MAZUR, CASPAR WESSEL, and H. VALENTINER. Then the curtain rises on a conversation taking place in November, 1813, between JOSEPH GERGONNE, JACQUES-FREDERIC FRANÇAIS, ARGAND, and FRANÇOIS JOSEPH SERVOIS, during which a cameo appearance is made by ADRIEN-MARIE LEGENDRE.

MARIA: [*ad lib greetings to audience, in English or French*] Ladies and gentlemen, my name is Maria. Welcome to our tribute to Joseph Gergonne ... In a few minutes we will hear him in conversation with three contributors to his journal. I am your Storyteller today, to give some historical background. Please welcome Emmy, who will bring the perspective of a mathematician and mathematics teacher … [*applause*]

EMMY: Thank you, Maria!

MARIA: Emmy, will you tell us – how are complex numbers introduced to students today?

EMMY: It may be done by expecting students to take on trust the – quote [*she makes hand gestures for quote marks*] – 'imaginary' number *i*, with square taken equal to minus one—

MARIA: That's more or less what Augustin-Louis Cauchy did!

EMMY: Or it may be done by using ordered pairs of real numbers to create the algebra of complex numbers.

MARIA: As first proposed by William Rowan Hamilton in 1835!

EMMY: But, in whatever way they are introduced, the exposition is always accompanied by their 'geometric representation' in what we call 'the Argand diagram', or 'the complex plane'.

MARIA: Why is this still found so helpful? Do mathematicians not see complex numbers forming an algebra that can stand alone without the picture?

EMMY: Well, even today, few mathematicians think about real numbers without imagining them on the 'number line'. We imagine complex numbers in the *plane*. It's not only that this has been found to make the new numbers more palatable, by involving some basic human intuitions, but also because the rich geometry of the plane has proved extremely fruitful – you could say indispensable – in developing complex analysis.

MARIA: Yet, believe it or not, complex numbers were used for over two centuries before being matched with this geometrical picture.

EMMY: That's amazing! I simply cannot imagine teaching or using complex numbers without the Argand diagram. Who was Argand? Was he the first to introduce the geometrical representation?

MARIA: Argand is rather an obscure person – generally supposed to be a Parisian bookseller called Jean-Robert, who published his ideas in 1806, but scholarly doubts have been shed on these assumptions, and about his dates. The historian Gert Schubring can testify to the extraordinary difficulties of uncovering historical facts two centuries later.

[*Cameo appearance*]

SCHUBRING: Unfortunately we do not have any primary evidence as to Argand's first name. Even knowing his address in Paris in 1813 does not help, as administrative and death registers were lost in the fights of the *communards* in Paris in 1870/71. Almost all we can state – in terms recalling medieval history – is that Argand 'flourished' in 1806, 1813, and 1814.[1]

[*Exit*]

MARIA: However, the earliest statements about picturing the imaginary numbers geometrically probably came from John Wallis in Oxford, well over a century earlier in spite of his statements about the absurdity of negative numbers, even. This was in 1673, a whole century after the Italian, Rafael Bombelli, first thrust the imaginary numbers into controversial prominence on the mathematical stage.

[*Cameo appearance, while equation appears on screen*]

WALLIS:[2] I say that $\sqrt{-bc}$ may be perceived as a mean proportional between b and $-c$, in that the following holds:

$-bc = \left(\sqrt{-bc}\,\right)^2$ gives $\dfrac{b}{\sqrt{-bc}} = \dfrac{\sqrt{-bc}}{-c}$

This may be exemplified in Geometry, by taking the root of the imaginary quantity as the point in the Plane distant $\sqrt{bc}$ above the line of real numbers.

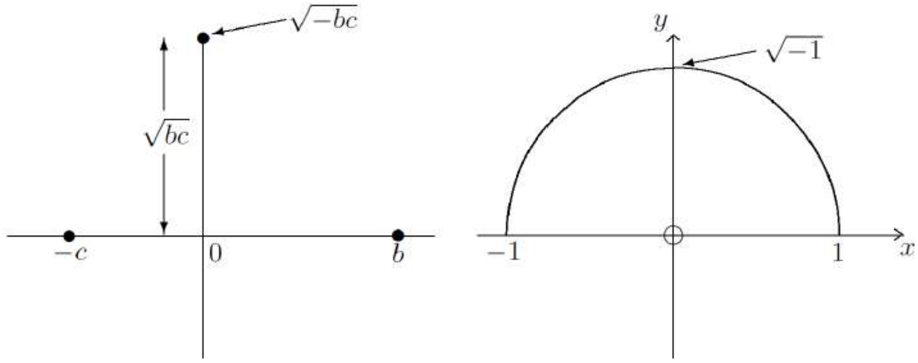
[*Exit*]

EMMY: [*diagrams and equations appear on a screen for live audience*] Here's the diagram on the left. Take $b = c = 1$, and use the modern Argand diagram on the right. Wallis says that the mean proportional between the two points *1* and minus *1* on the number line is $\sqrt{-1}$, which for us is the complex number *i*, and it corresponds to the point (*0*, *1*) in the plane, or a unit distance

[1] Paraphrased from (Schubring 2001), 132, which decribes his careful investigations, clarifying what is and is not known about Argand, as well as about Buée and some of the earlier work that influenced them. Much of what Schubring describes as 'the mystery' surrounding these obscure authors and their isolated achievements, still remains. Schubring also analyses (p. 132) how the original designation of Argand's contested biographical details arose from his 1874 editor Hoüel's unsubstantiated assumption that he originated in Geneva.

[2] Based on his *Algebra* of 1685; English translation of extract in (Smith 1959, 48).

up the perpendicular y-axis in our Argand diagram. It's easy to see this as a sort of half-way stage between the two points, thinking of rotations through a right-angle about the origin.



MARIA: Rotations! Wallis was onto something here, but he did not pursue it any further. And nobody of his generation was listening! What's the point, they might have asked? Tell me, Emmy, how would you convince your students that complex numbers are actually useful?

EMMY: Well, I would *love* to show them the algebraic solution of cubic equations! But, sadly, this is not in most mathematical curricula! Indeed, in many countries, high school students do not meet complex numbers at all. When I *have* discussed complex numbers with students, it has been when we reach De Moivre's theorem and its applications that I have caught some of them actually looking interested!

MARIA: Emmy, please state for us this theorem, once called Cote's theorem.

EMMY: The theorem says that, for any real number θ [theta] and any integer n,

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n$$

However, you know, it's when I have gone further and shown them the wonderful and unexpected connection, found by Leonhard Euler, between the real *exponential* function e^x and the real trigonometric functions *cosine* and *sine*, that a few have really shown excitement. I have even known some to say 'awesome!' at that point. Here it is:

$$\cos x + i \sin x = e^x$$

MARIA: From which De Moivre's theorem follows immediately, setting $\theta = x$, then $n\theta = x$.

EMMY: That reminds me of a piece of mathematical folklore: the shortest distance between any two points in the real domain lies through the complex domain.

MARIA: Euler's inspired manipulations gave eighteenth century mathematicians – at least the Continental ones – a growing confidence in the potency of these so-called fictions – these impossible numbers.[3]

EMMY: Did De Moivre discover his theorem long before Euler discovered his?

MARIA: Euler arrived at his theorem by his usual magic around the middle of the eighteenth century. But, as early as 1707, Abraham De Moivre had caught a glimpse of the analogy

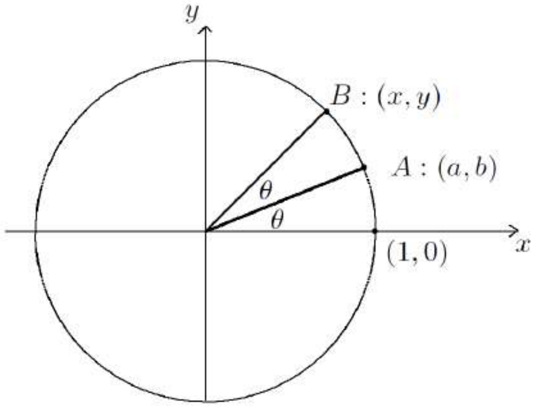
[3] (Euler 1748, 1749, 1770)

between two apparently quite different things: on the one hand, the powers of imaginary numbers, and on the other hand, the way coordinates of points in the plane change when a multiple angle is taken.

[*Cameo appearance*]

DE MOIVRE:[4] Consider a point A: (a, b) on a circle of unit radius in the plane, whose radius vector makes angle theta with the axis. Now double the angle, and let the new point on the circle be B: (x, y). Then it can easily be shown that the new point is given by

$$x = a^2 - b^2, \quad y = 2ab$$



But I observe that when I square the imaginary quantity $a + b\sqrt{-1}$, I obtain the imaginary quantity $a^2 + b^2 + 2ab\sqrt{-1}$:

$$\left(a + b\sqrt{-1}\right)^2 = a^2 + b^2 + 2ab\sqrt{-1}$$
$$\left(a + b\sqrt{-1}\right)^3 = a^3 - 3ab^2 + \sqrt{-1}\,(3a^2 b - b^3)$$

I observe a similar analogy when taking three times the angle, for the coordinates of the new point C on the circle are just the two expressions $a^3 - 3ab^2$, $3a^2b - b^3$ that I obtain by cubing the imaginary quantity $a + b\sqrt{-1}$. If I take the multiple n of the angle successively as 1, 2, 3, 4, 5, 6, &c., there will arise expressions of the same form as those obtained by taking the nth power of the imaginary quantity, although they are things of quite a different nature.

EMMY: Students today often find this connection with the multiple angle formulas of trigonometry the most compelling justification for accepting the imaginary numbers, for no other derivation is as simple. Take a point (a, b) on a unit circle $x^2 + y^2 = 1$, with polar angle θ [theta], so that $a = \cos\theta$, $b = \sin\theta$. Then double the angle of the radius vector; De Moivre observes that the coordinates of the new point will be $a^2 - b^2$, $2ab$, taking the real and imaginary parts of $\left(a + b\sqrt{-1}\right)^2$. But this point has coordinates $\cos 2\theta, \sin 2\theta$, so we immediately have two useful trigonometric formulas.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\cos\theta\sin\theta$$

[4] The idea is from De Moivre's paper, 'Reduction of Radicals to More Simple Terms', c. 1738, in (De Moivre 1809), but is expressed here in a form more accessible, if less intriguing, than his. He did not have unit radius, but this makes the analogy he observed clearer.

Similarly we could find the triple angle formulas by taking triple the angle, or indeed we could take any multiple $n\theta$, where n is a positive integer, and take real and imaginary parts of the corresponding power of $\cos\theta + i\sin\theta$, using the binomial theorem.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

MARIA: It is interesting that De Moivre's motivation went in the opposite direction, as shown by his title: 'Reduction of Radicals to More Simple Terms'. He used a knowledge of the way the sine of a triple angle was connected to the sine of the angle, to find the sorts of cube roots of imaginary quantities that arise when Cardano's Rule is applied to the irreducible case for cubic equations.

DE MOIVRE: I use this to extract the nth root of the impossible binomial, by using a table of sines. For example, I shall show by this method that the cube roots of the impossible quantity $81 + \sqrt{-2700}$ are:

$$\frac{9}{2} + \frac{1}{2}\sqrt{-3}\,, \quad \frac{3}{2} - \frac{5}{2}\sqrt{-3}\,, \quad -3 + 2\sqrt{-3}$$

There have been several authors, and among them Dr. Wallis, who have thought that those cubic equations, which are referred to the circle, may be solved by the extraction of the cube root of an imaginary quantity, as of $81 + \sqrt{-2700}$ without any regard for the table of sines. But that is a mere fiction; and a begging of the question; for on attempting it, the result always recurs back again to the same equation as that first proposed. And the thing cannot be done directly, without the help of the table of sines, specially when the roots are irrational; as has been observed by many others.

[*Exit*]

EMMY: To me it would seem a small step from De Moivre's perception of this analogy between 'impossible numbers' and ideas from geometry, to what we now call a 'geometric representation of the algebra of complex numbers.'

MARIA: Yet it would be many decades before this unexpected connection would be clarified and published in its fully explicit form. And remember, there was not yet such a thing as 'the algebra of complex numbers'. That would take more than a century to emerge!

EMMY: So for a century no-one conceived of what we now call 'the complex plane'? But after De Moivre's paper imaginary numbers would surely have been naturally associated with the trigonometric functions?

MARIA: Yes, it's true that De Moivre's insight was used by both Euler and Gauss.[5] But neither seemed to give explicit recognition of the big geometrical picture.

EMMY: Well, when was the geometrical representation finally made explicit?

MARIA: More surprises! The great idea was arrived at and published independently by five different, relatively obscure authors, over a period of nearly three decades: Wessel in 1799,

[5] Euler's version of what we call De Moivre's theorem appeared in Chapter 8, article 132, of (Euler 1748). Gauss made use of it in his proof of the Fundamental Theorem of Algebra in 1801.

Argand in 1806 (though his work was not published until 1813), Buée in 1806, Français in 1813, and finally John Warren in 1828.

EMMY: That seems really weird to me in the twenty-first century! Did Warren not know about the earlier publications?

MARIA: It seems that almost no British mathematicians read them – one of the symptoms of the political and intellectual gulf between Britain and the Continent over that period.

EMMY: And, I suppose, the language, and the particular forum or professional organ of publication – did they appear in international journals?

MARIA: The idea of a truly international journal was still in the future. But especially interesting is that two of the earliest accounts of this fundamental connection not only appeared in very obscure places, but were by complete mathematical outsiders.

EMMY: But why did no mathematician expound the connection? What was in the minds of those mathematical giants, Euler, Lagrange, and Gauss? You suggested they used the connection freely enough!

MARIA: Yes, they did. But as late as 1801, Gauss, in his proof of the fundamental theorem, dealt with real and imaginary parts separately, never talking about the combined whole as a point in a planar picture of these numbers.

EMMY: Maybe they didn't see it as a big issue. Maybe the great Gauss never actually *saw* it that way at all! And we serve it up to our students as if obvious!

MARIA: Barry Mazur, in his book *Imagining Numbers*, expresses well the novelty and importance of the connection:

[*Cameo appearance*]

MAZUR: What a dramatic act: to find a home in our imagination for such an otherwise troublesome concept!

[*Exit*]

MARIA: But he also asks the question whether this idea was in any sense 'in the air' for those leading mathematicians.

EMMY: And what *is* the answer?

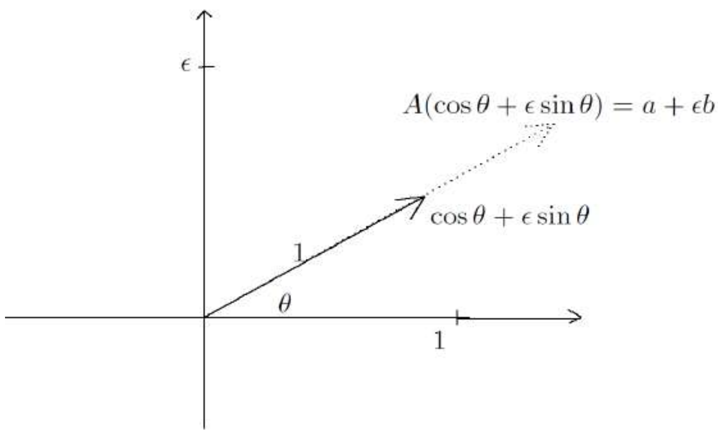
MARIA: Mazur doesn't claim to know. I don't think anybody can be sure. The fascinating fact is that the first to write clear accounts were a surveyer, and someone so obscure that there is doubt about even his first name and profession – a clock technician, or a bookseller. The earliest exposition was in a talk given to the Royal Danish Academy of the Sciences in 1797 by the Danish-Norwegian surveyer, Caspar Wessel.

[*Cameo appearance*]

WESSEL:[6] The title of my paper is 'On the Analytical Representation of Direction'. I am concerned to find a way of expressing, by means of algebra, the line segments (which I conceive as fundamental to surveying and navigation), that have both a given length and a given direction. The method I use employs the imaginary numbers.

Two straight lines are added if we unite them in such a way that the second line begins where the first one ends and then [we] pass a straight line from the first to the last point of the united lines. This line is the sum of the united lines. Now, in seeking a way of multiplying straight lines, I insist on three properties: First, the product of two lines must remain in the plane of the two lines. Second, the length of the product must be the product of the two lengths. And third, if all directions are measured as angles made with a positive unit line, which I call unity, and designate by the number 1, then the angle made by the product must be the sum of the angles made by the two lines.

Now, if I let ϵ [epsilon] denote the line with unit length perpendicular to the first line 1, and apply the three properties above, it is easy to see that $\epsilon^2 = -1$, and also $(-\epsilon)^2 = -1$. Thus a line of length 1 and direction θ [theta] can be designated as $\cos\theta + \epsilon\sin\theta$, and a line of length A and direction θ can be designated as $A(\cos\theta + \epsilon\sin\theta) = a + \epsilon b$.



Now I can add and multiply these algebraic entities as follows:

$$(a + \epsilon b) + (c + \epsilon d) = (a + c) + \epsilon(b + d)$$

$$(a + \epsilon b)\times(c + \epsilon d) = (ac - bd) + \epsilon(ad + bc)$$

and I easily show that the addition and multiplication both satisfy my geometrical properties laid down earlier.

[*Exit*]

EMMY: Amazing! – so his motivations also went in the opposite direction to ours! Caspar Wessel starts with the geometrical idea of directed line segments, generalising the notion of oppositely directed positives and negatives, and sets out to represent line segments

[6] This is a paraphrase of what he might have said in his talk, based on what he published two years later. A partial English translation is in (Smith 1959), 55–66.

algebraically. And he simultaneously provides a beautiful geometrical representation of the imaginary numbers!

MARIA: Yes. The history of mathematics is full of surprises for modern mathematicians. Unfortunately, very few mathematicians read Wessel's paper (it was written in Danish). It took a century for the French edition to come out. Here us one of the editors of that edition, Monsieur H. Valentiner:

[*Cameo appearance*]

VALENTINER:[7] It is surprising that a man can come up with a book as remarkable as the one before us, after having exceeded fifty years of age, and without producing any other scientific work, before or afterwards.

[*Exit*]

MARIA: The next major events in our story took place in 1806. In London, Adrien-Quentin Buée published a paper in the Transactions of Royal Society that he'd read to the Society the previous year.

EMMY: And did the British not sit up and take notice?

MARIA: No, it seems not. With the notable exception of George Peacock, about twenty years later. And in Paris, the same year, someone called Argand produced at least two copies of a manuscript in French: *Essai sur un maniére de représenter les quantités imaginaires dans les constructiones géométriques* (*Essay on a Way of Representing Imaginary Quantities by a Geometric Construction*).

EMMY: Was his approach different from Wessel's?

MARIA: Perhaps he was the first to clearly conceive of numbers as transformations. Like Wessel, he distinguished carefully between *rapport numérique* and *rapport de direction*, and showed how the imaginaries could be given a common construction with positives and negatives – combining and generalising the fundamental concepts of numerical or absolute value, and direction.

EMMY: His motive seems to have been slightly different, then: to bring the imaginaries into the geometric fold by extending the number line to the number plane. So the Parisian mathematicians at least must have taken note?

MARIA: Well, no. Which is not actually surprising. None of the leading mathematicians were aware of his work, except Legendre, because Argand approached him personally, and was presumably discouraged from formally submitting his manuscript.

EMMY: That's sad. So what first caught the attention of mathematicians – why is it called the Argand diagram today?

MARIA: The Continental mathematicians, at least, were alerted when a paper of J.-F. Français appeared in Joseph Gergonne's journal, *Annales de mathématiques pures et appliquées,* in

[7] Preface to (Wessel 1897).

1813.[8] Gergonne was an extremely pro-active editor, and enjoyed adding footnoted comments to articles, and challenging other mathematicians to respond. Significantly, he placed the article of Français and subsequent responses in a rather inauspicious section of his journal, called 'Philosophie Mathématique'. Let's listen in, now, to a conversation between Gergonne and three contributors.

CURTAIN RISES

GERGONNE: Monsieur Français, thank you for your very interesting contribution to the *Annales*. I have invited some others to join us, who have expressed views on this same subject you call *géométrie de position*. Allow me to introduce M. Argand ... and M. Servois …

[*they shake hands, saying appropriate things in early nineteenth century French style, and Gergonne ushers them to a table where they are all seated. A bottle of wine may be poured*]

FRANÇAIS:[9] [Thank you, M. Gergonne, I am honoured to meet you and colleagues. My article is merely a] sketch, very much abbreviated, of the new principles on which it is convenient, and even necessary, to base the theory of 'Geometry of Position' which I submit to the judgement of geometers. Since these principles are in formal opposition to current ideas about the nature of imaginary quantities, I expect numerous objections. But I dare to believe that a deep examination of these same principles will find them correct, and the consequences that I deduce from them, no matter how strange they may at first appear to be, will nevertheless be judged to conform to the most rigorous rules of dialectic.

GERGONNE: Far be it from me to deflate you, M. Français, but, you know, these ideas are not at all so strange as to be incapable of germinating in several heads at once.

FRANÇAIS: Indeed, M. Gergonne? May I ask who else has propounded them?

GERGONNE: Two years ago, in editing an article in the *Annales,* I was constrained to add a footnote which I thought would bring more clarity to the discussion; and in doing so I placed some imaginary numbers in a geometric configuration not dissimilar to that advocated by you.

FRANÇAIS: I do not claim that this idea is my own – I discovered it in a letter of Legendre, when going through the papers of my brother after his death.

GERGONNE: Ah – are you saying that M. Legendre is the discoverer?

FRANÇAIS: No – Legendre seemed not to regard this idea as worthy of publishing, nor was it his own idea. Indeed, he said that he obtained it as an object of pure curiosity from some other unnamed mathematician. But I believe it has more importance than he or its author gave it, and I hope that whoever it was that first produced the idea will make himself known and publish his theory in full.

[8] (Français 1813a).

[9] This speech, except for the square bracketed part, is a direct translation, by Barry Mazur, of a passage in the article in *Annales*, p. 70. The dialogue that follows is inspired by Mazur's discussion in Chapter 11 of (Mazur 2004), including his English translation of certain phrases.

GERGONNE: [*turning to Argand*] M. Argand, you have come forward with a claim to being this unknown author!

ARGAND: Yes, indeed! I am pleased to respond to the request of M. Français for the original author of these ideas to make himself known. I can confirm that M. Legendre first heard from me of the geometric construction by which I represent the imaginary numbers, and indeed, M. Français's wish for the full exposition of the theory has been answered already some years ago, in 1806.[10]

GERGONNE: [*sharply*] I have not seen that publication!

ARGAND: Unhappily, my manuscript attracted no attention at the time. I have now had some copies printed, taking the liberty of inscribing the original date of 1806. Should any mathematicians wish for one, they may write to me. And I have one here to present to you, M. Gergonne. [*hands it to Gergonne with a bow*]

GERGONNE: I am greatly obliged... [*scrutinises the cover page*] ... and you have dedicated it to me, I see – I am honoured, Monsieur! [*sighs deeply*] I fear the state of mathematical communication is deplorable! I hear from a colleague that a man called Buée published related ideas in London – also in 1806, but I have no access to the London Transactions. And regrettably Lacroix and the other Parisian mathematicians do not take the trouble to keep me informed![11] But did you not publish your original paper seven years ago?

ARGAND: I confess I simply distributed a few copies of my *écrit*. But I did pluck up the courage to show a copy to Legendre in order to find his opinion before I could dare submit to the *Institut*. He seemed sceptical, and I went away feeling very discouraged. However I left the manuscript with him to examine further, and imagined I would contact him or hear from him in due course. Unfortunately, nothing ever came of my visit.

[*Cameo appearance, aged about 60, reminiscing directly to audience*]

LEGENDRE:[12] [*wearing a nightgown, looking distressed, carrying a lighted candle*] I cannot sleep tonight. I see a face in my dreams ... I recall that face vividly. One day an unknown person requested to see me – a young man, who wished to show me some work of his. He did not explain his motives very well, but I understood that he considered the so-called imaginary quantities to be as real as other numbers, and he represented them by lines. After he left, I brought myself to take a closer look at his manuscript, and, quite contrary to my expectations, found some quite original ideas, leading to demonstrations of trigonometric formulas and Cote's theorem. He never returned, and I realised I did not have his name or address. I wonder what happened to him. I hope I was not rude to him.

[*walks on a few steps, before turning to audience again*]

---

[10] In November, Argand responded with a note (Argand 1813) in the *Annales* under the same title as his 1806 manuscript and its freshly printed copies.

[11] *Annales* 4, 1814, 367.

[12] Based loosely on an extract from a letter from Legendre to F.-J. Français, 2 November 1806, in (Schubring 2001), 129.

I was so sure at first that he was just another eccentric come to trouble me ... Yet the ideas were good, and I suppose I might have helped him. However, I did write to M. Français the elder – that's Francois Joseph Français – giving him a sketch of the work, and leaving him to judge its worth. Strange – I cannot shake off this feeling of having let the young man down ... or perhaps even been guilty of unfaithfulness to mathematics.

[*shakes his head and attempts to shrug off his unease*]

But if people don't leave their address, what can one do?

[*shuffles offstage and exits*]

FRANÇAIS: I cannot speak for M. Legendre's attitude, but I do regret, Monsieur Argand, that my brother Francois seems not to have given due attention to your excellent ideas. I got the impression that neither he nor M. Legendre knew the name of the author!

GERGONNE: [*shaking his head and frowning*] Never write a manuscript without your full name, address and affiliation! You know, Monsieur Argand, your previous letters to me [*scrutinizes again the paper of Argand*] – and, indeed, also this paper you have presented to me – do not include your full name! One would imagine you have some criminal past to hide!

ARGAND: [*stiffly*] I apologise Monsieur – and assure you that my life has been exemplary and singularly uneventful! My big regret is isolation from those who might understand my work, and [*now he shows warmth*] I am overjoyed to find myself in the present illustrious company, and have given, for your *Annales*, a full résumé of my theory.

GERGONNE: [*smiling*] I trust you inserted your full name there? The printers do not care, and I may have missed that detail.

ARGAND: Ah – I must check on that ... I am pleased to say, however, that I have given an application to simplification of a well-known proof of d'Alembert.

GERGONNE: Hmmm ... [*turning to Servois*] I wonder, M. Servois, if you would be good enough to respond to both M. Français and M. Argand, regarding this curious matter of the geometrical representation of imaginary numbers. I have shown you their submissions; have you looked through them?

SERVOIS: Indeed – I have read them with interest. But, firstly, I must point out that M. Argand has made some mistaken claims, in particular, his supposed simplification of d'Alembert's inadequate proof of the fundamental theorem first satisfactorily proved by Gauss. And also his assertion that the imaginary quantity $\sqrt{-1}^{\sqrt{-1}}$ cannot be given expression in the standard form. This latter has been shown by Euler to be expressible as a real number.[13]

ARGAND: [*shocked*] Monsieur –

GERGONNE: [*interrupts, exerting editorial authority*] Let us hear what else M. Servois has to say.

[13] This and his subsequent speeches are based on a letter to the editor in November 1813 (Servois 1814).

SERVOIS: [*turning to Français*] I must also point out to you, M. Français, that you do not adequately demonstrate much of what you assert.

FRANÇAIS: [*angrily, may improvise French expostulations*] Excuse me, M. Servois, but I really do not think –

GERGONNE: [*pacifying gestures*] Gentlemen, we stand before novel ideas here, let us judge and debate with patience and respect. M. Servois, if what you claim is true, readers of this journal will be in your debt for drawing attention to these points. But I think it would be greatly appreciated if you would give your opinion as to the usefulness of this new mode of perceiving, reasoning and calculating with imaginary quantities.

SERVOIS: I myself am very doubtful that this geometrizing of the imaginary quantities, giving rise to the peculiar notation of these authors, is of great value.

FRANÇAIS & ARGAND: [*emotional gestures and French exclamations*]

GERGONNE: [*waving hands*] Monsieurs! Monsieurs! Let us hear him out!

SERVOIS: In setting up the foundations of an extraordinary doctrine, somewhat opposed to received principles, in a science such as mathematical analysis, mere analogy is hardly a sufficient mode of reasoning. I see the so-called geometrical representation as nothing more than a geometrical mask, to be worn over the analytical forms as some sort of softening disguise. But I myself regard the calculation with imaginaries, by the usual fully understood laws, as simpler and more effective than resorting to geometry.

GERGONNE: M. Servois, do you really consider it unimportant to see, at last, algebraic analysis stripped of its unintelligible and mysterious forms – stripped of the nonsense that limits it and makes it, so to speak, a cabalistic science?

SERVOIS: I see the future of algebra as pure analysis of symbols, without any need to introduce geometrical disguises and analogies. If one wishes to make the algebra more palatable by this means then, of course, one is free to think with the aid of geometry. But algebra deserves to be set free to be a science in its own right, at least when practised by initiates who do not cavil at imaginary numbers as cabalistic symbols, but accept them on an equal footing with what we call real numbers.

GERGONNE: M. Français, what is your view of this – ah – visionary description of the progress of algebra?

FRANÇAIS: [*still seething*] I have published my exposition in order to draw attention to, and give reasons for, my important affirmation, given as Corollary 3, which it seems is precisely the view of M. Servois, although he does not give cogent reasons for his view. I say: 'Imaginary numbers are just as real as positive numbers and negative numbers, and only differ in their position, which is perpendicular to the latter.' The imaginary part of $a + \sqrt{-1}\,b$ is to be taken at right angles to the line, so that the quantity, otherwise without clear meaning, can be visualised and accepted without hesitation.

ARGAND: That is quite right – I have made the same assertion –

SERVOIS: [*airily, even scornfully*] I deny that there is anything essentially 'perpendicular' about imaginary numbers – that is just the geometric mode in which you choose to see them. Analogies may help some, but they can also distract from essential commonalities and distinctions.

GERGONNE: Monsieurs Français and Argand, what is your response?

FRANÇAIS: M. Servois, you merely *claim* equal footing for imaginaries and real numbers, but many will choose to disagree with you. By this geometric argument, it can be *shown* to be valid.

ARGAND: For my part, it was only when I saw that numbers can be regarded as transformations of points in the plane, that I found myself at peace with the imaginaries, which were previously unintelligible and mysterious to me. The number 1 leaves every point where it is; the number -1, which some have regarded as *reflecting* the line in its origin, can more fruitfully be seen as *rotating the plane* through 180 degrees, or two right angles. Whereupon it becomes clear that there can be a transformation that rotates the plane through one right angle, and that this perfectly concrete and intelligible transformation then has a square equal to the transformation -1. Indeed, there are two such transformations – another is that which rotates the plane through three right angles, or simply a negative right angle. Thus we have two square roots of -1. And three cube roots, etc.

GERGONNE: Well expressed, M. Argand! Monsieurs, we shall let the arguments rest for now – I invite you all for dinner at my favourite restaurant!

[*rises to his feet and waves his arms in all-embracing gestures*]

No doubt we may await developments in both M. Servois's pure algebra and also the geometric point of view. It will be interesting to see what fruit there is from this clarification of the mysteries of imaginary quantities. If the uniting by Descartes of algebraic equations and geometric curves has been so fruitful, may we not expect comparable fertility from this new union of the algebraic with the geometric? *Viva la mathématiques!*

[*Argand, Français and Servois all stand, and follow him offstage, engaged in vigorous and emotional debate, with dramatic gesticulations*]

CURTAIN FALLS

MARIA: Gergonne's prophecy would be wonderfully fulfilled in the development of complex analysis, beginning with Cauchy's work just a year later.[14] Servois is, like his English contemporary, Charles Babbage, a radical formalist – he believes in symbolic expressions as the sure carriers of mathematical thought, needing no justification or geometrical representation. His views are not representative of many others of his time. EMMY: [*laughing*] Fortunately! – thank goodness for the Argand diagram! But that statement by Français that 'imaginary numbers are just as real as positive numbers and negative numbers' is now a truism of algebra.

[14] Cauchy himself attributes priority in discovery of geometric representation of imaginaries to 'un savant modeste', Henri-Dominique Truel, who mostly kept the ideas to himself from about 1786. Details are in (Schubring 2001), 136.

MARIA: Yes, indeed, but at that time it was a remarkable insight, echoing Argand's own views. There would be few affirmations like it in the literature for many decades to come! We can truly admire the foresight of Gergonne in encouraging debate on radically new ideas, bringing together, in a corner of his journal, pure algebraic formalists and pragmatic geometric conceptualists, and envisioning the power of this new union of algebra and geometry!

EMMY: I see Gergonne as an early exemplar of the immense influence that journal editors can have, and the responsibility they share, in the propagation and cross-fertilisation of ideas.

MARIA: Absolutely, and in the reception of novelty. Gergonne demonstrates what a vital role an editor can play in shaping disciplinary discourse, and negotiating disciplinary boundaries.

EMMY: Yes, I agree that editors are important agents in filtering valuable ideas, encouraging correspondence, mediating disputes, and influencing research priorities. On the negative side, however, they can sometimes be responsible for discouraging promising young researchers and killing new ideas.

MARIA: Sounds like you have had, or seen, some bad experiences, Emmy! So, you obviously think editorial influence and responsibility is just as important today in the mathematical and scientific communities, even in the age of the internet?

EMMY: Oh, yes! The business of quality control and the process of overseeing selection, and bringing to wider attention the right articles among the rising tide of submissions, is crucial for the development of mathematics. And the right articles may not emanate from the big names only!

MARIA: Right – Gergonne might have ignored the offerings of Français and Argand. Our story is a fascinating example of the way advances are made by the cooperative efforts of many lesser-known and even unknown contributors.

EMMY: And also of the vital importance of the *go-betweens* in encouraging conversation and collaboration – mentors and sponsors, journals and editors, all the activities of scientific societies.

MARIA: How significant was it that Gergonne the editor was an excellent mathematician?

EMMY: For our story, highly significant, I think. He had mathematical foresight, he enjoyed the deep respect of his contributors and could engage with them technically. Even today I think journals can benefit greatly from editors who work, or have worked, in the field. Of course, most editors now have an editorial board and a stable of trusted reviewers.

MARIA: Emmy, what corresponds today to Gergonne's innovative 'Philosophy of Mathematics' section?

EMMY: I think it's even more important, today, for journals to provide a forum for review articles, correspondence, debate and interdisciplinary discourse, as researchers are forced to focus on ever more narrow specialisations.

MARIA: Then let us celebrate Gergonne, and his editorial heirs, for their great service to mathematics! And let us also applaud the contributors to Gergonne's journal whom we saw in debate, and their forerunners in developing the fruitful ideas they were discussing –

EMMY: – which, two centuries later, are part of the basic toolkit of mathematics – the heritage of all mathematics students!

[*Maria invites the cast on stage and leads applause, then holds hands with Emmy and the rest, and they take a bow*]

THE END

Acknowledgements

I wish to express my indebtedness to Barry Mazur, whose book *Imagining Numbers* first gave me the inspiration for a dialogue centred around Gergonne, and to Gert Schubring, who kindly drew my attention to his investigations of Argand and pointed to the drama inherent in Argand's encounter with Legendre.

REFERENCES

Argand (1806/1874). *Essai sur un maniére de représenter les quantités imaginaires dans les constructiones géométriques* (1806), anonymously printed. Republished Paris: J. Hoüel (1874).

Argand (1813). Éssai sur un maniére de représenter les quantités imaginaires dans les constructiones géométriques. *Annales de mathématiques pures et appliquées*, *4*(5), 133-148.

Buée, A. Q. (1806). Mémoire sur des quantités imaginaires. *Philosophical Transactions of the Royal Society of London. For the Year MDCCCVI, Part 1*, 23–88.

De Moivre, A. (1738/1809). Reduction of radicals to more simple terms. In C. Hutton, G. Shaw, & R. Pearson (Eds.), *Philosophical Transactions of the Royal Society of London from their commencement in 1665 to the year 1800; Abridged, Vol. 8*, 1735-43.

Euler, L. (1748/1988). *Introductio in analysin infinitorum.* 2 Vols. Lausanne: M. M. Bousquet (1748). Trans. J. D. Blanton, *Introduction to the Analysis of the Infinite*. New York: Springer (1988). For this and all Euler's works see the Euler Archives: http://eulerarchive.org

Euler, L. (1749/1751). Recherches sur les racines imaginaires des équations. *Histoire de l'Académie Royale de Sciences et Belles. Vol. 5.* Berlin (1749), pp. 222–228. Also *Mémoires de l'Académie des Sciences de Berlin 5* (1751). In *Opera Omnia, Ser. I, Vol. 6* (pp. 78–147).

Euler, L. (1770/1822/1984/2006). *Elements of algebra*. Longman (1822), 3rd English edition, trans. from the German *Vollstandige Anleitung zur Algebra* by F. Horner (who started the project but is not officially recorded) & J. Hewlett; reprinted New York: Springer-Verlag (1984); Part I re-printed: C. J. Sangwin (Ed.),Tarquin (2006). Euler Archives E387/8.

Français, J.-F. (1813a). Nouveau principes de géometrie de position, et interprétation des symboles imaginaires. *Annales de mathématiques pures et appliquées*, *4*(2), 61-72.

Français, J.-F. (1813b). Lettre sur la théorie des quantités imaginaires. *Annales de mathématiques pures et appliquées*, *4*(5), 222-230.

Mazur, B. (2004). *Imagining numbers (particularly the square root of minus fifteen)*. London: Allen Lane (2003); Penguin Books (2004).

Otero, M. H. (1997). *Joseph-Diez Gergonne (1771–1859): Histoire [logique] et philosophie des sciences*. Nantes: Centre Français Viète. Reviewed by G. Schubring, *Isis 90*, no. 3, September 1999, 603-604.

Schubring, G. (2001). Argand and the early work on graphical representation: New sources and interpretations. *Around Caspar Wessel and the geometric representation of Complex Numbers. Proceedings of the Wessel Symposium at the Royal Danish Academy of Sciences and Letters*, 125–146. Copenhagen, August 11–15 1998: Invited Papers. Matematisk-fysiske Meddelelser 46:2, J. Lützen & C. A. Reitzel (Eds.).

Servois, F. J. (1814). Essai sur un nouveau d'exposition des principes du calcul differentiel, et, en particulier, sur la doctrine des infiniment petits. *Annales de mathématiques pures et appliquées*, *5*, 141-170.

Smith, D. E. (Ed.) (1959). *A Source Book in Mathematics*. New York: Dover.
Wallis, J. (1685). *De Algebra Tractatus, Historicus et Practicus. (A treatise of algebra both historical and practical, shewing the original, progress, and advancement thereof, from time to time, and by what steps it hath attained to the heighth at which now it is.).* London. In: *Opera mathematica*, 3 Vols, 1693–99; reprinted in facsimile, Hildesheim: Olms, 1972, 1–482. Translation of extracts in (Smith 1959) p. 48.
Warren, J. (1828). *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities.* Cambridge.
Wessel, C. (1799/1897/1999). *Essai sur la représentation analytique de la direction*. Translation of: *Om Directionens analytiske Betegning. Préfaces du H. Valentiner et T.-N. Thiele*. Copenhague (1897). A partial English translation is in (Smith 1959), pp. 55–66. Full English translation: *On the Analytical Representation of Direction. An Attempt Applied Chiefly to Solving Plane and Spherical Polygons. Trans. Damhus F. With introductory chapters by B. Branner, N. Voje Johansen, & K. Andersen*. Branner B. & Lützen J. (Eds.), Matematiske-fysiske Meddelelser 46:1, Det Kongelige Danske Videnskabernes Selskab. Copenhagen (1999).

2. Discussion Groups

GEOMETRY IN THE SECONDARY SCHOOL CURRICULUM AND IN PROGRESSION TO UNIVERSITY

Discussion Group 1

Évelyne BARBIN, Leo ROGERS
IREM & Laboratory LMJL, University of Nantes, France
evelyne.barbin@wanadoo.fr
University of Oxford, United Kingdom
leorogers@appleinter.net

ABSTRACT

There has been considerable debate in recent years showing concerns about the place of geometry in the general school curriculum. The impression that the place of geometry in the curriculum, in programmes of study for school students, and in the courses for training teachers has significantly diminished in status. Furthermore, the impression the lack of a coherent geometrical education in school is a factor that is deleterious to many students' progress in their university education. One key factor in this context is the great concern over students' ability to understand the purpose and nature of 'proof' in mathematics.

In order to begin to understand whether, and to what extent, many of these concerns may be shared among colleagues, we devised and circulated a Questionnaire that was sent out to colleagues between August 2015 and March 2016. Until now, we have received 24 answers from 23 countries all other the world. The Questionnaire has two main areas of investigation where colleagues have been asked to comment upon the following themes.

1 Geometry teaching in school from 1970 to 2015

There have been considerable changes in most countries' school curriculum since 1970, and this has caused (or been caused by) political or institutional changes affecting school systems.

These changes have had an effect on the presentation of geometry in school text-books and also on the training of teachers.

2 Geometry in your country now

What kind of geometry is taught at different stages (or pupils' ages) in the school curriculum: for example, deductive geometry, transformation geometry, analytical geometry, vector-based geometry.

What software systems are used in the teaching and learning of geometry: for example, Cabri, Geometers Sketchpad, Geogebra.

Colleagues were also asked whether they felt that currently, students having little experience of geometry when they enter university, that this lack of geometrical experience is detrimental to their progress at university.

We thank all our colleagues who have already contributed to this enquiry; detailed analysis of the data will become available later.

If you are interested in participating in this debate, please write to Évelyne Barbin and Leo Rogers.

HISTORY OF MATHEMATICS IN TEACHERS' EDUCATION

Motivation for and activities of Discussion Group 2

Kathleen CLARK, Sebastian SCHORCHT
Florida State University, USA; University of Giessen, Germany
kclark@fsu.edu; s.schorcht@gmx.de

Discussion Group 2 (DG 2), in a sequence of two one-hour sessions, seeks to revisit the discussion about the role of history of mathematics in teachers' education – both for pre-service teachers and currently practicing teachers. There have been several efforts to address the role of history of mathematics in this way (e.g., Beutelspacher et al., 2011; Clark, 2014; Jankvist, 2009; Siu, 1997), and which have appeared in various forms, including articles, monographs, book chapters, conference presentations, and, during HPM 2012 in Daejeon, South Korea, a panel discussion. The panel "Why Do We Require a 'History of Mathematics' Course for Mathematics Teacher Candidates?" obviously focused on a particular use of history (in the form of a history course of some sort) with pre-service teachers. The activities of Discussion Group 2 will build on some of the impressions we gleaned from the small group discussions that occurred during the audience participation of the "Why Do We Require…" panel in 2012.

Participants of the "Why Do We Require…" panel were asked to discuss in small groups one or more of four given prompts, and we provide the prompts and a summary of responses from HPM 2012 for each.

Prompt 1: Identify one or two valuable aspect(s) of a "History of Mathematics" course (from either the perspective of having taken or taught such a course before).

- History of Mathematics (HM) helps students to gain true understanding of how mathematics developed.
- Mathematics is not just the results but also the process of discovery that led to the results. Mathematics was developed by the need of society. If students understand mathematics on a basis of developmental stages, they may think that it is valuable to teach.

Prompt 2: Identify one or two obstacles that may arise in implementing a "History of Mathematics" course. Describe ways in which the obstacles can be addressed.

- From an administrative perspective: We should persuade other faculty members to implement an HM course. That is, there are often too few persons able or willing to teach HM.
- There are few courses for HM and pedagogy; rather, pure mathematics is more likely to occupy the curriculum in teacher education programs.

Prompt 3: Describe the benefits to teacher candidates that requiring a "History of Mathematics" course may provide (again, based on actual experience or what you believe).

- If teacher candidates know HM, they may have access to create lesson plans for helping students with guiding them in a way of gradual mathematizing.

Prompt 4: With regard to the potential content and pedagogy of such a course, what are examples of tasks that we could require?

- If we let students create and perform plays or movies where they consider themselves a mathematician, it will provide them a positive experience in learning mathematics.

The aim of DG 2 is to provide a venue for deeper discussion on prompts 1 through 4, and to do so with respect to teachers' education more broadly. In particular, we hope for more robust examples of content and pedagogy (prompt 4) for use in mathematics teacher education to emerge from the DG activities. Thus, we plan to structure DG 2's activities in the following manner. First, we will summarize examples from the literature that propose benefits for teachers and their students when history of mathematics is utilized to teach (and, consequently, learn) mathematics. The remainder of the first hour of the discussion group (DG) will be spent on participants sharing experiences from their own contexts. The DG facilitators will capture the results of the discussion, noting concrete examples from university courses, professional development programs, and projects, and when possible, the observed and/or documented benefits derived from the use of history of mathematics resulting from these examples.

During the second hour of the DG, participants will focus on sharing and discussing specific tasks or activities, which may serve as examples for contexts that do not currently possess a strong history of mathematics dimension within mathematics teacher education programs, or which may provide new examples for those who do. A key product of the DG is to produce a document that contains a description of examples, notation of potential uses, and contact information for persons who either devised or implemented the sample task or activity. The compilation of the DG efforts will be disseminated via the HPM Newsletter or website at the conclusion of the HPM 2016 conference.

REFERENCES

Beutelspacher, A., Danckwerts, R., Nickel, G., Spies, S., & Wickel, G. (2011). *Mathematik Neu Denken: Impulse für die Gymnasiallehrerbildung an Universitäten.* Wiesbaden: Vieweg+Teubner Verlag.

Clark, K. M. (2014). History of mathematics in mathematics teacher education. In M. R. Matthews (Ed.), *International handbook of research in history, philosophy and science teaching* (pp. 755-791). Dordrecht: Springer Science+Business Media.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235-261.

Siu, M.-K. (1997). The ABCD of using history of mathematics in the (undergraduate) classroom. *Bulletin of the Hong Kong Mathematical Society*, *1*(1), 143-154.

ORIGINAL SOURCES IN THE MATHEMATICS CLASSROOM

Janet Heine BARNETT, Tinne Hoff KJELDSEN
Department of Mathematics and Physics, Colorado State University-Pueblo, Pueblo CO, USA
janet.barnett@csupueblo.edu
Department of Mathematical Sciences, University of Copenhagen, Copenhagen, Denmark
thk@math.ku.dk

ABSTRACT

This discussion group seeks to bring together individuals who are interested in the use of original sources in the mathematics classroom, from the perspective of a classroom teacher or a mathematics education researcher, for a discussion of issues and concerns related to their educational potential and effects. Each of the two sessions will focus on a different theme related to the use of original sources in the mathematics classroom. The two sessions will structured around a common framework but sufficiently independent of each other to allow interested individuals to participate in the second session, even if they did not participate in the first session. Both novice and more experienced users of original sources are strongly encouraged to participate in both sessions.

1 Overview: The value of an explicit-reflective framework

This discussion group will explore issues related to the use of original sources in mathematics classrooms as a means to create inquiry based learning experiences that invite students into (parts of) some (past) mathematicians' "work bench," as a means of allowing them to engage in learning processes that to some extent mimic what mathematicians do when they produce new mathematical knowledge. We will consider in particular the importance of (a) making the educational ambitions and learning objectives of such experiences *explicit* and (b) designing activities that promote student *reflection* on their learning experiences. The notion of an explicit-reflective framework[1] will thus serve as an organizing theme for a discussion of different strategies for using original sources with students as a means to teach them mathematics, to enhance their understanding of the nature of mathematics, to develop their historical awareness or to achieve more general educational goals. We will also consider the ways in which such a framework opens up the possibility of establishing complex learning environments and situations that may address combinations of these learning goals.[2] As part of this discussion, we will address the questions of "what, how, and why" regarding the potential roles, uses and benefits of integrating original sources into mathematics education, but with an emphasis on actual classroom teaching practices (versus an investigation of the general arguments for using original sources and integrating history into mathematics education[3]). Additionally, we will consider the value of basing the development of explicit-reflective frameworks for using original sources in mathematics classrooms on a theoretical

[1] We have borrowed this notion from Abd-El-Khalick's and Lederman's (2000) work in science education. See Kjeldsen (2014) for an exploration of the framework within an existing practice of mathematics education
[2] See Barnett, Pengelley and Lodder (2014) for concrete illustrations of such learning environments.
[3] For this we refer to Tzanakis et al. (2002), see also the survey paper by Jankvist (2011).

foundation drawn from the research areas of history and philosophy of mathematics and the teaching and learning of mathematics.

2 Session 1: Educational Potential of Original Sources

The theme of the first session is the educational potential of original sources and the notion of explicit-reflective frameworks. Questions related to this theme that will be discussed include:

- Why are we as mathematics teachers and/or researchers interested in using original sources with students? What are potential gains for students? For instructors?
- How are original sources being used in math teaching, by whom, for what purpose?
- Based on different pedagogical goals and learning objectives, what type of original source readings and reflective challenges for students are most appropriate?

3 Session 2: Resources and Research Needed to Support Classroom Use

The second session will focus on what is needed in terms of classroom resources and educational research to support the use of original sources in the classroom in order to promote their full potential for students. Questions related to this theme include:

- What do we know about the effect of learning mathematics from original sources? What would we like to know? How could we find that out?
- What resources exist/are needed to assist instructors in using original sources?
- Can we develop general principles for source selection and guides for creating explicit-reflective learning environments?

4 Overall structure of discussion sessions

Each session will begin with a brief introduction to that session's overall theme to set the scene for the session. Participants will then work together in smaller groups for 20-25 minutes, guided by discussion worksheets to be provided by the organizers, to examine issues related to that session's theme. A plenum discussion that brings all participants back together in order to exchange and synthesize ideas generated by the various groups will round out each session. Possible opportunities for and interests in forming future working groups will be explored and facilitated in each plenum discussion.

The two sessions will be sufficiently independent of each other to allow interested individuals to participate in the second session, even if they did not participate in the first session. Both novice and more experienced users of original sources are strongly encouraged to participate in both sessions.

REFERENCES

Abd-El-Khalick, F., & Lederman, N. G. (2000). Improving science teachers' conceptions of nature of science: A critical review of the literature. *International Journal of Science Education*, *22*(7), 665-701.

Barnett, J., Pengelley, D., & Lodder, J. (2014). The pedagogy of primary historical sources in mathematics: Classroom practice meets theoretical frameworks. *Science and Education*, *23*(1), 7-27.

Barbin, E. (1997). Histoire et enseignement des mathématiques: Pourquoi? Comment? *Bulletin de l'Association Mathématique du Québec*, *XXXVII*(1), 20-25.

Barbin, E. (2011). Dialogism in mathematical writing: historical, philosophical and pedagogical issues, In V. Katz, & K. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in Mathematics Education* (pp. 9-16). Washinngton: Mathematical Association of America.

Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., Idrissi, A., da Silva, C. M. S., & Weeks, C. (2002). The use of original sources in the mathematics classroom. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education: the ICMI study* (pp. 291-328). Dordrecht: Kluwer Academic Publishers.

Jankvist, U. T. (2009). A categorization of the 'whys' and 'hows' of using history in mathematics education. *Educational Studies in Mathematics*, *71*, 235-261.

Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation - History as a method for the learning of meta-discursive rules in mathematics. *Educational Studies in Mathematics*, *80*(3), 327-349.

Kjeldsen, T. H. (2014). Teaching with and about the nature of mathematics through history of mathematics: Enacting inquiry learning in mathematics. *Education Sciences, Special issue*, 38-55.

Tzanakis, C., Arcavi, A., Correia de Sa, C., Isoda, M., Lit, C.-K., Niss, N., Pitombeira de Carvalho, J., Rodriguez, M., & Siu, M.-K. (2002). Integrating history of mathematics in the classroom: An analytic survey. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education: the ICMI study* (pp. 201-240). Dordrecht: Kluwer Academic Publishers.

3. Panels

PANEL 1: THEORETICAL AND/OR CONCEPTUAL FRAMEWORKS FOR INTEGRATING HISTORY IN MATHEMATICS EDUCATION

Michael N. FRIED, David GUILLEMETTE, Hans Niels JAHNKE
Ben Gurion University of the Negev, Program for Science and Technology Education,
POB 653 Beer Sheva 84105, Israel
mfried@bgu.ac.il
University of Ottawa, Faculty of Education, 145 J-J Lussier, Lamoureux Hall (room 309),
K1N 8C8 Ottawa, Canada
david.guillemette@uottawa.ca
Universität Duisburg-Essen, Fakultät für Mathematik, Thea-Leymann-Str.9, 45127 Essen,
Germany
njahnke@uni-due.de

ABSTRACT

This panel considers theoretical rather than practical frameworks for the alignment of history of mathematics and mathematics education. Naturally, such a theoretical framework will have practical implications, but its main task is to set out a certain line of questioning. First of all, it must also ask, even if only implicitly, what it really means to be a theoretical framework in the first place. But, principally, it must ask the ends and the value, and, ultimately, the meaning of history of mathematics in mathematics education. Though all the authors agree that in any theoretical framework, history of mathematics, as such, must be taken as the starting point. That said, no claim is made that there must be a single theoretical framework. Nevertheless, each of the three parts of this paper emphasize similar themes. One of these is the relationship between readers and texts: what do readers bring to texts? How are they affected by the texts? How are their own mathematics selves shaped by their engagement with historical material?

1 The idea of a theoretical framework

Michael N. Fried

The task of this panel is not to set out specific practical approaches for bringing history of mathematics into the mathematics classroom: the aim is not to produce a catalog of options from which teachers can choose an approach fitting their needs, and it is not to argue for one particular ideal approach. Rather, its task is to try and delineate questions—some key questions at least—which any theoretical framework must embrace or suggest in the first place. For any theoretical framework is at bottom a set of questions giving rise to other questions.

Of these key questions, the very first, and maybe the most important question, is a reflexive one: what does it mean to be a theoretical framework in the first place? More concretely, what does it mean to be a theoretical framework, as opposed to a practical framework—what should its object be? What should it look like? Certainly, it cannot be of the form, "Do X, do Y, do Z," or even "Try X, Y, or Z." Such directives would imply a theoretical framework already in place. They imply that we know *why* we are to try X or Y, that we know what ends we are trying to achieve. Any theoretical educational framework ought to be cognizant of ends, of what goals one is aiming for and why they are good. It differs in this way from a theoretical framework for a natural science where the main concern is a description of how things are and why they are as they are: we would not expect a physical theory to say why it is better that like poles of a magnet repel than attract, that is, while it might say why like poles repel, it does say why they ought to repel. So a theoretical framework for history of mathematics in mathematics education should at least address the question of why we ought to learn history, what is to be gained by it, what value there is in it and what value there is in learning it in a certain way.

But it is not enough to have a reason for teaching history of mathematics. For one's end in bringing history of mathematics into mathematics teaching might have little to do with history. This is the case, surely, when teachers suggest using history of mathematics to "liven" their teaching, so that what the teacher really wants to teach, say, the quadratic formula, will be met by students more awake and more willing to learn. Let me say a little more about this motive, which, following Gulikers and Blom (2001), I have called elsewhere (Fried, 2014a), the "motivational theme."

Assume that history truly has this motivational effect on students and that, further, it is not merely the effect of novelty, that is, it persists even after it has become routine. For the motivational theme to be the basis of a theoretical framework for teaching history of mathematics in the mathematics classroom, one would have to ask what is it about history that touches students and moves them. What part of their intellectual lives is touched by history? Asking these questions is essential to the question of a theoretical framework; however, if we obtain an answer to them or even if we manage to bring out the force of them, it will be something quite different from a statement of the form, "history of mathematics increases students' motivation." The affective gain will, in this regard, be a kind of epiphenomenon only. To be sure, whether or not students enjoy mathematics or are motivated to study it with more vigor is not something to be ignored. The point is only that it itself does not provide a theoretical framework for introducing history of mathematics into mathematics teaching: in providing an end for the study of history, a theoretical framework must refer to history, not to pleasure, as important as that might be.

The purely instrumental role of history of mathematics, by pointing to ends foreign to history as such, can blind us in our search for a theoretical framework specifically for history of mathematics in mathematics education. The problem is that indeed those other ends are important goals for mathematics education. In this connection, a more difficult and subtle case in which ahistorical goals are made goals for history is that concerning students' understanding of mathematical content.

Whatever other knowledge we hope students will gain in their mathematical education—and I will return to this point shortly—one could hardly imagine excluding knowledge of concepts, propositions, and procedures central to modern mathematics, even if one can argue that this concept or this proposition or that procedure may be safely eliminated. Assuming the importance of mathematical ideas as they are understood today, history is often taken as a means of making these ideas more accessible. In the context of the motivational argument, more accessible only means more palatable; in the context of mathematical knowledge, *per se*, more accessible means more understandable or more deeply understandable, though these are not exactly the same.

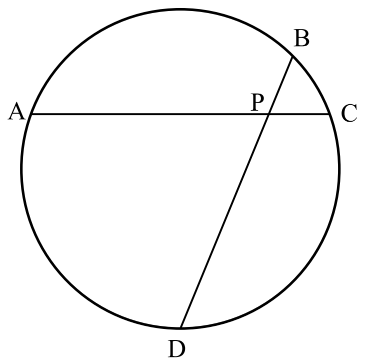
A book such as André Weil's *Number Theory An approach through history from Hammurabi to Legendre* (Weil, 2007) is good example of this use of history. The very title makes that clear: it is a book *about* number theory; its *approach* is through history. It is a good example too because Weil was a superb number theorist, and his knowledge of the relevant texts, particularly those Fermat, Euler, Lagrange, Legendre, and Gauss, is learned and thorough. He reads these texts through the eyes of a modern mathematician, through the eyes of the superb number theorist that he was. And he does so unapologetically. To take a random example, having described Fermat's judgment as to the difference between Diophantus and Vieta, Weil writes:

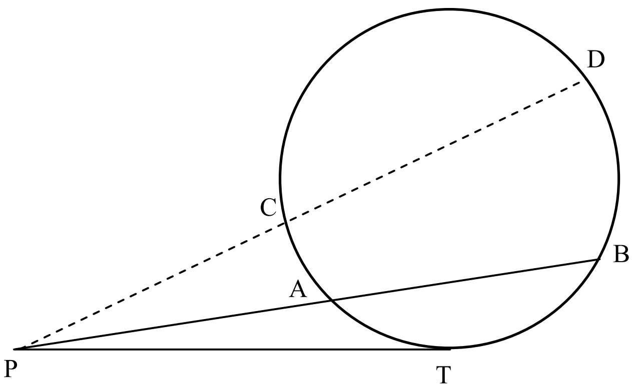
> From our modern point of view, things look somewhat differently. Firstly, since so much of Diophantus, and even more of Viète, remains valid over arbitrary fields, we would classify this primarily as algebra; of course Viète's algebra, both in its notations and in its content, is far more advanced, and much closer to ours, than that of Diophantus. Secondly, the distinction between rational numbers and integers is not as clearcut as the above quotation from Fermat would suggest; as he knew very well, it does not apply when one is dealing with homogeneous equations. Thirdly, as will be seen presently, there is much, in Diophantus and in Viète's *Zetetica*, which in our view pertains to algebraic geometry. Moreover, modern developments have led to a better understanding of the analogies (already dimly perceived on some occasions by Leibniz and Euler) between function fields and number-fields, showing that there is sometimes little difference between solving a problem in rational numbers and solving it in a field of rational functions (Weil, 2007, p.25)

The point is not to criticize Weil's book or to praise it. It is only to make clear that what guides the exposition is Weil's own modern understanding of number theory. For him there is no problem in this. He clearly sees that the mathematical content that interests him is, perhaps in an inchoate fashion, also guiding Fermat, Euler, and the others. For him, setting out these thinkers thoughts is precisely showing the steps in the modern understanding. Is he wrong? This depends on what ultimately is the nature of mathematics, whether it is a Platonic body of ideas or a cultural production. But if mathematical content guides history itself and an historical approach for teaching mathematics, then that Platonic core content, and not history as such, is at the bottom of one's educational framework. If one is to have a truly historical framework then the historical nature of mathematics must at least be put on the table as something to question: the nature of mathematics itself must be problematized. This requires distance, or, to use the potent phrase found in Evelyn Barbin's works, *dépaysement* (e.g. Barbin, 1997), a sense of foreignness, of "displacement," regarding the mathematical thought of the past.

To take a simple example, consider proposition 35 (and 36) in Book III of Euclid's Elements: "If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other." Following typical textbook formulations, such as that of Legendre, Steiner (1826) restates the theorem and combines it with Elem.III.36 as follows:

> From an arbitrary point in the plane of a circle M, let there be straight lines PAB, PCD,.. be drawn cutting the circle: then the product (rectangle) of the segments from point to the intersection points of the circle is a constant quantity, that is: PA×PB=PC×PD=... (p. 22).





By this means, Steiner, and Louis Gaultier (1813) before him, redirects our attention from the rectangles which interest Euclid (for it is not by accident that these propositions end the book that follows upon Book II concerning equal rectangles and squares) to a point with respect to a circle and a number associated with it. For Steiner and Gaultier, *this* is the content of Elem.III.35,36. We will never think otherwise if we do not allow ourselves to be "displaced" by way Euclid himself states these propositions and how and when he chooses to prove them (see Fried, 2004, for more details). So, to reiterate, if we are to have a framework for aligning history of mathematics with mathematics education, such a framework must invite us to question what we mean by mathematical content, and not assume that it is given in advance.

But if we are to avoid a dogmatic attitude towards mathematical content so as to leave room for history. we ought also avoid a dogmatic attitude towards history itself. Our own view of mathematics of the past must be problematized. Whether or not Weil's understanding of the historical nature of mathematics is true or not, it does represent a kind relationship with the past, and there are many possible relationships (this was discussed in Fried, 2014b). In asking about a theoretical framework for history in mathematics education, we ask about history, and that means, in large part, that we ask what it means to stand facing the past: our own posture towards the past must be explored.

This kind of exploration is a kind of exploration of ourselves. It echoes to a degree with what Collingwood saw as the heart of historical knowledge. As is well-known, Collingwood held that history involves the "re-enactment of past experience," and the object of history is historical inasmuch as it can be re-enacted in one's mind. What this means is not that we become Thomas Becket, to use his example, by thinking his thoughts, but that we come to know Becket, and that we perfectly aware that this is what we are trying to do. As he puts it:

> I do not 'simply' become Becket, for a thinking mind is never 'simply' anything: it is its own activities of thought, and it is not these 'simply' (which, if it means anything, means 'immediately'), for thought is not mere immediate experience but always reflection or self-knowledge, the knowledge of oneself as living in these activities. (Collingwood, 1993, p. 297)

Earlier—and repeatedly—Collingwood argues how this reflective aspect of historical knowledge makes historical knowledge simultaneously subjective and objective, and that this is the distinctive character of such knowledge:

> Historical knowledge is the knowledge of what mind has done in the past, and at the same time it is the redoing of this, the perpetuation of past acts in the present. Its object is therefore not a mere object, something outside the mind which knows it; it is an activity of thought, which can be known only insofar as the knowing mind re-enacts it and knows itself as so doing. To the historian, the activities whose history he is studying are not spectacles to be watched, but experiences to be lived through in his own mind; they are objective, or known to him, only because they are also subjective, or activities of his own. (p. 218)

If we consider that education involves not only informing students of what is known or how things are done but also forming students' thinking, their own self-awareness as intelligent beings, and their own relationships to their world, then an educational framework for history of mathematics in mathematics would have to attend to precisely this kind of mix of the objective and subjective. And, if Collingwood is correct about his view of history, then tending to history in the history of mathematics would be one way of leading to this more general educational goal. At the same time, it shows how history of mathematics aligns mathematics education with ends of general education without sacrificing attentions to its more circumscribed goals connected to mathematical concepts, ideas, and procedures.

The two sections which follow, one by Hans Niels Jahnke and one by David Guillemette, will present ways of thinking about the incorporation of history of mathematics from a theoretical point of view, as we have been speaking about it presently. We shall see that they are complementary, and in some ways both concern the relationship between students or teachers with their mathematical past and with the texts that have come down from the past.

2 The Hermeneutic Approach: Reflections and Extensions

Hans Niels Jahnke

In the following remarks, I shall present one of the more sophisticated approaches to history of mathematics in the classroom, the so-called "hermeneutic approach". It has been elaborated in several papers, as for example Jahnke (1994) and (2012), Glaubitz (2010) and (2011). In a second step I shall extend my discussion by pointing at two papers, Kjeldsen & Blomhøj (2012) and Arcavi & Isoda (2007) which elaborate on the issue of what a student can gain by the “experiences of difference” (see Fried 2001) which history provides. The whole discussion fits very well to the catch-word „experience of the subject“ proposed by David Guillemette.

2.1 A pragmatic categorical imperative

Frequently, Otto Toeplitz’s famous saying from 1926 that going to the roots of mathematical concepts would remove the dust of time and the scratches of long use from them (Toeplitz, 2015) is interpreted in a way that we should use history when we introduce a new mathematical idea or concept in the classroom. May be, this sometimes works. But in general, a first historical appearance of an idea is clumsy and difficult to understand to a modern mind. This is no wonder since, in general, new knowledge does need a long process of clarification before we really understand it. Thus, introducing a new subject through the door of history seems to me simply unrealistic, and we should not overload the enterprise “History and pedagogy of mathematics” by demands which necessarily must lead to failure. One can call this a pragmatic categorical imperative.

Thus, the hermeneutic approach grew first of all out of the idea that you should confine history to a local experience. Students are asked to examine a source in close detail and explore its various contexts of historical, cultural and scientific nature. The hermeneutic approach will not give you an overview. Rather, it is a hope that some pupils will like history and develop a certain interest in it which might motivate them to search for further reading.

The basic guidelines of the hermeneutic procedure can be summarized in 6 principles.

(1) Students study a historical source <u>after</u> they have acquired a good understanding of the respective mathematical topic in a <u>modern</u> form and a <u>modern perspective</u>. The source is studied in a phase of teaching when the new subject-matter is applied and technical competencies are trained. Reading a source in this context is another manner of applying new concepts, quite different from usual exercises.

(2) Students gather and study information about <u>context</u> and <u>biography</u> of the author.

(3) The historical <u>peculiarity</u> of the source is kept as far as possible.

(4) Students are encouraged to produce <u>free associations</u>.

(5) The teacher insists on <u>reasoned arguments</u>, but not on accepting an interpretation which has to be shared by everybody.

(6) The historical understanding of a concept is contrasted with the modern view, that is the source should encourage processes of reflection.

2.2 What then is hermeneutics?

Simply said hermeneutics is the “art or the science of interpreting texts.” It distinguishes systematically between the author and the reader of a text and their different perspectives. This means that the strong tension between the historical perspective and the modern conception of a mathematical topic should not necessarily be smoothed out or eliminated, rather it should be embraced as essential for how a historical text may contribute to the intellectual development of a person. . The whole enterprise of reading a source rests, then, on experiences of *dépaysement*, as the French say (Barbin 1994) or *Verfremdung* ("alienation"), as the German writer Bertold Brecht would have said. Sources introduce into teaching an unwieldy element. But how comes it that such unwieldy elements do not lead to failure? This is so only when they have anchor points. The student who deals with something that he already knows but that is presented in a radically different, unfamiliar way or an unknown guise, should be able to make connections to these anchor points. In hermeneutics you would say: His horizon merges with the horizon of the past. *Horizon merging* is a term that was coined by Hans Georg Gadamer (1900 – 2002). In the horizon merging the student may begin to wonder and to reflect upon what he possibly had never thought about before. In essence he begins to develop deeper awareness. This is in fact an instance of broadening one's horizon. And it does so by utilizing a strategy of dissonance. It is well known that this kind of incompatible information ensures greater retention and ease of retrieval from memory. But to do so, there must be a familiar reference frame. It is therefore applied only to subjects that students are already familiar with.

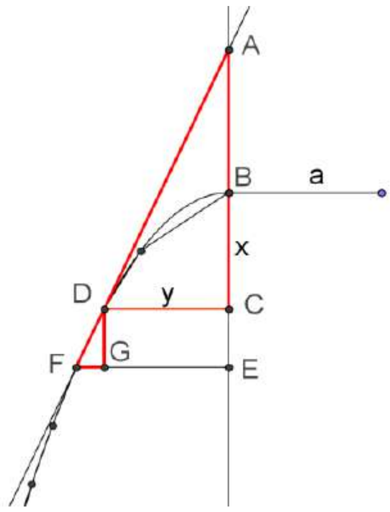
In hermeneutics the process by which the merging of horizons occurs is described by a spiral, the so-called '*hermeneutic circle*' which points to the necessity of already possessing an interpretation of a text in order to gain a new interpretation. For us as mathematics educators this appears not so strange as it might be for other people since we are used to reflect about spiral processes, the most prominent being the process of modelling.

You start with a certain image of the text reflecting your expectations about what it might be about. Then you read the text and realize that some aspects of your image do not agree with what is said in the source. Thus, you have to modify your image, read again, modify and so on until you are satisfied with the result or simply do not like to continue. Thus we have a spiral process like

$$\text{expectation/interpretation}_i \rightarrow \text{reading}_i \rightarrow \text{modififation}_i \rightarrow \text{interpretation}_{i+1}$$

In the case of a teaching unit on Johann Bernoulli's manuscript on the differential calculus (see figure 1), students started with the expectation that Bernoulli's text might be about determining tangents to and extremal values of curves, that the idea of a limit of an infinite process might be central to the subject, that concepts like derivative and slope of a tangent (a quotient) will frequently appear and that all is sort of algebra, with new rules, but symbolic in nature. After some windings of the hermeneutic spiral they had realized that the source was in fact about tangents to and extremal values of curves, but that there was no limit concept, instead there was the difficult concept of infinitely small quantity. Also, the slope of a tangent was less important than expected. It was used in the source to get rid of the infinitely small quantities, but the target object was a second point of the tangent. Thus, the status of a slope changed to that of an auxiliary object. And so on.

On a more basic level the hermeneutic circle can be considered as a process in which a *hypothesis* is put up, tested against the source, modified, tested again and so on until the reader arrives at a satisfying result. For example, the students were asked to infer from the source what a differential is. From their knowledge of calculus some students formed the idea that a differential is something similar to a derivative. With this hypothesis they studied Bernoulli's derivation of the product rule and realized that this cannot be true, since Bernoulli did not calculate a quotient. After some further attempts they saw that a differential is in fact a difference. In a similar manner they found out what a subtangent is.



Fugure 1. Bernoulli's parabola

Can one say then that students behave like historians of mathematics when they read a source? In principle, this is the case. When they enter the source they have questions similar to those a professional historian of mathematics would ask. Roughly spoken, these questions refer to the different meanings of concepts and the different conceptual structures at the time of Bernoulli and today.

The most important difference concerns the previous knowledge a historian and a student have at their disposal. For example, consider the segment in Bernoulli's sketch of the parabola representing the parameter a. A professional historian knows of course that the segment hints at the ancient geometrical definition of the parabola. Of course, the students do not know this. For the teacher it is a difficult question whether she should tell this to her students or simply let it as it is.

2.3 What students might gain: part 1

In the hermeneutic approach, mathematics enters at least in two ways. First, there is the experience of dissonance or alienation. Students learn something about their own mathematics by experiencing and *reflecting on the contrast between modern concepts and their historical counterparts*. And the point of the „hermeneutic circle" as understood here is that the reflection is in both directions, so that the students deepen both their understanding of history and of their own set of modern conceptualizations. Second, and equally important, is the fact that in reading a source (modern) mathematics itself is *applied as a tool* (Jankvist 2009). The task to think oneself into the situation of persons living at a time long ago requires to be able to argue from the assumptions of these persons, to use their symbols and methods of calculation. This poses completely new demands on the students' abilities to argue and to prove mathematically. Thus, reading a source deepens the mathematical understanding on both levels, on that of *doing mathematics* and on that of *reflecting about mathematics*.

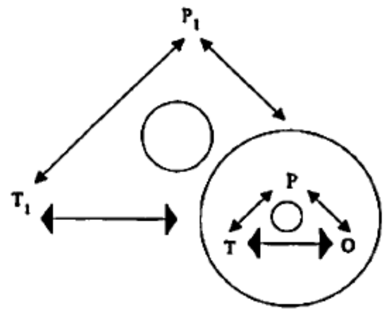


Figure 2. The 'double circle' (Jahnke 1994)

There is an important extra component students learn by such an hermeneutical experience. According to hermeneutics understanding a text consists in the merging of different horizons, the horizon of the reader and the horizon of the text/author. Different readers with their different backgrounds arrive at different interpretations. This situation becomes even more complicated when we take into account that the scientist herself can be considered as an 'hermeneutician'. Her texts are the problems she is studying and, obviously, she travels through a spiral process producing ever new images of the problem at hand. This generates a primary circle symbolized in the bottom right corner of figure 2. The primary circle is itself a component of a secondary circle (the big triangle in figure 2) through which the historian/student travels in the course of her work. This might cause a feeling of participation or solidarity. The insecurity a historian experiences when she is approaching an interpretation but has not yet arrive at a satisfactory state is mirrored by the insecurity of the scientist studying a not yet solved problem.

2.4 What students might gain: part ii—meta-discursive rules

In the following I present two papers which also reflect about what students might gain by the „experience of difference" which history provides. Kjeldsen & Blomhøj (2012) model the historical texts their university students study as „interlocutors". Reading a text amounts to enter a dialogue. Of course, when your interlocutor simply repeats what you already have in mind you might gain a feeling of security, but you will not learn anything new. Consequently, a text will be productive and teachable only when it imparts the 'experience of difference'. "This, of course, presupposes that the pitfalls of whig history are avoided" (p. 331)

If this condition is satisfied something like „meta-level learning" can happen. In order to describe what this type of learning might amount to Kjeldsen & Blomhøj take recourse to A. Sfard's theory of thinking (Sfard 2008). There again thinking is modeled as communicating, that means the thinking of an individual is considered as a special case of "interpersonal communication" or as a "combination of communication and cognition". In this way, mathematics itself becomes a type of discourse and as such an autopoietic system. Communicating is a rule-regulated activity, and Sfard distinguishes between rules "concerning the content of the discourse and discursive rules about the discourse." (p. 329)

The one are referred to as “object-level rules”, the other as “meta-discursive rules”. In regard to mathematics meta-discursive rules

> manifest their presence…in our ability to decide whether a given description can count as a proper mathematical definition, whether a given solution can be regarded as complete and satisfactory from a mathematical point of view, and whether the given argument can count as a final and definite confirmation of what is being claimed. (Sfard 2000, quoted in Kjeldsen & Blomhøj 2012, p. 329)

In its deep structure the process of learning mathematics amounts „to gradually modify the meta-discursive rules that govern a student’s mathematical discourse. As a consequence, an essential aspect of mathematics education is to create teaching and learning situations where meta-discursive rules are exhibited as explicit objects of reflection for students” (p. 330). There are not so many opportunities in the process of learning where students get a mirror of their own development in the development of other mathematical discourses. Consequently, to Kjeldsen & Blomhøj “history is, if not indispensable, then at least an obvious choice for learning meta-discursive rules in mathematics.” (p. 329) On a general level we find here again the idea that „experience of difference“ helps to “objectify the subjective”.

2.5 What students might gain (3): learning to listen

As a last example of reflection about what students might gain by the „experience of difference“ we present a paper by Arcavi & Isoda (2007) in which they focus on furthering the interpretative skills of students in the context of learning mathematics. One of the most important re-evaluations of teachers’ skills which flow from a constructivist approach to learning mathematics is the re-evaluation of “listening“. Arcavi & Isoda give a detailed analysis of what it should mean to teachers to listen to their students in a right way. They define “listening” as “giving careful attention to hearing what students say (and to see what they do), trying to understand it and its possible sources and entailments. ‘Listening,’ as we envision it, is not a passive undertaking,...” (p. 112) Listening in a right way benefits to the students as well as to the listener. It will lead to “a caring, receptive and empathic form ... of teacher-student conversations. If often modeled by teachers, students would feel respected and valued. Moreover, the listening modeled by the teacher may become internalized by students as a habit incorporated into their repertoire of learning techniques and interpersonal skills “ (p. 112).

On the other hand, “effective listening may influence ‘listeners,’ by making them reinspect their own knowledge, against the background of what was heard from others. Such re-inspection of the listener’s own understandings (which may have been taken for granted) may promote the re-learning of some mathematics or meta-mathematics.” (p.112) More concretely, Arcavi & Isoda identify three dimensions of listening.

(1) Listening should “unclog” automatisms which necessarily are connected with the “packaged knowledge” of expert mathematicians. Instead, a teacher should be able to set aside much of what he knows so well. “...an important part of learning to listen to students

would include, paradoxically, some kind of 'mathematics unlearning' on the part of the teacher" (p. 113).

(2) The right way of listening requires 'decentering capabilities'. "...we take it to mean, the capacity to adopt the other's perspective, to 'wear her conceptual spectacles' (by keeping away as much as possible our own perspectives), to test in iterative cycles our understanding of what we hear, and possibly to pursue it and apply it for a while. Such a decentering involves a deep intellectual effort to be learned and exercised.“ (3) As a consequence Arcavi & Isoda distinguish between two types of listening. "Evaluative listening" is focused on 'wrong' and 'right' and opposed to "attentive listening" or "hermeneutic listening". "The former, which is more common, consists of listening against the background of an expected correct answer. It implies a virtual 'measurement' of the 'distance' between the student's present state of knowledge and the desirable goal, ... However, such listening may tend to disregard the students' thinking, the sources of their idiosyncratic ideas and their potential as a source for learning" (p. 114).

As a consequence the authors propose the thesis: "History of mathematics can provide many solution approaches (to problems) which are very different from what is common nowadays. Such solution processes may conceal the thinking behind them. Thus, one has to engage in a 'deciphering' exercise in order to understand what was done, what could have been the reasoning behind it and what is the mathematical substrate that makes an unusual method/approach valid and possibly general. Engaging in such an exercise bears some similarities to the process of grasping what lies behind our students' thinking and actions. We do not claim that there may be parallels between the mathematics underlying primary sources and that of our students. What we do claim is that experiencing the process of understanding the mathematical approach of a primary historical source can be a sound preparation towards listening to students" (pp. 115-116).

Thus, it is a natural consequence that hermeneutical work with historical sources should become an important component of teacher training.

3 Concerning the lived experience of the subject, and the subject itself

David Guillemette

3.1 Introduction

For several years now that I have been involved in the training of secondary mathematics teachers. I often use historical texts reading activity and primary sources in the training classroom, partly because of my own passion and interest, but mostly because I strongly believe in its educational potential. The reactions of the students range from a total indifference to a powerful existential emotion. Indeed, some reactions are quite impressive, several students react very strongly to those activities and feel a strange vertigo from the temporal distance dividing them from the text and the author, distance which carries all the weight of history and culture. Others react dismissively, and just make fun through translation, highlighting atypical notations, reasoning or arguments. Finally some react with displeasure. These find the reading activities and the encounter with history to be an unpleasant ordeal. They feel uncomfortable, frustrated, sometime see themselves “hurt in their pride”, unable, as they have always been, to make sense with a mathematical discourse.

These elements, which concern specifically the lived experience of the learners, whether positive or negative, are not hazardous or inconsistent. They are not just epiphenomena, but are consubstantial of learning itself and should be, in my opinion, taken very seriously. Indeed, moving on to the lived experience of the learners, and confronting ourselves as researchers/trainers to those elements, brings us to what is inalienable concerning the encounter with history of mathematics. In a phenomenological perspective, my work has carried on the examination and the description of these experiences and such investigation has shed light “from within” the role of history in mathematics education and to reveal a fresh look and potentially new ways of thinking at its potential (Guillemette, 2015a). For my contribution to the panel, I will try to highlight various discourses concerning this lived experience of the learners in this context. These discourses, based on broad paradigms around mathematics and mathematics education, involve different perceptions, implicit or explicit, of the subject who is learning mathematics and is living the encounter of the history of the discipline. I’ll concentrate on two large tendencies or paradigm that I usually meet concerning theoretical and/or conceptual frameworks for integrating history of mathematics in mathematics education: the “humanistic” perspective and the “dialectic” perspective (in its modern Hegelian sense).

3.2 The lived experience from a “humanistic” perspective

It has been claimed that one of the main roles of the history of mathematics is to “disorient”. Indeed, history of mathematics, in a classroom or teachers training context, surprises and astonishes with the diversity the mathematical activity across cultures and the history of societies, which involves many considerations as to the form and use of mathematical objects. For many, these experiences could lead to a more cultural understanding of mathematics and invite to a historical-anthropological reflection on mathematical activity by repositioning the discipline as “human activity” (D’Enfer, Djebbar, & Radford, 2012). In other words, without excluding the development of mathematical understandings that history can play (cognitive pole), this exploration of historical and cultural dimension of the discipline, could brings the learners to take a critical look on the social aspect of mathematics, to understand the historical-cultural mechanisms of their production and to understand that, as mathematical as it is, there is no ideologically neutral knowledge, that all knowledge is part of an ethical issue for which we need to develop our sensitivity.

As Barbin put it many times, history could bring a “culture shock” in “immediately immersing the history of mathematics in history itself” (2012, p. 552, my translation). Therefore, the objective is not to read historical texts simply related to our (modern) knowledge, but rather in the context of the one who wrote them. This is when history can become a source of “epistemological astonishment” by questioning knowledge and procedures typically taken as “self-evident” (*ibid.*). For Barbin, as history invites the learners (specially here the preservice teachers) to stand out in that tone of “epistemological astonishment”, it may also invite them to investigate the following questions: “Why contemporaries did not understand such a novelty?" and "Why students do not understand?" (*ibid*.).

From a practical point of view, Barbin's claims suggest that history should allow prospective teachers to "understand the difficulties of students who are not like teachers, in a well-known country" and "better hear their questions or to better interpret their mistakes" (*id*., p. 548). More specifically, the "exotic aspects" of the history of mathematics can be a good way to "start thinking about the content taught and programs", "to sketch answers students' questions about the status of mathematical knowledge", to "avoid the fake concrete-abstract debate "and finally allow the teachers to change the way they teach, but also their educational relations" (Barbin, 1997, p. 24, my translation).

These ideas of Barbin implicitly give history a critical function in the context of learning mathematics students, a critical function that would find its point of tension in the confrontation with the history of science and objectivist philosophies that have been remained rooted in naturalism and scientific universalism. In contrast, history of mathematics, allows learners to realize their own understandings and perceptions and the particularity of their ways of dealing with mathematical objects, depending on their own contexts, cultures and experiences.

As Fried (2007; 2008) mentions, history of mathematics, in general, should be playing a central role in this quest to self-knowledge. For him it is a special contact with the history of the discipline that could make emerge in the learner some awareness of his own ideas, his individuality and his ability to confront constructively with those of others. Fried considers history, when it is taken as history and not as a means for something else, is able to contribute to the personal growth of individuals through the discovery of their own individuality. This individuality would not lead to a form of isolation, but, on the contrary, on the openness and the possibility to the exchange with the others and to understand the others. In this sense, mathematics education, through history must aim at mutual enrichment between knowledge, self-knowledge and knowledge of others.

Indeed, Fried stressed that the back-and-forth movement between the current understanding of mathematical objects and understandings from other eras brings learning to a deeper understanding of himself: "a movement towards self-knowledge, a knowledge of ourselves as a kind of creature who does mathematics, a kind of mathematical being" (Fried, 2007, p. 218). He proposes that this self-knowledge, that is to say, the knowledge of ourselves as a "mathematical-being", should be the primary objective that must give itself all forms of mathematics education based on the history of the discipline. Fried does not hesitate to emphasize the background of his thinking around these considerations by stating that: "[Education], in general, is directed towards the whole human being, and, accordingly, mathematics education, as opposed to, say, professional mathematical training, ought to contribute to students' growing into whole human beings" (p. 219).

In this perspective, the experience associated with the encounter with history of mathematics would be accompanied by an awareness and a growing movement. This would be a personal experience involving relation to ourselves (introspective element) through the history of mathematics, experience that supports the movement of growth, which is that of the learner. This "humanistic" perspective on the history of mathematics is also present, and developed, in many other speculative works in the field (e.g. Bidwell, 1993; Brown, 1996; Tang, 2007).

3.3 The lived experience from a "dialectic" perspective

Several researchers share, in large part, this view on history and its potential for teaching and learning mathematics. For example, for Radford, Furinghetti and Katz (2007), it is precisely in the highlighting and in the understanding of the links between past and current knowledge that the history of mathematics brings the most to the enrichment of the perception of discipline, and of the understanding of its genesis and its epistemology.

Above all, the authors put forward the importance of actions and events in the acquisition of knowledge, but also emphasize the crucial dimension of the possibility of introspection, confrontations and critical reflections about his own conceptions and knowledge. More specifically, history can show us how the particular meanings attributed to the mathematical objects are confined within our own experience. This limit can be exceeded only by the encounter with a truly foreign form of understanding. These theoretical elements are anchored in the dialectic tradition and specially based on the thought of philosopher Mikhaïl Bakhtin for whom "meaning only reveals its depths once it has encountered and come into contact with another, foreign meaning: they engage in a kind of dialogue, which surmounts the closeness and one-sidedness of these particular meanings" (Bakhtin, 1986, cited in Radford, Furinghetti, & Katz, 2007, p. 108).

In this perspective, the history of mathematics becomes itself a place where it is possible to overcome this particularity of our own understanding of mathematical objects, which is limited to our personal experiences. History is "a place to enter into a dialogue with others, and with the historical conceptual products produced by the cognitive activity of those who have preceded us in the always-changing life of cultures" (Radford, Furinghetti, & Katz, 2007, p. 109).

Through the discourse of social philosophers and psychologists, as Bakhtin (1986, 1986/2003), Ilienkov (1977) or Leontiev (1984), this perspective is marked by the Hegelian and Marxist dialectical developments. It is strongly tinged with social and historical dimension in the exploration of human life by radicalizing the very dialogic aspect of human being, of everything that a sense, a value in the world, and by placing the human being in a historical and ideological reality. Regarding the acquisition of knowledge, "action" plays a decisive role here. It is in the concrete action that the historical movement is created and that are elaborated reflections and changes. In this context, the student experience would be dominated by actions that are ways of getting in a form of "dialogue" with another culture, with different forms of understanding, in order to create the very actual social and historical tissue of the reality. In this sense, history of mathematics is a place, where it is possible to reconstruct and reinterpret the past in order to open new possibilities for learners and for future teachers.

However, as close as they seem, this “dialectic” perspective maintains some doubts about the “humanistic” one, especially with the idea that the learner must build himself free from any form of authority (understood here in both an ideological and in a pragmatic sense), a socio-culturally coded knowledge in a given socio-political context. On the one hand lies the subject conceived of itself as the master of its destiny; on the other hand, lie the regimes of truth, the discourses, and the significations of the sociocultural world in which the subject finds itself subsumed. As many other thinkers have shown, the difficult existence of the modern self unfolds against the unbearable backdrop of this dichotomy and its ensuing antinomies. The antinomies cannot be erased: they are part of the modern forms of production, the very same forms that define the modern subject (Radford, 2012).

In a way, this takes us back the still living question of Kant: “How to educate in freedom under authority?” (Kant, 1803/2004, p. 57). Indeed, the critic of the “dialectic” perspective concerns a conceptualization of emancipation that has its source in the Enlightenment. This conceptualization, built around a dualism and rationalism, is generally associated with the idea that the individual possesses somewhere in himself the conditions for personal growth, a certain intellectual potential or the possibilities of its complete socialization. In this perspective, the entourage of the learner is seen as a facilitator for the individual in his personal quest for growth.

But the “dialectic” thinkers do not attempt to resolve these contradictions and surely not propose a return to direct forms of education, but offers new avenues to rethink the very notion of emancipation. Through a sociocultural perspective, they first try to move away from a conception of a learner as “private” owners of knowledge. The learner is rather seen as an ethical subject (see Radford, 2011; 2012). In other words, they wish to avoid that education produces individuals emancipated side by side, both free and isolated. Their speech leads rather to put forward commitment, answerability and caring (Bakhtin, 1986/2003; Heidegger, 1927/1986; Levinas, 1971/2010) in the education relation, which implies a different conception of teaching-learning, as well as the classroom, and the role of the learners and the teachers.

In view of this, the history of mathematics can be seen as a place where it is possible to overcome the particularity of our own understanding of mathematics, understanding limited to our own personal experiences and sociocultural context in which we live. History of mathematics therefore offers, for Radford, and more generally for sociocultural thinkers, opportunities of meetings with ways of doing and being radically different in mathematics, distant historically and culturally. The lived experience of the students could, accordingly, be characterized by an experience of otherness in mathematics. An experience that focus not necessarily and exclusively on the relation to ourselves, but, more broadly, on a relation with the Other in mathematics, a movement toward the community.

Thus, the focus is not on an individual experiencing personal possibility of emancipation, but to the possibility for learners to discover new ways of being-in-mathematics, to open, with the others, the space of possibilities in mathematics and to respect the classroom as a politics and ethics entity, open to novelty and subversive issues (Guillemette, 2015b).

3.4 Two perspectives on the subject

Behind these two perspectives, it is possible to discern two different conceptualizations regarding the subject who is learning and who is living the encounter with the history of the discipline.

On the one hand, one can find through the "humanistic" perspective a subject that is perceived as "already given" in history. That is to say, a determined subject that is historically and culturally in search of an understanding of himself. Here, history of mathematics should contribute to the awareness of his origins, and enable the subject to understand his historical and cultural determinations. The subject, properly educated and intimately aware of its position in the world through history, will be able to deploy a certain freedom, making him responsible for his actions, improving his relation to the discipline and engaging him with lucid, rich and open activity in mathematics. A freedom that is made of independence, critical thinking, openness and curiosity, characteristics of any good scientist.

In this perspective, the learner needs to know and perceived himself in its mathematical context. This is why the experience is associated with a certain adventure, because the subject is taken, through the study of the history of mathematics, in the exploration of the intimacy of his being-in-mathematical. It is this recognition that could lead to the opening to the outside, opening to others as opportunity for mutual growth. The surprise, the doubts or the adversity, that punctuates this encounter with history, are as many experiences that characterize this awareness and this self-discovery.

This is why we can talk about a subject that is "already there". A determined subject (culturally, historically, linguistically, socially, etc.) that is waiting to be discovered throughout history, the same history that had made him this subject hidden, inaccessible or potentially not yet noticed. History thus becomes a self-recognition tool, much as language becomes an instrument in psychoanalytic therapy.

However, this perception of a subject "already there" that carries the prospect humanism is opposed to a perception of a subject which, in some way, "is not already there". Through "dialectic" perspective, the subject is not considered as completed, determined, giving himself to himself. On the contrary, the subject is conceived as "a becoming", constituted and constituting itself through history in a deep and fundamental ontological sense.

Consequently, history offers different experiences to this different subject. The difference can maybe be grasped through the notion of "emancipation". Emancipation is *emancipatio*, that is to say, literally, "the fact of being released of paternal authority", more generally to be released from an authority, a domination. In a sense, the history of mathematics holds the emancipatory elements, liberating the subject as it provides the key to formulating an answer to this quest for self-knowledge. On the other hand, the "dialectic" perspective involves a different way of understanding how one becomes free of authority. . Indeed, the class is not perceived here as a closed environment where students develop skills or a certain adaptability through negotiation process, but as the space of collaboration and cooperation so that the learners become part of the collective. The teacher therefore has the role to promote an idea of autonomy conceived as social engagement, to develop a (con)science, literally a knowing-with-others, following the idea that nobody liberates nobody, that nobody liberates himself and that that learners liberate themselves together by the means of the world (*cf*. Freire, 1974).

This perspective offers rather "ways of being and knowing as how students engage in group in their pursuit of cultural knowledge referred" (Radford, 2011, p. 15, my translation). In this context, the history of mathematics is to be the meeting place and experience where it is possible for the subject to form and to engage himself in the creation of its mathematical reality. Indeed, the acquisition of knowledge, must be taken in its etymological sense, that is to say the *adquaerere* meaning "to search". Learning could therefore be understood as a process of opening, searching for attitudes or ways of being. Learning is not a submission to a prevailing culture, much less a possession of cultural content, but rather a movement of openness to the world and to the others. With the introduction of the history of mathematics, this openness to the world is seen radicalize and takes an unusual turn. Indeed, ways of thinking and acting are multiplying around the learners in contact with the history of mathematics. These elements invite to introspection, to an awareness of our historical anchors. History puts learners in "research mode".

But is it not simply the reverse of the same coin? The subject constituting himself throughout history, but in opposite directions? On the one hand, a subject discovering himself, liberating himself and therefore growing in its powers and possibilities of mathematical development. On the other, a subject not released, but rather engaged in a creative activity embracing its historical and cultural dimension so often forgotten.

3.5 Different ways of being in research: open questions

Finally, the two positions on the subject necessarily involves different ways of conducting research. From a "humanistic" perspective, one could emphasize intimate lived experiences and the learners' new relation with the discipline. Evidence of those elements, highlighting how consciousness transforms itself in a reflexive manner can be found. On the other side, the "dialectic" perspective would try to grasp the world in common that emerges from the introduction of history into mathematics learning; it would try to explore emerging ways-of-being-in-mathematics. It would pursue this under the assumption that any movement of consciousness is itself dialogical, penetrated by and in dialogue with other movements of consciousness, and thus cannot be approached without consideration for other movements of consciousness to which it answers, and which it allows as an answer.

4 Concluding words

Michael N. Fried

Our goal in this panel, as stated in our introduction, was to set out some key questions for any theoretical framework concerning the alignment of history of mathematics with mathematics education. To this end, it was essential first to try to make clear what kind of thing a theoretical framework is, or, rather, what it is not. This is a matter that ought to remain a focus for reflection. Jankvist (2009) brought out the distinction between "history as a tool" and "history as a goal." These are not two options for a theoretical framework. History taken as something *used* rather than *studied* (Fried, 2001) subordinates history to other non-historical goals whose justifications are independent of history, even antithetical to it. Thus, "history as a tool," even if it actually serves as genuinely *useful* tool for something, cannot be the basis for a *theoretical* framework for history, for the simple reason that history taken this way is not really about history.

But one might argue that the contributions of this panel pose history as tool no less than those who would use it to motivate students to learn whatever it is the school program requires. Does not Jahnke speak of *using* history "to learn to listen"? Does not Guillemette stress the *use* of history in leading students to a "lived experience", "new ways of being-in-mathematics"? And even in the introduction, is not history conceived as something *used* as a means to "self-knowledge"? The crucial difference is that in all these cases, although one can introduce the word "use," in fact "the *uses* of history" here are expressions of the substance of history itself or of historical experience. Theoretical frameworks for history of mathematics in mathematics education, *as* theoretical frameworks, should be driven by questions centered on the historical character of mathematics, on the historical conditioning of our experience of mathematics, and, generally, the meaning of our relationship to the past.

The word "relationship" is the key one. When it comes down to it, history is not knowledge of the past if knowledge is understood as the knowing of "historical facts"—it is a fundamental historical insight that "historical facts" are always problematic. What is a fact is that we have a certain relationship to the past conditioned by texts, artifacts, and language. History is an exploration of that relationship, and it is that relationship which is the main concern of the theoretical frameworks presented in the contributions above. How those frameworks differ, how any such theoretical framework will differ from another, is in how they conceive that relationship with the past. Thus in a hermeneutic framework, which Jahnke has emphasized, one is understood to be limited in one's access to the past by one's own circumscribed experience and our unavoidable "prejudices," and, accordingly, Gadamar, the principal philosopher of modern hermeneutics, speaks of our existing within a horizon: the act of "understanding" is then one of merging horizons, that is, ours with others'. Guillemette emphasized yet another kind of relationship with the past in which one is an a dual position of looking towards "emancipation" from the authority of the past, while recognizing the creative possibilities of being bound up with the past. There are yet other kinds of relationships with the past (see Fried, 2014b). Investigation and elaboration of such relationships will continue to refine our theoretical ideas concerning history of mathematics and its place in mathematics education.

REFERENCES

Arcavi, A., & Isoda, M. (2007). Learning to listen: From historical sources to classroom practice. *Educational Studies in Mathematics, 66*, 111-129.

Bakhtin, M. (1986). *Speech genres and other late essays*. Austin: University of Texas Press.

Bakhtin, M. (2003). *Pour une philosophie de l'acte*. Lausanne: Édition l'Âge d'Homme. (Originally published in 1986).
Barbin, É. (1994). Préface. In Commission Inter-Irem Épistémologie et Histoire des Mathématiques (Ed.), *Quatrième université d'été d'histoire des mathématiques* (pp. ii–iii). Lille: IREM.
Barbin, E. (1997). Histoire et enseignement des mathématiques : Pourquoi? Comment? *Bulletin de l'Association mathématique du Québec*, *37*(1), 20–25.
Barbin, E. (1997). Histoire et enseignement des mathématiques: Pourquoi? Comment? *Bulletin AMQ (Association Mathématique du Québec)*, *37*(1), 20-25,
Barbin, E. (2012). L'histoire des mathématiques dans la formation : Une perspective historique (1975-2010). In J.-L. Dorier, & S. Coutat (Eds.), *Acte du congrès de l'Espace mathématiques francophone 2012* (pp. 546–554). Geneva: University of Geneva.
Bidwell, J. K. (1993). Humanize your classroom with the history of mathematics. *Mathematics Teacher*, *86*, 461–464.
Brown, S. I. (1996). Towards humanistic mathematics education. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education, part 2* (pp. 1289–1321). Dordrecht: Kluwer Academic Publishers.
Collingwood, R. G. (1993). *The idea of history*. Oxford: Oxford University Press.
D'Enfert, R., Djebbar, A., & Radford, L. (2012) Dimensions historique et culturelle dans l'enseignement des mathématiques – Compte-rendu du Groupe de Travail no. 4. In J.-L. Dorier, & S. Coutat (Eds.), *Acte du congrès de l'Espace mathématiques francophone 2012*. Geneva: University of Geneva.
Freire, P. (1974). *Pédagogie des opprimées*. Paris: Maspero.
Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science & Education, 10*, 391-408.
Fried, M. N. (2004). Steiner and Euclid: The Emergence of Mathematics as the Science of Patterns. *Convergence*. ARetrieved June 20 1016 from http://convergence.mathdl.org.
Fried, M. N. (2007). Didactics and history of mathematics: Knowledge and self-knowledge. *Educational Studies in Mathematics*, *66*(2), 203–223.
Fried, M. N. (2008). History of mathematics in mathematics education: A saussurean perspective. *The Montana Mathematics Enthusiast*, *5*(2), 185–198.
Fried, M. N. (2014a). History of Mathematics and Mathematics Education. In Michael Matthews (Ed.), *History, philosophy and science teaching handbook*, volume I (pp. 669-705). New York: Springer.
Fried, M. N. (2014b). Our relationship to the mathematical past. Lecture for the *MAA-AMS Joint Conference—Short Course on Historiography*. Baltimore, Maryland, USA.
Gaultier, L. (1813). Mémoire sur les Moyens généraux de construire graphiquement un Cercle déterminé par trois conditions, et une Sphère déterminée par quatre conditions. *Journal de l'École Polytechnique, XVI.ᵉ Cahier*, 124-214.
Glaubitz, M. R. (2010). The use of original sources in the clasroom. Empirical Research Findings. In E. Barbin, M. Kronfellner, & C. Tzanakis (Eds.), His*tory and Epistemology in Mathematics Education. Proceedings of the Sixth European Summer University ESU 6 (section 3.01)*. Wien: Holzhausen.
Glaubitz, M. R. (2011). *Mathematikgeschichte lesen und verstehen. Eine theoretische und empirische Vergleichsstudie*. Dissertation, Universität Duisburg-Essen. Retrieved from http://duepublico.uni-duisburg-essen.de/servlets/DocumentServlet?id=25416.
Guillemette, D. (2015a). *L'histoire des mathématiques et la formation des enseignants du secondaire : Sur l'expérience du dépaysement épistémologique des étudiants*. Doctoral thesis, Université du Québec à Montréal, Montréal, Canada. Retrieved from http://www.archipel.uqam.ca/7164/1/D-2838.pdf.
Guillemette, D. (2015b). Rôle de l'histoire des mathématiques dans l'enseignement-apprentissage des mathématiques : Le point de vue socioculturel. In C. Sabena, & B. Di Paola (Eds.), *Actes de la 67ᵉ réunion de la Commission internationale pour l'étude et l'amélioration de l'enseignement des mathématiques (CIEAEM 67) - Enseigner et apprendre les mathématiques : Ressources et obstacles* (pp. 607-618). Aosta, Italy: CIEAEM.
Gulikers, I. & Blom, K. (2001). 'A historical angle', A survey of recent literature on the use and value of history in geometrical education. *Educational Studies in Mathematics, 47*, 223-258.
Heidegger, M. (1986). *Être et temps*. Paris: Gallimard. (Originally published in 1927)
Ilienkov, E. (1977). The concept of the ideal. In *Philosophy in the U.S.S.R.: Problem of dialectical materialism* (pp. 71–99). Moscow: Progress Publishers.

Jahnke, H. N. (1994). The historical dimension of mathematical understanding - Objectifying the subjective. In *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education, vol. I* (pp. 139 – 156). Lisbon: University of Lisbon.

Jahnke, H. N. (1995). Historische Reflexion im Unterricht. Das erste Lehrbuch der Differentialrechnung (Bernoulli 1692) in einer elften Klasse. *Mathematica didactica*, *18*(2), 30-58.

Jahnke, H. N. (2014). History in mathematics education. A Hermeneutic Approach. In M. Fried, & T. Dreyfus (Eds.), *Mathematics & mathematics education: Searching for common ground* (pp. 75-88). Dordrecht: Springer 2014.

Jahnke, H. N., & Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., el Idrissi, A., da Silva, C., Weeks, C. (2000). The use of original sources in the mathematics classroom. In J. Fauvel, & J. van Maanen (Eds.), *History in Mathematics Education*, *New ICMI Study Series vol. 6* (pp. 291-328), Dordrecht/Boston/London: Kluwer.

Jankvist, U. Th. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*, 235–261

Kant, E. (2004). *Réflexion sur l'éducation*. Paris: Vrin. (Originally published in 1803)

Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation: History as a method for learning meta-discursive rules. *Educational Studies in Mathematics, 80*, 327-349.

Leontiev, A. (1984). *Activité, conscience, personnalité*. Moscow: Les Éditions du Progrès.

Levinas, E. (2010). *Totalité et infini : Essai sur l'extériorité*. Paris: Librairie Générale Française. (Originally published in 1971)

Radford, L. (2011). Vers une théorie socioculturelle de l'enseignement-apprentissage: La théorie de l'Objectivation. *Éléments*, *1*, 1–27.

Radford, L. (2012). Education and the illusions of emancipation. *Educational Studies in Mathematics*, *80*, 101-118.

Radford, L., Furinghetti, F., & Katz, V. (2007). Introduction: The topos of meaning or the encounter between past and present. *Educational Studies in Mathematics*, *66*, 107-110.

Sfard, A. (2008). *Thinking as communicating*. Cambridge: Cambridge University Press

Steiner, J. (1826/1971). Einige geometrische Betrachtungen. In K. Weierstrass (Ed). *Jacob Steiners Gesammelte Werk*, *vol I*. (pp.17-76). Bronx, New York: Chelsea Publishing Company.

Tang, K.-C. (2007). History of mathematics for the young educated minds: A Hong Kong reflection. In F. Furinghetti, S. Kaijser, & C. Tzanakis (Eds.), *Proceedings HPM 2004 & ESU 4 (revised edition)* (pp. 630-638). Uppsala: University of Uppsala.

Toeplitz, O. (2015). The problem of university courses on infinitesimal calculus and their demarcation from infinitesimal calculus in high schools. M. N. Fried and H. N. Jahnke, translators. *Science in Context, 28*, 297-310.

Weil, A. (2007) *Number theory: An approach through history from Hammurapi to Legendre*. Birkhäuser: Boston.

MATHÉMATIQUES EN MÉDITERRANÉE:
Réflexions autour de deux itinéraires

Marc MOYON, Eva CAIANIELLO, Mahdi ABDELJAOUAD,
XLIM, 123 avenue Albert Thomas - 87060 LIMOGES CEDEX, Limoges, France
marc.moyon@unilim.fr
Équipe Fibonacci, Université Federico II, Naples, Italie
eva.caianiello@gmail.com
(Retired Professor) University of Tunis, Tunis, Tunisie
mahdi.abdeljaouad@gmail.com

ABSTRACT

In this contribution, we propose two historical studies on mathematics in Mediterranean Countries. In the introduction, first of all, we intend to place our purpose at a right level precising our willingness to study those mathematics in clearly defined spaces and times, but not in their globality. After that, we focus on mathematics studied in Abbaco Schools in Mediaeval Italy and their relationship with works written by Fibonacci (d. after 1241) in the thirteenth century, one of the mathematicians who best represents mathematics in the Mediterranean Basin. Last but not least, we present transfers of both mathematics books and European Teachers to Southern Countries of the Mediterranean Sea that occurred during the Ottoman period of the eighteenth century. Less present in the historiography, we would like to show that they can't be omitted from the idea here illustrated of Mediterranean Mathematics.

RÉSUMÉ

Dans cette contribution, nous proposons deux études à caractère historique sur les mathématiques en Méditerranée. Dans l'introduction, nous voulons tout d'abord placer notre propos à la bonne échelle en précisant notre volonté d'étudier lesdites mathématiques dans des espaces et des temps clairement définis, et non pas dans leur globalité. Nous nous intéressons ensuite aux mathématiques des écoles d'abaque dans l'Italie médiévale et leur relation avec les travaux de Fibonacci (mort ap. 1241), un des mathématiciens qui représentent le mieux le bassin Méditerranéen. Enfin, ce sont les transferts, durant la période ottomane du XVIII^{e} siècle, des ouvrages mathématiques et des enseignants européens vers les pays du Sud de la Méditerranée qui sont développés. Peu représentés dans l'historiographie, nous voulons montrer qu'ils ne peuvent pas être omis de l'idée ici illustrée de mathématiques en Méditerranée.

MATHÉMATIQUES MEDITÉRRANEENNES: EN QUEL SENS?

Marc MOYON

Il est évidemment fort difficile d'avoir une vision d'ensemble des « mathématiques des pays méditerranéens » dans le temps et dans l'espace. Jamais dans son histoire l'ensemble du bassin méditerranéen n'a été (ré)uni même si de forts pouvoirs politiques s'y sont succédés. Il ne s'agit donc pas de le faire ici, et encore moins à propos des mathématiques ; l'entreprise serait vouée à l'échec tellement elle serait caricaturale. En effet, chacune de ses régions, chacun de ses Empires, chacun de ses pays, chacune de ses villes… montre des spécificités locales et temporelles qui confient au paysage méditerranéen de nouvelles végétations et de nouveaux reliefs.

Autour de la Méditerranée et sur un temps long, sont représentées certaines des plus grandes aires culturelles et scientifiques avec, entre autres, l'Égypte ancienne, la Grèce Antique, l'Empire Romain, les pays d'Islam, l'Empire Byzantin, l'Égypte Mamelouke ou encore l'Empire Ottoman. Malgré tout, nous avons envie de croire que l'idée de « mathématiques méditerranéennes » a un sens indéniable si l'on considère ses acteurs, en différents lieux et à différentes époques. Les mathématiques européennes ne sauraient alors négliger l'immense dette qu'elles doivent aux mathématiques méditerranéennes, quelles que soient les régions, les époques, les langues d'expression. C'est ainsi posé dans l'introduction de l'*Europe mathématique*, en référence à la seule Grèce Antique et à Euclide et Diophante, deux de ses mathématiciens : « Ainsi, accepter l'un ou l'autre de ces mathématiciens, ou les deux, parmi les pères fondateurs des mathématiques européennes, dépend d'une certaine vision des mathématiques. Mais aussi d'une certaine vision de l'Europe : il faut en effet y admettre Alexandrie[1]. »

Nous pensons que c'est dans le cadre de la Méditerranée que se révèle, sans doute le plus explicitement possible à une échelle locale et dans une perspective diachronique, l'universalité des mathématiques et de son questionnement. Inter- ou transculturelles, les mathématiques sont une œuvre collective qui se construit par couches successives grâce aux collaborations de tous : chercheurs, enseignants ou simples citoyens selon des modalités distinctes. En ce sens, les mathématiques méditerranéennes sont réunies car elles se construisent, certes au cours de plusieurs siècles, voire millénaires, à l'intérieur d'une même unité géographique dont le centre est la Mer Méditerranée. Dans cet espace, il y a une relative continuité de la construction mathématique[2] : pour reprendre la métaphore attribuée à Bernard de Chartres par son élève Jean de Salisbury (XII^e^ s.) dans le *Metalogicon*[3], chacun peut (et devrait) se considérer comme un « nain juché sur des épaules de géants » (que ces géants soient grecs ou syriaques pour les Arabes, qu'ils soient Arabes pour les latins du XII^e^ s., qu'ils soient européens pour les Ottomans du XVIII^e^ siècle…).

Depuis au moins l'Antiquité grecque, le bassin méditerranéen est marqué par de nombreuses acculturations et migrations d'individus isolés ou de populations entières. Elles sont pacifiques ou belliqueuses, au nom de Dieu(x) ou non, volontaires ou subies pour des raisons économiques ou/et politiques. Ces migrations, ces évolutions et bouleversements cultuels, culturels et linguistiques sont nécessairement suivis d'effets sur le partage et l'élaboration de la connaissance. Il est important, dans le cadre de l'enseignement ou de la formation d'enseignants, de faire émerger l'importance de ces relations humaines. En guise d'exemple, le territoire correspondant à l'Égypte d'aujourd'hui a vu naître ou évoluer de l'Antiquité au Moyen Âge : de nombreux scribes dont Ahmes (en référence au célèbre papyrus Rhind du Moyen Empire égyptien) et d'autres pour nous restés anonymes, Euclide (300 av. J.C.), Héron (1^er^ s.) ou encore Diophante (*ca.* 250 ?) tous trois sont dits d'Alexandrie, mais aussi le « calculateur égyptien » [*al-hāsib al-misrī*] Abū Kāmil (m. 930), Ibn al-Haytham (m.1039) ou encore Ahmad ibn Thabāt (m.1273) et encore d'autres savants andalousiens[4] venus étudier ou travailler au Caire comme Ibn al-Faradī (m.1012), Muhammad al-Jayyānī (m.1079), Abū l-Salt al-Andalusī (m.1134). Le témoignage du mathématicien pisan Fibonacci est aussi fondamental. En effet, il explique avoir appris l'arithmétique indienne au contact des Arabes dans la ville de Béjaïa (en Algérie actuelle), alors comptoir marchand de la république maritime de Pise. Il poursuit en citant ses voyages en Égypte,

---

1 (Goldstein, Gray & Ritter 1996, 3)

2 Voir (Moyon 2016) à propos de la division des figures planes.

3 (Salisbury, col.900)

4 À la suite des travaux de historiens d'*al-Andalus* (partie de la péninsule ibérique dirigée au nom de l'Islam), « andalousien » désigne tout ce qui se rapporte à cette région (pour marquer la différence avec l'Andalousie, province espagnole contemporaine) ; (Marín 2000)

en Syrie, en Sicile, à Constantinople ou encore en Provence[5]. Toute question d'identité nationale ou de nationalisme n'a pas lieu d'être. De nombreux autres exemples pourraient être énoncés, et même au-delà du Moyen Âge. C'est ce que propose la seconde partie de notre propos dédiée à « l'introduction des mathématiques 'européennes' dans l'Empire ottoman à partir du XVIII[e] siècle ». Elle permet aussi d'inverser l'itinéraire et ses modalités par rapport au Moyen Âge : cette fois-ci, le transfert se fait du Nord vers le Sud, et l'adaptation se fait à partir des principales langues européennes pour un monde arabophone.

L'histoire des savoirs et des pratiques scientifiques dans le basin méditerranéen est de plus en plus documentée grâce aux travaux des archéologues, des historiens et des historiens des sciences[6]. En particulier, même s'il reste insuffisant, le nombre d'artefacts, de manuscrits exhumés et analysés est de plus en plus important. Et, chaque étude permet de comprendre encore un peu mieux les pratiques locales archaïques qui sont restées inconnues pendant longtemps[7], l'appropriation de ces pratiques réalisée ça et là en fonction des (nouveaux) besoins de la population, qui peut être nouvelle ou renouvelée. Nous sommes aussi bien mieux renseignés sur les transferts d'une langue à une autre (du grec et du syriaque à l'arabe, de l'arabe au latin ou à l'hébreu pour se limiter aux langues de communication scientifique les plus significatives), sur la circulation des textes mathématiques[8]. En particulier, nous pouvons raisonnablement comparer (dans les grandes lignes) les impressionnants mouvements de traduction réalisés dans le Bagdad abbasside du IX[e] siècle et dans l'*Andalus* du XII[e] siècle, mouvements qui se révèlent essentiels pour comprendre la science dite européenne. L'un a permis de nourrir les hommes de sciences des pays d'Islam affamés devant l'immensité des *corpus* scientifiques grecs mais aussi indiens ou encore syriaques[9], ce qui démontre « l'importance et le poids des symbioses culturelles qui ont pu exister à des moments donnés et dans des espaces particuliers de cet immense empire » [10]. L'autre, *mutatis mutandis*, a largement permis de combler les lacunes des latins dans l'ensemble de la connaissance scientifique pour non seulement redécouvrir la science hellénistique mais aussi appréhender la science arabe dans ses aspects innovants, au moment où de nouvelles demandes se pressaient en Europe avec la fondation des premières universités (Paris, Oxford, Bologne) ou encore la naissance de la bourgeoisie, nouvelle classe sociale curieuse et cultivée. Les sciences des pays d'Islam (à partir du IX[e] siècle) et celles de l'Europe latine (à partir du XII[e] siècle), dans les deux cas, ont alors été suffisamment outillées pour permettre à leurs acteurs des développements originaux, autonomes et innovants en commentant, prolongeant les travaux des Anciens et en concevant de nouvelles disciplines mathématiques.

L'exemple le plus caractéristique est indéniablement celui de l'algèbre. Cette discipline est officiellement baptisée, peut-être à partir de pratiques locales anciennes, avec le *Mukhtasar fī hisāb al-jabr wa l-muqābala* [Abrégé du calcul par la restauration et la comparaison] d'al-Khwārizmī, entre 813

5 (Fibonacci 1857, 1)

6 Citons ici les très récents essais de synthèse de B. Vitrac et A. Djebbar concernant les *corpus* mathématiques grec, respectivement arabe, arrivés en occident. (Vitrac 2015 ; Djebbar 2015).

7 C'est notamment le cas, d'après J. Samsó, pour des pratiques astrologiques en *Andalus* ; (Samsó 2003).

8 Voir, par exemple, le cas du *Kitāb al-Istikmāl* [Livre de la perfection] d'al-Mu'taman ibn Hūd, roi de Saragosse en 1081 et 1085, qui circule dans tout le bassin méditerranéen accompagnant ses lecteurs successifs ; (Djebbar 2002).

9 (Gutas 2005)

10 (Djebbar 2004)

et 833 à Bagdad[11]. Elle s'appuie sur un vocabulaire spécifique, se dote d'objets particuliers avec des règles précises, d'une typologie de problèmes et de champs d'application. Elle sera largement reprise par les latins dès le XII[e] siècle soit directement grâce aux traductions de l'ouvrage d'al- Khwārizmī[12], soit indirectement par son utilisation dans la résolution de problèmes[13]. Mais, il faudra attendre la Renaissance italienne avec les mathématiciens bien connus, parmi lesquels Tartaglia, Cardan ou Bombelli, pour dénouer le problème initialement posé par al-Khayyām au XII[e] siècle à l'est de l'Orient musulman[14], de la résolution par radicaux de toutes les équations de degré inférieur ou égal à 3 : « Mais à la démonstration de ces espèces, si l'objet du problème est un nombre absolu, ni moi [al-Khayyām], ni aucun des hommes de cet art, ne sommes parvenus (peut-être d'autres, qui nous succéderont, sauront- ils le faire) que pour les trois premiers degrés qui sont le nombre, la chose et le carré[15] ». Par la même occasion, l'équation de degré 4 sera aussi résolue par radicaux : les algébristes de la Renaissance, formés à et par l'algèbre des pays d'Islam, cultivés de certains développements latins et vulgaires[16], sont murs pour s'autoriser des opérations impensables. L'algèbre des équations qui trouvera son accomplissement en Europe prend inévitablement ses racines dans l'algèbre arabe d'al-Khwārizmī, mais elle ne saurait s'y réduire. En effet, de plus en plus de problèmes du troisième degré vont se trouver résolus dans le contexte des mathématiques pratiques, marchandes de l'Italie médiévale, avant, à l'époque et après Fibonacci[17]. C'est tout ce contexte qu'Eva Caïaniello dépeint dans la première partie de notre contribution : « Léonard de Pise et les mathématiques de l'abaque ».

REFERENCES

Cardano, G. (1545). *Artis magnæ, sive de regulis algebraicis*, Nuremberg.

Djebbar, A. (2002). Le Manuscrit (retrouvé) de Saragosse. *Alliages 47*, 66-71. Disponible en ligne http://revel.unice.fr/alliage/?id=3812.

Djebbar, A. (2004). Universalité et localité dans les pratiques scientifiques des Pays d'Islam. *Alliages, 55-56*, 35-42. Disponible en ligne http://revel.unice.fr/alliage/index.html?id=3590#tocto1n2.

Djebbar, A. (2015). Quels sont les écrits mathématiques arabes et leurs contenus qui ont circulé dans l'Europe médiévale ? Un bilan provisoire. In É. Barbin, & J. L. Maltret (Eds.), *Mathématiques méditerranéennes. D'une rive et de l'autre* (pp. 89-108). Paris : Ellipses.

Fibonacci, L. (1857). *Il Liber Abbaci. Baldassarre Boncompagni* (Ed.). Rome : Tipografia delle scienze matematiche e fisiche.

---

[11] (Rashed 2007)

[12] Deux traductions sont réalisées au cours du XII[e] siècle grâce à Robert de Chester et Gérard de Crémone ; (Hughes 1986 ; Hughes 1989).

[13] C'est le cas par exemple des textes de géométrie pratique dans lesquels les auteurs vont utiliser l'algèbre comme méthode de résolution de problèmes anciens ; (Moyon 2012)

[14] Précisons qu'à la lecture de la *Muqadimma* d'Ibn Khaldūn, on peut supposer que le contenu du traité d'al-Khayyâm n'est toujours pas connu à l'Ouest de la Méditerranée avant la fin du XIV[e] siècle ; « Nous avons appris qu'un certain grand mathématicien de l'Orient a dépassé le nombre de six équations et est arrivé à plus de vingt espèces. Pour toutes ces équations, il a trouvé des solutions rigoureuses basées sur des démonstrations géométriques. » ; (Ibn Khaldūn 2002, 951). Il semblerait que le traité ne soit pas non plus connu des algébristes de la Renaissance italienne. À notre connaissance, aucun élément ne le suggère.

[15] (Rashed & Djebbar 1981, 16)

[16] J. Cardan le revendique dans son *Ars Magna* dès le premier chapitre en citant explicitement Muhammad ibn Mūsā d'al-Khwārizmī et Fibonacci : « *Haec ars olim a Mahomete, Mosis Arabis filio initium sumpfit. Et enim huius rei locuples testis Leonartus Pisauriensis est.* (…) » [Cet art a commencé avec Muhammad, arabe, fils de Moïse. Et, en fait, un témoin fiable de ce qu'est l'origine est Leonard de Pise (…)] ; (Cardano 1545, 3r).

[17] Voir, par exemple, (Franci 1985 ; Franci & Toti Rigatelli 1988).

Franci, R. (1985). Contributi alla risoluzione dell'equazione di 3° grado nel XIV secolo. In M. Folkerts, & U. Lindgren (Eds.). *Festschrift Für Helmuth Gericke* (pp.221-228). Stuttgart : Franz Steiner Verlag.
Franci, R., & Toti Rigatelli, L. (1988). Fourteenth-Century Italian Algebra. In C. Hay (Ed.). *Mathematics from Manuscript to Print 1300-1600* (pp.11-29). Oxford : Clarendon Press.
Goldstein, C., Gray, J., & Ritter, J. (1996) *L'Europe mathématique : Histoires, mythes, identités*. Paris : Maison des Sciences de l'homme.
Gutas, D. (2005). *Pensée Grecque, Culture Arabe*. Paris : Aubier.
Hughes, B. (1986). Gerard of Cremona's Translations of al-Khwarizmi's al-Jabr. *Mediaeval Studies*, *48*, 211-263.
Hughes, B. (1989). *Robert of Chester's Translation of al-Khwarizmi's al-Jabr*. Stuttgart : Franz Steiner Verlag.
Ibn Khaldūn, A. (2002). *Le Livre des exemples*, *A. Cheddadi* (trad.). Paris : Gallimard.
Marín, M. (2000). *Al-Andalus et les andalousiens*. Aix en Provence : Edisud.
Moyon, M. (2012). Algèbre & Practica geometriæ en Occident médiéval latin : Abū Bakr, Fibonacci et Jean de Murs. In S. Rommevaux, M. Spiesser, & M.-R. Massa-Esteve (Eds.), *Pluralité de l'algèbre à la Renaissance* (pp. 33-65). Paris : Éditions Honoré Champion.
Moyon, M. (2016). Mathématiques et Interculturalité : L'exemple de la division des figures planes dans l'histoire des pratiques mathématiques. *Repères-IREM, 103*, 5-22.
Rashed, R., & Djebbar, A. (1981). *L'œuvre algébrique d'al-Khayyām*. Alep : Imprimerie de l'Université d'Alep.
Rashed, R. (2007). *Al-Khwārizmī. Le commencement de l'algèbre*. Paris : Blanchard.
Salisbury (de). J. *Metalogicum*. In J. P. Migne (Éd.), *Patrologia Latina*. Disponible en ligne http://www.documentacatholicaomnia.eu/02m/1115-1180,_Joannis_Saresberiensis,_Metalogicus_[Metalogicum],_MLT.pdf.
Samsó, J. (2003). A social approximation to the history of the exact sciences in al-Andalus. In *Acte de La VII trobada d'historia de la ciencia i de la tecnica* (pp. 519-30). Barcelone: SCHCT.
Vitrac, B. (2015). La transmission des textes mathématiques grecs anciens : esquisse d'un problème. In É. Barbin, & J. L. Maltret (Eds.). *Mathématiques méditerranéennes. D'une rive et de l'autre* (pp. 3-26). Paris : Ellipses. Repris et augmenté dans *Quand ? Comment ? Pourquoi les textes mathématiques grecs sont-ils parvenus en Occident ?*, Disponible en ligne https://www.academia.edu/16162595/Quand_Comment_Pourquoi_les_textes_mathématiques_grecs_sont-ils_parvenus_en_Occident_.

LÉONARD DE PISE ET LES MATHÉMATIQUES DE L'ABAQUE

Eva CAIANIELLO

À Leonard de Pise, ou Fibonacci, on doit l'introduction d'une nouvelle mathématique, dont les principaux bénéficiaires sont les marchands italiens qui, à partir du XI^e^ siècle, sont les protagonistes de l'essor des échanges commerciaux entre les marchés du Nord et ceux de la Méditerranée orientale. Il serait impossible de comprendre l'apport de Léonard de Pise à la comptabilité et aux mathématiques du négoce sans évoquer l'influence sur sa formation de la révolution socio-économique qui s'accomplit au tournant des XII^e^ et XIII^e^ siècles. Celle-ci est marquée par le passage d'une économie féodale terrienne reposant sur le commerce local et de détail à une économie du crédit et du commerce de gros et d'outre-mer[18]. En fait, dès le XI^e^ siècle, les bases de l'évolution sociale et économique changent : le développement des villes et la croissance de la production agricole s'accompagnent de la multiplication et de l'extension des divers réseaux de commerce. Ainsi, les échanges avec l'Orient musulman et byzantin deviennent toujours plus fréquents. Sont exportés des tissus ouvrés, des métaux, de la laine,

[18] (Caianiello 2013, 218)

du bois, des objets en or et en argent, et sont importés des épices, de la soie, de l'alun (indispensable pour la teinture des draps), etc...[19]

L'Italie se trouve au milieu de ces échanges en raison de sa position géographique et de sa tradition commerciale ancestrale : les républiques maritimes telles que Venise, Pise, Florence, Gênes, Amalfi sont des centres importants de cette expansion.

Contexte d'instruction

L'expansion du commerce serait peu compréhensible en l'absence d'un savoir-faire de deux types : des activités de lecture, écriture, de calcul d'une part, et des compétences juridiques d'autre part. Cependant toutes les données disponibles sur l'organisation des écoles pour les marchands dans l'Italie médiévale sont relatives à une période postérieure à celle de Léonard : elles datent au mieux du début du XIV^e^ siècle. Mais, Pirenne, partant de la considération que tout commerce un peu développé, présuppose nécessairement un degré assez élevé d'instruction pour ceux qui l'exercent, a avancé l'hypothèse, en ce qui concerne l'Italie, qu'on peut bien aller plus en arrière[20]. En effet, en Italie, le niveau d'instruction des marchands du XIII^e^ siècle est bien supérieur à celui des régions du nord. À Pise, par exemple, la seule école attestée, sans doute laïque, est une école de droit, mentionnée dans la lettre d'un moine Victorin vers 1124/1127[21].

Pour la lecture et l'écriture en Europe, on fait alors usage de la langue latine, pour le calcul on utilise le système de numération romaine et l'abaque[22]. On sait qu'à Florence[23], dès la première moitié du XIV^e^ siècle, environ la moitié[24] des enfants apprennent à lire, écrire et les rudiments du latin entre 5 et 7 ans[25]. On sait en outre que cet enseignement est dispensé par des maîtres, appelés en latin *doctores puerorumi* pour les hommes, ou *doctrices puerorum* pour les femmes. De 10 à 11 ans, les fils de marchands (environ 10%) et souvent ceux de la noblesse, s'ils étaient orientés vers les affaires commerciales, fréquentent une école d'abaque, où ils apprennent l'arithmétique marchande. Vers 13-14 ans, ils sont prêts pour un apprentissage professionnel dans une maison de commerce. Un petit nombre de ceux qui ont la possibilité d'entrer dans le commerce international sont envoyés à l'étranger et voyagent.

Il y a, par la suite, les écoles appelées « de grammaire » où l'on approfondit la langue latine, les lettres, la rhétorique et la logique. Ces écoles s'adressent à ceux qui (environ 5%) se destinent aux professions civiles, mais elles sont fréquentées aussi par des élèves des écoles d'abaque, qui veulent

[19] Ces transactions sont bien documentées dans les chapitres VIII et IX du *Liber Abaci* pour ce qui regarde les liens entre Pise et ses partenaires commerciaux ; Voir (Moyon & Spiesser 2015, 2 et suivts).

[20] (Pirenne 1929, 19)

[21] (Dufour 1979, 504 et svts.).Voir aussi (Caianiello 2013, 230).

[22] Les algorismes (traductions latines et adaptations du *Calcul Indien* d'al-Khwārizmī perdu dans sa version originale) sont fondés sur le système de numération indo-arabe et sur le calcul arithmétique écrit dérivé. Ils se développent tout au long du XII^e^ siècle. Les chiffres indo-arabes commencent à être utilisés à la place des chiffres romains entre la fin du XII^e^ et le début du XIII^e^ siècle, mais dès son introduction et pendant des siècles (jusqu'aux XV^e^-XVI^e^ siècles), la consultation des manuscrits montre qu'ils ne sont pas nécessairement utilisés.

[23] Grâce a un passage très connu de G. Villani dans sa *Cronaca fiorentina* ; (Davidsohn 1965, 4 :211).

[24] (Davidsohn 1965, 211-15 ; Høyrup 2007, 27-28).

[25] Tout d'abord, les enfants apprenaient à lire, puis à écrire. Pour ce qui concerne l'apprentissage et l'utilisation du latin voir (Black 2007, 43 et svts.).

recevoir une culture humaniste approfondie[26] (fondée sur l'étude de la grammaire latine et sur la lecture des auteurs classiques et médiévaux). Pour ce qui a trait à la question de la laïcité des écoles destinées aux marchands[27], on sait qu'aux XII[e]-XIII[e] siècles, les écoles épiscopales ou paroissiales ne répondent plus aux besoins professionnels de la « bourgeoisie » naissante. Elles sont alors remplacées au fur et à mesure par des écoles communales, d'abord privées et ensuite financées par les gouvernements citadins, même si elles ne sont pas encore publiques selon le sens actuel du mot. Elles sont tenues par des clercs, mais souvent par des laïcs, dont la première activité est celle de notaires. Il paraît, en effet, que ceux-ci servent principalement de maîtres à côté des clercs dans les villes communales. On peut supposer que, dès le XI[e] siècle, dans l'Italie médiévale et à Pise en particulier, une grande corrélation existe entre l'étude du droit, l'activité notariale et le début de l'enseignement laïque. On ne peut pas, donc, douter de l'existence à Pise d'une ou plusieurs écoles pour les marchands où les jeunes peuvent étudier (ou auprès d'un précepteur privé) les rudiments de l'arithmétique commerciale et du latin.

Léonard entre Pise et Béjaïa

Fibonacci appartient à l'élite commerciale marchande de la ville de Pise, où il est né entre 1170 et 1180. La date de sa mort est inconnue, on la suppose postérieure à 1241[28]. À cette époque, Pise est à l'apogée de sa puissance commerciale. Après une guerre pirate très agressive, commencée au début du XI[e] siècle avec le Maghreb et, de manière plus générale, avec le monde islamique, la ville est entrée dans une deuxième phase où s'établit un flux commercial plus tranquille avec les anciens ennemis, flux réglé par de nombreux traités[29]. La ville, devenue cosmopolite, est fréquentée par des marchands musulmans et des esclaves, qui sont souvent de haut rang, avec lesquels des relations d'amitié et de coopération s'établissent. Sont institués des consulats pisans dans les endroits marchands du Maghreb et du bassin méditerranéen en général, dont celui de Bougie dans le Maghreb (Bejaïa en Algérie). C'est précisément à Bougie que le scribe (titre correspondant à l'époque à celui de consul[30]) pisan Guillaume appela son fils Léonard pour lui donner une formation adéquate (peut-être vers 1192) comme Léonard le raconte lui-même dans le « Prologue »[31] de la deuxième édition du *Liber Abaci* (1228).

Le séjour à Béjaïa a certainement largement influencé les mathématiques de Fibonacci. La ville est un carrefour entre l'Occident, le Maghreb occidental et le désert du Sahara, ainsi qu'un parcours obligé dans les échanges culturels entre l'Occident et l'Orient musulmans[32]. Devenue pôle intellectuel d'un haut niveau scientifique, Bougie est le siège d'une école renommée pour la science du calcul[33] et

[26] (Davidsohn 1965, 4 :211 ; Ulivi 2000, 87; Black 2007, 121-140, 226)

[27] (Pirenne 1929, 139 et svts)

[28] L'indication de l'année 1241 est déduite d'un arrêté de la municipalité de Pise contenu dans le *Constitutus usus Pisanae Civitatis* du 4 novembre 1242, qui montre qu'il est vivant. La date textuelle du document a été rédigée selon le calendrier pisan qui correspond au 4 novembre 1241 (année reportée dans Bonaini (1857, 241) selon le calendrier moderne. Voir (Ulivi 2011, 251 ; Caianiello 2012, 27 et suivts.)

[29] (Tangheroni 1994, 22-23)

[30] (Tangheroni 1994,16)

[31] (Boncompagni 1857-1862, I :1), voir aussi (Germano 2013, 14-17).

[32] (Aïssani &Valerian, 2003)

[33] « La 'science du calcul' dans les classifications de la tradition classique arabe englobait non seulement le calcul indien, le calcul digital et mental mais aussi les procédures de résolution des problèmes telles que la méthode des quatre grandeurs proportionnelles, les méthodes de simple et double fausse position et les algorithmes algébriques associés aux 6 formes canoniques d'al- Khwārizmī. et un ensemble de problèmes qui s'appellent " calcul des transactions " (Djebbar 2005, 76). Au regard des textes de science du calcul, voir aussi (Djebbar 2001, 327-344).

de l'algèbre[34]. Ici, Léonard a pu apprendre des éléments du calcul indien, de la notation des fractions de l'école maghrébine[35], les fondements de l'algèbre d'après la tradition d'al- Khwārizmī et d'Abū Kāmil[36] et grâce à son expérience de fils de marchand, les fondements de la science du calcul appliquée au négoce. À côté de la science du calcul, est aussi attestée une tradition pré-algébrique de géométrie pratique qui a eu la faveur des traducteurs latins[37].

Le résultat de cette intense activité d'études et de voyages est la rédaction, en rentrant à Pise vers 1200, du *Liber Abaci* en 1201-1202[38] (révisé en 1228). La *Practica Geometriae* est rédigée, quant à elle, vers 1220-21. Lorsqu'en 1226, l'Empereur Frédéric II de Hoenstaufen séjourne dans la ville, Fibonacci est introduit à sa cour par un certain Maître Dominique et prend part, en présence de l'Empereur, à des défis mathématiques. On suppose que les deux ouvrages : le *Flos* et le *Liber Quadratorum* sont composés après cette visite. Le manuscrit de l'*Epistola ad Magistrum Theodorum*, un philosophe de la cour impériale, avec les *Quaestiones avium*, n'est pour sa part pas daté[39]. Maccagni[40] suggère la datation suivante : *Liber Quadratorum* [1226-1228], *Flos* [1228/1234], *Epistola ad Magistrum Theodorum*, *phylosophum domini imperatoris* [après 1228]. Léonard aurait composé trois autres ouvrages[41], qui n'ont pas été retrouvés : la première édition du *Liber Abaci*, le *De minore guisa*, livre d'arithmétique commerciale et un commentaire au Livre X d'Euclide, où l'auteur envisage un traitement numérique des irrationnels. Ce dernier ouvrage, d'après quelques auteurs, pourrait correspondre au « chapitre XIV » de la seconde édition du *Liber Abaci*. Je m'arrête brièvement sur les deux ouvrages qui ont eu la plus grande influence sur les mathématiques de l'abaque des siècles suivants.

Le *Liber Abaci*, le projet de l'ouvrage

Composant le *Liber Abaci*, Fibonacci a voulu transmettre un ouvrage complet sur les nombres qui, tout en tenant compte de l'héritage euclidien, soit fondé sur le calcul indien et sur la méthode développée par la tradition des calculateurs des pays d'Islam que Fibonacci choisit comme la meilleure et qu'il aurait connue pendant son séjour au Maghreb et lors de ses voyages dans la Méditerranée [Prologue].

L'organisation du contenu, en 15 chapitres, rappelle, dans une certaine mesure, celle des manuels de « science du calcul » des pays d'Islam. Mais, comme le souligne Giusti, « même à l'égard de ses maîtres, Fibonacci compose un ouvrage unique, sinon par son originalité, du moins par sa

---

34 (Aïssani & Valerian 2003)

35 (Moyon & Spiesser 2015)

36 Conformément à la tradition d'al-Khwārizmī et d'Abū Kāmil, pour le Pisan, la validation géométrique des théorèmes algébriques est essentielle.

37 Du X^e siècle, on a retrouvé un texte d'Ibn ^cAbdūn, de Cordoue, proposant des problèmes de géométrie de la mesure avec des algorithmes arithmétiques mais non algébriques, qui ressemblent au type babylonien. (Djebbar 2005, 79-80, 107-110). Pour une étude détaillée, voir (Moyon 2016).

38 Sur la datation controversée des œuvres de Fibonacci voir (Maccagni 1988 ; Ulivi 2011, 250-254 ; Caianiello 2012, 18 et suivts.).

39 Les ouvrages susmentionnés ont été publiés dans (Boncompagni 1857-1862).

40 (Maccagni 1988)

41 Franci rapporte que, dans un manuscrit du XV^e de la Bibliothéque Nationale de Florence, ils sont cités comme le *Libro di merchaanti detto di minor guisa* et le *Libro sopra il 10° di Euclide* ; (Franci 2002, 303). Le premier traité est mentionné par Fibonacci au chapitre XI (Boncompagni 1857, 154): " *Est enim alius modo consolandi, quem in libro minoris guise docuimus* " (Il y a en effet une autre manière de faire l'alliage, que nous avons enseignée dans le *livre de minore guisa*). Quant au commentaire au Livre X d'Euclide*,* il y a une allusion que Fibonacci fait dans son *Flos* (Boncompagni 1856, 3).

dimension … »[42]. Dans son texte, coexistent : une section d'arithmétique entendue comme apprentissage du calcul, qui comprend la numération indo-arabe, opérations arithmétiques et calcul avec les fractions (chap. 1-7) ; la règle de trois et une vaste collection de problèmes à l'usage des marchands tels que le calcul du prix des marchandises, le troc, l'alliage, les compagnies mercantiles (chap.8-11) ; les questions erratiques c.-à-d. qui peuvent être résolus avec des diffèrent méthodes, parmi lesquels la règle de double fausse position. Il s'agit d'un large éventail de problèmes, à la fois d'ordre pratique et d'ordre théorique, qui sont actuellement classés comme problèmes récréatifs[43] et qui peuvent être résolus avec différentes méthodes, parmi lesquels la règle de double fausse position (chap. 12-13) ; une section de géométrie pratique suivie par un chapitre sur les irrationnels et une section d'algèbre, où les problèmes sont résolus à l'aide d'équations et d'inconnues (chap. 14-15). Dans le Prologue, Fibonacci se donne un objectif très ambitieux. En s'adressant à la *gens latina*[44], il souhaite toucher un vaste public. C'est sans doute une des raisons pour lesquelles il a écrit l'ouvrage en latin[45]. Si l'on se demande si le *Liber* est plus orienté vers la théorie plutôt que vers la pratique, la réponse est que c'est un ouvrage hybride. D'un côté, l'aspect théorique est toujours présent : dans la partie arithmétique, Fibonaccci utilise les énoncés d'Euclide pour vérifier l'exactitude de la pratique, il renvoie aux énoncés d'Euclide lorsqu'il énonce des propositions qui sont reprises des *Eléments*, mais démontre également lui-même, dans un style euclidien de nombreuses propositions ayant trait au domaine des nombres. De l'autre, Fibonacci ne perd jamais de vue les applications pratiques liées à l'activité des marchands. Tangheroni observe que :

> le *Liber Abaci* offre beaucoup d'espace aux problèmes de mathématiques, même s'il ne peut pas être considéré comme un manuel pratique à l'usage des marchands : le *Liber* n'offre pas de formules d'application immédiate, mais des problèmes et des solutions aux problèmes à travers des procédés mathématiques[46].

Enfin, la volonté d'une large accessibilité est très présente dans l'ensemble du traité. Ainsi Fibonacci présente les choses à ses lecteurs de manière systématique et cohérente. Il ordonne ses exemples, en général, du plus simple au plus compliqué. Tout au long de l'ouvrage, des diagrammes et des tableaux sont souvent disposés à côté du texte, afin de simplifier ou de schématiser l'explication, voire de la rendre mémorisable.

La *Practica Geometriae* (1220-1221)

La même orientation, théorique tournée vers la pratique, inspire le second grand ouvrage de Fibonacci, la *Practica Geometriae*, composé en 1220-1221. Fibonacci, comme il le dit dans la dédicace à Maître Dominique, veut composer un traité de géométrie valable pour les théoriciens comme pour les praticiens. Son objectif est de présenter des problèmes de géométrie pratique pour résoudre les difficultés quotidiennes des gens, mais aussi de méthodes géométriques avec un système de démonstration pour ceux qui ont des intérêts scientifiques. Largement inspiré des ouvrages du *ᶜilm al-*

---

42 (Giusti 2002, 60)
43 (Franci 2003, 35)
44 C'est- à dire à l'ensemble des peuples qui partageaient la culture latine- ce que nous pourrions traduire aujourd'hui comme les peuples européens.
45 Le latin était la langue des échanges internationaux, surtout avec les peuples de langue germanique (Pirenne 1985, 139 et suivts).
46 (Tangheroni 1994, 25)

misāha des pays d'Islam[47], on y trouve aussi des éléments de la géométrie pratique des *agrimensores* de la tradition romaine. Outre des références explicites aux *Éléments* d'Euclide, on peut y trouver des références implicites aux *Metrica* de Héron d'Alexandrie, au *Livre sur les divisions des figures* d'Euclide qui est perdu[48], ou encore aux travaux d'Archimède (notamment pour la méthode d'approximation de π). Le texte a été largement diffusé et a été transmis par une dizaine de manuscrits. En outre, il a été soumis à plusieurs traductions en langue vernaculaire, comme celle, par exemple, de Cristofano Gherardo di Dino, mathématicien de Pise au XV^e^ siècle[49].

Un aperçu sur les écoles d'abaque et les plus anciens textes d'abaque italiens des XIIIe-XIVe siècles.

De la deuxième moitié du XIIIe et jusqu'à la moitié du XVIe siècle environ, apparaissent, surtout en Italie, des écoles de mathématiques (écoles ou *botteghe* d'abaque) destinées non seulement aux marchands, mais aussi à tous ceux qui nécessitent l'apprentissage des mathématiques pour leur travail (comme les changeurs, les techniciens, les fonctionnaires des communes ou encore les architectes). Des maîtres y enseignent (maîtres d'abaque) et utilisent des manuels, appelés manuels d'abaque[50], en langue vernaculaire. Ces manuels sont très nombreux ; on les estime à quelques centaines. Comme le souligne Folkerts :

> The word *abbacus* in this context is confusing, because normally *abacus* is the name for the counting board, but the mathematics taught in the *libri d'abbaco* is not done with the help of a counting board, but with a pen on paper. The name *abbacus* was derived from the *Liber abbaci* written by Leonardo Fibonacci[51].

Les maîtres d'abaque avaient un statut social comparable à celui des petits marchands et des artisans. En effet, en langue vernaculaire (i.e. en italien), l'acception « maestro d'arte » ou « maestro di bottega » indiquait dans les Communes italiennes un expert artisan et le lieu d'apprentissage était l'atelier d'artisanat. Sur la base de plusieurs documents on a vu qu'ils ont été les premiers mathématiciens 'professionnels': souvent ces maîtres, à côté de l'enseignement, effectuaient un travail de consultation sur la base de leurs compétences spécifiques : réviseur de comptes, consultant pour l'estimation du prix des marchandises etc[52]. En Toscane, et en particulier à Florence – où l'enseignement a toujours été privé – il y a eu les écoles et les maîtres les plus prestigieux. La documentation relative à Florence et à d'autres centres est désormais très détaillée. Plus lacunaire apparaît aujourd'hui la documentation concernant Pise où le premier maître d'abaque (après peut-être le même Fibonacci) est un homme du nom de Magister Nocchus de abbaco, dont l'activité est fixée entre la fin du XIIIe siècle et la première partie du XIVe siècle. Lui et/ou son fils Bindo pourraient être les auteurs de deux traités d'abaque pisans anonymes datant environ de la même période : le ms

47 Pour un approfondissement des sources arabes voir (Moyon 2012).
48 (Moyon 2011)
49 (Arrighi 1966)
50 On trouve aussi d'autres titres comme: *Libro o trattato d'aritmetica*, *Libro o trattato d'Algorismo*, *Algorismo* ou encore *Libro di ragioni.*
51 (Folkerts 2011, 282)
52 (Franci 1988, 182-183)

anonyme LVI.472 de la Bibliothèque Intronati de Sienne (dont on verra plus loin) et le *Tractato de Arismetricha* (Bibliothèque Riccardienne, Florence, 25)[53].

Dès 1960, les éditions et les commentaires d'Arrighi et de l'école de Sienne[54], et en 1980, le catalogue des manuscrits de Van Egmond[55], ont permis la diffusion de ces manuels parmi les historiens des mathématiques. Ils ont aussi permis de dater le commencement de l'algèbre des équations des troisième et quatrième degrés en Italie au début du XIVe siècle. Ces manuels utilisent les chiffres indo-arabes, contiennent des instructions au sujet de l'arithmétique, les algorithmes pour les opérations avec les entiers et le calcul sur les fractions et résolvent des problèmes (*ragioni*), surtout à caractère commercial, par des méthodes variées : règle du trois, fausse et double position et parfois, par l'algèbre. À côté de ces textes, nous trouvons également des traités de géométrie pratique et, dans une moindre mesure, des traités purement algébriques et des recueils de jeux récréatifs. Par rapport à l'algèbre, ces manuels présentent des équations du troisième et du quatrième degré qui sont absentes dans les traités d'al-Khwārizmī (et de ses adaptations latines) et de Fibonacci. En outre, à la différence de ces derniers, le propos essentiel de l'algèbre de l'abaque des premières décades du XIVe siècle, est centré sur des réalisations concrètes, comme la résolution de problèmes de la pratique marchande. Les sources de l'algèbre de l'abaque italienne sont donc diverses et multiformes.

Les éléments essentiels de l'enseignement au sein des écoles d'abaque découlent des deux premières parties du *Liber Abaci* (Chap. 1-11) et de la *Practica Geometriae*, mais aussi des *Traités d'abaque* en langue vernaculaire qui présentent un choix de contenus plus accessibles. Selon Franci :

> Même si on ne peut pas nier que le traité de Léonard est l'archétype de ces textes, il faut remarquer, par contre, que chaque auteur a développé son travail de façon autonome, en choisissant les sujets et en les adaptant aux exigences de ses interlocuteurs (…)[56].

Même si l'algèbre n'est pas nécessaire pour résoudre la plupart des problèmes commerciaux, elle est néanmoins enseignée. La raison en serait qu'il y a un chapitre entier (le chap. XV) consacré à l'algèbre dans le *Liber Abaci*. En suivant Franci[57], on peut supposer trois niveaux d'enseignement. Dans le premier, on apprend la lecture et l'écriture des nombres, la représentation des nombres avec les mains et les calculs afférents, les algorithmes pour effectuer les opérations, le calcul avec les fractions, la règle de trois et ses applications aux calculs de l'intérêt, le système des monnaies, les poids et mesures, quelques notions de géométrie pratique. La formation est suffisante pour de petits artisans et commerçants. Ensuite, le deuxième niveau est prévu pour les marchands qui exercent leur profession dans le commerce international. Enfin, le troisième niveau est réservé aux amateurs des mathématiques et aux maîtres d'abaque. Parmi les matières enseignées dans les derniers niveaux : l'algèbre, la théorie des nombres et les problèmes marchands plus complexes. Il s'agit d'une tripartition qui, à bien des égards, reflète l'organisation du *Liber Abaci*.

---

[53] (Ulivi 2002, 2008 ; Franci 2015)

[54] L'école de Sienne est une équipe de travail qui, sous la direction de R. Franci et L.Toti Rigatelli, a poursuivi le travail de G. Arrighi. Elle a étudié et publié 26 traités d'abaque dans les *Quaderni del Centro Studi di Matematica Medioevale dell'Università di Sien».* Il ne s'agit pas, pourtant, d'éditions critiques des textes ou de leurs traductions dans une langue moderne.

[55] (Van Egmond 1980)

[56] (Franci 2003, 36)

[57] (Franci 1996)

Parmi les quelques témoignages relatifs à la structuration des matières dans une école d'abaque, on a retrouvé l'articulation des programmes d'enseignement selon la « méthode de Pise », dans le *Libbro d'abbaco* (1442) de Cristofano di Gherardo di Dino[58]. L'enseignement se divise en « mute », se déroule sur une journée entière, avec beaucoup de travail à faire à la maison. Cristofano propose dans deux sections appelées les « minori » et les « tredici » selon la « méthode de Pise »[59] avec des problèmes commerciaux extraits du chap. 8 du *Liber Abaci*, aussi contenus dans un texte d'abaque pisan écrit vers la fin du XIII[e] siècle[60]. Tout cela témoigne de la mémoire encore vivante de Fibonacci et de la continuité de l'enseignement depuis Fibonacci.

Conclusion

On peut considérer, à la suite de l'historiographie, le *Liber Abaci* comme le précurseur ou, pour le moins, un important représentant[61] d'une série de manuels d'abaque qui ont contribué à une formation mathématique massive, par la diffusion des chiffres indo-arabes et des techniques de calcul arithmétique et algébrique. On a vu, en particulier, l'influence qu'à deux siècles de distance l'œuvre de Léonard a continué à exercer sur les traités d'abaque d'origine pisane.

Si on se demande, cependant, quelles méthodes de Fibonacci ont été transmises et lesquelles ont été abandonnées, on s'aperçoit que[62]:

- Dans les écoles d'abaque italiennes des siècles suivants, l'enseignement des mathématiques se fait par problèmes (« ragioni »). La méthode sera plus prescriptive qu'explicative : « fais comme ça » sans explications ultérieures.
- L'aspect théorique sera négligé. Seules les règles comptent : la règle de 3, de simple et double fausse position etc. contrairement à Fibonacci.
- L'apprentissage se fera par imitation des problèmes résolus. On développera une mentalité mnémonique-opérative plutôt que logique-déductive.
- La démonstration sera remplacée par la preuve numérique des théorèmes, de l'exactitude d'un calcul. L'efficacité d'un algorithme sera une preuve suffisante pour le rendre « vrai ».

Bien que ces développements puissent apparaître en-deçà du modèle qu'est le *Liber Abaci*, l'activité des maîtres d'abaque a contribué, avec les écoles d'abaque, à la diffusion des mathématiques dans les activités professionnelles et à la consolidation – également grâce à la circulation de la pratique algébrique – d'une culture favorable aux grands exploits de l'algèbre italienne du XVI[e] siècle. Ceux-ci ont conduit à la résolution des équations des troisième et quatrième degrés par les mathématiciens tels que Tartaglia, Scipione del Ferro, Cardano et Ferrari qui ont eu une formation dans les écoles d'abaque. Leurs écrits sont aussi directement connectés avec ceux des maîtres des siècles précédents.

58 (Arrighi 1965-67).

59 Cette répartition est aussi présente dans la *Regula de Arismethica* (ms. Ash 576 de la Bibliothèque Laurentienne de Florence (1435)) et dans le *Trattato di Aritmetica e Geometria*, anonyme, (ms. LVI.46 de la Bibliothèque des Intronati de Sienne (*ca*. 1460)) ; (Franci 2015).

60 Il s'agit du manuscrit anonyme LVI.47$_2$ de la Bibliothèque des Intronati de Sienne ; transcription dans (Franci 2015).

61 Høyrup propose un renversement de perspective, qui fait l'objet d'un débat complexe que j'ai synthétisé dans (Caianiello 2013b) et dont je ne discuterai pas ici ; (Høyrup 2007).

62 (Gamba-Montebelli 1987)

L'importance des écoles d'abaque est encore plus significative si l'on considère que la pratique algébrique a été presque absente du curriculum universitaire jusqu'à la moitié du XVII^{e} siècle[63].

REFERENCES

Aïssani, D., & Valerian D. (2003). Mathématiques, commerce et société à Béjaia (Bugia) au moment du séjour de Leonardo Fibonacci (XII-XIII siècle). *Bollettino di storia delle scienze matematiche*, *XXIII*, 9-31.

Arrighi, G. (1966). *Leonardo Fibonacci, La Pratica di Geometria volgarizzata da Cristofano de Dino Cittacino Pisano, dal Codice 2186 Della Biblioteca Riccardiana di Firenze*. Pise: Domus Galilaeana.

Arrighi, G. (1965-67). Un 'programma' di didattica di matematica nella prima metà del quattrocento (Dal Codice 2186 della Biblioteca Riccardiana di Firenze). *Atti e Memorie della Accademia Petrarca di lettere, arti e scienze di Arezzo*, *38*, 117-28.

Black, R. D. (2007). *Education and Society in Tuscany from the 13th to the 15th Century c. 1250-1500*. Brill e-books.

Boncompagni, B. (1856). O*puscoli di Leonardo Pisano secondo la lezione di un codice della Bibliothèque Ambrosiana di Milan*. 2ème éd. Florence: Tipografia galileiana.

Boncompagni, B. (1857-1862). *Scritti di Leonardo Pisano matematico del secolo decimo terzo*. 2 vols.: I. Il Liber abaci pubblicato secondo la lezione del Codice Magliabechiano C. 1. 2616, Badia Fiorentina, n. 73; II. La pratica geometriae. Opuscoli: Flos, le Questiones avium e il Liber quadratorum. Rome: Boncompagni Tipografia delle Scienze matematiche.

Busard, H. L. L. (1968). L'algèbre au moyen age : Le 'Liber mensurationum' d'Abu Bakkr. *Journal des Savants*, *avril-juin*, 65-125.

Caianiello, E. (2012). La vita e l'opera di Leonardo Pisano, Appendice 1. In G. Matino (Eds.), *Forme e modi delle lingue dei testi tecnici antichi* (pp.54-60). Naples: D'Auria.

Caianiello, E. (2013). Leonardo of Pisa and the Liber Abaci. Biographical elements and the project of the work. In A. Bernard, & C. Proust (Eds.) *Scientific sources and teaching contexts throughout history: Problems and perspectives*. (pp. 217-246). Dordrecht-Heidelberg-New York-London: Springer.

Caianiello, E. (2013b). Les sources des textes d'abaque italiens du XIVème : Les échos d'un débat en cours. *Reti Medievali Rivista*, *14*(2), 189-209.

Davidsohn, R. (1956-68). *Storia di Firenze*, éd.it, 8 vols, IV (1965)-partie 3. Florence: Sansoni.

Djebbar, A. (2001). Les Transactions dans les mathématiques arabes : Classification, résolution et circulation. In CIHSO (Ed.), *Actes du Colloque International « Commerce et mathématiques du Moyen Age à la Renaissance, autour de la Méditerranée* (pp. 327-344).

Djebbar, A. (2005). *L'algèbre arabe. Genèse d'un art*. Paris: Vuibert.

Dufour, J. *et al*. (1979). L'attrait des 'Leges'. Note sur la lettre d'un moine victorin (vers 1124/27). *Studia et Documenta Historiae Iuris*, *45*, 504-529.

Folkerts, M. (2004). Leonardo Fibonacci's knowledge of Euclid's elements and of other mathematical texts. *Bollettino di storia delle scienze matematiche*, *24*, 93-113.

Folkerts, M. (2011). Review "J. Høyrup, Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture". *Annals of Science*, *68*(2), 282-284.

Franci, R. (1996). L'insegnamento dell'Aritmetica nel Medioevo. In P. Freguglia, L. Pellegrini, & R. Paciocco (Eds.), *Scienze matematiche e insegnamento in epoca medievale* (pp. 111-132). Chieti: Edizioni Scientifiche Italiane.

Franci, R. (2002). Il Liber Abaci di Leonardo Fibonacci: 1202-2002. *Bollettino della Unione Matematica Italiana*, *5-A*, 293-328.

Franci, R. (2003). Leonardo Pisano e la trattatistica sull'abaco nei secoli XIV e XV. *Bollettino di storia delle scienze matematiche*, *XXIII*(2), 33-54.

Franci, R. (2015). Un trattato d'abaco pisano della fine del XIII secolo. *Bollettino di storia delle scienze matematiche*, *XXXV*(1), 9-96.

Gamba, E., & Montebelli, V. (1987). La matematica abachista tra recupero della tradizione e rinnovamento scientifico. In *Giovan Battista Benedetti e il suo tempo* (pp. 169-202). Venise: Istituto Veneto di scienze, lettere e arti.

Germano, G. (2013). New Editorial Perspectives on Fibonacci's Liber Abaci. In G. Germano (Ed.), Studies on Fibonacci's Liber Abaci, *Reti Medievali Rivista*, *14*(2), 157-173.

Giusti, E. (2002). Matematica e commercio nel Liber Abaci. In E. Giusti (Eds.), *Un ponte sul Mediterraneo: Leonardo Pisano, la scienza araba e la rinascita della matematica in Occidente* (pp. 59-120). Firenze: Polistampa. Retrieved from http://php.math.unifi.it/archimede/archimede/fibonacci/catalogo/giusti.php.

[63] À ce sujet, voir (Moyon 2012).

Høyrup, J. (2007). *Jacopo da Firenze's Tractatus Algorismi and Early Italien Abbacus Culture*. Basel-Boston-Berlin: Birkhäuser.

Maccagni, C. (1988). Leonardo Fibonacci e il rinnovamento delle matematiche. In *L'Italia ed i paesi mediterranei: Atti del Convegno internazionale di studi*, *Pisa, 6-7 giugno 1987* (pp. 91-115). Pisa: Nistri-Lischi e pacini.

Moyon, M. (2011). Le 'De Superficierum Divisionibus Liber' d'al-Baghdâdî et ses prolongements en Europe. In A. Bouzari, & Y. Guergour (Eds.), *Actes du 9e Colloque Maghrébin sur l'Histoire des Mathématiques Arabes, Alger, 12-14 Mai 2007* (pp.159–201). Alger: Imprimerie Fasciné.

Moyon, M. (2012). Algèbre & Practica geometriæ en Occident médiéval latin : Abū Bakr, Fibonacci et Jean de Murs. In S. Rommevaux, M. Spiesser, & M.-R. Massa Estève (Eds.), *Pluralité de l'algèbre à la renaissance* (pp. 33-65). Paris : Honoré Champion.

Moyon, M., & Spiesser, M. (2015). L'arithmétique des fractions dans l'œuvre de Fibonacci : Fondements & usages. *Archive for History of Exact Sciences*, *69*(4), 391-427.

Moyon, M. (2016). *La géométrie de la mesure dans les traductions arabo-latines médiévales*. Turnhout : Brepols.

Pirenne, H. (1985). *Storia economica e sociale del Medioevo*. 2ème éd. italienne. Milan: Garzanti.

Stürner, W. (2009). *Federico II e l'apogeo del suo impero*. Rome: Salerno Editrice.

Tangheroni, M. (1994). Fibonacci, Pisa e il Mediterraneo. In R. Tangheroni, & M. Morelli (Eds.), *Leonardo Fibonacci, il tempo, le opere, l'eredità scientifica* (pp. 15-34). Pise: Pacini.

Ulivi, E. (2000). Le scuole d'abaco e l'insegnamento della matematica a Firenze nei secoli XIII-XVI. In P. Freguglia, L. Pellegrini, & R. Paciocco (Eds.), *Scienze matematiche e insegnamento in epoca medievale: Atti del Convegno internazionale di studio, Chieti*, *2-4 maggio 1996* (pp. 85-110). Naples-Rome: Edizioni Scientifiche Italiane.

Ulivi, E. (2011). Su Leonardo Fibonacci e sui maestri d'abaco pisani dei secoli XIII-XV. *Bollettino di storia delle scienze matematiche*, *XXXI*, 247-286.

Van Egmond, W. (1981). *Practical Mathematics in the Italian Renaissance: A catalog of italian abbacus manuscripts and printed books to 1600*. Florence: Giunti Barbera.

L'INTRODUCTION DES MATHÉMATIQUES « EUROPÉENNES » DANS L'EMPIRE OTTOMAN À PARTIR DU XVIIIe SIÈCLE

Mahdi ABDELJAOUAD

Dans la Turquie ottomane du XVIIIe siècle, les souverains et les élites dirigeantes sont confrontés à des échecs militaires et à des menaces étrangères grandissantes. Les armées encadrées par des janissaires utilisent des techniques devenues obsolètes et inopérantes face aux armées européennes. Géré depuis plusieurs siècles par les Ulémas selon un modèle médiéval, le système éducatif traditionnel transmet des connaissances ne répondant ni aux besoins militaires ni à ceux de la société civile. Ce n'est qu'après le traité de paix de Karlowitz (1699), désastreux pour les Ottomans, qu'un processus de transfert de connaissances se développe.

En voulant examiner la dynamique qui a permis la circulation des mathématiques européennes dans l'Empire ottoman et analyser les facteurs de son accélération ou la nature des obstacles rencontrés, nous cherchons à montrer que les principes, les contenus, les méthodes, les lieux et les acteurs du transfert ont été modifiés tout au long du siècle, jusqu'à atteindre le seuil du mimétisme institutionnel et le rejet dramatique.

L'étude de l'introduction des mathématiques « européennes » dans l'enseignement au Sud de la Méditerranée à partir du XVIIIe siècle a produit de nombreuses publications et, en particulier, l'important ouvrage de Pascal Crozet : *Les Sciences modernes en Égypte : Transfert et appropriation (1805-1902)*. L'auteur y montre[64] que :

[64] (Crozet 2008, 7)

> (i) le transfert des sciences modernes vers l'Égypte n'est pas engagé par les empires européens, mais s'effectue, dans un premier temps du moins, contre leur expansion ;
>
> (ii) la société égyptienne est héritière d'une tradition scientifique islamique qui partage un important patrimoine commun avec les sciences cultivées en Europe jusqu'au milieu du XVIIe siècle, sciences dans lesquelles s'enracinent précisément les sciences modernes.

Notre communication s'inscrit dans le cadre de ces thèses, mais en la focalisant sur la Turquie du XVIII^{e} siècle.

Des mathématiques immédiatement utilisables

Au XVIII^{e} siècle, les mathématiques traditionnelles continuent à être enseignées dans certaines *madrasas* [écoles] ottomanes par des ulémas [savants traditionnels] qui produisent des traités de géométrie pratique et d'arpentage. C'est le cas, par exemple, d'al-Dimashqī (m. 1749) auteur de *Sharh nukhbat al-tuffāha fī ^{c}ilmi l-misāha* [Commentaire sur <le poème> « Le meilleur de la science des mesures »], de Abu Sahl Nu^{c}mān (m. 1753) auteur d'un traité de géométrie pratique en turc : *Tabyīnu a^{c}māl al-misāha* [Clarification des arts de la mesure] ou de Muhammad Akkirmani (m. 1760), auteur de *Risāla fī ma^{c}rifat l-ab^{c}ād* [Epître sur la détermination des distances].

Ces auteurs traitent du mesurage (aires de surfaces planes rectilignes et non rectilignes et des volumes des solides usuels) et présentent l'arpentage utilisé respectivement pour installer des canalisations et agencer des édifices. Des instruments de nivellement sont décrits et la manière de déterminer la hauteur des bâtiments et la largeur des rivières est détaillée. Ils puisent leurs savoirs dans les ouvrages arabes anciens[65] et essayent d'améliorer les techniques décrites.

Les efforts de renouveau scientifique par les traductions

Cependant, la nécessité de se prémunir contre les menaces étrangères conduit une partie de l'élite ottomane à se rendre compte que les mathématiques et les techniques traditionnelles sont obsolètes et ne permettent plus de défendre l'Empire. Ils commencent à s'intéresser aux sciences et techniques européennes de la guerre. Cet intérêt se retrouve chez des bureaucrates exerçant dans l'administration centrale (secrétariat du grand vizir – affaires étrangères – gestions des conflits – fisc – monnaie) ou dans les provinces balkaniques. La plupart reçoit une formation de base soit à l'école du Palais, soit dans une madrasa traditionnelle et poursuivent leur spécialisation au domicile d'un savant ou dans un cabinet ministériel. Le profil d'autoformation d'un de ces fonctionnaires est décrit de la manière suivante par Vlahakis :

> He entered the office of the Council of State at the age of fourteen where he learned computation and syakat, a style of writing used in treasury accounts. Then, he started to work in a military office and had to participate in military campains. While in Istanbul, he frequented medrese teachers to learn astronomy and mathematics as well as theology, the Quran, the Hadith and logic. To build his erudition, he read biographical and bibliographical books.[66]

Leurs chefs directs – qui souvent ont suivi le même parcours – les encouragent à apprendre les langues étrangères et à fréquenter les Européens réfugiés ou expatriés, interprètes, officiers, ingénieurs et techniciens, souvent des convertis, recrutés pour aider à la formation des militaires. Certains de ces

[65] (Moyon 2016)
[66] (Vlahakis 2006, 89-90)

personnages sont même devenus Grand vizir après avoir été collecteur de taxe, responsable du bureau de la monnaie, chef comptable, *reiskuttab* (chef du secrétariat), ambassadeur, gouverneur régional ou ministre. On retrouve ce même profil dans les biographies des quelques savants ottomans ayant participé au processus de transfert de connaissances. Citons, en guise d'exemple, certains de ceux qui se sont intéressés aux mathématiques et à l'astronomie.

Avant de devenir Grand vizir, Rāghib Pācha (m. 1763) a suivi ce même parcours. Il codirige la commission ottomane qui négocie le traité de paix de 1740 réintégrant Belgrade dans l'Empire. Il est connu pour son désir de paix, ses réformes discrètes, sa grande culture et pour avoir encouragé les sciences et les arts. Lui-même est l'auteur d'une encyclopédie turque dont un chapitre traite de l'astronomie.

Fils du premier ambassadeur ottoman à Paris, Yenisehri Mehmed Said Efendī (m. 1767) maîtrise le français et les mathématiques. Grand vizir, il est ensuite lui-même nommé ambassadeur à Paris en 1742. Il est l'auteur de deux traités de géométrie pratique en turc, mélangeant l'inspiration traditionnelle aux apports étrangers : *Risalat al-misāha* [Epître sur le mesurage] et *Risāle-i sinüs li misāhat al-bu*ᶜ*d* [Epître de l'usage des sinus pour la mesure des distances][67].

Haut fonctionnaire de l'administration militaire, féru de science classique, Mustafa Sidqī (m. 1769) développe des recherches de haut niveau en géométrie et en astronomie et collabore avec Halife-zāde pour diriger une équipe de traducteurs et préparer un traité sur l'astrolabe à partir de « *l'usage des astrolabes tant universels que particuliers accompagné d'un traité qu'en explique la construction* » de Nicolas Bion (Paris 1702). L'équipe produit aussi des instruments astronomiques.

Çināri Ismaïl Efendī (m. 1788), mathématicien et astronome, traduit en 1765 les tables éphémérides de Cassini. Elles remplacent les tables élaborées par Ulug Beg au XVᵉ siècle à Samarkand et utilisées jusqu'à cette époque par les astronomes de l'Empire. Elles seront imprimées en 1772. C'est avec cet ouvrage que les logarithmes sont introduits pour la première fois en Turquie. Les travaux scientifiques de ces savants se caractérisent par leur volonté de s'approprier les nouveaux savoirs tout en les intégrant dans les cadres du savoir ancien et en cherchant à éviter les ruptures. Cette approche est parfaitement illustrée par le traité de géométrie d'un érudit provincial, interprète du gouverneur ottoman de Belgrade, ᶜUthmān al-muhtadī [Osman le converti]. Nous présentons maintenant ce personnage et son travail.

Un savoir hybride

ᶜUthmān al-muhtadī ou ᶜUthmān Efendī [Maître Osman] est connu comme ayant été, entre 1747 et 1784, l'interprète officiel et le traducteur des gouverneurs ottomans de Belgrade. C'est un érudit polyglotte confirmé, auteur de trois traductions en langue turque d'ouvrages scientifiques européens, le premier en géographie (1751), le second sur les plantes (1770) et troisième sur la distillation (1782). Il achève un traité de géométrie appliquée à l'art militaire, écrit en arabe, terminé en 1779, cinq années avant son décès à Belgrade. D'après l'auteur lui-même, *Hadiyyat al-muhtadī li-iqād al-sirāj al-muntafi* [L'Offrande du converti pour ranimer la flamme éteinte] s'inspire d'ouvrages français et allemands[68]. Cet ouvrage peut être considéré comme l'un des premiers vecteurs du transfert des mathématiques européennes vers la langue arabe.

---

67 Toutes les translittérations des titres d'ouvrages de langue turque sont prises de (Ihsanoglu & als. 1999).

68 L'analyse du texte est réalisée dans (Ageron & als. 2016). Notre étude s'appuie sur ce travail.

Lorsque l'on commence à lire l'ouvrage, rien ne le distingue d'un traité traditionnel arabe de mathématique. L'auteur commence par le *bismillah* (invocation du nom de Dieu) et par quelques vœux pieux, puis il introduit directement son projet sans s'attarder dans des éloges du prince. L'auteur précise qu'il a lui-même rédigé un texte de base surligné en rouge et un commentaire afin d'aider le lecteur à mieux comprendre les sujets étudiés :

> Lorsque j'ai constaté que dans certains traités de géométrie, on n'y examine que quelques notions et propositions où on n'y expose également ni leurs objectifs ni leur utilité, j'ai décidé d'écrire un traité de géométrie et notamment de mesurage contenant tout ce qui est indispensable, ajoutant certaines applications et négligeant ce qui y est rare, <sous la forme> d'un concis et d'un commentaire écrits dans une langue simple[69].

L'ouvrage contient plusieurs parties: (1) Objets de base de la géométrie euclidienne, (2) Éléments de géométrie euclidienne plane pratique suivi d'un appendice arithmétique, (3) Des instruments de mesures, (4) De la longimétrie et de l'altimétrie, (5) Des corps solides, (6) coniques, (7) De la science du mouvement galiléenne et ses applications au jet des bombes, (8) De la théorie des mines.

Sans jamais l'indiquer explicitement ou implicitement, l'auteur emprunte alternativement des définitions, des démonstrations, des commentaires, des descriptions d'instruments de mesure ou de dessin, à la version allemande du « *Nouveau Cours de mathématiques à l'usage de l'artillerie et du génie* » de Bélidor, aux « *Anfangsgründe der Geometrie* » de Wolff, à « *L'Art de jeter les bombes* » de Blondel, et tantôt à des passages d'origine indéterminée, composés à partir d'ouvrages ou d'instruments disponibles dans l'environnement de l'auteur, comme ceux signalés au paragraphe 2.

Presque nulle part dans la préface ou dans le texte lui-même, il n'est suggéré que le travail proposé contienne des sections traduites à partir de textes étrangers. L'auteur ne se réfère explicitement aux géomètres chrétiens [*muhandisī l-nasārā*] qu'une fois en écrivant « qu'en cherchant à faciliter la tâche des calculateurs et à préciser les mesures d'aires, ils ont choisi de décimaliser les unités de mesure », et il propose aux géomètres musulmans d'en faire de même. Dans le chapitre sur l'arpentage, il décrit plusieurs instruments de nivellement, mais il ne se réfère qu'une fois aux géomètres français [*muhandisī l-frank*] lorsqu'il décrit l'astrolabe simple servant à déterminer les angles et désignée comme étant « le demi-cercle ». Au chapitre sur les canons, il explique que les mécréants [*al-kafarah*] utilisent des petits canons bien mobiles et efficaces, inventés en Hollande et améliorés en Prusse puis après les avoir décrits il incite les militaires musulmans à les inclure dans leur armement. Parfois, il fait référence aux philosophes tardifs [*al-hukamā al-muta'akhkhirūn*] sans préciser s'il parle de savants musulmans ou chrétiens mais le contexte permet la plupart du temps de comprendre qu'il s'agit des seconds.

L'analyse minutieuse de l'ouvrage permet de constater que l'auteur n'a pas simplement cherché à traduire ses sources ou à leur emprunter définitions, descriptions et preuves, il a également tenté de comprendre les modes de raisonnement, d'assimiler les techniques et de s'approprier concepts et méthodes. Il a retravaillé le texte et l'a restructuré pour enfin lui donner une forme semblable à celle des traités traditionnels, accessible à des étudiants des *madrasas* ayant déjà reçu une formation en géométrie classique euclidienne comme celle que l'on trouve dans *Sharh ashkāl al-ta'sīs* [Commentaires sur le traité « Les figures de base »] de Qādhī Zāde al-Rūmī, connu pour avoir

[69] Manuscrit de Médine, Maktabat al-Malik ᶜAbd al-ᶜAzīz, ms. 2879, folio 1b. (notre traduction)

largement été enseigné dans l'Empire ottoman. Ce n'est qu'à la fin de l'ouvrage, que l'auteur précise qu'il l'a composé en traduisant des écrits en langue germanique [*nemjah*] ou en langue française [*frenjah*]. Quel est l'accueil reçu par cet ouvrage ? A-t-il été largement diffusé dans les écoles militaires nouvellement créées à Istanbul ? Nous proposons de répondre à ces questions dans le prochain paragraphe.

Cohabitation des savoirs

Considérant que les savoirs mathématiques et techniques européens sont à la source de la supériorité des armées européennes, le sultan Abdülhamid I (1774-1789) décide de les introduire dans la formation des cadres de la marine et de l'armée de terre. Une école, *Hendeshāne* [École de géométrie] est installée aux chantiers navals impériaux à Üsküdar. Elle est constituée d'une seule classe de 10 à 15 élèves-officiers. Des officiers français entrainent les cadets dans les sciences et techniques militaires. Le capitaine de vaisseau algérien Seyyd Hasan y enseigne les techniques navales et un second Algérien, ancien pilote de vaisseau, entretient les différents instruments et explique leur utilisation. Plusieurs *mudarris* [enseignants] des écoles traditionnelles, dont Ismail Gelenbevi Efendī, font également partie du corps enseignant. En 1784, l'école est transformée en *Mühendeshāne* [École d'ingénieurs] et on y ajoute la formation des officiers de l'armée de terre (artillerie, génie, mines, fortifications, topographie). Sept parmi les meilleurs cadets reçoivent une bourse pour se consacrer à leur formation. Outre leurs obligations d'encadrement des officiers de terrain aux techniques militaires modernes, des ingénieurs militaires français, en particulier les officiers André-Joseph Lafitte-Clavé (1740-1793) et Jean-Gabriel Monnier (1745-1818), assurent à l'école des cours théoriques et pratiques en sciences militaires, alors que les cours de mathématiques sont assurés par des *mulazīm* [assistants-professeurs], anciens diplômés de la *Hendeshāne*, encadrés par leur ancien professeur Ismail Gelenbevi, personnage caractéristique de la phase hybride de cohabitation des savoirs traditionnels et européens.

Ayant suivi le cursus traditionnel des ulémas ottomans, Ismail Gelenbevi (1730-1791)[70] réussit, en 1763, l'examen l'habilitant à enseigner dans les *madrasas*. Durant sa carrière d'enseignant, il publie en langue arabe plusieurs dizaines d'ouvrages dans diverses disciplines, telle que la philosophie, la logique et les mathématiques. À partir de 1775, il est recruté à *Hendeshāne* pour enseigner la géométrie théorique et pratique, tout en continuant à progresser dans sa carrière d'uléma [enseignant traditionnel], atteignant le grade suprême équivalent à celui d'enseignant à la Mosquée Suleymaniyya d'Istanbul et dans sa carrière de juriste en tant que *qādhī* [juge]. Il est certain qu'Ismail Gelenbevi s'intéresse aux sciences européennes et cherche à se les approprier, les utiliser et les enseigner ; nous listons ci-dessous les données qui l'attestent :

1) Un de ses collègues officiers français Lafitte-Clavé rapporte que Gelenbevi et d'autres *muderris* assistaient parfois aux cours qu'il professait.

2) Gelenbevi a eu entre les mains une traduction des *Tables portatives de logarithmes* de Callet, car il a ajouté sur le verso de la couverture la remarque suivante: « Cet ouvrage est étudié que par ceux qui sont capables d'apprécier les mathématiques élégantes et donc d'apprécier ces connaissances ». Ce manuel reste longtemps en usage dans les écoles d'ingénieurs turques.

[70] L'essentiel des informations que nous donnons figurent dans (Umut 2011, 46-71).

3) La rédaction par Gelenbevi lui-même, d'un manuel sur les logarithmes: *Sharhu Cadāvil al-Ansāb Lugaritma*. [Commentaire sur la table des logarithmes][71].

4) Lorsque le meilleur élève de l'école, ᶜAbdürrahamane Efendi, qui maîtrise bien le français, a traduit en turc des chapitres de certains de ses cours, son professeur, Ismail Gelenbevi, a révisé la traduction et en a rédigé une version finale. Ce cours pourrait être *Adla-i musallasat*, un traité de trigonométrie, attribué également à Gelenbevi.

5) Gelenbevi a rédigé une épître contenant des traductions en turc de larges extraits du chapitre sur les coniques du traité *L'Offrande du converti* de ᶜUthmān al-muhtadī[72]. Que Gelenbevi ait eu accès à cet ouvrage et qu'il l'ait lu ne surprend pas, car une copie de ce traité figure dans un inventaire de la bibliothèque de *Hendeshāne*, établi en 1788, et au moins une des copies que nous avons retrouvées a été retranscrite par Ibrāhīm Kami, un étudiant de cette école.

6) Gelenbevi avait suffisamment d'expertise en balistique pour intervenir sur le terrain et résoudre un problème par des calculs mathématiques. Cette initiative lui a valu d'être publiquement félicité par le sultan Selim III.

Compte tenu de son aura d'uléma et de sa renommée de mathématicien, Gelenbevi a contribué à la formation des premiers ingénieurs militaires ottomans capables de transformer les infrastructures militaires et civiles du pays et des premiers enseignants multilingues maîtrisant les sciences et les techniques européennes. Ces nouveaux lieux de formation des cadres militaires respectent les codes pédagogiques traditionnels par la présence, en tant que principal enseignant de géométrie et d'astronomie, d'un uléma comme garant de l'acceptabilité des savoirs enseignés. Non seulement, il participe à la formation des cadres militaires mais il côtoie également des officiers étrangers chargés de l'instruction militaire des cadets. Dans ces établissements, deux langues d'enseignement y sont privilégiées, le turc qui commence à acquérir un statut de langue scientifique et technique et le français. Le caractère hybride de ces institutions est bien souligné par Martykánová, qu'elle considère « quite similar to the École des Ponts et Chaussées in the first years of its existence », elle ajoute :

> The newly-created institutions were a mixture of Ottoman administrative-military traditions and madrasah ways of proceeding, blended with the innovations in contents and in conduct imported by foreign experts. They produced men of an original profile, in which the theoretical knowledge of a classical scholar combined with the status of a performance-oriented sultan's servant, without omitting a feature that would acquire an increasing importance: that of a bearer of modern "European" knowledge –including language skills- which was to become a gateway to a fast career in the Ottoman bureaucracy in the first two thirds of the nineteenth century[73].

Martykánová ajoute que ces écoles ne produisent pas des ulémas mais des *muhandis* [ingénieurs-géomètres] éduqués dans les sciences mathématiques européennes et sachant les utiliser dans les arts militaires. Ce modèle d'institution hybride cède rapidement la place à des lieux de transfert institutionnalisé des savoirs mathématiques et techniques formant des ingénieurs et laissant à la marge les sciences traditionnelles et les ulémas.

---

[71] On trouve dans les bibliothèques turques et arabes au moins 12 copies de cet ouvrage, dont une a été écrite en 1817 par un ancien élève de *Mühendeshāne,* Ibrāhīm Adham. Ce personnage est présenté plus loin au paragraphe 6.

[72] Communication privée de Mahmood Shahidī (février 2016).

[73] (Martykánová 2010, 65)

Institutionnalisation des transferts

La circulation des mathématiques et des techniques militaires européennes s'institutionnalise en 1793 lorsque le sultan Selim III (1793-1806), en accédant au trône, décide la mise en place d'une nouvelle armée (*Nizam-i Cedide*) organisée selon les normes européennes. Pour encadrer cette armée, il crée sur des bases nouvelles une école d'ingénieurs : Nouvelle École impériale d'ingénieurs (*Muhendeshāne Cedide*) formant des officiers artilleurs, topographes et sapeurs. Cette nouvelle école est caractérisée par :

1) La structuration en quatre classes de niveaux séparés dans lesquels des cours formels sont enseignés selon les niveaux.
2) Introduction des langues étrangères et plus particulièrement le français. Lorsque l'enseignant est un officier français, le cours est traduit en turc simultanément par un interprète ou par un élève d'une classe supérieure.
3) Les enseignants de mathématiques non étrangers sont des officiers-ingénieurs, diplômés des précédentes écoles. Ils enseignent en turc.
4) L'école forme un nouveau corps d'ingénieurs-architectes militaires appelés à encadrer la nouvelle armée. Ils s'exercent sur le terrain au cours de leur formation.
5) Une large bibliothèque de manuels européens est mise à la disposition des cadets.
6) L'imprimerie en caractères arabes est réactivée : elle publie des traductions de manuels européens et des cours de mathématiques et de technologie militaire.

Structuré sur le modèle européen, ce nouveau type d'établissement de formation suscite résistance et critique, comme en témoigne un jeune officier, Séid Moustafa, ancien élève de l'école devenu lui-même ingénieur-topographe dans l'armée de terre et enseignant-assistant à l'école. Dans un pamphlet écrit en français, *La diatribe de l'ingénieur*, il décrit l'hostilité rencontrée par les nouvelles méthodes d'enseignement :

> L'école fut établie et pourvue de maîtres et d'écoliers permanents et salariés. Je fus du nombre de ces derniers. Nous commençâmes à travailler en public ; c'était la première fois que le monde ignorant avait entendu à Constantinople des leçons de mathématiques et avait vu des géomètres en pleine assemblée ; la voie de l'impéritie et de l'ignorance s'éleva de tous côtés, on nous molesta, on nous persécuta presque, on criailla (*sic*) en disant : « pourquoi tirent-ils ces lignes sur le papier ? Quel avantage croient-ils retirer ? La guerre ne se fait point au compas et à la ligne[74].

Globalement, les réformes introduites par le sultan rencontrent l'opposition combinée des ulémas constatant que leur domaine de compétence était envahi par ces nouvelles et de divers corps de métiers (architectes, scribes et janissaires) menacés par la montée en puissance des nouveaux métiers (ingénieurs et officiers). En s'alliant aux ulémas et aux *tolbas* (élèves des madrasas), les janissaires organisent une révolte populaire et provoquent l'assassinat en 1807 du sultan Selim III et des principaux acteurs de la réforme (ministres, bureaucrates, officiers de la nouvelle armée, professeurs et

[74] (Séid 1803, 20)

cadets). Séid Moustafa est assassiné et son collègue, Ibrāhīm Edhem, échappe au massacre et s'exile en Égypte.

Transfert des savoirs vers l'Égypte

En affaiblissant le pouvoir central, le coup d'état d'Istanbul offre à Muhammad Ali Pacha, le vice-roi ottoman d'Égypte depuis 1805, l'opportunité de s'émanciper tout en retenant la leçon de l'échec de Selim III. En exterminant tous les chefs militaires autochtones d'Égypte, les mamelouks, il élimine toute opposition interne à ses projets d'autonomie et de réformes. Il peut, dès 1911, recruter des experts turcs et européens chargés de réorganiser complètement l'armée, l'administration et l'économie dans le cadre d'un large secteur étatique industriel, agricole et marchand. Il envoie également une première mission d'étudiants égyptiens en Europe pour y former des ingénieurs et des techniciens, militaires et civils. Dès le retour des premiers diplômés, il crée un collège de type européen pour les futurs cadres civils et militaires du pays et amorce ainsi la mise en place d'institutions éducatives parallèles et indépendantes du système traditionnel dirigé par les ulémas de la Mosquée al-Azhar. L'un des rescapés du massacre d'Istanbul et exilé en Égypte, Ibrāhīm Adhem (1785-1865) est recruté pour encadrer la nouvelle armée égyptienne. Son profil est caractéristique de la phase d'institutionnalisation du transfert des savoirs européens.

Connu également sous le nom égyptien d'Ibrāhīm Adham Pacha, il est un officier-ingénieur diplômé de *Mühandishāne* d'Istanbul, où il a également enseigné les mathématiques. Sa connaissance de la langue française lui permet de produire une seconde traduction en langue turque des *Tables de logarithmes* de Jean-François Callet. Ce travail terminé en 1806 est resté manuscrit[75]. En Égypte, il est chargé de la direction de l'intendance militaire et en particulier des usines d'armes et de canons de la Citadelle qui atteint son apogée en 1828. Parallèlement, entre 1820 et 1830, il forme en mathématiques des officiers de l'armée égyptienne travaillant dans l'Arsenal. Puis entre 1839 et 1863, il participe à la mise en place du système éducatif égyptien de type moderne, d'abord pour les besoins de l'armée, puis pour un secteur plus large de la population. Après le décès de Muhammad Ali Pacha en 1849, il tente de résister au démantèlement du nouveau système scolaire et, n'y parvenant pas, il retourne en 1863 en Turquie pour y terminer sa vie. Ses antécédents de traducteur de mathématiques françaises l'amènent à :

- réviser et publier, en 1836, à Bulāq la traduction turque du traité de mécanique : « La statique » de Bossut, réalisé à Istanbul avant 1836.
- traduire en turc et publier, en 1836, les *Éléments de géométrie* de Legendre (1752-1834), édition de 1823 à laquelle il ajoute des compléments personnels et des extraits puisés dans la *Géométrie* de Lacroix (1819). Cet ouvrage est ensuite traduit en arabe par son élève Muhammad 'Ismat et plusieurs éditions en seront publiées.
- écrire une épître sur les axiomes de la géométrie, publiée à Bulāq en 1836.

Ibrāhīm Adham Pacha est un produit et un acteur de la circulation mathématique institutionnalisée[76]. Il en maîtrise les codes, s'approprie une part importante des savoirs européens, les reproduit en les traduisant en langue vernaculaire, et participe à la mise en places de nouvelles

[75] (Ihsanoglu & als. 1999, vol. 1, 338)

[76] C'est-à-dire « la circulation de questions, de problèmes, de méthodes, d'explications, d'enseignements, de pratiques, de points de vue, de théorèmes mathématiques ou de métadiscours » (Nabonnand & als 2015, 7).

structures qui facilitent la réception des nouveaux savoirs (écoles nouvelles, imprimerie et bureaux de traduction des manuels). Une nouvelle phase de la circulation mathématique est atteinte lorsqu'on assiste « à l'insertion dans la société égyptienne de nouveaux savoirs et de nouvelles pratiques, dont les quelques travaux de recherches entrepris, la fondation d'écoles supérieures, ou les institutionnalisations des professions d'ingénieurs ou de médecins, portent suffisamment témoignage. »

Conclusion

Les élites traditionnelles religieuses (Ulémas et étudiants) et militaires (Janissaires et Mamelouks) perçoivent la présence européenne (experts et écoles militaires, imprimeries, modes diverses, savoirs nouveaux) comme un moyen insidieux et efficace de pénétration et de conquête, susceptible de détruire leur prestige, de leur ôter privilèges et fonctions au bénéfice d'une nouvelle classe européanisée. La résistance à cette pénétration est violente, elle entraîne des révoltes sanguinaires, reporte le processus de modernisation et le ralentit.

La circulation mathématique permet de suivre le processus de transfert des savoirs, des techniques et des méthodes européennes dans l'Empire ottoman. La science traditionnelle des pays d'Islam ne disparaît pas au XVIII^e siècle, les ulémas tentent d'abord de la rénover pour répondre aux besoins de l'armée, de la marine et des demandes civiles. La publication de traités de géométrie pratique en témoigne. Cependant, les guerres perdues incitent les gouvernants à chercher à acquérir les savoirs techniques des Européens par l'intermédiaire d'ambassades vers l'Europe, de recrutements d'experts étrangers – à condition qu'ils acceptent de se convertir à l'Islam – et l'encouragement des élites scientifiques à apprendre les langues étrangères. Les tentatives d'intégration de la science européenne restent cantonnées dans les cercles de spécialistes et les cabinets proches des états-majors et des ministères. Les changements s'accélèrent vers la fin du siècle car les besoins augmentent, la nécessité de former des cadres se précise et entraine la création d'écoles d'ingénieurs militaires. Ce sont, d'abord, des lieux hybrides similaires aux *madrasas* traditionnelles, mais dans lesquels ulémas et professeurs étrangers se côtoient et les savoirs se superposent. L'institutionnalisation d'un nouveau savoir importé est assumée par les décideurs : place est laissée aux enseignants et aux productions de cours traduits simultanément, puis directement enseignés par des ingénieurs-enseignants diplômés des premières écoles. Les travaux de ^cUthman al-muhtadī et la personnalité d'Ismail Gelenbevi illustrent la première phase, alors que les personnages de Séid Mustafa et d'Ibrāhīm Adham personnifient la seconde.

Le drame de 1806-1807, avec l'assassinat du sultan réformateur Selim III, marque la fin brutale de la tentative de réforme des institutions militaires ottomanes et en accélère la décrépitude. Cette expérience malheureuse reste dans la mémoire des successeurs qui pour réussir toute réforme ont dû exterminer les chefs des Janissaires et des Mamelouks et mettre au pas leurs alliés ulémas. En Égypte sous la férule du Vice-roi Muhammad Ali Pacha à partir de 1809, puis en Turquie même en 1834 (la réforme des *Tanzimāt*) sont créées des écoles militaires sur le modèle européen où on enseigne les mathématiques modernes dans des lieux confortables et avec des outils adéquats. Que ce soit à Istanbul ou au Caire, des mathématiques élémentaires et des sciences militaires sont d'abord enseignées en langue française ou italienne. Mais, les enseignants sont encouragés à traduire leurs cours en langues turque et arabe.

La circulation mathématique apparaît comme préalable à tout accomplissement du transfert des savoirs, méthodes et techniques européennes, mais leur appropriation par les communautés réceptrices nécessite d'autres études et analyses.

REFERENCES

Abdeljaouad, M. (2012). Teaching European mathematics in the Ottoman Empire during the eighteenth and nineteenth centuries: Between admiration and rejection. *ZDM Mathematics Education*, *44*, 483-498.

Ageron P., Abdeljaouad, M., & Shahidī, M. (2016). Émergence d'un savoir mathématique euro-islamique : L'Offrande du converti pour ranimer la flamme éteinte. *Philosophia Scientiae*, *20*(2).

Berkes, N. (1998). *The development of secularism in Turkey*. London: C. Hurst.

Crozet, P. (2008). *Les sciences modernes en Égypte. Transfert et appropriation, 1805–1902*. Paris: Geuthner.

Hitzel, F. (1995). Les Écoles de mathématiques turques et l'aide française. In D. Panzac (Ed.), *Histoire économique et sociale de l'Empire Ottoman et de la Turquie* (pp. 813–824). Louvain: Peeters.

Ihsanoglu, E. (1992). *Transfer of modern sciences and technology to the Muslim World*. Istanbul: IRCICA.

Ihsanoglu, E., Şeşen, R., & Izgi, C. (1999). *Osmanli Matematik Literatürü Tarih*. Istanbul: IRCICA.

Ihsanoglu, E. (2004). *Science, technology and learning in the Ottoman Empire: Western influence, local institutions and the transformation of knowledge*. Michigan, Burlington: Ashgate.

Martykánová, D. (2010). *Reconstructing Ottoman Engineers. Archeology of a profession (1789-1914)*. Pisa : Plus-Pisa University Press.

Moreau, O. (2009). *Réformes de l'État et réformismes au Maghreb au 19e siècle.* Tunis: IRMC.

Moyon, M. (2016). *La géométrie de la mesure dans les traductions arabo-latines médiévales*. Turnhout : Brepols

Nabonnand, P., Peiffer, J., & Gispert, H. (2015). Circulation et échanges mathématiques (18e – 20e siècles). *Philosophia Scientiae*, *19*(2), 7-16.

Séid, M. (1803). *Diatribe de l'ingénieur*. Istanbul: Scutari Imperial Printing house.

Umut, H. (2011). *Ismail Gelenbevi at the Engineering School: The Ottoman Experience of European Science Through Logarithms*. Master of Arts in History thesis. Istanbul Bilgi University.

Vlahakis, G. (2006), *Imperialism and Science. Social impact and Interaction*. New-York: ABC-Clio.

4. Oral Presentations

THE EMERGENCE OF THE IDEA OF IRRATIONALITY IN THEORETICAL MUSIC CONTEXTS IN THE RENAISSANCE

Oscar João ABDOUNUR
Instituto de Matemática e Estatística da Universidade de São Paulo, Rua do Matão, 1010, São Paulo, Brazil
abdounur@gmail.com

ABSTRACT

This presentation covers questions of how the relationship between mathematics and theoretical music throughout western history shaped modern comprehension of critical notions such as "ratio" and "proportion"; exploring the educational potentiality of such a comprehension. In order to do that, it will be consider a procedure taken by Erasmus of Höritz, a Bohemian mathematician and music theorist who emerged in the early 16th century as a German humanists very articulate with musical matters. In order to divide the tone, Erasmus preferred to use a numerical method to approach the geometrical mean, although his procedure did not recognize itself as an approximation of the true real number value of the geometric mean. The Early Modern Period saw the growing use of geometry as an instrument for solving structural problems in theoretical music, a change not independently from those occurred in the conception of ratio/number in the context of theoretical music. In the context of recovery of interest in Greek sources, Erasmus communicated to musical readers an important fruit of such a revival and was likely the first in the Renaissance to apply explicitly Euclidean geometry to solve problems in theoretical music.

1 Introduction

Although Erasmus also considered the tradition of De institutione musica of Boethius, he was based strongly on Euclid's The Elements, using geometry in his De musica in different ways in order to solve musical problems. It is this comprehensive geometrical work rather than the summary arithmetical and musical books of Boethius that serves Erasmus as his starting-point. However, Erasmus proposed a proportional numerical division of the whole tone interval sounding between strings with length ratio of 9:8, since it was a primary arithmetical problem. This presentation aims at showing the educational potentiality of the implications of such a procedure of Erasmus on the transformation of conception of ratio and on the emergence of the idea of modern number in theoretical music contexts. Under a broader perspective, it aims at show the implications on education of a historical/epistemological and interdisciplinary appraisal of theoretical music and mathematics.

In order to do that, it will be considered here a passage in chapter seventeenth of Book VI of Erasmus *De musica*, entitled *"Propositio decimaseptima Toni proportionem scilicet sesquioctavam in duas proportiones equales artificialiter et geometrice dividere*[1]*"*. It concerns the equal and proportional numerical division of the whole tone interval sounding between strings with length ratio of 9:8, a problem which confused the musical theorists from

[1] Proposition decimaseptima how to divide the sesquioctave ratio (that is, 9:8) of the whole tone into two equal ratios, artificially and geometrically.

Antiquity up the Renaissance and that played an important part in the historical process that led to the emergence of equal temperament. In this passage Erasmus seemed to be in a position to solve such a problem.

2 Division of the tone

The problem of the division of the tone arose from the Pythagorean discovery of numerical indivisibility of a superparticular or epimoric ratio, i.e., n : n+1, by its geometrical mean, in particular applicable to the division of the ratio 9:8. Given $p < x < q$, where p and q are integers and the ratio p:q is superparticular, x cannot be both an integer and at the same time fulfill the condition p:x=x:q, that is, be the geometric mean of p and q. Mathematically, the equal division of the tone 8:9 provides ratios involving surds or incommensurable ratios underlying musical intervals. These procedures were considered impossible by Pythagoreans in theoretical music, since these intervals could be determined only by ratios of integer numbers.

Attempts to divide the tone were already done, however, since Antiquity by Aristoxenus (fourth century B.C.), who conceived the theoretical nature of music as essentially geometric, understanding pitches, musical intervals and also distances as continuous quantities that should follow the rules of the Euclidean geometry and should be capable of being divided continuously, which inevitably raises questions concerning the nature of ratio in this context. Traditionally it is considered that Aristoxenian music theory rejected the position of the Pythagoreans in the sense that musical intervals should properly be expressed only as mathematical ratios involving whole numbers, asserting instead that the ear was the sole guide for musical phenomena (Winnington-Ingram, 1995, 592). It did not mean however that Aristoxenus' theory could not be put on a mathematical base related to the developments in Greek mathematics of his time. Aristoxenus preferred geometry to arithmetic in solving problems involving relations between musical pitches and believed in the possibility of dividing the tone into two equal parts, conceiving musical intervals and ratios as continuous magnitudes.

Such an idea unchained many reactions expressed for instance in the *Sectio Canonis* (Barbera, 1991, 125) and much later in the *De institutione musica* (Bower and Palisca, 1989, 88) of Boethius in the early Middle Ages, which stood for a strong Pythagorean tradition in theoretical music in the Middle Ages. Following the Pythagorean tradition, many medieval musical theorists maintained the impossibility of the equal division of the tone, which would mathematically lead to incommensurable ratios underlying musical intervals. Such a position begins to change in the 15th century and was eventually systematically overcome in the early Renaissance through scholars like Nicholas of Cusa, Erasmus of Höritz, Faber Stapulensis, Henricus Grammateus, Pedro Ciruelo, Juan Bermudo and others, who proposed the equal division of tone mostly by means of geometry. In his *Musica,* Erasmus of Höritz made use of an abstract numerical procedure to propose a solution for the problem of the equal division of the tone, expressing rather as a number the geometrical mean between the terms of the ratio 9:8 underlying the tone.

3 The *De Musica Speculativa* from Erasmus Horicius

Erasmus' *De musica* emerged in a time when the rediscovery, translation and publication of sources from Antiquity, such as the works of Euclid, Archimedes and Ptolemy, increased interest and development on number theory. Gaps in the Pythagorean numerical system was quite disturbing resulting in crisis and conceptual changes in the demarcation of the disciplines arithmetic and geometry. So ratios involving surds, i.e., incommensurable quantities, could only be discussed in the domain of continuous quantity and would request the unification of such two disciplines as well as the conquest of a number continuum on mathematical activity.

Particularly for Erasmus, Arabic and Hindu concepts were highly influential, since they promoted the development of Greek mathematics and handled entities such as negative and irrational numbers, as well as allowed, with the introduction of Hindu numerals by Fibonacci, the computation of unprecedented complexity as well as the development of extremely large numbers, being the latter an important component in Erasmus' division of the whole tone ratio, as it will be seen in the following.

In the Chapter Seventeenth of Book VI, Erasmus refers specifically to the division of the 9:8 ratio, which represents the musical interval of a whole tone. In the 4 previous chapters of book VI, Erasmus demonstrated incompletely the divisibility of other superparticular ratios into equal and proportional halves, like the octave (2:1), fourth (4:3), fifth (3:2) and minor third (6:5).

In Chapter Seventeenth, Erasmus proposed an abstract numerical procedure to find the geometrical mean between the terms of ratio 9:8 underlying the tone, expressing it as a number. He did not use the geometrical construction of mean proportional to two given straight lines from proposition 13 of Book VI of Euclid, as did for instance Jacques Lefèvre d'Etaples in 1496, using exclusively non-numerical Euclids methods capable of being carried out with straightedge and compass. He attempted rather to reach an expression for the ratio for the supposedly equally proportional halves of the whole tone interval using very large integer numbers. He did it first using proposition 15 of Book V of the Elements, which asserts, that a:b :: am:bm. Following his method, the half of the 9:8 ratio of a tone could be obtained by the geometric mean of its expansion into the term 34828517376: 30958682112. This ratio was derived directly from 9:8 by multiplying numerator and denominator by the factor 3869835264, a procedure guaranteed by proposition 15 of Book V for a = 9, b = 8 and m = 3869835264. The proportionality between the original ratio 9:8 and 34828517376:30958682112 allows a mapping between intermediate terms of the ratio 9:8, including the mean; and numbers between the terms of its expansion into a large number ratio, considering that the interval determined by the expansion is much more subdivisible and that the greater the distance between the terms in the large number ratio, the greater the precision one can get for the intermediate terms of the ratio 9:8, represented by the large number between the two terms of the large number ratio. Since there were not decimal fractions at this time, the proportional extended ratio is used for the purpose of extracting the square root with a high degree of precision, in this case associated with large integer numbers rather than with places after the decimal point. The bigger the distance between the terms in the large number ratio, the better the precision in which the geometrical mean is obtained. Nevertheless Erasmus seemed not to worry carrying out any computation in the text and he did not present his result as an approximation of the true real number. The text presents the square root of 72 as the geometrical mean of 8 and 9, without showing how he got this result. The procedure mentioned before induces one to think that it can be used to get such a result. But it seemed that Erasmus was not worried in showing it. He is the first author to propose an abstract numerical procedure for the given problem, expressing it as a number and avoiding using the construction of a geometrical line. Since it was a primary arithmetical problem, it could be solved "*artificialiter*", that is, numerically.

Erasmus asserts that "... in musical demonstrations we are forced to use all kinds of ratios ... since not all shapes of consonances and also dissonances are founded in rational ratios and for that reason we must not neglect the ratios of surds..." (Erasmus Horicius, [ca. 1500], fo. 61v). Erasmus considered here incommensurable ratios or irrational numbers in musical contexts. For music theoretical purposes, it would seem in principle that in order to make use of Eudoxus' theory of book V of Euclid's Elements on which theory of ratios of surds is based and which deals with abstract quantities with continuous nature; he established a link between continuous and discrete quantity[2]. Erasmus realized that the sought for a geometrical mean to the ratio underlying the whole tone could not result in a rational number and instead of changing the domain at this point from discrete quantity of numbers to continuous quantity of geometrical lines, he establish a link between continuous and discrete quantity, proposing a number continuum, although not explicitly, creating a very dense discrete point space between the original terms 9 and 8 by their expansion.

4 Concluding remarks

It is thinkable that if Erasmus really thought that he could divide the sesquioctave ratio in terms of a purely numerical operation, he must have possessed an at least rudimentary concept of the number continuum. Such an assumption is corroborated by a passage appearing later on in Chapter 17, where he seems to refer directly to the idea of such a continuum, mentioning Boethius as a prisoner of the Pythagorean doctrine of discrete integer number not accessing all ratios of numbers (Erasmus Horicius, [ca. 1500], fo. 67v). Just

[2] It is possible to identify similar ideas between the numerical division of the tone proposed by Erasmus and the Eudoxus´ definition V of Book 5 of the Elements. Whereas Erasmus confined a searched irrational number by using only integers, Eudoxus' definition corresponded, arithmetically speaking, to establish a proportionality of ratios through the confinement of ratios with integer terms. In these analogous procedures, Erasmus and Eudoxus got precision, to find an irrational number and to establish a proportionality between two given ratios, respectively, through ratios with big terms. Erasmus made use of "The Elements", nevertheless his source was the Campanus' translation, which had an arithmetical terminology that was not derived from the geometrical ratio theory of Book V of Euclid, but instead from a number of different sources including very likely the Arithmetic of Jordanus de Nemore from 13th century. Such a fact makes, on the one hand, not plausible that Erasmus would have had access to Eudoxus definition in the original sense and, on the other hand, very instigating the strong and curious structural analogy between both procedures.

before this passage, Erasmus asserts that the exact half of the whole tone interval would be provided by extracting the square root of the product of its terms 8 and 9, which would be sqrt (72) (Erasmus Horicius, [ca. 1500], fo. 67v). He did not however relate explicitly this result with the computations he presented. He got the large number ratio, but it was still needed to find the geometrical mean between the two terms. Since he presented the way to do this by extracting square roots of 9.8, one might ask why he did not do it from the ratio 9:8, or if he produced the proportional large number ratio, how could he use this representation to the extraction mentioned above and/or to approach the geometrical mean. It might be assumed that he left it to the reader. Such a method is structurally analogous to that used by Eudoxus to establish a criterion to compare ratios including incommensurable ones. In such an analogy, it is especially interesting and maybe the attribute which make this analogy structurally strong is that both procedures made use only of commensurable ratios in geometric and arithmetic contexts. This feature has educational potential, being an example of using history for epistemological purpose in the learning/teaching dynamics. Both procedures established by Eudoxus and Erasmus use only commensurable/rational ratios/numbers to introduce incommensurable/irrational ones, with a geometrical approach and an arithmetical one respectively and exemplify ways for introducing irrational numbers making use only of integers. Such historical analogous examples also allows to introduce a broader sense for the crisis of incommensurable, now presenting it in parallel to its musical version, in which Erasmus created a criterion to deal with such magnitudes making use only of commensurable ones as Eudoxus did.

Theoretically based on many geometrical propositions and, unusually, modeled on Euclidean style, *Musica* dealt with ratio as a continuous quantity, announcing perhaps what would emerge as an arithmetical treatment of ratios in theoretical music contexts during the sixteenth century, approaching ratio to a real number. Under an educational perspective, such two historical approaches for theories of ratio make music a favorable context for the differentiation between ratios, fractions and numbers, insofar as the semantic distinction between such two approaches stands out in this context. In music contexts, two musical intervals produced by two proportional ratios are clearly different although similar, whilst such a difference disappears in an arithmetical context, in which these ratios are identified with numbers. For instance, the ratios 2:3 and 4:6 produce musically two fifths with an octave difference. They are proportional, similar by not the same, noticeably not the same, whereas the difference between such ratios disappears in an arithmetical context, since 2/3 is equal to 4/6 arithmetically speaking. Interestingly, Erasmus could have easily solved the equal division of the tone making use of the proposition of Euclid's Elements which provide the geometrical mean as the height of a right angled triangle. Nevertheless, missing the concept of infinity, he preferred to use a numerical method to approach such a mean, although his procedure did not recognize itself as an approximation of the true real number value of the geometric mean. Erasmus provided a mathematical theoretical structure for a virtual pitch relation space, a continuum of rational numbers, that can be seen as an important step for laying the foundations for the real number system.

REFERENCES

Barbera, A. (1991). *The Euclidean division of the canon.* Lincoln: University Nebraska Press.

Bower, C. M., & Palisca, C. V. (Eds.) (1989). *Fundamentals of Music. Anicius Manlius Severinus Boethius*. New Haven & London: Yale University Press.

Erasmus Horitius (1500). *De Musica* [ca. 1500]. Reg. Lat. 1245. Città del Vaticano: Biblioteca Apostolica Vaticana.

Palisca, C. V. (1994). The Musica of Erasmus of Höritz. In C. Palisca (Ed.), *Studies in the history of Italian music and music theory* (pp. 146-67). Oxford: Clarendon Press.

Rommevaux, S., Vendrix, P., & Zara, V. (2011). *Proportions: Science, musique, peinture & architecture*. *Collection – études Renaissantes*. Brepols: Turnhout.

Winnington-Ingram, R. P. (1995). Aristonexus. In S. Sadie (Ed.), *The new Grove dictionary of music and musicians* (p. 592). London: Macmillan.

RÉCRÉATIONS MATHÉMATIQUES, GÉOMÉTRIE DE SITUATION… DE NOUVEAUX OUTILS POUR ENSEIGNER LES MATHÉMATIQUES À LA FIN DU XIX^{E} SIÈCLE

Jérôme AUVINET
Laboratoire de mathématiques Jean Leray, Université de Nantes, France
auvinet_jerome@yahoo.fr

ABSTRACT

At the end of the XIXth century, some mathematicians like Charles-Ange Laisant (1841-1920) or Édouard Lucas (1842-1891) develop new ideas about teaching mathematics using innovative tools or original visualizations. These ideas are linked to their interest for recreational games viewed through a mathematical background and other specific problems from combinatory, number theory, algorithms and the "géométrie de situation". We propose to give some examples of these mathematical representations and these ingenious games, explicitly presented by their authors in a didactical scheme. Moreover, many of these approaches could be presented nowadays to students of secondary schools. We discuss the role of visualizations through these experimentations and we point out the connections between diagrams and symbolisms. We study the need of arousing curiosity through striking results. We also underline the goals of representing processes in certain moments of mathematical learning and the benefits of these presentations, much more than simple recreational questions.

Les différentes situations que nous présentons ici sont abordées par plusieurs mathématiciens français de la fin du XIX^{e} siècle qui décèlent dans leurs propres études en théorie des nombres, arithmétique ou combinatoire des opportunités pour de nouveaux outils dédiés à l'enseignement des mathématiques. Cet enseignement scientifique est alors en plein renouveau sous la Troisième République jusqu'à la réforme de 1902[1]. Charles-Ange Laisant (1841-1920), Édouard Lucas (1842-1891) mais aussi Émile Fourrey (1869- ?) puis André Sainte-Laguë (1882-1950) sont autant de figures croisées au détour d'ouvrages portant sur ces récréations ou « curiosités » mathématiques. Si nos exemples sont fréquemment tirés de l'ouvrage de Laisant, l'*Initiation mathématique* (1906), il convient de souligner la proximité des approches de ces auteurs, en particulier à la suite de Lucas. Cette volonté de fournir une nouvelle base originale pour enseigner les mathématiques raisonne encore à notre époque : c'est pourquoi nous choisissons uniquement des exemples, aussi variés que possible, ayant été utilisés en classe ou qui pourraient l'être, notamment parce qu'ils apparaissent dans certains manuels actuels à destination des élèves de l'enseignement secondaire[2]. Nous les présentons dans la diversité de leurs utilisations pédagogiques : dans un premier temps, des situations introductives interpellant et mettant en activité l'élève. Suivent les possibilités d'accompagner visuellement la mise en place progressive d'une notion, ou d'illustrer de manière globale un procédé mathématique. Enfin, les différents changements de registres entre numérique et visuel permettent de faire vivre les connaissances et les techniques dans un cours.

[1] Voir par exemple (Belhoste, 1990).
[2] Nous renvoyons entre autre aux *Bulletins officiels spéciaux*, n° 9 (30/09/2010), n° 8 (13/11/2011).

1 Des récréations pour susciter la curiosité

Quand il publie son *Initiation mathématique* en 1906, le mathématicien C.-A. Laisant concrétise la nouvelle orientation donnée à son œuvre (Auvinet, 2013). Après avoir été le promoteur des équipollences et des quaternions, mais également député et homme fort de la presse mathématique, ce pédagogue souhaite, grâce à ce « guide » destiné aux éducateurs, libérer les jeunes enfants d'un enseignement rigide et dogmatique, peu formateur scientifiquement et peu utile socialement (Auvinet, 2015). Soucieux de s'appuyer sur la curiosité naturelle des enfants, il y reprend entre autres plusieurs récréations mathématiques étudiées par son ami Lucas, auteur d'une *Théorie des nombres* (Lucas, 1891) et chef de file de cette communauté de mathématiciens d'horizons divers traitant de manière originale de théorie des nombres, discipline peu représentée institutionnellement en France à l'époque.

« L'éventail mystérieux »[3], une des récréations présentée, se compose de cinq bandes de papier sur lesquelles sont inscrits apparemment de manière chaotique des entiers entre 1 et 31. Si une personne choisit secrètement un de ces entiers, puis signale les bandes (A, B, C ...) sur lesquelles il est inscrit, le détenteur de l'éventail est capable de retrouver immédiatement la valeur de l'entier en question. Laisant expose le principe de cet amusement en expliquant que les entiers sont considérés dans leur écriture binaire (fig. 1). Le premier carton rassemble alors les nombres se terminant par 1, le deuxième ceux dont le deuxième chiffre à partir de la droite est 1 etc. 25 = 11001 est donc inscrit sur les cartons A, D et E. Une fois ces cartons désignés, il suffit d'ajouter les nombres inscrits en tête de liste pour retrouver la valeur de départ. Ainsi, 1 + 8 + 16 = 25 car l'opération dans une écriture binaire s'écrit 1 + 1000 + 10000 = 11001.

1	1				
2	10	12	1100	22	10110
3	11	13	1101	23	10111
4	100	14	1110	24	11000
5	101	15	1111	25	11001
6	110	16	10000	26	11010
7	111	17	10001	27	11011
8	1000	18	10010	28	11100
9	1001	19	10011	29	11101
10	1010	20	10100	30	11110
11	1011	21	10101	31	11111

A	B	C	D	E
1	2	4	8	16
3	3	5	9	17
5	6	6	10	18
7	7	7	11	19
9	10	12	12	20
11	11	13	13	21
13	14	14	14	22
15	15	15	15	23
17	18	20	24	24
19	19	21	25	25
21	22	22	26	26
23	23	23	27	27
25	26	28	28	28
27	27	29	29	29
29	30	30	30	30
31	31	31	31	31

Figure 1. Construction de l'éventail mystérieux.

[3] (Lucas, vol. 1, 1891, p. 154) et (Laisant, 1916, p. 106). L'intérêt de Lucas pour les questions d'enseignement apparaît à maintes reprises, comme dans (Lucas, 1889 & 1895).

Si l'éventail mystérieux est utilisé par Laisant afin de consolider le calcul mental, il peut constituer une première approche de la numération binaire suscitant la curiosité des élèves voulant percer les mécanismes de ce procédé "magique". Le dispositif permet parallèlement une première pratique d'opérations simples dans ce système.

Le ressort didactique de ce procédé est bien de susciter la curiosité, voire l'étonnement et donc in fine l'intérêt. C'est aussi la démarche de Laisant lorsqu'il propose, dans la leçon 40 : « Un assez grand nombre » de son *Initiation*, de faire chercher le plus grand nombre que l'on puisse écrire avec trois chiffres 9 : soit le nombre

$$9^{9^9}$$

(et non 999, réponse immédiate des élèves). La surprise suscitée par le nombre conséquent de chiffres de son écriture décimale (neuvième terme de la suite de Joyce, calculé ici pour la première fois) reste une constante pour les élèves, à l'époque de l'*Initiation* comme à la nôtre. C'est pourquoi Laisant commente abondamment le temps d'écriture de ce nombre, la longueur de papier nécessaire etc.

D'autres « problèmes curieux ou amusants » sont également proposés par Émile Fourrey dans ses *Récréations Arithmétiques* (Fourrey, 1899). Comme l'explique son auteur, l'ouvrage profite d'un regain d'intérêt pour les récréations mathématiques. Cet engouement est dû à de nouvelles pratiques pédagogiques souhaitant introduire les notions mathématiques à partir d'« amusements » simples et attractifs. Fourrey constate d'ailleurs :

> il existe actuellement dans l'enseignement une tendance à ne pas faire aborder aux enfants l'étude des sciences par l'exposé de la théorie pure dont l'aridité peut les rebuter. À l'aide des quelques principes strictement nécessaires, on commence cette étude par d'amusantes applications qui intéressent les jeunes esprits et leur donnent le désir d'en connaître davantage. (Fourrey, 1899, p. VII)

Il signale aussi l'essor des jeux de réflexion dans la presse populaire de l'époque. Puis il reprend pour un large public un grand nombre de récréations arithmétiques, discipline souvent sollicitée dans ces jeux d'esprit (tout en développant particulièrement la question des carrés magiques). Par exemple, le problème « Écrire 31 avec seulement 5 chiffres 3 » soit :

$$31 = 3^3 + 3 + \frac{3}{3}$$

est un exercice simple de calcul mental mobilisant des stratégies diverses par les élèves.

Le problème du « Jeu du piquet à cheval » est un autre exemple de ces activités ludiques : deux joueurs somment successivement des entiers choisis entre 1 et 11 avant que le premier atteignant 100 ne remporte la partie (Fourrey, 1899, p. 48). Cette version simplifiée du jeu de Nim peut être étudiée à l'aide d'une progression arithmétique. La conception d'une stratégie gagnante à l'aide d'une suite est une illustration pertinente de l'attrait de l'outil mathématique, une mathématique des jeux que soupçonnent peu les élèves du secondaire.[4]

[4] (Rougetet, 2012 & 2014).

2 Suivre des yeux un trajet pour saisir une notion

Dans son article « Remarques arithmétiques sur les nombres composés» (Laisant, 1888), Laisant ne se contente pas de déterminer le nombre de décompositions d'un entier N en *k* facteurs. Il suggère un mode de figuration « fort simple » des entiers en précisant : « Il y aurait peut-être lieu d'en tirer parti pour l'enseignement des premiers principes élémentaires relatifs à la décomposition des nombres en facteurs premiers » (Laisant, 1888, p. 152).

À partir d'une ligne droite verticale (fig. 2)[5], il numérote des bandes horizontales par les nombres premiers 2, 3, 5, 7… et dessine sur chaque bande un nombre de cases égal à l'exposant où apparaît le facteur premier correspondant dans la décomposition de l'entier considéré. Ce nombre est ainsi représenté par la figure obtenue ou, au choix, par le contour extérieur de celle-ci. Ainsi, « Ce mode de représentation met en relief d'une façon saisissante la formation des diviseurs. » (Laisant, 1888, p. 153) Le nombre de diviseurs est en effet égal au nombre de chemins tracés sur le quadrillage intérieur de la figure pour aller de sa base inférieure à sa base supérieure. Quand il doit rechercher de façon exhaustive le nombre de diviseurs d'un entier, l'élève peut alors substituer à l'élaboration parfois laborieuse[6] d'un arbre tentaculaire de choix, la recherche des différents tracés intérieurs possibles.

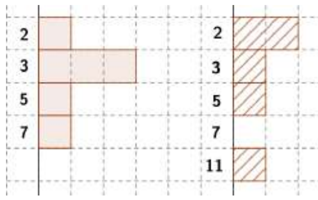


Figure 2. Représentations des entiers $N = 1890 = 2.3^3.5.7$ et $N' = 660 = 2^2.3.5.11$.

Ainsi, « Un assez grand nombre de propriétés connues peuvent avec cette figuration prendre un caractère intuitif » (Laisant, 1888, p. 154), intuitif car ici perceptible et sensible. Un nombre est multiple d'un autre lorsque le contour qui lui est associé contient le contour du second. En superposant la représentation de deux entiers sous cette forme, leur PGCD apparaît par la partie commune de leurs figurations et leur PPCM sera représenté par le contour extérieur de cette nouvelle figure (fig. 3). Deux nombres sont premiers entre eux si leurs contours respectifs n'ont aucun segment en commun.

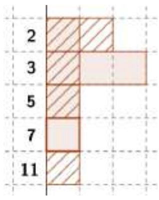


Figure 3. N et N', leur PGCD ($D = 2.3.5 = 30$) et leur PPCM ($p = 2^2.3^3.5.7.11 = 41580$).

En utilisant différentes couleurs pour la représentation de *n* entiers, on obtient la possibilité de déterminer à vue le PGCD ou le PPCM de deux d'entre eux et de vérifier s'ils

5 Comme dans plusieurs exemples qui vont suivre, la figure peut-être tracée sur un papier quadrillé dont Laisant souligne les avantages pratiques pour de telles constructions. Voir aussi (Sainte-Laguë, 1910).

6 C'est cette présentation qui apparaît habituellement dans un manuel scolaire : Le Yaouancq (dir.) (2012). *Collection math'x, Terminale S spécialité*, Les éditions Didier, p. 54.

sont premiers deux à deux. Cette visualisation a également l'intérêt de mettre en lumière le fait que le PGCD soit la plus grande partie commune aux représentations de ces entiers. De même que la recherche du PPCM consiste à déterminer le plus petit contour des figurations des multiples communs à ces deux entiers. Pour les élèves qui peuvent confondre les deux notions, c'est une aide précieuse que de leur montrer cette différence essentielle, de leur proposer une image éclairant ces acronymes. Une schématisation unique est ici exploitée comme support de distinction de deux notions liées.

Suivre un trajet sur une figuration est un procédé qu'on retrouve plusieurs fois dans les travaux de Laisant, comme chez d'autres mathématiciens travaillant à l'époque sur la théorie des nombres.[7] Sa communication « Sur la figuration graphique de quelques nombres combinatoires » (Laisant, 1894, fig. 4) a lieu au cours du congrès de l'Association française pour l'avancement des sciences (AFAS) de 1893, association créée en 1872 qui organise des réunions annuelles partout sur le territoire français afin de promouvoir la Science auprès d'un large public d'amateurs. Il y propose une figuration du nombre n ! en dénombrant les itinéraires reliant un point a_0 à un point a_n où, sur une même droite, le point a_i est relié au point a_{i+1} par $i + 1$ segments. De manière semblable, si on se place sur un quadrillage (fig. 4), le nombre $u_{n,\,p}$ d'itinéraires permettant de rejoindre, à partir de l'origine et en marchant dans le sens positif dans chacune des deux directions, le point de coordonnées entières (n, p) s'exprime à partir des coefficients binomiaux qu'on retrouve dans le carré de Fermat largement étudié et généralisé par Laisant. Cette représentation permet de compléter l'approche habituelle à partir d'un arbre de choix ; elle est d'ailleurs présente dans certains manuels actuels.[8] Comme Laisant le précise, elle relève d'une configuration classique en géométrie de situation : celle de l'étude de la marche d'une tour sur un échiquier. La figure de l'échiquier est ainsi volontairement insérée dans plusieurs manuels d'enseignement dont Laisant participe à la rédaction dans les années 1890.[9]

La combinaison des deux précédentes figures (fig. 4) permet également une généralisation à l'illustration des arrangements, nombres qu'il est alors naturel de considérer en suivant l'exposé de Laisant. La visualisation proposée soutient ici une progression graduée et naturelle dans l'introduction des notions.

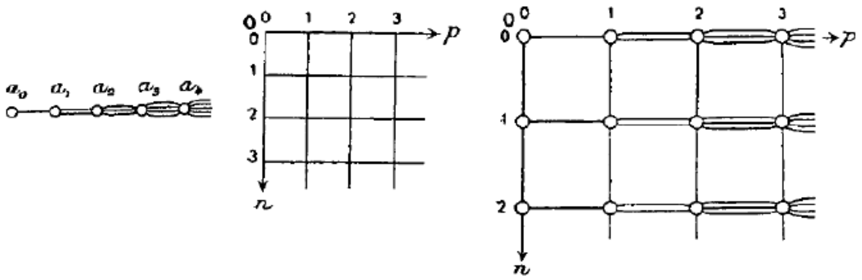


Figure 4. Figuration des nombres $n!, C_n^p, A_n^p$.

---

[7] Voir par exemple (Tarry, 1886).

[8] Le Yaouancq (dir.) (2015). *Collection math'x, Première S*. Les éditions Didier, p. 241. Une telle représentation est susceptible de mettre en lumière la relation $u_{n,p} = u_{n-1,\,p} + u_{n,\,p-1}$.

[9] Voir par exemple l'appendice de (Laisant & Perrin, 1892) ou (Laisant, 1916, p. 99).

3 Embrasser d'un seul regard un processus mathématique

Un exercice classique du chapitre « raisonnement par récurrence » du programme de terminale scientifique consiste à déterminer la somme des nombres entiers impairs consécutifs, soit :

$$S_n = \sum_{k=1}^{n} (2k - 1) = 1 + 3 + 5 + ... + (2n - 1) = n^2.$$

Cet exercice peut être complété par un dessin constituant une « preuve sans mot ».[10] On retrouve une telle preuve dans l'*Initiation mathématique* de Laisant à la leçon 28 sur les nombres carrés (Laisant, 1916, p. 28). Cette visualisation (fig. 5) est directement inspirée du « carré de choux » étudié par Lucas dans une communication de 1884 à l'AFAS (« Le calcul et les machines à calculer », Lucas, 1884).

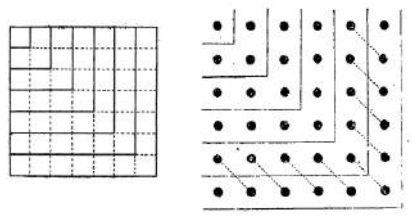


Figure 5. La représentation de la somme des impairs par Laisant,
le carré de choux de Lucas.

L'exercice précédent présente en effet certaines difficultés pour les élèves. La toute première est le symbolisme du "sigma" mal maitrisé en Terminale et qui peut freiner les élèves les plus fragiles, même si la formule "éclatée" permet de dépasser cet obstacle. Ici, le formalisme, même allégé, devient une barrière à la démonstration et à la compréhension d'une propriété qui, reformulée schématiquement, devient plus intelligible. L'intérêt de la preuve sans mot est de dépasser tout symbolisme puisque la figure est dénuée de toute écriture. L'étape d'hérédité dans la récurrence peut également être un obstacle technique gênant : l'acte technique peut alors s'appuyer sur la visualisation du « carré de choux ». La figure contient en effet en elle-même le principe de l'hérédité puisque pour passer d'un carré (de côté n) au carré plus grand (de côté $n + 1$), on ajoute le $n + 1^{\text{ième}}$ nombre impair.

D'un point de vue didactique, la preuve sans mot complète la démonstration rédigée pour en faire comprendre les ressorts : elle en est la parfaite traduction visuelle et une aide précieuse pour l'enseignant. Elle a de plus l'avantage de s'inscrire dans la mémoire des élèves qui assurent se souvenir plus volontiers d'un dessin coloré que d'une austère preuve par récurrence. La propriété est ainsi justifiée, comprise et retenue grâce à un support visuel attrayant que l'apprenant peut s'approprier immédiatement. Ce support suscite l'étonnement des élèves, peu habitués à cette démarche et doutant souvent du statut de preuve d'un tel

[10] Le Yaouancq (dir.) (2012). *Collection math'x, Terminale S*. Les éditions Didier, p. 31. Pour d'autres preuves sans mot, on peut consulter (Delahaye, 1998) et (Alsina & Nelsen, 2010).

dessin non formel. Si la visualisation ne semble pas prouver pleinement, elle les convainc pourtant aisément.

Une telle représentation peut de plus être généralisée à l'espace afin de déterminer la somme des cubes des entiers, un autre avantage perçu par les élèves. C'est aussi ce que propose Lucas dans une de ses récréations mathématiques (fig. 6) :[11]

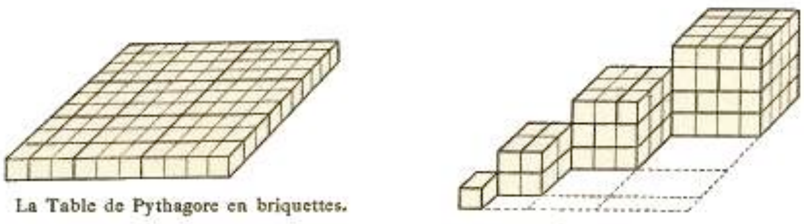


Figure 6. « la pile des cubes est le carré de la pile des nombres. » (Lucas, 1894, p. 65)

Déjà, en 1906, Laisant a conscience du potentiel didactique de telles figurations. La leçon « Le vol des grues » (fig. 7) est aussi inspirée de la même communication de Lucas à l'AFAS. Laisant y étudie les nombres triangulaires et détermine à partir d'un quadrillage la somme des n premiers entiers, formule incontournable pour les élèves de Première.[12]

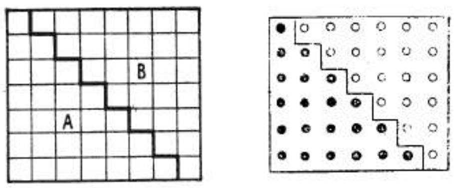


Figure 7. Le vol des grues de Laisant et de Lucas.

Il ajoute en particulier :

> Voilà des formules qui paraissent bien savantes et qui cependant n'exigent pas même le moindre calcul, puisqu'on les lit sur les figures, puisqu'on les voit, puisqu'on peut les construire de ses mains avec des petits carrés de bois. (Laisant, 1916, p. 88)

Dans un article consacré à l'étude des tables de multiplication et de division modulo un entier m qu'il mène avec le mathématicien amateur Gabriel Arnoux (1831-1913), il précise le statut qu'il confère à la preuve par la figuration :

> Dans beaucoup de questions, et particulièrement en mathématiques, la méthode graphique présente de grands avantages au point de vue de la clarté. Elle met en évidence la vérité qui n'apparaît que confusément sous les symboles, et quand on peut se contenter de dire : « Voyez », la démonstration approche de la perfection. On

[11] Exemple repris dans un manuel actuel de Terminale: Le Yaouancq (dir.) (2012). *Collection math'x, Terminale S.* Les éditions Didier, p. 28.

[12] Laisant en déduit également diverses formules sur les nombres triangulaires, comme par exemple :

$$2T_n - n = n^2 = T_n + T_{n-1}.$$

pourrait presque dire que l'art d'exposer est celui de faire des schémas. (Laisant, Arnoux, 1906, p. 36)

Laisant a conscience des applications variées de la visualisation de procédés mathématiques. En 1887 (Laisant, 1887), il répond à une question de Lucas posée dix ans plus tôt. Nous la formulons comme ceci (fig. 8) : dans un réseau (noté p_r) de points de coordonnées (n ; $a + n \times r$ (mod p)) avec r et p premiers entre eux, à quelles conditions les parallélogrammes ainsi formés sont-ils des rectangles, des losanges ou des carrés ?

Rang	0	1	2	3	4	5	6	7	8	9	10	11	12
Terme	0	3	6	9	12	15	18	21	24	27	30	33	36
Résidu	0	3	6	9	12	2	5	8	11	1	4	7	10

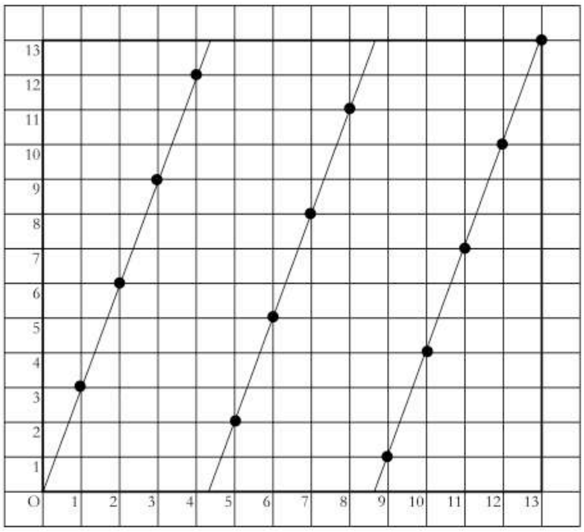


Figure 8. Figuration des résidus d'une progression arithmétique, réseau 13_3.

Ce questionnement fait suite aux recherches de Lucas sur la géométrie des tissus, notamment en lien avec une problématique industrielle.[13] Cette étude s'insère dans plusieurs travaux sur cette « géométrie des quinconces » impliquant Laisant, Lucas ou le mathématicien Ernest Laquière (1840- ?) pour qui elle représente « la peinture graphique de la théorie des nombres ».[14]

À l'aide de la méthode des équipollences, Laisant détermine les relations entre p, r, q et ρ (où $p = qr + \rho$) correspondant à chaque cas. Il souligne que « Les réseaux dont nous avons indiqué la construction peuvent être d'un grand secours dans bien des questions d'arithmétique. » (Laisant, 1888, p. 227) Outre la détermination graphique des résidus des termes $a + n \times r$ (mod p), un tel réseau permet de déterminer visuellement une solution particulière des équations diophantiennes de la forme $rx - pz = a$. L'équation $7x - 40z = 3$ trouve « à vue » une solution grâce au réseau 40_7 (fig. 9) : en ayant repéré le point d'ordonnée

[13] Lucas, É. (1867). *Application de l'arithmétique à la construction de l'armure des satins réguliers*. Paris, Reteaux.

[14] Laquière, E. (1879). Note sur la géométrie des quinconces. *Bulletin de la Société mathématique de France*, *7*, 85-92, p. 85.

$y = 3$ du réseau, on lit son abscisse $x = 29$; la valeur de z est ensuite déterminée, toujours d'un regard, en numérotant les droites parallèles à la droite d'équation $y = 7x$ qui peuvent être tracées jusqu'au point considéré.

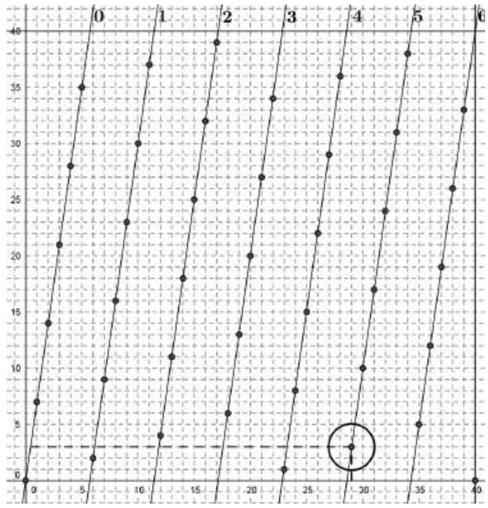


Figure 9. Réseau 40_7 utile à la résolution de l'équation $7x - 40z = 3$.

Laisant montre en s'appuyant sur ces réseaux que l'on peut résoudre les équations de la forme $rx + pz = a$ et donne de nombreuses autres applications (comme une « figuration des fractions périodiques »). Il résume : « Ce qu'il faut retenir surtout de cet emploi des échiquiers et des réseaux, c'est la figuration dans un espace limité de faits arithmétiques qui se reproduisent périodiquement dans l'étendue infinie du plan. » (Laisant, 1888, p. 228) C'est en effet un avantage d'une telle visualisation que de proposer un cadre graphique à la notion de congruence, parfois délicate pour les élèves. Elle induit un raisonnement particulier et la conscience précise de la périodicité des restes dans la division euclidienne. De telles représentations permettent de condenser dans l'espace d'une figure des processus qui s'étendent sur l'ensemble des entiers.

4 La visualisation comme variable didactique

Dans son *Initiation*, Laisant présente également une figuration des identités remarquables chères aux élèves de Collège pour lesquels la linéarité de la fonction carrée est souvent intuitivement naturelle.

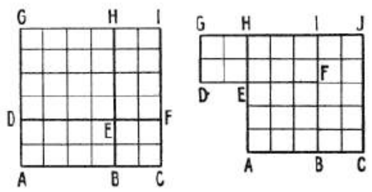


Figure 10. Figuration du développement de $(a + b)^2$ et $(a - b)^2$.[15]

[15] Sur la première figure, $(AB + BC)^2 = DE^2 + BC^2 + AB.AD + EF.FI = AB^2 + 2.AB.BC + BC^2$.
Sur la deuxième, $AB^2 = \text{aire}(ACJGDE) - 2.\text{aire}(BCJI) \Leftrightarrow (AC - BC)^2 = AC^2 + BC^2 - 2.AC.BC$.

Ayant tracé la première partie de la figure 10, il explique que « Le carré construit sur la somme de deux lignes est équivalent au carré construit sur la première, plus le carré construit sur la seconde, plus deux fois le rectangle construit sur ces deux lignes comme côtés » puis que « Le carré de la somme de deux nombres est égal à la somme des carrés de ces deux nombres, plus deux fois leur produit » ou encore $(a + b)^2 = a^2 + 2ab + b^2$. Il en conclut : « Voilà trois vérités dont on bourrera trois fois la mémoire des enfants non avertis, alors qu'elles n'en font qu'une seule, sautant aux yeux. Les apparences, les costumes sont distincts ; mais la personne est la même. » (Laisant, 1916, p. 82) Le même exemple est aussi traité par Fourrey dans ses *Récréations arithmétiques* (Fourrey, 1899, p. 65). Si Laisant présente le pendant géométrique d'autres identités remarquables (fig. 10), leur compréhension par les élèves est plus délicate et leur mémorisation moins évidente. Outre cette limite, il n'en demeure pas moins un support géométrique pertinent pour raisonner à partir des aires sur une figure plane et surtout une remédiation originale aux erreurs de développement grossières grâce à un changement de registre de manipulation, l'algèbre s'éclipsant devant la géométrie.

Nous proposons un dernier exemple de passage du champ algébrique au champ graphique. L'étude des graphiques représentant la distance parcourue par un véhicule en fonction du temps est classique dans beaucoup de manuels du secondaire.[16] Si le contexte de proportionnalité est rapidement repéré, la description d'un tel voyage est accessible et simple. Il semble intéressant mais plus délicat de faire construire de tels graphiques aux élèves pour résoudre des problèmes de rencontre entre deux voyageurs progressant à des vitesses différentes. Ce « problème des courriers » offre un contexte d'application particulièrement adapté à une visualisation graphique. On trouve des exemples de cette démarche dans les manuels, utilisant parfois l'outil informatique.[17] Il est cependant important de souligner qu'ici le cadre de résolution est fonctionnel. Le recours au "tout numérique" est aussi un réflexe des élèves dans cette situation présentée comme ouverte. La mise en équation s'y révèle d'autant plus périlleuse pour ceux qui s'engagent dans cette voie ; l'usage d'un graphique à la manière d'un abaque reste marginal bien qu'éclairant.

Nous choisissons de présenter la démarche de Laisant dans son *Initiation*, qui est tout autre. Le problème des « deux marcheurs » est une application de la leçon 46 au titre évocateur : « Les graphiques ; algèbre sans calcul ». À partir d'un graphique construit uniquement sur des considérations géométriques (fig. 11), le point de rencontre (heure et distance au point de départ) est rapidement repéré. Ce premier exemple est complété par des énoncés plus complexes. D'abord, lorsqu'un troisième voyageur vient à la rencontre des deux premiers. La situation suivante relève du cas de deux voyageurs disposant d'un seul moyen de locomotion (vélo, automobile) pour arriver à destination : le premier doit délaisser son véhicule une voire deux fois à certains endroits pour que le second, parti à pied, puisse en profiter. Enfin, le problème « Le chien et les deux voyageurs » examine la distance parcourue par un chien se déplaçant alternativement d'un marcheur à l'autre tandis que ces derniers poursuivent leur progression à des allures différentes. Laisant propose une évaluation

[16] Voir par exemple : Le Yaouanq (dir.) (2014), *collection Math'x seconde.* Les éditions Didier, p. 74.
[17] Malaval, J. (2014). *Hyperbole seconde.* Éditions Nathan, p. 49.

graphique des valeurs solutions, avec l'imprécision qui s'y rattache. Il souligne la multiplicité des situations concrètes pouvant amener à de tels graphiques. Il insiste également sur le potentiel de telles représentations pour donner sens immédiatement, au-delà de calculs sophistiqués ou d'un âpre symbolisme. Il écrit : « Il y a donc beaucoup de questions auxquelles s'appliquent avantageusement ces tracés, qui ont en outre l'avantage de parler à l'esprit par l'intermédiaire des yeux, de figurer les choses elles-mêmes. C'est là une qualité précieuse en matière de pédagogie. » (Laisant, 1906, p. 135)

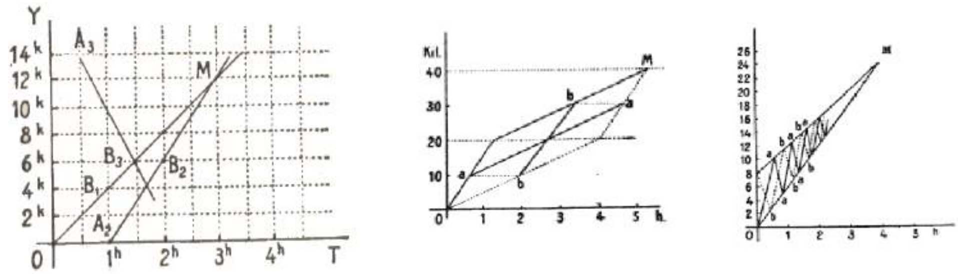


Figure 11. Le problème des courriers et ses variantes par Laisant.

Nous signalons également le problème suivant posé par Édouard Lucas (fig. 12). Sachant que chaque jour partent simultanément du Havre et de New York des bateaux en direction de l'autre port, la question est de savoir combien de bateaux va croiser un navire durant son voyage d'une semaine. L'illustration proposée par Laisant montre « la haute utilité des représentations graphiques » où se complètent le raisonnement et la perception visuelle de la situation. Il souligne les erreurs des notables scientifiques à qui la question avait été posée (probablement lors d'un congrès de l'AFAS) « raisonnant mais ne voyant pas » (Laisant, 1916, p. 141). Le rôle du regard du mathématicien prime ici de manière très appuyée.

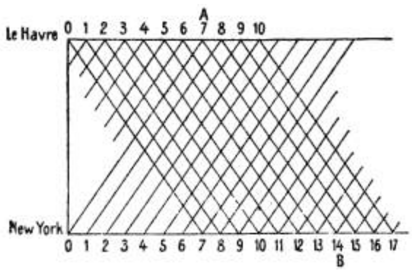


Figure 12. Le problème de Lucas.

5 Éléments de conclusion

Les quelques exemples précédents illustrent la richesse des ressources historiques pour diversifier l'activité mathématique en classe ; ce que les auteurs de la fin du XIX^e siècle avaient déjà bien saisi, les ouvrages de Laisant ou Lucas ne négligeant pas les références historiques. Leurs cadres d'applications sont multiples (approche d'une notion, justification, mémorisation, remédiation, généralisation etc.) tout autant que les possibles moments de leurs usages. Ils correspondent à plusieurs compétences à travailler chez les lycéens (chercher, représenter…)[18] et ancrent les notions abordées dans une réalité intelligible, parfois ludique.

[18] Voir *Les compétences mathématiques au lycée*, MEN/DGESCO-IGEN, novembre 2013.

Ils modifient la perception de l'activité mathématique chez les élèves (mathématiques stratégiques ou défis historiques). Beaucoup des situations présentées s'appuient sur l'importance du regard de l'apprenant dont la vision soutient l'entendement : elles s'inscrivent donc dans une pulsation entre discursif et visuel (Barbin, 1998). Mais c'est en citant La Chalotais et son *Essai d'éducation nationale* (1763) ou Jean Macé et son *Arithmétique du Grand-Papa* (1862) que Laisant et Lucas enracinent leurs démarches dans un renouveau nécessaire de l'enseignement. Le premier distingue notamment les situations proposées dans son *Initiation* comme « moyen pédagogique » des récréations du second, renvoyant par la même à une responsabilité de « l'éducateur » et à ses choix. La principale fonction de ces même approches pédagogiques demeure, à la fin du XIX^{e} siècle comme de nos jours, de susciter l'intérêt de l'élève, car comme l'explique Lucas : « Ainsi, vous le voyez, l'enseignement des Sciences doit être gai, vivant, amusant, récréatif et non froid, imposant, solennel. » (Lucas, 1895, p. 193)

REFERENCES

Alsina, C., & Nelsen, B. (2010). An invitation to proofs without words. *European journal of pure and applied mathematics*, *1*(3), 188-127.

Auvinet, J. (2013). *Charles-Ange Laisant. Itinéraires et engagements d'un mathématicien de la Troisième République*. Paris : Hermann Éditeurs.

Auvinet, J. (2015). Around a book dedicated to childhood friends: Initiation Mathematique by C.-A. Laisant. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), History and epistemology in mathematics education. *Proceedings of the seventh European summer university* (pp. 709-722). Danish Scholl of Education.

Barbin, É. (1998). La démonstration : pulsation entre le discursif et le visuel. *Publication de l'Institut de recherche mathématique de Rennes*, *S4*, 39-67.

Barbin, É. (2007). Les Récréations : des mathématiques à la marge. *Pour la science*, *30*, 22-25.

Belhoste, B. (1990). L'enseignement scientifique au tournant des XIX et XXe siècles. *Revue d'histoire des sciences*, *44*(4), 371-400.

Delahaye, J.-P. (1998). La preuve sans mot. *Pour la Sciences*, *244*, 100-106.

Fourrey, É. (1899). *Récréations arithmétiques*. Paris : Nony.

Fourrey, É. (2001). *Curiosités géométriques*. *Préface d'Évelyne Barbin*. Paris : Vuibert.

Laisant, C.-A. (1888). Remarques arithmétiques sur les nombres composés. *Bulletin de la Société mathématique de France*, *16*, 150-155.

Laisant, C.-A. (1888). Quelques applications arithmétiques de la géométrie des quinconces. *Compte-rendu du congrès de Toulouse*, *2*, 218-235.

Laisant, C.-A. (1894). Figuration graphique de quelques nombres combinatoires. *Compte-rendu du congrès de Besançon*, *2*, 298-303.

Laisant, C.-A. (1916). *Initiation Mathématique, ouvrage étranger à tout programme dédié aux amis de l'enfance. 2nd édition*. Paris : Hachette.

Laisant, C.-A., & Perrin, É. (1892). *Premiers principes d'algèbre à l'usage des élèves de l'enseignement primaire supérieur, des écoles normales primaires et des candidats aux écoles supérieures de commerce*. Paris : Ch. Delagrave.

Lucas, É. (1891). *Théorie des nombres. vol. 1*. Paris : Gauthier-Villars.

Lucas, É. (1889). *Jeux scientifiques pour servir à l'histoire, à l'enseignement et à la pratique du calcul et du dessin.* Paris : Chambon & Baye.

Lucas, É. (1891-1894). *Récréations Mathématiques*. *4 vol., 2nd édition*. Paris : Gauthier-Villars.

Lucas, É. (1884). Le calcul et les machines à calculer. *Compte-rendu du congrès de Blois*, *1*, 111-141.

Lucas, É. (1895). *L'Arithmétique amusante.* Paris : Gauthier Villars.

Popescu-Pampu, P. (2014). L'éventail mystérieux. *Images des Mathématiques*, CNRS.

Sainte-Laguë, A. (1910). Note sur les usages du papier quadrillé. *L'Enseignement mathématique*, *12*, 5-17.

Sainte-Laguë, A. (1990). *Avec des nombres et des lignes. Récréations mathématiques*. Paris: Vuibert.

Rougetet, L. (2014). *Des récréations arithmétiques au corps des nombres surréels et à la victoire d'un programme aux Echecs. Une histoire de la théorie des jeux combinatoires au XX^{e} siècle.* Thèse de doctorat en d'histoire des sciences et épistémologie, soutenue le 22 septembre 2014, Université de Lille.
Rougetet, L. (2012). Les multiples ancêtres du jeu de Nim. *Pour la Science*, *420*, 80-85.
Tarry G. (1886). Géométrie de situation : nombre de manières distinctes de parcourir en une seule course toutes les allées d'un labyrinthe rentrant, en ne passant qu'une seule fois par chacune des allées. *Compte-rendu du congrès de Nancy*, *2*, 49-53.

DES RÉCRÉATIONS POUR ENSEIGNER LES MATHÉMATIQUES AVEC LUCAS, FOURREY, LAISANT

Évelyne BARBIN, René GUITART
LMJL & IREM, Université de Nantes, France
evelyne.barbin@wanadoo.fr
IMJ-PRG, Université Paris Diderot, France
rene.guitart@orange.fr

ABSTRACT

À la fin du XIX[e] siècle, il existe un retour aux récréations mathématiques dans une communauté française de mathématiciens, enseignants et amateurs. La nouveauté de ce retour est la volonté des auteurs d'inscrire les récréations dans l'histoire des mathématiques, de rechercher leur valeur pour l'enseignement et de développer de nouvelles mathématiques pour résoudre les problèmes qui leur sont liés. Nous trouvons ces trois intérêts historiques, pédagogiques et mathématiques chez Édouard Lucas, Émile Fourrey et Charles-Ange Laisant. Les récréations choisies par eux indiquent un continuum entre les trois propos. Avec eux, les récréations ne sont pas seulement un moyen d'amuser et elles deviennent une matière sérieuse à enseigner, selon des conceptions mathématiques et éducatives que nous analysons dans cet article.

At the end of 19[th] century, there was a coming back to recreational mathematics into a French community of mathematicians, teachers and amateurs. The novelty of this coming back is the will of the authors to inscribe recreations into the history of mathematics, to investigate their value for teaching and to develop new mathematics to solve problems involved. We find these three historical, pedagogical and mathematical interests in Édouard Lucas, Émile Fourrey and Charles-Ange Laisant. The recreations chosen by them indicate a continuum between the three purposes. With them, recreations are not a way to amuse only and become a serious matter to learn and teach, according to mathematical and educational conceptions that we analyze in this paper.

1 Les Récréations chez Édouard Lucas : un continuum

Édouard Lucas était professeur de mathématiques spéciales au lycée Saint-Louis. Depuis *La Fasioulette* (1889) jusqu'à la *Théorie des nombres* (1891), en passant par les quatre tomes des *Récréations mathématiques* (de 1882 à 1894) et *L'arithmétique amusante* (1895), le spectre des destinataires de ses ouvrages est large, mais le propos est identique et les thèmes se retrouvent d'un ouvrage à l'autre. Dans tous ces ouvrages, Lucas accorde une part importante à l'histoire des mathématiques : elle occupe la trentaine de pages de la préface de la *Théorie des nombres* et elle sert d'introduction à ses *Récréations*. Cette histoire est orientée, elle vise à montrer que les mathématiciens du passé se sont intéressés aux récréations et qu'ils ont introduit, pour les aborder, des mathématiques qui semblent marginales en cette fin de siècle.

La portée éducative des récréations est fortement soutenue par Lucas. *La Fasioulette* est dédiée à ses enfants Paul et Madeleine : « j'ai donc fait ces petits livres pour vous récréer tout en vous apprenant des combinaisons arithmétiques et géométriques très difficiles » (Lucas, 1889, p. 4). L'ouvrage, paru dans la série « Jeux scientifiques pour servir à l'Histoire, à l'Enseignement et à la Pratique » dirigée par Lucas, est signé par le « Professeur N. Claus (de Siam), Mandarin du Collège Li-Sou-Stian » – pseudonyme qui ne devait pas cacher grand-chose. Des contenus sont communs aux différents ouvrages, comme les différentes manières de multiplier deux nombres, avec la multiplication rapide du *Liber Abbaci*, la « multiplication arabe par réseaux » et les bâtons de Neper, entre autres. Nous avons choisi de présenter plus précisément les problèmes de dominos et le jeu de taquin, mais les carrés magiques seraient aussi un thème de choix, car ce sont des manuscrits inédits de Fermat sur les carrés magiques qui ont d'abord attiré l'attention de Lucas vers l'histoire et vers les récréations.

1.1 Les problèmes de dominos

Dans *L'arithmétique amusante*, Lucas propose une « simple amusette », qui consiste à escamoter discrètement un domino d'un jeu de dominos, puis à demander à un ami de disposer tous les dominos suivant la disposition rectiligne de la règle du jeu. Votre ami sera étonné que vous puissiez prédire les points des deux extrémités. En effet, comme Lucas l'explique, lorsque l'on place tous les dominos suivant la règle du jeu, le jeu se termine toujours par un nombre final égal au nombre initial. Mais si vous avez subtilisé le domino avec les nombres cinq et trois, alors la disposition rectiligne aura pour extrémités cinq et trois. Les problèmes de dominos sont nombreux dans *Les tablettes du chercheur*, journal « des jeux d'esprit et de combinaisons » créé en 1890 (Barbin, à paraître).

Le problème de trouver le nombre de dispositions rectilignes possibles pour un jeu de dominos a été posé en 1859 par le Dr Reiss de Francfort et il intéresse la communauté des mathématiciens et amateurs qui fréquente l'Association Française pour l'Encouragement des sciences. La solution est trouvée par Gaston Tarry en 1886 (Barbin, à paraître) et elle fait l'objet de la récréation « La géométrie des réseaux et le problème des dominos » du tome IV des *Récréations mathématiques*. Elle repose sur une remarque de Laisant, à savoir qu'en supprimant les doubles, le problème se ramène à décrire d'un trait continu tous les côtés et les diagonales d'un heptagone en les empruntant une seule fois (Figure 1, gauche haut).

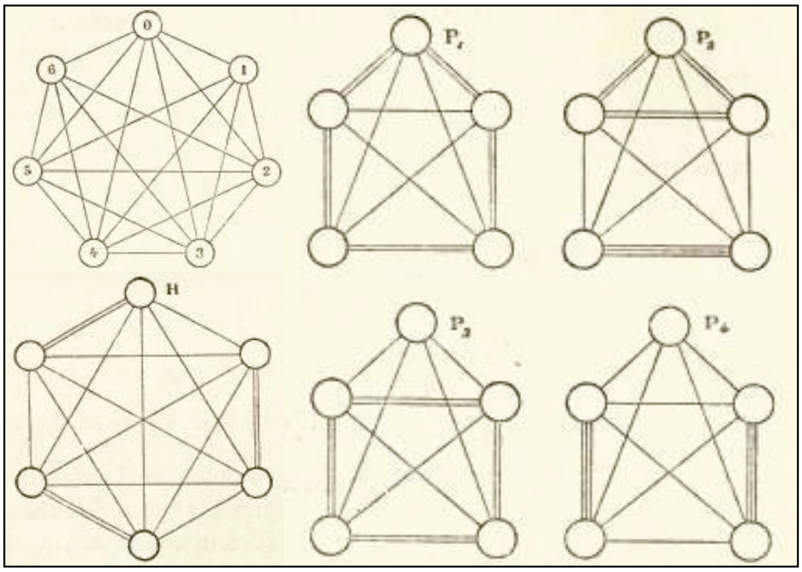


Figure 1. Problème des dominos : la solution de Tarry

Lucas appelle « réseau » et « carrefour » ce que nous nommons graphe et sommet. Il indique que Leonhard Euler a abordé ce type de problème dans son « Mémoire des Ponts de la Pregel » et a montré que tout réseau pair (pour chaque sommet le nombre d'arêtes est pair) peut être décrit d'un seul trait par un circuit fermé. Le problème « Le melon et la fourmi » illustre la situation : « un melon a douze côtes ; une fourmi visite successivement les douze vallons qui séparent les côtes et revient à son point de départ. Quelle est le nombre de manières d'accomplir ce voyage ? » (Lucas, 1894, p. 136). Lucas explique les raisonnements combinatoires qui ont permis à Tarry de trouver la solution en ramenant le problème de l'heptagone au même problème pour un hexagone, puis pour des pentagones, etc. Ainsi, le nombre de tracés N pour l'heptagone est égal à 15 fois le nombre de tracés H pour un hexagone (Figure 1, gauche bas). Ce nombre H est égal à $8\ P_1 + 4\ P_2 + 8\ P_3 + 4\ P_4$, où P_1, P_2, P_3, P_4 sont les nombres de tracés pour des pentagones (Figure 1, droite). Finalement, il trouve que le nombre de dispositions rectilignes pour le jeu de dominos est égal à 129 976 320.

Le problème des dominos est typique des récréations qui intéressent Lucas, en ce sens qu'elles sont au départ de nouvelles investigations mathématiques. Sa solution mérite alors de figurer dans un ouvrage proprement mathématique et elle se trouve en effet dans le chapitre VII de la *Théorie des nombres* sur « La Géométrie de situation » (Lucas, 1891, pp. 102-108).

1.2 Le jeu du taquin

Le dernier chapitre du volume I des *Récréations mathématiques* d'Édouard Lucas (1882) est consacré au taquin, jeu qui a été inventé en 1879 aux États-unis sous le nom de « Fifteen Puzzle » (Guitart, à paraître). Ce jeu y fait rapidement fureur, puis en France, Allemagne, Angleterre. Mis au courant dès 1880, notamment par James Sylvester qui a publié dans son journal deux auteurs d'analyses complètes du jeu, Lucas simplifie celles-ci, profite de remarques de ses amis Delannoy et Laisant (eux aussi amateurs de puzzles mathématiques), et publie sa présentation en 1881, reprise dans les *Récréations* en 1882. Lucas met l'accent sur l'usage d'une théorie importante, due à Leibniz et prolongée par Cauchy, qu'il nomme théorie des déterminants. Nous mettrions plutôt l'accent aujourd'hui sur l'aspect particulier de cette théorie qu'il emploie, à savoir le calcul des permutations, des dérangements, de la signature. En l'occurrence, pour lui, le jeu devient l'occasion « d'une sorte d'introduction à l'étude de cette partie de l'algèbre moderne » dont le jeu est alors « une représentation sensible » (Lucas, 1882, p. 190).

Dans une boîte 4×4 sont placés 15 petits blocs rectangulaires numérotés, laissant libre une des 16 places ; les blocs sont mobiles et peuvent seulement glisser, horizontalement ou verticalement, en profitant d'une case libre voisine. Le problème est de passer d'une configuration donnée à une autre, par exemple de la configuration de gauche à celle du centre (possible) ou à celle de droite (impossible) (Figure 2). Lucas montre que les configurations du centre « 1...13-15-14 vide » et de droite « 1...13-14-15 vide » ne peuvent se ramener l'une à l'autre, et que de plus, toute configuration peut être transformée en l'une de ces deux là. Il énonce un critère pour savoir si deux configurations peuvent se ramener l'une à l'autre, et si oui, il donne le moyen de le faire (Lucas, 1882, p. 207). D'une part, à toute configuration est associée une permutation des 16 premiers nombres (la case vide étant numérotée 16). par exemple dans la figure 2, les configurations de gauche, du centre et de droite déterminent les permutations (16 1 2 … 15), (1 2 … 15 14 16) et (1 2 … 14 15 16). D'autre part, si le jeu est colorié comme un échiquier en cases noires et blanches. Alors, pour deux configurations, si les cases vides sont de même couleur (resp. de couleurs différentes) alors il est possible de passer de l’une à l’autre si et seulement si les signatures des permutations associées sont les mêmes (resp. différentes).

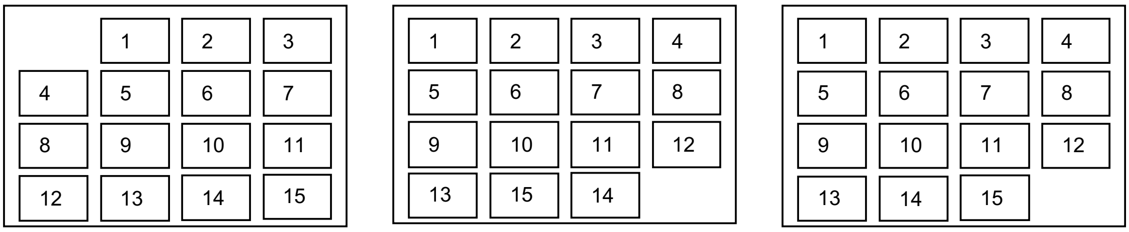


Figure 2. Problème du taquin : cas possible et impossible

Lucas poursuit par l’étude de jeux de taquin généralisés, de forme rectangulaire m × n, ou à barrières, obstacles, embranchements ou garages. Il énonce par exemple le théorème suivant: « Dans un taquin formé d’un nombre quelconque d’embranchements et d’un garage sans aucune tête de ligne, on peut passer d’une position quelconque à une autre position de même classe » (Lucas, 1882, p. 232). Il donne (Figure 3) un exemple d’un tel taquin. En fait, il utilise ici, pour ce théorème, le même lemme que lui a fourni Laisant et qu’il a utilisé déjà dans le taquin 4 × 4, qui ici se dira : par le moyen du garage on peut toujours faire franchir à un bloc quelconque deux blocs consécutifs, sans déranger l’ordre de tous les autres cubes.

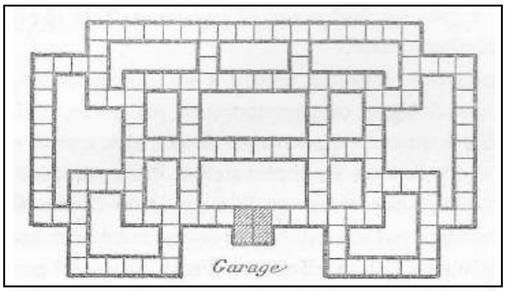


Figure 3. Taquin à embranchements et garage et sans tête de ligne.

Il existe aujourd’hui (Hordern, 1986) des centaines de tels jeux par glissements de blocs, avec de plus des blocs de formes différentes, voire des mouvements permis différents selon les blocs. Au contraire des jeux électroniques permis par les nouvelles technologies, ces jeux initient directement aux gestes mathématiques et leur pratique est, pourrait-on dire avec Lucas, une sorte d’introduction à cette partie importante de la mathématique moderne qui est la théorie des graphes, dont lesdits jeux sont donc des représentations sensibles.

2 L’arithmétique visuelle des récréations

Les mathématiques des récréations proposées par les auteurs sont visuelles, elles évitent les lettres et l’algèbre et elles préfèrent une figure à un discours. Nous en prendrons pour exemple les opérations arithmétiques, qu’elles soient amusantes chez Lucas ou récréatives chez Émile Fourrey. Ce dernier est professeur de mathématiques à l’École spéciale des travaux publics et auteur de plusieurs manuels d’algèbre, d’arithmétique, de stéréotomie. Il publie en 1899 des *Récréations arithmétiques* et en 1907 des *Curiosités mathématiques*, où l’histoire des mathématiques, qui connaît à l’époque un grand essor, fait l’objet d’une esquisse de 32 pages et de nombreuses notes.

2.1 Le testament du Nabab de Lucas

Le problème de *l'Arithmétique amusante*, intitulé « Le testament du Nabab », est présenté par Lucas comme un problème d'arithmétique indienne. Les mathématiques indiennes et arabes sont étudiées à la fin du XIX^e siècle, et beaucoup de textes sont alors traduits en langue française. Un Nabab laisse à ses enfants des diamants : le premier enfant prend un diamant et le 1/7 du reste, le second prend 2 diamants et le 1/7 du reste, le troisième prend 3 diamants et le 1/7 du reste, etc. Après le partage des diamants, toutes les parts se trouvent égales. On demande le nombre de diamants et le nombre d'enfants. Lucas considère un carré de 36 diamants qui sont noirs ou blancs pour « distinguer ceux sur lesquels nous porterons plus particulièrement notre attention » (Lucas, 1895, p. 145). La colonne de droite est déplacée en dessous le reste des diamants, alors en retirant le pion noir à droite et le 1/7 de ce qui reste (soit 5 diamants), la part du premier enfant est de 6 diamants (Figure 4, gauche). On recommence : le deuxième enfant prend deux diamants et le 1/7 de ce qui reste (soit 4 diamants) et il aura 6 diamants (Figure 4, milieu). Ainsi de suite, et il y a bien 6 enfants. etc. Lucas remarque que si on remplace 1/7 par $1/n$ alors le nombre d'enfants est $(n-1)$ et le nombre de diamants son carré.

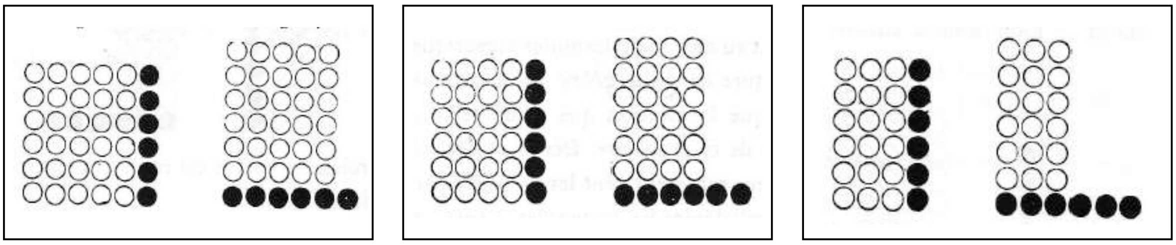


Figure 4. Le testament du Nabab

Lucas remarque que l'on donne habituellemnt la solution de ce problème à l'aide de formules algébriques, comme Euler dans son *Algèbre*. Pourtant la solution visuelle lui semble à l'origine du problème et elle correspond à la représentation des nombres avec des briquettes de l'Indien Aryabhata.

2.2 Les propriétés des opérations arithmétiques chez Émile Fourrey

Dans le chapitre IV des *Récréations arithmétiques* sur les « nombres polygonaux », Fourrey présente les nombres figurés à la manière pythagoricienne, depuis les nombres triangulaires aux hexagonaux. L'arrangement géométrique des nombres lui permet de montrer visuellement des propriétés, ainsi, l'octuple d'un nombre triangulaire plus l'unité est un nombre carré (Figure 5). Il indique que l'on en déduit un moyen de reconnaître si un nombre est triangulaire. Ainsi 21 est triangulaire car $(21 \times 8) + 1 = 169 = 13^2$ (Fourrey, 1899, p. 60).

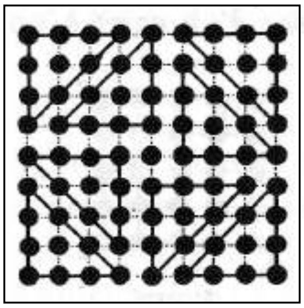


Figure 5. Arrangement de nombres triangulaires

Fourrey continue d'explorer les représentations géométriques dans le chapitre V sur les carrés (Fourrey, 1899, pp. 64-79). Il montre (figure 6, gauche) que « le carré de la différence de deux nombres est égal à la somme des carrés de ces nombres moins le double de leur produit » en juxtaposant les carrés de deux nombres (ici 5 et 3) et en décomposant la figure obtenue en un carré (ici de côté 2) et de deux rectangles (ici de côtés 5 et 3). Il montre encore (figure 6, droite) que « la différence des carrés de deux nombres est égale au produit de la somme de ces deux nombres par leur différence ». Il retire du carré d'un nombre (ici 5) le carré d'un nombre (ici 2) et décompose la figure restante en deux rectangles, puis il « fait pivoter » l'un des deux rectangles à côté de l'autre pour obtenir le rectangle demandé (ici de côtés 7 et 3).

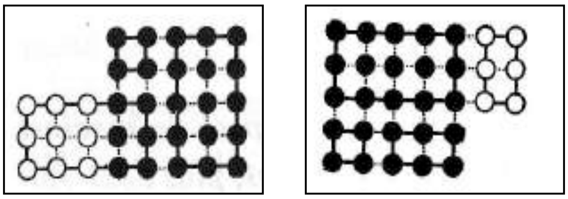


Figure 6. Les carrés dans les récréations de Fourrey.

Ce type de raisonnement évoque le Livre II et les Livres arithmétiques des *Éléments* d'Euclide, où les nombres sont représentés par des segments et leurs carrés par des figures carrées. Les propriétés énoncées par Fourrey correspondent à des propositions du Livre II sur des grandeurs et des figures géométriques. Ces propriétés sont ordinairement démontrées en utilisant les symboles algébriques et les propriétés des opérations de l'addition et de la multiplication. Mais chez Fourrey, l'opération d'addition correspond à une juxtaposition et la soustraction à un découpage, c'est-à-dire à des opérations géométriques. Évelyne Barbin propose à des futurs professeurs d'école, de démontrer par de tels découpages géométriques des identités remarquables et des propriétés sur les nombres. Comme Fourrey, elle énonce une propriété par une phrase et non par une formulation algébrique. La compréhension des phrases peut poser de vraies difficultés à certains étudiants. Mais une fois celles-ci surmontées, l'aspect ludique de la recherche de décompositions et de découpages de figures est manifeste. Les séances viennent en début d'année : elles ont pour but, non pas de travailler sur les nombres, mais d'opérer visuellement sur des figures. Les étudiants, qui n'avaient plus de pratique mathématique depuis quelques années ou qui n'avaient pas un bon souvenir des mathématiques, ont construit une nouvelle approche des nombres et de leurs opérations.

3 Les curiosités géométriques de Fourrey (1907)

Fourrey explique en avant-propos que son ouvrage est conçu dans le même esprit que les *Récréations arithmétiques* : "instruire en présentant la science par les côtés curieux" et pour cela, il lui apparaît comme indispensable de débuter par un historique de la géométrie. Alors que l'introduction historique de Lucas vise à montrer que les récréations ont intéressé des mathématiciens comme Euler ou Fermat, le souci de Fourrey est proprement historique : « nous avons voulu que le lecteur pût facilement associer ces documents à leur milieu, à la période qui les a vus naître » (Fourrey, 1907, np). Son « Esquisse de l'histoire de la géométrie » couvre une trentaine de pages, depuis l'Orient antique jusqu'au XIX[e] siècle, et elle s'appuie sur les travaux les plus récents, ceux de Moritz Cantor, Hieronymus Georg Zeuthen ou encore August Eisenlohr (Barbin, 1994). L'ouvrage s'appuie sur l'histoire de la géométrie, sans se limiter à la géométrie théorique, avec des chapitres sur les casse-tête géométriques (dont le loculus d'Archimède) et les paralogismes. La seconde partie concerne « la géométrie de la mesure », avec un chapitre sur les instruments de dessins et de topographie et un autre sur la division des figures planes, qui s'appuie sur les écrits de Aboul Wafa. L'approche historique est en phase avec le projet de Paul Tannery émis en 1894 d'introduire l'histoire dans l'enseignement des mathématiques (Barbin, 2007). Il est possible que Fourrey ait voulu répondre au besoin formulé par Tannery d'ouvrages historiques destinés aux enseignants.

1.3 Une approche historique : le théorème de Pythagore

Après un premier chapitre sur les définitions, le chapitre II est consacré au théorème de Pythagore, avec un historique sur Euclide citant Tannery et Cantor. Sur une trentaine de pages, Fourrey donne 24 démonstrations du théorème de Pythagore, historiquement référencées, depuis Euclide jusqu'en 1889. Elles ne suivent pas l'ordre chronologique, mais sont classées par « types », selon qu'elles sont basées sur l'équivalence des figures, sur des « transpositions d'éléments » ou sur l'algèbre. Le chapitre finit par des « variantes », comme le théorème de Pappus. Tout au long du chapitre, Fourrey, à l'instar de ses contemporains, modernise les démonstrations en les traduisant en symbolisme algébrique. Les différences entre celles-ci sont donc amoindries. Toutefois, l'uniformisation de l'écriture a le mérite de laisser plus de place à la considération des figures sur lesquelles reposent les démonstrations. Ce qui apparaît alors comme « curieux » est leur profusion et leur diversité. Fourrey fournit d'ailleurs six cas de figures pour la démonstration d'Euclide. Dans le cours évoqué plus haut, Evelyne Barbin a travaillé avec ses étudiants, futurs enseignants, sur des exemples grecs, chinois et indiens de démonstrations en essayant de garder l'esprit de leur époque. Ainsi, les étudiants ont découpé des figures sur des cartons colorés. À l'issue des cours, il était demandé aux étudiants de dire leur préférence. Cette question est apparue incongrue, et cette incongruité a été travaillée et approfondie avec une autre question : « Pourquoi les mathématiciens donnent-ils de nouvelles démonstrations quand ils en ont déjà une ? ».

Pour illustrer la démarche de Fourrey, nous prenons une démonstration indienne, donnée par l'orientaliste Aristide Marre en 1887, car elle sera reprise dans un manuel de géométrie, le célèbre *Traité de géométrie* de Rouché et Comberousse (édition de 1891). Étant donné le triangle rectangle *ABC* de côtés $BC = a$, $AC = b$, $AB = c$ (Figure 7), on construit les carrés *CDEB* et *AGFB* sur *CB* et *AB* et on montre que *E* est sur *GF*. Puis on mène par *D* la parallèle *DP* à *GF* et la parallèle à *AG* qui coupe le prolongement de *GF* en *H*. Fourrey montre que que *DHGP* est un carré de côté *b*, puis il explique que si on "enlève" du carré a^2 les triangles *PCD* et *ABC* pour les disposer autour du pentagone ombré *APDEB*, on forme la somme des carrés b^2 et c^2.

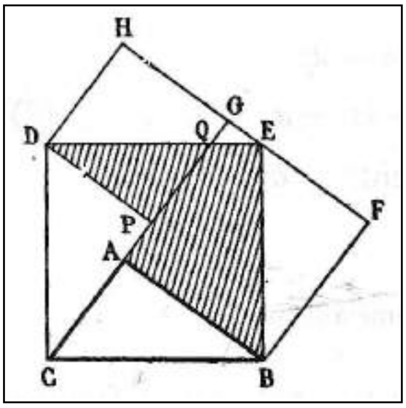


Figure 7. Une démonstration du théorème de Pythagore chez Fourrey

1.4 Résolutions de problèmes numériques par la géométrie

Le chapitre des *Curiosités géométriques* intitulé « Applications de la géométrie au calcul » commence par l'exécution des opérations géométriques par la géométrie, en particulier la racine carrée d'un nombre. Puis suivent des problèmes du mathématicien arabe Al-Khwarizmi, qui correspondent à des équations du second degré, et des problèmes du *Lilavati* et du *Vija Ganita* du mathématicien indien Baskhara. Une partie importante du chapitre concerne des sommations de progressions géométriques, dont plusieurs solutions sont parues dans les Congrès de « l'Association Française pour l'avancement des sciences », qui rassemblent mathématiciens et amateurs.

Fourrey résout trois problèmes par ce qu'il appelle des « procédés japonais ». Nous les avons trouvés dans les *Mémoires de l'Académie des sciences, inscriptions et belles lettres de Toulouse* sous la plume de Berson (Berson, 1891). Dans le troisième problème, « un officier de police passant sur un pont entend une bande de voleurs qui, sous le pont, procède au partage de quelques pièces de soies volées. 'Si nous donnons à chacun 7 pièces, il en restera 6, et si nous voulons en prendre chacun 8, il en manquerait 9'. L'officier peut-il deviner le nombre de voleurs et le nombre de pièces volées ? » (Fourrey, 1907, pp. 339-340). On considère un rectangle *ABCD* dont le côté *AD* représente le nombre de voleurs et *AB* vaut 7 (Figure 8). En lui accolant le rectangle *BHEF* de côtés *BH* égal à 1 et *EF* égal à 6, on obtient une aire égale au nombre de pièces de soie. Puis on considère le rectangle *A'H'G' D*' de côté *A'D'* égal à *AD* et *A'H'* égal à 8. En lui retranchant le rectangle *F'E'G'C'* de côtés *F'E'* égal à 1 et *E'G'* égal à 9, on obtient une aire *A'H'E'F'C'D'* encore égale au nombre de pièces de soie. On en déduit que *FC* égal à *F'C'* vaut 9 et que le côté *AD* vaut 15. C'est le nombre de voleurs et le nombre de pièces volées est (15 × 7) + 6 = 111.

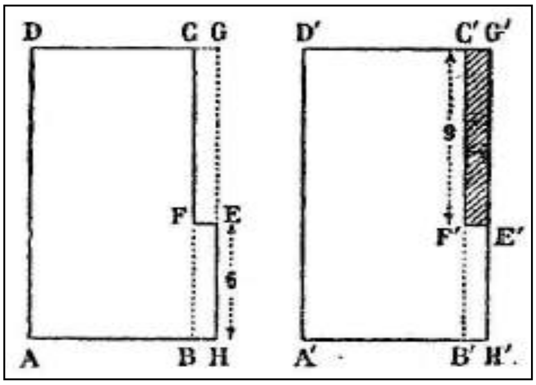


Figure 8. Le problème des voleurs et des pièces de soie.

Fourrey et Laisant appartiennent à un même mouvement en faveur d'un renouveau de l'enseignement de la géométrie. Fourrey termine son historique avec le manuel de Charles Méray paru en 1874, *Nouveaux Éléments de géométrie*, qui a eu un impact important plus tard pour les programmes de géométrie de la réforme de 1902. Le manuel est réédité en 1903 grâce à Laisant, celui-ci écrit combien l'auteur sort des sentiers battus et par cela même, « pique la curiosité du lecteur » (Laisant, 1901, p. 100-101).

4 L'initiation mathématique de Charles-Ange Laisant (1906)

La table des matières de l'*Initiation mathématique* pourrait laisser penser qu'il s'agit d'un ouvrage de récréations, avec des opérations curieuses, une « macédoine mathématique » et des « carrés magiques ». Pourtant Laisant précise que le livre « n'a rien de commun » avec les récréations mathématiques, où il est demandé d'appliquer à des sujets amusants les théories mathématiques déjà connues. Il écrit : « Ici c'est l'inverse, nous nous servirons de questions amusantes comme moyen pédagogique pour attirer la curiosité de l'enfant et arriver ainsi à faire pénétrer dans son esprit, sans efforts imposés, les premières notions mathématiques les plus essentielles. Et la diversité des questions, qui pourrait faire croire à un désordre apparent, cache une suite d'idées, voulues, utiles et complètement ordonnées » (Laisant, 1906, p. VI). L'ouvrage n'est pas « un tout didactique », mais un guide entre les mains de l'éducateur, dont il s'inspirera. Nous y trouvons, comme chez Lucas et Fourrey, une approche visuelle des mathématiques, comme les nombres figurés et le chapitre 46 sur les graphiques.

1.5 Les graphiques : une algèbre sans calcul

Au début du chapitre 46 « Les graphiques : une algèbre sans calcul », Laisant commente l'usage important des graphiques dans les journaux et aussi pour représenter la marche des trains et des phénomènes physiques. Il introduit alors le terme de « fonction », par exemple le chemin fait par un train est fonction du temps, car cette idée semblera « très naturelle » aux enfants en lui présentant le plus possible de graphiques (Laisant, 1906, p. 117). Pour lui, la fonction est associée à un graphique, alors que dans l'enseignement classique, on parle plutôt du graphe associé à une fonction. Les chapitres suivants contiennent des questions posées de manière piquante et résolues par des graphiques, comme la rencontre de deux marcheurs et la poursuite à vélo de deux cyclistes.

Le chapitre 49 raconte une anecdote « absolument authentique » qui est arrivée à Édouard Lucas au cours d'un congrès scientifique, à la fin d'un déjeuner où se trouvaient plusieurs mathématiciens connus, quelques uns illustres. Lucas leur posa la question suivante : « je suppose que chaque jour à midi, un paquebot parte du Havre pour New-York, et qu'en même temps un paquebot de la même compagnie parte de New-York pour le Havre. La traversée se fait exactement en sept jours, soit dans un sens, soit dans l'autre. Combien le paquebot qui part du havre aujourd'hui à midi rencontrera-t-il en route de navires de sa compagnie faisant la route opposée » (Laisant, 1906, p. 124). Quelques « illustres auditeurs » répondirent sept et d'autres se turent. Mais nous voyons sur le graphique que le paquebot rencontrera 13 navires, plus deux autres à l'arrivée et au départ, donc en tout 15 (figure 9).

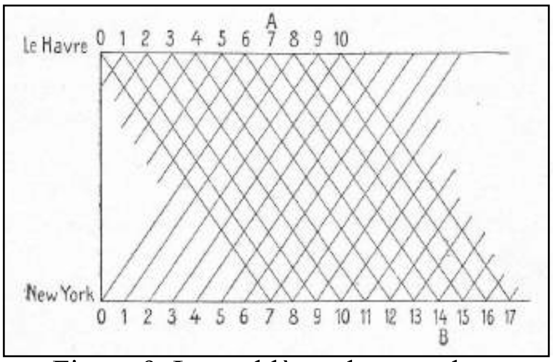


Figure 9. Le problème du paquebot

1.6 Le propos éducatif de Laisant

L'initiation mathématique propose une mise en œuvre des récréations dans un projet d'enseignement, où il ne s'agit pas de faire appliquer des contenus de programmes aux élèves mais de les initier aux mathématiques. Pour comprendre la portée éducative de cette mise en œuvre, il faut la rapprocher des propos de Laisant dans *La mathématique. Philosophie – enseignement*, ouvrage paru en 1898. Laisant écrit que « le problème de l'enseignement mathématique se pose dans des conditions toutes nouvelles dans notre civilisation actuelle et du développement industriel extraordinaire sans précédent, qui s'est accompli au cours du XIX[e] siècle et ne s'arrêtera pas au XX[e] siècle » (Laisant, 1898, p. 7). C'est en ce sens, qu'il est l'un des promoteurs de la réforme des mathématiques de 1902. Homme politique et mathématicien engagé à l'échelon international (Auvinet, 2013), il crée la revue *L'Enseignement mathématique* en 1899 avec Henri Fehr.

Dans l'ouvrage de 1898, Laisant présente sa vue générale sur l'enseignement, puis ses conceptions pour les différentes branches. Concernant les premières notions de calcul, il faut aux élèves de « véritables jeux » : dominos, lotos et billes pour apprendre à compter. Le jeune élève trouve un attrait aux exercices, quand ils sont présentés sous forme de récréations, "sa curiosité s'éveille, il désire aller chaque jour un peu plus loin que la veille" (Laisant, 1898, p. 201). L'apprentissage de l'arithmétique peut aller assez loin avec les récréations, mais à une condition cependant : « c'est de suivre une méthode rigoureusement expérimentale et de ne pas s'en départir ; de laisser l'enfant en présence des réalités concrètes qu'il touche et qu'il voit, faire lui-même ses abstractions » (Laisant, 1898, p. 203). Concernant la géométrie, il faut commencer avec de nombreux tracés de figures. « c'est peut-être en géométrie, plus que dans aucune autre des parties élémentaires de la mathématique, que la curiosité , l'esprit de la recherche peuvent être le plus facilement mis en éveil » (Laisant, 1898, p. 222).

Nous avons une idée de la réception de l'*Initiation mathématique*, grâce à l'avant-propos de Laisant dans la seconde édition de 1915. L'approbation s'est manifestée dans l'enseignement primaire, surtout dans les écoles normales d'instituteurs. Dans l'enseignement secondaire, des enseignants se sont offusqués de ses critiques vis-à-vis de l'administration. Il écrit son admiration pour ces enseignants en ces termes : « leur mérite est d'autant plus grand qu'ils ont à lutter contre une bureaucratie dont ils sont les premières victimes, contre un système séculaire de centralisation et de routine qui semble avoir pris à tâche de tuer les initiatives et d'empêcher la lumière de pénétrer dans les cerveaux » (Laisant, 1915, p. 4).

5 Conclusion : jeux ou récréations ?

Au tournant du siècle, la portée des récréations mathématiques est l'objet d'un changement important, qui se décline en quatre volets solidaires. 1) Les récréations ne sont plus seulement des divertissements, elles visent à faire entrer le lecteur dans une recherche mathématique, à diffuser des mathématiques non scolaires, à le cultiver en faisant connaître l'histoire et ainsi à enseigner. 2) Les récréations sont des occasions de faire des mathématiques directement, c'est-à-dire d'exercer à vif la combinaison des vues et de leurs modifications pour inventer des chemins vers une solution, des chemins exacts, c'est-à-dire ce que l'on appelle des preuves. 3) Dans des situations dont l'énonciation est simple, le lecteur est en prise avec l'objet même de la mathématique, la source de son plaisir : l'énigme d'un problème et la possibilité d'une preuve. 4) Il expérimente l'évidence comme fait mathématique, comme surprise. La curiosité est piquée par l'énoncé, mais bien plus, la solution conduit au ravissement et l'envie de réessayer.

Avec les récréations de nos trois auteurs, le lecteur est mis hors-pistes, hors l'ennui des programmes d'étude clos ou des règles et des logiques préconisées, il est directement au travail dans une représentation sensible par laquelle la rigueur des actes s'impose d'elle-même, par la pratique si l'on veut atteindre la solution. L'introduction ponctuelle de ces hors-pistes dans l'enseignement peut troubler les chemins balisés, mais pourtant si difficiles des programmes ou des conventions scolaires, sans pour autant produire les bénéfices escomptés. Nous en prenons pour preuve les raisonnements qui s'appuient sur des exemples, et qui sont repoussés parfois, sous le prétexte qu'ils seraient de simples cas particuliers. Cependant, l'exigence d'une généralité prématurée risque de conduire à des formulations algébriques ou à des discours logiques complexes, et donc de priver les élèves de l'usage de ce que nous appelons « exemples génériques » sur lesquels ils pourront exercer leur capacité d'induction. L'introduction des abstractions et la formulation des règles viendront plus tard, alors que les élèves en auront éprouvé la nécessité. Un autre hors-piste est l'exercice du mélange entre des disciplines que les programmes distinguent. Avec les récréations, l'algèbre, la géométrie, les lettres, les figures, les discours ne sont que des moyens, ils ne sont pas obligatoires, et même c'est souvent dans le passage entre ces registres que se fait l'invention et que se produit l'évidence. Nous l'avons vu avec l'arithmétique visuelle et les preuves japonaises de Fourrey, ou bien avec les graphiques de Laisant. Tous nos auteurs insistent sur la pratique de la visualisation. Les récréations sont des représentations sensibles de théories, que l'on assimile sans coup férir, et elles sont porteuses aussi de nouvelles théories à venir, comme avec les jeux de dominos ou le taquin chez Lucas.

Les jeux sont pratiqués aujourd'hui, mais souvent en dehors des classes, dans des clubs ou dans des concours. Ils rencontrent un succès manifeste et nous pouvons nous demander pourquoi ils ne rentrent pas plus dans les classes et les programmes. Laisant se prononçait, comme d'autres de ses contemporains, contre les programmes, mais il faut situer sans doute ailleurs sa proposition de *l'Initiation mathématique*. En effet, les hors-pistes ont pour conséquence la marginalité de jeux pris isolément, il fallait donc imaginer un enseignement où les récréations soient au centre et mises en ordre, où les pratiques offertes par les récréations conduisent à une recherche et à un savoir-faire authentiquement mathématiques.

REFERENCES

Auvinet. J. (2013). *Charles-Ange Laisant. Itinéraires et engagements d'un mathématicien de la Troisième République*. Paris : Hermann Éditions.

Barbin, É. (1994). Actualité de Fourrey. In É. Fourrey, *Curiosités géométriques* (rééd.) (pp. III-XVII). Paris : Vuibert.

Barbin, É. (2007). Les récréations : Des mathématiques à la marge. *Les génies de la science*, *30*, 14-17.

Barbin, É. (à paraître). Gaston Tarry et la doctrine des combinaisons. In É. Barbin, C. Goldstein, *et al.* (Éds.), *Les travaux combinatoires entre 1870 et 1914*. Limoges : PULIM.

Berson, M. (1891). Mémoires sur l'emploi des figures géométriques par les Japonais pour la résolution des problèmes d'arithmétique. *Mémoires de l'Académie des sciences, inscriptions et belles-lettres de Toulouse*, *9*(III), 268-271.

Decaillot-Laulagnet, A.-M., (1999). *Édouard Lucas (1842-1891) : Le parcours original d'un scientifique français dans la deuxième moitié du XIX^e^ siècle* (Thèse). Paris : Université René Descartes.

Fourrey, É. (1899). *Récréations arithmétiques*. Paris : Vuibert.

Fourrey, É. (1907). *Curiosités géométriques*. Paris : Vuibert.

Guitart, R. (à paraître). Les taquins d'Édouard Lucas à Richard Wilson. In É. Barbin, C. Goldstein, *et al.* (Éds.), *Les travaux combinatoires entre 1870 et 1914*. Limoges : PULIM.

Hordern, E. (1986). *Sliding Piece Puzzles*. Oxford: Oxford University Press.

Laisant, C.-A. (1898). *La mathématique. Philosophie – Enseignement*. Paris : Carré et Naud.

Laisant, C.-A. (1901). Une exhumation géométrique. *L'enseignement mathématique*, *3*, 98-105.

Laisant, C.-A. (1906*). Initiation mathématique*. Paris : Hachette.

Laisant, C.-A. (1915*). Initiation mathématique* (2d éd.). Paris : Hachette.

Lucas, É. (1882). *Récréations mathématiques* (Vol. I). Paris : Gauthier-Villars.

Lucas, É. (1883). *Récréations mathématiques* (Vol. II). Paris : Gauthier-Villars.

Lucas, É. (1889). *La fasioulette du Professeur N. Claus* (de Siam). Paris : Chambon & Baye, Édouard Lucas.

Lucas, É. (1891). *Théorie des nombres*. Paris : Gauthier Villars.

Lucas, É. (1893). *Récréations mathématiques* (Vol. III). Paris : Gauthier-Villars.

Lucas, É. (1894). *Récréations mathématiques* (Vol. IV). Paris : Gauthier-Villars.

Lucas, É. (1895). *L'arithmétique amusante*. Paris : Gauthier-Villars.

MONSTERS IN THE MATHEMATICS CLASSROOM:

Learning analysis through the works of Gaston Darboux

Janet Heine BARNETT
Colorado State University – Pueblo, 2200 Bonforte Blvd, Pueblo, CO USA
janet.barnett@csupueblo.edu

ABSTRACT

The drama of the rise of rigor in nineteenth century mathematical analysis has now been widely rehearsed. Notable within this saga is the appearance of functions with features so unexpected (e.g., everywhere continuous but nowhere differentiable) that contemporary critics described them as "bizarre", "pathological," or even "monsters." Among the "monster-makers," one of the most influential was Gaston Darboux (1842-1917). This paper reviews Darboux's mathematical and "backstage" contributions to the development of nineteenth century analysis, including some of his own favorite pet monsters, and explores the important role played by these "pathological functions" in the historical re-shaping of analysis. We then consider how these functions can be used in to help students develop a more robust understanding of modern analysis. We examine in particular how Darboux's proof of the result now known as "Darboux's Theorem" (i.e., all derivatives have the intermediate value property) in his 1875 *Mémoire sur les fonctions discontinues* can be used in today's analysis classroom.

1 Introduction

In 1904, Henri Poincaré wrote:

> Logic sometimes creates monsters. For half a century we have seen a host of bizarre functions which appear to be created so as to resemble as little as possible those honest functions which serve some purpose. More of continuity, or less of continuity, more derivatives, and so forth. ….. ….. In former times when one invented a new function it was for a practical purpose; today one invents them expressly to show the defects in the reasoning of our fathers, *and nothing more can be drawn from them*. (Poincaré, 1902, p. 263, emphasis added)

This paper explores the question of whether there *is* something more that today's student of analysis can learn from the strange functions that Poincaré so roundly condemned. We begin in the next section by examining the motivations of one of the foremost "monster makers" of the nineteenth century, Gaston Darboux (1843-1917). In Section 3, we then briefly describe an instructional approach for bringing Darboux's thinking to the classroom through the use of excerpts from his original works in the form of a guided reading module for students. In our closing section, we return to the historical drama and examine Darboux's overall influence on the re-shaping of analysis that began in the late nineteenth century.

2 The Historical Players

In this section, we provide some background information on Darboux, introduce some of the other major players with whom he interacted, and examine his personal motivations for bringing "pathological" functions into the study of analysis. The larger historical drama in which Darboux's story is set – namely, the story of the rise of rigor in analysis in the nineteenth century - has been well rehearsed; see for example (Chorlay, 2016), (Lützen, 2003) and (Hochkirken, 2003). For details concerning Darboux's part in this drama, we also draw extensively on (Gispert, 1983), (Gispert, 1987) and (Gispert, 1990).

2.1 Gaston Darboux: Student, Teacher and Editor *par excellence*

Born on 14 August 1842, Darboux attended Lycée first in Nimes, and later in Montpellier. In 1861, he was admitted to both the École Polytechnique and the École Normale Supérieur; he chose to attend the École Normale. While a student there, he published his first paper on orthogonal surfaces; his 1866 doctoral thesis (under Michel Chasles) was on this same topic. From 1866-1867, Darboux taught at the Collège de France before spending five years at the Lycée Louis le Grand (1867-1872) and another four at the École Normale Supèrieur (1872-1881). He then moved to the Sorbonne where he taught for the remainder of his life.[1] While at the Sorbonne, Darboux demonstrated his excellence as both a teacher and an organizer. For the last 17 years of his life (1900-1917), his talents as an organizer were also put to use in his capacity as the Secrétaire Perpétuel de l'Académie des Sciences.

Darboux also excelled as an organizer and promoter of mathematical research in France. Of particular relevance to the story being told in this paper was his role as a founding editor of the *Bulletin des Sciences*, sometimes referred to as "Darboux's *Bulletin*" in recognition of his role as its co-founder in 1870.[2] The *Bulletin* published lists of titles of research papers from journals from outside of France, as well as summaries of the contents of the more important works and, when possible, complete translations of those papers. In this way, the *Bulletin* sought to provide the French mathematical community with access to cutting-edge mathematical research being conducted elsewhere that, for a variety of issues related to financial and infrastructure, was difficult to obtain inside France at the time. Darboux was especially concerned that, without proper exposure to the new research methodologies and standards then evolving outside of France, the training of future generations of French mathematicians as researchers would be compromised. Echoes of these concerns are heard in an early letter to the co-founder of the *Bulletin,* in which Darboux asserted:

> … we need to mend our [system of] higher education. I think you agree with me that the Germans get the better of us there, as elsewhere. If this continues, I believe the Italians will surpass us before too long. So let us try, with our *Bulletin*, to wake the holy fire and the French understanding that there are many things in the world that they do not suspect, and that even if we are still the Grrrand nation, no one abroad perceives this. [As quoted in (Gispert 1987, p. 160)]

Before looking further at the correspondence between these two men, we briefly introduce the recipient of Darboux's letter: Jules Houël (1823-1886).

---

[1] Darboux's first academic position at the Sorbonne was as the assistant to Liouville who was then Chair of Rational Mechanics (1873-1878). From 1878-1880, he served as the assistant to Chasles in Chair of Higher Geometry, and moved into that position himself upon Chasles' death. From 1889-1903, Darboux also served as the Dean of the Faculty of Science at the Sorbonne.

[2] Recall that this was the same year that saw the start of the Franco-Prussian War, in which the French were defeated by a coalition of German states under the leadership of Prussia. Darboux resided in Paris throughout the period of this brief war (July 19, 1870–May 10, 1871).

2.2 Jules Houël: Translater *par excellence*

Senior to Darboux by twenty years, Houël completed Lycée at Caen, and then studied at the Collêge Rollin. Like Darboux, he received his initial mathematical training at the Ècole Normale Supérieur (entering in 1843) and his doctorate (in celestial mechanics) from the Sorbonne (in 1855). He then returned to his home town of Thaon for four years, pursuing mathematical research on his own despite an offer from Urbain Le Verrier for a post at the Paris Observatory. In 1859, Houël accepted the Chair of pure mathematics in Bordeaux, and remained in that position for the remainder of his life.

Prior to joining Darboux as co-founder of the *Bulletin*, Houël had already gained a reputation for excellence as a translator. An early proponent of non-Euclidean geometry – he expressed doubts about the parallel postulate as early as 1863, even before learning about the work of Lobachevski and Bolyai – Houël produced French translations of key papers by both these men, as well as other important works in non-Euclidean geometry by Beltrami, Helmholtz, and Riemann. After it was founded in 1870, Houël contributed numerous French translations to the *Bulletin*. Notable among these was his translation of Riemann's 1853 *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe*. First published in German in 1868, it was not until Houël's translation appeared in the *Bulletin* in 1873 that the contents of this important work, including Riemann's treatment of the integral, became generally known in France. That same year marked the beginning of an exchange between the *Bulletin*'s founding co-editors in which several "monster functions" made their debut as Darboux sought to convince Houël of the need for increased rigor in the latter's approach to analysis.

2.3 Monsters in the Darboux-Houël Correspondence

The impetus for the ten-year debate[3] concerning rigor in analysis in the Darboux-Houël correspondence was Houël's request for feedback on preliminary drafts of his intended textbook on differential calculus, eventually published as *Cours de Calcul infinitésimal* in 1878. Throughout this debate, Darboux offered various counterexamples in a (vain) attempt to convince Houël of the need for greater care in certain of his (Houël's) proofs. One such example [4]was the function given in modern notation by:

$$y = f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

As noted by Darboux, it is easy to see that $\lim_{x\to 0} \frac{y}{x} = 0$, so that $f'(0) = 0$. Thus, f is a differentiable function with derivative function:

$$y' = f'(x) = \begin{cases} \cos\left(\frac{1}{x}\right) + 2x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases},$$

[3] See (Gispert 1983) for an in-depth analysis of this debate.

[4] See (Gispert, 1983, pp. 55-56) for Darboux's actual discussion of this example in his letters to Houël.

Since the derivative f' becomes indeterminate near $x = 0$, the function f thus provides an example of a differentiable function for which the derivative itself is not continuous. Faced with Darboux's inevitable conclusion that derivatives are not necessarily continuous, Houël essentially responded by saying: *They are for all the functions that I consider!*

Throughout the course of the debate, one can hear Houël's increasing exacerbation with Darboux's examples in his description of them as "drôlatiques" (humorous), "bizarres" (bizarre), "dereglés" (disorderly), "saugrenues" (absurd), and "gênantes" (obstructive). Darboux too became increasingly vexed by Houël's apparent inability to understand the underlying purpose of these examples. The function $y = x^2 \sin(1/x)$, for instance, was put forward by Darboux in an attempt to explain to Houël that the proofs he proposed to include in his calculus textbook often relied on the assumption of *uniform* convergence, whereas only simple convergence had been assumed. Elsewhere in the correspondence, Darboux very explicitly explained this concern as follows:

In the expression

$$\frac{\mathrm{f(x+h)-f(x)}}{\mathrm{h}} - \mathrm{f'(x)} < \epsilon$$

ϵ is a function of two variables x and h, where h approaches zero with x remaining fixed ….. but if x and h vary as in your proof, then each new subdivision of the interval $\mathrm{x_1 - x_1}$ introduces a new quantity ϵ. [As quoted in (Gispert, 1983, p. 54)]

Darboux's real message to Houël – a message which he tried repeatedly to convey by means of counterexamples and other explanations – might thus be paraphrased as follows:

> Your *proofs* often (implicitly) assume some assumption [e.g., uniform differentiability]. That's fine … but then you need to (explicitly) verify this condition *before* applying your theorems, and you never do! Even if you did bother to do this, you would be better off changing your proofs altogether – why not use the Mean Value Theorem as your foundation? That would be less complicated than verifying, for example, uniform differentiability each time you invoke a theorem for which you gave a proof that depends on it.

Houël's responses over the years as illustrated by the following quotations from their correspondence[5] seem to reveal frustration not only with Darboux's insistence, but with his own inability to even fully understand what Darboux is trying to accomplish with his monsters.

- I completely reject the simultaneous variation. I consider only successive variations.
- Get rid of your preoccupation with this simultaneous variation that does not belong here.
- …. and then you raise fatal worries about the points I thought the best established …

[5] These four quotes are taken from (Gispert, 1983 p. 56), and appeared in letters written by Houël on the following dates in 1875 respectively: January 9, January 24, January 31, February 15.

- I understand what Cauchy said, or when I do not understand, I can see what I am lacking. But your doctrine simply amazes me, and your abyss[6] all the same makes me insane.

2.4 Monsters in Darboux's Published Works in Analysis

Many of the monsters presented in Darboux's private letters to Houël remained hidden away from public sight until the publication study of that correspondence by Helene Gispert between 1983 and 1990. But other of his monster creations appeared in Darboux's three published works in analysis. In this section, we consider the contents of only the most influential[7] of the three, his 1875 publication *Mémoire sur les fonctions discontinues*.

Darboux's strong grasp of recent developments in analysis in the hands of his German contemporaries is well documented throughout his *Mémoire*. His citations included works by Hankel, Gilbert, Klein and Thomae, as well as Riemann's 1853 *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe*. Darboux greatly admired Riemanns' concept of the integral, as the latter described it in a brief (5-6 page) discussion in his 1853 paper. In fact, a primary goal of Darboux's *Mémoire* was to provide a rigorous reformulation of the Riemann integral.[8] Key components of this reformulation included:

- the first clear distinction between the concepts *supremum/infinum* and *maximum/minimum*
- the first definitions of upper and lower integrals;
- the rigorous establishment of necessary and sufficient conditions for integrability;
- the separation of discontinuous functions into two classes: integrable and non-integrable;
- rigorous proofs of the properties of integrable functions, including the following:
  - Every continuous function is integrable.
  - $F(x) = \int_a^x f(y)dy$ is continuous in x.
  - If f is continuous at x_0,

    then $F(x) = \int_a^x f(y)dy$ is differentiable at x_0 with $F'(x_0) = f(x_0)$.

With a rigorous definition of the Riemann integral in hand, Darboux next offered a collection of new functions of his own creation which included in particular a specimen of each of the following "monsters":

- A continuous, no-where differentiable function[9]
  (via a sum of uniformly convergent series of continuous functions):

[6] Houël is responding here to Darboux's earlier attempts to explain the abyss that exists between simple convergence and uniform convergence.
[7] Darboux's other two analysis publications were (Darboux, 1872) and (Darboux, 1879).
[8] For a discussion of both Riemann's and Darboux's treatment of the integral, see (Hochkirchen, 2003).
[9] Darboux's example of such a monster was, of course, not the only one put forward around this time.

$$\sum \frac{\sin[1.2.3 \ldots (n+1)x]}{1.2.3 \ldots n}$$

- A continuous function that is neither increasing nor decreasing on any interval

$$\sum a_n \sin(nx\pi)^{2/3}, \text{ where } \sum a_n \text{ is absolutely convergent}$$

- A discontinuous function that satisfies the Intermediate Value Property

$$F(x) = \int_0^x f(y)dy = \sum \frac{a_n}{n} \phi(\sin nx\pi),$$

$$\text{where } \phi(y) = y^2 \sin\left(\frac{1}{y}\right) \text{ and } \sum a_n \text{ is absolutely convergent.}$$

Darboux's proof that this last example possesses Intermediate Value Property followed from a theorem that now bears his name – i.e., every derivative has the Intermediate Value Property. We include his proof of this theorem in the next section, in which we consider what role Darboux's monsters might play in today's analysis classroom.

3 Monsters in the Classroom?

In this section, we turn to the questions of why and how to bring Darboux's monster functions to the classroom. We accept as a given that these functions *do* belong in an analysis course. In fact, most modern undergraduate analysis textbooks already include example such as Darboux's $f(x) = x^a \sin(1/x)$. But missing from these modern treatments is a consideration of the historical context in which these examples were first considered. Why *were* these examples developed in the first place? What mathematical intuitions were refined and in what ways by studying them? Were they even accepted as legitimate examples of functions and, if not, why not? Because most students enter an analysis course with a general understanding of the calculus (and the concept of continuity in particular) that differs little from the views of mathematicians like Houël, exposing them to the historical context by sharing Darboux's motivations for considering such functions can be valuable in a number of ways.

One important role that the history of these functions can play in today's classroom is to help students to develop the more rigorous and critical view of the basic ideas of calculus that an introductory analysis course seeks to achieve. A companion goal is to help them to develop an understanding of the language, techniques and theorems of elementary analysis that developed when mathematicians adopted such a critical perspective in the nineteenth century. To achieve these two goals, it is especially important to focus students' attention on the underlying logical relationship of fundamental notions of analysis such a continuity and convergence, where both these concepts play pivotal roles in Darboux's monster functions.

How best to motivate this change of perspective on students' part and help them to develop an understanding of the mathematics that grew out of it? The answer proposed in this paper is to have students read the actual works of the mathematicians involved. Possible reactions to this proposal include a concern that the reading original sources is too difficult a task for students, and also one that is too far removed from the mathematical goals of the course; an analysis course is, after all, not a course in the history of mathematics. In reply to these legitimate pedagogical concerns, we propose placing the works in question within a "Primary

Source Project" (PSP), following a guided reading approach developed with support from the US National Science Foundation.[10]

The aim of this particular approach is to provide students with sufficient guidance to allow them to successfully read an original source, while still allowing them the excitement of directly engaging with the thinking of its author. This is accomplished by interweaving three essential elements within the PSP:

- excerpts from the relevant original source(s);
- secondary commentary that discuss the historical context and mathematical significance of these excerpts; and
- a series of tasks that prompt students to develop their own understanding of the underlying concepts and theory.

For example, by reading selected excerpts from the writings of the nineteenth century mathematicians who led the initiative to raise the level of rigor in the field of analysis – as well as those who resisted or misunderstood this initiative – students' own understanding of and ability to work at the expected level of rigor can be refined. Grappling with the examples presented in Darboux's work through completion of the accompanying tasks also supports students' ability to develop mathematical ideas on their own, as well as their ability to formally communicate those ideas through reading and writing.

Throughout a PSP, the emphasis remains on the *mathematical* concepts and techniques involved, with historical and biographical information on the source's author playing only a supporting role. In this particular case, Darboux's monster functions and Houël's reactions to them thus take center stage. To illustrate how this can be accomplished, we close this section with a brief description of a portion of this particular PSP which is based on the proof of the theorem that now bears Darboux's name – i.e., every derivative has the Intermediate Value Property – from his *Mémoire sur les fonctions discontinues* (1875). (The complete proof in the original French is reproduced below.)

This particular excerpt from Darboux serves as the culmination of the PSP, following students' introduction to several of his monster functions. Students are first prompted to read the proof (in English translation) with those examples in mind. In a series of PSP tasks, students are then asked to provide either proofs or counterexamples from Darboux's menagerie of monster functions as a means to further explore the interplay of continuity, the Intermediate Value Property and anti-differentiability. For example, is the converse of this theorem true; that is, can a function satisfy the Intermediate Value Property without being a derivative? Can

[10] Examples of this approach can be found in a compendium of projects developed and tested scientists for the teaching of topics in discrete mathematics since 2008 by an interdisciplinary team (including the present author) of mathematicians and computer. Elsewhere, members of our team have written about the general pedagogical design goals of our work and analyzed specific projects in the collection relative to those goals. See, for example, (Barnett, 2014), (Barnett, Pengelly & Lodder, 2013) and (Barnett et al, 2011). A new collaboration of faculty from seven US institutions of higher education was recently awarded a five-year collaborative grant entitled *TRansforming Instruction in Undergraduate Mathematic via Primary Historical Sources (TRIUMPHS)* from the US National Science Foundation to continue and expand this work; more information about TRIUMPHS is available at http://webpages.ursinus.edu/nscoville/TRIUMPHS.html.

a continuous function fail to be anti-differentiable? Can a derivative function fail to be continuous? As a final task, students are asked to critique Darboux's proof from the perspective of modern standards of rigor – how would Darboux's proof fare if he were a student in the course they are completing? – and to provide their own version of the same proof, filling in any missing details and/or adapting the language employed to their own personal proof style.

> Soit, en effet, $F(x)$ une fonction dont la dérivée existe pour toute valeur de x, mais soit discontinue. Supposons que, pour $x = x_0$, $x = x_1$, la dérivée prenne les valuers
>
> $$F'(x_0) = A \ , \qquad F'(x_1) = B$$
>
> Je dis que, si x varie de x_0, á x_1, $f'(x)$ passe au moins une fois par toutes les valeurs intermèdaires entre A et B. Soit, en effet, M une des ces valuers,
>
> $$A > M > B,$$
>
> et formons la function
>
> $$F(x) \ - \ Mx.$$
>
> Cette fonction continue aura, pour $x = x_0$, une dérivée $A - M$ positive et, pour $x = x_1$, une dérivée $B - M$ négative.
>
> Elle commencera donc parêtre croissant quand x variera de x_0, á x_1, puis elle finira parêtre dècroissant pour $x = x_1$. Donc elle aura un maximum qu'elle atteindra pour une certaine valeur
>
> $$x_0 + \theta(x_1 - x_0),$$
>
> et pour lequel sa dérivée sera nulle; on aura donc
>
> $$f'(x_0 + \theta(x_1 - x_0)) - M = 0.$$
>
> Ainsi tout nombre M intermèdaire entre A et B est une valeur de la dérivée.
>
> (Darboux, 1875, pp.109-110)

4 Epilogue: Darboux's influence on mathematics, in and out of France

In this section, we return to the historical drama and consider the morals that may be drawn from it by a modern spectator.

Thus far in the tale, we have encountered several important features of Darboux's work in analysis, in both his published papers and his private correspondence with Houël. These include the identification of important conceptual distinctions such as simple versus uniform convergence, supremum/infinum versus maximum/minimum, continuity versus the Intermediate Value Property. Note that each of these topics is typically first encountered by undergraduates in an introductory analysis course. Darboux's skillful use of well-crafted counterexamples to explore the boundaries of function classes thus has a natural place in such a course. As was the case in the nineteenth century, the ways in which monster functions are able to "exemplify the general" can also assist students in making the transition to the more abstract level of thinking required at this level.[11]

Another (as yet unmentioned) feature of Darboux's work that students first seriously encounter in an introductory analysis course is the study of the *pointwise* behaviour of functions. The appearance of this new focus, which was characteristic of the approach taken by the leading nineteenth century analysts (e.g., Riemann, Dirichlet, Darboux), marked a new phase in the study of the calculus. In contrast, the focus throughout the early history of the calculus was on the *global* behaviour of functions, shifting to a focus on *local* behaviour (i.e., across an interval) only with the work of Cauchy.[12] Here again monsters can be helpful by bringing this new focus clearly into view: encountering a function with a positive derivative at a particular point that is not increasing across any interval containing that point underscores well the sharp contrast between these two viewpoints.

Yet another role played by Darboux's monsters was that of a "dissection tool" for uncovering implicit assumptions in a proof. Darboux's expertise with this tool was made especially clear in his letters to Houël (even though Houël himself managed to miss the point). The carefully crafted definitions and proofs that distinguish his published works in analysis offers further evidence of Darboux's mastery of this technique. His rigorous reformulation of the Riemann integral – yet another topic first encountered in an undergraduate analysis course – provides an especially lovely example of this.

In short, Darboux's work in analysis serves as an excellent representation of the changes that occurred in analysis in the latter half of the nineteenth century. His 1875 *Memoire* in particular drew notice even at the time as an outstanding embodiment of the new spirit of analysis, at least outside of France. The Italian analyst Dini cited it alongside works by various German analysts in his own highly commended *Fondamenti per la teoria delle funzio ni di variabili reali* of 1878, but named no other French mathematician. The reason for this was simple: no other French mathematician at the time understood the new direction and standards of analysis sufficiently well to merit international notice.

In fact, Darboux's view of analysis received such a chilly reception from his French colleagues that he eventually gave up analysis as a field of research altogether. The comments from Houël noted in Section 2.3 of this paper illustrate one extreme of the reactions to the new trends in analysis. The extent to which Houël rejected Darboux's view is evidenced by the fact that Houël declined Darboux's advice to base the proofs in his (Houël's) 1878 *Cours de Calcul infinitesimal* on the Mean Value Theorem. By this time, the divergence of their viewpoints on

[11] The expression "exemplify the general" is borrowed from Renaud Chorlay; see pp. 7 - 8 of version of (Chorlay 2016), posted on-line at www.sphere.univ-paris-ziderot.fr/IMG/pdf/Chorlay_chapitre_mis_en_forme_V2.pdf

[12] Interestingly, Cauchy also studied the behaviour of the function $f(x) = x^3 \sin(1/x)$ in a neighborhood of *0*, but made no mention of its differentiability properties.

these issues was extremely clear, and Darboux in turn declined Houël's requests to consult on his infinitesimal calculus text.[13]

Even those who appeared to understand and share Darboux's goals appear not to have been directly influenced by Darboux's work. For instance, in the first edition of his *Cours d'analyse* of 1882, Jordan did not employ the Mean Value Theorem in the foundational role recommended to Houël by Darboux, yet in the second edition of 1887 he did do so. This change of heart was not a consequence of Darboux's views on this matter, however. Rather, Jordan followed the advice of Peano with whom he corresponded on issues in analysis between 1882 and 1884. As part of Peano's critique of the first edition of Jordan's text within that correspondence, the counterexample $f(x) = x^2 \sin(1/x)$ made yet another appearance. More than just a coincidence, this incident provides yet another indication of the role played by such monsters in helping heedful practitioners of the day understand the need for new standards of rigor in their proofs.

But while their creators viewed the existence of these monsters as a warning of the need for increased vigilance with respect to rigor, many French mathematicians continued to view them instead as evil omens to be avoided.[14] According to Lebesgue, Hermite once lamented to Stieltjes in a letter purportedly written in 1893

> I turn with horror and revulsion from this lamentable plague of functions that can have no derivative whatsoever. (Lebesgue, 1922, p.13)

As for how he himself was treated by his colleagues, Lebesgue further reported that:

> I became the man of the functions without derivatives …. whenever I tried to take part in a mathematical discussion there would always be an analyst who would say, "This won't interest you; we are discussing functions having derivatives." (Lebesgue, 1922, p.13-14)

Yet once set loose by Darboux and others, the monsters themselves could not be ignored. One important reason for this was described by Poincaré (who perhaps lamented their existence more than anyone) as part of the commentary with which this paper opened, quoted her in its entirety:

> Logic sometimes creates monsters. For half a century we have seen a host of bizarre functions which appear to be created so as to resemble as little as possible those honest functions which serve some purpose. More of continuity, or less of continuity, more derivatives, and so forth. *What's more, from the logical point of view, these strange functions are the most common, those that we came across without looking for them now appear to be but a particular case. They are left with but a small corner.* In former times when one invented a new function it was for a practical purpose; today one invents

---

[13] Differences of opinion had also begun to appear in terms of their editorial work as well, although Houël continued in this capacity until his death in 1883.

[14] The Latin root of the word "monster" is "monere", meaning "to warn". Interestingly, the English word "money" comes from this same Latin root.

them expressly to show the defects in the reasoning of our fathers, and nothing more can be drawn from them. (Poincaré, 1902, p. 263, emphasis added)

In other words, the monsters are everywhere!

In this paper, we have argued that bringing them into the classroom can provide today's students with a convincing motive for adopting a critical approach to the study of analysis, tools for constructing proofs that meet modern standards of rigor, important cognitive support for their efforts to develop an understanding of introductory analysis concepts, and more. As Darboux's student Emile Borel pointed out, there is at least one other reason to become familiar with the monsters:

> Until now, no one could draw a clearly line between *straightforward* and *bizarre* functions; when studying the first you can never be certain you will not come across the others; thus they need to be known, if only to be able to rule them out. (Borel, 1972, p. 120)

Borel's own contributions to the delineation of the boundaries of the small corner in which these straightforward functions reside, together with the works of Lebesgue and Baire, bear witness to the ultimate fulfillment of Darboux's hope for the rejuvenation of French analysis.

REFERENCES

Barnett, J. (2014). Learning Mathematics via Primary Historical Sources: Straight from the Source's Mouth. *PRIMUS: Problems, Resources and Issues in* Mathematics *Undergraduate Studies, Special Issue on the Use of History of Mathematics to Enhance Undergraduate Mathematics Instruction*, *24*(8), 722-736.

Barnett, J., Pengelley, D., & Lodder, J. (2013). The Pedagogy of Primary Historical Sources in Mathematics: Classroom Practice Meets Theoretical Frameworks. *Science and Education, Special Issue on History and Philosophy of Mathematics in Mathematics Education*, *23*(1), 7-27.

Barnett, J., *et al*. (2011). Designing student projects for teaching and learning discrete mathematics and computer science via primary historical sources. In C. Tzanakis, & V. Katz (Eds.), *Recent developments on introducing a historical dimension in Mathematics Education* (pp. 187-198). Washington, DC: Mathematical Association of America.

Borel, E. (1972). Notice sur les travaux scientifiques. In *OEuvres d'Emile Borel, I* (pp. 119-190). Paris: éditions du CNRS.

Chorlay, R. (2016). Questions of Generality as Probes into Nineteenth Century Analysis. In K. Chemla, R. Chorlay, & D. Rabouin (Eds.), *The Oxford Handbook of Generality in Mathematics and the Sciences*. Oxford: Oxford University Press. Also available on-line at *www.sphere.univ-paris-ziderot.fr/IMG/pdf/Chorlay_chapitre_mis_en_forme_V2.pdf*.

Darboux, G. (1872). Sur un théorème relatif à la continuité des fonctions. *Bulletin des Sciences Mathèmatiques, 3,* 304-313.

Darboux, G. (1875). Mémoire sur les fonctions discontinues. *Annales de l'Ecole Normale, 2ième série, 4*, 57-112.

Darboux, G. (1879). Addition au Mémoire sur les fonctions discontinues. *Annales de l'Ecole Normale, 8*, 195-202.

Gispert, H. (1983). Sur les fondements de l'analyse en France (à partir de lettres inédites de G. Darboux et de l'étude des différentes éditions du Cours d'analyse de C. Jordan). *Archive for History of Exact Sciences*, *28*, 37-106

Gispert, H. (1987). La corresondace de G. Darboux avec J Houël. Chronique d'un rédacteur (déc. 1869-nov. 1871). *Cahiers du séminaire d'histoire des mathématiques, 8*, 67-202.

Gispert, H. (1990). Principes de l'analyse chez Darboux et Houël (1870-1880), textes et context. *Revue d'histoire des sciences*, *43*(2/3), 181-220.

Hochkirchen, T. (2003). Theory of Measure and Integration from Riemann to Lebesgue. In H. N. Jahnke (Ed.), *A History of Analysis* (pp. 261-290). Providence, RI: American Mathematical Society.
Lebesgue, H. (1922). *Notice sur les travaux scientifiques*. Toulouse: Privat.
Lützen, J. (2003). The Foundation of Analysis in the 19th Century. In H. N. Jahnke (Ed.), *A History of Analysis* (pp. 155-195). Providence, RI: American Mathematical Society.
Poincaré, H. (1904). Les définitions générales en mathématiques. *L'enseignement mathématique, 6,* 257-283.

COMMENT INTRODUIRE L'ANALYSE?

Ce que nous apprennent les débuts de l'apprentissage du calcul des différences

Sandra BELLA
Laboratoire de Mathématiques Jean Leray, 2, rue de la Houssinière, 44300 Nantes, France
Laboratoire SPHERE, Université Paris Diderot, 4, rue Elsa Morante, 75013 Paris, France
bellusky@hotmail.com

ABSTRACT

Dans cet exposé nous nous tournons vers l'histoire pour comprendre les difficultés qu'ont représenté les fondements de l'analyse à leur début. A la fin du XVII^è siècle, le calcul différentiel est inventé simultanément par Newton et Leibniz. Les deux inventions reposent sur la notion d'infiniment petit.

Pour les mathématiciens français, le nouvel algorithme ne laisse pas indifférent. Dès sa réception, des questions sont posées : comment présenter les infiniment petits afin de les utiliser dans une pratique mathématique « rigoureuse » ?, en vue d'un enseignement ? Les réponses apportées sont nombreuses : ces entités sont tantôt interprétées comme des grandeurs, des "évanescents", des fictions, etc

Un des acteurs principaux de la réception du calcul leibnizien est Guillaume de l'Hospital. Bénéficiant de cours et d'échanges épistolaires avec Jean Bernoulli, il apprend les bases du calcul, puis l'approfondit et l´applique à la résolution de problèmes physico-mathématiques. Maîtrisant suffisamment le calcul il élabore un ouvrage sur le calcul différentiel qu'il publie en 1696: *Analyse des infiniment petits pour l'intelligence des lignes courbes*[1]. L´Hospital destine l´ouvrage à ses pairs, pour les initier au calcul, en leur montrant les avantages de son utilisation dans de nombreux domaines. Si certains peuvent louer les qualités de clarté de l´ouvrage, ils le jugent difficile d´accès. D´autres sont plus virulents, attaquant le calcul différentiel et la présentation qu'en fait l'Hospital. Leur critique se focalise sur le statut de ces infiniment petits, le manque d'évidence de leurs règles opératoires qui ne prolongent pas forcément celles reconnues de l'algèbre.

Des améliorations de présentation sont tentées par des savants, pour la plupart des enseignants. Vers 1710, un enseignant lausannois nommé Pierre de Crouzas s'initie au calcul des différences à travers l'ouvrage de l'Hospital. Mais il juge que la présentation ne facilite pas l'apprentissage du calcul, en particulier il trouve que les explications sont trop souvent laconiques. Cela l'amène à préparer un commentaire de l'*Analyse des infiniment petits*[2]. Crousaz est très concerné par la vocation enseignante. Lorsqu'il élabore son commentaire, il a déjà écrit plusieurs ouvrages de mathématiques. Ces livres sont le fruit d'une longue réflexion

[1] Hospital, G., *Analyse des infiniment petits pour l'intelligence des lignes courbes*, Paris, Imprimerie Royale, 1696.

[2] Crouzas, P., *Commentaire sur l'Analyse des infiniment petits*, Montalant, Paris, 1721.

sur l'éducation et sur l'enseignement. Nous exposerons les motivations de Crousaz pour éclaircir les infiniment petits et comment sa pensée de l'enseignement des mathématiques s'applique à présenter ces entités si controversées.

Si le calcul différentiel a été abandonné au profit de la notion de limite, nous pensons que la réflexion en amont faite par des mathématiciens, comme Crousaz, qui ont reçu et manipulé le calcul des différences leibnizien n'est pas inutile pour connaître le contexte dans lequel la notion de limite a été élaborée et est une possibilité pour aborder les difficultés rencontrées aujourd'hui dans l'apprentissage et/ou l'enseignement de l'analyse.

NEW-MATH INFLUENCES IN ICELAND

Selective Entrance Examinations into High Schools

Kristín BJARNADÓTTIR
School of Education, University of Iceland, Stakkahlid, Reykjavík, Iceland
krisbj@hi.is

ABSTRACT

The New Math was implemented in Iceland with the intention to facilitate understanding in the midst of increased demands for education for all. The article contains an analysis and comparison of typical papers in a selective entrance examination into high schools before and after the implementation period of the New Math during 1966–1968, and a discussion on the understanding it was expected to facilitate.

1 Introduction

An entrance examination to high schools in Iceland was established in 1946, intended to provide equal opportunities for education. By the mid-1960s it became considered a hindrance on young people's path to prepare for life. In the midst of increased demands for education for all, New Math was implemented, expected to facilitate understanding. The article contains an analysis and comparison of typical examination papers before and after the implementation period of New Math during 1966–1968.

The questions that arise concern educational expectations that can be gleaned from the examination papers: what content and performance expectations were considered optimal preparation for further studies; what changes did the implementation of the New Math bring, and did they promote better understanding?

2 Background

2.1 The history

Iceland belonged to the Danish realm from late 14th century until 1944 when the Republic of Iceland was established. From around 1800, one secondary level school was run in Iceland under the Royal Danish Directorate of the University and the Learned Schools. When Danish learned schools were split into language-history and mathematics-physics streams in 1877, Icelanders chose a language-history stream for their sole learned school. A mathematics-physics stream was established in 1919 and the teaching of Euclidean geometry was restored. In 1903, the Danish school system was split into a lower and upper secondary level, while this system alteration was not implemented in Iceland until 1946. The upper level will from now on be called high school. In the 1920s, educational opportunities in Iceland consisted of the sole six year school with a selective entrance examination, and several local technical schools. In the early 1930s, another high school and a number of lower secondary schools were established in towns and rural areas, providing general education such as arithmetic and languages, but without pathways to the high-school level.

In 1946, the new-born Republic of Iceland issued new education legislation (Law no. 22/1946). Its goal was to create a uniform educational system with eight-year compulsory education and equal access to high school education. A clear route from primary level to high-school level was created with a national high school entrance examination in eight main subjects. The examination was run in lower secondary schools all around the country. The high-school authorities were dissatisfied that their former six-year program was reduced to four years, and that they were deprived of selecting their students. As a compromise, regulations (no. 3/1937) for the former six-year schools' second year were chosen as a basis for that official examination.

2.2 The conditions of the national entrance examination to high schools

The goal of the examination was not officially defined at the outset in 1946, but was later analysed to be as follows: to ensure a certain and standardized minimum knowledge in a number of subjects; the selection of the fittest with respect to certain attributes, considered necessary for those who wanted to study in a high school and a university or other higher education; to offer all students and their relatives a certain assurance of an impartial assessment; and identical examinations for all students (Pálsson and Ólafsson, 1961). Assurance of an assessment by impartial persons was a reaction to rumours that the selection procedure had not always been on grounds of abilities.

The examination papers were official documents. The papers for the years 1946–1962 (Vilhjálmsson 1952; 1959; 1963) were published, first as an official report, while the latter two publications were made available for individual purchase. Later papers were copied in the schools and are available in the National Archives of Iceland.

2.3 The mathematics examination before New Math

For the first 20 years, the mathematics examination was divided into two parts with equal weight: seen problems and unseen problems, tested two days in a row (Vilhjálmsson, 1952; 1959; 1963; National Archives of Iceland). The seen problems were problems that students had previously solved in class under the supervision and assistance of their teacher. The content of the unseen part of the examination was typically 6–8 problems; 4–6 story problems on area, volume and proportions, some to be solved by setting up equations; and two rather complicated fraction problems with algebraic expressions in denominators. The story problems either described situations in contemporary daily life, or were versions of old problems, even from Leonardo Pisano's *Liber Abaci*, such as the problem of a fox chasing a dog (Sigler, 2003, p. 276). In the national examination's first year it became clear that examining all over the country in Euclidean geometry as prescribed in the former high-school regulations (no. 3/1937) did not work and was dropped thereafter.

2.4 The New Math and the mathematics entrance examination

The mathematics examination was run in a similar form each year until 1966 when the New Math was first introduced at all school levels in Iceland. The draft to a curriculum document (Landsprófsnefnd, 1968, pp. 56–60), preparing for its introduction to the country-wide examination, stated that the aim was to base school mathematics on the basic concepts of the set theory, which simultaneously were simple and general, and increase emphasis on the meaning and nature of numbers and of number computations. Clearly, these changes required a different approach in the national examination classes, where the basis was laid for algebra. Symbolic language of set theory allowed ideas and their relations to be expressed in an exact and clear way. It was desirable to delay the algebra of numbers (i.e. the conventional algebra) until students acquired mastery of the relations of sets and the introduction to set theory.

The only textbook at that time fulfilling the requirements on basic concepts of the set theory and the meaning and nature of numbers and of number computations was *Tölur og mengi*, [*Numbers and Sets*] (Arnlaugsson, 1966; Bjarnadóttir, 2015), specifically written for this purpose. In his forewords, the author stated that the basic concepts of logic and set theory would facilitate understanding, even for small children. The textbook *Numbers and Sets* was used for preparing to the national entrance examination during 1967–1975 together with a conventional textbook on algebra by Ó. Daníelsson (1951), first published in 1927. Set theory had however to be taught for a while as an isolated topic, unrelated to other topics in school mathematics contrary to draft curriculum recommendations, as the basics of number algebra had to be taught as soon as possible in the only one short academic year intended to prepare for the entrance examination and thus for academic studies at high schools and universities.

2.5 Problems and critique of the entrance examination

Until 1960, a constant rate of 20% of the cohort attempted the examination and 13-14% reached high-school-admission minimum grade. By 1969 the rates had risen to 34% vs. 21% (Bjarnadóttir, 2006/2007, p. 421). From 1966 onwards, the examination time was shortened, the seen problems were replaced by small problems, only testing one item each, and the number of problems rose to 50 and later 100 small and often unrelated items, presumably to simplify grading, but also to help the less able students to show basic competences.

After 1960, the examination came under growing attack from prominent persons and researchers. A longitudinal research in 1967 by psychologist Dr. Matthías Jónasson showed a correlation between results on IQ tests and the national examination. Shortly after, Jónasson (1968) wrote that an entrance examination to higher education had for a long time had the role of filtering or selecting, which was neither painless nor infallible. This could be justified in nations with educational institutions in a constant funding crisis, where channelling only the fittest students into higher education might seem the preferable utilization of available educational provisions. However, the preparation time for the national examination was far too short. The teachers needed more time to learn to know the capacity and the diligence of their students and have more opportunity to give them guidance. Moreover, the studies could be more carefully planned. One year only led to too tight a time schedule, pressure and hurried work which a youngster in a formative period could not easily sustain.

Jónasson's opinion, gradually shared by many influential people, was also that the national examination would have to be changed from the ground up. The host of incoherent details that the students were expected to remember was horrifying. Would the answers to such questions be the correct measure of the capacity of youngsters for higher education? What about inventiveness, judgement, reasoning and creativity? Jónasson's critique about incoherent details concerned the examination in eight school subjects as a whole, but could as well be applied to the mathematics examination as it began to develop from 1966 onwards.

2.6. Results in the examination

The entrance conditions were strict. Students had to reach 60% average out of the eight subjects examined. The native language had double weight. During the periods 1952–1955 and 1962–1966, the mathematics average was always lower than the average of all eight subjects. The national total is not available but a survey from 5–8 schools indicates an average difference of 5% from the total average of all subjects. However, exchanging the seen problems in 1966 for shorter problems did not make a difference in this respect. Data from years 1958–1962 for one school with a number of classes indicate that grades for the seen problems were on average about 12% higher than the unseen problems. The total national average is only available during 1968–1973. The difference between total average and mathematics average reduced slightly from 1970, and in 1972, the national mathematics average was higher than the total average by 2% (Bjarnadóttir, 2006/2007, pp. 200–205, 286–288).

3 Exploration of the examination papers

The 30-year period 1946–1975 of national examination may be divided into sub-periods, with different characterizations: the experimental period 1946–1950; the period of traditional mathematics 1951–1965; the transition period 1966–1968; and the New Math period 1969–1975 which divides into two sub-periods, 1969–1972 with one syllabus, and 1973–1975.

3.1 The period 1946 to 1950

The examination paper in 1946 contained six questions:

1. Three merchants buy and sell boxes of oranges. The questions concern all versions of percentage computation.
2. Simplifying a complex algebraic expression, containing all operations and e.g. a need to factorize $a^3 - b^3$ and $a^3 + b^3$.
3. A cylindrical water container, finding its volume and time to fill it.
4. Information about sums, differences and proportions between three groups from which to set up an equation.
5. To draw a circumscribable quadrilateral with the following given: two adjacent sides, the angle between them and a diagonal from that angle.

6. A is an obtuse angle. On one of its arms B lies between A and D, and on the other arm C lies between A and E. Prove that DE > DC > BC.

The examination results were reasonably good in the two former six-year schools but worse in the rural and small-town lower-secondary schools around the country. Euclidean geometry such as presented in questions 5 and 6 disappeared altogether from the examination papers. In 1947, they were replaced by a question that could be solved by arithmetic only, and another complex algebraic expression for simplification. Euclidean geometry had only been taught at high school level from 1919 and not all high school teachers had received such training.

3.2 The period 1951–1965

The examination paper in 1951 also contained six questions.

1. An arithmetic problem similar to that of 1947.
2. A percentage problem as in 1946.
3. A double cylinder problem, somewhat more complex than that from 1946.
4. An algebra problem, solvable by division.
5. An equation to be set up, similarly to the 1946 problem.
6. A fairly complex algebraic equation with two known variables and two unknown and a complex insertion after simplification.

The examination continued until 1960 in a similar format with 6–8 large unseen problems; two or three of composite algebra, an algebraic equation was often one of them; one or two large arithmetic problems; one or two stories to set up equations from; and a composite measurement problem of volumes of pyramids, cylinders, cones or spheres, often applying the theorem of Pythagoras.

From 1961 the unseen problems were 10 in total, and from 1962 the examination was given in two sessions, run in the morning and the afternoon of the same day, in order to provide enough time. The seen problems were posed in a special session the day before as earlier. This was continued through 1965 after which the seen problems were dropped. During the period 1957–1965, the ratio of problems formulated in a story, so-called "word problems", was 60-71% of the unseen problems but began to decrease from that time on (Bjarnadóttir, 2006/2007, pp. 426–427).

3.3 The period 1966–1968

This was a transition period, preparing for introducing the New Math for all undergoing this examination. In 1966, the seen problems were replaced by smaller problems, only testing one item each. In years 1967 and 1968, two different versions of the examination were presented, one based on former syllabus of arithmetic and algebra, and another one the New Math syllabus based on Arnlaugsson's (1966) *Numbers and Sets* as a half part against Daníelsson's (1951) algebra. The syllabus of the New Math contained elements from number theory and set theoretical concepts and corresponding operations. Also, in 1967, the proportion of the cohort attempting the examination reached 30%. One person had previously been external examiner and used intuitive evaluation methods of complex problems (H. Steinþórsson, personal communication, 2003) which had provided assurance of a uniform assessment. This was no longer possible due to the increased number of participants. This situation may have contributed to selection of a number of small problems with right/wrong answers. The ratio of story problems decreased also, especially in the New Math problems, 55% in 1967 and 45% in 1968 (Bjarnadóttir, 2006/2007, pp. 426–427).

3.4 The period 1969–1975

During 1969–1972, only one of the two syllabus options was offered: the combination of algebra and the New Math. The ratio of story problems dropped to 30–40%, and the problems were presented in 50 items. Comprehensive textbooks were introduced from 1973 to present the entire syllabus in one set-theoretical framework. During 1973–1974, one option out of two was the syllabus of 1969–1972, first presented in 1967, and the other option was based on a translated Swedish textbook series, composed on behalf of the Nordic Committee for Modernizing Mathematics Teaching, NKMM. In 1975, the third option was based on a domestic series by H. Lárusson (1974–1976), which remained in widespread use until around 1990. The problems were presented in 100 items, and the ratio of word problems was 20–34% (Bjarnadóttir, 2006/2007, pp. 426–427).

4 Analysis

4.1 Analysis according to the TIMSS framework

We shall analyse a selection of examination papers with unseen problems, as examples of the content and performance expectations in the national examination of its period. We choose 1953, when the examination had become established with traditional mathematics; 1966, right before the implementation of the New Math when the number of participants had grown considerably, and the seen problems had been removed; 1971 when the implementation of the New Math had been established; and one of the three examination versions of New Math from the final year 1975 (National Archives of Iceland).

The analysis is based on a curriculum framework for TIMSS (Robitaille, Mc Knight, Schmidt, Britton, Raizen and Nicol, 1993, pp. 61–66, 75–83). The examination papers differ in format and can only be compared qualitatively. The results of the analysis are presented in Tables 1 and 2. Numbers indicate the numeration of the problems posed in the examinations papers. A caveat is that a number of problems, especially from earlier dates, contain varied content and performance expectations, so they have been classified several times each.

In the paper of 1953 as a representative of the period 1951–1965, the main emphasis of the content was on fractions and decimals, including percentages; measurements, always including volume; and equations and formulas, which were the main examination topics. The content remained basically the same in the 1966 paper, of the transition period 1966–1968.

Table 1. Content of a selection of examination papers

	Year	1953	1966	1971	1975
1	**Content**	6 items	25 items	50 items	100 items
1.1.	**Numbers**				
1.1.1	**Whole numbers**				
1.1.1.1	Meaning (place value and numeration, ordering …)			9, 48-50	
1.1.1.2	Operations	1,3	16-17	24, 27-28	
1.1.2	**Fractions and decimals**				
1.1.2.1	Common fractions (meaning, representation, computations …)	2, 3, 4, 6	2	11, 12, 13	
1.1.2.2	Decimal fractions (meaning, representation, computations …)	3, 4, 6	11, 20-22	10	
1.1.2.3	Relationships of common and decimal fractions				
1.1.2.4	Percentages	3, 4, 5	4, 8, 10, 14, 23-25		1-2, 9-10, 49-52, 95-96
1.1.3	**Integer, rational and real numbers**				
1.1.3.1	Negative numbers, integers and their properties			17	
1.1.3.2	Rational numbers and their properties (terminating, recurring …)				5-6
1.1.4	**Other numbers and number concepts**				
1.1.4.1	Binary arithmetic and/or other number bases			21, 22, 23	
1.1.4.2	Exponents, roots, and radicals			15	
1.1.4.4	Number theory (primes and factorization, …)		1	16, 18-20	
1.1.5	**Estimation and number sense**				
1.1.5.2	Rounding and significant figures				3-4
1.2	**Measurement**				
1.2.1	Units	3	6, 12, 13		
1.2.2	Perimeter, area and volume	3, 4	6, 12, 13		65-68
1.3	**Geometry: position, visualization and shape**				
1.4	**Geometry: symmetry, congruence, and similarity**				
1.5	**Proportionality**				
1.5.2	Proportionality problems	1, 5	8, 20, 21, 22	44-47	7-8, 97-100
1.6.	**Functions, relations, and equations**				
1.6.1	Patterns, relations, and functions (number patters, operations on functions…			13, 24	11-12, 13-14, 17-18, 53-56
1.6.2	Equations and formulas (representation of numerical situations, …operations with expressions, factorization and simplification, linear equations …)	2, 4, 5, 6	3, 5, 7, 8, 9, 11, 14, 15, 16-17, 18-19, 20-22, 23-25	14, 15-20, 25-26, 27-28, 29-30, 31-33, 34-35, 36-38, 39-41, 42-43, 44-47, 48-50	15-16, 19-20, 21-24, 25-26, 27-30, 31-34, 35-40, 43-44, 57-60, 61-64, 65-68, 69-72, 73-76, 77-80, 81-84, 85-88
1.7	**Data representation, probability, and statistics**				
1.7.1	Data representation and analysis				45-48
1.7.2	Uncertainty and probability				89-94
1.9	**Validation and structure**				
1.9.2	Structuring and abstracting (sets, set notation)			1-8, 14, 15-20	11-12, 13-14, 17-18

The content changed considerably with the New Math in the paper of 1971, representing the period 1969–1972. It laid emphasis on numbers and number-theoretical items: whole numbers, negative number, recurring decimals, but no percentages, binary arithmetic and arithmetic in other bases, in addition to number patterns, and sets and set notation.

The content changed in 1975, too, with less emphasis on whole numbers and number bases. Area, volume, and percentages appeared again. Rounding numbers appeared. Number patters and sets and set notation continued, while statistics and probability were new topics.

The ratio between the categories does not emerge through the above categorization. It may however be seen that category 1.6.2., equations and formulas, described in details as many kinds of algebraic activities (Robitaille et al., 1993, pp. 78–79), are presented in more than 50% of all problems in all the examination papers.

The results of analysis of performance expectations are presented in Table 2. Numbers indicate the numeration of the problems posed in the examinations papers.

Table 2. Performance expectation in a selection of examination papers

	Year	**1953**	**1966**	**1971**	**1975**
2	**Performance expectations**	6 items	25 items	50 items	100 items
2.1	**Knowing**				
2.1.1	Representing	2, 4	1	9, 10, 11, 12	11-12, 13-14, 15-16, 17-18
2.1.2	Recognizing equivalents	2	2	8, 21, 22	
2.1.3	Recalling mathematical objects and properties	3, 4	6		
2.2	**Using routine procedures**				
2.2.2	Performing routine procedures	2, 4, 6	10, 14	10, 11, 12, 13, 23, 31-33, 34-35	1-2, 3-4, 5-6, 7-8, 9-10, 19-20, 21-24, 25-26, 27-30, 31-34, 35-40, 47-48, 49-52, 69-72, 73-76, 77-80, 89-90
2.2.3	Using more complex procedures	2, 5	3, 5, 6, 7, 10, 12-13	24, 25-26, 27-28, 29-30, 36-38, 39-41, 41-43	41-42, 43-44, 45-46, 53-56, 57-60, 61-64 81-84, 85-88, 91-94
2.3	**Investigation and problem solving**				
2.3.1	Formulating and clarifying problems and situations	1, 3, 4, 6	4, 5, 6, 8, 9	44-47, 48-50	65-68
2.3.2	Developing strategy	2, 3, 4, 5, 6	5, 6, 7, 8, 9, 11, 12-13, 15, 16-17	44-47, 48-50	95-96, 97-100
2.3.3	Solving	1, 2, 3, 4, 5, 6	5, 6, 7, 8, 9, 11, 12-13, 15, 16-17	14, 15-20, 44-47, 48-50	95-96, 97-100
2.3.4	Predicting				
2.3.5	Verifying	1, 3, 4, 6	15		
2.4	**Mathematical reasoning**				
2.4.1	Developing notation and vocabulary	1, 4, 6	20-22, 23-25		
2.4.2	Developing algorithms	1, 4, 6	20-22, 23-25		
2.5	**Communicating**				
2.5.1	Using vocabulary and notation			1–7	
2.5.2	Relating representations			9	

In all the examination papers, performance expectations were similar: knowing, performing routine procedures without using equipment, performing more complex procedures, and solving problems. Only in the two former papers were students expected to verify their solutions. Notably, what is classified as mathematical reasoning, i.e. developing notation and vocabulary, and developing algorithm, disappeared in the 1971 and 1975 papers. Students were assisted in problem solving situations in word problems by suggesting which concepts to choose as unknowns and to set up equations at a certain step in the solution process. This may have helped students as well as eased the grading process.

Generalizing, conjecturing, justifying, proving and axiomatizing were not yet included in the syllabus, and so not demanded at examination. Communicating was not emphasized either. Only using the New-Math vocabulary and notation was seen, while describing, discussing and critiquing were absent in examination papers.

4.2 Presentation of problems in examination papers

What may not be read from the above tables is how the topics, in particular the new topic of set-theoretical items, were examined. Comparison of the first pages of three examination papers reveals a difference:

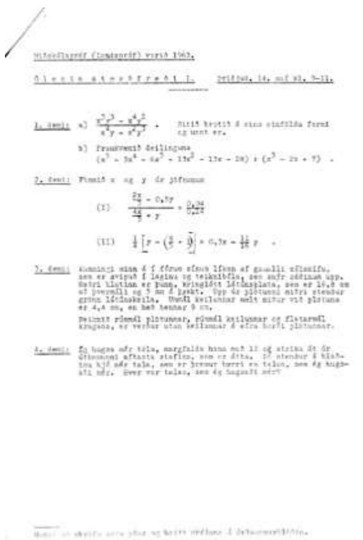


Figure 1. 1963 paper

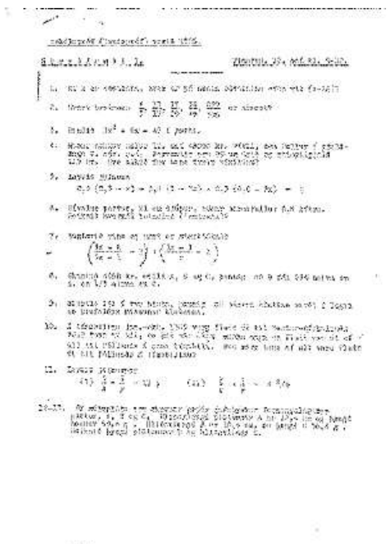


Figure 2. 1966 paper

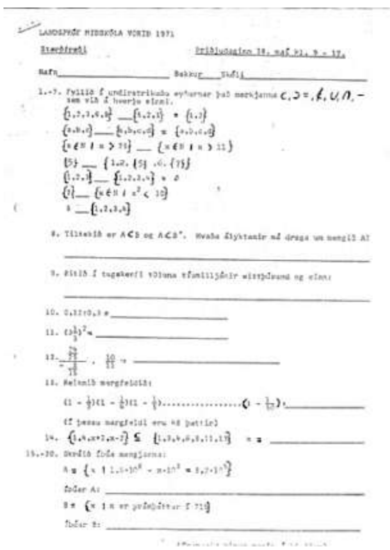


Figure 3. 1971 paper

The first page of the 1963 paper represents the first part of the unseen problems of the period 1946 –1965. All the problems contain either non-trivial algebra or story problems to be solved by setting up equations.

The content on the first page of the 1966 paper replaced the seen problems, and was considered straight forward: ordering fractions by size, finding interests, factorizing, solving an equation with decimal fractions, computing the diameter of a cylindrical pot, dividing two rational expressions, sharing two sums in different ways, solving two simultaneous equations with variables in the denominators, and computing the side and weight of a rectangular prism. The problems on this page count for half the grade of the examination.

Looking at the 1971 paper, the problems count for 16 out of 50 items of the paper. The first seven items, 14%, concern recognizing symbols: $\cup$, $\cap$, $\supset$. $\subset$, $\in$, $-$, $=$. In problems no. 14–20, set-theoretical notation is used to present seven open sentences, other 14%.

5 Understanding

Skemp (1978) distinguished between *instrumental understanding* and *relational understanding*. Instrumental understanding concerned knowing particular items without relating to previous knowledge, while in the case of relational understanding new concepts relate to a network of ideas and previous knowledge. There were also two kinds of mathematics: instrumental and relational. Within its own context, instrumental mathematics was usually easier to understand; the rewards were more immediate and more apparent; and one could often get the right answer more quickly. Relational mathematics, knowing not only what method works in a particular case but why, was however more adaptable to new tasks; easier to remember and relational knowledge could be effective as a goal itself, as the need for external rewards and punishments were greatly reduced. The difficulties in emphasizing relational mathematics and relational understanding lied e.g. in the backwash effect of examination, overburdened syllabi, and difficulty of assessment of whether a person understands relationally or instrumentally. The kind of learning which lead to instrumental mathematics consisted of the learning of an increasing number of fixed plans, by which students can find their way from a particular starting point (the data) to requiring finishing points (the answer to the question). In contrast, learning relational mathematics consisted of building up a conceptual structure (schema) from which its possessor could produce an unlimited number of plans for getting from any starting point within his schema to any finishing point. Skemp suspected that much of what was being taught under the description of 'modern mathematic' alias New Math, was being taught and learnt just as instrumentally as were the syllabi which have been replaced. This might happen due to mismatch between teachers whose conception of understanding is instrumental and aims implicit in the content.

Anna Sierpinska (1994, pp. 28–29) distinguished between *acts of understanding* and *processes of understanding*. An act of understanding was to relate mentally an *object of understanding* to another object. According to Sierpinska (1994, pp. 72–73), processes of understanding could be regarded as lattices of acts of understanding, linked by reasonings (explanations, validations). A relatively good understanding could be achieved if the process of understanding contained a certain number of especially significant acts, namely acts of overcoming obstacles specific to that mathematical situation (Sierpinska, 1994, p. xiv).

Another aspect of understanding mathematics was proposed by George Polya (1973) in his book *How to Solve It*. There, Polya suggested a four-step problem solving procedure: Understanding – Devising a plan – Carrying out the plan – Looking back. In this procedure, devising the plan is the hardest, and Polya suggested that one should try to think of a familiar problem having the same or similar unknown. Understanding consisted of realizing what was unknown, which data were available, and what was the condition.

6 Discussion

We have seen that the structure, content and performance expectations in the national high school entrance examination in mathematics developed drastically during the period 1966–1975, as did the population seeking admission. The examination paper in 1965 was similar to what had been conventional from next to the beginning in 1947. The reason was doubtlessly to ensure equality, not only from place to place, but also from year to year. The idea behind training students in seen problems may have been to provide an aid to industrious students to reach acceptable grade, but also to enhance problem solving skills, such as proposed by Polya (1973). The intention would then have been to aid the students in thinking of a familiar problem where a similar procedure could be used. It is also worth noting that most textbooks up through the 19th century did not contain exercises for the students to solve but published solutions attached to all examples for the students to learn from.

By the introduction of New Math, the content became more oriented towards whole numbers, number theoretical items and patterns, such as the draft to curriculum of 1968 suggested. Less emphasis was placed on large story problems demanding multiple approaches. The word problems became fewer and more abstract, and without a story. They were increasingly short, and the number of problems increased inversely with the shortness of the problems. The ratio of word problems to the total problems in the examination decreased markedly, i.e. from up to two-thirds of the examination to less than one-third, even as little as one-fourth. The word problems were replaced by short problems with one right solution, unrelated to each other. This leads to recalling Dr. Jónasson's remarks on the horrifying host of incoherent details that the students were expected to remember, and his doubts that answers to such questions, short of inventiveness, judgement, reasoning and creativity, were the correct measure of the capacity of youngsters for higher education.

Performance expectations became less oriented towards independent development of notation, vocabulary, and algorithms. The students were more often helped to choose variables in order to be able to form equations out of story problems. The seen problems were abandoned. The committee members may have expected that lower-ability students, presumably an increasingly large proportion of the examination candidates when a larger proportion of the year cohort attempted the examination, did better on the single item problems without stories. The needs of those students had previously been met by the seen problems in addition to generous grading for all first attempts at a problem, with increased demands when the scale came closer to full credit (H. Steinþórsson, personal communication, 2003).

The question if implementation of the New Math facilitated understanding is hard to answer. The role of set theory in the curriculum seems primarily have been to exercise notation in order to prepare the students for further studies. At this point it could only be used for minimum problem solving. Clearly there was not time in one academic year to postpone the introduction of algebra of numbers until the students had acquired mastery of the relations of sets as was proposed in the curriculum document of 1968. The role of set theory to increase clarity and facilitate understanding as Arnlaugsson (1966) had hoped was not relevant as yet.

Examination papers are not intended to be a learning material that could promote relational understanding as defined by Skemp, or Sierpinska's processes of understanding. The role of the national examination was to test if a certain and standardized minimum knowledge had been attained. The papers were, however, officially published in print until 1962, and in many schools, a great deal of the time in class during that only one preparation year was spent on reciting old examination problems. Jónasson remarked that the preparation time for the national examination was far too short. The teachers needed more time to learn to know the capacity and the diligence of their students and have more opportunity to offer guidance to their students. One year only led to overly tight a time schedule, pressure and hurried work. Skemp remarked that the backwash effect of examination, and overburdened syllabi promoted the more superficial instrumental understanding at the cost of relational understanding. Both Skemp's and Jónasson's reasoning point to conditions that could promote instrumental understanding. In this respect it is worth mentioning that the national entrance examination was considered of high importance, both for the students and for the individual schools, and the best qualified and most experienced teachers were chosen for instruction leading up to it. But even the best teachers' conception of understanding may have been instrumental and they may even have considered mathematics itself instrumental. As Skemp remarked, instrumental mathematics was usually easier to understand; the rewards were more immediate and more apparent; and one could often get the right answer more quickly.

One wonders if the long story problems from textbooks and previous examination papers could provide opportunity for teachers to delve deeply into composite problems together with their students and even create by them a lattice of acts of understanding, using Sierpinska's vocabulary. Possibly, obstacles specific to the mathematical situations presented in the examination papers were too difficult to overcome at that point of time in many youngsters' lives. Reality shows that the introduction of shorter problems moved the average of mathematics grades closer to the average grade in all school subjects. The mathematics examination thus ceased to be the blame for not reaching the desired goal, 60% average grade to be qualified for high school entrance. The question remains if this new trend of shorter problems with less concrete content and more diffused focus affected the students' attitude towards mathematics for the better or the worse.

REFERENCES

Arnlaugsson, G. (1966). *Tölur og mengi* [*Numbers and sets*]. Reykjavík: Ríkisútgáfa námsbóka.

Bjarnadóttir, K. (2006/2007). *Mathematical education in Iceland in historical context - Socio-economic demands and influences.* Reykjavík: Háskólaútgáfan. Retrieved June 202016 from http://rudar.ruc.dk/bitstream/1800/3491/1/IMFUFA_456.pdf

Bjarnadóttir, K. (2015). Tölur og mengi – Numbers and sets. *NOMAD*, *20*(3-4), 11-34.

Daníelsson, Ó. (1951). *Kennslubók í algebru*. Reykjavík: Ísafoldarprentsmiðja.

Jónasson, M. (1967). *Mannleg greind*. [Human intelligence]. Reykjavík: Mál og menning.

Jónasson, M. (1968). Landsprófið og vandi þess. [The national examination and its problems]. *Lesbók Morgunblaðsins*, May 5. Reykjavík. Retrieved June 20 2016 from

http://timarit.is/view_page_init.jsp?issId=241425
Landsprófsnefnd [National Examination Board] (1968). *Drög að námsskrá í landsprófsdeildum miðskóla [Draft to a curriculum for a national examination for middle schools]*. Reykjavík: Ministry of education.
Law no. 22/1946 on a school system and official duty to provide education.
National Archives of Iceland. National examination mathematics papers 1963–1975.
Pálsson, J., & Ólafsson, H. (1961). Athuganir á landsprófi miðskóla. *Skírnir, 135*, 195-210.
Polya, G. (1973). *How to solve it*. Sec*ond Princeton paperback printing*. Princeton: Princeton University Press.
Reglugerð fyrir Menntaskólann í Reykjavík [Regulations for Reykjavík High School], no. 3/1937.
Robitaille, D. F., Schmidt, W. H., Raizen, S., Mc Knight, C., Britton, E., & Nicol, C. (1993). *Curriculum frameworks for mathematics and science*. Vancouver: Pacific Educational Press.
Sierpinska, A. (1994). *Understanding in mathematics*. London: Falmer Press.
Sigler, L. (2003). *Fibonacci's Liber Abaci. A translation into modern English of Leonardo Pisano's Book of calculations*. New York: Springer.
Skemp, R. R. (1978). Relational and instrumental understanding. *Arithmetic Teacher, 26*(3), 37-50.
Vilhjálmsson, B. (1952). *Landspróf miðskóla 1946–1951*. Reykjavík: Fræðslumálastjórnin.
Vilhjálmsson, B. (1959*). Landspróf miðskóla 1952–1958*. Reykjavík: Bókaútgáfa Menningarsjóðs.
Vilhjálmsson, B. (1963). *Landspróf miðskóla 1957–1962*. Reykjavík: Bókaútgáfa Menningarsjóðs.

OF THE USE OF THE "ENGLISH SECTOR" IN TRIGONOMETRY: WHAT AMOUNT OF MATHEMATICAL TRAINING WAS NECESSARY IN THE 18TH CENTURY?

Mónica BLANCO
Universitat Politècnica de Catalunya – BarcelonaTech
ESAB, Campus del Baix Llobregat, c/ Esteve Terradas, 8, 08860 Castelldefels (Barcelona), Spain
monica.blanco@upc.edu

ABSTRACT

In 1723 Edmund Stone published *The construction and principal uses of mathematical instruments*, which was essentially a translation from the French of Bion's *Traité de la construction et des principaux usages des instrumens de mathématique* (1709). As the title of the book indicated, Stone annexed a number of instruments that had been omitted by Bion, in particular, those invented or improved by the English. Hence, after the translation of Book II, on the construction and uses of the "French sector", Stone added a chapter on the "English sector". In the 17th century there had been a number of debates concerning the amount of mathematical training required for the study of mathematical instruments. In the context of the study of mathematical instruments in the 18th century, it is worth exploring the link theory-practice in the books on instruments. The aim of this contribution is to explore the mathematical knowledge involved in the use and applications of the "English sector" in trigonometry in a number of 18th-century books on mathematical instruments.

1 Edmund Stone and the study of mathematical instruments

In 1723 Edmund Stone (1695?-1768) published *The Construction and Principal Uses of Mathematical Instruments*, one of the earliest general manuals on mathematical instruments in English. It was essentially a translation from the French version of Nicolas Bion's *Traité de la construction et des principaux usages des instrumens de mathématique*, first published in 1709.[1] In the Translator's preface, Stone defined mathematics both as a science, with regard to the theory, and as an art, with regard to the practice. Mathematical instruments connected these two sides of mathematics. Since the knowledge of mathematical instruments led to the knowledge of practical mathematics, the study of mathematical instruments could be regarded as one of the most useful branches of knowledge in the world, which, therefore, had to be spread. However useful it might be, Stone lamented the lack of a general treatise like Bion's in English. Although this could partly explain why Stone translated Bion's treatise, his book cannot be said to be merely a translation. As the title of the book indicated, at the end of each book Stone annexed a section on the construction and uses of a number of instruments that had been omitted by Bion, in particular those that were invented or improved by the English. "English instruments" added by Stone. Hence, for instance, after the translation of Book II, "Of the Construction and Uses of the Sector", Stone added a chapter on the "English sector".

[1] There is an account of Stone's book and its production in Blanco (2015).

Stone's work seems to be addressed to gentlemen, since, as it is stated in the translator's preface, the study of mathematics "made a part of the education of almost every gentleman" (Stone, 1723, p. v). From the late 16th century onwards, the study of mathematics was an essential element in the education of a gentleman "to maintain his position and engage in the activities traditional to his class" (Turner, 1973, p. 51), such as astronomical navigation, warfare, surveying or trade, among others. In the 17th century there had been a number of debates about the role of mathematical education in the training of gentlemen in mathematical instruments, e.g. the controversy that arose in 1632 between Richard Delamain (1600-1644) and William Oughtred (1574-1660) (Turner, 1973). While Oughtred emphasised the study of mathematics over mathematical instruments, Delamain put the stress on the practice, that is, on the use of instruments. Some fifty years later, John Aubrey (1626-1697) wrote a manuscript on education of gentlemen, which remained unpublished. In his manuscript Aubrey argued that not only a proper mathematical theoretical groundwork was essential in the training of a gentleman, but also the practical concerns had to be taken into consideration. To such and extent, that instruments could be used as pedagogical tools in mathematics education, all the more so since they could be regarded as something new and stimulating in the mathematics classroom. Today we find a similar approach in the works of Bartolini Bussi on how to use ancient instruments in the modern classroom, both as a physical experience and as an alternative way to develop the understanding of specific mathematical concepts (Nagaoka et al, 2000, pp. 343-350). As Greenwald and Thomley (2013) point out, the explicit connection between tools, problems, people and history can boost the teaching and understanding of mathematics. In turn, this connection can be developed further with the integration of traditional instruments with computers in the mathematics classroom.[2]

In the context of the study of mathematical instruments in the 18th century, it is worth exploring the link theory-practice in books on instruments. Was it necessary a sound mathematical groundwork in order to learn how to use an instrument? Or could it be enough to learn a simple set of mechanical rules? Taking inspiration from the debates described by Turner (1973), the aim of this contribution is to explore the mathematical knowledge involved in the use and applications of the "English sector" in trigonometry through a number of 18th-century books on mathematical instruments. To have a more comprehensive perspective, it is also worth having a brief look at how contemporary works on trigonometry presented the solution for the same kind of problems.

2 The 'English sector'

The sector was a mathematical instrument made of two legs of equal length joined to each other by a hinge, with scales on its sides.[3] It turned out to be useful for solving problems in proportion. Figure 1 shows how the instrument worked with the help of a pair of compasses. From Euclid's *Elements*, book VI, proposition 4, it is clear that, as a is to x, so is b to y. The lines drawn upon the sides of the sector (e.g. a, b) are called lateral. Parallel lines run from one leg of the sector to the other, in equal divisions from the center (e.g. x, y).

[2] See for instance Isoda's work in Nagaoka et al (2000, pp. 351-358).

[3] On the history of the sector, see Frémontier-Murphy (2013).

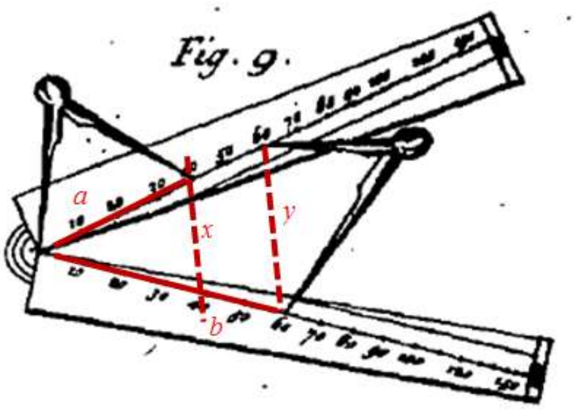


Figure 1. General use of the sector, in Stone (1723), plate VII, Fig. 9 [digitized by the Internet Archive, 2012, with funding from Gordon Bell, available at http://archive.org/details/constructionprin00bion]. Red lines added by the author.

In the second book Bion dealt with "De la construction et des usages du compas de proportion", which Stone translated as the "French sector".[4] At the end of this book, Stone added the section "Of the construction and uses of the English sector". The main differences between the "English sector" and the "French sector" were the nature and the number of lines on each instrument (Figure 2). Upon the faces of Bion's model, there were usually drawn six lines or scales: line of equal parts, line of planes, line of polygons, line of chords, line of solids and line of metals. When it comes to the "English sector", besides the principal lines (equal parts, chords, sines, tangents, secants and polygons), there were also the lines of artificial sines and artificial tangents, Gunter's line of numbers, and a line of inches. The artificial lines represented the logarithms of numbers, sines and tangents and were to be used as on Gunter's scale. In fact, in 1624 Edmund Gunter had produced a book on the sector and the cross staff, wherein he described a sector with up to twelve lines, including the trigonometric lines. At the end of this work, Gunter provided his tables of artificial sines and tangents. While in France Bion's model would be used until the end of the 18th century, sectors with trigonometric lines and artificial lines became very popular in England (Frémontier-Murphy, 2013).

The advantage of the "English sector" above the common scales, or rules, was that all its lines could be fitted to any radius, not exceeding the length of the legs. In Section II, "Of the general use of the line of chords, sines, tangents, and secants, on the sector", Stone illustrated this fact:

> Suppose the chord, sine, or tangent of 10 degrees, to a radius of three inches, is required. Take that three inches, and make it a parallel between 60 and 60 on the line of chords, then, as I have already said, the same extent will reach from 45 to 45, on the line of tangents, also on the other side of the sector, the same distance of three inches will reach from 90 to 90 on the line of sines so that if the lines of chords be set to any

[4] See, for instance, Stone (1723, p. 66).

> radius, the lines of sines and tangents are also set to the same. Now the sector being thus opened, if you take the parallel distance between 10 and 10 on the line of chords, it will give the chord of 10 degrees. Also if you take the parallel distance on the line of sines between 10 and 10, you will have the sine of 10 degrees. Lastly, if you take the parallel extent on the line of tangents, between 10 and 10, it will give you the tangent of 10 degrees. (Stone, 1723, pp. 67-68)

The sector was soon regarded as a universal instrument, "generally useful in all the practical parts of the mathematicks, and particularly contrived for navigation, surveying, astronomy, dialling, projection of the sphere, &c." (Harris, 1704-1710, entry *SECTOR*). Therefore, the sector could be employed to solve trigonometric problems, such as the solution of oblique triangles.

As mentioned above, Turner (1973) gives an account of the discussions about how necessary the teaching of a proper mathematical groundwork was for the education of gentlemen, in the context of the study of mathematical instruments in the 17th century. It is true that Stone (1723) was intended for gentlemen. Yet, other works dealing with the "English sector" also addressed those interested in practical mathematics (e.g. Gunter, 1624; Cunn, 1729; Webster, 1739), architects, engineers and artificers (e.g. Robertson, 1747) and learners in general (e.g. Rea, 1717). Widening the audience to all kind of readers and getting into the 18th century, did works on the "English sector" include the principles of trigonometry? And if so, how was the doctrine of triangles taught? Since this contribution is a work-in-progress, next I will show how a few works on the sector presented the solution of triangles.

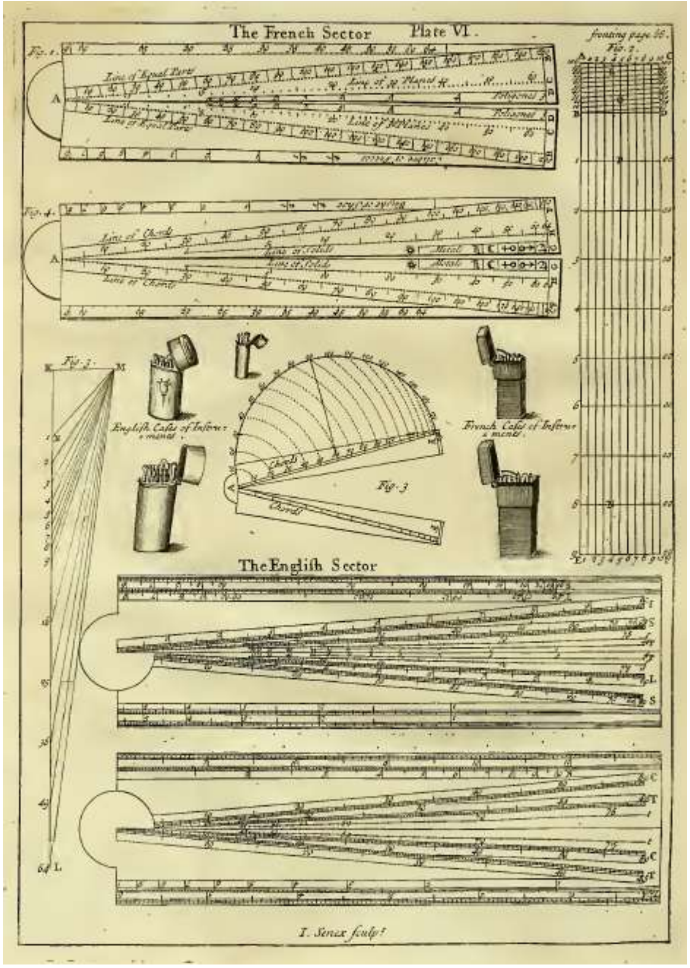


Figure 2. The French sector and the English sector, in Stone (1723), plate VI [digitized by the Internet Archive, 2012, with funding from Gordon Bell, available at http://archive.org/details/constructionprin00bion]

3 Of the use of the 'English sector' in trigonometry

In Section III Stone illustrated the use of the sector in trigonometry with the solution of several cases of plane triangles, both right-angled (uses I-VI) and oblique-angled (uses VII-IX), and spherical triangles (uses X-XI). The doctrine of triangles was involved in a number of practical applications. For instance, in surveying:

> How to find the distance of a fort, or walls of a city, or castle, that you dare not approach for fear of gun-shot; or the breadth of a river or water, that you cannot pass, or measure over it; made by two stations. (Sturmy, 1684, p. 20)

and in navigation:

> Two ships, one sails N. 36º 00' E. 50 miles, the other sails S. 50º 00' E. 70 miles; I demand their bearing and distance? Jones (1705, p. 28)

In Use VII Stone studied the case of an oblique triangle, when two angles and one side were known, and solved it with the help of the "English sector":

Use VII: *The angles* CAB*, and* ACB*, being given, and the side* CB*: to find the base* AB.

Take the given side CB, and turn it into the parallel sine of its opposite angle CAB, and the parallel sine of the angle ACB, will be the length of the base AB. (Stone, 1723, p. 69)

Despite giving no further explanation, it is evident that here Stone was using the following trigonometric result: in all plane triangles, the sides are in proportion one to another as the sines of the angles opposite to these sides, that is to say, what today we call the law of sines.[5] Stone simply showed the mechanical process, the instrumental way of solving the problem by means of the sector. We find a similar approach in Gunter (1624):

To find a side by having the angles and one of the other sides given.

Take the side given, and turn it into the parallel sine of his opposite angle; so the parallel sines of the other angle shall be the opposite sides required. (Gunter, 1624, p. 70)

and later in Rea (1717) and in Martin (1745), though not as concise:

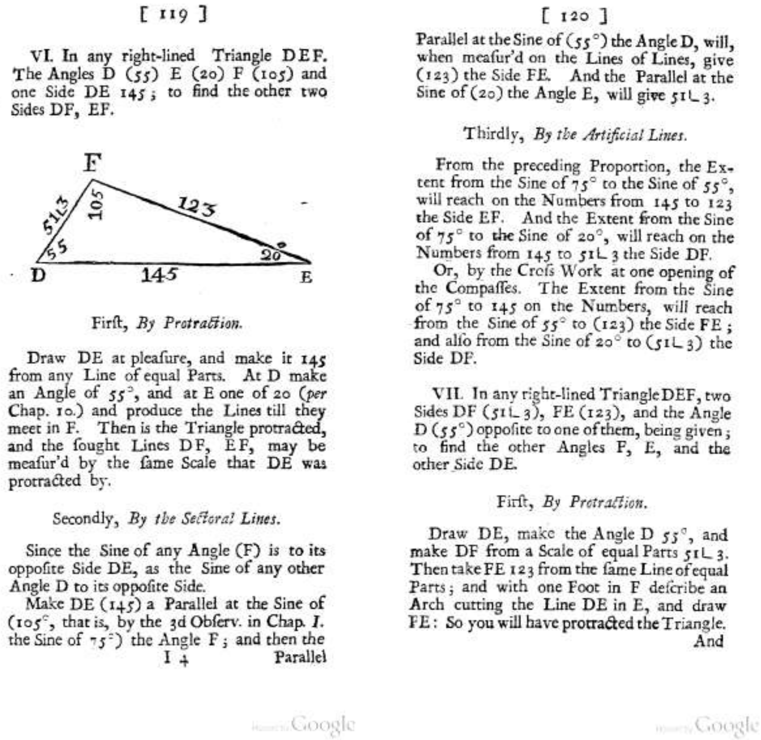
In the oblique right lin'd triangle ABC, there is given the angle ACB 115 deg. 24, the angle BAC 28 deg. 30, with the length of the side AC 75 lea. to find the quantity of the angle ABC, and the length of the two sides AB and BC. (…)

From the center of the line of lines take the length of the given side AC 75 lea. between your compasses and make it a parallel between the sines of its opposite angle ABC 36 deg. 6, (found by the substraction) the sector being continued at that angle, take the distance between the sines of the angle BAC 28 deg. 30 between your compasses, and measure that distance from the center of the line of lines, and it will be 61 lea. the length of its opposite side BC, and the distance taken between 64 deg. 36 (the complement of the angle ACB 115 deg. 24 to 180 deg.) it will be 115 lea. the length of the side AB, as was required. (Rea, 1717, p. 33)

However, in *The description, nature and general use of the sector and plain-scale* (1721), a book most likely written by Stone, the author formulated the law of sines verbally before solving the problem instrumentally by means of the sector: "As the sine of the angle ABC is to the length of the base AC, so is the sine of the angle ACB to the length of the side AB sought" (Stone, 1721, p. 30). A few years later, Cunn (1729), in a work published posthumously and carefully revised by Edmund Stone, proceeded similarly, after having solved the problem by protraction (Figure 3). Cunn went to the extent of presenting a third way of solving the problem, by the artificial lines. It would be worth analysing the use of the artificial lines in the doctrine of plane triangles, all the more so since other books on mathematical instruments, such as Rea (1717), Stone (1723), Webster (1739), Martin (1745) and Robertson (1747), explained how to use these lines in order to solve an oblique triangle. We can also find this alternative method in some books of trigonometry, as in Newton (1658)

[5] On the history of the law of sines, see van Brummelen (2009).

and Heynes (1701). Yet, a detailed discussion of the use of these lines is beyond the scope of this paper.



[119]

VI. In any right-lined Triangle DEF. The Angles D (55) E (20) F (105) and one Side DE 145; to find the other two Sides DF, EF.

Firft, *By Protraction.*

Draw DE at pleafure, and make it 145 from any Line of equal Parts. At D make an Angle of 55°, and at E one of 20 (*per* Chap. 10.) and produce the Lines till they meet in F. Then is the Triangle protracted, and the fought Lines DF, EF, may be meafur'd by the fame Scale that DE was protracted by.

Secondly, *By the Sectoral Lines.*

Since the Sine of any Angle (F) is to its oppofite Side DE, as the Sine of any other Angle D to its oppofite Side.

Make DE (145) a Parallel at the Sine of (105°, that is, by the 3d Obferv. in Chap. *I.* the Sine of 75°) the Angle F; and then the

I 4 Parallel

[120]

Parallel at the Sine of (55°) the Angle D, will, when meafur'd on the Lines of Lines, give (123) the Side FE. And the Parallel at the Sine of (20) the Angle E, will give 51L3.

Thirdly, *By the Artificial Lines.*

From the preceding Proportion, the Extent from the Sine of 75° to the Sine of 55°, will reach on the Numbers from 145 to 123 the Side EF. And the Extent from the Sine of 75° to the Sine of 20°, will reach on the Numbers from 145 to 51L3 the Side DF.

Or, by the Crofs Work at one opening of the Compaffes. The Extent from the Sine of 75° to 145 on the Numbers, will reach from the Sine of 55° to (123) the Side FE; and alfo from the Sine of 20° to (51L3) the Side DF.

VII. In any right-lined Triangle DEF, two Sides DF (51L3), FE (123), and the Angle D (55°) oppofite to one of them, being given; to find the other Angles F, E, and the other Side DE.

Firft, *By Protraction.*

Draw DE, make the Angle D 55°, and make DF from a Scale of equal Parts 51L3. Then take FE 123 from the fame Line of equal Parts; and with one Foot in F defcribe an Arch cutting the Line DE in E, and draw FE: So you will have protracted the Triangle.

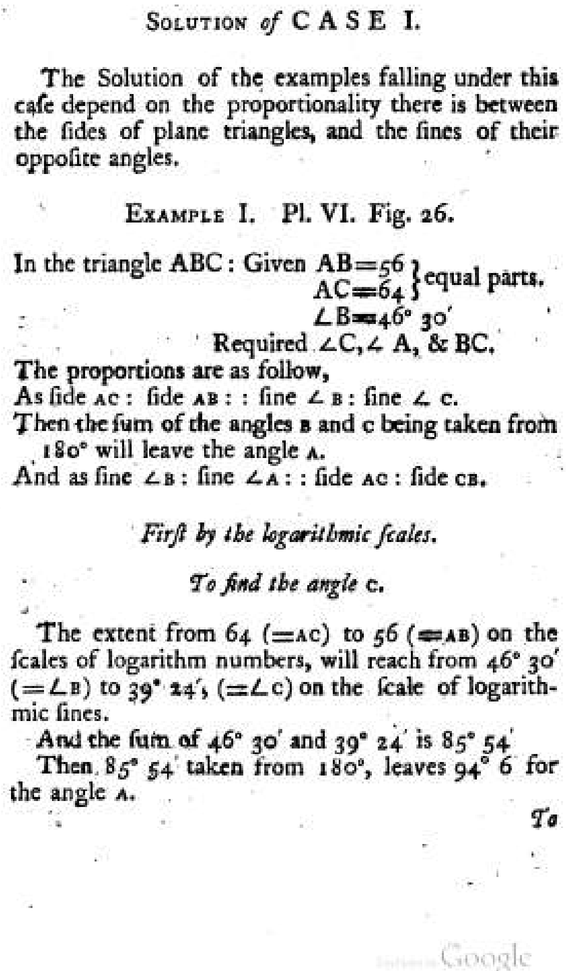
And

Figure 3. Cunn (1729) [digitized by Google Books].

Finally, Robertson (1747) expressed the law of sines in a symbolical way, though not fully developed (Figure 4). In treatises on trigonometry, the so-called law of sines was expressed either verbally: “As the side AB is to the side BC. So is the sine of the angle ACB to the sine of the angle BAC” (Pitiscus, 1614, p. 95), or symbolically, as in Newton (1659, p. 63), Heynes (1701, p. 20) and Harris (1703, p. 32):

CB . s. CDB :: CD . s. CBD

Thus, it seems that books on trigonometry started to use symbols to express the law of sines earlier than books on mathematical instruments.



SOLUTION *of* CASE I.

The Solution of the examples falling under this caſe depend on the proportionality there is between the ſides of plane triangles, and the ſines of their oppoſite angles.

EXAMPLE I. Pl. VI. Fig. 26.

In the triangle ABC: Given AB=56 } equal parts.
AC=64 }
∠B=46° 30′
Required ∠C, ∠A, & BC.

The proportions are as follow,
As ſide AC : ſide AB : : ſine ∠B : ſine ∠C.
Then the ſum of the angles B and C being taken from 180° will leave the angle A.
And as ſine ∠B : ſine ∠A : : ſide AC : ſide CB.

Firſt by the logarithmic ſcales.

To find the angle C.

The extent from 64 (=AC) to 56 (=AB) on the ſcales of logarithm numbers, will reach from 46° 30′ (=∠B) to 39° 24′, (=∠C) on the ſcale of logarithmic ſines.

And the ſum of 46° 30′ and 39° 24′ is 85° 54′

Then 85° 54′ taken from 180°, leaves 94° 6′ for the angle A.

To

Figure 4. Robertson (1747) [digitized by Google Books].

4 Some final remarks

From this preliminary study, three different ways of teaching how to use the sector for solving triangles can be identified: 1) instrumentally, with no explicit formulation of the law of sines; 2) with a verbal formulation of the law of sines before the instrumental solution; 3) with a symbolic formulation of the law of sines before the instrumental solution. The study of more works dealing with the “English sector” and its applications in trigonometry could contribute to determine a possible pattern in the transition from one way of teaching to another, especially in the first half of the 18th century. It would also be interesting to examine alternative ways of solving triangles by means of the artificial lines and of instruments other than the “English sector” and, consequently, the interaction between the mathematical development (theory) and the instruments (practice). Finally, from a pedagogical point of view it would be worth exploring the use of the sector to solve triangles in the mathematics classroom.

Acknowledgements

I am grateful to the staff of the British Library (especially of the Rare Books & Music Reading Room) and to Tony Simcock (Library and Archives, Museum of the History of Science in Oxford). Thanks to Stephen Johnston for his helpful comments regarding the collection of sectors of the Museum of the History of Science in Oxford. I also thank the Spanish Ministry of Economy and Competitivity for financial support (HAR2013-44643-R).

REFERENCES

Bion, N. (1709). *Traité de la construction et des principaux usages des instrumens de mathématique*. Paris: Chez la Veuve de J. Boudot, J. Collombat et J. Boudot fils.

Blanco, M. (2015). On gardeners, dukes and mathematical instruments. *Bulletin of the Scientific Instrument Society*, *125*, 22-28.

Cunn, S. (1729). *A new treatise of the construction and use of the sector*... Being a work of the late Mr. Samuel Cunn's, Teacher of Mathematicks, &c. Now carefully revised by Edmund Stone. London: Printed for J. Wilcox and T. Heath.

Frémontier-Murphy, C. (2013). A new mathematical vision for an innovative calculating instrument: Galileo's sector. *Bulletin of the Scientific Instrument Society*, *118*, 35-43.

Greenwald, S. J., & Thomley, J. E. (2013). Using the history of mathematics technology to enrich the classroom learning experience. *Electronic Proceedings of the Twenty-fourth Annual International Conference on Technology in Collegiate Mathematics*, paper S071. Retrieved from http://archives.math.utk.edu/ICTCM/i/24/S071.html.

Gunter, E. (1624). *The description and use of the sector. The crosse-staffe and other instruments*. London: Printed by W. Jones.

Harris, J. (1703). *The elements of plain trigonometry demonstrated: And its practice and use explained*. London: Printed by E. Midwinter.

Harris, J. (1704, 1710). *Lexicon Technicum: or, An Universal English Dictionary of Arts and Sciences*, 2 vols. London: Printed for D. Brown.

Heynes, S. (1701). *A treatise of trigonometry, plane and spherical, theoretical and practical*. London: Printed by J. Matthews

Jones, W. (1705). *A new epitomy of the whole art of practical navigation: Containing the elements of plain trigonometry, and its application to plain, Mercator's and middle-latitude sailing*. London: Printed by T. Mead.

Martin, B. (1745). *The description and use of a case of mathematical instruments; particularly of all the lines contained on the plain scale, the sector, the Gunter, and the proportional compasses.* London.

Nagaoka, R., Barrow-Green, J., Bartoloni Bussi, M. G., Isoda, M., van Maanen, J., Michalowicz, K. D., Ponza, M. V., & van Brummelen, G. (2000). Non-standard media and other resources. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education: the ICMI study* (pp. 329-370). Dordrecht: Kluwer.

Newton, J. (1658). *Trigonometria Britanica, or, The doctrine of triangles: in two books... the one composed, the other translated from the Latine copie written by Henry Gellibrand.* London: Printed by R. & W. Leybourn.

Newton, J. (1659). *Geometrical trigonometry, or, The explanation of such geometrical problems as are most useful & necessary, either for the construction of the canons of triangles, or for the solution of them*. London.

Pitiscus, B. (1595). *Trigonometry: or the doctrine of triangles*. First written in Latin, by Bartholomew Pitiscus. English Translation: 1614. London: Printed by E. Allde.

Rea, R. (1717). *The sector and plain scale, compared.* London: Printed by J. Cluer.

Robertson, J. (1747). *A treatise of such mathematical instruments, as are usually put into a portable case: containing their various uses in arithmetic, geometry, trigonometry, architecture, surveying, &c.* London: Printed for T. Heath, J. Hodges and J. Fuller.

Stone, E. (1721). *The description, nature and general use, of the sector and plain-scale, briefly and plainly laid down. As also a short account of the uses of the lines of numbers, artificial sines and tangents*. London: Printed for T. Wright.

Stone, E. (1723). *The construction and principal uses of mathematical instruments. Translated from the French of M. Bion, Chief Instrument-Maker to the French King. To which are added, the construction and uses of such instruments as are omitted by M. Bion*. London: Printed for J. Senex and W. Taylor.

Sturmy, S. (1669). *The Mariners Magazine.* 3rd ed: 1684. London: Printed by J. Playford.

Turner, A. J. (1973). Mathematical instruments and the education of gentlemen. *Annals of Science, 30*(1), 51–88.

Van Brummelen, G. (2009). *The mathematics of the heavens and the earth: The early history of trigonometry*. Princeton, Oxford: Princeton University Press.

Webster, W. (1739). *The description and use of a complete sett or case of pocket-instruments: Containing the construction of the several lines laid down on the plain-scale, and sector*. 2nd ed, corrected. London: Printed by A. Betteswroth, C. Hitch and D. Browne.

ARTICULATION OF MATHEMATICAL NOTIONS WITH QUECHUA NOTIONS ACROSS HISTORY OF MATHEMATICS AND DYNAMIC GEOMETRY

María del Carmen BONILLA
Universidad Peruana Cayetano Heredia, Apinema: Peruvian Association for Research in Mathematics Education, Lima, Perú
maria.bonilla.t@upch.pe, mc_bonilla@hotmail.com

ABSTRACT

Peru is a multilingual, multicultural country with a bilingual education system (now termed intercultural) that has developed greatly in recent years. Since 2014, the Peruvian government has offered scholarships to students from indigenous communities working to develop this education sector and cover an estimated deficit of 20,000 teachers. Prestigious universities in Lima have opened Undergraduate Programs in Intercultural Bilingual Education (IBE), which bring together students from different indigenous communities, specifically Quechua, Aymara (Andeans) and Shipibo (Amazonians). In an initial assessment, most students did not achieve a satisfactory level of performance in mathematics. This paper develops a proposal for integrating long-established Andean mathematical tools and dynamic geometry software into the learning and teaching process in order to improve attainment of mathematical competence in the intercultural education sector.

1 Research problem

Results from the 2007-2014 national census assessments indicate underperformance in mathematics by indigenous students (OMCA, 2015). The measurement curve for satisfactory performance (see Figure 1) from this time period shows a tendency toward lower performance in rural areas in comparison to the curve for urban areas. Indigenous communities are mainly located in rural areas. Over the years, the achievement gap has continued to widen. One factor that could explain this situation is that, although students study in their native language, the teaching and learning process does not consider the mathematical notions implied in the daily practices of their communities, thus divorcing school-taught mathematics from the logic of the bodies of mathematical knowledge pertaining to their cultures (González & Caraballo, 2015).

The research question in this paper is as follows: Is it possible to improve the level of numerical performance of students of Intercultural Bilingual Education using the *Yupana* for calculating basic operations, and improve their level of geometrical performance through the use of Dynamic Geometry software for designing their cultural objects?

2 Theoretical Framework

Oliveras (2015) indicated that there is a relativist paradigm in mathematics. In the same way, researchers suggest that the teaching and learning of mathematics is related to the history, language and culture of specific peoples (Barton 2008, cited by Sun & Beckmann 2015), (Ellis & Berry, 2005), (Ernest, 2005).

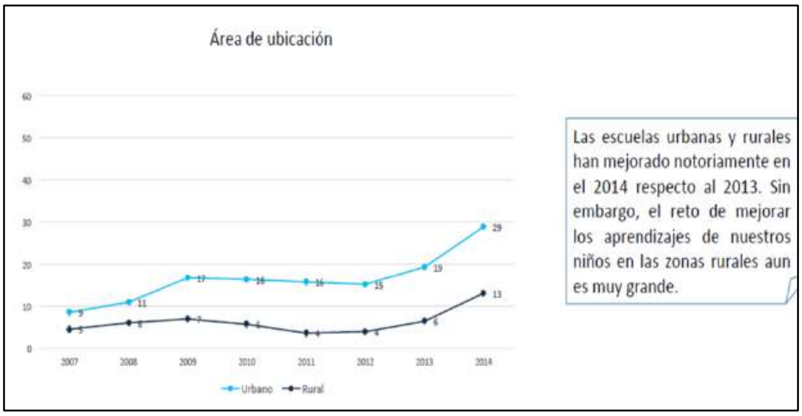


Figure 1. Percentage of students at the satisfactory level in mathematics 2007 – 2014. Rural and urban area. (OMCA, 2015)

From a cultural perspective, Cauty (2001) affirms that all cultures, create bodies of knowledge and systems for symbolically representing mathematical concepts. He believes that mathematics education should be taught in a way that conforms to the cultures of students, taking their culturally-specific mathematical conceptions into account, articulating the knowledge and skills you want students to acquire, without causing deculturation.

Over the past four decades, there has been a shift towards integrating the history of mathematics into the teaching of mathematics (Jankvist, 2009). Epistemological and didactic foundations support the introduction of the historical dimension in teaching mathematics (Barbin, 2012b), (D'Enfert, Djebbar and Radford, 2012). Similarly Barbin highlights the increasingly growing trend of teaching the history of mathematics in teacher training (2012a). Furthermore, the academic community of the HPM Group (History and Pedagogy of Mathematics) noted the need for more empirical research on integrating the history of mathematics into the teaching and learning of mathematics (Arcavi, quoted by Jankvist, Op.cit.).

The research into the history of mathematics has made use of Inca calculation and data-recording devices, such as the yupana and the quipu, respectively, to bolster arguments in favour of integrating culturally-relevant mathematical concepts into culturally-specific mathematics pedagogy. According to this research, the yupana and quipu were used as pedagogical tools to teach mathematics to children throughout Inca history (Villavicencio, 2011).

Concerning learning and teaching geometry, Dynamic Geometry System (DGS) is a *milieu didactique,* didactic milieu. DGS was created as a specific type of digital application that allows students to build and drag mathematical objects on the screen (Arzarello, Bartolini, Leung, Mariotti & Stevenson, 2012). The multiple computational tools have rejuvenated mathematics and mathematics education.

> Any DGS figure is the result of a construction process, since it is obtained after the repeated use of tools chosen from those available in the 'tool bar'… what makes DGS so interesting is the direct manipulation of its figures, conceived in terms of the

embedded logic system of Euclidean geometry (Laborde and Straesser, 1990; Straesser, 2001; cited by Arzarello, et al, 2012, p. 103).

The important thing is to know how to articulate the DGS with culturally-specific Amazonian and Andean mathematical knowledge.

3 Tools of history of mathematics

This research, taking into account the cultural and historical dimension in mathematics education, applies the tools of historical Inca mathematics, the yupana and quipu, to develop numerical skills in undergraduated students in Intercultural Bilingual Education. They also studied characteristics of Inca mathematics.

Likewise, it is necessary to develop the capability of the Cabri II Plus geometry application in undergraduated students to design Amazonian and Andean cultural objects.

Both proposals were tested in IBE initial teacher training.

3.1 Andean and Amazonic number systems

The Inca number system is decimal and positional, like most Western languages. The Aymara and Shipibo cultures originally had only five and three numbers, but under the Inca empire, along with the spread of the official Quechua (Runa Simi) language, the Inca number system was adopted (Figure 2).

3.2 Yupana

Yupana comes from Quechua word, to count. The Yupana is a type of abacus for performing arithmetic operations. They were made of clay, stone or wood. The numbers were represented with corn kernels, seeds, or pebbles. A yupana can be seen in the lower left corner of a drawing of "*Primer Nueva Corónica y Buen Gobierno*", a book written by the indigenous chronicler Felipe Guaman Poma de Ayala (Figure 3). The illustration suggests that the *quipucamayoc* (Quechua word meaning, the one who owns or masters quipu) calculated with the yupana and then recorded the data on the quipu.

The yupana was known and appreciated by colonial administrators. The Spanish priest José de Acosta wrote the following about this artifact in his book *Historia Natural y Moral de las Indias*:

> To see them use another kind of quipu with maize kernels is a perfect joy. In order to effect a very difficult computation for which an able calculator would require pen and ink for the various methods of calculation these Indians make use of their kernels. They place one here, three somewhere else and eight I do not know where. They move one kernel here and three there and the fact is that they are able to complete their computation without making the smallest mistake. As a matter of fact, they are better at calculating what each one is due to pay or give than we should be with pen and ink. Whether this is not ingenious and whether these people are wild animals let those judge who will! What I consider as certain is that in what they undertake to do they are superior to us. (Acosta, 1590; quoted in Leonard & Shakiban, 2010).

Número	Quechua Collao	Quechua Incahuasi Cañaris	Aimara	Shipibo Konibo
1	*huk*	*uk*	*maya*	*westiora*
2	*iskay*	*iskay*	*paya*	*rabe*
3	*kimsa*	*kimsa*	*kimsa*	*kimisha*
4	*tawa*	*ĉusku*	*pusi*	*chosko*
5	*pichqa*	*pichqa*	*qallqu*	*pichika*
6	*suqta*	*suqta*	*suxta*	*sokota*
7	*qanchis*	*qanĉis*	*paqallqu*	*kanchis*
8	*pusaq*	*pusaq*	*kimsa qallqu*	*posaka*
9	*isqun*	*isqun*	*llätunka*	*iskon*
10	*chunka*	*ĉunka*	*tunka*	*chonka*
11	*chunka hukniyuq*	*ĉunka uk*	*tunka mayani*	*chonka westiora*
12	*chunka iskayniyuq*	*ĉunka iskay*	*tunka payani*	*chonka rabe*
13	*chunka kimsayuq*	*ĉunka kimsa*	*tunka kimsani*	*chonka Kimisha*
14	*chunka tawayuq*	*ĉunka ĉusku*	*tunka pusini*	*chonka chosko*
15	*chunka pichqayuq*	*ĉunka pichqa*	*tunka qallquni*	*chonka pichika*
16	*chunka suqtayuq*	*ĉunka suqta*	*tunka suxtani*	*chonka sokota*
17	*chunka qanchisniyuq*	*ĉunka qanĉis*	*tunka paqallquni*	*chonka Kanchis*
18	*chunka pusaqniyuq*	*ĉunka pusaq*	*tunka kimsa qallquni*	*chonka posaka*
19	*chunka isqunniyuq*	*ĉunka isqun*	*tunka llätunkani*	*chonka Iskon*
20	*iskay chunka*	*iskay ĉunka*	*paya tunka*	*rabe chonka*
30	*kimsa chunka*	*kimsa ĉunka*	*kimsa tunka*	*kimisha chonka*
40	*tawa chunka*	*ĉusku ĉunka*	*pusi tunka*	*chosko chonka*
50	*pichqa chunka*	*pichqa ĉunka*	*qallqu tunka*	*pichika chonka*
60	*suqta chunka*	*suqta ĉunka*	*suxta tunka*	*sokota chonka*
70	*qanchis chunka*	*qanĉis ĉunka*	*paqallqu tunka*	*kanchis chonka*

Figure 2. Some Andean and Amazonian number systems
(Source: Ministerio de Educación de Perú, 2013)

Figure 3. Quipu and Yupana

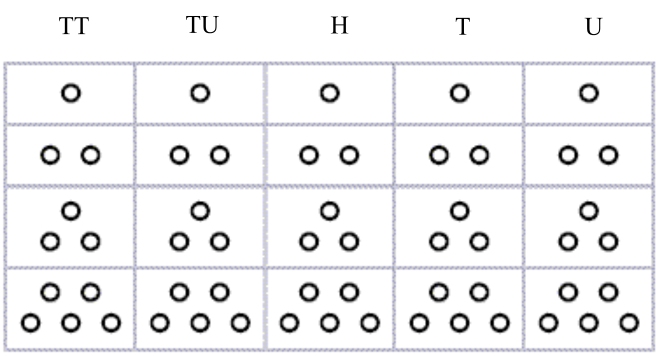


Figure 4. The yupana with a 90 degrees turn

Educators and social scientists have used the yupana in several educational research projects in Perú. Such is the case of Martha Villavicencio (2011) who worked in the Experimental Bilingual Education Project in Puno (1978-1988) and in the Andean Rural Education Program (1989-1996) in Ecuador. Likewise, the yupana was promoted as a teaching aid for the first elementary grades by the Peruvian anthropologist Daniel Chirinos (2010), in a bilingual education project in Loreto (peruvian amazon region) funded by Spanish Agency for International Development Cooperation (AECID) in Peru. 'There are research and testimonies certifying the educational capacity of the yupana in facilitating the understanding of the positional value of numbers in the decimal system and the execution of immediate arithmetic calculations' (Montalvo, 2012, p. 16).

The yupana (Figure 4) is composed of five rows and four columns. In the first column there are five circles in each row, in the second, three circles, in the third, two circles, and in the last column, one circle. The bottom row represents the units, the previous row is for the tens, the third from the bottom up represents the hundreds, the fourth, for the units of thousand, and the fifth, the ten-thousands (González & Caraballo, 2015). A corn kernel (seed or pebble) in a circle represents unit, ten, hundred, or thousand, according to its position in the yupana. It is used to do the arithmetic operations of addition, subtraction, multiplication and division with whole numbers in a concrete way.

Practical examples of operations with the yupana are detailed below (Mejia, 2011). Examples of addition with and without regrouping, subtraction with and without decomposing, and a multiplication with yupana are shown from Figures 5 to 8.

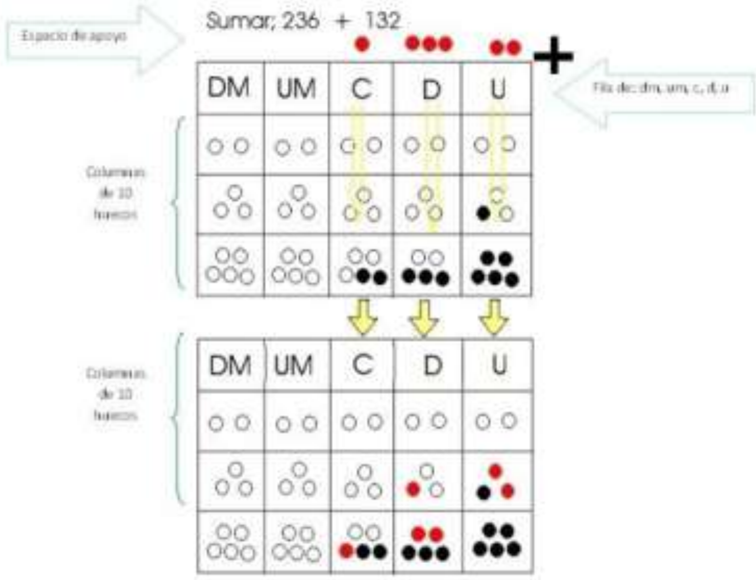


Figure 5. Sum with Yupana (Mejía, 2011)

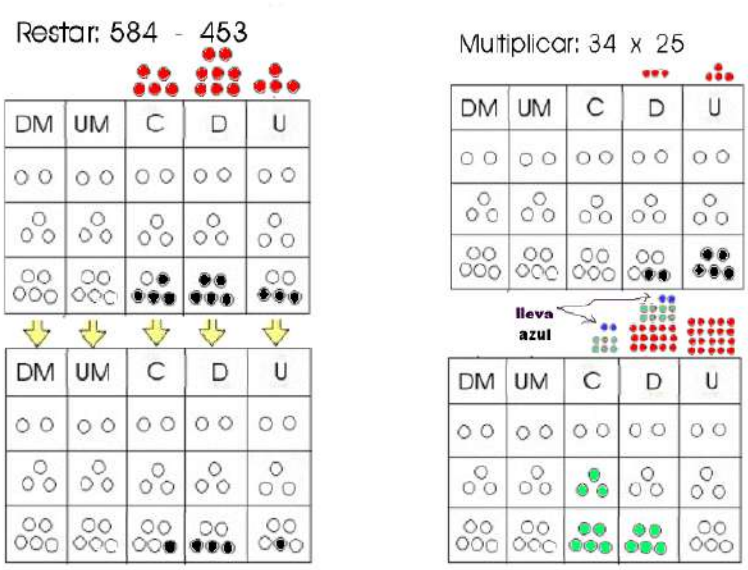


Figure 6. Subtraction and multiplication with Yupana (Mejía, 2011)

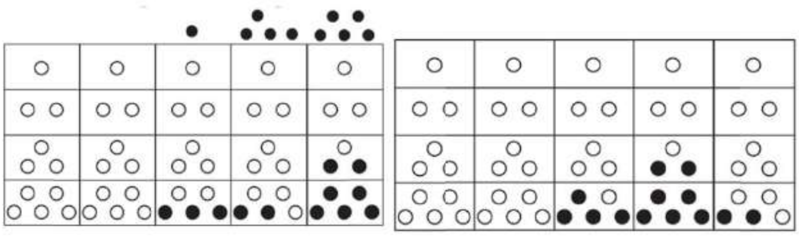


Figure 7. Sum with regrouping, 327 + 145 (González & Caraballo, 2015)

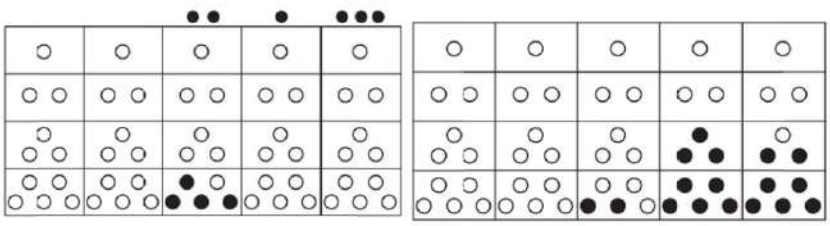


Figure 8. Subtraction with decomposing, 500 – 213 (González & Caraballo, 2015)

3.3 Quipu

Quipu is a Quechua word meaning knot. The Quipu was an Inca writing system containing statistical information, demographic data, and also a record of information of abstract topics, such as political systems, myths, etc. (Zapata, 2010). The Incas were not an exclusively oral culture. They knew how to preserve historical memory through an annotation system with knots. The quipu is not a phonetic writing system. It is a writing system using visual or tactile signals containing meaning. It is a system of three-dimensional annotations that makes sense through the combination of knots, colors, shapes and rope twisting.

The societies that spoke languages with same root, as root of Indo-European languages, developed a common path towards phonetic writing, despite being mutually unintelligible. In those nearby territories, where languages from various roots interacted, was developed ideographic writing, because the signs did not represent sounds, but concepts- such as the knots in the case of quipu - and it allowed people from different languages to read without translation. Hence, the information recorded in the quipu could be interpreted by different quipucamayocs, despite they spoke different languages.

The Quipu is formed by a horizontal main rope. Several vertical hanging ropes are tied to the main rope. In the hanging ropes are the knots that contain information. In the mathematical quipu, numerical information was coded using the Inca number system. The numbers were represented according to the turns of the knots and according to the position of the knot in the hanging rope. The knots farther away from the main cord are units, as they rise, represent the tens, the hundreds, the thousands and the ten-thousands.

3.4 Application of tools of history of mathematics in the pre-service training

As has been previously discussed, several projects have been attempted since the advent of the IBE in 1978 to incorporate culturally-specific mathematical concepts and ancient tools into the primary education. The novelty of the present proposal is that these methods have been used by undergraduates of IBE to achieve basic learning of numbers and operations. Undergraduate students in the IBE Program of the Faculty of Education at the Cayetano Heredia Peruvian University, in 2015 school year, in the course Mathematics 1 and Ethnomathematics, learned to add, subtract and multiply with Yupana. They also studied the Quipu and its relationship to the Yupana.

The efficacy of these tools can be seen in videos where students can add, subtract and multiply, both in Spanish and in their native languages, Quechua, Aymara and Shipibo. The videos are posted on YouTube and can be seen in the following link. https://www.youtube.com/channel/UCgPeZpHDhIEpnIWgNM5-Csw/playlists The videos were uploaded in March 2015.

4 Tools of Dynamic Geometry

The focus of our research in dynamic geometry was designing a didactic proposal that incorporates the geometric designs in Amazonian and Andean culture as teaching tools.

In learning geometry, the research question posed was: How do we get that indigenous students use the fundamental concepts of geometry to build their cultural designs? For solve the problem posed is more advisable that geometric concepts appear in constructed designs, in a dynamic and interactive learning environment.

The Quechua, Aymara and Shipibo communities make designs using lines, segments, polygons, applying geometric transformations, all without a theoretical knowledge, having learned these techniques as part of the handed-down cultural knowledge passed from one generation to the next. The students gained a theoretical understanding of the geometrical concepts underlying their cultural art works through using Cabri II Plus software. Students built their Amazonian (Figure 9) and Andean (Figure 10) geometric designs using polygons, lines, segments, geometric transformations. This activity reinforced their geometrical knowledge in a meaningful way that allows it to take root in their minds as a practical tool and not just an intellectual abstraction. The designs can be seen on the following link https://www.flickr.com/photos/63485986@N02/sets/72157649003841112/

The process that was used to develop these patterns with Cabri can be found in the short course presented at the XIV Inter-American Conference on Mathematical Education (Bonilla, 2015).

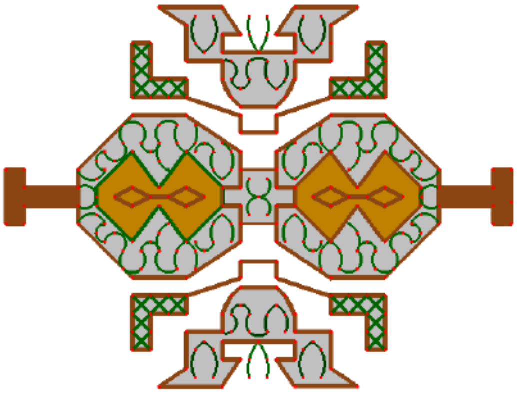


Figure 9. Shipibo design (Amazonian)

Figure 10. Quechua design (Andean)

5 Conclusions

At the end of the activities, tests were applied to measure numeracy skills in the development of basic operations, and to identify the knowledge and construction of the fundamentals of geometry, figures and geometric transformations. Students were able to increase their level of performance in the numeric domain and in the geometric domain with the proposed activities. The most important achievements are located in the affective and emotional field. The study of yupana and quipu, making the designs of their cultures, reaffirmed their identity, revaluing their cultural objects. The affirmation and approval of their work raises their self-esteem, injecting enthusiasm to develop new projects in their academic training, becoming aware of its potential.

REFERENCES

Arzarello, F., Bartolin, M., Lun, A., Mariotti, M., & Stevenson, I. (2012). Experimental approaches to theoretical thinking: Artefacts and proofs. In G. Hanna, & M. de Villiers (Eds.), *Proof and proving in mathematics education, The 19th ICMI study* (pp. 97-143). New York, USA: Springer.

Balacheff, N. (1994). Didactique et intelligence artificielle. *Recherches en didactique des mathematiques, 14*(1-2), 9-42.

Barbin, E. (2012a). L'histoire des mathematiques dans la formation : Une perspective historique (1975–2010). In J.-L. Dorier, & S. Coutat (Eds), *Enseignement des mathématiques et contrat social : enjeux et défis pour le 21e siècle – Actes du Colloque Espace Mathématique Francophone 2012* (GT4, pp. 546-554). Retrieved from http://www.emf2012.unige.ch/images/stories/pdf/Actes-EMF2012/Actes-EMF2012-GT4/GT4-pdf/EMF2012GT4BARBIN.pdf.

Barbin, E. (2012b). *What is epistemology (for)?* Powerpoint presentation. Colloque en hommage à Michèle Artigue. Paris.

Bonilla, M. (2015). Etnomatemática y Geometría Dinámica. *Proceeding of XIV Interamerican Conference on Mathematical Education.* Retrieved from http://xiv.ciaem-redumate.org/index.php/xiv_ciaem/xiv_ciaem/paper/viewFile/1479/729.

Cauty, A. (2001). Matemática y Lenguajes. ¿Cómo seguir amerindio y aprender de la matemática de la que se tiene y se tendrá necesidad en la vida? In A. Lizarzaburu, & G. Zapata (comps). *Pluriculturalidad y aprendizaje de las matemáticas en América Latina. Experiencias y desafíos* (pp. 49-87). Madrid: Ediciones Morata S. L.

Chirinos, D. (2010). *Los Quipus del Tahuantinsuyo*. Lima: Editorial Commentarios.

D'Enfert, R., Djebbar, A., & Radford, L. (2012) Dimensions historique et culturelle dans l'enseignement des mathématiques – Compte-rendu du Groupe de Travail n° 4. In J.-L. Dorier, S. Coutat (Eds.), *Enseignement des mathématiques et contrat social: Enjeux et défis pour le 21e siècle – Actes du colloque EMF2012* (GT4, pp. 523-528). Retrieved from http://www.emf2012.unige.ch/images/stories/pdf/Actes-EMF2012/Actes-EMF2012-GT4/EMF2012GT4CR.pdf.

Ellis, M., & Berry, R. (2005). The paradigm shift in mathematics education: Explanations and implications of reforming conceptions of teaching and learning. *The Mathematics Educator*, *15*(1), 7–17.

Ernest, P. (2005). Paradigm Shifts in Mathematics Education. In *The Mathematics Education into the 21st Century Project.* Malaysia: Universiti Teknologi. Retrieved from http://math.unipa.it/~grim/21_project/21_malasya_Ernest_05.pdf.

González, S., & Caraballo, J. (2015). Native american cultures tradition to whole number arithmetic. In X. Sun, B. Kaur, & J. Novotná (Eds.), *Conference Proceedings of The Twenty-third Study: Primary Mathematics Study on Whole Numbers* (pp. 90-98). Macau, China: University of Macau.

Jankvist, U. (2009). *Using history as a 'goal' in mathematics education*. [Ph.D. dissertation]. Department of Science, Systems and Models, Roskilde University, Roskilde, Denmark. Retrieved from http://milne.ruc.dk/ImfufaTekster/pdf/464.pdf.

Mejía, J. (2011). *Programa de operaciones aritméticas con base en la Yupana en el cálculo aritmético en estudiantes del primero de secundaria del Callao*. Tesis para optar el grado de Maestro en Educación con Mención en Psicopedagogía. Lima, Perú: Universidad San Ignacio de Loyola. Retrieved from http://repositorio.usil.edu.pe/wp-content/uploads/2014/07/2011_Mej%C3%ADa_Programa-de-operaciones-aritm%C3%A9ticas-con-base-en-la-yupana-en-el-c%C3%A1lculo-aritm%C3%A9tico-en-estudiantes-del-primero-de-secundaria-del-Callao.pdf

Ministerio de Educación de Perú. (2013). *Matemáticas en Educación Intercultural Bilingüe. Orientaciones pedagógicas*. Lima, Perú: Corporación Gráfica Navarrete.

Montalvo, J. (2012). Educational digital recycling: Design of videogame based on "Inca Abacus". In Dr Ioannis Deliyannis (Ed.), *InTech, open science open mind.* Retrieved from: http://www.intechopen.com/books/interactive-multimedia/educational-digital-recycling-design-of-videogamebased-on-inca-abacus-.

Oficina de Medición de la Calidad del Aprendizaje – Ministerio de Educación del Perú. (2015). *Resultados de la Evaluación Censal de Estudiantes 2014* (ECE 2014). Retrieved from http://www.minedu.gob.pe/opyc/files/presentacionresultadosECE2014.pdf.

Oliveras, M. L. (2015). El pensamiento creativo, la crítica y la comunicación en el ICEm5. *Revista Latinoamericana de Etnomatemática, 8*(2), 4-10.

Villavicencio, M. (2011). Las etnomatemáticas en la educación intercultural bilingüe de Perú: Avances y cuestiones a responder. *Proceedings of the XIII Inter-american Conference on Mathematical Education*. Retrieved from http://www.gente.eti.br/lematec/CDS/XIIICIAEM/artigos/2067.pdf.

Zapata, A. (productor). (2010). *Los Quipus [Sucedió en el Perú*]. Lima: TVPerú Televisión Peruana. Retrieved from https://www.youtube.com/watch?v=0slrSPT3_ws.

ARCHIMEDES AND VISUAL REASONING – HOW A GEOMETRIC GAME LED TO COMBINATORICS

George BOOKER
Adjunct Lecturer, Griffith University, Nathan Qld 4111, Australia
george.booker@optusnet.com.au

ABSTRACT

An exploration of the *Stomachion* dissection puzzle provides an extension to the geometric activities schools introduce with tangrams, using the pieces to make shapes that were reported in the game that aroused Archimedes attention. The puzzle can also be used to investigate rotations, symmetry, reflections, angles and areas. Replacing the pieces into the square in which they are contained leads to a discussion of the different ways that this can be done, the problem that Archimedes investigated. A concluding discussion will focus on the combinatorial analysis of this, drawing on research that accompanied the decoding of the Archimedes Codex.

1 Archimedes

Archimedes was the greatest mathematician of the ancient world, one of the greatest thinkers who ever lived. He approximated the value of π, developed the theory of centres of gravity, and (used this) to make steps towards the development of the calculus. (Netz & Noel, 2007

Archimedes was born, lived his life and died in Syracuse at the time the most powerful city-state in the Mediterranean. Through visits to Alexandria he came into contact with the Euclideans and Eratosthenes, the director of the famous library at that time, and kept up a continuous series of letters to these mathematicians that provide glimpses into his ways of working (*Arkimedeion catalogue*, 2011). Unfortunately killed in 212 BCE during a battle with Roman forces, his work known as *the method* led to two guiding principles for modern mathematics and science:

- the mathematics of infinity
- the application of mathematical models to the real world

Indeed, Galileo, Liebniz, Huygens, Fermat, Descartes and Newton all drew on these principles and Archimedes ways of working in a theoretical light, using the method of indirect proof or proof by induction and contradiction. His analysis of conic sections, spheres and cylinders, squaring the circle and parabola as well his practical inventions such as the Archimedes screw and the uses of parabolic mirrors are remarkably well illustrated in an interactive museum, the *Arkimedeion*, opened in Syracuse in 2011.

However, one aspect of his work is only hinted at there – his investigation of a 14 piece puzzle known as the *Stomachion*. Only recently has the mathematics behind his interest come to light in the painstaking uncovering of the palimpsest now known as the Archimedes Codex. An investigation of this puzzle and the way in which it led to profound mathematical thinking will be the focus of this presentation.

1.1 Visual reasoning

Archimedes thinking relied on the visual, using schematic diagrams that represent topological features of a geometrical object. In this, ideas and arguments are as well represented by a diagram as they are by language or equations, forming a valid part of the argument used in the mathematics of Archimedes time without the danger of errors based on visual evidence (Netz & Noel, 2007, p. 104).

In the method as it has been known and accepted for a long time, Archimedes used a notion of 'potential infinity' which avoided consideration of actual infinity utilised since the 19th century. A match between lines representing cuts in shapes such as parabolas, spheres and cylinders (Berggren, 2008) were made as big or as small as wanted, and the same thinking applied to any left over sections indefinitely, thus exhausting all the possibilities without referring to an actual notion of infinity. In this way, ratios among the lines could be determined, and lead to similar relationships among the shapes represented among the lines 'contained' within them. In particular, combining principles of mathematics and physics, Archimedes was able to reason that if a cone was contained in a hemishere which in turn was enclosed in a right circular cylinder, then the volume of the three solids are in the ratio 1:2:3 (Swetz, 1994, p. 180). In turn, this result led him to determine the volume of a sphere by considering a cylinder that would just enclose the sphere – the volume of the sphere was 2 thirds of the volume of the cylinder (in our terms, $4/3\pi r^3$).

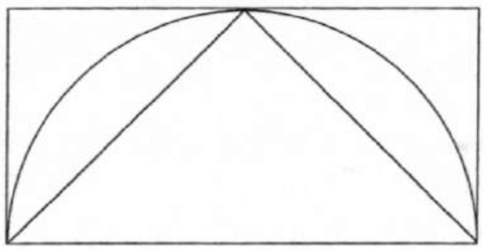


Figure 1. Slice of a cone, hemishere and cylinder used by Archimedes

However, the surprise in store for Noel along with his colleague Saito as they deciphered more of the lost codex was that Archimedes had gone beyond his original method to look at reults that were equal in magnitude. In other words, he had set up a one to one correspondence between the two infinite sets of magnitudes that he used, the same manner as the concept was structured in the 19th Century when set theory and the calculus were at last put on a firm foundation.

2 Stomachion

When the first, incomplete attempt to decipher Codex C, now known as the Archimedes Codex, by the philologist Heiberg in the early 1900s, the section relating to the Stomachion puzzle was not able to be read and the writing largely ignored. It was thought that Archimedes might be trying to determine the number of ways a particular shape might be made using some of the 14 pieces that made up the puzzle, but this seemed a curious past time, not a mathematical pursuit (most shapes could be made in many ways, perhaps an unlimited number of arrangements)

.

One of the most interesting shapes presented in Figure 2 above is the elephant which was probably the most popular among the Roman citizens at least who took up a keen interest in the puzzle. Not only was the shape an engaging one to construct, the fact that Syracuse had switched its alliance to Carthage from Rome, perhaps believing that Hannibal with his elephants that had come so close to conquering Rome was to be the new dominant power in the Mediterranean. Unfortunately, as Archimedes found to the cost of his own life, neither the power of the Carthiginians nor the formidable weapons constructed under his instruction and knowledge of mechanics was sufficient to keep the Roman army under Marcellus from taking Syracuse.

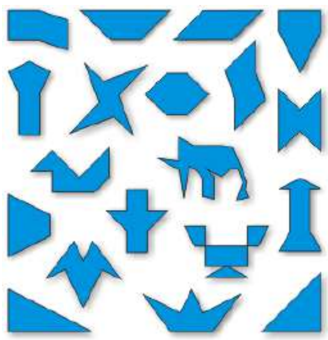


Figure 2. Arrangements that can be formed using some or all of the 14 Stomachion pieces (*NCTM Illuminations*, 2015)

Other investigations can be made with the puzzle. For instance it has been observed that when the puzzle is formed on a grid, the vertices of all pieces meet on the intersection of the vertical and horizontal lines which then gives rise to the area of each piece:

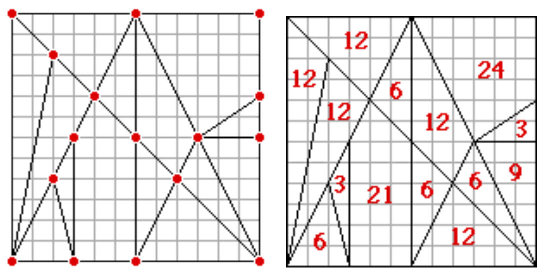


Figure 3. These can be further sinplified by noting each is a multiple of 3 (Archimedes Square, Kadon, 2004).

It is perhaps this unique construction among dissection puzzles that led to Archimedes interest in the geometric properties inherent in it.

2.1 Playing with the Stomachion puzzle

The pieces can also be rearranged to make various polygons in different ways, for example:

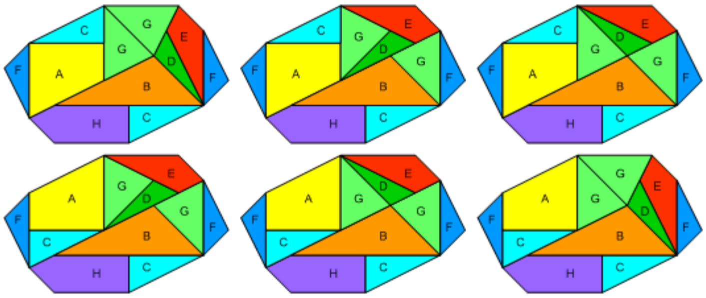


Figure 4. Examples by Pitici (2008)

After using the pieces to make different shapes, it is time to put them away and this is where we now know that Archimees became very interested in the problem this presents. It turns out that the pieces can be packed back into the original square in a great many different ways although this is indeed a finite number. There is as yet no report of the result that Archimedes found but the investigators of the Archimedes Palimpsest set a problem to mathematicians to solve and a total of 17 153 possibilities were determined, but when rotations and reflections were considered, this reduced to 536 unique solutions found by Bill Cutler a PhD graduate from Cornell University who set up a computer model to determine the result.

Figure 5. Some of the arrangements made by Cutler (2004)

This result was found independently by Fan Chang and her husband Ron Graham, mathematicians who used complex model theory for their result. They also noted that 3 pairs had to be always placed together, thus reducing the puzzle to one with a smaller number of pieces they called a stomach (Fan Chang & Graham, 2004)

While the determination of the number of possibilities is a problem for mathematicians working in the field of graph theory, it could be used as a motivation to explore combinatorics among students of all ages and provide colour and interest to a topic that often seems simply an application of number ideas.

REFERENCES

Arkimedeon Catalogue (2011). *The keys to the thinking of Archimedes.* Mannelli, Italy: Editinera.

Berggen, J. L. (2008). Book Review: The Archimedes Codex. *Notices of the American Mathematical Society, September,* 943-947.

Fan Chang, & Graham R. (2004). A tour of Archimedes' *Stomachion*. Retrieved in June 20 2016 from www.math.ucsd.edu/~**fan**/stomach/**tour**/dis.html

Kadon Enterprises (2004). The life of games Issue no 3, see *Archimedes Square.* Retrieved in June 20 2016 from http://www.gamepuzzles.com/

Netz, R., & Noel, W. (2007). *The Archimedes Codex.* London, UK: Weidenfeld & Nicolson.

Pitici, M. (2008). Geometric Dissections: Archimedes *Stomachion.* Retrived in June 20 2016 from http://www.math.cornell.edu/~mec/GeometricDissections/1.2%20Archimedes%20Stomachion.html

Swetz, F. (Ed) (1994). *From five fingers to infinity: A journey through the history of mathematics*. Peru, IL: Open Court.
Zordak, S. (2015). *Archimedes Puzzle. NCTM Illuminations October*. Reston, VA: National Council of Teachers of Mathematics.

ENDNOTE. A fascinating account of the unravelling of the Archimedes Codex presented by Willian Noel can be viewed at: www.ted.com/talks/william_noel_revealing_the_lost_codex_of_archimedes?language=en Retrieved in June 20 2016

MATH HIGH SCHOOL: A TEACHING PROPOSAL

Roberto CAPONE, Umberto DELLO IACONO, F. Saverio TORTORIELLO, Giovanni VINCENZI
University of Salerno, Italy
rcapone@unisa.it
udelloiacono@unisa.it
fstortoriello@unisa.it
vincenzi@unisa.it

ABSTRACT

The purpose of this paper is to describe Math High School (MHS), an experimental teaching project promoted by the research group of Mathematics Education at the University of Salerno. MHS was devised to surpass the dichotomy between scientific and humanistic cultures, responding to the daily, social, politic, national and world challenges. In this process, Mathematics is the common denominator between the two cultures. In this paper we describe the theoretical context of MHS, the educational innovations of the methodologies and the first year's experimentation activities.

1 Introduction

Math High School (MHS) project has its doctrinal roots on postmodern philosophical ideas in mathematics education; it embraces the educational program associated to the theory of complexity of E. Morin. It consists of additional in depth lectures with respect to regular school mathematics classes, aimed at expanding the educational level of the students in order to improve the development of critical skills and aptitude to scientific research (Morin, 1993).

MHS is based on nietzschian philosophical theories, with the belief of a school education tending to enlarge as much as possible the student knowledge, in order to develop critical abilities and the aptitude for scientific research. Nietzsche believed that a good humanistic culture does not prevent the scientific knowledge and vice versa. He hypothesized the creation of a free school where more creative, but therefore weaker, spirits, can find a space to develop their potentialities and art, Nietzsche (1872). Thus, the school in general and Mathematics in particular must offer the opportunity to human being of understanding both universal external world and its most interior part. In particular, he hypothesizes the birth of a organic structure, the "Gaia Scienza", where the art melts with the science through Mathematics, which becomes the connection between the scientific "will to truth" and the artistic desire of illusion, Tortoriello (2015).

One of the main ideas of MHS is the belief in investing in education by competences, aiming to promote a coordinate system of knowledge and abilities moved by the subject for a purpose (a task, a set of tasks or an action) supporting a good internal motivational and sentimental aptitude (Pellerey, 2003). As mentioned by D'Amore, competences cannot be reduced to a single subject; they presume and create knowledge and suggest new uses and masteries, i.e. "competences generate competences" (D'Amore, 2000). The objectives are to develop in students essential basic competences in a cultural education of citizens responding to ethic and social needs: suppose and solve problems, design and build models of real situations, adequately express information, grasp and imagine, make connections between different kinds of knowledge. The idea is of supplying contents usefully applicable also and especially outside the school's world, in daily life, as citizens rather than students: "competences must be an experience for the citizen (as abilities to solve problematic situations rather than knowledge, being able to choose proper resources, strategies and ways of thinking) (Arzanello, Robutti, 2002). Thus, it is needed to detect important contents constituting the basic core and then surround them by other supplementary contents. In Mathematics, besides the transmission of contents, it is needed to manage an aware and active reworked version of them, driven by motivation and volition, in order to allow their use and interpretation in problematic situations and to master the connections between different contents. When the student goes out of the usual classroom life, making connection between different pieces of knowledge, the idea of passing from mere knowledge to competence arises (Sbaragli, 2011).

Within the school of the competences, the educational purposes assume a social relevance: students must get the aptitude to organizing the knowledge. Not only is the teacher responsible of a correct acquisition from the students, but also of "teaching" involving emotional and motivational aspects. Moreover, he has the task of "educate": in this way, the school becomes education of human condition, apprenticeship to life, apprenticeship to uncertainty, education to European and global citizenry (Morìn, 2000). Education by competences aims to surpass the fragmented knowledge of multidisciplinarity on the side of pluridisciplinarity, which means that the same topic is proposed from the point of view of various disciplines through a planning by disciplinary environments. The transfert of knowledge in a multi-objective education by competences is realized through an education which surpasses also the pluridisciplinarity with interdisciplinarity. Nevertheless, the actual education revolution is realizable only with a transdisciplinary setting. As mentioned by Mauro Laeng, the term "transdisciplinary" describes the "interdisciplinarity in the strong sense", because at this level "the actual surpass of an epistemological barrier with the discovery of a new unite horizon" happens (Laeng, 1992).

An education by competences expects a bind between humanistic and scientific cultures, finding the most natural contextualization in historical research of mathematical contents and in their revival under a modern interpretation, following the last guidelines of the Italian high school reform. On the definition of a common European framework of qualifications and scholastic competences (G.U.E., 2008), the reform indicates in the connection between "scientific and humanistic cultures, …, through laboratorial practice" one of the main features of new lyceums (Miur, 2010): "students should have numerous and varied experiences related to the cultural, historical and scientific evolution of mathematics so that they can appreciate the role of mathematics in the development of our contemporary society and explore relationships among mathematics and the disciplines it serves: the physical and life sciences, the social sciences, and the humanities. It is the intent of this goal-learning to value mathematics - to focus attention on the need for student awareness of the interaction between mathematics and the historical situations from which it is developed and the impact that interaction has on our culture and our lives" (Swetz, 1995). Like Swetz suggests, we have done experiments in the measurement of PI, using Eratosthenes technique to obtain the circumference of the earth, Greek ruler and compass construction, and algebraic equations. The benefit of any historical math is that there is a great opportunity to reinforce skills and provide opportunity for enrichment.

MHS was born from cultural and social motivations. If the cultural roots of this educational path are fed by theory of complexity, the social motivations need have to overtake the cultural gap between Italy and other European states, as in the statement of the OECD-PISA surveys results. Indeed, the competences of Italian teenagers in Mathematics are under the average of OCSE. In particular, the results in Mathematics are worse than the total average of the south of Italy, in general characterized also for a more internal variability of results: over the national average north west and north east are collocated, and the centre of Italy is on the mean. Thus, the results of south of Italy and islands are lower than the national average. The results form national INVALSI survey are quite similar to OECD ones. Indeed, "both for Mathematics and Literature, the rank of the single regions emerging from two sources is similar. Also coherent is the position of single school which have participated to both surveys" (OCSE-PISA Rapporto Nazionale, 2012). Thus, there is the immediate need of a revision in the entire national territory, but in particular in the south of Italy where the results have been worse. The general aim is to supply to students the tools to deal with the complexity of the current society. On the one hand, it is necessary to surpass the fragmented knowledge in the same way as it was requested from the Gentile's reform of 1923. On the other hand, it is needed to increase the Mathematical Literacy, that "is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens" (Draft Mathematical Frameworks, 2015).

2 Math High School (MHS)

MHS is a research and educational project, devised by the research group of Mathematics Education at University of Salerno. The schools adhering to the project have made an agreement with the Department of Mathematic, to find the experts covering the supplementary courses in mathematics, as expected by the proposed theoretical framework. It is composed of additional advanced courses of Mathematics and other disciplines to it related. The activities are programmed in the five years of high school. In particular, 40 hours are expected for the students of the first year, 60 hours for those of the second, 70 hours the third, 80 hours for the fourth and 80 hours for the fifth year. The courses occur at single institutes and in a span of about seven months, from November to May, with one lesson per week.

We structured each educative module in order to improve some Mathematics competences.

The time schedule is the following (Table 1).

Table 1. Time schedule and subject

Subject	1st year	2nd year	3rd year	4th year	5th year
Mathematics and Italian Literature	0	5	15	15	15
Mathematics	10	10	10	15	15
Physics	10	10	5	10	10
Mathematics and Philosophy	0	0	10	10	10
Logic	20	25	10	10	10
Mathematics and History	0	5	10	10	10
Mathematics and Chemistry	0	0	5	5	5
Mathematics and Biology	0	5	5	5	5
TOTAL	*40*	*60*	*70*	*80*	*80*

3 Methodology

The MHS educative activities are inspired by constructivist learning. It is based on the active participation of the students to problem solving and it aims to the development of a critic thinking. The students "build" their own knowledge starting from a test and exploit their knowledge and previous experiences applying these new situations and adding intellectual constructs. In this constructivist education context, it has been also proposed, in some learning object, a flipped teaching (Hamdan et al., 2013) as a model of experimentation of the future classroom, using a revolution of the traditional methodologies, flipping the old system: an explanation time in classroom from the teacher, an individual study at home and a phase of test in classroom again. In this way, the time spent at school is more functional and productive to the teaching-learning process, investing the teaching hours to solve most complex problems, deepen topics, connect themes and analyzing the disciplinary contexts, produce team work in peer to peer mode in a laboratory driven context. The teacher becomes a driver and tutor, supplying his assistance in classroom to students in order to stress significant observations and considerations, through the use of exercises, shared researches and learning by doing.

Within the activated learning modules, the educative methodologies have been sharpened to cope with the objectives. We focused on the language and the skills improvement of students in debating, in order to ease the passage from an informal register to an advanced one (Ferrari, 2004). As observed by Bruner, for Vygotskij and Dewey every type of language is a way to self-manage our thoughts about reality and the thinking is a way to organize the perception and the actions (Bruner, 2005). In each proposed activity, the history of Mathematics played an important role and it has been applied in several fields: read aloud mathematical stories to the class, students write about mathematical history topics, students perform plays and skits or make videos about historical topics, hands-on experiences, through the arts, through the visual arts (Reimer and Reimer, 1995). Students develop and refine estimation strategies and develop an understanding of when to use algorithms and when to use calculators. Students learn when exact answers are appropriate and when, as in many life experiences, estimates are equally appropriate. The involved artefacts have been meant as both technical instruments and psychological instruments, that are "semiotics mediation instruments" (Bartolini et al., 2006). We did not omitted the "centrality of the role of operator, of the person building its own concepts on the base of knowledge not deriving from an eventful discovery, but from thoughts, buildings, aware and voluntary activities of other people" (Boero & Garuti, 1994). Students learn to make sense of the mathematical tools they use by making valid judgments of the reasonable of answers. Students reinforce skills with whole numbers, fractions and decimals through problem solving and application activities.

4 First Experimentations

MHS project has been experimented in academic year 2014/2015, with 14-15 years old students, attending the first year of Liceo Scientifico "P.S. Mancini" in Avellino, Liceo Scientifico "P. P. Parzanese" in Ariano Irpino (AV) and Istituto Superiore "A. Gatto" in Agropoli (SA). The activated modules since now have been three: Mathematics, Logic, Physics.

4.1 Mathematics Module

The Mathematics module has been meant as improvement of basic Mathematics knowledge, with particular reference to geometry and fundaments of Mathematics:

- Euclidean geometry insights: Napoleon's Theorem, Torricelli configuration, Euler line, continuous triangles, pedal triangles and pedal quadrilaterals, orthic triangles and orthic quadrilaterals, Van Aubel's Theorem, analysis of the characteristic processes of human thought (axiomatizations, definitions, proofs, generalizations), equality between geometric figures.
- numeric sets N, Z, Q, R: geometric model;
- basics in arithmetic: the principle of mathematical induction, recursive sequences, modular arithmetic, tests of divisibility, the finished structures; binomial coefficients; secrets of Pascal's triangle.

Some contents of geometry, such as those related to the construction with ruler and compass and other mathematical machines, were analyzed through a historical investigation and re-contextualized, also with the use of dynamic geometry software.

4.2 Logic Module

The Logic learning module was organized focusing on the historical path of logic-mathematics birth, from the origins to the present. With this aim, every content of the course has been historically contextualized. The relation between the language and the development of logic abilities. We noticed that the level of linguistic competence influenced the problem solving since first interventions. In a vygotskian picture, we managed to analyze how the language supported the development of a locic thinking (Vygotskij, 1934). The study of competences of problem solving opens a window on the people capacity in cognitive activities as basis for other cognitive approaches of general order, to face the life challenges (Lesh and Zaqojewski, 2007). In agreement with PISA 2012, we considered the "competence in problem solving" as the ability of an individual of starting cognitive processes to understand and solve problematic situations for witch the solution is not clear from the beginning. This competence is comprehensive of the own will to face with those situations to realize own potentialities as thinking citizens and with a constructive role (OCDE, 2013). In order to develop the ability to solve problems, logic surveys have been proposed to students, organized in cooperative teams. To each group, we assigned the task of answering to the survey, agree on and motivate a final group answer. Being the problem solving competence dependent from knowledge and specific strategies by topic (Mayer, 1192; Funke and Frensch, 2007), we proposed surveys to students whose solution did not required previous knowledge.

4.3 Physics Module

In the Physics module, the teaching was realized in laboratorial form, favoring the heuristic approach through an experimental-inductive method. Inquiry Based Science Education (IBSE) was the model, i.e. a teaching based on the investigation of problems, critical group discussion and search for new solutions in a constructivist perspective. We have proposed to students a problem from the observation of reality and asked them to identify it by formulating hypotheses (engage).

After that, they had to plan the survey exploring the variables (explore), they led the survey individually or in groups by documenting the results (explain). Then, together with the teacher who served as scaffolder in all the activities, the results (evaluate) and communicated by formulating new problems have been interpreted (extend). This investigative path of reality has encouraged the development of problem posing and problem solving skills. The path has also involved several aspects: psychological, perceptual, language and practical. Finally, we have always tried to activate the argumentation and conjecture processes to facilitate the transition from intuitive notions and operating levels to forms of deductive thinking and to abstract or virtual levels.

4 Conclusions and Future Work

From the feedback of customer care questionnaires performed by the students and teachers involved, it is clear the high level of satisfaction, pushing us to continue these activities. The education project of MHS collected approval of principal Italian Mathematics associations and caused interest in institutional authorities at both local and national levels. The school environment welcomed it with enthusiasm so that the next year all over Italy some other high schools are starting. The next year activities will be enriched with an e-learning platform in order to support flipped teaching. Furthermore, already two high schools will install a laboratory of mathematics machines. On the one hand, the involvement of these machines will be intensified as an instrument of semiotic mediation inside the learning process. On the other hand, there will be an attempt to improve the linguistic skills. We will try to create a permanent pair between internalization of thoughts through the improvement of a practical intelligence and the exteriorization process of thoughts through the improvement of the language, intended as a multimodal (including oral text, symbolic expressions and figurative representations) and multivariate (including a wide range of registers) system.

REFERENCES

Arzarello F., & Robutti, O. (2002). *Matematica*. Brescia: La Scuola.

Bartolini Bussi, M. G., & Mariotti, M. A. (2009). Mediazione semiotica nella didattica della matematica: Artefatti e segni nella tradizione di Vygotskij.. *L'Insegnamento della Matematica e delle Scienze Integrate*, *32*, 270-294.

Bartolini Bussi, M. G., & Maschietto, M. (2006). Gli strumenti meccanici: Le macchine per tracciare curve e realizzare trasformazioni. *Macchine matematiche: dalla storia alla scuola*, 1-32.

Boero, P., & Garuti, R. (1994). Approaching rational geometry: From physical relationships to conditional statements. In J. P. da Ponte, & J. F. Matos (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, *Lisboa* (Vol. 2, pp. 96-103). Lisbon: Departamento de Educaçao, Faculdade de Ciências da Universidade de Lisboa.

Bruner, J. (2005). *La mente a più dimensioni*. Laterza, Roma-Bari (*Actual minds, possibile worlds)*. Cambridge-London 1986: Harvard University Press.

D'Amore, B. (2000). La didattica della Matematica alla svolta del millennio: Radici, collegamenti e interessi. *La Matematica e la sua Didattica 3*, 407-422.

Ferrari, P. L. (2004). Mathematical Language and Advanced Mathematics Learning. In M. Johnsen Høines, & A. Berit Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, *Bergen* (Vol. 2, pp. 383-390). Bergen: Bergen University College.

Funke, J., & Frensch, P. A. (2007). "Complex problem solving: The European perspective – 10 years after". In D. H. Jonassen (Ed.), *Learning to Solve Complex Scientific Problems* (pp. 25-47). New York, NY: Lawrence Erlbaum.

G.U.E. (2008). *Raccomandazione del Parlamento Europeo e del Consiglio del 23 aprile 2008 sulla costituzione del Quadro europeo delle qualifiche per l'apprendimento permanente* (Testo rilevante ai fini del SEE) (2008/C 111/01).

Hamdan, N., McKnight, P., McKnight, K., & Arfstrom, K. M. (2013). *A review of flipped learning*. Retrieved from http://www.flippedlearning.org/cms/lib07/VA01923112/Centricity/Domain/41/LitReview_FlippedLearning.pdf.

Lesh, R., & Zawojewski, J. S. (2007). Problem solving and modeling. In F. Lester (Ed.), *The handbook of research on mathematics teaching and learning* (2nd ed.) (pp. 763-804). Charlotte, NC: National Council of Teachers of Mathematics, Reston, Virginia, and Information Age Publishing, (joint publication).

Laeng, M. (1992). *Pedagogia sperimentale*. Scandicci: Nuova Italia.

Mayer, R. E. (1992). *Thinking, Problem solving, Cognition* (2nd ed.). New York: Freeman.

MIUR (2010). Art.8 del *Regolamento recante "Revisione dell'assetto ordinamentale, organizzativo e didattico dei licei ai sensi dell'art. 64, comma 4, del d.l. 25 giugno 2008, n. 112, conv. dalla legge 6 agosto 2008, n. 133"*.

Morin, E. (1993). *Introduzione al pensiero complesso: Gli strumenti per affrontare la sfida della complessità*. Milano: *Sperling & Kupfer*.

Morin, E. (2000). La testa ben fatta. In *Riforma dell'insegnamento e riforma del pensiero* (pp. 115-117). Milano: Raffaello Cortina Editore.

Nietzsche, F. (1975). *Sull'avvenire delle nostre scuole* (Traduzione di G. Colli). Milano: Piccola Biblioteca Adelphi, 7ª ediz.

OCSE PISA 2012, RAPPORTO NAZIONALE (a cura di INVALSI) http://www.invalsi.it/invalsi/ri/pisa2012/rappnaz/Rapporto_NAZIONALE_OCSE_PISA2012.pdf

OECD (2013). *PISA 2012 assessment and analytical framework: Mathematics, reading, science, problem solving and financial literacy*. OECD Publishing.

Pellerey, M. (2005). Verso una nuova metodologia di ricerca educativa: La Ricerca basata su progetti (Design-Based Research). *Orientamenti pedagogici, 52*(5), 721-737.

Sbaragli, S. (2011). Le competenze nell'ambito della matematica. *Difficoltà in matematica*, *7*(2), 143-156.

Swetz, F. (1995). To know and to teach: Mathematical pedagogy from a historical context. *Educational Studies in Mathematics*, *29*, 73-88.

Tortoriello, F. S. (2015). Nietzsche e la matematica dionisiaca. *Archimede, 67*(2)*,* 82-87.
Vygotskij, L. S. (1934). *Thougth and language*. Moscow-Leningrad: Sozekgiz.

A TEXTBOOK IN PLANE AND SPHERICAL TRIGONOMETRY FROM 1834

Andreas CHRISTIANSEN
University of Bergen, P.O. Box 7800, N–5020 Bergen, NORWAY
andreas.christiansen@hsh.no

Bernt Michael Holmboe (1795–1850) wrote most of the textbooks in mathematics that were used in the learned schools in Norway between 1825 and 1860, and he was a very influential person in the development of school mathematics in this period. He wrote textbooks in arithmetic, geometry and stereometry, trigonometry and higher mathematics. Most of Holmboe's textbooks came in several editions, but the textbook in plane and spherical trigonometry came in only one edition (Holmboe, 1834).

I will in my presentation of the textbook in trigonometry focus on Holmboe's understanding and presentation of trigonometric functions and magnitudes, and his use of the unit circle and of trigonometric lines. I will also give a comparison between this textbook and some contemporary textbooks in trigonometry, and with the current understanding of these concepts.

The only approach to the trigonometric functions, used in Norwegian schools today, is to find their values between $0°$ and $90°$ by studying right angled triangles, and later to expand the range to $360°$ by using the same way of thinking on a unit circle. The trigonometric value will then be a ratio of magnitudes. The term "*trigonometric lines*" is not used in the Norwegian schools today.

Holmboe's approach is different. He defines *trigonometry* as that part of geometry that may be used for "solving the triangle" – some of the six magnitudes of a triangle, three sides and three angles, are known, and the remaining can be found. Holmboe starts by defining *trigonometric lines* to an arc, and with this definition, lines in spherical geometry also have trigonometric lines. Trigonometric values calculated by ratios of sides in right angled triangles are results, deducted from the definitions, and presented in this textbook as a theorem with proof. I will in my presentation demonstrate the construction of all the trigonometric lines, and a method for calculating their values with required accuracy, with a specified number of correct decimals.

REFERENCES

Holmboe, B. M. (1834). *Plan og sphærisk Trigonometrie*. Christiania: C. L. Rosbaum.

LES RÈGLES *D'UN QUART ET UN VINGTIÈME* ET *DES COMPTES DE FLANDRE* COMME MODÉLISATION DU RÉEL

Costa Clain, TERESA
Universidade de Aveiro, CIDMA - Centro de Investigação e Desenvolvimento em Matemática e Aplicações,
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal, costa.jesus.teresa@ua.pt

RÉSUMÉ

La Maison de l'Inde et le comptoir de Flandre sont à l'origine de règles spécifiques dont on ne trouve pas d'équivalent dans les traités d'arithmétique des autres pays. La *règle d'un quart et un vingtième* correspond à un prélèvement d'un quart plus un vingtième des trois quarts restants, c'est-à-dire $\frac{1}{4}+\frac{1}{20}\times\frac{3}{4}=\frac{23}{80}$ de la quantité initiale. La *règle des comptes de Flandre* correspond à une formule de conversion. Dans cette session, nous proposons une introduction à ces règles spécifiques du commerce des épices au Portugal à l'époque de la Renaissance, en tenant compte de l'interaction entre les Mathématiques, l'Histoire et l'introduction de la mentalité quantitative pour la modélisation du réel.

1 Les traités d'arithmétique portugais du XVI^e

Le Portugal connut une période de forte prospérité et de grande activité commerciale durant le XVI^e siècle. Les découvertes maritimes furent le moteur d'un formidable accroissement du développement économique et ont placé Lisbonne au centre d'un vaste commerce international. Le développement d'activités commerciales mettant en jeu des sommes considérables, et l'introduction d'un impôt lié au commerce des épices motivèrent et favorisèrent l'utilisation de nouvelles pratiques arithmétiques plus efficaces, répondant de manière plus adéquate aux nécessités d'un négoce sans cesse plus complexe et structuré. On assiste à quelques exemples de modélisation arithmétique utilisés à cette époque, comme furent les calculs des impôts de la Maison de l'Inde (la règle d'un quart et un vingtième), la règle des compagnies pour l'exécution des négoces et la confluence de diverses formes de *cabedal*[1] à la poursuite d'une entreprise particulière (les règles de troc).

Ce fut précisément à ce moment qu'apparurent les premières œuvres de référence en arithmétique telles que le *Tratado da Pratica d'Arismetica* de Gaspar Nicolas, publié pour la première fois en 1519 ; la *Pratica d'Arismetica* de Ruy Mendes de 1540 ; le *Tratado da arte de Arismetica*, de Bento Fernandes de 1555. Issue d'un besoin de formation des marchands et de la nécessité de construire des institutions royales adaptées au commerce international et à sa gestion (perception de l'impôt par exemple), la publication de traités d'arithmétique représente

[1] *Cabedal* – Fonds financiers pour assurer le commerce de l'Orient. Parfois, le *cabedal* se composait d'or, d'argent et de cuivre, en plus de la monnaie en vigueur. D'autre part les lettres de change voire même les marchandises, ceci dans le cas spécifique du *troc*, étaient considérées comme faisant aussi partie du *cabedal* [Almeida 1994, vol. II, p. 294].

une étape importante pour le développement et la diffusion d'une pratique mathématique liée au monde des affaires au Portugal.

Des trois traités imprimés durant ce siècle au Portugal, le plus populaire fut sans aucun doute l'œuvre de Gaspar Nicolas qui connut onze éditions s'étalant sur plus d'un siècle alors que le traité de Ruy Mendes ne fut publié qu'une seule fois en 1540 par le même éditeur Germão Galharde[2].

À l'image des traités italiens, les ouvrages portugais s'articulent autour de sujets liés au négoce. Une constante commune à tous ces traités est la présence de règles spécifiques liées à la Maison de l'Inde et aux impôts résultant du commerce à grande échelle. Néanmoins d'autres aspects des mathématiques que l'on classifie de «classiques» ou «traditionnelles» comme les problèmes sur les nombres, les racines carrées et cubiques, les progressions, sont aussi présents dans toutes les publications. La persistance de ces thèmes, qui ne sont pas les objectifs principaux des traités montre l'importance que les différents auteurs leur accordent.

1.1 Les règles d'un quart et un vingtième et des comptes de Flandre

La Maison de l'Inde se révèle être une des pierres angulaires de l'organisation commerciale du Portugal (et plus particulièrement de Lisbonne) au XVI^{e} siècle. Elle est à l'origine de règles spécifiques que l'on rencontre uniquement dans les traités d'arithmétique marchande lusitaniens, comme nous l'avons déjà mentionné.

La principale source d'information que nous avons sur cette institution provient d'un manuscrit décrivant le règlement de la Maison de l'Inde dont un exemplaire se trouve à la Bibliothèque Centrale de la Marine et un autre à la Bibliothèque Nationale du Portugal. Une transcription de ce manuscrit fut réalisée par Damião Peres en 1947 [Peres 1947] qui en donne une reproduction exacte. La Maison de l'Inde fut créée en 1503 sous le règne de Dom Manuel I afin de gérer le commerce international avec l'Orient et garantir un monopole sur le transit des marchandises en faveur du roi. Elle englobe aussi deux autres institutions : la Maison de Guinée et la Maison de Mina, liées au commerce avec la côte ouest de l'Afrique. La Maison de l'Inde avait pour mission de réaliser la manutention, le stockage et la vente des marchandises venant de l'Orient vers le reste de l'Europe (principalement les épices). On y maintenait une comptabilité sur les achats et les ventes, en particulier on y calculait l'impôt (la règle d'un quart et un vingtième que le marchand devait au roi). La Maison de l'Inde était localisée sur la rive nord du Tage dans l'actuel *Paço da Ribeira* ; ses bâtiments étaient disposés perpendiculairement au fleuve.

L'accroissement rapide du commerce avec le port de Lisbonne et l'augmentation du volume de négoce qui en a découlé sont les principales motivations de la création de l'institution et de son règlement évoqués dès les premières pages :

> Considérant les grandes choses que sont nos comptoirs de Guinée et des Indes, Dieu soit loué, et les avantages et bénéfices qui en résultent pour notre royaume, notre peuple et les autres parties de la chrétienté, étant donné que nous devons travailler pour la bonne

[2] Germão Galharde était un typographe français installé à Lisbonne à partir de 1519. Il adopta le patronyme de German Galharde ou Germão Galharte et possédait des imprimeries à Lisbonne et Coimbra.

gestion, gouvernance et préservation de ces affaires, il nous apparaît que le négoce est une occupation de la plus haute importance...[3]

De fait, en raison des découvertes maritimes, les routes commerciales avec l'Inde se trouvent modifiées en faveur de Lisbonne et la route du Cap permet un transport de marchandises plus rapide, plus fiable, garant d'une meilleure qualité et surtout plus économique que le transit par la route traditionnelle de l'est via les pays du Moyen Orient et l'Italie. Lisbonne devient une des principales plateformes d'un commerce mondial en pleine expansion, justifiant la création d'un outil de gestion adapté.

Un des principaux objectifs de la Maison de l'Inde était la perception de l'impôt. Deux articles du règlement sont dédiés aux règles de calcul de cet impôt spécifique à Lisbonne. L'article 68.° règlemente la valeur de la taxe [Peres 1947, pp. 56-58], basée sur la règle d'un quart et un vingtième pour toutes les marchandises, tandis que l'article 71.° prévoit l'enregistrement et le contrôle de cette taxe. La règle d'un quart et un vingtième est une des spécificités du traité de Ruy Mendes. Elle est issue d'un calcul des impôts appliqué au Portugal aux marchandises provenant du commerce avec l'Orient. Elle correspond à un prélèvement d'un quart plus un vingtième des trois quarts restants, c'est-à-dire $\frac{1}{4}+\frac{1}{20}\times\frac{3}{4}=\frac{23}{80}$ de la quantité initiale.

Nous en donnons le principe tiré de l'ouvrage de Mendes: « Étant donnée une quantité, et connaissant le quart et les trois quarts de cette même quantité, quel est le vingtième de cette dernière quantité ? Et retranché de cette dernière quantité, combien reste-t-il ? »[4]

L'auteur propose ensuite un exemple concret : « Le quart de 155 *cruzados* et le vingtième de ses trois quarts cela fera combien? »[5]. Nous reproduisons la résolution du problème telle quelle dans le traité par Ruy Mendes, mais traduit en langage mathématique actuel.

Considérons les fractions $\frac{1}{4}$ et $\frac{1}{20}$. Multiplions 4 par 20, cela fait 80. Effectuons maintenant les produits suivants $\frac{1}{4}\times 80 = 20$ et $\frac{3}{4}\times 80 = 60$. Le vingtième de ces $\frac{3}{4}$ [de 80] est 3, et $20 + 3 = 23$, qui est $\frac{1}{4}$ de 80 et le vingtième de ces $\frac{3}{4}$ [de 80].

Relativement aux 155 cruzados est appliquée la règle de trois

80 *cruzados* ______________ 23 *cruzados*

155 *cruzados* ______________ x

---

[3] « Considerando nos quam grandes couzas sam os nossos trautos de Guiné e das Índias, a Deos louvores, y quãto proveito delles se segue a nossos Regnos, e naturais delles, y assi a outras partes da Christandade, e como somos obrigados a trabalhar, quanto em nos for, de as taes couzas serem sempre bem regidas e governadas e conservadas, parecendo nos que por ho negocio ser grande e de munta importancia e ocupação… » [Peres 1947, p.3].

[4] « *Tendo uma quantia, saber* $\frac{1}{4}$ *e* $\frac{3}{4}$ *da mesma, a vintena quantos será*? *E tirados dela quanto ficará*? » [Mendes 1540, f. 80].

[5] « O quarto de 155 cruzados e a vintena dos seus três quartos, quanto será? » [Mendes 1540, f. 80].

$x = 44$ *cruzados* 2 *tostões* 1 *vintém* 5 *reais*[6].

Les trois auteurs portugais établissent un modèle (un algorithme) facile à appliquer à une grande variété de problèmes. Ce sont ces questions que nous allons traiter en classe.

La règle des comptes de Flandre correspond à une règle de conversion. En complément à la Maison de l'Inde, le comptoir d'Anvers était à l'époque sous juridiction portugaise et était dédié à la distribution en Europe des produits venant de l'Orient. La monnaie en usage à Anvers (*livra* (livre), *soldo* (sou)*, dinheiro* (denier), *mita* (mite)) étant différente de celle du Portugal sur la place de Lisbonne, une conversion entre les deux systèmes a conduit à la définition de règles spécifiques regroupées sous le nom de règle des comptes de Flandre.

1.2 La mathématique et l'histoire pour les étudiants du lycée au cours de sciences humaines

Durant la première année du lycée, est enseigné le cours de "Mathématiques appliquées aux sciences sociales", destiné aux élèves de Sciences Humaines. Il est recommandé de promouvoir l'approfondissement des connaissances scientifiques, techniques et humanistes qui constituent un soutien cognitif et méthodologique pour une étude plus approfondie et pour l'intégration dans la vie active. Un des sujets que nous enseignons est lié aux différents impôts. Les étudiants sont familiers avec la règle de trois et il est prévu de travailler avec différentes situations de proportionnalité directe. Une circulaire très importante nous recommande de faire connaître quelques dates importantes de l'histoire des mathématiques et de les relier à des moments historiques de forts impacts culturels ou sociétals.

Durant cette même année et en association avec le cours d'Histoire, il est recommandé de pratiquer l'analyse de sources sur l'époque des découvertes portugaises lorsque le professeur aborde en classe ce thème, en particulier le rôle du Portugal dans la navigation au XVI^e^ siècle et le déroulement de la science en tenant compte la mathématisation du réel. Selon la méthodologie proposée, nous trouvons pertinent d'enseigner les «taxes» prévues au Portugal au XVI^e^ siècle. Cette approche est particulièrement riche. Les étudiants connaissent la mathématisation du réel à travers des œuvres de l'époque, par la lecture de textes anciens. Les traités portugais sont particulièrement riches en informations, comme par exemple, les préoccupations des marchands dans les cas où la marchandise est perdue et, en même temps comment la Maison de l'Inde va y répondre au travers de la taxe prélevée.

1.3 Un exemple d'une activité en classe

Nous présentons ici un exemple à proposer aux élèves.

« La mathématisation du réel à travers des traités d'arithmétique du XVI^e^ siècle: les règles d'un quart et un vingtième et des comptes de Flandre »

Publique: les élèves en première année du lycée

[6] Nous utilisons la notation actuelle de la règle de trois simple et la monnaie portugaise de l'époque. Les valeurs utilisées par l'auteur sont les suivantes :
$1\ cruzado = 4\ tostões, 1\ tostão = 5\ vinténs, 1\ vintém = 20\ reais$ [Mendes 1540, f. 7].

Résumé:

1. Qu'est-ce que la mathématisation du réel?
2. La Maison de l'Inde et le comptoir de Flandre.
3. Les impôts: Les règles d'un quart et un vingtième et des comptes de Flandre (analyse des textes et résolution de problèmes proposés par les arithméticiens du XVI^e^)

1. En Histoire sont étudiés certains aspects de la mathématisation du réel. Vitorino Magalhães Godinho rapporte l'éclosion d'un état d'esprit quantitatif qui sera progressivement répandu dans tous les secteurs de la société. Commencez par lire le texte 1.

Texte 1

« La formation d'une mentalité quantitative trouve son origine dans deux motivations. D'une part, la construction progressive d'un état moderne se substituant aux liens de dépendance féodale, en passant du momentané, de l'occasionel pour le durable et le permanent. Nous mentionnerons trois exemples : la mobilisation militaire pour la constitution d'une armée permanente (…), les taxes avec des impôts généraux et permanents substituant les rentes royales (…), la comptabilité de différents services ainsi que la comptabilité publique (…).

D'autre part, durant ces deux siècles [XVe et XVIe siècles], se développe et s'enracine une économie de marché, essentiellement monétaire, basée sur une production à vendre dans le but de tirer de l'argent et sur les applications monétaires pour gagner encore plus d'argent. Les agents de la vie économique vont penser chaque fois plus en terme de quantité, de prix, de coût, valeur et stock d'argent. »[7].

1.1. D'après la lecture du texte, souligner trois facteurs de la vie politique, économique, sociale ou culturelle, qui, selon Vitorino Magalhães Godinho, expliquent la déclaration d'un état d'esprit quantitatif en Europe dans les XVe et XVIe siècles.

1.2. Vitorino Magalhães Godinho fait référence aux impôts. Un des principaux objectifs de la Maison de l'Inde était la perception de l'impôt. Les trois arithméticiens du XVIe présentent dans leurs traités de nombreux problèmes pour enseigner aux marchands les règles sur les impôts. Par exemple, Ruy Mendes, a écrit:

Texte 2

[7] « A formação da mentalidade quantitativa prende-se a duas ordens de razões. Por um lado, é a progressiva construção do Estado moderno, substituindo os laços de dependência pessoal e passando do momentâneo, do ocasional para o duradouro, para o permanente.
Enumeremos três pontos: a mobilização militar para construção de exércitos permanentes (...); a tributação, com impostos gerais e permanentes, substituindo as rendas do domínio real (...); a contabilidade dos vários serviços e a contabilidade geral pública (...).
Por outro lado, durante estes dois séculos [séculos XV-XVI], desenvolve-se e enraíza a economia de mercado, basilarmente monetária, assente na produção para vender, na venda destinada a obter dinheiro (...) e na aplicação do dinheiro a fim de ganhar mais dinheiro. Quer dizer que os agentes da vida económica vão pensar cada vez mais em termos de quantidades, preços, custos, valores e stocks de moedas. » [Godinho 1963-1971, p. 31].

« Que ôter un quart et un vingtième, selon la pratique de la Maison de l'Inde, n'est rien d'autre sinon de savoir de combien, d'une certaine quantité, du quart et des trois quarts dont on ne prend que la vingtième quantité, sera retiré de cette quantité et combien restera t'il? »[8]

1.3. Expliquer quel est le modèle proposé par l'auteur en ce qui concerne l'impôt selon la règle décrite dans le texte. Quel est le modèle pour le marchand? Et pour le roi?

2. Un problème qui aborde le sujet de la perte d'une partie d'une cargaison de poivre, ce qui était à l'époque une situation habituelle, est le suivant:

« Un navire part des Indes avec 500 quintaux de poivre et, arrivant au Portugal, on déplore une perte de 6 pour cent de la marchandise. On demande, premièrement, combien de quintaux arrivent au Portugal. On demande ensuite quel est le quart et le vingtième des trois-quarts. Et retranché de la somme précédente, combien reste t-il? »[9]

2.1. Compléter le tableau ci-dessous.

Le texte	**Le langage mathématique**
Sur 500 quintaux de poivre il y a une perte de 6 pour cent.	
Combien de quintaux arrivent au Portugal?	
Quel est le quart et le vingtième des trois-quarts de ce qui arrive au Portugal?	
Et retranché d'eux, combien reste t-il?	

2.2. Les arithméticiens portugais mentionnent que la perte de marchandise correspond entre 6% et 12% de la marchandise totale. Soit x la quantité de marchandise et y la valeur de la perte.

2.2.1. Trouver une expression qui donne la perte de marchandise.

2.2.2. Trouver une expression donnant la quantité de marchandise soumise à l'impôt.

3. Nous allons maintenant aborder la deuxième règle locale : la règle des comptes de Flandre qui correspond à une règle de conversion. En complément à la Maison de l'Inde, le comptoir d'Anvers (Flandre) était à l'époque sous juridiction portugaise et était dédié à la distribution en Europe des produits venant de l'Orient. La monnaie en usage à Anvers (*livra* (livre), *soldo* (sou), *dinheiro* (denier), *mita* (mite)) étant différente de celle du Portugal sur la place de Lisbonne, une conversion entre les deux systèmes a conduit à la définition de règles spécifiques regroupées sous le nom de règle des comptes de

[8] « Que tirar quarto e vintena segundo se tira na Casa da Índia nã he outra cousa salvo saber de hũa certa cãtia ho quarto e dos tres quartos della mesma a vintena quantos sera e tirados della quantos ficarã'» [Mendes, 1540, f. 80 v].

[9] « Hũa nao partio da India com 500 quintaes de pimenta: e chegando a Portugal achouse nella de quebra a razam de 6 por cento: preguntase primeiramente com quantos quintaes chegou a Portugal e esto sabido preguntase mais o quarto deles e a vintena dos seus tres quartos quantos seram: e tirados deles mesmos quãtos ficará » [Mendes 1540, f. 82]. Les produits transportés pouvaient souffrir d'un mauvais conditionnement entrainant leur détérioration ou bien disparaissaient complètement à cause des actes de piraterie ou d'un naufrage. Selon les auteurs, à la Maison de l'Inde il faut payer l'impôt comme síl n'y avais pas de perde.

Flandre. Nous reproduisons un problème tel qu'il est donné dans le traité par Ruy Mendes.

« Supposons que un *arroba* de Flandres à 25 *arráteis* ou livres, comme il le nomme là bas, et qu'un homme souhaite vendre là bas 16 arroba de sucre à 5 deniers le *arrátel*. On pose la question: combien de livre va-t-il obtenir? »[10]

3.1. Compléter le tableau ci-dessous (prendre en consideration l'information de la figure 1).

Le texte	**Le langage mathématique**
L'*arrova* de Flandre correspond à 25 *arratéis* ou *livres*.	
Un homme souhaite vendre en Flandre 16 *arrovas* de sucre à 5 *dinheiros* l'*arratal*.	
Combien sera le coût en livre?	

Livre (***Livra***)	**Sou** (***Soldo***)	**Denier** (***Dinheiro***)	**Mite**[11] (***Mita***)
1	20	240	5760
1/20	1	12	288
1/240	1/12	1	24
1/5760	1/288	1/24	1

Figure 1. Table monétaire au comptoir de Flandre

3.2.Une autre activité en Flandre était l'échange. À ce sujet Bento Fernandes a écrit :

« Pourquoi certains marchands sur le prendre et le donner de l'argent en échange à Anvers pour payer à Medina del Campo ou dans une autre foire en Espagne ou prenant en Espagne et donnant à Anvers ne sont pas ni experts ni même expérimentés sur cette règle de compte comme le sont les flamands et les italiens qui sont plus habiles sur ce sujet. Et pour une chose très nécessaire aux négoces et aux marchands de faire ici declaration pour savoir de quelle manière doit-on faire des comptes similaires pour que le donner et le prendre ne soit pas sujet à tromperie. »[12]

---

[10] « Na qual digo primeiramẽte assim: pode por caso que arrova de frãdes tẽ 25 arratés ou livras como la se chamã e que hũ homẽ qr vẽder laa 16 arrovas de açucar a 5 dinheiros o arratal. Pregũta se quantas livras se montaria nelas » [Mendes 1540, f. 83].

[11] La Mite fut émise en Flandre dès 1418.

[12] « Porque algũs mercadores sobre ho tomar ou dar do dinheiro ha cãbio ẽ Inves pera pagar ẽ Medina del Cãpo ou ẽ outra qualquer feira d'Espanha ou tomado e dado ẽ Espanha pera lhe respõderẽ ẽ Inves nã sã tam espertos nẽ esprimẽtados nesta cõta como ho sã os framẽgos e italianos que andã mais corrẽtes neste cõtratar. E por ser cousa muy necessaria aos tratãtes e mercadores farei aqui declaraçã pera saber a maneira que se ha de ter no fazer de semelhãtes contas e no dar e tomas do dinheiro que nã sejais enganados. » [Fernandes, 1555, f. 41].

Selon Bento Fernandes, que pouvons-nous conclure au sujet de la formation des marchands nationaux? Faire une recherche sur l'enseignement dispensé à l'époque ainsi que sur les autres domaines de la science au Portugal au XVI^{e} siècle.

Remerciements

L'auteur remercie le professeur Américo Costa pour sa contribuition dans cette ativité, les professeurs Helmuth Malonek et Luís Castro du laboratoire CIDMA, pour leur soutien, à Éliane et Louis Clain pour une révision linguistique, à la *Fundação para a Ciência e a Tecnologia* pour son soutien financier et à l'*Escola Secundária de Caldas das Taipas* pour son soutien actif à la promotion de la recherche en histoire des Mathématiques.

This work was supported in part by the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013.

REFERENCES

Almeida, A. (1994). *Aritmética como descrição do real* (1519-1679), vol. II. Lisboa: Imprensa Nacional, Casa da Moeda.

Clain, T. C. (2015). *A Matemática e o comércio em Portugal através das obras de três aritméticos do século XVI: Gaspar Nicolas, Ruy Mendes e Bento Fernandes.* Aveiro: Universidade de Aveiro.

Clain, T. C. (2015). La Pratica d'Arismetica de Ruy Mendes dans le contexte des arithmétiques ibériques. *Revue d'histoires des mathématiques*, SMF, *21*(1)*,* 5-39.

Fernandes, B. (1555). *Tratado da arte de Arismetica.* Porto: Francisco Correia.

Godinho, V. M. (1963-1971). *Os descobrimentos e a economia mundial*. Vol. I. Lisboa: Editorial Presença.

Mendes, R. (1540). *Pratica d'Arismetica*. Lisboa: Germão Galharde.

Nicolas, G. (1519). *Tratado da pratica d'arimetica*. Lisboa: Germão Galharde.

Peres, D. (1947). *Regimento das Casas das Índias e Mina*. Coimbra: Universidade de Coimbra.

THE LIGHT PREFERS THE SHORTEST

Physics and Geometry about Shortest Path Problems from Heron to Fermat

Roberto CAPONE, Immacolata D'ACUNTO, Maria Rosaria DEL SORBO, Oriana FIORE
Università di Salerno, Fisciano, Italy
rcapone@unisa.it
Università di Salerno, Fisciano, Italy
idacunto@unisa.it
IS "L. Da Vinci", Poggiomarino, Italy
marodel@gmail.com
Liceo Statale "P.E. Imbriani", Avellino, Italy
orianafio@gmail.com

ABSTRACT

Our work is a detailed report about an educational experience of a group of students of the first two years of high school. The starting point is the classical problem of the "Seven Bridges of Königsberg", but the whole historical background of our inquiry spans from Heron to Fermat, on the thread of the shortest path problem. In particular, starting from physical phenomena easily experienced in the daily life, the Fermat's least time principle is introduced. The behavior of the light is described in a geometrical optic approximation, in analogy with kinematics. The activities, based on laboratory teaching and learning, are focused on linking physics and mathematics: refraction and reflection experiments, compass-and-straightedge constructions and interactive open source geometry software, as Geogebra; recognition of isometries in problems and reasoning based on their features; working by artifacts to build knowledge and skills; translation of geometrical objects in a purely algebraic language.

1 Introduction

Educational research recommends to make the students aware about the historical and theoretic path of mathematical sciences, bringing them closer to the origins of scientific thought. It is strongly advisable to provide students with concrete references to everyday reality, in order to help them to accomplish their targets in the educational process. Most likely, one of the best ways to deal with "shortest path" problems and symmetries consists in recalling phenomena ordinarily and spontaneously occurring in nature, Arts, Physics and Biology. For an instance, Physics offers to Mathematics countless and significant opportunities to observing and experimenting about the "shortest path". In detail, geometrical optics and classical theories on the behavior of the light are two central topics: on one hand they are tightly linked to the real-world and on the other hand they are a perfect "shortest path" problem's examples.

The activities proposed in an action-research approach, such as Mathematics and Physics laboratories to learn by discovering, can be considered a part of the cultural background of a skills-based education.

In this educational process, interdisciplinarity is a useful way to facilitate the transition from knowledge to action. In fact, "(skills) cannot be reduced to a single discipline; they assume and create connections between knowledge and suggest new uses and mastery, which means 'skills beget skills'" (D'Amore, 2000).

Learning by doing is very important because "when we experience something we act upon it, we do something; then we suffer or undergo the consequences. We do something to the thing and then it does something to us in return: such is the peculiar combination. The connection of these two phases of experience measures the fruitfulness of experience…". (Dewey, 2007)

We are going to show a teaching unit starting from a historical problem, whose resolution aims to activate more skills at the same time, in a holistic approach. The ability to integrate knowledge and ways of thinking coming from many different disciplines establishes areas of expertise to produce a cognitive advancement – such as explaining a phenomenon, solving a problem or creating a product – in ways that would have been impossible or unlikely through a single discipline. Skills produce skill in creating connections between knowledge and suggest new uses and settings of known elements.

Following a constructive methodology, we suggest to start from concrete states and to prefer an inductive experimental hypothetical-deductive approach. Everything we can know is the product of an active construction of the subject; as well as constructivists, we believe that "the learning isn't the result of the development, learning is the development".

Learning by doing is our point: in addition to theoretical explanation, we essentially introduce many experimental activities about optics.

In order to overcome a fragmented knowledge "closed" within a single discipline, all the activities were structured through an interdisciplinary teaching action, which could allow students to get new points of view. Our proposal does not focus on the subject, but on methodological aspects. Students are placed at the very core of the whole process of teaching and learning. In this way, they are allowed to acquire a general attitude in asking questions, in treating problems and in mastering knowledge. Indeed, a skills-based education aims to promote "a coordinated system of knowledge and skills put in action by the party, in connection with a purpose (i.e., a task, a set of tasks or an action) generating interest and promoting good internal motivational and affective attitudes" (Pellerey 2005). Aiming to making easier the transition from knowledge to action, an interdisciplinary expertise has been proved to be the best choice. As part of an interdisciplinary design, the concept of symmetry has offered theoretical suggestions in other disciplines such as biology, chemistry, mineralogy. Pursuing skills, with this respect, imply to exploit the knowledge in an integrated way, in order to address real and concrete problems. Carrying out the project has called upon different knowledge expertise and personal resources to handle situations, while building new knowledge and skills, always with the ultimate purpose of the education of men and citizens (Tornatore, 1974).

Our inspiring model was the constructivist “situated learning” (Lave, 2006) or “anchored learning”. This choice was straightforward, because the most part of learning depends on the context and on the place where the cognitive experiences were located in authentic activities, through project-based learning. The constructivism proposed the idea that learning occurs more efficiently if the learner is involved in the production of concrete objects. Our theoretical framework relies on a constructionist foundation. " The word constructionism is a mnemonic for two aspects of the theory of science education underlying this project. From constructivist theories of psychology we take a view of learning as a reconstruction rather than as a transmission of knowledge. Then we extend the idea of manipulative materials to the idea that learning is most effective when part of an activity the learner experiences as constructing a meaningful product". (Papert, 1980)

2 Educational Activities

The classical “Seven Bridges of Königsberg” problem is the starting point of our teaching activities. It is a problem inspired by a real city and a concrete situation. Konigsberg, formerly in East Prussia, today a Russian exclave on the Baltic known as Kaliningrad, is crossed by the Pregel River and its tributaries. In these rivers, there are two large islands connected with each other and with the two main areas of the city by seven bridges. Over the centuries it has been repeatedly proposed the question whether it is possible with a walk to follow a cyclic path that crosses each bridge just only once, returning towards the starting point. In 1736, Leonhard Euler faced this problem, showing that this hypothetical walk is actually impossible.

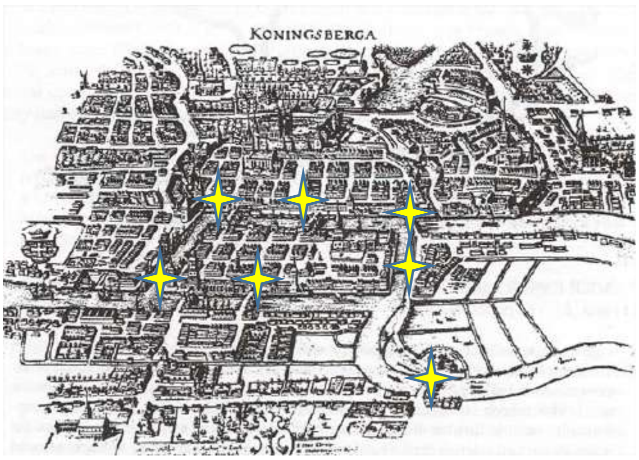


Figure 1. Königsberg’s map. The stars highlight the seven bridges’ placement.

The educational activities were broken up into steps:

Step 1 – Paper and ruler

Give students a sheet with a straight line and two points A and B, and asked them to:

1. Choose some points D, E, F, G on the straight line;
2. measure with the ruler the length of the polyline ADB, AEB, AFB and AGB;
3. sort the measures in ascending order and write the smallest measure.

Figure 2. straight line with two points, A and B

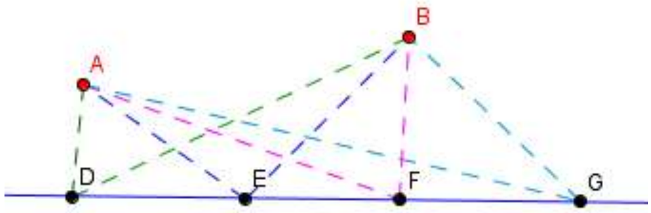


Figure 3. straight line with points and distance measures

Step 2 – Geogebra and table

Students are required to project a Geogebra file to represent the straight paths from A to B that touch the line in the point Q and then to tabulate the values found as a function of the position of the point Q on the straight line; unlike the previous phase, now, the tabulation of the points is performed by the software Geogebra, as well as the determination of the minimum.

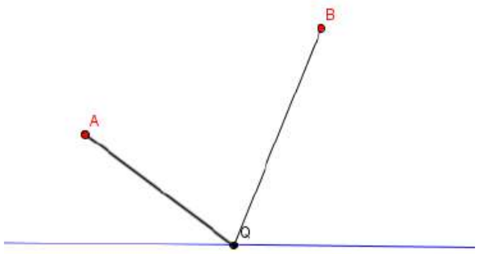


Figure 4. straight path between A, Q and B

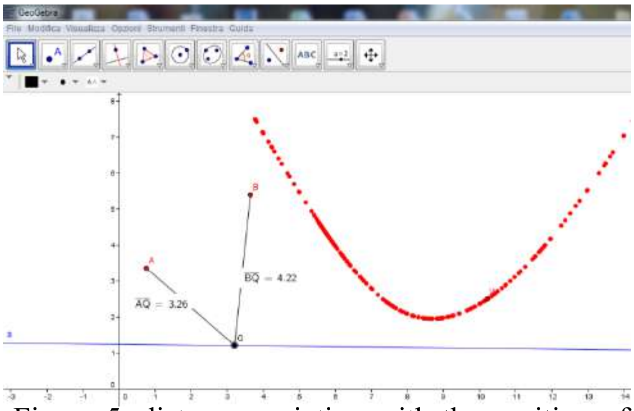


Figure 5. distance variation with the position of Q on the straight line

Step 3 – Locus of points

Students are required to design Geogebra files to graph the trend of the path length. In this step students:

1. build a locus of points, learning to deduce information by the graph;
2. conjecture that there is at least one point on the straight line Q corresponding to the shortest path.

The history of optics is closely linked to the history of geometry. Hero of Alexandria observed that the light travels in such a way that it goes to a mirror and to other points running the shortest possible *distance*.

Students are involved in a simple but concrete experience, using flat mirrors for observing the behavior of the light and the reflection of the images and guessing the geometric characterization of the shortest path.

Step 4 – Mirror and Physics laboratory

Learners are asked to analyze the image of the reflection of two points drawn on a paper sheet in front of a mirror and then perpendicular to it. The sequence of actions is:

1. simulate the mirror using two paper sheets;
2. draw the segments between the points;
3. measure the segments.

Students should consider that the segments AB' and A'B intersect in a point P; The segments AA' e BB' look like they are perpendicular to the sheet's edge placed near the mirror; point P looks like it belongs to the reflection axis.

After measuring the segments, point P comes out to be the midpoint between A and its image A' as well as from B and its image B' (Figure 6).

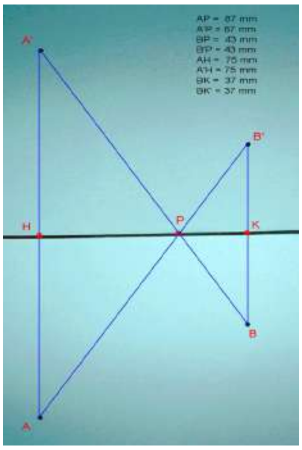


Figure 6. simulation of a mirror reflection

After that, students are asked to repeat the construction using Geogebra software to prove and verify that their measurements and observations are correct.

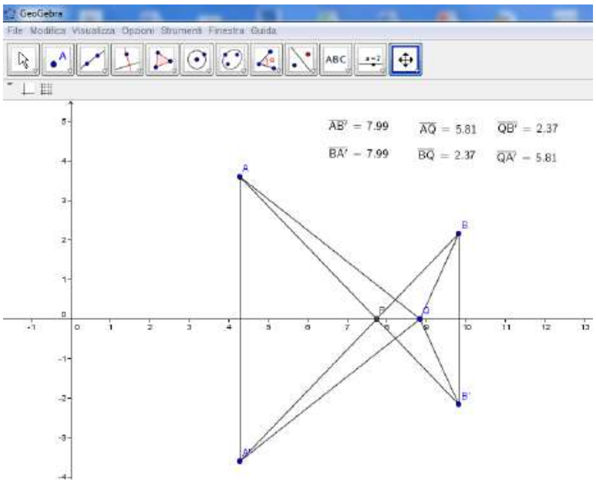


Figure 7. Geogebra simulation of reflection experiment

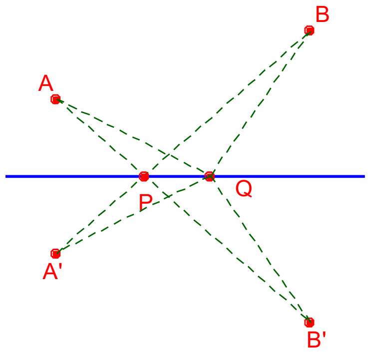


Figure 8. Drawing's details

At this point students are required to find out the characteristic properties of the point P of the straight line minimizing the length of the path APB.

Students are provided with the following directions:

1. Consider on the reflection axis a point Q different from the point P and link it to the points A, B and B';
2. Measure the polylines APB, AQB, AQB';
3. Describe your observations;

This exploration/discovery phase is supported by Geogebra software; at the end students are required to prove or disprove the conjecture: the point corresponding to the shortest path is the intersection between the segment AB' and the reflecting axis.

According to the properties of the axial symmetry, the length of the polyline AQB' is equal to the length of the polyline AQB and the length of the polyline APB' is equal to the length of the polyline APB.

Observing the triangle AB'Q we can infer that, because of the triangle inequality, the path APB' equal to APB is the shortest path, compared to any other path passing through a point P on the straight line different from P (Figure 7 and 8).

Step 5 – Angles

Students are asked to compare the magnitude of the angles between reflection axis and the segments AQ and BQ (Figures 9 and 10) and to verify the conclusions of the previous step. Students observe that, if Q overlaps P, the angles are equal.

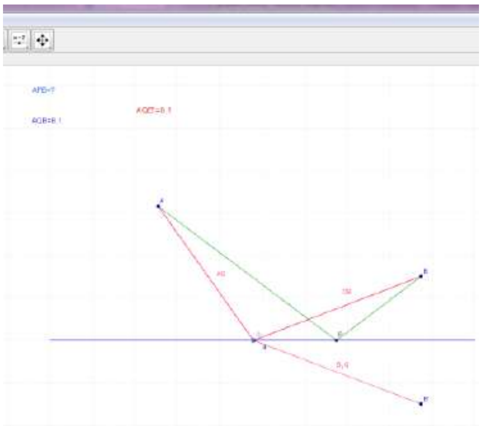


Figure 9. Angles in Geogebra simulation

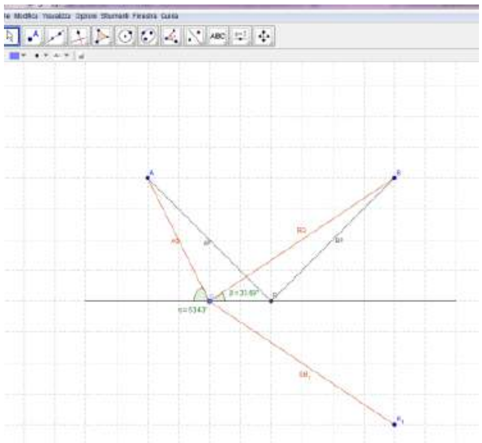


Figure 10. Details of angles

Then they make a conjecture: Angles APX and BPY are equal.

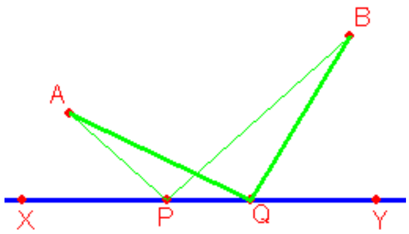


Figure 11. Further details of angles

The magnitude of the angle APX is equal to the magnitude of the angle XPA’, as long as they are correspondent in a axial symmetry. The magnitude of the angle XPA’ is equal to the magnitude of the angle BPY they are opposite angles. Thus, by the transitivity, the amplitude of the angle APX is equal to the amplitude of the angle BPY.

Step 6 – Law of reflection

At this stage the students can observe the phenomenon of light reflection using, for an instance, a mirror and a laser pointer and asking them to describe this experience as a mathematical expression. A deep connection between the “minimum problem” and the light’s behaviour. Students observe that the light “chooses”, the shortest of many paths (heron), the shortest polyline. Moreover, they also observe that the path composed by the incident ray and the reflected ray (Figure 12) links the two points in the shortest time.

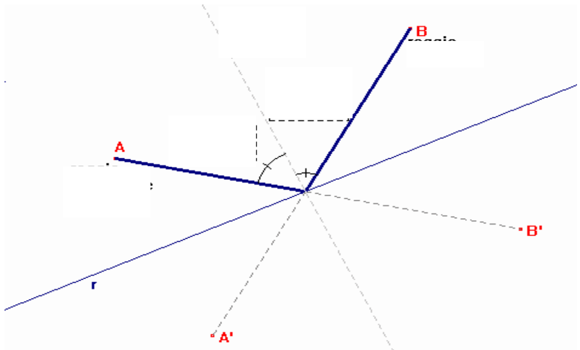


Figure 12. Law of reflection

Step 7 – Physics experiment: brachistocrone.

Students are required to guess which ball arrives first, falling down on different trajectories between two points as in the picture shown in Figure 13. It's easy to show, using an exhibit or an applet that the minimum path for a little ball moving from the point A to the point B is a curved line and not, as students can suppose in advance, a straight line. Observing this evidence, students are usually amazed.

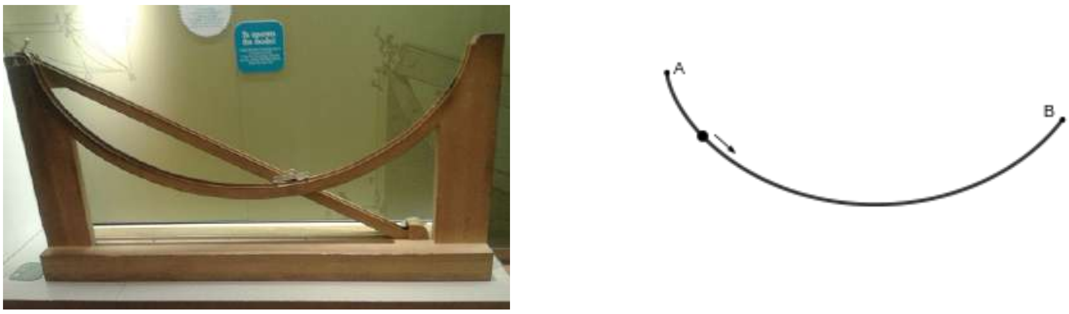


Figure 13. picture of Galileo's brachistocone and a cycloid (Galilei's museum, Firenze)

Using geometrical methods, in 1602, Galileo showed that a body subject to gravity takes less time to fall along the arc of a circumference between two points than along the corresponding segment of straight line, notwithstanding that the latter is shorter. Galileo did not realize that the "brachistochrone path" of a body leading down between two points is an arc of a cycloid, and not an arc of a circumference. The mathematical proof of brachistocronism of the cycloid was provided by Jacques Bernoulli in 1697.

Step 8 – Testing

Given the point A(0,1) and B(1,2), detect the point A' correspondent to A in an axial symmetry whose axis is the horizontal axis in a Cartesian coordinates system; then write down the equation of the straight line passing through points A' and B. Finally, find the point P as the overlap of the straight line r on the x-axis.

Chosen freely the coordinates of the points A and B, detect the coordinates of the point P as in the previous case.

Chosen freely the coordinates of the points A and B, belonging to two opposite half-planes with respect to horizontal axis (e.g. A(0,1) and B (1,-2)), detect the abscissa of the point P corresponding to the shortest path. Translate in a correct mathematical language all the choices of the previous steps.

Given two locations A and B on opposite sides of the bank of a straight river, locate where it is better to place a bridge on the river to minimize the length of the path that connects the location A to location B. (It is assumed that the banks are parallel and that the bridge is built perpendicularly to the shore).

Step 9 – Minimal path: Fermat and the refraction

It's a daily experience for students to see objects "broken" by refraction of light passing from one medium to another, typically from air to water. The minimum path principle of Hero of Alexandria works fine for light passing through homogenous media with the same refraction index. When light passes from one medium to another with a different refraction index, its velocity changes.

Figure 14. Refraction - Picture of a rope and its refracted image

Fermat was inspired from Heron to explain this phenomenon: due to refraction, light obviously does not choose the path of shortest *distance* but it prefers the shortest *time.* Thus the statement that the angle of incidence is equal to the angle of reflection is equivalent to the statement that the light goes to the mirror in such a way that it comes back in the *least possible time*.

Geometrical optics is just an approximation, but it is very relevant by a technical point of view and of great historical interest: the real behavior of light was discovered by Fermat about in 1650, it is called *the principle of the least time*, or *Fermat's principle*. Although we highlight to the students the well know double nature of light, our discussion is limited to the geometrical optics region, where we ignore the wavelength and the photonic character of the light. In fact, when we run reflection and refraction experiments, the wavelengths involved are very small compared to the dimensions of the equipment available for their study;

furthermore, the photon energies, using the quantum theory, are small compared with the sensitivity of the equipment.

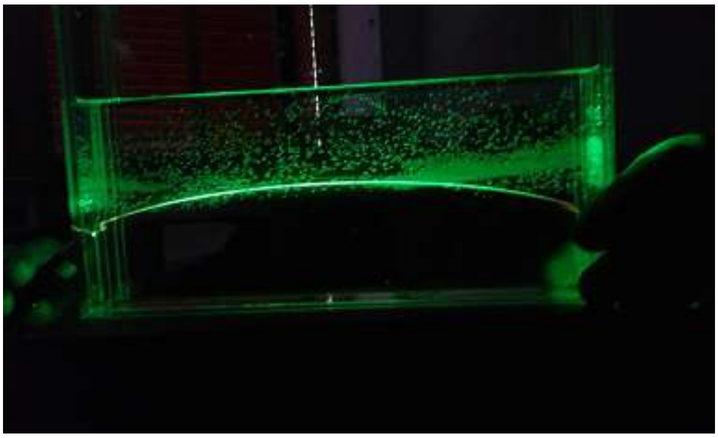


Figure 15. Experiment with a laser beam passing through water and glycerol

It is not difficult to design and build experiments in order to investigate the trajectory of the light, namely laser beam, as it goes through different media: glass, water, Plexiglas, transparent oils, glycerol, alcohol, etc. Students can see that the light propagates in a straight line. But, due to the Fermat's principle, if the light passes from one medium to another with a different refraction index, it changes direction. In the Figure 15 is shown an experiment with a laser beam passing through water and glycerol, two nonmiscible media: the beam curves!

3 Methodology

One of the most important inspiring models for our educational experiment was Inquiry Based Science Education (IBSE) model, which proposes teaching activities based on the investigation about the problems, the critic debate in groups and the research of innovative solutions, in a constructive outlook. Students deal with a problem originated from the observation of real world and teachers ask them to identify the problem making hypotheses (engage). After that, they had to plan the inquiry, exploring the variables (explore), to conduct an investigation on their own or in groups, documenting the outcomes (explain); Then, together to the teacher, who, in all the activities, operated as a scaffolder, the results were interpreted (evaluate) and communicated by formulating new problems (extend).

The students were given some assignments to be carried out using artifacts within a cycle that promotes the use of specific signs in relation to the use of special tools or artifacts, such as work in pairs or in small groups, with the artifact that promotes social exchange, together to words, plans, gestures. Students have been involved individually in different semiotic activities, especially affecting written productions. For example, after using an artifact, the students were often asked to write an individual report of their experience and their reflections, including doubts and questions arisen.

The collective discussions, finally, have been an essential part of the teaching-learning process, at the heart of the semiotic process, where the teaching-learning is based on. The whole classes were collectively engaged in a mathematical debate to answer a mathematical question, usually launched by the teacher, who explicitly formulates the topic of discussion.

During the meetings, students have always been given a task to accomplish. They were asked questions about the procedure of the activities (e.g., “how are you going to do it?”, “what are you doing?” “how did you do it?”); we tried to make the student aware of the significance, function, methods and potential of their knowledge. This process of metacognition was constructed through reflection and reconstruction process of how a student learns. Experimental practice was always been followed by theoretical and group activities. In fact, in the same way, it is not possible simply teaching in an abstract, formal and theoretical way, without a context, we cannot also leave students at the very early stage of experience and “doing”.

GeoGebra software turned out to be a resource for both displaying of concepts and proof. In accordance with national indications that "the student will be able to switch from one representation register to another (numerical, graphical, functional) even using IT tools for data representation", Duval (2006), the paths have been made of minimum distance with the help of the software and compared with the paths of a light beam.

Students can see the objects handled in a dynamical geometry software in two different ways: as simple pictures, i.e. relying on the perceptive aspects of the observation, or as schemes linked to a theory, that is, relying on conceptual aspects, Fishbein (1993). In our case, rather than to speculate and formulate, students were able to take advantage of the perceptive aspects of the figures. It was not here the case to choose whether to build using ruler and compass or to exploit the potential of the dragging, because the two things are not excluding each other. In a sense, the epistemological potentialities included the verification of theoretical models and empirical data at the same time.

4 Conclusions

The purpose of the educational path shown in this work was the realization of geometric constructions with simple tools such as ruler and goniometer and open source software, to recognize, in problematic situations, isometries, to explicate them using their properties and to decode, in algebraic language, the "geometric objects."

At the end of the learning path, students learned to work in groups and socialize / share their experiences; they also learned to use knowingly the dynamical geometry software Geogebra and to deduce, infer and understand aspects of the real world through the mathematical formalism and through abstract models; besides, they learned to highlight possible links and / or analogies setting them in a unitary context.

The ruler and compass constructions and the Geogebra software as semiotic mediation tools encouraged communication, stimulated discussions and facilitated the sharing of knowledge. This methodology, as a final outcome, simplified and accelerated the understanding of the properties of geometric transformations. It has also led to a shift from a colloquial language register to an advanced language, suitable to formalize definitions and properties.

REFERENCES

Bartolini Bussi, M. G. & Mariotti, M. A. (2008), Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English (ed.), *Handbook of International Research in Mathematics Education* (second edition). Routledge.

Bern, M. W., & Graham, R. L. (1989). The shortest-network problem. *Scientific American*, *260*(1), 84-89.

D'Amore, B. (2000). La Didattica della Matematica alla svolta del millennio: Radici, collegamenti e interessi. *La matematica e la sua didattica*, *3*, 407-422.

Dewey, J. (2007). *Democracy and Education*. Teddington*:* Echo Library.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*, 103-131.

Feynman, R. D. (2006). *QED, The strange theory of light and matter*. Princeton science library.

Fishbein, E. (1993) "The theory of figural concepts." *Educational studies in mathematics*.

Kourant, R., & Robbins, H. (1941). *What is Mathematics*. Oxford University Press.

Lave, J., & Wenger, E. (2006). *L'apprendimento situato. Dall'osservazione alla partecipazione attiva nei contesti sociali*. Edizioni Erickson.

Mushkarov, O. (2010). "A*lla ricerca dei cammini minimi*". Retrieved from http://www.math2earth.v3d.sk/publications/08_In%20search%20of%20shortest%20paths.pdf

Papert, S. (1990). *Constructionist learning. A 5th Anniversary Collection of Papers, Reflecting Report Research, Projects in Progress, and Essays by the Epistomeology.* Cambridge: Learning group MIT Media Laboratory, 3.

Papert, S. (1980). *Mindstorms: Children, Computers, and Powerful Ideas*. New York: Basic Books.

Pellerey, M. (2005). Verso una nuova metodologia di ricerca educativa: La Ricerca basata su progetti (Design-Based Research). *Orientamenti pedagogici*, *52*(5), 721-737.

Radford, L. (1997). On psychology, historical epistemology, and the teaching of mathematics: Towards a socio-cultural history of mathematics. *For the learning of mathematics*, *17*(1), 26-33.

Tornatore, L. (1974). *Educazione e conoscenza*. Loescher.

INITIER LES ETUDIANTS A LA DISTINCTION ENTRE VERITE DANS UNE INTERPRETATION ET VALIDITE LOGIQUE EN S'APPUYANT SUR LA THEORIE DU SYLLOGISME FORMEL D'ARISTOTE

Viviane DURAND-GUERRIER
Université de Montpellier, Place Eugène Bataillon, 34000 Montpellier, France
Institut Montpelliérain Alexander Grothendieck, UMR CNRS UM 5149
viviane.durand-guerrier@umontpellier.fr

RESUME

Dans cette communication, je présenterai tout d'abord brièvement les éléments permettant de soutenir la thèse selon laquelle les principaux concepts fondamentaux de la sémantique logique sont déjà présents chez Aristote. Dans une deuxième partie, je présenterai les éléments consacrés à la théorie du syllogisme formel d'Aristote dans le cadre d'un module intitulé « Analyse logique des énoncés et des raisonnements mathématiques - Aspects épistémologique et didactique » proposé de 1994 à 2007 à des étudiants de licence et repris pour partie et adapté dans le cadre d'un module d'épistémologie en 2012-2013 à partir d'extraits des livres II (*De l'Interprétation*) et III (les *Premiers analytiques*) de l'*Organo*n.

1 Introduction

La distinction entre vérité dans une interprétation et validité logique est un enjeu crucial pour l'apprentissage de la preuve et du raisonnement (Durand-Guerrier, 2008). Dans le *Tractatus logico philosophicus*, Wittgenstein (1921) élabore une version sémantique du calcul des propositions et clarifie cette distinction en introduisant la notion de tautologie. Pour le calcul des prédicats du premier ordre qui prend en compte les notions de variables, de propriétés, de relations et de quantification, il faut attendre les travaux de Tarski (1936) pour une définition des notions de conséquence logique et de validité universelle d'un point de vue sémantique. Les résultats de recherche en didactique des mathématiques montrent que la plupart des étudiants et des enseignants sont trop peu familiers avec les concepts de la logique du premier ordre pour aborder avec profit ces questions. Ceci m'a conduit à proposer dans différentes formations une approche de cette distinction en appui sur la théorie du syllogisme formel d'Aristote. En effet, bien que l'on ait souvent reproché à juste titre au système formel d'Aristote d'être trop pauvre pour les besoins logiques de l'activité mathématique, de nombreux logiciens contemporain (Largeault, 1972) soulignent le fait que ses intuitions géniales lui ont permis de construire un système formel appuyé sur les concepts fondamentaux de la moderne sémantique logique, comme nous allons le montrer dans ce qui suit. Dans une première partie je présente brièvement les éléments permettant de dire qu'Aristote est un précurseur de la sémantique logique contemporaine. Dans la deuxième partie je présente l'usage de la théorie du syllogisme comme première rencontre avec les concepts fondamentaux de la sémantique logique.

2 Aristote précurseur de la sémantique logique contemporaine

La logique aristotélicienne nous est connue par le traité de l'*Organon*, qui peut se traduire par Instrument. Pour Aristote la logique est un outil pour distinguer entre les raisonnements concluants (nous dirions valides aujourd'hui) et ceux qui ne le sont pas. C'est ce qu'il développe dans les *Premiers Analytiques* qui constitue le livre III de l'*Organon*. Auparavant, dans le livre II, intitulé de l'Interprétation, il met en place les catégories logiques qu'il mobilisera dans le livre III[1].

2.1 Catégories logiques, quantification et modalités d'opposition

Aristote introduit la distinction entre *termes singuliers* et *termes universels :*

> Puisqu'il y a des choses universelles et des choses singulières (j'appelle universel ce dont la nature est d'être affirmée de plusieurs sujets et singulier ce qui ne le peut: exemple homme est un terme universel et Callias un terme singulier) nécessairement la proposition que telle chose appartient ou n'appartient à un sujet s'applique tantôt à un universel, tantôt à un singulier.

Cette distinction est essentielle pour la suite de son propos ; il va en effet s'en servir pour proposer une classification des énoncés de la langue naturelle. Il propose quatre catégories d'énoncés prenant en compte les formes affirmatives et négatives: 1/ les énoncés singuliers, affirmatif ou négatifs : « Socrate est blanc » - « Socrate n'est pas blanc » ; 2/ les énoncés universels pris universellement : « tout homme est blanc » - « nul homme n'est blanc » ; 3/ les énoncés universels non pris universellement : « quelque homme est blanc » - « quelque homme n'est pas blanc » ; 4/ les énoncés indéfinis : « L'homme est blanc » – « l'homme n'est pas blanc ».

Dans les énoncés précédents, il y a des formes affirmatives et des formes négatives ; pour autant, pour ce qui concerne les énoncés portant sur des universels ces formes syntaxiques ne correspondent pas nécessairement à la négation conçue comme un opérateur échangeant les valeurs de vérité. Rappelons que pour Aristote, ce sont les énoncés qui portent le vrai ou le faux : « dire de l'être qu'il est et dire du non-être qu'il n'est pas, c'est le vrai » (*Métaphysique*, G, 7, 111b)

Aristote appelle contradiction cette modalité d'opposition pour laquelle il faut faire porter la négation sur le terme universel et changer la quantité, comme dans les exemples : « Tout homme est blanc / Quelque homme n'est pas blanc » et « Nul homme n'est blanc / Quelque homme est blanc ». Pour ces deux paires d'énoncés, comme pour la paire « Socrate est blanc / Socrate n'est pas blanc », Aristote souligne que nécessairement l'une est vraie et l'autre fausse, ce qui n'est pas le cas pour les deux autres paires. La paire « Tout homme est blanc / Tout homme n'est pas blanc » est une paire de contraires : les deux énoncés peuvent être tous les deux faux ; tandis que la paire « Quelque homme est blanc / Quelque homme

[1] Dans ce texte, j'utilise la traduction de Jean Tricot publiée aux éditions Vrin (nouvelle édition 1989-1992).

n’est pas blanc » ne correspond pas à une opposition car les deux énoncés peuvent être simultanément vrais.

Cette étude sur les modalités d’opposition pour les énoncés comportant une quantification est tout à fait éclairante, et met en valeur un aspect essentiel de la négation qui dans une langue donnée doit articuler des critères syntaxiques et des critères sémantiques. Au début du 20ème siècle, Russell (1903) écrit que l’erreur consistant à croire que la négation d’un énoncé universel de la forme « Pour tout x, $P(x)$ » est l’énoncé « Pour tout x, non P(x) » est facile à commettre (p. 241). Il pourrait encore l’écrire aujourd’hui. En ce qui concerne le français, ceci est renforcé par le fait que selon la norme linguistique la négation d’un énoncé de la forme « Tous les A sont B » est l’énoncé « Tous les A ne sont pas B »[2].

2.2 Vérité et validité dans la théorie du syllogisme formel

Dans les *Premiers Analytiques*, Aristote s’intéresse aux syllogismes démonstratifs. Il commence par définir ce qu’il appelle « prémisse » pour le syllogisme démonstratif, en reprenant les types d’énoncés introduits au livre II, ainsi que ce qu’il appelle « terme » :

> La prémisse est le discours qui affirme ou nie quelque chose de quelque chose, et ce discours est soit universel, soit particulier, soit indéfini. (I,1 , 24a, 20)
>
> J’appelle terme ce en quoi se résout la prémisse, savoir le prédicat et le sujet dont il est affirmé, soit que l’être s’y ajoute, soit que le non-être en soit séparé. (I-1, 24b, 15)

Ceci étant posé, il définit le syllogisme

> Le syllogisme est un discours dans lequel, certaines choses étant posées, quelque chose d'autre que ces données en résulte nécessairement par le seul fait de ces données.[...] J'appelle syllogisme parfait celui qui n'a besoin de rien autre chose que ce qui est posé dans les prémisses, pour que la nécessité de la conclusion soit évidente; et syllogisme imparfait celui qui a besoin d'une ou plusieurs choses, lesquelles il est vrai résultent nécessairement des termes posés, mais ne sont pas explicitement énoncées dans les prémisses. (I-1, 24b, 20)

Les syllogismes parfaits sont les syllogismes concluants de la première figure qui est caractérisée par le fait que le moyen terme est successivement sujet et prédicat :

> Si A est affirmé de tout B, et B de tout C, nécessairement A est affirmé de tout C
>
> Si A n’est affirmé de nul B et si B est affirmé de tout C, il en résultera que A n’appartiendra à nul C.
>
> Si A est affirmé de tout B et si B est affirmé de quelque C, nécessairement A est affirmé de quelque C.
>
> Si A n’est affirmé de nul B et si B est affirmé de quelque C, nécessairement A n’appartient pas à quelque C.

---

[2] Ceci est développé dans (Durand-Guerrier & Ben Kilani, 2004).

Exemple d'interprétation du syllogisme universel de la première figure :

> Si mortel est affirmé de tout homme et si homme est affirmé de tout grec, alors mortel est affirmé de tout grec.

Pour chacun des syllogismes de la première figure qui ne sont pas parfaits, il donne un exemple montrant que deux conclusions différentes peuvent être tirés selon les termes choisis, si bien qu'aucune des deux conclusions n'est nécessaire. Comme termes d'attribution universelle prenons par exemple : *animal, homme, cheval*; et de non attribution universelle : *animal, homme, Pierre. » (p.14)*

Animal est affirmé de tout homme Homme n'est affirmé de nul ***cheval*** ***Animal*** est affirmé de tout ***cheval***	**Animal** est affirmé de tout homme Homme n'est affirmé de nulle ***pierre*** **Animal** n'est affirmé de nulle ***pierre***

Figure 1. Exemples de syllogisme aristotélicien non concluant

Après avoir posé les résultats concernant la première figure, Aristote propose les autres figures qui consistent à modifier la position du terme qui est présent deux fois (le moyen terme). Lorsque le syllogisme obtenu est concluant, il le prouve en le ramenant à un syllogisme parfait de la première figure en utilisant les deux règles de conversion ci-dessous:

> R1 : « B est affirmé de quelque A » peut être remplacé par « A est affirmé de quelque B ».
>
> R2 : « B n'est affirmé de nul A » peut être remplacé par « A n'est affirmé de nul B ».

Pour les syllogismes non concluants, il procède comme dans l'exemple ci-dessus**.** Il fait noter que les deux seuls énoncés quantifiés qui peuvent être converti par modification de l'ordre des termes sont les deux énoncés ci-dessus. En effet il est possible qu'une interprétation de « B est affirmé de tout A » soit vrai, sans que l'interprétation de « A est affirmé de tout B » ne le soit ; de même, il est possible que l'interprétation de « A n'appartient pas à quelque B » soit vrai sans que l'interprétation de « B n'appartient pas à quelque A » ne le soit (par exemple avec A : Grec et B : Homme).

Comme le souligne Blanché (1970, pp. 49-50), les syllogismes concluants d'Aristote sont des lois logiques qui garantissent la validité de l'inférence, dans la mesure où ils donnent lieu à un énoncé vrai quelque soit l'interprétation des termes A, B et C. On retrouve ici l'idée de validité universelle telle que définit par Quine (1950). Pour montrer qu'un syllogisme n'est pas une loi logique (n'est pas concluant), il suffit de trouver une interprétation dans laquelle les prémisses sont vraies et la conclusions fausse. Aristote distingue explicitement les vérités nécessaires obtenues comme conclusion d'un syllogisme concluant à prémisses vraies des vérités *de facto* en accord avec les faits. En procédant de la sorte, il marie les méthodes syntaxiques et sémantiques et met en lumière la distinction entre validité logique et vérité dans une interprétation telle qu'elle est développée en logique du premier ordre dans la suite des travaux de Wittgenstein (1921) et Tarski (1936)[3].

Nous pensons avoir montré dans cette brève présentation ce qui justifie le jugement des logiciens contemporains qui considèrent que les intuitions géniales d'Aristote lui ont permis de construire un système formel appuyé sur les concepts fondamentaux de la moderne sémantique logique : classification et modélisation des énoncés de la langue ordinaire par des énoncés formels (énoncés singuliers, généraux universels ou particuliers) ; définition des propositions ; introduction de lettres de termes pour caractériser la forme des énoncés et construire les figures du syllogisme ; notion d'interprétation des énoncés formels ; notion de validité logique des syllogisme ; distinction entre vérité de facto et vérité nécessaire ; mise en œuvre conjointe de méthodes syntaxiques et sémantiques pour établir ce qui forme ou non un syllogisme concluant parmi ceux pouvant être construits dans les quatre figures possibles de son système.

Ceci m'a conduit depuis de nombreuses années à proposer un travail autour de la théorie du syllogisme formel dans différents modules en licence.

3 La théorie du syllogisme formel d'Aristote : une première rencontre avec les concepts fondamentaux de la sémantique logique

Il est clair que la théorie du syllogisme formel d'Aristote est insuffisante pour les besoins de l'activité mathématique. Néanmoins, nous avons fait le pari que cette théorie pouvait permettre une première rencontre avec les concepts fondamentaux de la sémantique logique qui jouent un rôle central dans la preuve et le raisonnement mathématique. Nous présentons ci-dessous un bref aperçu des contenus et des modalités de travail proposés aux étudiants de licence et nous donnerons pour l'année 2012-2013 quelques éléments extraits d'une évaluation. Je précise que ces données n'ont pas été recueillies suivant un protocole de recherche mais dans le cadre de l'enseignement mis en place.

3.1 Insertion dans des modules d'épistémologie en licence de 1994 à 1997

De 1994 à 2007, j'ai proposé un module optionnel intitulé « Analyse logique des énoncés et des raisonnements mathématiques - Aspects épistémologique et didactique » à des étudiants de première année d'université, tout d'abord à Valence (Université Grenoble 1) de 1994 à 1998 puis à l'Université Lyon 1 de 1999 à 2007. J'ai repris des éléments de ce module dans le cadre d'un module optionnel de licence pluridisciplinaire intitulé « Epistémologie » de 2010 à 2013. Dans chaque cas, la théorie du syllogisme formel a été introduite comme support pour une première rencontre avec la distinction entre vérité et validité.

L'initiation à la théorie du syllogisme formel d'Aristote s'inscrit dans le projet de sensibiliser les étudiants aux apports pour les apprentissages mathématiques de l'épistémologie, discipline qui étudie le développement des connaissances à travers l'histoire, en s'attachant moins aux évènements qui entourent la naissance et le développement des

[3] Ceci est développé dans (Durand-Guerrier, 2008).

notions, concepts ou théories scientifiques[4], qu'à la genèse et à l'évolution de ces concepts ou de ces théories, du point de vue des connaissances en jeu, au cours du temps. En particulier, nous souhaitons que les étudiants prennent conscience de ce que l'éclairage épistémologique nous permet de mieux comprendre la forme achevée du concept ou de la théorie. Dans le cas qui nous intéresse ici, il s'agit d'éclairer la distinction nécessaire, pour les activités mathématiques, entre *vérité dans une interprétation* et *validité logique*. L'initiation à la théorie du syllogisme formel constitue une première rencontre avec ces notions, avant d'aborder les apports des auteurs du renouveau de la logique moderne (Frege, Russell, Wittgenstein, Tarski) et d'introduire la déduction naturelle à la manière de Copi (1954) qui fournit un outil de contrôle des raisonnements, parmi lesquels les syllogismes aristotéliciens (Hottois, 1989)[5]. En effet, comme nous l'avons évoqué dans la première partie, la logique contemporaine du premier ordre (calcul des prédicats) est en quelque sorte un prolongement et un achèvement de la logique d'Aristote. Dans ces modules, nous faisons travailler les étudiants essentiellement sur les outils développés par Aristote pour garantir la correction des raisonnements ; ces outils nous servent de référence lorsque l'on aborde ensuite les auteurs contemporains.

3.2 Un travail en appui sur quelques textes

La consigne générale donnée aux étudiants est la suivante :

> Vous allez avoir à travailler sur ces textes, qui sont des traductions du Grec ancien. Il faudra faire un effort pour s'approprier ce langage. Vous devrez essayez de mettre ce que vous lirez en relation avec des choses connues, en particulier des exemples pris dans le domaine mathématique. Il ne s'agit évidement pas pour vous de devenir des spécialistes de la logique antique, mais de voir en quoi les textes proposés éclairent les questions que vous pouvez vous poser aujourd'hui sur le raisonnement mathématique.

Les textes retenus pour le travail sont extraits de l'*Organon*, principalement des extraits des livres II (De l'interprétation) ; III (*Les premiers Analytiques*) dans la traduction de Jean Tricot publiée aux éditions Vrin (Nouvelle édition 1989-1992). Les textes issus des *Premiers Analytiques* qui portent spécifiquement sur la théorie du syllogisme formel sont donnés en annexe. Les étudiants sont invités à lire les textes proposés accompagnés d'un questionnement visant à favoriser leur appropriation : reformulation en langage moderne ; identification des notions en jeu ; mise en relation avec des énoncés mathématiques ; preuve à l'intérieur du système (la théorie du syllogisme formel d'Aristote).

Dans ces modules, les étudiants devaient en outre faire un mémoire sur un thème de leur choix. Dans la plupart des cas, les questions de l'articulation entre vérité et validité, bien que n'étant pas la question centrale se trouvaient engagées ; les travaux montraient une prise de recul que l'on rencontre rarement, même chez des étudiants plus avancés (Durand-Guerrier,

---

[4] Ceci même si la dimension historique est un aspect important des études épistémologiques que nous avons conduites sur les questions de logique.

[5] Un travail était également proposé sur la logique des stoïciens, mais nécessairement plus limité compte tenu de ce que l'on n'a pas d'écrit équivalent à l'*Organon*, mais seulement quelques textes *a posteriori* (Blanché 1970)

2008). Néanmoins, ceci n'a jamais fait l'objet de recherche ; ce que nous présentons ici relève seulement d'observations naturalistes dans le cadre de la mise en œuvre de ces modules au fil des années.

3.3 Une reprise dans un module d'épistémologie en licence de 2010 à 2013.

Ce module optionnel a été proposé de 2010 à 2013 aux étudiants de première année du portail Curie rassemblant les étudiants des filières Mathématiques, informatique, et Physique de la faculté des sciences de l'Université Montpellier 2. L'équipe pédagogique comportait deux enseignants de chacune des trois disciplines, le responsable était Thomas Hausberger.

Dans la présentation du module, nous avions écrit :

> Un module pour prendre du recul et donner du sens : vous voulez faire de la science...mais ça veut dire quoi ?
>
> L'épistémologie est la discipline qui étudie comment se construisent les connaissances : qu'est ce que la science ? Quels problèmes la science se pose-t-elle, quels sont ses objets, ses méthodes ? Qu'est-ce qu'un théorème, une loi, un modèle ? Le mathématicien, l'informaticien et le physicien raisonnent-ils tous de la même manière ?

Le module comportait 25h30, dont 12h de cours magistral et 13h30 de Travaux dirigés, et était évalué sur projets réalisés par groupes de 4 (50%) complété par une épreuve sur table (50%). Les groupes de Travaux dirigés étaient organisés par domaine disciplinaire.

Deux séances de 1h30 de cours magistral était consacrées à la logique. La première séance étant consacrée à l'antiquité, essentiellement la logique d'Aristote : le cours consistait en une présentation et un commentaire des textes avec quelques questions du type de celles posées dans les modules de Lyon et Valence. La seconde séance était consacrée à l'implication avec un accent important sur la distinction vérité/validité. Les étudiants inscrits dans mon groupe de Travaux dirigés ont ensuite retravaillés sur la négation en appui sur un texte de Frege (1970), en référence aux catégories d'opposition d'Aristote. Certain d'entre eux ont choisi pour leur projet un sujet relevant de la logique (principe d'induction, implication, nécessité et contingence, relations et quantification, démonstration naturelle).

Une des questions de l'examen écrit du 17 décembre 2012 consistait à étudier la validité de deux syllogismes, l'un étant valide (concluant), l'autre non, en justifiant soigneusement la réponse.

a) Si quelque A est B et si tout B est C, alors quelque A est C.
b) Si tout A est B et si quelque B est C, alors quelque A est C

Aucun document n'était autorisé ; 11 étudiants ont passé l'épreuve – tous ont traité cette question, avec plus ou moins de réussite. Quatre étudiants on reformulé « Tout B est C » en « B = C » et « Tout A est B » en « A = B ». Presque tous les étudiants proposent une preuve pour justifier qu'un syllogisme est concluant (preuve ensembliste[6] ou preuve par l'absurde, ou preuve par substitution pour les quatre étudiants ayant identifié « Tout B est C » à « B = C » et « Tout A est B » à « A = B » qui concluent que les deux syllogismes sont concluants). Un seul étudiant a proposé un contre-exemple pour justifier que le deuxième syllogisme n'est pas concluant. Pour le premier syllogisme, il montre qu'il est concluant en supposant *pour simplifier* (ses propres mots) que « B = C ». Il explicite le fait que lorsque les deux prémisses sont vraies, la conclusion est nécessaire, qui correspond à la notion de nécessité introduite par Aristote (c.f. première partie de ce texte). Pour le second syllogisme, il justifie qu'il est non concluant en donnant un contre-exemple. Il attribue des termes à A, B et C : A : Homme, B : Blanc, C : Idiot. Il considère le syllogisme obtenu: « Si tous les hommes sont blancs et quelques blancs sont idiots, alors quelque homme est idiot » et écrit : la conclusion n'est pas nécessaire car blanc n'est pas forcément un homme. « Quelques blancs sont idiots » peut référer à d'autres êtres blancs et idiots.

Un des objectifs du module en ce qui concerne la distinction entre vérité et validité au sens d'Aristote consiste à mettre en valeur le fait que pour établir qu'un syllogisme n'est pas valide (concluant), il faut proposer un contre exemple, c'est-à-dire ici une interprétation des lettres du syllogisme pour laquelle les deux prémisses sont vraies et la conclusion fausse. Un seul étudiant a mis en œuvre cette méthode et l'a mené à son terme. Les autres étudiants qui pensent que le syllogisme n'est pas valide cherchent à donner une preuve de type syntaxique de cette non validité, adapté de la preuve faite pour établir la validité.

4 Conclusion

En appui sur l'expérience du module « Analyse logique des énoncés et des raisonnements mathématiques- Aspects épistémologique et didactique » conduit de 1994 à 2007, je peux témoigner que la logique d'Aristote offre un contexte favorable pour aborder les concepts fondamentaux de la logique contemporaine avec des étudiants de mathématiques. Son éloignement des pratiques mathématiques contemporaines est un élément favorisant un questionnement de type logique qui peut ensuite être réinvesti, constituant ainsi une bonne introduction pour étudier la sémantique logique développée à la suite de Frege par Wittgenstein et Tarski. L'expérience de l'adaptation sous la forme d'un cours magistral des contenus pour le module « Epistémologie » conduit à Montpellier a confirmé la nécessité d'un temps long pour une appropriation adéquate de la distinction entre validité logique et vérité dans une interprétation. Il n'y a actuellement aucun module en licence à l'Université de Montpellier pour faire vivre un tel travail.

REFERENCES

Aristote (1989). *L'Organon, livre II - De l'interprétation* ; traduction Jean Tricot. Paris: Vrin.
Aristote (1992). *L'Organon, livre III - Les premiers analytiques* ; traduction Jean Tricot. Paris : Vrin.
Aristote (2000). *Métaphysique*, tome 1, (Traduction Jean Tricot). Librairie philosophique. Paris: J. Vrin.
Blanché, R. (1970). *La logique et son histoire d'Aristote à Russell*. Paris: Armand Colin.
Copi, I. (1954). Symbolic logic (2nd edition, 1965). New York: The Macmillan Company.

[6] Aucun des étudiants utilisant la représentation ensembliste n'a donné un argument complet.

Durand-Guerrier, V. (2008). Truth versus validity in mathematical proof. *ZDM The International Journal on Mathematics Education*, *40*(3), 373-384.
Durand-Guerrier, V., & Ben Kilani, I. (2004). Négation grammaticale versus négation logique dans l'apprentissage des mathématiques. Exemple dans l'enseignement secondaire Tunisien. *Les Cahiers du Français Contemporain, 9*, 29-55
Frege, G. (1970). *Ecrits logiques et philosophiques*. Paris: Editions du Seuil
Hottois, G. (1989). *Penser la logique, une introduction technique, théorique et philosophique à la logique formelle.* Bruxelles : De Boeck-Wesmael.
Largeault, J. (1972). *Logique mathématique*. *Textes*. Paris: Armand Colin
Quine, W. V. O. (1950). *Methods of logic, Holt, Rinehart & Winston* (Traduction française). Paris: Armand Colin, 1972.
Russel, B. (1903) *The principles of mathematics.* Cambridge : Cambrige University Press (Traduction française partielle par J.-M. Roy dans *Écrits de logique philosophique*. Paris: PUF, 1989)
Tarski, A. (1936) Sur le concept de conséquence. In *Logique, Sémantique et Métamathématique*, *1* (pp. 141–152). Paris: Armand Colin, 1972.
Wittgenstein, L. (1921) *Tractatus logico-philosophicus,* In *Annalen der naturphilosophie, 24*. Leipzig: traduction française G. G. Granger. Paris : Gallimard, 1993.

Annexe

Le document ci-dessous était propose dans les modules de première année de licence de Valence, de Lyon et de Montpellier:

Analyse logique des énoncés et des raisonnements mathématiques. Aspects épistémologique et didactique.

Les étudiants ont travaillés auparavant sur des extraits du livre II (*De l'interprétation*) et du livre V (*Les topiques*).

Nous avons introduit dans ce cadre la notion classique pour les propositions portant sur des sujets universels par les lettres A, E, I, O[7]:

A universelle affirmative: Tout homme est blanc

E universelle négative: Nul homme n'est blanc (tout homme est non-blanc)

I particulière affirmative: Quelqu'homme est blanc.

O particulière négative: Quelqu'homme n'est pas blanc

Les textes présentés ci-dessous concernent spécifiquement la théorie du syllogisme formel.

Document 3 : Aristote - L'*Organon* III - *Les Premiers Analytiques*

Texte 1: Ce qu'est la prémisse

“ La prémisse est le discours qui affirme ou qui nie quelque chose de quelque chose, et ce discours est soit universel, soit particulier, soit indéfini. J'appelle (....) indéfinie, l'attribution ou la non-attribution faite sans indication d'universalité ou de particularité: par exemple, les contraires rentrent dans la même science ou le plaisir n'est pas le bien. ” (pp. 2-3)

Texte 2: Ce qu'est le syllogisme

[7] Nous rappelons que cette notation n'est pas présente dans l'*Organon.*

“ Le syllogisme est un discours dans lequel, certaines choses étant posées, quelque chose d’autre que ces données en résulte nécessairement par le seul fait de ces données. Par le seul fait de ces données: je veux dire que c’est par elles seules que la conséquence est obtenue; à son tour, l’expression c’est par elles seules que la conséquence est obtenue signifie qu’aucun terme étranger n’est en sus requis pour produire la conséquence nécessaire. J’appelle syllogisme parfait celui qui n’a besoin de rien autre chose que ce qui est posé dans les prémisses pour que la nécessité de la conclusion soit évidente; et syllogisme imparfait, celui qui a besoin d’une ou plusieurs choses, lesquelles il est vrai résultent nécessairement des termes posés, mais ne sont pas explicitement énoncées dans les prémisses. ” (pp. 4-5)

Texte 3: le syllogisme catégorique de la première figure

“ Quand trois termes sont entre eux dans des rapports tels que le mineur soit contenu dans la totalité du moyen, et le moyen contenu ou non contenu dans la totalité du majeur, alors il y a nécessairement entre les extrêmes syllogisme parfait. J’appelle moyen le terme qui est lui-même contenu dans un autre terme et contient un autre terme en lui, et qui occupe une position intermédiaire; j’appelle extrêmes à la fois le terme qui est lui-même contenu dans un autre, et le terme dans lequel un autre est contenu.. Si A est affirmé de tout B, et B de tout Γ , nécessairement A est affirmé de tout Γ. (...) De même, si A n’est affirmé de nul B, et si B est affirmé de tout Γ, il en résulte que A n’appartiendra à nul Γ. (...) Soit donc A appartenant à tout B et B à quelque Γ, (...) nécessairement A appartient à quelque Γ. Et si A n’appartient à nul B, et que B appartienne à quelque Γ, nécessairement A n’appartient pas à quelque Γ. (...) dans un syllogisme de cette figure, les termes doivent être en rapport comme nous l’avons indiqué, autrement aucun syllogisme n’est possible. Il est évident que tous les syllogismes rentrant dans cette figure sont parfaits (car tous reçoivent leur achèvement des prémisses originellement posées), et que toutes les conclusions peuvent être démontrées au moyen de cette figure, universelles aussi bien que particulières, affirmative aussi bien que négatives. J’appelle une telle figure la première. ” (pp 13-20)

Une autre traduction possible du premier syllogisme est “ Si tout B est A, et si tout C est B, nécessairement tout C est A ”. En utilisant les lettres associées aux propositions, on désigne ce syllogisme par A.A.A.

Question 1: Faire la même chose avec les trois autres syllogismes cités. Puis donner un exemple de chacun de ces syllogismes en choisissant les termes que vous attribuez aux lettres A, B et Γ.

Les syllogismes proposés par Aristote sont appelés classiquement syllogismes concluants; ceux dont il dit dans ce texte qu’ils ne sont pas possibles seront dits non concluants.

Question 2: On peut construire d’autres arrangements de propositions en respectant les règles données par Aristote pour la première figure. Proposez un tel arrangement et essayez de justifier le fait qu’il ne soit pas concluant.

Dans la seconde figure le moyen joue deux fois le rôle de prédicat; voici un exemple de syllogisme concluant de la deuxième figure:

“ si nul N n’est M, et si tout Σ est M, nécessairement nul Σ n’est N. ”

Aristote réduit ce syllogisme en convertissant la prémisse “ nul N n’est M ” en la prémisse équivalente “ nul M n’est N ”, il se ramène ainsi au syllogisme concluant de la première figure

“ si nul M n’est N et si tout Σ est M, nécessairement nul Σ n’est N. ”

Question 3 : Donnez le syllogisme de la seconde figure en A.A.A. et justifiez le fait qu’il ne soit pas concluant.

Question 4 : Donnez le syllogisme de la seconde figure en E.I.O. Est-il concluant? si oui, le réduire; si non, donner un exemple.

Texte 4 : Vérité des prémisses; vérité de la conclusion

“ Il peut se faire que soient vraies les prémisses qui forment le syllogisme; il peut se faire aussi qu’elles soient fausses, ou encore que l’une soit vraie et l’autre fausse. La conclusion, elle, est nécessairement ou vraie, ou fausse. De prémisses vraies, on ne peut tirer une conclusion fausse, mais de prémisses fausses, on peut tirer une conclusion vraie, avec cette réserve qu’elle portera non sur le pourquoi, mais sur ce qui est en fait. C’est que le pourquoi ne peut faire l’objet d’un syllogisme à prémisses fausses. ” (pp 209-210)

Question 5 : A partir du syllogisme en E.I.O. de la première figure, et des termes: pair, premier, carré parfait, illustrer le texte ci-dessus.

CE QUE NOUS DIT L'ÉTUDE D'UNE RECHERCHE DE KEPLER DE L'INFLUENCE D'UNE RELATION SINGULIÈRE AUX OBJETS NATURALISÉS

Mathias FRONT
ESPE de l'Académie de Lyon, et S2HEP, 38 boulevard Niels BOHR Université Lyon 1
F-69622 Villeurbanne Cedex, Lyon, France
mathias.front@univ-lyon1.fr

ABSTRACT

La question des articulations entre histoire, épistémologie et didactique est suffisamment riche pour être source inépuisable de réflexion. Nous souhaitons proposer ici une approche qui étudie l'activité du savant cherchant et les liens potentiels avec des situations didactiques où la recherche de problème est centrale. Nous explicitons dans un premier temps comment une vision philosophique, mettant en avant l'agir en mathématique, peut légitimer l'étude de l'activité du savant cherchant jusque dans sa relation aux objets familiers. Cette approche nous amène ensuite à étudier, à titre d'exemple générique, la relation aux objets de Kepler explorant les pavages archimédiens du plan. Nous présentons pour finir des retombées de l'enquête historique et épistémologique sur un travail d'ingénierie qui étudie la construction de situations lors desquelles les élèves peuvent, dans des allers-retours entre objets familiers et objet nouveau, élaborer de nouveaux savoirs.

1 Quelle enquête historique pour rendre compte de l'agir constructif en mathématiques ?

1.1 L'histoire ne trace pas de chemin

Le lien entre une étude méticuleuse du passé et des projets d'action contemporains, en particulier en éducation, ne peut être immédiat et direct. L'histoire ne trace aucun chemin. Toutefois il serait dommage de ne pas considérer l'accumulation des « expériences » passées qui peuvent faire retour dans notre actualité. Mais comment alors concevoir, accueillir ou récuser les analogies ?

Le premier point de vue que nous retenons consiste à personnaliser, contextualiser l'étude des faits mathématiques. Nous cherchons donc en premier lieu à atteindre ce que Veyne appelle des *tranches de vie* :

> Les faits n'existent pas isolément, en ce sens que le tissu de l'histoire est ce que nous appellerons une intrigue, un mélange très humain et très peu « scientifique » de causes matérielles, de fins et de hasards ; une tranche de vie, en un mot, que l'historien découpe à son gré et où les faits ont leurs liaisons objectives et leur importance relative. (Veyne, 1996, p.51)

Alors, la compréhension de l'importance de ce tissu historique nous confirme qu'il est vain de vouloir définir la scientificité en général. Les travaux de Simon (1979, 2008) sur Kepler l'ont ainsi obligé à réintroduire des savoirs jusque-là écartés car jugés a posteriori non scientifiques, et plus globalement à considérer l'ensemble des objets de savoir du savant, pour tenter d'approcher la structure de sa pensée. Comprendre les productions de Kepler nécessite de comprendre le paradigme képlérien, sa rationalité propre, indépendamment d'une rationalité (re)construite a posteriori.

Dans un deuxième point de vue, philosophique et épistémologique, cela revient à placer l'activité du savant au centre de la réflexion sur la production scientifique. Et cela amène à envisager, pour comprendre les objets en jeu dans une recherche, un regard pragmatiste qui :

> accorde à la pratique de la construction, de l'application des règles de fabrication le long du temps opératoire, une sorte de primauté par rapport aux choses prétendant à un statut d'objet, constructions ou symboles. L'objet de la mathématique, soit comme symbole soit comme construction, selon cette vue, « provient » de l'agir mathématique, constructif en l'espèce. (Salanskis, 2008, p.72)

Dans ce paradigme, et dans une perspective très locale d'émergence des savoirs, les recherches historiques peuvent engendrer des questionnements didactiques. En effet, il s'agit, dans tous les cas, d'identifier, dans des agirs constructifs qui permettent l'émergence de nouveaux objets mathématiques, des caractéristiques de l'activité du chercheur. Et, confrontés à un même problème, dans une même position instable non finalisée par un savoir pour l'heure absent, nous pouvons faire l'hypothèse que des sujets différents vont développer à l'égard des objets familiers[1] des relations singulières mais comparables car caractérisant ces phases de recherche. Avant de mettre en évidence ces relations par une étude historique singulière, nous revenons sur une caractérisation possible de l'activité mathématique dans les phases d'émergence d'objets nouveaux[2].

1.2 La dimension expérimentale des mathématiques comme grille de lecture

En mathématiques, un des modes de l'agir constructif est un mode empirique qui permet l'émergence d'objets. Ce mode empirique se traduit par :

> la multiplication des expériences, en appui sur des objets, des méthodes et des connaissances naturalisées pour le sujet, [qui] favorise l'élaboration de nouveaux objets conceptuels et de leurs propriétés, de résultats nouveaux et de leurs preuves, et contribue de manière essentielle au processus de conceptualisation (au sens de Vergnaud). (Durand-Guerrier, 2010, p.5) [3]

Ce mode d'émergence et la démarche associée caractérisent la dimension expérimentale des mathématiques. De nombreux auteurs, Chevallard (1992), Perrin (2007), ont décrit tout l'intérêt de cette dimension expérimentale dans des argumentaires qui amènent par exemple Perrin à formuler une maxime : « *Deux expériences valent mieux qu'une démonstration fausse* », (Perrin, 2007, p.25). Il souhaite ainsi mettre en évidence que dans des phases d'élaboration de nouvelles connaissances, la maîtrise formelle est souvent incertaine pour s'assurer de la vérité d'une conjecture, alors qu'un aller-retour supplémentaire entre objets

[1] Familier signifie « dont la manipulation est devenu facile, aisée, par la pratique, l'habitude ».

[2] Objet nouveau désigne un objet mathématique que le sujet est en train d'explorer et de faire advenir.

[3] Dans ce même texte Durand-Guerrier définit les objets « naturalisés » comme des objets « suffisamment familiers pour que les résultats des actions soient considérés comme fiables […]. Ils permettent donc de valider les hypothèses ou les prévisions et constituent donc en ce sens un domaine d'expérience pour le sujet […] ».

familiers et objet nouveau permet souvent des progrès dans la conceptualisation. Cette dimension expérimentale est une des grilles de lecture de l'activité mathématique, utile quand cette activité relève de la construction d'un modèle, de l'élaboration d'une théorie locale, qui rendent compte des actions sur les objets et des premières conjectures.

L'usage de cette grille de lecture peut orienter les travaux de recherche dans plusieurs directions. Ouvrier-Buffet (2013) et Gardes (2013) ont, par exemple, mis en évidence des invariants dans l'activité mathématique effective d'un sujet confronté à la résolution d'un problème de recherche. Chez ces deux auteurs, les invariants retenus caractérisent essentiellement la méthode mise en œuvre par les sujets et donnent des « indices du progrès » de la recherche. L'approche que nous proposons vise, quant à elle, à identifier dans la relation aux objets des causes des modifications de l'action.

Mais alors que certaines études peuvent s'appuyer sur des échanges avec des chercheurs contemporains, comment envisager, lorsque c'est indispensable, l'étude de l'activité de chercheurs anciens ? L'étude de ce qui est pensable, l'étude de la relation aux objets manipulés sont-elles alors possibles ? Le travail de l'historien peut-il accompagner le travail de l'épistémologue et identifier les tenants et les aboutissants de l'activité à dimension expérimentale en cours ?

2 Kepler et la recherche des pavages archimédiens du plan

Ce travail d'étude du savant cherchant nous l'avons entrepris à partir de l'activité de Kepler à la recherche des pavages archimédiens du plan, c'est-à-dire des pavages stricts du plan dont les tuiles sont des polygones réguliers convexes et tels que les nœuds sont tous identiques.

2.1 Une enquête historique qui interroge l'activité du savant cherchant

L'enquête historique menée vise donc à analyser l'activité de Kepler à la recherche de ces pavages, dans un contexte qui a été par beaucoup jugé a posteriori non scientifique. En tenant l'équilibre entre un familier mathématique qui nous donne aujourd'hui de trop nombreuses grilles de lecture et un inconnu à définir à partir de bien peu d'éléments, il s'agit de comprendre la relation de Kepler aux objets familiers dans son exploration d'un nouvel objet.

Figure 1. Planches de l'Harmonices Mundi[4]

Entre 1595, date annoncée par Kepler du début de ses recherches sur « la belle harmonie des choses immuables » et 1619, date de la parution de l' « Harmonices Mundi », Kepler a, au milieu de ses grandes avancées astronomiques, produit un résultat simple, mais jusque là non encore explicité par quiconque : *il existe exactement 11 pavages archimédiens du plan*[5]. Notons brièvement que les écrits précédents de Pappus sur la question[6] se limitaient à la présentation et la preuve de l'existence des 3 pavages réguliers du plan, et que les écrits suivants, de Badoureau, beaucoup plus tardifs, datent de 1881 et sont indépendants de ceux de Kepler. Badoureau, qui ne mentionne pas Kepler, retrouve plus de 250 ans plus tard un résultat longtemps tombé dans l'oubli. Sans nous attarder sur le fait que ces genèses sont totalement différentes, nous allons tenter de mettre à jour quelques particularités de l'activité d'un savant, Kepler, qui a su dépasser les cadres qui contraignaient ses prédecesseurs.

2.2 Quelques éléments généraux sur la pensée keplérienne

Kepler est Chrétien. Et en tant que savant, il cherche à comprendre l'harmonie du monde, à comprendre le modèle qui aurait servi au créateur comme archétype de la Création. Cette recherche s'appuie sur une formation classique puisque, une fois admis au grand séminaire, Kepler suit un enseignement basé sur le trivium : grammaire, rhétorique et dialectique et sur le quadrivium : arithmétique, géométrie, musique et astronomie. Il n'est donc pas surprenant que ses premières recherches mélangent harmonies planétaires, musicales, polygonales et que les analogies soient très présentes. Mais Kepler est un esprit curieux et il devient autre chose que le dernier des grands anciens :

4 (Kepler, 1940).

5 Pour une démonstration du résultat, nous renvoyons à (Front, 2010).

6 La collection mathématique, (Pappus d'Alexandrie, 1933a,b).

> Dans tous les domaines qu'il aborde, il rompt de manière consciente avec ses prédécesseurs. Même quand il émerge le plus manifestement d'un passé révolu, dans les savoirs que nous considérons comme fossiles, il est profondément novateur. (Simon, 1979, p.449)[7]

Et novateur, il l'est également malgré des positions philosophiques qui resteront identiques à elles-mêmes, tout au long de sa vie.

> Loin d'être desservi par ses a priori métaphysiques, Kepler fut dans sa tâche guidé par eux. […] Il fut constamment aiguillonné par la conviction que rien dans le monde n'est laissé au hasard, et que tout au contraire y obéit au principe du meilleur. La pensée d'une création faite à l'image du Créateur lui servit à la fois de guide et de garant. Mieux même, la forme métaphorique qu'elle prit [...] alimenta ses intuitions les plus profondes. (Simon, 1979, p.453)

Loin d'une image lisse de la science, Simon met ici en évidence la construction de savoirs en appui sur des paradigmes qui aujourd'hui surprennent. Mais il est vrai que Kepler ne se contente pas du socle métaphysique. Il est en effet remarquable qu'il sait, au-delà de ses a priori, produire des théories variées et les soumettre aux retours de l'expérience, celle de son époque, qui est toute observation. Et, formé au contact de Tycho Brahe à l'importance des données astronomiques qui l'ont plusieurs fois amené à revoir ses modèles, Kepler sait être pragmatique. C'est ainsi, par exemple, qu'après avoir longtemps cru à l'identité des harmonies musicales et des aspects, sa soumission aux données l'a progressivement amené à changer d'avis :

> Il est juste sur ces matières aussi d'écouter le témoignage de l'expérience : car c'est elle qui comme pour le reste fait la conviction première, avant les raisons. Kepler cité par (Simon, 1979, p.169)[8]

Ces quelques éléments mettent ainsi en évidence la capacité de Kepler à faire des allers-retours entre théorie en cours d'élaboration et observations, la capacité à mettre en accord la théorie et l'expérience en reprenant aussi bien la première que les résultats de la seconde.

Nous revenons maintenant sur la constitution du domaine d'expérience de Kepler et l'effet de la création d'une théorie structurant l'ensemble des polygones réguliers.

2.3 Kepler et les polygones réguliers

Par la mise au point d'un opérateur conceptuel sur les polygones, Kepler va attribuer aux différents êtres géométriques, par l'intermédiaire de démonstrations géométriques, une plus ou moins grande dignité ontologique. Ceci l'amène ainsi à définir des « êtres connaissables » :

---

[7] Pour cette sous partie nous nous appuyons sur les analyses de Simon qui a su éclairer la rationalité de Kepler et sa capacité à mettre en accord, par un travail patient et inventif, la théorie et l'expérience. Simon a non seulement remis en valeur les travaux de Kepler mais plus fondamentalement par cette étude réaffirmer l'intérêt d'une histoire des savoirs.

[8] Il est notable que certaines traductions de Simon, sont plus pertinentes que celle de (Kepler, 1980), c'est pourquoi nous les préférerons à deux occasions ici.

Est dit connaissable, ce qui est mesurable par le diamètre ou immédiatement par soi-même, soit une ligne, soit une surface par son carré ; ou bien au moins ce qui est formé par une raison sûre et géométrique de telles quantités qui dépendent d'un enchaînement, de n'importe quelle grandeur, enfin pourtant du Diamètre ou de son carré. La démonstration de la quantité, ou à décrire, ou à connaître, est une déduction à partir du diamètre au moyen des intermédiaires possibles. (Kepler, 1980, p.9)

Et, en cohérence, la théorie produit des « non-êtres » ;

Car ici, quant à nous, nous nous occupons d'êtres connaissables ; et nous affirmons à juste titre, que le côté de l'heptagone fait partie des non-êtres ; j'entends connaissables. Kepler cité par (Simon, 1979, p.155).

Cette approche engendre donc, dans le cadre de cette ontologie képlérienne, des modes d'existences des polygones réguliers spécifiques. Et Kepler ne retiendra, dans les « limites de la Notion, de la Science, de la Détermination, de la Description » qu'un nombre réduit de polygones ayant des démonstrations « appropriées ». Ce sont les polygones convexes à 3, 4, 5, 6, 8, 10, 12 côtés auxquels il faut ajouter les polygones étoilés associés au pentagone, à l'octogone, au décagone et au dodécagone. Il nous reste maintenant à voir comment cette organisation des objets, désormais « naturalisés », agit sur la recherche de Kepler des pavages archimédiens du plan.

2.4 La relation de Kepler aux objets naturalisés et l'émergence d'objets nouveaux

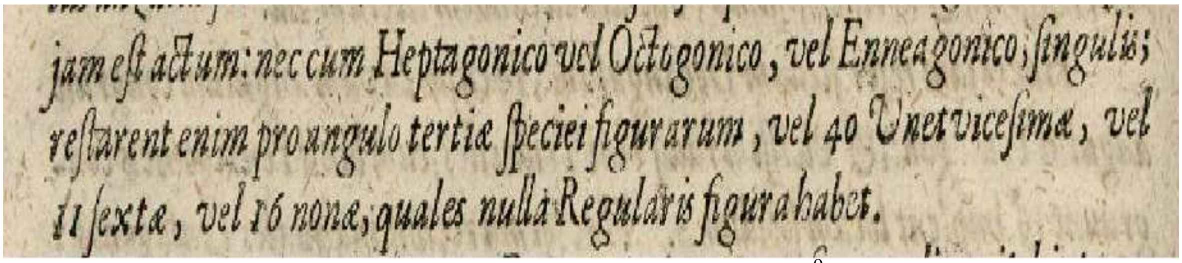

jam est actum: nec cum Heptagonico vel Octogonico, vel Enneagonico, singulis; restarent enim pro angulo tertiæ speciei figurarum, vel 40 Unetvicesimæ, vel 11 sextæ, vel 16 nonæ, quales nulla Regularis figura habet.

Figure 2. Une erreur de Kepler[9].

Kepler travaille depuis 1595 sur les harmonies planétaires, musicales, numériques et astrologiques. Ainsi quand l'harmonices Mundi parait en 1619, nul doute que le savant a eu le temps de peaufiner cette œuvre de maturité sous tous ces aspects. Rappelons également que Kepler est un calculateur hors-pair, particulièrement familier avec les angles des polygones réguliers. Aussi quand, lors d'un passage où il recherche les pavages constitués avec des *angles plans de trois espèces*, il ne « reconnait » pas en 40 vingt-et-unièmes, ou 11 sixièmes, ou 16 neuvièmes [de droit] les angles des polygones réguliers à 42, 24 et 18 côtés, ce n'est certainement pas pour des raisons internes au raisonnement qu'il mène. On peut au contraire y voir une influence du milieu de la recherche et de la relation de Kepler aux non-êtres qui font que certains objets sont de fait exclus des combinaisons possibles. Il est fort probable que la

[9] Un angle du triangle n'est pas joint [...] ni avec l'angle de l'Heptagone ou de l'Octogone ou du polygone de neuf côtés chacun en particulier en effet il reste pour l'angle des figures de troisième espèce ou 40 vingt-et-unièmes, ou 11 sixièmes, ou 16 neuvièmes [de droit], tels que n'a aucune figure Régulière. (Kepler, 1980, p.68)

figure Régulière que Kepler évoque doit avoir, pour lui, d'autres caractéristiques que celles des polygones réguliers que nous connaissons et doit, de façon plus ou moins consciente, renvoyer aux *connaissables*.

Il se trouve que les candidats pavages (3,7,42), (3,8,24), (3,9,18) ne permettent pas, au-delà de l'assemblage local autour d'un nœud, de paver le plan. Cette « erreur » de Kepler ne portera ainsi pas à conséquence. Notons, s'il fallait mettre davantage en évidence la familiarité de Kepler avec les polygones réguliers et les pavages, que sa relation particulièrement forte lui permettra de ne pas passer à côté de la possibilité de réaliser deux pavages énantiomorphes (cf. figure 3) avec l'assemblage (3,3,3,3,6). Notons que cette possibilité ne sera pas perçue par Badoureau qui développera, lui, une approche beaucoup plus formelle.

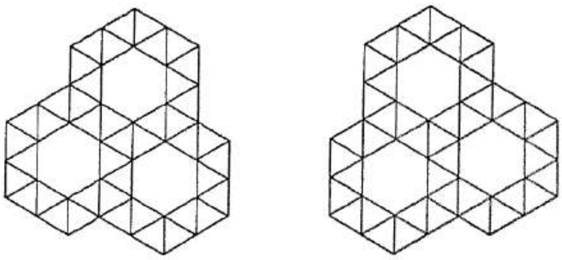


Figure 3. Deux pavages énantiomorphes

L'ensemble des sources montre que les dimensions ontologique et pragmatique se cotoient chez Kepler du début de la recherche (1595) à la rédaction de l'œuvre (1619). L'enquête que nous avons menée, met alors en évidence différents impacts d'une telle relation aux objets. Le positionnement ontologique de Kepler engendre par exemple dans la recherche des pavages une réduction de l'ensemble des objets à considérer et facilite, de façon abusive, une étude exhaustive[10]. Mais par ailleurs, en générant des objets « marginaux », il induit potentiellement des retours d'expériences qui pourraient troubler ultérieurement la nouvelle théorie locale en cours d'élaboration. Kepler a montré qu'il était capable d'intégrer dans la théorie de tels retours d'expérience. Reste à voir si des chercheurs moins expérimentés auront eux aussi cette capacité.

3 Étude historique et ingénierie didactique en vue d'élaborer des situations didactiques de recherche en classe

Le point de vue que nous développons renvoie fondamentalement à l'activité mathématique singulière et en aucune façon à une thèse qui chercherait à faire un parallèle entre la construction du concept au fil des siècles et les possibles élaborations d'élèves. Les genèses singulières, sans lien, présentant chacune des particularités et des imprécisions, nous renvoient beaucoup plus au fait que les connaissances nouvelles produites ont toutes un caractère local. Dès lors, concernant l'élaboration de situations didactiques pour la classe, les questionnements produits par ces retours du passé nous ont amenés à considérer

[10] On retrouve ici l'idée que toute idée « fausse » n'est pas nécessairement, et tout au moins dans un premier temps, un obstacle à la découverte.

particulièrement deux axes. En premier lieu une réflexion sur l'activité qui amène à penser le problème comme essence d'une situation didactique, en garantissant une activité de l'élève à forte dimension expérimentale. En second lieu l'observation de cette activité et particulièrement l'analyse de la relation des élèves aux objets en situation.

3.1 Des situations didactiques où la recherche de problèmes est première

Nous nous centrons, dans l'élaboration de situations didactiques, sur les phases adidactiques à fort caractère expérimental. Dans des situations didactiques de recherche de problèmes construites pour favoriser l'émergence d'objets nouveaux dans des allers-retours avec des objets rendus familiers par l'usage et la construction de savoirs[11], nous étudions alors particulièrement le rôle des relations aux objets. L'étude des pavages archimédiens du plan se révèle doublement favorable, d'une part par l'appui possible sur des objets familiers à un grand nombre d'élèves (les polygones réguliers), et d'autre part par la possibilité d'émergence relativement aisée d'éléments structurant potentiellement le nouvel objet. Il s'agit par exemple de lois locales (*la somme des angles autour d'un noeud vaut un plein*, *les côtés sont nécessairement de même longueur*, *...*) mais également de contigences globales, avec par exemple le fait que « *la somme des angles autour d'un noeud vaut un plein* » n'est pas une condition suffisante pour la réalisation d'un pavage.

La mise en oeuvre d'une situation didactique explorant les pavages archimédiens du plan permet donc la vie de la dimension expérimentale de l'activité mathématiques des élèves. Des expérimentations permettent alors de constater, qu'en fonction de relations différentes aux polygones réguliers, des élèves vont s'orienter sur des voies différentes, complémentaires dans la détermination des pavages archimédiens du plan. Les uns, peuvent s'engager sur la détermination d'une formule donnant l'angle d'un polygone régulier convexe et déterminer l'ensemble des pavages réguliers du plan alors que d'autres développeront des résultats à un niveau plus global.

Nous pouvons alors affiner l'étude du rôle de la relation aux objets et en particulier observer d'éventuelles difficultés à intégrer certains polygones dans le domaine d'expérience, voire à refuser quils y entrent au nom d'une théorie, l'usage d'analogies, de métaphores, d'icônes de l'objet manipulé, ...

3.2 Analyses de productions d'élèves explorant les pavages archimédiens du plan

De la même façon que les productions de Kepler à la recherche de pavages harmonieux montrent le plaisir du savant à multiplier la variété des constructions possibles, et une jubilation à mettre en valeur la richesse combinatoire des figures géométriques, les productions d'élèves ou d'étudiants dans le cadre d'une situation didactique de recherche de problèmes fondée sur ce problème sont elles aussi très riches.

[11] Au sens ici de Conne : « Lorsque le sujet reconnaît le rôle actif d'une connaissance sur la situation, pour lui, le lien inducteur de la situation sur cette connaissance devient inversible, il sait. Une connaissance ainsi identifiée est un savoir, c'est une connaissance, utile, utilisable, dans ce sens qu'elle permet au sujet d'agir sur la représentation ». (Conne, 1992, p. 235)

Les figures 4 et 5 montrent deux extraits de la production d'un étudiant de terminale S, à l'issue d'une recherche individuelle puis en groupe.

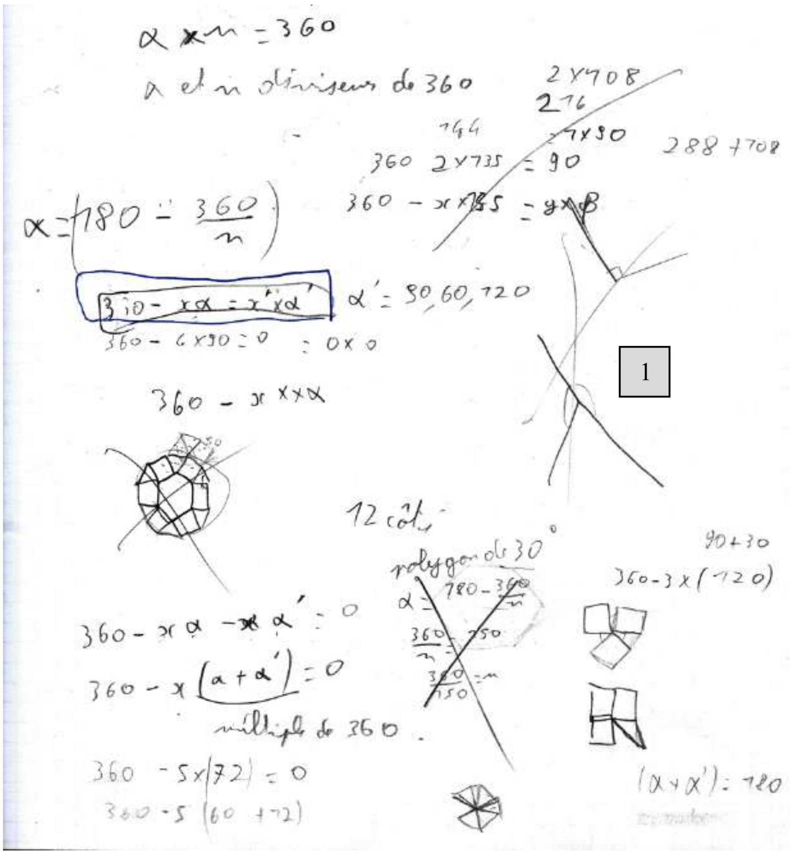


Figure 4. Extrait 1 de la production de Thimoté.

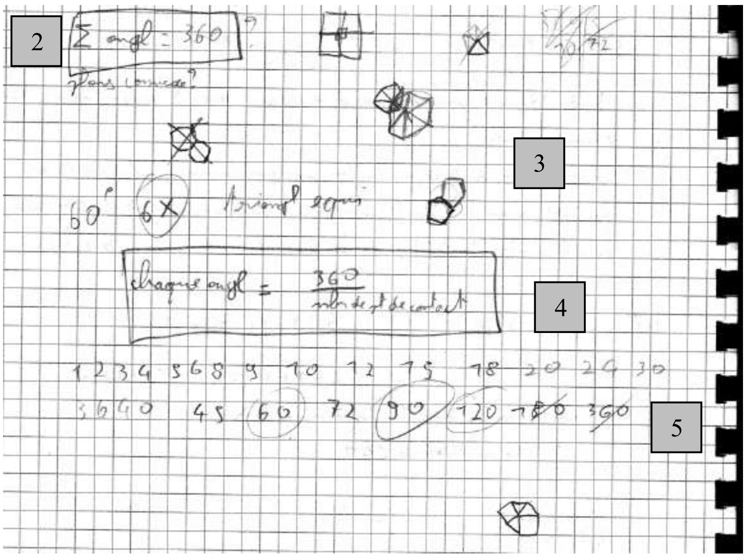


Figure 5. Extrait 2 de la production de Thimoté.

L'analyse de productions de ce type nécessite un outil qui puisse rendre compte de la dimension dynamique de la production et de l'évolution de la recherche vers une structure non prédéfinie. Nous ne développons pas ici la technique utilisée qui s'appuie sur la sémiotique de Pierce. Elle permet, par exemple pour la production de Thimoté, de mettre en évidence la succession de significations émergentes (sémiose). Ainsi, initiée par un premier geste de la main[12] faisant exister un noeud imaginaire, la sémiose se poursuit par le tracé d'un noeud[13], comme le montre le repère 1 de la figure 4. L'évolution se poursuit par l'apparition de signes régis par la symbolique mathématique (repère 2 de la **figure 5**) qui montrent l'entrée de la sémiose dans une dimension formelle. Le repère 3 montre une nouvelle étape avec le changement de statut des relations mathématiques lors d'interactions avec les objets familiers. Les repères 4 et 5 montrent un travail dans le nouveau modèle, travail qui permet, in fine, la détermination des pavages réguliers. La sémiose se poursuivra désormais (figure 4) dans le contexte général de pavages utilisant plusieurs types de polygones.

3.3 Une similitude

Dans cette élaboration d'un objet nouveau, ce qui est également particulièrement intéressant c'est d'identifier dans les interactions de Thimoté aux objets des phénomènes similaires à ceux identifiés dans la recherche de Kepler. Il est ainsi possible de constater en début de recherche la mise à l'écart du pentagone régulier. Cette mise à l'écart intervient quand Thimoté ne parvient pas à construire ce polygone malgré un appui sur un travail préliminaire ayant produit une liste des angles des premiers polygones réguliers et une liste des polygones constructibles à la règle et au compas. Ainsi, des essais malheureux finiront pas exclure le pentagone du milieu objectif, « *Ça n'existe pas, cette figure n'existe pas* », sans que cela pénalise la recherche. La réintroduction du pentagone interviendra après la détermination des pavages réguliers, quand l'association de deux polygones de types différents amène Thimoté à considérer des relations de la forme $360 - x\alpha - x'\alpha' = 0$. L'introduction du décagone ($x = 1$ et $\alpha = 144$) fait alors dire à Thimoté : « Ça fait 2 fois 108 ? Donc ça veut dire que si on a un décagone, on peut mettre deux figures à 5 cotés autour. Les 5 côtés c'est un peu bizarre mais ... ». Ainsi le pentagone, exclu dans un temps du domaine d'expérience, le réintègre quand le modèle en cours d'élaboration le nécessite.

Cette similitude dans le « jeu » avec les objets nous renvoie en premier lieu à l'intérêt de considérer ce jeu dans notre réflexion sur l'agir en mathématique, et donc également à l'importance de construire des situations didactiques qui permettent ce jeu et les constructions de savoirs associés. Elle nous conforte aussi dans notre approche historique et dans la pertinence d'études qui s'astreignent à atteindre ce degré de granularité dans l'observation de l'émergence des savoirs en mathématiques.

4 Conclusion sur les apprentissages

Le modèle d'analyse que nous utilisons pour analyser les processus d'élaboration d'un nouvel objet mathématique croise une approche sémiotique, qui rend compte de la dynamique

[12] Qualisigne iconique rhématique.
[13] Sinsigne iconique rhématique.

récursive de construction de l'objet exploré, et la dialectique entre cet objet et les objets naturalisés. La dialectique objets naturalisés-objet nouveau évolue au fil de l'avancée de la recherche. Dans un premier temps, elle est portée par la relation première aux objets naturalisés, relation qui influe fortement sur la direction prise et les premières structurations de l'objet à explorer. Puis la dialectique s'enrichit et peut produire une évolution de la relation aux objets.

C'est l'existence de la relation singulière aux objets et l'influence qu'elle peut avoir sur l'avancée de la recherche que nous venons de donner à voir aussi bien chez Kepler que chez un élève de terminale impliqué dans une situation didactique de recherche de problème. Dans tous les cas nous mettons en évidence des relations très personnalisées et qui portent en elles leviers et obstacles à la résolution du problème. Ces relations, constatées dans l'enquête historique et présentes en situation didactique, sont résistantes aux tentatives superficielles de modifications et doivent être intégrées à la réflexion sur l'élaboration de situations didactiques. S'en servir comme levier revient donc à les laisser vivre et agir, parce que localement elles permettent l'avancée de la recherche. Éviter qu'elles ne deviennent des obstacles, c'est permettre la confrontation et la régulation des points de vue en classe.On l'aura compris, notre approche s'appuie sur une vision constructiviste qui souhaite donner la possibilité aux élèves de s'engager dans une recherche et de construire des savoirs sur un objet à explorer. Thimoté, et les autres élèves considérés dans les expérimentations réalisées, ont construit, dans des dynamiques dans un premier temps très personnalisées, des savoirs, pour certains conformes aux savoirs institués, pour d'autres erronés ou imprécis, comme ont pu l'être en leur temps les savoirs produits pas Kepler et Badoureau. Dans la classe, donner le temps aux savoirs de vivre avant institutionnalisation, c'est favoriser une nouvelle vision des savoirs, c'est casser cette idée du savoir-absolu, pré-existant à l'exploration, qui prive élèves et enseignants de toute liberté dans l'acte de recherche. Si un autre parallélisme entre Kepler et Thimoté peut être fait, c'est bien celui qui met en évidence la richesse de l'imagination pour peu que les conditions d'une réelle activité mathématique soient réunies.

REFERENCES

Chevallard, Y. (1992). Le caractère expérimental de l'activité mathématique. *Petit x*, *30*, 5–15.

Conne, F. (1992). Savoir et connaissance dans la perspective de la transposition didactique. *Recherche en didactique des mathématiques*, *12(*2.3), 221-270.

Durand-Guerrier, V. (2010). La dimension expérimentale en mathématiques, enjeux épistémologiques et didactiques. In *Expérimenter des problèmes de recherche innovants en mathématiques à l'école (cédérom)*. INRP.

Front, M., & Legrand, P. (2010). Pavages semi-réguliers du plan. *Bulletin de l'APMEP*.

Gardes, M. (2013). *Étude de processus de recherche de chercheurs, élèves et étudiants, engagés dans la recherche d'un problème non résolu en théorie des nombres.* Thèse de doctorat, Université de Lyon.

Kepler, J. (1940). *Harmonices Mundi*. München: C. H. Beck.

Kepler, J. (1980). *Harmonie du monde. Harmonices Mundi. Traduit du latin avec un avertissement et des notes par Jean Peyroux.* Blanchard.

Ouvrier-Buffet, C. (2013). *Modeling of the defining activity in mathematics and of its dialectic with the proving process Epistemological study and didactical challenges*. Paris VII : Habilitation à diriger des recherches, Université Paris-Diderot.

Pappus d'Alexandrie (1933a). *La Collection mathématique, avec une introduction et des notes de Paul Ver Eecke. Tome premier*. Paris and Bruges : Desclée De Brouwer. Réédition Blanchard, Paris 1982.

Pappus d'Alexandrie (1933b). *La Collection mathématique, avec une introduction et des notes de Paul Ver Eecke. Tome second*. Paris and Bruges : Desclée De Brouwer. Réédition Blanchard, Paris 1982.
Perrin, D. (2007). L'expérimentation en mathématiques. *Petit x*, *73*(6), 34.
Salanskis, J.-M. (2008). *Philosophie des mathématiques*. Vrin.
Simon, G. (1979). *Kepler astronome astrologue*. Gallimard.
Simon, G. (2008). *Sciences et histoire*. Bibliothèque des histoires, ISSN 0768-0724.
Veyne, P. (1996). *Comment on écrit l'histoire : Texte intégral*. Points. Histoire. Seuil.

ANALYSIS AND GEOMETRY IN THE DEVELOPMENT OF THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER IN THE 18thAND THE 19thCENTURY

An Example of Ideas which don't appear in undergraduate Mathematics

Gerard Emile GRIMBERG
Institute of Mathematics, Federal University of Rio de Janeiro, Av. Athos da Silveira Ramos 149, Centro de Tecnologia - Bloco C Cidade Universitária - Ilha do Fundão. Caixa Postal 68530 21941-909 Rio de Janeiro - RJ - Brasil
gerard.emile@terra.com.br

ABSTRACT

In an attempt to develop a new way of teaching mathematics, we can use the history to find the important ideas that articulated the now classical mathematical theories. My goal is to illustrate through an example, the Partial Differential Equations of the first order, how the different disciplines interact within a theory. This contribution aims indeed to show how the PDEs of first order constitute a point of convergence between geometry, algebra and analysis in the 19th century. Differential calculus of several variables has its origin in the study of the geometric properties of curves and some mechanical problems. Resolutions of PDEs use first analytical methods until Lagrange and Monge who provide a geometrical interpretation. The research about generalization of PDEs of first order in n variables oblige the mathematicians to use uniquely analytical methods in the first half of 19th century (Pfaff, Cauchy, Jacobi). The development of projective geometry, the birth of theory of groups, the geometrical vision of algebraic theory of invariants give the conditions for reinterpreting the general PDEs with geometrical methods (Lie, Klein). A few observations upon the teaching of Lie's Theory will follow by way of conclusion.

1 Introduction

In history many of mathematicians (including Descartes and Leibniz) stressed the difference between the *Ars inveniendi* and the *Ars expoendi* in mathematics. Concern for rigor and formalization inherited from Bourbaki has reinforced this trend to the point that the exhibition of the mathematical theories of the curriculum of graduation became dogmatic, separating the disciplines (algebra, analysis, linear geometry algebra, etc.) and deleting connections and ideas that were at the origin of their development. In an attempt to develop a new way of teaching, we can use history to find the important ideas that articulated the now classical mathematical theories. My goal is to illustrate through an example, the Partial Differential Equations (PDEs) of the first order, how the different disciplines interact within a theory.

In view of the very short time available, I will limit my talk to the development of PDEs of the first order in the 18th and 19th centuries. I pretend to show that the development of these theory is a great testimony of the close relations between geometry, algebra and analysis. For that purpose I will describe the historical conditions this theory has crossed and follow the different stages of its development which are: 1) The birth of this theory; 2) The first methods of resolution of such equations; 3) The geometric interpretation of Monge; 4) The development of analytical methods in the first half of 19th century; 5) And the great synthesis (geometric, algebraic and analytic) realized by Lie in the decade of 1870's.

2 The birth of PDEs

In accordance with (Engelsman, 1982; Grimberg, 2009), partial differentiation arose in the decades 1680-1720 from the problems enunciated by Leibniz and John Bernoulli. In this time the concept of function does not exist. Curves are described through their equation, what we would call today implicit functions. And differential calculus operates on every variable involved in the equation. Partial differentiation is then already included in the leibnizian differential algorithm and appears naturally from the problems Leibniz and Bernoulli began to treat analytically in this period, such as the envelop of parametrized curves, orthogonal trajectories, brachistochrone curves, and isoperimetrical problems. In these problems the solution involves the parameter of a family of curves and differentiation according to the parameter which was called "differentiation from curve to curve". These problems bring out also the concept of function and Euler in the 1730's reorganizes the differential calculus around the basic concept of function of one or several variables. He defines implicit functions and discusses the problem of conditions for a differential form to be complete. Independently Fontaine and Clairaut arrive to the same results (Grimberg, 2009). In this time what we call now partial differentials appear only as coefficients of a differential form. For instance, A and B in $Adx + Bdy$. The first application in mechanics appears in 1743 when Clairaut (1743) shows that a necessary equilibrium condition of a Fluid submitted to force field of components (P,Q,R) is that the force field is conservative.

In the same time the first PDE appears in the D'Alembert's 1743 *Traité de dynamique* through the study of compound pendulum. Other PDEs appear in D'Alembert's memoirs (1747, 1749a, 1749b, 1749c). The D'Alembert's methods of resolution consists in linear change of variables or what we call today Lagrange multipliers, and in the case of vibrating cords, the method of separation of variables.

The second stage is the various contributions of Euler in the same problems of mechanics, other methods of resolution, and Euler's Fluid equations. In this time the PDEs appear from the study of geometrical infinitesimal properties of the problems led by the geometric diagram. The complete analytical formulation of this type of problem will be so obtained with the *Mécanique analytique* of Lagrange where Lagrange affirms bravely in his preface that

> in this treaty you will not find diagrams, the methods I expose require neither constructions, nor geometric, nor mechanics reasoning, but only algebraic operations submitted to a regular and uniform running.

2.1 A first stage in analytical resolution of PDEs of first order.

The methods of resolution of PDEs follow the process of algebraization of the problems of mechanics. The first important stage in this process is realized by Euler in his treaty of integral calculus were he considers the partial differential equations of first order as implicit functions of 5 variables, the three variables of space were the third z is function of x and y and the two other variables are the partial differentials of z.

Lagrange elaborates a new conception of the nature of solutions. Considering an implicit function $V(x, y, z, a, b) = 0$ of three variables x, y and z, and two parameters, a and b, z being function of x and y, Lagrange shows that we can interpret such a function as what he calls a "complete integral" of a PDE of first order. The PDE $Z = 0$, originated from $V(x, y, z, a, b) = 0$, is indeed obtained resolving the system $V = 0$, $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, where a and b are eliminated (Lagrange, 1774, p. 239). From this fact, he deduces

> that all complete integral of all first order PDEs in three variables have to contain two arbitrary constants.

In this memoir, the relation between equation $V = 0$ and $Z = 0$ is based on a geometrical vision of the problem. The equation $V = 0$ representing a parametrized family of surfaces verifying the PDE, the envelop of this family verifies also the equation, but this time the particular equation $Z = 0$ without the parameters a and b. But this geometric insight established by Lagrange do not lead to the analytical method of resolution, even if we can see here the geometrical way of thinking of Lagrange.

2.2 The geometrical interpretation of Monge

The geometrical interpretation of first order PDE had to wait the works of Monge elaborated in the decade 1780 and gathered in his 1807 treaty *Application de l'Analyse à la Géométrie*. In this treaty, Monge realized a general study of surfaces and curves by analytical means and characterized developable surfaces and surfaces of revolution with PDEs, he defines the tangent plane and the normal of a surface, and characterizes the surface by the radius of curvature and as envelop of osculating circles. Monge introduces the concept of characteristic curves in the resolution of first order PDE's. In his book, the geometrical interpretation of PDEs of Lagrange, by simple analogy, turns out to be a crucial point of the theory (Monge, 1807, p. 369).

3 The analytical methods of resolution in the first half of the 19th century

The deep reason why the first further developments of the methods of resolution of PDE were analytical consists in the fact that the geometry for four and more dimensions was yet to be done while a few problems of mechanics which appeared from the works of Lagrange, Poisson, Jacobi involved more than three variables and consequently PDEs with many variables. Then a geometrical interpretation was not in this time possible. It explains the analytical way of Pfaff, Cauchy and Jacobi. A huge development of geometry, algebra and analysis will be necessary to join the conditions of the great synthesis realized by Lie.

Pfaff was then the first in a memoir dated of 1814 to begin an elaboration of methods of resolutions of PDEs involving n variables. He shows how to eliminate one by one the variables (Pfaff, 1814). And then he considers the equation $f(x_1, x_2, ..., x_n, z, p_1, ..., p_n) = 0$, the p_i being the n partial differentials of z, sets down $p_n = \varphi(x_1, \ldots, x_n, z, p_1, \ldots, p_{n-1})$, and integrates the equation:

$$dz - p_1 dx_1 - p_2 dx_2 - \ldots - p_{n-1} dx_{n-1} - \varphi(x_1, x-2...., x-n, z, p_1, p_2, ..., p_{n-1} dx_n) = 0.$$

The demonstration was not complete and it will be achieved by Cauchy and Jacobi. Demidov observes that this equation leads to the resolution of n systems, one of which is (Demidov, 1982, p. 334):

$$\frac{dx_i}{\frac{\partial f}{dp_i}} = \frac{dz}{\sum_{x=1}^{n} p_k \frac{\partial f}{\partial p_k}} = \frac{dp_i}{-\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial z} p_j}.$$

Cauchy comes back to the general resolution of PDEs of three variables (Cauchy, 1819), and show how initial conditions lead to a solution. Cauchy constructs also solutions by means of characteristic curves even if he does not use this term. Considering an equation involving three variables x, y, z, he uses a change of variables, initial conditions, and builds solutions consisting of characteristics passing trough the curve $(x = x_0,\ z = \varphi(y))$.

Finally Jacobi had elaborated two methods to solve a system of equations, the first in an article (Jacobi, 1837), but the second is more interesting even if it was edited after his death (Jacobi, 1862) by Clebch. This is the second method we want to describe.

Jacobi considers a system of n equations $f_i(x_1, x_2, ..., x_n, p_1, p_2, ..., p_n) = h_i$ were the parameters h_i are arbitrary constants. The p_i are functions of variables x_i with the initial equation $f_0(x_1, x_2, ..., x_n, p_1, p_2, ..., p_n) = 0$. The p_i are functions of the x_i and are partial differentials which verify the condition that $p_1 dx_1 + p_2 dx_2 + ... + p_n dx_n$ is a total differential. For this, the necessary and sufficient condition is:

$$(f_i f_k) = \sum_{l=1}^{n} \frac{\partial f_i}{\partial x_l}\frac{\partial f_k}{\partial p_l} - \frac{\partial f_i}{\partial p_l}\frac{\partial f_k}{\partial x_l} = 0, \quad i, k = 0, 1, ..., n.$$

Jacobi uses indeed the Poisson Brackets which turn possible an expression of the system which he can solve beginning from the known function f_0 and successively determining the functions f_1, f_2, etc. (Demidov, 1982, p. 337-339).

In this method Jacoby is dealing with what we call today differential operators as $A(f) = \sum_{i=1}^{n} A_i(x_1, x_2, ..., x_n)\frac{\partial f}{\partial x_i} = 0$, the condition determining solutions of the system of equation, as $A(B(f)) - B(A(f)) = 0$ which leads to the so-called Jacobi's identity, Jacobi being the first relating this property to the Poisson Brackett (Hawkins, 2000, p. 48): f, g, h, being function of $2n$ variables, x_i, p_j,

$$((f, g), h) + ((g, h), f) + ((h, f), g) = 0.$$

4 Sophus Lie

The following stage of the theory represents a great revolution which is also related to the reorganization of mathematics, especially the new vision of geometry realized by Klein in the same time of elaboration of Lie's theory. But before explaining the importance of Lie's works we have to explain the context and what was the background of Lie and Klein. In the second half of 19th century, the works of Plücker were edited in Europe offering an analytical view of projective geometry. The idea of homogeneous coordinates allows the generalization of the concept of projective space. And especially to pass from the real projective space to the complex projective space and then work in space of higher dimension than 3. In the same time the works of Grassmann, Hamilton, Cayley developed new connections between geometry, algebra and analysis. They realized the condition for a geometric investigation of n-dimensional geometry.

For instance Cayley elaborates the theory of invariants using the homogeneous coordinates defined by Plücker and uses this geometrical view to relate projective and euclidean geometry, especially in his famous 1859 Sixth Memory which became the most important source of inspiration for Klein in his deduction of non-euclidean geometry from the projective geometry and further elaboration of Erlangen's Program.

4.1 Sophus Lie and Felix Klein in Paris

The theory of groups with Silow and Jordan became also a basis of the reflexion of Klein and Lie by 1870. Lie indeed met Klein in Berlin in 1869, and visited France with him in 1870, he traveled to England and came back in Göttingen in 1872 were he was also with Klein. It's very difficult to really separate the reflexion of the two mathematicians in this period (Hawkins, 2000, p. 10-30). It was just after this period that Lie elaborated his theory of n-variable partial differential equations.

Berlin was the center of analytical research with Weierstrass, Kummer and Kronecker but the source of inspiration was more in Göttingen, with Plücker (Clebsh and Klein edited the posthumous work). Staying in Paris, Klein and Lie studied the works of Jordan and had long discussions with French mathematicians such as Darboux (Klein, 1892).

The first works of Lie were about sets of lines in three dimensional projective complex space. The approach of the tetrahedron in this space is really in the spirit of the last works of Plücker. Lie's investigation, following (Hawkins, 2000, p. 2-6) consists in the consideration of a tetrahedron Δ of the complex projective space. A tetrahedron line complex Δ is determined by 4 planes. Each line meets Δ in four points. Each line intercepts in four points. Then he considers the set of lines T for which the cross ratio is the same. Lie considers then the set Θ of all projective transformations which let the vertices of the tetrahedron Δ invariant. Then for any given line he studies the orbit of this line under the projective group Θ. Another object of common research with Klein was the discovery and the study of what they called W-curves.

Two other crucial concepts in the investigation of Lie were the concept of infinitesimal transformation and contact transformation. An infinitesimal transformation is a function $x \mapsto x + dx$ which is defined by a system of linear differential equation in $\mathbb{R}^4$. These transformations form a commutative group and Lie and Klein used this tool in the study of W-curves. The concept of contact transformation is really close to Lie's method of resolution of PDEs as we will see now.

4.2 Lie's method of resolution

Lie (1872, 1873b, 1873a) interprets the equation $f(x, y, z, p, q) = 0$ as a four dimensional manifold of $\mathbb{R}^5$. The integration of equation means the determination of all manifolds $M_k, k \leq 2$ whose points satisfy

1. the equation;
2. the condition $dz - pdx - qdy = 0$.

With this interpretation, Lie can give a geometrical vision of solutions in terms of manifolds (Demidov, 1982, p. 343). In this theory the contact transformations play an important role.

Defining a contact transformation as a transformation which preserves the tangent, Lie shows that there exists a contact transformation which transforms a PDE equation in any given PDE. As Demidov observed, the demonstration of Lie was not completely rigorous. But this vision of PDEs was entirely new, and Lie's geometrical insight was crucial in this conception. The Lie conception also leads by mean of infinitesimal transformation to the identity of Jacobi, which represents indeed the beginnings of Lie' s Algebra.

In the following years, Lie will try to apply his method to second degree PDEs, and this theory will be developed by other mathematicians especially by Elie Cartan, but this is another story.

The history of PDEs of first order is a good example how a problem suffers many transformations in the time, both in the theories they required and the terms in which the problem is posed. I would wish to use the beautiful metaphor that Berger (2013) use in his book, *Geometry revealed*, that of Jacob's Ladder. To arrive to his theory, Lie had to go up in the ladder posing the problem in the new terms of differential geometry in 5-dimensional space, developing theory of contact and infinitesimal transformations. But perhaps this growth in the ladder did not turn him nearer God, because as a great mathematician of 20th century said: "we don't know if God exists, but the Book certainly does", and Lie, certainly too, wrote a few pages of the Book.

5 Conclusion

What can we deduce from this story for the elaboration of a new program of undergraduate mathematics ? A first idea is the connection between all disciplines, algebraic, analytical and geometrical methods are going together in solving problems and the exposition of graduate mathematics have to deal with this fact.

A second observation is related to the geometrical vision the student have to cultivate, even in more abstract algebra or analysis, a way Sobczyk (2013), for instance, had worked out in mathematics, and before Giaquinto (2007) tried to give an epistemological response. We have to break with the walls which separate disciplines in graduate mathematics if we want to train mathematicians and not scholastic students.

We have to emphasize finally that this path was already indicated in (Howe, 1983). In this article, as Howe enhances that Lie's Theory was taught only at graduate level, he insists too on the possibility to teach this topic in the undergraduate level (*loc. cit.* p. 601):

> While a complete discussion of Lie's Theory does require fairly elaborate preparation, a large portion of its essence is largely accessible on a much simpler level, appropriate to advanced undergraduate instruction.

More recently, Dresner (1999) wrote a text book which expounds the basics of Lie's theory of ordinary and partial differential equations. This book shows that it is possible to teach these ideas without waiting until the graduate level.

REFERENCES

Berger, M. (2013). *Géométrie vivante ou l'échelle de Jacob.* Cassini.

Cauchy, A. L. (1819). Note sur l'intégration des équations aux différences partielles du premier ordre à um nombre quelconque de variables. *Bulletin de la Société Philomatique de Paris*, *Année 1819*, 10-21.

Clairaut, A. C. (1743). *Théorie de la figure de la terre; tirée des principes de l'hydrostatique*. Paris.

D'Alembert, J. (1747). *Réflexions sur la Cause générale des vents*. Paris.

D'Alembert, J. L. R. (1743). *Traité de dynamique*. Paris.

D'Alembert, J. L. R. (1749a). Recherches sur la courbe que forme une corde tendue, mise en vibration. *Hist. de l'Ac. des Sc. de Berlin*, *3*, 214-219.

D'Alembert, J. L. R. (1749b). Suite des recherches sur la courbe que forme une corde tendue, mise en vibration. *Hist. de l'Ac. des Sc. de Berlin*, *3*, 220-229.

D'Alembert, J. L. R. (1749c). *Theoria resistentiae quam patitur corpus in fluidomotum, ex principiis omninò novis et simplissimis deducta, habitâ ratione tum velocitatis, figurae, et massae corporis moti, tum densitatis et compressionis partium fluidi, publisher=Manuscript in Berlin-Brandenburgische Akademie der Wissenschafte Akademie-archiv-I-M478*.

Demidov, S. S. (1982). The Study of Partial Differential Equations of the First Order in the 18th and 19th century. *Archives for History of Exact sciences*, *26*(4), 325-350.

Dresner, L. (1999). *Applications of Lie's Theory of Ordinary and Partial Differential Equations*. Institute of Physics Publishing, London.

Engelsman, S. B. (1982). *Families of Curves and the Origins of partial diferenciation*. North Holland.

Giaquinto, M. (2007). *Visual Thinking in Mathmatics, An epistemologica Study*. Oxford University Press.

Grimberg, G. E. (2009). Elaboração do cálculo diferencial leibniziano, A Gênese da teoria das funcões de várias variáveis no século XVIII. In D. Flament & W. Barroso (Eds.), *Dualidade Álgebra Geometria* (p. 59-137). Casas das Musas Brasilia.

Hawkins, T. (2000). *Emergence of the Theory of Lie Groups*. Springer.

Howe, R. (1983). Very Basic Lie Theory. *The American Mathematical Monthly*, *90*(9), 600-623.

Jacobi, C. G. (1837). Über die Reduction der integration der partiellen Differentialgleichungen Erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systems gewöhnlicher Differentialgleichungen. *J. reine und angew. Math.*, *17*, 1-189.

Jacobi, C. G. (1862). Über die Reduction der integration der partiellen Differentialgleichungen Erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systems gewöhnlicher Differentialgleichungen. *J. reine und angew. Math.*, *60*, 1-181.

Klein, F. (1892). *Über Lie's und meine Arbeiten aus den Jahren 1870-72 Von F. Klein und S. Lie. Über Lie's und meine ältere geometrische Arbeiten, von F. Klein (Erster Borkumer, Entwurf von August 1892)*. Manuscript, NSUB Göttingen, Cod. Ms f. kLEIN 22 GII, 24 p.

Lagrange, J. L. (1774). Sur les integrales particulières des équations différentielles. *Nouveaux Mémoires de l'Académie royale des Sciences et Belles-Lettre de Berlin*, *5*, 197-275.

Lie, S. (1872). Zur Theorie partieller Differentialgleichungen erster Ordnung, inbesondere iiber eine Klassifikation derselben. *Gottinger Nachrichten*, *1872*, 473-489.

Lie, S. (1873a). Partielle Differentialgleichungen erster Ordnung, in denen die unbekannte Funktion explizite vorkommt. *Forhandlinger Christiania*, *1874*, 52-85.

Lie, S. (1873b). Uber partielle Differentialgleichungen erster Ordnung. *Forhandlinger Christiania*, *1874*, 1-51.

Monge, G. (1807). *Applications de l'analyse à la géométrie*. Paris.

Pfaff, J. F. (1814). Methodus generalis, aequationes differentiarum particularum, necnon aequationes differentiales vulgares, utrasque primi ordinis inter quotcumque variabiles, complete integrandi. *Königlichen Akademie der Wissenschaften zu Berlin*, *1814-1815*, 76-136.

Sobczyk, G. (2013). *New Foundations in Mathematics, The Geometric Concept of Number*. Birkhaüser.

THE APPLICATION OF HPM VIDEO CLIPS IN MATHEMATICAL TEACHING IN MIDDLE SCHOOL

Teaching the application of linear equation with one unknown

Hong Yan-jun, Chen Ping
Department of Mathematics, East China Normal University, Shanghai, China
hyj_tea@sina.com
Akesu No.3 Middle School, Xinjiang, China

ABSTRACT

The history of mathematics is of great importance to mathematical teaching. To give full play to its educational value and to solve the dilemma faced by middle school teachers, the authors shoot some video clips with math-related historical materials and bring them into classroom. It proves that these video clips not only play a guiding role in the classroom, but also broaden students' horizon of knowledge. It's constructive to reach aim of teaching in the syllabus. The goal of teaching is to master knowledge and skills, it also helps to understand the process and methods, attitude and values are to be judged. Meanwhile, it provides a new way of thinking for the study on integrating history of mathematics into mathematical teaching.

1 Introduction

One of the major studies in contemporary mathematics education focuses on the relations between History and Pedagogy of Mathematics (HPM). Recent years have witnessed a great number of researches on applying history in mathematical teaching. The HPM study group at East China Normal University believes that the historical materials selected for mathematics teaching must comply with the five principles of being interesting, scientific, effective, learnable, and innovative (Gao & Hu, 2014). The group classifies all HPM teaching methods as the approach of complementation, replication, accommodation, and reconstruction (Wang, 2012, see table 1). In his idea, there is no good or bad between these four methods. We may choose one or more methods according to the curriculum standards and students' aptitude. They also set up a standard teaching process, i.e. choosing a teaching subject → investigating related history → selecting suitable materials → analyzing classroom requirements → designing classroom activities → implementing teaching design → evaluating the course.

Table 1. Approaches of using history of mathematics in teaching

Approaches	Description
Complementation	Display mathematicians' pictures, give an account of related stories, etc.
Replication	Directly using historical problems, methods, etc.
Accommodation	Problems adapted from historical ones or based upon historical materials
Reconstruction	Genesis of knowledge based on or inspired by the history of mathematics

The improvement of HPM related theoretical researches and the development of related cases have caught the attention of middle school math teachers. They would like to have a try in their own teaching practice. However, they often feel perplexed about how to acquire reliable historical materials and how to effectively integrate these materials into class teaching. There's a seemingly insurmountable valley between academic researches and their application in middle school classrooms (Zhang & Wang, 2009).

As information technology advances rapidly, multimedia video learning has been witnessing a sound momentum in terms of development and utilization. Among which, the video clip teaching, with the characteristics of short and exquisite, lively and convenient, and dynamic and repeatable, has been applied in many of the education and teaching levels, and is gradually becoming the hotspot and frontier of education and teaching reform.

High School Mathematics Curriculum Standards (Provisional) promulgated by the Chinese Ministry of Education states that promoting the integration of information technology and mathematical teaching would have a great impact on course contents, teaching and learning, and enable students to understand the nature of mathematics (He,2012). The purpose of this research is to make fully use of video clips, to resolve teachers' perplexities, and to provide a freer and broader platform for integrating the history of mathematics into practical teaching. The authors took math-related historical materials in some video clips which has educational values. It helps and expands the students' knowledge and reach aim of teaching in the syllabus. The goal of teaching is to master knowledge and skills, it also helps to understand the process and methods, attitude and values are to be judged. These short clips are called HPM video clips.

2 The teaching application of HPM video clips in *the Practical Problems and Linear Equations with one unknown*

The Practical Problems and Linear equations with one unknown is a teaching content in Mathematical Textbook for the 7th graders, they are between the ages of 12-14. Prior to this course, the students have learnt algebraic expressions, simple equations and solutions to linear equations with one unknown. This course is aimed at solving matching problems between the number of people and workload, which not only consolidate the knowledge students have learnt, but also serves as an extended application of integrating theory with practice.

In 2014, the Chinese Ministry of Education issued an *Opinion on Comprehensive Deepening Curriculum Reform and Implementing the Fundamental Task of Setting High Moral Values and Cultivating Persons*, in which it states clearly that too much emphasis on subject contents in the current curriculum standards should be gradually changed and the educational mode also needs to shift. As to this teaching practice, the concept of equation is first introduced in primary education. Some teachers try to enlighten their classrooms by integrating the history of mathematics into practical teaching. However, due to either unfamiliar with or not profoundly understanding of the historical materials, their classrooms are still lack of liveliness. Similarly, they can't display the cultural charms of the history of mathematics or the values of moral education.

To further the 7th graders' understanding of and love for mathematics in cultural dimension, and to strengthen their perception and application of equations, the authors incorporate some video clips into the teaching design of this course and deliver it in classroom.

2.1 Selecting and processing math-related historical materials

Going back in history, we can find the problems of linear equations with one unknown in ancient Egyptian papyri and Babylonian clay tablets. And in China the term equation first appeared in *the Nine Chapters on the Mathematical Art*. This book endows great historical and cultural connotations to linear equations with one unknown. It echoes the idea of *from life to mathematics and from mathematics to life* advocated by the new curriculum standards. It is also helpful to cultivate students' exploring spirit, practical ability and application awareness. It is different from the modern definition of *equation*, but they have the same ideas of solving the problem. Therefore, all historical materials in this teaching design are chosen from *the Nine Chapters on the Mathematical Art.*

2.1.1 Linear equations with one unknown recorded in *the Nine Chapters on the Mathematical Art* (Wang,2007a;2007b)

In *the Nine Chapters on the Mathematical Art*, there are altogether 246 mathematical problems, among which are five types of linear equations with one unknown, i.e., the problems of four basic arithmetical operations, travel, cooperation, fixed sum, and remainder (see figure 1).

Word Problems	Description	Original Question
Four Basic Arithmetical Operation	Linear equations with one unknown can be written as $x+a=b$ or $ax=b$.	Now given a rectangular field whose width is 1 ½ *bu*. Assume the area is 1 *mu*. Tell: what is its length?
Travel Problem	Meeting and catching-up word problems.	Now a wild duck flies from the south sea to the north sea in 7 days, and a wild goose flies from the north sea to the south sea in 9 days. Assume the two birds start at the same moment. Tell: when will they meet?
Cooperation Problem	A number of people, n, accomplish a task individually need $a_i (i = 1,2,\cdots,n)$ days. Find out how many days they need if they cooperate to finish the task.	Now given a cistern which is filled through 5 canals. Open the first canal and the cistern fills in ⅓ day. With the second, it fills in a day; with the third, in 2½ days; with the fourth, in 3 days; with the fifth, in 5 days. Assume all of them are opened. Tell: how many days are required to fill the cistern?
Fixed Sum Problem	Find out the given number which, after	Now given a person carrying gold through 5 passes. At the first pass he

	adding some times of its own, fractions of its own and a certain known number, is equal to another known number.	pays a tax of one part in 2. At the second pass, one part in 3; at the third, one part in 4; at the fourth, one part in 5; at the fifth, one part in 6. Assume the total tax at these five passes is just 1 *jin*. Tell: how much gold is carried originally?
Remainder Problem	Find out the given number whose remainder, after subjecting several fractions, is a known number.	Now given a person carrying cereal through 3 passes. At the outer pass, one-third is taken away as tax. At the middle pass, one-fifth is taken away. At the inner pass, one-seventh is taken away. Assume the remaining cereal is 5 *dou*. Tell: how much is carried originally?

Figure 1. Word Problems of linear equations with one unknown recordedin *the Nine Chapters on the Mathematical Art*

2.1.2 Selection of math-related historical materials and production of HPM video clips

In order to introduce the matching and work problems, the original exercise is revised and divided into two questions with the same background. This approach employs the method of accommodation to introduce the history of mathematics, which is in line with the principles of being innovative and scientific for selecting historical materials. As the main thread, the historical materials chosen from *the Nine Chapters on the Mathematical Art* are used through the whole teaching process, which is consistent with the current textbook, well-matched with students' cognitive levels, and also in agreement with the principle of being learnable. With the history of mathematics as background, the teaching enhances the reaching of teaching goals through arousing and resolving doubts, which is compatible with the principle of being effective.

To ensure a smooth teaching process, the authors use screen recorder software to make three short video clips of mathematical history. The first video is a three-minute introduction about *the Nine Chapters on the Mathematical Art*, lively and emotionally displaying the wisdom and achievements of the ancient mathematicians. It enables the students to get an idea about this ancient and most complete Chinese mathematical classic. In the situation creation section, the second video clip lasting for three minutes tells about the history of linear equations with one unknown. It also borrows the original problems from the book to deepen the concept of equation in students' mind. And the last video clip, two minutes in length, is used to exhibit the original solutions to the problems in *the Nine Chapters on the Mathematical Art*. By comparing the ancient and modern solutions, it aims at assisting students to imperceptibly understand the development and evolution of mathematics.

2.2 Teaching Sections

2.2.1 Introduction

The teacher started the class not with the typical pedagogy, but by playing a video clip introducing the book *Nine Chapters on Mathematical Art*. The author noticed that all students were drawn to the screen. They watched the original book of *the Nine Chapters on Mathematical Art.* They learned that it's the Chinese who first introduced the concept of negative numbers and the addition and subtraction algorithm. They got to know that the solutions to linear equations are almost the same as the elimination method in Algebra, which is 1500 years earlier than the Europeans. Their eyes were filled with wonder and endless longing.

The video clip conveys a lot of information within a short time in a lively way. It helps the students immersing in an information setting to feel the need for personality growth and the power of spiritual growth. It's a wonderful prelude to the intensifying and expanding of the teaching process.

2.2.2 Lecturing

Why should we study equation? Upon the end of the video, this question raised by the teacher brought the students back to reality. They started whispering.

Student A: Equation can solve problems.

Teacher: Why can they solve problems? What problems can they solve?

Student B: We have learnt that an equation is created by two expressions set equal to one another. And we only need to solve the unknown.

Student C: If we set the unknown, every problem can be solved.

Teacher: Yeah. An equation is a mathematical model created by two expressions set equal to one another. It is a useful approach to describe social and natural phenomena. That's why today we are going to learn how to create equations and find solutions to word problems.

The teacher wrote the topic for this lesson on the blackboard, i.e. practical problems and linear equations with one unknown. And here came the next part, playing the video about the history of the linear equations with one unknown, and the problem recorded in the book *Nine Chapters on Mathematical Art* appears in the video. (Shen Kang-shen,1999) *Now one person makes 38 prostrate tiles or 76 supine tiles a day. Assume he makes an equal number of both kinds of tiles a day. Tell: how many tiles of each kind can he make?* (see figure 2).The video showed and explained the pictures of supine tiles, prostrate tiles, and related ancient buildings. To leave suspense which set the stage for the following teaching, there was not give solution to the problem in this video.

Figure 2. The pictures of Supine tile and Prostrate tile

When the video was over, the teacher displayed the following problem in multimedia courseware.

Problem A: Now there are 34 people. One can make 38 prostrate tiles or 60 supine tiles a day. Assume some tiles are needed, among them the supine tiles are two times of the prostrate tiles. Tell: how many people are needed to make prostrate tiles and supine tiles respectively?

The textbook offers a matching-up problem as follows. Now there're 22 workers in a workshop. One worker can make 1200 screws or 2000 nuts a day. Assume a screw goes with two nuts to make a set. Tell: in order to produce exact sets of screws and nuts in everyday, how many workers should be assigned to make screws and nuts respectively?

By analyzing the given information, the teacher led the students to discover that the ratio of prostrate tiles to supine tiles was 1:2. Assume x as the number of people needed to make prostrate tiles, and the equation would be $2\times 38x=60\times(34-x)$. It's easy to work out x=15. Therefore the people needed to make supine tiles were 18. The teacher gave a detailed solving process on the blackboard.

Then, the teacher presented the second problem for the students.

Problem B: Now a number of people are needed to make prostrate and supine tiles. It will take one person 10 days to finish the work. Assume in the first period some people make tiles for 2 days, and then one person joins them to finish the work together in 2 days. Tell: how many people with the same working efficiency and working style are needed in the first period?

The problem in the textbook goes like this. Now it takes one person 40 hours to arrange some books. Assume some people arrange books for 4 hours and then 2 more people join them. And they work together to finish the job in 8 hours. Tell: how many people with the same work efficiency are needed in the first place?

The teacher suggested that the total amount of the work be regarded as 1. With the equation that workload = productivity × the number of people × time, the students assumed that there were x who started the work in the first place and had the equation $\frac{2x}{10}+\frac{2(x+1)}{10}=1$.

Hence x=2.

These two examples, adapted from the original problems in *the Nine Chapters on Mathematical Art*, shared the same problem solving strategy with the matching problems in the textbook. The author noticed in class that solving the mathematical problems under the same historical background made the teaching environment lively and natural, and greatly raised students' learning initiative. This was also backed up by the post-class survey.

2.2.3 In-class exercises

In the next section, the teacher displayed the original problem recorded in *the Nine Chapters on Mathematical Art* and required the students to solve it.

Teacher: Please think about it and discuss with your tablemates.

Student C: Teacher, this problem is simpler than what you have just taught us. It can be solved by creating an equation related to the time which will take.

Teacher: OK. You may have a try and tell me the result.

Student C: It's weird. The result is not an integer.

Student D: 25.33333 tiles! How can they make it?

Student E: Teacher, how did the ancient people solve it?

……

Teacher: Great. You've done it correctly. Now, let's watch a video clip to see how the ancient people solve it.

And the video was played, displaying the solution used in *the Nine Chapters on the Mathematical Art.*

Student F: I See. The ancient people were smart, but the current solution seems much simpler and clearer.

……

Those three HPM video clips were integrated into the practical teaching perfectly, just as *the wind sneaked into the night and moistened everything silently* that is from the Tang poem by DU Fu.

2.2.4 Induction and expansion

Amid the ancient charms of the mathematical history, the class came to an end. The teacher asked the students to generalize the basic process of solving practical problems by creating linear equations with one unknown. And then they came up with a process as: set an unknown→create an equation→solve the equation→check the result→finalize the answer (see figure 3). The teacher stressed repeatedly that the key to solve the problem was to find *the two expressions equal to one another*.

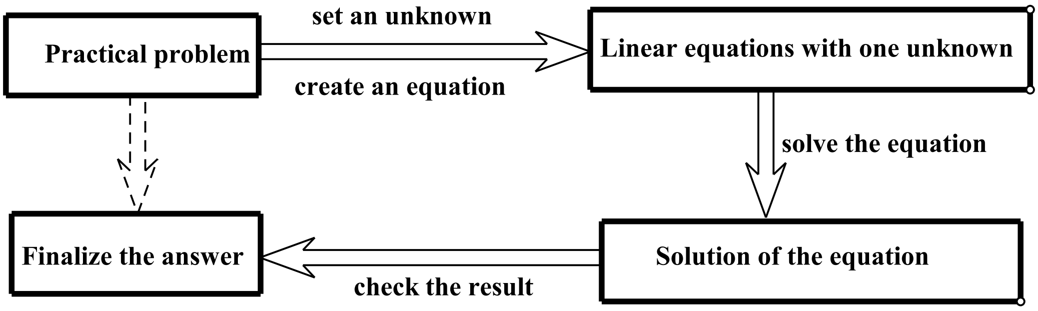


Figure 3. Class generalization

In the end, the teacher asked the students to go over the problems in the textbook and assigned an extended word problem in the textbook as after-class practice so as to reinforce their consciousness of applying the linear equations with one unknown. This would enable the students *to learn while practicing* and *to think while studying*.

Exercise:

Please collect some data concerning the important issues like climate, energy-saving and economy, make an analysis and create a problem that can be solved with linear equations with one unknown. State clearly the problem and its solving process.

2.3 Student Feedbacks

This teaching was carried out in a common class with 61% minority students in western China. The researchers conducted a post-class survey among the 42 students who attended the class. 45% of the students said *they completely understood the ancient problems after teacher's lecture*. 22% of the students said *they completely understood the teaching*. 29% of the students said *they partly understood.* 4% of them said *they did not understand*. As to the opinion on *teaching integrated with the history of mathematics*, only 4% of the students said *they did not like it*.

As to the question *which forms you like most to integrate history of mathematics in class,* the students' choice goes like this: HPM video clips, stories told by teacher, courseware display, pre-class reading materials, in-class reading materials, and after-class reading homework.

As to the subjective question *what impressed you most in this course*, nearly all students talked about how they liked the HPM video clips. They said that they learnt a lot about ancient Chinese mathematics, had a great interest in ancient mathematics, and were proud of ancient Chinese mathematicians. Here are some answers from the students.

(1) On *the Nine Chapters on the Mathematical Art*

Student G: This video let us get to know the history of ancient Chinese mathematics.

Student H: The mathematical problems illustrated in the videos interest me a lot and push me to think about them.

Student I: After watching these videos, I wanna have a look at the book.

(2) On Ancient mathematical problems

Student J: What the teacher said about the ancient mathematical problems got my attention. They may be used to solve real life problems.

Student K: The ancient mathematical problems are quite interesting. It's good for our brain, activate it.

(3) On the class integrated with HPM video clips

Student L: This class tells me that our method to solve equations is 1500 years earlier than Europeans. I get to know a lot of mathematicians, learn how to solve some ancient problems listed in *the Nine Chapters on the Mathematical Art*, and have great interest in the history of mathematics.

Student M: This class vividly tells me the history of equations and let me love to solve the practical problems and linear equations with one unknown.

Student N: The HPM video clips are funny. It seems we were back in the ancient society. We, modern people, can solve the ancient problems.

3 Conclusion and inspiration

From students' active in-class performance and simple answers in the survey, we can see that HPM video clips play an important role in guiding the teaching progress, expanding students' horizon of knowledge, and enhancing the achieving of knowledge and techniques, process and methods as well as attitude and value. Besides, HPM video clips also help to solve middle school teachers' perplexities in the following ways:

(1) Presenting historical materials lively, it's time saving time and effort saving.

Within a short time, HPM video clips can display complete mathematical information through rich forms. It enables the teachers who even don't know much about the history of mathematics to teach in a flexible way. It assists teachers to help students enrich their knowledge and understand mathematics better. It is of great support for the integration of mathematical history with practical teaching. If applied properly, it can be conducive to the harmony for the teaching factors.

(2) Exploring the teaching from diversified levels, it arouses students' interest to study.

HPM video clips present the study task to students in a way that suits their cognitive characteristics. It not only creates a lively learning environment for students, but also helpful to interest and motivate them. As the videos can be played repeatedly without the limitation of time and place, it rectifies the traditional teaching approach that what a teacher says is all, creates personal and diversified learning paths for students to explore by themselves, and provides them with more opportunities to think and make a decision.

(3) It catches students' attention and advances their study.

The teaching practice shows that HPM video clips can attract students' attention in a short time. On the one hand, they meet students' curiosity; on the other hand they help to improve their knowledge construction and application, technique forming and consolidation, and the betterment of study strategies. And what's more, the effect of study and memory lasts longer.

This practical teaching with integration of HPM video clips brings us some inspirations as follows.

(1) Professional production and application promotion

The use of HPM video clips can improve students' understanding and internalization of related knowledge points, and is good for them to carry out complex mathematical research activities. To realize this objective, the producers should not only have a complete knowledge of the history of mathematics, but also master relatively advanced shooting and production techniques. Therefore, it's necessary to organize a technical team specifically for the selection and production of historical materials of mathematics. The HPM video clips produced by this special team may serve as a handy tool for all teachers as natural as the use of paper and pens. This will solve the perplexities felt by middle school teachers lacking proper knowledge of mathematical history, eliminate the limitation brought by techniques, and promote the application and development of HPM video clips.

(2) Learning from others and developing flexibly

At present, the research and application of HPM video clips in China is still in the exploring stage. On the one hand, we need to learn from the foreign experience and search for new ideas and new techniques, on the other hand we should initiate reasonable application blueprint based on our own teaching practice. For example, we may focus on the development of video production techniques from the angles of opening and exploratory, and at the same time build a complementary resource library of mathematical history so as to provide a lasting guarantee for researchers and educators.

All in all, the study on the history of mathematics with orientation of mathematical education and the exploration of HPM video clips will be important research directions in the future (Wang & Zhang, 2006).

REFERENCES

Gao, Y., & Hu, Y. (2014). Teaching the area of a circle from the perspective of HPM. *Shanghai Middle School Mathematics*, *5*, 1-4.

He, K. H. (2012). Studying the interpretation of the ten-year plan for education informationization. *China Educational Technology*, *12*, 19-23.

Institute of Textbook Researches, Middle School Mathematics Textbook Research and Development Center (2014). *Mathematics Textbook for the 7th Graders, First Semester [M]*, People's Education Press., pp. 100-111.

Kang-shen, S., Crossley, J., & Lun, A. W. C. (1999). *The nine chapters on the mathematical art: Companion and commentary*.

Wang, X. Q. (2007a). The Questions of Linear equations with one unknown in History (I). *Mathematics Teaching in Middle Schools (Junior Middle School)*, *11*, 51-53.

Wang, X. Q. (2007b). The Questions of Linear equations with one unknown in History (II). *Mathematics Teaching in Middle Schools (Junior Middle School)*, *12*, 54-56.

Wang, X. Q. (2012). HPM: Present studies and future expectation. *The High School Mathematics Monthly*, *2*, 1-5.

Wang, X. Q., & Zhang, X. M. (2006). Researches on HPM: Contents, methods and examples. *Journal of Mathematics Education*, *15*(1), 16-18

Zhang, X. M., & Wang X. Q. (2009). Integrating the history of mathematics into mathematics teaching in senior high schools: An action research. *Journal of Mathematics Education*, *18*(4), 89-92.

MATHEMATICS EDUCATION IN COLONIAL TAIWAN AS SEEN THROUGH A STUDENT'S MATHEMATICAL NOTEBOOK

Mei-Lun HUANG, Jyun-Wei HUANG
National Taiwan Normal University, No.88, Ting-chou Road, Sec. 4, Taipei City, Taiwan.
colorguardmary@gmail.com
Taipei Municipal Heping High School, No.100, Wolong Street, Da-an District, Taipei City 106, Taiwan.
austin1119@gmail.com

ABSTRACT

In this article, we will discuss the mathematics pedagogy in the school, Taihoku Kodo Gakko (Taipei High School), established in 1922-1945 with a student's mathematics notebook that was used in 1944-45. The student's mathematics "professor" was likely Kato Heizaemon (1891~1975), a researcher of *wasan*, which is Japanese mathematics in the Edo period. Due to Kato's background, the contents in the notebook show characteristics of traditional Japanese mathematics. In this study, we will take the topics of the binomial theorem, polynomials, series and approximations as examples to explain how the teacher's emphasis on numerical approximations reflects the need for practical purposes in mathematics teaching and learning. For instance, the topic of polynomials is devoted more to the methods of estimation and calculation than to the properties of algebraic operations.

1 Introduction

Because of the *Treaty of Shimonoseki* signed after the First Sino-Japanese War, Taiwan was ceded by Japan. The Empire of Japan governed Taiwan from 1895 to 1945 and reformed the education system and recommend Higher education system which included the westernized mathematical knowledge in this period. In 1922, the first High School[1], Taihoku Kodo Gakko (臺北高等學校, Taipei High School), was founded in Taiwan, and the first university, Taihoku Teikoku Taigaku (臺北帝國大學, Taipei Imperial University), was founded in 1928 (Syu, 2012) The Japan government abolished *wasan* (和算), the traditional mathematics of Japan in Edo period, after the Meiji Restoration and recommend the westernized mathematical knowledge[2]. Under the influence, the westernized mathematical knowledge was transferred to Taiwan, and the mathematical teachers of Taihoku Kodo Gakko and Taihoku Teikoku Taigaku at that time were all Japanese.

[1] High schools in old education system (before 1950) of Japan is different form new education. High schools in old education system of Japan is part of higher education, aiming to educate precollege students, and High schools in new education system of Japan is part of secondary education.

[2] Wasan was influence by the mathematics of Song and Yuan Dynasties and had developed into a unique mathematical culture and mathematics activities. Although Wasan and the traditional Chinese mathematics shared the use of Chinese character and some old mathematical questions, but the content of this notebook was mainly western mathematics knowledge not the traditional questions of ancient Chinese.

From the curriculum guidelines at that time,[3] we could know that the contents of mathematical learning in Taihoku Kodo Gakko were indeed the westernized mathematical knowledge. Fortunately, we got a notebook written by a student at 1944-1945. Through this notebook, we can know the real mathematical materials in the class. The notebook is an important primary material of mathematical teaching—between professor and students in the classroom—at that time, and was the only one written by student we saved after World War II so far. In this research, we will discuss the background of Taihoku Kodo Gakko first, then sketch the educational institution, the teachers, the curriculum guidelines, and finally focus on the mathematical notebook by which we can discuss the mathematical education characteristics and the related educational purposes of Taihoku Kodo Gakko

2 The background of Taihoku Kodo Gakko

After the Japan government took over Taiwan, they continuously popularized Japanese for the purpose of controlling Taiwan, and pushed Taiwanese to learn Japanese by different kinds of ways (Huang, Jhang, & Wu, 2011). At that time, Taiwanese who used Japanese bad couldn't learn higher mathematical knowledge through the official education system. In 1922, the Japan government announced the second edition Taiwan Education Instruction (台湾教育令). This document regulated students who possess of the ability of Japanese (including Taiwanese and Japanese students) can obtain the qualification of entering all kinds of schools in Taiwan. The Instruction issued in 1922 decreased the gap between Japanese and Taiwanese students, but still there were discriminations between Taiwanese and Japanese students except students among Taihoku Kodo Gakko. The figure.1 shows the educational system at that time.

Taiwan students who were unable studying abroad must study in the Taihoku Kodo Gakko first in order to enter the only university, Taihoku Teikoku Taigaku. Since Taihoku Kodo Gakko was the only High school in Taiwan, the competition of the entrance examination was very intense (Admission rate of Japanese: 13-40%, Admission rate of Taiwanese: 1%-10%). Most of all students came from cities of Taiwan, 80% of students came from Taipei, and 20% came from others, which included Taichung, Tainan, Kaohsiung, etc. Few of them came from Japan. Moreover, only 9.7% students of all graduates were from public school[4] which had been established for Taiwanese (Syu, 2012).

Mathematics classes were took 12% to 15% of all classes in jinjouka[5] (尋常科, junior high level) of Taihoku Kodo Gakko. The level is the same as the students who finished the four years curriculum in middle school. The students in koutouka[6] (高等科, advanced level) could choose to major in science or liberal arts, and also could choose to learn German or French as second foreign language. The students who major in science had to take 12%~14% mathematics class (Taipei High School of Office of Governor-General of Taiwan, 1936). All mathematics teachers

[3] http://hdl.handle.net/2298/1136

[4] Public schools in the colonial period was founded for Taiwanese students (or for students who couldn't use Japanese well), at the same time, Elementary schools was founded for Japanese students (or for students who could use Japanese well).

[5] Jinjouka was settled for first 4 year education in 7-year program of High school in old education system of Japan.

[6] Koutouka was settled for last 3 year education in 7-year program of High school in old education system of Japan.

were Japanese. The mathematics teachers of jinjouka were called *kyouyu* (教諭, teacher) and of koutouka were called *kyouju* (教授, professor) (Syu, 2012). It's worth mentioning that many students would compare the reading quantity to each other under the climate for learning.

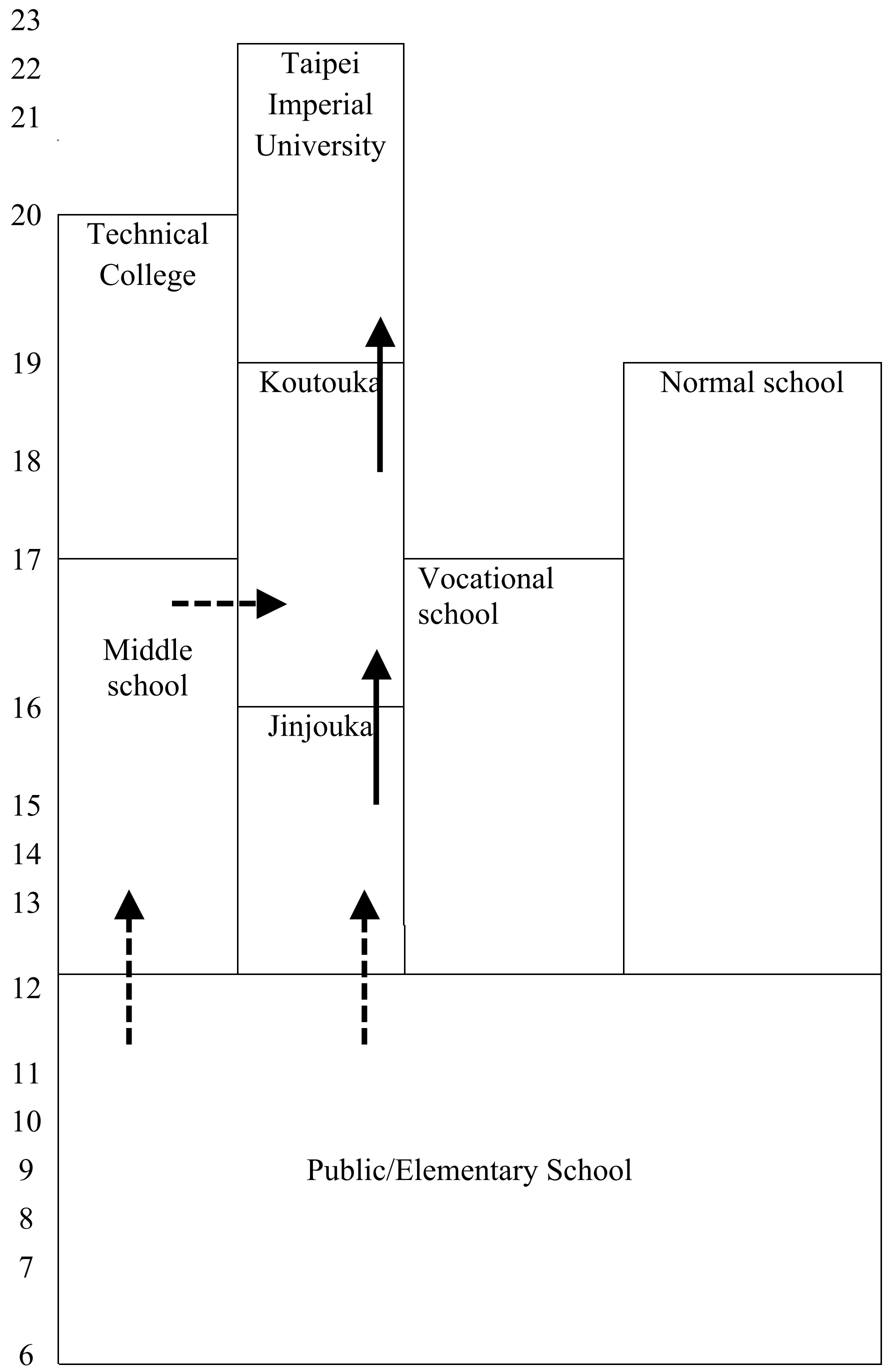


Figure 1. Education system of Taiwan in 1922-1945 (Syu, 2012, p. 17)

Mathematics classes were took 12% to 15% of all classes in jinjouka[7] (尋常科, junior high level) of Taihoku Kodo Gakko. The level is the same as the students who finished the four years curriculum in middle school. The students in koutouka[8] (高等科, advanced level) could choose to major in science or liberal arts, and also could choose to learn German or French as second foreign language. The students who major in science had to take 12%~14% mathematics class (Taipei High School of Office of Governor-General of Taiwan, 1936). All mathematics teachers were Japanese. The mathematics teachers of jinjouka were called *kyouyu* (教諭, teacher) and of koutouka were called *kyouju* (教授, professor) (Syu, 2012). It's worth mentioning that many students would compare the reading quantity to each other under the climate for learning.

3 Mathematics Education of Taihoku Kodo Gakko in koutouka

The content of mathematic curriculums in elementary school and public school at that time were mainly arithmetic which contained daily arithmetic problems about the four fundamental operations of integer, fraction, and problems about area, circumstance and angle of plane figure, calculation of time, and volume of podetium (Huang, 2012). The curriculum of middle school[9] was developed for five years education, the curriculum of mathematics at first year contained arithmetic, algebra, and geometry; contained algebra and geometry at second and third year; contained algebra, geometry and trigonometry at fourth and fifth year. The level of mathematical knowledge above almost equaled to Taihoku Kodo Gakko jinjouka.

Most of the students of koutouka in Taihoku Kodo Gakko graduated from jinjouka without taking upgrading exam. Other koutouka students graduated from other middle schools, those middle schools were the best in their own cities. To study in Taihoku Teikoku Taigaku well, the mathematical course in koutouka was settled for connecting with the courses of Taihoku Teikoku Taigaku and for the need of application (Syu, 2012).

From 1925 to 1946, eight teachers had taught mathematics in this school, three of them majored in mathematics (Syu, 2009). The figure.2 shows the teaching periods of these teachers.

We concern now that Kai had the experience of being *kyouyu* and he served as an investigator of mathematics when Taiwan Association of Education investigated the middle schools in 1931. Minewaki served as a *kyouju* of Taihoku Kodo Gakko, and he also served as a music *kyouju*. Besides, Sudo who was not major in mathematics was knew as his research about folk mathematics (Syu, 2012 & Huang, 2012).

The mathematical notebook discussed in this research came from a student of Taihoku Kodo Gakko, Mr. Wang, who was born in 1925. After he graduated from public school, he studied in the Taipei second middle school. He entered Taihoku Kodo Gakko in 1944 and became the 19th graduate in 1946.

[7] Jinjouka was settled for first 4 year education in 7-year program of High school in old education system of Japan.

[8] Koutouka was settled for last 3 year education in 7-year program of High school in old education system of Japan.

[9] Middle schools in old education system of Japan are 5-year program for boy students who had graduated from public schools or elementary schools.

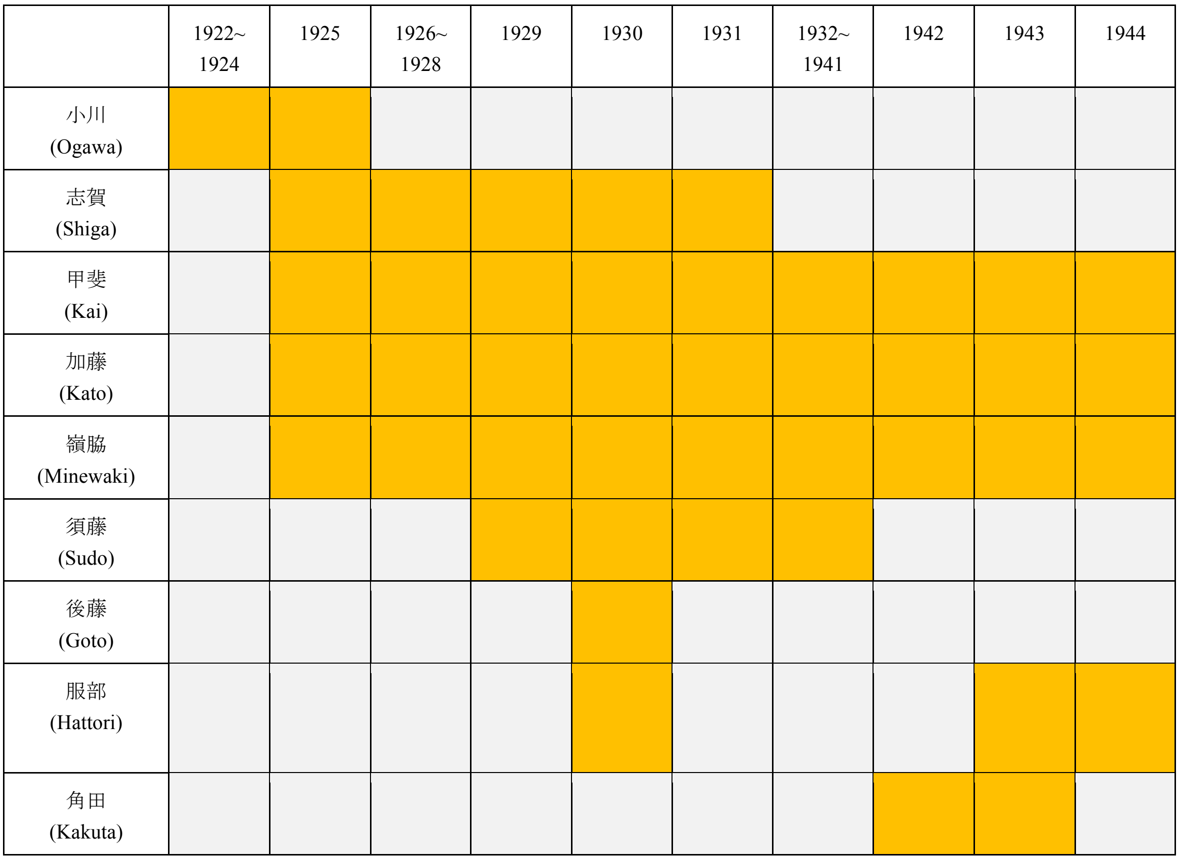


Figure 2. Mathematics teachers at Taihoku Kodo Gakko

According to the notebook, the mathematical teacher of Mr. Wang should be Kato Heizaemon (1891~1976) who served as a mathematical teacher in this school from 1927 to 1947, and left Taiwan back to Japan in 1947. Even though he received the westernized mathematical education, he was interested in researching Wasan, especially in the mathematical research of Seki School. He published papers about research of Wasan in *Taiwan Education* (臺灣教育) and the school magazine of Taihoku Kodo Gakko, *Taihoku* (臺高).

4 The content and characteristic of the notebook

First, we introduce the characteristics of the notebook. Including the catalog, this notebook totally contains 67 pages, which were mainly written by Katakana and Chinese character. The content of this notebook contains the basic probability, equations, polynomial, permutation and combination, series, analytic geometry, space geometry, the foundation of integral, and simple harmonic motion. Figure3 shows the catalog written in the first page.

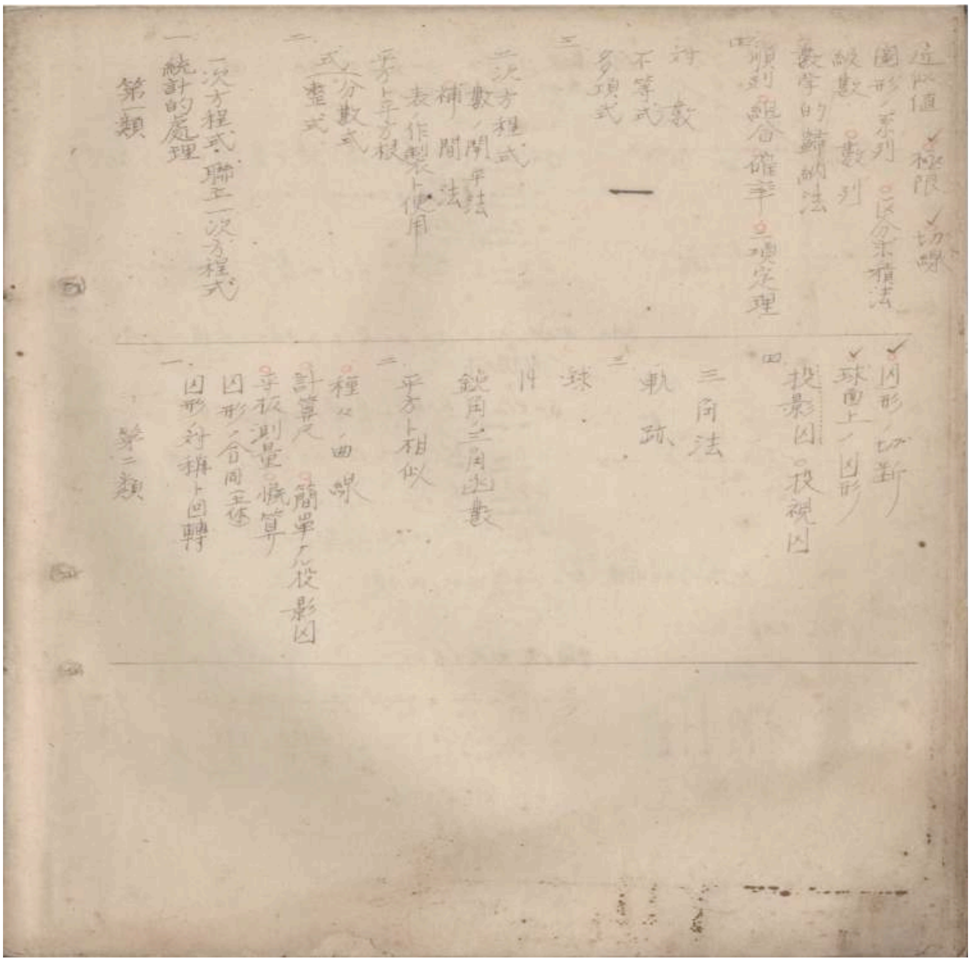


Figure 3. The catalog of the notebook

The contents listed in the catalog should be the reserved schedule of the mathematics course, but the subjects showed in notebook are different to the catalog. The real subjects taught contained method of integration by parts (区分求積法), mathematical induction, conic section (analytic geometry), permutation, combination, binomial theorem, probability, miscellaneous series, figures on the sphere surface, triangle on the sphere surface, the generalization of binomial theorem, polynomial, approximation, simple harmonic motion, miscellaneous problems, advanced algebra, continuity and variety, important problems in new textbooks, special methods for the summation of series and curve formed by the locus of points.

The notebook did not stipulate a clear teaching sequence and did not accord a series logic structure, but did present many scattered subjects. Overall, no matter what in the catalog or in the notebook, there were no subjects about the number theory or the theory of vector. This is very difference with the course guideline in force. Besides, this notebook contained calculation about the limit of series and space geometry that were belong to the course of *koutouka.*

The figure 4(a) shows the method of finding the formula of sphere surface area written in the notebook. And the figure 4(b) shows the method in the book *Solid Geometry* written by Kato Heizaemon. Through our researching, the method of proof in the notebook is the same with that in the book *Solid Geometry*, and it seems more streamlined and simplified in the notebook. This also shows that the teacher of the mathematics course should be Kato Heizaemon.

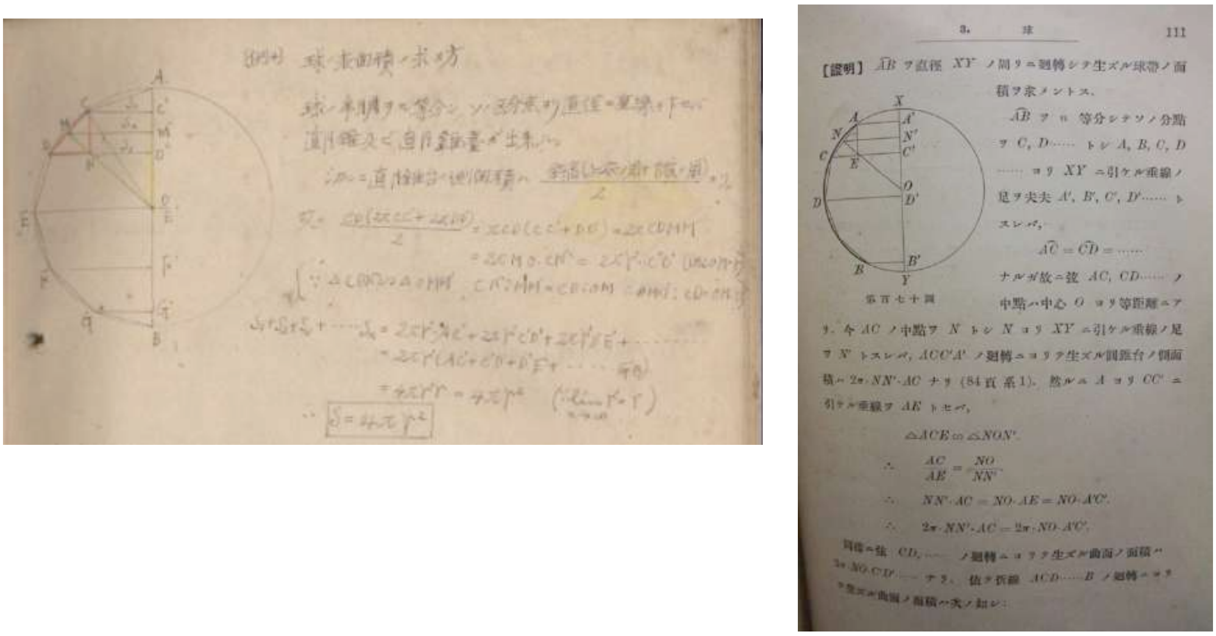


Figure 4. The diagram and method of finding the formula of sphere surface area written in the notebook and in the book *Solid Geometry*

Furthermore, the method of proof isn't the same as integral method of present day, but seems like the method used by the *wasan* mathematicians in the 19th century that developed for the purpose of solving all kinds of problems about arc length, surface area, and volume. First, he cut the sphere surface area into n pieces that are approximated by n trapeziums. Then, he used some basic properties about similar figures and circle to present the area of all trapeziums. Finally, he summed the areas and toke the limit of the summation, then he got the formula of sphere surface area. This method should relate to the studying background of Kato Heizaemon who is familiar with *wasan.*

There are some mathematical properties and theorems that were not proved in the notebook, and there are also many examples and mathematical problems. In other words, these properties and theorems are adhered to the purpose of problem solving and application, and are not discussed deeply even about the connection of the concepts of mathematics. Because of the pages are limited, we can't explain and introduce all contents of the notebook, and we will take the subjects about algebra, such as polynomial and the approximation, as examples to explain the characteristics of the notebook in this paper.

Comparing with the traditional Chinese mathematics and *wasan*, the contents of this notebook are completely terms and new concepts of westernized mathematics, and show that the mathematics education at that time focused on application, calculation, and problem solving. We refer to students' handbook learning guideline:

To learn math, learners should understand math and science and know well calculating and application. Furthermore, learner should think accurately. Liberal arts students will be taught main points of all kinds of math knowledge. Science students will be taught algebra, solid geometry, trigonometric rules, basic analytic geometry, basic calculus and basic mechanics (Taipei High School of Office of Governor-General of Taiwan, 1936).

There existed a subject about simple harmonic motion which is mathematical knowledge about simplifying $a\sin\theta + b\cos\theta$, but pages about related phenomenon of physics were only 3 lines in the notebook. The pages about permutation, combination, classical probability were more.

Application and problem solving are both important aspects in this notebook that specially contained many units about estimating the approximate. For example, the figure5 shows an table about interpolation method which was used for estimating.

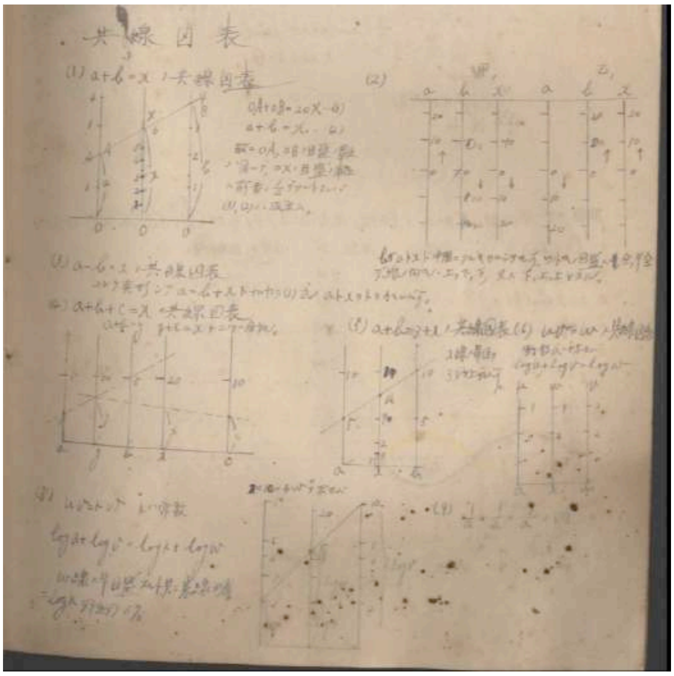
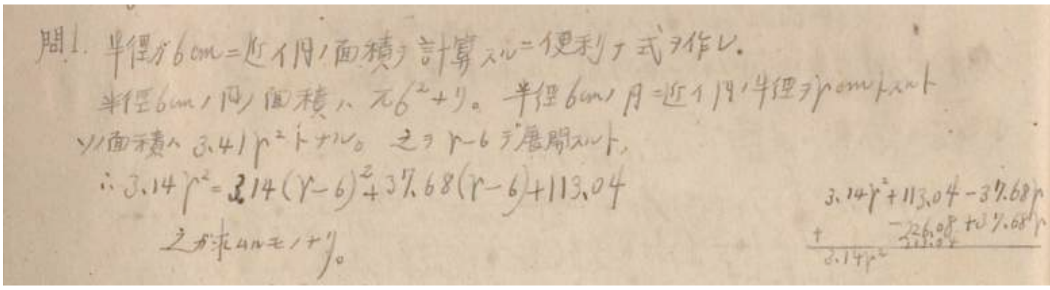


Figure 5. Contents about Interpolation method in the notebook

The subjects about polynomial showed a characteristic of application. Due to the limit of calculation tool, mainly abacus and pen, using the expansion of polynomial to calculate values is the necessary skill in mathematical course. The contents about polynomial did not emphasize on the definitions, properties and related mathematical structure, but was about evaluating the approximate.

The examples in the notebook were about using the expanding of polynomial to estimate the approximate. The method of expanding the polynomial is unrelated with differential, but is the synthetic division (組立除法) in the notebook. This method is similar to the traditional one in Wasan.

From the examples in the notebook, it was a matter of learning science. There were examples about data of experiment, such as estimate the boiling point under different atmospheric pressure. Besides, there is an exercise about constructing a formula to calculate the area of a circle which radius is close to 6cm.

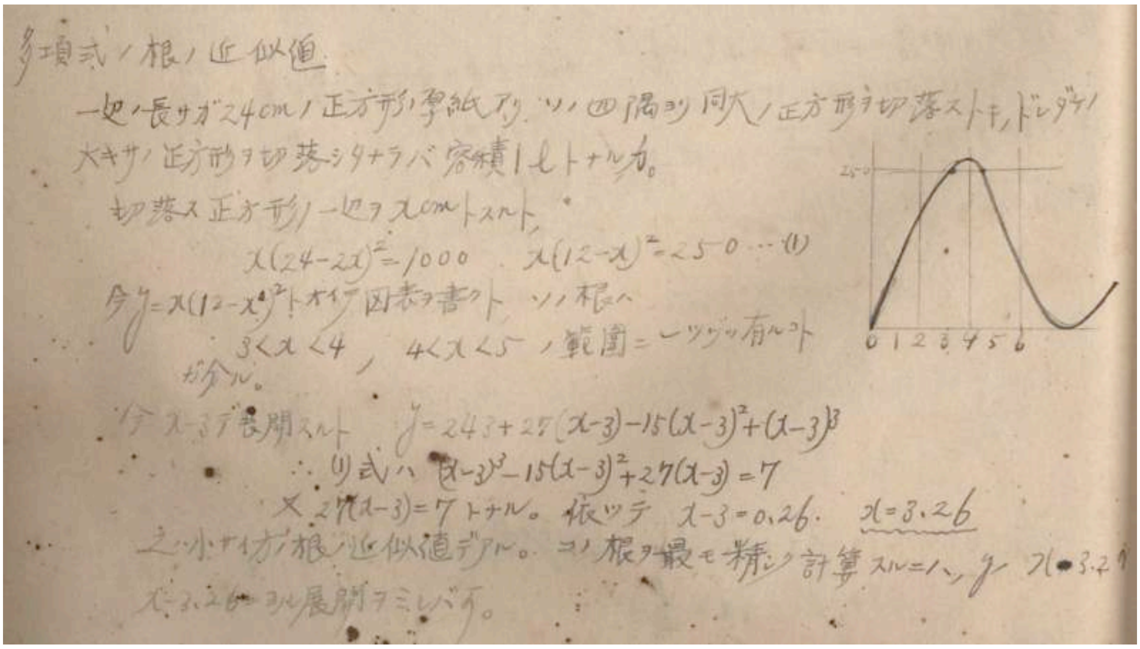


問1. 半径ガ6cm=近イ円ノ面積ヲ計算スル=便利ナ式ヲ作レ.
半径6cmノ円ノ面積ハ $\pi 6^2$ ナリ。半径6cmノ円=近イ円ノ半径ヲr cmトスルト
ソノ面積ハ $3.41r^2$ トナル。之ヲ $r-6$ デ展開スルト,
$\therefore 3.14r^2 = 3.14(r-6)^2 + 37.68(r-6) + 113.04$
之ガ求ムルモノナリ。

Figure 6. The exercise and formula of calculating the area of a circle which radius is close to 6cm

The purpose of this problem is to construct a formula which could be used to calculate the, approximate area of a circle, and the method used in the notebook is synthetic division. Take another example, the teacher used the synthetic division again to extract the approximate of the roots of a cubic equation as figure 7. The method of extracting the roots is similar to the method of Wasan.

On the other hand, there was no theory discussed and introduced in the topic of "maximum and minimum" in the notebook, but only a typical example. The method used in the notebook to deal the problem of extreme was not about differentiation, but about the traditional method which was related to synthetic division in Wasan. In the process of problem solving, He expressed the function to be the form of a polynomial of *x*-2, which was used to solve out the extreme as figure 8.



多項式ノ根ノ近似値
一辺ノ長サガ24cmノ正方形ノ厚紙アリ. ソノ四隅ヨリ同大ノ正方形ヲ切落ストキ, ドレダケノ大キサノ正方形ヲ切落シタナラバ容積1ℓトナルカ。
切落ス正方形ノ一辺ヲ$x$cmトスルト,
$x(24-2x)^2 = 1000 \quad x(12-x)^2 = 250 \cdots (1)$
今 $y = x(12-x)^2$ トオイテ図表ヲ書クト ソノ根ハ $3<x<4$, $4<x<5$ ノ範囲ニ一ツヅツ有ルコトガ分ル。
今 $x-3$ デ展開スルト $y = 243 + 27(x-3) - 15(x-3)^2 + (x-3)^3$
$\therefore$ (1)式ハ $(x-3)^3 - 15(x-3)^2 + 27(x-3) = 7$
又 $27(x-3) = 7$ トナル。依ッテ $x-3 = 0.26$. $x = 3.26$
之ハ小サイ方ノ根ノ近似値デアル。コノ根ヲ最モ精シク計算スルニハ, y ヲ $x=3.26$ ヨリ展開シテミレバ可。

Figure 7. Extracting the approximate of the roots of a cubic equation

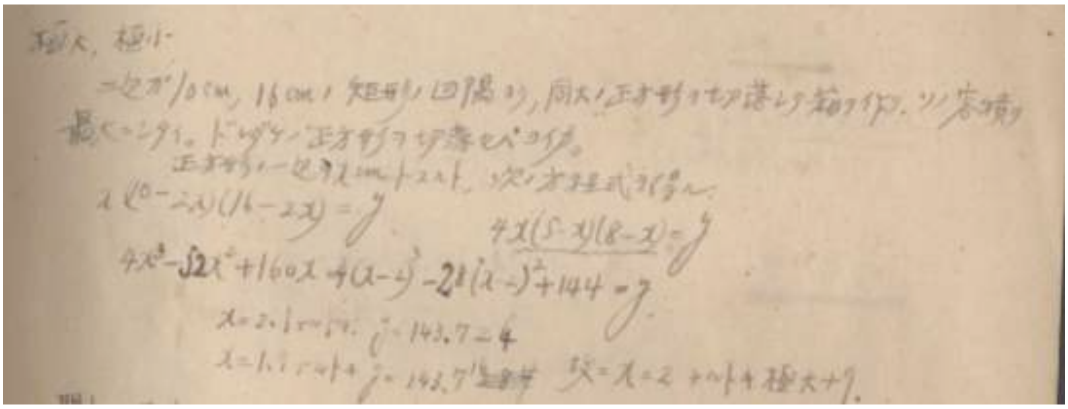


Figure 8. An application problem and the method about extreme

Besides, the notebook listed many approximate algebra equations in the subjects about application, and these equations could be used to estimate the approximate. Four approximate formulas were introduced, and the written used the signal "=" in the notebook which should be"≈":

$$\frac{1}{1+x}=1-x$$

$$(1+x)(1+y)=1+x+y$$

$$\frac{1+x}{1+y}=1+x-y$$

$$\frac{1}{(1+x)^2}=1-2x+3x^2-2x^3+x^4$$

In conclusion, the subjects about algebra in this notebook contains polynomial evaluation, extra maximum value and extra minimum value, the calculation and estimate of approximation, interpolation method, that are all emphasize on the purpose of calculation and estimate the value, and also focus on the problems of application, such as probability, permutation, combination, and physics. These characteristics are maybe related to the social needs of practice. On the other hand, there are many exercises and examples in the notebook, from which we can know that mathematics teaching at that time mainly focused on problem solving.

5 The reflection and conclusion about HPM

This research is devoting to describe the situation of Taihoku Kodo Gakko, and elaborate the background of mathematics learning, and explain the characteristics of mathematics education by a primary mathematical notebook at that time. This paper is the first mathematics edcation reseach that focus on Taiwan higher mathematics education of Japanese-Occupied Period. Then, we hope to enrich the contents of the history of mathematics education in Taiwan.

The curriculum guidelines, the title of the lessons and the mathematical symbols transcribed in the notebook were all westernized mathematical knowledge and were not traditional Chinese mathematics or Wasan[10]. In this notebook, many subjects were contained in mathematical curriculum currently in effect, and some were not. This notebook included many additional problems and emphasized the purpose of calculating and application. The teacher at that time should be Kato Heizaemon who interested in researching Wasan. Influenced by the background of the teacher, the social practical needs and the war, the content of this notebook was not consistent with the curriculum guidelines.

The curriculum guidelines of Taihoku Kodo Gakko announced that mathematics education had an orientation towards practical. Take the notebook as example; it indeed included subjects such as combination, probability, approximation, and simple harmonic motion. The subjects about algebra focused in this paper stressed on problem solving and calculating approximations, such as using binomial theorem and some equations for calculating approximations, but not on the mathematical theorems and proofs. The notebook also included many problems for students to practice. From these we can know that application and solving practice problems were important for mathematics learning at that time.

REFERENCE

Cia, J. T., & Syu, S. K. (2013). *Biology of Huang Bo Chao*. Taipei: Avanguard press. (In traditional Chinese).

Department of Mathematics, National Taiwan University (1998). *Interview record of Syu Jhen Rong*. *Introduction of Department of Mathematics, college of Science in National Taiwan University*. Taipei: D. Math, NTU. (In traditional Chinese)

Huang, M. L. (2012). *Primary Research about Mathematical materials of Taipei High School: Take notebook as an example*. Master's thesis. Taipei: National Taiwan Normal University. (In traditional Chinese)

Huang, S. J., Jhang, S. Y., &Wu W. S. (2011). *History of Taiwan.* Taipei: Wu-Nan Book Inc. (In traditional Chinese)

Lin, M. S. (1929). Translate by Yong Mei Lin (2000). *Public Education in Formosa under the Japanese Administration – A Historical and Analytical Study of the Development and the Cultural Problems*. Taipei: Thirdnature. (In traditional Chinese)

Syu, S. K. (2009). Research of Taipei High School in Japan colonial period (Doctoral dissertation). Taipei: National Taiwan Normal University. (In traditional Chinese)

Syu, S. K. (2012). *Taipei High School in Japan colonial period and elite cultivating*. Taipei: NTNU press. (In traditional Chinese)

Taipei High School of Office of Governor-General of Taiwan (1936). *Curriculum of subjects, institution and textbooks*. *Students' handbooks of Taipei High school*. Taipei City: Office of Governor-General of Taiwan (In Japanese)

Takasu, T., & Kato, H. (1931). *Solid Geometry*. Tokyo: Syokwabo and Company. (In Japanese)

[10] After Meiji Restoration, Japan government abolished wasan education in governing system. Relation within traditional Chinese mathematics, Wasan and western Mathematics is a really big issue that we can't explain within the pages.

INTEGRATING HISTORY OF MATHEMATICS WITH FOUNDATIONAL CONTENTS IN THE EDUCATION OF PROSPECTIVE ELEMENTARY TEACHERS

Ana MILLÁN GASCA, Elena GIL CLEMENTE
Università Roma Tre, via Milazzo 11/B, Roma, Italy
anamaria.millangasca@uniroma3.it
Universidad de Zaragoza, Pedro Cerbuna 12, Zaragoza, Spain
elenagil@unizar.es

ABSTRACT

The proposal presented here relates to the integration of the mathematical syllabus content, including foundational issues, with selected aspects of history of mathematics in the education of prospective elementary teachers.

An analysis of the needs and cultural profile of prospective elementary teachers in Italy was the original motivation of this study. The new syllabus has been applied for several years in Rome, with good results in mathematical learning and self-confidence regarding the performance as a future mathematics teacher. Students also show an ability to use history of mathematics in the classroom with good feedback from the children.

The historical contents were used in Spain to offer a stronger mathematical background to a group of teachers and prospective teachers collaborating in an experimental workshop for children with Down´s Syndrome. Once again the result was improved appreciation of their own skills and greater autonomy and spirit of initiative.

1 The historical background: from optimism to an age of uncertainty in children's mathematical education and its impact on primary school teachers' education

Both in Italy and in Spain prospective elementary school teachers show fear and rejection in the face of mathematics, which is largely explained by a cultural atmosphere of pessimism about the difficulties of the task of teaching maths to children and also by confusion about how to carry out such a task.

A loss of certainty about methods and contents of children's mathematical education spread throughout much of the industrialized world after the Second World War. In a context in which a long, solid tradition of calculations and elementary training was thoroughly applied around the world, the pestalozzian optimism of many innovators between the second half of the 19th century and the first decades of the 20th century, most of them mathematicians (Fröbel, Mary Everest Boole, Grace Chisholm Young, Jean Macé, Charles Laisant, Maria Montessori, Rodolfo Bettazzi), was overwhelmed by a wave of pessimism.

The crisis of European science due to political evolution and the war was one main cause, because the network of international contacts, cultural centres, publishing houses and general scientific journals had suffered a severe blow. Deep contributions based on the practical experience of educators when confronted with a didactical tradition rooted in the medieval abacus schools had marked an age of confidence in the powers of the mathematical mind to be exploited in the "preparation of the child for science". However, after many editions and translations of Laisant's *Initiation to mathematics* and Macé's *Grand-papa's arithmetic*, these booklets addressed to educators became forgotten books. In his 1938 *Mathematics in society and culture* Federico Enriques, referring to Johann Pestalozzi and Friedrich Fröbel's inspiring contributions, wrote "the formative value of mathematics can also be seen in the first grades of children's education and working-class education; because mathematical intelligence is quite precocious", referring to Johann Pestalozzi and Friedrich Fröbel's inspiring contributions. However, 1938 was the year of the race laws in Italy and Enriques was expelled from the university. Even Maria Montessori's optimism in her *Psychoarithmetics* and *Psychogeometry*, published in 1934 in pre-civil war Spain was confined to the circle of montessorian schools in the following decades.[1]

The impact of the psychological experimental approach, from Edward Thorndike to Piaget's prominent books with Alina Szeminska and Bäber Inhelder on number and concepts of space in the child can be summarized in the transition from an approach of confidence in children's intuition to a focus on the difficulties of learning mathematics – dominant through the 1950s to the 1960s. The above mentioned innovators, pedagogues and mathematicians had criticized rote learning and boring and mechanical methods of 19th century primary schools and they tried to put forward a different style of teaching specially designed for children. Alternatively, the psychological world believed it had found an obstacle to learning mathematics in the young child's mind, a kind of illogical child, not able to make logical inference similar to how Lévy-Bruhl had spoken of a primitive illogical mentality. In the 1970s and 1980s Edinburgh psychologists led by Margaret Donaldson offered a radically different vision of the strength of children's mind (Donaldson 1978, Hughes 1986). Nevertheless, in the meantime such a context of pessimism and loss of almost every deep rooted certainty on children's mathematical education led to questioning of the on-going education of young future primary schools teachers.

This was the case in many European countries in Western Europe and in the United States, in contrast with the continuity in China or the USSR and Eastern European countries. The long term effects of this sharp contrast was aptly described by Liping Ma in her 1999 *Knowing and Teaching Elementary Mathematics: teachers understanding of fundamental mathematics in China and the United States*, which considers both the effects on children's education and on the cultural and emotive position of primary teachers.

The spread of psychological views on early maths education combined with *new math*. Meanwhile, ideas on children's first steps in mathematics had deeply changed. For the first graders, new math meant sets, trying to find intruders in sets, exercises of equipollence, and 0 as cardinal of the empty set; for pre-schoolers, down and up, inside and outside and those so called "topological notions". Geometry had never had a real relevance in primary education, but innovators had shared the feeling of the crucial role that geometry could have in transforming and improving primary mathematical education. All of this was completely

[1] Montessori wasn't a mathematician but a medical doctor, however, in Rome she had attended the discussions of mathematicians deeply concerned with basic concepts of mathematics and geometry, and her optimism was a legacy of the indefatigable Séguin, the author of the 1875 *Report on education*.

forgotten as a result of the rejection of Euclidean geometry throughout the compulsory and secondary education system.[2]

What kind of teacher should have taught new math to children? It's hardly surprising that this new trend had a deep impact on the image and profile of primary teachers. René Thom had perceptively anticipated this long-term effect in his 1972 lecture at the Exeter conference on "Developments in mathematical education" (International Commission for Mathematical Instruction, Howson 1973), emphasizing the fact that the change in mathematical programmes or contents in primary school was intended also, if not mainly, to force primary teachers to change their pedagogical methods, in order to obtain a heuristic teaching that was direct, free and constructive, that could awaken the child's interest. Twenty years later, in 1991 he remembered as a witness the dramatic impact in the 1970s France:

> Old white haired teachers, who taught elementary calculations with sticks, have been seen to be forced to come up-to-date. They were told: Sirs, what you do is ridiculous; you know nothing about set theory, and one cannot do arithmetic without understanding it. And those old teachers were forced to sit down at school desks to listen to pretentious youngsters who explained to them that they had understood nothing about numbers!" (Thom 1991).

Thom himself, wrote, that he had been a modernist *ante litteram* (he entered the Paris *École Normale Supérieure* in 1943), and was fond of the foundations of mathematics, logics, and set theory, but there have been excesses of zeal and a deep misunderstanding: that "by making the implicit mechanisms or techniques of thought conscious and explicit, one makes these techniques easier" (Thom 1973, p. 197). However, this is exactly what became the norm and set theory, Boolean algebra and topological structures were considered as the basic mathematical background of a teacher in order to work with illogical children.

The impact on primary teacher's education, life and work, and self-image as a result of the coalition of psychology with "modernism" in mathematics, as Thom put it, was severe. In many countries around the world this education lasted until very recently outside the university, in secondary education institutions, which moreover were considered of lower rank if compared with the classical language high school. In addition, his or her point of view on mathematics was that of a young student, not that of a prospective teacher. Eventually the idea of post-secondary normal schools devoted to the primary teachers education spread, or alternatively there were specific degrees in primary education in colleges of education. A modern math teacher for children aged 3 to 10 should have a psycho-pedagogical background, for some; or in other cases should learn logic and the foundations of mathematics; or a mixture of both new elements depending on the institutional set.

2 The challenge of self-confidence among prospective teachers

[2] In fact, this did not get rid of geometry, because 3D physical didactical materials designed and commercialized in those years (Dienes, Cuisenaire) were eventually introduced in primary schools to try to solve an entangled situation, which implied appealing to children's geometrical intuition (for example comparing and adding solids).

In Italy and Spain, young students, mainly women, who enter the university to become teachers, have an experience of math courses in compulsory and secondary school lasting about twelve years. This experience often involves feelings such as annoyance or distress and a vision of mathematics reduced to pure calculation and mechanical procedures.

An easy way to face up to this situation is to give up to teach more mathematics to them, and focus on courses on mathematics education, with the hope that, combined with the maths learned in secondary school and training in psychology and pedagogy, it will be enough to tackle the mathematical education of children.

In Spain, in 1999 Miguel de Guzmán organized a European conference on the training and performance of primary teachers in mathematics at the Academy of Sciences in Madrid. A year later he spoke about the risk that the adoption of an option such as the former would entail for mathematical instruction:[3]

> our system is depriving the vast majority of people who are being trained to teach, of the possibility of a slow, calm and joyful learning of the huge wealth that the mathematical task provides for the building of thought; depriving them of the chance of appreciating maths properly through the awareness of its usefulness and its ubiquity in everyday life.
>
> It is depriving them of the useful and practical knowledge of the appropriate tasks for proper learning of mathematical topics that would be required at this level; it is depriving them of the satisfaction that comes from playing and from the aesthetic pleasure that can balance, as happens with any worthwhile activity, the many humdrum and laborious efforts we have to do to reach joyful practice …
>
> Is it possible for these people to approach teaching without transmitting to their pupils the fears, and the worries, conditioning their attitude to the subject?[4]

This lack of useful and practical knowledge joined with an environment of low confidence on children´s chances to understand numbers as a result of the influence of Piaget´s conceptions of the building of the idea of number (Arnau 2011).

In Italy, during the first 15 years after the introduction of a university degree in primary education,[5] a significant number of women who had already graduated in literary subjects enrolled in these studies. These women had given up mathematics for periods of up to 10 years, and this fact amplifies their unpleasant memories and fossilized vision of mathematics as simple calculus. Until 2010, the number of hours devoted to mathematics depended on the university, ranging from nothing to a course of sixty hours. Almost 30 hours were usually

---

[3] In Spain, from 1970 primary teachers' education consisted of a three-years course of studies in university colleges (Escuela de Magisterio) and from 2009 it became a 5 year masters degree.

[4] Guzmán, 2000.

[5] In Italy, until 1996, the education of prospective teachers was entrusted to five-years secondary schools (Istituti Magistrali, students aged 14 to 19), where a mathematics high school programme was developed, but further education proposals focused mainly on foundations

allocated to mathematics education. The current curriculum includes more than 150 hours of mathematics and mathematics education.

Children are the main interest of prospective teachers, and they frequently think of their role in terms of caring, playing and perhaps their literary education. As a result of their own school experience, the task of providing an introduction to mathematics and science is viewed as a kind of punishment they don´t want to inflict on children. Increasing social pressure on the value of mathematics adds fear to this rejection, because learning mathematics shows clear difficulties. Mass media contribute to spread of the results of comparative research, for instance PISA o TIMSS, where countries such as Italy or Spain are in positions under the OCDE mean.

There is an internationally shared core on the basic mathematical instruction a child should received in their primary training: it focuses on numeracy, which includes measure and interpretation of tables and graphs; the differences in national curricula lie mainly in the introduction of intuitive geometry programmes.

In spite of this agreement and as a consequence of the historical evolution that we have just outlined, prospective primary teachers in countries such as Spain and Italy develop their training in a cultural environment marked by a lack of confidence about the actual possibilities of every children to learn mathematics and disorientation about the best way to reach this learning.

It is known that one's own teacher is a crucial reference model (Margolinas 2007). However, if this model is negative what can the student do?

2.1 Exploring the reasons for primary school teachers' fear and lack of selfconfidence toward learning and teaching of mathematics: an experience in Italy

Concern over mathematics as a course of study and the lack of confidence about their future role as mathematics teachers was identified by the first author of this paper in Roma Tre University as one of the possible reasons for poor performance in the basic maths course.

Between 2010-2012 four workshops aimed at pupils who show a radical rejection to the basic maths course, graduate students taking a second degree in primary education and supply teachers in primary education or kindergarten were carried out (Millán Gasca, Marrano, Spagnoletti Zeuli 2015)[6]. The goals of the workshops were to show the participants that children can enjoy mathematics and are able to learn deep mathematical concepts, and to put across a different vision of school mathematics. To achieve these goals we started from the initial interest of all of the participants in children.

During the workshops, sixty hour projects carried out in the school apprenticeship period were discussed by the students who had designed them: a brief discussion of the theoretical framework was followed by *narration of the living experience* (Van Manen 1990), through words and images. Narration was proposed as an alternative to didactic

[6] Each workshop consisted of five sessions lasting three hours. There were 25-30 participants in each one.

models (Grégnier et al 1991). Story-telling and images were planned to achieve an effect of virtual immersion in a classroom, inspired by the Japanese *lesson study* (Isoda 2007), and to take advantage of peer coaching.

Later analysis of the experience showed that the main goal achieved was in fact increased confidence in their own actual potential as teachers or prospective teachers. This was possible thanks to a *mimetic effect*, or abandoning *feeling like the 'other'* (Scaramuzzo 2011), much more than a rational discussion on methods or pedagogical models. Because of this, the workshops dedicated progressively more time to the individual presentations of each person, sharing and analysis of school memories, discussion of fears and concerns and to reasons for hope and for assuming responsibility for children's maths education. The main focus was the formative value of mathematics that the participant had seen in the projects: participants had found a human value in mathematics.

Let us close this section with the dilemma regarding education for praxis of prospective elementary teachers in the subject of mathematics. A false dichotomy has become rooted in the mathematical education of primary teachers; between the mathematical concepts that prospective primary teachers should learn[7] and the pedagogy, psychology or didactic they have to manage. The later are sometimes understood – especially for those who will work with children – as an ability that turns them into professionals providing them with self-confidence, planning proficiency, leadership and knowing how to manage a classroom. However, these abilities alone don't work when we refer to mathematics and as a consequence mathematics is taught in primary schools with fear and this fear is transmitted from one generation to another. Hence, the origin of the need to work on the view of mathematics that prospective teachers have in order to transform them from learners to teachers. Mathematics needs to be understood as a result of human thought in its contact with the world, and history of mathematics is what can be used to show this human face of mathematics.

3 An historical approach in the training of prospective primary teachers as a source of mindfulness and interest in mathematical education

In Italy, the organization of university programmes for prospective elementary teachers has, in the past 10 years, offered the opportunity to rethink their mathematical education. Efforts have concentrated on contents, the introduction of informal language (Di Sieno, Levi, 2005, 2010), didactical methods and key issues and the connections with practice. History of mathematics had little or no role. In fact, little attention is devoted internationally to the role of history of mathematics (see for example the contributions in the volumes of the 2008 *International Handbook of Mathematics Teacher Education*), with the exception of the introduction in some textbooks of a few historical "facts" regarding for example number systems (Beckmann 2012).

[7] There has been a recent controversy in the Spanish society (March, 2013) when it became known that 83% of the applicants to be primary teachers would not be able to pass the test of knowledge and essential skills of the curriculum for twelve year old children

The first author of this paper[8] and Giorgio Israel (both historians of mathematics) have developed a syllabus with a textbook (Israel, Millán Gasca 2012) and online materials based on integration of mathematical concepts and theorems (elementary arithmetic, geometry, probability and some ideas of calculus) together with historical analysis and epistemological reflection, in order to place mathematics as part of culture as a whole[9]. The historical contents are illustrated in the classroom using images and the two hours lessons include exercises. In the written exams, students write integrated discussions of mathematical, historical and epistemological aspects.

Introducing history has a double purpose. On the one hand, to restore the contact point between mathematical concepts and reality and mankind, so that the tasks teachers design have a human sense for children and support their intuition (Donaldson 1978). On the other hand, to offer future teachers a humanistic view of mathematics which awakens the desire for sharing it with children as an integral part of the educational action, strengthening their autonomy and self-confidence in the mathematics classroom.

The role of an historical approach for achieving both objectives is a leitmotiv in Federigo Enriques' educational work (Enriques 1924-27; Enriques 1921) and more recently in the work of Miguel de Guzmán (2007). Both were a source of inspiration, along with the historical work of Cajori (1917, 1928-29). The epistemological approach of Enriques, Poincaré (1902) and Husserl (see Israel 2011) integrated the axiomatic presentation of arithmetic and geometry. This was also thanks to recent research on ancient mathematics.

3.1 Primitive concepts in arithmetic and geometry

The axiomatic approach to arithmetic and geometry has thrown an interesting light on elementary maths concepts with regard to their teaching and learning, particularly in the early stages of children's education. Dedekind and Peano explicitly referred to this in connection with their work on the concept of number (Israel, Millán Gasca 2012, Ferreirós 2008). In particular, the identification of objects and primitive relations appears helpful for identifying starting points of pathways to number and geometry. A remarkable, essential aspect is the consistency between Peano's axioms (successive, induction principle) and educational research on the central role of counting in developing number concept between 2 and 8 years and on how counting supports early additions of natural numbers (Fuson 1988).

The nature of primitive concepts in axiomatic theories is described only through the relationships among them established by the axioms. Axiomatic systems developed in the late 19th century are a culmination of the process of detachment of mathematics from reality and human mind-body experience. Therefore, it may seem paradoxical to link them to children's

---

[8] A short history of mathematics aimed at prospective elementary teachers by Millán Gasca (2009) was published in the series "Quaderni a quadretti" of mathematics for primary education edited by Simonetta di Sieno.

[9] In addition, the historical evolution of children's mathematical instruction was also considered in order to put the pedagogical issues in context.

intuition: in fact, the third element is to track them in history, in Euclid's definitions and in the historical roots of these definitions.

Recent research on the ancient origin of oral and written number offers useful elements for unveiling this connection (Schmandt Besserat 1997; Nissen, Damerow, Englund 1993; on Egyptian number words see the discussion in Cartocci 2007). Moreover, Peter Damerow (2007) has considered links with neuroscience research on human cognition; Denise Schmandt Besserat has linked her thesis on the origin of symbolic thinking with anthropological research on number words and symbols in several cultures; and the historian and mathematician Enrico Giusti, has linked number, point and straight line to ancient technical experience in his essay *Hypothesis on the nature of mathematical entities* (1999). Schmandt Besserat (1999) and Giusti (2011) have even adventured to present this evidence to children in an appropriate language and with great attention to images. Cerasoli (2012) is another example of transferring results of historical research to children.

Our point of departure was that the study of the current discussion on the origin of numbers combined with a knowledge of axiomatic of Peano, including the role of recurrence (Poincaré 1902) and of repetition (Lafforgue 2010), provides prospective elementary teachers access to a deeper insight into mathematical ideas.

We want to emphasize that our proposal to discuss Peano's axioms does not mean a re-propositioning of logic and set- theoretical formal presentation of arithmetic divorced from human beings and the physical world (Kline 1977). Instead, the primitive concepts identified by the axiomatic system are studied together with historical research data on the ancient world and with epistemological reflection; and moreover other elements are brought into the framework: historical linguistics reflection on the systems number words (Greenberg 1978; see Fuson, Kwon 1992) and data from anthropological research (Cassirer 1923, Menninger 1958 and recent contributions from ethnomathematics).

This approach is more far-reaching than merely introducing a few examples of numerical systems without reference to the historical, epistemological and anthropological context or the epistemological problem. It could be considered too ambitious for an elementary school teacher. However, the fact is that when a teacher begins to work with children from 2 to 8 years old, he gets involved in the deep epistemological and anthropological problems of number, as is clear in the research of Martin Hughes (1986) on the child's contact with mathematical symbols when entering school.

A similar approach has been developed in the presentation of the study of geometry. Enriques and his school devoted great attention and many essays to the continuum and the concept of straight line and point. Recent historical research on primitive geometrical concepts and relations in the text of *Elements* by Euclides (Giusti 1999); on geometrical concepts and measurement systems in ancient civilizations (see for example Hoyrup 2002) and on prehistoric archaeological material (Keller 1998) offer many elements for examining the root in human action – technique, art, architecture – and in human perception of basic geometric concepts. Epistemological reflection by Husserl (Israel 2011) and Poincaré is the third element in a triangulation useful for a better understanding of children's first steps in geometry. In order to apply these ideas to the exploration of children's naïf conceptions and to the design of geometrical activities different from recognition of three or four forms (usually triangle, circle, square and rectangle), we based our research on Thom's views on the *continuum intuition* in the emergence of consciousness in the young child. This is consistent with research on context of a child's acquisition of language (Tomasello 2003). In the case of geometry other inputs can come from history of art (Focillon 1942, Gombrich 1995) and history of techniques.[10]

Of course the main challenge is to offer an in-depth presentation including aspects of current research in history of mathematics and presenting the open research problems; and at the same time, choosing examples and issues and communicating them in a way that can meet student's expectations and enlarge their cultural horizon. As already mentioned with regard to arithmetic, the point is that teachers actually face deep issues when working with children in their first steps with number and forms, with measure, with probability. Mathematical education is not only a question of the training in skills.

This approach has already been implemented with prospective teachers in Rome from 2011 to 2015 (about 150 students each year). The result was an improvement in commitment and performance in the basic mathematics course. Moreover, the approach has also been implemented with in-service teachers in 2014-2015 with good feedback also in terms of changes in their classes; and in a training course for prospective middle-school maths and science teachers aimed at graduate students (mathematics, physics, chemistry, biology and geology). On each occasion, the positive reactions have focused on the contrast with student's school experience and on the awakening of self-confidence regarding the possibility of enjoying teaching and learning maths with children.

3.2 Other historical elements: the confluence of several ideas in the modern concept of rational number; measures in the ancient and modern worlds; probability

The approach to rational numbers aimed at elementary school teachers often uses the school vision of an entangled mixture of fractions, decimals, percentuals, time fractions, and ratios. The modern construction of Q from Z has a power of unification and it's conceptually helpful for developing a higher point of view. Again, the formal approach is combined with a consideration of several ideas – from ancient to modern times – behind the modern concept of rational numbers as an extension of integers. This means considering both symbols (for example, fractional notation in Egypt and in the Arab and Latin middle ages, sexagesimal positional notation in ancient and Greek astronomy, decimal positional position from Stevin) and concepts (geometrical ratios, arithmetical ratios and proportionality). We cannot quote historical scholarship fully, but we would

[10] "…the geometric continuum is the primordial entity. If one has any consciousness at all, it is consciousness of time and space; geometric continuity is in some way inseparably bound to conscious thought". (Thom 1971, p. 698). This vision of geometry has already inspired three educational projects with pre-scholar children (Schiopetti 2013; Colella 2014) and a PhD Thesis on mathematics with children with Down´s Syndrome (Gil Clemente 2016).

mention Cajory's still fundamental *History of mathematical notations* and Grattan-Guinness 1996.

As to probability, the axiomatic approach was presented thanks to a discussion on the ideas of chance and the origins of probability theory and the modern definition of probability (Laplace, von Mises, de Finetti). Finally, the study of measure was also integrated with recent research on metrological systems in antiquity; considering Husserl's discussion of the idea of precision as a pathway to the Greek concept of mathematics; and a discussion of the cultural Enlightenment project of the Decimal Metric System and the force against and in favour of its spread in the late modern period is presented. It's remarkable that frequently, completely ahistorical comments accompany the introduction of the decimal metric system to children in elementary schools; and of course this is one of the most hated and annoying parts of the syllabuses.

4 Learning history of mathematics for teaching mathematics to children with Down´s Syndrome: an experience in Spain

During research for a PhD Thesis on mathematics with children with Down´s Syndrome[11] (Gil Clemente, 2016) a workshop was carried out between 2014 and 2015 with eight of these children supported by a team consisting of five young teachers from 21 to 28 years old. All of them had attended sudies related to education (Therapeutic Pedagogy, Hearing and Language, Special Education ...). In these specialities, in the 1995 Spanish study plan the percentage of credits corresponding to mathematics subjects reached only 2.5% [12].

As it was clear after a discussion-group maintained with them, previous to the development of the workshop, all of them, except one, had positive memories about maths in their school years and this was the reason why they decided to voluntarily collaborate in the experience. In spite of this, they assumed that mathematics they had studied was mechanical, with a lot of arithmetic and algebraic calculations and although they had enjoyed them, the lack of real problems had made the subject useless in relation to their lives. None of them mentioned reasoning or development of thought when speaking about the contributions that the study of maths had made to their global training.

Despite their working experience with disabled children, their great vocation for education and their commitment to collaborate in the project, it was clear from the beginning that their knowledge about maths and maths education was limited. Consequently, a training course was provided for them based on the historical approach we have presents here.

The course consisted of two phases. In the first we held two sessions before the workshop and the second took place alongside the workshop.

[11] Down´s Syndrome is the most frequent cause of an intellectual disability ranging from mild to moderate. The difficulties of people with this syndrome to learn calculation-based mathematics are well known (Buckley, 2007; Faragher, 2014). These children learn slowly and need to break up the tasks into small steps. It is for this reason that an approach based on the axiomatic approach to arithmetic and overall geometry was shown to be suitable

[12] From the analysis of this curriculum, we can observe there are only 4.5 credits of mathematics over 200 credits

The first session was devoted to arithmetic. We began by introducing the teachers to the cognitive problem of counting (Gelman, Gallistel 1978) in relation to which we outlined the importance of oral counting (Fuson 1988) and how the historical origin of number can be located in this human action. Following this we tackled the axiomatic system of Peano, stressing how he reflects on his primitive concepts (one, successive) the close relationship between number and counting. Using this triple approach which is deep and fruitful we tried to give the teachers confidence in their chances as teachers.

The second session was dedicated to geometry. Initial reflexion was focused on the strong relationship that geometry has with reality and on historical and philosophical explanations for its origin. We introduced them to the idea of representative space – visual, motor and tactile – (Poincaré 1902) as the basis of the three kind of strategies we were going to use with the children to help them to explore their surroundings: sensory activities, mimetic activities that use the power of movement and symbolic activities like drawings that allow them to begin to develop their abstract thought. From this reflexion we introduced the primitive concepts and relationships in Hilbert axiomatics–point, straight line, planes, go through, to be between– and how we can build all geometry from these concepts.

These two training sessions were well accepted by the teachers. Their knowledge of geometry was even poorer than what they knew about arithmetic, therefore, they valued the ideas we presented to them on this topic. In spite of this initial lack of knowledge they were quick to realise the power of geometry for children's education, because what they learned connected with their own experience as teachers.

The next phase was developed to run alongside the ten months of the workshop. Before each session, the team met, analysed the activities and looked deeply at the mathematical content of each one. The objective was to help them to understand the goal of each activity, and to think about which attitude we should adopt as teachers to allow concepts to arise in children's minds. This kind of reflection notably improved their ability to observe mathematical processes that unfolded in each of the later session.

The main achievement of this training was the way the team members gradually began to actively collaborate in the design of their own activities. All of them had great imagination and creativity but they suffered a lack of confidence as a consequence of their poor mathematical education. The historical and epistemological approach helped to ensure that what they invented actually helped the children to learn mathematics. Knowledge of Hilbert's axiomatic was particularly fruitful in order to design geometrical activities suitable for Down´s Syndrome children.

It may seem that to work with children with an intellectual disability it isn't necessary to know a lot about maths concepts, but that it's more important to know about methodological resources adapted to their cognitive characteristics. However, one of the aspects most valued by the teachers in this training was the way in which it helped them to approach the significant concepts and make them easy for the children to understand. The historical approach and the study of axiomatics was of great help for understanding the root and origin of concepts and this, coupled with their knowledge of children, increased their conviction in the children's chances of learning and allowed them to confidently lead the children to what they wanted tem to learn.

REFERENCES

Arnau, D. (2011). Presencia y ausencia del número natural en la educación infantil. *Summa, 66*, 7-15.

Bartolini Bussi, M. G., Ramploud, A., & Baccaglini-Frank, A. (2013). *Aritmetica in pratica. Strumenti e strategie dalla tradizione cinese per l'inizio della scuola primaria*. Trento: Erickson.

Beckmann, S. (2012). *Mathematics for Elementary Teachers with activities*. *4th ed*. Boston: Pearson.

Bell, M. S., Fuson, K. C., & Lesh, R. A. (1976). *Algebraic and arithmetic structures: A concrete approach for elementary school teachers*. New York: Free Press.

Buckley, S. J. (2007). Teaching numeracy. *Down Syndrome Research and Practice*, *12*(1), 11-14.

Cajori, F. (1917). *A History of Elementary Mathematics, with hints on methods of teaching*. *2nd ed*. London-New York: MacMillan.

Cajori, F. (1928-29). *A history of mathematical notations*. Chicago: Open Court Pub.

Cartocci, A. (2007). *La matematica degli Egizi, I papiri matematici del Medio Regno.* Firenze: Firenze University Press.

Cassirer, E. (1923). *Philosophie der symbolischen Formen*. Oxford: Bruno Cassirer.

Cerasoli, A. (2012). *La grande invenzione di Bubal*. San Dorligo della Valle: Emme Edizioni.

Colella, I. (2014). *Un catalogo di attività matematica prima della scrittura. Relazione finale di tirocinio in Scienze della Formazione Primaria*. Roma: Università degli Studi Roma Tre.

Cuomo, S. (2007). *Technology and culture in Greek and Roman Antiquity*. Cambridge: Cambridge University Press.

Damerow, P. (1996). *Abstraction and Representation: Essays on the Cultural Evolution of Thinking*. Dordrecht, Boston: Kluwer.

Damerow, P. (2007). The material culture of calculation. A theoretical framework for a historical epistemology of the concept of number. In U. Gellert, & E. Jablonka (Eds.), *Mathematisation and demathematisation. Social, philosophical and Educational ramifications* (pp. 19-56). Rotterdam: Sense Publishers.

Di Sieno, S., & Levi, S. (2010). *Aritmetica di base*. Milano: Mac Graw Hill (1st ed., 2005).

Donaldson, M. (1978). *Children's minds*. London: Fontana Press.

Enriques, F. (1921). Insegnamento dinamico. *Periodico di matematiche*, 4th series, *1*, 6-16.

Enriques, F. (Ed.) (1924-1927). *Questioni riguardanti le matematiche elementari. 3rd ed*. Bologna: Zanichelli (reprinted Bologna, Zanichelli, 1983).

Enriques, F. (1938). *Le matematiche nella storia e nella cultura*. Bologna: Zanichelli.

Faragher, R., & Clarke, B. (Eds.) (2014). *Educating children with Down Syndrome.* New York: Routledge.

Fauvel, J., & Van Manen, J. (Eds) (2000). *History in mathematics education*. Dordrecht, Boston: Kluwer Academic Publishers.

Ferreirós, J. (1999). *Labyrinth of Thought. A History of Set Theory and its Role in Modern Mathematics*. Basel-Boston-Berlin: Birkhäuser.

Focillon, H. (1942). *Le life of forms in art*. New Haven: Yale University Press.

Fuson, K. C. (1988). *Children's Counting and Concepts of Number*. New York: Springer.

Fuson, K. C., & Kwon, Y. (1992). Learning addition and substraction: Effects of number words and other cultural tools. In J. Bideaud, C. Meljac, & J. P. Fischer (Eds.), *Pathways to number: Children's developing numerical abilities* (pp. 283-305). Hillsdale, NJ: Lawrence Erlbaum.

Friberg, J. (2007). *Amazing traces of a Babylonian origin of Greek mathematics*. Singapore: World Scientific.

Gelman, R., & Gallistel, R. (1978). *The child's understanding of number.* Cambridge, MA: Harvard University Press.

Gil Clemente, E. (2016). *Didáctica de las matemáticas para niños con síndrome de Down a partir de una vision integrada de la aritmética y de la geometría elementales*. Unpublished PhD thesis. Zaragoza: Universidad de Zaragoza.

Giusti, E. (1999). *Ipotesi sulla natura degli enti matematici*. Torino: Bollati Boringhieri.

Giusti, E. (2011). *Awa insegna a contare*. Firenze: Il Giardino di Archimede.

Gombrich, E. H. (1984). *The Sense of Order: A Study in the Psychology of Decorative Art*. *2nd ed*. London: Phaidon press.

Gombrich, E. H. (1995). *The Story of Art*. London: Phaidon Press Ltd.

Greenberg, J. H. (1978). Generalizations about Numeral Systems. In J. H. Greenberg, C. A. Ferguson, & E. A. Moravcsik (Eds.), *Universals of Human Language* (pp. 249-295). Stanford: Stanford University Press.
Grenier, J. Y., Grignon, C., & Menger, P.-M. (2001). *Le modèle et le récit*. Paris: Editions de la Maison des Sciences de l'homme.
Grattan-Guinnes, I. (1973). Not from nowhere. History and philosophy behind mathematical education. *International Journal of Mathematical Education in Science and Technology*, *4*(4), 421-453.
Grattan-Guinnes, I. (1996). Numbers, magnitudes, ratios, and proportions, in Euclid's Elements: How did he handle them. *Historia mathematica*, *23*, 355-375.
Guzmán, M. de (2000). La educación matemática en riesgo. Retrieved from http://www.mat.ucm.es/catedramdeguzman/drupal/migueldeguzman/legado/educacion.
Guzmán, M. de (2007). Enseñanza de las ciencias y la matemática. *Revista Iberoamericana de Educación*, *43*, 19-58.
Hughes, M. (1986) *Children and Numbers. Difficulties in Learning Mathematics*. Oxford & New York: Basil Blackwell.
Hoyrup, J. (2002). *Lengths, Widths, Surfaces. A Portrait of Old Babylonian Algebra and its Kin*. Springer: New York.
Isoda, M. (Ed.) (2007). *Japanese lesson study in mathematics: Its impact, diversity and potential for educational improvement*. New Jersey: World Scientific.
Israel, G., & Millán Gasca, A. (2012). *Pensare in matematica*. Bologna: Zanichelli.
Israel, G. (2011). *La natura degli oggetti matematici alla luce del pensiero di Husserl*. Genova-Milano: Marietti.
Keller, O. (1998). Questions etnographiques et mathématiques de la préhistoire. *Revue de synthèse*, 4th series, *4*, 545-573.
Kline, M. (1973). *Why Johnny can't add: The failure of the new mathematics*. New York: St. Martin's Press.
Kline, M. (1977). *Why the professor can't teach: Mathematics and the dilemma of university education*. New York: St. Martin's Press.
Kline, M. (1980). *Mathematics: The loss of certainty*. New York: Oxford University Press.
Lafforgue, L. (2010). *L'importance du calcul et de la géométrie à l'école primaire*. Journées Trans-Maître: Instruir aujourd'hui à l'école primaire. Retrieved from http://www.ihes.fr/~lafforgue/textes/ImportanceCalculGeometrie.pdf.
Levy Bruhl, L. (1923). *Primitive mentality*. London: George Allen, New York: Macmillan.
Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers understanding of fundamental mathematics in China and the United States*. Mahwah-London: Lawrence Erlbaum Associates; new ed. 2010, New York-Abingdon, Routledge.
Margolinas, C. (2007). What mathematical knowledge does the teacher need? *La matematica e la sua didattica*, *21*(1), 21-28.
Millán Gasca, A. (2016). *Numeri e forme. I bambini e la matematica*. Bologna: Zanichelli, forthcoming.
Millán Gasca, A., Marrano, M., & Spagnoletti Zeuli, E. (2014). Labo-B: la sfida della fiducia nell'aula di matematica. Poster, Giornata di studio *Diventare maestri: il tirocinio di Scienze della Formazione Primaria tra università e scuola*. Roma, University Roma Tre, 9 maggio 2014.
Nissen, H. J., Damerow, P., & Englund, R. K. (1993). *Writing and Techniques of Economic Administration in the Ancient Near East*. Chicago: University of Chicago Press.
Poincaré, H. (1902). *La science et l'hypothèse*. Paris: Flammarion.
Robson, E., & Stedall, J. (Eds.) (2009). *Oxford Handbook of the History of Mathematics*. Oxford: Oxford University Press.
Schmandt-Besserat, D. (1997). *How writing came about*. Austin: University of Texas Press.
Schmandt-Besserat, D. (1999). *The history of counting*. New York: Morrow Junior Books.
Schiopetti, V. (2013). *Le concezioni geometriche ingenue e la Scuola dell'Infanzia*. Relazione finale di tirocinio. Roma: Università degli Studi Roma Tre.
Sierra, M., & Rico, L. (1996). Contexto y evolución histórica de la formación en Matemática y su Didáctica de los profesores de primaria. In J. Giménez, S. Linares & V. Sánchez (Eds.), *El proceso de llegar a ser un profesor de primaria. Cuestiones desde la educación matemática*. Granada: Comares.
Thom, R. (1971). Modern mathematics: An educational and philosophic error?. *American Scientist*, *59*(6), 695-699.
Thom, R. (1973). Modern mathematics: Does it exist? In A. G. Howson (Ed.), *Developments in mathematical education* (pp. 194-209). Cambridge: Cambridge University Press.
Thom, R. (1991). *Prédire n'est pas expliquer*. Paris: Eshel.

Tomasello, M. (2003). *Constructing a language. A usage-based theory of language acquisition*. Cambridge, Mass.-London: Harvard University Press.
Van Manen, M. (1990). *Researching lived experience*. Albany, NY: State University of New York Press.
Wiese, H. (2003). *Numbers, language, and the human mind*. Cambridge: Cambridge University Press

MATHEMATICAL KNOWLEDGE FOR TEACHING TEACHERS

The case of history in mathematics education

Uffe Thomas JANKVIST, Reidar MOSVOLD, Kathleen CLARK
Aarhus University, Denmark
utj@edu.au.dk
University of Stavanger, Norway
reidar.mosvold@uis.no
Florida State University, USA
kclark@admin.fsu.edu

ABSTRACT

Teaching mathematics in school has been researched by many, with Ball, Thames, and Phelps (2008) and their practice-based theory of mathematical knowledge for teaching (MKT) primary among them. However, the work of teaching mathematics in teacher education has been much less researched. An emerging theory of mathematical knowledge for teaching teachers (MKTT; Zopf, 2010) is of particular interest in our current work. This paper deals with part of a *Didactics of Mathematics* course given to future mathematics teacher educators at the Danish School of Education, and asks the question of how to develop these future teacher educators' MKTT in relation to history of mathematics in mathematics education. We share the key components of the theoretical constructs underlying our work and illustrate these by means of the students' own mini-project reports, which address cases or topics ranging from analysis of the inclusion of history in mathematical textbooks, to the development of an activity for pupils – or for student teachers – which include original source materials.

1 Introduction

The paper addresses the question of how to introduce future mathematics teacher educators to the discussion of history in mathematics education, and how to prepare them theoretically for a potential use of history of mathematics in their own future practice. The "answer" presented to this question is one by example, since the paper reports on a concrete design and implementation of a course. The theoretical framework adhered to in the paper is a further development of the practice-based theory of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008) into an emerging theory of mathematical knowledge for teaching teachers (Zopf, 2010) [1]. At the core of this theory is a particular conception of teaching – being a plausible conception of the professional practice of teachers – and the work of teaching can further be defined through the mathematical tasks that teachers do in order to facilitate students' learning of mathematics (Hoover, Mosvold, & Fauskanger, 2014). Tasks of teaching can be seen as a decomposition of the work of teaching, and mathematical knowledge for teaching can thus be described as the "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 399). Whereas many have investigated the work of teaching mathematics in school, researchers have much less investigated the work of teaching (future) mathematics teachers. Illustrative examples of students' reports from the course will be displayed and discussed after presenting the educational setting of the course and the design of the six sessions related to the topic of “history in mathematics education.” Hence, the aim of this paper is to employ aspects of this evolving framework for MKTT in order to describe the future teacher educators’ development of MKT and MKTT in relation to the use of history of mathematics in mathematics education.

2 Mathematical knowledge for teaching (teachers)

In the following, and before we present some recent attempts to investigate what can be referred to as mathematical knowledge for teaching teachers (MKTT), we first describe some foundations of MKT. Far too often, descriptions of MKT in the research literature are limited to a presentation of “the egg” and the sub-categories of MKT that are depicted in it (figure 1).

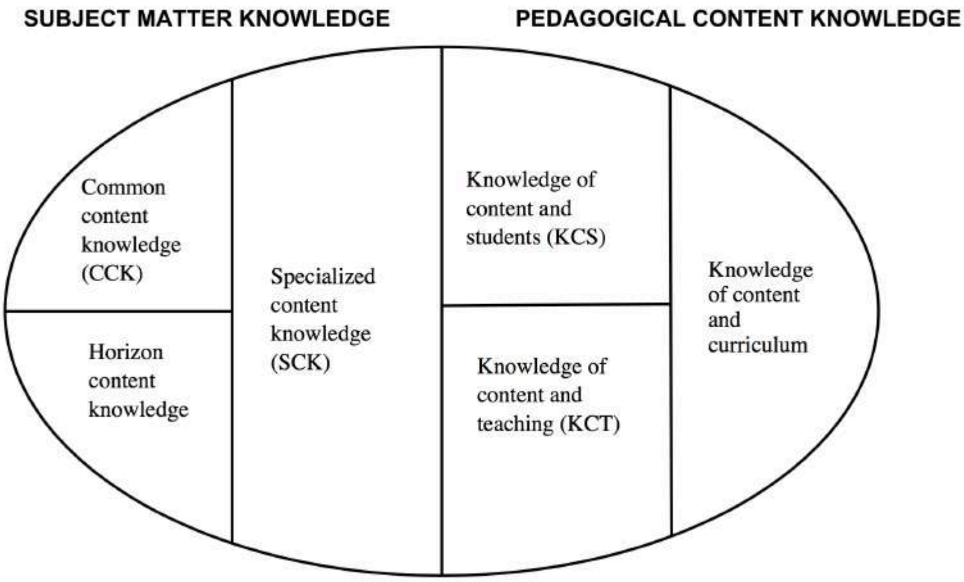


Figure 1. The common representation of MKT (Ball, Thames, & Phelps, 2008, p. 403)

This representation of categories and sub-categories of MKT might have initially served a purpose for representing one version of the forms of knowledge a teacher might employ or draw upon in teaching, but it is not necessarily considered as the core of the practice-based theory of MKT. In their discussion of the assumptions that are underlying the development of MKT at the University of Michigan, Hoover and colleagues (2014, p. 11) emphasized: (1) the role of the discipline of mathematics in and for teaching; (2) the meaning of the term “teaching” in the phrase “for teaching”; and (3) the mutual importance of both conceptual work and the validation of proposed conceptualizations in advancing early-stage research. The first assumption highlights the role of the discipline of mathematics, and this means that there must be a commitment not only to students’ thinking, but also to mathematics as a discipline. The latter is important and can easily be forgotten. This emphasis on the discipline of mathematics is also what represents a significant development of Shulman’s (1986) more general ideas about teachers’ professional knowledge – on which the theory of MKT is based. Second, the term “teaching” is important, and Hoover and colleagues proposed that, “teaching is seen as a *plausible* conception of *professional* practice” (p. 11, original emphasis). Such conceptions – in the work of Deborah Ball and colleagues – are based on careful analyses of the work of teaching mathematics. The aim of such analyses is to identify “what is entailed mathematically in that teaching” (Hoover et al., 2014, p. 12). The focus on identifying or decomposing the work of teaching mathematics is strongly connected with the third assumption. Although many seem to associate MKT with “the egg” or multiple-choice measures as were developed as part of the Learning Mathematics for Teaching (LMT) Project, the research of Ball’s group can be described as strongly conceptual and analytic. Thus, the aim of this research is to develop professionally grounded concepts that can be meaningful and usable (Ball & Bass, 2003; Hoover et al., 2014). At the core of the research efforts from Ball and her colleagues is the “focus on the mathematical tasks that teachers have to deal with in the work they do that have significant mathematical entailments” (Hoover et al., 2014, p. 13). The question about what knowledge demands are entailed in the teaching of mathematics, for these researchers, thus becomes a question of identifying recurrent mathematical tasks of teaching mathematics.

Ball et al. (2008, p. 400) presented the following list in their attempt to conceptualize several core tasks of teaching: presenting mathematical ideas; responding to students’ “why” questions; finding an example to make a specific mathematical point; recognizing what is involved in using a particular representation; linking representations to underlying ideas and to other representations; connecting a topic being taught to topics from prior or future years; explaining mathematical goals and purposes to parents; appraising and adapting the mathematical content of textbooks; modifying tasks to be either easier or harder; evaluating the plausibility of students’ claims (often quickly); giving or evaluating mathematical explanations; choosing and developing useable definitions; using mathematical notation and language and critiquing its use; asking productive mathematical questions; selecting representations for particular purposes; and finally, inspecting equivalencies. This list, which was not meant to be definitive, has later been subject to further investigation, and attempts have been made to extend (e.g., Delaney, 2008) and criticize (e.g., Ng, Mosvold, & Fauskanger, 2012) it. The multiple-choice items that resulted from the LMT Project – often referred to as “the MKT items” – can be seen as attempts to operationalize these mathematical tasks of teaching. The few attempts that have already been made to investigate mathematical knowledge for teaching teachers (MKTT) build upon these foundational ideas behind MKT.

In their investigations of MKTT, both Zopf (2010) and Kim (2013) focused on investigating the tasks of teaching mathematical knowledge for teaching in teacher education. Zopf (2010) argued that the work of teaching MKT in teacher education entails a number of recurrent tasks of teaching. The following three are highlighted in particular: “selecting interpretations and representations, selecting examples, and managing the enactment of mathematical tasks for the work of teaching mathematical knowledge for teaching” (Zopf, 2010, p. 199). She suggested that there is a distinct domain of mathematical knowledge that is needed for teaching MKT in teacher education, and she referred to this as MKTT. Zopf proposed that this MKTT includes a specialized knowledge of MKT as well as a solid knowledge of the discipline of mathematics. The latter includes “knowledge about mathematical structures such as definitions, properties, theorems, and lemmas and how these are used to do mathematics; knowledge about descriptions, explanations, justifications, and proof and how these are used for mathematical work" (ibid.). Kim (2013) concurred with this in her study, and further developed a framework for teaching MKT in teacher education. This framework consists of two interrelated entities: mathematical work of teaching and knowledge about mathematics. The latter category of knowledge appears to coincide with Zopf's (2010) concept of disciplinary knowledge of mathematics, and it is particularly interesting for our study since it "is about the nature of knowledge in the discipline, such as where it comes from, how it changes, and how truth is established" (Kim, 2013, p. 12). In our reading, this points to the history of mathematics.

Mosvold, Jakobsen and Jankvist (2014) argue that history of mathematics can be useful for pre-service as well as in-service mathematics teachers, e.g. "history of mathematics can be useful for the teachers as a means to increase knowledge and awareness of possible misconceptions, obstacles and impediments related to various mathematical concepts and ideas" (ibid., p. 58). They also argue that investigations into the history of mathematics, for instance by studying historical sources, have the potential to increase the mathematical knowledge of the student teachers. As seen above, from the emerging literature on MKTT it appears evident that solid disciplinary knowledge of mathematics is required from the teacher educators (Superfine & Li, 2014; Zopf, 2010). From their review of literature on mathematical knowledge for teaching, Hoover, Mosvold, Ball and Lai (2016) argue that development of MKT in teacher education requires close connection between mathematical content and the work of teaching. Mosvold et al. (2014) suggested that history of mathematics has a lot to offer for mathematics teacher education, but the introduction of history also placed some demands on the teacher educators. In this paper we dig deeper into this phenomenon when we investigate further how history of mathematics can be introduced to future mathematics teacher educators, and how they can be prepared for using history of mathematics in their future teaching practice.

3 Educational setting and background

To become a mathematics teacher educator of primary and lower secondary teachers in Denmark, it is often favored by teacher training colleges that the educators hold a master's degree in mathematics education [2], of which the Danish School of Education at Aarhus University is the only provider in the country. To enter the master's program the student must already have a university bachelor degree, e.g. in mathematics, or a vocational bachelor degree, e.g. as a primary and lower secondary mathematics teacher. The two-year master's program consists of courses in mathematics, courses in general didactics, and a course in didactics of mathematics, several of these involving student projects, and finally a master's thesis.

The course of our interest here is *Didactics of Mathematics*, as implemented in the years of 2014 and 2015. In this course, each of the four mathematics educators within the department are given the opportunity to teach in a mathematics education topic of their own choice. One of the ideas behind this is that students in this way are also confronted with recent research, in which the mathematics educators themselves are involved. The course counts 10 ECTS (European Credit Transfer System), and each topic consists of six sessions of two to three hours of instruction and supervision each, along with group work, etc. For each of the topics, groups of students must submit a mini-project. Based on a random selection, at the end of the course the student groups are examined in one of the four mathematics education topics.

4 Design and function of the six lessons

We now describe the content and purpose of the six sessions related to history in mathematics education. For each session the students were to read a collection of texts (primarily research papers), with which they were to work during a given session. Additionally, supplementary texts were provided. Students' previous experiences with history of mathematics vary from a superficial exposure to full undergraduate course work.

In session 1, students were to familiarize themselves with different arguments for and against the use of history (and epistemology) in mathematics education, as well as potential dilemmas, and of course different approaches to involving history. The assigned texts included Fried (2001), Jankvist (2009) and Niss and Højgaard (2011). The purpose of this collection of texts was to enable the students to more qualifiedly discuss concrete uses of history at different educational levels including teacher training. Session 2 focused on the role and use of theoretical frameworks in empirical studies related to a use of history in mathematics education. The students were presented with two studies (Jankvist, 2011; Kjeldsen & Blomhøj, 2012), which served as cases, and they then were to discuss the use of Sfard's (2008) framework of commognition within these two cases (the students were already somewhat familiar with this framework). As supplemental literature for this lesson, students were encouraged to examine Jankvist and Kjeldsen (2011) and the use of the Danish competency framework (Niss & Højgaard, 2011) in this.

Next, session 3 addressed the use of original sources in the teaching and learning of mathematics as well as different approaches to involve such sources (e.g. Barnett, Lodder, & Pengelley, 2014; Jankvist, 2013). Here again, one purpose was to prompt the students to argue for and against a potential use of original sources in a particular educational setting. Supplementary texts for session 3 included Glaubitz (2011) and Jankvist (2014). The topic of session 4 was that of history in mathematics teacher education and not least teachers' potential benefits of being introduced to elements of the history of mathematics. Drawing on the topics of the previous lessons, the students were to compare an older empirical study (Arcavi, Bruckheimer, & Ben-Zvi, 1982) with a newer one (Clark, 2012), and discuss the outcomes of these in the light of interpreting results by means of the MKT framework (e.g. Ball, Thames, & Phelps, 2008; students were already somewhat familiar with this), drawing also on a reading of Mosvold, Jakobsen, and Jankvist (2014).

Session 5 was designed as a workshop in which the students were to further relate the read texts to each other as well as to a constructing a concrete case of their own choice. This work eventually resulted in a submitted mini-project report (approximately 12 normal pages, plus appendices) for each student group. These reports were then presented and discussed during session 6, where each student group would also have read the report of another group in order to provide constructive feedback and to also receive feedback themselves. (In Spring 2015, the third author of this paper was present at the course in sessions 4 and 5.)

5 Students' mini-projects

In this section, we first present a list of the students' mini-projects in order to provide an overview of the topics and issues that the students themselves have chosen to address from the point of view of using history in mathematics education. Next we describe two of the mini-projects from this list in more detail; one dealing with the changing notions of the concept of function through the 18th and 19th century; and another that applies the presented theoretical constructs of the course literature in an analysis of the inclusion of history of mathematics in a secondary school mathematics textbook.

5.1 List of students' mini-projects (Spring 2014 and Spring 2015)

- The use of history in a mathematics textbook system for primary and secondary school (see illustrative example 1), asking to the use being one of history as a goal or tool, Whig history, and the use of excerpts from original sources.
- The use of original sources in relation to the introduction of the concept of function in grade 9 of secondary school (see illustrative example 2)
- Students' development of overview and judgment (Niss & Højgaard, 2011) concerning the historical development of mathematics, exemplified by the history of number systems and the number 0.
- A discussion of different mathematical discourses (cf. Sfard, 2008) in selected primary sources concerning calculation of π, and the change of teaching discourses between addressing in-issues and meta-issues (Jankvist, 2009).
- The use of Babylonian tablets for teaching 2nd-degree equations in upper secondary school as an example of using original sources in teaching (Barnett et al., 2014).
- An analysis of a HAPh-module on Boolean algebra and circuit design (Jankvist, 2013), drawing on the course literature.
- Designing a teaching module around different proofs of the Pythagorean theorem (Euclid; Liu Hui; and a modern textbook one) focusing on aspects of history as a goal in relation to proofs and proving (Jankvist, 2009).
- Using history of mathematics in mathematics teaching education (Clark, 2012), illustrated by means of Mayan mathematics (the codexes from Paris, Dresden, and Madrid).
- A discussion of historical parallelism between the coming into being of the number 0 and pupils' mathematical learning difficulties of this particular number.
- How the history of negative numbers may potentially assist pupils in their reification (Sfard, 1991) of negative numbers as mathematical objects.
- A discussion of how the history of mathematics, exemplified by the history of π, may assist pupils in developing the mathematical competencies (Niss & Højgaard, 2011) of reasoning, representation, and problem tackling.
- The history of probability theory and its potential use in the teaching of mathematics.
- Analysis of the use of history in a mathematics textbook system for primary and secondary school and the ministerial orders for the school subject of mathematics when the textbook system was published.

- The potential use of Al-Khwarizmi's *Algebra* in original source as a means for inclusion of pupils with Arab ethnicity in Danish classrooms.
- Analysis of a recent textbook system for upper secondary school, and a discussion of how this system explicitly or implicitly invites the reader (or teacher) to use history of mathematics in the teaching of the subject.

5.2 Illustrative example 1: Textbook analysis (Spring 2014)

One group focused on a Danish textbook system called *Sigma*, and in particular they looked at the books for 8th grade, which consist of one textbook for the pupils and one for the teacher. For their initial analysis of the books' use of history, the group relied on the constructs of Fried (2001) and Jankvist (2009). Firstly, the group discussed the textbook authors' purpose of using history and the degree to which they find this realized:

> In the teachers' textbook [...] we find the following statement in the chapter on Numbers and Algebra: "We believe it to be important that the pupils learn about the development of mathematics and in particular that of numbers. Although such knowledge may not have a direct yield, it assists in providing the background for a part of the world, which we live in today. Without this knowledge, mathematics [...] appears as if it has always existed in the form we are introduced to today" [Sigma 8, teachers' textbook, p. 6]. Here, knowledge of the history of mathematics is viewed as relevant in itself. Hence, generally speaking, we have to do with a goal argument [of using history]. The interesting thing then is how this is reflected in the pupils' textbook [...]. Through the entire chapter, we see a large focus on the history of mathematics. 10 out of 24 pages are dedicated entirely to history of mathematics, where the pupils are informed about the historical development of the numbers from hieroglyphs over Roman numerals to negative numbers and the number 0. Occasionally, the historical account is replaced by traditional mathematics exercises. However, there is almost no connection between the historical information and the exercises, since these can be solved without having read the historical account. Hence, the intention from the teachers' book is not clearly implemented in the pupils' book. (Færch et al., 2014, p. 3)

The group also provided another example, one on the Pythagorean Theorem, where the teachers' textbook provided an extensive account of Pythagoras, his school, and the presumed Babylonian origin of the theorem. Again, the implementation of this piece of history in the pupils' textbook is reduced to a cartoon drawing, a picture of a marble bust of Pythagoras, and several examples accompanied by modern-day notation. As pointed out by the group, the book missed an opportunity of applying excerpts from original sources here. Original sources, however, are part of the teachers' book, but as remarked by the group the book authors' intention with this remains unclear:

> ... in the teachers' book [...] six excerpts from original sources on the proof of the Pythagorean Theorem are shown [...], but no suggestions as to how the pupils may be brought to work with these sources are provided – actually, there is no mentioning of

> the sources themselves, so it is unclear why they are included in the first place. (Færch et al., 2014, p. 9)

In further relation to the discussion of purpose of using history versus approaches to using it, the group pointed out that despite it being difficult to realize 'history as a goal' through mere 'illumination approaches' (Jankvist, 2009), this appeared to be what happened in the *Sigma* system (ibid. p. 5). They continued:

> The teachers' textbook [...] contains quite a number of test exercises, but history of mathematics is not a part of any of these. In the teachers' book it is clear that history is used as a 'spice' and seen as a tool, not as a goal. In the notion of Fried (2001), what we are dealing with is a 'strategy of addition.' (Færch et al., 2014, p. 6)

With continued reference to Fried (2001), the group went on to argue that the book system had a somewhat Whig approach to using history, particularly in the pupils' textbook. Following this, the group discussed the missed opportunities of the book system in relation to fostering Sfard's commognitive conflicts, with reference to Kjeldsen and Blomhøj (2012):

> In the teachers' textbook [...] it is stated that the pupils should become acquainted with the Roman numerals, although not to a very large extent: "The positional number system should – once more – be examined carefully with the pupils, while the Roman numerals should not be examined as much – they merely serve to illustrate the advantages of the numeral system we apply today" [Sigma 8, teachers' textbook, p. 7]. The authors' intention here is that of having one numeral system meet another in order to illustrate clearly the good idea of one of them. It is exactly in this meeting between two different discourses that the opportunity for learning arises, since the difference between the two discourses is made clear by the advantage of one of them. The intention here is for students to discover the ineffectiveness of addition in the Roman numerals as compared to our current positional system. Unfortunately, as seen before, this idea is not pursued in the pupils' textbook, which only contains little information about the Roman numerals, and not a single exercise where pupils are to work with these. (Færch et al., 2014, pp. 7-8)

5.3 Illustrative example 2: Concept of function (Spring 2014)

The case of the second group was four different definitions of the concept of function, more precisely Euler's definitions from 1748 and 1755, respectively, Dirichlet's from 1837, and a modern definition relying on the notion of sets (e.g. see Kjeldsen & Petersen, 2014). The group aimed to construct a small module to be implemented in 9th grade of secondary school, since they found that the concept of function is one that is troublesome for pupils at this grade and the beginning of upper secondary school. Hence, an assumption of the group was that such a module might assist in easing the transition phase between the two educational levels (Jankvist, 2014), meaning that they aimed at using history as a tool (Jankvist, 2009):

> We intend a half-half relationship between mathematics and history, and we use the history of mathematics as a means to teach the pupils the concept of function, i.e. our

> 'why' is history as a tool. We use it [history] as a motivating and cognitive tool by offering different ways to introducing the concept. (Hansen et al., 2014, p. 8)

In terms of approach, Group 2 intended a four-session module relying on the hermeneutic approach (Glaubitz, 2011):

> We find that the hermeneutic approach fits our case, because it is the contrasts between past and present that are to be examined consciously, and because it is the embracing of these tensions that provides the deeper understanding of both the mathematics itself and the history of mathematics (Barnett et al., 2014). Since we choose the hermeneutic approach we first present the pupils with the modern definition of the concept of function and afterwards the original sources. (Hansen et al., 2014, p. 9)

The idea, the group explained, is that the pupils must relate the early definitions to the modern one. In relation to Dirichlet's definition, they said, the point of that and the modern one is actually the same, but the associated concepts have changed over time, e.g. set theory was not available at the time of Dirichlet. And, by relating the modern definition and Dirichlet's to those of Euler, the pupils must obtain an idea of why Euler's concept of function is insufficient for us. Following this explanation, the group addressed the potential benefits of relying on original sources:

> One of the advantages of original sources is that they promote the reader's abilities to think like the author, and another is an understanding of the different context in which the sources are written (Barnett et al., 2014). If the pupils become aware of the historical context and try to understand what the author did, there is a chance that they also try to understand the mathematics. [...] Other advantages are, among others, to bring the pupils closer to experiencing the creation of mathematics and see the road of mathematical development, flow, errors, and success (Barnett et al., 2014). (Hansen et al., 2014, p. 9)

Finally, Group 2 touched on the discussion of having a Whig approach to history, and even though they admitted that their purpose of using history as a tool may have such consequences, the important issue is that they did so in an informed and conscious manner:

> In our case we have chosen not to use all of the original sources, because even in the Danish translations they appear too difficult. Hence, we have chosen to use only the definitions, which are what the pupils get as 'original sources'. [...] We have found ourselves in the dilemma that either the original sources were too difficult, or we had to face that it was not possible to avoid being a 'little' Whig in our approach. Hence, we are aware of the fact that our approach is a little Whig, but we have found this difficult to avoid when the target group is secondary school. (Hansen et al., 2014, p. 10)

6 Concluding discussion

We now turn our attention to the discussion of the students' yield in terms of 'history in mathematics education' as part of their future practice as teacher educators. The list of mini-projects bear witness to the way the future teacher educators attempted to connect the two primary types of knowledge pointed to for possessing MKTT: disciplinary knowledge of mathematics and mathematical work of teaching (Kim, 2013; Zopf, 2010). On the one hand, the students had the opportunity to familiarize themselves with a small piece of history of mathematics, sometimes also involving mathematics with which they were not already familiar (e.g. the group that studied Boolean algebra). On the other hand, they were expected to carefully reflect upon the way in which this history entered into a teaching and learning situation (relying on course literature), and in this process also reflect upon several of Ball and colleagues' (2008) core tasks of teaching, e.g. presenting mathematical ideas (through history) and recognizing what is involved in using a particular (historical) representation. That disciplinary knowledge and specialized knowledge of MKT make up two major components of MKTT is perhaps not so surprising. A teacher educator should possess a good portion of the six types of MKT, but they need a knowledge of MKT that is special to the work of teaching teachers. This specialized knowledge of MKT is strongly related to the deep disciplinary knowledge that is required to teach (future) mathematics teachers. Two types of MKT appear to be more in focus in relation to MKTT, namely horizon content knowledge (HCK) and knowledge of content and teaching (KCT). The two illustrative examples seem to support this.

In the first illustrative example, the students found that the textbook system authors declared that their use of history is one of history as a goal, while the students' analysis of the textbook system, exercises, tasks, etc. showed otherwise. Furthermore, the use of history in the textbook system, mainly relying on an illumination approach, often appears rather Whig, and according to the students the textbook authors missed several opportunities to include the history in a sensible manner. The group's textbook analysis did require knowledge about the curriculum in primary and secondary school (KCC) as well as the actual mathematics (CCK) and the pupils at this educational level (KCS). However, their assessment of the textbook system not adhering to its declared use of history as a goal required them to activate their HCK and KCT to a much larger extent. The observation of the textbook authors being Whig in their approach was a matter of the students' knowledge of the historical development of mathematics, i.e. related to HCK, while their observation of missed opportunities drew on HCK, KCT and SCK. In relation to MKTT, we observed an overuse of disciplinary knowledge and mathematical work of teaching. When discussing disciplinary knowledge of mathematics in MKTT, it is important to emphasize that this is a knowledge of mathematics that goes beyond common content knowledge and mathematics taught in 'regular' university courses in mathematics (cf. Hoover et al., 2016). Disciplinary knowledge of mathematics in MKTT involves knowledge of the nature of mathematics and how it has developed (Kim, 2013) – as illustrated in the discussion of this first example – and we therefore suggest that knowledge of history of mathematics and its use in teaching constitute a significant part of the knowledge that is special to the work of teaching (future) mathematics teachers.

In the second illustrative example, the students intentionally chose history as a tool to develop pupils' concept of function. They adhered to a use of original sources (the various definitions of the concept of function) through a hermeneutic approach. These students knew from their own experience that the concept of function was a difficult one for pupils (KCS), yet a central one in mathematics (CCK) and in the 9th grade curriculum (KCC). The essential practice for this group, however, is that they use their newly developed knowledge of how the concept of function has evolved (HCK) to design a teaching activity putting this into play from a cognitive perspective (KCT). If these students, in their future profession as teacher educators, were to present this activity to their student teachers, it seems plausible that it would be the aspects of HCK and KCT and the MKTT-related disciplinary knowledge of mathematics and mathematical work of teaching that would be in focus.

As argued by Smestad, Jankvist and Clark (2014), common content knowledge (CCK) and knowledge of content and curriculum (KCC) change over time with the introduction of new reforms, new curricula, etc., and such times of change teachers – and teacher educators – may rely on their HCK and KCT. As pointed to by several (e.g. Mosvold et al., 2014; Smestad et al., 2014), the history of mathematics is less prone to change, and hence offers a stable foundation for teachers in terms of HCK. A similar argument seems obvious for teacher educators, since the history of mathematics provides valuable input to developing a teacher educator's knowledge about the discipline of mathematic. In the students' mini-projects, questions about mathematical structures (definitions, properties, etc.), explanations, justifications, symbolism, formalism, proofs and proving, etc. occurred as a natural part of the process of doing these projects. And every time the students' mathematical knowledge was challenged, they also had to consider this from a pedagogical and didactical point of view relying on the course literature. Hence, we argue that the history of mathematics has an obvious role to play in the education of mathematics teacher educators as well as in illustrating the two identified components of the emerging framework of MKTT.

NOTES

1. The course described in this paper has previously been analyzed from the point of view of developing teacher competencies (Niss & Højgaard, 2011). This analysis is available in the proceedings from TWG-12, CERME-9.

2. Alternatively, the educators may hold a master's degree in mathematics. Note that in Denmark teacher educators are not required to possess a Ph.D.

REFERENCES

Arcavi, A., Bruckheimer, B., & Ben-Zvi, R. (1982). Maybe a mathematics teacher can profit from the study of the history of mathematics. *For the Learning of Mathematics, 3*(1), 30-37.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis, & E. Simmt (Eds.), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton: CMESG/GCEDM.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Barnett, J. H., Lodder, J., & Pengelley, D. (2014) The pedagogy of primary historical sources in mathematics: Classroom practice meets theoretical frameworks. *Science & Education, 23*(1), 7-27.

Clark, K. M. (2012). History of mathematics: Illuminating understanding of school mathematics concepts for prospective mathematics teachers. *Educational Studies in Mathematics, 81*(1), 67-84.

Delaney, S. (2008). Adap*ting and using U.S. measures to study Irish teachers' mathematical knowledge for teaching* (Unpublished doctoral dissertation). Ann Arbor: University of Michigan.

Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education, 10*(4), 391-408.

Færch, J. V., Hansen, M. E., Møller, M. T., & Østerby, H. L. (2014). *Non-published exam report on "The history, application, and philosophy of mathematics in teaching"*. Aarhus University.

Glaubitz, M. R. (2011). The use of original sources in the classroom: Empirical research findings. In E. Barbin, M. Kronfellner, & C. Tzanakis (Eds.), *History and Epistemology in Mathematics Education Proceedings of the 6th European Summer University* (pp. 351-362). Vienna: Holzhausen Publishing Ltd.

Hansen, S. L., Mikkelsen, L. N., & Skovsgaard, N. (2014). *Non-published exam report on "The history, application, and philosophy of mathematics in teaching"*. Aarhus University.

Hoover, M., Mosvold, R., Ball, D. L., & Lai, Y. (2016). Making progress on mathematical knowledge for teaching. *The Mathematics Enthusiast, 13*(1–2), 3-34.

Hoover, M., Mosvold, R., & Fauskanger, J. (2014). Common tasks of teaching as a resource for measuring professional content knowledge internationally. *Nordic Studies in Mathematics Education, 19*(3–4), 7-20.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235-261.

Jankvist, U. T. (2011). Anchoring students' meta-perspective discussions of history in mathematics. *Journal of Research in Mathematics Education, 42*(4), 346-385.

Jankvist, U. T. (2013). History, applications, and philosophy in mathematics education: HAPh – A use of primary sources. *Science & Education, 22*(3), 635-656.

Jankvist, U. T. (2014). On the use of primary sources in the teaching and learning of mathematics. In M. R. Matthews (Ed.), *International handbook of research in history, philosophy and science teaching* (pp. 873-908). Dordrecht: Springer.

Jankvist, U. T., & Kjeldsen, T. H. (2011). New avenues for history in mathematics education – Mathematical competencies and anchoring. *Science & Education, 20*(9), 831-862.

Kim, Y. (2013). *Teaching mathematical knowledge for teaching: Curriculum and challenges* (Unpublished doctoral dissertation). Ann Arbor: University of Michigan.

Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation: History as a method for learning metadiscursive rules in mathematics. *Educational Studies in Mathematics, 80*(3), 327-349.

Kjeldsen, T. H., & Petersen, P. H. (2014). Bridging history of the concept of function with learning mathematics: students' meta-discursive rules, concept formation and historical awareness. *Science & Education, 23*(1), 29-45.

Mosvold, R., Jakobsen, A., & Jankvist, U. T. (2014). How mathematical knowledge for teaching may profit from the study of history of mathematics. *Science & Education, 23*(1), 47-60.

Ng, D., Mosvold, R., & Fauskanger, J. (2012). Translating and adapting the mathematical knowledge for teaching (MKT) measures: The cases of Indonesia and Norway. *The Mathematics Enthusiast, 9*(1&2), 149-178.

Nielsen, D. S., Bøhme, K. M., & Andersen, L. (2014). *Non-published exam report on "The history, application, and philosophy of mathematics in teaching"*. Aarhus University.

Niss, M., & Højgaard, T. (Eds.). (2011). *Competencies and mathematical learning – Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (IMFUFA tekst nr. 485/ 2011). Roskilde University.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1-36.

Sfard, A. (2008). *Thinking as communicating*. Cambridge: Cambridge University Press.

Smestad, B., Jankvist, U. T., & Clark, K. (2014). Teachers' mathematical knowledge for teaching in relation to the inclusion of history of mathematics in teaching. *Nordic Studies in Mathematics Education*, *19*(3-4), 169-183.

Superfine, A. C., & Li, W. (2014). Exploring the mathematical knowledge needed for teaching teachers. *Journal of Teacher Education, 65*(4), 303-314.

Zopf, D. (2010). *Mathematical knowledge for teaching teachers: The mathematical work of and knowledge entailed by teacher education* (Unpublished doctoral dissertation). Ann Arbor: University of Michigan.

ENACTING INQUIRY LEARNING *IN* MATHEMATICS THROUGH HISTORY[1]

Tinne Hoff KJELDSEN
Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark
thk@math.ku.dk

ABSTRACT

We explain how history of mathematics can function as a means for enacting inquiry learning activities in mathematics as a scientific subject. It will be discussed how students develop informed conception about i) the epistemology of mathematics, ii) of how mathematicians produce mathematical knowledge, and iii) what kind of questions that drive mathematical research. We give examples from the mathematics education at Roskilde University and we show how (teacher) students from this program are themselves capable of using history to establish inquiry learning environments in mathematics in high school. The realization is argued for in the context of an explicit-reflective framework in the sense of Abd-El-Khalick (2013) and his work in science education.

1 Introduction

The Rocard (2007, p. 12) report emphasizes inquiry-based teaching methods as a means to increase students' interest in mathematics and science. Inquiry-based teaching aims at inviting students into the workplace of scientists and mathematicians. The idea is that if students are engaged in activities and learning processes similar to the way scientists and mathematicians produce knowledge, the students will develop a deep understanding of science and mathematics, as well as of the epistemology and more broadly the nature of these subjects. Problem solving and mathematical modelling play a major role in inquiry-based mathematics education. In such activities the focus is often on mathematics as a tool for solving problems outside of mathematics – not on the production and validation of mathematical knowledge. In the present paper we point towards history of mathematics as a means for enacting inquiry learning activities in which students gain experiences with research processes in mathematics. To be more specific, how students through history can develop informed conception about i) the epistemology of mathematics, ii) of how mathematicians produce mathematical knowledge, and iii) what kind of questions that drive mathematical research.

The analyzes take point of departure in an existing study program in mathematics at Roskilde University (RUC) in Denmark, where students get to work with history of mathematics through problem oriented project work. We present some of the students' project work and discuss in what sense these projects fulfil the intentions of inquiry-based learning in

[1] Some of the content of this paper is also presented in Kjeldsen (2014). It appears here with the permission of the journal.

mathematics as a scientific subject. We discuss and explain the realization of the problem oriented project work in history and philosophy of mathematics in the context of an *explicit-reflective framework* in the sense of Fouad Abd-El-Khalick (2013) and his work in science education. Finally, we illustrate by an example how (teacher) students[2] from this program are themselves capable of establishing inquiry learning environments *in* mathematics in upper secondary school by using history – and point to some of its benefits for high school teaching.

2 Inquiry teaching in mathematics through history of mathematics

How can students in higher mathematics education get first-hand experiences in a meaningful way with research practices in a field that is so specialized and operates with such abstract notions as mathematics? Usually, students are not invited into a research environment until they become Ph.D. students. There is not much help to gain from their textbooks. Mathematics textbooks very rarely discuss or indicate where the mathematical objects they are dealing with come from, why they look the way they do and why they are interesting. On the contrary, mathematical objects are usually introduced as timeless entities that appear in textbooks seemingly out of nowhere. Most of the mathematics that students are introduced to at bachelor level and in the first year of master programs has not been developed recently. For most students, observing how mathematicians work when they do research will not be a feasible way to gain insights into inquiry processes in mathematical research. However, students can examine such processes of how and why mathematicians have generated the ideas, constructed the concepts and produced the mathematics they read about in their textbooks by studying the history of the content matter of mathematics. In the following we will illustrate and discuss how students in the mathematics program at RUC through project work in history of mathematics are challenged to reflect explicitly upon concrete aspects of the nature of mathematics like the ones listed in the introduction.[3]

3 The PPL implementation of history in higher education in math at RUC

History is implemented in the mathematics program at RUC through Problem-oriented Project Learning (PPL). PPL is the pivotal pedagogical principle underlying the organization of *all* the study programs at RUC (Andersen & Heilesen (2015). It refers to the principle of problem-oriented participant-directed project work as it has been developed and is practiced at RUC. Common for all study programs is that in each semester the students use half of their time on regular course work. The other half of their study time they work in groups of two to eight students on a project which objective is to solve (or analyse or formulate or make solvable) a problem of the group's own choice under supervision of a professor.

The students document their project work during the semester through outlines and discussion-papers produced by the students. These are discussed and planned for at weekly meetings with the group's supervisor. The group writes a coherent report of 50-100 pages, in which the students state and argue for their problem and its relevance, their methodology and

[2] In Denmark, upper secondary mathematics teachers have a university degree in mathematics.

[3] See also Barnett, Pengelley and Lodder (2014) for uses of history in higher mathematics education in the USA.

its validity, their choice of theories, experiments (in the science program students often design experiments), data-collections, analyses, results and conclusions. They make a critical evaluation of their results. The students consult textbooks, research literature and experts in their field(s) of investigation. The project work is evaluated at an oral group-examination with an external examiner. All the students in a group are responsible for every aspect of the project work. They are supposed to be able to reflect explicitly and concretely about how their analyses and solutions depend on their methodology including their choices of theory and possible experiments. The students are in charge of the project work from beginning to end, supported by their supervisor and various milestones throughout the semester.

The students have opportunities to work with history of mathematics at the bachelor level and at the master level. However, we will only describe the study regulation for the master level, since it also fulfils the requirements for the relevant projects at the bachelor level. The project at the master level is defined by the theme: 'mathematics as a scientific discipline'. In the study regulation it is described in the following way:

> The project should deal with the nature of mathematics and its "architecture" as a scientific subject such as its concepts, methods, theories, foundation etc., in such a way that the nature of mathematics, its epistemological status, its historical development and/or its place in society gets illuminated.

Every mathematics student at RUC at the master level will work on a project that fulfils these requirements. In the next section we will we present two of them in more detail to illustrate how the students 1) gained experiences with inquiry processes that resemblance how mathematical research is done, and 2) developed informed conception of the production and validation of mathematical knowledge.

4 Examples of Students' History of Mathematics Projects at RUC

The list of selected titles in Figure 1 gives an idea of the variety of subjects and issues the students have dealt with in the PPL history of mathematics project over the years at RUC.

The Contribution of Galois to the Development of Abstract Algebra (1986)
Euler and Bolzano: A mathematical analysis in an epistemological perspective (1993)
The Intuitionistic Mathematics of Hermann Weyl (1995)
Hamilton's Quaternions – an assessment of their applicability (1996)
Paradoxes in Set Theory and Zermelo's 3th. Axiom (1996)
$\sqrt{-1}$ *a multiple discovery?* (1997)
Cayley's Problem: a historical analysis of the work on Cayley's problem 1870-1918 (1998)
What Mathematics and Physics did for Vector Analysis (1999)
Generalizations in the Theory of Integration (2001)
Euler's Differential Calculus compared with the modern one (2000)
Hilbert's Philosophy of Mathematics (2001)
The Solving of Equations Algebraically from Cardano to Cauchy (2002)
Fourier and the Function Concept: From Euler's to Dirichlet's concept of a function (2003)
Foundations of Mathematics – assessed using Euclidean and non-Euclidean geometry (2003)
D'Alembert and the Fundamental Theorem of Algebra (2003)
Infinity and 'Integration' in Antiquity (2004)
The Real Numbers – Constructions in the 1870ties (2004)
Physics' Influence on the Development of Differential equations (2005)
Haar's contribution to generalized Fourier theory (2005)
Holomorph Dynamics – a Historical Perspective (2010)

Figure 1. List of titles of students' history of mathematics projects and its year

In the following we will present two of these PPL-projects in more detail. They illustrate how the students through history 1) gained experiences with inquiry processes that resemblance how mathematical research is done, and 2) developed informed conception about: i) the epistemology of mathematics, ii) how mathematicians produce mathematical knowledge, iii) what kind of questions that drive mathematical research.

4.1 Generalizations in the Theory of Integration: An Investigation of the Lebesgue Integral, the Radon Integral and the Perron Integral

The 75 page long project report Generalizations in the Theory of Integration: An Investigation of the Lebesgue Integral, the Radon Integral and the Perron Integral was written by two students who were curious about the "need" in mathematics for further integrals than the Riemann integral. In the textbook of their first analysis course they had stumbled over a footnote in which it was pointed out that there are other types of integrals e.g., the Lebesgue integral. The students soon realized that there are many integrals, e.g. the Denjoy, the Perron, the Henstock, the Radon, the Stieltjes and the Burkill integral. They wrote:

All these integrals are most often described in the literature as *generalizations*, and sometimes as *extensions*, of either the Riemann or the Lebesgue integral. This gave rise to questions such as: What do these integrals do? Why have so many types of integrals been developed? Why is it always the Lebesgue integral we hear about? What is meant by generalizations in this respect? In what sense are the various integrals generalizations of former definitions of integrals? Are the generalizations of the same character? (Timmermann and Uhre, 2001, p. 1, italic in the original)

The students chose to investigate these questions by looking into the history of mathematics. Supported by literature from historians of mathematics, the students traced and read mathematical papers and books of Lebesgue, Perron and Radon in which they developed, or worked with mathematical ideas that motivated them to develop the integrals that bear their names. The students analysed and compared the sources with respect to 1) the motivation of the mathematicians, 2) why they created these generalized integral concepts, 3) the differences and similarities between the characteristics and scope of the generalizations.

The students based their investigations of Lebesgue's motivation and development of his integral concept on some of his notes in *Comptes Rendus de l'Académie des Sciences de Paris* and his thesis *Intégrale, Longueur, Aire*. The students found that Lebesgue's search for a new integral concept primarily was motivated by the (lacking) symmetry in Riemann's integral concept of what Lebesgue called 'the fundamental problem of integral calculus', i.e. to find a function when its derivative is known. The students based their argument on Lebesgue's own words and on their analysis of the first part of his thesis, where the students found that the theorems were organized in a hierarchy of theorems. (p. 33-34). They concluded that for bounded functions, the Lebesgue integral is an extension of the Riemann integral, but that the Lebesgue integral is not the same as the Cauchy-Riemann integral (improper integral) for unbounded functions. Lebesgue's integral is based on generalization of the measure concept. For Perron, the students found that he introduced a new definition of the integral, which he found to be more elementary than Lebesgue's. It was based on the concepts of upper and lower adjoint functions for a bounded function *f* defined on an interval [*a*, *b*]. The adjoint functions can be interpreted as approximating anti-derivatives of *f* . Perron's integral is then based on the supremum and infimum of their values at *b*. The students concluded that Perron with "elementary" meant the avoidance of measure theory and they discuss his integral concept with respect to this. They argued that Perron's integral is more abstract than Lebesgue's e.g. the intuitive conception of the integral as an area or a measure is lost, and the definition does not provide a method for constructing the integral. However, as they wrote, it has the didactical advantage that the definition is based on the anti-derivative. Regarding Radon's motivation for generalizing the integral concept, the students concluded that he needed it for his work with integral equations. These insights into mathematicians' motivations for defining a new or extending and existing integral concept made them realize that "There have been different motivations associated with the development of integral concepts [..]. This illustrates that […] there may well be several reasons involved in the creation of new mathematics. This also illustrates that mathematics does not necessarily evolve along a beaten path and that perhaps it is only in hindsight that a certain approach appears to be the most natural." (p. 57)

On the one hand, by focusing on understanding the mathematicians' motivation for generalizing the integral concept when reading the historical sources, the students became engaged with mathematical inquiry analogue to some research processes in mathematics as they are carried out by working mathematicians. The students' work was guided by historical and philosophical questions. They answered these questions through analyses of the mathematical content, definitions, theorems, proofs and techniques that were stated, treated and worked out in the sources they consulted. In this way, the students gained first-hand experiences with research processes in the production of mathematical knowledge by studying the masters, so to speak.[4] On the other hand, by focusing on analysing and identifying the characteristics of the various generalizations of the integral concept, the students came to reflect upon their inquiry investigations from within an epistemological framework. This part of the students' investigations structured their critical reflections about the function, assessment and significance of the generalizations of the integral concept.

4.2 The Real Numbers – Constructions in the 1870ties

The project on the constructions of the real numbers was carried out by a group of six students. They wrote a report of 56 pages in which they answered the following question:

Why did a need for a construction of the real numbers emerge around the 1870ties?

These students were puzzled when they realized that mathematicians had worked with the real numbers on an intuitive foundation for years and years before it (the foundation) 'all of a sudden' became a problem that needed to be solved. The students explained their own motivation in their report as follows (Wandahl et. al., 2004, p. 1-2):

---

[4] A warning needs to be issued here: Reading sources does not in itself guarantee that students will gain experiences with research processes in mathematics. For this to happen, they need to study historical processes.

> Basically we had the idea that one mathematician saw that it was a problem that a construction of the real numbers was lacking, solved it and presented it [the construction of the real numbers] as a solved problem. However, we quickly discovered that this was not the case. There was not just one mathematician who believed that there was a problem, but several. This indicated that the lack of a construction of the real numbers had become a problem in connection with developments of mathematics up to then.

The students' motivation and their research question (the problem that guided their project work) are concerned with a fundamental aspect of the nature of mathematics as a scientific subject, namely what do mathematicians wonder about? How do the problems that mathematicians struggle to solve emerge? How are they connected with ongoing research in mathematics? When, why and by whom are they deemed so important, that they become essential problems of the field in need of a solution? These questions structured the students' analyses of the mathematical sources and their reflections of how mathematical knowledge was generated in this particular episode in the history of mathematics.

To place the discussion in the historical context, the students studied the paper by Bernard Bolzano from 1817: *Rien analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegensetztes Resultat gewären, wenigstgens eine reelle Wurzel der Gleicheung liege* in which he proved the mean value theorem, and discussed methodological and epistemological issues in mathematics. The students considered Blozano's work as an indication that such concerns were present in mathematical research during the nineteenth century. In order to answer their own research question for the project work, they chose to investigate the work of Dedekind and Cantor. They realized that these two mathematicians had different reasons for constructing the real numbers with Dedekind realizing a need for a rigorous construction in a teaching situation, whereas Cantor ran into the problem of the construction of the real numbers in his work with trigonometric series.

In order to place their analyses of the historical sources in a broader historical context, the students discussed the various conceptions of the arithmetization of analysis in the 1870'ties based on the work of historians of mathematics. They realized that there were various conceptions of mathematics among mathematicians in the 1870'ties. With reference to Epple (2003) they discussed their analyses and conclusions with respect to the traditional conception, the arithmetical conception and the formal conception. This categorization made the students realize that what counts as a validation or knowledge in mathematics depends on these conceptions. They quoted Dedekind: "For myself this feeling of dissatisfaction [with geometrical arguments] was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis" (p. 48).

The students' were invited into the mathematical 'workbench' of Bolzano, Dedekind and Cantor among others in this project work. Through their work with following the historical development, they became engaged with mathematical inquiry. The requirement for the "mathematics as a scientific subject"-project and the specific problem-orientation provided opportunities for the students to reflect upon their inquiry investigations of the mathematical content of the sources from within an epistemological framework of how this particular mathematical knowledge was generated, validated and discussed at the time.

The two project reports illustrate how the students acquired informed conceptions about the nature of mathematics coming from different perspectives: In the first project, the students obtained concrete experiences with processes of forming new concepts in mathematics with reference to already existent concepts through generalizations. They investigated how different processes of generalization can be distinguished with respect to whether the new concept, that is generated, is an abstraction of the already existent one or whether the new concept is an extension in the sense, that it contains the concept it is a generalization of. In the second project, the students experienced how problems of foundational and epistemological nature arose with respect to the foundation of the real numbers, how this generated new research in mathematics and gave rise to discussions and debates among mathematicians – aspects which are important parts of mathematical inquiry, but rarely occur in traditional teaching. However, as the examples illustrate, such aspects can be brought forward and made explicit objects of students' reflection through history of mathematics.

5 The significance of an explicit-reflective framework

Research from science education has documented that "while inquiry might serve as an ideal context for helping students and teachers develop informed NOS (nature of science) views, it does not follow that engagement with inquiry would necessarily result in improved understandings." (Abd-El-Khalick, 2013, p. 2089). As Abd-El-Khalick and Lederman (2000) have argued for the development of NOS, an explicit-reflective framework is required in order to achieve such understandings. By 'explicit' they mean that some specific learning objectives related to students' understanding of NOS must be included in the curriculum; by 'reflective' they refer to instructions aimed at helping students to reflect upon their experiences with learning science from within an epistemological framework.

The problem orientation of the project work at RUC together with the study regulation of the 'mathematics as a scientific subject' guaranties that the students work with a specific problem that address (aspects of) the nature of mathematics, in such a way that it is anchored in the subject matter of mathematics (its concepts, methods, theories, foundation etc.). Through this anchoring, the students' reflections become contextualized and concretized. The problem orientation together with the curriculum description provides structured opportunities for the students to reflect upon learning experiences from within an epistemological framework. The students gain their 'mathematics as a science' learning experience from the concrete attachment of their problem within the subject matter of mathematics. The aspects of the nature of mathematics that their problem is addressing provide the epistemological context in which the students examine their 'mathematics as a science' learning.

It goes without saying that teachers in order to be able to enact such learning environments must themselves have informed conception of the nature of mathematics and (processes of) mathematical research. The question now is whether students from the PPL program in mathematics at RUC are capable of designing and implementing such learning environments. In order to discuss this we use the notions of teaching *with* and teaching *about* the nature of mathematics in the sense of Abd-El-Khalick for teaching *with* and *about* NOS:

> Teaching *about* NOS refers to instruction aimed at enabling students to achieve learning objectives focused on informed epistemological understandings about the generation and validation of scientific knowledge and the nature of the resultant knowledge. […] Teaching *with* NOS entails designing and implementing science learning environments that take into consideration these robust epistemological understandings about the generation and validation of scientific knowledge. (Abd-El-Khalick, 2013, p. 2090).

If we adopt this framework to mathematics, teaching *about* the nature of mathematics means to teach towards learnings objectives of the following kind: To make students able to investigate and analyze the methods mathematicians use to generate knowledge, to discuss, criticize and asses the epistemological status of this knowledge, to investigate and analyze the role of proofs in mathematics, to investigate and discuss mathematical objects' ontological status, to investigate and discuss the relationship between mathematics and other sciences, to discuss and critically asses the distinct nature of mathematics, as well as its development historically and in interaction with culture, society and other sciences.

In order to couple the development of students' informed conceptions about the nature of mathematics with inquiry teaching in mathematics in a meaningful way, mathematics teachers must 1) have knowledge and informed conceptions of and *about* the nature of mathematics themselves, 2) come to understand how inquiry is in fact conducted in mathematical research, 3) be able to design inquiry based learning environments *in* mathematics teaching, and 4) be able to teach *with* the nature of mathematics in the sense explained above in the case of NOS.

6 Teaching with the nature of mathematics through history in an inquiry learning environment in upper secondary school in Denmark

The nature of mathematics (NOM) is explicit in the mathematics curriculum for upper-secondary school in Denmark. One of the learning goals for mathematics is that the students should be able to demonstrate knowledge of how mathematics has developed in interaction with the historical, scientific and cultural development. Another learning goal is that students should demonstrate knowledge of the identity and methods of mathematics.

In this section we shall see an example of how a mathematics (teacher) student from RUC used her knowledge about the nature of mathematics that was formed within the explicit-reflective framework which was unfolded and illustrated above, to teach with and about the nature of mathematics in a Danish high school. She used her knowledge and experiences from her own mathematics education at RUC to enact an inquiry learning environment that invited her 17 year old high school students (11^{th} graders) into inquiry processes that bore some resemblance with authentic mathematical practices. She created the learning environment by having the students read excerpt from original sources from the development of the function concept. By setting up explicit learning goals for the students that addressed historical and mathematical questions related to the development of the function concept, and by having the students analyze the original sources to answer these questions, she designed a learning environment in which she taught both *with* and *about* the nature of mathematics. Altogether she spent 13 lessons of 50 minutes each with the students in the classroom – and the students were expected to spend an equal amount of time working on the tasks at home.

The intentions for the students' learning outcome reflect the explicit-reflective framework (Petersen, 2011):

> The students will acquire an understanding of what is involved in the modern definition. of a function, and the role of the concept of domain in the modern function concept.
>
> The students will come to reflect upon the concept of a function – what is a function?
>
> The students will acquire an understanding of our modern function concept as a result of a historical developmental process.
>
> The students' will come to reflect upon the role of proofs.[5]
>
> The students will gain insights into the role played by the human actors [former mathematicians] in the development of the function concept.

The teaching module was designed in two steps: In step 1 the students were divided into four so-called basic-groups that had specific but different tasks. Group 1 worked with various historical definitions of a function, group 2 worked with the debate of the motion of a vibrating string primarily between Leonhard Euler (1707-1783) and Jean le Rond D'Alembert (1717-1783), group 3 situated the main actors (Euler and Peter G. L. Dirichlet (1805-1859)) in their respective societies and the time in which they lived, and group 4 worked on the modern concept of a function. Each group received a working sheet from the teacher with explicit requirements for their work. Group 1, e.g., was given Danish translations of an extract from Euler's (1748) book *Introductio in Analysin Infinitorum* with the definition of a function, and an explanation of how Euler later in 1748 extended his original function concept, and an extract from Dirichlet's paper (1837) *Über die Darstellung ganz wilkürlicher Functionen durch Sinus- und Cosinusreihen*. Based on these three texts the students received the following task (Petersen 2011, Appendix B):

> Explain what a function is according to Euler's original definition of a function in *Introductio in Analysin Infinitorum*, Euler's extended definition of a function and Dirichlet's definition of a function. Describe how these three definitions of a function are different from each other and in what ways they are similar. Explain what the

[5] This learning objective is linked to the investigation of whether history can be used to exhibit meta-discursive rules of mathematics and make them explicit objects for students' reflections, which was also a part of the experiment (see Kjeldsen & Petersen, 2014).

principle of the generality of the variable is all about and the relationship between this principle and the principle of the generality of the validity of analysis.[6]

The task was followed by eight questions which were meant both as a help for the students to 'de-code' the task and to make the requirements and the expectations of the teacher for the students' work more explicit, and to qualify the students' reflections and structure their work:

> (1) What are the central concepts in Euler's definition of a concept? (2) Which principle characterizes a variable according to Euler, and what is this principle called? (3) What is the principle of the generality of the variable all about? (4) What are the similarities between the principle of the generality of the variable and the principle of the generality of the validity of analysis? Consider why both principles have been given names that contain the word "general".[7] (5) How does Euler's extended concept of a function differ from his original concept, and what are the similarities? (6) Find three ways in which Dirichlet's concept of a function differs from Euler's definition. (7) Explain from text 3, what Dirichlet must have thought about the generality of the variable. (8) On page 10 there are four pictures. Which of these pictures are graphs of functions according to Euler's definition in *Introductio in Analysin Infinitorum*, Euler's extended definition and Dirichlet's definition respectively? (Petersen 2011, Appendix B)

In step 2 new groups were formed in such a way that each group had at least one participant from each of the four basic-groups. This meant, at least in principle, that all the knowledge that had been developed in the basic groups was present in each of the new groups. In contrast to the basic-groups, the new groups, also called the expert-groups, all worked on the same assignment. They were asked to write an essay about a fictional debate between mathematicians, where one part is claiming that mathematical concepts are static, timeless entities that exist independent of humans and society. In opposition to this, the other part reinforces that mathematical concepts develop over time, that they are the results of research processes. The students received a made-up invitation from the journal NORMAT to contribute to this debate by submitting a paper for the journal on this issue. The students were required to argue for their own opinions based on the collected work that had been done in the basic-groups. The teacher had formulated four issues that the students had to address and discuss in their paper: Euler's, Dirichlet's and our current concept of a function; the two meta-rules i.e., the generality of the variable and the general validity of analysis, the raison d'etre behind domain, range of image and proofs; sociological factors that had influenced the development of the function concept; and human factors.

The analysis of data that was collected during the teaching shows that the high school students realized that the concept of a function, they read about in their textbook, was the

---

[6] These principles refer to historical meta-rules of mathematics (see footnote 5).

[7] These questions address the issue of what Sfard (2008) calls meta-discursive rules in mathematics. For a discussion of this aspect of the teaching module see Kjeldsen and Petersen (2014).

result of a historical development, and that they also gained more specific knowledge about some of the key elements of this development. The students became immersed in mathematical inquiry processes by tracing (parts of) the path of the masters, as illustrated by the following quotes from one of the essays written in the expert-groups. The quotes show that these students became aware of at least one source for mathematical research questions as well as of discussions among mathematicians that relate to the validation of mathematical knowledge:

> The reason why Euler began to work with [Euler-]discontinuous functions was because of a debate between contemporary mathematicians. The debate concerned the fact that the functions the mathematicians worked with could not describe a vibrating string.
>
> […] the development of the concept of a function was among other things due to human attitudes and interpretations, which were important factors. For example, some of Euler's contemporary mathematics colleagues were of the opinion that Euler's extended function concept should not be used because it went against the principle of mathematics. They thought it was cheating. This meant that Euler's extended function concept never came to be used as intended, and a new function concept was developed by Dirichlet. (Petersen, 2011, Appendix C)

We will not go further into the results regarding the learning of the high school students, for this we refer to Kjeldsen and Petersen (2014). Here we will restrict ourselves to pointing out that the teacher's knowledge *about* the nature of mathematics and her informed epistemological conception about the production and validation of mathematical knowledge enabled her to design and implement the course described above in high school. A course where she, through history of mathematics, taught both *with* and *about* the nature of mathematics, i.e., she taught in a way that immersed the high school students in mathematical inquiry processes and made them capable of forming epistemological understandings of how mathematical knowledge is generated and validated. It is also noteworthy to keep in mind that mathematical argumentation and reasoning is very much present in the core of this teaching experiment. The ability to teach *with* and *about* the nature of mathematics in inquiry based teaching with history provides an opportunity to strengthen proofs and deduction in high school mathematics.

7 Conclusion

The problem-oriented project work in the master's program in mathematics at Roskilde University provides an explicit-reflective framework in the sense of Abd-El-Khalick (2013). The analyses of the two student projects on generalizations in the theory of integration and the construction of the real numbers respectively, demonstrate how the students in this program through working with mathematically rich and thick episodes from the history of mathematics, can develop informed conception about the epistemology of mathematics, of how mathematicians produce and validate mathematical knowledge, and what kind of questions that drive mathematical research, while experiencing an inquiry learning environment that to some extent gives them insights into authentic mathematical practices. In the course of their problem-oriented project work they came to reflect upon and criticize the way mathematicians generate and validate mathematical knowledge, i.e., inquiry processes in mathematical research were made explicit objects for the students' reflections within an epistemological framework. The analysis of the design and implementation of the teaching experiment in high school exemplifies how the RUC program in mathematics enables its graduates to use their integrated understanding of history and philosophy of mathematics to enact inquiry learning *in* mathematics through history.

REFERENCES

Abd-El-Khalick, F. (2013). Teaching *with* and *about* nature of science, and science teacher knowledge domains. *Science & Education*, *22*(9), 2087-2107.

Abd-El-Khalick, F, & Lederman, N. G. (2000). Improving science teachers' conceptions of nature of science: A critical review of the literature. *International Journal of Science Education*, *22*(7), 665-701.

Andersen, A. S, & Heilesen, S. (2015). *The Roskilde model: Problem-oriented learning and project work*. Ccharm. Heidelberg, New York, Dordrecht, London: Springer.

Barnett, J., Pengelley, D., & Lodder, J. (2014). The pedagogy of primary historical sources in mathematics: Classroom practice meets theoretical frameworks. *Science & Education, 23*, 7-27.

Dirichlet, P. G. L. (1837). Über die Darstellung ganz willkürlicher Funktionen durch sinus- und cosinus-Reihen. *Repertorium der Physik*, *1*, 152-74. Also in Dirichlet's *Werke*, *1*, 133-60.

Epple, M. (2003). The end of the science of quantity: Foundations of analysis, 1860-1910. In H. N. Jahnke (Ed.), *A History of Analysis* (pp. 291-323). History of Mathematics, Vol. 24, American Mathematical Society, London Mathematical Society.

Euler, L. (1748a). *Introductio in Analysin Infinitorum* (2 volumes). In Euler's *Opera Omnia* (*1*) (pp. 8-9). Lausanne: Bousquet.

Kjeldsen, T. H. (2014). Teaching *with* and *about* nature of mathematics through history of mathematics: Enacting inquiry learning in mathematics. *Education Sciences*. *Special Issue,* 38-55.

Kjeldsen, T. H., & Petersen, P. H. (2014), Bridging history of the concept of function with learning of mathematics: Students' meta-discursive rules, concept formation and historical awareness. *Science & Education*, *23*, 29-45.

Petersen, P. H. (2011). *Potentielle vindinger ved inddragelse af matematikhistorie i matematikundervisningen*. Master Thesis in mathematics. Denmark: Roskilde University.

Rocard, M. (2007). *Science Education NOW: A renewed pedagogy for the future of Europe*. Brussels: European Commission. Retrieved June 20 2016 from http://ec.europa.eu/research/science-society/document_library/pdf_06/report-rocard-on-science-education_en.pdf.

Sfard, A. (2008). *Thinking as communicating*. Cambridge: Cambridge University Press.

Timmermann, S., & Uhre, E. (2001). *Generalizations in the theory of integration – An investigation of the Lebesgue integral, the Radon integral and the Perron integral*. (In Danish). IMFUFA, text 403, Roskilde University.

Wandahl, D., Terkelsen, R., Jørgensen, L., Andersen, L., Hansen, H., & Lassen, L. (2004). *The real numbers – constructions in the 1870'ties.* (In Danish). IMFUFA, Roskilde University.

THE CONCEPT OF MATHEMATICAL COGNITIVE TRANSGRESSION IN EXPLORING LEARNERS' COGNITIVE DEVELOPMENT OF NONDETERMINISTIC THINKING

Ewa LAKOMA
Military University of Technology, Faculty of Cybernetics, Institute of Mathematics and Cryptology, ul. Gen. S. Kaliskiego 2, PL-00-908 Warsaw, Poland
ewa.lakoma@wat.edu.pl

ABSTRACT

The concept of 'mathematical cognitive transgression' (MCT) has been introduced by Zbigniew Semadeni (2015) to investigate mathematics development from phylogenetic as well as ontogentic perspective. The author discusses many examples of the concept of MCT but none of them is directly connected with the development of nondeterministic thinking. Careful analysis of historical development of probabilistic and statistical concepts which led to hypotheses concerning regularities of cognitive development of probabilistic thinking (Lakoma 2000) became a point of departure of my current research in an area of understanding randomness and probability – based on an overview through the lens of the concept of MCT. The aim of this article is to discuss how the concept of MCT can be helpful in recognizing main levels in learners' cognitive development of probabilistic and statistical thinking and to show that it can be useful in constructing an approach to teaching and in arranging an effective process of learning.

1 Introduction

In recent years we are witnessing and participating in civilization changes progressing at enormous pace. They are related in particular to an omnipresence of information and communication technology in every field of science, technology and social life. Mathematics is a natural support for technological environment, therefore areas of human activity are structured in a formal way, in order to be suitable for explorations based on mathematics and information technology applications. This implies strong social demand for getting mathematical knowledge than ever before.

Students, in their process of learning, must have an opportunity to develop mathematical thinking in comprehensive way, to shape their own skills to mathematical modelling of phenomena and to be able to communicate by means of mathematics language. Students must also gain readiness to self-learning and using mathematical skills in future adulthood and professional life.

Recently randomness, probabilistic and statistical methods are present in methodologies of many various scientific disciplines, i.e.: physics, biology, economy, linguistics, history, psychology, engineering, information technology, and their applications. Therefore developing nondeterministic thinking is demanded as one of fundamental aims of education at all levels.

On the other hand, observing ways of probability and statistics learning in today classroom provides with many examples of students' difficulties and epistemological problems, even at the level of higher education. Learning probability theory and statistics seems to be more difficult for students than other parts of mathematics. These difficulties have often their roots not only in nondeterministic aspects of reasoning but also in misunderstanding of other fundamental mathematical concepts.

Creating a device for recognizing some symptoms of understanding mathematical concepts and for diagnosing student's cognitive difficulties could be very helpful for a teacher in supporting a process of students' learning as well as in constructing a didactical approach to teaching, more effective and more suitable for students, from their cognitive development point of view.

A useful support in this area of investigations seems to be obtained on the one hand by exploring the historical ways of mathematical thinking and – on the other hand – by careful studying students' individual ways of mathematical reasoning.

In this paper, after presenting the concept of mathematical cognitive transgression and its examples given by Semadeni (2015), I will present current results of my research in an area of randomness and probability – based on an overview through the lens of the concept of mathematical cognitive transgression.

2 The concept of mathematical cognitive transgression

The concept of 'mathematical cognitive transgression' has been introduced by Zbigniew Semadeni (2015) to investigate mathematics development from phylogenetic as well as ontogenetic perspective. The idea to introduce this concept to mathematics education was an effect of the author's inspiration by the book of Jozef Kozielecki (2004) who defined the concept of 'transgression' in the field of social psychology.

The mathematical cognitive transgression (MCT) is defined as an act of exceeding some previously existed cognitive limitations of knowledge which makes an individual or scientific community able to gain a new knowledge and to 'open the door' to think and reason at the more advanced and mature level than it was before (Semadeni, 2015). This transition is possible to appear as a result of active actions. The author argues, that MCTs expose some momentous changes in mathematics development. They consist in overcoming limitations of possessed prior knowledge and can be recognized in the history of mathematics when following ways of reasoning of people in the past (phylogenetic aspect) – or today when observing individual's process of mathematical development (ontogenetic aspect). Usually a MCT is an effect of long-term reasoning and investigations of a person or a group of scientists or learners. Sometimes it can be revealed suddenly (Semadeni, 2015).

However, in the author's opinion, in order to be able to judge a given transition as MCT the following necessary conditions must be fulfilled: this transition concerns a broadly defined mathematical idea; the difficulty of this transition is inherent, lies in the nature of structure of this idea, and is directly linked to it how a person or community recognizes this idea; this transition strongly counts as important for development of this idea and for other concepts related to it; this transition leads from a lower level into a new level, higher and well defined; this transition is a result of an action - conscious although possible to be unintentional - of a person or scientific community to get to know something new, to understand something that was not yet possible to understand; this transition is accompanied by a kind of surprising cognitive dissonance (Semadeni, 2015).

Semadeni (2015) discusses many examples of MCTs but none of them is directly connected with a development of nondeterministic thinking. In the next part of this paper I will present these examples of MCTs, analysed by Semadeni, which are not directly related to probabilistic concepts, but their lack can be an important difficulty for students in learning probability and statistics.

3 Examples of mathematical cognitive transgressions

Zbigniew Semadeni in his work (Semadeni, 2015) presents and analyzes several examples of mathematical cognitive transgressions. Among them I have chosen those that seem to be related with ability to construct models of random phenomena.

When we follow and analyse students' mathematical reasoning we are able to recognize often easily a lack of some sorts of MCT among students' solutions containing errors or misunderstandings. For a teacher who carefully observes a student's process of learning, there is an opportunity to pose a diagnosis what kind of support a student needs in order to be able to experience a transgression which is necessary to reason correctly at the more mature level.

3.1 The transition from processes to objects represented by symbols

From the ontogenetic point of view, there is possible to observe many examples of so called mini-transgressions in the process of children mathematical thinking. For example: the symbol ‚3+4‘ has two basic meanings depending on the context: as a process of adding two numbers: 3 and 4 or as an object which is a result of this process and is represented by this symbol: ‚3+4‘. Children initially are able to recognize this symbol just as a task to calculate; when they become able to treat this symbol ‚3+4‘ as an object we can agree that they already experienced the MCT under consideration (Semadeni, 2015).

This kind of transgression appears not only in arithmetic, in algebra or mathematical analysis, but also in probability theory and statistics. For example, one of the most fundamental notions in probability theory – the distribution function which is defined as a function whose set of arguments is the set of real numbers and a function is defined for each real argument x as the probability that the random variable X has values smaller than an argument x – is treated by students as one of the most difficult notions. One of reasons of this feeling seems to be a lack of transgression 'from processes to objects represented by symbols': there are many students, even at academic level, who when considering this function:

for each real number x: $F(x) = P(X < x)$

try at first 'to solve' inequality inside parentheses and they are not able to understand this formula as an object represented by symbols.

Another example of a lack of this MCT we can meet when exploring students' solutions of probabilistic problem of 'waiting for the first success': two players shoot to the target in turns; who will reach the target as the first is a winner and the game is over; calculate probability to win for the player who begins this game and probability to win for the second player. Frequencies of getting a target in one trial are the same and equal to 0.5.

A solution of this problem leads to calculations of a sum of the following series:

for the first player: $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots$ for the second player: $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots$

Even if students are able to present the solution in the formulas above, they are very often not able to find the correct answer; they try to add numbers but they cannot see these formulas as objects and consider both series from this perspective: some of students are able to compare appropriate components and to notice that in each pair the number 'belonging to the first player' is two times greater than the number 'belonging to the second player', so the whole sum expressing a probability to win by the first player is two times greater than probability for the second player, thus probability to win for the first player is 2/3 and for the second one: 1/3.

Experiencing by a student the MCT 'from processes to objects represented by symbols' gives an opportunity to solve this problem in very simple way. The problem presented above is also connected with another example of MCT – related to the concept of infinity.

3.2 Transgressions related to infinity in its ‚increasing' aspect

The transition from large numbers, such as a thousand or million, to a consciousness that for each number it is always possible to give even bigger number is a mental process which was carefully described by Aristotle and is known as ‚potential infinity'. Aristotle also distinguished the ‚actual inifinity' which means that the process of infinitely many steps is completed, and is realized in the form of a single act (Semadeni, 2015).

Aristotle distinguished two types of the ‚potential infinity': inifnity in view of adding – increasing and infinity in view of dividing. Understanding the ‚actual infinity' is a symptom of experiencing the MCT. In the example above we can observe both kinds of the ‚potential infinity': when students try to summarize numbers and to find a sum of series, also when students try to use a geometric representation of these calculations – to present numbers as appropriate parts of a geometric figure whose area is equal to 1. Anther example of the MCT related to infinity is an ability to treat a set of all natural numbers as a single object which is a final result of an endless process of increasing the number of objects (Semadeni 2015). My observations of students' efforts when they use basic probability distributions as the models of random phenomena reveal often in students' reasonings a lack of this kind of MCT when considering sets of natural numbers.

3.3 Transgression from reality perceptible by senses to ideas perceived by the intellect

Semadeni (2015) presents this kind of MCT from the phylogenetic perspective: he evokes historical mathematical reasoning based on the concept of symmetry, Platon's distinguishing reality and ideas, and Euclid's considering abstracts only. On the other hand, when we take into account the ontogenetic perspective, we can give several examples of students' ways of thinking in which problems with distinguishing empirical observations of considered phenomena and reasoning related to their mathematical models are evident. It seems that a lack of this kind of MCT is one of the fundamental problems when students learn probability and statistics. These problems can appear even in very unexpected moments. For example, students at academic level (management field of studies) had to solve the following probabilistic task: to calculate an expected value of discrete random variable X which has three values: -1, 0, 1 with appropriate probabilities: 0.1, 0.4, 0.5. The following solution (Fig.1) reveals these problems: a student at first recognized this task as theoretical one (in a model), calculated the expected value of this random variable, but afterwards he decided to treat the values of the random variable as statistical data with frequencies equal to given probabilities and to calculate an arithmetic mean, but when he tried to obtain it, he reminded that these values could be equally frequent and he divided the result obtained into 3 (three values).

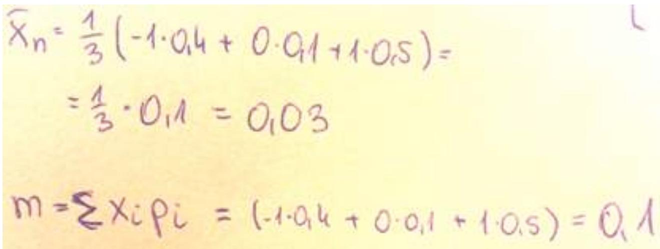


Figure 1. Student's incorrect calculation of the expected value of discrete random variable.

Another student's solution of the problem above is an example of sophisticated combination between the notion of arithmetic mean and the notion of expected value (the sum of probabilities is eual to 1, like number n – total amount of data in the formula of arithmetic mean (Fig.2), even the symbol of expected value is borrowed from arithmetic mean).

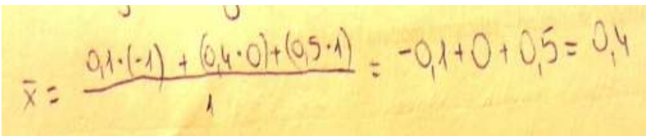


Figure 2. Student's calculation of the expected value of discrete random variable.

It is worth to notice, that this kind of reasoning is not rare among students, and strongly needs to be taken into account by teachers.

3.4 Transgression from axioms understood as certainties to the axioms understood as assumptions

Among several examples of MCTs, Semadeni (2015) has chosen the MCT which has a great importance in a field of probability theory. Semadeni argues, considering this transgression from the phylogenetic point of view, that until the discovery of non-Euclidean geometry mathematicians understood the term ‚axiom' as a ‚certainty', ‚self-evident truth', that means a statement which doesn't require to be proved. This kind of axioms were fundamental for mathematical theories until the early twentieth century, such as Euclidean geometry or arithmetic of real numbers, arithmetic of natural numbers. In the twentieth century axiomatic theories appeared in which ‚axioms' are understood as ‚assumptions' – ‚postulates'.

Kolmogorov's theory of probability belongs to this kind of axiomatic theories (Semadeni 2015). In the process of probability learning experiencing this transgression by a learner is very important for constructing his mental structure of probability concepts and models.

4 Searching for mathematical cognitive transgressions in the area of probability and statistics

Careful analysis of historical development of probabilistic and statistical concepts led to hypotheses concerning regularities of cognitive development of probabilistic thinking (Lakoma 2000). Following the opinion of Ian Hacking (1975) the history suggests that the concept of probability has a dual nature. Hacking distinguishes two aspects of this concept. One of them - epistemological – is implied by the general state of our knowledge concerning a considered phenomenon – related to the degree of our belief, conviction or confidence, which arise in connection to an argument – related to this phenomenon – and are supported by this argument. The other aspect – aleatory – is related with the physical structure of the random mechanism and with the admission of its tendency to produce stable relative frequencies of events. The first aspect gives the basis for 'chance calculus' and the other – for the 'frequency calculus' (Lakoma, 2000). The phylogenetic analysis shows that both these aspects became inseparable and pierced each other starting from the time of Pascal and Fermat. Before that time they were developed independently. Conscious sticking them together, subtle contrasting and verbalising, done by Pascal, Fermat and also Huygens, has caused an act of illumination in the way of probabilistic thinking. The history points that in order to acquire the probability concept it is necessary to make conscious its dual nature (Lakoma 2000), thus there is possible to pose hypothesis that making conscious the dual nature probability is an example of MCT. In time of Huygens, Pascal and Fermat there is also possible to notice that before that historical period, all probabilistic problems, which appeared, could be characterised by finite probability spaces. The first problem, which was solved, using an infinite probability space, was found in the work of Christiaan Huygens (1657). The history suggests another possible example of MCT: transgression from finite probability space to infinite probability space. However in this case there is a need of more careful explorations: distinguishing between discrete infinite case and continuous case modelled by continuous random variables.

Observations of students' solutions of probabilistic problems lead to important conclusion that students in their initial naive trials evoke the concept of symmetry of the random mechanism under consideration. This naive approach – using a model of even chances leads sometimes to surprising arguments. For example, I will present the solution of the following problem: The height of teenagers is modelled by random variable X which has a normal distribution $N(165, 15)$, that means the expected value is 165 and standard deviation 15. Calculate probability that two of randomly chosen teenagers will have their height above the mean. Figure 3 shows the following tree model of this problem: student (management field of studies) distinguished two steps of the random experiment, at each of them he distinguished three possibilities: height above the mean, height less than the mean, height equal to the mean. To two of them he assigned equal probabilities 82/165: 'below' and 'above' and to the third one: 'equal'- probability 1/165. The solution was led according to the rule of adding at multiplying probabilities. (Fig.3). It seems that the discrete model, and in particular the model of even chances (certainly except of the third possibility) was so strong as an device to model random phenomena that possibilities to model random phenomena by continuous random variables was rejected in this case.

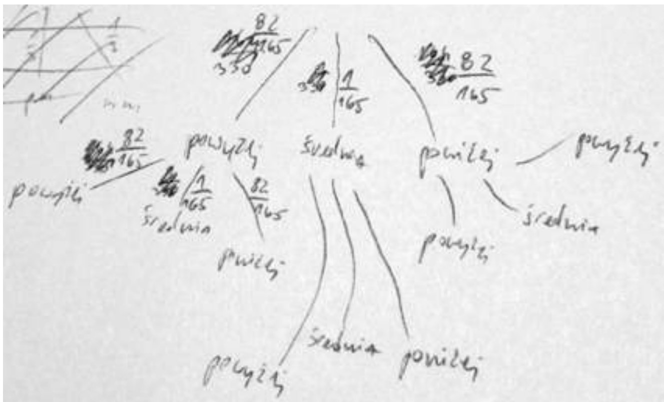


Figure 3. Student's incorrect solution of the example modeled by normal distribution.

4 Final remarks

The overview of probabilistic reasoning – in phylogenetic context as well as an ontogenetic - brings many examples of situations which reveal a presence of mathematical cognitive transgressions necessary for students to experience in the way of their process of developing mathematics. It is symptomatic that exploring students' ways of mathematical thinking there is possible to recognize a lack of some MCTs. Incorrect solutions often are not a result of inattention, they have roots which reveal deep misunderstanding of the fundamental mathematical ideas. Probability and statistics are these parts of mathematics which are especially sensitive of a lack of MCTs coming from different areas of mathematics and are necessary to use in tasks unexpectedly for a student. Therefore probabilistic and statistical problems can serve also as a tool to recognize the level of students' progress in mathematics and possible gaps necessary to fulfil.

REFERENCES

Hacking, I. (1975). *The emergence of probability*. Cambridge: Cambridge University Press.

Kozielecki, J. (2004). *Transgressive society, chance and risk* [in Polish: *Społeczeństwo transgresyjne, szansa i ryzyko*]. Warsaw: 'Zak'.

Lakoma, E. (1992). *Historical development of Probabilit*y [in Polish: *Historyczny rozwój pojęcia prawdopodobieństw*]. Warsaw: CODN-SNM.

Lakoma, E. (2000). Stochastics teaching and cognitive development. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education. The ICMI Study* (pp. 74–77). Dordrecht: Kluwer.

Semadeni, Z. (2015). Cognitive transgressions as an essential feature of the development of mathematics [in Polish: Transgresje poznawcze jako istotna cecha rozwoju matematyki]. In R. Murawski (Ed.), *Philosophy of mathematics and computer science* [*in Polish: Filozofia matematyki i informatyki*] (pp. 65 –90). Cracow: Copernicus Center Press.

HISTORY OF MATHEMATICS: AN ORIGINAL PERSPECTIVE FOR BEGINNER TEACHERS

Frédéric LAURENT
IREM de Clermont-Ferrand, Campus universitaire des Cézeaux, Aubière, France
frederic.laurent@univ-bpclermont.fr

Since the last two years, the French teaching education system has changed and students who want to be teachers have to graduate a master, called master MEEF ("métiers de l'enseignement, de l'éducation et de la formation").

In the French region Auvergne – main town Clermont-Ferrand – a specific component (taught unit) of history and epistemology of mathematics has been included in the course structure.

That's the reason why, I and other teachers working at the IREM of Clermont-Ferrand and interested in the use of history of mathematics in teaching, decided to gather and start thinking about the way history of mathematics could change the students' points of view about the mathematical notions to be taught and help them understanding the inherent complexities of this school subject. We created a research group within the IREM of Clermont-Ferrand, named AHMES ("Apports de l'Histoire des Mathématiques aux Enseignants du Secondaire"), and started to develop activities for the future teachers' education.

The first topic we tackled is algebra: a first activity was completed and experimented in 2015. In this activity, we aimed to focus on the intrisinc teaching difficulties of algebra but also to point out that the way the French national curriculum of studies introduces what is called "calcul littéral" doesn't make things easy for the teachers. Trying to work on the students' misconceptions, we use historical sources and mathematical methods to show how a common problem can be solved differently, so that history of mathematics becomes for us a didactic and pedagogical tool.

A second activity has been completed and experimented in 2016 dealing with the problem of introducing the tangents to a curve and the link with the derived function.

In this oral presentation, I'd like to present the teaching materials we used with the students, to describe their reactions and, more generally, our teaching experiments, but also to consider what can be the educational use and effects of the history of mathematics in such courses.

USING MATHEMATICAL FICTIONS AS A MEANS FOR REVEALING STUDENTS' IMPLICIT BELIEFS OF MATHEMATICS

Po-Hung LIU
National Chin-Yi University of Technology, Taiping, Taichung, Taiwan
liuph@ncut.edu.tw

ABSTRACT

Beliefs are mental constructs representing the codification of people's experiences and understandings as beliefs. It has attracted expanding inquiry for its potential influence on students' perceptions and decision-making, which in turn affect their learning strategies and responses to behaviors in the classroom. Conventional instruments used in educational settings for probing students' beliefs are questionnaire, interviews, notes, and behavior analysis. Though the validity of these scientific tools have been established, relevant studies also indicate that belief systems are much like icebergs , in that it is hard to directly depict their depths and characters. Therefore more alternative methods are required for revealing students' beliefs about mathematics. The relationship between mathematics and narrative has been discussed in recent decades. The purpose of this study was to encourage college students to create their own mathematical narratives and, through interpreting their narratives, to investigate their implicit beliefs or images of mathematics. This study was conducted in two courses called "History of Mathematics" and "Mathematics in Ancient Civilizations". The former introduces the development of mathematics from ancient to modern times; the latter only focuses on mathematics in several ancient civilizations including Egypt, Babylon, Greece, Arabia, India, and China. Students handed in mathematical fictions created on their own as their final projects. Themes for mathematical fictions are flexible but they were required to use topics which they had learned in the course as the ingredients of their creations. Several components including (1) the themes of mathematical fictions, (2) the mathematical topics most likely appear in the fictions, (3) the historical episodes most likely addressed, (4) the mathematicians most likely mentioned in the fictions, (5) the characters of figures in the fictions, and (6) the role of mathematical topics and figures in the fictions, are analyzed to meet the purpose. The results indicate: (a) students were more likely to see mathematics as a problem-solving activity; (b) a mysterious image of mathematical knowledge was generally held by these college students; (c) adventurous and smart are the two main characters of figures in their fictions.

ANTHYPHAIRESIS, THE "ORIGINARY" CORE OF THE CONCEPT OF FRACTION

Paolo LONGONI, Gianstefano RIVA, Ernesto ROTTOLI
Laboratorio didattico di matematica e filosofia. Presezzo, Bergamo, Italy
ernerott@tin.it

ABSTRACT

In spite of efforts over decades, the results of teaching and learning fractions are not satisfactory. In response to this trouble, we have proposed a radical rethinking of the didactics of fractions, that begins with the third grade of primary school. In this presentation, we propose some historical reflections that underline the "originary" meaning of the concept of fraction. Our starting point is to retrace the anthyphairesis, in order to feel, at least partially, the "originary sensibility" that characterized the Pythagorean search. The walking step by step the concrete actions of this procedure of comparison of two homogeneous quantities, results in proposing that a process of mathematisation is the core of didactics of fractions. This process begins recording the act of comparison by a pair of natural numbers, and is realized in the Euclidean division. Classroom activities ensure that children perceive the Euclidean division as the icon of their active process of learning. The Euclidean division becomes the core of many feedback loops along which the teaching process is developed.

1 Introduction

The scientific literature we have seen for a period of about fifty years, reports that, in spite of the efforts both in research and in practice, results of teaching and learning fractions are not satisfactory and difficulties are widespread and persistent.[1]

Some causes of these difficulties, widely discussed in scientific literature, are: (a) The multi-faced structure of rational numbers.[2] This feature is well represented in the scheme of the "sub-constructs of the construct of rational number" proposed by Kieren.[3] (b) Another obstacle: research and teaching practice have proposed teaching-learning processes that are often based on an unique pattern of action. This approach implies some difficulties in transferring the acquired knowledge to other situations. In teaching practice, the situation of division, associated to the part-whole substructure, is used in most cases.[4] It is important to

---

[1] Among the many quotes, we choose the following two, distant in time. "The concept of fraction has manifested itself in education as a refractory one" [Streefland, 1978]. "It is now well known that fractions are difficult concepts to learn as well as to teach" [Tunç-Pekkan, 2015].

[2] "Rational numbers should be a mega-concept involving many interwoven strands" [Wagner (1976) in Kieren, 1980].

[3] The five Kieren's sub-constructs are: part-whole, quotients, measure, ratios, operators. Other possible subconstructs are: proportionality, point on the number line, decimal number and so on.

[4] The scientific literature has largely confirmed that the situation of division has limited effectiveness [Nunez & Bryant, 2007].

stress that this teaching choice may also develop inhibitions.[5] (c) The natural numbers bias causes confusion between the features related to natural numbers and those related to fractions.[6] In scientific literature there are two attitudes. The one supports continuity between natural numbers and fractions.[7] The other considers the universe of fractions as a new universe, with its own rules and properties.[8]

The prolonged poor results in teaching and learning fractions have produced different conducts. Sometimes the purpose of teaching is to master the rules for calculating with fractions, without making an explicit connection between calculation and conceptual understanding.[9] Another option has been to postpone the teaching of the fractions.[10]

In Italy we have experienced a paradoxical situation: while the unsatisfactory results highlighted by research are widely acknowledged in middle and in high school, on the contrary, in primary school teachers mostly percept the teaching and learning fractions as easy.[11]

2 Historical reflections

These considerations, along with the direct experience in teaching, led us to a radical rethinking of the didactics of fractions. The resulting unusual project begins with the third grade of primary school and is characterized by five key points. (1) It is a process of familiarization with fractions rather than a process of teaching and learning them; this point, taken from Davydov[12], must be interpreted within the ZPD.[13] (2) From the didactic point of

---

[5] "Not only 'part of a whole' diagrams are possibly misleading, but, more seriously, it will be argued later that their use may well inhibit the development of other interpretations of a fraction …". [Kerslake, 1986]. The development of inhibitions has received not adequate attention both by researchers and by teachers.

[6] "The natural number bias is known to explain many difficulties learners have with understanding rational numbers." [Van Hoof, Verschaffel & Van Dooren, 2015].

[7] "The major hypothesis to be tested was that children could (and should) reorganize their whole number knowledge in order to build schemes for working with fractional quantities and numbers (the rational numbers of arithmetic) in meaningful ways." [Behr, Harel, Post & Lesh, 1992].

[8] "Kieren argues that rational number concepts are different from natural number ones in that they do not form part of a child's natural environment" [Kerslake, 1986].

[9] This choice had already been confuted in the Erlwanger's seminal article: "Benny's case indicates that a mastery of content and skill does not imply understanding." [Erlwanger, 1973].

[10] "Instruction in rational numbers should be postponed until the student has reached the stage of formal operations." [Kieren, 1980].

[11] This perception is explained by the fact that the only sub-construct proposed to children is the part-whole one. This choice allows structuring a feasible proposal and building evaluation processes with satisfactory results on average. However, this type of proposal can produce some inhibitions that will burden on the later learning process of rational numbers, causing the difficulties highlighted by research.

[12] In the 60s of the last century, Davydov has experienced an interesting approach to the concept of fraction in some schools at Moscow. This approach is distinct from those favoured in Western school: Davydov researches the "objective origin of the concept of fraction" and he identifies it in the measure (as in the measure he identifies the objective origin of the concept of multiplication) [Davydov, 1991]. Following this approach, he builds a practical, unitary and coherent teaching proposal, that we have experienced in our classrooms and that has contributed significantly to the construction of our proposal.

view, fractions are a new universe, with its own rules and properties, distinct from the universe of natural numbers. (3) The process of mathematisation begins with the act of identifying the comparison between two homogeneous quantities with a pair of natural numbers, and characterizes itself as "elementary and fundamental". (4) The measure of a quantity is defined as the comparison between the quantity and the "whole", while the term "unit" is assigned to indicate the common unit between quantity and whole. (5) The dialogy[14] among the activities, that consists in choosing the most appropriate manipulative to introduce a specific property, should create a "polyphony" among the different activities; so each of them finds meaning in the others and gives meaning to them.

A more detailed structure of the project is presented elsewhere.[15] Here we intend to do a historical reflection, because history is "the site" where our project has arisen and has found its structure; in this site we have looked for the "originary"[16] meaning of the concept of fraction and we have found some foundational aspects that allow to rethink its didactics.

Our research has begun with a suggestion emerged thanks to the speech of Imre Toth at the conference held in Bergamo in 1999.[17] He exhorted to listen to hidden meanings that could still be kept in the Pythagorean mathematics and in the mathematics of the Platonic Academy. That's how we discovered the anthyphairesis. In the Appendix 1 we present the procedure of the anthyphairetic comparison. This method of comparison did not last long. Its crisis came with the discovery of incommensurable quantities[18] and its difficulties were contrasted by the effectiveness of the Euclidean algorithm. This latter overshadowed the anthyphairetic comparison, which was consequently forgotten.[19]

---

[13] In our activity of familiarisation with the concept of fraction we provide children with "a broad base of experiences both practical and linguistic" [Nunez & Bryant, 2007] and we assess the actions and reactions of children properly guided.

[14] The word "dialogy" is taken from Bakhtin. It has been preferred to the word "dialogue" because it keeps some characteristics of the Bakhtin's thought, as "voices", "formative interaction", "polyphony", that better pinpoint the sense of our proposal. There are two ways of expressing this word: "dialogy" and "dialogism". The second is partially compromised by the excessive use of the suffix "ism" made in the twentieth century.

[15] We presented a first version of our project at Cieaem 67 in Aosta [Alessandro, Bonissoni, Carpentiere, Cazzola, Longoni, Riva, & Rottoli, 2015]. An updated version will be presented at ICME 13 in Hamburg.

[16] The word "originary" is not an English word. Nevertheless, some authors (Roth & Radford, 2011) are beginning to use it. In this way the wealth of meaning possessed by the corresponding term "originario/originaire" that is used in continental philosophy, is recovered.

[17] "Matematica, Storia e Filosofia: quale dialogo nella cultura e nella didattica?" Bergamo, 1999, May 19.

[18] "Consideration of the non terminating anthyphairesis of incommensurable magnitudes would lead to serious philosophical problems and technical mathematical difficulties… . When the ratio theory, based on anthyphairesis, was abandoned for Book V – style proportion theory, the interest in anthyphairesis as a mathematical procedure would greatly diminish, and the details of its erstwhile connection with ratio would be forgotten." [Fowler, 1979].

[19] The effectiveness of the Euclidean algorithm results from its direct operating on numbers and from its characterization as a much faster multiplicative process. The additive–subtractive feature of the Pythagorean procedure was thereby lost; an additive-subtractive feature we bring to light by retracing step by step the procedure.

Walking step by step the concrete actions of the anthyphairetic process, we have met some indications stored in it and still potentially significant: indications of historical type, as this walking allows to listen again to not secondary aspects of Greek philosophy;[20] indications of mathematical type, as it highlights a "physical" language for rational numbers[21] and it enables an unusual outlook on intrinsic reciprocity;[22] indications of pedagogical type, which have moved our project.

3 Pedagogical indications

As shown in Appendix 2, the anthyphairetic comparison produces a binary string and a pair of numbers. The binary string is a "physical" language for rational numbers; the pair of numbers is the "logos",[23] that makes "effable"[24] the comparison. So the Pythagorean statement "all is number" acquires the characteristic of a search for a "scientific" procedure in the description of nature, and the anthyphairetic procedure is a primitive form of mathematical knowledge of it. This is the indication from which our educational project has began. The objective of our slow retracing step by step this archaic procedure of comparison, was not the trivial

---

20 The additive-subtractive feature that characterizes this procedure reflects the time of thinking of the Pythagoreans. To listen to this time permits to feel, at least in part, their astonishment in front of the number; the astonishment that led them to say that everything is number. In fact, if you think you establish an unit of measure among homogeneous quantities, each of them has its own additive-subtractive structure expressed by a logos. Overcoming the lack of homogeneity of the real world by the "arché", the unit becomes the Parmenidean One, with which to compare every quantity. So every quantity has its own additive-subtractive structure in its comparison with the One; a structure expressed by a pair of numbers: everything is number.

21 In the Appendix 2 we show how, marking the sequence of actions of the comparison by modern symbols, it is possible to build a binary string. In this way each rational number can be expressed by a binary string. While the correspondence between the set of natural numbers and the set of binary strings is the foundation of the contemporary sciences of information and computers, the correspondence between the set of rational numbers and the set of binary strings seems not to be the subject of adequate attention.

22 Retracing the steps of the anthyphairetic comparison, the reciprocity between the two compared quantities comes evident. The role of reciprocity in teaching and learning fractions is under discussion in scientific literature [Thomson & Saldanha, 2003]. But we believe that, by retrieving the reciprocity with the characteristics that it shows in the anthyphairetic comparison, the investigation could be provided with useful new indications. For example, the intrinsic reciprocity of anthyphairesis suggests the possibility to go sometimes beyond the usual definition of quantity: " … a quantity is completely determined in mathematics when a set of elements and the criteria of comparison are determined… The comparison is usually traced back to the application of the relation "equal to", major to" or "minor to" [Davydov, 1991]. Intrinsic reciprocity might enrich the criteria of comparison and, therefore, the concept of quantity and the process of measure. We like to think that the fact of bringing reciprocity at the heart of the measure may provide insights into the challenges that complexity today presents, especially in the presence of quantities of dual nature.

23 The Greek word "logos" that the Pythagoreans use for denoting the pairs of numbers obtained by anthyphairetic comparison, is a polysemic word that acquired different meanings in the historical course of ancient Greece: word, speech, talk, oration, discourse, ratio, logic, cause, rationale. At the Pythagoreans, it looks like a wonderful synthesis of two different meanings: the one comes from the verb "légein", that is "to bind", "to relate"; the other is contained in the meaning of "voice", "speech", that, already at that time, the word "logos" had taken. Its translation in the Latin word "ratio" has originated the wording "rational numbers".

24 The word "effable" conveys, in addition to the meaning of "capable of being expressed", also the effort of the search and the wonder of the discovery: what is indescribable and could not be adequately expressed in words, becomes expressible thanks to the logos. The word "effable" hints also at the broad discussion on "commensurability" that troubled the world of the Pythagoreans.

knowledge of the procedure; according to Toth's indications, our aim was to retrace the procedure in order to feel, at least partially, the "originary sensibility"[25] that characterized the Pythagorean search.

3.1 The comparison

Consequently, differently from how we acted time ago [Rottoli & Riva, 2000], when we proposed the anthyphairetic procedure in some classrooms of a high school, in our present project this procedure is not directly used. Instead we have tried to make operating the originary feature of this basic form of mathematical knowledge: the act of comparison is a pair of natural numbers. This feature becomes the starting point for teaching the concept of fraction in primary school.

To this end, we have proposed to the children of two third classes of primary school, numerous and diverse activities of recording the act of comparing two homogeneous quantities by a pair of natural numbers. The children have worked with discrete and continuous quantities. As regards the activities with discrete quantities, the decisive choice of the teacher in order to motivate the act of comparison, has been to associate it to a game, the game of multiplication tables.[26] As regards the comparison of continuous quantities, we have made use of the activities with water, proposed by Davydov. The children have represented all the comparisons by squares or segments and by a formula of the type A;B = 13;8 : "the comparison between the quantities A and B is the pair of numbers 13;8".

[25] The wording "originary sensibility" is taken from Roth and Radford, 2011, but it has here a different nuance of meaning. While in Roth and Radford its interpretation is "achieved as part of a categorical reconstruction of the human psyche on evolutionary grounds", here we give it the meaning of sensory experience obtained by retracing step by step the actions that constitute the process of the anthyphairetic comparison. It is this sensibility that, in our case, makes effable the "originary". According to the philosophical reflections about the "arché" (see note 20), the "originary" as substantive, is identifiable with the true substance and is referable only to the absolute One, which is "ineffable". What becomes "effable" is its adjectival transformation, that is its transformation in being, linked to the bodily-historical sensibility.

[26] The teacher prepares a special deck. A multiplication is written on each card. The class is divided into two groups. The teacher plays a card and reads the multiplication. The group who first gives the product wins a candy that is put in the basket of the group. If the answers of both groups are almost simultaneous, each group wins a candy. At the end, there is the comparison of the candies won by each group.

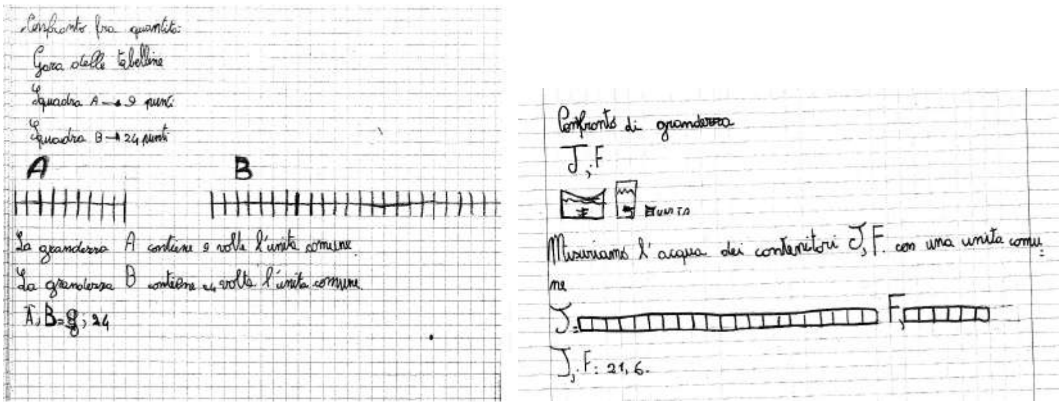


Figure 1. Examples taken from the exercise books: on the left the comparison between discrete quantities; on the rigth the comparison between continuous quantites

In order to understand the contribution that the historical reflection has given to our project, it may be interesting to compare the evocative/indicative meaning of the word "logos",[27] used in our approach, with the formalized meaning of the term "ratio" used by Lachance and Confrey.[28] In their introduction of the concept of fraction, starting from the subconstruct ratio, they look in the direction of the "broader" subconstruct, which would contain all the other subconstructs. We refer instead to history in order to "e-vocate", by an endeavour to listen to "originary sensations", and to investigate towards "indications" we receive from this listening.

3.2 The measure

[27] Leopardi, in the "Zibaldone", underlined the difference between the meanings of "word" and "term". Differently from the scientific, rigid meaning associated with "term", "word" has evocative value: "evocative" because it brings to light some meanings that have belonged to other poetic contexts. We interpret "evocative" as opening to new directions of investigation.

[28] "Using multiple contexts and experiences, we hope to guide students to first explore and understand a broad construct such as ratio and use that understanding to explore more specific instances of that construct such as fraction, decimal and percent." [Lachance & Confrey, 2002]. Certainly the search in the direction of "the broader", characterizes many investigations of the twentieth century. But we hope, by the search in the direction of "originary sensations", to be entered in resonance with one of the many echoes that come from the world of originary.

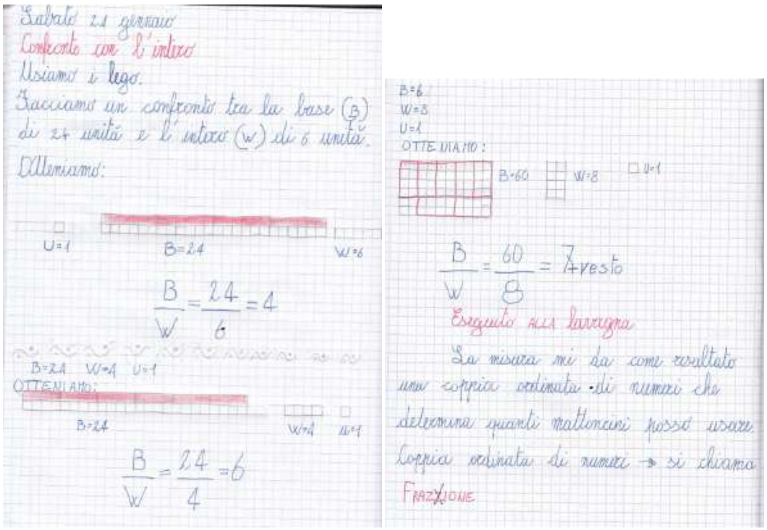


Figure 2. Examples of activities of measure

The procedure of anthyphairetic comparison highlights features that the usual process of measure leaves aside: the reciprocity between the two compared quantities and the search for a common unit. With regard to the role of reciprocity, we refer to what previously said. The search for a common unit characterizes the introduction of the concept of measure within our teaching process. Here the measure is defined as the comparison between a quantity and the “special” quantity called “whole”. The term "unit" is reserved to indicate the common unit. The children yet make use of discrete and continuous manipulative: egg boxes, lego, packages of candies, picture cards; water, cakes, stripes of paper and so on. The special nature of the whole is highlighted by the special symbol “W” and by colouring. Also in this situation, all comparisons are represented by squares or segments. In order to write the comparison/measure, a special symbol is used: B/W = 25/6. This pair of numbers is the fraction: “The measure of the quantity B with respect to the whole W is the fraction 25/6”.

3.3 Euclidean Division as core of the didactic activity

An important achievement of our proposal is that the children arrive naturally to write, already in the third grade, the formula of Euclidean division: Z/W= 16/5 = 3+1/5.

Figure 3. Euclidean division in a exercise book.

This achievement completes the process of mathematisation concerning the didactics of fraction: it starts by associating the act of comparing with a pair of natural numbers and is realized in the Euclidean division. Thanks to this approach, the Euclidean division is experienced by the children not as a formula to be memorized, but as the icon of their active process of learning.

This mathematisation, which differs from modelling,[29] characterizes itself as elementary and fundamental. It is elementary because turned to “the originary elements”, but also by reason of the "lightness" [Calvino, 1988] with which the children have lived these activities: their answer has been quiet, serene, with adequate results. Il is fundamental because it determines the structure of the new universe of fractions; a structure that has the Euclidean division as core.

As the Euclidean division is the core of the universe of fractions, their teaching follows a particular proceeding: teaching is not a linear path of accumulation of consecutive learning; it is composed of many feedback loops: they start from the core, immerse in the contexts that are characterized by the different subconstructs, and return to the core. It is in this sense that we affirm that all “subconstructs of the construct of rational number”, have their roots in the Euclidean division and find unity in it.

4 Concluding remarks

“In classroom activities, the children's answer was light, that is quiet, serene, with adequate results.”[30]

The teachers have found many difficulties: only 10% continued the started activity. Notwithstanding common manipulative are used mostly, the teachers are asked to change their way of seeing and thinking, in order to discover objects, properties, operations that belong to the new universe. According to Thomas Kuhn[31], they are called for a change of “paradigm”, for a “revolution” that upsets their habitual way of seeing and thinking fractions. This change should produce acquisition of awareness[32], reshaping the way of teaching proceeding[33], structuring the mutual interaction among teacher, class and research[34].

---

[29] “Mathematisation” differs from “modelling”: while “modelling applies a fragment of mathematics to a fragment of reality” [Israel, 2002], mathematisation has an “universal” meaning, because it guides the structuring of the universe of fractions as a new universe, distinct from the universe of natural numbers.

[30] Here some examples of adequate results: - Children, already in the third grade, consider the pair of numbers that form the fraction as a number (the number of packs). - Immediately and by themselves, children connect the fraction to the division. - In fourth grade children naturally know that in a decimal number (for example 3.7) 3 indicates the wholes and 7 indicates the decimal part. (d) Children can put themselves properly in front of some problems containing concepts not yet unfolded. In the experiment of the candle: Teacher: “We know that the oxygen constitutes 21% of the air; how much the water grows in the glass when the candle goes out?” “It is difficult to divide the container into 100 parts; we can divide it into ten parts.” So children calculate roughly where the water comes. The topic “percentage” has not yet been presented in the classroom.

[31] We make an analogical use of the Kuhn’s concept of “scientific revolution”.

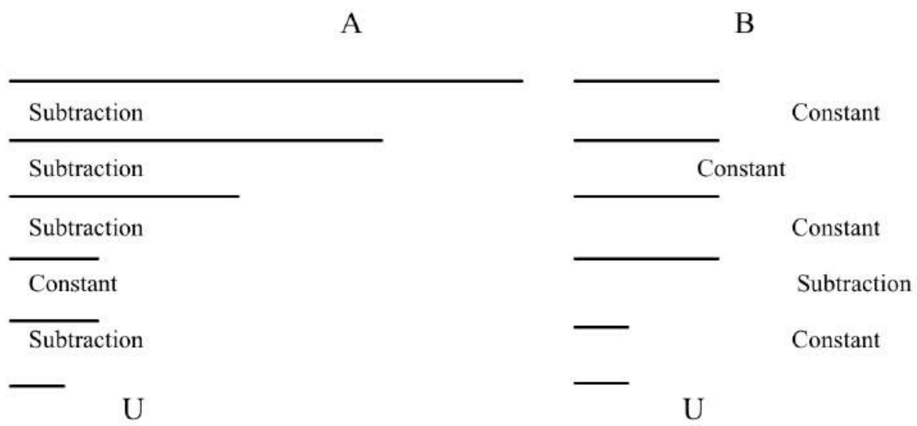
[32] A double awareness: the awareness that comes from the at least partial recovery of the "originary sensibility" that gives meaning to this mathematisation process; the awareness that some concepts are fundamental in “scaffolding” the universe of fractions.

[33] As mentioned above, the central importance of Euclidian division requires that didactic proceeding is structured in the form of feedback loops.

Appendix 1: Anthyphairesis

The anthyphairesis of two homogeneous quantities is a method of repeated removing and consists in subtracting the smaller of the two quantities from the larger one; after each removing, the larger quantity is replaced by the excess, while the smaller one stays unchanged. The process continues until the excess obtained by removing, is equal to the unchanged quantity.[35]

If you consider, for example, two segments, you can compare them in the following way:



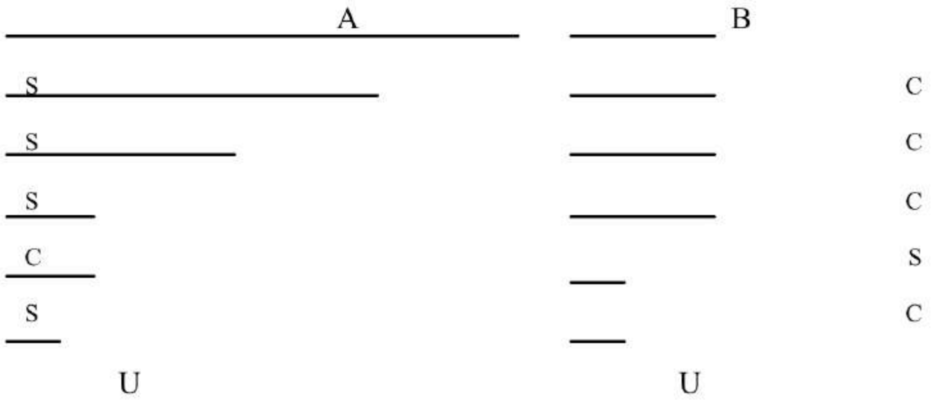
The result of the comparison (if the process terminates) is the unit U, common to both segments. Each segment contains the common unit a certain number of times: A = nU; B = mU.

[34] In the interaction between teacher and class, teacher listens to and is guided by the class in the development of teaching proposal; in the interaction between teacher and researcher, the effectiveness of the didactic path is continually rethought.

[35] The word anthyphairesis comes from the ancient Greek and its etymology is the following: anti–hipo–hairesis / reciprocal–sub–traction. Aristotle uses for the same procedure the term antanairesis: anti–ana–hairesis / reciprocal–re–traction. [Zellini, 1999]

Appendix 2: Logos

Our going step by step along the ancient procedure of anthyphairetic comparison, is characterized by the modern attitude of using symbols to indicate the action taken in each step. For example, the previous comparison between the quantities A and B looks like this:



The following chart uniquely represents the comparison:

S	C
S	C
S	C
C	S
S	C

To get how many times the quantities A and B contain the common unit, it is enough to turn upside down the chart and to retrace it, reading "S" as "Sum" rather than "Subtraction":

1 U			1U
	S	C	
2 U			1 U
	C	S	
2 U			3 U
	S	C	
5 U			3 U
	S	C	
8 U			3 U
	S	C	
11 U			3 U
A			B

So A = 11U and B = 3U

The comparison has produced the chart, which may be expressed by the binary string SSSCS, and the pair of numbers (11;3). The binary string is a "physical" language for rational numbers. The pair of numbers is the "logos".

REFERENCES

Alessandro, S., Bonissoni, P., Carpentiere, S., Cazzola, M., Longoni, P., Riva, G., & Rottoli, E. (2015). Familiariser avec les nombres fractionnaires : Ressources et obstacles. In B. Di Paola, & C. Sabena (Eds.), *Teaching and learning mathematics: resources and obstacles, Proceedings of CIEAEM 67* (Aosta), *Quaderni di ricerca didattica, 25*(2) (pp. 109-114). Palermo: University of Palermo.

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 296-333). NY: Macmillan Publishing.

Calvino, I. (1988). *Lezioni americane*. Milano: Garzanti Editore.

Davydov, V. V., & Tsvetkovich, Z. H. (1991). On the objective origin of the concept of fractions. *Focus on Learning Problems in Mathematics*, *13*(1).

Erlwanger, H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children Mathematical Behavior 1, 7-26*.

Fowler, D. H. (1979). Ratio in early Greek mathematics. *Bulletin (New Series) of the American Mathematical Society. 1,* 807-846.

Israel, G. (2002). *Modelli matematici*. Roma: Franco Muzzio Editore.

Kerslake, D. (1986). *Fractions: Children's strategies and errors*. A Report of the strategies and errors in Secondary Mathematics Project (pp. 1-139). Windsor: The NFER-NELSON Publishing Company Ltd.

Kieren, T. E. (Ed.) (1980). *Recent research on number learning.* Columbus, Ohio: ERIC Clearing house for Science, Mathematics, and Environmental Education.

Kuhn, T. S. (1978). *La struttura delle rivoluzione scientifiche*. Torino: Einaudi.

Lachance, A., & Confrey, J. (2002). Helping students build a path of understanding from ratio and proportion to decimal notation. *Journal of Mathematical Behavior. 20, 503–526*.

Leopardi, G. (1949) *Zibaldone di pensieri*. *A cura di Francesco Flora*. Milano: Classici Mondadori.

Nunes, T., & Bryant, P. (2007). Understanding rational numbers and intensive quantities. In *Key understandings in mathematics learning*, *Paper 3* (pp. 1-31). Nuffield Foundation.

Roth, W. M., & Radford, L. (2011). *A cultural-historical perspective on mathematics teaching and learning*. Rotterdam / Boston / Taipei: Sense Publishers.

Rottoli, E., & Riva, G. (2000). The antiphairesis: A site of an educational dialogue among mathematics, history and philosophy. In W.-S. Horng, & F.-L. Lin (Eds.), *Proceedings of the HPM 2000 conference: History in mathematics education: Challenges for a new millennium. Vol. II* (pp. 94-102). Taipei: Department of Mathematics, National Taiwan Normal University

Streefland, L. (1978). Some observational results concerning the mental constitution of the concept of fraction. *Educational Studies in Mathematics, 9*,51-73.

Thomson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 95-114). Resto, VA: National Council of Teachers of Mathematics.

Toth, I. (1997). *Aristotele e i fondamenti assiomatici della geometria. Prolegomeni alla comprensione dei frammenti non-euclidei nel «Corpus Aristotelicum» nel loro contesto matematico e filosofico.* Milano: Vita e Pensiero Editrice.

Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2015). Inappropriately applying natural number properties in rational number tasks: Characterizing the development of the natural number bias through primary and secondary education. *Educational Studies in Mathematics, 90*, 39-56.

Tunç-Pekkan, Z. (2015). An analysis of elementary school children's fractional knowledge depicted with circle, rectangle, and number line representations. *Educational Studies in Mathematics, 89*, 419-441.

Zellini, P. (1999). *Gnomon. Un'indagine sul numero*. Milano: Adelphi Edizioni.

THE DEVELOPMENT OF VIEWS ON MATHEMATICS EDUCATION AS SEEN IN MATHEMATICS EDUCATION CONTROVERSIES IN JAPAN

Naomichi MAKINAE
University of Tsukuba, 1-1-1 Ten-nou-dai, Tsukuba-city, Japan
makinae@human.tsukuba.ac.jp

ABSTRACT

It can be said that mathematics education in Japan was started in 1872 when the school system was established. Since that establishment era, controversies have emerged time and again in mathematics education in Japan. Through these controversies, debates have been held on views on mathematics education such as how mathematics ought to be taught and what constitutes knowledge concerning numbers, quantities, and shapes that is desirable for students to acquire. In this paper, I shall look back at how views on mathematics education in Japan have developed since the Meiji era from the perspective of such controversies on mathematics education. As the controversies on mathematics education, the four phases are picked up. The first is Theoretical Mathematics and the Enumeration Principle. The second is Controversy over Formal Building. The third is Conventional Teaching of Mathematics and the Creation of Mathematics. The forth is Relationship between Daily Life and Mathematics.

1 Introduction

The purpose of this paper is to identify the historic process of mathematics education in Japan, and thereby focus on the controversies emerged time and again in mathematics education. As recent studies that introduced the history of mathematics education in Japan, we can mentioned Baba et al. (2012), Ueno (2012), Kota (2012) and so on. In this paper, I focus on another view point from these previous studies. Through the educational controversies, the debates have been held on views on mathematics education in elementary and lower secondary level about its subject objectives and curriculum structure, teaching methods etc. When we look back history of mathematics education in Japan, we can pick up four phases for these controversies. In following sections, I will illustrate such controversies.

2 Theoretical arithmetic and the enumeration principle

2.1 Beginning of the modern school education system and mathematics education

The beginning of the modern school education system in Japan can be traced back to the school system in 1872. Aiming at building a modern state, the Meiji government began incorporating the technology and culture of Europe and the United States, and the school system came to be developed in this process. In mathematics education, a policy was adopted in which Western mathematics based Arabic numerals and the base-10 positional notation system would be taught in addition to traditional Japanese mathematics. More time was devoted to the teaching of the four operations in written calculation, and such textbooks as "*Hissan Kunmo* (Learning Written Calculation)," "*Yozan Hayamanabi* (Quick Learning of Western Arithmetic)," and "*Shogaku Sanjutsusho* (Elementary Arithmetic Textbook)" were published. In the decade between 1877 and 1886, "*Sugaku Sanzendai* (Three Thousand Math Problems)" by Seikyu Ozeki who was an Aichi Prefecture samurai, was published (first edition in 1880). "Three Thousand Math Problems" is a textbook in the format of a collection of problems. Although it returns to the principle of seeking answers since "*wasan* (Japanese mathematics)," the textbook became widely used at the time. An explanation on calculations appears at the beginning of the textbook, followed by 3,000 problems in accordance with its title.

2.2 Theoretical arithmetic

Firstly, "*Chuto Kyoiku Sanjutsu Kyokasho* (Secondary Education Arithmetic Textbook)" (published in 1888) by Hisashi Terao who was a professor of Tokyo University, can be cited as a textbook that is representative of theoretical arithmetic. The basic idea of theoretical arithmetic is stated at the beginning of the textbook.

> Observing the method of teaching arithmetic at the school where I am in charge of secondary education, everyone basically neglects theory, seemingly simply solving problems...(passage omitted)...Arithmetic is essentially a type of science, and it is simply not art, regardless of what they call it. Even if I were to yield an inch and call arithmetic an art, this art, like medicine and architecture, cannot form a firm foundation unless it is based on theory. Therefore, trying to give a lecture on arithmetic while neglecting theory is like trying to teach surgery without learning about anatomy. (Hisashi Terao 1888, p. 10)

It is explained here that arithmetic is not simply calculation skills but rather that there is a legitimate theory behind it. Terao asserts that calculations hold true because they are supported by theory. He emphasizes that it is necessary to teach theory in mathematics education. Under this concept, such items as "definition," "principles," "laws," and "caution" are listed in the textbook to explain the theory of arithmetic. After the explanation, calculations using specific numbers are given as examples. In the explanations in this textbook, an approach is taken to lead to new contents based on definitions that have already been learned rather than simply describing calculation.

A characteristic of theoretical arithmetic is the concept of not regarding mathematics education as simply learning calculation methods but rather as teaching the underlying theory. It is based on mathematics education based on a strict French-style theory that Terao learned while studying abroad. In his theoretical arithmetic, Terao pursued arithmetic as a discipline and looked toward theory as its cornerstone.

2.3 Enumeration principle

It was Rikitaro Fujisawa, a professor of Tokyo University, who criticized theoretical mathematics and expounded a different argument. "*Sanjutsu Jomoku Oyobi Kyoju-ho* (Mathematics Rules and Teaching Methods)" can be cited as a work in which Fujisawa straightforwardly expounded his views on mathematics education. Here, he lists the two following points at the purpose of mathematics education.

> The purpose of mathematics education is to provide phased preliminary mathematical

knowledge and to nurture mathematical philosophy, i.e., spiritual discipline (Fujisawa, 1895, p. 2)

The first point refers to providing knowledge of mathematics that will serve as the necessary foundation for advancing to higher grades and studying mathematic. Fujisawa mentions an aspect of substantial discipline in mathematics education. The second point refers to strengthening mathematical thinking and philosophy through mathematics. Fujisawa mentions an aspect of formal building in mathematics education. Thereupon, Fujisawa rejects the argument of theoretical arithmetic, asserting that there are definitions, laws, and explanations in arithmetic, and arithmetic is not biased toward only seeking answers. He goes on to distinguish between arithmetic and algebra, asserting that the theory of arithmetic is algebra and proof should not be taken up in arithmetic. Moreover, he argues that in view of the current situation of school education and the developmental stages of children, theory should not be taken up.

In criticizing theoretical arithmetic, the enumeration principle is what Fujisawa advocated as a replacement guidance principle. Among textbooks based on the enumeration principle is "*Sanjutsu Sho Kyokasho* (Arithmetic Textbook for Elementary Schools)" (published in 1898). Fujisawa was introduced to the enumeration principle by his teacher L. Kronecker at the time of his return from studying in Germany. It is based on the principle advocated by R. Knilling and others. The enumeration principle is based on the fundamental idea that acquisition of the concept of numbers becomes possible by counting. For this reason, textbooks based on the "enumeration principle" start with the introduction of numerals and how to count numbers. The concept of numbers is said to be acquired by associating numbers that are abstract with numerals that are the names of the numbers when counting and also by counting those numerals out aloud.

2.4 Views on mathematics education seen in the conflict between theoretical arithmetic and the enumeration principle

With regard to the conflict between theoretical arithmetic and the enumeration principle, debates were held over which policy to adopt in the process of editing the first government–designated textbook "*Jinjo Shogaku Sanjutsusho* (Ordinary Elementary School Arithmetic Textbook)," which was published in 1905. The conflict was ultimately settled when it was decided that Fujisawa's enumeration principle would be adopted. Attention must be paid to the fact that this conflict was taking place in an era when it was questioned what constitutes mathematics education in introducing the modern education system in Japan.

Those who advocated theoretical arithmetic were concerned that mathematics education would get buried in simple teaching of calculations and seeking answers; they placed priority on the underlying theory. Asserting that calculations are possible by using the rules of calculations that have been proven by theory, they sought the essence of mathematics education in the underlying theory rather than in the calculations themselves. In contrast, those who advocated the enumeration principle placed priority on the concept of numbers and the understanding of the rules of calculations. While asserting that it is necessary to take up theory in studying more advanced mathematics, they sought the essence of mathematics education not only in the understanding of mathematics itself but also in disciplining one's mind and acquiring a way of thinking by recognizing the concept of numbers and through specific calculations.

Although people get the impression that the theory of mathematics and the concept of mathematics are the same thing, there is a big difference. The two viewpoints take on a completely different appearance with regard to developing teaching methods and textbooks/teaching materials. Theory refers to axiomatic development in accordance with mathematics as a system of learning. It can be considered a difference between the standpoint of those who want students to acquire this theory and the standpoint of those who assert that even if the students forget what they learned about mathematics, they would not forget the concept of mathematics, which they acquired through studying, and this is what they want the students to acquire.

3 Controversy over formal building

3.1 Period of adjustment of school education and review toward formal building

After the school education system was established, mathematics education in Japan entered a period of adjustment under the influence of Taisho democracy and the liberalism movement and as the mathematic education reform movements in Europe and the United States were conveyed. Ever since the Meiji period, mathematics education was promoted for the purpose of both substantial discipline and formal building, but in actual teaching, lessons were held centering on teaching calculations. What was actually sought of children in mathematics education was for them to be able to solve problems in textbooks, and it was a period when this kind of teaching was criticized and improvements were sought. Amid calls for education centering on children that is at the foundation of the improvements, a major controversy occurred in mathematics education.

During this period, Fujisawa retained his influence with regard to the purpose of mathematics education. The following phrase remains intact in the Elementary School Ordinance Enforcement Regulations: "The gist of arithmetic is achieving proficiency in daily calculations, providing knowledge required in life, and at the same time making thinking accurate." The concept of formal building that aims at making thinking accurate had been placed at the basis of mathematics education. However, at the time, this idea has contained a misunderstanding that lead to a so-what attitude in teaching mathematics. In other words, there spread an idea that solving mathematic problems would train thinking, resulting in acquiring the ability to think logically. As a last resort when it was not possible to respond to the question of whether mathematics taught in school is useful or what meaning there is in learning mathematics at school, the concept of formal building was used as the theoretical background in answering that mathematics taught in school was for training the process of thinking. Teaching mathematics was justified simply asserting that as long as students solved mathematical problems, they would be able to train their thinking process.

In response to mathematics education that took comfort in the concept of formal building, doubts that were presented against formal building triggered the controversy over formal building.

3.2 Arata Osada's disavowal of the formal building theory

In 1922, a meeting of the "*Zenkoku Chutogakko Sugaku Kyoju Kenkyu-kai* (National Research Society of Secondary School Mathematics Teachers)" was held in Hiroshima. At this meeting, Arata Osada, a professor of Hiroshima Higher Normal School gave a lecture titled "Recent Controversy over Formal building." Here, he totally disavowed the traditional theory of formal building, going on to introduce a doctrine that it is harmful to approve of the formal building theory. Osada described formal building as follows:

> In other words, formal building tries to discipline mental capacity itself by using mathematics teaching materials as an expedient. When we study not only mathematics but other things as well, it can be said that the effects are all universal. (Osada, 1923, p. 61)

For example, when solving a geometry problem, some kind of psychological effects remain in the student. Osada says these effects are beneficial in solving other geometry problems, other mathematical problems, or even problems other than mathematical problems. This is known as *transference* in psychology. When we think this way, the power of reasoning is trained by studying mathematics, and, therefore, mathematics is worth learning.

3.3 Kinnosuke Ogura's "Fundamental problems in mathematics education"

"Fundamental Problems in Mathematics Education" by Kinnosuke Ogura who was a researcher of Siomi Institute of Physical and Chemical Research, mathematician, was published in 1924. This is a book that criticized mathematics education at the time, asserted that improvements were necessary. Here, Ogura also supported disavowal of the formal building theory advocated during the 5th annual meeting of the Secondary Education Mathematical Society of Japan, and made a similar argument in the book.

Ogura asserts reform of the traditional mathematics education with regard to pre-war mathematics education. In addition to morals, religion, and the arts, science is necessary for humans to create a lifestyle. What can be learned from science include various matters such as the facts of biology and physical and chemical phenomena, but the most fundamental thing is to learn about scientific views, scientific concepts, and the scientific spirit. When there are more than two phenomena, a close examination of their causes is conducted on the basis of empirical fact, and a determination is made as to whether there are any cause-and-effect relationship between the phenomena, and if so, what kind of relationship that is. The efforts made and spirit for discovering that is scientific spirit. Development and creation of human lifestyles and human ideals require nurturing this scientific spirit. Similar to the development of natural sciences, the development of mathematics was born from nature, and mathematics has the similar aim as the natural sciences. For this reason, "the significance of mathematics education lies in the development of the scientific spirit" (Ogura, 1924/1953: 159). The "concept of functions" speaks of the scientific cause-and-effect relationship, and at the same time, it is most widely and deeply related to human life.

Ogura expanded the grounds and structure of his argument against formal building in a similar manner as the aforementioned lecture by Osada. After describing formal building, he disavowed formal building by negating the existence of the ability, which is the prerequisite of formal building. From the standpoint of disavowing formal building, Ogura has pointed out the need for the "concept of functions." Although the contents of teaching pure mathematics are an ad hoc, unrealistic event, functions are related to everyday life. Functions are learned not simply from formulas and graphs but also by linking them with actual phenomena and experiences, and that must be done. Although general formal building cannot be approved, transfer is allowed when identical elements are included, and therefore, teaching material must be socialized. When socializing mathematics teaching materials, teaching materials that are related to actual daily life shall be selected. This is because the selection of teaching material from events in daily life in which transfer is valid and is most likely to occur is derived from disavowing formal building. What links daily life with mathematics as such is the "concept of functions."

3.4 Rebuttal by Motoji Kunieda

In 1924, Motoji Kunieda who was a professor of Tokyo Higher Normal School, presented a counterargument against Osada and Ogura in a lecture at the 6th annual meeting of the Secondary Education Mathematical Society of Japan. At the time, Kunieda was vice chairman of the Secondary Education Mathematical Society of Japan, and as an expert in mathematics education, he had taken a position to recognize formal building. First, he says that arguments by Osada and Ogura introduce only parts of the debates on formal building, and although it may appear that formal building has been completely disavowed in Europe and the United States, that is not true. He went on to assert that many psychologists in the United States admit the significance and effects of formal building.

While reviewing the results of experiments and the theories of eight psychologists, Kunieda introduced the fact that formal building was recognized. Specifically, he reported such matters as that investigating formal building of memorization abilities from a survey on changes in the time it takes to memorize a book revealed that some effects were found with children, and that the habit of writing answers neatly was transferred not only to the relevant subject but also to other subjects. In addition, as Chairman Tsuruichi Hayashi who was a professor of Tohoku University, stated in his opening speech, a survey report on formal building was released in the United States, Kunieda stressed that even if there were some differences the effects of formal building were recognized[1].

3.5 Views on mathematics education seen in the wake of the conclusion of the controversy over formal building

In the controversy over formal building, both the opponents and proponents based their

[1] In his speech, Hayashi stated that the argument against formal building cannot be taken at face value, that the importance of mathematics education would not disappear as a result of the argument against formal building, and that the effects of formal building are recognized in "The Reorganization of Mathematics in Secondary Education," a report edited by J. W. A. Young and released by the Mathematics Committee on Mathematics Requirements. (Hayashi, 1923)

arguments on overseas theories and survey results. They did not state their view based on their own survey results. The controversy was not settled in the form of one view being adopted while the other view was discarded in the education policy and the editorial policy concerning government-designated textbooks. Regardless of the pros and cons regarding formal building, assertions regarding the objectives of mathematics education and what is desirable for children to acquire are at the basis of the arguments. The arguments against formal building served as criticism against teachers who taught mathematics as is as a discipline in mathematics education and became stepping stones toward improvements.

Ogura advocated learning the scientific spirit, while Kunieda advocated learning mathematical common sense adapted to the culture of the times. In either case, regardless of the pros and cons regarding formal building, they asserted the significance of studying mathematics. Here, the arguments include assertions concerning improving mathematics education to train the thinking process rather than regarding the ability to solve math problems as mathematics education. In fact, Kunieda also asserted incorporating experiments and actual measurements in geometry, incorporating intuitive handling, and incorporating teaching materials on functions. Here, we can sense their desire to see children being able to understand mathematics and being able to think.

At the time, the Secondary Education Mathematical Society of Japan was organized in 1918 for the purpose of "studying matters related to mathematics and the method of teaching mathematics in secondary schools and taking initiatives to achieve progress and make improvements," and research in mathematics education began to be promoted in mathematics education circles in Japan. The controversy over formal building was the first of an academic controversy here, and an issue that started the sprouting of academic research on mathematics education. In addition, it can be said that at its root, it was an issue calling into question the goal of mathematics education.

4 From conventional teaching of mathematics to the creation of mathematics

4.1 Mathematics education reconstruction movement

Affected by mathematics education reform movements in Europe and the United States, mathematics education in Japan progressed gradually in 1930's and 40's. In particular, in elementary education, government-designated textbooks were drastically revised by Naomichi Shiono who was a compiler of national textbooks in Ministry of Education, and the effects of the reform movement were clearly evident at the government-level educational administration. Reform of mathematics education in secondary schools was inevitably sought when children who received this new mathematics education advanced to the secondary school level. A movement that occurred in response to this was the mathematics education reconstruction movement – a movement to drastically reconstruct mathematics education ranging from the objectives of teaching to the contents of teaching and the methods of teaching. This movement culminated in the revision of the syllabus of teaching for secondary schools and the compilation of "Mathematics," an authorized textbook.

In the mathematics education reconstruction movement, proponents asserted the introduction of new teaching contents such as analytic geometry, calculus, descriptive geometry, statistics, and dynamics, as well as teachers' own curriculums and teaching methods, without being bound by the constraints of the system of mathematics as a discipline. A study group was established centering on the Secondary Education Mathematical Society of Japan, and the research contents were reported in journals and at conventions. In response to these research results, the Ministry of Education carried out a revision of the syllabus of teaching in 1942.

4.2 Revision of the syllabus of teaching for secondary school

With regard to the revision of the syllabus of teaching for secondary schools, various teaching contents and the curriculum were subject to studies from the idea that mathematics education is intended for students to discover and create mathematics on their own rather than for teaching conventional mathematics. The guidelines for this revision are shown below.

(I) Without adherence to the existing academic system, adopt the appropriate system for extending the students intellectual abilities.

(II) Carefully select teaching materials across the board, and in particular, adopt them by taking the points at the left into consideration.

1. Matters that are effective and appropriate for the daily lives of the people
2. Matters that are for industries and national defense
3. Matters that should contribute to long-term culturing of insights

(III) Designate specific operations such as observation, actual measurement, and construction as the foundation of learning, and along with training for knowledge and action, make efforts to nurture the ability to discover and create.

(IV) In addition to placing priority on intuition, make efforts to train the capability to think abstractly, analyze, and integrate.

(V) Make teaching matters as precise as possible, while adding the teaching policy and adopting a system that clarifies the objective of teaching

(Ministry of Education, 1942, p. 2)

Compared with the conventional syllabus of teaching, there were the following changes. Firstly, the teaching contents were separated into Type 1, which concerns quantity, and Type 2, which concerns diagrams. Secondly, although only mathematical contents such as "ratio," "quadratic equations," and "similar figures" were listed in the conventional syllabus, what kind of teaching is to be conducted was written in the new syllabus before listing the teaching contents. In other words, mathematical contents and the objectives and activities in the teaching process were combined as the teaching content.

4.3 Government-designated textbook: "Mathematics"

Following the revision of the syllabus of teaching for secondary schools, publication of

"Mathematics," a new mathematics textbook created under this spirit, started in 1943. Corresponding to the syllabus of teaching, two volumes – Class-1 and Class-2 – were issued for each grade from 1st grade to 5th grade. The editorial prospectus of this textbook contains the following editorial guidelines, which plainly indicate the characteristics of the textbook.

1. Stop introducing conventional mathematics and guide the students so that they can discover mathematical principles on their own in line with events.
2. Incorporate many concrete materials in problems, mathematize events, and give priority to training for processing them.
3. Ensure that students sufficiently understand mathematical principles in line with specific examples; then abstracted and formalize them and train so that they can be freely applied to concrete events.
4. Give priority to the handling of similarities in mathematical expressions of events.
5. Make it a rule to give the definitions of terms and symbols after their concepts have been developed.

(Secondary School Textbook Co., Ltd., 1943, p. 2-3)

Textbook materials are introduced from the scene of specific problems. Through the course of processing them through mathematization, a procedure has been taken in which the concept of mathematics and the method of processing are abstracted and formalized. Students are expected to write between the lines on the textbooks what they notice here, or definitions are written in small type so that they are not conspicuous. In this way, the textbooks reflect the intention to have the students create mathematics on their own rather than teaching them mathematics that has already been completed.

Experiments, actual measurements, and functions are often used for thinking about such scenes of specific problems. Data is actually collected through experiments and actual measurements, and quantitative relationships concerning events are regarded as functions. They are processed by expressing them as graphs, drawing graphs, and expressing them as numerical formulas. New formulas are derived and new rules are found through this process. In this way, the textbooks are designed so that new mathematical contents intended as the teaching contents are created. It is the aim of the textbooks to have the students conduct such activities on their own, and here in lies an argument that leads to the "syllabus residue theory" advocated during debates on the syllabus of teaching.

4.4 Views on mathematics education seen in criticisms of the syllabus of teaching and textbooks

However, the syllabus of teaching and textbooks of the time were not necessarily fully accepted at the field of education and in mathematics education circles. As a social situation, the effects of World War II extended to school education itself, and teaching requiring such time and effort had become impossible. In addition, under such circumstances, the aim and intentions that substantially differed from those of conventional teaching was not accurately transmitted to the field of education and thus not understood.

As a more fundamental problem, there existed conflicts over how mathematics education ought to be. With regard to teaching material that were introduced from scenes of specific problems, mathematicians of the time, while acknowledging their diversity, pointed out that the contents were too wide-ranging. In the backdrop of this situation was the concept that the utilization and application of mathematics to specific scenes rested on the understanding of the basics of mathematics. While not denying the utilization and application of mathematics, the fact that this has become the center had become the problem.

5 Relationship between daily life and mathematics

5.1 Postwar educational reforms and empirical approach to mathematical education

Right after the end of World War II, occupation by the Allies and the fundamental social reformation took place in Japan. The reform also took place in the field of education. The educational system, contents, and instructional methods were modified in conformity with the idea of pacifism and democracy. In this process, education centered on the needs of children was sought instead of prewar teacher-centered uniform instructional method. Especially, the unit learning method that was contextualized in relation to children's experience or day-to-day activities was widely adopted. Curriculum structure that incorporated daily life, not just math as a subject, was also introduced to math education. The following chart from the draft of the Course of Study for Lower and Higher Secondary Schools, which was published in 1951, describes mathematics instructional content based on "living experience."

This curriculum model reflects the criticism against pre-war mathematics education, which overly emphasized teaching mathematics as a discipline and did not correspond with the children's real learning conditions. It was claimed that in order to incorporate children's needs in mathematics education, examples and problems discussed needed to be driven from everyday situations so that children could learn mathematics through solving those problems. The Allied Forces made this claim to the Japanese Ministry of Education, and this idea was also supported by a group of core curriculum proponents who had been promoting research in the field of pedagogy regarding the curriculum in the United States. In addition, it was claimed that the standard of mathematics education did not match the children's comprehension ability, and after 1948, standards were lowered by one level in all grades, and instructional contents that met the new standards were implemented.

5.2 "Mathematics for middle school students," a model textbook for unit learning

In 1949, the Ministry of Education published its "Mathematics for Lower Secondary School Students," a model textbook for unit learning based on empirical education discussed above. The textbook uses examples from children's daily lives and aims to teach mathematics through problem solving. The problem takes up a scene in daily life and unfolds in such a way that relevant mathematical contents are taken up. Taking the structure of daily objects into consideration, it can be acknowledged mathematical concept and knowledge.

The textbook was a collection of these materials and it showed how to teach in classroom. The areas covered in this textbook were the number and calculation, quantity and geometry, the textbook picked up the children's daily life experience that corresponds to each.

5.3 Criticism against unit learning

These initiatives to promote unit learning came under criticism after the conclusion of the (San Francisco) Peace Treaty, along with the objection to the occupation policy, and declined rapidly. The strongest point of the criticism was the falling of children's academic standards. Compared to surveys on computational problems conducted before the war, the results of the survey conducted at the time showed a clear drop in the academic standards among children. Of course, there certainly must have been many other factors responsible for the fall in academic standards such as the period of turmoil in education during and after the war, reduction in teaching contents, and changes in the grade system. However, unit learning solely was targeted as the cause of the fall in children's academic standards.

In reaction to the criticism, teachers and schools asserted the significance of student-centered education and the necessity of solving problems. In 1958, however, on the occasion of the revision of the course of study, the instructional contents were presented based on the system of subjects, and since then, systematized learning was adopted.

5.4 Perspective on mathematics education during post war educational reforms

If based on empiricism, priority is placed not on the assumption that the contents of mathematics that ought to be learned will become useful someday in the future, but rather on the fact that they are necessary in the situation that students are currently facing. Since this is often mixed up with the notion of practicality in the familiar sense, caution should be exercised. Here, practicality refers to a condition in which something is operating and functioning on the spot, and it is asserted that students should learn mathematics that is actually useful in this sense. It does not mean simply mastering mathematics that students can utilize and apply. Furthermore, it does not mean that if students take a mathematical approach, they can discover relevant mathematics in their daily lives. Rather, it means to capture mathematics that exists in our daily lives, or on the contrary, to capture our daily lives that exist because of the existence of such a function. This is about learning mathematics that exists on top of people's experiences rather than mathematics that exists as an abstract idea in our minds. The objective here is totally different from the contents as in the case of criticism against unit learning, and it is also not restricted to the principle of teaching that humans first gain recognition and understanding by experiencing. It is an issue of how to look at mathematics itself as a subject of learning.

Through four phases of controversies, we can see the changes in views on mathematics education in Japan. In establishment era, it was important to introduce western mathematics to school. It meant they regarded mathematics as discipline. In controversies, the debates have been held on views on mathematics education such as how mathematics ought to be taught. At that time they became to regard mathematics as school subject. We can recognize the history of mathematics education in Japan as development of educational philosophy of school mathematics.

REFERENCES

Baba, T., Iwasaki, H., Ueda, A., & Date, F. (2012). Values in Japanese mathematics education: Their historical

development. *ZDM – The International Journal on Mathematics Education, 44*, 21–32.
Chuto Gakko Kyokasho Kabushikigaisha (1943). *Sugaku Hensan Shuisho (Mathematics Editorial Prospectus)*. Tokyo: Chuto Gakko Kyokasho Kabushikigaisha. (in Japanese).
Fujisawa, R. (1895). *Sanjutsu Jomoku oyobi Kyojuho (Arithmetic Rules and Teaching Methods)*. Tokyo: Dainippon Tosho Publishing. (in Japanese).
Fujisawa, R. (1898). *Sanjutsu Sho Kyokasho (Arithmetic Textbook)*. Tokyo: Dainippon Tosho Publishing. (in Japanese).
Hayashi, T. (1924). Kaikai no ji (Opening Speech), *Nippon Chuto Kyoiku Sugakukai-shi (Journal of the Secondary Education Mathematical Society of Japan), 6*(4, 5), 177-181. (in Japanese).
Knilling, R. (author), & Sasaki, K. (translation and commentary) (1906). *Sanjutsu Kyoujuho Shinzui (Essence of Teaching methods for Arithmetic (Die Naturgemasse Methode des Rechen-Unterrichts)*. Tokyo: Dobunkan (in Japanese).
Kota, O. (2012). Teaching and learning of functions in modern Japan. *HPM 2012 The Satellite Meeting of ICME-12 Proceeding Book 2*, 659-670.
Kunieda, M. (1924). Sugaku Kyoiku Zakkan (Miscellaneous Thoughts on Mathematic Education). *Nippon Chuto Kyoiku Sugakukai-shi (Journal of the Secondary Education Mathematical Society of Japan), 6*(4, 5), 131-150. (in Japanese).
Ministry of Education (1942). Chugakko Kotojogakko Sugaku oyobi Rika Kyoju Yomoku Kaisetsu Yomo (Commentary for Syllabi of Mathematics and Science in Secondary Schools). *Monbu Jiho (Bulletin of Ministry of Education), 759*(2), 1-25. (in Japanese).
Ministry of Education (1949). *Chugakusei no Sugaku (Mathematics for Lower Secondary School Students)*, Tokyo, Chuto Gakko Kyokasho Kabushikigaisha. (in Japanese).
Ministry of Education (1951). *Chugakko Kotogakko Gakushu Shido Yoryo Sugaku-hen (Shian) (Course of Study for Lower and Upper Secondary Schools: Mathematics (Tentative version))*. Tokyo: Chubu Tosho Publishing. (in Japanese).
Ministry of Education (1958). *Chugakko Shido Yoryo (Course of Study for Lower Secondary Schools)*. Tokyo: Printing Bureau, Ministry of Finance. (in Japanese).
Ogura, K. (1924). *Sugakukyoiku no Konpon Mondai (Fundamental Problems of Mathematics Education)*. Tokyo: Idea Shoin. (in Japanese).
Osada, A. (1922). Keishikitoya ni Kansuru Saikin no Ronso (Recent Controversies over Formal Building). *Nippon Chuto Kyoiku Sugakukai-shi (Journal of the Secondary Education Mathematical Society of Japan), 5*(2), 60-74. (in Japanese).
Ozeki, S. (1880). *Sugaku Sanzendai (Three Thousand Math Problems)*. Tokyo: Narumido Publishing. (in Japanese).
Secondary School Textbook Co., Ltd. (1943). *Sugaku Hensan Shui Sho (Compilation Prospectus for Mathematics)*. Tokyo: Secondary School Textbook Co., Ltd. (in Japanese).
Terao, H. (1888). *Chuto Kyoiku Sanjutsu Kyokasho (Secondary Arithmetic Textbook)*. Tokyo: Keigyosha. (in Japanese).
Ueno, K. (2012). Mathematics teaching before and after the Meiji Restoration. *ZDM – The International Journal on Mathematics Education, 44*(4), 473-481.

THE COGNITIVE VALUE OF HISTORICAL CASE STUDIES FOR TEACHING AND LEARNING

The case of leibniz's text *de quadratura arithmetica* (1675-1676)

Gustavo MORALES, Matias SARACHO, Erika ORTIZ
National University of Cordoba, Cordoba, Argentina
gust.914@gmail.com
matias.m00@gmail.com
erikarortiz@gmail.com

With the aim to provide students with deeper understanding of the processes involved in doing mathematics, some scholars (Pengelley 2011) suggest to ground teaching activities in the of use of primary historical sources as it is common practice in the humanities. However, given the great variety of primary historical sources the selected texts need to be scrutinized in the context of the corresponding mathematical cultures where they originated. Moreover, the required contextualization of primary sources to be used in mathematics education brings to light further issues such as the question about the way in which the research mathematician organizes the presentation of his results at a given moment in history within the standards of the community of experts, a concern which is also at the center of current debates in the philosophy of mathematical practice. According to the standard view, western cultures of mathematics ever since Euclid have been setting the emphasis on axiomatic-deductive models for organizing the presentation of research results. This idea crystallizes in the modern axiomatic conception of rigor which requires that only those aspects concerning the deductive structure underlying the validity of results ought to be made explicit in textbook presentations. Accordingly, the specificity of problem solving activities underlying other ways of articulating results have been neglected with a view to attaining higher levels of abstraction required by systematic theory construction. From a historical perspective, however, some mathematical texts composed by leading mathematicians bring to light a different way of articulating results. In our presentation we shall consider a different style of textbook presentation in mathematics, a tradition which for many scholars started only with 17th century mathematical analysis. In this tradition the mathematician aims to make explicit some of his most fundamental cognitive strategies as well as the design of innovative working tools and procedures for problem-solving**.** This is the case with Leibniz's *Quadrature Aritmétique du Cercle, de l'Ellipse et de l'Hyperbole* (from now on, DQA), a text composed towards the end of his mathematical studies in Paris (1672-1676), in which Leibniz aims to provide a general method to solve quadrature problems for curvilinear figures at the same time that he analyzes some fundamental notions concerning the origin of infinitesimal calculus. The style of presentation proposed by Leibniz (DQA) organizes results following the specific cognitive strategies he conceived for his research so that the study of this text ought to provide the reader with interesting details about the newly designed working tools and the pattern of reasoning the author followed. We find this case study particularly interesting as Leibniz intends to engage the reader with the complexity of his reasoning, even if the organization of results and the arguments displayed may be not evident at all for the modern student.

In this paper we aim to provide a study of the style of argumentation present in DQA that sheds light upon the methodological and epistemological aspects underlying the presentation of results with a view to motivate its inclusion in the classroom. With this in mind, we follow current suggestions by Grosholz (2007) and Chemla (2003) concerning different styles of argumentation in mathematical texts to be found throughout history which appear to be guided by two different "ideals" or "epistemic preferences". The first one points to a preference for "abstraction" which guides the "axiomatic exploration of truths" leading to axiomatic presentation, the second one is best described as the search for "generality" of algorithms and procedures on the basis of "paradigmatic problems" finding expression in more historical presentations. We argue that Leibniz organized his text following the "ideal of generality" showing that DQA is designed so as to provide a general method to solve quadrature problems for curvilinear figures where the results obtained are organized around two fundamental problems, the squaring of the circle and the hyperbola. The intermediary reasoning steps involved exhibit a procedure that aims to precise determination of the area of the circle and the hyperbola, at the same time that it indicates how the procedure might be extended to a new family of problems, the squaring of "transcendental curves". Leibniz displays the reasons why the results presented are correct but also shows the way successful procedures can be extended. We discuss the significance of Leibniz´s text presentation for teaching and learning, some of its difficulties as well as requirements to contribute to make the selected primary source accessible to the student.

REFERENCES

Chemla, K. (2003). Generality above abstraction: The general expressed in terms of paradigmatic in Ancient China. *Science in Context,16*(3), 413-458.

Grosholz, E. (2007). *Representation and productive ambiguity in mathematics and the sciences*. New York: Oxford University Press.

Leibniz, G. W. (2004). *De Quadrature arithmétique du cercle, de l'ellipse et de l'hyperbole et la trigonométrie sans tables trigonométriques qui en est le corollaire*. French trans. with commentary, by M. Parmentier. Paris: Vrin.

Pengelley, D. (2011). Teaching with primary historical sources: Should it go mainstream? Can it? In V. Katz, & C. Tzanakis (Eds.), *Recent developments in introducing a historical dimension in mathematics education* (pp. 1-8). Washington, DC: Mathematical Association of America.

TOWARDS A CULTURALLY MEANINGFUL HISTORY OF CONCEPTS AND THE ORGANIZATION OF MATHEMATICS TEACHING ACTIVITY

Vanessa Dias MORETTI , Luis RADFORD
Federal University of São Paulo, Guarulhos, São Paulo, Brazil
vanessa.moretti@unifesp.br
Université Laurentienne, Sudbury, Ontario, Canada
Lradford@laurentian.ca

ABSTRACT

The research aims to investigate possible implications that the relationship between the phylogenesis and the ontogenesis in the organization of the teaching of mathematics may have in teacher education. We seek to establish links between a pedagogical approach to mathematical concepts, the history of the concept, and its cultural signification, basing the relationship between human activity, social practice, and the history of concepts from a historical-cultural perspective. We suggest that the sense of a problem can achieve a new dimension by involving elements from the history of mathematics – as a support for the development of potentially triggering situations of learning, the concept of problem-solving classroom activity, and the symbolic systems of cultural signification. We argue that the history of mathematics allows the recognition of social practices related to the historical and cultural production of concepts. It also allows teachers to understand the limits of mathematical problems that can be formulated and the necessary mediation, in order to help students become aware of theoretical ways of thinking mathematically.

1 Introduction

In this research, we investigate possible implications that the relationship between the phylogenesis and the ontogenesis in the organization of the teaching of mathematics may have in teacher education. We seek to establish links between a pedagogical approach to mathematical concepts, the history of the concept, and its cultural signification. In order to do so, we draw on historical-cultural perspective (Vygotsky, 2002; Leontiev, 1983; Kopnin, 1978) and of the Cultural Theory of Objectification (Radford, 1997, 2006, 2013, 2014).

In this theoretical context, we understand that the learning process of teachers involves the encounter and grasping of concepts, the practices in which these concepts are subsumed, the values that the concepts convey, and the ways of acting and reflecting that encompass and endow with meaning the target concepts. The encounter and grasping of such concepts and their theoretical constellations produced historically and socially is what we term objectification. According to Vygotsky, the encounter of historical concepts takes place in human activity involving signs and tools, in a dialectical movement between inter- and intrapsychic processes (Vygotsky, 2002), and results in the production of sense that relates to a change of motive in the activity developed by the individuals, based on a certain need (Leontiev, 1983).

The concept of need, understood from a dialectic perspective, goes beyond the immediate relationship between individual, need, and objective. Need, from a dialectic perspective bears an ontological meaning that Fraser (1998), drawing on the works of Hegel and Marx, connects to ethical, social, and aesthetical dimensions. Human beings here are seeing as beings of need. It is in the ontological constitution of the individual to find the bases of her existence beyond herself: in nature, in society, and in others. The distinction between natural needs and socially created ones indicates the change in the way every person satisfies their needs (Fraser, 1998, p.125). The concept of human need in Marx relates, indeed, to the realization of the human essence mediated by consciousness (Fraser, 1998, p.143).

The relationship between the learning of teachers and the concomitant process of consciousness —i.e., the process of becoming conscious of cultural meanings (mathematical and others)— implies the transformation of sense and thus the transformation of the needs in the individual's teaching activity.

2 Human Activity, Social Practice, and History of Mathematics

From a historical and cultural perspective (e.g., Vygotsky, 1989, 2002; Leontiev, 1983, 2001; Moura, 2007; Radford, 1997, 2006, 2013, 2014), mathematical concepts are understood as human productions that aim at meeting the needs of individuals at a certain historical time and place.

One example of the relationship between the production of mathematical knowledge and their corresponding human activities and social practices can be found in the study developed by Høyrup (1994) on the history of measure, number, and weight in the cultures of Mesopotamia and Greece. By overcoming a platonic understanding of mathematics, Høyrup shows how cultural institutions mediate the influence of general sociocultural forces on individuals at the same time those individuals also contribute to modelling the interaction with the sociocultural forces. In order to make explicit such mediation (even if the latter is not recognized as such, that is, in its historical-dialectic context), Høyrup identifies the sense of the work of the scribes in their social and historical context. More specifically, Høyrup shows that, in spite of the demand concerning the immediate needs of everyday life, the scribes' motivation to solve problems went through a social *recognition* and the *professional identity* of that activity, so that "scribal practice transposed from the region of practical necessity into that of virtuosity" (Høyrup, 1994, p. 66), which was only possible in a certain society that valued and encouraged it – and, therefore, in an imbricated way or subsumed to social systems of the production of sense.

Another aspect explored by Høyrup is the constitution of mathematics as an entity and a field of knowledge as the "point where preexistent and previously independent mathematical practices are coordinated through a minimum of at least intuitively grasped understanding of formal relations" (Høyrup, 1994, p.67-68).

Such understanding of the relationship between social practices and abstract mathematical knowledge has also been undertaken by Kopnin (1978) on the historical and logical aspects of concepts. The concept of history, as Kopnin (1978) adopts it, as it differentiates from a positivist view on history, meets the concept proposed by the historical-cultural perspective. According to Marx and Engels (1976),

> The first historical act is thus the production of the means to satisfy these needs, the production of material life itself. And indeed this is an historical act, a fundamental condition of all history, which today, as thousands of years ago, must daily and hourly be fulfilled merely in order to sustain human life. […] The second point is that the satisfaction of the first need (the action of satisfying, and the instrument of satisfaction which has been acquired) leads to new needs; and this production of new needs is the first historical act. (Marx and Engels, 1978, p.48)

The concept of history is understood as an ontological category —which constitutes the human— directly connected to the way individuals produce their life and their existence through the production of new needs that overcome the natural needs. Those new needs, intrinsically human, are social, cultural, and historical.

Immersed in this dialectic comprehension of history, Kopnin (1978) contends that the historical movement of the production of concepts, particularly the mathematical ones, as it was developed by humans in sensuous and material activity, constitutes *the logical aspect of the concep*t.

Within this context, the production of mathematical ideas is understood in unity with its signification manifested in social practices in a culturally specific environment. The further production and refinement of mathematical ideas (e.g., the concept of number or the concept of geometric figure) constitute their phylogenesis, that is their historical development *in the dialectic sense of history* mentioned above. Their ontogenesis is the development of these ideas in the course of the individuals' life. But ontogenesis is not the mere repetition of the historical path of the concepts (Furinghetti and Radford, 2008; Radford and Puig, 2007). It is only, as Vygotsky (1989) noted, in the organic realm that such a repetition may take place. In organic development, phylogeny is repeated in ontogeny. In cultural development

> there is a real interaction between phylogeny and ontogeny: man [sic] is not necessary as a biotype: for the human fetus or embryo to develop in the mother's uterus, it is not necessary for it to interact with a mature biotype. In cultural development, this interaction is the principal driving force of all development (adult and child arithmetic, speech, etc.).

It is hence in the unity between phylogenesis and ontogenesis that we find the driving force of cultural development; the former "revives" the latter in the logical sense of the concept, as proposed by Kopnin (1978). From this perspective, the concept is understood in the unity between logic and its (previous, but also new and creative) use in human activity. That is to say, the concept is the unit between phylogenesis and ontogenesis.

Drawing on these key ideas of dialectic materialism, we have already stressed the importance of the interaction between sociocultural history and the ontogenetic development of culture and its individuals. As pointed out in a previous article "to understand conceptual developments we need to place the cognizer and the whole mathematical activity under study within his or her cultural conception of mathematics and of science in general". (Radford, 1997, p.28). From this viewpoint, it is necessary to recognize the importance of studying concepts in their process of production, along with the cultural significations intrinsic to the culture in which they are inserted, since, ontogenetically, human thinking is subsumed into a cultural reality (Radford, 1997, 2006, 2013, 2014).

The concept of objectification (Radford, 2002) is an attempt to try to understanding knowledge as a cultural objective entity and its relation to individuals as they encounter such a knowledge and try to grasp and to make sense of it.

> Objectification is precisely this social process of progressively becoming aware of the Homeric *eidos*, that is, of something in front of us—a figure, a form—something whose generality we gradually take note of and at the same time endow with meaning. It is this act of noticing that unveils itself through counting and signalling gestures. It is the noticing of something that reveals itself in the emerging intention projected onto the sign or in the kinaesthetic movement which mediates the artefact in the course of practical sensory activity, something liable to become a reproducible action whose meaning points toward this fixed eidetic pattern of actions incrusted in the culture which is the object itself. (Radford, 2007, p. 1791)

3 Implications for Teaching and Learning Activity

The dialectic relationship between phylogenesis and ontogenesis, as it is manifested in the relationship between human activity and the socially and historically constituted mathematical knowledge, indicates the potential of the history of mathematics as a support for the teaching organization that aims at developing the theoretical thinking of students —within everyday life problems, but also *beyond* the focus on everyday life situations. Thus, we propose that the sense of mathematical school problems take a dimension that associates elements from the history of mathematics (its historicity in the dialectic understanding of the term), the concept of classroom activity as that which ensures the dialectic unity of phylogenesis and ontogenesis, and what we term *symbolic systems of cultural significations* (Figure 1). Symbolic systems of cultural significations refer to a supra-symbolic dynamic structure where we find cultural conceptions about mathematical objects, their nature, the social standards of meaning production, the manner in which mathematical investigations are supposed to occur, etc. Symbolic systems of cultural significations organize, at a symbolic level, classroom teaching and learning activity, in particular through the modes of knowledge production and the forms of human collaboration that are nurtured in the classroom.

> [...] is through social practice that [men] produce their ideas (mathematical or otherwise), it is clear that social practice does not operate autonomously by itself: the social practice is steeped in symbolic systems which organize it in different ways. These symbolic systems we call semiotic systems of cultural significance. (Radford, 2014, p.10).

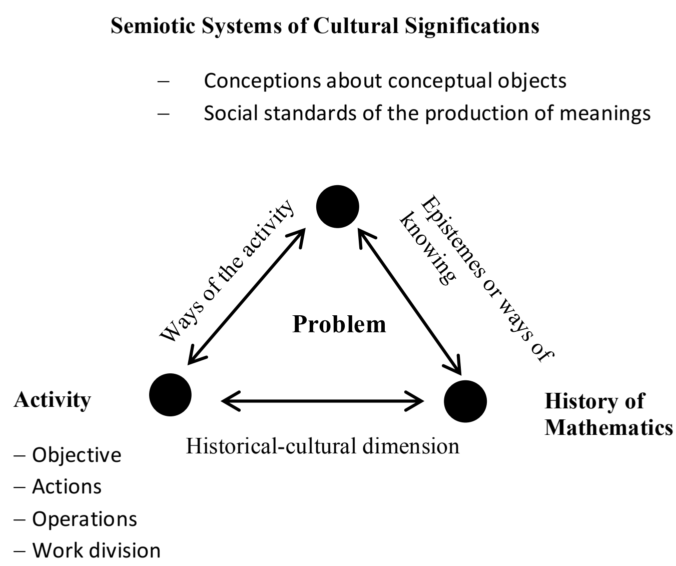


Figure 1. Dimensions of the Problem: History of Mathematics, Activity, and Semiotic Systems of Cultural Significations (adapted from Radford, 2006, p. 109).

Let us turn to an example reported in Janßen and Radford (2015). The example deals with linear equations in the classroom and the manner in which two teachers position themselves vis-à-vis mathematical knowledge and students. Two episodes are discussed. In the first episode the first teacher draws on a mode of knowledge production that can be characterized as *subjectivist*. That is to say, the students are supposed to produce their own knowledge by engaging with the equation that the teacher has chosen for them. The equation is presented in a kind of abstract form, through concrete materials: boxes and matches (see Figure 2). There were 5 boxes and 4 matches on the left side of the equation and 2 boxes and 19 matches on the right side. In symbolic notations, the equation would be translated as $5x + 4 = 2x + 19$.

Figure 2. The teacher (in the middle) discusses with two students an equation expressed through matches an boxes (From Janßen and Radford (2015).

The recourse to concrete material is intended as a means to simplify the epistemological density of the target knowledge. The rules of simplification of the equation, that is the rules of al-gabr and al-muquabala of Al-Khwarizmi (for a discussion of these rules, see Radford, 1993), should appear in their simplicity through the conceptual transparency of the matches and the boxes to the Grade 8 (13–14-year-old) students. As the classroom episode reveals, this is not the case. Yet, the mature biotype (to use Vygotsky's term), that is to say the teacher, is constrained by the very forms of knowledge production she draws on and that lead her to refrain herself from fully interacting with the students. The teacher undergoes a painful process in the course of which she suggests some guilty hints. She manages to utter:

> Well what can one change here for example, so that it stays (*briefly holds both her hands above the two tables*) the same. (*multiply taps the tables with all of her fingers—see Fig. 2, right image*) that must always stay the same that is very important.

The teacher remains imprisoned within the confines of some "gestures, (un)allowable hints, and the unsayable [mathematical] matter" (i.e., that which would be *improper* to mention by the teacher) (Janßen and Radford, 2015). The unit of phylogenesis and ontogenesis is not achieved. The dialectic interaction between phylogenesis and ontogenesis does not happen. The production of the concept was not possible. It took indeed a daring utterance by the teacher to move things a bit forward: against her visible beliefs, talking to the students, she uttered "take away something."

In the second episode reported in the study, the second teacher resorts also to concrete material. But this time, the problem is formulated as a story and the teacher fully interacts with the students, Here is the story problem:

> Sylvain and Chantal have some hockey cards. Chantal has 3 cards and Sylvain has 2 cards. Her mother puts some cards in three envelopes making sure to put the same number of hockey cards in each envelope. She gives 1 envelope to Chantal and 2 to Sylvain. Now, both children have the same amount of hockey cards. How many hockey cards are in an envelope?

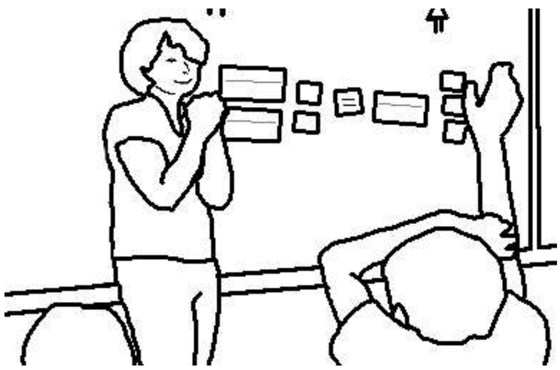


Figure 3. The story problem is expressed through envelopes and hockey cards.

The story problem is translated by the teacher in front of the Grade 2 class (7–8-year-old students) and expressed as an equation made up of hockey cards and envelopes (see Figure 3).

In alphanumeric symbols, the equation would be $2x + 2 = x + 3$ (a card with the equals sign on it divides the "two sides" of the equation).

The teacher asks for ideas and engages with the ideas that the students offer. She follows the still not fully linguistically articulated actions of Cheb and Cheb's pointing gestures, by moving the concrete envelopes and cards on the blackboard. The reported dialogue (Janßen and Radford, 2015) goes as follows:

1. Teacher: I'll go with the isolating strategy, Ok? Cheb? (*see Fig. 2*)
2. Cheb: Umm… you remove one of Sylvain's envelopes and one of (*the teacher has already put the hand on the envelope, yet she stops to wait for the next part of C's utterance, turning her head towards C)* Chantal's envelopes
3. T: Is it important to remove the same thing from each side of the equal [sign]? (*she makes a two-hand gesture around the equal sign moving the hands to the bottom of the blackboard, where envelopes and cards have been moved, to indicate that removing action is happening in both sides of the equality*)
4. C: Yes. And you can remove the other envelope… Oh non! One of Sylvain's cards and one card from Chantal's *(the teacher removes one card from Chantal's, see Fig. 3, left image*).
5. T: Aw! Again, one envelope, we remove one envelope (*see Fig. 3, centre image, where the teacher points to the removed envelopes*), one card, [and] one card (*see Fig. 3, right image, where the teacher touches the two removed cards*) …
6. C: You remove one of Sylvain's cards and you remove one of Chantal's cards (*the teacher moves the cards towards the top of the blackboard*)
7. T: We remove another card of Chantal's cards. Then, that gives us…
8. C: The answer!

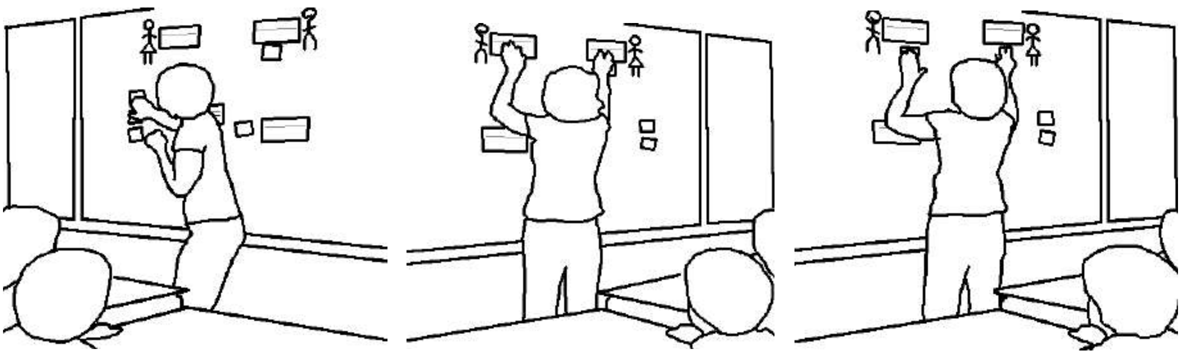


Figure 3. The teacher follows the still not fully linguistically articulated actions of Cheb.

The concrete material is not enough to ensure the encounter of the students with a historically constituted algebraic knowledge. The epistemological density of algebraic knowledge cannot be made transparent by the use of artefacts. A meaningful and challenging situation for the students has to be envisioned. Furthermore, the full participation of the teacher is required. The mature biotype has to participate with the students in order to bring to consciousness the intricate cultural mathematical meanings that underpin algebra and to ensure the unit of phylogenesis and ontogenesis.

The teachers in the short episodes discussed above draw on different semiotic systems of cultural significations. The first teacher draws on learning as an individual and subjective endeavour. The second teacher draws on learning as a social and collective endeavour. They promote different modes of classroom knowledge production and different forms of human cooperation. Needs appear differently. In the first case, need is subjective. In the second case, need is collective. Need is the collective phenomenon driven by the desire to get the problem solved *together*.

4 Concluding Remarks

In this article we have suggested that a meaningfully cultural history of mathematical concepts in mathematics education includes a critical stance towards history. History is not the mere succession of events (Radford, 2016). A meaningfully cultural history of mathematical concepts in mathematics education also includes the recognition of the importance of taking into account the production of mathematical ideas in unity with its signification manifested in social practices in a culturally specific environment—both at the phylogenetic and ontogenetic levels. Within this context, studying the history of mathematics should allow the recognition of the social practices related to the historical and cultural production of concepts as well as the educator's recognition of the limits and the qualitative changes of those practices—which can indicate a theoretical thinking about the practice, without which the production of the concept would not be possible. In the classroom examples presented in this paper, studying history for educational purposes could involve a discussion of the distinction between arithmetic and algebraic methods to solve linear equations. Such a discussion could involve historical problems discussed with teachers to enhance their content knowledge. Knowing the history of mathematics "gives us an idea of the epistemological density of knowledge" and allows us to understand that, for every kind of knowledge, "there is always a possibility already built to think about it", which does not mean repeating it (Radford, in Moretti, Panossian, and Moura, 2015, p. 254).

Such knowledge allows the elaboration of situations that trigger learning and potentially move the students towards a collective need for the concept, as they demand a theoretical thinking on the practice and the recognition of certain historically and culturally signified ways of knowing. Such need is not necessarily related to real historical problems and can emerge from different types of problem situations, such as "a game, a contextualized problem, or even a problem of logical compatibility within mathematics itself" (Moretti and Moura, 2011, p. 443). Needs, as they arise in the classroom, may not be directly related to the real historical problems. Yet, they are deeply entangled with the desires that motivate activity. As Leont'ev noted, "Behind the object [of activity], there always stands a need or a desire, to which [the activity] always answers" (Leont'ev, 1974, p. 22).

Another aspect related to the contribution of the history of mathematics for the teaching organization, from a historical-cultural perspective, concerns the teacher's recognition of a historical and epistemological perspective of knowledge, without which

[...] we risk not understanding the difficulties that many students may undergo as they meet those condensed ways of reflecting and acting, and we also miss chances to generate sophisticated designs for the activities we wish to bring to the classroom. (Interview with Luis Radford in Moretti, Panossian, and Moura, 2015, p.254)

Finally, focusing on the training of the mathematics educator, we understand that there is no formulation of mathematical problems that can bring out a certain concept or some knowledge *by itself*. The proposition of problems based on the history of mathematics, as we see it, can only be a learning trigger through a *joint work* with the teacher. In this sense, the history of mathematics is clarifying, since it allows the teacher to understand the boundaries of the mathematical problems that can be formulated, as well as the mediation necessary for the student to become creatively aware of theoretical ways of thinking mathematically.

Acknowledgments

Vanessa Moretti expresses her gratitude to FAPESP, Brazil, for the financing granted to Visiting Researcher Professor and to Laurentian University, Canada, where the research was conducted under the supervision of Luis Radford.

REFERENCES

Fraser, I. (1998). *Hegel and Marx: The concept of need*. Edinburgh University Press.

Furinghetti, F., & Radford, L. (2008). Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics. In L. English (Ed.), *Handbook of international research in mathematics education (2nd edition)* (pp. 626 - 655). New York: Taylor and Francis.

Høyrup, J. (1994). *In measure, number, and weight: Studies in mathematics and culture*. SUNY Press.

Janßen, T., & Radford, L. (2015). Solving equations: Gestures, (un)allowable hints, and the unsayable matter. In *Proceedings of the ninth congress of the european society for research in mathematics education (CERME)* (pp. 419-425). Prague: Charles University.

Kopnin, P. V. (1978). *A dialética como lógica e teoria do conhecimento* [Dialectic as logic and theory of knowledge]. Rio de Janeiro: Editora Civilização Brasileira.

Leont'ev, A. N. (1974). The problem of activity in psychology. *Soviet Psychology*, *13*(2), 4-33.

Leontiev, A. N. (1983). *Actividad, Conciencia, Personalidad*. Ciudad de La Habana: Editorial Pueblo Y Educación.

Leontiev, A. N. (2009). The Theory of the Development of the Child's. In A. N. Leontev, *The Development of Mind: Selected Works*. Marxists Internet Archive.

Marx, K., & Engels, F. (1978). *The German Ideology*. New York: International Publishers.

Moretti, V. D., & Moura, M. O. de (2011). Professores de Matemática em Atividade de Ensino: Contribuições da Perspectiva Histórico-Cultural para a Formação Docente [The teaching activity of Mathematics teachers: contributions of the cultural-historical perspective for teachers training]. *Ciência & Educação*, *17*(2), 435-450.

Moretti, V. D., Panossian, M. L., & Moura, M. O. de (2015). Educação, educação matemática e teoria cultural da objetivação: Uma conversa com Luis Radford [Education, mathematics education and the cultural theory of objectification: A conversation with Luis Radford]. *Educação e Pesquisa*, *41*(1), 243-260.

Moura, M. O. de (2007). Matemática na infância [Mathematics in childhood]. In M. R. Migueis, & M. G. Azevedo (Eds.), *Educação Matemática na infância: Abordagens e desafios* [Mathematics education in childhood: Approaches and challenges] (pp. 39-64). Vila Nova de Gaia: Gailivro.

Piaget, J., & Garcia, R. (1987). *Psicogénese e história das ciências*. Lisboa: Dom Quixote.

Radford, L. (1993). L'émergence et le développement conceptuel de l'algèbre [The emergence and conceptual development of algebra]. In E. Barbin (Ed.), *Proceedings of the first european summer university* (pp. 69-83). Montpellier: IREM de Montpellier.

Radford, L. (1997). On Psychology, Historical Epistemology and the Teaching of Mathematics: Towards a Socio-Cultural History of Mathematics. *Forthe Learning of Mathematics*, *17*(1), 26-33.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, *22*(2), 14-23.

Radford, L. (2006). Elementos de una teoría cultural de la objetivación. *Revista Latinoamericana de Investigación en Matemática Educativa*, *Special Issue on Semiotics, Culture and Mathematical Thinking*, 103-129.

Radford, L. (2007). Towards a cultural theory of learning. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the fifth congress of the european society for research in mathematics education (CERME – 5)* (pp. 1782-1797). Cyprus: Larnaca.

Radford, L. (2013). Three key concepts of the theory of objectification: Knowledge, knowing, and learning. *Journal of Research in Mathematics Education*, *2*(1), 7-44.

Radford, L. (2014). Cultura e historia: Dos conceptops dificiles y controversiales en aproximaciones contemporaneas en la educación matemática [Culture and history: Two difficult and controversial concepts in current approaches to mathematics education]. In I. A. Mendes, & C. A. Farias (Eds.), *Cultura, Práticas Sociais e Educação Matemática* (pp. 49-68). São Paulo: Livraria da Física.

Radford, L. (2016). Epistemology as a research category in mathematics teaching and learnig. In B. Hodgson, A. Kuzniak, & J. Lagrange (Eds.), *The didactics of mathematics: Approaches and issues* (pp. 31-41). Switzerland: Springer.

Radford, L., & Puig, L. (2007). Syntax and meaning as sensuous, visual, historical forms of algebraic thinking. *Educational Studies in Mathematics*, *66*, 145-164.

Vygotsky, L. S. (1989) Concrete human psychology. *Soviet psychology*, *27*(2), 53-77.

Vygotsky, L. S. (2002). *A formação social da mente* [The social formation of mind]. 6 ed. São Paulo: Martins Fontes.

ON THE RELATIONS BETWEEN GEOMETRY AND ALGEBRA IN GESTRINIUS' EDITION OF EUCLID'S *ELEMENTS*

Johanna PEJLARE, Staffan RODHE
Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE-412 96 Gothenburg, Sweden
johanna.pejlare@gu.se
Department of Mathematics, Uppsala University, SE-751 05 Uppsala, Sweden
staffan.rodhe@math.uu.se

ABSTRACT

In 1637 the Swedish mathematician Martinus Erici Gestrinius contributed with a commented edition of Euclid's *Elements*. In this article we analyse the relationship between geometry and algebra in Gestrinius' *Elements*, as presented in Book II. Of particular interest are Propositions 4, 5, and 6, dealing with straight lines cut into equal and unequal parts, and the three kinds of quadratic equations Gestrinius associates with them. We argue that Gestrinius followed Clavius translation of the *Elements*, but was influenced by Ramus to include algebra.

1 Introduction

Martinus Erici Gestrinius (1594–1648) was a Swedish mathematician and became a professor of geometry (then named *professor Euclideus*) at Uppsala University in 1620. He was born in Gävle as the son of the parson and in 1611 he became the student of Claudius Opsopæus, the professor of Hebrew at Uppsala. As a student Gestrinius repeatedly visited Germany and some of its universities, which made him familiar with German mathematics. In 1614 he travelled to Helmstedt and in 1615 to Wittenberg. In 1618 he received his university degree at Greifswald before he returned to Uppsala to start teaching at the university. In 1630, 1638, and 1643 he was the vice-chancellor of the university and in 1633 he was one of the university's delegates at the Parliament.

Uppsala University, founded in 1477, is the oldest university in the Nordic countries and was the only university in Sweden until King Gustav II Adolf in 1632 founded the university at Dorpat, which today is the University of Tartu in Estonia. The university had grown out of an ecclesiastical centre and had been charted through a papal bull by Pope Sixtus IV. During the turbulent times of the Reformation in the 16th century, however, there was very little activity at the university. In 1593, at *The meeting of Uppsala* – the most important synod of the Lutheran Church of Sweden – Lutheran Orthodoxy was established in Sweden, and the Duke Charles (later King Charles IX) gave new privileges to the university, which reopened in 1595.

It is known that Gestrinius lectured at Uppsala University on the significance of arithmetic for the bourgeois life, algebra, astronomical calculations, and geometry. He seems to be the first in Sweden to use the symbols + and − and he was the first Swede to use square roots. One of his greatest mathematical achievements was that he introduced algebra into Swedish mathematics. In 1637 he contributed with the textbook *In Geometriam Euclidis Demostrationum Libri Sex* – a commented edition of the first six books of Euclid's *Elements*. This is the first edition of Euclid's *Elements* published in Sweden, and it was used as a textbook at the university at least until the beginning of the 18th century. Gestrinius probably had knowledge of Campanus', Clavius' and Peletier's editions of the *Elements*. However, contrary to these three interpreters of Euclid´s *Elements*, Gestrinius included algebra in Propositions 4, 5 and 6 of Book II in his edition of the *Elements*.

In this paper we will consider the relations between geometry and algebra in Gestrinius' *Elements*. In order to do this we will in detail consider the connections between Propositions 4, 5, and 6 of Book II and the three kinds of quadratic equations Gestrinius associates to them, as well as the different ways in which Gestrinius solves the equations. We will also investigate who influenced Gestrinius' work. Even if Petrus Ramus is not mentioned in Gestrinius' edition of the *Elements*, as we shall see there are indications that he inspired Gestrinius: They use the same notation and they also classified the quadratic equations in the same way. However, an important difference is that Gestrinius explained the solutions of the equations geometrically, by utilizing diagrams.

2 Book II of Gestrinius' *Elements*

In the introductory text of Book II in Gestrinius' version of the *Elements*, he mentions the multiple uses of the contents of Book II – not only in geometry, but also for cossic algebra, geodesy and astronomy. He also mentions that he solves algebraic equations and uses surd numbers in certain propositions of Book II. Gestrinius' version of Book II contains 14 propositions divided into twelve theorems and two problems. He establishes all of the propositions in a traditional manner. He also exemplifies the propositions, using numbers.

In Gestrinius' edition of the *Elements*, Propositions 4, 5 and 6 of Book II are of particular interest (Gestrinius, 1637, pp. 117–129). The propositions consider straight lines cut into equal and unequal segments. Gestrinius presents traditional geometrical proofs of these propositions, as well as exemplifying them through the solution of the following quadratic equations associated with the proposition:

$$1q + 10l\ \text{æ}quat.119$$

$$1q + 20\ \text{æ}quat.12l$$

$$1q\ \text{æ}quat.12l + 64$$

Gestrinius describes the way to solve these equations in three different ways: rhetorically, with tables, and geometrically. We will consider them in detail below.

2.1 Proposition 4 of Book II

In Gestrinius' edition of the *Elements*, Proposition 5 of Book II is formulated in a traditional geometrical manner:

Proposition 4: If a straight line is cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

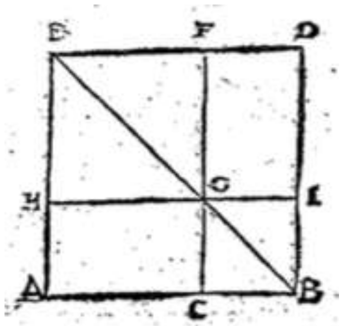


Figure 1. Geometric interpretation of Proposition 4

According to the proposition, the straight line AB is cut at the point C. The square on AB is equal to the sum of the squares on AC and CB together with the two rectangles with sides AC and CB.

Gestrinius establishes the proposition in a purely geometrical manner, similar to the traditional proof of Euclid. Thereafter he exemplifies the proposition with numbers, letting the length of the straight line be 12 and the segments 7 and 5. Gestrinius' conclusion, with modern notation, is that $12^2 = 7^2 + 5^2 + 2 \cdot 7 \cdot 5$.

Gestrinius now associates a quadratic equation of the first kind ("Æquatio Algebraica secunda primi generis") to the proposition: $1q + 10l$ $æquat.$ 119. He uses the symbol q for the unknown square (quadratum), l for the unknown side (latus) and the abbreviation *œquat.* (æqualitatem) for the equality. In this equation the sum of the "maximum" and the "medium" terms is equal to the "minimum". Gestrinius proceeds by verbally describing the solution of the equation:

> Half of the medium 10 is 5, its square is 25, composed with the minimum 119 is 144, is from its side 12 the half of the medium 5 is removed it is remained the value 7 of one side.

Gestrinius tests the received value 7: $10l$ gives the value 70 and $1q$ gives the value 49, the total equals 119. This verbal solution could be summarized with modern notation as follows:

$$\sqrt{(\frac{10}{2})^2 + 119} - \frac{10}{2}.$$

Now he summarizes the solution in a table (see Figure 2):

1q + 10l æquat. 119.

Dimid. medij	5.
□. dim. medij	25.
Minimus	119.
Composit.	144.
Latus compos.	12.
Dimid. med. subtr.	5
Resid. & val. 1l	7.

Figure 2. The table summarizing the solution of the equation associated with Proposition 4

The reason for including the table is probably to show how to find the solution to the equation through a more algorithmic-like process.

Gestrinius solves the equation one more time, this time using geometry, and referring to Figure 1. Translated into English, Gestrinius' text proceeds as follows:

> This equation is applied to the present proposition and the truth of the equation will be evident. To the square AD the gnomon is IEC. Now $1q + 10l$ equals the gnomon IEC; therefore it equals 119. Moreover $1q$, which is HF, is the square of the gnomon: wherefore $10l$ is equal to the two rectangles AG, GD and the plane $5l$ is equal to the rectangle whose length $1l$ is the side of the square HF already placed, and 5 is the side of the remaining square CI. Therefore, 25 is the remaining square CI, and added to the gnomon IEG, which is 119, it makes the square AD complete to 144. From the side 12 of the square AD the side 5 of the square CI is removed, remaining 7, which is the side of the square HF and the value of $1l$.

The gnomon IEC is easily found in the equation $1q + 10l = 119$, where $1q$ corresponds to the square HF of the gnomon and $10l$ corresponds to the two rectangles AG and GD. The crucial part of the proof is the squaring of the gnomon, which is done by finding the side 5 of the remaining square CI. After this has been done, Gestrinius easily finds the side of the square HF, i.e., the square root of $1q$.

2.2 Proposition 5 of Book II

In Gestrinius' edition of the *Elements*, Proposition 5 of Book II states:

> *Proposition 5*: If a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

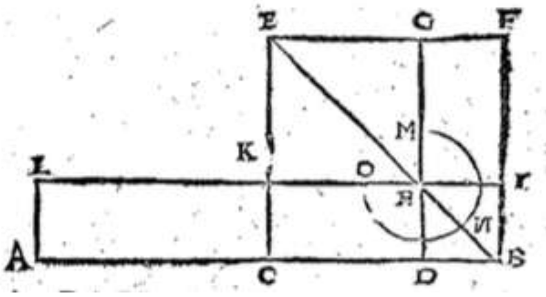


Figure 3. Geometric interpretation of Proposition 5

The straight line AB is cut into equal parts at C and in unequal parts at D. The sum of the rectangle AH (the rectangle contained by the unequal segments) and the square KG (the square on the straight line between the points of section) equals the square CF (the square on the half).

After establishing the proposition in a geometrical manner similar to the traditional proof of Euclid, Gestrinius exemplifies the proposition with numbers. He lets the length of the straight line AB be 12. It is divided in equal parts at C such that $AC = 6$ and $CB = 6$, and in unequal parts at D such that $AD = 10$ and $DB = 2$. The length of the straight line between the point of section is $CD = 4$. He concludes, translated into modern notation, that $6^2 = 10 \cdot 2 + 4^2$.

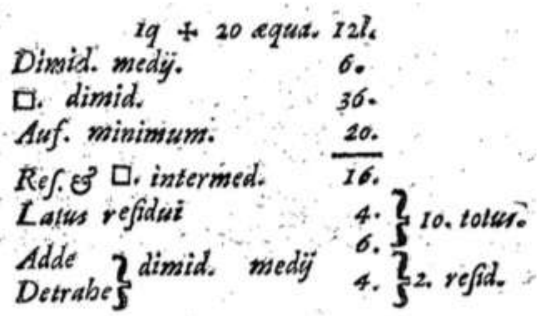
Gestrinius now associates a quadratic equation of the third kind ("Algebraica Æquatio secunda tertij generis") to the proposition: $1q + 20 = 12l$. In this equation the sum of the "maximum" and the "minimum" terms is equal to the "medium". Gestrinius proceeds by describing the solution of the equation verbally:

> Half of the medium is 6, its square is 36 from which the minimum 20 is subtracted, remaining 16, if its side 4 is added to half of the medium 6 it will make 10, if withdrawn it will make 2. Therefore $6 + 4$ and $6 - 4$, i. e., 10 and 2, will be the values of $1l$.

In modern notation this verbal solution could be summarized to:

$$\frac{12}{2} \pm \sqrt{(\frac{12}{2})^2 - 20}\,.$$

Just as in Proposition 4 Gestrinius now summarizes the solution in a table (see Figure 4):



1q + 20 æqua.	12l.
Dimid. medij.	6.
□. dimid.	36.
Auf. minimum.	20.
Ref. & □. intermed.	16.
Latus residui	4. } 10. totur.
	6.
Adde } dimid. medij	4. } 2. resid.
Detrahe	

Figure 4. The table summarizing the solution of the equation associated to Proposition 5

If an equation of the third kind has a positive root it always has one more. This makes it difficult for Gestrinius to give a geometrical solution that is well connected to the equation. Despite this fact, Gestrinius uses geometry to solve the equation one more time:

> The diagram makes this equation visible. For if AB 12 is divided into unequal segments AD 10 and DB 2, then AH 20 is the plane made up of the unequal segments AD and DB. And therefore if this plane is reduced from 36, the square of the half segment CB, i.e., from the square CF, there remains 16, the square KG of the intermediate segment CD, whose side 4 together with the half 6 gives 10, the longest segment AD, for the value of $1l$. The same subtracted from 6, leaves 2, the smallest segment DB, which in the same manner is the value of $1l$. And both operations depend however on this proposition by addition and subtraction. Since the plane of the whole and the longest segment, is equal to so many times the longest segment, as there are unities in the whole. And similar is the plane of the whole and the smallest segment equal to so many smallest segments, as there are unities in this. As in the same example the plane 120 out of 12 and the longest segment 10 is equal to $12l$, which contains 100, the square of the longest segment 10, and 20 the plane of the unequal segments. Thus the plane 24 out of the whole 12 and the smallest segment 2 is in a similar way equal to $12l$, which contains 4, the square of the smallest segment, and the plane 20 of the same unequal segments, as the demonstration of the operation shows. And therefore there will be a free choice to use sometimes addition and subtraction, sometimes just one of them, as it will be manifested by the algebraic problems.

To be able to tackle the problem, Gestrinius had to give the roots of the equation beforehand. One problem is that the diagram only corresponds to one of the solutions of the equation ($1l = DB$). Since Gestrinius did not use a minus sign he could also not easily find a gnomon that can be squared. For example, the gnomon MNO could be described with the expression $12l - 1q$. To square this gnomon, the "square on the straight line between the points of intersection", i.e. 16, has to be added. Then the square on the half is equal to $20 + 16 = 36$, i.e., the half is 6, which gives $6 - 4 = 2$ as a root. A similar diagram would give the root $6 + 4 = 10$.

2.1 Proposition 6 of Book II

In Gestrinius' edition of the *Elements*, Proposition 6 of Book II states:

> *Proposition 6*: If a straight line is cut into equal segments and a straight line is added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

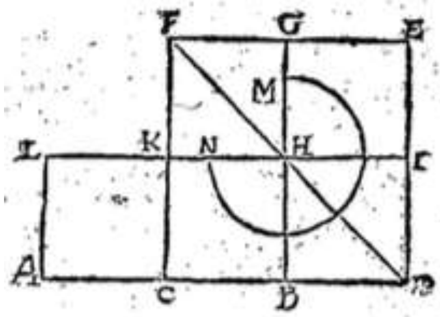


Figure 5. Geometrical interpretations of Proposition 6

According to the figure, AB is the straight line and it is bisected at C and the straight line BD is added. The proposition states that the rectangle AI (the rectangle contained by the whole with the added straight line and the added straight line) together with the square KG (the square on the half) equals the square CE (the square on the straight line made up of the half and the added straight line).

Gestrinius proves also this proposition in a traditional geometrical manner before exemplifying it with numbers. He lets the length of the straight line AB be 12. It is divided in equal parts at C such that $AC = CB = 6$. The added straight line is $BD = 4$. He concludes that since the whole and the added straight line equals 16 and the half together with the added straight line equals 10, it follows from the proposition that $10^2 = 16 \cdot 4 + 6^2$.

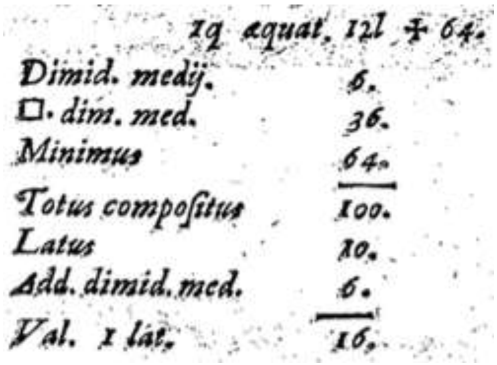
Gestrinius now associates a quadratic equation of the second kind ("Secunda Æquatio Algebraica secunda generis") to the proposition: $1q = 12l + 64$. In this equation the "maximum" term is equal to the sum of the "medium" and the "minimum" terms. Gestrinius proceeds by describing the solution of the equation verbally:

> Half of the medium is 6, its square is 36, add the minimum 64 which gives 100, whose side is 10. To this add half of the medium 6, the total will be 16 which is the value of $1l$.

He concludes that $12 \cdot 16 + 64 = 16^2$. In modern notation this solution could be summarized in:

$$\sqrt{(\frac{12}{2})^2 + 64} + \frac{12}{2}.$$

Just as in the other propositions Gestrinius now summarizes the solution in a table (see Figure 6):



	1q æquat. 12l + 64.
Dimid. medij.	6.
□. dim. med.	36.
Minimus	64.
Totus compositus	100.
Latus	10.
Add. dimid. med.	6.
Val. 1 lat.	16.

Figure 6. The table summarizing the solution of the equation associated to Proposition 6

Since an equation of the second kind only has one positive root, it is easier to illustrate the solution with the help of the diagram in Figure 5. Gestrinius uses geometry to solve the equation in this way:

> The proposed equation is applied in this way: The minimum is the plane made up by the composed and the added, and the square of the half of the middle form is the square of the half; and therefore the sum of these is equal to the square of the half and the added, whose side if the half is added is the side of the great square. As in the same example $1q$ æ$quat.12l + 64$. The plane of the composed and the added AD, DB, i.e., rectangle AI, is 64, added to the square of the middle CB, i.e., square KG, which is 36, gives 100. This number 100 is equal to the square CE, out of the half and the added CD. Its side is 10, together with 6, the half AC, gives the side of the great square.

Even though Gestrinius does not mention any gnomon to be squared, it is involved in the solution. The root $1l$ is equal to "the whole with the added straight line", i.e., AD. The rectangle AK is equal to the rectangle HE, so the gnomon composed by the rectangle CH, the square BI and the rectangle HE is equal to the rectangle AI. The square KG, which equals to 36, completes the square.

3 Influence on Gestrinius

In the introduction of his *Elements*, Gestrinius mentions four post-classic mathematicians: Johannes Kepler (1571–1630), Campanus of Novara (1220–1296), Jacques Peletier (1517–1582), and Christopher Clavius (1538–1612). Kepler did not publish his own edition of the *Elements*, but he is mentioned in connection to the treatment of the five regular polyhedra and his Platonic solid model of the solar system. However, Campanus, Peletier, and Clavius published versions of the *Elements.* We will now explain the connection between these three versions of the *Elements* and the connection to Gestrinius' edition, as well as other possible influences on Gestrinius. In particular, we argue that Gestrinius was influenced by Petrus Ramus.

3.1 Campanus, Peletier, Clavius, and Commandino

Campanus' Latin version of the *Elements,* based on Arabic translations, was completed between 1255 and 1259 (Corry, 2015). In 1482 it appeared as the first printed edition of the *Elements*, and thus became the standard source until the 16th century, when other editions based on direct translations from the Greek were published. Campanus' treatment of the *Elements* contains many modifications, additions and comments, as he was making efforts to present the text in an as self-contained form as possible. In Book II Campanus remained within a geometric context, but he also stated that the first 10 propositions of Book II were true for numbers as well as for lines.

Gestrinius mentions the two 16th century mathematicians Peletier and Clavius in connection with Proposition 16 in Book III and their debate regarding the angle of contact. Peletier's edition of the *Elements* was published in 1557 with the title *In Euclidis Elementa Geometrica demonstrationum, Libri sex*. Its title is very similar to Gestrinius', but Peletier's edition is a work on geometry and includes no algebra. Clavius' edition of the Elements was published in 1574 with the title *Euclidis Elementorum Libri XV. Accessit liber XVI*. Book II of Clavius' *Elements* includes many problems solved numerically. The wording of the propositions and the proofs of Book II of Clavius are very similar to Gestrinius', but Clavius did not include any algebra.

One of the most important Latin translations of the *Elements* is the 1572 version of Frederico Commandino (1509–1575) (Heath, 1956). Commandino followed the original Greek more closely than Clavius. Gestrinius mentions Commandino when he in Book V besides the traditional 25 propositions included 9 extra propositions in an *Appendix Ex Commandino*. However, Commandino in his *Elements* of 1572 only included 33 propositions of Book V, compared to the 34 included by Gestrinius, as well as by Campanus, Peletier and Clavius.

In 1632 Gestrinius lectured on geometry at Uppsala University. In the lecture notes we can see that the wording of the propositions is very similar to those of Clavius, indicating that Gestrinius followed Clavius. In the lecture notes Gestrinius also gave the same numerical examples, as he would later present in his *Elements*. However, algebraic applications are missing in the lecture notes. Even if Gestrinius followed Clavius' *Elements*, his idea of including algebra into Book II cannot come from Clavius. As we will argue below, there are indications that Gestrinius got this idea from Petrus Ramus.

3.2 Ramus

Petrus Ramus (Pierre de la Ramée, 1515–1572) is not mentioned in Gestrinius' *Elements*, but there are strong indications that Gestrinius knew of Ramus' work and was inspired by it. Ramus was the most important French algebraist before François Viète (1540–1603). As a mathematician Ramus is most famous for his ideas on the practical use of mathematics and the influence of his book on the education of mathematics. In his *Algebra* from 1560, he explains that "Algebra is a part of arithmetic" and he mentions that the two parts of algebra are "numeratio" ("arithmetical" handling of algebraic expressions) and "æquatio" (solving equations). In particular, Ramus showed how to treat quadratic equations. His notation was simple, with l for the unknown (latus=side) and q for the square of the unknown (quadratus=square). Instead of an equality sign he used the abbreviation "æqua." We recognise this notation from Gestrinius. Just as Gestrinius did, Ramus classified the quadratic equations into three types ("canon"), which he presented through the following examples (Ramus, 1560, pp. 13–19):

$$Canon\ primus: 1q + 8l \text{ æ}qua.\ 65$$

$$Canon\ secundus: 6l + 40 \text{ æ}qua.\ 1q$$

$$Canon\ tertius: 1q + 21 \text{ æ}qua.\ 10l$$

As we can see, Gestrinius and Ramus used the same notation, and also classified the quadratic equations in the same way. Just as Gestrinius, Ramus also illustrated the algebraic solutions of the equations with the help of the geometry in Propositions 4, 5, and 6 of Book II in the *Elements*, but he did it in an algebraic context, not using diagrams as Gestrinius did. Thus, Ramus used the three propositions to illustrate the three equations, that is, he used geometry to illustrate algebra. As we have seen, Gestrinius, on the other hand, used the equations to illustrate the propositions, that is, he used algebra to illustrate geometry.

In 1545 Ramus also published a version of Euclid's *Elements* (Heath, 1956). However, it only includes definitions and propositions, but no comments. Ramus also only included 25 propositions in Book V and not 34 as Gestrinius, as well as Campanus, Peletier and Clavius, did. This indicates that Ramus, in this sense, was not following the traditions of Campanus, Peletier and Clavius.

4 Concluding remarks

Gestrinius was the first Swedish mathematician to publish an edition of Euclid's *Elements* and to present algebra in a printed text. However, he was not the first mathematician to use algebra in a geometrical context. William Oughtred (1573–1660) was one of the first to exemplify theorems of classic geometry using algebra (Stedall, 2002). He demonstrated all of the 14 propositions of Book II of Euclid's *Elements* in his *Clavis mathematicæ* from 1631 with his analytical method, which means that he used algebra. He used Viète's algebraic notation, as presented in Viète's symbolic algebra, or the *Analytical Art.* Inspired by Diophantos' work during the end of the 16th century, Viète used capital letters instead of abbreviations as symbols for the unknown and known entities. This made it possible for him to present a general equation and to give a general method of solving it. Nevertheless, Gestrinius did not use Viète's symbols, and it is possible that he did not even know about them. Therefore it seems unlikely that he, at least in 1637, had knowledge of Oughtred's *Clavis*.

Many years before Oughtred's *Clavis*, Thomas Harriot (1560–1621) had written the propositions of Book II algebraically, but this text was never printed (Stedall, 2000). Also Harriot was influenced by Viète, but he used his own symbols, similar to Descartes'. Another important contribution was made by Pierre Hérigone (1580–1643) when he in 1634 published his *Cursus Mathematicus*, where he replaced the rhetorical language of Euclid's *Elements* with a symbolic language (Massa-Esteve, 2010). Hérigone's aim was to introduce a universal symbolic language for dealing with both pure and mixed mathematics.

We do not suggest that Gestrinius was an extraordinary mathematician even though he was the first professor of mathematics at Uppsala University who brought the subject to a more scientific study. Primarily he was an educator, and his importance lies in the fact that he transferred known mathematical theories to the following generations of Swedish mathematicians. His edition of the *Elements* was used at Uppsala University, as well as Clavius' edition of the *Elements*. For example, the Swedish mathematician and mathematics teacher Anders Gabriel Duhre (1680–1739 (possibly 1681–1739)), most well-known for his textbooks on algebra and geometry, most likely studied Gestrinius' version of the *Elements*. In his book on geometry, Duhre also connected geometry to algebra and proves the propositions of Book II of Euclid's *Elements* using algebra in Descartes' notation as well as in the notation of Wallis and Oughtred. Also Samuel Klingenstierna (1698–1765), the most well-known Swedish mathematician during the 18th century, learned Euclid by reading Gestrinius' edition of the *Elements*. Therefore, Gestrinius keeps an important position in the Swedish history of mathematics and mathematics education.

REFERENCES

Clavius, C. (1574). *Euclidis Elementorum Libri XV. Accessit liber XVI*. Romae.

Commandino, F. (1572). *Euclidis Elementorum Libri XV. Una cum scholijs antiquis.* Pisauri.

Corry, L. (2013). Geometry and arithmetic in the medieval traditions of Euclid's *Elements*: A view from Book II. *Archives for History of Exact Sciences, 67*, 637–705.

Gestrinius (1632). *Lecture notes by Olaus Stephani Helsingius Gnarpius from Gestrinius' geometric lectures at Uppsala University. Manuscript signum A 501*. Uppsala: University Library.

Gestrinius, M. E. (1637). *In geometriam Euclidis demonstrationum libri sex*. Uppsala.

Heath, T. (1956). *Euclid: The Thirteen Books of The Elements*, *Volume 1*. New York: Dover.

Massa-Esteve, M. R. (2010). The symbolic treatment of Euclid's *Elements* in Hérigone´s *Cursus mathematicus* (1634, 1637, 1647). In A. Heeffer, & M. Van Dyck (Eds.), *Philosophical Aspects of Symbolic Reasoning in Early Modern Mathematics*, Studies in Logic, *Vol. 26* (pp. 165–191). London, UK: College Publications.

Peletier, J. (1557). *In Euclidis Elementa Geometrica demonstrationum, Libri sex*. Lyon.

Ramus, P. (1560). *Algebra*. Paris.

Stedall, J. (2000). Ariadne's thread: The life and times of Oughtred's Clavis. *Annals of Science, 57*(1), 27-60.

Stedall, J. (2002). *A discourse concerning algebra – English Algebra to 1685*. Oxford University Press.

THE MATHEMATICS IN THE POLYTECHNIC ACADEMY OF PORTO

(Portugal, 1837-1911)

Hélder PINTO
CIDMA – University of Aveiro,
Departamento de Matemática, Campus Universitário de Santiago, Aveiro, Portugal
hbmpinto1981@gmail.com

ABSTRACT

The Polytechnic Academy of Porto (PAP, 1837-1911) was created in 1837, and replaced the Royal Academy of Navy and Trade Affairs of the City of Porto (RANTACP, 1803-1837). Its creation brought a new paradigm to higher education in Porto, because several engineering courses were then implemented – before that, the focus was on the formation of traders and navy sailors – and it was the first school in Portugal dedicated to engineering outside the military context (it should also be highlighted that Mathematics occupied a central role in its curricula). This work will be divided in three parts: The first part is a brief presentation of the RANTACP. In the second part, the Mathematics taught in the PAP will be presented, with special emphasis on three relevant moments of its history: its creation in 1837, the profound Reformation of 1885 which allowed its scientific apogee and its transition to the University of Porto in 1911. The third part will be a short presentation of the important Portuguese mathematician Gomes Teixeira (1851-1933) that arrived at PAP, as a professor, in 1884.

1 Introduction

The Polytechnic Academy of Porto (PAP, Academia Politécnica do Porto, 1837-1911) was created in 1837, and replaced the Royal Academy of Navy and Trade Affairs of the City of Porto (RANTACP, Academia Real de Marinha e Comércio da Cidade do Porto, 1803-1837). The creation of these two institutions was the beginning of superior teaching in the city of Porto. Both academies had an important curriculum in the field of mathematics and was the first time that the «higher» mathematics was taught outside the cities of Coimbra (where the single University (1290-) of the country at the time was located; note that the Faculty of Mathematics of this university was created in 1772) and Lisbon (with several academies within the military context, of which the most important was dedicated to the navy: the Royal Academy of Navy, Academia Real de Marinha (1779-1837)).

Both academies in Porto faced many difficulties during their implementation but, at the turn to the 20th century, the PAP was in its scientific apogee and in some aspects could be analogous to the University of Coimbra and to the Polytechnic School of Lisbon (which replaced the Royal Academy of Navy in 1837), the two most important schools in Portugal at that time. In fact, after the implementation of the Republic in Portugal in 1910, three similar faculties of sciences in the cities of Lisbon, Coimbra and Porto were created in 1911.

The most important scientist of PAP was, by far, the mathematician Gomes Teixeira. Note that Teixeira was the most important Portuguese mathematician of his time and the only Portuguese with some reputation among the international community. His arrival as a professor to PAP, as we will see, was one of the most decisive moments of the institution.

2 The Royal Academy of Navy and Trade Affairs of the City of Porto (1803-1837)

The official predecessor of PAP was the Royal Academy of Navy and Trade Affairs of the City of Porto, created in 1803. This institution was established to graduate and prepare navy pilots and traders. The need of the city of Porto to form good professionals in these two areas arises from the fact that, at the time, the trade (mainly in the area of wines and liquorish beverages) with Brazil and northern Europe was very intense and vital for the economy of the city and to the rest of the northern region of Portugal. The creation of RANTACP was based on two classes that had already existed at that time in the city: the Nautical Class (created in 1762) and the Drawing Class (1779). These two classes, as well as the RANTACP, were under the full responsibility of the General Company of Agriculture of the Vineyards of the Upper Douro (*Companhia Geral da Agricultura das Vinhas do Alto Douro*), except for the selection of professors, that only the King could do. Note that this private company had been created by the Marquis of Pombal in 1756, and had the monopoly of the wine trade in the Douro region (which included the famous Port wine), being controlled by influential personalities of the city. Note that it was the city itself that paid for these classes, by several taxes on wine and trade affairs, which was, at the time, a novelty in the Portuguese context. For the first time, a superior teaching institution was created without the financial support of the King or the state. We should also note that, in many situations like collecting taxes or constructing public roads, the Upper Douro Company substituted the state and managed many important aspects of the daily city life.

In addition to the two existing classes, the RANTACP added classes of Mathematics, Trade Affairs, English Language and French Language (Law of February 9, 1803). According to the Law Decree of July 29 of that same year, it was also decided to create a course of Rational Philosophy. What stands out of RANTACP statutes is the relevance of mathematics in the curriculum of this institution; this was evidenced by the existence of three mathematical years which were very similar to what was practiced in the Royal Academy of Navy from Lisbon:

- First year: Arithmetic, Geometry, Plane Trigonometry and their practical uses, and elementary principals of Algebra until the second degree equations;
- Second year: the continuation of Algebra and its applications to Geometry, Differential and Integral Calculus, followed by the fundamental principles of Statics, Dynamics, Hydrostatics, Hydraulics, and Optics;
- The third year: Spherical Trigonometry, and the Art of Navigation (theoretical and practical), followed by naval manoeuvre notions, and knowledge and practical uses of the astronomical and navy instruments.

Note that the second mathematical year, the most advanced one in terms of contents, was not mandatory for the pilots. The last year was very practical and should be complemented with reports about the navy voyages that the pilot students should perform to the northern Europe seaports.

The political and social context that accompanied the existence of this institution was very problematical[1], making its implementation very difficult and it never achieved the scientific level of its «similar» in Lisbon. In fact, the mathematical production of the professors that composed the RANTACP was not very extensive but it was very important to begin to break the mathematical «exclusivity» from Coimbra and Lisbon. With the RANTACP the (higher) education of mathematics in Porto began.

3 The Polytechnic Academy of Porto (1837-1911)

3.1 The creation of Polytechnic Academy of Porto

The PAP was created in January 13, 1837, by Passos Manuel (politician from Porto, who was at the time Minister of the Kingdom), replacing the ancient RANTACP. This new academy was created with substantially different objectives, focusing their studies on the several engineering courses then implemented:

> 155.º Article
> The Royal Academy of Navy and Trade Affairs of the City of Porto that now should be designated by – Polytechnic Academy of Porto –; with the special aim of teaching Industrial Sciences, and should graduate:
> 1. Civil Engineers of all classes, such as Mines Engineers, Constructors Engineers, Bridges and railroads Engineers;
> 2. Navy officers;
> 3. Sailors;
> 4. Trade business men;
> 5. Farmers;
> 6. Factory managers;
> 7. Artists. [2]

In fact, at that time, the socio-economic context in Porto had changed dramatically since Brazil had become independent in 1822 and the city began to attend a process of some industrialization. Note that, despite the changes introduced in 1837, the graduation of navy pilots and traders continued formally to be part of the studies. However, aspects related to navigation would lose relevance throughout the life of PAP, having completely disappeared, as we will see later, in the reformation of 1885. The trade affairs survived in the PAP curricula until the end of its life, but were always losing importance in comparison to engineering and other sciences such as mathematics, chemistry or natural history.

At the time of creation of PAP, eleven disciplines (divided in four sections) were introduced, namely:

---

[1] It should be highlighted the Napoleonic invasions (1807-1811) which occupied the city; the Portuguese Liberal Revolution of 1820 and the civil war of 1832-34 which was particularly violent to the city of Porto and which opposed the liberal troops of D. Pedro to the absolutist troops of D. Miguel (the siege of the city lasted more than a year).

[2] *Collecção…*, 1837; pp. 52-53; original is in Portuguese.

- Mathematical section: 1. Arithmetic, Elementary Geometry, Plane Trigonometry, Algebra until the second degree equations; 2. Continuation of algebra, and its application to geometry, Differential and Integral Calculus, Principles of mechanics; 3. Descriptive geometry, and its applications; 5. Spherical Trigonometry, Principles of Astronomy and Geodesy, Theoretical and practical Navigation; 6. Artillery and Naval Tactics.
- Philosophical section: 7. Natural history of the three kingdoms of nature applied to Arts and Crafts; 8. Physics and industrial mechanics; 9. Chemistry and mines; 10. Botanic, Agriculture and rural economics, Veterinary.
- Drawing section: 4. Drawing.
- Trade affairs section: 11. Trade Affairs and industrial economics.

This structure changed slightly over the time, with PAP reaching to thirteen disciplines in 1885 [Pinto, 2013, pp. 116-128]: elimination of Artillery and Naval Tactics (6th) in 1844; creation of the discipline of Political economy and principles of commercial and administrative law (12th) in 1857; creation of the discipline of Applied mechanics to civil constructions (13th) in 1868 and restoration of the 6th discipline in 1883 (now dedicated to Mineralogy, geology, metallurgy and mining). Note that some graduations of PAP were eliminated in 1868, leaving only the three engineering graduations (1.), the graduation of Pilots (3.) and the graduation of Traders (4.).

It should be noted that the creation of these academies in Porto was an attempt to replicate the similar schools that existed in Lisbon at the time. The Porto academies were strongly influenced by what was practiced in Lisbon and not (at least in a direct way) by the famous polytechnic academies that existed in France and in other European countries.

3.2 The Reformation of 1885

The 1885 Reformation of PAP was largely due to Wenceslau de Lima (1858-1919), who propose it in the parliament in his condition of deputy. Note that Wenceslau de Lima was a substitute professor of the PAP, nominated in 1882. Note also that the 1885 reformation began, somehow, in 1883 with the restoration of the 6th chair, which was proposed also by Wenceslau de Lima in the parliament; this discipline was assign to Wenceslau de Lima himself (*i.e.*, he was promoted to full professor). After this first step, on July 21, 1885, a more comprehensive and extensive plan for the reformulation of studies of PAP was approved by parliamentary initiative of Wenceslau de Lima. This reformation[3] included the redistribution of the topics of 3rd discipline by two disciplines; the same was done to the 6th and 9th

[3] "Artigo 1.º A geometria descritiva e suas applicações, mechanica geral e cinematica actualmente professadas por um só lente na 3.ª cadeira da academia polytechnica do Porto serão lidas d'ora avante em duas cadeiras; por igual fórma se procederá ácerca da mineralogia, geologia, metallurgia e lavra de minas (6.ª cadeira); e da chimica inorgânica e orgânica (9.ª cadeira); as disciplinas da 13.ª cadeira (mechanica applicada e construcções civis) serão distribuídas por três cadeiras.
§1.º O concelho academico procederá immediatamente á revisão dos programmas dos cursos legaes da academia polytechnica, ordenando e distribuindo as suas materias pelas dezoito cadeiras que ficam constituindo o seu quadro, ..." [*Collecção…*, 1886; p. 272]

disciplines; the 13th discipline was divided into three new disciplines. With these alterations, the number of PAP disciplines increased substantially, as five new disciplines were added to the thirteen previously existing. Note that the mathematical section of the PAP was one of the greatest beneficiaries of this reformulation of 1885, having enlarged from five to eight disciplines – the 3rd was divided into two, the 13th (the only discipline of PAP full dedicated to engineering itself) would be decomposed into three (12th, 13th and 14th). It should also be highlighted the fact that, for the first time in the academies of Porto, the PAP professors were responsible for deciding the programmes of the different disciplines, as well the structure of the graduations of the institution (previously, it was almost a copy from what was practiced in Lisbon).

Gomes Teixeira joined the PAP, as a professor, the year before, in 1884, having been assigned to the 2nd discipline (Differential and integral Calculus; calculation of differences and variations). In the first school year after the reformation, Gomes Teixeira also accumulated, temporarily, with the 4th discipline: Descriptive geometry. The name of Gomes Teixeira was already in the list of professors of PAP that had been sent in a representative letter to the parliament to seek approval of this very important reformation. On the other hand, it should be also noted that Gomes Teixeira had participated in the review of the discipline's programmes.

The disciplines approved by this reformation were the following:

- 1st: Analytic Geometry; Higher Algebra; Spherical trigonometry.
- 2nd: Differential and Integral Calculus; Calculation of differences and variations.
- 3rd: Rational mechanics; Kinematics.
- 4th: Descriptive Geometry.
- 5th: Astronomy and Geodesy.
- 6th: Physics.
- 7th: Inorganic Chemistry.
- 8th: Organic and Analytical Chemistry.
- 9th: Mineralogy, Paleontology and Geology.
- 10th: Botany.
- 11th: Zoology.
- 12th: Strength of materials and stability of buildings.
- 13th: Hydraulics and machines.
- 14th: Buildings and roads and railroads.
- 15th: Montanistic and docimasy.
- 16th: Economy policy. Statistic. Principles of public, administrative and commercial law. Legislation.
- 17th: Trade affairs.
- 18th: Drawing.

In this reformation the following graduations were still approved: 1. Civil Engineers: Public works, mines and industrial; 2. Trade affairs (the graduation of Pilots was eliminated).

Note that PAP, as previously, continued to do the preparatory studies to those who want to follow to the Army, to the military Navy (officers) or to the medical schools.

As can be easily seen, the engineering graduations also gained more importance in the statutes of PAP. Highlight also that each PAP engineering graduation had six years of duration while the graduation of trade affairs only had three. Note, however, that trade affairs did not last long in PAP and its correspondent discipline (17^{th}) was replaced, in 1897, by the discipline of "Industrial Technology", which further accentuated the feature of "engineering school" of PAP.

This reformation left out the graduation of "Pilots" that had been formalized in 1837. In fact, the reformation of 1885 eliminated all the remaining vestiges of the old navy academy. On the other hand, there was in the city the Industrial Institute of Porto (founded in 1864 to replace the former Industrial School, which had been established in 1852), an institution that was more focused and prepared than PAP to teach a more technical and practical education, as it was the case of, for example, the graduation of "Factory managers".

The importance of this reformation was, without a doubt, vital for the PAP, because it allowed its expansion and consolidation as an "engineering school", and to establish itself, definitely, as a higher educational and scientific institution (note that there had been some proposals to merge PAP with de Industrial Institute of Porto, with the last one in 1882). As stated by Magalhães Basto, perhaps with some exaggeration:

> The Polytechnic Academy was safe at the end of almost fifty years of struggle! It was now at the level of their counterparts in Portugal! (...) In short, it can actually well be said that a new life began on that date. [4]

The reformation of 1885 was, in fact, and almost at all levels decisive for the PAP and allowed it to reach its scientific apogee. The coincidence between this reformation – remember that it has brought an increasing number of disciplines, the significant expansion of subjects taught, as well as an improvement of the material conditions – and the arrival of new and important professors of which stood out, inevitably, the name of Gomes Teixeira, allowed it an important increase in their academic and scientific activities.

3.3 The transition to the University of Porto

With the reformation of 1885, the PAP formalized an approximation to the theoretical feature that existed in Coimbra and in Lisbon. Note that the Polytechnic School of Lisbon (school that replaced the Royal Academy of Navy in 1837), since its creation, was comparable to the University of Coimbra in terms of prestige and scientific level. For instance, in mathematics, there were many professors that were graduated in Coimbra but moved to the Polytechnic of Lisbon and were very active in the Royal Academy of Sciences of Lisbon (created in 1779).

[4] "Salvara-se, quasi ao cabo de cinquenta anos de lucta, a Academia Politécnica! Ela estava agora a par das suas congéneres em Portugal! (…) Em suma, pode na verdade bem dizer-se que vida nova começou naquela data!" [Basto, 1937; pp. 414 e 418].

The PAP just reached this kind of scientific level at the end of the century with the arrival of Gomes Teixeira and the reformation of 1885.

This enormous improvement of PAP allowed the creation, in 1911, of the Faculty of Sciences of Porto, institution that formally replaced the PAP. Note that this was done by the new Portuguese republican regime (implemented in October of 1910), that had treated similarly the three cities (Coimbra, Lisbon and Porto) in the reformulation of the Portuguese higher education, creating three faculties of sciences very similar to each other. The University of Porto was then implemented with two faculties: Sciences and Medicine (replacing the ancient Royal Medical-Surgical School of Porto created in 1836). The structure of the Faculty of Sciences was divided into three sections: 1st, Mathematical sciences (1st group: Calculus and geometry; 2nd group: Mechanics and astronomy); 2nd, Physical and Chemical sciences (1st group: Physics; 2nd group: Chemistry); 3rd, Historical and Natural sciences (1st group: Geological Sciences; 2nd group: Biological Sciences). The Faculty of Sciences of Porto had also attached a School of Civil Engineering that was transformed in an autonomous Technical Faculty in 1915 (in 1926 changed its name to Faculty of Engineering).

Note that this change in 1911 it was not as significantly as it had been the reformation in 1885. In fact, the majority of professors of the PAP remained in their disciplines (the majority of them remained with the same subjects as before) and it was only need to implement minor adjustments in the new context. In fact, in a way, the decisive step towards the creation of a university in the city of Porto was done in 1885.

4 The Portuguese mathematician Gomes Teixeira (1851-1933)

This part will be a short presentation of the important Portuguese mathematician Gomes Teixeira (1851-1933, Figure 1) that arrived at PAP, as a professor, in 1884 – just one year before of the implementation of the Reformation of 1885, in which he participated (also highlight that he was nominated Director of PAP in 1886).

Figure 1. Portrait of Gomes Teixeira published in (*Anais…*, 1933, p. 4)

Note that, Gomes Teixeira was a respected mathematician in his time; for instance, he was included in the Patronage Committee of the first volume (1899) of the *L'Enseignement Mathématique* – it should also be noted that “among the authors [of this journal] there were […] famous mathematicians (E. Borel, C. Bourlet, L. E. J. Brouwer, E. Czuber, G. Darboux, F. Enriques, M. Frechet, Z. G. de Galdeano, J. Hadamard, D. Hilbert, F. Klein, H. Lebesgue, B. Levi, C. Méray, P. Painlevé, H. Poincaré, F. Gomes Teixeira, H. Weyl, etc.)” [Coray et all, p. 33]. His international reputation was also manifested on the two honorary doctorates that he received from the Universities of Madrid in 1922 and Toulouse in 1923. He had also exchanged letters with some of the most important mathematicians of his time such as Levi Civita, Peano, Mittag Leffler and Hermite.

Gomes Teixeira made all his academic studies at the Faculty of Mathematics at the University of Coimbra (from 1869 to 1875, year in which he obtained his PhD degree). Gomes Teixeira enters the Faculty of Mathematics in 1876, first as substitute professor, having arrived in 1880 to full professor. However, he stayed shortly at the University, having been transferred in 1884 to the PAP (a short time after his transfer, he holds the position of Director of PAP; this kind of promotion could not have happened in Coimbra where the criterion was only the time of service). It should be noted that in those years Gomes Teixeira was deputy in the parliament having spent long periods in Lisbon (1879 and 1882-1884). Thus, it can be considered that Gomes Teixeira arrives to the PAP coming from the Parliament (the last session he attended was on May 17, 1884) rather than from the Faculty of Mathematics, despite its official disassociation with University of Coimbra just happened on that year.

In 1884, Gomes Teixeira asked and obtained his transfer to the PAP. In the words of Rodolfo Guimarães [Guimarães, 1918; p. 126], this happened due to family reasons. Almost all the existing literature on Gomes Teixeira makes exclusive reference to the family reasons as a justification for his transfer to PAP, although, in general, they are presented with no detail – the only exception is found in [Alves, 2004; p. 51] which states that, according to the testimony of his grandchildren, Gomes Teixeira “came to Porto to marry” (in fact, he married with a lady from a very important family from the city). It should also be noted that these reasons were presented in Gomes Teixeira lifetime and he could, if he wished, refute them. Since there is no knowledge of such an initiative by Gomes Teixeira, we can assume that, in fact, “family reasons” were one of the factors that led him to move to Porto, or alternatively, that Gomes Teixeira was comfortable with the public display of this reason (not creating any sort of embarrassment to Gomes Teixeira nor to the University and the Faculty of Mathematics; note that he showed, throughout his life, public recognition and appreciation for these two institutions).

Since decisions of this magnitude are taken almost always taking into account various factors, we must consider, potentially, other reasons that have contributed to this decision. In [Pinto, 2013; pp. 289-354] are presented two other possible factors: the existence of conflicts between the professors of the Faculty of Mathematics and the deep reformation that PAP suffered in 1885, by the political action of Wenceslau de Lima, colleague (from the political party and PAP) and friend of Gomes Teixeira.

Although there are several examples of conflicts into the Faculty of Mathematics, we are only presenting the most important that involved Gomes Teixeira. This episode happened in 1879 and was trigged by a proposed law in parliament with the intent to change the subjects of the 4th discipline (beyond the descriptive geometry, this discipline should also teach "Superior Geometry"). This law had been proposed in parliament by Rocha Peixoto and Gomes Teixeira (both members of the parliament and they were also the two youngest professors of the Faculty of Mathematics…). They presented this law without giving any advance warning to the others professors of the Faculty of Mathematics and, even worse, without asking for previous agreement of the professor of that discipline. As expected, the Faculty of Mathematics, by unanimity (the two proponents of this law were in Lisbon and did not participate in this decision), decided to complain by letter (with very harsh words) to the parliament against the proposal and this law was not approved (highlight the gravity of the situation: the Faculty strongly refused a law proposed by two of its members…). Note that this episode was a public disapproval of his conduct and should have been very unpleasant for Gomes Teixeira (note that Gomes Teixeira never said anything about this question). Three years later, in 1882, Gomes Teixeira tried again in the parliament, now alone, to introduce the Superior Geometry at the Faculty of Mathematics but again the law was not approved (although this time, we did not find any reaction from the Faculty of Mathematics).

A pertinent question is to know if these events have actually contributed in some way to Gomes Teixeira's decision to leave to the PAP. After what it was presented here, it seems clear that the Gomes Teixeira relationship with some of his colleagues would not be the best. The lack of courtesy to let them know of the proposed law and all subsequent reaction of the Faculty let foresee some problems within the Faculty. What was the intention of Gomes Teixeira to interfere in this manner in the affairs of a discipline that was not his and that had a well-defined owner? Would it be a test to his strength and his influence in changing the *status quo*? Or was it just a naive behaviour, due to youth, considering that his attitude would not be strongly rejected by the Faculty and, in particular, by the owner of the discipline?

Anyway, this episode is quite suggestive that, despite the scientific value of Gomes Teixeira as a mathematician (at the time, already far superior than his elderly colleagues), he would not have the permission, at least at this early stage of his career, to reform the Faculty without the consent and collaboration of his senior peers. Gomes Teixeira did not have to confront such problems in PAP: the new programs introduced in 1885 were produced by the PAP itself – Gomes Teixeira participated on that; the second discipline program was even «made for him», since it was based on his text of Infinitesimal Analysis (*Fragmentos de um curso de analyse infinitesimal*). In the following years, Gomes Teixeira wrote several didactic books, always in the field of Integral and Differential Calculus, his main scientific interest. These texts were used on his discipline, but also in the first discipline of PAP (Analytic Geometry, Higher Algebra and Spherical Trigonometry). On the first years he was also the professor of Descriptive Geometry (4th discipline). On the other hand, note that the majority of the mathematical professors of PAP were relatively young and must had been more receptive to Teixeira's ideas (symptomatic is the fact that it had been possible to appoint as Director of the PAP a young newcomer, as it was the case of Gomes Teixeira in 1886).

Another reason that could be pointed out in the transfer of Gomes Teixeira to Porto is his close relationship with Wenceslau de Lima. Wenceslau de Lima was elected to the parliament for the first time in 1883, having shared several parliamentary activities with Gomes Teixeira (for example, they proposed some laws together and belonged both to the Public Education Committee which was related to higher education). From this contact, as well as by Gomes Teixeira's letters to Wenceslau de Lima which contain several exchanges of favours, it is possible to infer that Gomes Teixeira had a close relation[5] with him and he was aware of the intention of Wenceslau de Lima to improve the PAP, which were materialized in 1885, shortly after the transition of Gomes Teixeira to Porto. This reformation, as already noted, significantly improved the PAP, and this institution recognized the role of Wenceslau de Lima in its political implementation. The gratitude was such that Wenceslau de Lima had the privilege of seeing his portrait (Figure 2) published on the first page of the yearbook of PAP (1885-86, the first after the implementation of reformation) – usually on the first page of yearbooks only portraits of deceased professors appeared (most were already retired at the time of his death and appear portrayed in old age).

Figure 2. Portrait of Wenceslau de Lima published in (*Annuario…*, 1885, p. 3)

The publication of a portrait of someone so young (under thirty years) in the yearbook, honoring a personality in life, is truly indicative of the importance that the PAP gave to the reformation sponsored by Wenceslau de Lima (Gomes Teixeira wrote in the following yearbook that this honor was well deserved by Wenceslau de Lima).

Although there is no categorical and definitive answer about the motivations of Gomes Teixeira for moving to PAP (note that there are no indications by Gomes Teixeira himself on this subject), it is likely that these three factors may have contributed in some way to the change: family reasons, the existence of some conflicts in the Faculty of Mathematics and the "guarantees" that he had, given his relationship with Wenceslau de Lima, about the development and improvement of PAP.

---

[5] There is also a reference in a text of Gomes Teixeira that he had done a trip to the Alps with Wenceslau de Lima in 1876 (note that both were very young at the time).

5 Conclusion

The Faculty of Sciences of the University of Porto (existing until today) succeeded two important institutions, the RANTACP and the PAP, which have always been linked to the city of Porto and to its real economic needs – the first institution formed traders and sailors when the commercial trades with Brazil were vital to the city; the second, an engineering school (since its foundation, a civilian one), was created to respond to the country's industrialization boost.

> The two academies played a major role in the educational formation of the city's youth and made excellent contributions to raising the cultural and scientific level of the city, and, in general, the northern region. Although devoid of the status of universities, for its pedagogical action as by its scientific value, both can be considered true university institutes[6]

Note that the PAP, as expected, has contact points with both its predecessor and its successor – at the beginning, PAP still has some traces linked to the navy and to trade affairs that, over the time, disappeared. At the end of its existence, PAP became an important school of engineering and sciences, characteristics that were maintained during its transformation into the University of Porto.

It should be highlighted that RANTACP was a unique case of local initiative in Portugal and an embryo of a decentralized higher education that only later would be fully accomplished, creating the foundations that allowed the later establishment of a true engineering school in the city of Porto.

The PAP has reached its scientific peak in the 1880s. The arrival of the mathematician Gomes Teixeira in 1884 allowed this institution to have, for the first time, a recognized top scientist, which gave it a scientific and institutional legitimacy that never before had been achieved. Another decisive event, almost simultaneous with the arrival of Gomes Teixeira to Porto, was the Reformation of 1885 where the PAP formalized, in many aspects, an approximation to a more theoretical school, similar to those that existed in Coimbra and in Lisbon. It should be highlighted again the name of Wenceslau de Lima by his action on the implementation of this reformation of PAP:

> A man of Porto [Passos Manuel] created the Polytechnic Academy; another man of Porto had given it, forty eight years later, the pedagogical elements that were most needed so that, with dignity, the Academy could fully accomplish its purposes.[7]

---

[6] "As duas Academias desempenharam um papel de relevo na formação educativa da juventude portuense e contribuíram de forma notável para a elevação do nível cultural e científico da cidade, e, de uma maneira geral, da região nortenha. Ainda que desprovidas dos estatutos de universidades, tanto pela sua acção pedagógica, como pelo seu valor científico, podem ser consideradas verdadeiros institutos universitários." [Azevedo, 1982; p. 148]

[7] "Um homem do Porto [Passos Manuel] criara a Academia Politécnica; outro Portuense lhe dera, quarenta e oito anos volvidos, os elementos pedagógicos que mais falta lhe faziam para que, dignamente, a Academia pudesse realizar os seus fins!" [Basto, 1937; p. 419].

These two events have enabled a significant increase, both in quantity and in quality, in the mathematical production of PAP. Consult [Pinto, 2013] for a comprehensive and detailed study of the PAP and, in particular, on the mathematics that were produced and taught at this institution.

Acknowledgments

This work was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), within project UID/MAT/04106/2013.

REFERENCES

2º Centenário da Academia Real da Marinha e Comércio da Cidade do Porto (2003). Porto: Reitoria da Universidade do Porto.

Alves, M. da G. D. F. (2004). *Francisco Gomes Teixeira, o homem, o cientista, o pedagogo*. PhD Thesis. Braga: Universidade do Minho.

Anais da Faculdade de Ciências do Pôrto, Vol. XVIII (1933). Porto: Imprensa Portuguesa.

Annuario da Academia Polytechnica do Porto (1877/1878-1910/1911). Porto: Academia Politécnica do Porto.

Anuário da Faculdade de Sciências da Universidade do Porto (1911/1912-1917/1918). Porto: Universidade do Porto.

Azevedo, R. A. (1981). «O Porto na época moderna» Da academia real da marinha e comércio do Porto à academia politécnica do Porto. *Revista de História, 4*, 133-150.

Basto, A. de M. (1937). *Memória Histórica da Academia Politécnica do Porto, precedida de Memória sobre a Academia Real da Marinha e Comércio pelo conselheiro Abreu Cardoso Machado*. Porto: Ed. Universidade do Porto.

Carvalho, A. S. G. de (1937). *A Matemática na Academia Politécnica do Porto*. Porto: Universidade do Porto.

Collecção de Leis e Outros Documentos Officiais publicados no 1.º semestre de 1837 (Sétima série – 1.ª parte) (1837). Lisboa: Imprensa Nacional.

Collecção Official da Legislação Portugueza, anno de 1885 (1886). Lisboa: Imprensa Nacional.

Coray, D., Furinghetti, F., Gispert, H., Hodgson, B. R., & Schubring, G. (Eds.) (2003). *One Hundred Years of L'Enseignement Mathématique*, Monographie n. 39 de *L'Enseignement Mathématique*. Geneve: L'Enseignement Mathématique.

Guimarães, R. (1918). Biografia de Francisco Gomes Teixeira. In *História e Memórias da Academia das Sciências de Lisboa*, *tomo XII, parte 2 (1910-1915)* (pp. 119-149). Lisboa: Imprensa Nacional.

Leite, D. (1933). Prof. Dr. F. Gomes Teixeira. In *Anais da Faculdade de Ciências do Porto, Vol. 18*. Porto: Imprensa Portuguesa.

Pinto, H. (2011). A Academia Real de Marinha e Comércio da Cidade do Porto (1803-1837). *Revista Brasileira de História da Matemática: An international journal on the History of Mathematics, 11*(21),13-43.

Pinto, H. (2013). *A matemática na Academia Politécnica do Porto*. PhD Thesis. Universidade de Lisboa.

Ramos, L. A. de O. (dir.) (2000). *História do Porto (3ª ed.)*. Porto: Porto Editora.

Santos, C. dos (1996). *Universidade do Porto, raízes e memória da instituição*. Porto: Universidade do Porto.

Silva, M. C., Duarte, A. L., & Sá, C. C. (2004). Francisco Gomes Teixeira. *CIM Bulletin 16, June*, 21-23. Retrieved from http://www.cim.pt/docs/4/pdf.

Vilhena, H. de (1935). *O Professor Doutor Francisco Gomes Teixeira (Elogio, Notas, Notas de Biografia, Bibliografia, Documentos)*. Lisboa.

SNAPSHOTS OF THE HISTORY OF THE SIGN SYSTEM OF ALGEBRA IN A COURSE FOR PRE-SERVICE SECONDARY TEACHERS OF MATHEMATICS[1]

Luis PUIG
Departament de Didàctica de les Matemàtiques, Universitat de València Estudi General, Av. Tarongers, 4, València, Spain

As part of the Bologna process, pre-service training of teachers of secondary education has become in Spain a Master degree. Unfortunately, it was decided to adopt the so-called 4+1 system instead of the 3+2 system, which has the consequence that this Master in teacher education lasts one year, including the period of practice spent in a secondary school. In this general framework, the University of Valencia started, in the academic year 2009-2010, a Máster de Formación del Profesorado de Secundaria, which includes, besides the part common to all areas of Secondary Education on psycho-pedagogical and social matters, the practice in schools and the final master thesis, 28 credits on mathematics education. Of these 28 credits, 1 is devoted to the history of mathematics. Having so little time allotted, among the reasons for using history of mathematics in courses for prospective teachers, we concentrate in challenging their conception of mathematics as a set of eternal truths, and in confronting them with the task to try to understand ways of dealing with problems which are familiar to them but that in history are expressed in systems of signs they do not know, and are conceptualised and solved somehow differently.

In this communication we will show a short teaching unit in which some snapshots of the history of the sign system of algebra and of the solving of polynomial equations are presented to the prospective teachers. In some cases the texts from history are presented in what Jens Høyrup has called "a conformal translation" (Babylonian cut and paste methods to solve algebraic problems, and al-Khwārizmī's al-jabr and al-muqābala operations to transform an equation in one of the canonical forms, and the algorithmic rules to solve them); in other cases selected pages from the original texts are presented (Descartes' *Géometrie*, Recorde's *The Whetsone of Witte*, Aurel's *Arithmetica Algebratica*, Viète's *De emendatione*, Bombelli's *L'Algebra*). We will discuss one of the tasks given to the prospective teachers and their performances. In this task a page from Bombelli's *L'Algebra*, which contains the solving of a quadratic equation, is given to the students. They have never seen before Bombelli's sign system (nor Chuquet's) but they have already worked with Babylonian, al-Khwārizmī's, Recorde's and Aurel's texts. Students are informed that in Bombelli's text the first line is an equation, transformed in the subsequent lines step by step to solve it, and they are asked to decipher the system of signs of Bombelli using their knowledge of the transformations which are used to solve a quadratic equation. They are also asked to compare the transformations used by Bombelli with the ones they would used to solve this equation and with the ones they have seen in previous historical text.

[1] This work has been partially supported by the Spanish Ministry of Economy and Competitiveness through projects EDU2012-35638 and EDU2015-69731-R.

THE RESEARCH ON THINKING OF VARIANT OF PYTHAGOREAN CHAPTER

Chunyan QI
Department of mathematics of East China Normal University

ABSTRACT

In this paper I consider the thinking of variant from the perspective of Pythagorean chapter in The Nine Chapters. Among the various elements to be considered I focus on the need to address five aspects to analysis the thinking of variation of Chinese classical works. The five aspects are solutions of a given problem, changes of a given problem, uses of a given problem, changes of a given diagram and changes of identity. Furthermore I shall analyze the thinking of variant from psychology of mathematics education in order to provide a reference for today's mathematics education research and practice.

1 Introduction

Variant teaching is a traditional and typical teaching methods in China. It has not only a wide range of experience, but also through the test of practice. Mathematics teaching mode in China is not "passive infusion" and "mechanical training", it is essentially a variable teaching. This thinking of variant has been reflected in the ancient classics in our country. This article sorts out the all thinking of variant about Pythagorean chapter in The Nine Chapters in order to provide some advice for today's mathematics education research and practice.

2 Variant method of section of Pythagorean in the Nine Chapters

Variant method is a method in our teaching and hiding in The Nine Chapters. For example, many solutions to a problem, variants of a problem, uses of a method, variants of a picture and identical variant. Now we show these methods of variant about the section of Pythagorean in The Nine Chapter.

The section of Pythagorean chapter consists of three parts: problems of three sides of triangle mutual obtained from the Pythagorean theorem , problems of the rectangle and the circle in the right-angle triangle respectively and the measurement problem by using the Pythagorean ratio and that of the Pythagorean numbers. There are 24 problems in the section of Pythagorean. It is mainly the application of the Pythagorean theorem from 1 to 14, the application of the Pythagorean scale from 17 to 24, how to solve Pythagorean number from 14 to 21. There are eight types about Pythagorean number in this chapter, it is rare in the mathematical literature of the ancient world. The Pythagorean theorem is also called GaoShang theorem, it is called the Pythagorean theorem in the west. In ancient China the short rectangular is called hook, long rectangular edge called share, hypotenuse called string in the right triangle. 3000 years ago, mathematicians GaoShang in the Zhou Dynasty put forward the form of the Pythagorean theorem, furthermore people found and proved that the right angled triangle trilateral relations: two right angle sides of the square and the hypotenuse is equal to the sum of the squared, namely: if two right angle sides of a right triangle are respectively a， b, the hypotenuse is c， then $a^2 + b^2 = c^2$. It can be transformed: $a^2 = c^2 - b^2$， $b^2 = c^2 - a^2$.

The variant of contents of the Pythagorean theorem is used flexibly in the section of Pythagorean, the thinking of variant is embodied both each content and between the content. Identical variant also runs through many uses of a method and many variants of a picture. Now list is as follows:

Table 1. Thinking of variant of chapter Pythagorean Distribution

Classification	Solutions ofa problem	Changes of a problem
Operation of Pythagorean		(1)changes among a ， b ， c ； (2) changes among c+b, c-b, c+a, c-a ； (3)If $a,c\text{-}b$,then b,c ； (4)If c, a -b（ or a +b）,then b, a ； (5)If $a,c+b$,then b ； (6)Ifc- a,c-b,then a,b,c; (7)If c- a,c-b,then d.
Square of the Pythagorean	Square of the Pythagorean	
Circle of the Pythagorean	Circle of the Pythagorean	
Measurement of city	Door of North-southor the city	

2.1 Solutions of a problem

The formula that if hypotenuse and the Pythagorean difference then Pythagorean is used In solving the problem for households higher than wide , the formula that string is known to be associated with the sum of Pythagorean and the Pythagorean difference and hypotenuse constructed a quadratic equation to calculate the Pythagorean .

2.2 Changes of a problem

Three Pythagorean relationships were changed by different problems in the operation the Pythagorean.

2.3 Uses of a method

Two Pythagorean triangle are first constructed, and then use the similar principle to solve practical problems in the section of measurement.

The prominent variant is the derivation of Pythagorean general formula, namely generalized Pythagorean representation.

2.4 Identical variant

The sum and difference among Pythagorean chord are used to the deformation of the identities in order to solve three sides of right-angle triangle.

3 Thinking of variant is analyzed from the perspective of mathematical education psychology

The teaching of mathematics activity experience is a kind of teaching method. Mathematics activity experience is usually embedded in the dynamic process of mathematics, level is the basic feature of mathematical activity process. This level can be expressed as a series of steps, but also as a kind of activity strategy or experience. Therefore, the main teaching meaning of process change is that students can solve the problem step by step and accumulate a variety of activities experience through the promotion of a level in the process of mathematical activities.

There are three developments of problem solving can be constructed the variants of Specific experience system: (1) a variety of changes of a problem, which includes both to pave the way for and all of the original problem extended (such as changing conditions and changing the conclusion and generalization); (2) a variety of solutions of a problem is that taking solving process of a problems as variant to link all the different solution; (3) the same method to solve a variety of problems is a particular method for a class of similar problems, which can produce some leads to reduction / inquiry strategy.

There are four process variant in the section of Pythagorean. (1)There are 7 cases in solving triangle by choosing either the number of known as triangle (edge) in the two right-angle sides, as well as the hypotenuse and difference; (2) There are 10 cases in solving triangle by choosing either the number of known as triangle (edge) in (1) and then choosing one condition in the product of two side; (3) There are 7 cases in solving triangle by choosing either the number of known as triangle (edge) in the sum and difference of sum and difference of chord and Pythagorean and then choosing one condition in the sum and difference of two; (4) There are 8 cases in solving triangle by choosing either the number of known as triangle (edge) in the sum and difference of sum and difference of hypotenuse and Pythagorean and then choosing one condition in the product of two.

4 Suggestions

We can also from the ancient classical mathematics to dig out more applicable to our teaching materials, such as changes of the graph in this chapter, changes of problem situation.

REFERENCES

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, N.J.: Lawrence Erlbaum Associates, Publisher.

Marton, F., Dall'Alba, G., & Lai, K. T. (1993). *The paradox of the Chinese Learner*. Occasional Paper 93. 1 ERADU, RMIT.

Marton, F., Dall'Alba, G., & Tse, L. K. (1996). Memorizing and understanding: The keys to the paradox? In D. Watkins, & J. B. Biggs (Eds.), *The Chinese Learner: Cultural, Psychological, and Contextual Influences*. Hong Kong, CERC and Melbourne, ACER.

ENGLISH MATHEMATICAL PRACTITIONERS AND THE NEW MECHANICAL PHILOSOPHY

Emerging epistemologies in the development of English Gunnery in the 16th and 17th Centuries: social contexts, technologies, implications for learning

Leo ROGERS
Oxford, UK
leorogers@appleinter.net

ABSTRACT

Tartaglia's *Nova Scientia* published in 1537 heralded a new approach to the problems of military technology, by adopting neo-platonistic principles to physical phenomena. However, his enquiry was limited by his Aristotelian viewpoint, and his final edition of 1558 offered the idea that nature could be explained by knowledge applied through Platonic philosophy. Tartaglia had accepted that the flight of the missile, hitherto thought to comprise distinct violent and natural phases, had to be of a mixed nature. While application of mathematics in a mechanical paradigm offered a practical validity, readers needed convincing by a formal proof. The problem of the Gunner was finding a reliable way of firing his shot with reasonable accuracy. Neo-platonist philosophy needed a convincing solution to this problem. In his Stratioticos (1579) Thomas Digges' chapter on gunnery proposed an extensive list of parameters for investigation, thus offering an experimental programme for the English Gunners to follow over the next century. Some thoughts on learning in communities and epistemological contexts are considered.

1 Introduction

Tartaglia's Nova Scientia published in 1537 considered as a Mechanical text heralded a new approach to the problems of military technology, by adopting neo-platonist principles of mathematical reasoning to physical phenomena. In the sixteenth century mechanics was seen as the study of phenomena that happen 'against nature', and inspired by the traditional anecdotes about Archimedes' achievements people believed that one could gain knowledge to achieve power over nature, and with this knowledge gained some 'epistemic authority'. By the use of mechanics, man would be able to master natural phenomena (Cuomo 1997).

After consulting with artillerymen, Tartaglia made some important modifications in his *Quesiti et invenzioni diverse* (1546). He stated that a body could possess violent and natural motion at the same time, so that unless the cannon was fired straight upward the projectile was bound to have a curved path. He proposed that this would obtain a maximum range at an angle of 45^0 elevation of the gun, but some insisted that the impetus given to a shot guaranteed that it would move in a straight line for part of its flight.

Tartaglia's enquiry was limited by his Aristotelian viewpoint and his final edition of the *Nova Scientia* in 1558 offered the idea that nature could be explained by a Euclidean pathway to knowledge giving access to mathematical discipline and true Platonic philosophy, accepting the gunners' experience that the flight of the missile, hitherto thought to comprise distinct violent and natural phases, had to be of a mixed nature. While the potential of the certainty of mathematics applied in a mechanical paradigm offered the belief in an epistemic potential that had practical validity, readers needed convincing that validity had to be proved formally, so while practitioners both praised and criticised the book, it was apparent that the lack of internal coherence was problematic.

The problem of the Gunner was finding a reliable way of aiming, ranging and firing his shot with reasonable accuracy. The neo-platonist philosopher had to find tools to provide a convincing solution to this problem. The sixteenth century English practitioners, particularly Thomas Digges, were pursuing similar aims first seen in his *Pantometria*[1] (1571) and in his chapter on gunnery in *Stratioticos* (1579) Digges prepared an experimental programme, listing a considerable number of parameters for investigation; the dimensions of the canon, its length, bore, and manufacture, the qualities of the powder, shape of the shot, air resistance, and the geometrical path of the missile, thus proposing an experimental programme for the English Gunners to follow.

This paper continues the theme of the community of practitioners in the new Common Wealth (Rogers 2012) and follows the development of the Art of Gunnery to the end of the seventeenth century; through improvement of metals technology, the uses in war, and gradual standardisation of the canon and the epistemological value of the development of instruments like the gunners' quadrant, gunsights, and other devices, described in the texts of William Bourne (1535-1582), Robert Norton (1575-1634), Nathaniel Nye (fl. 1647), Samuel Sturmy (1633-1669), an unknown author in 1672, and Robert Anderson (fl. 1668-1696).

1.1 The English gunners in the seventeenth century

There were a large number of manuals for practitioners of every kind published during the 17th century as a consequence of the work of the English mathematicians, astronomers and practitioners like Robert Recorde (1510-1558), John Dee (1527-1609), Leonard (1515-1559) and Thomas Digges (1546-1595) and many others who were regarded both as mathematicians and people who wrote about their practical work. Dee's *Mathematical Preface*[2] had claimed a large umber of pseudo-physical activities as minor branches of mathematics, thus creating an agenda for mechanical investigation (Culee, 1988; Rampling, 2011). The Preface had a major influence on the development of practical mathematics in England, stressing its many applications in navigation, architecture, geography, and even stagecraft. Dee also espoused the virtues of scientific method and vernacular language expressed in Francis Bacon's *Novum Organum* (1620)[3] where the intellect could pass beyond ancient arts and produce a radical revision of methods of gaining knowledge. By the dawn of the 17th century, London was a centre for instrument makers and practitioners of all kinds, and among this new social mix we find those who were writing manuals where gunnery was part of their profession. The men I

---

[1] Digges, Thomas (1571). *Pantometria: Longimetria,* First Book Ch. 30 (folios *Jiy* r – *Jv* v) discusses the problems of the table of Randons, the length of the piece, weight of bullet and force of the powder

[2] Dee, John. (1570) *The Mathematicall Praeface to The Elements of Geometry of Euclid of Megara*. Translated by Henry Billingsley.

[3] Francis Bacon (1520). *Novum Organum Scientiarum*, see https://en.wikipedia.org/wiki/Novum_Organum (retrieved June 20 2016)

mention here are some of the most well-known in the field at the time, often writing on practical skills as typical representatives of their craft and I focus on the parts of the works that are devoted to the development of the practice and theory of gunnery.

Thomas Digges clearly indicated problems with gunnery in his *Pantometria* of 1571, demonstrating that to achieve consistent results requires both the *experience of experiment and sound mathematical knowledge*. He then presented a list of problems in his *Stratioticos* (1579) where he devoted the final section[4] to artillery, and raised a number of questions about the efficacy of the canon, proposing to write on the subject, but it was never completed.

He claimed that the Primary problems to be investigated are (p. 181): The Powder mixture, its quality and quantity; the construction of the canon by the foundry, the length and dimensions of the cylinder; the Bullet, its material, (iron or stone or a mixture), weight and dimensions; and the *Randon* (later referred to as Random) which covers a number of effects like the angle of elevation and the effect of these on the range of the gun. Secondary causes he suggested are: the direction of the wind, density of the air; the gun mounting, the boring of the barrel, charging the gun; fitting of the bullet, and the temperature of the gun. Furthermore, he questioned whether the trajectory of the bullet be an ellipse, parabola or hyperbola and whether the trajectory varies continually with the range, and if the angle between the original elevation of the gun and the path of the shot was continually changing (pp.187/188). Accepting the trajectory of the shot comprised violent and natural motion, he insisted that those without practical experience should not make authoritarian statements about the flight of the bullet. Here, Digges had set out a 'research agenda' and his work was often referred to by gunners in the following century where some of these questions began to be approached, but any answer to the question of the range of shot continued to be a serious problem.

William Bourne (c. 1535–1582) was a gunner at Gravesend bulwark, defending the approach to London. In 1574, he produced a popular version of Martin Cortes de Albacar's (1510-1682) *Arte de Navegar*,[5] entitled *A Regiment for the Sea.* Bourne was critical of the original and produced a manual of more practical use to the seaman. He described how to make observations of the sun and stars using a cross-staff and how to plot coastal features from the ship by taking bearings using triangulation. He also published *The Art of Shooting in Great Ordnance* in 1578. He aims to rectify the faults of England's gunners, given their inability to determine relative ground heights, elevations of their pieces, and distances to their target (1578 folios Aiii r -Avi v). Bourne provides ten "Considerations" regarding great ordnance, and while important, they are simple maxims, providing qualitative explanations for missing the target or just an 'aide memoire' for the gunner (1578 pp. 1- 4). He covers topics in four categories: physical characteristics of ordnance (including gunpowder), numerical calculations, use of the quadrant and the foot rule, and the process of 'laying' a shot (estimating the range) and providing tables measuring elevation. His description of the flight

---

[4] Digges, Thomas (1579). *Stratioticos* Chapter 18 pp. 181 -189; revised and reorganised in 1590 pp. 361- 368.

[5] The *Arte de Navegar* translated in 1561. Bourne was critical of this book and published a more practical *Regiment for the Sea* in 1574. Translations of various European works continued throughout the century.

of the shot is in three parts: a straight line as long as the shot goes violently A; the second part circularly B; and the third part is at the highest distance above the earth C; and the fourth part downwards circularly towards the earth D. (Fig 1). He states that the maximum range is found at 45^{o} and he shows in a diagram that the flight is the same shape for 45, 30 and 15 degrees of elevation. Bourne, as an experienced gunner, had obviously accepted the fact that the flight of the projectile was some kind of curve, and abandoned the Aristotelian theory to use some kind of 'mixed' motion once the initial impetus was lost.

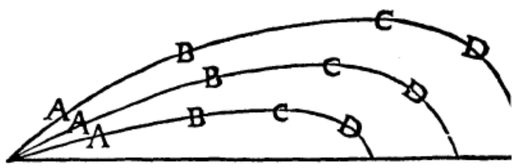


Figure 1. Bourne 1578 Chapter 9 p. 38

Bourne discusses the astronomical quadrant, introducing the measure of degrees as more profitable than the 'gunners quadrant' that had comparatively fewer makings.

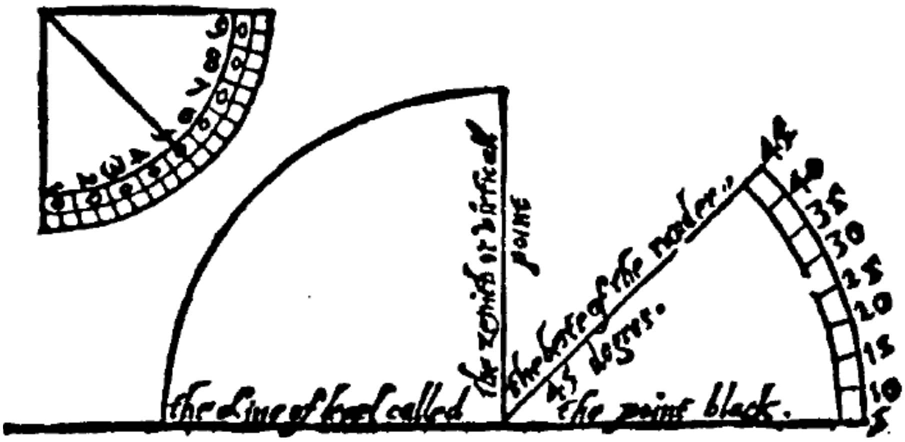


Figure 2. Bourne 1578 page 22 use of the quadrant

The diagram (Fig. 2) shows the vertical, the 'point blank' positions, and the 'best of the random' and he introduced a table comparing the fractions of an inch of a gunners level with the degrees on a quadrant. It is in Bourne's work that we find lists comparing different properties and quantities: diameters of shot to their weight; powder types for different shot, calibre of canon and material (stone, iron, etc.) and weight of shot; and many other comparisons. The lists are difficult to read as text, but he summarises some sections by introducing tables, an innovation which is taken up by succeeding practitioners. Bourne's was the first work in English which defined gunnery as a separate and independent branch of the Art of War by military writers.

Robert Norton (d. 1635) was a military engineer and gunner who studied under John Reynolds, master-gunner of England, and became a gunner in the Royal service. In 1627 he was sent to Plymouth as an engineer, to await the arrival of the Earl of Holland and to accompany him to the Isle of Rhé.[6] In the same year he gained the post of engineer of the Tower of London. He published *Of the Art of Great Artillery,* London 1624. Norton acknowledged Digges' work with expositions and answers of his own, and followed it with *The Gunner, showing the whole Practice of Artillery*, London 1628. Here, Norton supplied tables of various measurements and instructions in decimal arithmetic from Recorde's *Ground of Arts*. Norton had also published *Disme, the Art of Tenths or Decimal Arithmetic* London 1608[7], claiming that decimal fractions make calculation much easier. *The Gunners Dialogue.* London 1643, described the types of artillery[8] used at the start of the Civil War in England. This was also published in an edition together with his *Art of Shooting in Great Ordnance*. Here were more tables of shot sizes and weights, quality and mixtures of gunpowder and use of the gunners' level. The information is cumulative but provides no specific technical advance on earlier publications. Here we are beginning to see some improvements in presentation of data and the introduction new methods of calculation.

Nathaniel Nye (1624-1647?) living in Birmingham developed an interest in canon, the city's principal trade during the English Civil War (1642-1651) and tested cannon there in 1643. From 1645 he was master gunner to the Parliamentarian garrison, and in 1646 directed the artillery during the Seige of Worcester[9], recounting his experiences in his 1647 book *The Art of Gunnery*. For Nye, the 'Art of War' was also a science, and his other work focused on triangulation and cartography, fortification, and mechanics, as well as finding the ideal specification for gunpowder. Nye described the rules and directions in this book both with and without the help of arithmetic. He has the usual contents: description and maintenance of canon, composition of gunpowder and other fire-works, estimating heights and distances, plotting positions of targets and drawing maps of fortified places. He covers ranging and provides tables according to results of experiments with various canon, recounting an experiment (chapter 52) where (under the same conditions) he fired seven shots at a target in 50 minutes, and attempts some explanations of the difference in the distances of shot from the gun: the heat of the barrel, the dryness of the powder, and that the first shot parted the air allowing later shots to fly further. Here the use of tables include comparing a diameter of a canon's bore to the weight of powder, weights of various shot for different diameters, a cube root table to help with the required calculation, and another to compare quadrant degrees to the scale on the Gunners rule. In these tables we are beginning to see gunners focussing on recording and comparing simple experiments, but it did not progress the trajectory problem.

Samuel Sturmy (1633-1669) first published *The Mariners Magazine*, in 1669. It became the most extensive compendium of its kind in the latter 17[th] century. Subtitled the *'mathematical and practical arts'*, Sturmy had collected a considerable variety of information using instruments on Navigation, Surveying, Gunnery, Fire-Works, Fortification, Sundials,

---

[6] Norton took part in the expedition to support the Hugenots in the 1627 Anglo-French War.

[7] A translation of Stevin's *De Thiende* (1585).

[8] Civil War Artillery had adapted from heavy siege guns to mounted canon about 9-10 feet (2.7-3.0 metres) in length with bore 3-4 inches (8-10 cm) diameter firing a shot of 5-12 pounds (2.3-5.4 kg) in weight.

[9] The final Battle of Worcester in September 1651 was the last in the English Civil War.

and Spherical Astronomy. Subtitled the *'mathematical and practical arts'*, Sturmy had collected a considerable variety of information describing the use of instruments on Navigation, Surveying, Gunnery, Fire-Works, Fortification, Sundials, and Spherical Astronomy. Many topics had been addressed by others, but he shows he has command of the mathematics, with sections on geometry and the *'doctrine of triangles both plain and spherical'*. He describes the use of the quadrant, nocturnals, and other instruments, with problems resolved geometrically, and instrumentally.

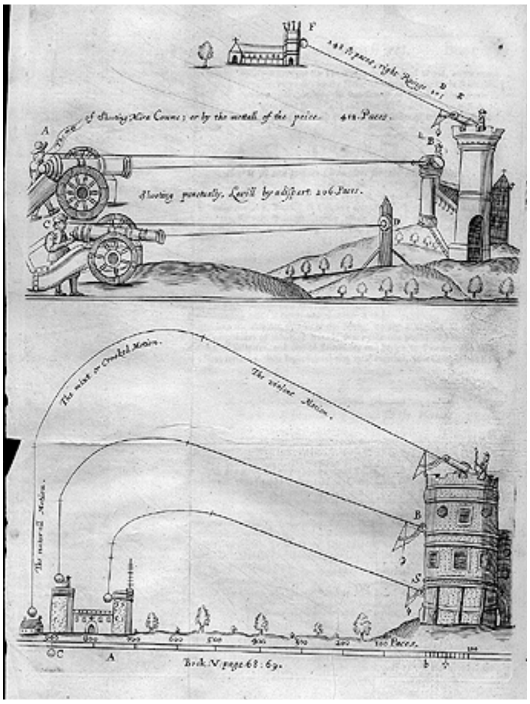


Figure 3. Source, MHS Oxford, Sturmy 1684 edition

Figure 3 shows the problem of the trajectory still unsolved. The upper part of the figure shows firing *Point Blank* where the path of the shot and the direction of the canon are parallel with different calibre canon and firing with different *Randons* below. While the ranges are different, the projectile paths are comparable consisting of violent straight motion, 'mixt or crooked motion' and natural straight motion. Sturmy's 1679 edition was *'diligently revised and carefully corrected'* by John Colson[10]. Included were tables of Sines and Tangents to every degree and minute of the quadrant, and Logarithms, showing their use in calculations. Sturmy's work also contains what may be one of the earliest complete explanations of the construction of a polar gnomonic chart, presenting a detailed example of a great circle route from the Lizard (SW England) to the Bermudas. His section on gunnery repeats the usual exhortation to gunners to make firing tables and by example, to continue other types of comparative tables.

[10] John Colson was Lucasian Professor of Mathematics at Cambridge, and a Fellow of the Royal Society.

An Anonymous 'W.T.' signed the preface to The Compleat Gunner in Three Parts. London 1672, claiming this is a collection of translations from various sources. The title page refers to material from "Casimir, Diego Uffano, Hexam[11] and other Authors" the author claims that he has translated material from French, German, Italian, and Dutch sources. Here we find many familiar topics, but there are some notable inclusions: Founding[12] and Casting with discussion of the composition of Metal and Examining Ordnance for flaws. There are Tables of diameter and weight for all ordnance used in England and tables for mixing stone and lead to equal iron canon balls. There is a section on preparing and clarifying saltpetre from Nitrous Earth[13] as well as detail on Proving[14], and refining the composition of gunpowders. There are descriptions of instruments, and the variety of instructions for training a gunner are quite significant: measuring heights and distances by using a quadrant or without instruments, showing many woodcuts borrowed from other authors[15]. The Circumferenter (page 68) is a surveying instrument for measuring horizontal angles.[16] There is a description of an improved Gunners Scale with multiple uses with a two-sided stepped scale to enable the gunner to measure the diameter of the bore, and know which type of canon he is dealing with (between pages 70 and 71). There are instructions on shooting to make a table of Randoms including measuring distances in Paces. Violent, crooked, and natural motion from discharge to target are discussed with reference to Tartaglia's Nova Scientia and there is an incomplete picture of shooting at Random from Sturmy 1684 (between pages 72 and 73). Much of this was copied form Hexham (1640). In the last part of the book an appendix, has two sections, the first being a description of "The Doctrine of Projects applied to Gunnery by those late famous authors Gallileus and Torricello, now rendered into English." "Together with Excellent Observations out of Mersennus and other famous Authors." It contains some 25 pages of mixed translations of selected parts of Galileo on motion with tables of sines and tangents of the angles of elevation taken in a semi-parabola (p. 78).[17] The proportional calculation is accompanied by the comparison of the angle in a semicircle with a cap of a parabola (Fig. 4).

[11] These people are: Diego Uffano (or d'Uffano), d. 1613 a Spanish military engineer; and Casimir or Kazimierz Siemienowicz (1600? – 1651) author of *Artis Magnae Artilleriae pars prima* (1650) which was used in Europe as a basic artillery manual, and Henry Hexham, author of *The Third Part of the Principles of the Art Millitarie* 1640 from which the 'Doctrine of Projects' was badly copied and misunderstood.

[12] The Blast Furnace and canon Foundries were well established in England from the late 15th Century.

[13] Nitrous Earth: Manufacture of saltpeter from nitrate deposits and from animal excrement, rotting bodies, and urine by extracting the chemicals ammonia, aluminium sulphate, and acid salts.

[14] 'Proving' is the verb describing the testing a canon for accuracy and the powder for its explosive force.

[15] For example, we can identify here the woodcuts from Digges *Pantometria*1571.

[16] An early Theodolite. See Turner 1973 (page 79 item 16). Leonard Digges *Prognostication Everlasting* 1564 showed an instrument called 'Theodoliticus' that became a standard instrument of surveyors and architects.

[17] Imagine an inverted parabola f(x,y) from $x_0 = 0$ to $x = x_1$ with a chord from x_0 to $y = (x_1/2)^2$; the angle of elevation is contained between this chord and the x – axis. Fig. 4.

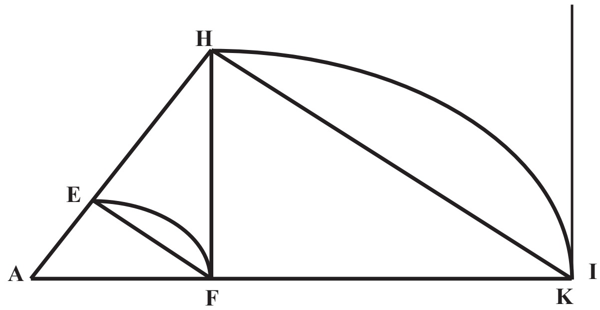


Figure 4. Anon 1672 Chapter XXI, p. 78

"AEF is like and proportional in like and crooked Ranges to HI and their distances or dead Ranges are AF unto AI." (the like Ranges are at Point Blank) The Author has clearly misunderstood the problem and is struggling to fit the results to a 'Euclidean' diagram.

Robert Anderson was helped by John Collins[18] with a loan of books and scientific equipment, and in *The Genuine Use and Effects of the Gunne, as well experimentally as mathematically demonstrated*, 1674 are fifty propositions on the use of all kinds of canon and mortar showing how to calculate ranges for different shot and powder based on his experiments, referring to a set of geometrical diagrams facing page 36. By these experiments he shows that the trajectory is a geometrically constructed curve, and that the ranges for equal elevations above and below 45^0 are equal. He justified this further by analogy, experimenting with water spurting out of two holes in a tube equidistant from the centre of the tube. The distance traversed by water jets is proportional to their horizontal speed and time of fall. Speed is proportional to the square root height of the water above the hole, whereas the time of fall is proportional to the square root of the height of the water below the hole, therefore both jets would hit the ground at the same time. (Anderson 1674: 26-27 and 30-31). The rest of the book contains many pages of tables showing his experimental results.

By this time, the most significant aspects of these publications is the emphasis they put on *accumulating data on all sorts of aspects of the canon and its' firing*. In parallel with this we have the *increasing use of instruments of various kinds, now being fashioned by the practitioner community, and just as important were the technological advances enabling the gradual standardisation of the canon*.

1.2 Gunnery: the general problem

Artillery gun-sights and levels were introduced in the 16th century accompanying the new cast bronze canons that came into use from the late 15th century. Ideally, the canons could be set at various elevations and by traversing around on its mounting, the gun could be aimed in any direction. Gunner's sights and levels were either separate or combined together. The quadrant was placed in the mouth of the gun, so that gunners could elevate the gun to the correct angle for the estimated range, often exposing the gunner to enemy fire, or the level was set up more safely at the touch-hole end. This instrument came in a variety of forms, with

[18] Anderson's book was sent to Newton by John Collins, apparently to provide him with more technical data.

a degree or a tangent scale graduated either in an arc or on a straight rule, usually with a plumb bob on a cord. The levels were set up along the longitudinal axis of the gun. There is often a confusion in the accounts of this instrument; the simple version used during the 16th century was the well-known proportional measure for sighting heights and distances[19]; the instrument used in the 17th century was the *adaption of the astronomer's quadrant marked in degrees of arc* (Fig.2). The early manuals often advised the gunner to make their own quadrant to suit an individual gun, but Bourne (1578) instructed his reader to use a quadrant marked in degrees. In the 17th century mounting the canon on wheels for field use meant that recoil, traversing and resetting the gun could be managed more easily. For maximum destructive force most cannon were fired at 'point blank' range, for which the gun, and hopefully the trajectory of the shot remained horizontal. But after Tartaglia's *La nova scientia* (1537) there was much discussion of the ranges that could be obtained by varying the elevation of the gun. Tartaglia insisted that the maximum range of the gun could be achieved at an elevation of 45^0, and Galileo showed later that a parabolic trajectory was theoretically the case,[20]but Aristotelian philosophers, and to some extent the gunners, relying on 'line of sight' observation[21] were in dispute.

2 Sources of innovation

2.1 The art of the gunner and the role of metallurgy

The emergent science of ballistics was the natural theoretical development following the spread of firearms from the fifteenth century onwards, and their deployment on the field and during sieges became an undeniable reality in the fifteenth century, but considering the level of technological development at the time ordinary gunners felt no need for a science of ballistics such as that formulated by Tartaglia.

Over the course of the fifteenth century, heavy artillery was used during sieges on fortresses and fortified towns. From the beginning, two categories of artillery were produced: one which was able to fire relatively light cannon balls (between 3 and 12 kilograms), and the other intended to destroy fortifications, and therefore capable of firing cannon balls weighing up to several hundred kilograms. Technological developments in the earlier part of the sixteenth century had concentrated primarily on the heaviest artillery, but by the middle of that century the balance became restored, when architecture was able to provide a response to the development of metallurgy, and the construction of the bastion succeeded in putting attack and defence onto a more equal footing, at least as far as sieges on fortresses were concerned.

Durer (1527)[22] had foretold the end of old fortresses, even if they had been readapted.

[19] Many examples of this instrument can be seen in Digges *Pantometria* 1571.

[20] Galileo (1638) *Two New Sciences*. Fourth Day; Theorem II Proposition II.

[21] From the gunners point of view, observing the shot go straight up and then fall (hopefully) on to the target does not provide clear evidence of any particular trajectory.

[22] In 1527 Albrecht Dürer (1471-1528) published a treatise on fortifications, Etliche underricht zu Befestigung der Stett, Schloss und Flecken. (Several instructions for fortifying towns, castles and small cities), which was the first printed work on the subject of permanent artillery fortification.

Now there was a need for new fortresses to be built, following a geometrical construction on the basis of the strategies of attack and defence made possible by the new firearms. In the sixteenth century, the attack developed during sieges focused on destroying one of the bastions to gain access to an area along the curtain wall where defence became weaker, and where it became possible to move in to attack. At this time, the technology had not yet been developed to avoid serious damage from the dangerous effects of the cannon's recoil. Furthermore, the process of loading the barrel of the cannon, the quality of the gunpowder and the quality of the cannon itself from the metallurgical perspective, made it impossible to fire shots that would follow rectilinear trajectories. Formerly, strategies had focused on destroying as much as possible of the inside of the fortress. A precise shot was not fundamentally important, and the experience of the gunner in battering the walls down was sufficient to achieve the required objectives. Due to developments in metallurgy and to innovations that increased the efficiency of hydraulic apparatus applied to run ventilation systems at the end of the fifteenth century, furnaces became capable of reaching much higher temperatures than before (Simmons 1992).

During the 16th century, metal technology improved considerably. The casting of barrels and smoothing of bores became more efficient, and higher furnace temperatures achieved provided cast iron[23] and bronze, much stronger metals, so that by the end of the 17th century both ammunition and the bore of the gun became more consistent and the sale of these armaments abroad led to furhter standardisation[24]. A new kind of cannon ball began to be produced in these furnaces, made of wrought iron or cast bronze. This innovation resulted in a revolution. Cast iron cannon balls of a relatively low calibre had a much higher capacity for penetration than those made of stone. Stone balls often disintegrated and crumbled on impact with the target, and only had a potential for destruction if they were very large and in free fall, while the low calibre cast-iron cannon balls finally made it possible to use smaller and lighter artillery that was easier to transport and cheaper to produce. This innovation led to a significant increase in the velocity of projectiles, which established artillery as essential to the art of war, such that it spread to a hitherto unimaginable extent. Over the following decades, the calibre and type of artillery were produced to the setting of standards, each of which was valid within at least one single country or princedom. There was a change in the role of the gunners who from this time became recognised as vital to the Renaissance army.

2.2 From the art of gunnery to ballistics

Aristotelian mechanics was concerned with natural and violent motion which, according to Tartaglia, failed to provide a sufficient answer to the fundamental question concerning the curvilinear segment of a trajectory. This question led to the formulation of an idea of "mixed motion," and thus formed the basis for a *concept of compositions of motions*. This idea is

[23] The English Iron foundries such as that in Horsmonden, Kent (1574 – 1685) set the highest quality for cast iron guns that were used by our ships and armies and were sold abroad until the development of steam power moved the industry to the Midlands in the 18th century. See http://www.horsmonden.co.uk/history/furnace/

[24] English canon were sold to the Dutch during their war of independence from Spain (1568-1648).

expressed in Digges' final chapter of *Stratioticos* (1579) and appears in the 1578 edition of *Nova Scientia.*

The gunner's problem was knowing how to hit a target with precision from as far away as possible, so during the 17th century gunners' manuals began to encourage the recording of firing tables to record the fall of shot for different angles of elevation. This meant that the gunner would find it easier to use fewer adjustment shots each time the target changed. Tartaglia's firing table (promised but not produced) would have been particularly relevant for the gunner, and the method for calculating his own table based on the data he would have obtained from a single shot would have been useful. As we have seen, firing tables began to appear in a wide range of publications, although there were great discrepancies in the values they showed. Generally, these results could only be useful to the person who had written them since only he was able, on the basis of his memory and accumulated experiences, to see their significance. But, believing that they would have some practical use, many people began to produce these tables *and in doing so gunners and others began to learn to read these tabular forms obtained through experiment*, and a series of records of angles of elevation, relating to one specific piece of artillery, firing similar projectiles and maintaining the same quality and quantity of gunpowder, would amount to a firing table. However, it took quite a long time for all these different conditions to be consistently achieved. While the accumulation of this data was of a very particular nature, it would have remained quite local and individual. While there was no organised collection of such data there was plenty of material to be found in the many published works that included 'gunnery'. There were too many varied parameters, but it gradually became possible to formulate *professional habits*, writing down the information became a custom, *it became what gunners do*, as part of their training, and from these general, practical rules the use of a more or less inductive method developed. The popularity of gunners' manuals during the 17th century to some extent depended on the promise that the next manual would have even better ways of solving the problem of consistent shooting.

Many other tables of measures appeared that provided early 'ready reckoners' for matching shot diameter and weight to canon; estimating weight of shot by size of stone, iron or mixed material; shot and powder for the length of a given canon; proportions for mixing components of gunpowder etc. Despite all of this data the problem of predicting the flight of a projectile remained unsolved. Matthew Bourne, whose picture of the trajectory was pretty well a complete curve (Fig. 1.) had clearly adopted the idea of 'mixed motion' while others like Samuel Sturmy (Fig. 3) and the Anonymous author of the *Compleat Gunner* …(1672) (Fig. 4.) were still clinging on to the old theory. Robert Anderson's trajectory of 1674 was the *cumulative result of labour* and the best collection of results before the eighteenth century.

Surveying techniques improved with better designed instruments; the rudimentary theodolite appeared, methods of measuring distances by triangulation became common, better surveys provided information on how defenders were positioned, and the efficiency of the artillery battery was considerably improved.

The quadrant, gunners level, and 'professional habits'

As we have seen, the quadrant found new applications, *randons* became *degrees* and a considerable amount of empirical data was accumulated by recording elevation in more accurate measurements. This became the first step in a process of abstraction, the *beginning of a theoretical reflection on the gunner's own actions*.

The quadrant, the gunners level, firing tables, and more accurate surveying, provided the physical instruments and records for the *intellectual milleu that became the epistemological spur that initiated a process of theoretical abstraction*, leadting to a better formulation of the gunners' question. During the seventeenth century, the accumulation of these data, formed the empirical basis from which the theory of ballistics emerged. In the early 17th century, special artillery schools for training gunners had appeared which created centres for the newly emerging science of ballistics. These advances in technology were attributable to the accumulation of small improvements, essentially empirical, collaborative and democratic, which were used by Francis Bacon (1561-1626) to demonstrate the manner in which intelligent application could lead to economic progress and intellectual advancement in his system of Natural Philosophy.

3 Social contexts, learning and epistemology

In the study of the practice of mathematics in 16th and 17th Century England, the epistemological varieties concern the views of the nature, use, and processes of the formation of the mathematics and its uses that were espoused by the different agents. As far as the authors of these texts are concerned, their views on the formation of mathematics depended on their ideologies, and were inevitably involved in the production of their texts. The purposes of mathematics that were perceived, and the uses to which it was put, to some extent determined the kind of mathematics that an individual was motivated to use and develop, and these needs were felt in different ways in particular sections of society. The mathematical practitioners of this period were influenced not only by the social context of enterprise, but also the belief that there was no particular barrier between practical mathematics and theoretical mathematics. The principal characters cited above and others like John Wallis, Thomas Harriot, and William Oughtred were all at some time or another involved in practical enterprise. This *Community of Practice*[25]became the basis of the generation of all kinds of specialised knowledge in the sixteenth and seventeenth centuries. With regard to the texts studied here, the aim of the authors of these books may be to produce an organised system of knowledge, and this assumes that their text, when read in the 'right' way provides a reader with a clearer understanding of the nature and purpose of their subject. This involved a process of investigating phenomena to establish new facts by developing the *method of induction*.[26] An important practical aspect of these investigations was introduced, and that was the process of *thinking with objects.* (Meli 2006; Rogers 2015).

By the seventeenth century, there were many instruments available for measuring distance and ranging the canon, and an important part of most gunnery manuals involved a

[25] The idea of a *Community of Practice* has been developed in sociology by Wenger (1999) and applied in mathematics education by Adler (2000)
[26] In particular as we know, in the case of Recorde, Dee and Digges.

section on the basics of arithmetic, geometry and surveying, with the use of proportional reasoning to solve the problems by organising range tables. The tables that we find in these gunnery manuals brought home the important idea that, if we could accumulate and refine enough data, we should be able to find an answer to our questions. The accumulation of this data formed the epistemic background, and in this atmosphere, new variations of the objects, the canons, the gunpowder and the instruments were experimented with. There is a reflexive relation between working with the object (material or theoretical) that provides new affordances (Gibson 1997) enabling new ideas to emerge and come into practical and theoretical use. The learning that takes place is shared in a community which emphasizes socialization, spreading values with not only the acquisition of skills and participation in activities, but a third stage where individual and collective learning goes beyond mere information given, and advances knowledge and understanding by a collaborative, systematic development of common objects of activity into shared *knowledge-creation.* (Paavlova & Hakkarainen 2005) Some of these attitudes, contexts and processes we learn from history, can be applied by reflective contemplation and adaptation to our current situations.

The next real innovation in the context considered here, will be the discoveries leading to the application of the new fluxional mathematics to the trajectory of the bullet in Benjamin Robins' *New Principles of Gunnery* (1742).

REFERENCES

Primary Sources

Anderson, R. (1674). *The genuine use and effects of the Gunne: As well experimentally and mathematically demonstrated*. With tables of projection, etc. exactly calculated and their use explained by Thomas Street. Oxford: The Queen's College.

Anon (1672). *The compleat gunner in three parts*. *Facsimile. EEBO editions*. London: Proquest.

Bourne, W. (1578). *The arte of shooting in great ordnaunce: Containing very necessary matters for all sorts of servitoures either by Sea or Land*. London: Thomas Woodcock. Facsimile. Massachusetts Centre for Renaissance Study.

Digges, T. *Pantometria* (1571); *Stratiaticos* (1579 and 1590). EEBO. Oxford: Bodleian Library.

Hexham, H. (1640). *The Third Part of the Principles Millitarie …*. EEBO. Oxford: Bodleian Library.

Norton, R. (1624). *Of the Art of Great Artillery.* London.

Norton, R. (1628). *The Gunner, showing the whole practise of Artillery*. London.

Norton, R. (1628). *The Gunners Dialogue With the Art of great Artillery.* London: John Tap.

Nye, N. (1647/8?). *The Art of Gunnery*. EEBO Bodleian.

Sturmy, S. (1669). *The Mariners Magazine.* London.

Sturmy, S. (1684). *The Mariners Magazine.* London

Secondary Sources

Adler, J. (2000). Social practice theory and mathematics teacher education. *Nordic Studies in Mathematics Education, 8*(3), 31-53.

Cuomo, S. (1997). Shooting by the Book: Notes on Niccolo Tartaglia's Nova Scientia. *History of S*cience*, 35*, 155-187.

Culee, N. H. (1988). *John Dee's natural philosophy: Between science and religion*. London: Routledge.

Gibson, J. J. (1977). The theory of affordance. In R. Shaw, & J. Bransford (Eds.), *Perceiving, acting, and knowing: Toward an ecological psychology* (pp. 67-82). Hillsdale, NJ: Lawrence Erlbaum.

Halliwell, J. A. (Ed.) (1842). *The private diary of Dr John Dee and the catalogue of his library of manuscripts*. London: The Camden Society.

Hall, A. R. (1969). *Ballistics in the seventeenth century: A study in the relations of science and war with reference principally to England.* Cambridge: Cambridge University Press.

Hill, K. (1997). Mathematics as a tool for social change: Educational reform in Seventeenth Century England. *The Seventeenth Century, 12*(1). 23-36.

Meli Bertoloni, D. (2006). *Thinking with objects: The transformation of Mechanics in the Seventeenth Century.* Baltimore: Johns Hopkins University Press.

Moran, B. (2005). Knowing how and knowing that: Artisans, bodies, and natural knowledge in the Scientific Revolution. *Studies in the History of Science, 36*, 577-585.

Paavola, S., & Hakkarainen, K. (2005). The knowledge creation metaphor – An emergent epistemological approach to learning. *Science & Education, 14*, 535-557.

Rampling, J. M. (2011). The Elizabethan mathematics of everything: John Dee's 'Mathematical praeface' to Euclid's *Elements. BSHM Bulletin, 26*(3), 135-146.

Rogers, L. (2012). Practical mathematics in 16th century England: Social-economic contexts and emerging ideologies in the new common wealth*.* In K. Bjarnadóttir, F. Furinghetti, J. M. Matos, & G. Schubring (Eds.), *"Dig where you stand" 2. Proceedings of the second "International Conference on the History of Mathematics Education"* (pp. 421-444). Caparica: UIED.

Rogers, L. (2015). Historical epistemology: Professional knowledge and proto-mathematics in early civilisations. *Proceedings of ESU 7 History and Epistemology*. Copenhagen. Electronic Publication

Simmons, J. (1992) Early modern wrought iron artillery: Microanalyses of instruments of enforcement. *Materials Characterisation, 29*, 129-138.

Taylor, E. G. R. (1963). *A regiment for the sea and other writings on navigation by William Bourne of Gravesend, a gunner (c. 1535-1582).* Cambridge: Haklyut Society.

Turner, A. J. (1973). Mathematical instruments and the education of gentlemen. *Annals of Science, 30*(1), 51-88.

Wenger, E. (1999) *Communities of practice: Learning, meaning and identity*. Cambridge: Cambridge University Press.

FATHER PADILLA'S *ARITHMETICA PRACTICA* (1732) IN ITS CULTURAL COLONIAL GUATEMALAN CONTEXT

Luis RADFORD
Laurentian University, Faculty of Education, Sudbury, Ontario, Canada
Lradford@laurentian.ca

ABSTRACT

In this article I focus on the oldest extant mathematics book published in the Colonial period in Central America: Padilla's (1732) *Arithmetica practica*. In the first part of the article, I briefly recount my encounter with Padilla's book and the contextual problems that I found in order to offer a facsimile edition of it. In the second part, I define the historiographic approach that I follow. I argue that to inquire about the history of mathematics is to inquire about mathematics as it was imagined, thought, and practised within a certain cultural context. But I also argue that we can go further: I contend that a certain cultural historical context can be understood only if we understand its intellectual life, and more specifically its mathematics. It is in this second line of thought that I wish to see how the book fit and responded ideologically to the constraints generated by the economic, political, and educational colonial structures. To do so, before dwelling on the book and its author, I discuss the cultural context of Padilla's *Arithmetica*.

1 Introduction

In 1732 the book "Noticia Breve de Todas las reglas mas principales de la Arithmetica practica con q ʃe puede deʃatar, no ʃolo las demãdas ordinarias, ʃino tãbien muchas difficultoʃas, que de otra ʃuerte ʃolo por la Algebra ʃe respondieran" [Short Notice of all main rules of practical arithmetic with which to solve not only the ordinary questions, but also difficult ones that otherwise would be solved by algebra] was published in Santiago of Guatemala. It was published by the printing press of Ignacio Jacobo de Beteta. The author of the book was Juan José de Padilla. Padilla's *Arithmetica* is the oldest extant mathematics book published in the Colonial period in Central America. There is, to my knowledge, only one copy of the book. This copy is kept at the *Museo del Libro Antiguo* in the colonial city of Santiago of Guatemala, known today as Antigua Guatemala—about 50 km from the current capital of the country.

My encounter with the book was rather accidental. One Sunday in the late 1980s, I was visiting Antigua Guatemala and ended up in the *Museo del Libro Antiguo*. While browsing the collection of old books I noticed Padilla's *Arithmetica*. The director of the museum was not there and I was unsuccessful in convincing the personnel of the museum to open the bookshelf door to have a look at the book's contents. I left a note addressed to the director and came back the next week. At the time, I was the director of the mathematics department of the Escuela de Formación de Profesores de Enseñanza Media (EFPEM, a High School Teachers Training Centre) of the Universidad de San Carlos. The historical prestige of the university in the intellectual development of the country was crucial element to how events transpired. When I arrived at the museum, the director greeted me, and unlocked and opened the shelf. I quickly went over the contents. After the visit to the museum, the idea of a facsimile edition of the book started taking place. The museum was part of the Ministry of Anthropology and

History. I drafted a collaboration letter between the Faculty of Humanities (of which EFPEM was a part) and the ministry. Once the letter was signed, I had access to the book, under certain conditions. EFPEM hired a professional photographer with whom I travelled to Antigua Guatemala several times a week. The book was made available to us if the climatic conditions were deemed suitable for having the book exposed. After that, each page was photographed and printed, and the material was prepared to go to the press. That was the second part of the project. Unfortunately, I was not able to secure the money to print the 200 copies that had been planned. At the time, I was moving to Canada for one year, to work at a research centre of the Université du Québec à Montréal. I decided to leave the material to a professor of the Universidad de San Carlos with the hope that he would continue with the project. I kept with me the photo rolls. A few years later, when CDs were invented, the photos were saved on CDs, and some years later, technology permitting, transformed into PDFs. The idea of a digital version of Padilla's book was finally possible. The PDF can now be downloaded (http://luisradford.ca/publications/).

2 Historiography: mathematics and the cultural context

The relationship between a cultural context and the ideas that emerge and evolve in such a context has been a crucial question in disciplines such as anthropology (Geertz, 1983), epistemology (Foucault, 1966, 1969), and psychology (Luria, 1931, 1934). Yet, supported by rationalist epistemologies, mathematical ideas have long been mainly considered as independent of their cultural context. While it is true that within this tradition (which, for obvious reasons, we may term *a*culturist), it is conceded that the environment may have some influence on the evolution of mathematics (e.g., by accelerating or deaccelerating it), the cultural context is, at the same time, considered as something that cannot modify the mathematical content. That is, the context cannot determine the essence of mathematics and its objective nature.

The aforementioned relationship between context and mathematical ideas has been theorized in different ways. For instance, the reconstruction of mathematical ideas has been considered as having its internal law, while the context in which mathematics emerges and develops is assumed to have a developmental law that is external vis-à-vis the internal one. This is the view that Lakatos adopts in *The methodology of scientific research Programmes.* In the famous *Internal and external history* section of this book, Lakatos argues that "Rational reconstruction or internal history is primary, external history only secondary" (Lakatos, 1978, p. 118). Piaget and Garcia (1989) took a similar path. Although it was recognized in their work that society provides mathematical and scientific objects with specific meanings, Piaget and Garcia traced a clear frontier dividing the social and the individual. For them, a distinction must be made between mechanisms to acquire knowledge and the way in which objects are conceived by the subject. In a concise and clear phrase, they said: "Society can modify the latter, but not the former" (1989, p. 267; for a more detailed discussion, see Radford, 2000; Furinghetti and Radford, 2008). The aforementioned distinction between the social and the mathematical finds, I believe, its most tremendous tension in the work of Glas (1993).

Recent historiographical approaches have nonetheless stressed the local nature of mathematics and the manner in which mathematics is conceptualized and practised. In these approaches, context and mathematics cannot be separated (e.g., Høyrup, 2007; Lizcano, 2009; Rowe, 1996). To inquire about the history of mathematics is to inquire about mathematics as it was imagined, thought, and practised within a certain cultural context. But we can also go further and argue the opposite, namely that a certain cultural historical context can be understood only if we understand its intellectual life, and more specifically its mathematics. This is the line of inquiry that I follow in this article. More precisely, I wish to see how Padilla's *Arithmetica* fit and responded ideologically to the constraints generated by the economic, political, and educational colonial structures. To do so, before dwelling on the book and its author, in the next section I discuss the cultural context of Padilla's *Arithmetica*.

3 The cultural context of Padilla's *Arithmetica*

Padilla published his *Arithmetica* in the city of Santiago de Guatemala, which was founded in 1543. Santiago de Guatemala was one of the main political and military centres of the Spanish colonies—the two other major centres were in Mexico and Peru. Santiago de Guatemala hosted the Audiencia de los Confines, the Capitanía General, and other institutions whose goal was to regulate and control life in the colony.

By the early 18th century, when Padilla published his book, the Spanish colonizing apparatus had reached an extreme degree of sophistication. In its beginning, such an apparatus was organized around the concept of "encomienda." The encomienda was a political concept of government. Its theoretical foundation rested on the alleged social and natural inferiority of the Native people. Its real practical basis was to repay the conqueror by making him the guardian of the conquered land (Barbosa-Ramírez, 1971, p. 43). Conquerors received a certain number of Native people to be placed under their tutelage. As a result, they became "encomenderos," or trustees of those assigned to them. An encomendero had to oversee the Christianization of "his" Native people and organize their work on the lands. In exchange, the Native people entrusted to an encomendero had to pay him an annual tribute in cash, fruits, products of the land, and personal work (Contreras, 2007). Guzman-Bockler and Hebert summarize the encomendero as follows: "grim character, who to save the natives' soul, Christianized Indians, then, against all Christian rule, treated them as if they were beasts" (1975, p. 43). The encomienda evolved later into a concept of "repartition of natives." Organized by the mayor or corregidor, the repartition of Native people ensured the supply of labour for the Spaniard entrepreneurs. The encomienda and its evolved form —the repartition of Native people—were at the centre of the relations of production, which appeared as a complex system of production and extraction of goods, consumption, distribution, importation, and exportation. They were part of a political mechanism that sought to legitimize the subjugation of the Native people and the appropriation and distribution of production.

Historians of the Spanish colonisation have argued that the colonisation was carried out within the Spanish medieval epic mindset of the simultaneous religious and military war against Muslims. Guzman-Bockler and Hebert note that in the American continent "the

Christian Spanish caste repeated this holy war, but this time to Christianize. The American conquest is the culmination of the [medieval] Spanish epic effort" (1975, p. 42). But at the end of the 15th century, with the "Reconquista"—that is, the end of Muslim rule in Iberia—a new relationship to the land emerged. The concept of property changed and land appeared no longer as a medieval lordship space but "as an instrument of production where the essential factor is what the land can produce" (Barbosa-Ramírez, 1971, p. 30). It is not surprising, then, that with the arrival of the Spaniards in the American continent, the land became a new mode of exploitation with new characteristics alien to its Native people. This concept of land as something to be owned and as an object of production in the emerging modern sense of the term, was indeed at odds with the Native people's concept of land. In their communities, the land was an object of "biological work" (Barbosa-Ramírez, 1971, p. 59); that is, something through which they responded to the needs of their group. In the pre-Columbian communities, "there [was] an intimate cohesion between the individuals and the land of each community. . . The Indians fought throughout the colonial period to safeguard this coherence against the attacks of all adverse factors, including the Spanish mentality of possession" (1971, pp. 59-60).

When Padilla's *Arithmetica* was published, Santiago de Guatemala had about 38,000 inhabitants—about 25,000 "gente ordinaria" (ordinary people), 5,500 Spaniards, about 1,000 clergy, and the rest were Native people. Around 1549, Santiago "consisted of a central core of Spanish households and church and Crown institutions ringed by barrios populated by newly emancipated Indians" (Lutz, 1997, p. 155). In the early 17th century, "Spaniards in the urban core were most often merchants, owners of rural agricultural estates, encomenderos, master craftsmen, and government officials" (p. 159). When Padilla's *Arithmetica* was published, the city was heading towards a "gradual disintegration of the barrios and associated institutions… [The] urban core and periphery [were becoming] more similar in socioracial composition" (Lutz, 1997, p. 156). In the surrounding barrios

> castas (including ladinos), free blacks, urban Indians, and poor Spaniards constituted a multiracial urban *plebe común* (commoners) or "laboring poor," living together as neighbors, spouses, in-laws, employees, employers, attending the same churches, even sending their children to some of the same schools. They drank, celebrated, and mourned together, endured low wages and high food prices together. (Lutz, 1997, p. 160)

At that time, the city had well-established political and economic structures. The political structure included a General Captain, the Real Audiencia (which was a justice court comprised of a president, judges, and other public servants), a cabildo (a city council), and "serenos" whose role was to patrol the city. While the Real Audiencia was the direct representative of the Crown's interests, the cabildo was the political body of the most important Spanish local group. The Real Audiencia and the cabildo were generally opposed to each other in a contradictory relationship that always had as its basis the appropriation and distribution of the colonial production surplus. The city council designed and enforced a power apparatus to control the Native people, the repartition of the conquered land, and the economic organization of the colony.

The economic structure included, at the local level, some markets, shops, artisanal workshops, etc. A complex system of food supply was in place to bring to the city the goods it required on a daily basis. “[H]igher-priced goods included maize, wheat, and meat, as well as sugar, tobacco, eggs, poultry, meat byproducts, cotton thread, cloth dyestuffs, and clothing” (Lutz, 1997, p. 143). These and other goods were distributed through regular and black market circuits. At the international level, the economic structure included an importation and exportation system. It is important to bear in mind that the international commercial activity was shaped by an important factor: the Spanish Crown banned its colonies to participate in international trade. Thus, commercial transactions in colonial Guatemala were made primarily with Mexico and Spain, and included cacao, indigo, and other highly regarded goods in the international market. The craft industry also occupied an important place, with exports of silverware, paintings, ceramics, textiles, and leather products. Imported goods included wine, iron, clothing, ink, olive oil, sweets, weapons, and religious objects (Polo, 1988).

Along with the political and economic structures was an educational structure. These three structures were, of course, deeply intertwined. A predominant role in the educational structure was played by the church. An impressive array of religious orders travelled to Santiago with the aim of evangelizing the Native people. To do so, the religious orders quickly acquired lands that were sown and harvested by indigenous labour. In this manner the religious orders became active agents of the new political and economic apparatus of the colonial system. Referring to the Dominicans, Pinto Soria (1969, p. 57) writes: “the Dominicans became another oppressive group [hiding] a disguised form of domination.”

The lands that the religious orders acquired were not only sites of agricultural production. On these lands the religious orders also erected churches to which the so-called escuelas de indios (Native people’s schools) were often appended. In these schools the clerics started teaching the conquerors’ language and some basic techniques of agricultural production. These schools were part of the dissemination of an ideology pulled by the internal contradictions of the spiritual worldview of medieval Christianism and nobility on the one hand, and the ambition of becoming rich through the possession of the land and the exploitation of the mines, on the other hand. As we can see, to reduce the role of the church to an evangelizing mission would miss the most important point. Pinto Soria (1969) notes that the church was instrumental in breaking the Native people’s insurrectional spirit and in the expropriation of their lands. The church was also instrumental in stripping away their cultural traits and replacing them with a foreign worldview and values. As Pinto Soria (1969, p. 62) argues, “The Spanish Crown had in the Catholic Church a great ally; without the Catholic Church’s presence the imposition and maintenance of the colonial domination is almost unthinkable.”

The religious orders also created the “schools of first letters,” where children learned to read, write and count, and the Christian doctrine. Counting does not mean knowing the "core operations" of arithmetic only, but also the resolution problems through the Rule of Three, and applying this rule, to various kinds of problems—e.g., revenue sharing among members of a mercantile corporation.

In addition to the escuelas de indios and the schools of first letters, the religious orders also created houses for orphans and maidens (doncellas), and "colegios mayores" (i.e., advanced schools frequented by the Spaniards' children, generally intended to produce clerics). In these advanced schools, the focus was on the teaching of grammar, canons and theology. Mathematics was not a part of such curriculum. The assumption was that the education provided in elementary schools (the schools of first letters), should allow the children of merchants to continue studying at home (with the help of tutors) the most advanced commercial applications (cf. Gonzalez Orellana (1970), pp. 95-96).

The first of these advanced schools was the school of Santo Tomás created in 1529 as part of the Dominican Convent to provide instruction to the children of poor Spaniards. The historian Contreras writes: "The most important [of the colegios mayores] were the school of Santo Tomás and the school of San Francisco de Borja. They conferred titles of bachelor and masters (maestros and licenciados) to those without religious affiliation" (Contreras, 2007, p. 48).

When Padilla published his *Arithmetica*, Santiago was not only an accomplished political and military centre; it also had an intellectual and cultural life for its elite. It already had a university—the Universidad de San Carlos de Borromeo (Rodriguez Cabal, 1976), created in 1676—with studies in theology, law, and medicine. Santiago's cultural milieu at the time of Padilla included the printing press, which arrived in 1660, and the first newspaper, called *Gaceta de Goathemala*, which started circulating in 1729—only three years before the publication of the *Arithmetica*.

4 Who was Padilla?

Juan José de Padilla was born in the city of Santiago, studied theology, and served as Master of Ceremonies of the Cathedral. The Guatemalan historian Domingo Juarros (1808) notes that Padilla taught himself mathematics, a discipline in which he made great progress "with a few books." Padilla died on July 17, 1749, when he was over 65 years old. Juarros also tells us that Father Padilla was an excellent watchmaker. We know that he built the clock of one of the towers overlooking the College of Christ—a clock that marked the flowing of hours by sounds (Gavarrete, 1980, 268 p.). The *Gaceta de Goathemala* for the month of February 1730 highlights the merits of Padilla and refers to him as "famous in the art of making clocks of all sizes."[1]

We do not know what books Padilla read to learn mathematics, nor do we know exactly what books inspired his *Arithmetica*. Probably one of those "few books" to which Juarros refers is Joseph Zaragoza's *Trigonometria hispana resolution triangulorum plani, & sphaerici* (1673), which is mentioned on page 32 of the *Arithmetica as Trigonometria* only. As a reviewer of this paper noted, tt is not clear, if Padilla refers to the Latin version or to the previous Spanish version of the book, *Trigonometria española: resolucion de los triangulos*

[1] As mentioned before, the *Gaceta de Goathemala* began to circulate in November 1729. It was the first newspaper published in Guatemala and was the second journal in the American continent, preceded only by a few years by the *Gaceta de Mexico*, founded in 1722 (González Orellana, 1970, p. 165). The *Gaceta de Goathemala* played an important role in the dissemination of new scientific ideas (Cf. Tate, 1978, p. 261 ff.)

planos y esfericos, fabrica y uso de los senos y los logaritmos (Mallorca: Francisco Oliver, 1672). To quote the reviewer: “I think it is more plausible that the Spanish version travelled to America than the Latin one.”

Be it as it may, in Chapter V of the *Arithmetica*, Padilla mentions Simon Stevin’s *Disme*. He also mentions the Jesuit Andrés Tacquet without mentioning, however, the title of the work. As we shall see in Section 6, it is reasonable to assume the (direct or indirect) influence of the *Arithmetica demostrada teorico-practica para lo mathematico y mercantil* of Juan Bautista Corachan (1699/1719) and the *Tratado de mathematicas en que se contienen cosas de arithmetica, geometría, cosmographia, y filosophia natural* of Pérez de Moya (1573).

5 The *Arithmetica*

Written in Spanish, Padilla’s *Arithmetica* contains 237 pages (see Figure 1). The size of the pages is 16 cm x 12 cm. Before Chapter I, there are three pages containing a dedication to Saint Gertrudes; a tribute by the author to Jesus, Mary, and Joseph; and a definition of arithmetic. Page 237 of the book, which closes the last chapter of arithmetic, is followed by an index of six pages and two pages of errata.

Figure 1. Photo taken from the cover page of the copy of the *Arithmetica* in the Museo del Libro Antiguo, in Antigua Guatemala.

The content of the book is as follows:

Definition of arithmetic.

Chapter I. Of the letters or characters of arithmetic and mode of numbering [modo de numerar].

Chapter II. Of the four rules of arithmetic.

Chapter III. Broken Numbers [i.e., fractional numbers]

Chapter IV. Of the four general rules with broken numbers.

Chapter V. Of decimal calculations.

Chapter VI. Of powers of numbers and their roots.

Chapter VII. Of proportions.

Chapter VIII. Of progressions.

Chapter IX. Of the Rule of Three.

Chapter X. Of the rules to find the measure of plane and solid [objects]

Chapter XI. Of the rules of combinations and permutations.

Chapter XII. Of other counts and things of Arithmetic.

Padilla's *Arithmetica* was intended for students attending the schools of first letters, while allowing the children of the Spaniards devoted to trade to continue the study of arithmetical methods. Those students probably took private lessons on the subjects addressed in the *Arithmetica*. It is likely that Padilla gave private lessons and that his book is the result of these lessons. What we do know for sure is that, in the early 19th century, the Bishop of Guatemala, Cayetano Francos y Monroy, recommended that the *Arithmetica* be used in his schools of first letters.

The *Arithmetica* is considered the first *pedagogical* treatise of Colonial Guatemala. The content of the books that were published in Santiago before the *Arithmetica* and even after were indeed of a religious or historical nature.[2] Thus, the first book published in Santiago was *Explicatio Apologetica,* written by Fray Payo Enríquez and published in 1663, followed by Roque Núñez' (1673) *Solemne Novenario*, both from the same publishing house that published the *Arithmetica*.

In the *Arithmetica*, Padilla deals with different types of problems of a commercial nature that were relevant in the economy of the colonial period. Let me mention three here.

A first type of commercial problem is the so-called "society problems." A number of people invest different amounts in a business and the problem is to determine how the profit should be distributed. This type of problem was solved using the Rule of Three.

A second type of commercial problem revolves around the mixing of products; that is, how to calculate the price of a product from the price of its components. In Section 12 of Chapter 9, a section entitled "From the rule of three to compound and mix prices and various other things" we read:" This rule teaches first to calculate an average price or value of a mixture: as if several portions of indigo ink of several prices are mixed, we have to find the price of the resulting mix"(Padilla, 1732/2013, p. 146).

A third type of problem deals with the calculation of the amount of ingredients of known prices to be mixed to obtain a mixture at a given price. This type of problem has a wide variety of applications: for example, mixing different qualities of wines, such that the price of the resulting mixture is attractive for sale. Another example is the mixture of precious metals in the craft industry or in minted coins.

[2] See https://issuu.com/informaticapatrimonio/docs/catalogo_museo_del_libro_antiguo.

Let us see in more detail how Padilla tackles the mixing problems (the aforementioned third type of commercial problems). Padilla observes first that the final price of the mixture must be chosen between the highest price and the lowest price of the ingredients of the mixture. The solution is based on proportional reasoning, which takes into account the difference between the price of the final mixture and the price of the ingredients. The numbers are placed around a cross, allowing for convenient and easy organization of data in order to apply multiple rules of three. Padilla states the rule to solve those problems as follows:

> Elegido el precio medio entre el menor, y mayor de todos los diverƒos, que ƒe han de meƒclar, ƒe ƒacará la diferencia, que hai del medio elegido â cada vno de los otros: y eƒtas diferencias en derecho de los precios; pero cada vna en derecho del precio opueƒto: eƒto es que las diferencias, que ƒe ƒacaren del precio medio â los infimos, ƒe pongan con los ƒupremos; y las que ƒe ƒacaran del medio â los ƒuperiores, ƒe pongan con los inferiores. Y quando los ƒuperiores ƒon mas, que los inferiores, ƒe repite el mas inferior; y ƒi al contrario los inferiores ƒon mas, que los ƒuperiores, ƒe repite el mas ƒuperior. (Padilla, pp. 148-149)
>
> [Having chosen a price between the lowest and highest [price] of the diverse [components] to be mixed, calculate the difference between the chosen price and each one of the other prices. Put the difference to the right of the prices; but each to the right of the opposite price: that is, the differences between the chosen price and the prices lower [than the chosen price] have to go with the prices that are higher [than the chosen price]; and the differences between the chosen price and the higher prices have to go with the lower prices. And when the higher prices are more than the lower prices, repeat the lowest price [as much as required]; and if by contrast the lower prices are more than the higher prices, repeat the highest price [as much as required. (Padilla, p. 146)]

Padilla gives the following example:

> Como ƒi ƒe han de meƒclar dos porciones vna del precio de â 2, y otra de â 7, y ƒe quiere que la meƒcla ƒalga â 5: ƒaqueƒe la diferencia de 5 â 2, y pongaƒe en derecho del precio 7: y la diferencia de 5 á 7 pongaƒe con el precio 2. (p. 149)
>
> [As if you have to mix two parts, one of a price of 2, and the other of a price of 7, and you want the mixture to be of a price of 5: calculate the difference between 5 and 2, and put it to the right of price 7: and put the difference between 5 and 7 to the right of the price 2. (Padilla, p. 149)]

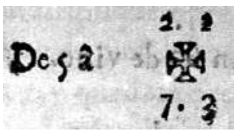


Figure 2. Shows the organization of data around the cross.

The distribution of numbers around the cross is the first step in solving the problem. Padilla spends some time explaining this step through two more examples. It is only when he has sufficiently explained the first step that he ventures into explaining the second step, which consists of a series of calculations with the numbers around the cross. He considers the following problem: to find the amount of water and wine to be mixed in order to produce 100 quartillos of wine to be sold at 3 reales/quartillo, knowing that the wine to be mixed costs 5 reales/quartillo. The calculations are expressed as follows:

> calculate the difference between 5 and 3, and the difference between 3 and 0 [the cost of water]. And put each one with the opposite price, and add, which gives 5. This will be the first term of the rule. The second term will be 100, the third 3 and 2. And to calculate each fourth term it will be better to divide 100 by 5 and then to multiply 20 [the result] by 3, and 20 by 2, and you get 60 quartillos of wine and 40 [quartillos] of water (p. 150)

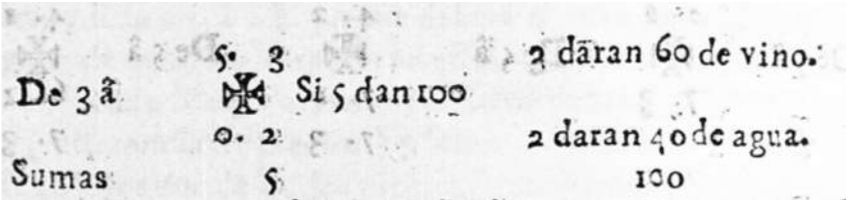


Figure 3. Padilla’s distribution of numbers and the two rules of three that solve the problem.

The known prices are placed on the first column (left of the cross). On the second column, the differences between the prices and the chosen price (3 reales in this example) are placed, but with the opposite prices. Therefore, the rule is: If 5 [the sum of differences] give 100, the first difference (the second difference, respectively) gives the amount of wine (the amount of water, respectively) to be mixed.

Padilla gives more examples, one with three ingredients to be mixed. Translated into modern symbolism and its concomitant ideas, the problem becomes an indeterminate linear system. By repeating the highest or lowest price [as much as required], the indetermination is removed and one possible solution is found (see Radford in Padilla 1732/2013).

6 Summary and concluding remarks: The ideology of the *Arithmetica*

I started this article with a short account of my encounter with Padilla’s *Arithmetica* and the contextual problems that I found in order to offer a facsimile edition of it. My hope was that making available the oldest known colonial book of mathematics of what is today called Central America would provide us with an interesting window through which to better understand the history of mathematics. From this line of thought, Padilla’s *Arithmetica* appears as a cultural artifact that refracts the mathematics that was practised in the colony. This is the argument that I submitted in Section 2, where I argued that to inquire about the history of mathematics is to inquire about mathematics as it was imagined, thought, and practised within a certain cultural context. I also argued the value of considering the opposite proposition, namely that the understanding of a certain cultural historical context can only be achieved if we understand its intellectual life, and more specifically, its mathematics. While in a previous article devoted to the *Arithmetica* (Radford, 2007; reproduced in Padilla (1732/2013)) I engaged with the artefactual view of the book, in this paper I moved to the second (opposite and dialectically complementary) view. I wished to see how the book fit and responded ideologically to the constraints generated by the economic, political, and educational colonial structures. I think that this line of investigation has not yet been explored in past or contemporary historiographical approaches to the history of mathematics (with perhaps the exception of Høyrup (2007), Restivo (1992, 1993) and a few other scholars). In the second view, mathematics and mathematicians are investigated as elements of an ideological apparatus. By ideological I do not mean something as a false consciousness or as a deception. I rather mean a system of cultural ideas in which mathematics and mathematicians unavoidably live, breathe, think, and act. Padilla's *Arithmetica* appears in this view as a book that conveys a worldview—the mercantilist view that started being shaped at the end of the Middle Ages and the dawn of the Renaissance in Europe and of which the *Tratado* of Perez de Moya (1573) and the *Arithmetica demostrada* of Corachan (1699) are two extraordinary examples. These books, which may have influenced Padilla, deal indeed with mixing problems in a manner that is similar to the one we find in Padilla's *Arithmetica* (see Perez de Moya, 1573, p. 290; Chorachan, 1699, p. 295). More research is required in order to ascertain the differences between Perez de Moya and Padilla's methods. But the point, along the lines of the second view, is that Padilla's *Arithmetica* comes to be part of and support an oppressive economic and political apparatus—one that distinguishes, for example, the education of Spaniards from that of the Native people; one that reaffirms the Spaniards and their children as the masters and the Native people as their slaves—even if "theoretically" by Crown law they are not slaves. By appearing as it does, the book naturalizes the oppression of the system. It helps to offer the practical knowledge required to maintain specific forms of the production of life and existence in the colony—both in its material, intellectual, and spiritual dimensions.

Does it mean that we should have expected Padilla to embrace the Native people's cause and fight for them, as the Dominican Bartolome de Las Casas did in the 16th century? While de Las Casas' countrymen saw in the Native people a formidable means to enrich themselves by occupying their lands and exploiting them as free labour, the Dominican priest made, through the presence of the Native people, the extraordinary cultural experience of *alterity*; that is, the encounter of the Other (de Las Casas, 1552/1994). Referring to his countrymen's actions vis-à-vis the Native people, de Las Casas notes: "I do not say that they [the Spaniards] want to kill them [the Indians] directly, from the hate they bear them; they kill them because they want to be rich and have much gold, which is their whole aim, through the toil and sweat of the afflicted and unhappy" (Cited in Todorov, 1984, p. 142). De Las Casas was confronted in an extraordinary new way by the problem of the Other, anticipating the current problems of social justice and equity with which contemporary societies are faced today and which are, to a large extent, sequels of colonialism.

Acknowledgments

This article is a result of a research programs funded by the Social Sciences and Humanities Research Council of Canada / Le conseil de recherches en sciences humaines du Canada (SSHRC/CRSH).

I would like to dedicate this short paper to the memory of my Mexican colleague and friend José Guzmán Hernández, who passed away unexpectedly on March 24 2016.

REFERENCES

Barbosa-Ramírez, A. (1971). *La estructura económica de la Nueva España*. 10th ed. Mexico: Siglo veintiuno.

Contreras, D. (2007). *Breve historia de Guatemala.* Guatemala: Editorial Piedra Santa.

Corachan, J. B. (1699). *Arithmetica demostrada theorico-practica para lo mathematico y mercantil.* Valencia: Jayme de Bordazar.

de Las Casas, B. (1552/1994). *Brevísima relación de la destrucción de las Indias.* Barcelona: Planeta.

Foucault, M. (1966). *Les mots et les choses [The order of things].* Paris: Éditions Gallimard.

Foucault, M. (1969). *L'archéologie du savoir [The archeology of knowledge].* Paris: Éditions Gallimard.

Furinghetti, F., & Radford, L. (2008). Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics. In L. English (Ed.), *Handbook of international research in mathematics education (2nd edition)* (pp. 626 - 655). New York: Taylor and Francis.

Gavarrete, J. (1980). *Anales para la historia de Guatemala*: *1497-1811*. Guatemala: Ed. José de Pineda Ibarra.

González Orellana, C. (1970). Historia de la educación en Guatemala. Guatemala: Editorial José de Pineda Ibarra. Reimpresión Imprenta Universitaria. Universidad de San Carlos (Cuarta edición revisada y aumentada, 1987).

Geertz, C. (1983). *Local knowledge.* New York: Basic Books.

Glas, E. (1993). Mathematical progress: Between reason and society. *Journal for General Philosophy of Sciences*, *24*(Part 1: 43-62 Part 2: 235-256).

Guzmán-Böckler, C., & Herbert, J. (1975). *Guatemala: Una interpretación histórico-social.* México: Siglo XXI.

Høyrup, J. (2007). The roles of mesopotamian bronze age mathematics tool for state formation and administration. *Educational Studies in Mathematics*, *66*(2), 257-271.

Juarros, D. (1808). *Compendio de la historia del reino de Guatemala*. Guatemala: Imprenta de Ignacio Jacobo de Beteta.

Lakatos, I. (1978). *The methodology of scientific research programmes. Philosophical papers, volume 1.* Cambridge: Cambridge University Press.

Lizcano, E. (2009). *Imaginario colectivo y creación matemática [collective imaginary and mathematical creation].* Madrid: Gedisa.

Luria, A. R. (1931). Psycological expedition to central Asia. *Science*, *74*, 383-384.

Luria, A. R. (1934). The second psychological expedition to central Asia. *Journal of Genetic Epistemology*, *41*, 255-259.

Lutz, C. (1997). *Santiago de Guatemala, 1541-1773.* Norman: University of Oklahoma Press.

Padilla, J. (1732/2013). *Noticia breve de todas las reglas mas principales de la arithmetica practica con la que se pueden desatar, no solo las demandas ordinarias, sino tambien muchas dificultades, que de otra suerte solo por el algebra se respondieran.* Santiago de Guatemala: Imprenta de Ignacio Jacobo Beteta.

Perez de Moya, J. (1573). *Tratado de mathematicas en que se contienen cosas de arithmetica, geometria, cosmographia, y philosophia natural.* Alcalá de Henares: Juan Gracian.

Piaget, J., & Garcia, R. (1989). *Psychogenesis and the history of science.* New York: Columbia University Press.

Pinto Soria, J. (1969). *Raices históricas del estado en centroamérica [Historical roots of the central american state].* Guatemala: Editorial Universitaria.

Polo, F. (1988). *Historia de Guatemala*. Editorial Everest: Guatemala.

Radford, L. (2000). The historical development of mathematical thinking and the contemporary student understanding of mathematics. Introduction. In J. Fauvel, & J. Maanen (Eds.), *History in mathematics education. The ICMI study* (pp. 143-148). Dordrecht: Kluwer.

Radford, L. (2007). La arithmetica practica del padre Padilla y los inicios de la matemática en Centro América en el periodo colonial. *Revista Brasileira da História da Matemática*, *7*(14), 193-211.

Restivo, S. (1992). *Mathematics in society and history, sociological inquiries.* Dordrecht Boston London: Kluwer Academic Publishers.

Restivo, S. (1993). The social life of mathematics. In S. Restivo, J. P. V. Bendegem, & R. Fischer (Eds.), *Math worlds* (pp. 247-278). New York: State University of New York Press.

Rodriguez Cabal, J. (1976). *Universidad de Guatemala.* Guatemala: Editorial Universitaria.

Rowe, D. (1996). New trends and old images in the history of mathematics. In R. Calinger (Ed.), *Vita mathematica. Historical research and integration with teaching* (pp. 3-16). Washington: NCTM.

Tate, J. (1978). *La ilustración en la Universidad de San Carlos*. Guatemala: Editorial Universitaria.

Todorov, T. (1984). *The conquest of America. The question of the other.* New York: Harper Perennial.

Zaragoza, J. (1673). *Trigonometria hispana. Resolutio triangulorum plani, & sphaerici, constructio sinuum, tangentium, secantium & logarithmorum, eorumque usus*. Valencia: Hyeronimum de Villagrassa

MACHINES DESIGNED TO PLAY NIM GAMES

Teaching supports for mathematics, algorithmics and computer science (1940-1970)

Lisa ROUGETET
Université Charles de Gaulle – Lille 3, Domaine Universitaire du Pont de Bois, rue du Barreau, Bâtiment F, Bureau F1-37, 59653 Villeneuve d'Ascq, France
lisa.rougetet@gmail.com

ABSTRACT

This article deals with Nim games and machines built to play against a human being between the 1940s and the 1970s. They were designed not only to entertain, but also to explain concepts in mathematics, algorithmics and computer science to a general public. Moreover, they were exhibited during fairs or science shows and manufactured later for personal use. Nim games are take-away games without chance whose winning strategy relies on the binary system, easily characterized by bistable circuits called flip-flops. The first electromechanical Nim player machine, called The Nimatron, was invented in 1940 and exhibited during the New York World's Fair. Its success led to the construction of another electromechanical machine (1951). Then electric or purely mechanical inventions were patented for their cheaper production cost and their pedagogical aspects to understand elementary level instructions in computers as well as the rules of the binary system and notions of Boolean algebra. This article provides examples of such machines and shows their pedagogical interest.

1 Introduction: what is a Nim game?

Nim game falls within the class of combinatorial games. In a combinatorial game, there are only two players, playing alternately. Usually, there are a finite number of positions and the information is complete – which means both players know what is going on at any moment of the game. There are no chance moves such as rolling dice or shuffling card and in the *normal* play convention the player who finds himself unable to play loses.[1]

Nim game is considered as a take away game: usually three (or more) piles of counters are set on a table. Each pile contains a different number of counters. Alternatively, both players select one of the piles and remove as many counters as they want: one, two… or the whole pile. The first player who takes the last counter(s) wins the game.

Nim game was first mentioned under this name in a 1901 article published in the *Annals of Mathematics* and written by a mathematician from Harvard: Charles Leonard Bouton (1869-1922). In this article, Bouton gives the complete mathematical solution to Nim (Bouton 1901), i.e. a strategy to win every game; first, each pile of counters must be written in the binary scale. Then these binary numbers are placed in three horizontal lines so that the units are in the same vertical column. The sum of each column is calculated and if all of them are congruent to 0 mod. 2, the position left on the table is called a *safe combination*. Such positions should be reached at each move in order to win the game, and as soon as we reach one, it is possible to obtain another one on our next move, but not for our opponent. Indeed,

[1] In the *misère* play convention the player who finds himself unable to play wins.

safe combinations verify the following properties: "I. if A leaves a safe combination on the table, B cannot leave a safe combination on the table at his next move. II. If A leaves a safe combination on the table and B diminishes one of the piles, A can always leave a safe combination" (Bouton 1901, p. 36). This use of the binary system is the basis of the functioning of the future machines designed to play Nim. Thus the game of Nim is particularly suited to the use of the usual binary computing elements, such as bistable circuits (flip flops). Later, *Geniacs*, simple electronic brain machines, was to be designed for educational purposes to translate from decimal notation to binary notation and vice versa, but also to add, multiply and compare numbers in the binary system (Geniacs 1955a).

Figure 1. A young lady playing against Nimatron (Condon, 1942, p. 330)

Bouton's article is considered as the starting point of the relatively recent (twentieth century) mathematical theory of combinatorial games. The theory was developed among the mathematical field and became a beautiful abstraction with John Conway's surreal numbers in 1976. Conway's construction admirably generalizes Dedekind cuts and his theory distances itself from its subject of study, namely games. Meanwhile, Nim game was introduced to the general public through machines designed to play against human players – and always win – but the initial ambition was to explain to people how machines operated, and which mathematics was behind. We will see that the first machines were so big that they were exhibited during fairs or science shows. Then, personal machines, totally mechanical or slightly electric, were designed, especially to explain basis of computer science (and also to reduce the production costs). Nowadays, these machines are of highly interest since the new French educational program in mathematics high school classes, which will come into effect in September 2016, recommends to develop the new theme "algorithmic and programing" through examples of games to help pupils to break down a problem, recognize ways of thinking, translate logical thinking to conditional instructions and create an algorithm that solves the problem.

2 Machines exhibited during fairs and science shows

In the spring of 1940, an electromechanical Nim player machine weighing a ton, called *The Nimatron* (see Figure 1.), was exhibited at the Westinghouse Building of the New York World's Fair and played more than 100 000 games (and won 90 000 of them). Two members of the Westinghouse Electric Company staff invented it during their lunch break. Condon, the signatory of the US Patent (Condon et al. 1942), underlined the entertaining purpose of the

Nimatron, but also specified that it could illustrate "how a set of electrical relays can be made to make a decision in accordance with a fairly simple mathematical procedure" (Condon 1940, p. 330).

The Nimatron was set to play Nim with 4 piles containing up to 7 counters. The human player began the game, and only 9 initial configurations were possible (due to space limitations), each of them was unsafe combination, so that the human player had a chance to win. The lamps (which can be seen on Figure 1) *a*1 to *a*7 (7 lamps for the 7 counters of the column *a*), *b*1 *to* b7, *c*1 to *c*2 and *d*1 to *d*7 were connected in circuits, which were controlled by relays A1 to A7, B1 to B7, C1 to C7 and D1 to D7 respectively, each of them controlled by a master relay A, B, C and D (one for each column). Other relays, AZ, BZ, CZ and DZ were actuated when the number of energized lamps in the corresponding columns *a*, *b*, *c* and *d*, respectively, contained a zero power of 2; the relays AF, BF, CF and DF were actuated when the number of energized lamps in the corresponding columns *a*, *b*, *c* and *d*, respectively, contained a first power of 2, and the relays AS, BS, CS and DS were actuated when the number of energized lamps in the corresponding columns *a*, *b*, *c* and *d*, respectively, contained a second power of 2 (the maximal number of lamps being 7, 111 in binary). To properly play the game, the machine had to determine whether any power of 2 was contained in the number of energized lamps in an even or an odd number of columns. If the three powers of 2 appeared in an even number – which meant that the human player left a safe combination – the machine played randomly, otherwise it analyzed in which column a change should be made to obtain an even number of the three powers of 2.

In 1942, the Nimatron was exhibited for the last time at the convention of the Allied Social Sciences associations in New York City under the sponsorship of the American Statistical Association and the Institute of Mathematical Statistics. Then the machine belonged to the scientific collections of the Buhl Planetarium in Pittsburgh (Condon 1940, pp. 330-331).

A few years later, Ferranti, the electrical engineering and defense electronics equipment firm, designed the first digital computer dedicated to play Nim, *The Nimrod*. It was exhibited at the Festival of Britain (Exhibition of Science) in May 1951 and afterwards at the Berlin Trade Fair (Industrial Show) in October of the same year. These exhibitions were a great success and many witnesses related that the most impressive thing about the Nimrod was not to play against the machine but to look at all the flashing lights which were supposed to reflect its thinking activity. It had even been necessary to call out special police to control the crowds (Gardner 1959, p. 156). This particular display was built for the purpose of illustrating the algorithm and the programming principles involved. The instructions followed by the Nimrod were written on the left side, as shown in Figure 2. More over, a booklet was available for visitors,[2] for the price of one shilling and six pence, which contained a lot of information about automatic digital computers in general. The introduction states: "the

[2] The booklet, released in 1951, *The Ferranti Nimrod Digital Computer*, is available at the following address: http://goodeveca.net/nimrod/NIMROD_Guide.html

machine has been specially designed to demonstrate the principles of automatic digital computers […] the booklet has been prepared for those persons who desire to learn a little more about computers in general and of Nimrod in particular." Explanations of main notions, such as "electronic brains", "calculating machines", "automatic computers", 'automatic sequence control" and characteristics of automatic computers (calculation, memorization and making decisions) are given in the first part of the booklet. The second part is devoted to Nimrod functioning ("some details of the machine", "the way in which Nimrod plays Nim"…). "Nimrod has been designed so that it can play the game of Nim with an opponent from the general public or it can be given a "split personality" so that for demonstration purposes it will play a game without an opponent" (Nimrod 1951).

For instance, the instructions run by the program while Nimrod plays Nim are the following:

1. Examine the column specified by the column counter. If this is even transfer control to 7.
2. Examine the digit specified by the column and heap counters. If this is 1 transfer control to 4.
3. Add 1 to heap counter. Transfer control to 2.
4. Examine heap selected memory. If a heap has previously been selected transfer control to 6.
5. Operate heap selected memory.
6. Substitute modifying number in heap specified by heap counter.
7. Deduct 1 from column counter. If this does not complete examination of columns transfer control to 1.
8. STOP.

This terminology is specific to algorithmic and programming: a set of instructions is provided, ordered in a logic way and described with conditional statements proper to Boolean data type. In a common language that everyone can understand, the procedure executed by the machine is thus explained (the earliest high-level programming languages with strong abstraction were written in the 1950s).

Nimrod booklet also contains a glossary with definitions of the terms used. The authors highlight the quick evolution of automatic digital computers during the 1950s and want to clarify the new terminology that arose for describing the machines. Once again, there was a real will to use the simplest possible explanations in order to embrace the widest field. We do not know for sure if the Nimatron or the Nimrod had a pedagogical impact in mathematics education. But few years later, smaller machines, cheaper to produce, appeared for pedagogical purposes, and were intended for a wide audience.[3] They are the subjects of the next section.

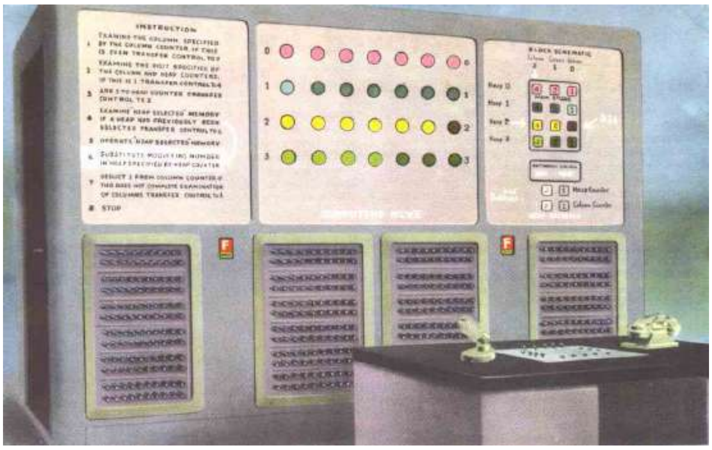


Figure 2. Drawing of the Nimrod, with instructions on the left side. “The display in front of the machine is used to demonstrate the process involved when Nimrod carried out a move” (Nimrod, 1951)

Figure 3. *Popular Electronics* cover, January 1958

Figure 4. *Radio Electronics* cover magazine, October 1950

[3] Other machines designed to play Nim were created between 1941 and 1958, but in a more mathematical sphere. In 1941, an assistant professor of mathematics at the University of California at Los Angeles (Gardner 1959, p. 156), Raymond Moos Redheffer, improved considerably the Nim-playing machine. (Redheffer 1948) To our knowledge, Redheffer’s machines were not exhibited to a broad public, consequently they were less known. In 1952, engineers from W. L. Corporation, Hubert Koppel, Eugene Grant and Howard Bailer, developed a lighter machine than Nimatron or Nimrod, as it weighed less than 25 kg and cost $2000 to build. We would like to note that machines mentioned in this note had no clearly expressed pedagogical or educational aspirations and were probably not much widespread. Nevertheless, Pollack’s DEBICON (1958) can be found on *Popular Electronics* magazine cover, which soon became the “World’s Largest-Selling Electronics Magazine”. (see Figure 3)

3 Machines designed for personal use

Since 1945 there has been interest in helping people understand how automatic machines reason, calculate, and behave.

> And we know that equipment that you can take into your hands, play with, and do exciting things with, will often teach you more, and give you more fun besides, than any quantity of words and pictures. (Geniacs, 1955a, p. 2)

This justifies the educational toy *Geniac* designed and marketed by Edmund Berkeley and Oliver Garfield between 1955 and 1958. Edmund C. Berkeley (1909-1988) was a mathematician, insurance actuary, inventor, publisher, and one of the founders of the Association for Computing Machinery (ACM) (Longo, 2015). In 1949, he published a book titled *Giant Brains or Machines That Think*, which was the first explanation of computers intended for a general readership.[4] In the 1950s, Berkeley developed mail-order kits for small, personal computers such as *Simple Simon* and the *Brainiac*. At a time when computer development was on a scale barely affordable by universities or government agencies, Berkeley took a different approach and sold simple computer kits for average Americans. He believed that digital computers, using mechanized reasoning based on symbolic logic, could help people make more rational decisions. These considerations show that the idea of handling objects to help the teaching of mathematical – here logical – concepts is not new. It has been shown (Brougère 1995) that until the 20th century, games were not considered as direct educational tools. "Recreation is essential but game has no status beyond it." (Brougère, 1995, p.135) This position has been evolving and now it is acknowledged that to construct a mathematical concept, a first phase of action is essential to build a mental representation (even if this handling phase alone cannot be enough to learn, and a mediation with the teacher – or someone who knows – is necessary). Construction kits such as *Simple Simon* and *Geniac* provide the necessary material to grasp mathematical and easy computer science notions and could be revisited nowadays in mathematic classes to illustrate the procedure of an algorithm.

When it was released in 1950, *Simple Simon* was the "World's Smallest Electric Brain" (see Figure 4). It weighed 39 pounds and showed how a machine could do long sequences of reasoning operations. "The machine itself has been demonstrated in more than eight cities of the United States."[5] (Geniacs 1955a, p. 1) "He will be useful in lecturing, educating, training and entertaining." (Berkeley & Jensen 1950, p. 29) By 1959, over 350 sets of Simon plans had been sold, but it cost over $300 for materials alone, and Berkeley and Garfield admitted: "it is therefore too expensive for many situations in playing and teaching." That is the reason why they worked four years long to develop a really inexpensive electric brain: *Geniac*, a construction kit costing less than $20.

---

[4] His journal *Computers and Automation* (1951-1973) was the first journal for computer professionals.

[5] Simple Simon was exhibited in New York, Seattle, Philadelphia, Boston, Washington, Detroit, Minneapolis, Pittsburgh, and other smaller cities. The fact that Berkeley could take Simon from place to place meant that students and other non-experts could have first hand contact with automatic computing equipment "for real".

3.1 Geniacs

The name Geniac stood for "Genius almost-Automatic Computer" (Geniacs 1955a, p. 2). The construction kit consisted in 30 small electric brain machines – each one being a Geniac – which could be made with very simple electrical equipment. The guide supplied with the kit first gave a general description of the material and the way the different components worked.

One of the proposed problems was to design a Geniac that could play Nim in normal convention with four piles of matches, containing respectively 4, 3, 2 and 1 matches. The solution of the wiring is shown Figure 5 and Figure 6.

Besides the construction of machines to play games such as Nim, Tit-Tat-Toe or to answer recreational mathematical riddles such as the Two Jealous Wives, the kit provided other Geniacs to illustrate more purely mathematical problems such as the adding machine, the multiplying and the dividing machines, or the machine for arithmetical carrying (Geniacs 1955a, p. 4). A 1958 advertisement explained all the interesting aspects of Geniac and highlighted its popularity, its pedagogical interest and its low price (see Figure 7).

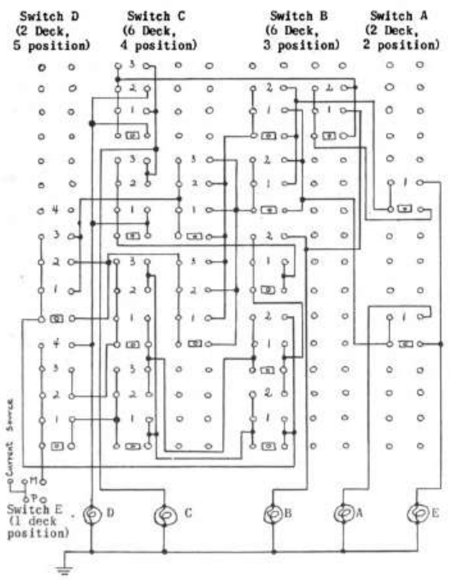


Figure 5. (Geniacs 1955a, p. 36)

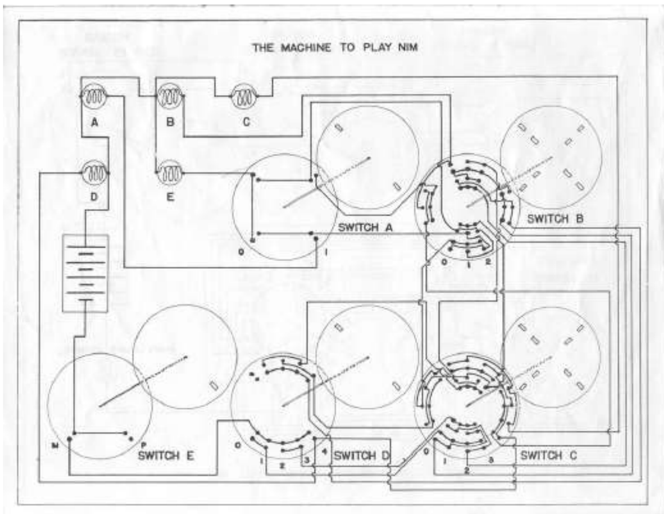


Figure 6. (Geniacs 1955b, p. 15)

For only $19,95, Geniac offered a complete course in computer fundamentals used by thousands of colleges, schools and private individuals. It seems that in October 1958, more than 30,000 Geniacs kit were in use by satisfied customers. The advertisement clearly underlined the pedagogical purposes of Geniac for understanding notions of mathematics and computer engineering:

> Geniac is a genuine electric brain machine, not a toy. The only logic and reasoning machine kit in the world that not only adds and subtracts but presents basic ideas of cybernetics, Boolean algebra, symbolic logic, automation, etc. So simple to construct that a twelve-year-old can construct what will fascinate a Ph.D. (Geniac 1958, p. 29)

Figure 7. Advertisement for Geniac, (Geniac 1958, p. 29)

Berkeley designed Geniac to be a tool for educators and it seemed it had some success in this area (Longo 2015). In 1958, the *Mathematical Gazette* published an article of a mathematics teacher, Martyn H. Cundy, who developed plans for a binary adding machine for classroom use. The machine could add two binary numbers of two digits or of three digits (with or without carry). Cundy credited his work to Geniac that taught him the fundamentals for building his own machine and justified that some knowledge of binary arithmetic should be part of the mathematical equipment of the normal grammar-school pupil (Cundy 1958, p. 272). In 1956, in the magazine about education *Phi Delta Kappan*, Daniel Davies[6] covered "breakthroughs" in educational administration, in which he detailed areas where new developments were having impacts on education. These included mathematics, for example game theory or binary numbers systems, and Davies claimed: "Boolean algebra is already at work in problem solving. One firm is advertising a kit for setting up an ingenious device known as Geniac which can quickly solve a wide range of problems involving multiple choices." (Davies 1956, p. 276) Moreover, as the 1958 advertisement stressed it: "In addition

---

[6] Daniel R. Davies was executive director between 1954 and 1959 of the UCEA (University Council for Educational Administration), an organization aimed to improve the professional preparation of educational administrators.

to its value as a source of amusement and education the kit exhibits certain technological features that may have widespread implications in other areas." (Geniac 1958, p. 28)

The use of Berkeley small electrical brain machines in classroom and how they impacted the teaching of mathematics is difficult to ascertain and this part of the work is still in progress. However, popularity of Geniac during the 1950s was an acknowledged fact, as a lot of electronics magazines or science journals can prove it.[7]

3.2 Machines or computer type devices patented

We have already mentioned that the main problem of electromechanical machines such as *Nimrod* and *Nimatron* was, first of all, their size, and also their expensive production cost: "Standard electronic computers […] have been both bulky and expensive." (Du Bosque 1962) That is why during the 1960s electric or purely mechanical inventions, in the same vein as Geniac,[8] were patented for their "durability and reliability in use." (Weisbecker 1968) For example, Joseph Weisbecker's invention related to a unique mechanism in the nature of a computer for use as a toy, game, puzzle or educational device, "say to illustrate computer operation and logical techniques […]" (Weisbecker 1967).

Moreover, these inventions permitted "the achievement of elementary level instructions in computers." (Godfrey 1968) "[…] the invention is an educational device for indicating the best play to be made […]" (Du Bosque 1962). "It is a principal object of the present invention to provide improved educational game apparatus which permits the learning of strategy techniques, logic methods, and mathematical systems." (Morris 1971) Authors of such patents also justified the interest of their machines by filling the gap left "in the ability of the student to understand and comprehend what a computer is all about" since the venue of high-speed electronic digital computers (Godfrey 1968). They emphasized the importance of presenting an invention that would provide a game and a teaching aid "so as to attract persons of higher intellectual level, while maintaining relative simplicity for attention arresting use as a toy by relatively young children." (Weisbecker 1968)

3.3 A marketed patent: *Dr. Nim*

Some of these inventions were marketed and sold as family parlor games. *Dr. Nim* is an example: it was manufactured by E.S.R. Inc., a company specialized in education toys during the 1960s, and was equivalent to a single pile Nim game where 1, 2 or 3 counters could be removed at each move. Dr. Nim was played by one player – against the machine – and offered several starting positions: the game could be played in normal or in misère convention (the last player to move loses) and the initial number of marbles could vary between 9 and 20 (see Figure 8).

---

[7] For example *Popular Electronics* or *Galaxy Science Fiction* magazines. A Ngram research in the English Google books corpus shows a net increase of the use of the term "Geniac" between 1955 and 1960: https://books.google.com/ngrams/graph?content=Geniac&year_start=1940&year_end=2000&corpus=15&smoothing=3&share=&direct_url=t1%3B%2CGeniac%3B%2Cc0

[8] By the way, during the early 1960s, Berkeley created other electric brain machines with Brainiacs and Tyniacs kits.

Dr. Nim device included a plurality of flip-flops – bistable circuits (see Figure 9) – being moved by marbles when they fell down, “so as to allow mathematical computations to be effected upon binary numbers to which the flip-flops are set.” (Godfrey 1968)

> Inclusive among the concepts which may be explained and understood by this computer invention are the following: the binary number system; the simplicity for machine design using binary arithmetic; […] the rule for binary counting and addition; modular arithmetic; the use of two’s complement arithmetic to achieve subtraction with only its add capability; […] binary multiplication. (Godfrey 1968)

Like the Nimrod, Dr. Nim provided a manual of fully detailed instructions (23 pages, A4 paper sized) with the rules of the game, its variations, how it was programmed… and also deeper considerations such as “can machines really think?” Few pages were devoted to the explanation of Boolean algebra in use behind flip-flops mechanisms. A capital letter was assigned to every flip-flop (A, B, C, D and E) and a bar was put over the letter when the flip-flop was open, for instance Ā. Then, every number of marbles left in the top row of the machine was written in the form of equation, for example: when 13 or 9 or 5 or 1 marbles are left in the top row, the corresponding flip-flop configuration is Ā B C D E. Instructions given to the machine could be expressed with Boolean algebra operations *and* and *or*.

Figure 8. Dr. Nim red plastic board with white flip flops, owned by the author

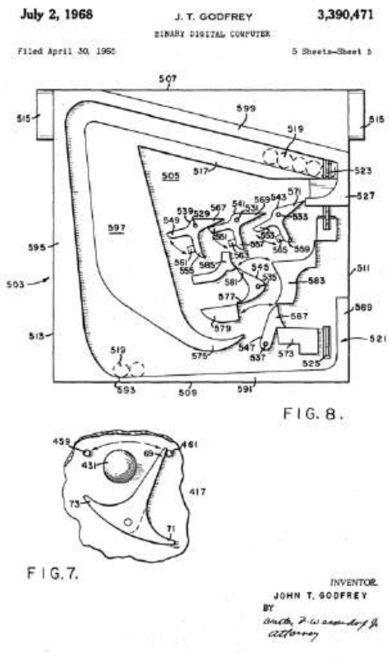


Figure 9. Patent of a binary digital computer (Godfrey 1968)

At the beginning of the 1970s, other inventions were patented and “relate to educational game apparatus”, for example a “computer-controlled apparatus for playing the game of NIM” (Morris 1971), but with the advent of electronic toys, the number of purely mechanical inventions declined. Moreover, the increase of personal computers favored the development of programs, and by the middle of the 1970s one could find game programming books.

4 Conclusion

In 1973 one of the first compilations of computer games in BASIC programming language was published: *101 BASIC Computer Games*, in which the Nim game was presented. The author, David Ahl, explained this interest in computer games by the expansion of minicomputers and timesharing networks that enlarged an emerging group of "computer hackers and of people who were furtively writing and playing game at lunchtime, before and after works on their employers' computer." (Ahl 1978, p. x) The study of first programs created to play Nim would go beyond the period covered in this article; however, it should be noted that the PCC (People's Computer Company),[9] created in the early 1970s, was one of the first organizations to recognize and advocate playing as a legitimate way of learning. PCC recognized the potential of BASIC and helped installing computers for children in libraries or schools to encourage hands-on learning approach.

In the 1980s, construction of mechanical Nim playing machines was still of interest, as "there are no commercial teaching materials that provide concrete modeling of Boolean algebra" (Cohen 1980) and that such mechanical models could render this algebra intelligible. Nowadays, Nim game is set for a comeback in French educational program in September 2016 as a recreational application to tackle algorithmic and programing in mathematic high school classes, and the creation of machines such as Geniacs or a simpler Dr. Nim could help pupils to acquire methods that build their algorithmic knowledge and develop skills in problem solving.

REFERENCES

Ahl, D. (1978). *BASIC computer games*. New York: Workman Publishing.

Berkeley, E. C. (1949). *Giant brains or machines that thinks*. New York: Science Editions, Inc.

Berkeley, E. C., & Jensen, R. A. (1950). World's smallest electric brain. *Radio Electronic*, *October*, 29-30.

Bouton, C. L. (1901). Nim, a game with a complete mathematical theory. *The Annals of Mathematics, 2nd ser*, *3*(4), 35-39.

Brougère, G. (1995). *Jeu et education*. Paris: l'Harmattan.

Cohen, H. A. (1980). Modeling Boolean algebra with Tinkertoy or Meccano: how to construct Nim playing machine. In Williamson, P. (Ed.), *Learn to love Mathematics*. Melbourne: Mathematics Association of Victoria.

Condon, E. U. (1942). The Nimatron. T*he American Mathematical Monthly, 49*(5), 330-332.

Condon, E. U. *et al*. (1940). *Machine to play game of Nim*. U.S. Patent Number 2,215,544, September 24th 1940. Washington, DC: U.S. Office.

Cundy, M. H. (1958). A demonstration binary adder. *The Mathematical Gazette,42*(342), 272-274.

Davies, D. R. (1956). The Impending Breakthrough. *The Phi Delta Kappan*, *37*(7), 275-281.

Du Bosque, C. Jr. (1962). *Instructional amusement device*. U.S. Patent Number 3,048,403, August 7th 1962. Washington, DC: U.S. Office.

Gardner, M. (1959). *Hexaflegons and other mathematical diversions. The first Scientific American book of puzzles & games*. Chicago: The University of Chicago Press.

Geniacs (1955a). *Geniacs: simple electric brain machines and how to make them*. Copyright 1955 by Oliver Garfield

Geniacs (1955b). *Supplementary wiring diagrams for the Geniac No. 1 electrical brain construction kit*. Copyright 1955 by Oliver Garfield

[9] In honor of Janis Joplin's rock group Big Brother and the Holding Company (Levy 1984, p. 136).

Geniac (1958). Geniac, an interesting kit builds circuits that solve problems and play games. *Popular Electronics, 9*(4), 27-29.
Godfrey, J. T. (1968). *Binary digital computer*. U.S. Patent Number 3,390,471, July 2nd 1968. Washington, DC: U.S. Office.
Levy, S. (1984). *Hackers heroes of the computer revolution*. New York, NY: Anchor Press/Doublebay.
Longo, B. (2015). *Edmund Berkeley and the Social Responsibility of Computers Professionals*. San Rafael, CA: Morgan & Claypool Publishers.
Morris, D. (1971). *Computer-controlled apparatus for playing game of Nim*. U.S. Patent Number 3,606,331, September 20th 1971. Washington, DC: U.S. Office.
Nimrod (1951). *The Ferranti Nimrod Digital Computer*, booklet available at the following address: http://goodeveca.net/nimrod/NIMROD_Guide.html
Pollack, H. (1958). Win at NIM with DEBICON. *Popular Electronics, January*, 37-41.
Projet de programme (2015). Projet de programme pour le cycle 4. Conseil supérieur des programmes, Ministère de l'Éducation Nationale, de l'Enseignement Supérieur et de la Recherche, available at : http://cache.media.education.gouv.fr/file/CSP/04/3/Programme_C4_adopte_412043.pdf
Redheffer, R. M. (1948). A Machine for Playing the Game of Nim. *The American Mathematical Monthly, 55*(6), 343-349.
Rougetet, L. (2015). Combinatorial Games and Machines. In R. Pisano (Ed.), *A Bridge between Conceptual Frameworks Sciences, Society and Technology Studies* (pp. 475-494). Dordrecht: Springer.
Schaaf, W. (1963). *Recreational mathematics. A guide to the literature*. Washington, DC: The National Council of Teachers of Mathematics, INC.
Shannon, C. (1950). Programming a computer for playing chess. *Philosophical Magazine, 41*(313), 256-275.
Weisbecker, J. A. (1967). *Computer-type device*. U.S. Patent Number 3,331,143, July 18th 1967. Washington, DC: U.S. Office.
Weisbecker, J. A. (1968). *Computer-type game and teaching aid*. U.S. Patent Number 3,338,483, June 18th 1968. Washington, DC: U.S. Office.

THE CONFLUENCE OF THE YELLOW RIVER AND THE MEDITERRANEAN:

Synthesis of European mathematics and Chinese mathematics during the seventeenth, eighteenth and nineteenth centuries

Man-Keung SIU
The University of Hong Kong, Pokfulam, Hong Kong
mathsiu@hku.hk

ABSTRACT

Transmission of mathematics between China and other parts of the world already went on in the ancient period. This presentation will however focus on a later period from the early seventeenth century onwards, when the transmission was mainly from the Western world into China.

1 Introduction

The title of this paper obviously carries a metaphoric rather than geographical meaning. The Yellow River in China does not flow into the Mediterranean, nor are they near to each other at all. It refers to the transmission of learning between the Eastern world and the Western world with a large span of land and sea in between. Such transmission has a very long history, with recorded accounts dating back at least to the Han Dynasty in China (the Former Han Dynasty from 202 B.C.E. to 9 C.E., and the Latter Han Dynasty from 25 C.E. to 220 C.E. with the Xin Dynasty of WANG Mang (王莽) from 9 C.E. to 23 C.E. in between). The famous Silk Road acted as the main trade route in Central Asia that established links between a cross-cultural mix of religions, civilizations and people of many different regions, and also enabled exchanges of learning and cultures of people of different races.

In mathematics transmission of learning, either directly or indirectly, between China and regions in Central Asia and the Middle East, India, the Islamic Empire and even Europe further to the West went on for many centuries from the Han Dynasty to the Yuan Dynasty (1279 to 1368). A well-known example often referred to is the Method of Double False Position, sometimes called by the name of "Rule of Khitai" (the term "Khitai", rendered as "Cathay" in English, means China, actually the Liao Dynasty in Northern China from the early 10th century to the early 12th century) (Shen, Crossley, & Lun, 1999). It should also be noted that the transmission of this method remains a debatable issue among historians of mathematics ever since the mid-20th century, with some historians putting it down to a linguistic misunderstanding in the Arabic term "hisab al-khata'ayn (reckoning from two falsehoods)". A more general view is that, despite the uncertainty about the time and way of its transmission, the origin of the method is that of *ying buzu* (盈不足 excess and deficit) explained in Chapter 7 of the Chinese mathematical classic *Jiuzhang Suanshu* [九章算術 The Nine Chapters on the Mathematical Art] that is believed to have been compiled between the 2nd century B.C.E and the 1st century C.E. In the 11th century it appeared in an anonymous

Latin book called *Book of Increase and Decrease*, and later in the famous treatise *Liber Abaci* of 1202 written by the Italian mathematician Fibonacci (also called Leonardo of Pisa). Another interesting item is a lost book with only its title remaining on record in the catalogue of the Library of the Astronomical Bureau of the Yuan Dynasty --- *Fifteen Books of Wuhuliedisi Baisuanfaduanshu* (兀忽烈的四擘算法段數十五部). Some historians surmise that this is a recension of Euclid's *Elements* by the title *Tahrir usul uqlidis* compiled by the Persian mathematician Nasīr al-Dīn Tūsī (1201-1274) in 1248. In the historical record of the Yuan Dynasty it is said that the Möngke Khan (蒙哥汗 1209-1259) of the Mogol Empire, a grandson of Genghis Khan (成吉思汗 1162-1227) and elder brother of Kublai Khan (忽必烈汗 1215-1294) who founded the Yuan Dynasty, understood some geometry through the study of this book. If this is indeed the case, then the book would mark the first transmission of Euclid's *Elements* into China, earlier than that through the Jesuits by three hundred and fifty years. Transmission of mathematical learning during that period is substantiated by the discovery in the late 1950s in the suburb of the city of Xian six iron plates that are 6 x 6 magic squares inscribed in Arabic numerals. More detailed information can be found in (Li 1999).

Obviously this long and intricate story of East-West transmission of mathematical learning is too vast a topic for a short presentation. We will therefore turn our attention to the latter part of the Ming Dynasty (1368 to 1644) when transmission of European mathematics into China on a much more systematic and larger scale began, and the subsequent three centuries. Even for that we can hardly offer a comprehensive account but can only sketch a few highlights.

2 First wave of transmission

The story started with the Christian mission in China of the Jesuits in the late 16th century. As a by-product of the evangelical efforts of the missionaries an important page of intellectual and cultural encounter between two great civilizations unfolded in history, the two most important protagonists of that period being the Italian Jesuit Matteo Ricci (1552-1610) and the Chinese scholar-official of the Ming court XU Guang-qi (徐光啟 1562-1633).

Early contact with Europeans in the 16th century, the first being the Portuguese who came in the double capacity of pirates and merchants, had left the Chinese people with a feeling of distrust and resentment. The brutal behavior of the Dutch, the Spaniards and the English who followed aggravated this uneasy relationship. In 1557 the Portuguese gained a permanent foothold by occupying Macao which developed into a settlement and centre of trade, through which the Catholic missionaries entered China. After studying at Collegio Romano in Rome, Ricci was soon afterwards sent on his China mission. He reached Macao in August of 1582 and proceeded to move into mainland China and finally reach Peking (Beijing) in January of 1601. To ease the hostile feeling the Chinese harboured against foreigners, many missionaries tried to learn the Chinese language, dressed in Chinese clothes and as far as possible adopted the Chinese way of living. Ricci, who adopted a Chinese name LI Ma-dou (利瑪竇), was a brilliant linguist, so he not only learnt the Chinese language but mastered it to such an extent that he could study Chinese classics. Coupled with his knowledge of

Western science he soon impressed the Chinese intellectuals who came into contact with him as an erudite man of learning, thereby commanding their trust and respect, becoming the most prominent Catholic missionary in China.

To Ricci, who studied mathematics under Christopher Clavius (1538-1612) at Collegio Romano, the treatise *Elements* of Euclid was the basis of any mathematical study. He therefore suggested to his Chinese friend XU Guang-qi that *Elements*, based on the version compiled by Clavius in 1574 (with subsequent editions), a fifteen-book edition titled *Euclidis Elementorum Libri XV*, should be the first mathematical text to be translated. Xu set himself to work very hard on this project. He went to listen to Ricci's exposition of *Elements* every day in the afternoon (since he could not read Latin, while Ricci was well versed in Chinese) and studied laboriously for four hours at a stretch every day, and at night he wrote out in Chinese everything he had learnt by day. According to an account by Ricci:

> When he [XU Guang-qi] began to understand the subtlety and solidity of the book, he took such a liking to it that he could not speak of any other subject with his fellow scholars, and he worked day and night to translate it in a clear, firm and elegant style. […] Thus he succeeded in reaching the end of the first six books which are the most necessary and, whilst studying them, he mingled with them other questions in mathematics. […] He would have wished to continue to the end of the Geometry; but the Father [Matteo Ricci] being desirous of devoting his time to more properly religious matters and to rein him in a bit told him to wait until they had seen from experience how the Chinese scholars received these first books, before translating the others.

Ricci reported that Xu agreed and they stopped the translation. The six translated chapters were published in 1607 under the title *Jihe Yuanben* [幾何原本 Source of Quantity]. However, in his heart Xu wanted very much to continue the translation. In a preface to a revised edition of *Jihe Yuanben* in 1611 he lamented, "It is hard to know when and by whom this project will be completed." This deep regret of Xu was resolved only two and a half centuries later when the Qing mathematician LI Shan-lan (李善蘭 1811-1882) in collaboration with the English missionary Alexander Wylie (1815-1887) translated Book VII to Book XV in 1857 based on the English translation of *Elements* by Henry Billingsley published in 1570.

How did *Elements* blend in with traditional Chinese mathematics? Let us first look at what Ricci said about traditional Chinese mathematics:

> The result of such a system is that anyone is free to exercise his wildest imagination relative to mathematics, without offering a definite proof of anything. In Euclid, on the contrary, they recognized something different, namely, propositions presented in order and so definitely proven that even the most obstinate could not deny them.

It is debatable whether it is true the notion of a mathematical proof was completely absent from ancient Chinese mathematics as Ricci remarked. We shall look at one example, which would have made Ricci think otherwise, had he the opportunity of having access to the commentaries of LIU Hui (劉徽) on *Jiuzhang Suanshu* in the third century. This is the following problem: Given a right-angled triangle *ABC* with *AC* as its hypotenuse, inscribe a square in it, that is, construct a square *BDEF* with *D* on *AB*, *E* on *AC*, and *F* on *BC*?

This problem does not appear in Euclid's *Elements*. Were it there, the solution would have probably looked like this: Bisect $\angle ABC$ by *BE* (*E* on *AC*) [Book I, Proposition 9]. Drop perpendiculars *ED*, *EF* (*D* on *AB*, *F* on *BC*) [Book I, Proposition 12]. Prove that *BDEF* is the inscribed square we want. The problem (in a more general version) appears as Added Proposition 15 of Book VI in *Euclidis Elementorum Libri XV*, which was translated by Ricci and Xu: Divide *AB* at *D* such that $AD : DB = AB : BC$ [Book VI, Proposition 10]. Draw *DE* parallel to *BC* and *EF* parallel to *AB*, (*E* on *AC*, *F* on *BC*). *DBFE* is the inscribed square we want.

Now that we know such an inscribed square exists we can ask what the length of its side is. It can be shown from the construction that the side x of the inscribed square in a right-angled triangle with sides of length a, b containing the right angle is given by $x = ab/(a+b)$.

The same problem appears as Problem 15 of Chapter 9 in *Jiuzhang Suanshu*, which says: "Now given a right-angled triangle whose *gou* is 5 *bu* and whose *gu* is 12 *bu*. What is the side of an inscribed square? The answer is 3 and 9/17 *bu*. Method: Let the sum of the *gou* and the *gu* be the divisor; let the product of the *gou* and the *gu* be the dividend. Divide to obtain the side of the square." (See Figure 1.)

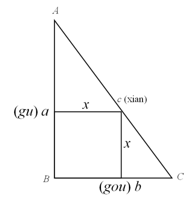


Figure 1. Problem 15 of Chapter 9 of *Jiuzhang Suanshu*

The line of thinking and style of presentation of the explanation by LIU Hui are quite different from that in *Elements*. Liu gave a "visual proof" of the formula $x = ab/(a+b)$ by dissecting and re-assembling coloured pieces. (See Figure 2.). Liu's commentary actually describes the coloured pieces so that were the original diagram extant it would provide the making of a useful teaching aid!

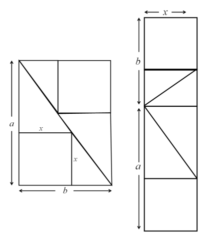


Figure 2. Explanation by LIU Hui

How did XU Guang-qi perceive Euclidean geometry which he newly learnt from Clavius' rendition of Euclid's *Elements*, and to what extent did he understand the thinking, approach and presentation of the book, which are so very different from those of traditional

Chinese mathematics that he was familiar with? Despite Xu's emphasis on utility of mathematics, he was sufficiently perceptive to notice the essential feature about *Elements*. In a preface to *Jihe Yuanben* of 1607 he wrote:

> As one proceeds from things obvious to things subtle, doubt is turned to conviction. Things that seem useless at the beginning are actually very useful, for upon them useful applications are based. […] It can be truly described as the envelopment of all myriad forms and phenomena, and as the erudite ocean of a hundred schools of thought and study.

In a preface to another book *Celiang Fayi* [測量法義 Methods and Principles in Surveying] of 1608, which is an adapted translation by Matteo Ricci and XU Guang-qi of parts of *Geometria practica* compiled by Christopher Clavius in 1606, he wrote:

> It has already been ten years since Master Xitai [西泰子 that is, Matteo Ricci] translated the methods in surveying. However, only started from 1607 onwards the methods can be related to their principles. Why do we have to wait? It is because at that time the six books of *Jihe Yuanben* were just completed so that the principles could be transmitted. […] As far as the methods are concerned, are they different from that of measurement at a distance in *Jiuzhang* [*Suanshu*] and *Zhoubi* [*Suanjing*]? They are not different. If that is so, why then should they be valued? They are valued for their principles.

He elaborated this point in an introduction to his 1608 book *Celiang Yitong* [測量異同 Similarities and Differences in Surveying] by saying:

> In the chapter on *gougu* of *Jiuzhang Suanshu* there are several problems on surveying using the gnomon and the trysquare, the methods of which are more or less similar to those in the recently translated *Celiang Fayi* (Methods and Principles in Surveying). […] The y*i* [義 principles] are completely lacking. Anyone who studies them cannot understand where they are derived from. I have therefore provided new *lun* [論 proofs] so that examination of the old text becomes as easy as looking at the palm of your hand.

In connection with this he wrote in a memorial submitted to the Emperor in 1629 in his capacity as the official in charge of the Astronomical Bureau:

> [not knowing that] there are *li* [理 theory], *yi* [義 principle], *fa* [法 method] and *shu* [數 calculation] in it. Without understanding the theory we cannot derive the method; without grasping the principle we cannot do the calculation. It may require hard work to understand the theory and to grasp the principle, but it takes routine work to derive the method and to do the calculation.

With this perception Xu tried hard to assimilate Western mathematics and to synthesize it with Chinese traditional mathematics. One example is his work on Problem 15 in Chapter 9 of *Jiuzhang Suanshu* and Added Proposition 15 of Book VI in *Euclidis Elementorum Libri XV* reported in his 1609 book *Gougu Yi* [勾股義 Principle of the Right-angled Triangle]. He explained this as Problem 4, which involves complicated reasoning that may seem rather round-about and unnecessary. Perhaps it indicates a kind of incompatibility between the two styles of doing mathematics so that it would be unnatural to force one into the mould of the other. However, despite its shortcomings this also indicates an admirable attempt of Xu to synthesize Western and Chinese mathematics. A more detailed discussion on this topic can be found in (Siu, 2011).

Despite the enthusiasm on the part of XU Guang-qi to introduce *Elements* into China, the Chinese in the seventeenth and eighteenth centuries did not seem to feel the impact of the essential feature of Western mathematics exemplified in *Elements* as strongly as he. The influence of the newly introduced Western mathematics on mathematical thinking in China was not as extensive and as direct as he had imagined. However, unexpectedly the fruit was brought forth not in mathematics, but in a domain of perhaps even higher historical importance. Study of Western science in general, and Western mathematics in particular, attracted the attention of some active liberal intellectuals of the time, among whom three prominent figures KANG You-wei (康有為 1858-1927), LIANG Qi-chao (梁啟超 1873-1929) and TAN Si-tong (譚嗣同 1865-1898) played an important role in the history of modern China as leading participants in the episode of "Hundred-day Reform" of 1898. The "Hundred-day Reform" ended in failure with Tan being arrested and executed in that same year, while Kang and Liang had to flee the country and went to Japan. This was one important step in a whole series of events that culminated in the overthrow of Imperial Qing and the establishment of the Chinese Republic in 1911.

Together with LI Zhi-zao (李之藻 1565-1630) and YANG Ting-jun (楊廷筠 1557-1627), colleagues and friends of XU Guang-qi, the three scholar-officials of high ranking in the Ming Court were hailed as the "three pillars of the Catholic Church in China". When Li got acquainted with Ricci in 1601 in Nanjing. he was deeply impressed by the map of the world that Ricci prepared, the *Kunyu Wanguo Quantu* [坤輿萬國全圖 Complete Map of the Myriad Countries of the World]. Li himself had prepared a map of the fifteen provinces of China at the age of twenty and thought at the time he had well mastered the knowledge of cartography so that he was all the more amazed by this work of Ricci.

Li collaborated with Ricci to compile the treatise *Tongwen Suanzhi* [同文算指, literally meaning "rules of arithmetic common to cultures"], which first transmitted into China in a systematic and comprehensive way the art of *bisuan* (筆算 written calculation) that had been in common practice in Europe since the sixteenth century. This treatise, accomplished in 1613, was a compilation based on the 1583 European text *Epitome Arithmeticae Practicae* (literally meaning "abridgement of arithmetic in practice") of Clavius and the 1592 Chinese mathematical classics *Suanfa Tongzong* [算法統宗, literally meaning "unified source of computational methods"] of CHENG Da-wei (程大位 1533-1606). In accord with a prevalent intellectual trend of the time known as *zhongxi huitong* (中西會通, literally meaning "synthesis of Chinese and Western [learning]) started by the dedicated work of the translation of *Elements* in 1607, Li also attempted to synthesize European mathematics with traditional Chinese mathematics by treating problems taken out of Chinese mathematical texts by the newly introduced method of written calculation. A more detailed discussion on this topic can be found in (Siu, 2015b). We give only one example on division here.

In traditional Chinese mathematics, calculation in arithmetic was performed using counting rods since very early times. The arithmetical operations were explained in mathematical classics such as *Sunzi Suanjing* (孫子算經 Master Sun's mathematical manual) of the fourth/fifth century. In the Western world there was a movement of contest in efficiency of reckoning between the so-called "abacists" and "algorists" towards the latter part of medieval time. In particular, a method known as the *gelosia* method, coming from the Islamic world, was commonly used at the time. (See Figure 3.) Written calculation did appear in some Chinese texts even before *Tongwen Suanzhi*, but not in a way as systematic and as comprehensive as in *Tongwen Suanzhi*. The *gelosia* method introduced into China in those texts was given a picturesque name of *pudijin* (鋪地錦, literally meaning "covering the floor with a glamorous carpet") by CHENG Da-wei.

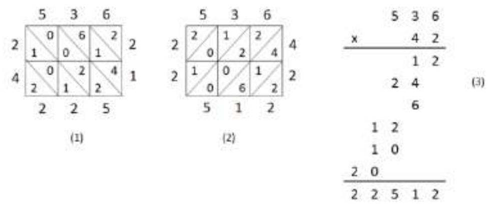


Figure 3. *Gelosia* method of multiplication

LI Zhi-zao seemed to prefer the more modern method to this pictureque *pudijin*. In *Tongwen Suanzhi* division is performed by the *galley* method, which was already quite well-known in the Western world, for instance, in the *Treviso Arithmetic* of 1478. (See Figure 4, with the last item in modern notation inserted for comparison.)

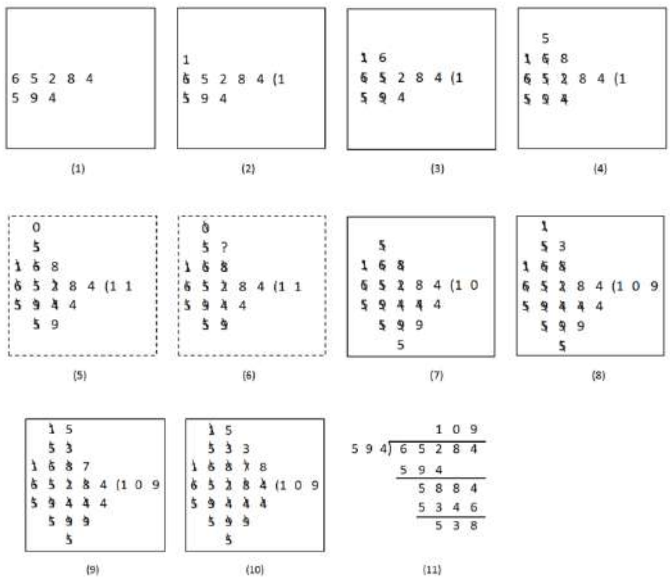


Figure 4. *Galley* method of division

3 Second and third waves of transmission

The translation of *Elements* by XU Guang-qi and Matteo Ricci led the way of the first wave of transmission of European science into China, with a second wave (or a wake of the first wave as some historians would see it) and a third wave to follow in the Qing Dynasty (1644-1911), but each in a rather different historical context with quite different mentality. The gain of this first wave seemed momentary and passed with the downfall of the Ming Dynasty. Looking back we can see its long-term influence, but at the time this small window which opened onto an amazing outside world was soon closed again, only to be forced open as a wider door two hundred years later by Western gunboats that inflicted upon the ancient nation a century of exploitation and humiliation, thus generating an urgency to know more about and to learn with zest from the Western world.

The main features and the mentality of the three waves of transmission of Western learning into China can be summarized in the prototype slogans of the three epochs. In the late-sixteenth to mid-seventeenth centuries (during the Ming Dynasty) the slogan was: "In order to surpass we must try to understand and to synthesize (欲求超勝必須會通)." In the first part of the eighteenth century (during the Qing Dynasty) the slogan was: "Western learning has its origin in Chinese learning (西學中源)." In the latter part of the nineteenth century (during the Qing Dynasty) the slogan was: "Learn the strong techniques of the '[Western] barbarians' in order to control them (師夷長技以制夷)." It is interesting to note the gradual and subtle change in the attitude and mentality on the Chinese side in learning from the Western world, from an open-minded enthusiasm with self-confidence to a strange mix of self-arrogance and resistance and finally to a feeling of urgency in the face of the precarious fate of their mother country.

The second wave came and lasted from the mid-seventeenth century to the mid-eighteenth century. Instead of Chinese scholar-officials the chief promoter was Emperor Kangxi (康熙) of the Qing Dynasty (reigned 1662-1722). Instead of Italian and Portuguese Jesuits the Western partners were mainly French Jesuits, the so-called "King's Mathematicians" sent by Louis XIV, the "Sun King" of France (reigned 1643-1715), in 1685.

This group of Jesuits led by Jean de Fontaney (1643-1710) reached Peking in 1688. An interesting account of their lives and duties in the Imperial Court was recorded in the journal written by one of the group, Joachim Bouvet (1656-1730), and published in 1697. Bouvet recounted how he and the other Jesuits conducted lessons in science and mathematics in the Imperial Court and how Emperor Kangxi studied with enthusiasm and diligence. More information and an in-depth analysis of this episode can be found in (Jami, 2012).

A main outcome was the compilation of a monumental one-hundred-volume treatise *Lüli Yuanyuan* [律曆淵源 Origins of Mathematical Harmonics and Astronomy] commissioned by Emperor Kangxi, worked on by a large group of Jesuits, Chinese scholars and official astronomers. The project started in 1713 and the treatise was published in 1722/1723, comprising three parts: *Lixiang Kaocheng* [曆象考成 Compendium of Observational Computational Astronomy], *Shuli Jingyun* [數理精蘊 Collected Basic Principles of Mathematics] and *Lülü Zhengyi* [律呂正義 Exact Meaning of Pitchpipes]. The treatise *Shuli Jingyun* includes both traditional Chinese mathematics, the part that was still extant and was understood at the time, as well as Western mathematics, highly likely from the "lecture notes" prepared by the missionaries for Emperor Kangxi.

Books 2 to 4 of *Shuli Jingyun* are on geometry, which is believed to be based on *Elémens de géométrie* by Ignace Gaston Pardies (1636-1673), first published in 1671 with a sixth edition in 1705. Books 31 to 36 are on solving algebraic equations known by the name of *jiegenfang* (借根方 borrowed root and powers), taught to Emperor Kangxi by the Belgian Jesuit Antoine Thomas (1644-1709), who had studied at University of Coimbra in Portugal and compiled *Synopsis mathematica*, based on the 1600 book *De numerosa potestatum ad exegesim resolution* (On the numerical resolution of powers by exegetics) of François Viète (1540-1603). Thomas later revised it as *Suanfa Zuanyao Zonggang* (算法纂要總綱 Outline of the essential calculations) and *Jiegenfang Suanfa* (借根 方算法 Method of borrowed root and powers), to be used as lecture notes for the mathematics lessons on solving algebraic equations in the Imperial Court.

The Chinese mathematician MEI Jue-cheng (梅瑴成 1681-1763) told the story on how he learnt this new method from Emperor Kangxi, who told him that the Westerners called it *aerrebala* (阿爾熱巴拉 algebra) that means "Method from the East". Mei suspected that the method resembled that of a traditional Chinese method of *tianyuan* (天元 celestial unknown) and studied it to clarify the matter, coming to the conclusion that despite the terminologies the two methods were the "the same, not just a mere resemblance." This would explain how the saying "Western learning has its origin in Chinese learning" got promulgated in those days. This is probably a tactic on the part of Emperor Kangxi to make his subjects willing to learn it and would not regard it as something opposing traditional value. Or, maybe he really thought that the method originated in older Chinese learning, without knowing that the art of solving algebraic equations was developed by Islamic mathematicians in medieval time. Indeed, similar methods were explained in Chinese mathematical classics of earlier days, most of which became less known by the Ming and early Qing period.

But when the French Jesuit Jean-François Foucquet (1665-1741) lectured on the "new method of *aerrebala*", which is symbolic algebra as explained in the 1591 book *Artem Analyticem Isagoge* (Introduction to the analytical art) of Viète, Emperor Kangxi reacted to it with strong resistance. I tend to believe that Emperor Kangxi was very diligent, determined and bright, but also studied hard not without vanity and intention to show off his knowledge with a political motive. Much as he left us with several sets of monumental compendia and a collection of books that benefit posterity, it has to be admitted (sadly) that owing to the limitation in his scope and motive this period of transmission was also a "missed opportunity" for China, because, being confined to a small group within the Imperial Court, it failed to exert the influence that would help the country to move forward and catch up with the Western world which had moved forward by leaps and bounds by the seventeenth century.

The third wave came in the last forty years of the nineteenth century in the form of the so-called "Self-strengthening Movement" after the country suffered from foreign exploitation during the First Opium War (1839-1842) and the Second Opium War (1856-1860). This time the initiators were officials led by Prince Gong (恭親王 1833-1898) with contribution from Chinese scholars and Protestant missionaries coming from England or America, among whom were LI Shan-lan and Alexander Wylie who completed the translation of *Elements*. In 1862 *Tongwen Guan* [同文館 College of Foreign Languages] was established by decree, at first serving as a school for studying foreign languages to train interpreters but gradually expanded into a college of Western learning, along with the establishment of other colleges of similar nature that sprouted in other cities like Shanghai, Guangzhou, Fuzhou, Tianjin, as well as the establishment of arsenals, shipyards and naval schools during the period of "Self-strengthening Movement" as a result of the fervent and urgent desire of the Chinese government to learn from the West in order to resist the foreign exploitation the country went through in the first and second Opium Wars. The slogan of the day, "learn the strong techniques of the '[Western] barbarians' in order to control them", reflected the purpose and mentality during that period. In 1866 the School of Astronomy and Mathematics was added to *Tongwen Guan*, with LI Shan-lan as its head of department. In 1902 *Tongwen Guan* became part of Peking Imperial University, which later became what is now Beijing University. A more detailed discussion on this topic can be found in (Chan & Siu, 2012).

4 An Epilogue in Montpellier

In 1810 the French mathematician Joseph Diaz Gergonne (1771–1859) established his own mathematics journal, officially called the *Annales de mathématiques pures et appliquées* but more popularly known as *Annales de Gergonne*, which was the first privately run journal wholly on mathematical topics. Geometry figured most prominently in this journal with many famous mathematicians of the time publishing papers there until the journal was discontinued in 1832 after Gergonne became the Rector of the University of Montpellier. To facilitate a dialogue between the Editor and the readership the journal posed problems regularly besides publishing papers. In the first volume of *Annales de Gergonne* the following problem was posed: "Given any triangle, inscribe three circles in such a way that each of them touches the other two and two sides of the triangle."

Soon after the problem was posed a solution appeared in a later issue of the journal and referred to a letter from a reader, the Italian mathematician Giorgio Bidone (1781-1839) in Turin, who pointed out that the original problem was posed by his compatriot Gianfrancesco Malfatti (1731-1807). Malfatti in 1803 asked, "Given a right triangular prism of any sort of material, such as marble, how shall three circular cylinders of the same height as the prism and of the greatest possible volume of material be related to one another in the prism and leave over the least possible amount of material?" Malfatti thought that the three non-overlapping circles inside the triangle occupying optimal space would be three "kissing circles". Actually this is never the solution, but it was only realized with the optimality problem fully settled as late as in 1994!

In the latter part of the 19th century some foreign missionaries, along with spreading Christian faith, worked hard to propagate Western learning in old imperial China through various means, one of which was publishing periodicals. The monthly periodical *Zhongxi Wenjian Lu* [中西聞見錄 Record of News in China and West] with English title *Peking Magazine*, founded in 1872, announced in the first issue that it adopted the practice and format of newspapers in the Western world in publishing international news and recent happenings in different countries, as well as essays on astronomy, geography and *gewu* [格物

science, literally meaning "investigating things"]. The fifth issue (December, 1872) of this magazine carried the following posed problem:

> A plane triangle (acute, right or obtuse) contains three circles of different radii that touch each other. We want to fix the centres of the three circles. What is the method? All students in *Tongwen Guan* retreated from trying this problem. Whoever can solve the problem should send the diagram [of the solution] to the School of Astronomy and Mathematics and would be rewarded with a copy of *Jihe Yuanben* [Chinese translation of Euclid's *Elements*]. The diagram [of the solution] would be published in this magazine so that the author would gain universal fame.

A solution submitted by a reader was published in the eighth issue (March, 1873), followed by a comment by another reader in the twelfth issue (July, 1873) together with an acknowledgement of the error and a further comment by the School of Astronomy and Mathematics of *Tongwen Guan*.

This kind of fervent exchange of academic discussion carried on in public domain was a new phenomenon of the time in China. In 1897 a book on homework assignments by students of *Longcheng Shuyuan* [龍城書院 Academy of the Dragon City], which was a private academy famous for its mathematics curriculum, contained two articles that gave different solutions to the Malfatti Problem with accompanying remarks by the professor. One solution is particularly interesting because it made use of a hyperbola, which is a mathematical object that was totally foreign to Chinese traditional mathematics and was newly introduced in a systematic way only by the mid-nineteenth century. It is not certain when the Malfatti Problem was first introduced into China. Apparently it was introduced by Westerners into China only two to three decades after the problem became well-known in the West, at a time when the Chinese were just beginning to familiarize themselves with Euclidean geometry, which was not part of their traditional mathematics. A more detailed discussion on this topic can be found in (Siu, 2015a). It is worth noting, from the active discussion generated around the Malfatti Problem, how enthusiastic the Chinese were in learning mathematics from Westerners in the late nineteenth century.

5 Endnote

In the preface as well as in two forewords to *Tongwen Suanzhi*, LI Zhi-zao and his friends and fellow official-scholars XU Guang-qi and YANG Ting-jun stressed the meaning of *tongwen* (literally meaning "common cultures"), adopted as part of the title of the book, which exhibits their open mind and receptive attitude to foreign learning, at the same time indicating a deep appreciation of the common cultural roots of mathematics despite different mathematical traditions. Let us look at some of their sayings to further illustrate this point.

XU Guang-qi said in the Preface at the printing of *Tongwen Suanzhi* (1613):

> The origin of numbers, could it not be at the beginning of human history? Starting with one, ending with ten, the ten fingers symbolize them and are bent to calculate them, [numbers] are of unsurpassed utility! Across the five directions and myriad countries, changes in customs are multitudinous. When it comes to calculating

numbers, there are none that are not the same; that all possess ten fingers, there are none that are not the same.

(In my primary school days we were discouraged from making use of this "unsurpassed utility" to aid in doing arithmetic. When the teacher spotted such an attempt of using the fingers to count, the pupil would be reprimanded for doing so. In order not to get a reprimand I did my finger-counting by hiding the hand in the pocket of my pants. My good friend and a mathematics educator in Hong Kong, LAW Huk-Yuen, jokingly dubbed this act the pre-historic version of a genuine "pocket calculator"!)

LI Zhi-zao said in the Preface to the reprinting of *Tianzhu Shiyi* (天主實義 The True Meaning of the Lord of Heaven, written by Matteo Ricci and printed in 1603 in Peking): "Across the seas of the East and the West the mind and reasoning are the same [同 *tong*]. The difference lies only in the language and the writing."

REFERENCES

Chan, Y. C., & Siu, M. K. (2012). Facing the change and meeting the challenge: Mathematics curriculum of Tongwen Guan in China in the second half of the nineteenth century. *Zentralblatt für Didaktik der Mathematik – The International Journal of Mathematics Education, 44*(4), 461-472.

Jami, C. (2012). *The emperor's new mathematics: Western learning and imperial authority during the Kangxi reign (1662-1722)*. Oxford: Oxford University Press.

Li, D. (Ed.) (1999). *A Compendium of the history of Chinese mathematics, volume 6 — Dynasties of Xi Xia, Jin, Yuan, Ming* (中國數學史大系, 第六卷 — 西夏金元明). Beijing: Beijing Normal University Press.

Shen, K. S., Crossley, J. N., & Lun, A. W. C. (1999). *The Nine Chapters on the Mathematical Art: Companion and commentary*. Oxford: Oxford University Press.

Siu, M. K. (2011). 1607, a year of (some) significance: Translation of the first European text in mathematics — *Elements* — into Chinese. In E. Barbin, M. Kronfellner, & C. Tzanakis (Eds.), *History and epistemology in mathematics education: Proceedings of the sixth European Summer University* (pp. 573-589). Vienna: Verlag Holzhausen.

Siu, M. K. (2015a). A geometric problem on three circles in a triangle (the Malfatti Problem) — A thread through Japanese, European and Chinese mathematics. In E. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and epistemology in mathematics education: Proceedings of the seventh European Summer University* (pp. 637-649). Copenhagen: Danish School of Education, Aarhus University.

Siu, M. K. (2015b). *Tongwen Suanzhi* (同文算指) and transmission of *bisuan* (筆算 written calculation) in China: From an HPM (History and Pedagogy of Mathematics) viewpoint. *Journal for History of Mathematics, 28*(6), 311-320.

PYTHAGORE ET LES ANGLES DROITS

des exemples d'ouvertures de classes de mathématiques sur le monde

Jean-Jacques SALONE
EA 4671 ADEF, Aix-Marseille Université, Marseille, France
jjsan@free.fr

RESUME

Ouvrir les classes sur le monde est une façon de redonner du sens aux savoirs enseignés et du plaisir d'apprendre aux élèves. Dans nos travaux de recherche, nous avons identifié trois directions d'ouvertures possibles. La première concerne les savoirs eux-mêmes: au delà de ceux qui sont exclusivement disciplinaires et officiels, d'autres peuvent exister dans les classes comme ceux que les élèves possèdent déjà à titre personnel, ou encore ceux issus de l'histoire de la discipline ou des disciplines associées. La seconde direction concerne l'ouverture des topos : proposer aux élèves de devenir de véritables acteurs de leurs apprentissages et du cours. La troisième est l'ouverture du milieu : activer des liens, numériques en particulier, au monde et à la société en impliquant d'autres personnes ou d'autres institutions scolaires ou extra-scolaires. Nous les déclinons en dispositifs didactiques et en activités d'étude et de recherche que nous proposons de présenter au travers d'une séquence en classe de quatrième autour du théorème de Pythagore.

1 Introduction

Quand on entre en quatrième[1] dans l'étude du théorème de Pythagore et des racines carrées, pourquoi ne pas rencontrer aussi leur célèbre inventeur et découvrir leurs usages anciens ou actuels? Les intérêts d'un tel contexte socio-historique pour l'apprentissage des mathématiques sont nombreux et plusieurs équipes de didactique des mathématiques se sont emparées de cette question. Au delà de ce seul contexte, les approches pédagogiques actuelles s'appuient sur plusieurs théories pour resituer les élèves au cœur d'une activité conjointe (Sensevy, 2011) et pour entrer dans un paradigme de questionnement du monde (Chevallard & Ladage, 2010). Avec le théorème de Pythagore comme fil conducteur, nous aborderons ici ces questions en introduisant d'abord le concept d'ouverture écologique des classes.

Puis la question de la mise en œuvre effective dans les classes de ces approches pédagogiques sera traitée. Comment concevoir des séquences et animer des séances ouvertes sur le monde? Avec quels rôles pour les élèves? Avec quelles ressources? Des activités avec un arrière-plan historique seront alors décrites et analysées. Fruits d'une ingénierie, elles ont été conduites sur plusieurs années par des enseignants dans leurs classes[2] et ont fait l'objet d'observations et d'analyses rapportées dans (Salone, 2015). Elles amèneront, au delà de la seule question des savoirs de référence, à s'interroger sur les topos accordés aux élèves et sur le milieu matériel mobilisé pour assumer une co-construction effective du cours et entrer dans une forme de pédagogie de l'enquête.

---

[1] Dans le système éducatif français, la niveau quatrième correspond à des élèves de 13-14 ans.

[2] Le terrain de recherche a permis de suivre 4 années de suite les classes d'un collège marseillais (France), dont trois classes de quatrième par an.

2 Les ouvertures écologiques des classes

Les classes sont des institutions didactiques ancrées dans des réseaux sociaux qui les légitiment. Notre postulat initial de recherche consiste à supposer qu'activer les liens sociétaux qui lient ses agents, élèves ou enseignants, et les savoirs enseignés au reste du monde a des conséquences bénéfiques sur les apprentissages. Cela autorise d'une part l'alternance des décontextualisations et des recontextualisations nécessaires pour la mise en œuvre de l'activité, et, d'autre part, cela semble contribuer à redonner de la motivation et du goût pour l'étude aux élèves.

2.1 Ouvertures des savoirs

La première ouverture écologique des classes que nous distinguons est celle qui concerne les savoirs. Ouvrir les savoirs, c'est activer dans la classe les liens qu'ils tissent entre eux. Ces liens sont d'abord internes à la discipline, comme le sont, pour initier notre exemple, ceux qui associent le théorème de Pythagore, l'angle droit et les racines carrées. Ainsi les programmes officiels de 2008 (Ministère de l'Éducation Nationale, 2008) situent ce théorème en classe de quatrième dans le domaine de la géométrie et le secteur des figures planes. Il y est associé aux trois théorèmes des milieux, au théorème de Thalès, au cosinus d'un angle, aux droites et aux cercles remarquables dans les triangles, aux notions de distance d'un point à une droite et de tangente à un cercle. L'étude de ce théorème s'inscrit donc officiellement dans un contexte praxéologique d'étude des triangles, avec une attention particulière portée aux triangles rectangles. Deux capacités sont associées, toutes deux exigibles pour le socle commun:

> Caractériser le triangle rectangle par l'égalité de Pythagore.
>
> Calculer la longueur d'un côté d'un triangle rectangle à partir de celles des deux autres.

Ainsi c'est davantage l'égalité de Pythagore qui est attendue dans les programmes officiels. Le théorème *stricto sensu* n'apparaît pas explicitement, ni le théorème réciproque ou sa contraposée. En outre la seconde capacité, calculer des longueurs, nécessite le recours aux racines carrées. Mais là aussi, en quatrième, il n'est nullement question de définir formellement ce que sont ces nombres nouveaux pour les élèves. L'étude de leurs propriétés algébriques est reportée dans la classe de niveau supérieur, en troisième. En quatrième il s'agit surtout de les découvrir de façon empirique et instrumentée. Les commentaires précisent que:

> On ne distingue pas le théorème de Pythagore direct de sa réciproque (ni de sa forme contraposée). On considère que l'égalité de Pythagore caractérise la propriété d'être rectangle. Calculer la longueur d'un côté d'un triangle rectangle à partir de celles des deux autres.

On peut encore ajouter que dans le corpus mathématique du XXe siècle, le théorème de Pythagore et sa réciproque sont valides dans un espace vectoriel E sur le corps $\mathbb{R}$ des nombres réels lorsqu'il est muni d'un produit scalaire et de la norme euclidienne associée.

Mais les liens gnoséologiques sont aussi d'ordre plus culturel, se fondant sur des pratiques et des théories à la fois forgées par les contextes historiques ayant présidé à leur élaboration et par les contextes actuels où ils sont pertinents. Ainsi le théorème de Pythagore amène d'abord à un personnage célèbre, haut en couleur, qui vécu approximativement de 580 à 495 avant JC, et que les élèves peuvent rencontrer avec plaisir et curiosité. L'enquête épistémologique que nous avons conduite repose en grande partie sur une compilation de textes historiques rassemblés par et al. (2007). Elle permet de rencontrer des œuvres mathématiques anciennes ainsi que des usages relatifs au théorème de Pythagore et des triplets homonymes. Ainsi des techniques de calcul de longueurs ou d'aires dans les triangles rectangles, ou de façon équivalente dans les rectangles, sont connues au moins depuis l'âge du bronze ancien (de 1800 à 1400 avant JC) en Mésopotamie. En attestent plusieurs tablettes babyloniennes de l'ancienne période (de 2000 à 1600 avant JC). Les tablettes Plimpton 322 (Collumbia University) et Yale Babylonian Collection 7289 (New Haven) sont certainement les plus fameuses. Des triplets pythagoriciens, c'est-à-dire les triplets de nombres entiers (a,b,c) vérifiant l'égalité de Pythagore $c^2 = a^2 + b^2$ et qui correspondent à des côtés de triangles rectangles, apparaissent dans ces tablettes, ainsi que des triplets de nombres décimaux non entiers vérifiant cette même égalité et une technique d'extraction des racines carrées. L'usage de ces triplets, en particulier du 3,4,5, est toujours d'actualité, avec par exemple des techniques en maçonnerie issues du Moyen Âge et utilisant une corde à 13 nœuds. D'autres tablettes, comme la British Museum 96957 (London) ou la Vorderasiatisches Museum 6598 (Berlin), utilisent des techniques similaires pour résoudre des problèmes de calcul de longueurs dans des constructions rectangulaires, comme des murs en briques, des portails, des tombes... Ces techniques de calculs dans les triangles rectangles sont également connues dans l'Égypte ancienne, vraisemblablement par diffusion depuis la Mésopotamie. La civilisation chinoise connaissait également l'égalité de Pythagore, et, au delà des techniques, en formulait un énoncé, le Gou-gu (Imhausen et all., p. 213): dans un triangle rectangle, le carré du 'gou' (la base) augmenté du carré du 'gu' (la hauteur) est égal au carré du 'xian' (l'hypoténuse). Régulièrement, les mathématiciens chinois compilent et reprennent les textes mathématiques anciens, comme le Zhou bi suang ji (les classiques mathématiques du gnomon des Zhou) au premier siècle avant JC, ou le Jiu zhang suan shu écrit par Liu Hui au troisième siècle après JC, et qui rassemble vingt-quatre problèmes dans son neuvième chapitre. Nous exploiterons plus loin l'un d'entre eux, celui du calcul du volume d'une mare à l'aide d'un roseau.

Mais, contrairement à ce que l'on peut supposer des mathématiciens mésopotamiens ou égyptiens, leurs homologues chinois ne se contentaient pas d'une simple règle technique utile pour la construction; le Gou-gu était en effet un théorème démontré. Liu Hui par exemple propose une démonstration du théorème de Pythagore reposant sur l'invariance des aires par ajouts ou retraits de surfaces. D'ailleurs une enquête sur le Web révèlerait que démontrer le théorème de Pythagore est et a été un défi maintes fois accompli par de nombreux auteurs, au delà de la seule communauté des mathématiciens. La majorité des démonstrations reposent sur des théorèmes de géométrie euclidienne, comme la supplémentarité des angles d'un triangle ou les égalités angulaires associées à des droites parallèles et une sécante, et sur des calculs d'aires, généralement de carrés. Parfois elles peuvent également mettre en jeu des techniques de calcul algébrique. Euclide, qui a vécu de 350 à 290 avant JC et qui est fameux pour son encyclopédie en treize volumes intitulée ‘les éléments’, en propose une à la fin du premier livre. Que reste-t-il donc à la gloire de Pythagore (de 580 à 495 avant JC) ? A-t-il eu des contacts avec les mathématiciens égyptiens ? A-t-il voyagé en Orient ? Les témoignages archéologiques ne permettent pas d’affirmer qu’il ait inventé et démontré le théorème dont il est aujourd’hui éponyme. Cependant, les pythagoriciens ont certainement découvert et démontré l’incommensurabilité de la diagonale et du côté d’un carré. Ce fut un secret lourdement gardé, car contraire à une des doctrines de l’école qui stipulait que toute grandeur pouvait s’exprimer à partir de nombres entiers.

Tableau 1. Catégorisation des sources de références (d’après Salone, 2015, p. 271)

Sources de références officielles	Sources de références scolaires
⅄ les élèves ou l’enseignant, pratiques issues de la vie courante ⅄ instances savantes de la discipline ⅄ instances savantes de l’histoire de la discipline ⅄ instances savantes des disciplines associées	⅄ instances savantes d’autres disciplines ⅄ instances de l’établissement scolaire **Sources de références externes** ⅄ instances du réseau social des élèves ou de l’enseignant ⅄ instances du réseau social de proximité de l’établissement scolaire

Revenons de façon plus générale à la question des ouvertures écologiques des savoirs dans les classes. Quelles sont les directions possibles? Une analyse du contenu des programmes officiels donne quelques pistes. Outre de se référer aux savoirs savants et à l’histoire de la discipline, ils préconisent aussi de s’intéresser aux autres disciplines, qu’elles soient scientifiques et donc en connexion directe, ou plus éloignées, comme les arts. Ils suggèrent également d’avoir recours à des pratiques sociales de référence (Martinand, 2005), issues de la vie courante ou de contextes professionnels. Le tableau 1 dresse une liste des ces diverses sources praxéologiques de référence. Dans une volonté d’avoir là un outil pragmatique susceptible d’aider les enseignants à concevoir des séquences ouvertes sur le monde, nous y indiquons des instances sociales, institutions ou personnes, scolaires ou non, qui pourraient fournir des ressources gnoséologiques, humaines ou matérielles. Nous y retrouvons d’abord les sources officielles sus-citées, puis en sont proposées d’autres, davantage tournées vers la société tout entière. Ainsi, les classes étant plongées dans des établissements scolaires, ces derniers offrent potentiellement d’autres possibilités pluridisciplinaires, notamment lorsqu’ils mettent en place des classes à thème (par exemple une des classes observées a une vocation scientifique) ou lorsqu’ils organisent des journées événementielles. Les enseignants des diverses disciplines sont alors autant de vecteurs possibles de savoirs, ainsi que les personnels non enseignants ou les parents d’élèves qui participent à ces systèmes didactiques auxiliaires. Nous décrirons ainsi plus loin comment un factotum a pu intervenir dans une des classes à propos de l’usage qu’il fait du triplet pythagoricien 3,4,5. Enfin le tableau 1 référence également des instances non scolaires qui peuvent être trouvées dans des réseaux sociaux de proximité des agents scolaires et qui pourraient être impliquées dans les apprentissages lors d'interventions dans les classes ou lors de sorties pédagogiques.

2.1 Ouverture des topos et du milieu matériel

Par topos d'un élève (Chevallard, 2007), nous entendons ici l'ensemble des tâches et des rôles qui lui sont dévolus par l'enseignant et dont a priori il s'empare lors des apprentissages en classe. Dans un esprit d'ouverture, ce topos est diversifié, allant de tâches purement disciplinaires à d'autres plus didactiques et sociales. Dans les classes observées les élèves se voient ainsi impliqués dans des activités contextualisées, avec des tâches individuelles ou coopératives variées et des rôles adaptés aux différentes situations didactiques (Brousseau, 1998) en jeu. Une façon de regarder leur topos est alors de suivre le devenir de leurs discours et de leurs travaux. À un premier niveau d'ouverture, leurs productions sont communiquées à la classe. Certains enseignants observés insistent ensuite sur la confrontation des idées, animant des débats publics dans leurs classes. C'est le second niveau. Puis, plus rarement, des enseignants conduisent aussi des négociations des savoirs, des pratiques et des énoncés. Les élèves occupent à ce niveau une position toute particulière dans la relation didactique: ils sont les coauteurs, avec l'enseignant, des textes du savoir qui constituent l'équipement praxéologique de la classe, partie écrite de sa mémoire didactique (Matheron, 2009) qui servira ensuite de référence locale (Sensevy et Mercier, 2007). Ainsi, par l'activité dialogique, se fonde une véritable communauté discursive. Accorder aux élèves un topos d'auteur du cours est, dans notre approche pédagogique, une condition *sine qua non* pour susciter leur adhésion à ce qu'il se fait en classe. Ainsi leurs discours publics sont repris dans certaines classes sous la forme de citations ou d'exercices modèles insérés dans le cours. Une classe est allée encore un peu plus loin dans cette ouverture: l'étude a été initiée par une enquête exploratoire conduite dans et en dehors de la classe (nous en rapportons une plus loin). Le topos des élèves est alors élargi: ils contribuent, avec l'enseignant, à une certaine programmation de l'étude. On peut synthétiser sur une échelle ces différents niveaux du topos des élèves relativement à la constitution de la trace écrite de la leçon:

- Productions communiquées
- Productions débattues et servant de référence locale
- Productions constitutives du cours
- Productions programmatiques de l'étude

D'autres dispositifs pédagogiques sont encore ponctuellement mis en œuvre. Il s'agit par exemple de '*sondages d'opinions'* lors d'émission de conjectures, avec recueils et analyses statistiques, ou de '*corrections en parallèles*' au cours desquelles plusieurs élèves présentent simultanément au tableau leurs solutions à un problème donné avant d'aboutir à une réponse commune, partagée et diffusée. Lors de moments d'exploration des types de tâches ou de travail des techniques (Chevallard, 2002), d'autres dispositifs tout aussi ouverts ont été observés: des temps d'*entrainement tutorés* dirigés par les enseignants au cours desquels les élèves s'entraident, et des temps de *révision en autonomie* relative. Dans ces deux cas, les élèves ont accès à des ressources, livres ou sites internet comme Labomep (www.labomep.net). Les topos des élèves s'ouvrent ainsi sur des compétences davantage didactiques.

Dans les exemples précédents, la composante matérielle du milieu, bien qu'en arrière plan, apparaît comme un élément essentiel. Comment en effet entrer dans une action conjointe si la classe ne disposent pas de ressources idoines? Une part importante de l'action didactique de l'enseignant est donc nécessairement de l'ordre du mésogénétique. Le premier niveau de structuration physique du milieu que nous avons observé est ainsi l'enrichissement des salles de classe. Les '*classes enrichies*' proposent des supports discursifs partagés (tableau, espaces d'affichage muraux), des moyens d'accès à des ressources multimédia (par exemple du matériel de vidéo-projection, une bibliothèque ou des ordinateurs). À un deuxième niveau, les classes deviennent des '*classes numériques'.* Toutes les classes observées sont ainsi connectées à internet, à partir d'accès partagés ou à partir de tablettes numériques personnelles, et les établissements scolaires offrent divers services via un espace numérique de travail, comme des cahiers de texte ou des cours en ligne, des espaces de partage de documents, des outils d'échange et de communication. Enfin, les ouvertures du milieu peuvent également consister à transporter la classe dans des lieux non classiques, internes à l'établissement scolaire, comme les centres de documentation et d'information ou les espaces récréatifs hors murs, ou extérieurs, comme tous ceux qu'offrent les espaces urbains ou les zones naturelles à proximité. Une des classes observées, la classe à vocation scientifique évoquée plus haut, a ainsi organisé tout au long de l'année des sorties pédagogiques en milieu naturel, le théorème de Pythagore et le cosinus d'un angle ayant alors servi pour calculer les profils topographiques des balades.

3 Un exemple de mise en œuvre

Dans chacune de leurs classes, les enseignants observés ont choisi librement la progression mathématique qu'ils souhaitaient adopter, l'ingénierie didactique ne leur fournissant que quelques thèmes d'activités et des descriptions des dispositifs pédagogiques évoqués plus haut. Dans l'exemple proposé ici, l'enseignant a choisi de ne pas aborder l'étude des racines carrées, en particulier de racine de 2, en même temps que le théorème de Pythagore. Étant informé de ce qui allait être observé, les ouvertures écologiques de sa classe, il met en œuvre quelques uns des dispositifs pédagogiques proposés. La progression retenue (Figure 1) commence l'étude par une enquête exploratoire de laquelle émergent des exercices conformes aux attentes des programmes officiels et qui seront ensuite travaillés dans le cadre d'un entrainement 'tutoré' avec Labomep puis qui seront cooptés pas la classe . L'étude se poursuit par l'intervention du factotum de l'établissement scolaire. Le triplet pythagoricien 3,4,5 est alors rencontré par la classe et est ensuite intégré au cours. La séquence s'achève finalement sur un nouvel entrainement 'tutoré' avec Labomep. Notons encore que dans cette progression aucune démonstration du théorème n'est étudiée.

L'enquête exploratoire est d'abord conduite par les élèves chez eux, avec les médias dont ils disposent dans leurs sphères familiales, puis en classe, avec les manuels scolaires et des accès à internet. Fermée par trois questions ciblées et une injonction de production d'énoncés, elle fait ressortir l'histoire du personnage, avec affichage dans la classe de quelques exposés, et les exercices officiels associés au théorème (annexe 1). En particulier les élèves rencontrent les types de tâches attendus: calculer une longueur dans un triangle rectangle et prouver qu'un triangle est rectangle ou non. Certains élèves, comme celui auteur de l'enquête en annexe 1, sont surpris par la diversité des usages du théorème dans la vie courante:

« Tous ceux qui travaillent dans la nature et qui ont besoin d'estimations des distances utilisent le théorème de Pythagore (militaires, géomètres, jardiniers). Les maçons utilisent 3,4,5 pour les angles droits des bâtiments. Les menuisiers pour découper, les géographes pour établir des diagnostics, les mathématiciens pour trouver des formules, etc... Je ne pensais pas que le théorème de Pythagore servait dans autant de métiers. »

- Calcul de longueurs dans un triangle rectangle

1. Enquête sur le théorème de Pythagore. Classe inversée. Questions de l'enquête: qui est Pythagore, qu'est-ce que le théorème de Pythagore, à quoi sert-il ? Trouver des énoncés d'exercices. L'enquête est conduite par les élèves hors de la Classe.
2. Travail de groupe. Production d'une synthèse sur les énoncés et les usages du théorème de Pythagore.
3. Entraînement tutoré avec Labomep.
4. Cooptation d'exercices modèles. À partir des exercices rencontrés par les élèves, des exercices d'application directe sont retenus et insérés dans les fichiers d'exercices modèles.

- Contrôles d'angles droits

5. Intervention du factotum. Le factotum contrôle les angles droits de la salle de Classe. Il montre également l'usage qu'il fait du triplet pythagoricien (3, 4, 5) pour construire des rectangles (le terrain de rugby).
6. Écriture de comptes-rendus d'observation. Travail individuel des élèves effectué à la maison.
7. Lecture de comptes-rendus et cooptation d'un exercice modèle.

- Construction de figures avec des angles droits

8. Enquête sur les triplets pythagoriciens et leurs usages. Ressources numériques de la Classe.
9. Cooptation d'une synthèse.
10. Entrainement tutoré à l'aide de Labomep.

Figure 1. Une programmation de l'étude du théorème de Pythagore

Cette ouverture écologique des savoirs amène donc d'une part à des compétences mathématiques officielles et, d'autre part, à des compétences transversales non routinières de l'ordre de l'enquête et de la production d'énoncés. L'enquête en elle-même aura eu pour fonction didactique de dévoluer et d'initier l'étude.

La phase d'enquête est suivie d'une production cooptée du cours. Les élèves, en groupes, produisent des énoncés du théorème, puis des énoncés d'exercices qui lui sont associés, puis de leurs solutions. Ces travaux sont ensuite communiqués et débattus et certains sont sélectionnés, après validation par l'enseignant, pour venir constituer le cours commun (Figure 2). Nous retrouvons là l'ouverture des topos qui consistent à octroyer aux élèves un rôle d'auteur.

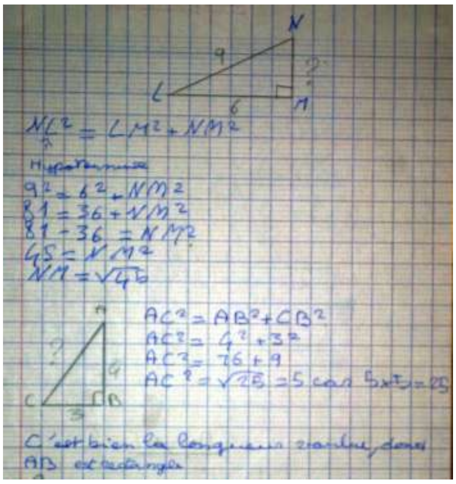


Figure 2: Exercices cooptés

Entre la phase de production d'énoncés et celle de cooptation du cours, le dispositif 'entraînement tutoré' est mis en œuvre à l'aide du site internet Labomep. Des énoncés d'exercices sont d'abord vidéo-projetés et les élèves cherchent individuellement des réponses. L'enseignant circule dans la classe et valide les résultats. Les premiers élèves qui réussissent sont alors désignés pour devenir des tuteurs: ils se rendent disponibles pour aider leurs camarades. Une solution commune est enfin cooptée et saisie avant de passer à l'exercice suivant. Ce dispositif assume ainsi pleinement son objectif d'ouverture des topos: les élèves mettent en œuvre leurs compétences didactique en devenant des aides à l'étude.

La séquence se poursuit ensuite par l'intervention du factotum, l'homme à tout faire de l'établissement scolaire de notre terrain de recherche. Celui-ci expose publiquement l'usage qu'il fait du triplet pythagoricien (3,4,5) et des cordes à 13 nœuds pour tracer le terrain de rugby de l'établissement et pour contrôler la perpendicularité des murs de la salle de classe par rapport au sol. Les techniques qu'il met en œuvre sont elles fort anciennes, déjà évoquées dans les tablettes babyloniennes et largement répandues au Moyen Âge lors de la construction des cathédrales. Les élèves sont alors invités à tracer de grands rectangles au sol ou sur le tableau et à effectuer des contrôles de perpendicularité. C'est l'occasion pour eux de manipuler des cordes à 13 nœuds et des instruments de mesure de longueurs qui ne font pas partie de leurs instruments classiques, comme le mètre en bois de l'enseignant et les décamètres de maçons. Ils ont ensuite pour tâche de produire un compte-rendu d'observation (annexe 2) de cette expérimentation. Toujours dans un esprit d'ouverture des topos, un de ces comptes-rendus est ensuite coopté et diffusé, puis un exercice 'modèle' est coproduit.

Dans cette phase de l'étude, l'ouverture des savoirs est donc poursuivie en direction d'une pratique sociale de référence fort ancienne et encore utilisée aujourd'hui par un professionnel non enseignant. Elle aura aussi conduit à un milieu physique et des types de tâches non routiniers. Elle aura maintenu l'ouverture des topos des élèves qui ont continué d'être auteurs du cours et qui ont eu des rôles d'expérimentateurs proches de ceux qui leurs sont octroyés dans les disciplines scientifiques.

La fin de la séquence suivra le même schéma, avec une ouverture des savoirs réalisée par une enquête en classe sur les triplets pythagoriciens, et une ouverture des topos autour d'une cooptation d'exercices 'modèles' puis un nouvel entraînement 'tutoré'.

4 Conclusion et perspectives

Dans cet exemple, le contexte historique a donc sous-tendu la progression de l'étude de l'égalité de Pythagore. Il a d'abord eu pour fonction de dévoluer et d'initier l'étude à partir d'une enquête multimédia, puis il a fourni des énoncés, des exercices et des solutions cooptés par la classe, et enfin, avec l'intervention d'un professionnel non enseignant, il a permis de rencontrer des usages anciens et actuels des triplets pythagoriciens. Ces ouvertures écologiques des savoirs ont été associées à des ouvertures des topos élèves qui ont été auteurs de l'équipement praxéologique de la classe et qui, lors des entrainements 'tutorés', ont mis en œuvre leurs compétences didactiques. Un milieu physique idoine a été nécessaire pour tout cela, enrichi d'espaces de communication partagés et de moyens numériques d'accès à des ressources.

Un tel contexte socio-historique est-il bénéfique pour les apprentissages ? Est-il efficace, plaisant et motivant pour les élèves ? Même si nous le postulons dans nos travaux et si nous nous inscrivons dans les paradigmes pédagogiques actuels, il nous semble difficile de l'affirmer ici. C'est pourquoi dans nos travaux de recherche actuels, nous tentons de réunir dans un collectif des enseignants, des chercheurs et des formateurs afin d'expérimenter et de concevoir des dispositifs pédagogiques et des activités d'étude et de recherche comme ceux et celles qui viennent d'être présentés.

REFERENCES

Bronner, A. (1997). *Étude didactique des nombres réels: Idécimalité et racines carrées*. Thèse de doctorat. Grenoble: Université J. Fourier (Grenoble 1).

Brousseau, G. (1998). *Théorie des situations didactiques*. Grenoble: La Pensée sauvage.

Chevallard, Y. (2002). Organiser l'étude, 3, écologie et régulation. In *Actes de l'a XIe école d'été de didactique des mathématiques* (pp 41-56). Grenoble: La Pensée Sauvage.

Chevallard, Y. (2007). Passé et présent de la théorie anthropologique du didactique. In L. Ruiz-Higueras, A. Estepa, & F. Javier Garcia (Eds.), *Sociedad, Escuela y Mathematicas: Aportaniones de la Teoria Antropologica de la Didactico* (pp 705-746). Baeza (Espagne): Universidad de Jaen.

Chevallard, Y., & Ladage, C. (2010). *Enquêter pour connaître: L'émergence d'un nouveau paradigme scolaire et culturel à l'âge de l'Internet* (Communication au Colloque Yves Chevallard). Liège, Belgique: Université de Liège.

Imhausen, A., Robson, E., Dauben, J., Plofker, K., & Lennart Berggren, J. (2007). Chapters in V. Katz (Ed.), *The Mathematics of Egypt, Mesopotamia, China, India, and Islam : A sourcebook*. Princeton, NJ: Princeton University Press.

Matheron, Y. (2009). *Mémoire et étude des mathématiques: Une approche didactique à caractère anthropologique*. Rennes: Presses Universitaires de Rennes.

Martinand, J. L. (2001). *Pratiques de référence et problématique de la référence curriculaire*. In A. Terrisse (Éd.), *Didactique des disciplines, les références au savoir* (pp. 17-24). Bruxelles: éditions De Boeck Université, Perspectives en éducation et formation.

Ministère de l'éducation nationale (2008). *Bulletin officiel spécial n°6 du 28 août 2008. Programmes de l'enseignement de mathématiques*., de physique-chimie, de sciences de la vie et de la Terre, de technologie pour les classes de sixième, de cinquième, de quatrième et de troisième du collège (RLR : 524-2a ; 524-2b ; 524-2c) arrêté du 9-7-2008 – j.O. du 5-8-2008 (NOR : MENE0817023A).

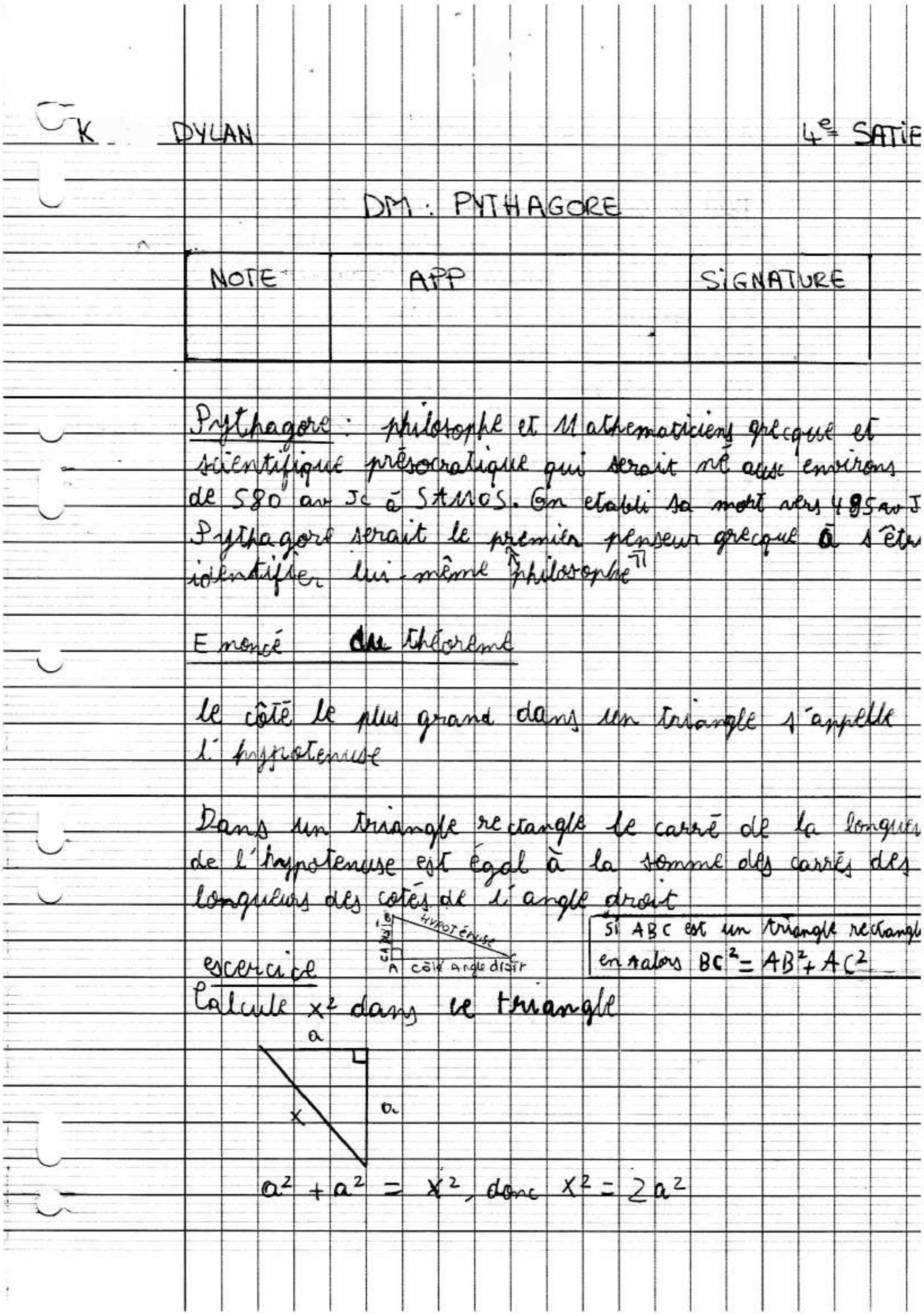
Salone, J. J. (2015). *Les références praxéologiques dans les systèmes didactiques*. Thèse de doctorat. Marseille: Aix-Marseille Université.

Sensevy, G. (2011). *Le sens du savoir, éléments pour une théorie de l'action conjointe en didactique.* Bruxelles: De Boeck.

Sensevy, G., & Mercier, A. (2007). *Agir ensemble: L'action didactique conjointe*. In G. Sensevy, & A. Mercier (Eds.), *Agir ensemble: L'action didactique conjointe du professeur et des élèves* (pp. 187-211). Rennes: Presses Universitaires de Rennes.

Annexes

Annexe 1: l'enquête exploratoire d'un élève



K DYLAN 4e SATIE

DM : PYTHAGORE

NOTE	APP	SIGNATURE

Pythagore : philosophe et Mathematiciens grecque et scientifique présocratique qui serait né aux environs de 580 av JC à SAMOS. On etabli sa mort vers 495 av J
Pythagore serait le premier penseur grecque à s'être identifier lui-même philosophe

Enoncé du théorème

le côté le plus grand dans un triangle s'appelle l'hypotenuse

Dans un triangle rectangle le carré de la longueur de l'hypotenuse est égal à la somme des carrés des longueurs des cotés de l'angle droit

Si ABC est un triangle rectangle en A alors $BC^2 = AB^2 + AC^2$

excercice

Calcule x^2 dans le triangle

$a^2 + a^2 = x^2$, donc $x^2 = 2a^2$

Annexe 2: compte-rendu d'observation d'un élève

A
Camilia
4e Satie.

Compte rendu de l'observation

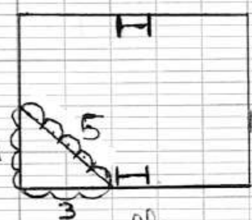
Intervenant :

Questions :

Que fait-il ?
Explication mathématique
Recherches internet sur des autres façons de faire.

Que fait il ?

M. Pelassy fait de nombreux travaux dans notre écol
par exemple c'est lui qui trace le terrain de rugb
pour effectuer ce travail il procède de cette magnèr



Un terrain de rugbi est un rectangl
donc il dois avoir 4 angles droit. Mais u
terrain de rugbi est beaucoup plus grand
qu'un carré sur une feuille, donc une éq
ne suffira pas pour tracer des angles droits. Pour tra
des angles droit M. Pelassy utilise une technique qui s'a
le 3-4-5 ou "La réciproque du théorème de Pythagores".
a dire qu'il trace un triangle en prenant la mesure d
quelque chose et pour faire les côtés du triangle il pren u
qui est égal à la chose X 3 un côté qui est égal à
la chose X 4 et un côté qui est égal à la chose X 5

1/1

à revoir

Comment verifier que ce triangle sera rectangle ?

D'après la réciproque de Pythagores:

$5^2 = 3^2 + 4^2$ — Ce triangle est bien rectangle
$25 = 9 + 16$ — la réciproque de Pythagores dis que si
$25 = 25$ — carré de l'hypoténuse est égale a la somm

THE ROLE OF CULTURAL ARTIFACTS IN THE PRODUCTION OF MEANING OF MATHEMATICAL KNOWLEDGE INVOLVED IN EMBROIDERY ACTIVITIES FROM THE HÑAHÑU CULTURE

Armando SOLARES-ROJAS[1], Erika BARQUERA[2]
National Pedagogic University (Mexico)
[1]asolares@g.upn.mx, [2]erikabarquera@hotmail.com

ABSTRACT

We identified mathematical knowledge involved in embroidery activities of the Hñahñu culture from Valle del Mezquital (Mexico) from a cultural-semiotic approach. The motivation that led to this research was to test the viability of a teaching model (Filloy, Rojano & Puig, 2008), which considers the social and cultural aspects of the embroidery activity. We resorted to the *Theory of objectification* (Radford, 2014) as it allows us to study the appropriation process of the mathematical knowledge involved in this activity, taking into account the cultural artifacts and the interaction between embroiderers. Among the results of this study, we can say that embroiderers establish a correspondence between the symmetries and patterns of the geometric motifs and the "symmetries" and patterns of the numerical sequence of stitches. We also found that they develop different strategies for counting and measuring the motifs, stitches and yarn strands.

Introduction

In this research we identified mathematical knowledge involved in the production of embroidery of Hñahñu culture (also called Otomí) in the region of Valle del Mezquital, Hidalgo State, Mexico. The question that guided our research was: What math knowledge is implicit in the production process of hñahñu embroidery?

We were specifically interested in the processes of acquisition and communication of this knowledge. For that purpose, we adopted a semiotic-cultural approach (Radford 2006, 2013, 2014), in which we considered that learning is comprised of active processes of acquisition of the meanings implicit in cultural artifacts and social interaction:

> It deals with making sense of conceptual objects that the student finds in his culture. The acquisition of knowledge is a process of active elaboration of meanings. It is what we will call later a process of objectification. (Radford, 2006, p. 113)[1]

We sought to answer this question considering that the findings could be useful for researchers on mathematics and culture as well as for teachers and designers of curriculum for indigenous education.

1 Previous research

In recent years, several theoretical approaches have been developed for considering the influence of context in mathematical knowledge learning. These approaches highlight

[1] Author's translation

the importance of social, cultural and historical aspects on the construction and dissemination of mathematical knowledge (D 'Ambrosio, 2002; Radford, 2006, 2013, 2014). They also consider that mathematical skills, knowledge, values and beliefs are essential part of any culture (Aroca, 2008; Oliveras & Albanese, 2012; Silva, 2006).

In this research, we made use of results of several studies that work with the mathematical knowledge involved in the construction of the geometric patterns of some embroideries (Gerdes, 2012; Gilsdorf, 2012; Sánchez, 2014; Soustelle, 1993). According to them, in carrying out this activity, embroiderers resort to various forms of the notion of symmetry; they combine aspects equally inspired by decorative aims, spiritual purposes and abstraction.

Some other studies dealing with the development of the geometry in several cultures conclude that mathematics and geometry are developed by all human group (Diaz Escobar & Mosquera, 2009; Fuentes, 2012; Micelli and Crespo, 2011; Scandiuzzi & Regina, 2008; Sufiatti, Dos Santos Bernardi & Duarte, 2013; Urban, 2010). They affirm that geometry arises from the survival needs of the people, and it is unfolded in the ways people organize and build their homes, in the ways they produce their basketry, ceramics and textile, and in all the activities they do for satisfying their needs.

Despite the large body of research dealing with the study of mathematical knowledge involved in the activities of specific social and cultural groups, it is still needed to address the acquisition of meaning of mathematical objects involved in such activities. Our study sought to contribute on this direction.

2 Theoretical approach

We adopted the theoretical approach of objectification (Radford, 2006; 2013; 2014) as it allowed us to study the acquisition of mathematical objects from a sociocultural perspective. According to this theory, the meaning of mathematical objects has two essential aspects: on the one hand, it is a subjective construction, linked to the history and experiences of each person; and, on the other hand, it is a cultural construction, previous to the subjective experience, in which the objects have their own values and cultural content.

Thus, in order to study the acquisition process of the meaning of mathematic objects is important to take into account two sources: the cultural artifacts with which people carry out their activities, and the social interaction. The artifacts are repositories of historical knowledge from cognitive activity of past generations. Radford argues that 'the human being is profoundly affected by the artifact: because of the contact with it, human being restructures its movements and creates new motor and intellectual capacities, such as anticipation, memory and perception' (Radford, 2006, p. 113). Regarding the interaction, he claims that "objects can not make clear the historical intelligence embodied in them. For this it is necessary to use them in activities, and the contact with other people who know 'read' such intelligence and help us acquire it" (Radford, 2006, p. 113).

3 Methodology

The methodological design had two phases. In the first one, we studied the "formal" mathematical knowledge that can be identified in a collection of 271 hñahñu traditional geometric motifs, which formed part of this research. We used the analytical tools proposed by Jaime and Gutiérrez (1995) to identify the *systems of generating transformations of figures* from 271 embroideries (Barquera & Solares, 2016).

In the second phase, we used an ethnographic methodology and developed field observations and interviews for analyzing the mathematical knowledge that women really use for the production of their embroideries. The field observation entailed accompanying the embroiderers during their daily work life, in the places and moments of the day when they usually embroider. We were interested in knowing how they learned to embroider, who taught them, what geometric motifs were the first that they learned to embroider, in what moments of the day and under what conditions they perform their work. We worked with 10 experienced embroiderers, who began with this work activity since childhood. For the interview we chose 4 of them, the more willing for being interviewed. The interview consisted of presenting one embroidery design and asking to the interviewee to reproduce it in a piece of fabric, according to a frequent custom of sharing geometric motifs. This phase is still ongoing.

4 Results

In this presentation we focus on some of the results of the second phase of the study. While research is still ongoing, the results presented show the kind of mathematical knowledge that hñahñus embroiderers put into action to carry out their activities.

4.1 Some results of phase 2: the ethnographic study

Among our results, we found that embroiderers structure their strategies considering both the characteristics of their work tools (size of fabrics, types and colors of yarns, etc.) and the characteristics of the geometric motifs that they construct. The uses of material resources (the work tools) and the conceptual resources (for instance counting, perception of symmetries and hñahñu cultural symbolism) determine strategies for building the geometric patterns of their embroidery.

Here we refer to the interviews of Isabel and Sabina, two older embroiderers, to show some strategies of thread counting resorted to when carrying out their activity. At the beginning of the interviews they were presented with the embroidery shown in Figure 1. This embroidery was chosen because is rich in geometric figures and has a number of symmetries. Isabel and Sabina were asked to reproduce it in a piece of fabric, as they do in the exchanges that commonly occur between the embroiderers.

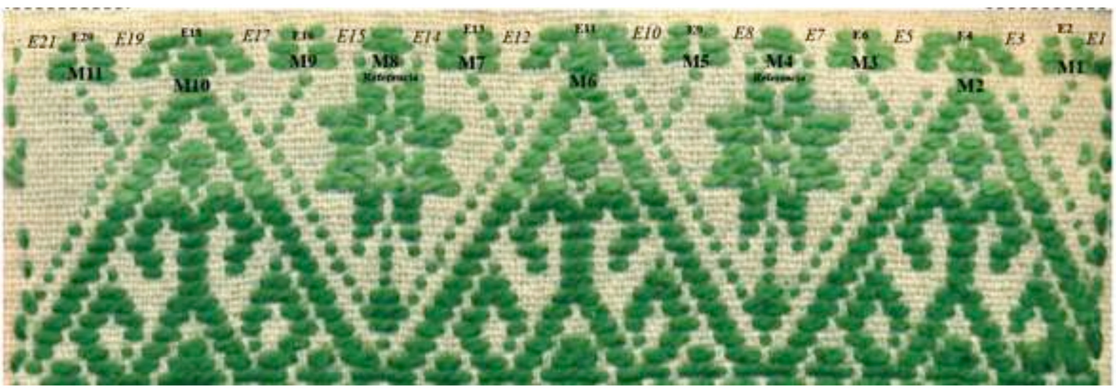


Figure 1. Embroidered motifs

In Figure 1 letter "M" indicates a geometric motif that forms part of the embroidery. In this case, we have 11 motifs, from which M2, M6 and M10 are rhombuses; M1, M3, M5, M7, M9 and M11 are motifs representing "leaves," as the embroiderers call them; M4 and M8 are "stars."

The distances between the motifs are counted as the number of fabric threads between them. In Figure 1, these distances are indicated by letter E. Distances E1 to E21 are measured by considering the first line of the embroidery. There are distances of two types: the first type separates parts of the same motif, for example, into the leaf M1 there is a distance of E2 between the two parts of the leaf; the second type measures the distance from one motif to another. In this case, E2, E4, E6, E9, E11, E13, E16, E18, E20 are distances of the first type and each measures 2 threads. E1, E3, E5, E7, E8, E10, E12, E14, E15, E17, E19 and E21 are of the second type, each measuring 10 threads.

Both embroiderers indicated that the count of the threads of the first line determines the success or failure of the reproduction of the given embroidery. The following episode shows what Sabina said about the importance of well counting in the first line:

> Interviewer: What is the hardest thing in embroidery?
>
> Sabina: The start. Once you have started, it's easier ... The start is more difficult because you have to count. In the second round you can just increase to go forming little leafs...

Isabel and Sabina counted by considering that the fabric is a grid. They proceeded line by line of the grid. Both Isabel and Sabina began embroidering the first line from right to left and, once reached the left side, they changed direction: now from left to right; and so on.

Isabel counted both the threads forming the motifs and the distances that separate them. Whenever she counted the threads of one motif, she verified that the similar motifs measured the same. Figure 2 shows the results of the count carried out by Isabel in the first line of this embroidery.

10 4 2 4106 2 6 10 4 2 4 10 6 10 4 2 4 10 6 2 6 10 4 2 4 10 6 10 4 2 410 6 2 6104 2410

Figure 2. Sequence of numbers obtained by Isabel in counting the threads of the embroidery first line

In this figure, the numbers with a line represent distances between motifs, measured by counting the number of threads of the fabric that are in sight. The distances of the first type (between parts of the same motif) are highlighted in green and the second type (between different motifs) in yellow. The numbers having no lines represent the motifs lengths, measured by counting the number of threads of the fabric under the stitches.

Isabel showed ease in producing the embroidery. She was meticulous in her counts and systematically verified them. In her explanations, she said that located similar motifs to one already counted and gave the corresponding stitches on piece of fabric. So, she did not need to do all the count, thread by thread. Moreover, she established that the "stars" (M4 and M8) were benchmarks, allowing her to look for patterns and symmetries. Figure 3 shows the sequence numbers determined by Isabel and its correspondence with the benchmarks.

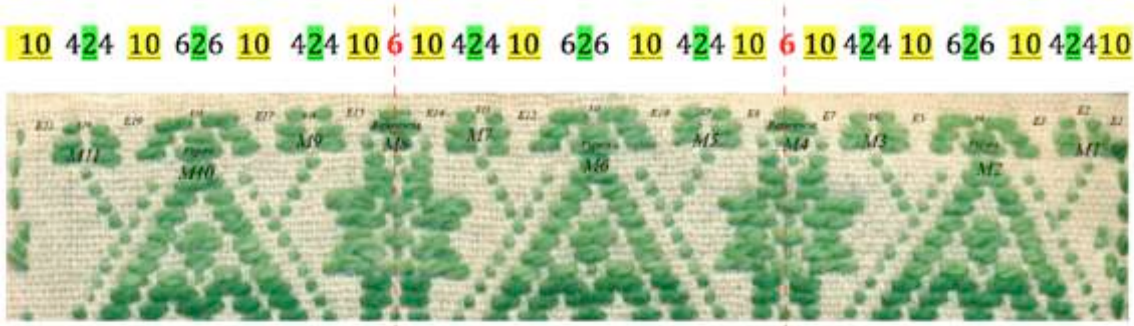


Figure 3. Symmetries and benchmarks in the embroidery

The embroidery symmetries allowed Isabel to reproduce it without counting all the motifs. We can say that Isabel established a correspondence between the geometric symmetries and patterns of the embroidery and the "symmetries" and patterns of the numeric sequence, line by line.

These symmetries also allowed Isabel to correct mistakes. For example, when embroidering the first three lines of the upper part of M2 and M6 motifs (rhombuses), Isabel counted as follows:

M6	M2
6 2 6	6 2 6
4 2 6 2 4	4 3 5 2 4
3 3 10 2 4	4 2 10 2 4

The red numbers indicate errors in stitching. To correct her mistakes, Isabel compensated them in the immediate following stitches. In M2, for example, to correct the stitch where she spent his needle beneath 3 threads (instead of 2), in the next stitch Isabel spent his needle above 5 threads (instead of 6). Thus, Isabel controlled the overall size of the motifs (motifs M2 and M6 measure 18 threads in the second line). Isabel considered not only the numeric characteristics of the motifs (number of threads), but also their symmetries and geometric regularities.

Final remarks

The practical knowledge of hñahñu embroiderers are deeply affected by artifacts, both material and conceptual, that they use to carry out their activities. By using tools such as threads, needles and fabrics, embroidery women mobilized and provide with meanings to their knowledge on counting, compensation through addition and subtraction, and numerical patterns and symmetries. They construct (and we have found that also share as a community) strategies that allow them to efficiently perform their activities.

Our research raises the possibility of characterizing a "practical geometry" used by the embroiderers, whose study can provide of teaching alternatives for school geometry. The richness and variety of meanings of mathematical objects found in embroidery activities show the educational potential of the contexts that can be exploited for the formal teaching of geometry in schools of hñañhu communities. We consider important that school recognize the practical geometry developed in practical activities, where geometric transformations and symmetries are used through embroidery tools not through educational tools, such as ruler and compass, or dynamic geometry software.

Finally, it is important to say that to unfold the details of the acquisition process of mathematical knowledge of hñahñu embroiderers is necessary to deepen the analysis of the relations between subjectification and objectification in this activity (Radford, 2014). This analysis is still ongoing (Solares and Barquera, forthcoming).

REFERENCES

Aroca, A. (2008). Análisis a una Figura Tradicional de las Mochilas Arhuacas. Comunidad Indígena Arhuaca. Sierra Nevada de Santa Marta, Colombia. *Boletim de Educação Matemática, 21*(30), 163-180.

Barquera, E. (2003). *Numeración y medida en la Cultura Otomí*. Unpublished master dissertation. Distrito Federal-México: CINVESTAV.

Barquera, E., & Solares, A. (2016). Conocimientos matemáticos involucrados en la producción de bordados de la cultura Hñahñu: Un análisis semiótico-didáctico. *Revista Latinoamericana de Etnomatemática: Perspectivas Socioculturales de la Educación Matemática, 9*(1), 26-48.

D'Ambrosio, U. (2002). *Etnomatematicas*. Bolonia: Pitagora.

Filloy, E., Rojano, T., & Puig, L. (2008). *Educational Algebra. A Theoretical and Empirical Approach.* New York: Springer.

Fuentes, C. (2012). La Etnomatemática como mediadora en los procesos de la reconstrucción de la historia de los pueblos, el caso de los artesanos del municipio de Guacamayas en Boyacá, Colombia. *Revista Latinoamericana de Etnomatemática*, *5*(2), 66-79.

Gerdes, P. (2012). *Etnogeometría. Cultura e o despertar do pensamento geométrico*. Belo Horizonte, Boane, Moçambique: Instituto Superior de Tecnologias e de Gestão (ISTEG).

Gilsdorf, T. (2012). *Arte Textil Otomí y Mazahua como una Expresión Femenina de las Matemáticas*. Instituto Tecnológico Autónomo de México.

Jaime, A., & Gutiérrez, A. (1995). Guidelines for teaching plane isometries in secondary school. *The Mathematics Teacher*, *88*(7), 591-597.

Micelli, M. L., & Crespo, C. R. (2011). La geometría entretejida. *Revista Latinoamericana de Etnomatemática*, *4*(1), 4- 20.

Oliveras, M. L., & Albanese, V. (2012). Etnomatemáticas en Artesanías de Trenzado: Un modelo metodológico para investigación. *Boletim de Educação Matemática, 26*(44), 1315-1344.

Radford, L. (2014). De la teoría de la objetivación. *Revista Latinoamericana de Etnomatemática, 7*(2), 132-150.

Radford, L. (2013). Three key concepts of the theory of objectification: Knowledge, knowing, and learning. *Journal of Research in Mathematics Education*, *2*(1), 7-44.

Radford, L. (2006). Elementos de una teoría cultural de la objetivación. *Relime, Número Especial*, 103-129.

Sánchez. J. (2014). *Iconología simbólica en los bordados populares toledanos*. Unpublished doctoral dissertation. Madrid-España: Facultad de Filología, Universidad Complutense de Madrid.

Scandiuzzi, P., & Regina, C. (2008). Simetrias e Assimetrias no Contexto de Povo Kadiwéu. *Revista Latinoamericana de Etnomatemática*, *1*(2). 4-26.

Soustelle (1993). *La familia otomí – Pame del México Central. Centro de Estudios Mexicanos y Centroamericanos.* México DF: Fondo de Cultura Económica.

Sufiatti, T., Dos Santos Bernardi, L., & Glavam Duarte, C. (2013). Cestaria e a história de vida dos artesãos indígenas da Terra Indígena Xapecó. *Revista Latinoamericana de Etnomatemática, 6*(1), 67-98.

Solares, A., & Barquera, E. (forthcoming). El papel de los artefactos y las interacciones en la construcción de significados de los conocimientos matemáticos involucrados en la producción de bordados Hñahñu.

UN PROBLÈME CLASSIQUE REVISITÉ : LA QUADRATURE DU YIN-YANG

Jean-Paul TRUC
École des Pupilles de l'AIR, Saint-Ismier 38332
jean-paul.truc@prepas.org

RÉSUMÉ

Cet article souhaite montrer comment un très vieux problème comme la Quadrature du cercle est toujours d'actualité pour les amateurs de géométrie et peut donner matière à des activités d'enseignement au Lycée. Après une courte introduction historique qui reprend quelques quadratures, nous exposons une nouvelle méthode de quadrature du cercle, découverte par un mathématicien amateur de Béziers, André Ferrandi (Ferrandi, 2015). Trois preuves différentes, venant de trois auteurs différents, en sont données. Elles utilisent la géométrie analytique, la géométrie classique du triangle, et la géométrie complexe. Deux d'entre elles au moins sont accessibles aux élèves de Terminale Scientifique des Lycées et peuvent faire des sujets de travaux dirigés. Nous l'avons vérifié avec la classe de Terminale S du Lycée d'Altitude de Briançon, grâce à son professeur, Hubert Proal.

1 Des origines à la Grèce antique

1.1 Les prémices

On peut considérer que la première tentative de quadrature du cercle apparaît sur un papyrus égyptien de 1650 av. J.-C. écrit par le scribe Ahmès, qui indique la méthode pour construire un carré qui ait la même surface qu'un cercle. Sa méthode (voir figure 1) consiste à diminuer le diamètre d'un neuvième pour trouver le côté du carré. Le calcul donne :

$$\pi\left(\frac{D}{2}\right)^2 = \left(\frac{8D}{9}\right)^2,$$

ce qui conduirait à la valeur approchée de π suivante : $\pi \simeq \dfrac{256}{81} \simeq 3,1604$. Bien sûr, le problème numéro 50 de ce rouleau est posé dans des termes beaucoup plus concrets : « Un champ circulaire a un diamètre de neuf Khet (environ 50 m), quelle est sa surface ? ». Il n'y a pas de méthode générale, juste des exemples numériques qui permettent de la deviner, et le but est avant tout opératoire. Enfin la notation π ne sera introduite par William Jones qu'en 1706 et fixée par Euler en 1737.

1.2 Les mathématiques grecques

Dès l'Antiquité, le problème est posé en des termes géométriques et pas seulement pour faciliter les calculs. D'après Plutarque, Anaxagoras (499 av. J.-C, 427 av. J.-C.) est le premier a avoir écrit sur ce sujet.

Hippocrate de Chios (vers 440 av. J.-C.) ouvre une voie indirecte mais importante avec ses lunules, car il réalise la première quadrature d'un objet géométrique à contours circulaires (par un triangle rectangle isocèle). Cette découverte va donner de l'espoir aux « quadrateurs » à venir.

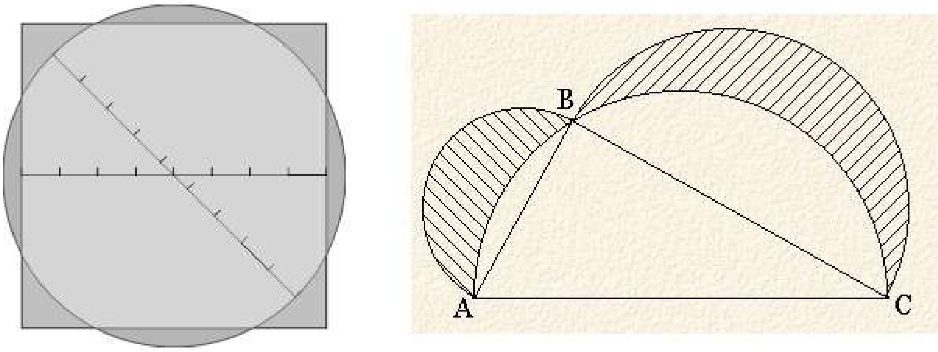


Figure 1 – La quadrature d'Ahmès et les lunules d'Hippocrate

D'autres mathématiciens vont utiliser des moyens détournés et en un certain sens « cinématiques » : Dinostrate (vers 350 av. J.C.), suivant les traces d'Hippias (vers 430 av. J.C.) utilise une courbe construite à l'aide d'un système mécanique (cette approche des courbes est générale à cette époque) : une droite horizontale se déplace vers le haut à une vitesse uniforme v, pendant qu'une autre droite tourne autour de O avec une vitesse de rotation uniforme ω. A l'instant $t = 0$ les deux droites sont confondues avec Ox. La *Quadratrice de Dinostrate* est la courbe décrite

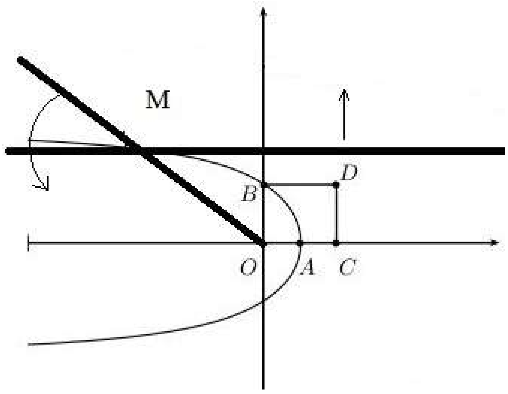


Figure 2 – La Quadratrice de Dinostrate

par le point d'intersection M de ces deux droites (voir figure 2). Déterminons l'équation de ce lieu en coordonnées polaires. À l'instant t, ce point M a pour ordonnée $y = vt$ et la droite qui tourne fait un angle $\theta = \omega t$ avec Ox. La distance $r = OM$ vérifie donc :

$$r^2 = x^2 + y^2 = v^2 t^2 + r^2 \cos^2 \theta = \frac{v^2 \theta^2}{\omega^2} + r^2 \cos^2 \theta,$$

d'où nous tirons :

$$r = \frac{v}{\omega}\frac{\theta}{\sin\theta}.$$

Cette courbe est une *quadratrice.* En effet, comme :

$$r(0) = \lim_{\theta\to 0}\frac{v}{\omega}\frac{\theta}{\sin\theta} = \frac{v}{\omega},$$

nous aurons donc : $r(\pi/2) = \frac{\pi}{2}r(0)$, et par suite (c.f. figure 1.2) : $OB = \frac{\pi}{2}OA$. Finalement si C est le symétrique de O par rapport à A, on a :

$$OC \times OB = \pi OA^2.$$

On a donc la possibilité, connaissant cette courbe, de construire un rectangle de surface égale à π en prenant par exemple $v = \omega$, mais la courbe n'est pas en elle-même constructible. D'après Proclus (Heath, 2013), le sophiste Hippias d'Elis aurait découvert cette courbe vers 420 av. J.-C. pour le problème de trisection de l'angle, et elle aurait ensuite été utilisée par Dinostrate vers 350 av. J.-C. pour la quadrature du cercle ; mais certains pensent que Hippias avait déjà introduit l'appellation de *quadratrice.*

Quoiqu'il en soit, il est déjà bien établi vers 300 av. J.-C. que « les cercles sont dans la proportion des carrés sur leurs diamètres » (Peyrard, 1804), le « sur » indiquant bien qu'il s'agit de comparer entre eux des objets géométriques, plus que des nombres.

1.3 Les quadratures d'Archimède

Du point de vue cinématique, Archimède propose une jolie solution dans son traité sur les spirales (ou cochloïdes), qu'il définit, toujours de manière « mécanique » comme les courbes engendrées par un point qui se déplace à partir d'un point central O de façon uniforme le long d'une droite qui tourne uniformément autour de O. Au bout d'un demi-tour, le point occupe la position M, d'angle polaire π. Traçons alors la tangente à cette spirale en M. Elle coupe en un point P l'axe Oy (voir figure 3). Archimède prouve alors que OP est égal à la demi-circonférence du cercle de rayon OM. En effet l'équation en coordonnées polaires de la spirale d'Archimède est de la forme $r = a\theta$ et il est bien connu que la tangente de l'angle fait par la tangente et le rayon polaire OM vaut :

$$\tan\widehat{OMP} = \frac{r(\theta)}{r'(\theta)} = \frac{a\theta}{a} = \theta$$

donc au point M, $\theta = \pi$ et :

$$\tan\widehat{OMP} = \pi = \frac{OP}{OM}.$$

Dès lors, $OM \times OP = \pi OM^2$ et on a bien construit un rectangle qui a pour aire l'aire du cercle de rayon OM.

Mais c'est dans le domaine des approximations, encadrant le cercle par des polygones convexes réguliers inscrits et ex-inscrits, que son apport sera sans doute le plus décisif (première et troisième propositions du traité sur la mesure du cercle), conduisant à l'encadrement :
$\frac{22}{7} < \pi < \frac{223}{71}$ soit $3.140845... < \pi < 3.142857....$

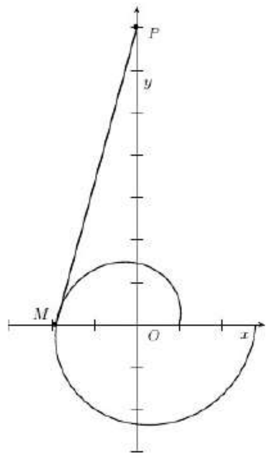


Figure 3 – La spirale d'Archimède

1.4 Une quadrature de la Renaissance : la quadrature d'Oronce Fine

La méthode d'encadrement d'Archimède sera améliorée, en nombre de polygones et en effort de calcul, durant le Moyen Âge et la Renaissance, avec parfois des variantes comme la quadrature de Francon de Liège (1050). François Viète en 1579 calcule π avec neuf décimales exactes (soulignons que Viète est un des premiers à penser que la quadrature géométrique exacte est impossible). Plus tard, le physicien Danois Snelle, par exemple, en 1621, calcule π par cette méthode avec 35 décimales.

Parallèlement à cette approche d'approximation, les tentatives de Quadrature géométriques continues, et sont réfutées l'une après l'autre. A titre d'exemple, nous allons détailler ici celle d'Oronce Fine.

Oronce Fine (qui signe de son nom latin : Orontii Finaei) a écrit un ouvrage dédié à la quadrature du cercle *Quadrature Circuli, tandem inuenta & clarrissimé demonstrata*, paru en 1544. Dans ce livre, il expose sa solution à ce problème classique, pour lequel Nicolas de Cues venait juste d'être réfuté et auquel Léonard de Vinci lui-même s'était essayé sans succès.

La critique de son travail se manifeste bientôt par la réfutation, en 1546, du mathématicien portugais Pedro Nunès, professeur à Coimbra, qui était son grand rival en cartographie. À sa suite, Jean Borrel, pourtant disciple de Fine le réfute également.

La quadrature d'Oronce est la suivante : considérons un cercle de centre E, dont AC et BD sont deux diamètres perpendiculaires. On joint par une ligne droite le point A à G, milieu de l'arc AD, on divise cette droite AG en moyenne et extrême raison (c'est-à-dire dans le rapport le nombre d'or, (de Lanascol, 1925)), GH étant le grand segment. Par H on mène à BD une parallèle HK qui rencontre AC en K : la longueur EK est le demi-côté du carré équivalent au cercle. Bien sûr, il s'agit encore d'une approximation. Comme on va le voir, cela donnerait comme valeur $\pi \simeq 3,15$ environ.

ORONTII
FINAEI DELPHINATIS,
REGII MATHEMATI-
CARVM LVTETIAE
PROFESSORIS,

Quadratura Circuli, tandem inuen-
ta & clarissime demonstrata.

De circuli mensura, & ratione circũferentiæ ad
diametrum, Demonstrationes duæ.

De multangularũ omniũ & regularũ figurarũ
descriptione, Liber hactenus desideratus.

De inuenienda longitudinis locorum differẽtia,
aliter quam per Lunares eclipses, etiam dato
quouis tempore, Liber admodum singularis.

Planisphærium geographicum, quo tum longi-
tudinis atq; latitudinis differẽtiæ, tum directæ
locorum deprehenduntur elongationes.

LVTETIAE PARISIORVM,

1544.

Cum priuilegio Regis.

Figure 4 – Page de garde de Quadratura Circuli de Fine

Détaillons les calculs de cette jolie quadrature **qui va relier π au nombre d'Or ϕ.** Les nombres complexes ont été découverts en Italie au XVIème siècle par des contemporains d'Oronce Fine, comme Tartaglia (1506-1557) et Bombelli. L'ingénieur mathématicien Albert Girard (1595-1632) les qualifie de *solutions impossibles*. Il n'est donc pas si anachronique de les utiliser dans notre calcul, bien que l'application des nombres complexes à la géométrie, par Argand et Gauss, ne date que du début du XIXème siècle. Reportons nous donc à la figure 5 ; notons ϕ le nombre d'or. Les affixes complexes des points A, G, et H sont reliées par la relation : $z_A - z_G = \phi(z_H - z_G)$, ou encore :

$$\phi z_H = z_A + z_G(\phi - 1). \quad (1)$$

Prenant un rayon $ED = 1$ et E comme origine, nous avons $z_A = i$ et $z_G = \frac{\sqrt{2}}{2}(1 + i)$. Nous tirons alors de la relation (1) l'ordonnée du point H, en prenant la partie imaginaire :

$$2y_H = 2\ EK = \frac{2 + \sqrt{2}(\phi - 1)}{\phi}.$$

Nous devons donc à Oronce Fine cette jolie approximation de $\sqrt{\pi}$ en fonction du nombre d'or (rappelons nous que $\phi^{-1} = \phi - 1$) :

$$\sqrt{\pi} \simeq 2\ EK = (\phi - 1)\Big(2 + \sqrt{2}(\phi - 1)\Big).$$

le calcul numérique ($\phi \simeq 1.618034$) nous donne $\pi \simeq 4\ EK^2 \simeq 3,1550623$.

1.5 Un coup d'arrêt

L'impossibilité de la quadrature du cercle par construction géométrique a été démontrée seulement par Ferdinand von Lindemann en 1882. Il faudrait pour cela que π soit un nombre

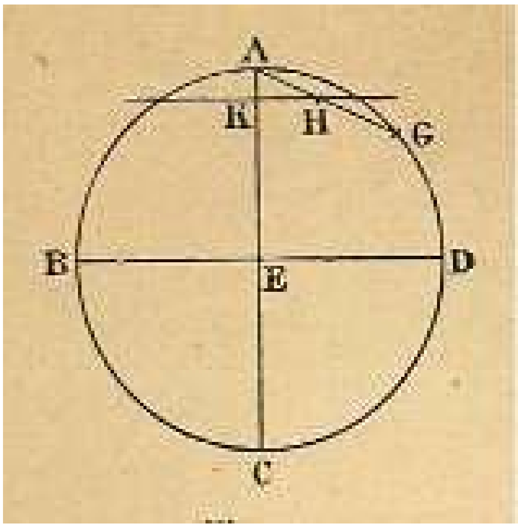


Figure 5 – La quadrature d'Oronce Fine

constructible, ce qui n'est pas le cas, car seuls les nombres algébriques le sont, à savoir les entiers, les rationnels et les racines d'équation algébriques à coefficients entiers. Les nombres qui ne répondent pas à ces conditions sont dits transcendants et π en fait partie. Depuis, les propositions d'article sur la quadrature du cercle sont le plus souvent le fait de mathématiciens autodidactes, et bien sûr rejetées dans la plupart des rédactions sans même être examinées en détails.

2 La quadrature du YIN YANG

2.1 La construction géométrique d'André Ferrandi

Dans le plan euclidien rapporté à un repère orthonormé Oxy, on considère (voir figure 6) un cercle (C) de centre O et de diamètre horizontal MN. On désigne par (Δ) la seconde bissectrice. Menons par le point N la parallèle à (Δ), notée (Δ').

Maintenant, construisons le cercle (ω) de centre N et de rayon MN. Il coupe la droite (Δ') en un point Q (voir figure 5). Nous traçons alors la demi droite OQ qui coupe le premier cercle (C) en un point K, puis par K nous menons la perpendiculaire à (Δ) qui coupe respectivement les axes Ox et Oy en des points A et B. Alors le rapport de l'aire du carré de côté AB sur celle du cercle (C) est voisin de 1.

Bien sûr, tout point K' de la droite (D) définit un cercle de centre O passant par K' et un carré de côté $A'B'$ dont le rapport des aires sera le même que précédemment. Dans ce qui suit, nous allons calculer une valeur exacte du rapport des aires de deux manières différentes.

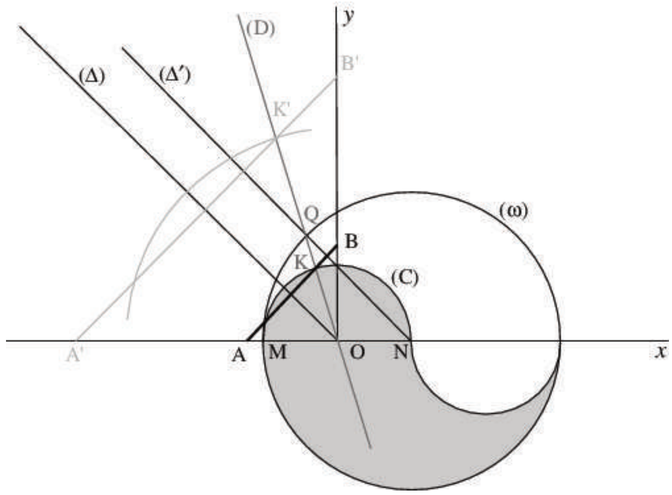


Figure 6 – La quadrature du Yin-Yang

3 Une méthode analytique

Sans perte de généralité, nous pouvons supposer que le cercle (C) est de rayon $R = 1$ (cela revient à un changement d'unité). Comme la bissectrice (Δ) a pour équation $y = -x$ et que le repère est orthogonal, l'équation de la perpendiculaire à (Δ) passant par le point K est de la forme $y = x + c$. La valeur de c suffit à déterminer l'aire du carré de côté AB. En effet A et B ont pour coordonnées respectives $(-c, 0)$ et $(0, c)$ ce qui fait que la longueur du segment AB est $\sqrt{2c^2}$ et l'aire $\mathcal{A}_2$ d'un carré de côté AB sera donc égale à : $\mathcal{A}_2 = 2c^2$.

Nous allons essayer de déterminer la valeur du paramètre c. Comme (Δ') a pour équation $y = 1 - x$, les coordonnées du point Q sont déterminées (sachant $y_Q > 0$) par le système d'équations :

$$\begin{aligned}(x_Q - 1)^2 + y_Q^2 &= 4\\ x_Q + y_Q &= 1\end{aligned}$$

On en tire facilement $y_Q = \sqrt{2},\ x_Q = 1 - \sqrt{2}$.

Remarquons maintenant que la droite (D) a une équation de la forme $y = tx$ et en écrivant qu'elle passe par Q, on calcule la valeur de t :

$$t = \frac{\sqrt{2}}{1 - \sqrt{2}} = -2 - \sqrt{2}. \tag{1}$$

Le point K étant l'intersection de (D) et de la perpendiculaire à (Δ), ses coordonnées vérifient

$$\begin{aligned}y_K &= t x_K\\ y_K &= x_K + c\end{aligned}$$

ce qui nous permet de calculer le paramètre c :

$$c = (t-1)x_K. \tag{2}$$

il reste à écrire que K est sur le cercle (C), ce qui nous donne :

$$x_k^2 + y_K^2 = (1+t^2)x_K^2 = 1.$$

On en tire successivement :

$$x_K^2 = \frac{1}{1+t^2},$$

puis grâce à (2) :

$$c^2 = \frac{(t-1)^2}{1+t^2}. \tag{3}$$

Nous sommes maintenant en mesure de calculer le rapport des aires :

$$\begin{aligned}\frac{\mathcal{A}_2}{\mathcal{A}_1} = \frac{2c^2}{\pi} &= \tfrac{2(t-1)^2}{\pi(1+t^2)} \\ &= \tfrac{2(3+\sqrt{2})^2}{\pi(7+4\sqrt{2})} \\ &= \tfrac{2(11+6\sqrt{2})^2}{\pi(7+4\sqrt{2})} \\ &= \tfrac{2}{17\pi}(29-2\sqrt{2}).\end{aligned}$$

La valeur numérique est d'environ 0.98008. Cette quadrature est moins précise que celle de Fine par exemple, puisqu'elle donne $\pi \simeq 2c^2 \simeq 3,08$, avec une erreur relative de 2% environ.

4 Une approche géométrique

Introduisons le point H, intersection de la bissectrice (Δ) avec AB. Revenons au cas général et notons r le rayon du petit cercle. Le grand cercle a ainsi pour rayon $2r$. Désignons par α une mesure de l'angle $\widehat{KOH}$. Dans le triangle rectangle OHK, nous avons :

$$OH = OK\ \cos\alpha = r\cos\alpha,$$

mais le triangle AHO étant isocèle (par construction), $AH = HO$. Par ailleurs, (Δ) est la médiatrice de AB donc $AB = 2AH$, d'où : $AB = 2r\cos\ \alpha$. Nous pouvons donc maintenant exprimer le rapport des aires du carré de côté AB et du disque de rayon r comme :

$$\frac{\mathcal{A}_2}{\mathcal{A}_1} = \frac{4\cos^2\alpha}{\pi}. \tag{4}$$

Ainsi, si on avait $\cos\alpha = \dfrac{\sqrt{\pi}}{2}$, la quadrature serait parfaite, comme l'indique d'ailleurs André Ferrandi dans son document original.

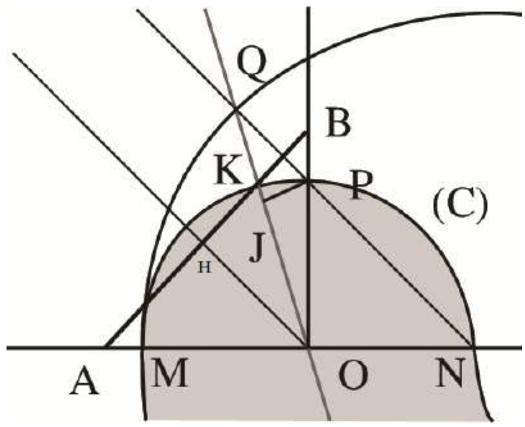


Figure 7 – La quadrature du Yin Yang (détail)

Calcul de $cos\alpha$

Soit P le point du petit cercle (ω) à la verticale de O, comme ndiqué sur la figure 7. Le triangle ONP est isocèle rectangle en O et $OP = ON = r$. Par le théorème de Pythagore $NP = r\sqrt{2}$. D'autre part $QP = QN - NP$ et $QN = 2r$, d'où finalement :

$$QP = r\sqrt{2}(\sqrt{2} - 1).$$

Abaissons la hauteur PJ issue de P dans le triangle OQP, comme indiqué sur la figure 7. Comme l'angle $\widehat{OQP}$ a aussi pour mesure α (angles alternes internes), cette hauteur a pour longueur $PJ = QP \sin \alpha$. Mais on peut aussi calculer PJ dans le triangle rectangle OJP ce qui donne :

$$PJ = OP \sin\left(\frac{\pi}{4} - \alpha\right).$$

Nous obtenons donc l'équation trigonométrique :

$$QP \sin \alpha = OP \sin\left(\frac{\pi}{4} - \alpha\right).$$

Remplaçant QP et OP par leurs expressions en fonction de r conduit à :

$$r\sqrt{2}\left(\sqrt{2} - 1\right) \sin \alpha = r\left(\frac{\cos \alpha - \sin \alpha}{\sqrt{2}}\right),$$

ou encore à :

$$\left(2(\sqrt{2} - 1) + 1\right) \sin \alpha = \cos \alpha,$$

d'où on tire :

$$\tan \alpha = \frac{1}{2\sqrt{2} - 1}. \tag{5}$$

Or $\cos^2 \alpha = \frac{1}{1+\tan^2 \alpha}$ donc, tous calculs faits :

$$\cos^2 \alpha = \frac{29 - 2\sqrt{2}}{34}. \tag{6}$$

Expression donnant le rapport des aires

Finalement, en reportant (6) dans (4), on retrouve le rapport des aires calculé précédemment :

$$\frac{\mathcal{A}_2}{\mathcal{A}_1} = \frac{2}{17\pi}(29 - 2\sqrt{2}) = C_{YY},$$

où on a introduit la notation C_{YY}, pour désigner cette constante (Ferrandi, 2015).

5 Une expérimentation au Lycée

Nous avons proposé ce sujet de réflexion à une classe de terminale S (niveau équivalent à : GB year 13, US 12th grade, Canada et Belgique 6ème sec.) en l'occurence la classe de Hubert Proal, professeur agrégé de mathématiques au lycée d'Altitude de Briançon, patrie d'Oronce Fine.

Le sujet a été abordé par Hubert Proal lors d'une séance de révisions sur les nombres complexes. La méthode relève donc d'un choix du professeur. Le sujet n'a pas été expliqué d'un point de vue historique dans un premier temps mais simplement présenté comme un exercice sortant des sentiers battus, à l'encontre des traditionnels exercices de révision pour le bac.

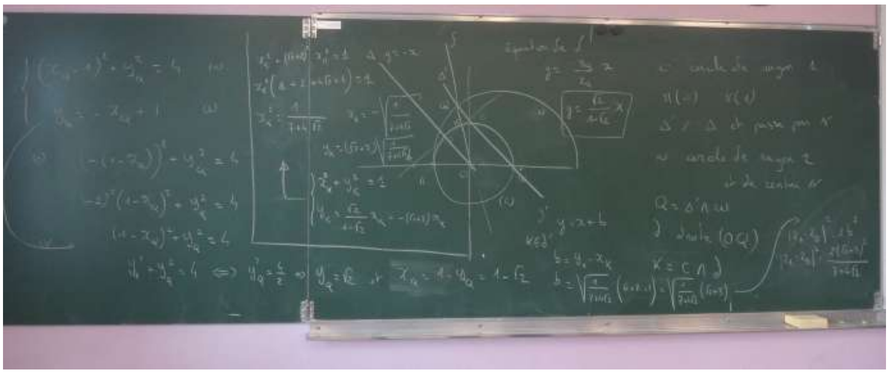


Figure 8 – Le problème au tableau dans la classe (Photo H. Proal)

La séance a eu lieu en classe entière (28 élèves). Les dessins ont été fait à la main au tableau, puis dans un deuxième temps le logiciel Geogebra, projeté sur la tableau blanc, a été utilisé pour fournir une base solide au raisonnement, tout en permettant d'annoter le dessin.

Dans cette classe, il y avait de bons élèves qui ont suivi et participé aux calculs, effectués à la main. L'exercice est abordé sous forme de discussion ; le professeur écoute les idées des élèves et regarde avec eux si cela peut marcher ou pas.

Nous détaillons ci-dessous les calculs effectués lors de cette séance :

Le problème est de calculer le rapport de l'aire du carré de côté AB sur celle du cercle (C), soit encore $\dfrac{|z_B - z_A|^2}{\pi}$ (voir figure 10). On commence par calculer l'affixe du point Q : $z_Q =$

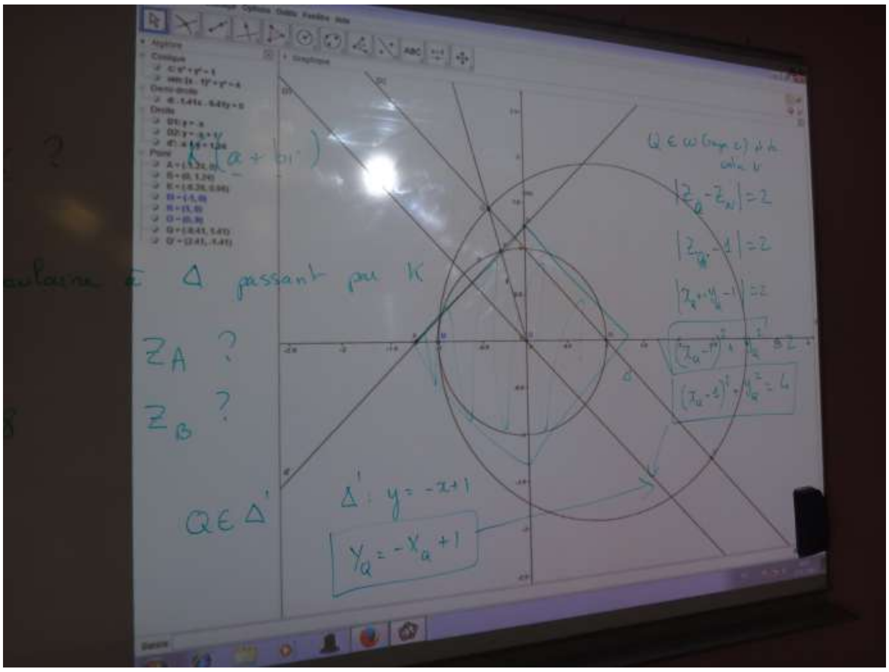


Figure 9 – Travail sur écran avec Geogebra (Photo H. Proal)

$1 + 2e^{\frac{3i\pi}{4}} = 1 - \sqrt{2} + i\sqrt{2}$. Puis, on calcule la distance OQ : $|z_Q| = \sqrt{5 - 2\sqrt{2}}$, ainsi que son inverse $\dfrac{1}{|z_Q|} = \dfrac{\sqrt{17}\sqrt{5 + 2\sqrt{2}}}{17}$. On en déduit l'affixe du point K du cercle trigonométrique :

$$z_K = \frac{z_Q}{|z_Q|} = \frac{-\sqrt{17(7 - 4\sqrt{2})} + i\sqrt{17(10 + 4\sqrt{2})}}{17}.$$

Il reste à calculer les affixes des points A et B. La figure 11 explique la façon dont procède le calcul. Remarquer que $z_A \in \mathbb{R}$ et $z_B \in i\mathbb{R}$.

6 En conclusion

Cette quadrature amusante peut-être l'occasion de faire un peu de géométrie élémentaire dans une classe de lycée. Elle n'est certes pas aussi précise que celle d'Hippias ou que celle d'Oronce Fine par exemple (Fréchet, 2013), mais elle a le mérite de fournir un petit problème de géométrie accessible au plus grand nombre, et de faire appel à des outils basiques, comme les équations de droite où la géométrie du triangle, tout en montrant les relations entre ces deux aspects du problème.

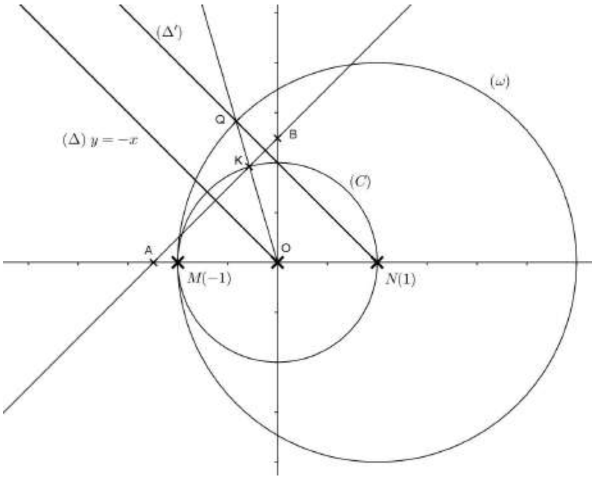


Figure 10 – La Quadrature par les complexes (Image H. Proal)

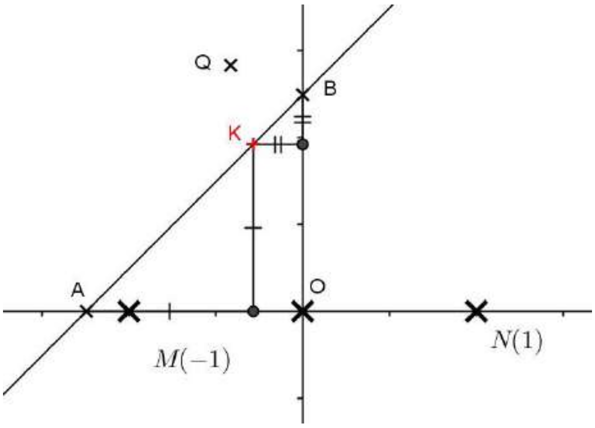


Figure 11 – La Quadrature par les complexes - détail (Image H. Proal)

RÉFÉRENCES

de Lanascol, Q. (1925). *La géométrie du compas*. Paris : Librairie scientifique A. Blanchard.

Ferrandi, A. (2015). La quadrature du Yin et du Yang. *Quadrature*, *97*, 12-14.

Fréchet, G. (2013). Oronce Fine, mathématicien et cartographe. *Quadrature*, *90*, 36–47.

Heath, T. (2013). *A history of Greek Mathematics*. Cambridge : Cambridge University Press.

Peyrard, F. (1804). *Les éléments de géométrie d'Euclide, traduits littéralement et suivis d'un Traité du cercle, du cylindre, du cône et de la sphère, de la mesure des surfaces et des solides, avec des notes*. Paris : F. Louis. Consulté sur `http://gallica.bnf.fr/ark:/12148/bpt6k110982q.r=.langFR`

Wilhelm Fiedler (1832 - 1912): the synthesis of descriptive and projective geometry

Klaus VOLKERT
Universität Wuppertal, Germany
klaus.volkert@math.uni-wuppertal.de

Wilhelm Fiedler made his career outside the universities – he never attended mathematical courses there. After a certain time as a teacher at technical schools in the eastern part of Germany he was called to the Polytechnical school at Prague (1864) as a professor for descriptive geometry and soon after to the ETH at Zürich (1867), where he spent the rest of his live. There Fiedler was responsible for the training of future teachers and of future engineers. In close contact to the famous engineer K. Culmann he developed a highly original synthesis of descriptive and projective geometry in order to provide a scientific base for the latter and an intuitive approach to the first. Fiedler also payed a lot of attention to the "Verwandtschaften" (relationships) following the adviser of his thesis, A. F. Möbius. During his lifetime this was a rather modern idea which could be used to structure geometry and its teaching.

In my talk I will present some of Fiedler's ideas and discuss their influence in the German speaking countries. In addition I will provide some information on his huge correspondence with mathematicians of his time. The publication of the letters exchanged by Fiedler and Cremona is announced by M. Menghini and others.

INTERROGATING HISTORICAL DEBATES OVER STANDARDS IN PEDAGOGICAL PRACTICE

Margaret WALSHAW
Massey University, Palmerston North, New Zealand
m.a.walshaw@massey.ac.nz

ABSTRACT

This paper seeks to make a contribution to a topic in the history of mathematics education, namely standards in pedagogical practice in schools. National standards in pedagogical practice have featured prominently in the news in many countries over a long period of time. Traditionally, they have been key rallying points not only for politicians, but also for administrators, parents, and educational researchers.

In this paper I provide an account of the powerful force of crisis voiced in relation to pedagogical standards. Utilising the historical period from post -Second World War to present time, in the context of the New Zealand public education system, the paper examines how debates are characterized by sharp divisions between professional and public opinion. It explores the competing conceptions of mathematics teaching and learning underpinning these divisions, illustrating the ways in which these conceptions combine to create a conviction that public education is in crisis, thus necessitating a decisive response from officialdom. The paper also examines the ways in which the debate obscures the fact that underpinning rival claims are radically different, possibly irresolvable conceptions of the purpose of education.

The historical data are drawn from a range of sources that informed a larger study within which this project is nested (see Openshaw & Walshaw, 2010). The dataset included archival accessions, publications and policy documents and was complemented by material from contemporary professional literature, and media sources. When this dataset was critically interrogated and triangulated, what was revealed were the ways in which notions of good mathematics pedagogy and sound mathematical benchmarks are historically constructed, and why they are so vigorously contested.

The paper contributes to informed historical debate over standards in mathematics pedagogy in two distinct ways. First, it highlights the reality of competing truths. Foucault (1980) has long argued for the social construction of knowledge, where truth itself is an historical production. Second, the paper aims to highlight the ways in which various interest groups utilize particular versions of "the truth" to shape the debate and to shift the parameters around wider processes relating to mathematics education. Thus, an important contention put forward is that mathematics curriculum, teaching and student achievement all need to be viewed within the context of alliances between key groupings of people.

REFERENCES

Foucault, M. (1980). *Power/knowledge: Selected interviews and other writings 1972-1977*. (Trans: Gordon, C. *et al*.). New York: Pantheon.

Openshaw, R., & Walshaw, M. (2010). *Are our standards slipping? Debates over literacy and numeracy standards in New Zealand since 1945.* Wellington: NZCER.

THE INFLUENCE OF MATHEMATICAL CLASSICS READING ON UNIVERSITY STUDENTS' MATHEMATICS BELIEFS

Jia-Ming YING, Jyun-Wei HUANG
Taipei Medical University, 250 Wuxing Street, Xinyi District, Taipei City 110, Taiwan
commandershran@gmail.com
Taipei Municipal Heping High School, No.100, Wolong Street, Da-an District, Taipei City 106, Taiwan.
austin1119@gmail.com

ABSTRACT

We report the influences of an undergraduate-level liberal-arts course, in which students in a medical university were guided to read through important parts of Euclid's *Elements* and taught about related cultural issues including logic and some Greek philosophy. Class questionnaires and reflexions showed that parts of students' beliefs changed after the course. Concerning the values of mathematics, students were more prone to believe that mathematics had real-world applications and improved one's sensibilities to beauty; for the nature of mathematics, they were more prone to believe proof was the only way to justify mathematical knowledge. The last change was expected but also interesting, since in another course in the same university about mathematics culture, students were confused about the context of discovery and that of justification. The course reported here was more helpful since it also had cultural connections but did not similarly confuse students, which is meaningful in a society where problem-solving, not logic, is considered the most useful aspect of mathematics.

1 Introduction

History is useful in mathematics education, as demonstrated in many scholarly works (e.g., Katz & Tzanakis, 2011), and it, too, could directly or indirectly shape how modern people view different academic disciplines, such as medicine or mathematics. The case of Taiwan is an interesting example. Medical practitioners – doctors, dentists, pharmacists …etc. – usually enjoy high social statuses Fernando Pessoa in modern Taiwanese society. Besides the facts that they have relatively high income, and that they help save human lives, there are also reasons rooted in Taiwan's history. Taiwan was colonised by Japan for half a century, since 1895 to 1945, during which period different laws were enforced in Taiwan from the 'Home Islands'. Post-secondary education, for instance, was highly restricted. For many reasons there were much less Taiwanese studying humanities and social sciences than in other disciplines in Taipei Imperial University, and most of the Taiwanese elites sent their sons and daughters to study medicine.[1] As a result, many of the leaders of social movements during the colonial period, and in later times before the democratisation of Taiwan, were originally medical doctors. Even today, a significant

[1] For discussions about higher education systems and practices in Taiwan's Japanese colonial period, refer to, for instance, (Zheng, 2002).

portion of Taiwanese politicians and social leaders have medical backgrounds, and this is also why medical practitioners generally receive relative high respects, and their opinion towards different aspects of social life, including education, are taken seriously by Taiwanese society. What does this have to do with mathematics education?

Since the admission to medical schools is highly competitive, medical students in Taiwan must have extremely high scores in examinations of basically every subject, including mathematics. However, after they have entered medical schools from secondary schools (medical education in Taiwan is in the undergraduate level), they only have to study very little mathematics, consisting of basic calculus and statistics. So what message does this practice send to the medical students? It seems that they would see mathematics as one of the tools to enter university, and that they *do not need* mathematics in their professional training and practices. Indeed, medical practitioners do not apparently use very much mathematics in their clinical practices as engineers, natural scientists or economists in their professions, but the logical, critical and reflexive thinking that could be improved in mathematics learning are also important for any profession. Since the beliefs of medical practitioners could influence public opinions, and some medical doctors may in fact become politicians and even educational policy makers in Taiwan, it is meaningful to do research in the country about mathematics beliefs held by medical students and try to bring them more diversified views on the values and nature of mathematics.

One of the authors of this paper has been doing research and teaching in a medical university in Taiwan, so one natural question following the previous paragraphs is: What kind of mathematics courses can influence medical university students' mathematics beliefs, and what changes can be observed? One possible kind of courses are selective liberal-arts courses about mathematics, since the lecturer has more space to talk about the aspects of mathematics other than problem-solving, such as its cultural and social connections. In fact, the authors had previously done an exploratory study about the influences of liberal-arts mathematics course with an emphasis on culture and history (see Literature review). Another possible kind of courses that may bring students deeper into the cultural roots of mathematics is classics reading, since in each civilisation there are always classics that carry its philosophical reflexions about the universe and human creations such as mathematics, and those reflexions have stood the test of time. Under the rationale, this study examined how a course about mathematical classics reading changed mathematics beliefs held by medical university students in Taiwan.

2 Aim of study

The main aim of this study was to explore how medical university students' mathematics beliefs changed after taking a liberal-arts course about mathematical classics reading. The authors wanted to explore how this course changed mathematics beliefs held by medical university students in Taiwan, especially about their beliefs toward the nature and values of mathematics. In what follows, relevant literature is reviewed, and a study about this topic and its results shall be reported.

3 Literature review

3.1 Mathematics beliefs

Many studies have shown that students' understanding and beliefs about the nature of mathematics do influence their mathematics learning. Over the course the mathematics education research late in the last century, belief had been considered a hidden variable in mathematics education (Leder, Pehkonen, & Törner, 2002). Alan Schoenfeld's research tells us related facts. For instance, after observing the problem-solving behaviours of high school and university students, he concludes in his article that a person's mathematics beliefs shape his or her ways to do mathematics (Schoenfeld, 1985). Also, Schoenfeld (1992; 1994) finds that inappropriate interpretations towards the nature of mathematics not only cause students' conflicting epistemological beliefs about mathematics, but also influence negatively on their mathematics learning effects. Besides, Cifarelli and Goodson-Espy (2001) find that university students' epistemological beliefs about mathematics do affect their mathematical learning.

Op 't Eynde, De Corte, & Verschaffel (2002) points out that mathematics-related beliefs, or students' subjective conceptions about mathematics that are implicitly or explicitly held true, have significant influences on their mathematical behaviours (Op 't Eynde, De Corte, & Verschaffel, 2002). In particular, Hong (2009) tells an interesting phenomenon about gender issues among students in universities of technology, that mathematics teaching environments unsuitable for female students would lower their willingness to pursue mathematics-related professions. For students' mathematics-related beliefs, Op 't Eynde, De Corte, & Verschaffel (2002) propose a framework, in which mathematics-related beliefs are described in students' class context, and are divided into three categories: beliefs about mathematics education (mathematics a subject, learning and problem-solving, and teaching in general), beliefs about self (self-efficacy, control, task value, and goal-orientation), and beliefs about the social context (social norms in the mathematics class). The questionnaire used in this study was developed based on this framework (see below).

The studies mentioned above tell us that the role mathematics beliefs play receive more and more attentions in recent years. This study also hopes to enrich research in this area, through exploring the effects of a course about mathematical classics reading.

3.2 History of mathematics, mathematics learning and mathematics beliefs

It is generally accepted that history helps mathematics teaching and learning. For instance, Liu (2003) proposes five reasons to use history of mathematics in school curricula: (1) history helps increase motivation and develop a positive attitude towards learning; (2) past obstacles in the development of mathematics helps explain what today's students find difficult; (3) historical problems helps develop students' mathematical thinking; (4) history reveals the humanistic facets of mathematics; (5) history gives teachers a guide for teaching. Also, history stimulates students' interests in mathematics (e.g., Furinghetti & Paola, 2003). In particular, historians of mathematics have been trying to use history to help students reflect upon the nature of mathematics (e.g. Jankvist, 2011). Besides, teachers may learn to appreciate mathematics as a cultural phenomenon through history (Tzanakis and Arcavi, 2000). Moreover, Horng (2000) finds that students may learn to appreciate the evolution mathematical knowledge through different solutions to the same problem in different times and locations. In the university level, Kjeldsen (2011) shows that using history as a means to teach tertiary-level mathematics helps students understand the special nature of mathematical thinking.

Following the aforementioned studies, the authors of this paper also tried to explore how history influences students' mathematics learning, especially in the tertiary level, because not few studies had been focused on this topic. In (Ying, Huang & Su, 2015), the authors of this paper and a colleague of theirs describe a study about the influences of a liberal-arts mathematics course, with an emphasis on culture and history, on the mathematics beliefs of medical university students. The results show that after studying through the course, the students were more prone to believe that, among other items, 'sensibility to beauty' and 'creativity' were both important values of mathematics. However, the results also revealed that the course did not clarify the difference between the 'context of justification' and the 'context of discovery' for students. A possible cause for this is that the contents of that teaching experiment were focused on the discovery of mathematical methods, but it did not talk very much about philosophy and logic. The authors then tried to find other ways to teach liberal-arts mathematics courses, and the teaching experiment described here is the outcome.

4 Research Setting

The study had a single-group pretest-posttest design. Research tools for this study included a liberal-arts course about mathematical classics reading, a questionnaire administered in the pre-test and the post-test, and students' reflexions. The same group of students took the pre-test, participated through the teaching experiment, wrote reflexions about class discussions and their reading, and finally took the post-test. A total of 40 students took the course, of which one of the authors was the lecturer. In what follows we shall describe the research setting in detail.

4.1 Subjects

The subjects in this study were a group of medical university students in Taiwan. They took an elective liberal-arts course about mathematics in the first semester of Taiwan's 2014-15 school year (September 2014 to January 2015) that was designed to be the teaching experiment in this study. Students' majors included medicine, dentistry, pharmacy, nursing, health care administration, medical technology, respiratory therapy, dental technology, and gerontology. Most of the students in the course were in their first or second year in university, and only one was in her third year.

4.2 Research tools

The research tools of this study included a teaching experiment, students' reflexions, and a 20-question Likert-scale questionnaire. We shall first explain how the teaching experiment went, including students' free reflexions, and then describe the questionnaire.

The course used as the teaching experiment was taught in a sixteen-week semester, with two hours of class time each week. In this course of mathematical classics reading, the main text used was Euclid's *Elements* (its modern English and Chinese translations). During the course, students were guided to read through various parts of the *Elements*, including the entirety of Book I, the so-called "geometrical algebra" in Book II, important propositions about the circle and regular polygons in Book III, theory of similarity in Book VI, essential propositions of number theory in Book VII to IX, basics of incommensurability in Book X, and finally, propositions about solid geometry leading to the five regular polyhedra in Book XI to XIII. During teaching, the lecturer might also talk about relevant Greek philosophy and history if necessary, including the philosophy of the Pythagorean School, Aristotelian logic, Plato's dialogue *Meno*, and ideas about the five classical elements. Students were required to write reflexions about their reading and about contents taught in class. The lecturer also discussed in class with students about their reflexions, so they might go deeper into the questions they were pondering upon. The reflexions were later used to confirm their belief changes seen from the questionnaire analysis. Since the flow of the course depended as much on the syllabus as on students' reflexions and discussions, the lecturer had a flexible way of teaching of the course, letting multiple topics be discussed in the same week if necessary. Table 1 shows the topics discussed in each week of the teaching experiment.

Table 1. Topics in each week of the teaching experiment

Week	Topics
1	Introduction to the course and Greek mathematics.
2	The *Elements* Book I; Aristotelian logic.
3	The *Elements* Book I; neutral geometry.
4	The *Elements* Book I; compass-and-straightedge construction; Athenian democracy.
5	The Pythagorean School and the Pythagorean Theorem; structure of Book I.
6	Brief history of Euclidean and non-Euclidean geometry.
7	The *Elements* Book II to III; Plato's dialogue *Meno*.
8	The *Elements* Book II to IV; geometrical algebra; regular polygons.
9	Midterm break; self-reading and reflexion.
10	The *Elements* Book II to VI; the laws and sine and cosine; similarity.
11	The *Elements* Book VII to IX; number theory.
12	The *Elements* Book X; incommensurability; Pythagorean mysticism.
13	The *Elements* Book XI and XII; solid geometry.
14	The *Elements* Book XIII; regular polyhedra; classical elements.
15	Conic sections; three famous classical problems of construction of antiquity.
16	The influence of Euclid's *Elements* in the Western civilisation.

As for the questionnaire administered in the pre-test and post-test, the authors used the same one developed in their previous research (Ying, Huang & Su, 2015). It was first modified from the framework provided in (Op 't Eynde, De Corte, & Verschaffel, 2002), and was designed to explore the changes in students' beliefs towards the (epistemological) *nature* of mathematics and the (social and personal) *values* of mathematics, which can be considered as two independent dimensions of mathematics beliefs (e.g., Goldin, 2002). The questionnaire consists of 20 Likert-scale questions, each of which is a declarative sentence about mathematics. The responses from which students could choose, for each statement, are typical seven-level Likert-items: 'strongly agree,' 'agree,' 'slightly agree,' 'neither agree nor disagree,' 'slightly disagree,' 'disagree,' and 'strongly disagree.' After the administrations of the questionnaire, their responses were transformed into scores for analysis, with the highest score 7 for 'strongly agree' and lowest score 1 for 'strongly disagree.' The questions in the pre-test and the post-test were identical, so the changes in students' beliefs could be revealed. Table 2 shows the structure of the questionnaire.

Table 2. Structure of the questionnaire.

Dimensions	Questions
Nature of mathematics	1. Mathematics is a discipline described with symbols. 3. Mathematics is a discipline composed of procedures and formulae. 5. Mathematics is a discipline that finds general principles from individual facts. 7. The general principles found in mathematical research can be applied to all kinds of situations that fit the conditions of the principles. 9. Mathematics is a discipline that constructs common models from complex phenomena in reality. 11. Results calculated with mathematics can accurately explain many phenomena. 13. Mathematics is a rigorous and logical discipline. 15. Proof is the only method that justifies mathematical knowledge. 17. If an assumption in mathematics fits most situations, then it is justified. 19. Mathematics is objective truth.
Values of mathematics	2. The research results of mathematics can help human beings describe phenomena of the real world. 4. The research results of mathematics can be used as tools to solve problems in other fields of study. 6. Learning mathematics can help us solve problems encountered in daily lives. 8. Learning mathematics can cultivate our logic and reasoning. 10. Learning mathematics can cultivate our creativity. 12. Learning mathematics can cultivate our observing ability. 14. Learning mathematics can increase our knowledge. 16. Learning mathematics can elevate our ability to judge things around us. 18. Learning mathematics can improve our sensibility to beauty. 20. Learning mathematics is helpful to our career development.

The questions were given an iterated order according to their dimensions (odd-numbered questions for the nature of mathematics and even-numbered ones for the values of mathematics) to reduce the disturbance that might be caused by similar wording of the statements in the same dimension (Ying, Huang & Su, 2015, pp.14-16). In the next section, we shall discuss our findings seen from the questionnaire and students' reflexions.

5 Results and discussions

A total of 35 students went through the whole teaching experiment, that is, they took the pre-test, stayed in the course through the whole semester, wrote several reflexions, and finally took the post-test. Paired-Samples *T* Test ($\alpha = 0.05$) were used to compare the averages of two paired samples which contained these 35 students. Several interesting results were found and are discussed below. Analysis showed that students have significant changes in the scores of the following six questions: question 2 (from 5.00 to 5.89, $p = 0.002$), question 6 (from 5.24 to 5.92, $p = 0.000$), question 11 (from 5.13 to 5.70, $p = 0.045$), question 15 (from 3.74 to 4.73, $p = 0.011$), question 18 (from 4.11 to 5.32, $p = 0.000$), and question 20 (from 4.84 to 5.51).

For the values of mathematics, statistics show that answers for four of the questions have significant changes. After taking the course, students were more prone to believe that mathematics could help human beings describe phenomena of the real world (q.2), that learning mathematics could help us solve problems encountered in daily lives (q.6), that learning mathematics could improve our sensibility to beauty (q.18), and that learning mathematics could help their career development (q.20).

The changes of beliefs about the connections between mathematics and real world, and between mathematics and beauty can be confirmed from their reflexions. For instance, a Ms H (first-year medicine major) was surprised when she learned that the form of the *United States Declaration of Independence*, a real-world political document of great importance, is related to that of the *Elements*. Students did pay much attention to the utility of mathematics, which can be seen from the high scores of q.2 and q.6 in the pre-test, and from the changes of the two questions in the post-test.[2] Moreover, several students, including a Ms X (second-year dentistry major), used mathematical concepts such as parallel lines or circles as metaphors to write proses in their reflexions. This could very well be the reason that they saw the connections between mathematics and beauty.

Results for the dimension of the values of mathematics are similar to those reported in the authors' previous study, showing that liberal-arts mathematics courses with contents in the history and culture of mathematics can usually strengthen students' beliefs that mathematics is useful in the real world, but it also lets them be more prone to believe that mathematics is linked to creativity and sensibility to beauty (Ying, Huang & Su, 2015, pp.19-20).

For the nature of mathematics, statistics show that answers of only two of the questions have significant changes. After taking the course, students were more prone to believe that

---

[2] It seems that the students believed mathematics could solve real-world problems, although they did not have so much related experience in their professional training. Beside the examples they saw in class that could slightly influence their beliefs, our guess for the explanation to this phenomenon is that it could be due to certain 'indoctrinations' they received from their primary and secondary education, that mathematics is somehow useful, even though they did not have so much related experience. This could be the topic of future studies.

results calculated with mathematics could accurately explain many phenomena (q.11), and that proof was the only method that justifies mathematical knowledge (q.15).

The result about question 15 was naturally expected, since students had read a classic that was organised according to a strict logical sequence. That result is also very interesting compared to the authors' previous study. We found that although a liberal-arts course in mathematics with an emphasis on history and culture had many advantages, it might also confuse students about the context of discovery and that of justification, because they were also more inclined to believe that both proof and exemplification could justify mathematical knowledge stated in questions 15 and 17 (Ying, Huang & Su, 2015, pp.18-19). In the present case, however, students' beliefs had significant changes for question 15 but not for question 17, showing that they were not more inclined to believe that exemplification could justify mathematical knowledge. In other words, they were not as confused as those in the previous study.

This kind of beliefs about logic and justification can also be confirmed from students' reflexions in the present study. For instance, a Mr S (second-year pharmacy major) wrote:[3]

> I felt great when the professor used some kind of 'map' to show the context of Book I and the relations among the theorems. […] The *Elements* is a book linked by logic and has its own thinking contexts. A small change can affect everything.

Another student, a Mr T (first-year medical technology major) also wrote:

> I have gradually accepted the rigorous, un-intuitional mathematical method of the *Elements*. I was originally reluctant to try to prove propositions that could be 'taken for granted.' I felt that was only constantly circling some known facts; there would not be any new discovery so mathematics would stop improving. Later I have gradually felt that those things about foundations have their necessities to exist […].

Therefore, it can be seen that reading the *Elements* did help students see the necessity and advantages of logic.

If we try to consider Taiwanese students' typical mathematics-learning experiences, we might also understand why reading *Elements* can bring changes in their beliefs. In the culture of problem-solving and examinations, students rarely have the opportunities to appreciate the logical structure of certain mathematical theories. Learning to use problem-solving techniques and earing high scores in tests are the most important, if not the only, goals for mathematics learning. Besides, geometrical topics in Taiwanese high school textbooks are mainly treated with analytical approaches, so students are still using algebra to solve problems. Reading Euclid's *Elements* gives students a chance to slowly understand and appreciate the structure and beauty of an axiomatic-deductive system, and thus could bring the changes reported in this article.

---

[3] The students in this course usually wrote in Chinese, and the English translations of their words were done by the authors.

To conclude, we believe that in a liberal-arts course about mathematical classics reading, with careful choices of reading material and related cultural backgrounds for teaching, and providing students with opportunities for discussions and reflexions, it may not only be easier for university students to see the connections between mathematics and the real world, and between mathematics and aesthetics, it may also help students see the differences between the context of discovery and that of justification in mathematics, which is especially meaningful in a society where problem-solving, not logic, is believed by many to be the most useful aspect of mathematics. We are hoping that this kind of courses can bring medical university students in Taiwan more diversified views about mathematics, and then they may, in turn, influence the general public in Taiwan.

REFERENCES

Cifarelli, V., & Goodson-Espy, T. (2001). The role of mathematical beliefs in the problem solving actions of college algebra students. In *Proceedings of the 25th Conference on The International Group for The Psychology of Mathematics Education*, *2*, 265.

Furinghetti, F., & Paola, D. (2003). History as a crossroads of mathematical culture and educational needs in the classroom. *Mathematics in School, 32*(1), 37-41.

Goldin, G. A. (2002). Affect, meta-affect, and mathematical beliefs structures. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* (pp. 59–72). Dordrecht, the Netherlands: Kluwer.

Hong, H. C. (2009). *A study about gender-stereotypes threat and mathematic-related professional development for female students major in engineer at universities of Science and Technology in Greater Kaohsiung Pintung Area*. Unpublished master's thesis. Kaohsiung: National Kaohsiung Normal University. (In Chinese).

Horng, W.-S. (2000). Euclid versus Liu Hui: A pedagogical reflection. In V. J. Katz (Ed.), *Using history of mathematics in teaching mathematics* (pp. 37-48). Washington, D.C.: MAA.

Jankvist, U. T. (2011). Students' beliefs about the evolution and development of mathematics. In V. Katz, & C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education* (pp. 123-131). Washington, DC: MAA.

Katz, V., & Tzanakis, C. (Eds.) (2011). *Recent developments on introducing a historical dimension in mathematics education*. Washington, DC: MAA.

Kjeldsen, T. H. (2011). History in a competence based mathematics education: A means for the learning of differential equations. In V. Katz, & C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education* (pp. 165-173). Washington, DC: MAA.

Kloosterman, P., & Stage, F. K. (1991). Relationships between ability, belief and achievement in remedial college mathematics classrooms. *Research and Teaching in Developmental Education, 8*(1), 27-36.

Liu, P. H. (2003). Do teachers need to incorporate the history of mathematics in their teaching? *Mathematics Teacher*, *96*(6), 416-421.

Op 't Eynde, P., De Corte, E., & Verschaffel, L. (2002). Framing students' mathematics related beliefs: A quest for conceptual clarity and a comprehensive categorization. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 13-37). Dordrecht, The Netherlands: Kluwer.

Schoenfeld, A. H. (1985). A framework for the analysis of mathematical behavior. In A. H. Schoenfeld (Ed.), *Mathematical problem solving* (pp. 11-45). New York: Academy Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Schoenfeld, A. H. (1994). Reflections on doing and teaching mathematics. In A. H. Schoenfeld (Ed.), *Mathematical Thinking and Problem Solving* (pp. 53-75). Hillsdale, NJ: Lawrence Erlbaum Associates.

Tzanakis, C., Arcavi, A. *et al*. (2000). Integrating history of mathematics in the classroom: An analytic survey. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education* (pp. 201-240). Dordrecht: Kluwer Academic Publishers.

Ying, J. M., Huang, J. W., & Su, Y. W. (2015). An exploratory study on influences of a mathematical culture course on university students' mathematics beliefs – The case in a medical university. *Taiwan Journal of Mathematics Education*, *2*(2), 1-24.

Zheng, L. L. (2002). *The establishment and development of imperial universities in colonies – A perspective centred in Taipei Imperial University*. Taipei: Ph.D. Thesis in National Taiwan Normal University. (In Chinese).

"INTERPRETING WITH RESPECT TO WHAT?" THE HERMENEUTIC APPROACH IN CLASSROOM EXPERIENCES

Adriano DEMATTÈ
Liceo Rosmini, Via Malfatti 2, Trento, Italy
adriano.dematte@fastwebnet.it

ABSTRACT

Interpreting a historical text is a complex task which involves different cognitive aspects regarding previous knowledge of students. From the classroom it emerges that the most relevant factors are not strictly cognitive: they involve didactical contract and are heavily connected with personal students' motivation. With the aim to facilitate students' learning, teachers can highlight specific points of an original and express the required performances. In this way the generic task "interpret the historical document" is clarified but at the meantime it becomes more complex.

Students are not alone in front of historical documents, being accompanied by teachers. I mention *Gadamerian philosophical hermeneutics* and its analysis of history and tradition. I like to consider that teachers have a role inside tradition, as a bridge between historical document and students. Making the choice to use a specific text in class, teachers show to have elaborated a preliminary type of interpretation about: mathematical content, educational goals, level of difficulty, students' involvement that it might produce.

Even if teachers do not insist on accepting an interpretation which has to be shared by everybody, introducing class activities they can orient students' interpretation of the original. I will also consider *Task Design in Mathematics Education* as a research perspective in order to analyse connections between historical document and students' performances.

LES MATHEMATIQUES DANS L'HISTOIRE DE LA CONSTITUTION DU SYSTEME DES BEAUX-ARTS ET DE L'ELABORATION DE L'ESTHETIQUE PHILOSOPHIQUE

Caroline JULLIEN
LHSP-Archives Henri Poincaré, Université de Lorraine 91 avenue de la Libération 54000 Nancy, France
cjullienwalter@free.fr

Que les mathématiques entretiennent avec l'art une relation privilégiée est un lieu commun largement répandu et qui connaît par ailleurs un regain sensible d'intérêt depuis une décennie. Au cours de cette présentation, je resituerai cette relation dans son évolution historique. Il s'agira en particulier de montrer que le rôle des mathématiques vis-à-vis de l'art, dans un sens large, ne se réduit pas à un rôle de discipline de service.

En effet, les mathématiques ont certes joué et jouent encore un rôle au niveau de la production artistique concrète, que l'on pense aux harmoniques musicales, au nombre d'or, à la théorisation de la perspective, etc. Mais, ce qui est certainement moins connu, et qui sera le point principal de cette présentation, c'est le rôle crucial des mathématiques dans l'histoire de la constitution du système des beaux-arts et dans l'élaboration de l'esthétique en tant que discipline philosophique autonome. Je présenterai en particulier le rôle fondamental tenu par les mathématiques dans l'œuvre de Hutcheson, philosophe irlandais, disciple de Shaftesbury, dont l'ouvrage *Recherche sur l'origine de nos idées de la beauté et de la vertu* (1725) est considéré comme l'un des ouvrages fondateurs de l'esthétique.

A MULTIDISCIPLINARY APPROACH TO TEACHING MATHEMATICS AND ARCHITECTURAL REPRESENTATION: HISTORICAL DRAWING MACHINES. RELATIONS BETWEEN MATHEMATICS AND DRAWING.

Laura FARRONI, Paola MAGRONE
Dipartimento di Architettura, Università Roma Tre, Largo G. B. Marzi, Roma, Italia
laura.farroni@uniroma3.it
Dipartimento di Architettura, Università Roma Tre, Largo G. B. Marzi, Roma, Italia
paola.magrone@uniroma3.it

ABSTRACT

The aim of this paper is to highlight some of the results of the interdisciplinary work carried out by the authors in the School of Architecture of Roma Tre University which unifies research and didactics showing a contemporary approach to the disciplines of drawing and mathematics. They conduct a course to develop a methodology of study that foresees three aspects: the construction of a mathematical drawing machine, the understanding of the analytic representations of the curve that the machine tracks, the consultation of historical sources. Case studies extracted from treatises on history of architecture, mainly from the nineteenth century, can show how geometrical issues have been resolved over time and interpteted in dfferent ways. The goal is to reinforce in students the awareness of the relationship between physical objects, their representation, and the mathematical theory underlying, through the study of historical cases.

1 Introduction

This work aims to highlight some of the results of the interdisciplinary teaching implemented in the elective course "*Mathematical drawing machines: historic drawing from a parametric point of view*", born from the encounter between the courses on the Science of Architectural Representation and those of Mathematics at the Department of Architecture of Roma Tre University (Italy)[1].

The course aims to compare themes which are common to both disciplines and is active from the academic year 2014/2015. In the cultural context, typical of Schools of Architecture, these two disciplines support the training of future architects, and in the specific case of Roma Tre they are part of the study plans of all degree programs. In recent years, the teachings of both disciplines have been object to many considerations and transformations. The path of the first 10 years of teaching mathematics in the School of Architecture of Roma Tre is described in the article by Pagano-Tedeschini Lalli[2], then continued by the activities of "formulas.it"

[1] The two authors are researchers of Drawing and Representation (Farroni), Mathematics (Magrone) in the Department of Architecture of Roma Tre University.

[2] Pagano, & Tedeschini Lalli, L. 2005

maths Laboratory[3]. As for the teaching of the Representation of Architecture, it has been transformed with the introduction of digital and information technology. These transformations have posed a set of problems, not only in the management of the contents of drawing , but also on how to transfer knowledge to learners. Those questions were posed in all specific areas, such as Descriptive Geometry, Drawing of Architecture, Survey and Representation Techniques.

A great inspiration has come from the work of Mariolina Bartolini Bussi[4], from her book[5] on drawing machines, along with the Mathematical Machines Association[6] and also from the studies by Riccardo Migliari[7], who for years has been dealing with these issues, linking them to the evolution of the teaching of Descriptive Geometry and shape recognition through the survey of built architecture. For this reasons the elective course "*Mathematical drawing machines: historic drawing from a parametric point of view*"[8] fits into a fertile environment of research of new methodologies and teaching strategies. It aims to put the student in a position to verify the same knowledge by comparing the two points of view. The course is intended for students of all degree programs of the School of Architecture of Roma Tre University. Relying on prior knowledge that we will explain in detail later, the course offers a methodological approach to the study of some cases found in the treatises of the history of architecture, mainly from the nineteenth century, combining the analytical geometrical treatment, the hands-on approach and representation.

The study of nineteenth-century treatises leads us to investigate issues that over time have been resolved and interpreted in different ways, leading students towards a greater understanding of the relationship between "curva figura e curva oggetto" (Gay 1999) that is "(curve) shape and (curve) object". Precisely in the nineteenth century we witness a systematization of the typologies of geometric constructions, the geometric drawing procedures and instruments. In the historical texts which were selected there is a description of elements of architectural construction whose shape is taken into consideration from the point of view of its geometric genesis, along with the design procedure and drawing. Not less important in these historical texts in some cases we find the description of the methods of construction made possible through the use of the mathematical machines machines

The compulsory courses in mathematics of the first three years of studies already educate students to visualization via analogical models such as drawing machines. In fact, in

---

[3] A group of mathematicians and architects, whose purpose is the dissemination of scientific culture, educational innovation, informal teaching of mathematics. In particular the web site gathers all the courses with mathematical content of the Department of Architecture of Roma Tre University.

[4] Full Professor of Didactics of Mathematics at Modena and Reggio Emilia University, Italy.

[5] Bartolini Bussi, Maschietto (2006) op.cit., Tedeschini Lalli (2009) op.cit.

[6] www.macchinematematiche.org , the site offers pictures, animations and explanatory pages on many drawing machines. Much attention is devoted to conicographs, which therefore constitute a valid first approach for students considering the huge amount of teaching materials available.

[7] Full Professor of Descriptive Geometry in Sapienza, University in Rome.

[8] See also Farroni & Magrone (2014) where the authors talk about the very first experience of this cross-disciplinary teaching.

the obligatory course of the second year[9], students can choose to build a machine to draw a conical curve and discuss it in the oral exam. This has further encouraged the authors to propose the course. Probably also inspired some of the students, encouraged by the typical traditional laboratory approach adopted by schools of architecture, where students often build physical models to scale.

Acknowledgment: the authors wish to thank Prof. Ana Maria Millan Gasca (Roma Tre University) for the stimulating discussions.

2 Course objectives, content and teaching strategies

The goal of this course is to lead students to acquire and make explicit the relationships, which has always existed, between the graphical and the analytical representation in the interdisciplinary approach to architectural drawing and mathematics. Specifically, the course addresses the problem of the tracking of curved objects and the process of abstraction, which is fundamental in both disciplines. This process is necessary to switch from a real object to its representation. For this purpose the authors offer the tools to expand the knowledge of plane curves, their genesis, their construction and tracking. The mathematical knowledge required as prerequisites are Euclidean geometry, which is studied in secondary schools, and differential calculus in one variable, which is acquired during the first year course. As for the drawing, the prerequisites consist of the knowledge of the elementary geometric constructions and the use of graphical instruments such as compass and ruler, which are part of the first year course called "Drawing of Architecture".

The advent of information technology, both in the setting of mathematics and architectural representation, has produced softwares that allow to easily manage both simple and complex shapes. Often the creation of a shape and its modification and transformation is implemented automatically and unconsciously. To educate the skill of creating the desired shapes with autonomy and consciousness, the authors propose the direct hands-on manipulation of the data and the subsequent identification of the parameters in the analytical representation as the main tool in learning contemporary instruments and theories dedicated to the formal project. To this end, each student is assigned a mathematical machine to design, implement and test.

During the lessons the analytical equations are related to the specific graphic constructions with ruler and compass, through the construction and use of machines such as some ellipsographs, parabolographs, and hyperbolographs. The workshop sessions require that students take back knowledge gained in the course of drawing, aquire new expertise, build prototypes of drawing machines for continuous tracking and then use them. The interdisciplinary teaching objectives are to develop in students the ability to foresee the main characteristics of the figurative project on a two-dimensional support, at the very moment of its drawing; to introduce the scientific basis for facing the digital modeling; consolidate the ability to relate shapes and formulas.

[9] The course is “Calculus (several variables) and Geometry 2”, led for the current year 2015-2016 by Valerio Talamanca and Laura Tedeschini Lalli

The choice of teaching strategies is connected to some considerations. The curve tracking highlights the problem of the continuity of the sign. Some of the methods of graphic constructions, although rigorous, allow detection of a curve by points instead of the continuous tracking. Therefore the problem arises of drawing a curve that joins the identified points and also find tools that directly allow the continuous tracking, without passing through the discretization. The mathematical machines meet this need: they draw the curves with continuity, with the exception of only a few points (only in some cases) where the tracer should be disconnected physically from the two-dimensional support in order to overcome the physical barrier of the mechanism and then proceed to the tracking of the sign. Therefore it has been decided, for educational purposes, to show the mathematical machinery presenting both continuity and discontinuity tracking during the lessons, so that students themselves can grasp this aspect.

In the early teaching units much space is devoted to conic curves, for reasons that will be explained below. A first consideration concerns the recognition of geometric loci, which is a key capability in the training of an architect. This recognition turns out to be a more immediate process when referred to conic curves. All the curves that learners are supposed to study and draw during the course are two dimensional, and can be interpreted as sections and profiles useful for the future management of three-dimensional objects (both analogical and digital) whether in the field of architectural design or in recognition of the built forms. Their detection, both graphical and analytical, leads to the ability to associate a geometric entity to a shape, in order to be able in a second moment to edit and modify it. The proposed working method aims then to start a continuous check between constructive and analytical procedure. For this purpose, the conical curves are the ideal starting point: they are familiar objects to most students, their analytic representation is not difficult.

Last but not least the focus is on the concept of parameter: the cartesian equation of a conic curve contains some parameters, whose change affects the shape of the curve. The curve drawn with a machine, puts the student in the position to lead actively the plotting and the corresponding analytical representation, by setting and then changing the initial data. Tracking a curve with the machine whose setting is reflected in the respective equation, and further manipulate the machine by observing the change of the curve, educates the process of abstraction that relates the shape with the analytical equation. This process goes through the drawing machine. This acquisition of awareness, implemented on known curves, characterized by simple equations, can be transferred at a later time to the study of more complex curves.

The first sessions are devoted to “thread conicographs”, because their mode of operation embodies the geometric locus corresponding to the curve and therefore constitute the ideal tool for trasmitting its comprehension.

The reading of some passages from the *Trattato teorico e pratico dell’arte di edificare* (Theoretical and practical art of building Treatise) by Giovanni Rondelet, 1832[10] has also been of help to show learners how, in the nineteenth century, a systematization of knowledge

[10] Rondelet 1832 op. cit.

and know-how, was carried out, in other words how theoretical knowledge was verified in the practice of building. In fact, Rondelet reports, (Third book, Stereotomy, first section, first part) a few paragraphs dedicated to the useful curves in architecture. In particular in *Cenno sulle curve che possono servire alla superficie interna delle volte* [11], the graphical and analytical-geometrical description *Delle curve chiuse* e *Delle curve aperte* [12] can be found by referring to suitable instruments for plotting. The decision to propose this to learners lies on the setting of the Treatise itself. Actually it highlights that the geometric drawing is a fundamental tool for the art of building, in the light of the theories of Monge[13].

In the text of Rondelet considerable theoretical sections of analytic geometry can be found. For example, in the section mentioned above, the ellipse is described as a projection of the circle, as a geometric locus of the plane, and at least three graphic methods to identify the curve by points are provided (see fig. 1); furthermore two machines for the continuous tracking are described: the "thread" machine (see fig 2) and the one that embodies the graphical method called " the paper strip" (see fig. 3). Again, various graphical methods are described to show how to draw ovals, i.e. the polycentric curves, of different typology depending on the numbers of centers, and finally the cassinoide curve and the cycloid are mentioned. Only after an extensive theoretical treatment of the above curves, Rondelet exposes the motivations that can push an architect to choose either one or another:

> La sensibile differenza che offrono queste curve fra loro, considerate come curvature di volte, influisce molto sulla loro solidità. La teoria d'accordo coll'esperienza prova che nelle volte schiacciate, più è curva l'arcatura del mezzo, minore è la sua spinta […] d'onde risulta che se si ha in vista la solidità conviene scegliere una curvatura che si avvicini più alla cicloide che alla cassinoide. Nondimeno quest'ultima, che è più aperta, presenta in certi casi una forma più aggradevole che si accorda meglio coi piediritti a piombo; ma essa agisce con più forza ed esige una maggior grossezza di sostegni. L'ellissi, la cui curvatura è media, unisce la solidità alla regolarità, e perciò dev'essere sempre preferita; tanto più che ha la proprietà di poter servire per tutte le altezze di volte, mentre la cassinoide ha dei limiti, e la cicloide non conviene che ad un caso solo[14]…
>
> The significant difference between these curves, regarded as vaults bending, greatly affects their solidity. The experience agreeing with theory shows that in flattened vaults , the more curved the arch in the middle part, the lower its pressure [...] whence it results that if you look for the strength, you should select a curve more similar to the cycloid than to the cassinoid. Nevertheless, the latter, which is more open, presents in some cases a more pleasant shape that harmonizes better with the pillars; but it acts with more force and requires a greater thickness of support. The ellipses, which has an average curvature, combines solidity to regularity, and therefore must always be preferred; furthermore the ellipses has the property of being able to serve for valuts of

---

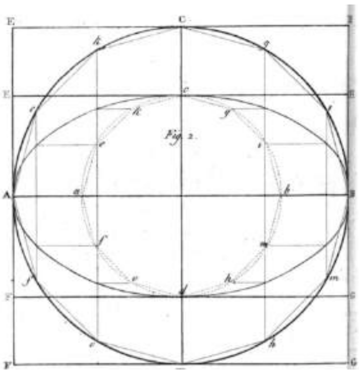
[11] Hint on the curves that can serve to the inner surface of the vaults

[12] About closed curves, about open curves

[13] In this regard it is interesting to see the work plan of the treatise by Rondelet, that here we chose to leave out.

[14] J.B. Rondelet, 1832, op. Cit. Third book, Stereotomy, first section, first part

all heights, while the cassinoid is limited, and the cycloid is not convenient except for one case […]



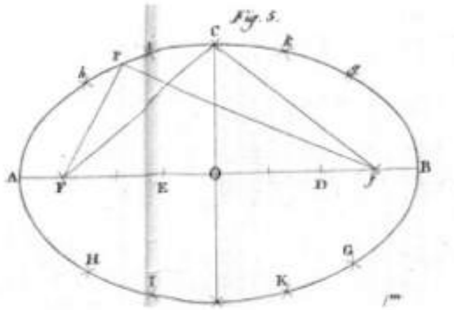


Figure 1-2. Rondelet G.B. (1832) "Trattato teorico e pratico dell'arte di edificare", Plate XIX. On the left epresentation of the graphic method to draw an ellipse by points, called the "concentric circles". On the right the so-called "gardener's ellipse", for drawing this curve in a continuous way; in the same Figure, reference is made to a graphical method to track an ellipse by points.

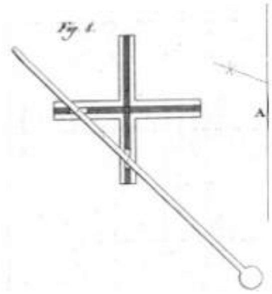


Figure 3. Rondelet G.B. (1832) "Trattato teorico e pratico dell'arte di edificare", Plate XIX ellipsograph of Proclo

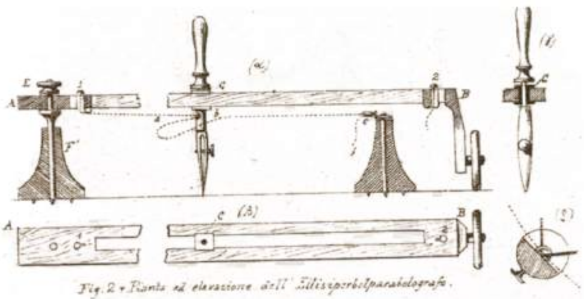
Since the beginning of the course a bibliography and webliography is suggested to the learners so that they can immediately begin to direct themselves autonomously on the topic for the final exam. To this end it is required that each student builds a drawing machine and understands how it works; knows the equations of the curve that is represented by the mechanism; studies a geometrical problem based on a historical text applied to architecture involving the studied curve. After the first in-depth sessions on conic and related machines, a the second part of the course starts, which is dedicated to individual insight. The individual progresses are often discussed with all the group so that everyone can contribute with questions and interventions. It is through these group discussions and the ongoing confrontation with the instructors, as well as individual work, that the historical research is narrowed and refined allowing the arrival to the final product.

3 History as a source of case studies

The proposed study methodology proceeds on three routes: the construction of the machine, the understanding of the analytic representations of the curve that the machine is used to track and the shape-formula comparison along with the consultation of historical sources. Geometrical problems that in the course of time have been interpreted in different manners and case studies can be extracted from treatises. The aim is to develop in students an awareness of the relationship between physical constructed reality, representation, and underlying mathematical theory, right through the study of historical cases. The nineteenth-century machines make this issue clear and explicit, which on the other hand may be less obvious to the future architect who designs and studies the forms through the computer. Here are some case studies taken into consideration because they highlight how "the accuracy of the geometry" comes to the "aid" in the definition of some architectural shapes in the nineteenth century. [15]

3.1 The “Pillet[16]” machine

This case study shows an example of an “ellipse-hyperbol-parabolograph” (fig. 4, 5, 6), a mechanism which encloses in a single object the three thread machines to draw conic curves. A first prototype was designed and built as part of the work of a post-graduate fellowship[17]. In order to get the students accostumed the theoretical/practical approach, during the course they make diagrams and graphical representations of the machine and a prototype whose beam was 60 cm long, suitable to draw, for example, an ellipse whose major axis can be about 45 cm.



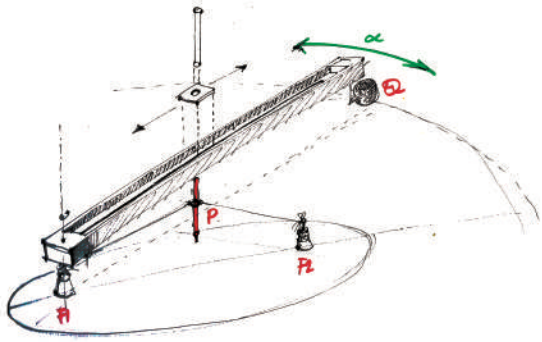


Figure 4. Plate from L’architettura pratica Disegni degli edifizi rispondenti ai bisogni moderni, (1891): scheme of the Pillet Machine. Figure 5. Hand made sketch for the construction of the first prototype of the Pillet machine, by Enrico Mele.

[15] Codazza (1844) op. Cit.

[16] Jules Pillet (1842-1912) was a professor of descriptive geometry at Ecole Normale des beaus Arts in Paris. The first time the authors learned about this machine was from the book of Migliari (1991).

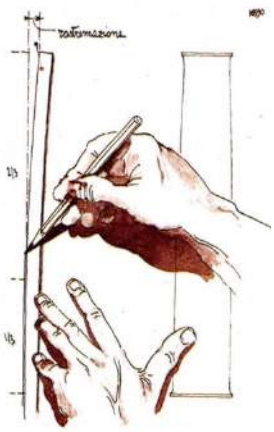
[17] The post graduate fellow was Enrico Mele, who is also the author of the sketch in fig. 5.

Figure 6. Prototype of ellipse-hyperbol-parabolograph, beam length 150 cm, made by the student Osvaldo Liva

Then one student chose this to be his product for the final exam, and then continued to deepen the knowledge of the machine, creating a model which measured 150 cm, therefore suitable to draw an ellipse with major axis length of about 120 cm [18].

3.2 Tracking the entasis[19] curve

The entasis curve of the columns and its tracking is a case study which foresees various solutions including the use of the Conchoid of Nicomedes.



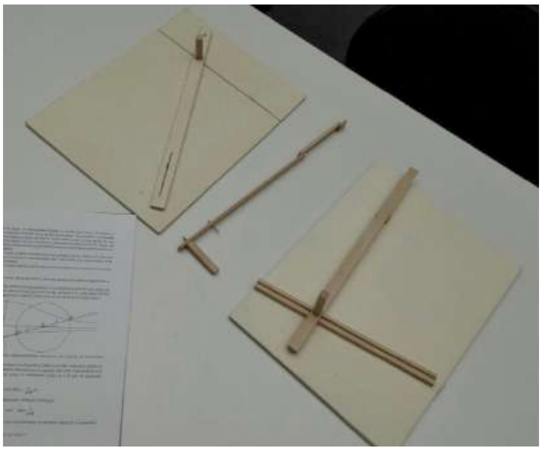


Figure 7. Migliari (1991) “Il disegno degli ordini e il rilievo dell'architettura classica: Cinque Pezzi Facili”. Sketch by Riccardo Migliari, in which the author reports the method for tracking the entasis by means of a “flexible wooden rod”. Figure 8. Picture of the drawing machine to track the conchoidal curve, design and implementation by the student Valerio Del Ferraro.

The conchoid curve was introduced to the students as a source of different insights: it can be used to resolve the trisection of the angle, one of the three Greek unsolved problems of antiquity. The curve possesses two branches, which leads to consider why we should use the

[18] Work performed by the student Osvaldo Liva

[19] The entasis indicates a narrowing of the shaft of the column, starting from one-third of its height, upwards.

parametric equation, rather than cartesian one. After the detailed mathematical study the work of the student continues with the analysis of the practical application in architecture, comparing it to the different procedures for tracking the entasis curve, in order to produce an inventory of the different methods. The student in charge of the project, proceeded to read and interpret the original source. Then he built two prototypes of drawing machines: one to draw the conchoid of Nicomedes, and a second one, which is cited by Peter Nicholson (in the text of 1867 "Carpenter's New Guide") which consists of a simple flexible wooden rod (see fig. 7,8). The setting of the work has helped the student to develop a critical attitude so that after reading the original sources he also whished to become acquainted with the state of the art of contemporary studies (with the help of the bibliography provided during the course), like those of Riccardo Migliari (Migliari 1991). Later, elaborating critically all the information, he proceeded to verify the working principles of the two prototypes he already made. This case study gave the authors the opportunity to and underline to their students the importance of understanding the relation between the “geometric object and the real object”:

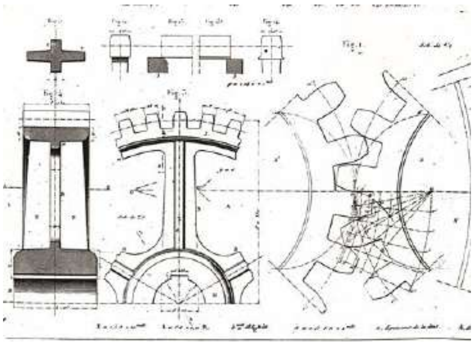
> Vige sempre un’incolmabile differenza ontologica tra il *cerchio* del muratore e quello de *‘l geometra*; mentre il primo conta il numero dei mattoni circonferenti come tre volte quello che sarebbe occorrente per il diametro, il matematico mette oggi nel rapporto tra diametro e perimetro, in luogo della malta del muratore, un miliardo di cifre decimali e dimostra che il suo lavoro contabile non finisce lì. Il profilo concoidale rifinito dallo scalpellino nell’entasi delle colonne non è propriamente la stessa concoide considerata da Nicomede per risolvere la mitica duplicazione del volume dell’altare cubico del tempio di Apollo […] (Fabrizio Gay (1999), pag 76 op. Cit.)

An increasingly unbridgeable ontological difference forever endures between the *circle* of the mason and that of the *geometer* (in this case we think that this term refers to the person who is in charge of controlling the project according to science, so someone who has a deep knowledge of theoretical geometry T.N.); while the mason counts the number of bricks to form a circle as three times those which would be needed for the diameter, the mathematician puts a billion decimal places today in the relationship between diameter and perimeter, instead of the mason mortar, and he also proves that his accounting job does not end there. The conchoidal profile of the column entasis, finished by the stonemason, is not exactly the same conchoid considered by Nicomedes to solve the legendary duplication of the volume of the cubic altar of the temple of Apollo […]

3.3 The study of a nineteenth-century gear

This study led to the deepening of epicycloid curves. This case differs from the others described because the application is not architectural, but instead consists of a nineteenth-century engineering product. The gear shown in fig 10 led the graphical depiction which was part of the work for the student’s final examination. At the theoretical level the student has studied the equations of epicycloids curves; then she designed and built some gear wheels, with holes to insert the tracer to draw different kinds of epicycloids. The first prototypes were cut from cardboard, while the conclusive gears were made of plexiglass and cut with a

Computer Controlled Cutting (CNC) machine of the “model and prototypes laboratory” of the Architecture Department of Roma Tre University.



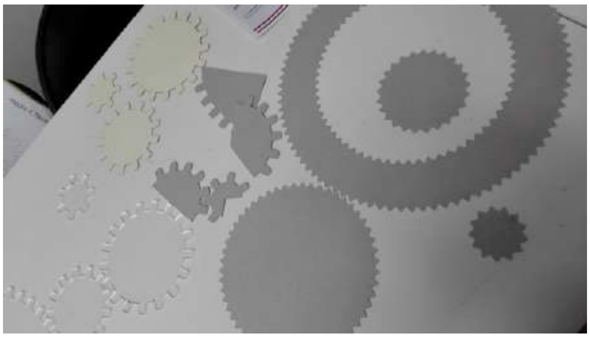


Figure 9. Plate from Armengaud, J.E, & Armengaud, C. A (1854) “The practical draughtsman's. Book of industrial design, and machinist's and engineer's drawing companion: forming a complete course of Mechanical, Engineering, and Architectural Drawing” showing the gears. Figure 10. gears in cardboard and plexiglass, design and implementation by the student Matilde Panascì

4 Conclusions and future developments in teaching and research.

The study of ancient drawing machines, as part of the training program for students of architecture, allows them to grow culturally and scientifically. If the use of the machines and the tracing of curves is accompanied by a continuous analytical verification, it strengthens the ability to recognize the geometry of curved objects, and to identify the key points of each curve. Furthermore the contextualization of the mathematical graphical problem, namey the study of the historical and cultural framework of the machine and the verification of the original sources, leads to the acquisition of analytical and critical skills in solving the issues that students may encounter in the professional world. From now on the objective is to widen the case studies and to create a collection of prototypes of different machines together with a bibliographic and iconographic filing of the original sources and of the built products.

This type of teaching has led to an experimental development and exchange between teaching and research in which both complement each other. In fact, the results obtained in the first edition of the course have been studied in depth by a student awarded with a post-graduate fellowship aimed at drawing up bibliographies related to the relationship between geometry and construction in architecture. This scientific product was then proposed for the second edition of the course in the form of lectures and as available material for in-depth study.

The construction of the historical machines highlights how each one is specialized in displaying certain properties. For example, in the case of ellipsographs, thread machines can be used to show the geometrical locus, while some linkages-type machines lead to the study of geometric transformations. So new study paths are created. The change of scale, experienced in the tracking of the conic curves with the Pillet machine, suggests reflections on how to manage curves at large. Also curved objects on an urban scale can be taken into consideration in order to introduce the use of ancient survey instruments for the collimation of points, useful for the definition of the designed curve. It has come to our attention that the structure of the study course allows to test the previous achievemnts and explore new persectives.

REFERENCES

L'architettura pratica. Disegni degli edifizi rispondenti ai bisogni moderni, (1891), Anno II, Fascicolo VII 26-27. Torino : Tip. Camilla e Bertolero.

Armengaud, J. E, & Armengaud, C. A. (1854). *The practical draughtsman's. Book of industrial design, and machinist's and engineer's drawing companion: forming a complete course of Mechanical, Engineering, and Architectural Drawing. Translated from the French of M. Armengaud, the Eleder and MM. Armengaud, The Younger, amouroux.* Rewritten and arranged, with additional matter and plates, selections from and examples of the most useful and generally employed mechanism of the day by William Johnson. New York: Stringer & Towsend.

Bartolini Bussi, M. G. C., & Maschietto, M. (2006*). Macchine matematiche : dalla storia alla scuola*. Milano : Springer Verlag Italia.

Codazza, G. (1844). *Nozioni teorico pratiche sulle centine delle volte*. Pavia: Tipografia Bizzoni.

Cresci, L. (1998). *Le Curve Celebri*. Padova: Franco Muzzio Editore.

Farroni, L., & Magrone, P., (2014). Mathematical drawing machines: Historic drawing from a parametric point of view. The case of conic curves. *Libro de actas del V Congreso Internacional De Expresión Gráfica. XI Congreso Nacional de profesores de expresión gráfica en ingeniería, arquitectura y áreas afine Egrafia 2014, Rosario, Argentina, I*, 130-137.

Gay F. (1999*). Intorno agli omolografi: strumenti e modelli per la geometria descrittiva*, Quaderni IUAV.12.2000 DPA. Venezia : Istituto universitario di architettura di Venezia.

Migliari, R. (1990). Il disegno degli ordini e il rilievo dell'architettura classica: Cinque Pezzi Facili. *Disegnare: Idee, Immagini*, *2*, 49-66.

Nicholson, P. (1808). *Carpenter's New Guide being a complete book of lines for carpentry and Joinery.* London: printed for J. Taylor.

Pagano, A., & Tedeschini Lalli, L. (2005). Università di Roma Tre, 1995-2005: Architecture and Mathematics. *Nexus Network Journal*, *7*, 89-97.

Pillet, J. (1887). *Traité de géometrie descriptive, ligne droite et plan, polyèdres, surfaces. Texte et dessins*. Paris: Librairie Ch. Delagrave.

Rondelet, G. B. (1832). *Trattato teorico e pratico dell'arte di edificare*. *Prima traduzione italiana sulla sesta originale con note e giunte importantissime, per cura di Basilio Soresana, seconda edizione*. Mantova: Fratelli Negretti.

Tedeschini Lalli, L. (2009). Mathematical machines: A laboratory for mathematics, Interview to Mariolina Bartolini Bussi by Laura Tedeschini Lalli. *Nexus Network Journal*, *11*, 317-324.

5. Workshops

L'ARITHMOMÈTRE DE THOMAS : SA RÉCEPTION DANS LES PAYS MÉDITERRANÉENS (1850-1915), SON INTÉRÊT DANS NOS SALLES DE CLASSE

Pierre AGERON
LMNO & IREM, Université de Caen Normandie, 14032 Caen, France
ageron@unicaen.fr

ABSTRACT

Thomas' Arithmometre was the first industrially produced mechanical calculator. It was developed as soon as 1820, then manufactured and sold from 1850 to 1915. This contribution first examines its reception amongst French teachers of mathematics : although some of them were very enthusiastic about it, using an arithmometer as a pedagogical device was not considered at the time. Some hints are given at why and how a virtual arithmometer could nowadays be used in the classroom. The mathematical content of the different editions of the user instructions booklet is then detailed, including Töpler's purely additive method for extracting a square root, which was specifically invented for arithmometre. The second part of the paper deals with the circulation of Thomas' arithmometre in Mediterranean countries. In particular, an hitherto unidentified Arabic manuscript is presented, that happens to be a freely translated version of the arithmometre's instructions booklet. This document, written in Morocco in 1875, is commented and selected parts of it are translated into French.

RÉSUMÉ

L'arithmomètre de Thomas fut la première machine à calculer produite industriellement. Elle fut mise au point dès 1820, puis fabriquée et commercialisée de 1850 à 1915. Cette contribution examine d'abord sa réception chez les professeurs de mathématiques français : malgré l'enthousiasme de certains d'entre eux, il ne fut jamais envisagé à l'époque d'utiliser l'arithmomètre comme outil pédagogique. Quelques indications sont données sur l'intérêt et les modalités de l'utilisation, de nos jours, d'un arithmomètre virtuel en classe. Le contenu mathématique des différentes éditions de la notice d'utilisation est détaillé, notamment la méthode purement additive de Töpler pour extraire une racine carrée, spécifiquement inventée pour l'arithmomètre. La seconde partie de l'article concerne la circulation de l'arithmomètre de Thomas dans les pays méditerranéens. On y présente notamment un manuscrit arabe jusqu'ici non identifié, qui s'avère être une traduction libre de la notice d'utilisation de l'arithmomètre. Ce document, écrit au Maroc en 1875, est commenté et des parties choisies en sont traduites en français.

1 L'arithmomètre de Thomas

L'arithmomètre de Thomas fut la première et très longtemps la seule machine à calculer fabriquée industriellement. Elle fut mise au point dès 1820 par un assureur parisien, Charles-Xavier Thomas, dit de Colmar (1785-1870), qui avait été administrateur militaire pendant les campagnes napoléoniennes d'Espagne et du Portugal. Le mécanisme d'entraînement qu'il conçut s'apparentait à celui imaginé jadis par Leibniz, sous forme de cylindres à cannelures inégales. Thomas ne commercialisa son invention qu'à partir de 1850 ; elle remporta alors un vif succès. À sa mort, cette activité fut poursuivie par son fils Louis Thomas de Bojano (1818-1881), puis par son petit-fils, aidés par l'ingénieur Louis Payen. En 1888, ce dernier reprit l'affaire à son compte ; après sa mort en 1901, elle perdura jusqu'en 1915.

1.1 L'arithmomètre, un objet pédagogique ?

L'arithmomètre, machine dont la fabrication demandait beaucoup de temps et de soin, était très coûteux. Le marché visé était celui des administrations, banques, compagnies d'assurance, bureaux d'étude et établissements industriels, mais aucunement celui des lycées ou des facultés. Certes, quelques professeurs ont eu la chance de l'expérimenter. En 1854, les *Nouvelles annales de mathématiques* de Terquem et Gerono y font une allusion, uniquement en termes de temps gagné[1]. En 1867, Rivière, professeur de physique au lycée de Rouen, fait une démonstration de l'instrument devant l'Académie de cette ville[2]. Vers 1872, Justin Bourget, professeur de mathématiques de la faculté des sciences de Clermont, se déclare « satisfait des services que lui rend journellement l'arithmomètre Thomas »[3]. En 1878, Édouard Lucas, professeur de mathématiques au lycée Charlemagne à Paris, obtient de l'Association française pour l'avancement des sciences une subvention pour l'achat d'un arithmomètre, et présente au congrès de cette association, où Payen vient de présenter sa machine, une communication « Sur l'emploi de l'arithmomètre Thomas dans l'arithmétique supérieure » – il s'agit de congruences et de nombres premiers[4]. En 1910 enfin, on signale un arithmomètre dans les facultés des sciences de Marseille et Montpellier[5]. Mais pendant tout le temps où il fut fabriqué, aucun professeur ne rapporta avoir utilisé un arithmomètre dans le cadre de son enseignement : le temps n'en était pas venu.

Aujourd'hui, il me semble que l'arithmomètre pourrait être aujourd'hui un outil pédagogique intéressant. C'est une machine séduisante, insolite et ludique, mais qui, contrairement à nos calculettes électroniques, ne fait pas tout : les seules opérations entièrement automatisées sont l'addition et la soustraction, qui se pratiquent au moyen d'une manivelle et d'un inverseur de marche permettant de choisir entre l'une et l'autre. La multiplication, la division, l'extraction de racines s'en trouvent bien sûr facilitées, puisqu'elles ne sont au fond que des additions ou soustractions itérées, mais le principe des algorithmes utilisés sur le papier demeure, préservant ainsi ce qui fait le prix de la numération de position et lui donne tout son sens. Pour multiplier 72909 par 35 par exemple, on porte 72909 sur la platine, on tourne la manivelle 5 fois, on décale la platine d'un cran vers la droite et on tourne encore 3 fois : le résultat apparaît dans les lucarnes du totalisateur[6]. On voit comment l'arithmomètre matérialise les algorithmes de l'arithmétique, de sorte que leur expérience n'est plus seulement intellectuelle, mais en quelque sorte kinésthésique, vécue dans les mouvements du bras, qui produisent tant la rotation de la manivelle que la translation de la platine. Le caractère local des calculs, c'est-à-dire le fait qu'ils n'impliquent qu'une tranche définie de chiffres, devient évident lorsqu'une partie de la platine dépasse de la machine, se trouvant visiblement « hors-jeu ». Auprès de lycéens ou d'étudiants, l'arithmomètre devrait aider à redonner sens à la numération de position et à réactiver des algorithmes appris, mais de moins en pratiqués. C'est l'occasion aussi de leur faire découvrir un intéressant algorithme, au strict sens étymologique du mot puisqu'il fut décrit par al-Khuwârizmî, qui n'est

[1] Terquem et Gérono (1854).
[2] Académie impériale des sciences, belles-lettres et arts de Rouen (1867), pp. 116-117.
[3] [Thomas de Bojano] (1872 ?), p. 3.
[4] Décaillot (1998), pp. 219-220.
[5] [Payen] (1910), p. 4.
[6] Pour mieux visualiser, nous conseillons les vidéos de Valéry Monnier sur son excellent site arithmometre.org.

plus enseigné en France depuis 1962 : l'extraction chiffre à chiffre de la racine carrée. Cet algorithme simple dont on ne leur a jamais parlé, mis en œuvre sur une étrange machine, devrait attirer leur attention. Il peut être leur être introduit en s'appuyant sur le texte latin du XII^e siècle *Dixit Algorizmi*, probablement très proche de l'original arabe perdu du IX^e siècle : la fin de ce texte qui concerne la racine carrée, elle-même longtemps perdue, retrouvée à New York il y a une vingtaine d'années, contient un exposé général difficile à suivre auquel on préférera le traitement de l'exemple de l'extraction de la racine de 5625, plus parlant et bien détaillé[7].

La question du matériel ne peut être éludée. Produit à quelques milliers d'exemplaires dont quelques centaines subsistent, pas toujours en bon état, l'arithmomètre de Thomas est aujourd'hui un objet rare recherché par les musées et les collectionneurs. Il faut compter au moins 1500 euros pour en acquérir un. Les modèles les plus anciens ou les plus luxueux atteignent en salle des ventes des prix affolants : on a vendu en 1997 un arithmomètre pour 166 000 livres sterling, un autre en 2013 pour 233 814 euros, en 2015 un troisième pour (seulement !) 46 700 euros. Comme au XIX^e siècle, son usage scolaire est donc exclu, sauf à négocier le prêt exceptionnel d'une machine. Deux solutions sont envisageables. L'une d'elles est de lui substituer un arithmomètre plus récent, d'utilisation plus ou moins analogue : le plus courant est l'arithmomètre d'Odhner dont des clones ont été fabriqués jusque dans les années 1970. Le prix d'une telle machine, en bon état de fonctionnement, est de l'ordre d'une cinquantaine d'euros, ce qui la rend accessible même s'il reste difficile d'en acquérir suffisamment pour mettre au travail un groupe d'élèves. Une autre possibilité, que j'ai testée en janvier 2016 avec des professeurs-stagiaires, est d'utiliser l'arithmomètre virtuel mis au point par Stephan Weiss, un Allemand passionné de l'histoire du calcul. Il est disponible sous forme d'une animation Flash à l'adresse réticulaire suivante :

www.mechrech.info/workmod/thomas/ThomasMod.html

Pour l'utiliser en classe, il suffit de disposer d'une connexion wi-fi et d'une tablette ou d'un ordinateur par élève, de sorte que chacun puisse expérimenter. Bien sûr, et on peut le regretter, ce ne sont plus ici les bras qui travaillent, mais les doigts : on *clique* donc sur la manivelle ou sur les différents boutons. Cependant, on *voit* bel et bien la manivelle tourner, on *voit* la platine se déplacer, on *voit* les chiffres tournoyer dans les lucarnes. Et on *entend* aussi tout cela, car les bruits mécaniques caractéristiques émis par l'appareil lors des différentes manipulations sont parfaitement reconstitués. Il faut d'ailleurs prendre garde, comme sur un arithmomètre réel, à ne pas tourner la manivelle trop vite, mais à bien attendre que chaque tour soit terminé, au risque de provoquer des erreurs de retenue. Il est un peu décevant que l'arithmomètre virtuel de Stephan Weiss ne soit qu'un « modèle réduit », dont le totalisateur ne peut afficher que huit chiffres : dans la réalité, les différents modèles construits et commercialisés par Thomas et ses successeurs affichaient dix, douze, seize ou vingt chiffres – la fabrication de l'espèce à dix chiffres ayant d'ailleurs cessé assez tôt. Cependant, cette limitation ne constitue en aucune façon un obstacle pédagogique. Et après tout, la plupart des calculettes électroniques actuelles ne peuvent elles aussi afficher que huit chiffres.

---

7 Folkerts (1997), pp. 99-100.

1.2. L'*Instruction pour se servir de l'arithmomètre*

Chaque arithmomètre vendu était accompagné d'une brochure d'une trentaine de pages, sans nom d'auteur, faisant office de manuel d'utilisation : l'*Instruction pour se servir de l'arithmomètre*. De 1850 à 1915, au fil des perfectionnements techniques de la machine, elle connut de nombreuses éditions[8]. Mais les efforts d'amélioration ne visaient clairement pas la pédagogie : sur toute cette longue période, malgré les rééditions nombreuses, les explications mathématiques demeurèrent telles qu'elles avaient rédigées en 1850. Les exemples aussi restèrent inchangés ; en voici la liste. Addition : $9 + 6 + 8 = 23$ (disparaît dès 1852) ; $307 + 785 = 1092$. Soustraction : $757 - 689 = 68$. Multiplication : $9 \times 6 = 54$; $35695 \times 29072 = 1037725040$. Division : $4300 \div 357 = 12$ avec reste 16 ; $3264566 \div 6242 = 523$ sans reste. Extraction de la racine carrée : $\sqrt{897650000} = 29960$ avec reste 48400 (de 1850 à 1895).

À trois reprises, des adjonctions ont cependant été apportées à l'*Instruction*.

La première adjonction apparaît dans l'édition de 1868, où on lit « Il a été découvert un autre mode d'extraction de la racine carrée ». Cet « autre mode » est une procédure purement additive, donc mieux adaptée à l'arithmomètre que la méthode traditionnelle. Elle repose sur le fait que le carré d'un entier n est la somme des n premiers nombres impairs. Expliquons-la en considérant le cas, facile à généraliser, d'un nombre A de quatre chiffres. Par soustractions successives, on cherche le plus grand entier b_1 tel que $R = A - (1 + 3 + 5 - ... + (2b_1 - 1))\ 10^2 \geq 0$: c'est le chiffre des dizaines de $\sqrt{A}$. Puis on cherche le plus grand entier b_0 tel que $R - ((20b_1 + 1) + (20b_1 + 3) + (20b_1 + 5) + ... + (20b_1 + 2b_0 - 1) \geq 0$: c'est le chiffre des unités de $\sqrt{A}$. Ce principe et sa pratique à l'arithmomètre furent exposés en 1865 par Franz Reuleaux (1829-1905), professeur de mécanique à Berlin et ardent partisan de l'arithmomètre[9]; il en attribua l'invention à son collègue physicien et chimiste allemand August Töpler (1836-1912), alors en poste à Riga. Les noms de Reuleaux et Töpler sont absents de l'*Instruction*, mais la filiation est évidente : les exemples choisis y sont, à un détail, près identiques à ceux qu'on trouve dans l'article allemand. Le premier est celui de la racine carrée de 2209 : ce nombre s'écrivant $(1+3+5+7)10^2 + (81+83+85+87+89+91+93)$, elle vaut exactement 47. Le second est celui de la racine carrée de 41621, qui n'est pas entière, mais dont le calcul est poursuivi selon le même principe jusqu'au troisième chiffre après la virgule : puisque $41621 = (1+3)10^4 + 0.10^2 + (401+403+405+407) + 0.10^{-2} + 40801.10^{-4} + (408021+408023)10^{-6} + 103856.10^{-6}$, elle vaut 204,012 à 10^{-3} près par défaut. Une nuance cependant : dans l'article allemand, la seconde racine extraite était celle du nombre décimal 4,1621, qui vaut 2,04012. La deuxième adjonction apparaît dans l'édition de 1873. Il s'agit de l'extraction d'une racine cubique, non traitée auparavant. Elle est expliquée sur deux exemples : la racine cubique de 79507, qui vaut exactement 43, et celle de 564375686432, qui vaut 8263 à l'unité près avec un reste de 201438985. La troisième adjonction apparaît en 1906, et se substitue aux sections antérieures sur les racines carrées et cubiques. Il s'agit d'une méthode d'extraction présentée

[8] [Thomas de Colmar] (1850, 1852, 1856, 1860, 1865, 1868, 1873, 1878, 1884, 1895, 1902, 1906, 1908).

[9] Reuleaux (1865).

comme « exclusive », « simple et extra rapide », mais qui n'est plus autonome puisqu'elle requiert la table des carrés et cubes des entiers de 1 à 999. Par ailleurs, l'édition de 1906 incorpore aussi au texte des précédentes des astuces pour diminuer le nombre de tours de manivelle dans certaines additions et multiplications.

2 L'arithmomètre en Méditerranée

Pour faire connaître sa machine, la stratégie de Thomas de Colmar fut, au début des années 1850, d'en offrir un exemplaire à de nombreux souverains ou chefs d'État. Certaines de ces machines prestigieuses, présentées dans de magnifiques boîtes en marqueterie dédicacées, ont subsisté jusqu'à nos jours. Beaucoup d'autres ont disparu, mais leur existence peut être déduite des distinctions étrangères conférées à Thomas entre 1851 et 1854[10]. La majorité des souverains dont on peut ainsi établir qu'ils furent bénéficiaires d'un arithmomètre régnaient sur des États de la rive nord de la Méditerranée : le roi de Grèce, le roi des Deux-Siciles et son frère le comte de Trapani, le pape, le grand-duc de Toscane, la régente du duché de Parme, le roi de Piémont-Sardaigne, le prince président de la République française, la reine d'Espagne, et, en poussant jusqu'à l'Atlantique, le roi du Portugal. En Europe du nord, les seuls cadeaux de ce type dont on a connaissance concernent le tsar de Russie, le duc de Nassau et le roi des Pays-Bas. Le modèle d'arithmomètre offert variait : c'est la version à dix chiffres qui fut offerte au comte de Trapani, à la régente de Parme et au roi du Portugal, quand le roi des Deux-Siciles et le tsar de Russie eurent droit à une machine à seize chiffres.

Après la mort de Thomas de Colmar en 1870, son fils Thomas de Bojano souhaita reprendre cette pratique de cadeaux royaux. En 1872, il offrit un arithmomètre à Don Pedro II, le très éclairé empereur du Brésil, probablement lors de l'étape parisienne du long voyage que celui-ci fit en France : il semble ne pas avoir été emporté par l'empereur[11]. Nous montrerons plus loin qu'un arithmomètre fut très probablement offert au sultan du Maroc vers 1873 : la notice fut alors traduite en arabe.

Nous allons maintenant donner quelques détails sur les circonstances et la réception de l'arithmomètre dans quatre pays méditerranéens : l'Espagne, l'Italie, la Tunisie, le Maroc.

2.1 L'arithmomètre en Espagne

Après que Thomas eut offert un arithmomètre à la reine Isabelle II, la réaction fut rapide. En 1854, l'Académie des sciences de Madrid reconnut le mérite de l'inventeur, souhaitant cependant que l'instrument restât d'un prix élevé pour ne pas déshabituer la société de la pratique du calcul[12]! La même année, une revue technique destinée aux ingénieurs civils

---

[10] Jacomy-Régnier (1855), p. 99 ; Moigno (1854), p. 663.

[11] Il appartenait en 1909 à un M. Gaillard, actuaire, qui le prêta en 1920 pour une exposition de machines à calculer organisée à Paris à l'occasion du centenaire de l'invention de Thomas.

[12] Moigno (1854), pp. 660-661.

y consacra une étude enthousisaste[13]. Et en 1856 parut à Barcelone une traduction espagnole de l'*Instruction pour se servir de l'arithmomètre*[14], fidèle à l'original . Son auteur était un professeur de langues, un certain Pedro Saver auquel on doit par ailleurs un manuel d'espagnol pour francophones et un manuel de français pour hispanophones.

2.2 L'arithmomètre en Italie

Lorsque Thomas commença à commercialiser son arithmomètre, l'unité italienne n'était pas encore réalisée : aussi est-ce à une multitude de souverains qu'il en offrit un exemplaire. La machine fut brièvement chroniquée dans les années 1854-1855, mais semble avoir été perdue de vue dans les années troublées de la marche vers l'unité. Ce n'est que bien plus tard que parurent deux textes longs et importants consacrés à l'arithmomètre. Le premier, paru en 1880, est l'œuvre d'Agostino Callero, titulaire de la chaire de machines à vapeur et chemins de fer à l'École d'application des ingénieurs de Turin et président de l'Institut royal technique de Turin. Son long article sur l'invention de Thomas en détaille les principes techniques, mais aussi, ce qui nous intéressera ici davantage, les usages arithmétiques[15]. L'algorithme usuel d'extraction de la racine carrée y est présenté sur l'exemple assez étonnant de la racine de 5035061463,982204. Ce nombre formé de seize chiffres, ce qui est « supérieur au nombre 8 des coulisses de l'arithmomètre », a été « choisi à dessein pour traiter immédiatement un des cas les plus compliqués » ! C'est à l'***Instruction pour se servir de l'arithmomètre***, édition de 1873, que Callero dit emprunter un « autre procédé d'extraction », par soustractions répétées de nombres impairs – celui de Töpler-Reuleaux donc. Il le présente sur un exemple nettement plus simple que le premier : celui de √ 378225, égale exactement à 615. L'article de Callero dut être apprécié, puisqu'il fut aussitôt traduit en français[16]. Le second texte, en 1885, est l'article « machines à calculer » de l'***Enciclopedia delle arti e industria*** : très bien documenté, il consacre vingt-deux de ses quatre-vingt quinze pages à l'arithmomètre de Thomas. Son auteur, l'ingénieur Giuseppe Pastore, y décrit minutieusement la mécanique subtile de l'appareil, mais aussi les algorithmes arithmétiques qu'on y met en application[17]. Les deux méthodes précédentes d'extraction de la racine carrée sont à nouveau présentées, mais cette fois toutes les deux sur le même exemple : celui de la racine de 915348,5, demandée avec deux chiffres après la virgule. Le résultat, qui est 956,73, est plus vite obtenu avec la deuxième méthode qu'avec la première. Bien que l'article de Reuleaux apparaisse dans sa bibliographie, Pastore n'indique pas à qui est due cette deuxième méthode. Enfin, l'***Instruction*** elle-même ne fut traduite en italien qu'à l'orée du vingtième siècle[18]: sans doute l'existence des deux articles précités avait-elle conduit à penser que la traduction d'un opuscule beaucoup plus limité était inutile.

2.3 L'arithmomètre en Tunisie

[13] F. C. (1854).
[14] [Thomas de Colmar] (1856).
[15] Cavallero (1880).
[16] Cavallero (1880b).
[17] Pastore (1885).
[18] [Thomas de Colmar] (*ca*. 1900).

On n'a pas de preuve directe de la présence de l'arithmomètre de Thomas en Tunisie, mais il paraît plus que probable que son inventeur en ait offert un exemplaire au bey de Tunis en 1851. On apprend en effet dans un ouvrage tout à la gloire de Thomas[19]: « Au mois de décembre 1851, S. A. le bey de Tunis envoya à M. Thomas de Colmar son Nichan en diamants de deuxième classe, qui correspond au grade de commandeur ».

Le bey de Tunis était alors Ahmed Bey, qui gouverna de de 1837 à 1855. Formellement vassal du sultan d'Istanbul, il régnait en fait sur un pays pratiquement autonome, qu'il s'était donné pour objectif de moderniser à l'européenne, sur les modèles turc et égyptien. Dès 1840, il avait ainsi créé l'École militaire polytechnique du Bardo, dirigée par un Piémontais. Lors d'une longue visite à Paris, du 5 novembre au 31 décembre 1846, il fut frappé par les multiples applications du génie industriel et convaincu de la nécessité de développer l'industrie en Tunisie. Il est donc parfaitement logique que Thomas ait songé à lui envoyer un arithmomètre, et qu'en guise de remerciements et de félicitations, Ahmed Bey ait décidé son admission dans le *Nîshân al-iftikhâr* [Ordre de la fierté] tunisien, créé dès 1832, mais dont il fut le véritable promoteur et organisateur. Remarquons la précocité de la distinction, qui précède toutes les autres dont fut honoré Thomas. Remarquons aussi que, dans l'état de nos connaissances, le bey de Tunis est le seul dirigeant non européen auquel Thomas ait envoyé un arithmomètre. Il serait cependant étonnant qu'il n'en ait pas réservé un au sultan ottoman Abdülmecit Ier, voire au gouverneur égyptien 'Abbâs.

2.4 L'arithmomètre au Maroc

Dans la première moitié du XIXe siècle, le Maroc était un des pays les plus fermés au monde ; son retard scientifique et technique était considérable. Les sultans Muhammad IV (de 1859 à 1873) et Hasan Ier (de 1873 à 1894) tentèrent, avec un succès mitigé, une politique d'ouverture et de modernisation. Des étudiants marocains furent envoyés en Europe, et des ouvrages savants français, anglais ou italiens furent traduits[20]. C'est dans ce contexte qu'un arithmomètre fut, très probablement, offert au sultan Hasan, après son accession au trône en septembre 1873. Nous avons en effet découvert à la bibliothèque privée du roi du Maroc à Rabat, dite bibliothèque Hasaniyya, un manuscrit arabe anonyme, daté au colophon du 5 *dhû al-hijja* 1291 (13 janvier 1875), qui s'avère être une traduction libre de l'*Instruction pour se servir de l'arithmomètre*[21]. Ce manuscrit comprend onze pages de texte en écriture maghrébine, à 14 lignes à la page, et une planche dépliante avec un dessin à la main. Sur ce dessin, on reconnaît sans peine le modèle d'arithmomètre breveté en 1865 et commercialisé presque sans changement jusqu'en 1890. Sa légende, *arîtmûmîtr ay al-hisab al-miqyas*, associe une translittération du français et un essai d'explication des formants du mot : « le calcul – l'instrument de mesure ». Comme dans les éditions d'époque de l'*Instruction pour se servir de l'arithmomètre*, c'est la version de base avec totalisateur de douze chiffres qui est représentée, mais il apparaît à la lecture que le traducteur marocain a travaillé sur la version à seize chiffres. Compte tenu des adjonctions

19 Jacomy-Régnier (1855), p. 98.
20 Ageron (2013).
21 [al-Suwayrî ?] (1875)

successives à l'*Instruction* que nous avons analysées plus haut, il nous est possible de préciser que l'édition qui lui a servi de base est celle de 1868. À la fin du texte se trouve un sceau au nom de Ahmad bin 'Abdallâh. Un érudit marocain, qui avait naguère examiné ce manuscrit sans pouvoir l'identifier[22], a supposé que c'était celui de Ahmad bin 'Abdallâh al-Suwayrî (v. 1811-1902), l'inamovible grand maître de l'artillerie et ingénieur en chef des sultans Muhammad IV (de 1859 à 1873) et Hasan I^{er} (de 1873 à 1894) : nous avons pu le confirmer en retrouvant le même sceau sur une lettre de la main d'al-Suwayrî. Ce dernier jouissait au Maroc d'une réputation de bon mathématicien, avait rédigé en 1861 le premier traité marocain sur le calcul logarithmique[23], puis avait appris le français : il est très plausible que ce soit à lui que le sultan ait confié le soin d'examiner l'étrange machine qu'on venait de lui offrir et qu'il en ait lui-même traduit la notice.

Le traducteur marocain a recomposé et réorganisé le texte de l'*Instruction*, lui donnant un plan plus simple, et a éliminé ce qui, selon lui, n'avait « pas d'utilité ». Son texte commence par les formules religieuses de rigueur, avec la citation d'une parole attribuée au Prophète de l'islam qui se serait dit « envoyé aux Noirs et aux Rouges », c'est-à-dire à l'humanité entière : manière sans doute de justifier la traduction d'un ouvrage étranger « de la langue française à la noble langue arabe ». L'introduction historique à la gloire de Thomas a disparu, et le nom même de l'inventeur est absent du texte arabe, ce qui correspond à l'habitude des traducteurs de l'époque. L'introduction reprend le principe de la liste des « Noms et usage des pièces qui servent aux opérations », constituant la légende de la figure. Mais le traducteur l'a fusionnée avec ce qui, dans l'*Instruction*, était groupé à part sous le titre « Principe de la machine », ainsi que quelques précautions d'usage qui clôturaient l'opuscule.

Viennent ensuite cinq sections sur les opérations fondamentales : addition, soustraction, multiplication, division et extraction des racines carrées des entiers. Dans les quatre premières, les procédures ne sont décrites qu'abstraitement, de manière générale. Les nombreux « exemples » fournis n'ont pas pour but des les illustrer pas à pas, comme dans l'*Instruction*, mais ont plutôt le statut d'exercices dont seul le résultat final est donné. Ce sont : 954 + 786 = 1740 ; 1740 – 954 = 786 ; 154936 $\times$ 8 = 1239488 ; 10604 $\times$ 6 = 633624 ; 1306 $\times$ 524 = 684344 ; 1024 $\times$ 2050 = 2099200 ; 154936 : 8 = 19367 avec reste 0 ; 1296 : 38 = 34 avec reste 4 ; 17204 : 34 = 506 sans reste. Aucun ne venant de l'opuscule français, ils reflètent sans doute des essais du traducteur, cherchant à s'approprier l'usage de la machine.

Le cas du chapitre sur l'extraction des racines des nombres est différent. Le traducteur marocain a voulu décrire la procédure en toute généralité, ce que ni l'***Instruction***, ni les auteurs italiens n'avaient tenté ; le résultat est, de manière prévisible, assez confus. Conscient sans doute de cela, il donne ensuite un exemple, de son cru, celui de la racine carrée de 69696, cette fois calculée pas à pas. Il évoque ensuite un autre exemple, celui de 897650000, précise que c'est celui qui est dans l'original français, et considère qu'il s'agit d'un choix « bizarre » de la part de l'auteur, puisque ce nombre a neuf chiffres tandis que l'arithmomètre, avec ses huit coulisses, ne permet a priori que d'inscrire que des nombres de huit chiffres au plus. Il remarque

22 al-Manûnî (1973), p. 152-153.
23 Ageron (2016).

cependant, et s'attribue un peu naïvement le mérite de cette observation, qu'en déplaçant la platine mobile, il est possible d'inscrire le nombre en deux fois. Il ne semble pas avoir conscience que le nombre aurait pu aussi être inscrit directement au moyen des petits boutons jouxtant les lucarnes, sans utiliser coulisses et manivelle. Du point de vue didactique pourtant, sa critique n'est pas injustifiée ; curieusement, elle prend exactement le contrepied de la démarche de l'Italien Giuseppe Pastore, qui, on l'a vu, dira, dans le même contexte, avec le même matériel, avoir voulu « à dessein » donner l'exemple le plus compliqué possible : un nombre à seize chiffres ! L'auteur marocain propose ensuite deux autres exemples d'extractions de racines, ne donnant que le résultat : la racine de 2209, égale à 47 sans reste, et la racine de 41621, qui est 204 avec un reste de 5. Comme on le vérifiera, ces exemples sont directement tirés du texte français. Pourtant, ils sont sortis de leur contexte : l'*Instruction* les proposait pour illustrer la méthode par soustraction de nombres impairs – ils venaient d'ailleurs initialement, nous l'avons dit, de l'article allemand de Reuleaux. Or le traducteur marocain ne dit mot de cette méthode, manifestement désireux de rester dans la stricte et maigre tradition arithmétique enseignée au Maroc. Tout se passe donc comme s'il avait voulu s'assurer que ces deux exemples pouvaient être traités avec la méthode traditionnelle, rendant toute innovation inutile. Bien plus : les nombres décimaux étant oubliés depuis des siècles dans les pays d'Islam où ils avaient pourtant vu naissance, il ne donne de la racine de 41621 que la partie entière 204, et précise qu'il y a un reste de 5, là où l'*Instruction* calculait trois chiffres après la virgule : 204,012. Il fait de même pour un dernier exemple soumis au lecteur : la racine de 9339200 est égale à 3054 et son reste est 64. Dans ce monde de nombres entiers, les « virgules d'ivoire » fournies par le constructeur pour séparer partie entière et partie fractionnaire n'auraient-elle aucune utilité ? L'auteur marocain leur trouve non sans astuce une autre raison d'être : elles matérialisent la séparation des chiffres d'un nombre par tranches de deux, étape préalable à l'extraction de sa racine carrée.

Le manuscrit se termine par ce qu'on pourrait appeler une *fatwâ*, c'est-à-dire un avis juridique argumenté émis par un spécialiste à la demande, en général, d'une autorité qui lui soumet un problème particulier. Dans ce cas, elle ne repose pas sur une accumulation de citations d'auteurs anciens, mais sur l'examen attentif et objectif de la machine soumise. L'avis émis est entièrement positif : les musulmans qui le peuvent et le veulent pourront utiliser l'arithmomètre. Cependant, la suppression de ce qui lui a paru « sans utilité » dans le texte français en restreint implicitement l'utilisation au corpus de connaissances arithmétiques traditionnellement enseigné au Maroc. Nous ignorons quel fut l'écho de ce texte, mais doutons que le moindre arithmomètre ait jamais été vendu au Maroc.

Nous proposons maintenant une traduction et une translittération de larges extraits du manuscrit[24].

> Louange à Dieu seul (*al-ḫamd li-Llâh waḫdahu*). Que la prière et le salut de Dieu soient sur notre Seigneur Muḫammad, sa famille et ses compagnons (*wa-ṣallâ Allâh wa-sallam ʻalâ sayyidinâ Muḫammad wa-âlihi wa-ṣuḫbihi*).

[24] [al-Ṣuwayrî ?] (1875)

Louange à Dieu qui a établi la clef des sciences mathématiques (*al-hamd li-Llâh alladhî ja'ala miftâh al-'ulûm al-riyadiyyât*) : la science du calcul, indispensable au vieux comme au jeune (*'ilm al-hisâb alladhî lâ ghanâ' li-shaykh 'anhu wa-lâ shâbb*) pour fonder sur elle toutes les transactions et en tirer profit (*li-btinâ' jamî' al-mu'âmalât 'alayhi wa-l-iktisâb*). Que la prière de Dieu soit sur notre Seigneur et Maître Muhammad (*wa-sallâ Allâh 'alâ sayyidinâ wa-mawlânâ Muhammad*), qui a été envoyé pour les noirs et les rouges parmi les non-Arabes et les Arabes[25] (*al-mab'ûth li-l-aswad wa-l-ahmar min 'ajam wa-'arab*), sur sa famille, sur ses compagnons et sur ceux qui les ont suivis sur le droit chemin, gens doués de sagacité (*wa-'alâ âlihi wa-suhbihi wa-tâbi'îhim al-muhtadîn ûlî al-albâb*).

Après quoi (*wa-ba'd*) : ceci est la traduction d'une épître sur une machine à calculer (*fa-hâdhihi tarjamat risala 'alâ âla hisabiyya*) de la langue française à la noble langue arabe (*min al-lugha al-faransawiyya ilâ al-lugha al-sharîfa al-'arabiyya*). Je l'ai traduite sur le fond (*tarjamtuhâ bi-l-ma'nâ*), et j'ai procédé à quelques adaptations (*wa-tasarraftu fîhâ ba'd al-tasarruf*) : j'ai écarté ce qui ne présentait pas d'utilité (*bi an hashshaytu mâ lâ tâ'il tahtahu*), j'ai ajouté des exemples supplémentaires dans l'explication (*wa-zidtu amthila ziyâdatan fî al-îdâh*), j'ai attiré l'attention sur le manquement que constitue leur omission dans l'original (*wa-nabuhtu 'alâ nakth aghfalahâ fî al-asl*) et je me suis cantonné à ce qui est purement prise en main et maîtrise (*wa-iqtasartu 'alâ mujarrad al-tahsîl wa-l-ijâda*). Je l'ai organisée en une introduction et cinq chapitres (*wa ratabtuhâ 'alâ muqaddima wa-khamsat fusûl*) : l'introduction est sur les dessins de la machine (*al-muqaddima fî rusûm al-âla*) et les chapitres sur les cinq fondements de l'arithmétique (*wa-l-fusul fî usûl al-hisâb al-khamsa*) – l'addition, la soustraction, la multiplication, la division et les racines des nombres entiers (*al-jam' wa-l-tarh wa-l-darb wa-l-qisma wa-judhur al-a'dâd al-sahîha*).

Introduction (*al-muqaddima*) sur l'explication des dessins de la machine (*fî sharh rusûm al-âla*)

La première chose est marquée par une lettre *nûn* grecque, comme ceci : N[26] (*wa-l-awwal mu'allam bi-harf nûn yunânî hakadhâ* N). C'est une poignée ou une manivelle de cuivre (*wa-huwa miqbad aw yad min nuhâs*), terminée par une bague d'ivoire (*fî ra'sihâ halqa min al-'âj*) que tu saisis pour la faire tourner (*tamsuk bihâ li-l-idâra*) au début des opérations de calcul (*'inda al-shurû' fî a'mâl al-hisâb*) pour mettre ses rouages en mouvement (*li-tahrîk nawâ'irihâ*). Mais on la fait toujours tourner de la gauche vers la droite, pas autrement (*wa-innamâ tudâr abadan min al-shimâl li-l-yamîn lâ ghayr*). Et quand son mouvement devient difficile (*wa-matâ ta'assarat harakatuhâ*), tu abandonnes afin d'examiner l'affaire (*tatruk hatta tundhar fî amrihâ*) en regardant à

[25] L'auteur reprend ici un *hâdith* de la tradition musulmane, c'est-à-dire une parole attribuée au Prophète : *J'ai été envoyé aux Rouges et aux Noirs*. Les Rouges désignant traditionnellement les non-Arabes (*al-'ajam*) et les Noirs les Arabes, ce *hadîth* exprime l'universalité de la mission muhammadienne. En choisissant de citer ce *hadîth*, l'auteur annonce et justifie d'avance sa traduction d'un ouvrage étranger.

[26] L'auteur ignore manifestement que l'alphabet grec diffère de l'alphabet latin.

l'intérieur de la machine (*bi-l-nadhr li-dâkhil al-âla*) pour qu'il ne s'y trouve pas une avarie (*li-lâ yaqa' fîhâ fasâd*).

La seconde est dessinée avec une lettre *alif* grecque A (*al-thânî marsûm bi-harf alif yunânî* A). Ce sont des boutons de cuivre (*wa-huwa azrâr min al-nuhas*) qui sillonnent les rainures ouvertes dans la ligne A (*li-akhidd fî al-shuqûq al-maftûha fî satr alif*) : il y coulissent vers le haut et le bas (*tazluq fîhi li-l-a'lâ wa-l-asfal*) pour identifier le nombre demandé (*li-tashkhîs al-'adad al-matlûb*).

La troisième (*al-thâlitha*) : un *sîn* grec C (*sîn yunânî* C) est la marque de la ligne de seize trous (*'alam 'alâ satr al-thuqûb al-sitta 'ashr*) où apparaissent les « figures de poussière »[27] au début de l'opération (*allatî tadhhar minhâ ashkâl al-ghubârî 'inda al-shurû' fî al-'amal*), dans le dessus de la platine mobile (*fawq al-lawh al-mutaharrik*), tout contre les boutons de remise des chiffres à zéro qui font face aux trous (*wa 'alâ al-azrâr allatî bi-izâ' al-thuqûb li radd al-ghubârî sifran*).

La quatrième (*al-râbi'a*) : un *dâl* grec D (*dâl yunânî* D) est la marque de la ligne de neuf trous (*'alam 'alâ satr al-thuqûb al-tis'a*) qui est sous la ligne C de la platine supérieure (*allatî taht satr sîn min al-lawh al-a'lâ*) dans lesquels apparaissent les résultats de la division et des racines (*yadhar minhâ khawârij al-qisma wa-l-judhûr*). C'est aussi de cette ligne qu'on apprend les positions du résultat de la multiplication, de la division et des racines lorsqu'on fait glisser la platine mobile vers la gauche pour que chaque rang soit vis-à-vis de ce qui lui correspond (*wa min al-satr aydan tu'lam marâtib khârij al-darb wa-l-qisma wa-l-judhûr 'inda izlâq al-lawh al-mutaharrik jihat al-yumnâ li-muqâbala kull rutba li-nadhîratihâ*).

La cinquième (*al-khâmis*) : un *mîm* grec M (*mîm yunânî* M) est la marque de la platine supérieure mobile, nommée *iblâtîn mûbîl*, c'est-à-dire platine mobile (*'alam 'alâ al-lawh al-a'lâ al-mutaharrik al-musammâ iblâtîn mûbîl ay lawh mutaharrik*). Son mouvement se fait un peu vers le haut (*wa harakatuhu li-l-a'lâ qalîlan*) en manipulant les boules de droite et de gauche remettant à zéro les lignes D et S après le début de l'opération (*wa-tahrîk al-kuratayn al-yumnâ wa-l-yusrâ li-radd ashkâl satray dâl wa-sîn asfâran qabl al-shuru' fî al-'amal*), ceci est indispensable (*wa-lâ budd min dhâlik*), et elle se déplace vers la droite lors des opérations de multiplication, de division et de racines (*wa-yuharrak li-l-yumnâ 'inda a'mâl al-darb wa-l-qisma wa-l-judhûr*).

La sixième (*al-sâdis*) : un *wâw* grec O (*wâw yunânî* O). Une boule de bois à droite de la plaquette mobile (*kura min al-'ûd 'alâ yamîn al-lawh al-mutaharrik*) que l'on actionne pour ramener à zéro les figures de la ligne C (*tuharrak li-radd ashkâl satr sîn asfaran*).

[27] Appellation traditionnelle des chiffres arabes maghrébins, remontant au temps des algorithmes à effacement et des tablettes de sable.

La septième (*al-sâbi‘*) : un *pâ’* grec P (*pâ’ yunâniyya* P) avec la nuance de sonorité entre le F et le B (*bi-l-ishmâm bayn al-fâ’ wa-l-bâ’*)[28]. Une boule à gauche de la plaquette mobile (*kura ‘alâ yasâr al-lawh al-mutaharrik*), faisant pendant à celle de droite (*wa yamîn al-nâdhir*), pour ramener de la même façon à zéro les figures de la ligne D (*li-radd ashkâl satr dâl asfâran ka-dhâlik*). Avec ces deux excroissances, on soulève la platine mobile lorsqu'on veut la mettre en mouvement (*wa-bihâtayn al-‘uqdatayn yur-tafa‘ al-lawh al-mutaharrik ‘inda irâdat harakatihâ*).

La huitième (*al-thâmin*) : un *bâ’* grec B (*bâ’ yunâniyya* B). C'est une boule d'ivoire dans l'ouverture sur la gauche de la ligne D (*wa hiya kura min ‘âj fî fatha ‘alâ yasâr satr dâl*). On la fait coulisser vers le haut au début pour opérer la multiplication et l'addition (*tuzallaq li-l-a‘lâ ‘ind al-shurû‘ fî ‘amalihi al-darb wa-l-jam‘)* et vers le bas pour opérer une division et une soustraction (*wa-li-l-asfal fî ‘amalihi al-qisma wa-l-tarh*).

[...]

Section sur la prise des racines des nombres (*fasl fî akhdh judhûr al-a‘dâd*)

[...]

Par exemple, si on demande la racine carrée de ce nombre : 6˙96˙96˙ (*mithâluhu idhâ tuliba jidhr hâdhâ al-‘adad wa huwa* 6˙96˙96˙), inscris-le sur la ligne A, mets la boule sur la ligne de l'addition et tourne la manivelle (*fa-tushakhkhisuhu fî satr alif wa-ij‘al kurat bâ’ fî satr al-jam‘ wa-adir al-yad*) : il est alors transporté à la ligne C (*fa-yuntaqal li-satr sîn*). Ensuite, fixe dans les petits trous qui sont en face de la ligne C (*thumma athbit fî al-thuqab al-raqîqa allatî izâ’a satr sîn*) les boutons d'ivoire indiquant les positions de la racine (*azrâr al-‘âj ta‘lîman li-marâtib al-jidhr*) : tu en trouveras trois, au-dessus des trois six (*fa-tajiduhâ thalâtha fawq al-sittât al-thalâtha*). Ensuite, efface le nombre de la ligne A (*thumma umhu al-‘adad min satr alif*). Ramène la boule B vers la ligne de la soustraction (*wa-radd kurat bâ’ li-satr al-tarh*). Tire la platine M vers la droite (*wa-ukhruj al-lawh mîm yamînan*) jusqu'à ce que le dernier 6 soit vis-à-vis de la première rainure de la ligne A (*hatta takûn al-sitta al-akhîra muqâbila li-l-futha al-ûlâ li-satr alif*). Recherche sa racine : c'est 2 (*wa-ibhath ‘alâ jidhrihâ fa-yakûn 2*). Pointe sur elle le bouton dans l'ouverture (*‘allim ‘alayhâ al-zirr fî al-futha*). Tourne la manivelle deux fois (*wa-adir al-yad marratayn*) : tu vois le 6 dans la ligne C qui est revenu à 2 (*tarâ al-sitta fî satr sîn ‘âdat 2*). Double la racine 2 en 4 (*da‘‘if al-jidhr 2 bi-arb‘a*), marque-la avec le bouton dans la deuxième rainure (*‘allim ‘alayhâ al-zirr fî al-futha al-thâniya*). Rentre la platine jusqu'à ce que le bouton d'ivoire corresponde à la première rainure (*wa adkhil al-lawh hattâ yuqâbil al-zirr al-‘âjî al-futha al-ûlâ*) : 29 est alors au-dessus de 4, le double [de 2] (*fa-yakûn 29 ‘alâ ra’s 4 al-di‘f*). Divise [29] par celui-ci (*aqsim ‘alayhâ*) : le quotient correspondant est 6 (*yakûn al-khârij al-muwâfiq* 6) et c'est la racine partielle 2 (*wa huwa al-jidhr al-juz‘iyy 2*). Marque le 6 dans l'ouverture A

[28] Le son [p] n'existe pas en arabe, d'où la tentative de description phonologique de l'auteur. La lettre *pâ’* a cependant été ajoutée à l'alphabet arabe pour écrire le turc ottoman ou le persan.

(*'allim 'alâ al-sitta fî al-futha alif*) et tourne la manivelle six fois (*wa-adir al-yad sitt marrât*). Alors le nombre dans la ligne C devient comme ceci : 2096 (*fa-yasîr al-'adad fî satr sîn hakadhâ* 2096)[29]. Ensuite, double la racine 6 et fais descendre le chiffre des unités du double, 2, dans la deuxième ouverture (*thumma da"if al-jidhr 6 wa-inzil âhâd al-di'f 2 fî al-futha al-thâniya*). Ajoute ses dizaines, 1, au premier double 4, ce qui fait 5 comme double dans l'ouverture 3 (*wa-udif 'asharâtahu 1 li-l-di'f al-ûlâ 4 takûn 5 di'fan fî al-futha 3*). L'image du double vient dans la ligne A, comme ceci : 520 (*wa-ta'tî sûrat al-di'f fî satr alif hakadhâ 520*). Ensuite fais rentrer la plaquette M (*thumma idkhal al-lawh mîm*). Il vient alors le six laissé par le nombre devant l'ouverture A (*fa-ta'tî al-sitta al-bâqiyya min al-'adad amâm al-futha alif*). Cinq, le double, est vis-à-vis du nombre 20 (*wa-l-khamsa al-di'f muqâbalatan li-'adad 20*). Divise alors [20] par lui : le quotient de la division est 4 (*fa-aqsim 'alayhâ yakûn khârij al-qisma 4*). Pointe sur lui dans la première ouverture *('allim 'alayhi fî al-futha al-ûla*) et tourne la poignée de 4 tours (*wa adir al-miqbad 4 dawrât*) : la racine vient dans la ligne D et c'est 264 (*fa-ya'tî al-jidhr fî satr dâl hâdhâ 264*). Chacune de ses positions est vis-à-vis d'un des boutons d'ivoire dans la ligne S et son double est dans la ligne A comme ceci : 528 (*kull martaba minhu muqâbala li-zirr min al-azrâr al-'âjiyya fî satr sîn wa-yakûn di'fuhu fî satr alif hakadhâ 528*). Et il n'y a pas de reste parce que le nombre est de racine rationnelle (*wa lam tabqa baqiya li-kûn al-'adad muntaq al-jidhr*).

Un autre exemple est comparable à celui-ci (*wa 'alâ hâdha al-mithâl yuqâs ghayruhu*) : si on avait demandé la racine de ce nombre 897650000 (*fa-law tuliba jidhr hâdhâ al-'adad wa-huwa 897650000*) et suivi pour ce nombre la procédure précédente (*wa-tu-tubbi'a fihî al-'amal al-mutaqqadim*), sa racine aurait été comme ceci : 29960 (*la-kâna jidhruhu hakadhâ 29960*).

Remarque (*tanbîh*). Quand le nombre dont on demande la racine a plus de huit positions (*matâ kânat marâtib al-'adad al-matlûb jidhruhu akthar min thamâniya*), ce qui est le nombre des rainures de la ligne A (*allatî hiya 'iddat futhât satr alif*), alors, certes, il n'est pas possible d'y inscrire le nombre (*fa-innahu lâ yumkin tashkhîs al-'adad fîhi*). La procédure dans un tel cas *(wa-l-'amal fî mithl dhâlik*) est que tu fasses dépasser ou que tu tires la platine M vers la droite (*an tufîqa aw (?) tukhrij al-lawh mîm li-yamînan*) jusqu'à ce que la ligne C sorte d'un nombre de trous égal à celui des positions excédentaires du nombre dont on demande la racine par rapport aux huit rainures (*hattâ yakhruj min thuqabihi satr sîn minhu ka-'adad al-marâtib allatî zâda bihâ al-'adad al-matlûb jidhruhu 'alâ al-thamânî futhât*) : ces positions extérieures aux rainures de la ligne A se joignent [aux autres] pour compléter les positions du nombre dont on demande la racine (*fa-tudâf tilka al-marâtib al-khârija li-futhât satr alif takmîlan li-marâtib al-'adad al-matlûb jidhruhu*). Ensuite H [*sic*] est inscrit dans la ligne A et transporté dans la ligne C (*thumma yushakhkhas hâ' fî satr alif wa-yunqal li-satr sîn*), et les positions supplémentaires par rapport à la ligne A, qui n'ont pas été transportées à partir d'elle (*wa-l-marâtib al-zâ'ida 'alâ satr alif allatî lam tunqal minhu*), sont inscrites en tournant les boutons

[29] Les six tours de manivelle retranchent 6 fois 4600 à 29696, ce qui donne 2096.

qui sont en face jusqu'à ce qu'ils correspondent à ce qui est voulu (*tushakhkhas̱ bi-idârat al-azrâr allatî bi-izâ'ihâ ẖattâ tawâfaq al-maṯlûb*). À ce moment là, le nombre est complètement inscrit dans la ligne C (*bi-ẖîna'idhin yakmul tashkhîs al-'adad fî saṯr sîn*). Ce qui est bizarre, vu la clarté de l'original, c'est : comment se fait-il qu'il n'ait pas donné d'explications sur ce point (*wa-l-'ajab min wâḏiẖ al-as̱l kayf lam yubayyin 'alâ hâdhihi al-nukta*), alors que c'est ce même exemple qu'il a pris pour modèle (*ma' annahu maththala bi-hâdha al-mithâl nafsihi*) ? Le reste dans cet exemple est le suivant : 48400 (*wa-baqiyat hâdha al-mithâl hâdhihi 48400*) ; il est visible dans la ligne C (*turâ fî saṯr sîn*).

Si on avait demandé la racine du nombre 2209, elle aurait été ceci : 47, sans reste (*wa-law ṯuliba jidhr 'adad 2209 la-kâna hâdhâ 47 wa-lâ baqiyya lahu*). Et de même, la racine de ceci : 41621, est égale à 204, et le reste est 5, dans la ligne C (*wa-kadhâ jidhr hâdhâ 41621 yusâwî 204 wa-baqiya 5 fî saṯr sîn*). Et la racine de ce nombre : 9339200, est égale à 3054, visible dans la ligne D, et son reste est celui-ci : 64, dans la ligne C (*wa-jidhr hâdhâ al-'adad 9339200 yu'âdil 3056 yurâ fî saṯr dâl wa-baqiyatuhu hâdhihi 64 fî saṯr sîn*).

Ceci achève le contenu de l'épître de cette machine concernant les opérations fondamentales du calcul entier, fractionnaire et irrationnel (*wa hâdhâ âkhir mâ iẖtawat 'alayhi risâlat hâdhihi al-âla min us̱ûl a'mâl al-ẖisâb al-s̱ahîh al-munṯaq wa-l-mughlaq*. Qui a l'aptitude et la possibilité d'en disposer dans la science pour son besoin (*wa-man lahu malaka wa-iqtidâr 'alâ tas̱arrufihi ? fî al-'ilm li-i'wâzihi*) l'utilise dans toutes les opérations du calcul, en matière d'entiers, de fractions, de conversion, de proportionnalité, de fausse position, d'algèbre, de jurisprudence de Dieu et des musulmans, lorsqu'il le souhaite et que cela l'arrange (*isti'malahâ fî majmû' a'mâl al-ẖisâb s̱ahîhan wa-kasran wa-s̱arfan wa-tanâsuban wa-khaṯâ'an wa-jabran wa fiqhan Allâh wa-l-muslimîn lammâ yuẖibbuhu wa-yurḏâhu*). Sa traduction en arabe a été achevée le mercredi 5 du mois sacré de *dhû al-ẖijjâ* de l'année 1291 de l'Hégire du Prophète [= 13 janvier 1875], sur lui soient la prière la meilleure et la salutation la plus pure (*wa kâna al-firâgh min ta'rîbihâ yawm al-arbi'â' khâmis dhî al-ẖijja al-ẖarâm 'âm iẖdâ wa-tis'în wa-ithnay 'ashr mi'a min al-ẖijra al-nabawiyya 'alâ s̱âẖibihâ afḏal al-s̱alât wa-azkâ al-taẖiyya*).

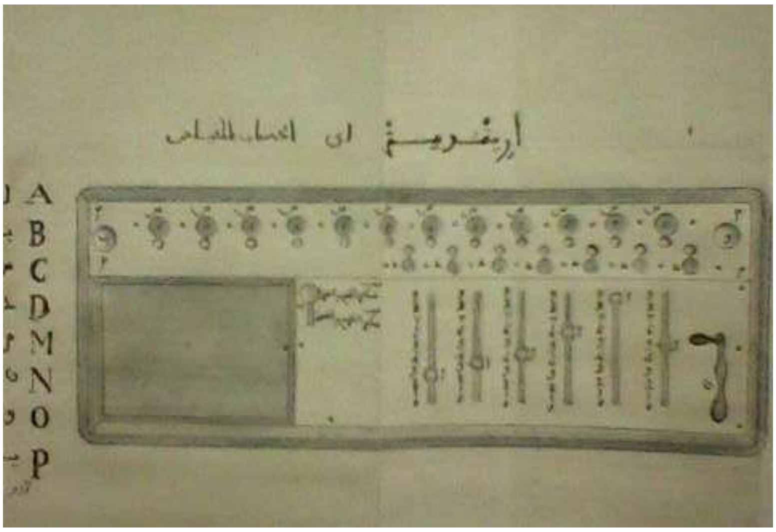


Figure 1. Dessin de l'arithmomètre dans le manuscrit 1738 de la bibliothèque Hasaniyya à Rabat.

2.5 L'arithmomètre en Syrie ?

Terminons ce tour de Méditerranée du côté est, avec un des tout derniers arithmomètres produits, conservé au Musée des arts et métiers à Paris. C'est une machine à 16 chiffres, fabriquée en 1915 ; elle était la propriété de la « Société ottomane du Chemin de Fer de Damas-Hamah et prolongements » et provient du bureau parisien de cette société, bien française malgré son nom. Ainsi, même s'il n'a vraisemblablement jamais été emmené en Orient, un arithmomètre de Thomas a joué un petit rôle dans la construction d'un réseau ferré dont la partie libanaise a disparu pendant la guerre civile (1975-1990) et dont la partie syrienne ne survivra sans doute pas à celle qui ravage le pays et condamne sa jeunesse depuis 2011.

REFERENCES

Sources (par ordre chronologique)

Le site www.arithmometre.org, créé et régulièrement enrichi par Valéry Monnier, comporte une vaste bibliothèque numérique où sont téléchargeables beaucoup des sources ci-dessous (manuscrit arabe exclu).

[Thomas de Colmar, C.] (1850, ...,1908). *Instruction pour se servir de l'arithmomètre*. Première édition : 1850, Paris : Chaix. Rééditions : 1852, Paris : Blondeau ; 1856, Paris : Blondeau ; 1860, Paris : Guérin ; 1865, Paris : Guérin ; 1868, Paris : Malteste, 1873, Paris : Malteste ; 1878, Paris : Jousset ; 1884, Paris : Jousset ; 1895, sans nom d'imprimeur ; 1902 (version allégée en 4 pages), Paris : Dupont ; 1906, Amiens : Imprimerie picarde ; 1908, Amiens : Imprimerie picarde.

F. C. (1854). Aritmómetro : Máquina para calcular, inventada por Mr Thomas, de Colmar. *Revista de obras públicas*, *II*(12), 159-160.

Terquem, O., & Gérono, C. (Ed.). (1854). Temps employé par un calculateur exercé pour faire diverses opérations arithmétiques. *Nouvelles annales de mathématiques, Journal des candidats aux écoles Polytechnique et Normale*, *XIII*, 257-259.

Moigno, F. N. M. (1854). Académie des sciences, séance du 11 décembre : Rapport sur l'arithmomètre de Thomas, etc. Cosmos. *Revue encyclopédique hebdomadaire des progrès des sciences et de leurs applications aux arts et à l'industrie, 3e année*, *5*(23), 660-663

Jacomy-Régnier (1855). *Histoire des nombres et de la numération mécanique*. Paris : Chaix.

[Thomas de Colmar, C.] (1856). *Instruccion para servirse del aritmómetro, máquina para calcular, inventada por M. Thomas de Colmar, premiado y privilegiado, traducida al español por D. Pedro Saver, profesor y traductor de idiomas.* Barcelona : Ginesta.

Reuleaux, F. (1865). Prof. Toepler's Verfahren der Wurzelausziehung mittelst der Thomas'schen Rechenmaschine. *Verhandlungen des Vereins zur Beförderung des Gewerbfleißes in Preußen*, *44*, 112-116.

Académie impériale des sciences, belles-lettres et arts de Rouen (1867). *Précis analytique des travaux de l'Académie pendant l'année 1866-67*. Rouen : Boissel.

[Thomas de Bojano, L.] (1872 ?). *Arithmomètre - Machine à calculer inventée par M. Thomas de Colmar*. Document publicitaire.

[al-Suwayrî, A. ?] (1875). *Tarjamat risâla 'alâ âla hisâbiyya,* manuscrit arabe 1738 de la bibliothèque Hasaniyya à Rabat (Maroc).

Cavallero, A. (1880). Aritmometro di Thomas, suo principio, descrizione ed uso. *Annali del Reale Instituto Tecnico Industriale E Professionale di Turino*, *9*, 93-125.

Cavallero, A. (1880b). Arithmomètre de Thomas. Principe fondamental, description et usage. *Revue universelle des mines, de la métallurgie, des travaux publics, des sciences et des arts appliqués à l'industrie*, *VIII*, 309-346.

Pastore, G. (1885). Machine da calcolare. In R. Pareto, & G. Sacheri (Dir.), *Enciclopedia delle arti e industria* (vol. 5, pp. 482-575). Torino : Unione tipografico-editrice.

[Thomas de Colmar, C.] (*ca*. 1900). *Aritmometro : macchina per calcolare*. Torino.

[Payen, L.] (1910). *Arithmomètre Payen*. Document publicitaire.

Études, sources publiées (ordre alphabétique)

Ageron, P. (2015). Des ouvrages mathématiques européens dans le Maroc du XIX^e^ siècle. In É. Barbin, & J.-L. Maltret (Dir.), *Les mathématiques méditerranéennes d'une rive et de l'autre* (pp. 247-265). Paris : Ellipses.

Ageron, P. (2016). Les savants marocains face aux mathématiques européennes : Les chemins du logarithme. In A. Bouzari (Dir.), *Actes du XI^e^ colloque maghrébin sur l'histoire des mathématiques arabes* (pp. 35-59). Alger : École normale supérieure de Kouba.

Décaillot, A.-M. (1998). L'arithméticien Édouard Lucas (1842-1891) : Théorie et instrumentation. *Revue d'histoire des mathématiques*, *4*, 191-236.

Folkerts, M. (1997). *Die älteste lateinische Schrift über das indische Rechnen nach al-Hwârizmî. Edition, Übersetzung und Kommentar.* München : Bayerische Akademie der Wissenschaften.

al-Manûnî, M. (1973). *Madhâhir yaqadhat al-Maghrib al-hadîth*. Rabat : Matba'at al-umniya.

Plantevin, F., Cargou, C., & M.-P. (2012). *Multipliez ! Instruments de calcul de la multiplication. Catalogue de l'exposition.* Brest : IREM.

L'HISTOIRE DES MATHÉMATIQUES ET DE LEUR ENSEIGNEMENT, EN CLASSE, POUR LUTTER CONTRE LES STÉRÉOTYPES DE SEXE

Using the history of mathematics and of mathematics education in the classroom to fight sex gender stereotypes

Anne BOYÉ
IREM des Pays de la Loire, Faculté des sciences et des techniques, 2 rue de la Houssinière, Nantes, France
anne.boye@neuf.fr

Nous essaierons, en prenant appui sur l'histoire, de comprendre comment se sont construits certains stéréotypes, en sciences et plus particulièrement en mathématiques. Nous essaierons de comprendre comment l'impossibilité pour les filles d'accéder à un enseignement de bon niveau en mathématiques, comment la non reconnaissance ou la méconnaissance des contributions de très grandes qualités que certaines femmes ont malgré tout apporté aux mathématiques, ont nourri la fabrique des préjugés, des idées reçues, la croyance en une certaine répartition « naturelle » des qualités intellectuelles entre les sexes. Évoquer en classe ces questions peut permettre de renouveler l'image des mathématiques aussi bien du côté des filles que des garçons. Nous nous appuierons sur des experiences menées en classe de mathématiques de l'enseignement secondaire (15-18 ans), en France, à partir de textes de mathématiciennes ou de commentaires de leurs contemporains-nes. L'atelier permettra aussi de comparer les situations dans différents pays.

We shall try, through history, to understand how gender stereotypes have been constructed in sciences and mainly in mathematics. We shall try to understand how the impossibility for girls to access a high level of teaching in maths, how the disregard of, and the failure to recognize the high quality contributions that some women have, nevertheless, brought into mathematics, have all nourished the construction of prejudices, of preconceptions, and the conviction that there is a natural separation between the intellectual qualities of females and males. To bring these questions into the classroom may allow the transformation of the image of mathematics in girls' minds and as well as in boys' minds. The workshop will be based on experiences in high school, (students age 15-18), using some women's mathematical writings. The workshop will be also an opportunity to compare the situation in different countries.

REFERENCES

Cartwright, M. L. (1970). The mathematical mind. *Applied Probability Trust*, *37*.

Cartwright, M. L. (1970). Mathematics and thinking mathematically. *American Mathematical Monthly, 77*(1), 20-28.

Case, B. A., & Leggett, A. M. (Eds.) (2005). *Complexities, women in mathematics*. Princeton: P.U.P.

Du Châtelet, E. (1735). Préface. *La fable des abeilles, de Robert Mandeville* (trad.). Manuscrit BNF, 135-136.

Dumur, J. P. (1912). Écoles secondaires de jeunes filles, Îles britanniques. *L'Enseignement Mathématique, 14*, 246-249.

Germain, S. (1896). Correspondance. In *Oeuvres philosophiques, suivies de pensées et de lettres inédites precedées d'une étude sur sa vie et ses oeuvres, par Hippolyte Stupuy*. Paris: Librairie Firmin Didot & C.
Kovalewsky, S. (1895). *Souvenirs d'enfance*. Paris: Hachette & C.
Lorey, W. (1910). Les femmes et les sciences mathématiques. *L'Enseignement Mathématique, 12*, 67-70.
Masson, R. (1910). La réorganisation de l'enseignement mathématiques dans les écoles supérieures de jeunes filles en Prusse. *L'Enseignement Mathématique, 12*, 59-67.
M asson, R. (1912). L'enseignement des jeunes filles en France. *L'Enseignement Mathématique, 14*, 65-68.
Mlodzievsky, B. (1915). L'enseignement mathématique aux écoles supérieures de femmes à Moscou. *L'Enseignement Mathématique, 17*, 365-370.
Rousseau, J. J. (1762). *Emile ou de l'éducation*. Paris: Nicolas Bonaventure Duchesne.
Smith, D. E. (1908). L'enseignement des mathématiques dans les écoles secondaires aux USA. *L'Enseignement Mathématique, 10*, 272-289.
Somerville, M. (1874). *Personal recollections, from early life to old age*. Boston: Roberts brothers.

TEACHING ARITHMETIC WITH HISTORICAL SOURCES

Mersenne's numbers

Martine BÜHLER, Anne MICHEL-PAJUS
Irem, University Paris-Diderot, Paris, France
annie.pajus@orange.fr

ABSTRACT

In France, students in the last year of high school majoring in mathematics (*Spécialité Mathématiques*) have two hours more than the others (who have 6 hours) in mathematics. For these two hours, the program contains some arithmetic, *based on problem solving* (the words in italics are taken in the official program syllabus used since September 2012). In particular, it contains *an investigation into prime numbers : primality tests, special prime numbers (Fermat , Mersenne)*. The content includes *congruences in Z* and *prime numbers*. Students have to know some theorems (Gauss, Bezout, Fermat) and master some methods: proof by induction, disjunction of cases, use of the least element of a non-empty subset of N. These are used as a "toolbox" for solving arithmetical problems. We'll show some examples of test papers.

During the workshop, which we'll organize as a "scale model" of our training sessions for teachers, we'll take the thread of the primality of Mersenne's numbers. After a quick presentation of the source texts, the participants will form reading groups, choose a text, and search which tools are used. The discussion in the workshop will focus on the use that can be made of these texts in the classroom. We'll provide few examples already experienced in the classroom.

Source texts (provided in French and in English) :

- Extract of a letter from Fermat to Mersenne ,1640
- Extract of Euler : *Theorems on residues obtained by the division of powers* (E262), 1755
- Extract of Euler : *Theorems on divisors of numbers* (E134), 1747-8
- Extract of Legendre : *Theory of numbers* , 1830

ANALYSIS OF PUPILS TASKS IN THE USE OF HISTORY IN MATHEMATICS TEACHING

Thomas DE VITTORI
Univ. Artois, EA 2462, Laboratoire de Mathématiques de Lens (LML), F-62300 Lens, France
thomas.devittori@espe-lnf.fr

ABSTRACT

After several years of video acquiring and analysis, this paper gives a synthetic view on the different tasks that appear in sessions blending history and mathematics. Based on a typology summarized by the acronym SaMaH (Specific, a-Mathematical and a-Historical tasks), this first reading grid enables the identification of the respective contributions of mathematics and history in a given school activity. This framework also makes visible the particular specific tasks in which both fields are involved. Then, the analysis of different forms of specific tasks will lead to a new a gradation between H-weak and M-weak sub-domains. This paper mainly gives a quick view on the most important theoretical results at this time only available in author's French publications.

1 Context and issues

Facing to the difficulties in interesting pupils at science and particularly at mathematics, many new pedagogical approaches have been engaged and the use of history in science teaching is one of them. Nowadays, the potential benefit of history of mathematics in secondary school teaching is not anymore a topic of debates. As for the acquiring of a humanist culture as for the fine understanding of scientific results, numerous authors have raised the interest of the historical perspective in teaching and in teacher training. In France, this movement is hardly linked to the national history of education. Through some major changes in the institutional demand since 1970 (modern mathematics, new teaching programs) and by the creation of new structures (creation and extension of the IREM, Research Institutes on Mathematics Teaching), this kind of practice, at the beginning developed by passionate teachers, tends to extend to any educator at any teaching level. Worldwide, many countries have followed the same history and lots of them have now introduced an institutional request for the use of history in science teaching and in teacher training. All local situations are slightly different but many books like History of Mathematics in Education – The ICMI Study (Fauvel & van Maanen, 2002), Recent Developments on Introducing a Historical Dimension in Mathematics Education (Katz & Tzanakis, 2011) or the last special issue of Science&Education 23-1 (2014) give a very good view on the importance of this question as a research topic. Back to the specific French context, man can notice that a massive institutional demand appears in secondary teaching programs as well as in students and teachers curricula. For instance, the teaching program of secondary school mentions at multiples levels the need of "coherence" in mathematics learning by the use "some problems based on historical elements" (Bulletin Officiel spécial n°6, 2008), or the reading of texts in order to "understand the genesis and the evolution of certain concepts (Bulletin Officiel Hors Série n°7 annexe, 2000, Bulletin Officiel spécial n°9, 2010). Each times, the aim is to give pupils a "scientific culture" in order to help them to get into the contemporary mathematics knowledge and become, for part a them, the new tomorrow scientists. In order to apply these recommendations, the teacher training program has been changed as well and mentions now the fact that any teacher should be able to "situate his topics in its history and in its epistemological issues" (Bulletin Officiel n° 30, 2013). A first exploratory study has been led during two years (june 2012 – june 2014) with the support of a regional grant ("Programmes émergents", Région Nord Pas-de-Calais, grant amount 10k€). This first program has helped in the creation of a video corpus that has enabled the elaboration of a new theoretical framework for the didactic analysis of such pedagogical practice. A large part of the results (short videos and comments) is available on the website of the program[1]. Taking into account both levels of this introduction of an historical perspective, the present paper aims to analyse the didactic effects of such a pedagogical approach in secondary school practice and their consequences for teacher training situations.

At the international level, specialists in the use of history in mathematics teaching agree on the following double observation: the literature gives lots of good examples of historical approaches but what makes them pertinent rarely goes beyond the subjective considerations of the author himself. For instance, in one of his papers in 2009 (Relime vol.12 n°1), after a large review of what had been published on this subject, the Danish researcher U.T. Jankvist explains that "the literature does, however, offers a variety of arguments on why and how to use history in mathematics education" and M-K. Siu and C. Tzanakis (proceedings ICME 10, 2004) write as well that "what is needed now are empirical investigations on the effectiveness of using history." The analysis of sessions blending history and mathematics does not easily enter the traditional didactics theoretical frameworks that have been mainly developed for a single topic. Whatever is the chosen entry point, such studies are blocked by numerous obstacles. Thus, a study on the use of history of mathematics in teaching incites to create and adapt new didactic concepts. According to the results of the previous step of the program, the analysis of sessions in which to fields are at stake requires the taking into account of each of them in order to point out the real *in situ* relationship between them. Until now, the researches on this topic have been centered either on mathematics learning (mathematics didactic studies) or on the choice of a good historical resource (epistemological and historical studies). In considering right away both fields, this research program deals mainly on their interactions during the teaching process and the way they can help pupils in understanding abstract concepts.

The main theoretical choice made in the proposed framework is to consider simultaneously history and mathematics and try to explain their interactions in specific student tasks. It should seem to be obvious that, in a session in which history and mathematics are involved, both have to be taken into account, but it is not from a theoretical point of view (Guillemette, 2011). History and mathematics are largely distinct in their epistemological bases. Thus, the teaching aims different and they could not be summarized to the application of one field to an other (de Vittori, 2012). As said previously, except in some recent works, the literature is mainly related to the analysis of one domain and not both. The most advanced project appears probably the last Mosvold article in which he tries to situate the interest of history within the mathematics teaching contents (Mosvold&al, 2014). As the analysis in this paper will lead to a theoretical framework at the student tasks level, it is complementary to these studies focused on the teacher level. Before entering into the details of such a model, it is important to say that this short paper will not deal with the way an activity blending history and mathematics has been chosen or created. The main reason is that, in France, lots of resources are available, as much as in specific publications as in a major part of primary or secondary school textbooks. Some examples will appear below.

[1] Website, http://eduhm.univ-artois.fr An identification is required: login: region-npdc, password: region-npdc

2 Tasks identification: the SaMaH typology

By the co-presence of two fields in a same session, each one of them produces its own tasks and some of them are new when man compares them to a more usual mathematics teaching situation. As a session blending history and mathematics is focused partly one this second field, it is easy to identify tasks that are quite the same in a more traditional mathematics course. For those, mathematics didactics have given lots of useful theoretical frameworks. Each of them, depending the topic, the age, the tools, etc. are totally pertinent when the research is pointing its interest only on mathematics learning. Could these mathematical tasks been ignored? Of course not and the main reason is that these elements anchor the session in a school context. As said before, the use of history in mathematics teaching is a pedagogical approach. It means that parts of the purposes are related to the usual mathematics teaching programs and they could not be suppressed. Even if mathematics are deeply linked to their history, some tasks that do not especially involve history can be identified. They will be called a-Historical tasks (aH). Lots of examples are available in literature and, within them, these tasks are obviously the easiest to identify. For instance, here is a part of the chinese ancient book *Neuf chapitres de l'art mathématiques* from which a secondary school teacher asks his students to solve the problem the way they want (Figure 1).

« Une ville dont on ne connait pas la taille est entourée d'un mur de forme carrée. Chaque côté du mur mesure un nombre entier de pas.
Au milieu de chaque côté il y a une porte. A vingt pas de la porte Nord se trouve un arbre. Si on sort par la porte Sud, qu'après quatorze pas on tourne à l'ouest, il faut faire 1 775 pas pour voir cet arbre.
On demande combien de pas fait le côté de la ville. »

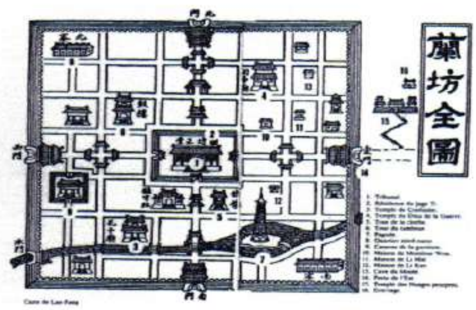


Illustration tirée du roman « les enquêtes du juge Ti : le fantôme du temple » de Robert Van Gulik, Editions 10/18.

Figure 1.

As the teacher does not give any indication, the pupils enter into the task only by the means of their previous mathematical knowledge (de Vittori, 2015). Temporarily, the historical content and context disappear in order to let the task become purely mathematical in a modern meaning. In the corpus analysed during our study, almost all activities propose some aH tasks in which pupils are mainly supposed to simply do some mathematics. For instance, in the textbook *Math 3e* (collection Prisme, Hachette, 2008, year 10 or 9th grade), two short exercises can quickly exemplify this phenomenon. In the first one (n°81 p.76), after a few words on Léon Foucault's pendulum, pupils have to calculate its period T by the means of the given formula and the length of the rope. The same way, in a second exercise (activity "Pour les curieux" p.99), pupils have to check the general formula of some Pythagorean triples by an algebraic calculus far away from the Greek mathematics. All these activities and exercises comprise a mathematical part that can be taken into account by the aH type which is one of the three categories in the SaMaH typology. The limits of such a pedagogical choice will be discussed later. Let's first have a look at the two others.

In parallel of these mathematical tasks, the history present in the session gives its own things to do. Contrary to mathematics in which objects tends to be pure abstraction, historical contents have to be situated in their real context. In order to achieve such an historical situation, some time and cultural markers should be given. For the students, these historical tasks can be done through the reading of a text, the listening of a story, a film, a webquest, and so on. There is a lot of possibilities in which the teacher implication varies but in each case, the task is only related to history and/or historical methods. As the mathematical content temporarily disappears, such tasks will be called a-mathematical task (aM). A name, a date, a biography… can be totally ignored by pupils if no work is engaged in the history field. Any task inspired by the work of professional historian helps the history to take its place in the session. One of the most common examples of such a practice is the use of a short introducing text about the author or the context. For instance, in the beginning of an activity on second degree equation, a short text about Al-Khwarizmi is given (Figure 2).

__Al Khwarizmi__ est un des premiers mathématiciens et astronomes du monde arabe.
Il vit entre 780 et 850 environ à Bagdad dans une époque très brillante.
Il fut le premier à répertorier de façon systématique des méthodes de résolution d'équations en classant celles-ci.

Figure 2.

Pupil’s task simply consists in reading the text and talking about it with the teacher. It is important to say that the role of the teacher is crucial at this time. For instance, in an activity about the Rhind mathematical papyrus (Odyssée 6e, Hatier 2014, p.101, year 7 or 6th grade), two third of the page is used by a comics which gives the story of two young Egyptians talking about one of the problems discovered in the ancient document. A short paragraph (100 words) explains where and when the Rhind papyrus has been found and, then, are given the questions. The first one (« À l’aide d’une recherche sur internet et auprès de ton professeur d’histoire, trouve quel pouvait être le pharaon au temps d’Ahmès ? ») proposes an inter-disciplinary webquest helped by the History teacher. The purpose of such a task is to clarify the epistemological and historical background of the topic. Nonetheless, without a discussion, some mistakes on the value of the historical elements shown can be done. All of this requires a little expertise from the teacher in order to make the contextualization tasks works correctly. This point is related to teacher training which have been discussed by many authors as said in the introduction of this paper.

The consideration of the first two mono-disciplinary aspects (aH and aM) of a mathematics session using history enables the identification of the coexistence of both fields (history and mathematics) but it gives only a few information on the way they interact. As mentioned in the beginning of this text, one of the main specificities of the EDU-HM program is to take into account, simultaneously, both fields in order to identify the place where they interact together. Considering this, the analysis of the corpus (about twenty sessions) has shown a third type of task which is totally specific (S) to an activity blending history and mathematics. Neither purely mathematical nor historical, these tasks create a link between both fields in which the epistemology of each one enters into an effective conjunction. Deeply analysed in a previous paper (Barrier & al, 2012), an example of an activity based on Indian

Sulbasutras in which pupils use a rope in order to draw figures in the playground shows the way the mathematical contents (student's knowledge about the circle) and the historical context (Indians use ropes) interact. It is obvious that in order to be efficient in its synthetic dimension, the S task has to anchor itself in both fields. Each one of them appears by the means of its own tasks (aH and aM) and together they make an S task possible. One can see such a situation in the following activity given at multiple levels (secondary school, age 14 and 15, university L3). The main topic in this session is the extraction of a cubic root by the means of a rule and two sets square (Figure 3. This activity is deeply explained in de Vittori, 2015).

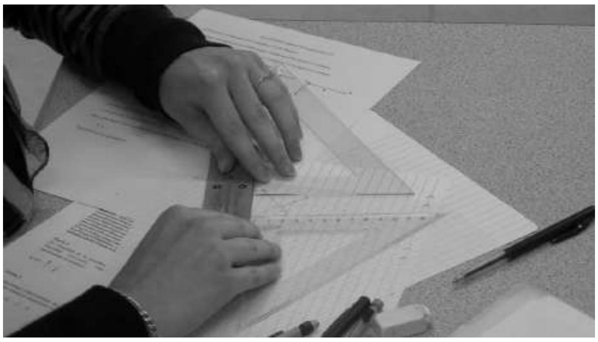
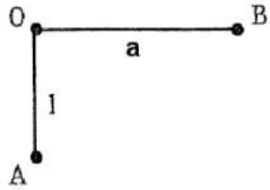


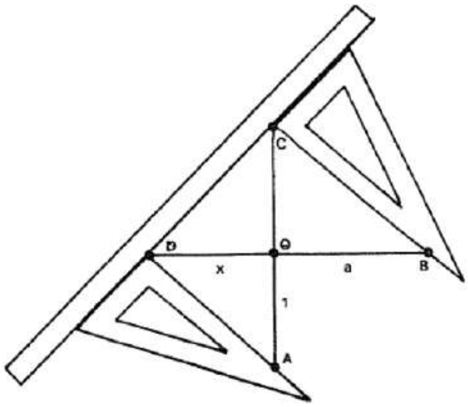
Figure 3.

Depending the level, the mathematical aim differs. In secondary school the activity is mainly related to integer powers and geometric construction programs, whereas in university it has been done in a pre-service teaching course about numbers that are geometrically constructible or not. Whatever the level, part of the session includes an aH time where the mathematics contents is involved and an aM time about the historical context. In this case, the method is a way to duplicate a cube; a very well-known class of problems since the Antiquity. What are the students supposed to do? Here is the situation (Figure 4):

Au IV^e siècle, un mathématicien grec propose la méthode suivante :

- Soit a le nombre dont on cherche la racine cubique.
- Tracer un segment [OA] de longueur 1 cm.
- Tracer un segment [OB] perpendiculaire à [OA] de longueur a.
- Placer les instruments « correctement » et on aura alors $x = \sqrt[3]{a}$.





Vérifier cette méthode en cherchant la racine cubique de 8.

Figure 4.

In order to extract the cubic root following this Eutocius', or more recently Lamy's, method (Hebert, 2004), students have to draw the segments [OA] (length 1) and [OB] (length *a*, perpendicular to the first one) and then they have to place a rule and two sets square into the good configuration. The cubic root of *a* is the length of the segment [OD]. The tools are well known but the way they are used is new. Indeed, in order to find the final position, the three tools have to be moved together by some rotations and variations of the gap. It is funny to notice that, as the use is unusual and a bit difficult, some students simply conclude that "it is impossible!". As in the activity with the rope in the playground, the co-presence of history and mathematics creates a new game on the constraints in the use of geometry tools. In such a context, the aH and the aM tasks both contribute to the justification of the S task and render actual the double anchoring.

Examples of specific tasks are at least as varied as the first two genres. Nonetheless, they generally are not part of the usual toolbox of the teacher, even if he has acquired some knowledge in history. Many times during interviews with teachers or master students, reluctance to engage in this kind of practice refers to the difficulty in conceiving the links with usual learning goals. Indeed, unless one considers them only as aH type, that is to say emptied of their historical content, the S tasks appear almost never outside this form of education. The deepest reason probably lies in the relationship between S and aM. Indeed, while some constructions or methods may seem usable beyond a historical context, the work of historians warns us against some too modern readings. For example, according to many students, an ancient figure from Euclid's time is the same as ours but, in fact, a line in the *Elements* is not this modern continuous object given in an abstract space. Similarly, in some equations, the unknown, the *thing* in medieval Arab mathematics, is not our *x* that can be multiplied by itself as many times as one wants. These epistemological distinctions are subtle and often beyond the reach of students, but they are real and they give its substance to the S tasks.

The identification of the different tasks helps clarify the issues of a session blending mathematics and history. In such an activity, part of the time is devoted to mathematics without any explicit link with history (aH), another focuses on pure historical knowledge without the use of mathematical content (aM). These two dimensions are complemented by specific tasks where the two fields explicitly join (S). In almost all sessions analysed in the EDU-HM corpus, the three aspects coexist in a whole that can be summarized by the acronym SaMaH.

3 From practise analysis to didactics engineering

Based on practice analysis, SaMaH typology allows at first to account for the richness and complexity of the various sessions studied. In the ordinary experiments and practices that have been observed, the three types of tasks are present in varying proportions. Thus, one can question the triptych S, aM, aH when an element tends to appear weak or misses. From the three types, three sectors can be defined: SaM, SaH, and aMaH (see figure 5 below).

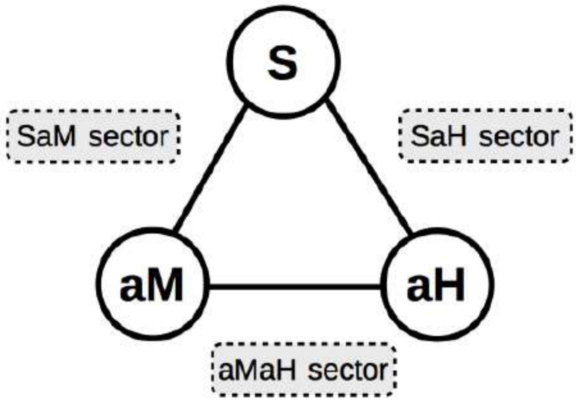


Figure 5.

What are the theoretical implications of the lack of one of the poles of SaMaH? In the case of SaM activity, the disappearance of the aH type carries with it the disappearance of the link with its anchor domain, namely mathematics. The result is an activity in which students have to deal with historical knowledge and do some specific tasks but all of this is unrelated to mathematics contents. It is therefore a disconnected session from school issues whose relevance could be justified in the context of extracurricular events (Science Festival for instance) but which cannot meet the current programs in a true educational context. Here, the S part is separated from its mathematical dimension that gives it its meaning. Moreover, if it can exist without its second pillar, the S task may be impossible to carry out due to difficulty in engaging some mathematics knowledge. Somewhat symmetrically, the SaH sector can account from another form of historically incomplete activity. In this second case, the aM pole is missing, namely the link with history. The activity is decontextualized and historical contributions are absent. As for the SaM case, the S type tasks are seriously compromised in this situation. As the advantage of the S task is to enlighten an ancient mathematical content by historical knowledge, it can be difficult or even impossible to do if the aM pole is not engaged. The last sector in this analysis regards the use of only the two poles aM and aH. There is no need to look far for examples of activities of this type because there are many situations in which historical elements are simply juxtaposed to a mathematics exercise. In these cases, when, for instance, the portrait of an author or a copy of an ancient manuscript just illustrates a chapter, the history appears as a varnish without any work. Through the juxtaposition of the two fields, the aim is to make a nice layout or to evoke an ancient culture. Without being unfounded, this approach does not enter the type of sessions effectively combining mathematics and history.

4 Developed version of the SaMaH model: the S task as a conjunction point of two epistemologies

In their modern definition history and mathematics have too little in common to create anything by simply taking place together. So how the S tasks enable a link between history and mathematics? In a recent paper (de Vittori, 2015), S tasks in an activity based on Arabic medieval mathematics have been deeply analysed in order to find a first answer to this question. Here is a quick view. In a secondary school textbook, a problem uses a text from Al-Khwarizmi in which one has to find the length of the side of a square inscribed in an isosceles triangle (base 12, side 10). The exercise proposes three questions. In the first one, using only the ancient text and the figure, pupils have to explain the meaning of the words "la chose" (thing or root) and "le bien" (wealth or property) which refer to the geometrical objects segment and square. In the second question, pupils have to solve the problem using only these historical "concepts". In those two first questions, the main point is that pupils are not supposed to use modern mathematics: they work in a specific old epistemological context. The modern mathematics reappear only in the third question in which the students have to associate a letter "l" to the "thing", calculate and find the solution another way. It is important to notice that pupils are not used to manipulate algebraic quantities (year 7, 8th grade). This activity is designed to create betterment through an historical based thinking. With S tasks, the connection is made at an *in situ* epistemological level. The use of history of mathematics in the school context justifies the extraction of an epistemological part of history in order to develop a new pedagogical approach. This one creates a local modification of the epistemology of mathematics facing the students. Qualitatively, the magnitude of this local epistemological change varies from an activity to another. Thus, an ancient text in which the word "carré" is simply written "quarré" but where everything else follows a style very conform to contemporary mathematics involves a small introduction of history in the field of mathematics. Named H-weak thereafter, this type of approach produces only a little inflexion in modern epistemology implemented by students. In a quite symmetrical way, the application of a division algorithm in which one just allows the use of modern numbers notation creates a small modification of the old epistemology underlying. Such S tasks will be called M-weak. At an extreme point, if a task does not enable any change in the manipulated concepts nor by a historical contribution, nor by modern mathematics supplements, it will refer to the aH, respectively aM, tasks mentioned above. The diagram below (Figure 6) gives the overall organization of the various sub-domains. Referring to M-weak and H-weak types, the tasks which perform a full effective implementation of a double content is called S-strong.

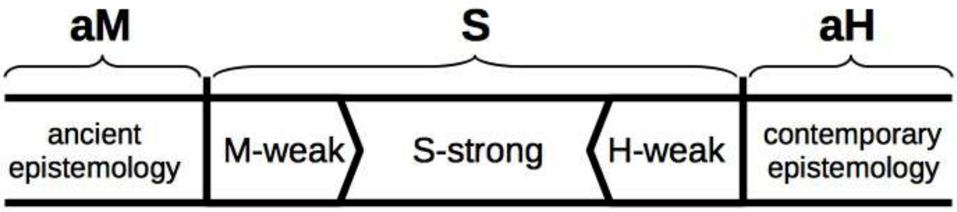


Figure 6.

At this point, one can notice that the SaMaH model enables too the localization of the potential contributions of different forms of historical research. It is clear that the local epistemological change required in S-strong tasks refers to a history of mathematics focused on the changes in the way mathematicians think about their objects. However, this does not imply the rejection of knowledge about the social development of science, on the biography of some scholars, or on the links with the political or economic history. These fields are far away from the mathematics contents, but they can predominantly appear in the aM tasks in which they take a true value in context explanation. Like mathematics, history is multiple and this quality cannot be a handicap.

5 Conclusion

The structure that has been presented here does not claim to completeness but strives to explicit the potential wealth of interactions between mathematics and history in a school context. Some examples have been given in order to illustrate certain patterns that have been observed in the corpus which has led to the development of the SaMaH model. This typology in three types (and sub-types) has been developed in order to create a framework for both the *a priori* and the *a posteriori* analysis. In an *a priori* analysis, some specific exercises or even some large textbooks collection can be explored in order to raise the most crucial points teachers should be aware of. Once some experiments have been done, the *a posteriori* use of the SaMaH framework can give a focus on the way the activity “works or not”. The main point is that the characterisation in aH, aM and S tasks helps the identification of the place where new things happen. Mathematics’ didactics is a well-recognized field in which the aH tasks can be analysed. On the other side, the aH tasks are deeply linked to the development of the research in history of science. As said previously, the S tasks, that make such pedagogical practice so special, have not been so much studied. The work on a didactic engineering specifically aiming the use of history in mathematics education is still in its beginning. The second part of the EDU-HM program mainly deals with the elaboration of activities where S tasks are clearly identified. Some experiments are already engaged and will be, hopefully, give new results soon.

REFERENCES

Barbin, É. (1991). The reading of original texts: How and why to introduce a historical perspective. *For the learning of mathematics*, *11*(2), 12-14.

Barbin, É. (1997a). Histoire et enseignement des mathématiques : Pourquoi ? Comment ?, (History and mathematics teaching: Why? How?), *Bulletin de l’AMQ* (Association Mathématique du Québec), *XXXVII*(1), 20-25.

Barbin, É. (1997b). Sur les relations entre épistémologie, histoire et didactique des mathématiques. *Repères-IREM*, *27*, 63-80.

Barbin, É., Stehlíková, N., & Tzanakis, C. (Eds.) (2008). *History and epistemology in mathematics education,*

Proceedings of the 5th European summer university ESU5.
Barrier, T., Mathé A.-C., & de Vittori, T. (2012). Des séances ordinaires comportant une dimension historique : Quels enseignements ?, *Petit x*, *90*, 5-33.
Brousseau, G. (1986). Fondements et méthodes de la didactique des mathématiques. *Recherches en didactique des mathématiques*, *7*(2), 33-115.
Brousseau, G. (1998) *Théorie des situations didactiques*. La pensée sauvage.
Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en didactique des mathématiques*, *19*(2).
De Vittori, T., & Loeuille, H. (2009) Former des enseignants à l'histoire des sciences : Analyse et enjeux d'une pratique en mathématiques. *Petit x*, *80*.
De Vittori, T. (2012). History in mathematics teaching: Current problems and new proposals. *Almagest*, *III*(1), 62-77.
De Vittori, T. (2015). Les tâches des élèves dans une activité mathématique à dimension historique. *Petit x*, *97*.
Dorier, J.-L. (2000). Recherche en histoire et en didactique des mathématiques sur l'algèbre linéaire – Perspectives théorique sur leurs interactions. *Les cahiers du laboratoire Leibniz*, *12*.
Fauvel, J., & van Maanen, J. A. (Eds.) (2000). *History in Mathematics Education – The ICMI Study*. Springer.
Guillemette, D. (2011). L'histoire dans l'enseignement des mathématiques : Sur la méthodologie de recherche. *Petit x*, *83*, 5-26.
Hebert, E. (dir.) (2004). *Les instruments scientifiques à travers l'histoire*. Ellipses.
Jankvist, U. T. (2009). On empirical research in the field of using history in mathematics education. *Revista Latinoamericana de Investigación en Matemática Educativa*, *12*(1), 67-101.
Jankvist, U. T. (2010). An empirical study of using history as a 'goal'. *Educational studies in mathematics*. *74*(1), 53-74.
Jankvist, U. T., & Kjeldsen, T. H. (2011). New avenues for history in mathematics education: Mathematical competencies and anchoring. *Science & Education*, *20*, 831-862.
Jankvist, U. T., & Møllegaard, S. (2014). 'Whys' and 'Hows' of using philosophy in mathematics education. *Science & Education*, *23*, 205-222.
Katz, V., & Tzanakis, C. (Eds.) (2011). *Recent Developments on introducing a historical dimension in mathematics education*. MAA (Mathematical Association of America).
Mosvold, R., Jakobsen A., & Jankvist, U. T., (2014). How mathematical knowledge for teaching may profit from the study of history of mathematics. *Science & Education*, *23*, 47–60.
Raichvarg, D. (1987). La didactique a-t-elle raison de s'intéresser à l'histoire des sciences ?, *Aster, 5*.
Reuter, Y. (Ed.) (2007). *Dictionnaire des concepts fondamentaux des didactiques*. de Boeck.
Rosmorduc, J. (1995). L'histoire des sciences dans la formation scientifique des maîtres de l'école élémentaire. *Didaskalia*, *7*.
Siu, M., & Tzanakis, C. (2004). History of mathematics in classroom teaching – Appetizer? Main course? Or dessert? *Mediterranean Journal for Research in Mathematics Education*, *3*(1-2), v–x. Special double issue on the role of the history of mathematics in mathematics education (proceedings from TSG 17 at ICME 10).
Tzanakis, C., & Arcavi, A. (2000). Integrating history of mathematics in the classroom: An analytic survey. In J. Fauvel, & J. van Maanen (Eds.), *History in Mathematics Education, Chapter 7. The ICMI Study* (pp. 201-240). Dordrecht: Kluwer Academic Publishers.
History, Philosophy and Mathematics Education (2014). *Science & Education*, *23*(1).
Website: EDU-HM (Études Didactiques de l'Utilisation de l'Histoire des Mathématiques en classe et en formation) : http://eduhm.univ-artois.fr.

LES FORMULES DANS LE DEVELOPPEMENT HISTORIQUE DE LA TRIGONOMETRIE. ANALYSE ET COMPARAISON DE DIFFERENTES PREUVES DE SINUS (*a*+*b*)

Abdellah EL IDRISSI
CFIE, Baba Tamesna, CP 4543, Rabat, Morocco
A_elidrissi@hotmail.com

Les formules trigonométriques ont souvent constitué une composante importante dans l'enseignement de la trigonométrie et ont également accompagné son histoire. Les preuves et les outils conceptuels alors utilisés dans les différentes preuves proposées n'échappent pas à cette règle et reflètent globalement les développements successifs qu'a connus le développement des mathématiques. A travers cet atelier, nous présenterons, analyserons et comparerons avec les participants différentes preuves « historiques », depuis les grecs à nos jours et ayant permis de justifier la formule donnant sinus (a+b). Nous aborderons, dans la limite du temps alloué, les possibilités d'exploitation des propos et des constats relevés dans l'enseignement ou dans la formation. D'ailleurs, certaines preuves seront extraites de manuels scolaires récents en usage dans différents pays. Auparavant, nous dresserons un survol rapide sur l'histoire de la trigonométrie et sur d'autres formules apparentées essentielles. Ainsi, nous évoquerons les preuves élaborées par Ptolémée, Al-Biruni, Abu-Louafa, Regiomontanus, et bien d'autres. Nous nous arrêterons également sur les preuves relativement récentes et qui font référence au calcul vectoriel, au calcul matriciel, aux transformations, aux nombres complexes, etc. Les preuves seront présentées et analysées avec les participants, ce faisant, nous veillerons à fournir les éclairages nécessaires autant au niveau des contextes culturels d'apparition des dites preuves que du point de vue des outils mathématiques utilisés. Ce qui permettra de débattre de la continuité et des échanges entre les civilisations et cultures et aussi de mettre en relief les aspects originaux de chaque preuve. Les supports qui seront investis sont essentiellement des documents historiques ou extraits de manuels, accompagnés de consignes et de commentaires.

REFERENCES

El Idrissi, A. (1998). *L'histoire des mathématiques dans la formation des enseignants: Étude exploratoire portant sur l'histoire de la trigonométrie*. Thèse de Ph.D. présentée à l'UQAM, Canada.

El Idrissi, A. (2005). L'histoire de la trigonométrie arabe: Conséquences pour l'enseignement. In E. Laabid & A. El Idrissi (Eds.), *Actes du7ème colloque Maghrébin sur l'histoire des mathématiques arabes* (Marrakech mai, 2002) (pp. 121-134). Marrakech, Maroc: Ecole Normale Supérieure.

El Idrissi, A. (2006). What is really an original source? What should be an original sources approach? In F. Furinghetti, S. Kaijser, & C. Tzanakis, (Eds.), *Proceedings HPM 2004 & ESU 4 – Revised edition* (pp. 186-188). Iraklion, Greece: University of Crete.

El Idrissi, A. (2009). Les modes d'introduction de l'histoire des mathématiques dans les manuels scolaires. In N. Bednarz & C. Mary (Eds.), *Actes du Colloque EMF2006 L'enseignement des mathématiques aux défis de l'école et des communautés* (Sherbrooke, mai 2006). Sherbrooke, Canada: Editions du CRP. récupéré de http://emf.unige.ch/files/1614/5389/0052/EMF2006_GT3_ElIdrissi.pdf

El Idrissi, A. (2011). La trigonométrie chez al-Biruni: Le théorème de la corde brisée. In *Actes du 10ème colloque maghrébin sur l'histoire des mathématiques arabe*s (Tunis, mai 2010) (pp. 153-165). Tunis, Tunisie: Edition de l'Association Tunisienne des sciences Mathématiques.

Jahnke, H. N. *et al*. (2000). The use of original sources in the mathematics classroom. In J. Fauvel, & J. Van Maanen (Eds.), *History in mathematics education: The ICMI Study* (pp. 291-328). Dordrecht: Kluwer.

PLAYING WITH INFINITY OF RÓZSA PÉTER

Problem series in a Hungarian tradition of mathematics education

Katalin GOSZTONYI
Bolyai Institute, University of Szeged, Hungary & Laboratoire de didactique André Revuz, Université Paris Diderot, France
Aradi vértanúk tere 1, 6720 Szeged, Hungary
katalin.gosztonyi@gmail.com

ABSTRACT

In this paper I present a workshop dedicated to in-service teachers about a special "Hungarian tradition" of mathematics education, focusing on Rózsa Péter's *Playing with infinity*, a book popularising mathematics (Péter 1944/1961). The workshop in question is part of a three-days teacher training we offer every year since 2012 to secondary teachers of the Parisian region; the teacher training itself is related to an interdisciplinary research project in history of sciences and to a working group for teachers.

First I will present this complex project which offers the context of my workshop. I will detail the construction of the workshop, and then I resume its content: the historical context of 20th century Hungarian mathematics education, and some elements of the analysis of Péter's text. In the conclusion, I will talk about recent perspectives of the presented workshop, namely about the work of a young French mathematics teacher inspired by the presented texts.

1 The context of the workshop: a complex project associating historical research, text edition, teacher education, and a working group for teachers

In a previous HPM paper (Bernard & Gosztonyi, 2015), we presented a complex project including historical research, text editing, in-service teacher training and a working group for teachers. Every part of this project is focused on the study of some historical texts having the form of a collection of questions and answers. As we explained in the aforementioned paper, the term "problem" has to be understood in this case in a very broad sense, as referring to any kind of verbal challenge: this includes mathematical or scientific problems in the usual sense, but also riddles (*enigmata*) or questions, in general any kind of practical, pedagogical or intellectual "task". The main originality of the research project consists in focusing not on individual 'problems' but on the principles, the characteristics and the possible roles of their collection in a certain order.

The research project entitled "Series of problems at the crossroad of cultures"1, developed with the collaboration of researchers from several, mostly Parisian research laboratories2, gathers around 15 researchers, including master and PhD students, from various disciplines: history, epistemology and anthropology of sciences and of literature, history of texts, cultural history, and educational studies. The "crossroad of cultures" can be understood here in several ways: it refers to the various origins of the examined texts (from Mesopotamia

[1] For the research blog of the project see http://problemata.hypotheses.org.
[2] In the frame of the "laboratoire d'excellence" HASTEC. See http://www.hesam.eu/labexhastec/partenaires/ (accessed 21 February 2016).

and Greek antiquity, through Middle Age and Renaissance, until 20th century's Hungary), to the diversity of scientific and professional cultures treated (from mathematics through marketing until theology), but also to the interdisciplinary character of our researches.

The project have recently led to a first collective publication, where each contributor presents the interest of studying series of problems from the point of view of his or her own research (Bernard 2015). In the following, we would like to proceed to the editing of a sourcebook about series of problems: the book would present extracts from the analysed texts with research commentaries. We also hope to add commentaries addressed to teachers (and also written by teachers in a part of the cases) which would make this book a useful source not only for research but also for education and for teacher training.

Since 2012, a three days in-service teacher training is associated to this research project that we organise every year in Paris. The public of these training sessions are in-service secondary teachers, most of them specialist of mathematics, but we also invite teachers of literature and history. The main aim of this training is not to give ready-to-use teaching material for the participants, rather to offer them cultural and historical knowledge and to stimulate reflections about their own teaching practice through the "meeting" with the texts.

Since 2014 autumn, we invite the participants of the training to continue their experience in the frame of a working group. The group, integrated in the structure of the IREM Paris Nord, basically follows the principle of the IREM workshops[3]: the collaboration of researchers and teachers at various levels (primary, secondary or university), looking for personal professional development, for the preparation of resources and the organisation of teacher trainings for colleagues. We encourage the participating teachers to develop projects answering to their own professional problematic, which projects are inspired by the texts discovered during the training session. At this moment, we follow the projects of three secondary teachers; one of them uses the documents of the Hungarian teaching tradition, presented below, to develop a problem-based, dialogical teaching practice in her classrooms.[4]

2 The construction of the in-service teacher training

The three-day teacher training is constructed by six three-hour sessions. The first one or two consist in a general introduction into the notion of "series of problems" and into the principles and the reasons of their study. The later sessions are respectively led by different contributors of the "series of problems" research project, presenting in each case one or more texts, their historical context and some results of our researches about them, susceptible to interest in-service teachers. In each case, we intend to leave enough time to reading and discussion and look for the emergence of teachers' professional reflection related to the analysed texts.

[3] "IREM" is the acronym for "Institut de Recherche sur l'Enseignement des Mathématiques". See http://www.univ-irem.fr/spip.php?article6 (accessed 21 February 2016) for a presentation (in French) and (Fauvel & van Maanen 2002, pp. 96-97), in English.

[4] For more details about the construction and the principles of the project, see (Bernard & Gosztonyi, 2015)

However it is difficult to leave enough places, in three hours, to an adequate presentation of the elements of historical context allowing the interpretation of the text, *and also* to deep discussions between participants. Therefore we experiment with different forms of organisation each year. At the beginning, we introduced each workshop by lectures, before reading collectively the texts. Later we tried to start with guided readings and furnish elements of context during the discussion. The novelty of the 2016's session is the aforementioned publication, at the end of 2015, of the first collection of articles related to the "series of problems" research project: thus, we are trying to invite the training's participants to read extracts from the related articles before each workshop of the training.

3 The workshop about *Playing with infinity* and Hungarian teaching traditions

The workshop I present every year in this frame is based on my PhD researches about the Hungarian "New Math" reform and its historical context (Gosztonyi 2015a). The Hungarian reform movement of mathematics education, led by T. Varga between 1963 and 1978, is closely related to the international New Math movement; at the same time, it is reputed in Hungary as an important representative of Hungarian traditions of mathematics education, focused on problem solving and on the discovery of mathematics. In my thesis, I compared Varga's reform to the French "mathématiques modernes" reform to show that, despite some common elements deriving from the common international context of the New Math movement, there are important differences between the two reforms. I argued that the epistemological background of the reforms is one of the key elements causing these differences: mathematicians play an important role in the conception of both reforms, but, while they represent a "bourbakian" kind of epistemology in the French case, mathematicians behind the Hungarian reform promote rather a "heuristic" epistemology of mathematics, close to the conception of Pólya or Lakatos.

The mathematicians in question wrote barely explicit philosophical text about the nature of mathematics. Nevertheless, their conception on the epistemology of mathematics can be reconstructed from their writings about mathematics education and especially from their books popularising mathematics. In the frame of the "series of problems" workshop, we read an extract from one of these books, namely the *Playing with infinity* of Rózsa Péter.

3.1 The historical context

Hungarian mathematical research culture passed through a spectacular development at the turn of the 19th and 20th century: until the last decades of the 19th century, hardly any significant mathematical research was led[5] in the country, but from this period, a number of internationally important mathematicians appear in Hungary[6]. 20th century Hungarian mathematical culture and mathematics education is reputed as focusing on problem solving and heuristic methods in mathematics.

[5] The two Bolyais, Farkas and János represent rather an exception.

[6] About the possible reasons of this progress, see (Császár, 2005; Frank 2012; Hersch & John-Steiner, 1993).

In my research, I concentrate on Tamás Varga's reform movement of the 1960's and 1970's. Varga, although participating in the international discourse about the New Math movement, emphasises similar ideas as many other Hungarian mathematicians, concerning the importance of discovery processes in mathematics education. I attempted to show that a group of Hungarian mathematicians, namely L. Kalmár, R. Péter, A. Rényi among others, exercise important influence on the conception of Varga's reform. Not only did they actively support Varga's movement, but since the 1940's Varga participated with some of them in an interdisciplinary group discussing questions of education. The young I. Lakatos, future philosopher of sciences as well as Á. Szabó, future historian of mathematics, where also related to this group[7].

3.2 The epistemological background

The above mentioned mathematicians express their ideas about the nature of mathematics and about its teaching in some books popularising mathematics and some lectures about mathematics education. An analysis of these texts shows that they represent a quite coherent conception about the epistemology of mathematics, with important consequences on teaching.

They present mathematics as a constantly developing and changing creation of the human mind, this development being guided by series of problems. According to them, the source of mathematics is intuition and experience; mathematical activity is basically dialogical and teaching mathematics is a joint activity of the students and of the teacher, the teacher acts as an aid in the students' rediscovery of mathematics. Excessive formalism is discouraged; formal language being also seen as a result of a development. They present mathematics as a creative activity closely related to playing and to the arts[8].

One of the analysed texts is Rózsa Péter's *Playing with infinity*: this is what I chose to treat more in detail during the teacher training workshop, because of its accessibility (translated in French as well as in English, easy to read) and because of the palpable presentation of a teaching situation in the book.

3.3 Playing with infinity of Rózsa Péter: a modelling in terms of series of problems

Playing with infinity (1944/1961) is a book popularising mathematics written to the non-mathematicians, especially to the people "of literature, of arts, of the humanities" (p. v), to "pass on something of the feel of mathematics", of the "joy of mathematical creation" (pp. xii-xiii). The book is written in a literary style, in a quasi continuous form almost without formulae; however, it leads the reader from the simplest mathematical notions, through different themes of high school mathematics until the theorems of Gödel and Church, important contemporary results of mathematical logic, special research domain of the author.

---

[7] For more details about the historical context of Varga's reform, including political, socio-economic, institutional and cultural aspects, see (Gosztonyi 2015a). About the relationship between Kalmár, Péter, Rényi, Lakatos, Szabó, Varga and the leader of the mentioned group, the pedagogue, psychologist and Calvinist pastor S. Karácsony, see (Gurka, 2001; Máté, 2006).

[8] For more details about this epistemological analysis, see (Gosztonyi, 2015; Gosztonyi, forthcoming).

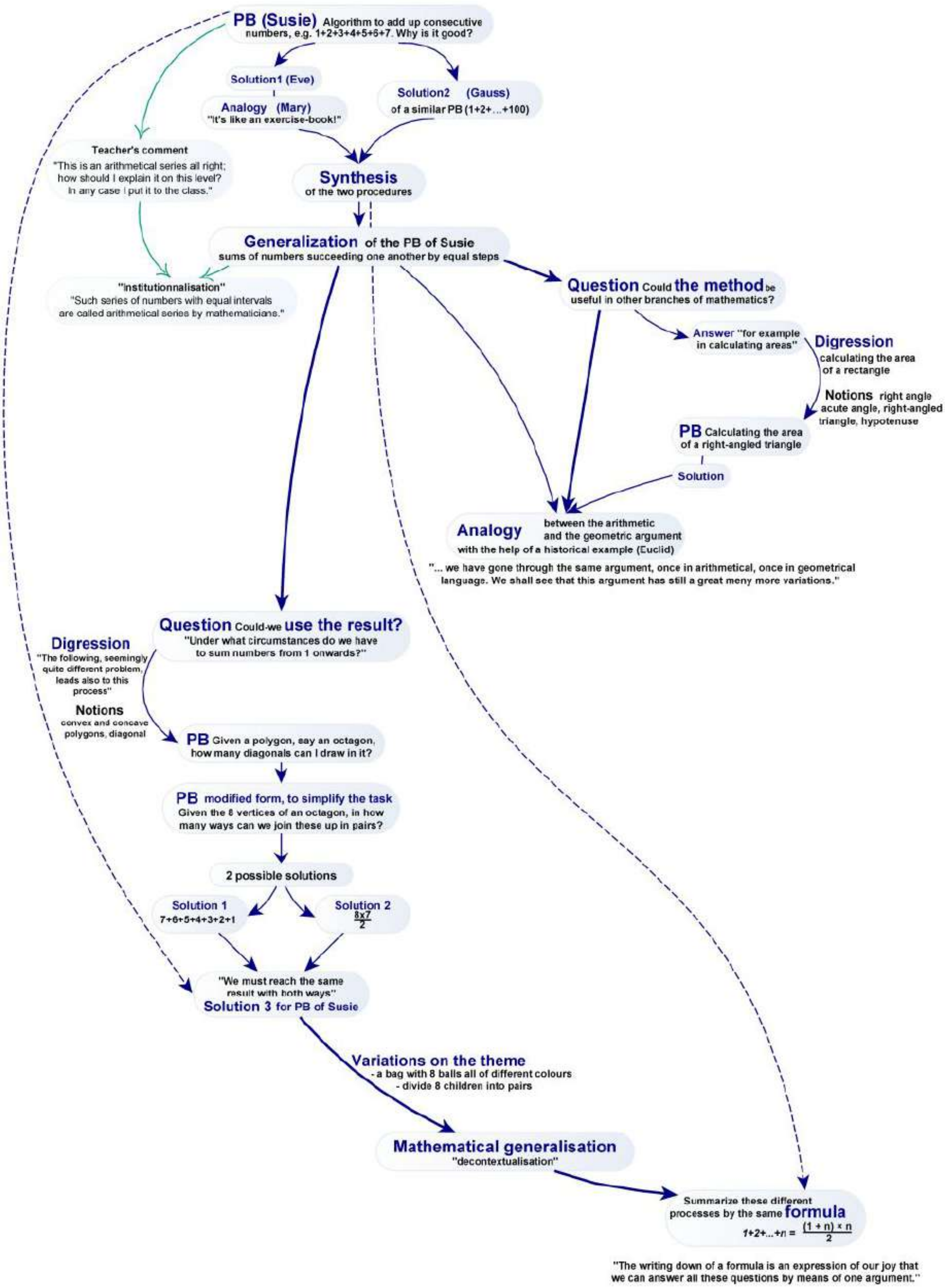


Figure 1. The modelisation of *Playing with infinity,* chapter 4 and 5, as a series of problems

During the workshop, we analyse two chapters of this book (chapter 4 and 5), describing a classroom experience of the author[9]. We look for a modelisation in terms of series of problems: I try to show that the two chapters are built on a sophistically organised series of problems, and on the dialectic of questions, attemps at answers, and new questions emerging from these answers (see Figure 1). The described classroom experience takes the form of teacher-student dialogues where the participants are partners in a collective discovery project, and the teacher plays the role of an experienced guide during this research.

The analysis of series of problems helps to understand how this discovery project is organised. Problems are related by using similar methods or similar mathematical results; new problems are created by analogies or generalisation among others. Mathematical knowledge develops progressively by the synthesis, the successive generalisations of the treated problems and their solutions. When, at the end of chapter 5, we arrive to a formula summarizing the solutions of different problems treated, the author concludes: „The writing down of a formula is an expression of our joy that we can answer all these questions by means of one argument" (p. 33).[10]

4 Conclusion

Teachers participating at our teacher training arrive with various motivations: some of them are looking for new tools, for example problems or texts that they can use in classrooms. Others are appealed to by the idea of the „crossroad of cultures": they hope to find tools to reveal the interest of their multicultural classes[11]. Many of them come to enrich their personal culture concerning history of science, or to develop an interdisciplinary approach for the teaching of mathematics. Of course, they are quite often attracted to also by the question of teaching by problems.

The workshop I propose in the frame of this training lets them discover a mathematics teaching tradition relatively close in time and space to the French one, however quite different in their characteristics. The presentation of Péter's book in its historical context and with its epistemological background leads to interesting discussions about the cultural and historical determination of mathematics education. The lecture and the analysis of the text often provokes vivid reactions: some teachers recognise in it teaching principles close to their own ideals, and find in the book a useful resource to develop a teaching practice based on problem solving.

One young teacher, a participant of the 2013/2014 session of the training continues to use this book in the frame of our working group. She tries to develop a dialogic teaching practice based on series of problems, and looks for inspiration in the Hungarian resources I presented. Our actual project concerning the teaching of Pythagoras' theorem is principally

[9] Although acknowledged researcher in mathematics, Rózsa Péter didn't obtain any academic position until the end of the Second World War and therefore she worked in a middle-school.

[10] For more details about the modelling, see (Gosztonyi, 2015b).

[11] Most of the participating teachers work in schools of the Parisian *banlieue*, with students from various social and cultural origins.

inspired by Péter's book, by Varga's textbooks and teacher's handbooks. The project is then utiliezed, not only in her classes but also during this year's session of the teacher training: she just presented her project after my workshop in April, hoping to receive useful feedback to further development, but also to inspire some more teachers to investigate in similar projects in the future.

REFERENCES

Bernard, A. (Ed.) (2015). *Les séries de problèmes, un genre au carrefour des cultures.* EDP-Sciences, SHS web of Conferences, 22. Retrieved June 20 2016 from http://www.shs-conferences.org/articles/shsconf/abs/2015/09/contents/contents.html

Bernard, A., & Gosztonyi, K. (2015). Series of problems, at the crossroad of research, pedagogy and teacher training. In E. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and epistemology of mathematics education. Proceedings of the seventh European Summer University ESU 7* (pp. 141-152). Aarhus : Aarhus University. Retrieved June 20 2016 from http://conferences.au.dk/ESU-7/.

Fauvel, J., & van Maanen, J. (Eds.) (2002). *History in Mathematics Education: An ICMI Study*. Dordrecth : Kluwer.

Gosztonyi, K. (2015a). *Traditions et réformes de l'enseignement des mathématiques à l'époque des « mathématiques modernes »: Le cas de la Hongrie et de la France.* PhD thesis. University of Szeged and University Paris Diderot.

Gosztonyi, K. (2015b). Séries de problèmes dans une tradition d'enseignement des mathématiques en Hongrie au 20e siècle. In A. Bernard (2015). Retrieved June 20 2016 from http://dx.doi.org/10.1051/shsconf/20152200013.

Gosztonyi, K. (forthcoming). Mathematical culture and mathematics education in Hungary in the XXst century. In B. Larvor (Ed.), *Mathematical Cultures*. Basel: Springer Birkhauser.

Gurka, D. (2001). Kalmár László szerepe Lakatos Imre matematikafilozófiájának alakulásában. In G. Palló (Ed.), *Recepció és kreativitás*. Retrieved June 20 2016 from http://www.phil-inst.hu/recepcio/htm/3/310_belso.htm.

Császár, Á. (2005). Education and research in mathematics. In Horváth, J. (Ed.), *A Panorama of Hungarian Mathematics in the Twentieth Century*. *Bolyai Society Mathematical Studies 14* (pp. 555-562). Budapest Berlin [etc.], János Bolyai Mathematical Society: Springer-Verlag.

Frank, T. (2012). Teaching and learning science in Hungary, 1867–1945: Schools, personalities, influences. *Science & Education, 21*(3), 355-380.

Hersch, R., & John-Steiner, V. (1993). A visit to Hungarian mathematics. *The Mathematical Intelligencer, 15*(2), 13-26.

Máté, A. (2006). Árpád Szabó and Imre Lakatos, Or the relation between history and philosophy of mathematics. *Perspectives on Science, 14*(3), 282-301.

Péter, R. (1961). *Playing with infinity* (trad. by Z. P. Dienes). New York: Dover Publications. (Originally published in 1944 as *Játék a végtelennel*. Budapest: Dante Könyvkiadó.)

SPACE, STRUCTURING, FIGURE

A prehistoric legacy

Olivier KELLER
Irem de Toulouse, France
autolycos@orange.fr

Before Euclid's *Elements* (*circa* 300 BCE), there were the Babylonian tablets and the Egyptian papyri of the first half of the second millennium before our era. But what about before that? How did the notions of figure, line, point, etc. – which are implicitly presented as self-evident in Middle Eastern documents[1] – develop and take shape? With the help of a few insights, I hope to convince the reader that certain fundamental geometric concepts underwent a veritable *gestation* period in prehistory, from the time when the human race first emerged 2.5 million years ago. Traces of this gestation survive today, in "inert" form in archaeology, and in "living" form in the ethnography of illiterate peoples.[2]

But, first of all, what might "geometry" have meant in a prehistoric context? Certainly not surveying, and still less a deductive system with definitions, axioms and propositions. Here geometry is taken to mean the general capacity to isolate, within a surrounding environment, a *space* (for example, an initial pebble, a fragment of a cave wall). From this space one then "extracts" a preconceived object, the *figure*, through a planned, methodical process, or plan of work, that is the *structuring* of this space:

Initial object	Plan of work	End product
Space	Structuring	Figure

The gestation of geometry, from the simple debitage of the early days of the *Homo* species,[3] to Euclid's *Elements*, consisted in the emergence and development of this fundamental blueprint. This means that, far from simply copying natural forms, or abstracting certain elements from them, humankind *imposed* figures on nature, starting with that most rebellious of materials: stone.

1 Geometry of the Palaeolithic stone industry

Early geometry, it seems, mainly consisted in the *debitage*[4] of pebbles or stones ("*spaces*") in order to obtain flakes with sharp edges. There is a clear distinction between a space, i.e. the

[1] See also http://www.mathunion.org/fileadmin/ICMI/docs/HPM2004Proceedings.pdf, pp. 83–98.

[2] Used correctly, the archaeological and ethnographic sources shed mutual light on one another. On the significant issues posed by this rapprochement, see Keller, O. (2004). *Aux origines de la géométrie. Le Paléolithique et le monde des chasseurs-cueilleurs* Vuibert, Chapter 1; and Keller, O. *L'invention du nombre. Des mythes de création aux* Eléments *d'Euclide*. Classiques Garnier (forthcoming), Annex B.

[3] In reality, the creators of the first tools may well have been australopiths. But it is certain that no living animal species is capable of implementing the basic schema outlined above in its entirety. Even the humble beginnings of debitage are out of reach of our cousins the chimpanzees.

[4] The term "debitage" is used when the flakes are the desired end product. If the desired tool is not the flake itself, but an altered nucleus achieved by removing flakes, the term used is "shaping".

initial pebble or chunk of rock to be worked on (known as the "nucleus"), and the tool that is used to do this, i.e. the stone hammer. This deliberate set of gestures loosely structures the raw material, evidenced by the fact that the flakes were often removed "in layers", as reassemblages have shown. The figure, in the sense of a preconceived finished product, is not the flake itself (because its form is not predetermined), but rather its – possibly retouched – edge.

The process is reversed when it comes to the fabrication of choppers, in that the edge is obtained by shaping the nucleus itself, by removing a few flakes from one or both side(s), thereby producing an edge. The removal of the flakes is itself a process of structuring, while the figure, in turn, is the part of the nucleus which is worked on.

Such was Oldowan industry[5] in the Archaic Palaeolithic period. These were the beginnings, and, though certainly humble in comparison with later developments, they nevertheless represent a fundamental qualitative rupture when compared with the capacities of other animal species.

Oldowan industry was followed, in the Lower Palaeolithic, by Acheulean industry[6], which was characterized by the *systematic shaping* of the nucleus to produce the era's famous *handaxes* (Figure 1).

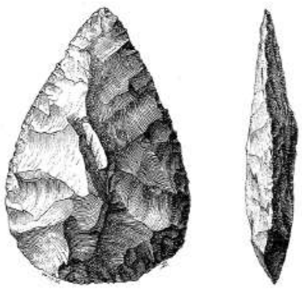


Figure 1. Handaxe found near Aurillac. Unspecified date.

The structuring consists in the removal of flakes to obtain a plane sharp edge. The volume is progressively reduced so that, when viewed in profile, it presents two symmetrical surfaces that provide a plane edge. The handaxe, when viewed front-wise, can have several possible (though probably standardized) forms, and, invariably, an axis of symmetry.[7] In the case of the handaxe – and in contrast with the figures of earlier epochs – this figure represents an initial step towards the modern-day meaning of the term, by virtue of the decisive importance of its internal relationships, namely equal magnitudes (symmetries) and definite ratios of magnitudes (possible standardized forms).

[5] From the Olduvai Gorge in Tanzania. Archaic Palaeolithic: in Africa, 2.5–1.5 million years BCE. These dates and those below should be taken only as an indicative order of magnitude.

[6] From the site at Saint-Acheul in the Somme. Lower Palaeolithic: in Africa, 1.6–400,000 million years BCE; in Europe, from 600,000 BCE onwards.

[7] For each period, I describe the typical tool in its completed form, leaving aside intermediary forms and the ongoing use of previous techniques. The production of handaxes obviously did not entail the abandonment of the fabrication of flake tools.

Oldowan industry was followed, in turn, by the Levalloisian[8] and laminar industries of the Middle and Upper Palaeolithic. These were characterized by a return to debitage, but, more precisely, to *systematic debitage*. In Levalloisian debitage, the volume of the pebble or chunk of rock is worked on layer by layer in order to progressively extract one or several flake(s) (Figure 2), possibly in a predetermined shape, such as a triangular point.

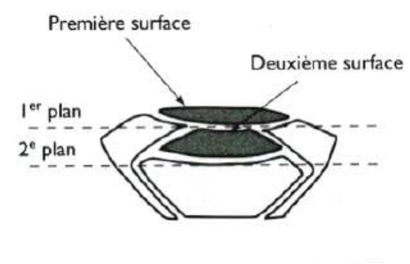


Figure 2. The concept of Levalloisian debitage.

If need be, the edge of each flake could then be retouched. In other words, the plan governing the production of a given flake was structured into three successive stages: the volume was prepared to produce a surface for debitage (in a sense, the structure supporting the flake); the flake was removed; and the edge was then retouched. Volume, *then* surface, *then* line. Furthermore, since the successive layers are parallel, it is clear that the overall plan of work reflects a *preliminary* and conscious structuration of the raw material, i.e. the initial pebble or chunk of rock, in accordance with its three dimensions. Geometrically speaking (in terms of the definition we are using here), the phenomenon is barely different from the debitage of blades during the Upper Palaeolithic and even beyond: the nucleus is prepared so that a series of blades can be extracted from it, each time with a single blow, and the edges of the blades are then retouched to produce a great variety of forms, depending on the function of the tool.

If this slow (over two million years!) appropriation of local space through work – the broad outlines of which we have described above – was certainly a form of geometry, in the sense of a spatial structuring to produce figures, it was nevertheless only an embryonic form of the future science. The vocabulary we have used (volume, surface, line, symmetry, plane, etc.) is linguistically convenient, but should not be misconstrued. In reality, it simply evokes the different states of the reworked stone in relation to physical movements. By becoming images and habits, and by separating the processes into distinct stages with a distinct product at each stage, as in the case of Levalloisian debitage, the states and movements in question were no doubt schematized in one way or another. But they were certainly not conceptualized. In the hunter-gatherer societies that produced stone artefacts, learning involved copying others and did not require an overarching vocabulary or technical description. If there was description, it was mythical, or possibly ritualistic, such as, for example, the incantations to the stone performed by certain aboriginal Australians prior to the debitage of blades. This is a far cry from the moment when surface, line, straight line, etc. would become not only ideas entirely disassociated from any material, but, above all, would be placed in abstract relation to one another (axiomatics), without reference to a technical process of any sort.

---

[8] From the Levallois-Perret quarries.

2 Geometry of cave and mobiliary art

On becoming *Homo sapiens* some 200,000 years ago, humans began to draw, paint and engrave.[9] They also sculptured statuettes and admirable bas-reliefs, and produced bodily ornaments. Here, however, we will limit our discussion to the most significant new development compared with earlier periods: two-dimensional representation, or, more exactly, what remains of this activity, given that all sorts of drawings (on the body, animal skins, bark, etc.) are obviously no longer extant.

The novelty was twofold. On the one hand, the object to be worked on is two-dimensional, i.e. a surface; on the other, as ethnography attests, the figure thereby obtained conveys thought: it is an active symbol, or sometimes even the very truth, of the original, and its function is entirely independent of its materiality as charcoal, ochre or engraving. This was a considerable revolution compared with the production of figures in the stone industry, which, in principle, were inseparable from their function as tools and therefore from their materiality. I say "in principle" because the beauty of certain handaxes (Figure 1), which show no signs of having been used, and the magnificent Solutrean leaf-shaped points, which were doubtless too fragile to be used as tools, suggest a momentary reversal of values in which the aesthetic function may have outweighed the technical function, and the pleasure of the object's form may have prevailed over the constraints of its function.

Whatever the case may be, this revolution was no doubt the culmination of a long process of maturation, for if *Homo sapiens* emerged *circa* 200,000 BCE, the first drawings (Figure 3a) did not appear until *circa* 77,000 BCE (engraved ochre pieces, Blombos Cave, South Africa) and *circa* 60,000 BCE (hatchings on ostrich eggshells, Diepkloof Rock Shelter, South Africa).

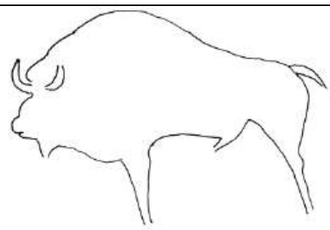
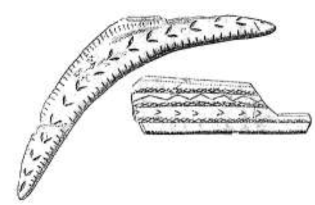
		
3a. Engraved ochre baton. Blombos, South Africa, *c*.77,000 BCE.	3b. Engraved outline: absolute profile and twisted perspective of the horns. La Grèze Cave, Dordogne, *c*.18,000 BCE.	3c. Examples of friezes. Engraved reindeer antler, Laugerie-Basse, Périgord, *c*.13,000 BCE.

Figure 3.

Homo sapiens, then, turned to surface as a new space on which to work. *Surface*. Really? How can one be sure that there was a genuine awareness of the idea of "that which has length and breadth only"?[10] To answer this question, one can first adduce numerous ethnographic

[9] A few rare traces of this phenomenon might suggest far older graphic activity. But the phenomenon becomes incontestable and massive only with the modern human, *Homo sapiens*.

[10] Euclid's *Elements*, definition 5. Of course, "length" and "breadth" should not be interpreted as measurements.

testimonies evincing the conception of the rock-face as a site of contact and transition between the real and supernatural worlds. For example:

> According to the Aboriginal Elders, the landscape was formed by the actions of Ancestral Beings […] At many locations the Beings entered directly into the landscape after their travels and creative acts, *often leaving their images behind on the rock surface*. […] In western Arnhem Land, Aborigines made seasonal camps in the shelters located at the base of the escarpment and adorned the walls and ceilings with thousands of hand or hand-and-arm stencils, monochrome and polychrome paintings. By stencilling and painting they bonded more closely with specific sites and *tapped directly into the power of Ancestral Beings represented in animal, human or mythical form*.[11]

Yet, as a site of *contact*, the rock-face clearly lacks breadth. Secondly, the idea of surface also exists in negative form, since in order to represent an object on it one has to remove one of its dimensions, hence the need for an authentic construction in the literal sense of the term. This can be a simple *section*, which gives an "absolute profile" (Figure 3b), or a *projection*, in which case one obtains a trompe-l'œil, with what appears to be a foreground and background. But the truly specific construction of prehistoric art, in my opinion, consists in the rabatment of elements that are judged to be indispensable, such as horns or the undersides of hooves. Abbé Breuil[12] called this phenomenon "twisted perspective" (Figure 3b) and, in later prehistoric periods, it can extend to "flattened perspective", that is to say technical drawing of sorts, if you will forgive me the anachronism. In practice, the various modes of construction coexisted. Lastly, and thirdly, the idea of *a surface structured in accordance with its two dimensions* emerges very clearly in mobiliary art on bone, deer antler or ivory. This art is made up of friezes (Figures 3c and 4b), that is to say a motif running along the "length" of the object, possibly with a symmetry about the perpendicular direction. It has been mathematically demonstrated that there are seven types of friezes, and all of them can be observed in prehistoric mobiliary art.

This is the evidence in favour of the existence of the idea of surface – not, as we have seen, as an empty abstraction, but rather as a motivation for predetermined constructions, and as the site of these constructions: section, projection, rabatment and structuring, in two orthogonal directions. If we remove one further dimension of space, we now come to the issue of *line*.

Let's take an easily recognizable figure, such as the representation of a bison. If it is complete, this figure structures the rock-face into two adjacent parts, the inside and the outside, and as a consequence determines a boundary between the two: a *line*. When the entire animal is painted, or when "negative handprints"[13] are used, the line is certainly present, but only in our minds – imbibed as they are with contemporary topology. Its existence is not shown with a specific mark. In contrast, when the representation is reduced to an outline (Figure 3b), the latter

---

[11] Taçon, P. S. C. The power of stone: Symbolic aspects of stone use and tool development in western Arnhem Land, Australia. *Antiquity*, 65 (1991), 192–207. Italics mine.

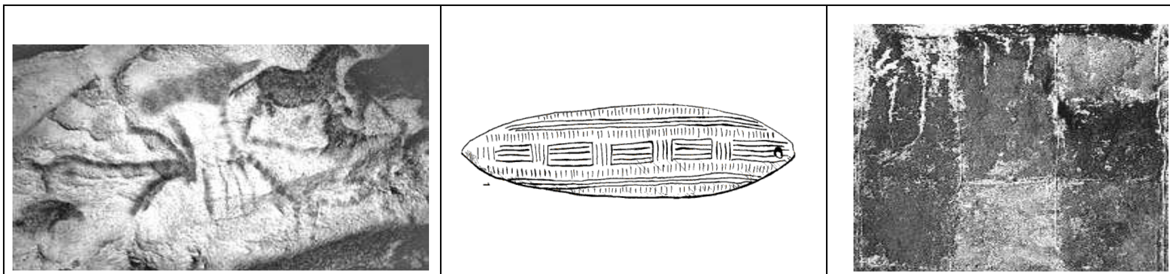
[12] Abbé Henri Breuil (1877–1961), known in his lifetime as the "pope of prehistory".

[13] Obtained by leaning one's hand against the rock-face and spit-spraying a pigment over it. These negative handprints can be found at a great many prehistoric sites around the world. In the case of modern-day hunter-gatherers, there is sometimes proof of an act of contact with the world of ancestral beings.

is, objectively, a *specific sign of line*, in that it serves as a boundary mark between two adjacent surfaces. Though it is possible to consider a fully painted representation using skilful nuances of colour and minimal trompe-l'œil as a talented copy of a spontaneous visual impression, the same cannot be said of representation using outline alone. No one ever "sees" an outline and, consequently, that outline can be only an interpretative code, an artificial sign of a boundary invented by our ancestors. We can therefore unequivocally conclude that the latter not only invented surface, but also line.

This analysis produces the following observation: we can be sure that a stroke is the sign of a line only if this stroke produces a *recognizable* outline, or fragment of an outline. But, to take the example of the strokes incised on the ochre baton from the Blombos Cave (Figure 3a), there is no reason to describe these strokes as lines, in the sense of "breadthless length" (*Elements*, definition 2). They may be nothing more than traces of meaningless entertainment, and this is perhaps also true of the very numerous meanders ("macaronis") traced by one or several fingers in the damp clay of the rock-face.

By removing one further dimension, we finally arrive at the geometric existing nothingness, "that which has not part" (*Elements*, definition 1): the *point*. Alignments of points are common in decorated caves. We talk of alignments because by following them with our eyes, we scan a line. But is the word "line" appropriate? We talk of points because these are little spots of paint made using a stamp or fingertip. But is the word "point" appropriate? I would suggest the following: since, as we have established, the complete outline of a recognizable figure is a line, a dotted outline of this same figure (Figure 4a) serves to suggest this line through a series of its "constituent parts". Yet, to my knowledge, these "constituent parts" are always the product of a *contact* with the rock-face, using either a stamp or a finger, and not dashes on the rock-face, in which case we would be dealing with fragments of line. The evidence therefore suggests that the "constituent parts" of the outline should be considered as *signs of the idea of a point* – yet another invention that we can credit to these ice-age artists.



4a. Ibex drawn with dotted lines. Lascaux Cave, *c.*15,000 BCE.	4b. Frieze of rectangles. Engraved reindeer antler, La Roche de Lalinde, Dordogne, *c.*11,000 BCE?	4c. Engraved and painted rectangles. Lascaux Cave.

Figure 4.

Surface, figure, line, point. Each of these fundamental ideas can be detected in the signs invented at least 40,000 years ago, and which, over time, would morph into the concepts of a future science. Prehistorians, for their part, think "geometry" only when they find themselves confronted with patterns whose meaning escapes us but whose form evokes classic figures: triangles, rectangles, circles, etc. This, admittedly, is a very superficial conception of geometry, yet it poses a substantial problem: did prehistoric peoples invent these classic figures? It is reasonable to use this term in at least one case, that of the rectangle (Figures 4b and 4c), and it is on this subject that we will conclude. The "reason" is the following: if we accept that, without a genuine idea of orthogonal symmetry, it would have been impossible for our *erectus* ancestors to produce handaxes; if we accept that our *sapiens* ancestors used the same idea to create friezes on bone, ivory or deer antler, and, moreover, structured the decorated surface in two perpendicular directions; if we accept all that, while also bearing in mind that symmetry can be verified by sight, by imagining a fold, then all the conditions are in place to make veritable rectangles. How? Simply imagine forming a rectangle by making successive folds in a sheet of paper of any given shape: starting from a given point, two suitable folds are all that is needed to obtain the four vertices – or rather, with four suitable folds, the fold axes produce the four sides. No theories or definitions are involved. Is it unthinkable that prehistoric peoples were able to mentally conceive of this construction ("in their head"), since all that was required was the practice of symmetry?

At the end of the Palaeolithic, geometry still had a longACKNO way to go before it would become a science. In the Neolithic, a new type of space emerged, as well as new two- and three-dimensional figures and combinations of figures.[14] In the time of the first empires, the association of number and figure, enabled by the invention of systematic measurement,[15] prompted a considerable leap forward and gave rise to the first treatises. Finally, the philosophical state of mind, in the sense of thought seeking its own premises, was at the origin of that astonishing mathematical introspection to which we owe Euclid's *Elements*.

Translated from the French by Helen Tomlinson

[14] For a detailed presentation, see http://www.apmep.fr/IMG/pdf/P2-27-compte-rendu.pdf (in French), and Keller, O. (2006). *Une archéologie de la géométrie*. Vuibert.

[15] Measurement, in the sense of a *general system* associating number and magnitude, does not exist in societies without writing. At most one finds measurements associated with very specific needs, and which are very restricted in scope and unrelated to one another.

THE INVENTION OF NUMBER

Olivier KELLER
Irem de Toulouse, France
autolycos@orange.fr

The Dao produced One; One produced Two, Two produced Three; Three produced all things.

Laozi, *Daodejing* (6th century BC?)

And the ancients, who were better than we and lived nearer the gods, handed down the tradition that all the things which are said to exist are sprung from one and many and have inherent in them the finite and the infinite.

Plato, *Philebus*

Arithmetic existed before all the others [methods] in the mind of the Creating God like some universal and exemplary plan, relying upon which as a design and archetypal example the Creator of the universe sets in order his material creations and makes them attain their proper ends.

Nicomachus of Gerasa, *Introduction to Arithmetic* (2nd century AD?)

ABSTRACT

The origin of integers 1, 2, 3 etc. is often attributed to activities of book-keeping, measurement, or goods exchange. In fact ethnographic evidence shows a lot of archaic societies inventing sophisticated numerical systems alongside poor goods exchange networks functioning without any help from these systems. The concept of integer is unthinkable without the dialectical, contradictory concept of the « one (intrinsically) many », and ethnography as well as ancient oriental mythologies show how this concept was handled by the archaic way of thinking as the main tool for modelling creative energy, and therefore the world and its genesis. But if the "one-many" accounts for the multiplicity of created things, it can't as such explain their variety; therefore it has to be determined, leading to what I call quanta, quantitative expressions of quality by one-to-one correspondences with alleged key elements of the world (cardinal directions for instance). The possibility of integers and some occasions for their actual constitution as a system emerge from that sort of mythic-ritual quantification.

During this workshop we will present and discuss some documents as evidence in favour of the following thesis: number is the culmination of a long process,[1] initiated within prehistoric societies, and it is related to a conception of the world that is specific to these societies, the broad outlines of which ethnography can reconstruct.[2] No doubt but that maths teachers and students are likely to be curious about this hypothesis of an ideological origin of number, as opposed to a purely technical one. In the paper we will focus on the main points of our thesis on the origin of numbers, having room only for a few documents ; later, in the workshop, we will have time to read and discuss more documents.

---

[1] The thesis that number is the result of a *process of invention* contrasts with the thesis of a purportedly innate "sense of number", which, it is argued, is shared by many other animal species and notably our cousins the chimpanzees. For a detailed critique of the theses of this current of thought, represented in France by Stanislas Dehaene, see www.apmep.fr/IMG/pdf/P1-28-compte-rendu.pdf (in French), access 3/05/2016.

[2] Used correctly, the archaeological and ethnographic sources shed mutual light on one another. On the significant issues posed by this rapprochement, see Keller, O. (2004). *Aux origines de la géométrie. Le Paléolithique et le monde des chasseurs-cueilleurs*. Vuibert, Chapter 1; and Keller, O. *L'invention du nombre. Des mythes de création aux* Eléments *d'Euclide*. Classiques Garnier (forthcoming), Annex 2.

One immediate problem is that of sources. When it comes to geometry, the archaeological material provides much food for thought and analysis. Alongside cave art (decorated caves) and mobiliary art (decorated objects), Palaeolithic stone tools (dating from roughly 2.5 million years to 10,000 years BCE) – which have been thoroughly indexed and analyzed, and, from a technical point of view, are well understood – have much to teach us.[3] Yet when it comes to number, things become trickier. For, though it is acceptable to infer an early conception of line from an outline of a mammoth in the Rouffignac Cave, it is an all too common error to interpret groups of various markings inscribed on prehistoric documents (notches on bones, dotted lines, parallel dashes, etc.) as numerical signs.[4] Yet ethnography is an effective safeguard in this very respect, as it shows that such signs are, at best, examples of one-to-one correspondence. And, as every teacher of mathematics knows, a bijection is not a number: you don't need to know how to count to make a dash each time you tally a ballot paper! We will see through a series of slides showing prehistoric, ethnographic and even historical documents, how false interpretations come easily when we do not know the context ; and while we are at it I think it would be interesting to discuss the reason for these spontaneous interpretations.

We are therefore reduced to the ethnographic sources. Obviously, the majority of recent or contemporary illiterate peoples knew or know numbers. But, by examining their myths and ritual practices, it is possible to identify certain fundamentals, which, precisely because they are, as such, subject to a strong inertia, reveal former "strata" of the formation of number within primitive societies. Such is our working hypothesis.

1 One-many

The most striking characteristic of the thinking of primitive peoples, hunter-gatherers or farmers without writing and state organization, is perhaps its anthropocentrism, or, more exactly, its anthropization of the world. An Australian Aborigine, for example, will be convinced that he is somehow identified with a given spot if it was there that his mother, penetrated by a "spirit-child", realized she was pregnant. *At the same time*, he will be convinced that he is Pelican if he belongs to the Pelican clan, *and* Kangaroo if the Kangaroo clan is a sub-clan of the Pelican clan, and so on and so forth. There can be up to several hundred of these characterizations within a single group.

In other words, the reality of traditional anthropocentrism is a *reconstruction* of the surrounding world by projection and by the *multiplication* of the individual or the group. Inversely, the multiplicity thereby obtained is "one" in the sense that it is a manifestation of a given individual or group.

This situation is found in sublimated form – and with sublimated human characters (demiurges, ancestors, etc.) – in the myth *par excellence* that is the myth of Genesis. If the world of familiar places, animals and objects is the multiplication of the members of the group and of

[3] See "Space, structuring, figure: A prehistoric legacy" in the present proceedings.

[4] A classic example is the famous bone discovered in Ishango in the Democratic Republic of Congo and dated to 20,000 years BCE. The bone is generally presented as the oldest arithmetical instrument in history. For a detailed critique, see https://www.bibnum.education.fr/sites/default/files/ishango-analysis_v2.pdf, access 3/05/2016.

the groups themselves, the world as a whole (as a concept) will, in one way or another, be the multiplication of the sublimated human in the person of one or several demiurge(s), ancestor(s), etc. Here again, the multiplicity thereby obtained is "one" in the sense of a manifested demiurge, but this is almost always accompanied by the effective reunification of the multiples by a return to the original unity (end of the world), which in turn is the point of departure of a new cycle (end of the world… but provisionally).

In short, the multiplication of the one and the oneness of the multiple was the only way for archaic thought to apprehend the movement (pulse) of the creative force. Admittedly, *there is as yet no notion of number here*. *But there certainly is its fundamental premise*, i.e. the multiplicity-one in the form of what remains unstructured plurality: the mere "many".

Here are a few examples to be read and discussed during the workshop. The first, particularly striking example concerns the beliefs noted by two doctors in the first half of the twentieth century among different Bantu tribes in the Kasaï region (modern-day Democratic Republic of Congo):

> Le créateur, Maweja Nangila, engendra les Esprits "par une métamorphose de sa propre personne, *en la divisant magiquement et sans qu'il en perde rien*. C'est pourquoi les Esprits participent de la nature divine de Maweja Nangila". (Fourché & Morlighem, 1973, p. 11). De même, les grands initiés d'autrefois étaient "réputés pour le pouvoir qu'ils avaient de métamorphoser leur personne, en apparaissant sous l'aspect de divers personnages, dans le même temps et dans un même ou dans différents lieux". (*id.*) C'est encore par démultiplication de sa propre personne, "sans qu'il en perde rien", que Maweja Nangila crèe : les animaux célestes, l'eau et le feu, le ciel et la terre, la lumière et les ténèbres, le soleil et la lune, les étoiles etc. (Keller, 2016, chap. 5)

This extremely widespread archaic theme contains innumerable variations, which differ in their degree of completion and rigour but no doubt account for a great deal of the charm of primitive mythologies. It runs the gamut from the Navajo "mist people" and the "neither flesh nor blood but shadow of things" of the Jicarillas[5] – fine images of original individuals reduced to the state of pure, undifferentiated multiplicities – to demiurges who decide to multiply themselves either by the force of their thought or by cutting themselves up into pieces (Vedic India, among other examples).

Among the Arunta of Australia, originally there was neither man nor woman, but rather beings called *Inapwerta*, who "presented the appearance of human beings all doubled up into a rounded mass in which just the outline of the different parts of the body could be vaguely seen" (Spencer & Gillen, 1899, p. 388). This is a way of underlining their dual nature: undifferentiated, on the one hand, but on the other, human beings *in the making*. This potentiality was realized by demiurges equipped with stone knives, who literally sculpted these amorphous *Inapwerta* one after another, thereby turning them into men and women. What's more,

[5] The Navajo and Jicarilla are Amerindians from south-western North America.

> These *Inapertwa* creatures were in reality stages in the transformation of various animals and plants into human beings, and thus they were naturally, when made into human beings, intimately associated with the particular animal or plant, as the case may be, of which they were the transformations – in other words, each individual of necessity belonged to a totem the name of which was of course that of the animal or plant of which he or she was a transformation (*id*., p. 389).

This emphasizes that all creatures, and not only humans, were originally identical, simple "ones", and that, in its way, totemism celebrates this original fraternity.

Among the Iqwaye of Papua New Guinea, one finds both the image of the world created by the metamorphosis of the body of the demiurge Omalyce,

> Omalyce's eyes […] became the sun and moon. […] Omalyce vomited semen and blood. As he was vomiting, he created all things in this world (Mimica, 1988, p. 75),

and the image of identical multiplication,

> Omalyce is himself, but he is also others. Taqalyce (my informant) explained this by means of a vivid example. He placed a bamboo stick on the palm of his left hand. The stick represented the creator, and the five fingers the five mud-men. The informant slowly revolved the stick, commenting as he did so: "Now he (i.e. Omalyce = the bamboo stick) turns his face to Neqwa (thumb, i.e. the first man), and the two are the same …" (and so on until the fifth finger). Then he stated that there are five fingers on his hand, but they are all one, their father Omalyce (*id*., p. 81).

Since the things thereby created result only from the multiplication of the same thing, they are, in principle, mere abstract plurality. This entails that ritual, as the actualization of the creative force, cannot occur without the concomitant invention of concrete signs of this abstract plurality. *Hence the invention of a great many signs denoting the repetition of the same*: graphic signs (dotted lines, notches, parallel dashes, etc.), corporal signs (dance steps, repetitive gestures, etc.), vocal signs (endless repetitions of verses or even syllables stripped of meaning), and objects (knots in a cord, bundles of sticks).

The eminence of verbal repetition in traditional rituals is well known:

> The investigation of primitive narrative as well as of poetry proves that repetition, particularly rhythmic repetition, is one of its fundamental, esthetic traits (Boas, 1955 [1927], p. 310). […] In a tradition of the Kwakiutl of Vancouver Island, the same formula is repeated forty times together with the description of the same ceremonial (*id*., p. 312).

Indeed, according to Franz Boas:

> In poetry rhythmic repetition of identical formal units are frequent. These occur in all *songs without words, consisting of vocables only* (*id*., p. 314, italics mine).

Jacques Roubaud underlines the importance of conserving « la magie de la répétition ainsi que les syllabes dites non-signifiantes qu'ethnologues et linguistes allègrement supprimaient de leurs études » (Delay & Roubaud, 1988, p. 10). For this repetition, he argues, is a "fondement" and not meaningless verbal decoration:

> Pour nous qui ne recevons de cette totalité [d'un rituel donné] qu'une description, que les mots de quelques chants (une trentaine sur trois cent vingt-quatre), ce qui frappe d'étonnement, c'est la *splendeur obsessive de la parole. Si la répétition est un des fondements de la pensée indienne*, les Navajos ont poussé son emprise à l'extrême, en donnant aux images parfois violentes et étranges de leur vision une structure rythmique extraordinairement hiérarchisée et maintenue dans l'ordre de la rigueur (*id*. p. 204, italics mine).

The "foundational" nature of repetition is entirely explicit in the mythology of Vedic India, with its cult of the famous syllable OM:

> Divinité universelle
> elle est tout à la fois
> et le toi, et le moi, et tous les êtres
> et tout ce qui existe […] (Varenne, 1981, p. 133)

As a consequence, the constant repetition of OM is a reproduction of the creative force, and thereby delivers from evil.

Multiplicity-one, ritual repetitions and signs of repetition: these are the elements we have observed up until now. Yet there is another, underlying element: since the plurality of the things created is, in essence, only the plurality of a given sign (the syllable OM, for example), there is *one-to-one mapping (bijection) between the things and the multiplicity of a given sign*. This idea spread widely in apparently secular domains. The message sticks of the Aborigines of Australia, for example, are small planks of wood marked with various groups of notches. On a single plank, one group of notches might represent specific individuals, another a list of foodstuffs, and a third the "camps" separating the sender from the recipient. One last list might take up the whole edge of one side of the plank, signifying that the recipient's community in its entirety was invited to a ceremony. These messages are apparently banal and lacking in mystery. Yet, notwithstanding the fact that the separation of the "secular" and the "sacred" was unknown in traditional societies, the message sticks are sacral objects, visible only by certain persons and subject to precise rites.

2 Quantum

Whether practised in ritual or in everyday life (two domains which are not disconnected in traditional societies), repetition is necessarily finite. It has to end somewhere. By necessity, the abstract multiplicity thus recreated is determinate, and as a consequence what emerges is a diversity of multiplicities of a same sign, such as the various groups of notches on the message stick. Contrary to what one might think, primitive thought did not spontaneously seize on this diversity of multiplicities of a same sign in order to organize them into numbers, that is to say a self-referential system independent of any concrete meaning. On the contrary: *if this quantitative diversity was acknowledged, it was taken as a sign of qualitative diversity*. This gave rise to strange, contradictory objects that are neither numbers nor qualities, but rather the products of a quantitative technique for grasping the qualitative, which I propose to call *quanta*. To distinguish these from numbers, we will call them monad, dyad, triad, tetrad, etc. A few typical examples will clarify this idea, and some others will be presented to the participants in the workshop.

A classic belief among the Amerindians is that everything "must go in fours". This is a sign of completion and harmony as a reference to the apparent movements of the sun, which govern the rhythm of animal and plant reproduction and determine the moments of the day–night cycle and the cycle of the year, and, by analogy, the moments of human life. But *it is important to recognize that what is at stake here is not the number four, but an act of bijection with the cardinal (or possibly solstitial) directions*, a quantum that I will call "tetrad". When the Hopi[6] circle their *kiva* (ceremonial building) four times, they do not, strictly speaking, count the number of circuits, but rather reproduce the completion and harmony incarnated by the tetrad,[7] that is to say, a bijection with the east, the south, the west and the north. Yet, on the other hand, among these same Hopi, everything must *also* go in sixes, in homage to the deceased ancestors, the "Six-Point-Cloud-People". Here the reference is to the cardinal points, plus the zenith and the nadir. This is further proof that these are not numerical determinations, in the sense that if "four" were the sign of completion, "six" would be an even more complete completion, which is absurd.

The Oglala Sioux refer to an impressive pantheon, which in reality proves to be a plurality of identicals:

> Every object in the world has a spirit and that spirit is *wakan*. Thus the spirit of the tree or things of that kind, while not like the spirit of man, are also *wakan*.
>
> *Wakan* comes from the *wakan* beings. These *wakan* beings are greater than mankind in the same way that mankind is greater than animals. They are never born and never die. […] There are many of these beings but all are of four kinds. The word *Wakan Tanka* means all of the *wakan* beings because they are all as if one. (Walker, 1917, p. 152).

A plurality of identicals, then, but beings who do not vary on an individual basis but rather according to a quaternary rhythm ("all are of four kinds"), in reference to a Genesis by the "Wind", who sent his four sons to the four cardinal points. This is what the calumet ritual – offered successively to the four "Winds", in the order west, north, east and south – reproduces: it makes a tetrad, a bijection with four definite things. The *Wakan* also proceed using tetrads: first Sun, Sky, Earth, Rock, and then their respective associates Moon, Wind, Beautiful Woman and the Winged. The fact that these are merely tetrads in the sense of a rhythm of deployment, and not the numbers four and eight, becomes apparent in the following extraordinary dialogue between Walker and his informer:

> Then there are eight *Wakan Tanka*, are there? No, there is but one.
>
> You have named eight and say there is but one. How can this be? That is right. I have named eight. There are four, *Wi*, *Skan*, *Inyan*, and *Maka*. These are the *Wakan Tanka*.

---

[6] South-western North America.

[7] An omnipresent leitmotif. Here is some precious Hopi advice on seduction: "D'abord, elle dira non et t'enverra promener, mais ce ne sera peut-être pas sérieux. Attends encore quatre jours et redemande lui ; elle dira non, mais elle t'engueulera moins ; quatre jours encore, elle aura peut-être l'air indécis, mais à la quatrième demande, elle dira probablement oui. Si elle reste froide, fous-lui la paix." (Talayesva, 1982, p. 390)

> You named four others, the Moon, the Wind, the Winged, and the Beautiful Woman and said they were *Wakan Tanka*, did you not? Yes. But these four are the same as the *Wakan Tanka*. The Sun and the Moon are the same, the *Skan* and the Wind are the same, the Rock and the Winged are the same, and the Earth and the Beautiful Woman are the same. These eight are only one. The shamans know how this is, but the people do not know. It is *wakan* (a mystery) (*id*., p. 154–155).

Among the Bambara of West Africa (Dieterlen & Cissé, 1972 ; Dieterlen, 1988), one of the stages in Genesis is the emergence of undifferentiated signs represented by dashes. This results in a highly important ritualistic figure to be shown to the participants in the workshop, composed solely of dashes, which is used for initiation and solstitial rites, and even has curative virtues. This figure shows (among other things) the dash representing the original monad, with three dashes for masculinity and four for femininity: the triad is in effect a sign of masculinity, because the male sexual organ has three parts, while the tetrad is a sign of femininity because a woman has four lips. Here monad, triad and tetrad are signs of qualities: when something is performed in a tetrad to signify feminity, it is not "four" that is actualized, but the female lips, and likewise for the triad. Furthermore, these signs do not imply any numerical-type relationship[8] of the kind "four is greater than three therefore woman is superior to man", or "one plus three equals four therefore woman is the fruit of the union of the original monad and man".

3 Number

If the forms of traditional thought briefly sketched out above do not as yet result in number, they nevertheless create the *fundamental conditions* that have been identified in passing: the multiplicity of identicals and the unity of this multiplicity; bijective mapping; the general organization of gestures, words and things into defined multiplicities (in classes of tetrads, for example, so that "everything goes in fours"); and, finally, the expression of these multiplicities in interchangeable and juxtaposable sign types.

To progress to number, an essential rupture is required. The multiplicities must in effect mutate and make a clean break from their function as the traces of bijections with specific referents, of which they are in a sense the soul. They must sever their qualitative ties with the external world and instead refer only to themselves, so as to form a *system*. How? By the simple but brilliant idea of the periodical return of the multiplicity to the unity, that is to say the invention of the *higher-order unit*. We will present to the participants at the workshop a few primitive number systems from different parts of the world, some of which will look rather strange to a modern reader.

Now, the main problem is: which circumstances occasion the formation of this system? Can they exist within primitive societies, before writing, surveying, accountancy and commercial exchange? It is possible to adduce certain concrete elements that are suggestive of the higher-order unit, such as the lunar month in comparison with the day and the year in comparison with the lunar month. Yet though these cycles were well known in traditional societies, there was very

[8] However, the formation of number entailed the proliferation of these kinds of relations within vast *numerology* corpora.

little concern to number them. One can also adduce the structure of the human body, with its "fives" of fingers – but this is merely a technique, not a motivation. And one can also adduce repetition, that profoundly meaningful act in traditional societies, because it naturally leads to rhythmic repetition, which, in turn, ultimately culminates in musical measure and poetical metre (with its great ritualistic importance in Vedism, for example). This is an area to be explored, but to my knowledge ritualistic elements are not found in the *techniques* used to construct number: for example, among the Amerindians, the "sacred" character of four tends to coexist with the systems in tiers of five, ten and twenty.

There is a context, I believe, in which one can observe the formation of number: the world of exchange, and, in particular, the ceremonial exchanges particular to primitive societies, which have very little to do with barter or commerce and much to do with creative multiplication. The broad outlines of the idea are as follows: within a given traditional society, there is no selling, buying or rationing.[9] Friendship and the "heart" are at the centre of things; gifts and counter-gifts are the norm (Godelier, 1996 ; and the magnificent Mauss, 2009 [1923]). But, in giving, one projects oneself; something of oneself circulates and thereby affirms its presence. This crucial phenomenon has been clearly delineated by Maurice Godelier:

> Les objets précieux qui circulent dans les échanges de dons ne peuvent le faire que parce qu'ils sont des doubles substituts, des substituts des objets sacrés et des substituts des êtres humains. […] [Ils ne circulent] pas seulement dans les potlatch, dans des échanges compétitifs de richesses contre des richesses, mais également à l'occasion des mariages, des décès, des initiations, *où ils fonctionnent comme des substituts d'êtres humains* dont ils compensent la vie (mariage) ou la mort (guerrier allié ou même ennemi mort au champ de bataille). (Godelier, 1996, p. 101, italics mine)

Thus, in multiplying the number of gifts *one also multiplies oneself, and therefore evinces power in the manner of a demiurgic creator*. Hence the highly ritualized economy of gift exchange, under the attentive surveillance of all, and in particular the competitive exchanges over the course of the famous potlatches[10], during which the gradual accumulation of gifts reflects the giver's "greatness of name".

Since what is at stake is either sealing a link through *equal* exchange, or affirming a *superiority* of power by giving more than one's partner, the ceremonial exchange naturally leads to direct pairings. These are independent of the ritualistic sense of the paired multiplicities, and in that sense lead to embryonic numerical systems. While such ceremonies take many varied forms and are particularly well documented among North Amerindians and in Papua New Guinea, the following schema provides, I think, a general idea of what they involve. Two adversaries, A and B, confront and publicly defy one another, after each has at length recalled the mythical genesis of their tribe, with festivities and feasts thrown in. Then A gives a gift, and B gives a larger gift, and so on and so forth until one of the competitors gives up. Since the aim is to publicly evoke a sense of escalation over the course of the competition – an auxiliary is on

[9] In the face of groups considered as enemies, pillage and murder were on the contrary recommended and highly valued.

[10] A potlatch is a competition of gifts.

hand for that very reason – there is a need for signs (objects, words, drawings, etc.) denoting purely quantitative accumulation, to ensure that this accumulation is easily discernible at every stage. Yet this cannot be achieved by simply recording each gift with the simple repetition of the same sign: beyond a certain point, perception becomes impossible, hence the invention of the principle of the higher-order unit. The larger units are frequently announced out loud during the tally and symbolized by objects, such as two stones for ten pairs, followed by a stick with five lines when ten stones have been collected.[11]

With this seemingly very modest first step, a decisive advance has been accomplished, since with higher-order units, the multiplicities as such are expressed in relation to the others, and therefore form a *system*. The progress made can be summarized by recalling the first definitions of Book VII of Euclid's *Elements*:

> Def 1. A unit is that by virtue of which each of the things that exist is called one.
>
> Def 2. A number is a multitude composed of units. (Euclid, 1956 [around 300 BCE], vol 2, p. 277)

But *that* "by virtue of which each thing is called one" is exactly the same for each thing; it is the same "virtue", and not two distinct "virtues", which makes me say that this tree, or this forest, is "one". As a consequence, the unit is unique: how could there be a multitude of units? *In reality, what the second definition sets out is not number, but the fundamental contradiction on which it is founded*: the unicity and plurality of "one", the One-Many (§ 1 of this paper).

It is in the third definition that Euclid moves on to what is truly distinctive about number and to that which, in turn, makes arithmetic possible – namely the internal relationship of the multiplicities as such:

> Def 3. A number is a part of a number, the less of the greater, when it measures the greater.
>
> Def 4. But parts when it does not measure it. (*Id.*)

In the above example of the potlatch, the stone (ten) "measures" the stick (one hundred), but if for instance in the end the total amounts to two sticks and one stone, the stick does not measure that total: it is "parts" of it.

Translated from the French by Helen Tomlinson

REFERENCES

Boas, F. (1897). The Social Organization and the Secret Societies of the Kwakiutl Indians. In *Report of the U.S. National Museum for 1895* (pp. 311-737). Washington: Government Printing Office.

Boas, F. (1955) [1927]. *Primitive art*. New York: Dover Publications.

Delay, F., & Roubaud, J. (1988). *Partition rouge. Poèmes et chants des Indiens d'Amérique du Nord*. Paris: Seuil.

Dieterlen, G. (1988). *Essai sur la religion Bambara*. Bruxelles: Éditions de l'université de Bruxelles.

[11] Example taken from a Kwakiutl potlatch cited in (Boas, 1895).

Dieterlen, G., & Cissé, Y. (1972). *Les fondements de la société initiatique du Komo*. Paris: Mouton.

Euclid. (1956). *The thirteen books of the Elements, translated with introduction and commentary by Sir Thomas Heath*. New York: Dover.

Fourché, T., & Morlighem, H. (1973). *Une bible noire*. Brussels: Max Arnold.

Godelier, M. (1996). *L'énigme du don*. Paris: Fayard.

Keller, O. (2004). *Aux origines de la géométrie. Le Paléolithique et le monde des chasseurs-cueilleurs*. Paris: Vuibert.

Keller, O. (2014). *Existe-t-il un sens du nombre ?* Retrieved from www.apmep.fr/IMG/pdf/P1-28-compte-rendu.pdf.

Keller, O. (2015). *The fables of Ishango, or the irresistible temptation of mathematical fiction*. Translated from the French (2010) by Helen Tomlinson. Retrieved from https://www.bibnum.education.fr/sites/default/files/ishango-analysis.

Keller, O. (2016, forthcoming). *L'invention du nombre. Des mythes de création aux* Éléments *d'Euclide*. Paris: Classiques Garnier.

Mauss, M. (2009) [1923]. Essai sur le don. Forme et raison de l'échange dans les sociétés archaïques. In M. Mauss*, Sociologie et anthropologie* (pp. 145-279). Paris: PUF.

Mimica, J. (1988). *Intimations of infinity. The mythopoeia of the Iqwaye counting system and number.* Oxford: Berg.

Spencer, B., & Gillen, F. J. (1899). *The native tribes of Central Australia*. London: Macmillan and Co.

Talayesva, D. C. (1982). *Soleil Hopi*. Paris: Plon.

Varenne, J. (1981). Bahvricha Upanishad. In *Sept Upanishads, traduction commentée précédée d'une introduction aux Upanishads par Jean Varenne* (pp. 131-134). Paris: Seuil.

Walker, J. R. (1917). The sun dance and other ceremonies of the Oglala division of the Teton Dakota. In *Anthropological Papers of the American Museum of Natural History, 16*(2). New York: Published by order of the Trustees.

HOW CAN 64=65? WHY IS THE SERIAL NUMBER PRINTED TWICE ON CURRENCY?

An Exploration of Vanishing Area Puzzles from Lewis Carroll, Martin Gardner, and Others

Stuart MOSKOWITZ
Humboldt State University, Arcata, California, United States
stuart@humboldt.edu

ABSTRACT

If a mechanical puzzle is difficult to solve, the problem solver will try multiple strategies until a solution is found. This is exactly the skill we want for our students. Our focus will be vanishing area puzzles. We will trace their history back 500 years to Serlio's Architettura. We will learn how, in the late 1800's, the great puzzlemaker Sam Loyd popularized them and author (and mathematician) Lewis Carroll studied them. We'll bring them into the 20th century with Martin Gardner, and we'll see how the Internet has modernized them for the 21st century. The variety of designs appeals to everyone from third graders and elementary teachers, to college students and faculty. They are easy to reproduce, but difficult to figure out, yet explainable with fraction arithmetic, beginning algebra, and geometry. In our workshop we will integrate history with mathematics education as we cut out and then puzzle over how an 8 by 8 square can be rearranged into a 5 by 13 rectangle, how leprechauns and rabbits can vanish right in front of us, and how nine $100 bills can be cut up and turned into ten $100 bills (don't get too excited; we'll also learn how this form of counterfeiting was stopped). We'll make a Martin Gardner Dollar Bill vanish and we'll visit Lewis Carroll's Cheshire Cat. As time allows, we will explore other puzzles from Lewis Carroll. We'll use scissors and abstract reasoning to keep you engaged and puzzled! Scissors will be provided; you bring the abstract reasoning.

UN COURS *MUSIQUE ET MATHÉMATIQUES*

Stefan NEUWIRTH
Laboratoire de mathématiques de Besançon, Université de Franche-Comté,
25030 Besançon CEDEX, France.
`stefan.neuwirth@univ-fcomte.fr`

RÉSUMÉ

Cet atelier propose de rendre compte d'un cours de la troisième année de la licence de mathématiques de l'université de Franche-Comté, *Musique et mathématiques*, organisé en 2015-2016 et 2016-2017 en collaboration avec un musicien, Olivier Toulemonde, deux mathématiciens, Aurélien Galateau et Martin Meyer, une chargée de projets culturels, Lucie Vidal, et une secrétaire, Mahdya Debayle.

Notre projet se concentre sur la musique improvisée, expérimentale et contemporaine. Ce choix rend compte de nos gouts musicaux, mais découle aussi du souci d'aller au-delà du contenu mathématique de l'acoustique, de l'harmonie et du rythme.

Le cours est construit sur des rencontres avec des musicien.ne.s et compositeur.e.s, suivies d'un concert financé dans le cadre d'un projet du service Sciences, arts, culture de l'université. Le programme est disponible sur `http://epiphymaths.univ-fcomte.fr/musique&maths` et une trace numérique du cours se trouve sur `http://moodle.univ-fcomte.fr/course/view.php?id=7321`.

Dans cet atelier, je voudrais rendre compte de cette expérience pédagogique où la table ronde fait concurrence au cours magistral, où la perception de la musique s'approfondit par un état d'esprit scientifique, et où l'activité mathématique retrouve sa nature de création artistique. Ce cours permet d'articuler des problématiques très profondes grâce à la juxtaposition de deux domaines de recherche éloignés.

Je proposerai d'écouter quelques extraits musicaux pour faire émerger les questionnements de notre cours : les langages mathématiques et les langages musicaux, la liberté de la création, la précision et l'exactitude, le déterminisme, l'espace et le temps, l'émotion mathématique, la matérialité du son.

Les considérations historiques se révèlent très utiles dans cette entreprise : le développement des symboliques mathématiques est mis en parallèle avec le développement des formes musicales ; le développement de l'algèbre et l'arithmétisation des grandeurs illustrent les concepts de créativité et de liberté ; l'histoire des mathématiques nous a aussi aidé.e.s à réfléchir à deux thèmes particuliers : l'erreur et le détournement des outils.

Je montrerai aussi comment la littérature permet de jeter un pont entre musique et mathématiques.

1 Introduction

Notre projet a consisté en un cours hebdomadaire selon une des deux modalités suivantes :

- une rencontre avec des compositeur.e.s et des musicien.ne.s centrée sur leur démarche, suivie d'un concert : Jonas Kocher et Olivier Toulemonde, Tom Johnson, Tony Di Napoli, Julia Eckhardt et Dafne Vicente-Sandoval, Andreas Stauder ;
- des exposés des étudiant.e.s suivis d'un cours.

Les séances de cours ont elles-mêmes adopté l'un ou l'autre des deux formats suivants :

- une table ronde pour réfléchir ensemble en amont et en aval des concerts (sept séances) ;
- un cours magistral (quatre séances).

Il y a eu en sus deux conférences, l'une de Franck Jedrzejewski dans le cadre d'un colloquium de mathématiques, *Tresses et improvisations libres*, l'autre de Michaël Parisot dans le cadre du Conservatoire du Grand Besançon, *Tempéraments et fractions*.

La musique est d'abord une durée consciente et structurée de production, de transmission, d'audition et de perception du son. Elle relève donc tout autant de l'acoustique et de la physiologie que des sciences du cerveau. Les mathématiques interviennent à chacune de ces quatre étapes, mais aussi dans leur articulation.

Je propose une revue de sept thèmes pour lesquels des concepts mathématiques nous sont parus pertinents dans le cadre de notre cours, et je vais montrer en quoi l'histoire des mathématiques est utile pour approfondir ces thèmes. Mais commençons par l'écoute d'un extrait du concert d'ouverture, une improvisation par Jonas Kocher à l'accordéon et Olivier Toulemonde aux objets sonores. Je vous propose d'en faire une écoute consciente en essayant de saisir avec des mots ce que vous entendez.

Voici mon compte rendu :

> une note ténue très haute – un grésillement – une voix de bébé – un roulement de bille – un relief : parfois ça gratte et le son monte et descend – l'accordéon entre en scène et en ressort – interférences entre l'accordéon et la bille qui tourne comme si elle tanguait – l'accordéon monte encore et encore – trois objets sonnent en même temps et l'accordéon s'impose dans l'espace sonore – il vibre d'un accord très bas – il va chercher la note de la bille qui tourne et lui tourne autour – un accord qui descend – note tenue et on n'entend presque plus les objets – la note dure comme dure la rotation de la bille – tout s'écroule, l'accordéon aussi – bruits rythmiques, clics, creusements, note haute tenue.

Ce compte rendu est tiraillé entre deux approches concurrentes de l'écoute. L'une s'abandonne à la force évocatrice de la musique : des images et des émotions surgissent. L'autre se concentre sur la matière sonore en tant que telle.

2 Le son physique

Le son consiste en la vibration longitudinale de couches d'air contigües. Par vibration, on entend un mouvement périodique d'aller et de retour. Nous percevons ces périodes dans le son et appelons *fréquence* le nombre de périodes par seconde. Brook Taylor (1713), dans sa solution du problème de la corde vibrante, fait le lien entre fréquence et onde stationnaire sinusoïdale. C'est Jean Le Rond D'Alembert (1747) qui a compris que le son produit par une corde vibrante peut être décrit comme une *superposition* de telles vibrations sinusoïdales : ainsi débute l'analyse harmonique. La figure 1 montre un exemple issu de D'Alembert (1747, tab. VII).

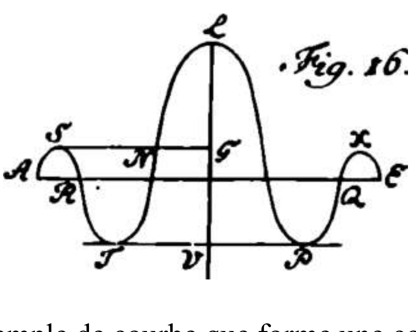


Fig. 1 : Exemple de courbe que forme une corde tendue

La nature ondulatoire du son fait qu'il est soumis au principe d'incertitude de Heisenberg et qu'en particulier il n'a pas de début ni de fin nets et clairs ; il y a aussi une limite inférieure à sa durée selon sa fréquence. Les sons révèlent de cette manière leur nature continue.

L'ajustement et l'organisation des sons selon leur hauteur remonte à la nuit des temps, et l'accordage des instruments a certainement préexisté à la théorisation des gammes. La première description connue d'une gamme remonte à Philolaos de Crotone (−470-−385, philosophe pythagoricien de la deuxième génération, Grande Grèce).

> La grandeur de l'harmonie comprend la syllabe [« la prise », c'est-à-dire la quarte] et la dioxie [« à travers les aigües », c'est-à-dire la quinte]. La dioxie est plus grande que la syllabe du rapport 9 : 8 [c'est-à-dire d'un ton]. En effet, une syllabe sépare l'hypate [c'est-à-dire la corde du haut, le mi] de la mèse [c'est-à-dire de la corde du milieu, du la] ; une dioxie la mèse de la nète [c'est-à-dire de la corde du bas, du mi] ; une syllabe la nète de la trite [c'est-à-dire de la corde tierce, du si] ; et une dioxie la trite de l'hypate. Entre la trite et la mèse il y a le rapport 9 : 8. La syllabe a le rapport 4 : 3, la dioxie 3 : 2 et le diapason [« à travers toutes », c'est-à-dire l'octave] 2 : 1. Ainsi l'harmonie comprend cinq rapports 9 : 8 et deux dièses [« laissés passer », c'est-à-dire demi-tons], la dioxie trois rapports 9 : 8 et un dièse, et la syllabe deux rapports 9 : 8 et un dièse. (Philolaos, 1988, p. 504)

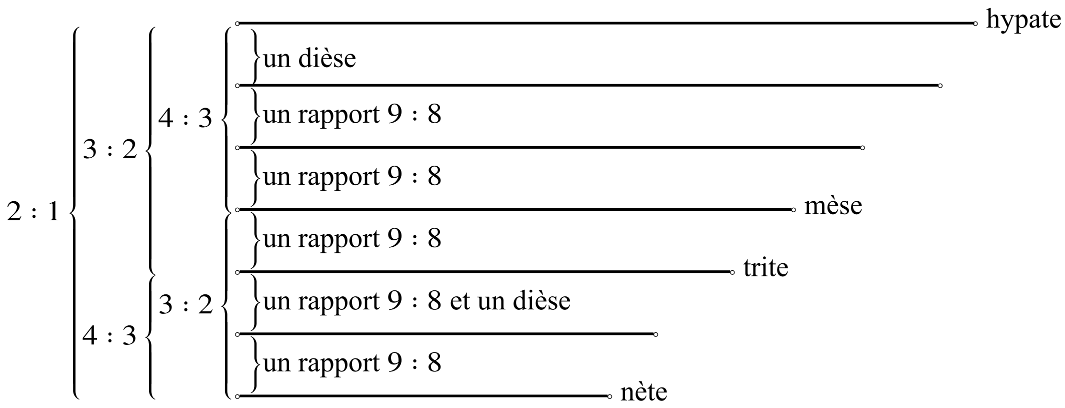


Fig. 2 : L'heptacorde de Philolaos

Cette description fait intervenir des rapports de nombres épimores, c'est-à-dire de la forme $(n+1):n$: les rapports 2 : 1, 3 : 2, 4 : 3 et 9 : 8. Cela fait de la musique le paradigme de la constatation suivante de ce philosophe :

> Et de fait, tout être connaissable a un nombre : sans celui-ci, on ne saurait rien concevoir ni rien connaitre. (Philolaos, 1988, p. 503)

La musique d'aujourd'hui se passionne pour la microtonalité, c'est-à-dire pour des intervalles de fréquences bien en deçà du rapport 9 : 8. Par exemple, Tony Di Napoli a accordé dix lames rectangulaires de calcaire de Vinalmont à l'intérieur de ce rapport, c'est-à-dire au neuvième de ton, avec les deux lames les plus graves à un demi-Hertz l'une de l'autre ; pourtant, l'oreille les discerne et l'accord des deux lames produit un battement de $1/0,5$ Hz $= 2$ s par leur superposition.

Selon la manière dont on fait sonner une lame, la répartition des fréquences qu'elle produit change. On appelle la plus basse la *fondamentale* et les autres les *harmoniques*. Pour les instruments à cordes, les harmoniques sont les multiples de la fondamentale, et c'est ce qui fonde la théorie musicale de Philolaos ; mais pour des lames rectangulaires, comme pour des cloches, leur relation est beaucoup plus complexe. En disposant du sable sur les lames, on peut faire apparaitre des figures de leur vibration, parce que le sable a tendance à se disposer le long des nœuds de la vibration, c'est-à-dire les lignes où la lame reste immobile au cours de son mouvement. Ces figures correspondent aux fonctions propres stationnaires de l'équation des ondes associée à la lame. Leur étude phénoménologique remonte à Ernst Florens Friedrich Chladni (1809) et dans la figure 3, on peut observer les figures qu'il a observées pour une lame carrée (Chladni, 1809, tab. IV).

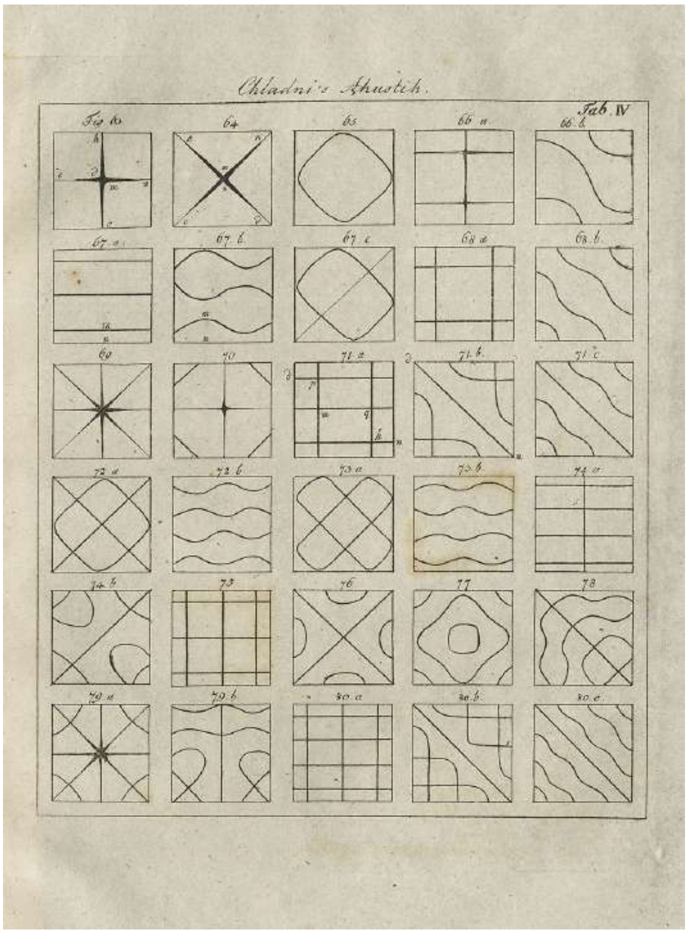


Fig. 3 : Figures de Chladni pour une lame carrée

La transmission des vibrations sonores dépend aussi de la géométrie du lieu où elles sont produites. En particulier, un son qui parcourt l'espace entre deux murs distants d'un multiple impair de la demi-longueur d'onde donne lieu à une onde stationnaire, de sorte que la perception du son change à chaque endroit.

3 Lissité et rugosité du son

Commençons par la lecture d'un extrait de *Nous autres*, de Ievgueni Zamiatine, où Д-503 se réveille dans un état de stupeur.

> Je ne distinguais plus le rêve de la réalité. Des quantités irrationnelles traversaient l'espace solide à trois dimensions et, au lieu de surfaces lisses et dures, il n'y avait plus autour de moi que des formes toutes tordues et velues. (Zamiatine, 1979, p. 103)

Les notes de la musique classique sont la contrepartie de ces « surfaces lisses et dures », alors que la musique d'aujourd'hui s'intéresse toujours plus à des sonorités « toutes tordues et velues », c'est-à-dire sales voire parasites. Cela mène les compositeur.e.s et les interprètes à découvrir des possibilités toujours nouvelles de leur instrument de musique. Par exemple, la bassoniste Dafne Vicente-Sandoval nous a expliqué comment elle y travaille : elle doigte une note fondamentale et essaie de produire une harmonique en envoyant l'air très vite ; la résistance de l'anche et la position des lèvres sont déterminantes. Ou bien elle produit un son diphonique qui peut être instable. La réitération d'un geste ne produit plus le même effet, de sorte qu'un maillage de l'espace sonore par des sons très proches se met en place. Chaque note requiert un ensemble d'actions microscopiques pour sa production et son maintien.

Cette recherche éloigne les musicien.ne.s des procédés de la mélodie et du rythme pour les plonger dans la matérialité du son. Ce saut est comparable à celui qui mène du concept de trajectoire à celui d'application, de celui de longueur à celui de coupure : fonction et nombre réel deviennent des objets autonomes de la science chez Dirichlet/Lobatchevski et chez Dedekind ; c'est alors que l'on peut envisager de construire des fonctions continues nulle part dérivables (Weierstrass) et d'appliquer la méthode diagonale pour prouver l'existence de nombres transcendants (Cantor).

Terminons sur un autre extrait, tiré des *Leçons Américaines* d'Italo Calvino.

> Le poète du vague ne peut qu'être un poète de la précision, dont l'œil, l'oreille, la main sont toujours prêts à saisir avec justesse la sensation la plus ténue. (Calvino, 1992, p. 104)

4 Le signe

Une composition musicale peut être décrite à l'aide d'un système de signes : on parle alors de partition. Ce système est discret par définition : il consiste à aligner des signes discernés les uns des autres. Or nous avons constaté que le son est un phénomène continu. Cette opposition fait que pendant longtemps la partition n'a pas été considérée comme une description mais comme une indication parfois très sommaire du thème. Ce n'est qu'à l'époque romantique qu'elle a acquis la prétention de constituer l'œuvre musicale elle-même. Les compositeur.e.s contemporain.e.s comme Andreas Stauder sont à la recherche de manières toujours nouvelles de dépasser les limitations d'une écriture composée de signes.

Le signe primitif des mathématiques est celui qui désigne l'unité, I, et selon Errett Bishop, c'est à partir de lui que toutes les mathématiques se construisent.

> La préoccupation première des mathématiques est le nombre, et cela veut dire les entiers positifs. Nous pensons des nombres de la façon dont Kant pensait de l'espace. Les entiers positifs et leur arithmétique sont présupposés par la nature même de notre intelligence et, nous sommes tentés de le croire, par la nature même de l'intelligence en général. Le développement de la théorie des entiers positifs à partir du concept primitif de l'unité, du concept de l'ajout d'une unité et du processus d'induction mathématique emporte une conviction complète. Dans les mots de Kronecker, les entiers positifs ont été créés par Dieu. (Bishop, 1967, p. 2, ma traduction)

Le désir de rendre compte de la structure profonde de pièces musicales a mené Franck Jedrzejewski à appliquer le concept mathématique de tresse à l'*Étude 1 : Désordre* de György Ligeti pour y déceler la distribution des accents et à *Modes de valeurs et d'intensité* d'Olivier Messiaen pour la distribution des figures sérielles. La description de Larry Ochs d'un de ses *Dispositifs et stratégies pour l'improvisation structurée* peut convaincre de la pertinence de cette approche.

> Dans *torque* (1988) et d'autres pièces composées après, j'ai ajouté la règle suivante pour chacun des trois joueurs lors de ces triples solos : commencer à partir d'un motif écrit initial (« l'idée A ») ; improviser sur l'idée A jusqu'à ce qu'elle s'établisse vis-à-vis de la version de l'idée A des autres joueurs ; introduire alors progressivement une nouvelle idée (« l'idée B »), qui peut être n'importe quel motif musical adapté à l'humeur de l'aire sonore d'ensemble à ce moment-là ; détruire progressivement l'idée A d'origine. Jouer l'idée B seule en la variant jusqu'à ce qu'elle s'établisse clairement vis-à-vis de la musique des autres joueurs (c'est-à-dire : jouer l'idée B jusqu'à ce que la relation entre votre idée B et la musique du groupe puisse être entendue des autres joueurs [et des auditeurs]) ; introduire progressivement une « idée C » ; détruire progressivement l'idée B. Et cetera. (Ochs, 2000, p. 326, traduction de Franck Jedrzejewski amendée)

On perçoit dans cet extrait l'importance de la clarté et de la précision dans la musique. Il ne s'agit pas d'un art du compromis, mais d'un art de l'affirmation d'une singularité prête à affronter d'autres singularités.

5 Modèle et détournement

Certaines compositions musicales sont des modèles de concepts voire de théories mathématiques. C'est le cas de certaines compositions de Iannis Xenakis : *Pithoprakta* est basée sur la théorie cinétique des gaz et pose le problème d'une interprétation acoustique de la température thermodynamique.

Tom Johnson, quant à lui, part de la simplicité mathématique de l'opération de comptage et de sa complexité dans les langues du monde pour composer ses *Musiques à compter*. Ses pièces ont vocation à fournir la description exhaustive d'un phénomène. Cela cause aussi un plaisir tout mathématique. Par exemple, dans *Mocking*, il énumère les différentes possibilités de marquer 2, 3, 4, 5 ou 6 temps parmi 8 temps donnés d'une mesure. Il y en a respectivement $\binom{8}{2} = 28$, $\binom{8}{3} = 56$, $\binom{8}{4} = 70$, $\binom{8}{5} = 56$, $\binom{8}{6} = 28$, mais alors se pose le problème de l'ordre dans lequel on les joue. En d'autres mots, cette énumération de possibilités a une structure qui ne se réduit pas à une succession. Johnson résout ce problème en présentant cet ensemble de mesures comme un graphe défini par la relation de différence minimale, qui donne aussi lieu à une distance entre deux mesures. Cette matière combinatoire prend vie lors de l'interprétation par la grande concentration qu'elle requiert de l'interprète ; la transparence des principes de composition a pour corollaire l'évidence de toute erreur éventuelle !

Le principe de cette musique libère l'auditeur de la recherche d'un sens plus profond. Elle illustre comment l'écoute attentive et bienveillante de toute œuvre sonore lui confère sa logique et sa beauté interne. Il s'agit là d'une caractéristique de notre sensibilité qui organise les sons perçus selon le contexte et qui est aussi capable de les investir dans d'autres formes de notre entendement comme par exemple celle de l'espace. Cette tendance à rechercher un sens est inverse de la démarche de la matérialité du son décrite dans la section 2.

La musique peut aussi profiter de stratégies de détournement, comme celle de détourner des objets de leur usage commun pour en faire des instruments de musique. Cela peut être mis en parallèle avec un fait observé maintes fois dans l'histoire des mathématiques, lorsque des contre-exemples, comme la fonction e^{-1/x^2} qui a permis à Cauchy d'exhiber une fonction dont le développement en série de Taylor au point 0 est nul, deviennent un ingrédient incontournable d'une autre théorie, comme celle des distributions.

6 Langage et logique

Le terme de langage est devenu un mot commode pour désigner un système complexe de communication qui n'a pas nécessairement la vocation d'universalité des langues naturelles. Il en est ainsi des langages et des dialectes musicaux.

Les musicien.ne.s improvisateur.e.s font du développement d'un langage clair et intelligible le principe même de leur démarche : connaitre son instrument, l'explorer et le maitriser jusqu'à en formuler une syntaxe avant de partir à la rencontre du public et de ses partenaires.

Interrompons notre propos avec une citation de *Rhétorique spéculative* de Pascal Quignard.

> Fronton écrit à Marcus : « Il se trouve que le philosophe peut être imposteur et que l'amateur des lettres ne peut l'être. Le littéraire est chaque mot. D'autre part, son investigation propre est plus profonde à cause de l'image. » L'art des images – que l'empereur Marc Aurèle nomme, en grec, icônes tandis que son maitre, Fronton, les nomme le plus souvent, en latin, images ou, à quelques reprises, en grec philosophique, métaphores – à la fois parvient à désassocier la convention dans chaque langue et permet de réassocier le langage au fond de la nature. (Quignard, 1995, p. 11)

Il en est ainsi aussi en mathématiques, parfois de manière explicite lorsqu'une révolution intérieure mène à la reformulation de pans entiers de la science : la rigueur en analyse a mené au langage des ε et des δ; la théorie des idéaux a mené à la théorie des ensembles ; la courbure des surfaces a mené aux variétés riemanniennes. Mais en fait, chaque mathématicien.ne développe un langage privé à l'égard des concepts mathématiques dont il.le se fait le gardien, et c'est ce langage qui lui permet de deviner ce qui est valide et ce qui est impossible, et pourquoi. Cependant, ce langage ne trouve pas en général le chemin des écrits des mathématicien.ne.s, et cela concourt au rôle très important du folklore transmis oralement, dont la maitrise est nécessaire pour participer à la recherche mathématique.

Si je peux me permettre une comparaison hardie, les partitions sont les théorèmes de la musique, et ils ne constituent qu'un simulacre de la pratique musicale.

La reconstitution de ces langages est un défi permanent pour l'histoire des mathématiques, et cette difficulté peut expliquer les querelles récurrentes entre historien.ne.s, comme celle des infinitésimaux chez Fermat et chez Cauchy, ou celle de l'impact de la philosophie éléate sur la constitution des *Éléments* d'Euclide.

La constitution d'une syntaxe musicale fournit aussi aux musicien.ne.s la contrepartie des règles d'inférence en logique mathématique. Le vécu des improvisateur.e.s est que dès la première seconde, leur performance se développe selon la forme de la nécessité, alors que juste auparavant il.le.s sont encore dans la plénitude de leur liberté. C'est comme si cette première seconde fournissait les axiomes nécessaires au développement de toute une théorie *ad hoc* et éphémère.

Il y ici une analogie avec l'attaque d'une note, que les synthétiseurs échouent à simuler alors qu'elle contribue beaucoup à l'identité des instruments de musique. De même, la phase transitoire pendant laquelle une corde se met en vibration se modélise très difficilement en mathématiques.

Cependant, des musicien.ne.s expérimentateur.e.s comme Catherine Christer Hennix posent la question du rapport d'une telle théorie au réel, et plus généralement de l'adéquation des logiques musicales à la réalité du son. Il.le.s développent alors une philosophie de la musique, ultra-intuitionniste pour Hennix, à partir de la musique populaire nord-américaine, indienne et africaine pour Henry Flynt.

7 Topologie

Les concepts de la géométrie différentielle et en particulier de la théorie des catastrophes ont eu un grand succès chez les musicien.ne.s : les notions de voisinage, de stabilité et d'instabilité, de bifurcation, de singularité, de surface leur parlent, en particulier lorsque des philosophes comme Gilles Deleuze et Félix Guattari s'en saisissent.

Elles permettent de décrire un phénomène bien connu des musicien.ne.s improvisateur.e.s : lorsqu'il.le.s arrivent sur un plateau sonore, qu'il.le.s s'y installent et y prennent leurs aises, il.le.s savent que ce plateau annonce l'irruption d'un bouleversement dont il faut saisir le germe.

C'est ainsi que le temps d'un plateau sonore peut être mis en parallèle avec le temps d'une époque entière des mathématiques comme celle des séries entières (qui croyait donner une forme définitive à l'analyse au 18^{e} siècle) avant qu'une catastrophe mène à la théorie des fonctions (au 19^{e} siècle) ; de même, la théorie de la représentation des fonctions par des séries de Fourier (au 19^{e}) a précédé celle de leur convergence par resommation (au 20^{e}).

8 Le hasard

Le hasard a été une des grandes affaires de la musique de la deuxième moitié du 20^{e} siècle, à la suite de Iannis Xenakis (écouter par exemple la musique poissonnienne d'*Achorripsis*) et de John Cage (écouter par exemple le silence rempli de bruits fortuits de *4′33″*). L'aléatoire apporte une objectivité au son, une indépendance par rapport aux affects et aux émotions qui est inaccessible autrement, et cela a été un de ses grands attraits. Le hasard permet aussi de modéliser la tension entre le son en puissance et le son en acte, provoquée par le libre arbitre chez les improvisateurs, et à l'origine d'une perception multidimensionnelle du temps : avant qu'un son ne soit produit, musicien.ne.s et auditeur.e.s vivent dans la potentialité simultanée de tous les sons possibles, et, comme le dit Mallarmé, « un coup de dés jamais n'abolira le hasard ».

La musique contemporaine est aussi à la recherche de l'indétermination due au chaos, par exemple l'exposition d'un.e interprète à une partition dont la complexité dépasse ses capacités techniques.

9 Conclusion : le dispositif pédagogique

Les quatre membres de l'équipe pédagogique se sont réunis six fois en amont du cours, deux fois pendant le semestre et deux fois en aval. Ces réunions ont pris la forme d'une table ronde dans laquelle chacun rend compte de son expérience musicale et mathématique, et propose des phénomènes et des concepts à explorer.

À l'issue du cours, nous nous sommes aperçus d'une forte dissension quant à la forme du dispositif pédagogique.

- Deux parmi nous ont simplement transposé la forme de ces réunions aux séances de cours, avec les tables disposées en carré, et chacun.e invité.e à prendre la parole, à réagir et à contribuer à la recherche.
- Les deux autres estiment que les contributions de chacun méritent d'être préparées sous la forme d'un exposé présenté aux autres, tant pour les enseignants que pour les étudiant.e.s.

De fait, le travail demandé aux étudiant.e.s relevait de ces deux formes :

- la participation active aux tables rondes et
- des exposés sur la base d'un mémoire écrit d'une dizaine de pages.

Les interrogations soulevées par l'opposition entre ces deux approches relèvent pour partie de conceptions différentes sur la finalité d'un tel cours.

- Les premiers, bien que détenteurs d'un savoir, estiment qu'il s'agit d'abord de faire germer des questions, et que le début d'un mouvement de questionnement chez les participants est aussi important que les réponses.
- Les deuxièmes ont pour but de donner une vue d'ensemble des rapports entre musique et mathématiques, et de transmettre une vision culturelle et diachronique du sujet.

Un enjeu du cours de 2016-2017 sera de trouver une synthèse de ces deux approches qui satisfera tous les quatre.

Pour les étudiant.e.s, il s'est agi d'une rare occasion de se voir traité.e.s à l'égal des enseignants, et ces derniers ont fait de leur mieux pour les encourager à partager leurs impressions et leurs interrogations. Ceu.lles qui avaient plus d'aisance à s'exprimer et à articuler leur pensée étaient visiblement privilégié.e.s. Le moyen le plus efficace pour permettre à tou.te.s de participer à la recherche a été de faire régulièrement un tour de table.

Le contenu du cours a déstabilisé les étudiant.e.s, dont une moitié avait une culture musicale, mais dont aucun.e ne connaissait les musiques présentées. La musique expérimentale américaine a provoqué chez la plupart une résistance et des émotions très fortes. Tant le travail de Tom Johnson que celui de Tony Di Napoli a rencontré un vif succès.

Voici pour conclure le retour de trois étudiant.e.s.

> J'ai, tout d'abord, beaucoup apprécié l'intérêt des enseignants pour le sujet ainsi que l'investissement qu'ils ont placé dans le projet et celui de tous les artistes et intervenants qui y ont pris part.
>
> Le programme était très intéressant et très complet, très dense peut-être trop. Certaines notions méritaient peut-être d'être approfondies d'avantage ce qui n'a pas été possible dans le temps imparti. D'un autre côté, il était intéressant d'aborder tous ces aspects de la musique (expérimentale, improvisée, contemporaine), c'est difficile de trouver un équilibre entre le contenu et le temps passé sur chaque notion.
>
> L'organisation de l'unité (rencontre avec les artistes suivie d'une représentation puis discussion en cours) permettait d'avoir du recul. J'ai d'ailleurs particulièrement apprécié la prestation de Tony Di Napoli avec qui j'avais échangé pour mon exposé.
>
> Cette unité n'exigeait aucun prérequis de la part des étudiants. Cependant, sans parler de notions de solfège poussées, il était nécessaire, selon moi, d'avoir eu une formation musicale générale pour pouvoir s'ouvrir à l'écoute de musiques différentes, nouvelles, peu médiatisées. Et certaines interventions exigeaient un niveau de connaissances théoriques relativement élevé. Je pense notamment à la présentation du compositeur Andreas Stauder ou même à la conférence sur les tempéraments.
>
> Personnellement, j'ai trouvé que cette unité était un bon complément de ma formation au conservatoire et de ma formation mathématique. Elle m'a permis, comme plusieurs unités ce semestre, de percevoir une image plus globale du monde des mathématiques. J'ai pu, ainsi, comprendre une partie des liens qui unissaient ces deux domaines et approfondir l'idée que les mathématiques sont une forme de culture et non seulement une science théorique.
>
> . . . .
>
> J'ai trouvé le cours et les rencontres avec les différents musiciens intéressants. Cependant, comme je n'ai aucune notion en musique, j'ai eu du mal à comprendre plusieurs notions abordées. Je trouverais plus intéressant de revoir en début d'année les notions de base en musique pour que ce soit clair pour tout le monde et pour mieux comprendre la suite du cours.
>
> . . . .
>
> En ce qui concerne le cours, j'ai trouvé cela très intéressant mais assez difficile d'accès. J'ai pas toujours trouvé ça évident de rentrer dans les différents concerts, et d'y trouver ma place. Cependant, j'ai apprécié d'assister à des concerts variés puisque j'ai moi-même une formation classique. J'ai trouvé que c'était une bonne ouverture musicale et cela m'a fait découvrir diverses musiques et ce qui s'ensuit.

RÉFÉRENCES

Bishop, E. (1967). *Foundations of constructive analysis*. New York : McGraw-Hill.

Calvino, I. (1992). Exactitude. In *Leçons américaines : aide-mémoire pour le prochain millénaire* (p. 97-129). Paris : Gallimard. (Traduction par Y. Hersant.)

Chladni, E.-F.-F. (1809). *Traité d'acoustique*. Paris : Courcier.

D'Alembert, J. L. R. (1747). Recherches sur la courbe que forme une corde tendue mise en vibration. *Histoire de l'Académie royale des sciences et des belles lettres de Berlin*, *1*, 214-249. Consulté sur `http://bibliothek.bbaw.de/bibliothek-digital/digitalequellen/schriften/anzeige?band=02-hist/1747&seite:int=00000243`

Ochs, L. (2000). Devices and strategies for structured improvisation. In J. Zorn (éd.), *Arcana : musicians on music* (p. 325-335). Granary books/Hips road. Consulté sur `http://www.rova.org/foodforthought/strategies-in-rovas-music.html`

Philolaos. (1988). B. Fragments. In J.-P. Dumont, D. Delattre, & J.-L. Poirier (éds.), *Les présocratiques* (p. 502-513). Paris : Gallimard.

Quignard, P. (1995). Fronton. In *Rhétorique spéculative.* Paris : Calmann-Lévy.

Taylor, B. (1713). De motu nervi tensi. *Philos. Trans. Roy. Soc. London*, *28*, 26-32. (Traduction en anglais dans *Philos. Trans. (abridged)*, *6*, 14-17 (1809).)

Zamiatine, E. (1979). *Nous autres*. Paris : Gallimard. (Traduction par B. Cauvet-Duhamel.)

ISAAC NEWTON: CONTAINS AWE

Working with students and teachers in period costume

Peter RANSOM
Bath Spa University, Newton Park, Newton St Loe, Bath, BA2 9BN, UK
p.ransom@bathspa.ac.uk

ABSTRACT

This workshop will be delivered by me, dressed as Sir Isaac Newton in period costume. It describes the work I have done with various groups of students and teachers over the past three years based on Newton's work. I explain how the use of digital technology has helped with some groups. The groups of students that have participated in sessions with Newton cover ages 13/14 years, 14/15 years and 16-18 years. The teachers who have attended sessions range from those in initial teacher education, newly qualified teachers and heads of mathematics in the UK. The work includes a thought provoking geometry problem from a piece of artwork featuring Newton that is accessible to both students and teachers and a brief history of Newton. It covers demonstrations of some of Newton's laws and provides some classroom materials on developing the binomial theorem with young people. Mention is made of Newton's work on optics which leads on to developing participants' mental geometry skills. Participants will get the opportunity to make themselves a set of proportional dividers and use some ink made according to Newton's own recipe using his original manuscript. All the classroom materials will be put on a CD-ROM and given to participants so they can use the materials with students and teachers at a later date.

1 The inspiration

I incorporate events from history into mathematics lessons because I find it very interesting to see the practical applications of mathematics set into the period when it was used. I chose to write about this episode since it has been developed over the past three years and is very rich in mathematical history.

Tzanakis & Thomaidis (2011) classify the arguments and methodological schemes for integrating history in mathematics education and my episode fits into the two-way table mainly as History-as-a-tool and Heritage though there are overlaps into the History-as-a-tool and History cell. The over-riding concept in my work is History-as-a-tool.

The reliability of this work in the sense of reproducibility by someone else is impossible to quantify, since teachers use such episodes in different ways with different students and probably not in costume! Every session I do with students or teachers is different according to local conditions and the knowledge people bring to the sessions, so nothing stays the same: that is one of the great joys of working with them. It is important that teachers recognise this so that they are flexible in their approach with how they present materials to students.

2 The evolution of the episode

This work came about after an invitation to do a 90-minute talk to around 250 students aged 16-18 and staff at local schools in the county of Somerset. I had accepted the invitation and being something for such a large group of people it meant I would not have enough workshop equipment to do what for me is the normal way I work with students. Therefore, I had to combine a lecture-style presentation with some paper-and-pen based activity, rather than a more active session. I did not want to do one of the sessions I have developed over the past decade, as it was time to do something new. In what follows I describe what has happened with all the people with whom I have worked: teachers and students aged 12-13, 13-14 and 16-18

2.1 Images of Newton

Since I had been using the British Library in the past year I had noticed that a modern statue of Newton had appeared on site (see Figure 1) and this was the inspiration I needed.

Figure 1. Statue of Newton by Sir Eduardo Luigi Paolozzi (1995)

It is based on a picture by William Blake (1757-1821: a seminal figure in the history of the poetry and visual arts of the Romantic Age) of 1795 (see Figure 2) and that features an interesting mathematical diagram that I have used ever since. If you examine Figure 2 you will see Newton working on a diagram in the bottom right corner.

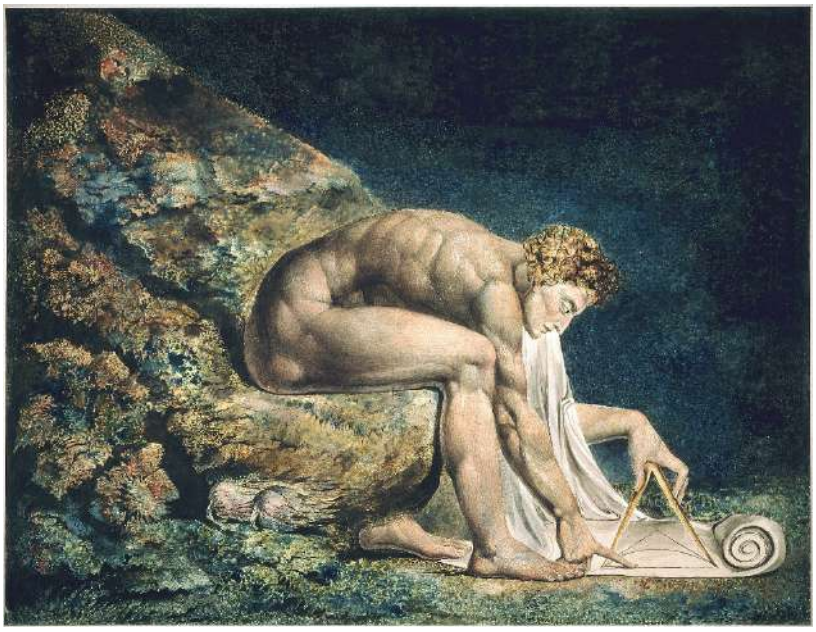


Figure 2. Newton by William Blake (1795) (see Fauvel, Shortland, & Wilson, 1989, p. 226)

This diagram appears to be a segment of a circle in an equilateral triangle and I set participants the problem of finding exactly what fraction of the triangle is covered by the segment assuming that two sides of the triangles are tangent to the arc of the circle. Now this puts students and (many) mathematics teachers in an unfamiliar situation as there seems to be a complete lack of information and it is interesting to see how people progress with this problem. (Some of teachers are not very confident with their mathematics as they may be starting in the profession or lacking in subject knowledge. Unfortunately, in the UK there is a shortage of mathematics teachers and some people have had to retrain from other subjects to fill vacancies.) Some teachers just seem to sit back in resignation with comments such as 'I was never any good at geometry', or 'I don't know what to do'. Others seize on the problem with enthusiasm, working on their own or with one or two others: again, it is fascinating to see the dynamics of social and mathematical interaction as people remember some long-forgotten facts about circle geometry or the exact value of sin 60°. I offer a prize of a £1 note featuring Sir Isaac Newton (see Figure 3) for the first correct solution as an incentive to keep people going as well as to raise awareness of our country's currency heritage as that is also an important part of the history of mathematics. These notes were in circulation between 1978 and 1988 and are no longer legal tender.

Figure 3. £1 note featuring Sir Isaac Newton

Students and adults seem pleased to be faced with the possibility of winning this piece of history, possibly to some because it brings back memories of the time before we had £1 coins. Now with younger students who may not have met much geometry or surd manipulation I tend to use Texas Instruments-Nspire CX wireless hand-held technology as this allows them an empirical approach by drawing the diagram and capturing data dynamically. Figure 4 shows a series of screen shots of what the students draw and the data capture.

In the first screen shot the student has drawn an equilateral triangle and using the fact that two sides of the triangle are both tangential to the arc of the circle, draws perpendiculars at the two vertices. The centre of the circle is therefore where these two perpendiculars intersect. Then the circle can be drawn as its centre is known as well as points through which it passes. The properties of the diagram are examined and students come to realise that if the original triangle is reflected in its base line then its top vertex will lie on the circumference of the circle as shown. Using the measurement tools on the hand-held device, variables such as the radius of the circle (rad), the area of the circle (cir) and the area of the triangle (tri) are determined and captured in a spreadsheet on a new page (shown on the second screen shot). By grabbing and moving one of the triangles vertices the values of these variables are recorded and many items of data collected. The third screen shot shows the area of the circle plotted against the area of the triangle and the last screen shot has the linear regression line superimposed from one of the menus available. Since all measurement is approximate there is a very small intercept on the y-axis due to rounding errors. The area of the segment can be found as there are three congruent segments (I expect the students to give an explanation of why they are congruent), so subtracting the area of the triangle from the area of the circle and then dividing by 3 gives the area of the segment. Students can then explore other relationships between these areas. It never ceases to amaze me how swiftly students cope with this technology: the session with 30 13/14 year olds had not used this technology before, yet were exploring the relationships as if it was second nature (which it probably is to these digital natives).

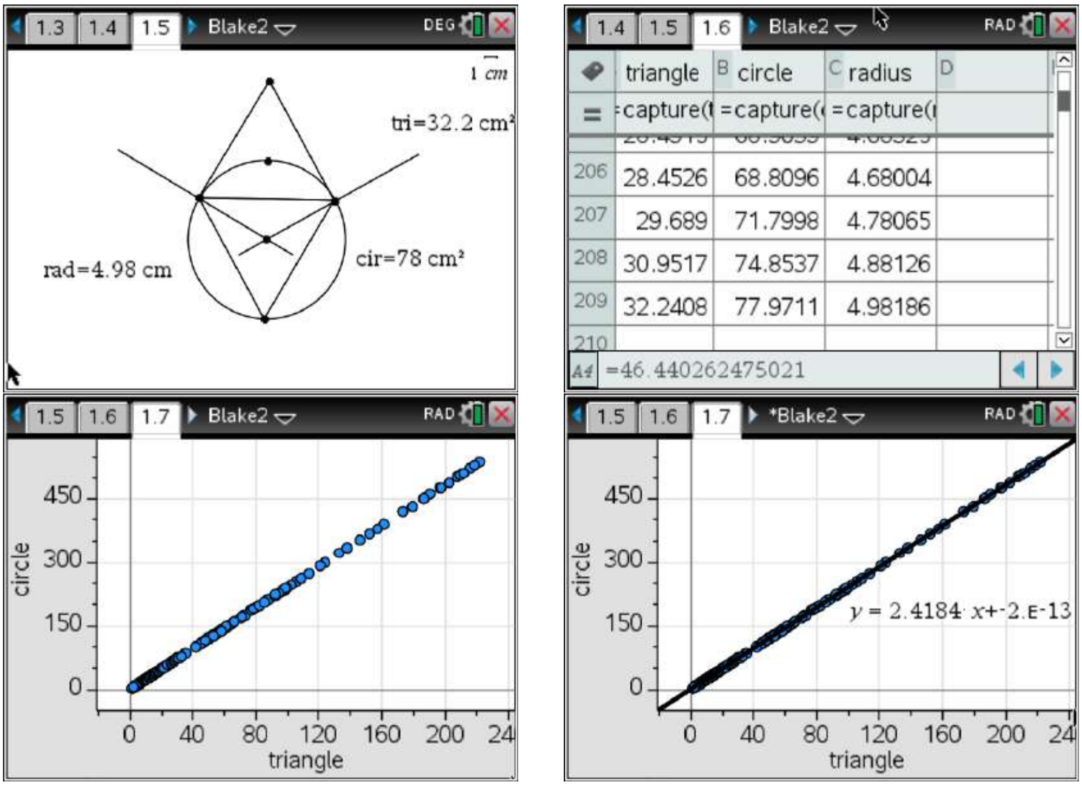


Figure 4. Screen shots from a student's use of hand-held technology

2.2 Introducing the history

After this introductory exercise I deal with some of Newton's history and his discoveries. In my teaching career spanning nearly four decades I have always found it good pedagogy to actively engage learners before talking for any length of time and Newton deserves a lot of time considering all that he achieved. In a two-and-a-half-hour masterclass however I tend to talk about some of the interesting facts so that students get a feel for some of his works and hopefully follow this up by reading more about him in their own time. After a brief five-minute talk about Newton's early years I mention how he discovered the generalised binomial theorem in 1665 and this forms the basis of the next piece of work for students, no matter what their age. Rather than dealing with the expansion of $(x + a)^n$ we work with $(x + 1)^n$ as this fits in well with the students' concrete experience of long multiplication. The pedagogical advantage here is that students will move from their concrete knowledge of multiplying whole numbers to the abstraction of algebraic multiplication of polynomials. Figure 5 shows the worksheet I use with students (and teachers who do not feel confident about algebraic manipulation).

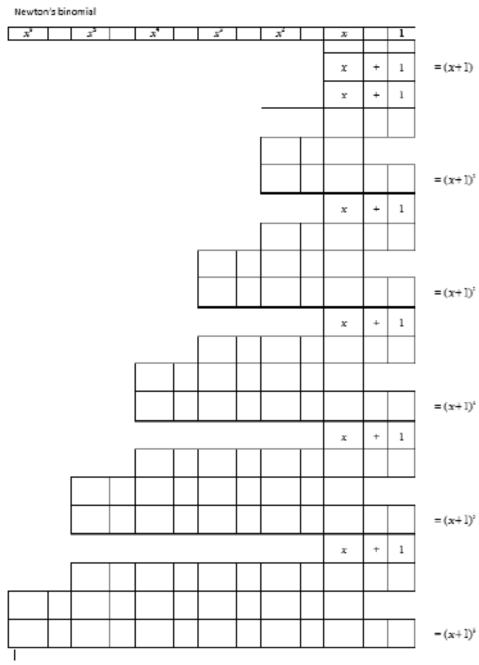


Figure 5. Worksheet used to expand $(x + 1)^n$

Here we use column headings of 1, x, x^2, x^3, … rather than **U**nits (10^0), **T**ens (10^1), **H**undreds (10^2), **Th**ousands (10^3) etc. which are used when multiplying whole numbers. Figure 6 shows what the first two results look like when complete.

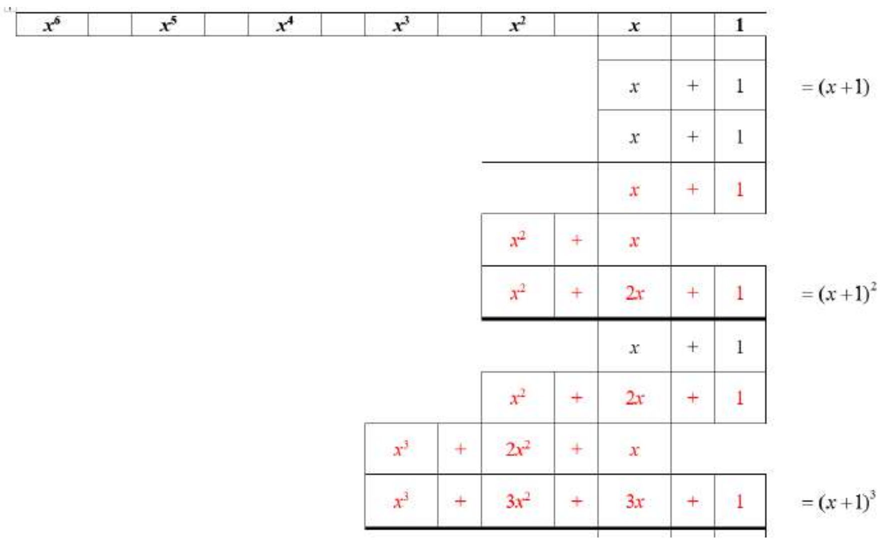


Figure 6. Partly completed worksheet

It takes a little while for students to feel confident filling in the cells, but once they have completed a few rows they swiftly get a sense of rhythm and satisfaction. I generally have them working in pairs so that they help each other, which gives me time to work with those who need some support. The students are encouraged to write down what they notice and some who have met Pascal's triangle in the past remember that. There are many opportunities here for extension work and teachers who have been observing this session often remark on how well the students have worked and how impressed they are with some of the student discoveries. I am keen to point out that this is not a normal classroom situation however: these students are here because they come voluntarily and I am unknown to them. Students react differently when working with strangers rather than their regular teachers, so that needs to be taken into account.

2.3 Newton's laws of motion

After the mental geometry activity, it is time for some more talk and a demonstration. I show a picture of *Principia* (see Figure 10) and talk about Newton's first law of motion: An object either is at rest or moves at a constant velocity, unless acted upon by an external force. This is demonstrated by balancing an egg on a stand which rests on a horizontal card on a glass of water. We discuss the forces acting on the egg (its weight and the normal reaction) and then I knock the card away and we watch the resulting motion.

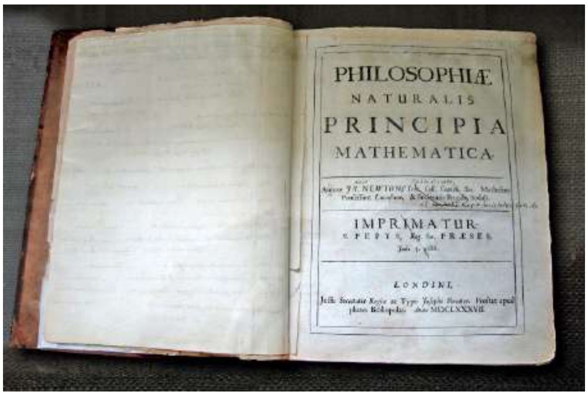


Figure 7. Newton's Principia, first published on 5 May 1687

I mention that this is just one example of Newton's laws, but there is never enough time to discuss or demonstrate the others. Anyway, that can be left to the physics teachers. It has always been my intention to whet students' appetites with some excitement of an experiment that may not work out exactly as planned and to realise that mathematics is not just a purely theoretical subject but that it involves experiments and practical work.

2.4 Newton's sundials

Since 1991 I have been obsessed with using sundials in school mathematics and it is an absolute delight to know that Newton made sundials from an early age (Ransom, 1998, (pp. 44-49). One of these (see Figure 8) was taken out of the wall of Woolsthorpe Manor, Newton's home, in 1844 and presented to the Museum of the Royal Society, where it is carefully preserved.

Figure 8. Newton’s sundial in the Royal Society, London

Another (see Figure 9) is in the church at Colsterworth, a village near Newton’s home. The Rev. John Mirehouse thought he would make a search at Woolsthorpe Manor and see if the second dial which Newton was known to have carved could be found. His effort was rewarded with success. The old stone was found in its original position on the south wall, covered up by a small coal house, and the relic was given by the owner of Woolsthorpe to the church. The disc is 11 inches wide at the top, and nearly 6 inches deep; it has been enclosed in a frame of alabaster and placed on the north wall of the Newton Chapel, with the following inscription:

> Newton: aged 9 years, cut with his penknife this dial: The stone was given by C. Turner, Esq., and placed here at the cost of the Rt. Hon: Sir William Erle, a collateral descendant of Newton, 1877

Figure 9. Newton’s sundial in Colsterworth Church

I believe that in teaching mathematics we should show our rich mathematical heritage to students so they are aware of the impact it has on so many lives in many ways. Unfortunately, there is never the time in these sessions to delve deeply into sundials, but I have covered that in the first European Summer University on History and Epistemology in Mathematical Education at Montpellier back in July1993 (Ransom, 1995).

2.5 The kissing problem

In 1694, a famous discussion between two of the leading scientists of the day - Isaac Newton and David Gregory - took place on the campus of Cambridge University. They wanted to know how many identical spheres can kiss (that is 'touch') the one in the centre?

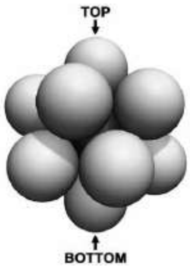
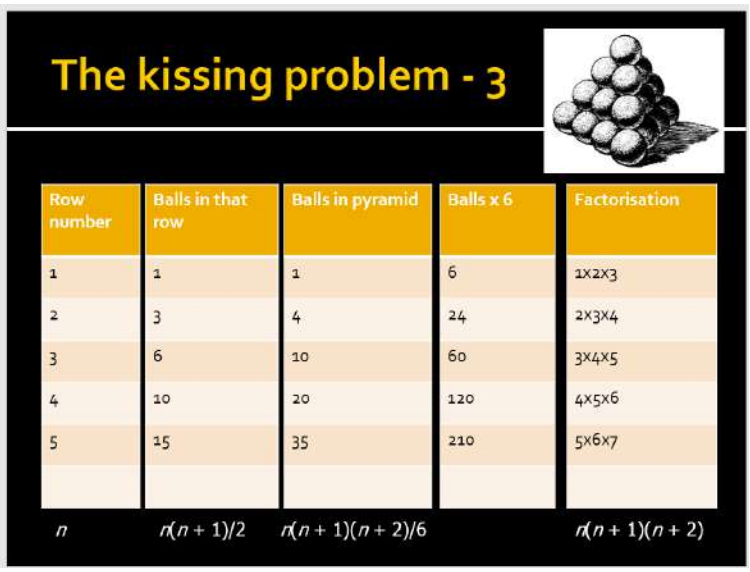


Figure 10. Kissing spheres

You can see (Figure 10) that 12 is possible, but in fact there is space for nearly 15 spheres! It took until 1953 for a proof that 12 is the limit. I give students a worksheet and small spheres to pile them up into a triangular based pyramid to investigate a similar problem, this time looking for a formula that links the number of layers with the total number of spheres in the pyramid. Figure 11 shows the results and formulae that students find and then we extend this by looking at the patterns in the formulae and extrapolate that to four dimensions, five dimensions and then to *n*-dimensions. Again, it is important for students to work with manipulatives before moving to the abstraction of algebra.



Row number	Balls in that row	Balls in pyramid	Balls x 6	Factorisation
1	1	1	6	1x2x3
2	3	4	24	2x3x4
3	6	10	60	3x4x5
4	10	20	120	4x5x6
5	15	35	210	5x6x7
n	$n(n+1)/2$	$n(n+1)(n+2)/6$		$n(n+1)(n+2)$

Figure 11. Results and formulae found when piling spheres

Conclusion

It is impossible to get through all these activities in two and a half hours (which includes a break) and as mentioned earlier, I select the activities to suit the audience and what I hope to achieve in the time allocated (which with teachers can be just one hour). Presenting in period costume (and here I must pay tribute to my mother, Mrs Joyce Ransom, who researched and made the costume at the age of 86) is not necessary, but for some students it adds an extra interest I believe as it helps bring our subject to life in ways that can appeal to not just the future mathematician, but mathematical historian. Teacher participants always receive a CD-ROM with all the classroom materials and PowerPoint presentations ready to use with a host of other relevant historical materials from other episodes I have presented to students and teachers over the past two decades. It is my hope that you, dear reader, will try at least one of these activities when the time is right.

REFERENCES

Fauvel, J., Flood, R., Shortland, M., & Wilson, R. (Eds.) (1989). *Let Newton be!* Oxford: Oxford University Press.

Ransom, P. (1995). Navigation and surveying: Teaching geometry through the use of old instruments. In F. Lalande, F. Jaboeuf, & Y. Nouazé (Eds.), *Actes de la première université d'été Européenne. Histoire et épistémologie dans l'éducation mathématique* (pp. 227-239). Montpellier, France: IREM de Montpellier, Université Montpellier II.

Ransom, P. (1998). *A Dozen Dials*. A book about sundials connected with mathematics and its history. Southampton: Privately published

Tzanakis, C., & Thomaidis, T. (2011). Classifying the arguments & methodological schemes for integrating history in mathematics education. In M. Pytlet, T. Rowland, E. Swoboda (Eds), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 1650-1660). Rzeszow, Poland: University of Rzeszow.

CONNECTING PAST TO PRESENT:

An approach to teaching topology and more via original sources

Nicholas A. SCOVILLE, Janet Heine BARNETT, Kathleen CLARK, Daniel E. OTERO, Diana WHITE
Ursinus College, Collegeville, PA, USA
nscoville@ursinus.edu
Colorado State University-Pueblo, Pueblo CO
janet.barnett@csupueblo.edu
Florida State University Tallahassee, FL, USA USA
kclark@fsu.edu
Xavier University Cincinnati, OH, USA
otero@xavier.edu
University of Colorado Denver, Denver, CO, USA
Diana.White@ucdenver.edu

This one-hour workshop will introduce participants to a guided reading approach for bringing history to the classroom in the form of "Primary Source Projects" (PSPs). Each PSP consists of excerpts (in English translation) from original sources that are historically related to core topics in today's curriculum, interwoven with exercises to guide students' active engagement with the mathematics in each excerpt. PSPs also contain commentary on the historical authors and the problems they wished to solve. At appropriate junctures, students are introduced to present-day notations and terminology, and asked to reflect on how today's definitions evolved to capture concepts that arose from attempts to solve those original problems.

Workshop participants will experience this teaching avenue by putting themselves in the role of the student as they work in groups on a specific PSP: *Connecting connectedness*. This 1-2 day PSP begins with a discussion of Cantor's efforts to determine the nature of a continuum. Students then wrestle with Cantor's original 1883 definition of connectedness, a definition which appealed to a metric. They next carefully examine Jordan's more nuanced understanding of connectedness, still with a metric. Finally, students encounter the first purely set-theoretic definition, given by Schoenflies, before arriving at the current definition provided by Lennes in 1911. By tracing the concept of connectedness and its evolution over a 30 year period, this PSP exposes students to the original motivations behind the esoteric concept of "connectedness" in a way that makes this concept both more meaningful and more accessible to them. Following this opportunity to grapple with original sources within a guided reading format, workshop participants will discuss classroom implementation issues. An overview of the pedagogical benefits of this approach will also be provided.

Workshop facilitators are members of a seven-person grant team with funding from the US National Science Foundation (NSF). A collection of PSPs from two prior NSF grants that supported the development of this approach is available at www.cs.nmsu.edu/historical-projects/. Through the current *Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources* grant, at least 50 new PSPs of different lengths, on a wide variety of undergraduate mathematics topics, will be created and tested. A list of new PSPs to be developed is located at http://digitalcommons.ursinus.edu/triumphs/.

MATHEMATICAL NARRATIVE: FROM HISTORY TO LITERATURE

Yi-Wen SU, Wann-Sheng HORNG, Jyun-Wei HUANG, Yuh-Fen CHEN
Department of Mathematics, University of Taipei, No.1, Ai-Guo West Road, Taipei, Taiwan
yiwen@uTaipei.edu.tw
Department of Mathematics, National Taiwan Normal University,
Taipei Municipal Heping High School,
New Taipei Municipal Mingder High School

ABSTRACT

Mathematical narrative is a concept that has gradually been attracting the attention of school teachers and mathematics educators, especially in recent years. The term 'mathematical narrative' in this workshop proposal refers to a form of narrative that is used to communicate or construct mathematical meaning or understanding. By introducing certain metaphors, a narrator may induce or promote learners' or listeners' mathematical understanding. This teaching tool can be used, for instance, to encourage further reflection upon some mathematical concepts or for alternative remedial learning, especially after conventional classroom teaching. Moreover, this can be applied in liberal arts courses at the university level. One straight forward way to introduce mathematical narrative into teaching is to use stories, arguments, problems, and their solutions from history, but one can also use literary metaphors to improve understanding. In this workshop, two school teachers and two mathematics educators will share their experiences in using mathematical narrative in their teaching, and also exchange teaching ideas with other participants in the workshop. The four presenters and the topics they will be discussing are described below.

Presenter 1: **Yi-Wen Su**, Department of Mathematics, University of Taipei.

The starting point of HPM is always history, and so is the beginning of this workshop. The first presenter is also the organiser of this workshop, and she shall share her experiences on how to design an activity to help improve undergraduate mathematics majors' reading comprehension using pre-modern mathematical arguments. The main material in this activity is the original proof of Heron's formula by Heron, and also another procedure that is equivalent to Heron's formula found in a thirteenth-century Chinese mathematical text written by Qin Jiushao.

Presenter 2: **Jyun-Wei Huang**, Taipei Municipal Heping High School.

The second presenter shall discuss his teaching in an elective mathematics course in senior high school in Taiwan (10^{th}–12^{th} grade). The core of this course is the use of the five most beautiful mathematical equations. The teacher tries to integrate mathematics discourse and mathematics writing in his teaching, and guides students in creating artistic works such as mathematical fictions, poems, games, and movies. In this way, he can help his students discover the beauty of mathematics and enhance their motives in learning.

Presenter 3: **Yuh-Fen Chen**, New Taipei Municipal Mingder High School.

The third presenter mainly teaches junior high school in Taiwan (7^{th}–9^{th} grade). She will share many activities designed by the team of mathematics teachers in her school, including publishing a school mathematical magazine, engaging students in mathematical games they created, decorating the school with 'mathematical tablets' that have their origins in the mathematical writings from Japan's Edo period, and encouraging students to write popular science novels after some mathematical reading.

Presenter 4: **Wann-Sheng Horng**, Department of Mathematics, National Taiwan Normal University.

Last but definitely not the least, history is transformed into narratives. The final presenter will introduce his experience in combining literary metaphor in teaching for undergraduate liberal art courses in mathematics. The main example he shall use is a questionnaire he employs to help students reflect upon mathematical concepts. He uses a metaphor quoted from Eli Maor's popular work, *The Pythagorean Theorem: A 4,000-Year History*. Maor

uses the metaphor of 'democracy' to talk about a theorem related to chords in a circle. The presenter's questionnaire guides students to reflect upon many aspects of Euclidean geometry and helps many university students, who are relatively weak in mathematics, to gain a new perspective about the discipline.

1 Reading and mathematical narrative

Reading ability and lifelong learning are closely related. In recent years, many countries increasingly put more of their educational efforts on students' reading ability. To understand how students read, these countries have participated in international reading evaluations and comparisons such as the Programme for International Student Assessment (PISA) and Progress in International Reading Literacy Study (PIRLS), the results of which have served as reference criteria according to which all countries can improve the teaching of reading. In Taiwan, the Ministry of Education continues to promote reading education and develop practical programs for teaching reading to help students develop an interest in reading. The Ministry seeks to promote reading not only in (Chinese) language and literature education but also in various disciplines, such as mathematics reading, social studies reading, and science reading (Executive Yuan, 2010). This article attempts discuss methods to add historical reading material in mathematics teaching, to help students have a cultural perspective on mathematics.

To add historical material into mathematics teaching, we need to first explain the term 'mathematical narrative.' By 'Mathematical narrative,' we mean a form of narrative that is used to communicate or construct mathematical meaning or understanding. It is a concept that has gradually attracted the attention of mathematics educators and school teachers in recent years. To communicate or construct mathematical meaning or understanding, a narrator usually has to introduce some metaphors to induce or promote learners' or listeners' mathematical understanding. This teaching tool can be used to encourage further reflections upon some mathematical concepts or for alternative remedial learning, especially after conventional classroom teaching in the primary or secondary level. Certain related approaches have been suggested by educators for the solution of underachievement in mathematics (Solomon & John O'Neill, 1998). Moreover, mathematical narrative can be applied in liberal arts courses at the tertiary level. One direct way to introduce mathematical narrative into teaching is to use stories, arguments, problems, and their solutions from history, but one can also use literary metaphors and fictions to improve understanding. Besides, mathematical narrative is also suitable for assessments.

In this article, two mathematics educators and two school teachers will share their experience pertaining to the use of mathematical narrative in their teaching. Their teaching practices are described below.

2 HPM and mathematical narrative practices

2.1 Practice for mathematics majors in the tertiary level

Why do we talk about the history of mathematics when teaching mathematics? There are many suggestions regarding this matter, from local and international circles. For example, Furinghetti and Paola (2003) explain that teaching history in the classroom not only stimulates the interest of the students in mathematics, but also makes it possible to arrange the classes according to the main mathematical contexts or concepts, thus, helping to achieve the objective of learning mathematics. By contrast, Barbin (2000) considers that the most common reason we teach the historical dimension of mathematics is to enable us to think about its raison d'être, thus, helping us to comprehend the concepts and theories of the subject. Tzanakis and Arcavi (2000) state that including history in mathematics teaching could help in the learning process, allowing teachers to learn the nature of mathematics and its development; in this way, history would enhance the understanding of teachers, who would then view mathematics as part of the cultural development of knowledge. By integrating the history of mathematics into the teaching of mathematics, educational goals can be reached, covering affective, cognitive, and cultural aspects. What role does the history of mathematics play in the current national mathematics curriculum? Fasanelli (2000) analyses 16 countries that implemented this strategy, and observed an improvement in the affective level among the students in China, Greece, Italy, the Netherlands, and Poland. An increase in the affective level can lead to greater learning motivation and interest from the students, and also inspire them through learning about the biographies of mathematicians. History in mathematics learning allows the students to compare different solutions or directions of thinking through mathematical texts, freeing them from a single way of thinking. Regarding cultural activities, students from Brazil, Denmark, France, New Zealand, Norway, and the United States, achieved a meaningful learning experience mainly through the understanding of the mathematical development of all ethnic groups, thus cherishing their own mathematical culture. In the nine-year compulsory education system of Taiwan, the importance of history for learning mathematics has also been proposed (Ministry of Education, 2008). Thus, introducing the history of mathematics into mathematics instruction is necessary.

This project is carried out with qualitative research method, based on such comprehension as retrieving and accessing, interpreting and integrating, reflecting and evaluating (OECD, 2010), by reading Heron's Formula texts, to re-produce in classrooms the lively thoughts among mathematicians of different eras. The main material in this activity is the original proof of Heron's formula by Heron, and also another procedure that is equivalent to Heron's formula found in a thirteenth-century Chinese mathematical text written by Qin Jiushao. It is hoped that, through the design of history of mathematics reading materials, the students will be able to discuss the differences between Greek and Chinese mathematics cultures. A total of 39 university students of mathematics participated in this study. These students were at 3rd and 4th grade. Each participant read the proofs of Heron's Formula and finished the worksheet which was designed by the teacher. These worksheets were collected and analysed.

This study has discovered the following:

1. Retrieving and accessing process: All students could illustrate by themselves the most beautiful part of Heron's original proof.

2. Interpreting and integrating process: All students could understand Heron's original proof, only parts of the students can use Triangle area formula and The Pythagorean Theorem to derivate Qin Jiushao's formula.
3. Reflecting and evaluating process: The students would reflect over the way mathematicians solve these problems. And this includes the idea that mathematicians can use various approaches to solve the same problem and the idea that Greek mathematics and Chinese mathematics exhibit different styles of argumentations and expressions.

2.2 Practice in the higher secondary level

In this part, we discuss the teaching in an elective course of mathematics in the senior high school level in Taiwan (10^{th}-12^{th} grade). This was a six-week (twelve-hour) course. In order to show different characteristics with other schools' mathematics courses, the course focused on 'polyhedron and geometry' and the allegedly five most beautiful mathematical statements:

$$e^{i\pi}+1=0,$$

$$V-E+F=2,$$

There are exactly five regular polyhedra,

There are infinite many prime numbers,

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

Since the last equation is too hard for 10^{th} -grade students, this equation was only showed but not discussed in depth the course.

At the beginning, some historical stories and snippets about geometry were told to students, then, the primary sources in Euclid's *Elements* were used to introduce the origin of the five Platonic polyhedra. The teacher also used the software GSP to draw and introduce Platonic polyhedra *and* Archimedean solids and their developed views. Then, the teacher asked students to design and create models of Platonic polyhedra *and* Archimedean solids, and then they used these models to guess and discuss the Euler formula '$V - E + F = 2$'. After that, the teacher designed a historical package about the mathematician Euler, his life and great works. Finally, the teacher guided students to understand how to prove that there are just only five Platonic polyhedra.

After that, the teacher let students watch the movie *The Professor's Beloved Equation* (*Hakase no aishita sûshiki*, 2006) which is about the formula '$e^{i\pi} + 1 = 0$', and discussed the mathematics they learned from this movie and the reflections about this movie. The teacher also told some historical stories and snippets about prime numbers, perfect numbers and friendly number pairs that appeared in the movie. Students in the class also discussed the correspondence between the main roles in this movie and the constants in the '$e^{i\pi} + 1 = 0$'. Through this course, the teacher combined the abstract mathematics with the image of literature and combined amusement with mathematics learning.

In the third part of the course, the teacher introduced the mathematics history and related mathematic culture about ancient Greece, ancient China and Wasan, the traditional mathematics of Japan in the Edo period and also introduced Greek text *Elements*, Chinese text *Nine Chapters* and some Wasan texts. The teacher used the *reductio ad absurdum* to prove that there are infinite many prime numbers, discussed many kinds of proof about the Pythagorean Theorem in each ancient text, introduced the proofs of many kinds of area formulae, that of sphere volume and some intuitive arguments by graphs in the *Nine Chapters*, and finally the teacher introduced the Wasan mathematician, Seki Takakazu, and his interesting works, along with the mathematic culture of Sangaku (mathematical tablets in shrines and temples) and some interesting mathematical problems written on the Sangaku.

The final part of the course was the assessment, which was held in the form of an achievements exhibition. Students could read a popular mathematics book and show their reflections, and they could also design and display creative works about mathematics or about all kinds of geometry models and report their ideas and motivations to all students in the class. Some students wrote poems about mathematics, and one of whom used an ancient mathematics problem in *Sun Tzu Suanjing* to create a poem. Another student designed a mathematics puzzle which combines the awareness of environmental protection with a mathematics problem.

Finally, students showed their reflections and gains in a questionnaire survey about this course. The teacher summed up and analysed the answers of all students, then concluded that there were four major aspects they mentioned in the questionnaire:

1. History and culture of mathematics
2. Learning motivation and the interests in mathematics
3. Diversity in mathematics
4. Mathematical knowledge

Table 1. The average satisfaction degree of this course scored by students

	The subjects of each week	Average score (Max 7)
Week1	The five Platonic *polyhedrons and* Archimedean solids	5.68
	The g history of *polyhedrons*	5.59
Week2	Design the regular *polyhedrons*	6.11
	The Euler formula about *polyhedrons*	5.89
	The most beautiful mathematical equations	6
	The popular mathematics books and movies about mathematics	6.11
Week3	The developed views of *polyhedrons*	5.95
	The intersection of mathematics and aesthetics	5.98
Week4	The Housekeeper and the Professor	6.55
Week5	Geometry in ancient China, ancient Greece, and Wasan	5.61
	Pythagorean theorem and the proofs	5.55
Week6	Achievements Exhibition	5.34

Table 1 shows the average satisfaction level of this course scored by students in a 7-point scale. From this table, we can see that all items are above the average score 3.5. This shows that students were satisfied with this course in the learning process.

Students' favourite parts of this course were the movie, the developed views of polyhedra, the course of designing and creating models of polyhedra, and the introduction of the popular mathematics books and movies, the score of which are all above 6.

For most senior high school students in Taiwan, mathematics is the most abstract and most difficult subject. But this elective course of mathematics can serve as a supplement to the regular course of mathematics. Through more diversified teaching methods, including explaining and discussing, working in small teams, designing models, watching the movie and the achievements exhibition, and through the intersection of mathematics and aesthetics, literature, art, history, culture…etc, the teacher designed and combined various teaching materials in order to help students experience the interesting and useful aspects of mathematics and also expand more extensive learning views. In this way, we hope to help students discover the beauty of mathematics, and enhance their motives in learning.

2.3 Practice in the lower secondary level

Junior high school students in Taiwan seldom do any extracurricular reading about science and mathematics. According to our survey in one rural secondary school in Taiwan, although 38% of the students have extracurricular reading habits (which is already low enough percentage), less than 1% read popular science or mathematics. This may not represent the general situation in Taiwan, but it does tell us that we need to do something to enhance students' interests in mathematics. Some teachers in that school began to promote popular mathematics reading and writing from students' perspectives.

In a span of more than five years, the teachers in that school tried to use several approaches to enhance students' interests in mathematics and their aptitudes in reading and writing about mathematics. Their strategies are listed below.

1. **Introducing popular mathematics publications in class**. Mathematics teachers classified popular mathematics publications into four categories and introduced them to students, helping them to find their interests in reading. The four kinds are: visual attractions, personal experiences, curiosity orientations, and narrative creations.
2. **Introducing popular mathematics publications with social media and their website**. They created an internal website for teachers and students in their school to discuss popular mathematics and mathematical knowledge.
3. **Creating reading environments in classrooms and campus**. They made an 'ema pavilion' outside of their mathematics laboratory. 'Ema' is a kind of small wooden plaques for Japanese Buddhists and Shinto worshippers to write their wishes. They put mathematical problems on the ema, and students could try to answer them, put them own solutions for public examination, and win rewards.

4. **Organising annual events: 'mathematical week'**. Students could use their mathematical knowledge to solve problems in personal and inter-class challenges. This gave them motivations and opportunities of self-learning.
5. **Publishing school magazine *Mathematics Fast Food***. This magazine has been published for more than 15 years. It has columns such as 'Teachers Column', 'Student Creations', 'Mathematics in Life', and 'New Horizons in Mathematics'. Students could appreciate the omnipresence of mathematics through this magazine.

With these approaches, teachers in this school successfully encouraged students in their mathematics reading and learning. Students wrote their reflexions about their reading, drew cartoons, created poems and mathematics riddles, and some even wrote novels. Some students had bigger projects such as making polyhedral lanterns and Escher-style tessellations. Although students did sometimes have some difficulties in reading pre-modern texts or even modern publications, these teachers' efforts have made mathematics more accessible to students, and created smiles on students' faces when they read and write about mathematics.

2.4 From history to literature – a practice in liberal-arts mathematics

In his *The Pythagorean Theorem: A 4000-Year History*, Eli Maor explains a paramount criterion for a theorem or the proof thereof to be called beautiful is symmetry. His example is the proposition that 'the three altitudes of a triangle always meet at one point'. Analogously, he also mentions the same truth holds for the medians and the angle bisectors. 'This statement,' he comments, has certain 'elegance to it, with its sweeping symmetry: no side or vertex takes precedence over any other; there is a complete democracy among the constituents.' Yet, the most interesting proposition he raises to me is as follows: If through a point *P* inside a circle a chord *AB* is drawn, the product $PA \times PB$ is constant – it has the same value for all chords through *P*. The literary metaphor he uses to illuminate the meaning of the proposition is as insightful as convincing: 'Again we have the perfect democracy: every chord has the same status in relation to *P* as any other.'

For years Prof Wann-Sheng Horng has used the metaphor to examine if students who attend his liberal study course at National Taiwan University, 'Mathematics and Culture: An approach of reading mathematical fiction,' benefit by thinking of mathematics in terms of literary metaphor. So at the first class of the course, he would encourage the students to answer a questionnaire with the following queries:

• Recall junior high school mathematics; try to prove the following theorem:

If through a point P inside a circle a chord AB is drawn, the product $PA \times PB$ *is constant.*

Hint: Through the point *P* draw any other chord *CD*, prove that $PA \times PB = PC \times PD$.

(1) Eli Maor comments on the proposition in his *The Pythagorean Theorem: A 4000-Year History* as follows:

‘*[Again] we have perfect democracy: every chord has the same status in relation to P as any other.*’

Question: Does the metaphor have any relation to the content of the proposition? Try to explain briefly.

(2) How does the argument of the proof benefit your understanding the metaphor? Try to explain briefly.

(3) On the contrary, does his metaphor enhance your understanding of the proof and its meaning? Try to explain briefly.

(4) Is there any similar task in your past experience of learning mathematics? If yes, in what kind of situations (say classroom)? Explain.

• Instead of the above questionnaire I also give an alternative version like the following:

Recall junior high school mathematics; try to prove the following theorem:

If through a point P inside a circle a chord AB is drawn, the product $PA \times PB$ *is constant.*

Hint: Through the point *P* draw any other chord *CD*, prove that $PA \times PB = PC \times PD$.

(1) Eli Maor comments in his *The Pythagorean Theorem: A 4000-Year History* that ‘*[Again] we have perfect democracy: every chord has the same status in relation to P as any other.*’

Regarding the metaphor Maor has employed for the geometrical proposition, please try to explain his popular writing approach.

(2) There are two main aspects of mathematical activities: interesting and useful. Which aspect is more likely to engage you in mathematics without assessment pressure like examination? Try to explain your reason.

In his presentation, he uses the questionnaires, among others, given in the liberal study course as both the analysing and explanatory tool in order to show how college students can be enhanced in their understanding of mathematics by means of literary metaphor. The results show that students who were relatively weak in mathematics had gained different views about mathematics.

3 Concluding remarks

From the above practices, we found that by introducing students to the history of mathematics, they have learned to realise the relation of mathematics to human activities, which have inspired the development and creativity of cultures. We hope that this article can act as a reference in providing this experience to the HPM community.

REFERENCES

Barbin, E. *et al*. (2000). Integrating history: Research perspectives. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education* (pp. 63-90). Dordrecht: Kluwer Academic Publishers.

Fasanelli, F. *et al*. (2000). The political context. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education* (pp. 1-38). Dordrecht: Kluwer Academic Publishers.

Furinghetti, F., & Paola, D. (2003). History as a crossroads of mathematical culture and educational needs in the classroom. *Mathematics in School, 32*(1), 37-41.

Ministry of Education (2008). *Grade 1-9 Curriculum Guidelines*. Taipei: Ministry of Education.

Executive Yuan (2010). Taiwan PISA 2009 report. Retrieved from http://www.ey.gov.tw/eycc/News_Content.aspx?n=DF52F83A5B7D2A47&sms=114B66117B4BF117&s=D0C45A32CD5F9B3B.

OECD (2010). PISA 2009 Assessment framework: Key competencies in reading, mathematics and science. Paris, France: OECD Publishing.

Solomon, Y., & O'Neill, J. (1998) Mathematics and narrative. *Language and Education, 12*(3), 210-221.

Tzanakis, C., & Arcavi, A. *et al*. (2000). Integrating history of mathematics in the classroom: An analytic survey. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education* (pp. 201-240). Dordrecht: Kluwer Academic Publishers.

CONIC AND CUBIC NOMOGRAMS

Dominique TOURNÈS
University of La Réunion, Laboratory of Computer Science and Mathematics
PTU, 2 rue Joseph-Wetzell, F-97490 Sainte-Clotilde, France
dominique.tournes@univ-reunion.fr

ABSTRACT

Conic and cubic nomograms were created in the first years of the 20th century by the civil engineer C. E. Wolff and the mathematician J. Clark, who were professors at the Cairo Polytechnic School. After a historical introduction, I will present the theory of these particular graphical tables and the participants of the workshop will experiment how to construct and use them, both on paper and with geometry software. These nomograms, which have many applications in mathematics and other domains, can be fruitfully exploited in the last years of secondary schools and first years or universities in an interdisciplinary context. Some activities involving them will be presented, as I could test them with students and future teachers. Such activities mix geometry, algebra and calculus, and so they permit constant interplay between frameworks.

REFERENCES

Clark, J. (1907). Théorie générale des abaques d'alignement de tout ordre. *Revue de mécanique, 21*, 321-335, 576-585; *22*(1908), 238-263, 451-472.

D'Ocagne, M. (1921). *Traité de nomographie. Étude générale de la représentation graphique cotée des équations à un nombre quelconque de variables, applications pratiques.* 2e éd. Paris: Gauthier-Villars.

Soreau, R. (1921). *Nomographie ou Traité des abaques*, 2 vols. Paris: Chiron.

Tournès, D. (2011). Une discipline à la croisée de savoirs et d'intérêts multiples : La nomographie. In P. Ageron, & É. Barbin (Eds.), *Circulation Transmission Héritage* (pp. 415–448). Caen: Université de Caen-Basse-Normandie.

Tournès, D. (2012). Calculer avec des hyperboles et des paraboles. In É. Barbin (Ed.), *Des mathématiques éclairées par l'histoire. Des arpenteurs aux ingénieurs* (pp. 131–148). Paris: Vuibert.

Wolff, C. E. (1903). *Diagrams for Egyptian Engineers*. Cairo: Al-Shaab Printing Office.

INTEGRATING THE HISTORY OF MATHEMATICS INTO TEACHING: CASES FROM CHINA

Jiachen ZOU, Yanjun HONG, Zhongyu SHEN and Xiaoqin WANG
Department of Mathematics, East China Normal University,
500 Dongchuan Road, Shanghai, China
jczou@math.ecnu.edu.cn

ABSTRACT

With further development of instruction designs on integrating the history of mathematics into mathematics teaching, more teachers and mathematics education researchers are paying more attentions on how to integrate the history of mathematics into mathematics teaching. In this workshop, our discussion will focus on the following topics:

- the definition of function;
- the sum of the angles of a triangle;
- the definition of prism.

These topics are the teaching contents of current curriculum in middle school in China. Many mathematics teachers find it difficult to teach these topics and many students also find it difficult to learn them. Thus we try to use the history of mathematics to get a better approach to teach these topics and we put it into practice. How to transform historical materials into didactical materials on these topics? In this workshop, we will address these questions by following the following steps:

Step1: we will discuss and analyze some original materials, and choose some of them to integrate them into the teaching of these topics.

Step2: we will discuss how to transform original materials into teaching materials and make instruction designs on these topics from the perspective of HPM.

Step3: we will show some pieces of classroom experiments about these topics as implemented in Shanghai and Zhejiang, China.

In order to get historical materials on these topics, we refer to some original documents on the history of mathematics and textbooks in history.

REFERENCES

Lagrange, L. L. (1797). *Théorie des fonctionsanalytiques*. Paris: De L' Imprimerie de la République.

Lacroix, S. F. (1797). *Traité du CalculDifférentielet du CalculIntégral* (*T. 1*), Paris: J. B. M. Duprat.

Cournot, A. A. (1841). *TraitéElémentaire de la Théorie des Fonctionset du CalculInfinitésimal* (*Tome 1*). Paris: L. Hachette.

Leibniz, G. W. (1694). Nova calculi deifferentiali sapplicatio & usus, ad multiplicem linear umconstructionem, ex data tangentium conditione. *ActaEditorum*.

Euler, L. (1988). *Introduction to Analysis of the Infinite* (*translated by J. D. Blanton*). New York: Springer-Verlag.

Wallace, W. (1823). Function. In *Encyclopaedia Britannica* (*the 6th edition, Vol. 9*), Edinburgh: Archibald

Constable & Company.
Hutton, C. (1815). *A Mathematical & Philosophical Dictionary*. London: S. Hamilton.
De Morgan, A. (1838). Function.*The Penny Cyclopaedia of the Society for the Diffusion of Useful Knowledge* (*Vol. 11*). London: Charles Knight & Co.
Beman, W. W., & Smith, D. E. (1900). *Elements of algebra*. Boston: Ginn & company.

6. Posters

TRANSFORMING INSTRUCTION IN UNDERGRADUATE MATHEMATICS VIA PRIMARY HISTORICAL SOURCES

Janet Heine BARNETT, Kathy CLARK, Dominic KLYVE, Danny OTERO, Nicholas A SCOVILLE, Diana WHITE
[1]Colorado State University – Pueblo, 2200 Bonforte Boulevard, Pueblo, USA
Janet.Barnett@csupueblo.edu
[2]Florida State University, 1114 West Call Street, Tallahassee, USA
kclark@fsu.edu
[3]Central Washington University, 400 E. University Way, Ellensburg, USA
klyved@cwu.edu
[4]Xavier University, 3800 Victory Parkway, Cincinnati, USA
otero@xavier.edu
[5]Ursinus College, 601 E. Main Street, Collegeville, USA
nscoville@ursinus.edu
[6]University of Colorado Denver, 1201 Larimer St., Suite 4118, Denver, USA
Diana.White@ucdenver.edu

ABSTRACT

Mathematics faculty and education researchers increasingly recognize the value of the history of mathematics as a support to student learning. There is an expanding body of literature in this area which includes direct calls for the use of primary historical sources in teaching mathematics. The current lack of classroom-ready materials poses an obstacle to the incorporation of history into the mathematics classroom. Transforming Instruction in Undergraduate Mathematics via Primary History Sources (TRIUMPHS) is a seven-institution collaboration that will design, implement, test, and publish curricular materials based on primary historical sources, train approximately 70 faculty and graduate students on their development or implementation, and conduct and evaluation-with-research study. TRIUMPHS will employ an integrated training and development process to create and test at least 20 full-length primary source projects (PSPs) and 30 one-day "mini-PSPs."

Project Rationale and Overview

In addition to the general benefits of inquiry-based learning, particular advantages of this approach of incorporating primary sources include providing context and direction to the subject matter, honing students' verbal and deductive skills through reading the work of some of the greatest minds in history, and the invigoration of undergraduate mathematics courses by identifying the problems and pioneering solutions that have since been subsumed into standard curricular topics. By working collaboratively to develop PSPs while training faculty across the country in their use, TRIUMPHS will ensure these materials are robustly adaptable to a wide variety of institutional settings, while simultaneously developing an ongoing professional community of mathematics faculty. Additionally, our evaluation-with-research study will directly contribute to a greater understanding of (a) how student perceptions of the nature of mathematics evolve,

(b) how students’ ability to write mathematical arguments changes over time, and (c) how to support faculty in developing and implementing this research-based, active learning approach in undergraduate STEM education.

PANDEY'S METHOD OF CUBE ROOT EXTRACTION: IS IT BETTER THAN ARYABHATA'S METHOD?

Deepak BASYAL
University of Wisconsin Colleges, Wisconsin, USA
Deepak.Basyal@uwc.edu

1 Gopal Pandey and Aryabhata's method for cube root extraction

Gopal Pandey (?1847-1921) occupies an important place in the history of Nepalese mathematics. His book *Vyakta Chandrika* (1884) is the first mathematics book written in Nepali (Basyal, 2015).

Pant (1980) observes that Aryabhata's method (AM) (499) for finding the cube root is adopted by Brahmagupta (628), Sridhara (~750), Aryabhata II (~950), Sripati (1039), Bhaskara II (1148) and Narayana (1356), and the rules given by Chandrashekhara (1869), and Pandey's teacher Bapudeva Shastri (1883) are slight modifications of the AM. On the other hand Parakh (2006) points out that the root extraction rules taught today in schools are essentially an extension of AM. To extract the cube root of N using AM, the binomial expansion of $N = (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ is reversed. Briefly, AM separates N into groups of three digits starting from the unit place, subtracts the biggest possible cube from the left-most group to produce digit a in the root, and divides the next group in N by $3a^2$. The quotient thus produced will be an estimate say $\hat{b}$ for b. We observe $\hat{b} \geq b$. An example of finding the cube root by AM can be found in p. 126 of Plofker (2009).

1.1 Pandey's method (PM) uses the Rule of Three

The rule of three provides three quantities A, B and C and seeks $\hat{b}$ such that $\frac{A}{B} = \frac{C}{\hat{b}}$. The PM sets $A = 1000[(a + 1)^3 - a^3]$, $B = 10$, and $C = (Two\ leftmost\ groups\ in\ N) - (10a)^3$, and seeks $\hat{b}$. Unlike in AM, the PM produces $\hat{b}$ such that $b - 2 < \hat{b} < b + 1$.

Thus, we note that PM may round $\hat{b}$ down (as in AM) or up to get b. To handle this situation PM suggests rounding up or down according to $\hat{b} < 5$ and $\hat{b} > 5$ respectively. This trick works most of the time, however, it does not always guarantee b. So what is the value of discussing PM? We note that sometimes $\hat{b}$ produced by AM is no better than $\hat{b}$ produced by PM because the former is not bounded above as nicely as the latter. I plan to publish a detailed analysis of PM in the near future.

REFERENCES

Basyal, D. (2015). *A 1933 Nepali mathematics and astrology book Śiśubodha Taraṅgiṇī II: Translation and commentary on mathematics chapters*. Doctoral dissertation, New Mexico State University.

Pandey, G. (1895). *Vyakta Chandrika (Second edition).* Mumbai, India: Nirnayasagar Chapakhana.

Pant, N. (1980). *Pandita Gopal Pandey ra unako ghanamula lyaune riti*. Kaṭhmandu, Nepal: Rajakiye Pragya-Pratishthan.

Parakh, A. (2006). Aryabhata's Root Extraction Methods. *arXiv preprint math/0608793*.

Plofker, K. (2009). *Mathematics in India*. Princeton, NJ: Princeton University Press.

HOW TO USE PROGRAMMING TOOLS TO ANALYSE HISTORICAL MATHEMATICAL TEXTS: A CLASSROOM EXPERIMENT CONCERNING THE NEWTON-RAPHSON METHOD

Mónica BLANCO
Universitat Politècnica de Catalunya – BarcelontaTech
ESAB, Campus del Baix Llobregat, c/ Esteve Terradas, 8, 08860 Castelldefels (Barcelona), Spain
monica.blanco@upc.edu

The Newton-Raphson method is a well-known numerical method for finding approximations to the real roots of a real-valued function. It is named after Isaac Newton (1643-1727) and Joseph Raphson (1668-1715), who, towards the end of the 17th century, elaborated their methods for finding the approximate roots of polynomial equations. However, if we look at the original sources, we soon realise that both methods differ not only from each other, but also from the algorithm used at present. The main differences concern the approach, the algorithmic efficiency and the role played by the differential expressions. It was actually Thomas Simpson (1710-1761) who, in 1740, published the method in its current form. This contribution focuses on the design, implementation and assessment of a classroom activity concerning the historical development of the so-called Newton-Raphson method. Although the activity was originally designed for the course on History of Mathematics (Bachelor's Degree in Mathematics, School of Mathematics and Statistics, Universitat Politècnica de Catalunya-BarcelonaTech), it could also be put into practice in courses on advanced mathematics in other contexts.

The aim of the activity is to explore the application of programming tools to analyse and compare the methods of Newton, Raphson and Simpson, as they were published originally. After having read and discussed the original sources, students have to develop routines to execute the three methods with the help of suitable mathematical software. From the routines developed, it is worth comparing aspects such as the accuracy of the results, the algorithmic efficiency and the scope of each method. Besides, this activity allows discussing the connection between algebra and calculus, at a time when the latter was not yet solidly articulated. The comparative analysis of the original sources help students understand the historical development of the method of Newton-Raphson. Yet, the discussion can even go beyond this specific method, leading towards critical reflection about the use, and misuse, of the mathematical reinterpretation of a historical result from today's standpoint. In short, the use of programming tools can shed new light on the analysis of the historical mathematical texts involved in the classroom activity presented.

QUAND EMMA CASTELNUOVO RENCONTRA ALEXIS-CLAUDE CLAIRAUT

L'histoire d'un esprit novateur dans l'enseignement de la géométrie pour des jeunes élèves

Valentina CELI
Université de Bordeaux, ESPE d'Aquitaine, Lab-E3D, France
valentina.celi@u-bordeaux.fr

Alexis Claude Clairaut (1713-1765), géophysicien et mathématicien français, publie en 1741 ses *Éléments de géométrie*. Désapprouvant les auteurs qui conçoivent le savoir mathématique comme un produit fini et non pas comme un processus (Barbin, 1991) et s'opposant à l'ordre traditionnel de présentation des contenus, il propose de construire un savoir géométrique en réponse à des problèmes concrets : c'est ainsi que, par exemple, les enclos de maisons à bâtir sont prétexte pour introduire les polygones et les canaux pour identifier des droites parallèles.

Quelque deux cents ans après sa publication, l'ouvrage de Clairaut éclaire la pensée d'Emma Castelnuovo (1913-2014), enseignante italienne de mathématiques qui s'interroge sur les effets d'un cours magistral de géométrie sur les apprentissages de ses élèves. Et elle va plus loin (Celi, 2014) : en plus d'élaborer un ordre nouveau dans la présentation de contenus géométriques (Castelnuovo, 1959), elle propose une méthode constructive pour laquelle *on est obligé d'avoir recours à des bases concrètes* (Castelnuovo, 1958).

En mettant en regard quelques pages des *Éléments* de Clairaut avec des extraits de ressources pédagogiques d'Emma Castelnuovo, nous souhaitons montrer que cette dernière ne se limite pas à une *exposition magistrale à partir de problèmes* (Glaeser, 1983) mais met entre les mains de ses élèves du matériel dynamique : la manipulation des objets et l'observation de leurs modifications encouragent alors l'apprenant à s'affranchir du concret pour s'orienter vers le formel et l'abstrait (Castelnuovo, 1958 ; 1964).

REFERENCES

Barbin, E. (1991). Les éléments de géométrie de Clairaut : une géométrie problématisée. *Repères-IREM*, *4*, 119-133.

Castelnuovo, E. (1958). L'objet et l'action dans l'enseignement de la géométrie intuitive. In Gattegno & *alii*., *Le matériel pour l'enseignement des mathématiques* (pp. 41-59). Neuchâtel : Delachaux & Niestlé.

Castelnuovo, E. (1959). *Geometria intuitiva*. Firenze: La Nuova Italia Editrice.

Castelnuovo, E. (1964). *Costruiamo la geometria. Scatola di materiale utilizzabile per la costruzione e lo studio di poligoni articolabili*. Firenze: La Nuova Italia Editrice.

Celi, V. (2014), La géométrie intuitive d'Emma Castelnuovo. *IREM de Grenoble, Petit x*, *95*, 71-82.

Clairaut, A. C. (1920). *Éléments de géométrie*. Gauthier-Villars et Cie éditeurs.

Glaeser, G. (1983). À propos de la pédagogie de Clairaut. Vers une nouvelle orientation dans l'histoire de l'éducation. *Recherches en Didactique des Mathématiques*, *4*(3), 332–344.

FIBONACCI NUMBERS IN NETWORKS AND OTHER SYMMETRIES

Roberto CAPONE, Maria Rosaria DEL SORBO, Oriana FIORE
University of Salerno, via Giovanni Paolo II, Fisciano, Italy
rcapone@unisa.it
madelsorbo@unisa.it
oriana.fiore@istruzione.it

Symmetries are very present in nature. The regularity of some objects, both at macroscopic and microscopic levels, is studied in many disciplines such as Biology, Chemistry, Mathematics and Physics. Already with the ancient Greeks, the study of some solids with particular symmetries (i.e., "Platonic" and "Archimedean" solids) aroused interest. Moreover, the characteristics of the "divina proportione" were well known and employed in order to construct temples and buildings and to sculpt statues. During the 12th century, the Italian mathematician Fibonacci found a numeric sequence, which takes his name, in an attempt to model a mathematical law describing the growth of rabbit populations. Furthermore, the Fibonacci sequence can be found in many objects representing regularities.

In this work, we describe the experience of two Italian High Schools in which some electrical networks, where the Fibonacci sequence is present, have been analyzed, together with some other networks that present particular symmetries. The calculation and measurement methods adopted have shown that the subject can be easily introduced as a lecture and as a laboratory session in High School.

By making use of particular symmetries of regular networks, the equivalent resistance, as measured by connecting two leads to two nodes of the network itself, can be calculated with no much more effort than in usual cases. Starting from a single elementary cell of equivalent resistance $2R$ and adding a second elementary cell, an equivalent resistance equal to $5/3\,R$ is obtained. Continuing the process of adding elementary cells to the initial one, we obtain a sequence that resembles the Fibonacci one. The equivalent resistance of our network with n cells can be then generalized to a_{2n+2}/a_{2n+1}. If $n \to \infty$ this ratio tends to the known golden ratio number of 1.618 ...

Furthermore, we measured the equivalent resistance in some specific nodes of Platonic solids both with numeric and experimental methods.

REFERENCES

Van Steenwijk, F. J. (1998). Equivalent resistors of polyhedral resistive structures. *American Journal of Physics*, *66*(1), 90-91.

Bapat, R. B., Gutmana, I., & Xiao, W. (2003). A simple method for computing resistance distance. *Zeitschrift für Naturforschung A*, *58*(9-10), 494-498.

D'Amore, B., Godino, D. J., Arrigo, G., & Fandiño Pinilla, M. I. (2003). *Competenze in matematica*. Pitagora Editrice Bologna.

PLIMPTON 322

A video documentary that uses history to motivate students to study mathematics

Laurence KIRBY
Baruch College, City University of New York, 1 Bernard Baruch Way, New York, USA
laurence.kirby@baruch.cuny.edu

ABSTRACT

This is an introduction to the 33-minute film *Plimpton 322: The Ancient Roots of Modern Mathematics*, which motivates college and high school students – especially minority students – to pursue mathematics through an exploration of its history.

The mathematics that drives our modern world owes its origins to ancient cultures in the Middle East, Asia and Africa. Set against a backdrop of today's New York City, the film explores the extent of our debt to this tradition. Along the way we meet up close some precious and revealing ancient artefacts that now have their homes in New York, notably a controversial cuneiform tablet from Mesopotamia known as Plimpton 322. We witness ancient mathematical ideas still playing crucial roles in 21st-century society and technology. This film celebrates the diversity underlying our mathematical culture. Teaching at a large, urban, multiethnic university, I have found it useful in encouraging students in courses ranging from mathematics for liberal arts students to history of mathematics.

The film incorporates brief introductions to two mathematical topics, positional number notation and Pythagorean triples, which can be developed in the classroom.

The purpose of my presentation is to bring this freely available resource to the attention of educators and to discuss with anyone interested its use in the classroom. As well as a poster, I shall display excerpts from the film on a tablet computer.

The film is at http://faculty.baruch.cuny.edu/lkirby.

ENGINEERING ACADEMIC PROGRAM BASED ON THE CURRICULAR INTEGRATION OF MATHEMATICS, PHYSICS AND COMPUTER TOPICS

Rubén SANTIAGO-ACOSTA, Lourdes QUEZADA, Ernesto HERNÁNDEZ-COOPER
Tecnológico de Monterrey, Campus Estado de México, Carr Lago de Guadalupe km 3.5, Col: Margarita Maza de Juárez, Atizapán de Zaragoza, Estado de México, México.
ruben.dario@itesm.mx
lquezada@itesm.mx
emcooper@itesm.mx

ABSTRACT

In this poster, we present an engineering academic program based on the curricular integration of mathematics, physics and computer topics called Principia Program. The main purpose of this program is to create a scientific and technological culture that will enable students to analyze and solve complex problems. For the time being, the program has been planned and implemented for junior engineering students. Some of the basic tools used in this program; are problem based learning (PBL), challenge based learning (CBL) and heavy use of computer technology. The math topics are focused on calculus of one or more variables and differential equations. These math topics are used for modeling physical phenomena related with classical mechanics, thermodynamics and electromagnetism. In many cases, our students need to apply numerical methods in several computer languages, to find viable solutions to the physical problems at hand.

1 The Principia Program

Within the design of this program, we have considered five fundamental principles: a) Curriculum integration for mathematics, physics, and computer sciences. b) Collaborative learning. c) Teamwork. d) Emphasis on mathematical modeling. e) Use of technology in the learning process (Polanco, Calderón & Delgado, 2004). On the other hand, PBL and CBL are the most frequent activities we use in order to develop the required skills in our students. The goal of these learning techniques is to confront students with complex, usually multidisciplinary problems, to promote teamwork. Problems should be sufficiently complex so that student's prior knowledge and conceptual frameworks become insufficient to solve them. Therefore, during the initial discussions, the problems should trigger questions that guide student's search for information and self-directed learning. Under these conditions, learning is motivated by the student's questions.

In this work, three examples of CBL or PBL activities will be shown, and the student's solutions will be analyzed. In order to solve and analyze the assigned problems about simple object's motion, students must use Newton's Second Law, some basic calculus ideas and Euler's Method. In the reports provided by our students, we can observe the evolution of their mathematical reasoning. In this poster, we will share our experiences with the students enrolled in this engineering program and we will show some statistical and comparative results, linked to the development of our student's competences and skills.

REFERENCES

Polanco, R., Calderón, P., & Delgado, F. (2004). Effects of a problem-based learning program on engineering students' academic achievements in a Mexican university 1. *Innovations in Education and Teaching International*, *41*(2), 145-155.

MATHEMATICS AT THE ROYAL DANISH MILITARY ACADEMY OF 1830

Asger SENBERGS
Roskilde University, Aboretvej 1, 4000 Roskilde, Denmark
asse@ruc.dk

Based on an analysis of source material from the foundation of the Royal Danish Military Academy of 1830, it is concluded that mathematics was chosen as the most important subject, because it was seen as a necessity in order to learn the other subjects, and because it was perceived as enhancing the students' cognitive abilities.

The Royal Danish Military Academy was established in 1830 as a further education of officers, so they could match the growing army and use the advancing weapon systems. The main idea for the academy was most likely inspired by the French academy Ecole Polytechnique, which is described in a military anniversary book from 1855, proclaiming that most conditions are copied from Ecole Polytechnique, but adapted according to the Danish economy and army size. The academy consisted of two classes, which each took two years to complete – the youngest and the oldest class. During the youngest class the students were taught in the "purely scientific subjects" and in the oldest class they were taught the "applicable subjects". Mathematics, which consisted of analytics, rational mechanics and descriptive geometry, was taught at the youngest class and these mathematical subjects made up 25% of the entire education. Through my analysis of the main plan for this academy it has become clear that the intention was to make an academy that educated not only good and effective soldiers on the battlefield, but also men with a high moral standard, general knowledge, scientific insights, and with a logical approach in all matters of life. Therefore, these officer students were taught in subjects spanning from mathematic, physics and chemistry to Danish literature, military history and geography in the youngest class, and subjects about mines, artillery, fortifications, and march in terrain in the oldest class. On the one hand, mathematics was seen as a foundation for most subjects taught at the oldest class, and on the other hand the mathematical method was seen as a way to train the mind in approaching problems in a logical and systematic way suited for an officer. Still, the high level of mathematics taught at the academy does suggest an intention of having the students learn mathematics for the sake of the mathematical subject and not only mathematics applications. Indeed it seems that mathematics served three purposes: (1) the subject in itself; (2) as a foundation for other subjects; and (3) the mathematical method as a cognitive tool.

REFERENCES

Bülow, F. C. (Ed.) (1830). *Plan til den kongelige militaire Højskole*. København.

Caroc, F. C. V., Bjerring, V. J., Reich, C. E., & Købke, J. P. (1855). *Udsigt over undervisningen ved de specialle corps før 1830 og den kongelige militære højskoles virksomhed fra den oprettelse til 1855*. København: Schultziske Bogtrykkeri.

HUMANITIES AND SCIENCES ON MACH'S IDEAS FOR A HIGHER EDUCATION

Luciana ZUCCHERI, Verena ZUDINI
Dipartimento di Matematica e Geoscienze, Università di Trieste
Via Valerio 12/1, 34127 Trieste, Italy
zuccheri@units.it
vzudini@units.it

Mach and mathematics education

Mach's position on humanities/sciences dichotomy regarding young people's education is illustrated, stressing his ideas regarding the importance of mathematics teaching.

According to the famous Austrian physicist and philosopher Ernst Mach (the centenary of whose death falls in 2016), studying science – in particular mathematics – proves to be fundamental in helping man to observe and understand the world around him and thus to act in an "economic" way (see, e.g., Mach 1889, pp. 577ff.); with this viewpoint, scientific education should be strongly pursued. There are in fact illuminating examples of application of Mach's ideas in mathematics education of his time (see Zuccheri & Zudini 2008).

The formative value of mathematics and sciences

Über den relativen Bildungswert der philologischen und der mathematisch-naturwissenschaftlichen Unterrichtsfächer der höheren Schulen is the significant title of a conference held in 1886 by Mach and contained in his *Populär-wissenschaftliche Vorlesungen* (Mach 1896, pp. 338-374). Mach shows himself to be a very modern scholar in his treatment of the relationship between humanities and sciences and their formative value. He argues that, within the cultural development of his time, humanities cannot be considered any longer to be the only (nor even the better) means to offer a higher education.

Mach counters the usual arguments in favor of the supremacy of humanistic culture with the greater value and effectiveness of teaching mathematics and science (which for certain formative aspects cannot be separated). He gives specific examples, claiming their superiority with regard to educational aims and for the development of ability in observation and logic (see Mach 1896, pp. 344ff.). He provides a series of directions to be implemented for an improvement of mathematics and science education (see Mach, 1896, pp. 364ff.).

REFERENCES

Mach, E. (1889). *Die Mechanik in ihrer Entwicklung historisch-kritisch dargestellt. 2. Edition*. Leipzig: Brockhaus [English translation *The science of mechanics: A critical and historical account of its development*]. Chicago: Open Court, 1989.

Mach, E. (1896). *Populär-wissenschaftliche Vorlesungen*. Leipzig: Barth. English translation: *Popular scientific lectures.* Chicago: Open Court, 1898.

Zuccheri, L., & Zudini, V. (2008). The “Jacob Method”: An example of application of cognitive theories in the period of the introduction of calculus in Austrian secondary mathematics instruction. *The International Journal for the History of Mathematics Education*, *3*(2), 57-64.

7. Exhibitions

GERGONNE IN MONTPELLIER, THE UNPUBLISHED MANUSCRIPTS

Elizabeth DENTON, Christian GERINI
Bibliothèque Interuniversitaire de Montpellier (Sciences), Université de Montpellier, Campus Triolet, bât.8, Place Eugène Bataillon, 34095 MONTPELLIER CEDEX 5
elizabeth.denton@umontpellier.fr
Université de Toulon & laboratoire GHDSO (Université Paris 11 – Orsay), France
gerini@univ-tln.fr

This exhibition was put together especially for the Congress "History and Pedagogy of Mathematics", and is based on a series of unpublished manuscripts preserved in the Science University Library (Bibliothèque Interuniversitaire de Montpellier) since the nineteenth century. Since Gergonne was the chosen figurehead of this convention, it was deemed necessary to evoke the story of this important character in the history of mathematics.

The collection held at the University Library is composed both of manuscripts and books that Gergonne used in his personal library. A selection of these documents is showcased during the convention, together with the very famous portrait that is preserved also by the University of Montpellier. A few letters that have been received by him are interesting for the study of the *Annales de Mathématiques pures et appliquées.* They are shown together with some interesting extracts of the scientific lectures he gave as Astronomy and Mathematics professor. His correspondents are of various kinds: both eminent mathematicians from Paris, but also college teachers or engineers from the Ponts et Chaussées, as well as some of his colleagues in Montpellier, all linked together by their readings and comments on the *Annales.* The scientific conversation that occurs in these letters shows the leading role of the *Annales* as a major publication of that time.

Gergonne also started a particular kind of lecture about the philosophy of science; it was probably the very first time such a topic was lectured in a French university. He thought it was essential that the students learn to think about why and how they were making scientific research. This approach looks very modern, especially as the University of Montpellier has started a few years ago to hold an annual study day on epistemology. The manuscript of his thoughts on the subject is shown together with the summary of some of his lectures. As many of his contemporaries, Gergonne had a wide range of interests, some unusual like his study on stenography or musical writing, as our collection shows. It was also necessary to evoke – although briefly – his administrative work as dean of the Faculty (1820-1830) then rector of the académie of Montpellier (1830-1844); the relevant documents are preserved by the Archives départementales de l'Hérault.

This exhibition is supplemented by a digitization of the whole collection that is accessible online via the Bibliothèque Interuniversitaire's online Digital Library **Foli@**: http://www.biu-montpellier.fr/redir/folia. It will enable historians to acquaint themselves better with his work as a leading French mathematician and on his teaching methods.

Mathematical Viewpoints: Mediterranean Routes

IREM Marseille
Aix Marseille Université, Case 901, 163 avenue de Luminy, F-13288 MARSEILLE CEDEX 9
regards@irem.univ-mrs.fr

The exhibition "Mathematical Viewpoints: Mediterranean Routes" is a journey through the Mediterranean which explores the circulation of scientific knowledge, those who made the formulation of ideas and concepts, that are still alive in today's mathematics, possible.

Images, texts, objects and instruments are displayed around maps that show how mathematical science has developed and has been disseminated. A timeline also serves to situate the mentioned events in their historical context, highlighting the considerable time frame over which this adventure played out.

Thanks to the researches of recent decades, we now have a better grasp of the respective contributions of the different civilizations around the Mediterranean, from the historical perspective of the interactions and transmission of knowledge. The Babylonians laid down the beginnings of calculus and geometry and provided the experimental basis for astronomy. Ancient Greece saw a significant development of these concepts and the birth of scientific reasoning. As was shown by the end of the twentieth century, the Arab scholars of the Middle Ages were not only the founders of modern algebra, but especially the mediators between ancient science and medieval Europe. It was then left to Renaissance scientists to synthesize this impressive body of knowledge which made the emergence of modern science possible.

Without claiming to be exhaustive, this exhibition presents some particularly significant historical events, grouped together in five areas:

• *Reckoning*: How the numerals travelled around the Mediterranean, Writing numbers, Multiplication around the Mediterranean, Mathematics in Occitania

• *Measuring*: A Babylonian tablet IM55357, Archimedes – at the origin of calculus, The Mediterranean origins of trigonometry, Aristarchus and Eratosthenes - measuring Earth and Heaven, Measuring instruments - the astrolabe, the geometric square

• *Locating*: Pytheas measures the obliquity of the ecliptic plane, Stories of spheres, The early days of astronomy in the Mediterranean, Greek astronomy - philosophical and geometrical, Arabic and Persian astronomy - mathematical and religious, Galileo - a new dimension to the observation of the Heavens

• *Representing*: Plato's five solids, The three great problems of Antiquity, The adventure of conics, Perspective geometry, Tilings

• *The emergence of Mathematics*: A founding text of mathematics - Euclid's elements, Routes taken by Euclid's elements, Greece - birthplace of mathematical proof, Algebra ... before letters, Diophantus of Alexandria - the father of algebra?, Arabic algebra, Meeting the Italian algebraists - Cardan and Tartaglia, At the origin of modern science – Galileo, The modern circulation of knowledge - Gergonne's Annals

Inline documents: http://www.irem.univ-mrs.fr/expo2013

List of registered participants as of June 30, 2016

Abdeljaouad Mahdi, Abdelli Mouloud, Abdounur Oscar, Achour Samia, Ageron Pierre, Alexandre Raphaël, Attilio Ferrini, Auvinet Jérôme, Barbin Evelyne, Barnett Janet, Basyal Deepak, Bella Sandra, Bessot Didier, Bjarnadóttir Kristín, Blanco Monica, Bond Denise, Bonilla María Del Carmen, Booker George, Boyé Anne, Broglio Annie, Bühler Martine, Burdick Bruce, Caianiello Eva, Capone Roberto, Carlyle Ariana, Celi Valentina, Chang Pingying, Chen Yuh-Fen, Chevalarias Nathalie, Chorlay Renaud, Christiansen Andreas, Clark Kathy, Cortella Anne, Costa Clain Teresa, De Oliveira Marcus Aldenisson, De Vittori Thomas, Del Sorbo Maria Rosaria, Dello Iacono Umberto, Demattè Adriano, Ducel-Fages Yves, Durand-Guerrier Viviane, El Idrissi Abdellah, Fiore Oriana, Fitzsimons Gail, Fried Michael, Front Mathias, Furinghetti Fulvia, Gerini Christian, Gil Elena, Gosztonyi Katalin, Grimberg Gérard, Guichard Jacqueline, Guichard Jean-Paul, Guillemette David, Guitart René, Hausberger Thomas, Heine George, Hernández Cooper Ernesto Manuel, Hitchcock Gavin, Hong Yanjun, Horng Wann-Sheng, Huang Mei-Lun, Jahnke Hans Niels, Jankvist Uffe Thomas, Jullien Caroline, Katz Victor, Keller Olivier, Kirby Laurence, Kjeldsen Tinne Hoff, Lakoma Ewa, Languereau Hombeline, Lanser Peter, Laptyeva Nataliya, Laurent Frederic, Liao Hui I, Liu Po Hung, Magrone Paola, Makinae Naomichi, Marquez Mckenzie, Mazzeo Elio, Michel-Pajus Anne, Mikkelsen Line, Modeste Simon, Moretti Vanessa, Morey Bernadete, Moskowitz Stuart, Moyon Marc, Neuwirth Stefan, Oladejo Michael, Oswald Nicola, Otero Daniel, Pejlare Johanna, Pinto Hélder, Puig Luis, Qi Chunyan, Quezada Batalla Ma. De Lourdes, Radford Luis, Ransom Peter, Rogalski Janine, Rogalski Marc, Rogers Leo, Rottoli Ernesto, Rougetet Lisa, Saby Nicolas, Salone Jean-Jacques, Santiago-Acosta Ruben-Dario, Saracho Matias, Schorcht Sebastian, Scoville Nicholas, Senbergs Asger, Siu Man Keung, Smid Harm Jan, Su Yi-Wen, Tournès Dominique, Truc Jean-Paul, Tzanakis Constantinos, Van Den Bogaart Theo, Van Den Bogaart Desiree, Volkert Klaus, Voronel Yael Joelle, Walshaw Margaret, Weiss-Pidstrygach Ysette, White Diana, Ying Jia-Ming, Zhongyu Shen, Zorn Paul, Zou Jiachen, Zudini Verena

Author Index