HPM2012

HISTORY and PEDAGOGY of MATHEMATICS * HPM

The HPM Satellite Meeting of ICME-12

DCC, Daejeon, Korea
July 16～20, 2012

PROCEEDING

BOOK 2

International Study Group on the Relations Between
the HISTORY and PEDAGOGY of MATHEMATICS
An Affiliate of the International Commission on
Mathematical Instruction

Contents

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2. History and epistemology implemented in mathematics education: classroom experiments & teaching materials.

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Workshop

4. Mathematics and its relation to science, technology and the arts: historical issues and educational implications.

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6. Topics in the history of mathematics education.

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7. Mathematics from Eastern Asia.

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Schedule

[Day 1] MONDAY, JULY 16

08:00–19:00	Registration & Information	
09:00–10:00	Opening Ceremony **[Sangki Choi]**	**R107**
	Welcoming Remarks: Evelyne Barbin(Chair of the HPM Group), Sunwook Hwang:(Chair of HPM 2012 LOC). **Introduction of Honored Guests** **Information for Excurssion** to Gongju City Tour **Welcoming Performance**: Dan-ga <Song of four seasons>, Pansori <Sim-chŏng ga> sung by Seo-eun Wang (Department of Musicology, The Graduate School of Seoul National University)	
10:00–11:00	Plenary Lecture**(Theme 2) [Man-Keung Siu]**	
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11:00–11:30	Break	
11:30–12:30	Oral Presentation**(Theme 2)**	
	Session 1 [Evelyne Barbin] Uffe Thomas Jankvist: *A Historical Teaching Module on "The Unreasonable Effectiveness of Mathematics"—the Case of Boolean Algebra and Shannon Circuits.* Jerry Lodder: Historical Projects in Discrete Mathematics.	**R101**
	Session 2 [Sunwook Hwang] Jin Ho Kim* & In Kyung Kim: *Future Research Topics in the Field of Mathematical Problem Solving: Using Delphi Method.* Kyunghee Shin: *Harriot's Algebraic Symbols and the Roots of Equations.*	**R102**
12:30–14:30	Lunch Break	
	Official Photographing at the entrance of DCC on 12:30	
14:30–16:00	Oral Presentation**(Theme 2)**	
	Session 1 [Kristin Bjarnadottir] Shu Chun Guo: *A Discussion on the Meaning of the Discovery of Mathemetics in the Worriers and the Han Dinasty.* Mustafa Alpaslan*, Mine Isiksal & Cigdem Haser: *Relationship Between Pre-service Mathematics Teachers' Knowledge of History of Mathematics and Their Attitudes and Beliefs towards the Use of History of Mathematics in Mathematics Education.* Kathleen M. Clark: *The Influence of Solving Historical Problems on Mathematical Knowledge for Teaching.*	**R101**

	Session 2 [Michael Fried]	**R102**
	Yoichi Hirano & Yoshihiro Goto: *An Essay on an Experiment in Mathematics Classroom – the Golden Ratio Related in the Form of the Nautilus Shell.*	
	Man-Keung Siu & Yip-Cheung Chan*: *On Alexander Wylie's Jottings on the Science of the Chinese Arithmetic.*	
	Jeyanthi Subramanian: *Indian Pedagogy and Problem Solving in Ancient Thamizhakam.*	
	Session 3 [Young Wook Kim]	**R103**
	Guo-qiang Li & Li-hua Xu: *Analysis of Mathematics Teaching in the View of the History of Mathematics.*	
	Toshimitsu Miyamoto: *Mathematics Education and Teaching Practice to Bring up History of Mathematics Culture Richly.*	
	Nobuki Watanabe: *Sundial and Mathematics: Analysis of Oldest Horizontal Sundials in Japan by Mathematics.*	
16:00–16:30	Break	
16:30–17:30	Workshop**(Theme 2)**	
	Janet Heine Barnett, Jerry Lodder & David Pengelley: *Projects for Students of Discrete Mathematics via Primary Historical Sources: Euclid on His Algorithm.*	**R101**
	Renaud Chorlay: *The Journey to a Proof: If f' is Positive, then f is an Increasing Function.*	**R102**
	Frederic Metin: *Using History for Mathematics in EFL Teaching.*	**R103**
	Peter Ransom: *Fortifying France: Les Villes de Vauban.*	**R104**
17:30–18:30	Poster Session	**Lobby**
	Moonja Jeong: *The Trends on Mathematics in Novels.*	
	Sung Sook Kim: *Orthogonal Latin Squares of Choi Seok-Jeong.*	
	Sang-Gu Lee, Kyung-Won Kim & Jyoung jenny Lee: *Teaching History of Mathematics by Creating Your Own Deck of Card and Poster of World Mathematicians (Including Your Fellow Countrymen).*	
	Alejandro Rosas & Leticia Pardo: *Arithmetical and Geometrical Progressions, and Numerical Series in China before 14th Century*	
	Hye-Soon Yun: *Chosun Mathematician Lee Sang Hyuk's Genealogy.*	

	Sang-Gu Lee, Jae Hwa Lee* & Hyungwoo Byun: *First Study on a Joseon Mathematics Book SUHAKJEOLYO(數學節要, 수학절요), which was Written by Jong-Hwa An in 1882.*	
18:00–19:00	Preparatory Session 1 for Asian HPM **[Sung Sa Hong]**	**R107**
	Opening Remark: Sung Sa Hong(Korea)	
	Chang-Koo Lee (Korea): *Chinese Mathematics in Chosun*	
	Morimoto (Japan): *Three Authors of the Taisei Sankei.*	
19:10–20:30	Welcoming Banquet	
9:00–19:00	Exhibitions	**Lobby** (Continues during the conference.)
	Jangjoo Lee: *An Eight-Fold Folding Screen Using the 24 Finest Scenes of Mathematics of Joseon Dynasty.*	
	Sang-Gu Lee & Yoonmee Ham: *Modern Mathematic Books by Korean Authors during 1884–1910.*	
	Zhe Zhu: *Textbook Preparation Based on the Conception of "Integrating the History of Mathematics into Mathematics Curriculum".*	
	Yongmei Liu & Tingting Liu: *A Study on 'Mathematization' of Obstacles and Solving Appoaches*	

[Day 2] TUESDAY, JULY 17

8:00–19:00	Regisration & Information	
9:00–10:00	Plenary Lecture**(Theme 3) [Sung Sook Kim]**	**R107**
	Janet Barnett (USA): *Bottled at the Source: The Design and Implementation of Classroom Projects for Learning Mathematics via Primary Historical Sources.*	
10:00–11:00	Plenary Lecture**(Theme 1) [Sung Sook Kim]**	
	Tinne Hoff Kjeldsen (Denmark): *Uses of History for the Learning of and about Mathematics: Towards a Theoretical Framework for Integrating History of Mathematics in Mathematics Education.*	**R107**
11:00–11:30	Break	
11:30–12:30	Oral Presentation**(Theme 1)**	

	Session 1 [Yoichi Hirano] Mi Kyung Ju, Jong Eun Moon & Ryoon Jin Song: *Ethnomathematics and its Educational Meaning: A Comparative Analysis of Academic Discourse and Educational Practice of Mathematics History in Korea.* Immaculate K. Namukasa: *History of Mathematics Implemented in Mathematics* Education Programs: *The Development, Implementation and Evolution of a University Course.*	**R101**
	Session 2 [Uffe Jankvist] Toshimitsu Miyamoto: *Theoretical Framework Concerning Drawing Method in Mathematics Education.* Kyeonghye Han: *The Historico-Genetic Principle and the Hermeneutical Methode as the Theoretical Background of Using History of Mathematics in Lesson.*	**R102**
	Session 3 [Frédéric Métin] David Guillemette: *Bridging Theoretical and Empirical Account of the Use of History in Mathematics Education? A Case Study.* René Guitart: *Misuses of Statistics in a Historical Perspective: Reflexions for a Course on Probability and Statistics.*	**R103**
	Session 4 [Maria del Carmen Bonilla] Patricia Baggett* & Andrzej Ehrenfeucht: *The "Ladder and Box" Problem: From Curves to Calculators.* David Pengelley: *Teaching Number Theory from Sophie Germain's Manuscripts: A Guided Discovery Pedagogy.*	**R104**
12:30–14:30	Lunch Break	
	Korean Traditional Music Performance: (12:30–13:00) Five musics performed by Daejeon Korean Traditional Orchestra	**R107**
14:30–16:00	Oral Presentation**(Theme 4)**	
	Session 1 [Mustafa Alpaslan] Oscar João Abdounur: *The Division of the Tone and the Introduction of Geometry in Theoretical Music in the Renaissance: a Historic-Didactical Approach.* Sunam Cho: *The Formation of Mathematics Curriculum Characteristics by Augustus de Morgan in University College, London: On the Boundary between Mathematics and Natural Philosophy.* George Heine: *The Flattening of the Earth: Its Effect on Eighteenth Century Mathematics.*	**R101**

	Session 2 [Youngmee Koh]	**R102**
	Woosik Hyun: *Mathematical Foundations of Cognitive Science.* Taewan Kim & H. K. Pak: *Analysis on Lines and Circles in Secondary School Mathematics Textbooks according to the Types of Concept.* Ho-Joong Lee: *On the Eigenvalues of Three Body Problem in the Early 20th Century.*	
	Session 3 [Renaud Chorlay]	**R103**
	Leonardo Venegas: *The Topological Intuition of Leonardo da Vinci* María del Carmen Bonilla: *Visualization of the Mechanical Demonstration to Find the Volume of the Sphere Using Dynamic Geometry.* Shuping Pu* & Xiao-qin Wang: *How to Integrate History of Mathematics into Mathematics Textbooks: Case Study of Junior High School Textbooks in China and France*	
16:00–16:30	Break	
16:30–17:30	Workshop**(Theme 3)**	
	Bjørn Smestad: *Teaching History of Mathematics to Teacher Students: Examples from a Short Intervention.*	**R101**
	Man-Keung Siu* & Yip-Cheung Chan: *Chinese Arithmetic in the Eyes of a British Missionary and Calculus in the Eyes of a Chinese Mathematician: Collaboration between Alexander Wylie (1815–1887) and LI Shan-Lan (1811–1882).*	**R102**
	Anne Michel-Pajus: *Historical Algorithms in the Classroom and in Teacher-Training.*	**R103**
17:30–18:30	Poster Session	**Lobby**
	Moonja Jeong: *The Trends on Mathematics in Novels.*	
	Sung Sook Kim: *Orthogonal Latin Squares of Choi Seok-Jeong*	
	Sang-Gu Lee, Kyung-Won Kim & Jyoung jenny Lee: *Teaching History of Mathematics by Creating Your Own Deck of Card and Poster of World Mathematicians (Including Your Fellow Countrymen).*	
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	Sang-Gu Lee, Jae Hwa Lee* & Hyungwoo Byun: *First Study on a Joseon Mathematics Book SUHAKJEOLYO(數學節要, 수학절요), which was Written by Jong-Hwa An in 1882.*	
18:00–19:00	Preparatory Session 2 for Asian HPM **[Chang Kyoon Park]**	**R107**
	Anjing Qu (China): HPM in China.	
	Evelyne Barbin (Chair of HPM): TBA.	

[Day 3] WEDNESDAY, JULY 18

8:00–12:00	Regisration & Information	
9:00–10:00	Plenary Lecture**(Theme 4) [Janet Barnett]**	**R107**
	Dominique Tournè's(France): *Mathematics of the 19th Century Engineers: Methods and Instruments.*	
10:00–10:30	Break	
10:30–12:00	Panel Discussion 1 **[Kathleen Clark]**	**R107**
	Theme 1: Why Do We Require a "History of Mathematics" Course for Mathematics Teacher Candidates? (And What Might Such a Course Look Like?).	
	Panelists: Mustafa Alpaslan (Turkey), Sang Sook Choi-Koh (Korea), Kathleen Clark (USA), Frédéric Métin (France).	
12:00–18:30	Excursion	

[Day 4] THURSDAY, JULY 19

8:00–18:30	Regisration & Information	
9:00–10:00	Plenary Lecture**(Theme 5) [Tinne Hoff Kjeldsen]**	**R107**
	Anne Michel-Pajus (France): *A Voyage into the Litterary-Mathematical Universe*	
10:00–11:00	Plenary Lecture**(Theme 6) [Tinne Hoff Kjeldsen]**	**R107**
	Johan Prytz (Sweden): *Social Structures in Mathematics Education. Researching the History of Mathematics Education with Theories and Methods from Sociology of Education*	
11:00–11:30	Break	
11:30–12:30	Oral Presentation**(Theme 6)**	
	Session 1[David Guillemette] Evelyne Barbin: *The Role of the French Association of Mathematics Teachers APMEP in the Introduction of Modern Mathematics in France (1956–1972).* Michael N. Fried: *Book XIII of the Elements: Its Role in the World's Most Famous Mathematics Textbook.*	**R101**

	Session 2[Sanwook Ree] Chun Chor Litwin Cheng: *The Mathematics Development of the Book Sea Mirror of Circle Measurements (Ceyuan Haijing).* Guo-qiang Li* & Li-hua Xu: *On Mathematics Teachers' Quality and Its' Advance in Mathematical History.*	**R102**
	Session 3 [Dominique Tournés] Hoyun Cho: *Common Core State Standards Movement in U.S Mathematics Curriculum.* Lorena Jimenez Sandoval & Gustavo Martinez Sierra: *Social Construction of the Algebraic Structures. A Model for Its Analysis.*	**R103**
12:30–14:30	Lunch Break	
14:30–16:00	Panel Discussion 2 **[Uffe Thomas Jankvist]**	**R107**
	Theme 2: Empirical Research on History in Mathematics Education: Current and Future Challenges for Our Field.	
	Panelists: Uffe Thomas Jankvist (Denmark), Yi-Wen Su (Taiwan), Isoda Masami (Japan), David Pengelley (USA).	
16:00–16:30	Break	
16:30–17:30	Oral Presentation**(Theme 6)**	
	Session 1 [Chun Chor Litwin Cheng] Toshimitsu Miyamoto: *History of Arithmetic Textbook and Composition of Content Based on Count Principle Method.* Osamu Kota: *Teaching and Learning of Functions in Modern Japan.*	**R101**
	Session 2 [Kathleen Clark] Kristín Bjarnadottir: *18th Century Mathematics Education: Effects of Enlightenment in Iceland.* Andreas Christiansen: *Geometry Textbooks in Norway in the First Half of the 19th Century.*	**R102**
	Session 3 [Bjørn Smestad] Nicla Palladino: *The Issue of Mathematics Textbooks in the Correspondence of Giovanni Novi to Enrico Betti during the Unification of Italy.* François Plantade: *After the Gösta Mittag-Leffler & Jules Houël Correspondence, Their General and Particular Thoughts on Mathematical Teaching.*	**R103**
17:30–18:30	HPM Session **[Evelyne Barbin]**	**R107**
	Opening Remarks: Evelyne Barbin (Chair of the HPM Group). Introducing New Chair of the HPM Group. Announcement of HPM 2016.	

[Day 5] FRIDAY, JULY 20

8:00–15:00	Regisration & Information	
9:00–10:00	Plenary Lecture**(Theme 7) [Tsang-Yi Lin]**	**R107**
	Sung Sa Hong (Korea): *Theory of Equations in the History of Chosun Mathematics.*	
10:00–10:30	Break	
10:30–12:30	Oral Presentation**(Theme 5)**	
	Session 1 [Jinho Kim] Chun-yue Stanley Lee & Mei-yue Christine Tang: *A Comparative Study on Finding Volume of Spheres by LIU Hui(劉徽) and Archimedes: An Educational Perspective to Secondary School Students.* Chang K. Park & Sun Bok Bae: *Cultural Prime Numbers: 2, 3 and 5.* Young Hee Kye: *Math and Art in View Point of Perspective Drawing of the West and East.*	**R101**
	Session 2 [Anne Michel-Pajus] André Cauty: *Invitation to Revisit the Mesoamerican Calendars. The Count That is Called Real Calendar.* Gregg De Young: *A Colorful Case of Mistaken Identities.* Albrecht Heeffer: *Dutch Arithmetic, Samurai and Warships: The Teaching of Western Mathematics in Pre-Meiji Japan.*	**R102**
	Session 3 [Evelyne Barbin] Yoichi Hirano: *Remark on the Notion of Golden Ratio—Concerning "Divine Proportion" in the Renaissance.* Leo Corry: *Euclid's Proposition II.5: A View through the Centuries-Geometry, Algebra and Teaching.* Qing-jian Wang: *The New "Curriculum Standard" and the New Mathematics—the Union of History of Mathematics and Mathematics Education*	**R103**
12:30–14:30	Lunch Break	
14:30–16:00	Oral Presentation**(Theme 7)**	
	Session 1 [David Pengelley] Sung Sa Hong, Young Hee Hong & Chang Il Kim: *Chosun Mathematician Hong Jung Ha's Least Common Multiples.* Sung Sa Hong, Young Hee Hong & Young Wook Kim: *Liu Yi and Hong Jung Ha's KaiFangShu.* Sung Sa Hong, Young Hee Hong & Seung On Lee: *Yang Hui's NaYin QiLi.*	**R101**

	Session 2 [Sang-Gu Lee] Michael Kourkoulos* & Constantinos Tzanakis: *An Experiment on Teaching the Normal Approximation to the Symmetric Binomial Using De Moivre & Nicholas Bernoulli's Approaches.* Youngmee Koh: *Educational Meaning of the Theory of Rectangular Array in the Nine Chapters on the Mathematical Art.* Sangwook Ree: *Meaning of the Method of Excess and Deficit.*	**R102**
	Session 3 [Sangki Choi] Yuzi Jin & Young Wook Kim: *Research on the Muk Sa Jib San Beob.* Hae Nam Jung: *A Study on "GuSuRyak" of Choi Seok Jung.* Toshimitsu Miyamoto: *Historic Investigation of Legendre's Proof about the 5th Postulate of "Elements".*	**R103**
16:00–16:30	Closing Ceremony **[Jinho Kim]**	**R107**
	Closing Remarks: Evelyne Barbin(Chair of the HPM Group), Luis Radford(New Chair of the HPM Group), Sunwook Hwang(Chair of HPM 2012 LOC)	

A VOYAGE INTO THE LITERARY MATHEMATICAL UNIVERSE

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ABSTRACT

This paper sketches a joint history of the mathematics and literature of the western world, from the perspective of our modern times. As we progress along our road, we shall comment on literary processes (literal insertions, analogies, mathematical structures, characters doing mathematics) and on how literary art can enhance the motivation for exploring, understanding and memorizing mathematical concepts.

First stop: in a Greek theatre of the fifth century B.C.E.

Mathematics, to the best of my knowledge, makes its entrance onto the western literary stage in the Greek theatre of the 5th century B.C.E. Someone called Meton appears, encumbered with his instruments: a compass and a flexible ruler. He offers his services to Pisthetairos, one of the principal characters of *The Birds* (414 B.C.E., Aristophanes). This Athenian, weary of his city rotted by corruption and demagogues, has decided to create another town (Νεφελοκοκκυγία, Cloudcuckooland) in the sky, with the help of the birds.

PISTHETAIROS[1]
For heaven's sake, who are you?
METON *[shocked]*
Who am I? I'm Meton, famous throughout Greece and Colonus.
PISTHETAIROS
What are these things you've got?
METON
Rods to measure air. You see, the air is, in its totality, shaped like a domed pot cover. . . Thus. . . and so, from up above I'll lay my ruler. . .it bends. . .thus. . Set my compass inside there. .You see?
PISTHETAIROS
I don't get it.
METON
With this straight ruler here I measure this, so that your circle here becomes a square—and right in the middle there we have a market place, with straight highways proceeding to the centre, like a star, which, although circular, shines forth straight beams in all directions . . . Thus . . .

[1] Aristophanes, *The Birds*, Translation by Ian Johnston, 2003, on line.

P I S T H E T A I R O S
This man's a Thales !

Pisthetairos immediately chases off after this charlatan, just as he had previously chased off after a poet and a prophet. Meton disappears from the scene once and for all.

So what do we find in this popular comedy written in 414 B.C.E.? A reference to Meton, an astronomer (which gave his name to a famous astronomical cycle, circa 432 B.C.E) and to Thales, who lived at least a century earlier, appearing here in his capacity as a paradigm of geometry. Note that another century will elapse before the arrival of Euclid.

Although it is doubtless too farfetched to see an allusion to the quadrature of the circle in the above, nevertheless problems of comparison of perimeters or of ruler and compass constructions can be read into it. If the public found this amusing, it was because there was already a strong mathematical culture.

Twenty-five centuries later: A discussion occurs during a meeting of the Transnoctial Discussion Group in the novel by Thomas Pynchon *Against the Day*, 2006. This group will never appear again.

"Time moves on but one axis," advised Dr. Blope, "past to future—the only turnings possible being turns of a hundred and eighty degrees. In the Quaternions, a ninety-degree direction would correspond to an *additional axis* whose unit is $\sqrt{-1}$. A turn through any other angle would require for its unit a complex number."

"Yet mappings in which a linear axis becomes curvilinear—functions of a complex variable such as $w = e^z$, where a straight line in the z-plane maps to a circle in the w-plane," said Dr. Rao, "do suggest the possibility of linear time becoming circular, and so achieving eternal return as simply, or should I say complexly, as that."

Is this funny? Is the question of linear time being transformed into circular time by functions of a complex variable more understandable by modern readers than that of a circle becoming a square with a bent rule? Pynchon's novels are sprinkled with fragments of culture (not only mathematical, but the list of mathematical themes cited is impressive [Koehler]).

When a text involves terms, expressions, and more or less coherent mathematical statements and names of mathematicians—but without a real relationship to the unfolding of the narrative I will call this literary mode "literal insertion".

The effect of this kind of fragments certainly depends on the readers, and on their mathematical knowledge. They might either be amused, moved, impressed or will simply skip such a digression. This kind of reference only shows that the author has some acquaintance with the subject. The mathematical coherence of a fragment of mathematics cannot be easily determined. Coming back to Aristophanes, different translators have interpreted him differently, some seeking to make the text coherent at all costs.

But so, these fragments can arouse interesting discussions about their coherence or relevance: what can a flexible ruler be used for? Measuring? You could introduce the theme of the rectification of curves. Is a straight ruler (or a straight line) best adapted for work on a sphere? Could Meton have discovered "non Euclidean" geometry before Euclid? In a more general context, the question of "The

Straight and the Curve"[2] has given rise to many works in the history of mathematics. And further, is it necessary to employ quaternions (complex numbers) to give analytical representations of a straight line and a circle? What role does time play in Hamilton's "*Essay on Algebra as the Science of Pure Time*"? If it is a question of modelling time, shouldn't you be interested in its psychological aspect, its speeding up or slowing down, etc.

Such discussions allow for lively teaching in the most interesting way. They let us make concepts precise, bringing the imagination into play, in the double sense of stimulating creativity and creating mental images. Thus we can make mathematics not only more pleasant but also easier to use and to memorize. It also gives an opportunity to enrich everyone's general culture, by making them interested in the mathematical and literary context. In some experiments, after a decent amount of exposure to such ideas students are asked to write poems or short stories[3].

So I am inviting you to take a little voyage to examine some of the tracks left by mathematics in "literary texts" (i.e. all those which are not claimed to be "mathematical texts"). Each of us can trace our own route through the ocean of citations that can be found in our own reading or on the web[4]. A good number of us let ourselves be guided by mathematical themes in order to find examples adapted to our teaching curriculum, but, since we are historians of mathematics, I want to lead you through time (complex!), bringing together mathematics, literature and history. Alas, if I know a little mathematics and a little of the history of mathematics, I know very little of the history of literary cultures other than mine. So this is why I limit myself to here, at the same time asking anyone who wishes, to help me broaden this study.

Second stop: In the world of the Romans and of the Gods of the Vth century C.E.

We don't find much mathematics in the fiction of the Roman World, but, at Carthage Martianus Capella published *De nuptiis Philologiae et Mercurii* (On the Marriage of Philology and Mercury). He recounts here the marriage of the god, Mercury and of the mortal, Philology, whose name speaks volumes for his love of words. For this union she must mount up into the Milky Way. In order to lighten herself, Philology starts by vomiting up books which lie heavy on her breast, but her thirst for Knowledge will be satisfied by her new husband, who offers her seven heavenly teachers, seven young girls who will teach her everything. They present themselves in order: Grammar, Dialectic, Rhetoric, Geometry, Arithmetic, and Harmony. The first three present the three literary areas that Boethius will call the trivium, and the last four the mathematical sciences (quadrivium). This division of knowledge will become the standard formulation of academic learning, right up to the Renaissance.

This work is thus both an encyclopedia and a work of *popularization*[5]. It carefully explains the mathematical knowledge of the author (limited alas), but in a pleasing form: flowery prose, rich in oddities and metaphors, passages versified in many different ways, a mixing of the serious and the grotesque. The objective is to teach while entertaining. It goes back to the Greek concept of *σπουδογέλοιον* (seriousness under the laughter). As wrote Lucrèce (341-271 B.C.E.) "To make children drink bitter absinthe,

[2]Cf Barbin
[3]Cf Lipsey
[4]See for instance the excellent website : `http://kasmana.people.cofc.edu/MATHFICT/`
[5]I don't like this word, but I didn't find a better one.

begin by coating the rim with pure golden honey".

Mathematics is definitely in a minority among the other sciences in this type of literature, probably because it is difficult to understand mathematics without really doing it. However, this genre is flourishing today. It is a difficult genre for learning mathematics without guidance, but many teachers in the world today use this kind of literature successfully with their students.

The historians of mathematics will be interested in Capella's book : his geometry speaks mainly of geography, and his arithmetic is mainly interested in the "qualitative" properties of numbers. The renewal of western mathematics will not come from this source but from the work of Al Kwharizmi (translated in the XIIth century). Practical arithmetic, in particular, will be revolutionized by the system of decimal place values, by the introduction of "Indian" calculation replacing calculations with the abacus or counters, as Roman numeration is incompatible with the least arithmetical calculation.

Third stop: in a chateau of the Middle Ages

In order to fill in their evenings, noble knights and gentle ladies listened at great length to poems set to music about the exploits of valiant knights, or to short sophisticated poems which sang of a platonic and impossible "courtly" love for a beautiful lady. From the 12th century on, the authors of chivalric poems delighted in playing with large numbers which the new decimal system allowed them to express easily. They also present many artificially concrete, small arithmetical problems which use "The Rule of Three" or "The Golden Rule".[6] This kind of arithmetical calculation will remain common for a long time, for example, in the philosophical tales and satires of the 18th century (e.g. *Gulliver's Travels*, *Swift*, *1726. Micromegas*, Voltaire, 1752). And they continue to be found right up to the present day....in more or less relevant calculations, which also make excellent exercises for young students.

We even see them turn up in the history of mathematics: first of all in French in the great epic poem *Le Roman de la Rose* (Guillaume de Loris et Jean de Meun (1237–1288), then in English in The *Book of the Duchess*, (circa 1370) by Geoffrey Chaucer, who also translated part of *Le Roman de la Rose*. Argus is the Latinized name of Al Khawarizmi.

Old French	*English translation of old French:*[7]	*Old english*
Se mestre Argus li bien **contens** I vo sist bien metre ses cures E venist o ses dix figures Par quoi tout certifie et nombre, Si ne péust-il pas le nombre De grans **contens** certefier Tant seust bien monteplier	If the Master Argus, who is so good at counting/ reckoning Would take care of it And come with his ten figures/ By which he can certify and count everything yet he can't certify the number of big conflicts/ pleasures So good were he at multiplying	That thogh Argus, the noble contour Sete to rekene in hys contour And rekene with his figure ten For by tho figures mowe al ken Yf they be crafty, rekene and noumbre And telle of every thing the noumbre Yet should he fayle to rekene even The **wonders** in my sweven

The French text plays on the word "contens" which can mean pleased, counting, recounting, reck-

[6] Cf Mira

[7] The English translation of all the French texts was made, unless otherwise mentioned, by my friends Stuart et Pam Laird.

oning as an adjective, and conflict, pleasures, happy people as a noun. The English translation must use more words “rekene/ contour”, ”rekene / noumbre”. This property of being both metaphorical and polysemous plays an important role in literature, above all in poetry, but it is evidently closely linked to the language being used. Here we see all the problems posed by translation. I will not labour this point here. In an interdisciplinary study, students can additionally study language and rhetorical effects.

In the previously mentioned *Roman de la Rose*, the author in dealing with expressing countless quantities even introduces the idea of a potential infinity: ”it is not wealth which equate to the value of a friend, for it could not attain a level so high that the value of a friend is not higher still.”

As for the poets of the south of France, the troubadours, they enjoyed systematically exploring the configurations allowed by the constraints of different versifications. For example, Arnaut Daniel (end of the XII century), invented the ”sextine”. This poem of six stanzas of six lines uses six rhyming words. The order of their entrance in the stanza changes in a circular permutation from one stanza to the next. The poem is terminated by a stanza of three lines ending in three couplets of rhyming words.

So it is through poetry that combinatorics make its entry into western mathematics again. The use of *constraints of a mathematical type* in the structure of a literary work will be the foundation of the works of OuLiPo[8] in the twentieth century.

Jacques Roubaud’ for instance, published a collection of poems with the rather enigmatic title: “∈” (this is the symbol of set membership but it is read *epsilon*). In the introduction we read: “This book is composed, on the principle of 361 texts which are the 180 white pieces and the 181 black pieces of a game of Go.” Then: “The text or pieces belong to the following varieties: sonnets, short sonnets, interrupted sonnets, quotations, illustrations, grids, whites, blacks, poems, prose poems⋯” As the *Encyclopedia Universalis* explains: “Four modes of reading are possible for these generally brief texts (or pieces), which are always preceded by numbers, signs or symbols and which reflect the diverse systems of succession, of regroupings, of correspondences and of separation, focusing on the way symbols group or on their continued development, following the movement of a game of Go or taking each element in its singularity.

Hortense, a trilogy of novels by the same author, is based on essentially combinatorial rules which are more or less explicit. All the works of Jacques Roubaud are interlaced with Mathematics. *Mathématique : (récit)* occupies a special position. In it we find diverse mathematical recollections and reflections of the author as a student at the beginning of “modern mathematics” and an attempt at modelling memory through neighbourhoods topology. It is followed by *Impératif Catégorique* where the title and the content play on the words at several levels. ‘Catégorique’ refers at Category Theory and at its meaning as it relates to Kant (the categorical imperative). Roubaud also writes that it is related to a literary mode in the Japanese language.

Fourth stop: in the Parisian salons of the 17th and 18th centuries

Euclid’s *Geometry* is now studied by all the well educated boys. Charles Perrault (1628–1703), a writer latercelebrated for his Tales (*Contes*), found inspiration in it. At the age of thirteen, he wrote a poem

[8]Cf Audin

called *The Loves of the Ruler and the Compass*. In this poem, the ruler (female in french) “of serious bearing, full of majesty, unbending and observing of fairness above all” resists all efforts at seduction by the compass (male in French), up to the moment when :

> ···The compass immediately stood erect on one foot
> And, with the other, in turning a large circle traced.
> The ruler was overjoyed, and suddenly came and laid herself down
> In the middle of the circle, and formed the diameter.
> Her lover embraced her, having her at his mercy,
> Now extending and now contracting,
> And so came to be born from their learned postures
> Triangles, squares and a thousand other figures

The mathematicians of this period are also philosophers and talented writers. Above all they pursue research into methods, in mathematics, just as in philosophy. But, in the Parisian Salons, serious discussions would become freely gallant.

> ”Madam···since we are inclined to always mingle the follies of gallantry with our most serious conversation, mathematical reasoning is made like love. You cannot permit the smallest thing to a lover that you have to allow more afterwards, and that ends up by going a long way. Similarly, grant the least principle to a mathematician; he will draw you a conclusion from it, that you will also have to grant and from this conclusion yet another, and in spite of yourself, he will lead you so far that you will scarcely be able to credit it.”

In this extract we can see the application of a mathematical concept to something else. I will call this literary process ”conceptual transfer” or ”analogy”.

This transfer of mathematical reasoning to amourous seduction is extracted from the *Entretiens sur la pluralité des Mondes*, 1686–1687, a work of *popularization* which is presented as a dialogue between the author, Fontenelle, and a marquioness. Fontenelle was a philosophe—mathematician, author for instance, of the ”*Eléments de la géométrie de l'infini*”.

Speaking about Pynchon, the literary critic John O. Stark, suggests that “by drawing nonmathematical conclusions from mathematics and by incorporating them into his literary works, he points out a feature that its technological applications might easily obscure: mathematics is metaphoric because it describes universals.”

The infinitesimal calculus was doubtless still too new for non mathematicians to take up, but the infinite inspired awe and wonder in everyone. So much so that they dedicated poems to it in their mathematical works, like Jacques Bernoulli, in connection with the summation of infinite series (written in latin before 1689)[9]. We can notice the geometrical structure of the poem: parallelism and symmetry.

[9]Translated from latin in Struik D.J., A Source book in Mathematics, 1200–1800,Harvard University Press, 1969.

> Just as a finite little sum embraces the infinite series
> And a limit exists where there is no limit
> So the vestiges of the immense Mind cling to the modest body
> And there exists no limit within the narrow limit
> O say, what glory is to recognize the small in the immense!
> What glory to recognize in the small the immensity of God

We find commonly, at this time, the algebraic analogy between finite and infinite justified in mathematics by mean of Metaphysics. For example, Newton, in (*De Analysi per Aequationes Numero Terminorum Infinita*), published in 1711, but which circulated from 1669.)

> And whatever the common Analysis performs by Means of Equations of a finite number of Terms (provided that can be done) this new method can always perform the same by means of infinite Equations. So that I have not made any Question of giving this the name of *Analysis* likewise. For the Reasonings in this are no less certain than in the other, nor the Equations less exact; albeit we Mortals whose reasoning Powers are confined within narrow Limits, can neither express, nor so conceive the Terms of these Equations as to know exactly from thence the Quantities we want.

Conversely, the philosophical and apologetic work of Pascal (1623–1662) is often underpinned by mathematical concepts. For example, he transfers the notion of calculation of probabilities to religion, with his famous"Bet" (Pari) concerning the existence of God, and the calculus of the infinite to the place of man in the world :

> For, finally, what is man in nature? He is nothing in comparison with the infinite, and everything in comparison with nothingness, a middle term between all and nothing. He is infinitely severed from comprehending the extremes; the end of things and their principle are for him invincibly hidden in an impenetrable secret; he is equally incapable of seeing the nothingness from which he arises and the infinity into which he is engulfed.

Actually, analogy plays an important heuristic role at this epoch in all matters that touch mathematical infinity. Has all this been successfully swept away by modern, rigorous mathematics? Here is what the mathematician André Weil thinks (1960)[10]:

> There is nothing more profound, all mathematicians know it, than these obscure analogies, these murky reflections from one theory to another, these furtive caresses, these inescapable contretemps; nothing also gives more pleasure to the researcher. A day arrives when illusion is dissipated, premonition changes into certitude; twin theories reveal their common source before disappearing; as the *Gita* teaches knowledge and detachment are attained at the same time. Metaphysics has become mathematics, ready to form the material for a treatise whose cold beauty will no more know how to move us⋯.

[10]*Complete Works*, Volume 2, p.408:

In the same way still, we see analogies between the calculus of finite differences and the differential calculus serving as a guide to Leibniz, Taylor and Euler, during the course of the heroic period during which Berkeley could write, with as much humour as relevance, that "believers" in the infinitesimal calculus were little qualified to criticize the obscurity of the mysteries of the Christian religion, the former being at least as full of mysteries as the other. A little later, d'Alembert, enemy of all metaphysics in mathematics besides, upheld that the true metaphysics of the infinitesimal calculus were nothing else than the notion of a limit, in the entries of the Encyclopedia. If he did not himself, extract all that could be drawn from this, he would be justified by the developments of the following century; and nothing known to be clearer can be found today, nor, it must be said, more tedious, than a proper exposition of be elements of the differential and integral calculus.

Long after mathematicians had settled the problem, writers have continued to confront it. Here is a more recent example: *The Aleph*, Jorge Luis BORGES (1945)

"For the rest, the central problem is unsolvable: the enumeration, even if only partial, of an infinite set [⋯] What my eyes saw was simultaneous; what I shall transcribe is successive. Nevertheless I shall cull something of it all.

In the lower part of the step, towards the right, I saw a small iridescent sphere, of almost intolerable brilliance. A first I thought it rotary; then I understood that this movement was an illusion produced by the vertiginous sights it enclosed. The aleph's diameter must have been two or three centimeters, but cosmic space was in it, without diminution of size. Each object, (the mirror's glass, for instance) was infinite objects, for I clearly saw it from all points in the universe"[11]

In the same way that we can speak of analogy, we can speak here of a mental image, or more sketchily of the imagination. As wrote D'Alembert in the Preliminary Discourse to his *Encyclopédie*, (1751):

Imagination is not less active in a geometer who creates than in a Poet who invents

The developments in Algebra and in the infinitesimal calculus appear a little later in a curious work, worthy of a paper all to itself: *Le manuscrit trouvé à Saragosse* (The Manuscript found in Saragossa) of Comte Jan Potocki (1761—1815). This erudite, cosmopolitan, fervent admirer of the philosophers of the enlightenment would have learnt mathematics in order to teach one of his sons. A first (partial) edition of the novel (in French) comes from St. Petersburg in 1805. Partial manuscripts multiplied. They were lost, translated or plagiarized and the most complete version dates from 2006[12]. A crowd of characters appear within a sophisticated structure of interlocking histories. One of them, Velasquez, is a mathematician-philosopher. Here is a summary of his story, intertwined with lots of others:

The father of the hero, a mathematician, attributes his reverses of fortune to his love of mathematics. He swears that his son will not learn it, but rather learn to dance, which he judges to be more

[11] translated by Anthony Kerrigan in *A Personal Anthology by Jorge Luis Borges*, 1967 Grove Press

[12] If you want to read it, check that you find in your book at least 60 Journées (Days)

socially useful. But our hero cannot remember the simplest type of dance, as “there is neither generating rule nor formula to memorize the different figures”! To punish him, his father shuts him up in a storehouse. It is thus (I summarize in modern language) that by beginning counting the square panes in the windows, he discovers the usefulness of multiplication, and its commutativity (for the integers). Continuing with fractions of squares, he discovers the distributivity of addition and multiplication. Moved, his father allows him to continue and proposes different cases to him, even with the lines of the squares ”infinitely small”. In the end, his father exclaims to himself ”Oh my god, look at it, he has discovered the binomial law, and if I let him, he will guess the differential calculus.”

The father then lets him have access to ”The Universal Arithmetic of the Chevalier Don Isaac Newton”. The study of mathematics continues in full swing, right up to the night when his aunt Antonia comes to see him ”almost undressed” under the pretext of making him teach her geometry. Velasquez begins by docilely showing her the first two propositions of Euclid. Antonia is vexed and replies ”So has geometry not taught you how babies are made.” This launches Velasquez into a deep reflection on the way ”to apply the calculation to the entire system of nature”.

In order to refresh his spirits, he leaves on a journey, but lost in his thoughts and his writing materials, he finds himself surrounded by hostile nomads. He offers a ransom, but the sheik takes him for a madman, and so protected by God, and puts him back on his way. What consternation for our hero: ”What’s that? I say to myself, following the tracks of Locke and Newton, I would have arrived at the furtherest reaches of human intelligence, applying the principles of calculation one after another. I would have secured several steps into the abyss of metaphysics, and what comes back to me? To be numbered among the mad, to pass for a degraded being who no longer belongs to humankind. Perish the differential calculus and all the integrations to which I had attached my fame”.

Reassure yourselves, the calculus remains alive in the novel, and the young man will learn how babies are made. In the course of these encounters, and with much humour, Velasquez will present many conceptual transfers to us. Here are several themes I have noted in the order of their appearance in the book (divided into days):

- 18^{th} day: The law of falling bodies on an inclined plane; the solution of a complicated arithmetical problem.
- 19^{th} day: ”Women cannot understand the first elements of science”; quarrels (the Bernoulli brothers, the isoperimetric problem, Jean Bernouilli and the Marquis de l’Hopital, Newton and Leibniz).
- 20^{th} day: A selection of conceptual transfers: the increase in impatience is in inverse proportion to the square of the inertia of the person concerned; the pursuit of happiness compared to the solving of equations, some with imaginary roots; the question of love “goes back into maxima and minima and the problem can be represented by a curve”.
- 22^{nd} day: The representation of human actions and passions by geometrical figures.
- 23^{rd} day: The story of our hero (the counting of panes); the hero discovers how to use the logarithmic tables of “Baron Napier”; Antonia’s attempt at seduction and what follows from it.
- 25^{th} day: From the quadrature of the circle to the quadrature and rectification of curves. Isochrones; women’s bodies and osculating curves.
- 29^{th} day: The analogy between the structure of a novel and of infinite series; the designation of unknowns by letters in algebra.
- 32^{th} Day: The analogy between the structure of a novel and recursive relations.

- 33^{rd} day: The analogy between love/hate, the rule for signs and the binomial formulae: "Yes, my dear, the binomial formula, invented by Chevalier Don Isaac Newton, must be our guide in the study of the human heart as in all calculations."
- 37^{th} day: Velasquez's ideas on religion, the infinitely small and the infinitely large.
- 39^{th} day: Abstraction and deduction; ideas and the senses; "the difference between minds is in the quantity of images and in the facility of combining them; effective calculation of combinations.
- 41^{st} day: A proof by calculus that a lake is a crater lake.
- 45^{th} day: The curve of human life.

This type of work is really exhilarating, above all for a historian of mathematics. The author had learned mathematics in order to teach it to his son. He does not explain mathematics, but causes us to reflect on this subject (among lots of other subjects), while mixing up the transfer of concepts with much humour. Finally, the construction itself refers back to the mathematics involved, and more generally to the multiplicity of points of view, which is a useful tool in mathematics.

I would like to mention two novels of the 20^{th} century, *Ratner's Star* by Don Delillo (1976), where mathematics operates simultaneously in the structure and in the themes, and Brazzaville Beach (1990) by William Boyd, with a lot of recent mathematical subjects and analogies.[13]

Fifth stop : From the old to the new world, the XIXth century

The teaching of mathematics has spread more widely. The literary fashion is to express oneself, and the authors willingly recount how they are inspired by mathematics (or their teachers). This brings new elements onto the trail of *the image of mathematics and mathematicians in society*.

Here is the young Stendhal in his autobiography *The life of Henry Brulard*, 1836

> "In my opinion, dishonesty was impossible in mathematics [$\cdots$]. I worked that out when I became aware that no-one could explain to me how it happened that $(- \times - = +)$? . My teacher replied '.. my son, you will understand that later'"

And the young Daniel Deronda in *Daniel Deronda*, by George ELIOT, 1876 :

> "[Daniel] applied himself vigorously to mathematics, and the favourable opinion of his tutor determined him to try for a mathematical scholarship in his second year. But he felt a heightening discontent with the wearing futility and enfeebling strain of a demand for excessive retention and dexterity without any insight into the principles which form the vital connection of knowledge."

Poets speak of it with greater exultation!

[13] Cf Michel-Pajus A. & Spiesser M. , HPM 2004

Victor HUGO, *Contemplations*, 1831 [⋯] I was then tortured by mathematics. I was wrung from wingtip to beak , On the frightful rack of X and Y ; Alas, I was whacked under the maxillary bones. By the theorem adorned with all its corollaries. Geometry! Algebra! Arithmetic! Zone Where the invisible plane cuts the indistinct cone, Where the asymptote searches, where the hyperbola flees! Crystallization of the night Sea where the polyhedron is the frightful madrepore ; Where the universe evaporates in calculations Where the vast and sparse fluid filling all Is no more than an hypothesis, and trembles and dissolves [⋯]	Isidore DUCASSE(Comte de Lautréamont) *The songs of Maldoror*, 1869 « O austere mathematics, I have not forgotten you, since your learned lessons, sweeter than honey, filtered into my heart, like a refreshing wave. [⋯] Arithmetic! Algebra! Geometry! Grand Trinity! Luminous triangle! Who does not know you is insensible. ! He deserves to undergo the ordeal of the greatest tortures; for there is blind contempt in his ignorant carelessness; but one who knows and appreciates you is worth no less than the goods of the earth; satisfied with your joyful magic; and carried on your dark wings, desiring nothing more that to ascend, in weightless flight, in constructing an ascending helix, towards the spherical vault of heaven.[⋯]

A particular discovery shook writer’s minds: that of non-Euclidean geometries. Reversing the argument of the infinitesimal calculus, Ivan Karamazov concludes that, since he cannot comprehend a non-Euclidean universe where parallel lines can meet, he cannot comprehend that which concerns God. (*The Brothers Karamazov*, Fyodor Dostoevsky)

> “But you must note this: if God exists and if He really did create the world, then, as we all know, He created it according to the geometry of Euclid and the human mind with the conception of only three dimensions in space. Yet there have been and still are geometricians and philosophers, and even some of the most distinguished, who doubt whether the whole universe, or to speak more widely, the whole of being, was only created in Euclid’s geometry; they even dare to dream that two parallel lines, which according to Euclid can never meet on earth, may meet somewhere in infinity. I have come to the conclusion that, since I can’t understand even that, I can’t expect to understand about God.”[14]

In the works of Edgar Allen Poe, the mechanics of thought are a fundamental theme. The faculty of analysis is carried to its highest point in Detective Dupin. For him it surpasses that which is called mathematical analysis, since this one functions solely by calculation: “to calculate is not in itself to analyze.”[15] While human analysis takes into account a multitude of balanced facts through a *calculation of probabilities* (the more improbable a fact is, the more important it is) and, above all through the possibility of “entering into the mind of an adversary”, a “pure machine” is not able to do this. In Maelzel’s *Chess Player* (1836), he deduces *mathematically* that an automaton supposed to be playing chess is, in fact, a hidden human being, from the fact that it is impossible for a machine to be able

[14]Translation from the website : `http://fyodordostoevsky.com/etexts/the_brothers_karamazov.txt`
[15]The Murders in the Rue Morgue, *Poetry and tales*, New York, Library of America, 1984, p.39

to play chess. "It is quite certain that the operations of the automaton are regulated by *mind*, and by nothing else. Indeed this matter is susceptible of a mathematical demonstration, *a priori*."

This statement is astonishing for us. It arises from the fact that Poe takes for his model Babbage's Machine. (These one was never constructed, however, it was described in The *Southern Literary Messenger*, in July 1834). Here is his proof:

> "Arithmetical or algebraic calculations are, from teir very nature, fixed and determinate. Certain data being given, certain results necessarily and inevitably follow. These results have dependence upon nothing, and are influenced by nothing but the data originally given. And the question to be solved proceeds, or should proceed, to its final determination, by a succession of unerring steps liable to no change, and subject to no modification. This being the case, we can without difficulty conceive the *possibility* of so arranging a piece of mechanism, that upon starting it in accordance with the *data* of the question to be solved, it should continue its movements regularly, progressively, and undeviatingly towards the required solution, since these movements, however complex, are never imagined to be otherwise than finite and determinate. But the case is widely different with the Chess-Player. With him there is no determinate progression. No one move in chess necessarily follows upon any one other. From no particular disposition of the men at one period of a game can we predicate their disposition at a different period. "[16]

If Edgar Allen Poe could not imagine that a machine could be programmed to incorporate new data, such as the position of pieces on a chessboard, as they arose, it is because he is a prisoner to the image of a machine as envisaged by the state of science of his time. This vision of a machine, tied blindly to strictly deterministic acts, is found again in his Essay *The Philosophy of Composition* (1846), where he explains that his celebrated poem, *The Raven* "emerged from a deliberate and conscious process that progressed with the precision and rigid consequence of a mathematical problem".[17]

The end of the voyage

Our voyage is going to come to an end at the threshold of the XXth century. It seems to me that mathematics is more and more fashionable in literature, cinema, and in TV series. This shows that mathematics play a role in the culture and the real life. One may like or not like the image of mathematics or of mathematicians which is portrayed. If these fictions lead the young to do mathematics, so much the better! If a coating of honey lets them take it up with more pleasure or creativity, so much the better! But the essence and the beauty of mathematics remains enclosed in mathematics itself and I do believe that a firm and competent hand will always remain indispensible to guide and encourage learners on the paths leading to it.

REFERENCES

– Barbin E., Itard G., 1993, Le courbe et le droit, in *Histoires de problèmes. Histoire des mathématiques*. p. 113–137. Ellipses, Paris.

[16]Edgar Allan Poe, "Maelzel's Chess-Player" [Text-02], Southern Literary Messenger, April 1836, 2:318–326. On line : www.eapoe.org/works/essays/maelzel.htm

[17]cfCassou-Noguès and Levy

- Audin M., *L'oulipo et les mathématiques, une description*, `http://www-irma.u-strasbg.fr/~maudin/ExposeRennes.pdf`
- Cassou-Noguès P., 2010, Poe Descartes et la cybernétique, `http://www.epistemocritique.org/spip.php?article118`
- Deledicq A., Deledicq J.C., Casiro F.,1996, *Les maths et la plume*, ACL éditions, Paris.
- Koehler D.0., 1982, Mathematics provides writers with a rich source of themes, images and metaphors, in *Mathematics and Literature*, vol 5, No2. `http://www.jstor.org/discover/`
- Lévy S., 2008, *Poe Expérience de pensée, la pensée comme expérience*, `http://www.epistemocritique.org/spip.php?article45`
- Lipsey S.I.,Pasternack S.B.,Mathematics in Literature, `http://math.unipa.it/_`
- Michel-Pajus A.,Spiesser M., Mathematics And Mathematicians Under The Writer's Pen, *Universitetstyckeiet*, Uppsala, 2004
- Mira C.,2000, L'usage du nombre dans la Littérature médiévale (XIIème et XIIIème siècles), in *Si le nombre m'était conté*.. p. 144–172. Collection : IREM—Epistémologie et Histoire des Mathématiques. Ellipses Paris.
- Paquin J.,2008, *L'intertexte scientifique en poésie*, `http://www.epistemocritique.org/spip.php?article66`
- Preece D.A., 2009, *Mathematics in Literature*, `http://www.kent.ac.uk/smsas/staff/dap/70thtalk.pdf`
- Sriraman B., Beckmann A., 2007, Mathematics and Literature :Perspectives for interdisciplinary classroom pedagogy `http://www.ucalgary.ca/ihpst07/proceedings/IHPST07%20papers/125%20Sriraman.pdf`
- Wilburne J.M., Napoli M., *Connecting Mathematics and Literature : an analysis of Pre-sevice Elementary School Teacher's Changing Beliefs and Knowledge, IUMPST : The Journal*. Vol2, 2008
- Zalmanski A. (dir.), 2006, *Mathématiques et littérature, une fascination réciproque*, Editions POLE, Paris.

A COMPARATIVE STUDY ON FINDING VOLUME OF SPHERES BY LIU HUI (劉徽) AND ARCHIMEDES

An Educational Perspective to Secondary School Students

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ABSTRACT

The formula of finding the volume of a sphere was discovered independently in China and Greece. It was well-known that Archimedes of Syracuse (287?–212? B.C.) discovered and proved the formula of finding the volume of a sphere by comparing spheres with right circular cones and circular cylinders. Credited to *LIU Hui*'s (劉徽，225?–295? A.D.) commentaries on *Jiu Zhang Suan Shu* (九章算術，or Nine Chapters on the Art of Mathematics), ZU Geng (祖暅，480–525 A.D.) also found out the same formula by comparing spheres to a special solid called *Mouhefanggai* (牟合方蓋), which is also known as Bicylinder Steinmetz Solid named after Charles Proteus Steinmetz in late 19th century. While understanding the formula of finding the volume of a sphere is included in Hong Kong's Mathematics Curriculum in Key Stage 3, i.e. junior secondary education, it is worthwhile to compare the mathematical methods used in the discoveries and study the possible impact to students' development in the ability to 'think mathematically'.

1 Introduction to Archimedes and Liu's work

In the scope of the history of mathematics, Archimedes and *LIU Hui* were two giants in Greek Heritage and Confucian Heritage cultures respectively. Though little of their lives were known, some of their works were remained. In this article, three classical texts, namely *The Method of Archimedes Treating of Mechanical Problems—To Eratosthenes and On the Sphere and Cylinder, Book 1* by Archimedes and *Liu's Commentaries to Jiu Zhang Suan Shu* (九章算術劉徽注) would be particularly mentioned and discussed.

1.1 Archimedes' works

Discovering the formula of finding the volume of a sphere was one of Archimedes' proudest achievements in mathematics that legend told that Archimedes requested that a diagram representing the

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relationship between the volume of a sphere and that of a cylinder be sculpted on his tomb stone. Up to date, two different proofs to the formula by Archimedes were remained in the mentioned texts.

In the treatise to Dositheus in *On the Sphere and Cylinder, Book 1*, Archimedes demonstrated a rigorous proof in the standard of classical Greek geometry. Using the concept of *reducio ad absurdum* and the method of exhaustion, Archimedes deduced the volume of a sphere from its surface area.

On the other hand, in the rediscovered text by Heiburg in 1906 of *The Method of Archimedes Treating of Mechanical Problems—To Eratosthenes*, an innovative approach of using a lever system to derive the volume of a sphere directly from the volumes of a cone and a cylinder was introduced.

Sir Thomas L. Heath's edition in *The Works of Archimedes and The Method of Archimedes* were referred in this article in the discussion of Archimedes works.

1.2 Liu's works

His commentaries to *Jiu Zhang Suan Shu* were most of Liu's remaining works. However, it should be highlighted that *Jiu Zhang Suan Shu* was already a classical text to the contemporary of Liu. Containing 246 arithmetic application problems and the numerical solutions, and 202 *Shu's* (術，or algorithms), the year of compilation of *Jiu Zhang Suan Shu* could be traced as early the Han Dynasty in the first century A.D.

In the original text of *Jiu Zhang Suan Shu* only included the numerical solutions and the algorithms to the problems. Liu's commentaries were great treasures in the development of mathematics in Confucian Heritage Culture because his commentaries provided the mathematical reasons on how the algorithms could indeed solve the problems stated which were also the important sources for the successors to decrypt this major classical mathematical text in ancient China.

In Chapter 4 of *Jiu Zhang Suan Shu*, a problem was stated to find the diameter of a sphere given its volume. The algorithm in the book implied that there had been a formula in finding volume of a sphere. However, Liu pointed out that the formula was wrong and disproved it by a cleverly designed counter-example. Though he could not derive a correct formula, his contribution was followed by *ZU Geng* about 200 years later and completed the search of the formula of finding the volume of a sphere in ancient China.

In this article, *Liu's Commentaries to Jiu Zhang Suan Shu* referred to the edition in *Qin Ding Si Ku Quan Shu* (欽定四庫全書，or *Imperial Collection of Four Treasures*) compiled in Qin Dynasty in 18th century. In this edition, besides Liu's commentaries there were also additional remarks given by *LI Chunfeng* (李淳風，602 – 670 A.D.) who had recorded the completion of works by Zu.

2 The approaches to find the volume of a sphere

In the following section, mathematical details will be discussed briefly. The article did not intend to give the complete proofs, but some mathematical insights would be highlighted as the points for further discussion in Section 3.

2.1 Archimedes' 'mechanical proof'[1]

Stated as Proposition 2 in *The Method of Archimedes Treating of Mechanical Problems*, Archimedes first considered three solids: a sphere; a cone with height the same as the diameter of the sphere whereas the diameter of the base doubled that of the sphere and a cylinder of same base and height to the cone.

Next, he compared the areas of the cross sections of the solids parallel to their bases. Cut the solids at the same distance from the top of the solids along the height. One could prove that the ratio of the area of the cross section of the cylinder to the sum of the area of the cross section of the cone and the sphere is equal to the ratio of the height of the solids and the distance that was fixed. Denote $A(X, h)$ as the area of the cross section of solid X at distance h from the top of the solids, one could derive the following:

$$A(Cylinder, h) \times h = [A(Cone, h) + A(Sphere, h)] \times 2R$$

where R is the radius of the sphere.

The most interesting part of this proof was that Archimedes treated the above expression in a way analogous to the weight (i.e. force) and the moment arm. Hence, he 'balanced' the weight of the cylinder to the cone and the sphere with an aid of a lever system. It should also be noted that the moment arm of the side of the cone and the sphere was fixed but that of the cylinder varied as the distance of the cut.

Lastly he treated all these circles as thin slides that would add up to the solids. With proper arrangement (the cone and the sphere placed vertically at the distance of $2R$ away from the pivot on one side and the cylinder placed horizontally along the lever from the pivot to the distance of $2R$ on the other side), one could find out the volume of the sphere by considering the position of the centre of gravity of the cylinder, which lied on the position which was at a distance of R from the pivot.

2.2 Archimedes' 'formal proof'[2]

In *On the Sphere and Cylinder, Book 1*, Archimedes somehow extended his idea on finding the area of circle by comparing the area of the inscribed and circumscribed regular polygons. For the case of sphere, he compared the solid of revolution of the inscribed and circumscribed regular polygons.

Archimedes claimed that the volume of a sphere was four times of the volume of a cone with the base equal to the greatest circle and height equal to the radius of the sphere. To prove the claim, he constructed two regular polygons with sides of $4n$. One of which was inscribed in the greatest circle of the sphere whilst the other circumscribed the circle. With a suitably large n, one could control the ratio of the sides.

Moreover, with the similarity property of the solids of revolution of the polygons, one could acquire the ratio of volumes of the solids as the cube of the ratio of the sides. Archimedes then went on using *reducio ad absurdum* to remove the possibility that the volume of a sphere was greater or smaller than four times of the volume of a cone with the base equal to the greatest circle and height equal to the radius of the sphere.

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[1] Heath, T. (ed.), 1912, *The Method of Archimedes: A Supplement to the Works of Archimedes 1897*, Cambridge: the University Press, pp. 18–21

[2] Heath, T. (ed.), 1897, *The Works of Archimedes*, Cambridge: the University Press, pp. 41–44.

Even though the proof was logically sounded, one would be aware that if Archimedes did not know the exact ratio between the volume of a sphere and that of a cone, the proof would have broken down because it would be impossible to choose the number n to control the ratio of the sides. Hence, it was reasonable to believe that Archimedes did find out the method of finding the volume of a sphere somewhere before he worked out a formal proof in *On the Sphere and Cylinder, Book 1*.

2.3 Liu's work and Zu's completion[3]

In the last question in Chapter 4 entitled "*Shao Guang*" (少廣，or "*The Study of Unknown Breadth*"), it stated "Given a sphere with volume 4500 units, find the measure of the diameter" (今有積四千五百尺，問□立圓徑幾何). The provided algorithm to the solution was (1) Multiply $16/9$ to the volume (置積尺數以十六乘之九而一); and then (2) take the cubic root of the product (所得開立方除之□圓徑). In other words, the formula of finding the volume of a sphere could be rearranged as:

$$V = \frac{9}{16}d^3 = \frac{3}{2}\pi r^3$$

taking π equal to 3, which was a common practice in ancient China.

Liu gave a possible reason on the formulation of the above relation by considering a sphere which was inscribed by a cylinder and the cylinder was inscribed by a cube. He believed his precedents took the cross sections at the greatest circle of the cube horizontally and vertically and compared the sphere to the cylinder and then the cylinder to the cube. The following table illustrated the shapes of the solids:

	Horizontal cross section	Vertical cross section
Sphere	Circle	Circle
Cylinder	Circle	Circumscribed square
Cube	Circumscribed square	Circumscribed square

Since the cross sections of the sphere and the cylinder were in one way the same and in the other way a circle to its circumscribed square, meaning that the ratios of cross section areas were $1 : 1$ horizontally and $\pi : 4$, or $3 : 4$, on vertically. Multiplying, the ratio of volume between a sphere and a cylinder was hence $3 : 4$. Similarly, the ratio of volume between a cylinder and a cube was also $3 : 4$. The product of these ratios deduced that the ratio of volume between a sphere and its circumscribed cube was $9 : 16$.

Liu was well aware the logical loopholes of this argument: the cross sections of the solids were not necessarily always touched each other but the cross section of a sphere must lie in the interior of that of the cylinder, and that of the cylinder must also lie in the interior of that of the cube. Hence, this method was an over-estimate of the volume of the sphere. But since approximating π by 3 was an under-estimate, Liu concluded that the formula suggested was just accidentally a reasonable approximation (是以九與十六之率偶與實相近).

He went on to construct the special solid *Mouhefanggai* which circumscribed a sphere to demonstrate that the estimation was better in estimating *Mouhefanggai* than the corresponding sphere.

[3] *LIU Hui, LI Chunfeng* (eds.) (劉徽注，李淳風注釋), *Qin Ding Si Ku Quan Shu: Jiu Zhang Suan Shu*.

Given a cube with side length d, its inscribed *Mouhefanggai* was constructed by the perpendicular intersecting surface of two cylinders of diameter d, i.e. the segments of axes of the cylinders with end points at the bases were perpendicular bisectors to each other. Hence, the shape of all the cross sections of *Mouhefanggai* in the direction perpendicular to any one of the axes were all circles of diameter d whereas that of the cube were all squares of side length d. Therefore, Liu claimed that the approximation method was more suitable for finding the volume of *Mouhefanggai* instead because the cross sections really touched each other everywhere along the axes. However, since a sphere was inscribed by *Mouhefanggai*, the approximation was an over-estimate to a sphere (而丸猶傷多耳).

Noted that if one considers all the cross sections of the sphere and *Mouhefanggai* in the plane parallel to the plane containing the axes of cylinders in the construction of *Mouhefanggai*, all the cross sections of *Mouhefanggai* were the circumscribed squares of the cross sections of the sphere which were circles everywhere along the axes. Hence, Liu pointed out that the exact ratio between the volume of a sphere to the corresponding *Mouhefanggai* was $\pi : 4$, or approximately $3 : 4$. However, Liu failed to discover the method of calculating the volume of *Mouhefanggai*.

It took two hundred years to completely solve the problem in Zu. Zu' works were almost lost. The only remaining results were remarked by Li in *Jiu Zhang Suan Shu*. Zu was recorded to be the earliest mathematician to state explicitly the *ZU's Axiom* (夫疊□成立積 ,緣冪勢□同 ,則積不容異), or known as Cavalieri's Principle.

Instead of directly found the volume of *Mouhefanggai*, Zu picked out one-eighth of *Mouhefanggai* and the cube and studied the difference of the cross sectional areas of the cube and the part *Mouhefanggai* along the the plane parallel to the plane containing the axes of cylinders in the construction of *Mouhefanggai*. He figured out that the difference was equal to the cross section of an inverted square pyramid with base length and height of $d/2$ at the same height. Hence, he found out the volume of *Mouhefanggai* by subtracting the volume of pyramid from the cube and also completely discovered the precise formula on finding the volume of a sphere.

3 How do these related to students in secondary school?

Without any knowledge of calculus for students in Key Stage 3 in Hong Kong who need to understand and use the formula of finding the volume of a sphere[4], it is important for teachers to explore the means to account for the formula. It is thus reasonable to study how the formula was originated and how can such historical findings be integrated in students' learning. Three approaches have been discussed. In this section, some comparison will be made in terms of the cultivation of students' development of 'mathematical mind'.

One may begin with whether the discovery of the volume of a sphere cohered with the geometric intuition. Interestingly, Archimedes and Liu both adopted the use of ratio to represent the volume of a sphere, though the objects of comparison were different.

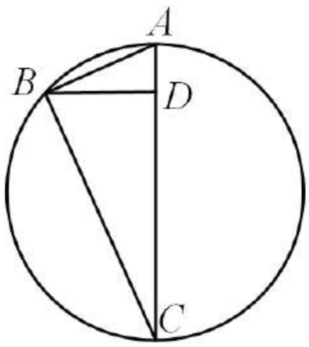
In Archimedes' 'formal proof', solids of revolution of the inscribed and circumscribed regular polygons were compared with the sphere. Such comparison is natural because it is visually well accepted that a circle can be infinitesimally approximated by a regular polygon. Most students might

[4]The Curriculum Development Council, 1999, *Syllabuses for Secondary Schools: Mathematics (Secondary 1–5)*, Hong Kong: The Education Department, p. 20.

have been equipped with similar techniques on finding the area of a circle which they were usually taught to approximate the area of sectors by the area of triangles.

However, the major difference between the treatment of the area of a circle and the volume of a sphere is that it is easy to rearrange the triangles in a way that students could easily find the area by simple calculation, but finding the volume of the solid of revolution is somehow as difficult, if not more, as calculating the volume of the sphere directly to students. Indeed, Archimedes did not directly calculate the volume of the solid of revolution either. Instead, he planned to complete the proof with more than 20 propositions which pointed to transform the solid of revolution into an isosceles cone of the same volume with known base and height. The procedure was far more complicated than the case of finding area of circles.

On the other hand, Archimedes reached the finish line on finding the volume of a sphere in a much faster way with the aid of a lever system. Not entirely mathematically, it is worthwhile for students to experience that one can 'borrow' the idea in the fields of studies other than in mathematics to solve mathematical problem. The starting point of formulating this approach might be a simple geometric observation that in the following figure, $\triangle ABC$ was similar to $\triangle ADB$:



where ABC is a circle with diameter AC, and BD perpendicular to AC. Then it is easy to come to the relation $AD^2 + BD^2 = AC \cdot AD$. And the most intelligent part was how Archimedes transformed the lengths stated in the relation to a physically sounded setting for the lever system. Though the transformation of the sides was quite artificial, it may give some insights for students to read through a basic geometric property to solve a more complicated situation.

Compared with Archimedes' approaches, Liu and Zu's approach relied on another geometric intuition, namely Zu's Axiom. Liu and Zu studied the sphere by the set of cross sections like Archimedes did in his 'mechanical proof'. The shape of *Mouhefanggai* was unfamiliar, but obviously Liu constructed it in the sense of refining an approximation. Knowing the limitation of the approximation given in *Jiu Zhang Suan Shu*, that a cube or a cylinder was too large to touch the sphere, he refined the shape of inscribing the sphere by intersecting another cylinder in another direction with the consideration that a cylinder touched a sphere at the two end points of the small circles other than the greatest circle. Hence, the intersections of two perpendicular cylinder touched the sphere as a square circumscribed a circle.

The idea of refinement was important in mathematical studies. Students in secondary schools might have an illusory thought that all mathematical problems could be solved directly starting from the very beginning, and if one could not solve the problem in such a way he/she failed in the whole problem. Students could be demonstrated with such an example that in the history of mathematics, mathematical breakthroughs were highly valued even though such breakthroughs may not solve the entire problem. Since most of the mathematical contents in the Mathematics Curriculum in Hong Kong are well developed mathematical facts and the emphasis on the process of development of mathematics is yet to be put, more accounts on the mathematical achievements in different stages in the history of mathematics could plant the humanity root back to students rather than considering mathematics as an artifact.

Zu's work demonstrated another advanced way of thinking. The idea of observing the exterior of *Mouhefenggai* was another breakthrough in attacking the problem. It is also a vital technique that a student in mathematics should be well equipped with, and it is another way to transform a mathematical problem to another 'solvable' problem. By a wise application of *Gou-gu* Theorem (勾股定理), or Pythagoras Theorem, Zu related the difference of areas of two squares with a third one. Zu's work could serve as a demonstration on inter-relating various mathematical knowledge.

Educationally, the three approaches were both valued means in demonstrating how a mathematician attacked a mathematical problem with diverse emphasis. Studying across different approaches to the same problem by comparison and contrast could create a more comprehensive mathematical view. It is note-worthy that finding the volume of a sphere is just one of the remarkable examples in the history of mathematics. More similar comparative studies can be done to enhance students' development in 'thinking mathematically'.

4 Conclusion

What Archimedes, Liu and Zu had accomplished elegantly demonstrated the transition from intuition to abstract mathematical concept in different angles. The inclusion of these historical significance may interest students in mathematics to explore the development of mathematics in a humane and comprehensive way instead of learning mathematics through mechanical calculation. The mathematical ideas, the historical perspectives and the pedagogical measures should enhance each other in the long journey of developing a mathematical mind.

REFERENCES

- Curriculum Development Council (The), 1999, *Syllabuses for Secondary Schools: Mathematics (Secondary 1–5)*, Hong Kong: The Education Department.
- He, J., 2004, "Zu-Geng's axiom vs Cavalieri's theory", *Applied Mathematics and Computation* **152**, 9-–5.
- Heath, T. (ed.), 1897, *The Works of Archimedes*, Cambridge: the University Press.
- Heath, T. (ed.), 1912, *The Method of Archimedes: A Supplement to the Works of Archimedes 1897*, Cambridge: the University Press.
- *LI Ji-min* (李繼閔), 1992, *Jiu Zhang Suan Shu Ji Qi Liu Hui Zhu Yan Jiu* (《九章算術》及其劉徽注□究, or *The Studies of Jiu Zhang Suan Shu and its commentaries by LIU Hui*), Taipei: *Jiu Zhang Publishing* (九章出版社).
- *LIU Hui, LI Chunfeng* (eds.) (劉徽注，李淳風注釋), *Qin Ding Si Ku Quan Shu: Jiu Zhang Suan Shu* (欽定四庫全書：九章算術, or *The Imperial Collection of Four Treasures: Nine Chapters on the Art of Mathematics*).

CULTURAL PRIME NUMBERS: 2, 3 AND 5
'문화적' 소수 : 2, 3, 5

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ABSTRACT

수학에서 하나의 소수는 1과 자기 자신 외에 약수를 가지지 않는 수로 자기와 다른 소수로는 표현되지 않는 독립성을 갖는다. 수학에서 1 보다 큰 자연수는 소수로 분해되듯이, '문화적 소수' 로 문화를 분석하게 된다면 그 문화의 정체성을 명료하게 드러낼 수 있을 것이다. 그러나 문화의 성격상 문화를 자연수처럼 소수로 분해하는 것은 쉽지 않다. 그렇다고 불가능한 것은 아니다. 왜냐하면 문화적 소수들은 역사적 사상적으로 '상대적 독립성' 을 가지기 때문이다. 본고에서는 음양론을 표상하는2와 샤머니즘과 연관된 삼재론을 표상하는 3, 그리고 오행론을 나타내는 5를 문화적 소수로 간주한다. 그리고 이들을 통해 10 천간과 12지지 등을 해명한다. 특히 백두대간을 중심으로 샤머니즘에 뿌리를 갖는 삼재론이 음양오행론과 습합하여 한국문화의 원류를 형성하였다는 우실하의 주장에 근거하여 2, 3, 5라는 문화적 소수들 가운데 한국문화에는 3이라는 수에 보다 큰 질적 가중치가 실현되었음을 보이려고 한다.

1 들어가는 말

문화에서 수학이 차지하는 비중은 일반적으로 과소평가되어 왔다. 이러한 저평가는 수학이라는 학문의 특성에도 기인하지만 보다 근원적으로는 문화와 수학과의 관계에 대한 몰이해 때문인 것으로 보인다. 특히 동양문화에서 수학은 서양과 비교하여 상대적으로 더 위축되어있다고 느껴지는데, 이는 유클리드의 <원론> 이 서양문화에 끼쳤던 영향과 비교한 결과 때문이라고 여겨진다. <원론> 은 성경 다음으로 많이 출간되었고 학문의 전형으로 인식되었으며 수학의 영역을 넘어서 서양문화를 세운 하나의 축으로서 평가되고 있는데 반해, 동양문화에서 수학은 매우 실용적인 목적으로 탐구되었고 문화전반에 어떤 큰 지배력을 행사한 것과는 거리가 멀었다고 볼 수 있다. 그러나 동양문화를 주의 깊게 살펴보면 동양문화를 구성하는 중요한 요소들은 수로 표상되어 있었고 이렇게 표상된 수는 풍부한 문화적 의미를 담지하여 사람들의 의식 혹은 무의식을 상당히 지배한 것으로 보인다.

수학에서 소수는 1과 자기 자신 외에 약수를 가지지 않는 수로 정의된다. 따라서 하나의 소수는 자기와 다른 소수로는 표현되지 않는 독립성을 가지고 있고, 모든 자연수들은 소수들로 분해되어 표현가능하다는 것은 잘 알려진 사실이다. 즉 1보다 큰 모든 자연수는 소수의 곱으로 환원될 수 있다. 괴델이 불완전성정리를 증명하는 데에도 이러한 성질을 활용하는데, 소수의 중요성은 자연수를 한 가지 방식으로 '소인수분해' 한다는 데에 있다. 소수가 서로 독립성을 가지고 자연수를 표현하는 것과 같이, 문화를 분석하는데 있어서도 '문화적 소수' 로 문화를 '소인수분해' 하여 취급하게 된다면 그 문화의 정체성은 명료하게 드러나게 될 것이다. 그 문화를 구성하는 요소들은 마치 소수와 같은 역할을 하게 될 것이고 이 구성요소들의 결합으로 그 문화는 표현되고 이해될 것이다. 그러나 이러한 시도는 수학에서처럼 그렇게 단순하지 는

않을 것이다. 왜냐하면 문화란 역동적이고 역사적 맥락을 가지고 있으며, 문화에서 사용되는 시원적 개념 자체가 애매하고 모호할 수 있기 때문이다. 따라서 문화를 분석하는데 사용되는 개념이나 구성요소가 상호 독립적인 것인가에 대한 문제가 제기가 있을 수 있다. 문화를 구성하는 요소의 개념적 독립성이 수학처럼 보장될 수는 없지만 역사적 사상적 맥락에 따라 독립성을 가진 것으로 간주할 수 있는 길이 전혀 없는 것은 아니다. 또한 문화가 가진 역동성으로 인해 문화를 분석하는 '문화적 소인수분해' 는 수학과 같이 반드시 한 가지 방식으로 되어야 한다고 고집할 필요가 없다.

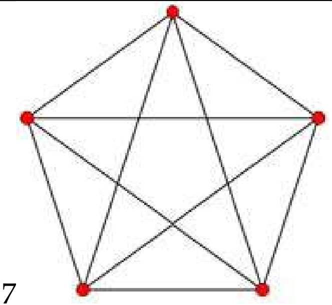
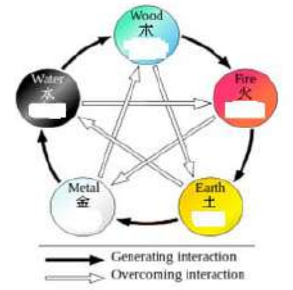
수학에서와는 달리 문화 분석에서는 적어도 두 가지 면에서 어려움을 가진다고 여겨진다. 즉 문화 분석의 구성요소들 간에 개념의 독립성이 완전히 확보되기가 쉽지 않고, 개념적 독립성을 확보한다고 할지라도 그 문화를 개념적 독립성을 가진 구성요소들로 분해하는 것의 유일성이 보장되기는 어렵다. 이러한 한계에도 불구하고 문화를 소인수분해하는 시도가 무의미 하지는 않다고 본다. 왜냐하면 적은 요소들을 가지고 많은 문화적 상황을 설명할 수 있고 이를 통해 분석하고자 하는 문화에 대하여 어느 정도의 정리된 그림을 그려볼 수 있기 때문이다. 문화적 소수는 수학적 소수와는 달리 다른 또 하나의 성질을 갖는다. 그것은 문화를 분석하는 것이 질적인 평가라는 측면에서 당연하기도 한 것이지만, 문화적 소수가 문화권에 따라 질적 가중치를 지닌다는 것이다. 즉 문화적 소수들은 수학적 소수처럼 동등한 자격을 가지는 것이 아니라 비록 같은 문화적 구성요소를 지닌다 할지라도 문화권이 다르면 질적인 가중치가 달라질 수 있고 어떤 문화적 소수가 다른 것 보다 더 중시될 수 있다는 것이다. 이를 수학적으로 표현한다면 특정 소수의 지수가 큰 것으로 생각할 수 있을 것이다.

수학에서 가장 작은 소수는 2이며 그 다음은 3이고, 5는 세 번째 소수이다. 그런데 이 세 소수들은 문화적으로도 중요한 의미를 가진다. 2는 음양론을, 3은 샤머니즘과 연관된 삼재론을 그리고 5는 오행론을 각각 표상한다. 본고에서는 이들 세 수가 수학에서 소수인 것처럼 문화적으로도 서로 환원될 수 없는 어느 정도 고유한 독립성을 가지고 있고 따라서 다른 여러 문화현상들을 해명해 준다고 주장한다. 그러한 점에서 2, 3, 5를 '문화적 소수' -라고 명명하고, 특히 한국문화는 이 문화적 소수들 중 3의 질적 가중치가 높은 즉 양적으로 표현하면 3의 지수가 크다는 것을 보이려고 한다. 이 글은 우실하의 저서『전통문화의 구성 원리』에서 주장한 한국인의 "문화 구성 원리" 에서 크게 도움을 받았으며, 그의 주장에 대한 수학적 이해라 해도 과언이 아니다.

2 문화적 소수 : 2, 3, 5

전술한 바와 같이 2와 3과 5는 음양론과 샤머니즘과 연관된 삼재론, 오행론을 각각 가리킨다. 이 세 수가 표상하고 있는 바가 오랫동안 고대 동양문화를 지배했다고 해도 중과부언이 아니다.

음양론에서 2의 소수체계는 참과 거짓, 어둠과 밝음, 낮과 밤, 남자와 여자 등 오직 두 가지의 기본적인 논리적 가치 만에 기초하면서, 그들의 상징적 기능에 따라서 지속적으로 하나의 완벽한 조합에서 전개되고 용해된다. 음양의 이가 체계를 처음으로 발견한 복희씨는 음--과 양— 이 두 가지 효만으로 ☰ ☱ ☲ ☳ ☴ ☵ ☶ ☷ 라는 팔괘를 창조하였다. 근대의 라이프니츠는 복희씨를 이 상징들의 수학적 의미를 최초로 알고 있었던 인물로 평가하였다. 라이프니츠는 왜 팔괘로서 우주를 구성하였는지에 대하여서는, 아마도 신이 세계를 7일 만에 창조하였듯이, 복희씨도 그러한 계획의 비밀을 간파했기 때문에 팔괘를 고안하였을 것이라는 재미있는 추측을 하였다. 팔괘의 주기는 0, 1, 2, 3, 4, 5, 6, 7이라는 수열의 질서가 반복되므로 생긴 수리체계이다. 팔괘가 이러한 순환과정으로 생긴 것은 피타고라스의 옥타브 음이 도, 레, 미, 파, 솔, 라, 시, 도에서 순환하므로 생기는 것과 같은 의미이다. 복희씨의 중첩된 팔괘로 64괘의 알고리즘이 도출되는 것은 자명한 이치였다.

펜타곤	5행설
 7	

삼재론이란 고대 동아시아 사회에 한반도로 이동한 민족 집단이 샤머니즘과 더불어 형성된 삶과 죽음과 세계, 그리고 하늘과 땅과 사람 사이를 매개하는 우주의 궁극적 실재(ultimate reality)에 대한 믿음의 형식이다. 고대 수렵사회는 원시사회로 삶과 죽음 사이의 현생에 대한 구분의 방안으로 삼재, 곧 하늘과 땅과 사람의 세 원소가 필요하였다. 현생에서 삼원소 사이의 독립적 경계는 무당에 의하여 중재되는데, 무당은 현생과 영계를 대화로 잇는 중재자이다. 즉 무당은 살은 자와 죽은 자 사이를 화해시키는 이야기를 이끌어내는 중재자 역할을 한다. 토테미즘과 샤머니즘은 대부분 써지지 않은 구술과 비전의 전승형식으로 보존되어 카발라 주의나 신비학적 성격을 띠고 있기 때문에 문헌학적 접근에는 한계가 있다. 그러나 민속학적 문화인류학적 관점에서 보면 삼재론은 우주에 대하여 하늘과 땅과 사람이라는 삼가 논리로 접근하고 있다. 삼재론이 다원적 수리논리의 가치에 기초한다고 보는 이유는 바로 무당 역할에서 볼 수 있듯이 현생과 영계 사이의 화해로서 이야기를 이끌어내기 때문이다. 삼재사상이 고대문명사회에서 승리할 수 있었던 요인은 현생에서의 이러한 영적 세계의 질서를 편입하므로 삶의 긍정적 가치를 더할 수 있었다는 점일 것이다. 프로이트의 심리학이 인간의 심층심리구조를 파헤치고 융이 인간의 집단무의식의 존재를 부각시켰듯이 삼재사상 역시 오랜 동양사유의 전통에서 자신의 독립적인 존립기반을 가지는 것은 부자연스러운 일은 아닐 것이다.

오행론은 원 안에서 동서남북 중앙이라는 다섯 방향으로 배열될 수 있어서 공간의식을 잘 반영하고 있다. 오행이란 우주를 표상하는 금, 수, 목, 화, 토라는 다섯 가지 기본원소의 질료적 변형과정을 나타내고, 금성, 수성, 목성, 화성 그리고 토성이라는 천체현상을 공간적으로 표상하고 있다. 오행은 각각 그 배열과 방향에 있는 상호작용에 따라 우리가 펜타곤을 형성하는 선분을 바탕으로 보았을 때, 생성기능과 파괴기능을 갖는다. 이미 잘 알려진 바에 따르면, 서양과학문화에서는 전통적인 프톨레마이오스 천동설이 코페르니쿠스, 갈릴레이, 케플러, 뉴턴에 의하여 각 행성들, 즉 떠돌이별들의 운행궤도의 위상학적 위치배열을 태양 중심에서 정확하게 계산하므로 태양 중심체계로 변형 된지는 약 500년이 채 못 된다. 이에 반하여 오행체계는 인간관계, 건강, 건축, 문화디자인, 음악, 언어, 정치, 비즈니스, 등 다양한 영역에서 일어나는 관계를 풀이한다. 그들의 논리적 가치는 복수이고, 다르고, 다원적이어서, 그들 오행의 형이상학적 토대는 생성하는 상호작용을 얻을 때, 조화로운 통일에 놓인다.

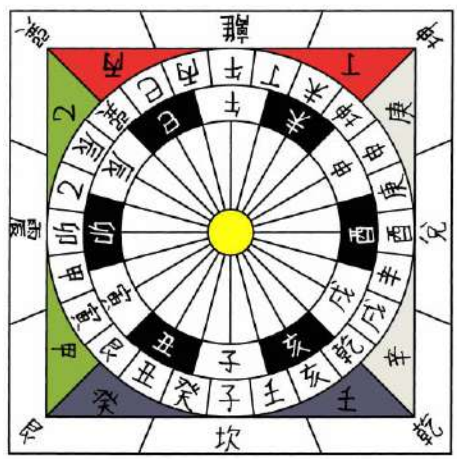
음양론, 삼재론, 오행론을 각각 표상하는 2와 3과 5가 문화적 소수가 되려면 음양론과 삼재론 그리고 오행론 사이에 서로 환원되거나 회귀될 수 없는 어떤 독립적인 요소가 있어야 한다. 물론 이 때 독립성은 수학처럼 그렇게 엄격한 것은 아닐지라도 어느 정도 독립적인 세계관적 맥락을 가지고 있어야 한다는 것을 의미한다. 상호 독립적이라는 것을 보이기 위해 모든 경우를 다 검토해야 하나 관심의 대상이 되는 경우만을 검토하기로 한다.

우선 삼재론과 오행론이 가장 작은 수로 표상되는 음양론에로 환원이 가능한 것이 아닌지를 살펴보자. 그러나 이러한 시도는 삼재론의 천, 지, 인중에서 양에 해당하는 천과 음에 해당하는 지를 제외하고 양과

음 어디에도 귀속되지 않는 인이 남게 됨으로써 삼재론을 음양론으로 완전하게 환원시킬 수는 없다. 마찬가지로 오행론도 양과 음으로 균등하게 분해하면 하나 즉 토가 남게 된다. 그런데 오행론이 2와 3 즉 음양론과 삼재론으로 분해된다는 견해는 문화적 소수의 독립성을 주장하는 입장에 대한 상당한 위협이 된다. 오행론을 음양론과 삼재론으로 분할할 수 있다는 주장을 수학적으로 말하면 2☆3=5인데 이 때 +대신 ☆를 넣은 것은 오행이 음과 양에 단순히 삼재인 천, 지, 인을 더한 결과가 아닌 어떤 특별한 연산이라는 뜻이다. 즉 이 연산에서는 목, 화, 토, 금, 수의 오행 중 양에 목과 화를, 음에 금과 수를 각각 배당하고 오행의 주재자내지 원리로서 토를 삼재론의 인에 해당하는 것으로 간주함으로써 오행을 음양과 삼재로 분할할 수 있다고 하는 것이다. 그러나 이는 오행론에 대한 피상적 관찰이라고 할 수 있다. 비록 오행론에 음양론과 삼재론이 습합되어 있다는 것을 부인하지는 못한다 해도 오행론에는 음양론과 삼재론으로 해명할 수 없는 '오행 상생의 원리' 와 '오행 상극의 원리' 가 있기 때문이다. 또한 1에서 출발하여 4, 8, 16, 32, 64로 분화하는 2수 체계의 음양론과는 달리 삼재론은 0에서 출발하여 1, 3, 9, 27, 81로 분화되는 과정에서 2를 건너뛰는 3수 체계를 가지고 있다. 한편 오행론에서는 주로 5수의 관계성을 가지고 삼라만상을 해명하려 한다. 이런 점에서 문화적 수로서 2, 3, 5는 어느 하나가 다른 것으로 완전히 환원되거나 둘이 합하여 다른 것을 설명하는 것이 어려운 서로 독립적인 성격을 가지고 있다고 볼 수 있다. 따라서 2, 3, 5를 '문화적 소수' 라고 부르는 것은 정당성을 가진다.

그러면 이들을 문화적 소수로 해명할 수 있는 것은 무엇인가? 우선 쉽게 생각할 수 있는 것은 10 천간과 12 지지와 같이 주로 수와 관련한 것이다. 10 천간은 기원전 1250년경 상 시대의 10일을 일주일 단위로 놓던 고대중국의 기수체계에서 유래하였다. 상 시대에는 10일(旬)을 주기로 10개의 태양이 나타난다고 믿었고, 천간은 각각의 날에 희생으로 받친 죽은 자의 명칭으로서 갑, 을, 병, 정, 무, 기, 경, 신, 임, 계 라는 명칭은 신전에 새겨진 황제들의 이름에서 따온 것으로 추정되고 있다. 10 천간은 목성괘도를 관측하므로 형성된 12 지지 체계와 결합되어서 사용된다. 12 지지는 목성의 12년의 괘도를 모사하는 기술로서 정확하게는 11.85년에 해당되는 명칭으로서 자, 축, 인, 묘, 진, 사, 오, 미, 신, 유, 술, 해를 갖는다. 목성 주기의 12년은 일 년 12달, 12동물, 방향, 계절, 달, 24시를 분할하는 의미를 갖는다. 그래서 한반도에도 오랜 천문관측의 전통이 있어왔으며, 고대 동아시아의 천문체계로서 12 지지 체계를 수용하여 내려오고 있다. 달력은 고대 동아시아 세계에 하늘과 땅의 상호작용에 관한 문제의 표현이다. 여기에는 중국달력이나 한국달력도 모두 동일한 12 지지 체계에 속하고 있다. 간지는 갑자에서 출발하여 계해에서 끝난다. 이러한 방식에서 12와 10을 결합하여 갖는 60갑자가 생겨난다.

간지체계



갑, 을 병, 정, 무, 기, 경, 신 임, 계로 이루어진 10 천간은 오행과 음양으로 분해할 수 있다. 즉 오행의

관점에서 보면 갑과 을은 목에, 병과 정은 화에, 무와 기는 토에, 경과 신은 금에, 임과 계는 수에 각각 해당하고, 오행의 요소에 해당하는 각 쌍 중 앞에 것을 양으로 뒤에 것을 음으로 간주함으로써 10 천간은 해명된다. 12지지의 경우에도 목에 인과 묘를, 화에 오와 사를 금에 신과 유를 수에 자와 해를 각각 배당하고 각 쌍의 전자를 양으로 후자를 음으로 하되 토에는 축과 진을 양으로 미와 술을 음으로 간주함으로써 해명된다고 본다. 음양론은 오행론과 결합되어 음양오행론으로 발전하였는데 샤머니즘과 관련된 삼재론은 음양오행론과 달리 상대적으로 독자적 행태를 보인다. 이에는 사상적인 요소뿐 아니라 지정학적 측면도 작용한 것으로 보인다.

3 한국문화의 문화적 소수

한국문화와 역사전통은 오랫동안 동아시아 문명사회를 지탱해온 2, 3과 5의 소수체계를 공유하고 있다. 이 모든 소수체계가 고대 동아시아 문명세계를 형성한 수학적 토대가 될지라도, 한국문화의 원형적인 소수체계는 특히 3을 중심으로 한국인의 고유한 심성에 뿌리 박혀 발전되어왔다. 3수 중심 수리사상은 한반도의 백두대간의 동쪽을 중심으로 전래되었고, 음양오행으로 대표되는 2와 5 수리체계는 백두대간 서쪽으로 들어와, 역사진행과 더불어 3 수 중심의 2와 5의 습합으로 완결된 원형수리사상을 형성하게 되었다. 2는 5와 더불어 인간만사와 자연현상의 선험적 구조를 설명하는데 필요한 정형화된 사유패턴에 적합한 수리체계인 반면에 3은 중앙아시아 시베리아에서 발원하여 한반도에서 심층적으로 발전된 샤머니즘에서 내려오는 수리체계이다. 3의 소수체계는 수렵과 유목생활에서 형성되어온 문화에 근거를 두고 있다면, 2와 5의 소수체계는 농경문화에 적합하게 수용되어 왔다. 2, 3, 그리고 5의 소수체계는 서로가 뒤섞이지도 않고 착종되지도 않는 독립성을 갖는다. 수렵문화에서 3을 바탕으로 하는 한국문화의 원형적 소수체계는 2와 5에 의하여 형성된 농경문화를 습합하여 발전하므로 동아시아 문명세계의 근원적인 가치형성에 기여하였다.

소수체계의 2와 5는 각각 음양론과 오행론으로 대표되는 점을 잘 알려진 사실이나, 한국문화와 한국인의 심성에 자리 잡은 원형적인 3수의 형이상학적 표상과 세계관의 논리적 함축은 무엇인가? 2와 3의 소수체계는 고대 동아시아 세계의 근원적인 논리 형이상학적 구성의 산물이다. 이들 체계는 고등문명의 과정에서 인간존재와 우주의 궁극적 실재를 파악하는데 상호조절이 되었던 개념적 도구였다. 음양론이 오직 그들의 상징적 가치 음--과 양— 만을 인정하였음에 반하여 오행론은 어떤 다른 원자적 존재를 고정시켰던 것이 아니라 자연의 보편적 과정, 인간관계, 재산, 건강, 길흉화복 등을 기술하는 수단이었다. 그러나 한국 샤머니즘에서 뿌리를 박고 한국인의 심성에 자리 잡고 천, 지, 인을 대표하는 삼재사상의 3수 체계는 삶과 죽음의 경계에서 3가지 우주적 상징가치를 실현한다. 샤만, 즉, 무당은 이생의 삶을 저승의 죽음을 연결하는 중재자로서 삶도 죽음도 넘나드는 경계인이다. 수렵과 유목에 의한 삶의 방식에 오랫동안 남아있는 토테미즘이나 정령주의적인 사고방식이 이러한 샤머니즘에 내려왔기 때문에, 한국인의 심성에서도 삶과 죽음의 엄격한 구분을 인정하지 않는 조화로운 통일적 세계관이 한반도의 산과 바다와 땅을 중심으로 남아있다. 모든 일생일대의 삶의 계기에 죽음을 걸고 있지만 이 경계를 쉽게 넘어서 새로운 생명의 도약을 시도하는 이러한 수렵과 목축에 의한 유목민의 세계관은 3수 소수체계로 잘 표현되고 있다. 그러나 3수 소수체계는 길이, 높이, 넓이에서 처음과 중간과 마지막, 위와 중간과 아래, 크고 작고 가깝고 멀고를 조율하는 다원적 가치를 인정하므로, 오행과정의 경계를 확정하고, 음양론의 약점을 중첩적으로 보강한다. 음양의 전체의 64괘는 팔괘의 중첩으로 구성되는데, 여기에도 3수 체계가 기여하고 있다. 팔괘의 각 괘는 음--과 양— 각 효가 상, 중, 하에 조합이 되어 $2 \times 2 \times 2 = 8$을 형성한다. 3수 체계의 상, 중, 하는 각각 천, 지, 인이라는 삼재사상을 표현하였고, 팔괘의 종횡의 조합은 $8 \times 8 = 64$괘를 만들어냈다.

4 문화적 소수 3의 표현형식

한국전통 석기문화에서 이러한 삼재사상은 결정적이라 할 수 있으며, 건축, 전례의식, 색상, 고분양식, 무덤양식, 문양, 설화, 등 다양한 문화디자인에서 표출되어있다. 문자가 처음부터 고정성에 결부된 것이라면, 돌에 표출이 되어있는 상징 언어에는 되물을 수 없는 과거의 흔적에서 현재에 이르기까지 그들의 전파의 질료성의 담지의 강도를 보증하는 한, 한반도의 전역에는 이러한 문물의 표현형식을 보여준다. 고려고분에는 주로 삼족오가 문화적 소수 3의 표현형식을 갖고 있으며, 조선의 전형적인 왕실의 창살문에는 삼 태극문양으로 표현되어 있다. 대한민국 국보 1호인 숭례문 현판식의 서까래와 기둥 문에도 삼 태극문양을 관찰할 수 있다.

일연의『삼국유사三國遺事』는 한반도에서의 국가와 민족의 기원에 대한 설화를 전하고 있는데 여기에는 문화적 소수의 3이 인상적으로 각인되어 표현되어 있다. 곰과 호랑이 태양을 숭배의 대상으로 삼는 태양숭배와 토테미즘은 중국북동지역과 극동러시아 지역에 산포되어있는 믿음체계이다.『삼국유사』에 따르면, 기원전 2333년에 중국 북동쪽 요령지방과 한반도에서 고조선을 건립한 전설적인 한국민족의 창시자는 단군왕검이다. 단군왕검에 관한 설화는 고려『삼국유사』이외에도 이승휴의『제왕운기帝王韻紀』, 중국 위수(魏收)의『위서魏書』에 각각 전해진다. 단군의 조상은 천손(天孫)으로 그의 할아버지 환인(桓因)은 상제(上帝)이다. 환인의 아들 환웅(桓雄)은 산과 계곡이 있는 땅에서 살고 싶어 하여서, 아들의 마음을 읽은 아버지는 풍사(風師), 우사(雨師), 운사(雲師)를 거느리는 3000 무리를 딸려서 태백산에 보내어 신시(神市)를 열어 법과 제도를 정비하고 기술, 의술, 농법, 뜸질과 침술을 가르치게 하였다. 여기서 호랑이와 곰이 인간이 되게 해달라고 환웅에게 기도하자, 환웅이 그들에게 쑥과 마늘을 주고 100일 동안 햇빛을 보지 말고 이것만 먹으라고 하였더니, 호랑이는 20일 후에 포기하고 동굴을 떠났고, 곰은 남았더니 여자로 변하였다. 여기서 웅녀(熊女)는 박달나무 곁의 신단수(神檀樹)에서 아이를 갖게 해달라고 기도하였는데 환웅이 웅녀를 아내로 삼아서 단군을 낳아 드디어 왕위에 오른 단군은 태백산의 아사달에 도읍지를 정하여 고조선을 건국하였다는 이야기이다. 이러한 설화를 전한 일연은 당시 원나라의 수도에 들어온 성 프란시스코 수도사들의 기독교 사상에 영향을 받아서, 단군을 처녀의 몸에서 탄생하게 하고, 단군신화에 드러나는 신적인 3수의 의미가 삼위일체와의 연관에서 나온 것이라는 주장도 있다. 이 신화는 고대한국인은 하늘세계를 박차고 지상의 세계로 내려와서 삶을 그려내고 있다는 점에서 삶과 죽음이라는 이분법적 구조를 지양하고 하늘과 땅 사이를 매개하는 현실세계를 그려낸다는 점에서 삼재사상을 반영한다.

한국 태권도에서도 운동을 수행하는 주재자가 단군 형, 팔괘에 따른 품세를 그려낼 수 있게 만들어져 있다. 운동수행의 처음과 마지막 사이의 과정은 태극의 음양 동정과 강유를 반영한다. 운동수행의 출발점과 도착점에 이르기 까지 중심에서 사방을 향한 동작을 사람의 주먹과 다리와 발 세 가지를 사용한다는 점에서 이 역시 3을 중요시 여긴다. 대한민국 국기의 태극디자인은 흰 바탕에 검은색의 건곤과 감리가 맞보고 있고 둥근 원에 위로는 양의 기운을 나타내는 적이 있고, 아래에는 음의 기운을 품는 청이 있다. 태극기 디자인에서 색깔은 홍, 청, 흑이다. 건은 하늘, 봄, 동을 지칭하며, 곤은 땅, 여름, 서이며, 감은 달, 겨울, 북을 지칭하고, 리는 해, 가을, 남을 지칭한다. 오행 방위는 목(동, 청), 화(남, 적), 토(중앙, 황), 금(서, 백), 수(북, 흑)이며, 인체의 비장(肝臓), 심장(心臓), 위장(胃臓), 패장(폐臓) 그리고 신장(腎臓)을 담고 있다. 삼재사상이 음양 오행론을 가장 학문적으로 습합한 결실은 한글창제에 잘 드러나 있다. 한글에는 삼재원리가 •(하늘), ㅡ(땅), 그리고 ㅣ(사람)라는 부호로 나타난다. 수 가운데 짝수를 양성모음과 인, 의, 예, 지, 신의 오상의 하늘, 홀수를 음성모음과 간, 심, 비, 폐, 신의 오장을 갖춘 땅에 배합시키면, 사람인 ㅣ만이 자리(位)와 수(數)가 없이 "무극의 참과 이오의 정기가 묘하게 엉켜" 초중성의 발현을 구현한다.

5 문화적 소수 3의 다가 논리

음양론을 아리스토텔레스 이가 원리에 상응하는 논리구조를 갖는다고 간주하면, 삼재론은 다치 논리학의 한 부류에서 파악될 수 있다. 음양론이 음과 양이라는 형이상학적 기본가치를 설정한다는 점에서 이가 논리의 참과 거짓이라는 논리적 가치설정과는 같은 방향으로 간다. 아리스토텔레스에 의하여 창시된 이가 논리학은 서양논리학의 핵심적 가치로서 서양문명의 근본적 토대로서 20세기 초에 이르기까지 흔들림이 없이 발전해왔다. 그러던 중 2차 세계대전을 경과하면서 폴란드에서 나치 독일군의 바르샤바 침공을 앞두고 삼자배척의 원칙 앞에서는 다치 논리학적 선택에 직면하므로 다치 논리학이 태동하였다. 2000년 이상을 지배한 이가 논리도 다치 논리학의 요구 앞에 한 걸음 비켜나간 것이라면, 동아시아문명권에서는 음양론이 삼신사상을 수용하느냐 마느냐의 기로에서, 역으로 삼신사상이 음양론을 습합하느냐의 문제 앞에 손쉬운 논리를 제공한다. 오랜 동아시아 전통은 양자병행 통합으로 진행되었음을 보여준다. 한반도의 경우는 백두대간 동쪽으로 삼신사상이 전래되었고, 서쪽으로는 음양사상이 유행하였지만, 삼신사상과 음양오행의 결합은 자연스럽게 한국전통문화의 문화 융합의 원형적 요소로 자리 잡았는데, 서양의 이가 논리학에 직면한 다치 논리학의 탄생배경에 또 다른 해결의 전형을 보여준다.

동아시아의 이가 원리는 아주 견고하게 음양의 순환의 원리를 문화적 문맥에서 적용하므로 삶과 죽음 그리고 우주에 대한 변화를 설명하였다. 철학적 이론으로서의 이가 원리는 실제로 종교, 문화, 과학의 영역에서 나타나는 다양한 현상들을 이원론적 세계관으로 환원시키므로, 나와 너, 주어와 대상, 정신과 물질, 삶과 죽음 등의 경계를 분명하게 발전시킬 수 있었다. 그 점에서 음양의 이가 원리는 자연, 의학, 건축, 음악, 문화, 정치, 경제, 등 다방면에 적용되었으며 오늘날 디지털문명의 이진법의 원형적 뿌리를 이룬다고 평가되기도 한다.

이가 원리의 형이상학적 배경은 무극으로 거슬러 올라간다. 無極生有極, 有極是太極, 太極生兩儀, 卽陰陽: 兩儀生四象, 卽 少陰, 太陰, 少陽, 太陽. 四象演八卦, 八八六十四卦이다. 주역 64괘의 토대가 되는 팔괘를 창안한 복희씨는 우주의 근본적 실재를 상징하는 乾 兌 離 震 巽 坎 艮 坤의 팔괘에 陰爻와 陽爻라는 이가원리를 삼중주를 쌓아서 단괘, 내지 소성괘라는 명칭을 갖는 상징체계를 창조하였다. 팔괘의 상징체계는 ☰ ☱ ☲ ☳ ☴ ☵ ☶ ☷에 해당되며 이에 상응하는 자연현상은 天, 澤, 火, 雷, 風, 水, 山, 地이다. 팔괘를 東이 離, 西가 坎, 南이 乾, 北이 坤이고, 그 외에 나머지는 각각 45도 방위를 유지하도록 방향에 따라 배열할 수 있다. 이러한 복희씨의 팔괘를 일컬어 선천 도(Earlier Heaven)이라고 한다. 훗날 주문왕은 사물의 자연적 질서에 따른 복희씨의 선천 도에 인간의 심성을 반영한 東이 震, 西가 兌, 南이 離, 北이 坎인 방위를 갖는 후천 도(Later Heaven)를 작성하였다. 복희씨에서는 원 중심에서 천을 가장 꼭대기에 놓고 반시계 방향으로 45도 방위를 유지하면서 돌아가면 사방이 나온다. 그러나 문왕의 후천 도에서는 이러한 완벽한 원 중심의 대칭관계는 무너지고 대신 엄격한 마방진의 사방이 등장한다.

무극에서 이가 논리를 토대로 64괘가 형성되었음에도 불구하고 여기에는 음양 이외에 하늘과 사람과 땅에 해당되는 삼신을 기본으로 팔괘가 나왔다. 왜 음양을 삼중주로 쌓아 팔괘를 낳았고 2^3, 또 팔괘를 중첩하여 8^2, 육십사괘를 만들었는지는, 마치 십 천간과 십이지간이 합하여 육십갑자가 생겨난 이유에 대한 물음과 맥락을 같이 한다. 이러한 질문은 동아시아의 고대문명의 원형 세계상 구성에서, 한반도에서의 삼신사상이 음양오행 사상을 습합하는 문명과정과도 동일한 맥락에서 세워진다.

오행은 의학, 음악, 점성술, 군사전력 등에 사용되었던 사물의 '운동', '경과', '단계', '원소' 등의 의미를 지니는 개념으로 각각 금, 수, 목, 화, 토라는 순환적 상징체계를 갖는다. 오행은 일종의 상호관계를 통하여 진행되는 사이클로 하나는 상생주기와 다른 하나는 파괴주기를 갖는다. 상생주기는 금→수→목→화→토, 파괴주기는 금↛목↛토↛수↛화이다. 회남자(淮南子)는 오행에 방향을 주었는

데, 그 방향은 금(서), 수(북), 목(동), 화(남), 토(중앙)이고, 색깔은 금(백), 수(흑), 목(청), 화(적), 토(황)이다. 오행은 팔괘의 방위로 나타낼 수 있으며, 오행의 방위는 색의 배치로 정돈될 수 있으며, 그것은 한국 전통문화의 원류로 삼 태극사상으로 자리 잡게 된 것으로 보인다.

6 나가는 말

음양론은 전통적 동양과학의 토대로서 주역의 수리이론과 결합하므로 한국의 문화적 전통의 원형형성에 크게 기여하였다. 오행 우주론은 17세기를 전후로 서양의 코페르니쿠스 이래의 태양 중심 세계관에 기초한 케플러 행성모델의 우주론에 경쟁적으로 대치되다가 급격하게 퇴조하여 지금은 문화 상징적 원형으로만 남고 있다. 오행이론의 강점은 색과 방향에 대한 탁월한 혜안을 제시하는 천문체계를 반영함과 동시에 전통 동양의학체계의 해명에 크게 기여한다는 점이다.

한반도의 삼재론은 전통적인 중국의 음양오행사상을 습합하면서 다치 논리학적 기초 위에서 다원적인 조화로운 세계관 형성에 기여하였다. 음양사상이 2를 소수로 하는 이가 원리를 조합법에 의한 토대에서 64괘를 완성하였다면, 오행사상은 전통적인 천문학적 관측체계인 천간과 지간의 조합에 의한 60갑자법과 결합하여 문명의 흥망성쇠와 세상만사의 길흉화복에 대한 점괘를 점지하는 역할을 하였다. 3에 대한 철학적 의미는 노자의『도덕경』42장에서 道生一一生二二生三萬生物萬物負陰而抱陽沖氣以爲和이라는 문구에서 보듯 만물의 화(化)를 향한 우주정형의 원형적 패턴으로 사용되고 있다. 삼재론은 음양오행을 보다 다원적이면서 조화로운 사상형성의 중심역할을 감당하였다. 대한민국의 문화원형적 소수체계는 2, 3, 5로 이루어져 있지만 특히 3에서 그 독특성을 보이며, 3에 의하여 음양오행이 보다 더 완벽한 형태로 발전할 수 있었다.

근대과학문명은 아리스토텔레스의 이가 논리를 수용하여 영육, 정신과 물질, 죽음과 삶의 이분법만 인정하고 죽음을 몰아내므로, 기계론적 물질문명을 구축하였다. 음양론은 음양차서도(陰陽 次序圖)에서 볼 수 있듯이 이진사유에 의한 세계만물의 변화를 쉽게 설명하는 아리스토텔레스 이가 원리에 부합될 수 있었다. 삼재론의 다원적 논리구조는 고전논리를 딛고 물질적 우주에 대한 다원적 논리적 가치의 접근으로 새로운 문명형성의 가능성을 개진할 수 있다는 점에서 중요한 의미를 지닌다.

2와 3과 5가 문화적 소수라고 해서 서로 완전 독립적이거나 모든 문화현상을 설명할 수 있다고 주장하는 것은 아니다. 문화적 소수들 간의 독립성은 역사적 사상적 맥락에서 '상대적 독립성' 을 지니며 그래도 의미 있게 문화를 해명한다. 수학에서 가장 기본적이고 오랜 역사를 가지고 있는 소수에 대한 연구가 아직도 진행 중인 것처럼 '문화적 소수' 를 활용한 문화에 대한 분석도 지속될 수 있을 것이다.

REFERENCES

- G. Guenther, Idee und Grundriss einer nicht-Aristotelischen Logik, Hamburg 1991.
- J. Lukasiewicz, Aristotle's Syllogistic from the Standpoint pf Modern Formal Logic, Oxford University Press, 1951, Reprinted in 1987.
- R. F. Matzka, Klassische Kenogrammatik, Eine semiotische Ortsbestimmung, 2010.
- P. H. Richter, Science and Calendars in China and West From Clavius to Xu Guangqi and Schall, Bremen 2008.
- en.wikipedia.org/wiki/"Celestial stem", "Earthly Branches", "Fu Xi", "Yin Yang", "Ba qua", "Korean Mythology", "Chinese calendar"

- 우실하, 『전통문화의 구성 원리』, 소나무, 1998.
- 이기동, 『주역강설』, 성균관대학교출판부, 2010.
- 일연(김원중 역), 『삼국유사』, 민음사, 2008.
- 후웨이홍(이은미 역), 『왼손에는 명상록 오른손에는 도덕경을 들어라』, 라이온북스, 2010.

MATH AND ART IN VIEW POINT OF PERSPECTIVE DRAWING OF THE WEST AND EAST

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ABSTRACT

In this study, we consider perspective drawing of the West and East. In particular, we are pay attention to the Western perspective (one point perspective), and the Eastern perspective(multi point and parallelism) Moreover, we research the reason: why the Western perspective drawing had not developed after its drawing was introduced in the East?

Keywords: Key words: math, art, one point perspective, perspective drawing, Western perspective, multi point perspective, parallelism perspective, projective geometry,

0 Introduction

We know that Ancient Greek math with logical geometry of deductive system had developed proofs, on the other hand the East Chinese math with calculate skill was developed by administrative management. Because Euclidean geometry's proofs systems was developed to persuade every mathematicians, and the East Chinese math proofs systems was not serviced in math. It was studied only measurement for national land and king's reign. In this study, we are going to pay attention to perspective drawing of the West and East. Also we consider issue: Why the Western perspective had not developed in the East?

1 Difference way of thinking in the West and East

Why one point perspective drawing was developed in the West? The Western people want to free away from the monotonic Medieval times paintings, and to acquire more realistic and 3-dimensional paintings. We can find the change from multi vanishing points to one vanishing point in the pictures of painter such as Duccio, Dieric Bouts and Leonardo da Vinci. The perspective is exact mathematical ratios in only one 3-dim space. Finally, Renaissance perspective artists had developed projective geometry. What is the influences of perspective in the West? Western people have had some self-concentrated thinking, so point of view is just my eyes. In Chinese culture, people have a philosophy such as theory of Bulgazi: It means that mankind couldn't understand the reason and principle of the world. Their point of view is long distance far away from me. In the East, Chinese artists had used parallelism and multi vanishing points in their drawings. Figure 1. is a wall painting of tunnel in Donhuang where the Chinese west cities. There are many vanishing points that located in the central line. Figure 2. is a parallelism drawing of Chinese in 17thcentury. So I think that one point perspective is a influence of European individual personality: Subjective self-consciousness is strong. On the other hand, many vanishing points and parallelism drawing is a unique Eastern way of thinking: Human could not understand the nature and universal principle, and human is only a part of nature.

2 Western perspective was introduced to China

In 1601, a missionary Matteo Ricci of Society of Jesus, arrived at China with 3 holy drawings and several books. Then the Western perspective was introduced Chinese court painters. An Italian painter and missionary, Castilione had taught to Chinese court painters about perspective and shading in painting. Hence they added perspective to Chinese traditional painting style. Actually, in 11th century, there was a unique Chinese perspective: artist Kuo Hsi invented Samwon-Bub i.e. High Perspective, Deep Perspective, Wide Perspective. This Chinese perspective drawing was introduced to Korean artists(Figure 3, 4, 5). In 1698, a priest Giovanni Gheradini had painted some wall pictures and ceiling painting of Beijing Sanctuary. Moreover, in 1735 Chinese artist Yeon(年希堯) translated a book『Method of Perspective』which had written by Baroque artist Andrea Pozzo. The book was published in the name of『Sihak』: It means theory of seeing.

3 Why the Western perspective was perished from China?

Figure 6 is a Chinese famous painting of some spring evening party in the garden of peach flower. The Western perspective was a little bit accepted. But not perfect! Even though some missionaries of society Jesus had introduced and taught the one point perspective into Chinese people. But the Western perspective had not set up and then perished. Firstly, those missionaries were banished by Portugal. Secondly, I think because of absence of geometric proofs. On the other hand, in the West, the perspective had become a great paradigm of art history until 19th century.

4 The Western perspective landed into Korea

In 1636, Korean prince Soheon received the Western drawing of God from Adam Schall at Beijing. In 1720, a noble man Lee Keeji wrote an essay for Beijing:『Essay of Western Painting』. In 1765, mathematician and scholar Hong Dae-yong visited Beijing and then discussed with German Hallerstein and Gogeisl. At that time, He described the characteristics of the Western perspective such as sense of distance, realistic and stereoscopic painting. In Choson dynasty, Korean scholar and noble man Hong Dae-yong wrote math book: there were ratio rule, reduction of fraction, measure of area and cubic volume. But the book had not proofs such as Euclidean geometry. A Choson noble man Lee Ik had come back to home country with『Element of Geometry』of Matteo Ricci.

In Choson painting, the one point perspective was accepted aggressively. Figure 7 was painted artist Kang Hee-on. However, the one point perspective was disappeared in Korean drawing.

5 Conclusion

In the 17th century, influenced by a Catholic missionary, Korean and Chinese painters accepted western perspective. But, its drawing had stayed in a short period of time in the Eastern. In the West culture, sprit of proofs brought about perspective notion and then it changed paradigm of art. Finally it had dominated painting style during several hundreds, moreover it had influenced into Projective Geometry. On the other hand, in the East(China, Korea), even though the Western perspective was accepted and applied in their drawing, it was disappeared because of lack of sprit of geometry.

REFERENCES

- Morris Kline, 1972, *Mathematical Thought from Ancient to modern Times*, Oxford University Press: New York.
- Sung M. Lee, 2000, *Western Drawing of the Choseon Dynasty Period Paintings*, Daewonsa: Seoul
- Y. W. Kim & Y. K. Kim, 1996, *The History of Chinese Mathematics*, Mineumsa: Seoul.
- Y. B. Song, 2009, "On the meaning of European scholarship introduced by Matteo Ricci", Korea Realism Study **17**, 7-41.
- W. M. Irvins, Jr., 1964, *Art and Geometry: A Study in Space Intuitions*, Dover Pub.: New York.
- S. W. Lee, S. K. Kim & H. K. Ahan, 2009, *The History of Eastern Art*, Doori Media: Seoul.
- 撗地 淸, 1995, 數學文化의 遍歷, 林北出版株式會社 : 東京.

Figure 1

Figure 2

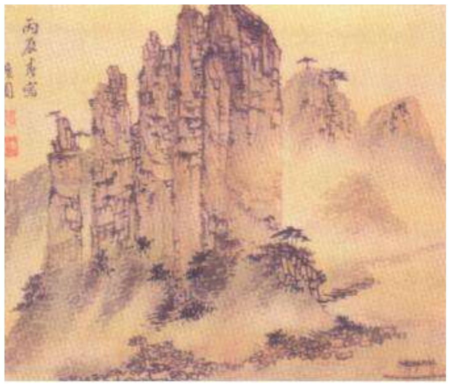

Figure 3

Figure 4

Figure 5

Figure 6

Figure 7

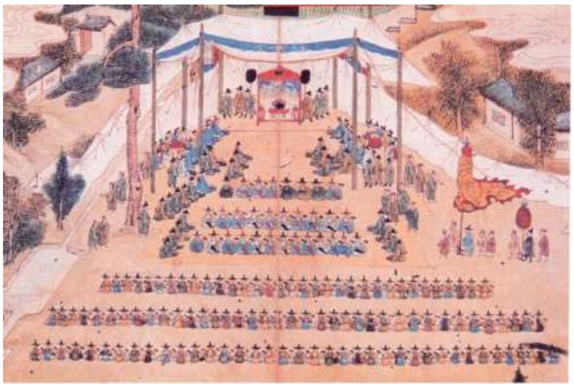

Figure 8

CALL TO REVISIT MESOAMERICAN CALENDARS
The One That is Called the Real Calendar

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ABSTRACT

If necessary, we shall review the numerations used by the Mayas and the Aztecs. We will propose a survey of Mesoamericans' arithmetic and time calculations. And, we will present and discuss a main thesis.

The thesis in focus has two major aspects: historical and epistemological. The presentation will take up and criticize the traditional belief that affirms that Mesoamerican peoples shared the same type of calendar, the main characteristics of which are the two following: a) this calendar would have been obtained by the combination of the almanac and of the solar vague year, two cycles respectively of 13×20 days (dated by expressions of the form $\boldsymbol{\alpha X}$) and of $18 \times 20 + 5$ days (which the Mayas dated by 365 expressions of the form $\boldsymbol{\alpha Y}$, and b) this combination should have produced, in the Mayan case: a Calendar Round not of 260 x 365 = 94 900 days but rather of only 18 980 days dated $(\boldsymbol{\alpha X}, \boldsymbol{\beta Y})$, or, in the Aztec case: an Aztec century of 52 years distinguished by expressions of the form $\boldsymbol{\alpha X_P}$. From an epistemological point of view, we will survey the why and how of some historical misunderstandings of the most original creations made by the ancient Amerindians, and the fact that certain colonial documents asserted that the Indians had a 'real' calendar, that is to say a calendar in sync with the annual course of the sun.

In contrast to the thesis b), my second objective is not really a proposition or a conjecture to be demonstrated, but a presentation of some Mayan and Aztec creations in arithmetic and time computations. These creations that could advantageously enter the curriculum of the classes of mathematics to widen educational horizons by teaching students to include, accept and understand the real problem of translation among foreign thoughts and cultures. Certainly, some ancient Mesoamerican creations may be taken up directly by mathematics classes, on the condition that teachers will first be trained in this very uniquely-evolved cultural domain. This, indeed, is a necessary condition for anyone wishing to avoid both the impulsive projections or interpretations which led Caramuel (1670) to produce a 'monstrous hybrid' as called by Hernández Nieto (1978) and which led Waldeck (1838) to see elephants on the text of the central panel of the Temple of the Inscriptions of Palenque.

1 Invitation to look mathematics made outside of the Occidental realm

My presentation invites researchers, teachers and didacticians of mathematics to step outside the disciplinary monologues and to open the windows of the classroom and of students' minds to diversity in the cognitive world.

1.1 Open your mind

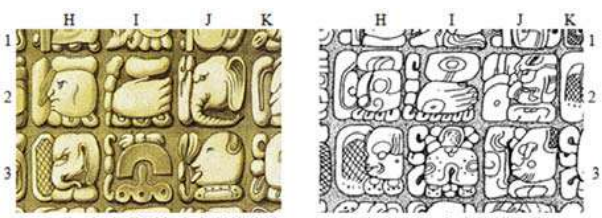
For being mathematically educated is also to be able to escape the routines and to have a free thought, it is also to know how to reject arguments based on authority or faith. This type of education is obtained by exposing one' s self to the problems of others, interactively and in proportion to the openness of each to the questions that call out to men of all times, places, cultures and languages. To more clearly focus on the essential, my presentation is based on the comparative analysis of calendars used in the Ancient Mesoamerica by Mayan and Aztec peoples. This presentation will examine the regards cast upon the mathematical practices that were made and unmade outside of the Occidental realm. It will be shown, in particular, how the Europeans caused, before having even understood it, the loss of the Mesoamerican expertise concerning calendars and computation. It also gives credence to those, who, in the Colonial period, claimed to have and to use a calendar that they qualified as "real", a calendar that did not lie and did not need to be reset. In doing so, we will discover that a wall of incomprehension looms before anyone who engages in a genuinely original or profound thought, even when it is as simple as a child' s expression, a student' s question or a first step into Mayan arithmetic knowledge.

Inviting the creations of Mesoamerican scribes into math class may contribute to the development of the capacity to translate and to break down the walls of incomprehension, and it is also an homage to the memory of the Numbers, the Numerations, the Calendars··· forgotten in the shadows cast by the expansion of the European Enlightenment.

1.2 Shock between Ancient and New World

When a people conquers another, exploits them and imposes its language and its currency, the cultural values of the indigenous peoples react brutally to the slightest decisions of the actors engaged in these historic circumstances. At the beginning of the colonization process, the conquerors deny any valuation of the immaterial productions of the peoples whom they are in the process of vanquishing, overcoming and exploiting[1]. Later, when the atrocities committed begin to be known, voices are raised denouncing all kind of ethnocide. Often in vain. On the one hand because the mass of the colonizers will not heed these voices, and, on the other hand, because the weight of time makes it difficult to see and to admit the existence of ways of thinking which are so radically different.

For this reason, even the most understanding cannot avoid acting in their own self-interest and finish, like the explorer Waldeck, by hallucinating elephants while copying the glyphs from the central panel of the Temple of Inscriptions of Palenque. Here are two fragments of the panel showing the glyphs J2 and H3 respectively drawn by Waldeck and by Schele (an expert in Maya epigraphy):



Jean-Frédéric Waldeck (1766/1875) **Linda Schele** (1942/1998)

Beyond provoking a smile, Waldeck' s approach bears a lesson: sure of the architectural prowess of their pyramids, deserted for centuries, but ignoring virtually everything of the prestigious past of the Mayas who had built them, Waldeck sought to explain their presence and their beauty. A credi-

[1] If pressed, they sometimes finish by explaining their exactions by providing justifications which are as false as they are sectarian and partisan: since we are defeating them, their gods have clearly abandoned them, and even if their gods have fled, it is because their works are the fruit of the devil or of their savage nature. In either case, they must thus be destroyed and eradicated, and their authors must be punished or educated; in short, indoctrinated or reconditioned.

ble civilizing force was thought to be found in the great African civilization of Egypt. In doing such, Waldeck placed himself within the small circle of a handful of scholars and explorers who fought against the *doxa* of the period which affirmed that Mesoamerican cultures were inferior to those of the ancient world. And the star civilization of the period was Egypt, following its revelation to the world through the embarkation of 68 scholars in Bonaparte's 1799 campaign. In spite of the open mindedness of Waldeck and of a few other scholars, the Mayas (and the Aztecs) were viewed as savages or, at best, as half-civilized. The title, *Notes on the* ***semi-civilized*** *nations of Mexico, Yucatan, and Central America* by Albert Gallatin (1845) perfectly illustrates the ambiguous, touching and arrogant nature of these scholars, both sufficiently open to study the facts of indigenous cultures and insufficiently open enough to grant them the title of "civilized" and to cease describing them as children, primitives or irrational beings. During the XIXth century, Mesoamerican values were relegated to the secondary market of semi-values. Well beneath those of the Greek or Roman classics, and even inferior to more prized exotic values such as those of the Egyptians or the Chinese.

The same type of incomprehension of the other has been described by Hernández Nieto (1978) under the appellation of "monstrous hybrids". This concept is clearly exemplified by the study of the works of a Spanish Cistercian of Czech origin, Juan Caramuel de Lobkozitz (1606/1682). Concerned with "penetrating with an open spirit the arithmetic in use in ancient Mexico", Caramuel refuted the erroneous affirmation of Brother Alonso de Molina stating that Aztec numeration did not exceed 8 000[2]. On the other hand, he also rejected the exact and recognized theory of the vigesimal nature of Nahuatl numeration. Regarding this, Caramuel presents his own interpretation of the numeral expressions that he has "manipulated" to the point of perverting the Nahuatl numeration into a purely quinary system[3](base 5 numeration). In other words, Caramuel's vision, which is both curious and benevolent, produced a hybrid – the purely quinary "Caramuelian" system mixes Aztec numerals and those invented by Caramuel – the parthenogenetic fruit of his solitary manipulation of Nahuatl numeric expressions[4]. Caramuel's hybrid is monstrous, first because he fuses groups of numeric expressions which were conceived and which develop in worlds having virtually no points in common: a numeration that is both Aztec and quinary simply does not exist. Second, because Caramuel's chimeras are still-born which never have had any existence outside the study by H. Nieto (1978) of the Caramuel's manuscript. Otherwise stated, Caramuel allowed himself to be carried away by an "impulsive interpretation" (Luria;1966).

Incomprehension of the thoughts of others obstructs, via the prejudices it provokes, the work of scientists, whether that of the epigraphist or the historian. Diego de Landa, one of the very first framework familiar to him: that of Latin alphabetic writing. Incapable

[2]Caramuel wrote "but I show that the Aztecs arrived at the number of 31 250, or even further" (p. 93). Limited to 160 000 for the mathematician Geneviève Guitel (1975).

[3]"Según el sistema defendido por nuestro autor las operaciones serían todas dentro de un orden quinario, en el cual cinco rayas darían una unidad del orden superior, y cada posición equivaldría a 5^2" (p. 91).

[4]Which constitute – in the opinion of all witlessness' who have testified on the witness stand of History since the beginning of the Conquest – a vigesimal system of numeration of a "well-organized variety" in the terminology of Guitel (1975).

[5]For the (poor) reason of being able to better combat "idolatry" in the Mayan texts.

'monk-ethnologists', provides a perfect illustration. In trying to understand[5]Mayan writing, he was unsuccessful in escaping the only of imagining another type of writing, he attempted to force Mayan writing into the alphabetical mold. For centuries, his "alphabet" blocked epigraphists who finally managed to see that Mayan writing is of the logo-syllabographic type.

Whether called a "monstrous hybrid", "Waldeck's elephant", or even the "self indulgence of the ethno-X", the concept reflects not the ineluctable fact of projecting its own frames of reference and its own forms of knowledge upon the foreign work that it is trying to understand, but rather the failure to submit all readings and interpretations to systematic and collective criticism, criticism that is at the least interdisciplinary and interethnic or intercultural.

Whatever the motivations, in face of the Occidental productions, both material and immaterial,

pushed by the Spanish colonists, occasionally the curiosity and desire of the Autochthones matched that of the Europeans regarding their diffusion and adoption. And vice-versa. With the exception that the inequality which is inherent to the condition of servitude has strongly limited the transfers in the Autochthon-European direction to the simplest forms. For example, the inescapable fact that only the Autochthons are forced to learn the language of the latter. Thus, the Amerindian languages are in danger everywhere, while in every State of the continent the official European languages have been enriched over five centuries by a more or less important number of lexicological and grammatical contributions which, of course, distance them somewhat from the languages of their actual neighbors and ancient metropolises. Today for example, the Spanish spoken in Merida (Yucatan) is not exactly that spoken in Merida (Venezuela). Nor that you hear in Spain or on the *Zocalo* of Mexico.

The elements of history of the numerations presented in this talk show that the "soft" form of colonization is often proved to be more "effective"[6]in the long term than the brutal form of those who seek the pure and simple eradication of the Autochthon's creations and the death sentence for their native creators. As I stated in beginning this introduction, the result of a half-millennium of European colonization is very clear as far as numerations are concerned. The current inhabitants of Mesoamerica all use[7]in their day-to-day lives: both the metric system imposed by the Revolutionaries of 1789 as well as the numeration of decimal position and Indo-Arabic numerals that they exhibit, for example, on the license plates of the vehicles driven today by certain descendants of the Mayas, the Aztecs, and others. Otherwise stated, the meeting between the two worlds unleashed a chain reaction leading to a numerical deculturation – if we may so say – that is nearly complete. A catastrophic evolution (in the modern sense of determinist chaos theories) which leads to the disappearance of the vigesimal numerations of two of the great American cultures: the numeration of position of the

[6]Thus, from a certain point of view, the "soft" is more menacing and dangerous for the indigenous productions. History should not forget that the Republican institution of "public, secular, free and obligatory" schooling, in France at the beginning of the XXth century, had as a collateral effect the disappearance from the public sphere, in only three generations, of the majority of regional languages, of which the most vigorous, like Basque or Breton, owe their survival to the force of will of militants who succeeded in integrating them into the school system and a select few other public spaces such as radio or TV waves.

[7]Even the menus at Chinese restaurants note their prices in Indo-Arabic numerals!

Mayas and the additive numeration of the Aztecs[8].

Like the French (and others!) who still know and use upon occasion the numeration in roman numerals, the Mayas or the Aztecs may still write numbers in the vigesimal numeration of days gone by, provided that they master the system and know its particularities[9]. But these numbers in ancient writings do not leave the context of private or semi-public use, and are prohibited on identity cards, passports, checks, or again in scientific or technological articles. These pseudo-vestiges thus produced at present (for instance in the bicultural schools) do not allow a return without risk towards the past, and suffice only quite imperfectly in the quest for the lost numbers of the Mayas.

For spoken numerations, the current situation is a bit more nuanced, for it also depends on the linguistic resistance of the Mesoamericans, varying according to the peoples, the circumstances and the period. Thus, we observe a continuum of multilingual situations which simultaneously confronts those who only (or only still) speak Spanish with those who speak only one or more Indian languages. In any case, an indigenous speaker may know all or part of the spoken numeration of the Indian languages which have evolved[10]and which are themselves more or less open to borrowing and to mixing. The most frequent case at present is to hear mixes in which small numbers are most often spoken in the Indian language and the greater (especially when they relate to the globalized world) in Spanish. One reads, for example, in a text relating a marriage ceremony "tehuan quinequi ***dos ochenta*** uan ***chicuase*** totolme" or "we want **two hundred and eighty** pesos (in Spanish) and six **turkeys** (in Nahuatl)" (Dehouve;1978:190).

1.3 A brief survey

As they have disappeared, the modern readers can no longer debate face-to-face with the scribes of the classical period. In order to use the corpus of equations that utilize the lost numbers, the modern reader may, however, count on the collective capacity of the scholars who discover one by one the documents left by History, who decipher them and who translate them. It is through them that we can hope to enter into the cognitive universe – *a priori* strange and objectively foreign –: **the Mayan Arithmetic Intelligence**.

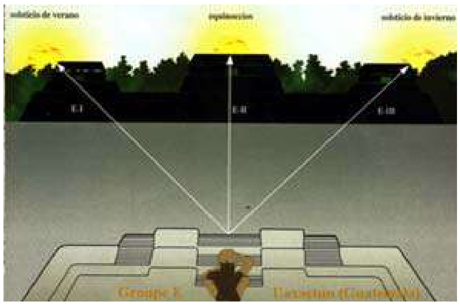
[8]It must be noted that the initial conditions of these two dramas were quite different. On one side, an Aztec Confederation with a centralized political system, hierarchical and strongly bonded by a very ritualized religion. On the other, dispersed Mayan populations among which one of the principal points in common was the continued divinatory use of the *tzolkin*, the "week or year" whose 260 dates allowed thousands of numerological inferences lending themselves to the acts of divination.

[9]For example, the best known irregularity is that of an 18-month year.

[10]The proactive formations characteristic of colonial Yucatec, are no longer used (and sometimes no longer even understood) by the speakers of the tens of Mayan languages still practiced.

Our advantage in this race for lost numbers is that our champion, the Mayan scribe of the Classical period, is without contest the only Mesoamerican prior to the meeting of the two worlds, who has left a treasure of mathematical vestiges. Sometimes this remained treasure is petrified in the ruins of astronomical observatories or in the alignments of monuments like the Temples of the group E of Uaxactun (Petén, Guatemala).

Sometimes in the mathematical tools painted in the codex in order to serve in the resolution of the problems of the mastering of time. The deciphering of the Dresden Codex confirms this observation and reveals certain instruments developed by the scribes: several systems of written numeration, several calendars, a forest of cycles, a system of units of measurement for time, multiplication tables for all of the cycles, tables of dates that are invariable by this or that multiple of these cycles.



In order to succeed in their activities, inextricably divinatory and astronomical, the Mayas needed writing (vase Kerr 1185), spaces for discussion and for calculation (vase Kerr 1196), and diverse arithmetical-calendarial tools such as tables of multiples and tables of dates (Dresden Code p. 24). They also needed astronomical observations. Before beginning, here are the principal tools of the Mayan astronomer/astrologist:

1) a spoken numeration of protractive-type
2) a logo-syllabographic writing system (Hoppan;2012)
3) a written positional and vigesimal numeration with zero that distinguishes between the cardinal and ordinal faces of the integers, and thus opposes the times and dates
4) a vigesimal system of units of time measurement (called the periods, $\mathbf{P_i}$): the **tun** “year” (360-d) plus the open list of its multiples like **katun** and **baktun**, and two submultiples (**uinal** “month of 20 days” and **kin** “day”)
5) astronomical observatories
6) a graphic system which produced tables of multiples and charts of dates
7) **four intertwined calendars**, namely:
    a. the Long Count, **CL**, theoretically open and isomorphic to the set of natural numbers, which dates the day by their distance $\sum(\mathbf{c_i P_i})$ at the origin of Maya chronology (the 12/08/-3113 in concordance GMT with the constant 584 284)
    b. the *tzolkin* “divination week” of 13 × 20 days, which provides 260 dates $\boldsymbol{\alpha X}$
    c. the *ha'ab* “solar year” of 18 months of 20 days and 1 rest of 5 days, which provides 365 dates $\boldsymbol{\beta Y}$
    d. the Calendar Round, **CR**, which results from the combination of *tzolkin* and *ha'ab*, and which provides 18 980 dates of the form $(\boldsymbol{\alpha X}, \boldsymbol{\beta Y})$
8) ephemeris to track the phases of Moon and Venus, the return of eclipses⋯

1.4 Main thesis: Mayas and Aztecs had different manners of writing dates

Unlike the Mayas, the Aztecs did not have four intertwined calendars. They were using **two calendars**[11]: the first dates the days, and the second serves to identify the years. The first, called tonalpohualli in nahuatl, is the twin of Mayan tzolkin of 260 dates $\boldsymbol{\alpha X}$. The second is a cycle of 52 dates $\boldsymbol{\alpha \underline{X}_P}$ that distinguish and define 52 years (forming what it is called the Aztec century, SA) through the intermediary of an agreement. One particular day, for instance the first day of the first month of the year, is distinguished and its tonalpohualli date $\boldsymbol{\alpha \underline{X}_P}$ is used as the "proper name" of the year. In other words, that day is, by definition, the day **eponymous of the year**; it is also said the **Year Bearer**.

Only a detail separates Mayanists and Nahualists. For the former, the Maya calendar contains exactly 18 980[12] **kin** 'days'. For the latter, the Aztec calendar consists of 52 *xihuitl* years whose total in days – determined by the type of the year – is supposed to be the same: 52×365. A supposition to be confirmed or invalidated. Showing that the Maya CR of 18 980 days and the Aztec *xiuhtlalpilli* of 52 years are two different types[13]of calendars, this presentation invites the reader to reconsider the traditional thesis according to which the sharing of a mutual calendar is a defining trait in the concept of Mesoamerica (Kirchhoff, 1943).

All Mesoamerican calendars seem to be defined by a common structure of a product determined by cycles whose root is the combination of an almanac of divination of 13×20 days and a vague year of $(18 \times 20) + 5$ days[14]. The calendar expressions in fact combined in the Mesoamerican *hic et nunc* are, however, visibly different between the two peoples. Each city followed its own rules for dating events and distinguishing days.

Common to all Mesoamericans, the 'almanac' dates are confirmed[15]from the middle of the VII$^{\text{th}}$ century B.C. until the colonial period. These are expressions of the form $\boldsymbol{\alpha X}$, where $\boldsymbol{\alpha}$ follows a cycle of 13 successive wholes, and $\boldsymbol{X}$ a cycle immutably ordered of 20 day-glyphs. In other words, a cycle which is the product of 13 x 20 = 260 dates, established with the order: $\boldsymbol{s(\alpha X) = [s_1(\alpha), s_2(X)]}$ where the 's' are the 'successor' functions of the cycles being considered.

[11]Like all peoples, the Aztecs have been subjected to the seasons, to the variations of the received solar energy. They divided the solar year into 18 months but they did not do an annual 365-calendar and they wrote no date of the form $\boldsymbol{\beta Y}$ (Cauty:2012).

[12]GCM (260, 365). Is one fifth of the product $tzolkinxha'ab = 94900$. It is also 949 **uinal** 'months', 73 *tzolkin* or 52 *ha' ab*. Its double, 2 CR, equals 65 cycles of Venus.

[13]In spite of a common origin and important similarities such as those that can be made between the Mayan 52-cycle of the Bearers and the Aztecan 52-cycle of the Eponyms.

[14]That is to say: Mayan tzolkin or Aztec tonalpohualli of 260 days and Mayan ha' ab or Aztecan xihuitl of 365 days.

[15]The oldest are anthroponyms (Urcid, Pohl and others).

Nearly absent with the Aztecs, the vague year dates are later and are only clearly attested to with the Mayas[16]. These are of the form βY, wherein β follows the cycle $(0, 19)$ and Y the ordered cycle of 18 named[17]Months, *veintena*, of 20 days, and of the named compliment **Uayeb**. Thus a set of 365 days/dates for the vague year solar *ha' ab*, equipped with the order s' :

$$s'(\beta Y) = [s_3(\beta), Y] \text{ for each } Y \neq Uayeb \text{ and } \beta < 19$$
$$s'(19Y) = [s_3(19), s_4(Y)] = [0, s_4(Y)] \text{ for each } Y \neq Uayeb \text{ and } \beta = 19$$
$$s'(\beta Uayeb) = [s_3(\beta), Uayeb] \text{ for each } \beta < 4$$
$$s'(4Uayeb) = [s_3(4), s_4(Uayeb)] = 0Pop$$

2 The product $tzolkin \times ha'ab$ of the Maya and the cycle of Bearers

The CR[18]Maya is the ordered product $tzolkin \times ha'ab$ whose elements are couples $(\alpha X, \beta Y)$. The study of the mathematical properties of the ordered products of ordered cycles has shown (Cauty;2009:20-30) that the conjunction of three factors[19], along with the fact that 260 and 365 are multiples of the same number[20], has two consequences. First, to limit the number of CR dates to 18 980. And to produce, in addition, cyclic events[21]that are resumed by the **theorem**:

Whatever the integer P, the almanac date of the P^{th} day of the vague year is of the form αX_P, where α follows the cycle of 13 integer almanac dates, and where X_P follows a class, modulo 5, of four X day names.

Each day of the vague year is thus associated with $13 \times 4 = 52$ almanac dates who characterize it and succeed one another year after year according to the law: $s(\alpha X_P) = s(\alpha)s(X_P) = [(\alpha + 1), (X + 5)]$. The value $P = 0$ distinguishes and defines the 1st day of the 1st month of the Mayan year, the New Year. Applied to this day, the theorem states, first, that the Mayan New Year is associated with four[22]tzolkin X_P day names. And, additionally, that each New Year date αX_P distinguishes and defines a ha' ab year in the group of 52 years that make up the CR. In other terms, the system of dates αX_P supplied a practical means[23]to distinguish and name the years of a CR: by making αX_P the eponym for the year.

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[16]The ***0 Yaxkin*** of the Leyden plaque (17/09/320 greg.) is among the first evidence of βY dates. The latter, from the Post Classic, are in the codex (Dresden and Paris).

[17]In colonial Yucatec, for example: Y covers the following (***Pop, Uo, Zip, Zodz, Tzec, Xul, Yaxkin, Mol, Ch' en, Yax, Zac, Ceh, Mac, Kankin, Muan, Pax, Kayab, Cumku***) that closes the compliment ***Uayeb***. To know the date βY of a day is equivalent to knowing its position γ in the *ha' ab*: the cycle of the ha' ab dates ordered by s' is isomorphic to a group of 365 natural integers fitted with the natural order of integers.

[18]Its Mayan name is unknown. But its value of **2.12.; 13.0.** is established. Mayanists use the expressions: *Calendar Round, Wheel Calendar, Ritual Calendar.*

[19]F1: the rules for formulating the expression composed of dates (*tzolkin, ha' ab* and CR), F2: the type of enumeration of the pairs $\alpha X \beta Y$, and F3 : the relative position (determinable by the origin date **4 Ahau 8 Cumku**) of tzolkin and ha' ab at the moment of starting the CR.

[20]Their GCD 5, greatest common divisor.

[21]Occasion to celebrate, each year among the Maya, the change of year Bearer; and every 52 years among the Aztecs, the Binding of the years (*xiuhtlalpilli*) and the New Fire.

[22]The entities associated with these 4 names are called the 4 Year Bearers: **Ik, Manik**, Eb and **Caban** in classic Mayan.

[23]Probably not used in the classic period because the dating methods were particularly redundant. Especially in the solemn public use of steles and monuments recounting the glory of Mayan cities and leaders in the enameled texts of dates given in the CR system, but also in the Long Count $\sum c_i \underline{P}_i$, in Lunar Series, and other cycles as well. For example: CL **13-baktun 0-katun 0-tun 0-uinal 0-kin** and the date CR **4 Ahau 8** ***Cumku*** from the Stele E of Quirigua (Guatemala, 771 A.D.).

3 *Tonalpohualli, xihuitl* and Aztec eponyms

Like all Mesoamericans, the people submitting to the Aztec Triple Alliance used the αX dates of the almanac. Like everyone else, the Aztecs underwent and observed diurnal and seasonal variations of the solar radiation and they used, as all Mesoamerican peoples, a year (not a calendar) we refer to as 'festive'. The festive year has 18 months of 20 days each and 1 residue of n days. Theoretically: each period has a proper name, the residue ***Nemontemi***[24]has 5 days, and all 19 periods are immutably ordered. So, we can refer to the festive year by the ordered list ***I***, ***II***, ***III***, etc., ***XIX*** of its periods.

But up to here, the pre-Colombian documents delivered **no Aztec date of the shape** βY distinguishing (as our January 1st or as the Mayan ***0 Pop***) the days of the solar year by their rank in their month Y. The fact is that: prior to the arrival of the Spanish, the Aztecs have not left written βY dates with the help of the glyphs of the indigenous pictographic writing. From the Spanish Conquest, Mesoamericans were all forced to follow the European calendar, a solar calendar divided in 12 months. At the same times, we observed a very few documents which transcribe into the Latin alphabet the detailed forms for a handful of event[25]dates that were critical for both Worlds.

In these conditions, it is possible to reconstruct an Aztec βY date[26]; and it is possible to reconstruct some complete expressions like "the day **8 Ehecatl** ***9 Quecholli*** of the year **1 Acatl**" for Cortez' entry into Mexico City. Usually not recorded, the rank within the month remains uncertain when we have it available. The reference ***9 Quecholli*** for example do not allow confirmation that the day corresponding to this date was, indeed, at the position ***9***, ***1***, ***10*** or ***20*** of the month Y. Why is this ?

Because the sources do not say how the Mesoamericans enumerated and counted their days. We don't know for instance if they all began with the same number. However, we are certain that the ways of counting were diverse: the Mayas wrote the numbers from ***0*** to ***19***, the Spanish numbering the days from ***1*** to ***31*** and the Tlapanèques from ***2*** to ***14***.

From which are derived two deductions:

1) The CR of 18 980 days distinguished and dated by as many expressions $\alpha X \beta Y$ is not a tangible reality in the space/time of the Triple Alliance.
2) We can however, from a colonial date like ***9 Quecholli*** for example, undertake the reconstruction of the 365 dates βY of a xihuitl[27].

[24]Except for those who attribute to Mesoamericans the use of a 366-day leap year, sources state that the ***Nemontemi*** contained 5 unlucky days, unnamed, sleeping··· In *Historia general de las cosas de Nueva España*, Sahagún gives a list of 18 expressions traditionally accepted as that of the 18 month names although they are "very different from the point of view of their syntactical structure: 'there are gifts of flowers', 'the trees sit up straight', 'little watch', 'skinning people', etc." (Launey;2009:personal communication). Sahagún also gives the associated divinities, the name of the period *Nemontemi*, as well as the position of the period in the Julian calendar. The second *veintena*, ***Tlacaxipehualiztli***, went from March 4 to 23; dedicated to Xipe Totec, it was characterized by the skinning of people. The list is attested to by multiple sources, *modulo* differences: the number of days and the position in the year of the period *Nemontemi*, the month that begins the year and subsequently the numeration of the months. The month ***Quecholli***, for example, is generally the 14th month, but it is the 13th for Rámirez. In two texts (manuscript 215 and *Historia antigua de Mexico*) Ixtlilxochitl begins/ends the year in: ***Atlcahuallo/Nemontemi*** and ***Atemoztli/Panquetzalztli*** (Roulet; 1999).

[25]The most credible eyewitness reports concern Cortez' entrance into Mexico City (08/11/1519), the Night of Sorrows when he is driven out, and the destruction of the Temple of Mexico (Tena:1987;ch. IV).

[26]To which we may add more vague indications stating, for example, that the twenty monthly ceremonies took place at the beginning, the middle or at the end of the *veintena*. Usually not recorded, the rank within the month remains uncertain when we have it available.

[27]In a more or less credible manner according to the suppositions retained, beginning with that of the number of days attributed to the *xihuitl* (365 or 366?). Or to reconstruct the βY dates of the 52 years of a *xiuhtlalpilli*, but this with yet more uncertainty.

Besides the formula of xihuitl and its invisible dates[28], the Aztecs inherited, on one hand, knowledge of the duration (52, in number of years) of the CR and, on the other hand, of the effects of the conventions that structured it and which began the Bearer cycle. This heritage takes into account the representations of successions of xihuitl and most importantly the habit of distinguishing and noting the years by the ordered succession of their eponyms $\boldsymbol{\alpha \underline{X}_P}$[29]like that of the folio 2r of Mendoza going from the year **2 <u>Calli</u>** to the year **13 <u>Acatl</u>** in passing by **2 <u>Acatl</u>** signaled as the year of the celebration of the New fire. Thus, as we had seen (date on the Dedication Stone of the Templo Mayor), the Aztec date for the day of an event is an expression $(\boldsymbol{\alpha X}, \boldsymbol{\alpha \underline{X}_P})$, where $\boldsymbol{\alpha X}$ is the tonalpohualli date of the day of the event and XP that of the eponym day of the year in which it occurs. This mode of dating does not result in 18 980 dates as in the case of the Mayan CR, but only 260 x 52 = 13 520 possible different expressions[30]. It is, by its construction, ambiguous: 260 dates $\boldsymbol{\alpha X}$ are not sufficient to distinguish the 365 days[31] of a xihuitl; and 13 520 pairs $(\boldsymbol{\alpha X}, \boldsymbol{\alpha \underline{X}_P})$ are not sufficient to date the days of an Aztec Century of 18 993 days – in verdadero (Sahagún) and Julian calendars – or of 18 980 days if one decides to identify it[32] with the Mayan CR.

4 Difficulties and differences

Identifying Mayan and Aztec calendars is a habit[33]whose principal fault is to hide a specific difference: only the Classical Mayas commonly wrote $\boldsymbol{\beta Y}$ dates of the annual *ha' ab* calendar. When it is noticed[34], the difference – the inscription of the eponym *vs.* the *ha' ab* date – is sometimes denied by an interpretation that reduces it to a question of abbreviation or preference[35]. But no: small cause, big consequence. This "preference" would give18 980 dates $(\boldsymbol{\alpha X}, \boldsymbol{\beta Y})$ to the Mayas; and 13 520 dates $(\boldsymbol{\alpha X}, \boldsymbol{\alpha \underline{X}_P})$ to the Nahuas. This is not a stylistic nuance. The Aztecan expression "8 ehécatl of 1 ácatl" and the Mayan "4 aháu 8 cumkú" are fairly different; and they are not, as Tena (2000) wrote, two "abbreviations" which denote similar or identical conceptualizations of the same "fact

[28]The bY dates of type 9 Quecholli that we never see written in indigenous glyphs.

[29]$\boldsymbol{X_P}$ are the 4 days, linked to the Bearers among the Mayas. Among the Aztecs, these are the days: **Calli, Tochtli, Acatl** and **Tecpatl** whose Mayan equivalents are **Akbal, Lamat, Ben** and **Edznab**, witnessed in the Dresden Codex, but which are neither **Ik, Manik, Eb** and **Caban** presented in Classical Mayan nor **Kan, Muluc, Hix** and **Cauac** recorded in the Colonial Period by Landa and the Madrid Codex.

[30]Not a proper part of the product, though 52 and 260 are divisible by 4 (see the CR), because the law of succession ('linear enumeration', type: 1 January, 2 January, etc.) defined in *tonalpohualli x xiuhpohualli* – $\boldsymbol{s(aX, aXP) = [s(aX), aXP]}$ as long as the xihuitl is not passed, otherwise $\boldsymbol{s(aX, aXP) = [s(aX), aXP + 1]}$ – is not the same as that defined in $tzolkin \times ha'ab$ ('diagonal enumeration', type: Monday 1, Tuesday 2, etc.).

[31]It is always possible to render the calendar unambiguous by taking an additional cycle, for example, that of the 9 Lords of the Night.

[32]Decision leading to the creation of 52×105 double dates $\boldsymbol{\alpha X}$ to add, in an order to be specified, to the 260 already used in each *xihuitl* year of the Aztec century.

[33]"El calendario mesoamericano era el resultado de la combinación entre un ciclo de 365 días, llamado en náhuatl *xiuhpohualli* o "cuenta del año" (*ha' ab* en maya), y otro ciclo de 260 días, llamado en náhuatl *tonalpohualli* o "cuenta de los días" (*tzolkin* in Maya) [···] Se requería el transcurso de 18 980 días nominales, equivalentes a un "siglo" de 52 anos, para que se agotaran todas las posiciones posibles de un día cualquiera del *tonalpohualli* dentro del *xiuhpohualli*, y viceversa" (Tena;2000:5).

[34]Which it is not always, because the habit of transcribing everything (*tzolkin* date, *ha' ab* date, period, etc.) in the same manner neutralizes and renders invisible many of the differences noted by the scribes.

[35]"Tanto los nahuas como los mayas utilizaban una fórmula **abreviada** para los fechamientos, pues ordinariamente no se mencionaban en forma completa todos los elementos que intervenían en una fecha, a saber: el día del *tonalpohualli*, el ordinal del día dentro de la veintena, y el año. Los nahuas **preferían** enunciar sólo el día del *tonalpohualli* y el año; decían, por ejemplo, *8 ehécatl* de *1 ácatl*. Los mayas, en cambio, sólo enunciaban el día del *tzolkín* y el ordinal de la veintena; decían por ejemplo: *4 aháu 8 cumkú*." (Tena; 2000: 5. AC underlined in bold).

of calendar". These are two types of dates to be analyzed //**8 Ehecatl**/**1 Acatl**// and //**4 Ahau**/**8 *Cumku***//, of very different components. The 1rst does not introduce any difference: for both an Aztec and a Maya, it is a $\boldsymbol{\alpha X}$ almanac date. But the 2nd component is, both in nature and form, quite different. In Nahua: another date $\boldsymbol{\alpha X_P}$, but in Mayan: a date $\boldsymbol{\beta Y}$. There is no isomorphism[36]. But rather a Type B elder who led the Classical Mayas to write dates $(\boldsymbol{\alpha X}, \boldsymbol{\beta Y})$ and to imply the eponym $[\boldsymbol{\alpha X_P}]$; and a Type A who, during the Postclassical, led people to write dates $(\boldsymbol{\alpha X}, \boldsymbol{\alpha \underline{X}_P})$ and to imply the date $[\boldsymbol{\beta Y}]$. The formulas are:

Type $B = (\boldsymbol{\alpha X}, \boldsymbol{\beta Y}), [\boldsymbol{\alpha X_P}]$ among the Mayas[37]
Type $A = (\boldsymbol{\alpha X}, \boldsymbol{\beta Y}), [\boldsymbol{\beta Y}]$ among the Aztecs[38].

5 Reflections and conclusions

The non-isomorphism of the Aztec and Mayan calendars possibly has its root in the fact that only the Mayas used the long Count $\sum \boldsymbol{c_i}(\boldsymbol{\underline{P_i}})$ and the CR $(\boldsymbol{\alpha X}, \boldsymbol{\beta Y})$ in a joint and immutable manner. This usage led, in effect, to the rigorous synchronous maintenance of the tzolkin and ha' ab cycles and to never change the number of days of the vague year or of any other unit. The real benefit of this rigor was, of course, the possibility of making precise calendar calculations[39] with the help of the Multiplication Tables[40] and the Date Tables that we find so abundantly in the Codex and which served to accomplish the challenges of modular Mayan arithmetic[41].

Without this rigor and these tools, the scribes would undoubtedly not have been able to simulate the return of eclipses, correct the shortcoming of the Venusians leap year or record the embellished narrative tales of a network of dates and numbers of distance that celebrated the grandeur of gods, cities, kings.... Otherwise stated, it is indeed the rule ROCm (Cauty; 2009b:10-12), consequence of the circumstances described above, and cause of the specificities expressed in § 2 by way of the **theorem** which serves as the basis of the possibility of calculating to the day, and which puts it into practice with the constraints of co-occurrence imposed on couples of the product $tzolkin \times ha'ab$. It will be further shown that the redundancy brought by the use of the CL[42]acted as a code detecting, even correcting, errors.

[36]In consequence of this non-isomorphism: the contrast between the Mayan facility to find the eponym $\boldsymbol{\alpha X_P}$ from the date $\boldsymbol{\alpha X \beta Y}$ of an event, and the difficulty for a Nahualist to find the date $\boldsymbol{\beta Y}$ from the date $(\boldsymbol{\alpha X}, \boldsymbol{\alpha X_P})$. A Mayan date $\boldsymbol{\alpha X \beta Y}$ defines one and only one day of the CR. The position in the ha' ab of the day $\boldsymbol{\beta Y}$ is "$\boldsymbol{\beta \to Y}$" in spoken protractive numeration. Subsequently, the eponym (*tzolkin* date of the day ***0 Pop***) sought is given by $\boldsymbol{\alpha X_P} = \boldsymbol{\alpha X} - (\boldsymbol{\beta \to Y})$. For example: **7 Eb** is an eponym for the year of **4 Ahau 8 Cumku**. Another facilitating factor for the Mayanists is the richness of redundant elements in the calendar expressions.

[37]Whose calendar system imposed the constraint of co-occurrence on the marked components (*tzolkin* date and *ha' ab* date) and implying the redundant eponym.

[38]Whose calendar system imposed writing the eponym $\boldsymbol{\alpha X_P}$ of the year, but no constraint of co-occurrence on the *tonalpohualli* dates (marked) or the *xihuitl* dates (never written).

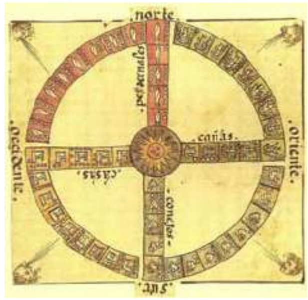
[39]In practice, to 1 day. The fact that the scribes used all sorts of cycles may be interpreted by saying that they calculated in rings Z/nZ or in classes of appropriate integers modulo n. Clocks give us a familiar image of this type of calculation which lowers by n any number that reached or surpassed this value, because the hour hand makes additions modulo $n = 12$: if it starts from 7 o'clock, for example, and ten hours must be added, it will not mark 17:00 hours, but 5 o'clock.

[40]Containing at times, in the position of intruder, non-multiple numbers serving to correct the calendar deviation in vague years, like that of Venus of the Dresden Codex.

[41]Given 2 dates $\boldsymbol{x}$, $\boldsymbol{y}$ (of the Mayan CR to clarity things) and one translation $\boldsymbol{t}$ (in whole number of days), solves the 3 equations $\boldsymbol{t}(\boldsymbol{x}) = \boldsymbol{y}$ according to whether the unknown is $\boldsymbol{x}$, $\boldsymbol{y}$ or $\boldsymbol{t}$.

[42]But also the Numbers of distance, lunar Series, and other cycles, like that of the Lords of the Night (or patrons of the Otherworlds).

For the Aztecs, the product $(\boldsymbol{\alpha X}, \boldsymbol{\alpha \underline{X}_P})$ of the dates *tonalpohualli* $\times$ *xiuhpohualli* is not limited by the factors F_1, F_2 and F_3 because the $\boldsymbol{\beta Y}$ dates were no longer written. Also, in the absence of the Long Count[43], eventual discrepancies or other calculation errors become readily unapparent. The Aztec century, *xiuhtlalpilli*, was thus freed from the functional obligations which the ROCm Rule imposed upon the product *tzolkin* $\times$ *ha'ab*. Due to this, the cycles $\boldsymbol{\alpha X}$ and $\boldsymbol{\alpha X_P}$ were like "free spinning gears" in relation to one another. The Almanac had sorted out its 260 $\boldsymbol{\alpha X}$ dates, but it needed something or someone to increment and count the eponyms $\boldsymbol{\alpha X_P}$ and to maintain the 52-year cycle no longer rigidly tied to the *xihuitl*.



Colonial or not, the sources do not explain how Mesoamericans without $\boldsymbol{\beta Y}$ dates knew when to increment the eponym. In principle an Aztec was not obliged to change *xihuitl* like a Maya would change *ha' ab*, that is to say: on the passage from the 365th and last day of the year $n-1$ dated ***4 Uayeb*** to the 1rst day of the year n dated ***CHUM/0 Pop***[44]. Numerous figures revealing the persistence of the number and of the order of succession of the 52 years of a *xiuhtlalpilli* prove that the tradition of the vague year of 365 days was maintained (Durán Codex).

However, the insistent indication – stating, without proof, that the Natives corrected the vague year, that they possessed a "real" (*verdadeiro*) calendar to which they added a 366th day every four years – proves two things relative to the Postclassic and Colonial calendar customs. Firstly, that they knew the vague year of 365 days. Secondly, that it seemed to remain in phase with a tropical year of 365.25 days or with a vague year of variable length[45].

Of course, the Aztecs could recognize the changing of seasons or of years. A reasonable hypothesis is thus to suppose that the changing of a year and of the eponym $\boldsymbol{\alpha X_P}$ could be decided according to the appearance of a designated natural or astronomic sign: the passage of the Sun at the Zenith of a site[46], a Solstice, a passage at the meridian of *Miec/Tianquitzli* (Pléiades), the Bridge of Turtles...

In *Los observatorios subterráneos*[47], Rubén B. Morante López evaluates the research and the measures taken since the eighties by Aveni and Hartung (1981), Anderson (1981), Broda (1986, 1991) and Tichy (1980, 1992) in the subterranean observatories, notably that of Xochicalco[48]. The authors recorded the days upon which rays of light enter into the chamber of the observatory, and the days upon which they do not. The results show that those who conceived the observatories constructed

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43 The "small" generative capacity of the Aztec written numeration and its repetitive form are elements unfavorable to the writing of large numbers and the composition of tables of multiples, that partially explains this absence of Aztec Long Counts.

44 Which amounts to changing the manner of counting the days, following the end of the installation of the New Year. Between ***0*** and ***1 Pop*** or between ***1*** and ***2 Pop*** or ***2*** and ***3 Pop*** according to the strategies of enumeration of the ranks $\boldsymbol{\beta}$ and the counting of days that we know to have started at ***0***, ***1*** or ***2*** according to the peoples and the periods.

45 Consequently, the dates (eponym, n^{th} day of the month, New Year, etc.) do not stray in the seasons like the Mayan ***0 Pop*** should – in spite of the affirmations by Landa who fixed the Mayan New Year on the Christian July 16th, and asserts that the scribes added a 366th day to the year every four years.

46 In the intertropical zone, the Sun passes twice at the Zenith (before and after the Summer Solstice), and each passage is easily noted by the absence of shadow for objects held vertically (steles, for example). One may also follow the rising and setting of the Sun in relation to reference points of the horizon or of the city, etc. All of this provides the means to elaborate an annual calendar in phase with the tropical year.

47 On-line: `http://www.uv.mx/dgbuv/bd/pyh/1995/2/html/pag/index.htm`, the article appeared in 1995 in *Lapalabra y el hombre*.

48 Are noted (Morante Lopez;2001:48) the subterranean observatories of Teotihuacan (200 AD), Monte Alban (400 AD) or Xochicalco (700 AD).

the chimney in such a way as to divide the year in two: one part during which the chamber received the Sun's rays, and a part in which it did not. According to the measurements taken in 1988 – 1992, the first period began on April 30 (once on May 1) and the second on August 13. These dates divide the year into a portion of 105 days and one of 260 days (261 in leap years). The 105 day portion is centered on the Solstice[49] of June 21: April 30 + 105 = August 13, August 13 + 260 = April 30[50], April 30 + 52 = June 21, June 22 + 52 = August 13.

The Aztecs thus disposed of a sort of clock or of calendar giving, live and continuously, the progression of the days and seasons of the tropical year:

	Solstice			
01/05	21/06	12/08	August 13	April 30

The interest of the experiment of Xochicalco is to reveal the following points:

Certain Mesoamericans constructed heliographs (markers of rays of light) that gave the beginning and end of two periods.

A lighted period that includes the principal bearings of the tropical year (Summer Solstice and passages at the Zenith) and whose length of 105 days enjoyed undeniable numerological properties. For example: 105 = 5 x 20 + 5 = 2 x 4 x 13 + 1.

A period of shadow lasting 260 days equaled the length of the divinatory almanac and catches up with, in four years, the delay of one day that the vague year has on the tropical year. The period of shadow lasts 260 days in a normal year and 261 days[51] in leap years.

Vague Year No. 1		Vague Year No. 2		Vague Year No. 3	
105 days	260 days	105 days	260 days	105 days	260 days
Vague Year No. 4		**Vague Year No. 5**		**Vague Year No. 6**	
105 days	**261 days**	105 days	260 days	105 days	260 days

Disposing of such a heliograph, the kings and priests have no need of a calendar of the vague year, nor even to mark the 365 passing days. Because, in order to know at which point was the tropical year, it was sufficient to go to the observatory to read and interpret what the Sun's rays revealed in this sacred place. And to decide, for example, to increment the Book of Years or to celebrate the New Fire. Without possible error. But in a manner quite distant from the calendar and computational habits of the Classical Mayas[52].

But not everyone disposes of an underground observatory. During the Postclassic and Colonial period, most of the peoples contented themselves with the dates $(\alpha X, \alpha \underline{X}_P)$ and to follow in parallel the course of the months of year and the rhythm of the 20 monthly celebrations. In the areas mixed by contact with the Spanish, the Natives had every interest in hiding their attachment to the sacred

[49] Framed by the 2 passages at the Zenith whose date depends on the latitude of the site.
[50] August 13 + 261 = April 30, during leap years.
[51] Because d(August 13, April 30/May 1) = 260/261 according to whether February counts 28/29 days.
[52] At best linked by a cryptomorphism the calendars $(\alpha X, \beta Y)$ and $(\alpha X, \alpha X_P)$ do not even speak the same language. The Aztec Century is an adjustable simulation of the solar year while the Mayan CR is an untouchable arithmetic model made to distinguish and to define each of the 18 980 days of the most typical Mesoamerican temporal cycle. At the cost of losing dates during seasons without making any claims for a "true calendar"

almanac and to the eponym cycle, largely stigmatized as Satanic works. One perceived way to do so consist of becoming a user of the vague solar year calendar. *Ha' ab* and *xihuitl* are, indeed, much close to the calendar of the Spaniards, even if they contain 18 months of 20 days instead of 12 months of 30 days on average. Or, the Colonial sources contain Indian calendars that seem to stem from this state of things. These are the annual calendars that respond like the Maya *ha' ab* or the Aztec *xihuitl* to the formula (18 x 20) + 5. But, that differ from it in the manner of writing the 20 dates of a $\boldsymbol{Y}$ month. At a first analysis, these calendars reveal three types of different situations:

1) The dates of the Mayan months of the Classic Period are of the form $\beta\boldsymbol{Y}$ and, without going into the details of the representation of ***Uayeb***, we may represent the *ha' ab* calendar by a table of columns labeled by the 18$\boldsymbol{Y}$ names[53] of the moths and learned by the sequence of the twenty first natural integers which semiotize the twenty $\boldsymbol{\beta}$ of the days in the month[54].
2) For the Mayas of the Postclassic or Colonial periods, the months remained unchanged and continued to label the columns of the chart, but the $\boldsymbol{\beta}$ positions are no longer written[55]. The monthly columns are, however, learned by the sequence of the 20 $\boldsymbol{\alpha X}$ dates of the days of the month, to which the rows $\boldsymbol{\alpha}$ bring a strong character of *trecena*. More precisely, the columns are indicated by the dates $\boldsymbol{\alpha_i X_j}$ where the references i and j vary from 1 to 20, modulo 13 (for the $\boldsymbol{\alpha}$ rows) and modulo 20 (for the $\boldsymbol{X}$ names)[56]. In the example of the *Calendario de los Indios de Guatemala*, the 20 lines of the days of a month are additionally numbered from 1 to 20 and specifying that it consists of the position of the day (Día) in the month of twenty days. This is evidently a notation made by/for a Spaniard, and not to assign to the $\boldsymbol{\beta}$ rows of the days in the Mayan month of the Classic Period (whose variable interval ranged from ***0*** to ***19***).
3) For the non-Mayas of the Post Classic or Colonial Periods, the months are not identified by their $\boldsymbol{Y}$ name[57], but by a scene or a description which seems to vary locally. A bit like the how expression *Le temps des cerises* allows a French person to identify the month of June[58]As in 2), the columns of months are indicated by the dates (*tonalpohualli*) of the form $\boldsymbol{\alpha_i X_j}$.

As shown by the examples of Annexes, it results from 1) and 2) that the Mayan *ha' ab* remained isomorphic, which amounts to saying that the ROCm rule remained in application. No longer recording the $\beta\boldsymbol{Y}$ dates allowed, however, the tolerance of minor deviations: to change the initial positions of the *tzolkin* and *ha' ab* cycles in relation to one another at the starting moment of the CR. One such deviation reveals itself by a change in the role of the Bearers. But it does not modify the organization of the CR, which remains a calendar of 18 980 dates. In this case, it would be legitimate to reconstruct, from a fragment, the 365 dates of the annual calendar, or the 18 980 of a CR.

In the cases 2) and 3), taking into account the bias introduced by the Colonizer/ Colonized contacts, and the imprecision already signaled concerning the eventual length of the *xihuitl* (365? 366?

[53]The $\boldsymbol{Y}$ names of the months evidently change with the languages and the periods, but all of the lists contain, with the exception of one translation, the same months in the same order of the type ***Pop, Uo***, etc., ***Cumku, Uayeb***.

[54]These 20 rows vary from **20/0** (***TI' HA' B/CHUM***) to **19**.

[55]Sometimes doubled by the number y of its position in the succession of the months of the *ha' ab*. Sometimes translated into Nahuatl or another language, and sometimes doubled by a description in Spanish or Latin. For example, in the *Calendario de los Indios de Guatemala, 1685, Cakchiquel*, `http://famsi.org/research/mltdp/item57` we have: *Mes n° 10 Rucactoeie.*

[56]Reading horizontally, the Xj are constant, and the ai in arithmetic progression with a common difference of 7 (modulo 13), so that, in 1 out of 2 columns, they are in the natural order of integers.

[57]By a glyph of the month, or by the $\boldsymbol{y}$ position of the month in the succession of the 18 months of a year.

[58]To cite an example: `http://www.lunacommons.org/luna/servlet/view/all/who/Tovar`.

365.25?), the dating ($\alpha X, \alpha \underline{X}_P$) does not, on its own, allow the reconstruction, to the day, of the 18 980 dates of a Mayan CR cycle[59]or the dates of the days of an Aztec Century of 52 years. From there results the question of a possible corrective[60]of the delay of the *ha*' *ab* calendar or *xihuitl* over the tropical year.

Revisiting the Mesoamerican calendars would consist of crossing a typology of the calendars[61] with a typology of the situations and users[62] .

This work could cast new light upon the diversity of the roles of the Bearers, eponyms and calendars, and show that the kings of cities with heliograph could have, without writing βY dates, maintained in phase the succession of the *xihuitl* and that of the tropical years. Otherwise stated, here or there, the *Calendario verdadero* may well have had a pre-Hispanic reality. Certainly, different both from the reality of Julian and Gregorian calendars and that of reinterpreted Mesoamerican calendars, in the *mestizo* zones of interaction between the two Worlds, by and for the Colonial and Evangelist institution of the Europeans.

REFERENCES

- Alva Ixtlilxochitl, F, 1977, in *Obras Históricas*, Tomo 2, O' Gorman (eds.), México, UNAM, Instituto de Investigaciones Históricas, p. 565.
- Aveni, A., 1981, "Tropical Astronomy", in *Science*, Vol. 213, n° 4504, p. 161-171.
- Cauty, A., 2009a, *Des Porteurs mayas aux éponymes aztèques*, http://celia.cnrs.fr/FichExt/Etudes/Maya/Cauty_DatesAnneeVague.pdf
- Cauty, A., 2009b, *Y a-t-il des années surnuméraires mayas ?* http://celia.cnrs.fr/FichExt/Etudes/Maya/CautyAnsMayasSurnumeraires.pdf
- Cauty, A., 2012, *Xihuitl, l' année vague solaire vue par Durán*, http://celia.cnrs.fr/FichExt/Etudes/Maya/Xihuitl_Duran.pdf
- Edmonson, M., 2000, "Los calendarios de la Conquista", in *Arqueología mexicana*, vol. VII, n° 41, Mexique, p. 4045.
- Hernández Nieto, H., 1978, "Una interpretación diversa de la aritmética nahualt según un manuscrito de Juan Caramuel", in *Journal de la Société des Américanistes*, Tome LXV, Paris, pp. 87-101.
- Hoppan, J.-M., 2012, *Parlons maya classique*, Paris, L' Harmattan, à paraître.
- Kirchhoff, P., 1943, "Mesoamérica. Sus Límites Geográficos, Composición Étnica y Caracteres Culturales' ", in *Acta Americana* **1** (1), p. 92–107.
- Luria, A, 1966, *Les fonctions corticales supérieures de l' homme*, Paris : PUF.

[59]Marked by 52 αX_P dates in the p. 34-37 of Madrid (dates X_P in p. 54-57 of the Dresden).

[60]Because correcting the vague year amounts to increasing the duration of a period and redistributing the normal/increased periods. This provokes a substantial delay likely to explain, for example, the diversity of the roles of the year Bearers or the dates $(\alpha X, [\beta] Y)$ of the Chilam Balam de Tizimin which contradicts the ROCm of the Classic.

[61]On the basis of respect of the ROCm or of the equation $x–\beta = C$ of the deviations of these rules, we can distinguish, for example: calendars of the type Aztec Century or *calendario verdadero* of 18 993 days (52 *xihuitl* of 365/366 days), calendars of the type CR Mayan Classic of 18 980 days or its four clones, characterized by the roles P_0, P_1, P_2, P_3 and P_4 of the Year Bearers, even the total of 94 900 $(\alpha X, \beta Y)$ possible pairs. A deviation from the ROCm rule may be provoked by a change in the variation interval of the rows β, a change of synchronization $tzolkin \times ha'ab$ leading to start **Ahau (Chicchan, Oc, Men)** out of $\mathbf{0(5, 10, 15), 1(6, 11, 16), 2(7, 12, 17), 3(8, 13, 18)}$ or $\mathbf{4(9, 14, 19)}$, or a change in the duration of a period (day, month, year, century···.) to correct the delay of the vague year over the tropical year.

[62]For example: 'experts' (i) Mayan or (ii) Aztec $vs.$ 'amateurs' (iii) Mesoamerican, *Mestizo* or Spaniard, of the Post Classic and Colonial Periods, etc.

DATES αX OF THE 20 DAYS OF AN AZTEC MONTH (*TECUILHUITL*, TOVAR CODEX)

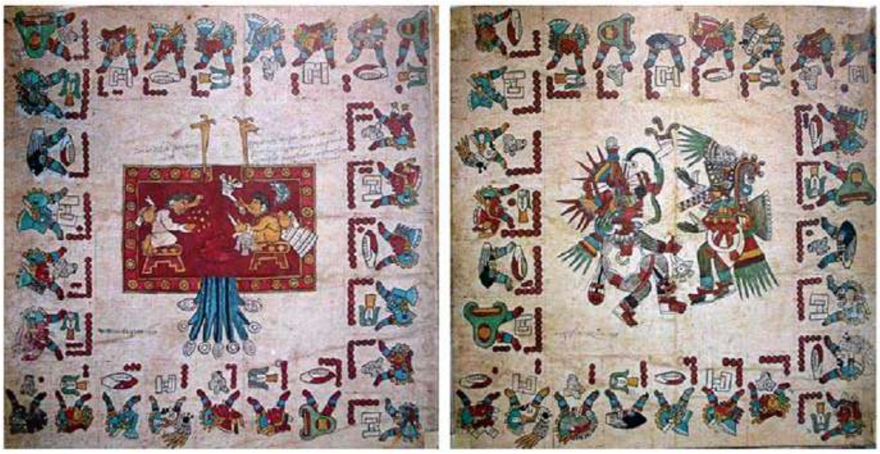
4 Cipactli	1 Ozomatli
5 Ehecatl	2 Malinalli
6 Calli	3 Acatl
7 Cuetzpalin	4 Ocelotl
8 Cohuatl	5 Cuauhtli
9 Miquiztli	6 Cozcacuauhtli
10 Mazatl	7 Olin
11 Tochtli	8 Tecpatl
12 Atl	9 Quiyahuitl
13 Itzcuintli	10 Xochitl

- Marcus, J., 2000, “Los calendarios prehispánicos”, in *Arqueología mexicana*, vol. VII, n° 41, Mexique, p. 12-19.
- Morante Lopez, Ruben B., 2001, “Las cámaras astronómicas subterráneas”, in *Arqueología mexicana*, vol. VIII, n° 47, Mexique, p. 46-51.
- Polh, M., Pope, K., Nagy, C. Von, 2002, “Olmec Origins of Mesoamerican Writing”, in *Science* 6, Vol. 298. no. 5600, pp. 1984 – 1987.
- Queixalos, F., 1986, “Autobiographie d’une néonumération”, in *Amerindia*, 11, Paris : Association d’Ethnolinguistique Amérindienne, p.155-164.
- Roulet, E., 1999, “Une page curieuse sur les fêtes aztèques dans un manuscrit du XVIII[e] siècle”, in *Journal de la Société des Américanistes*, 85, p. 367-373.
- Tena, R., 1987, *El calendario mexica y la cronografía*, México: Instituto Nacional de Antropología e Historia.
- Tena, R., 2000, “El calendario mesoamericano”, in *Arqueología mexicana*, Vol. VII, n° 41, p.4-11.
- Urcid Serrrano Javier, 2001, “Zapotec hieroglyphic writing”, in *Studies in Pre-Columbian Art and Archaeology*, n° 34, Dumbarton Oaks Research Library and Collection, Washington, D.C., 487 p

ANNEXES

The Mesoamerican year was a vague solar year composed of 19 periods/ months of 20 days, and a remainder of several days. According to the sources, the location, or the period, the remainder numbers 5 or 6 days. And there are two distinct ways to individually determine the days of the year. Like the Classical’ Mayas, sequencing them by their rank β in the Y period, or like the Aztecs, distinguishing them by their date αX in the tonalpohualli (and year αX_P).

DATES $\boldsymbol{\alpha X_P}$ (EPONYMS) OF THE 52 YEARS OF THE AZTEC *XIUHTLALPILLI*

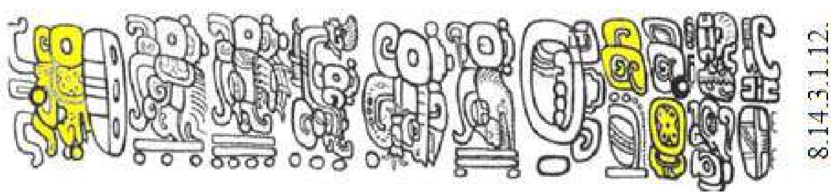


X_P = (Tochtli, Acatl, Tecpatl, Calli) codex Borbonicus, 19/21-20/22

THE 365 DATES βY OF THE MAYAN YEAR, HA' AB

Pop	Uo	Zip	Zodz	Tzec	Xul	Yaxkin	Mol	Ch'en	Yax	Zac	Ceh	Mac	Kankin	Muan	Pax	Kayab	Cumku	Uayeb
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	
6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	
7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	
8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	
9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	
10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	
11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	
12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	
13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	
14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	
15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	
16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	
17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	
18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	
19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	

0 Pop, *1 Pop*, *2 Pop*, etc. *19 Pop*, *0 Uo*, *1 Uo*, etc., *4 Uayeb*.



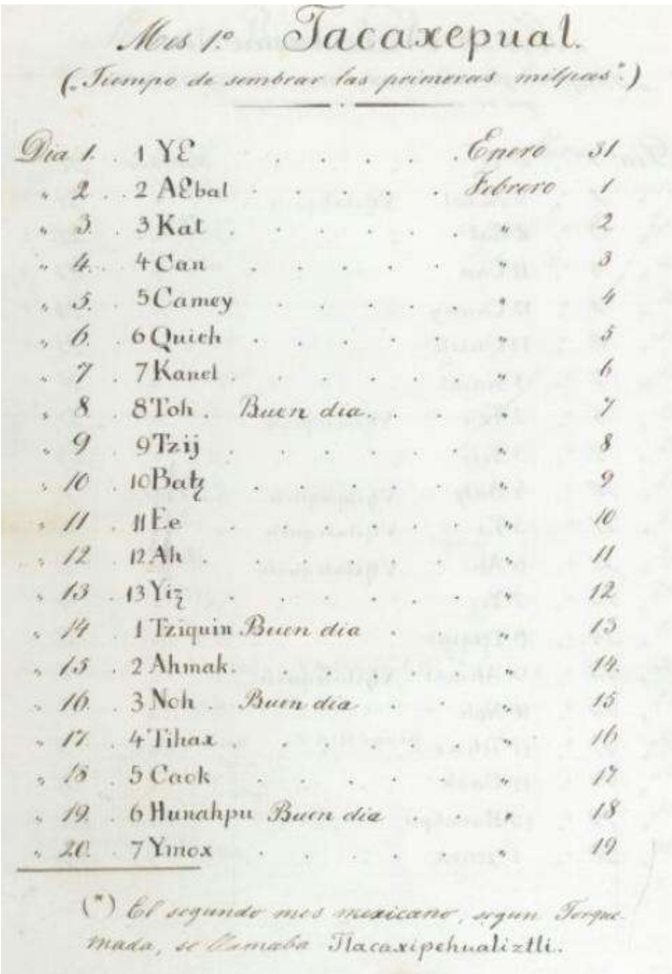
8.14.3.1.12.
1 Eb *0 Yaxkin*
(17/09/320)

THE 260 + 105 DATES αX OF THE 365 DAYS OF A 19 PERIODS MAYAN *HA' AB*

		I	*II*	*III*	*IV*	*V*	*VI*	*VII*	*VIII*	*IX*	*X*	*XI*	*XII*	*XIII*	*XIV*	*XV*	*XVI*	*XVII*	*XVIII*	*XIX*
		Pop	*Uo*	*Zip*	*Zodz*	*Tzec*	*Xul*	*Yaxkin*	*Mol*	*Ch'en*	*Yax*	*Zac*	*Ceh*	*Mac*	*Kankin*	*Muan*	*Pax*	*Kayab*	*Cumku*	*Uayeb*
XX	**Ahau**	13	7	1	8	2	9	3	10	4	11	5	12	6	13	7	1	8	2	9
I	**Imix**	1	8	2	9	3	10	4	11	5	12	6	13	7	1	8	2	9	3	10
II	**Ik**	2	9	3	10	4	11	5	12	6	13	7	1	8	2	9	3	10	4	11
III	**Akbal**	3	10	4	11	5	12	6	13	7	1	8	2	9	3	10	4	11	5	12
IV	**Kan**	4	11	5	12	6	13	7	1	8	2	9	3	10	4	11	5	12	6	13
V	**Chicchan**	5	12	6	13	7	1	8	2	9	3	10	4	11	5	12	6	13	7	
VI	**Cimi**	6	13	7	1	8	2	9	3	10	4	11	5	12	6	13	7	1	8	
VII	**Manik**	7	1	8	2	9	3	10	4	11	5	12	6	13	7	1	8	2	9	
VIII	**Lamat**	8	2	9	3	10	4	11	5	12	6	13	7	1	8	2	9	3	10	
IX	**Muluc**	9	3	10	4	11	5	12	6	13	7	1	8	2	9	3	10	4	11	
X	**Oc**	10	4	11	5	12	6	13	7	1	8	2	9	3	10	4	11	5	12	
XI	**Chuen**	11	5	12	6	13	7	1	8	2	9	3	10	4	11	5	12	6	13	
XII	**Eb**	12	6	13	7	1	8	2	9	3	10	4	11	5	12	6	13	7	1	
XIII	**Ben**	13	7	1	8	2	9	3	10	4	11	5	12	6	13	7	1	8	2	
XIV	**Hix**	1	8	2	9	3	10	4	11	5	12	6	13	7	1	8	2	9	3	
XV	**Men**	2	9	3	10	4	11	5	12	6	13	7	1	8	2	9	3	10	4	
XVI	**Cib**	3	10	4	11	5	12	6	13	7	1	8	2	9	3	10	4	11	5	
XVII	**Caban**	4	11	5	12	6	13	7	1	8	2	9	3	10	4	11	5	12	6	
XVIII	**Edznab**	5	12	6	13	7	1	8	2	9	3	10	4	11	5	12	6	13	7	
XIX	**Cauac**	6	13	7	1	8	2	9	3	10	4	11	5	12	6	13	7	1	8	

13 Ahau, 1 Imix, 2 Ik, 3 Akbal, etc. 13 Ben, 1 Hix, etc., 6 Cauac, 7 Ahau, etc., 13 Cimi, etc., etc. 13 Kan.

DATES αX OF THE 20 DAYS OF A 1ST MONTH OF A MAYAN *HA' AB* (COLONIAL TIMES)



Mes 1º Tacaxepual.

(Tiempo de sembrar las primeras milpas(*))

Dia 1	1 Yꜫ		Enero	31
" 2	2 Aꜫbal		Febrero	1
" 3	3 Kat		"	2
" 4	4 Can		"	3
" 5	5 Camey		"	4
" 6	6 Quich		"	5
" 7	7 Kanel		"	6
" 8	8 Toh	Buen dia	"	7
" 9	9 Tzij		"	8
" 10	10 Batz		"	9
" 11	11 Ee		"	10
" 12	12 Ah		"	11
" 13	13 Yiz		"	12
" 14	1 Tziquin	Buen dia	"	13
" 15	2 Ahmak		"	14
" 16	3 Noh	Buen dia	"	15
" 17	4 Tihax		"	16
" 18	5 Caok		"	17
" 19	6 Hunahpu	Buen dia	"	18
" 20	7 Ymox		"	19

(*) El segundo mes mexicano, segun Torquemada, se llamaba Tlacaxipehualiztli.

Calendario de los Indios de Guatemala, 1685, Cakchiquel
http://www.famsi.org/research/mltdp/item57/

A COLORFUL CASE OF MISTAKEN IDENTITIES

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ABSTRACT

In 1594, the Medici Press printed an Arabic version of Euclid as part of its overall publication program. It was one of the earliest European attempts to print Arabic from moveable type and the result was elegant indeed. But the book has long been the victim of a mistaken identity. Not only has the book itself been misidentified, but the entire history of the treatise (which we can now trace for more than four centuries) has been repeatedly influenced by mistaken identities. My paper aims to clear up at least most of these mistaken identities. This historical research lays the groundwork for informed use of this historical episode in our teaching of mathematics.

Keywords: Pseudo-Ṭūsī, Medici Press, Euclid's *Elements*

1 Introduction

In 1594, the Typographica Medicea, which had recently been established by Ferdinand de Medici in Rome, issued an elegant edition of an Arabic treatise on Euclidean geometry with the title *Kitā Taḥrīr Uṣūl li-Uqlīdis* [*Redaction of the Elements of Euclid*]. The publication, one of the earliest attempts to typeset Arabic mathematics in Europe, represented the highest standards of workmanship, expertly blending type and diagrams. Because of the clarity of its typeface and ease of access (as compared to manuscripts) the treatise quickly became a standard source for scholarship on the history of Euclidean geometry in the classical Islamic period. Although the motivation for this publishing experiment is still somewhat unclear, the press directors obviously had several potential markets in mind, for the treatise was issued with different title pages—some were in Arabic only and some were bilingual Arabic and Latin (Cassinet 1993, 20-21). They apparently also expected a fairly high demand for the edition because they printed 3000 copies. These expectations were clearly incorrect—nearly a century later, almost two thirds of these copies were still in the storerooms of the press (Jones 1994, 108). In this paper we survey the later history of this remarkable edition and its influence in mathematical circles—a history long clouded by a mistaken identity.

2 Mistaken identity

The title pages (both Arabic and Arabic-Latin) of this remarkable edition have produced one of the more long-lasting errors of historical studies of mathematics because they attribute the authorship of the treatise to Naṣīr al-Dīn al-Ṭūsī (597 / 1204 – 674 / 1274), one of the best-known and most influential of the mathematicians of the medieval Islamic world.[1] His redactions of the Arabic translations of important Greek mathematical treatises, beginning with Euclid's *Elements*, progressing through several smaller Greek tracts (known collectively in Arabic as the *Mutawasiṭāt*, or intermediate books) and culminating with Ptolemy's *Almagest* laid a new foundation for mathematical studies that continued to form the core of the curriculum in mathematics education until the nineteenth century.

Despite the bold title page statement that the Rome 1594 treatise was none other than the Taḥrīr Kitāb Uqlīdis by Naṣīr al-Dīn al-Ṭūsī, the most influential Euclidean text of the Arabic / Islamic world, it was a different treatise — different in style and diction, different even in mathematical content (at least as far as its "demonstration" of Euclid's parallel lines postulate is concerned). Since the printed text differs in important features from the many surviving manuscripts of the *Taḥrīr Kitāb Uqlīdis* that carry al-Ṭūsī's name, it was often initially assumed to be only a re-editing of the text by the author. This hypothesis seemed plausible initially but became untenable when scholars discovered a note in a manuscript copy of the treatise, manuscript Or. 50 in the Laurenziana Library in Florence, stating the date of completion as 698 / 1298 (Sabra 1969, 18). Since al-Ṭūsī died in 1274, to continue to ascribe the treatise to his authorship would require us to elevate "ghost-writing" to an entirely new level!

It was early recognized that text the Rome 1594 edition differed in some ways from that found the genuine redaction by al-Ṭūsī. For example, the Rome edition contains only 13 books, while that of al-Ṭūsī contains in addition the two apocryphal books (numbered fourteen and fifteen) that are ascribed to Hypsicles in the Arabic Euclidean tradition.[2] The erroneous ascription on the title page, however, inclined many historians to believe that it represented only a re-editing of the original text. This hypothesis becomes considerably less likely when one looks at the texts in some detail. For example, the redaction of al-Ṭūsī contained nearly 200 notes added by the author. Some discuss mathematical questions raised by Euclid's text, others substitute direct proofs in place of Euclid's indirect demonstrations, still others offer alternative demonstrations for various propositions.[3] These added notes are not found in the Rome edition, except for a few editorial notes describing differences between the Arabic translations of al-Ḥajjāj and Isḥāq ibn Ḥunayn. Furthermore, the Rome edition incorporates into its text many explicit references to definitions and earlier propositions as justification for points in the mathematical argument. Neither Euclid himself nor al-Ṭūsī included such references. There are many differences in diction and in style as well. For example, for statement of the problem in proposition I, 2 (I translate from the Arabic):

T = We want to extend from a given point a line equal to a bounded line.

PT = We are to add to any specified point a straight line equal to a bounded straight line on condition of the two of them being in a single plane.

[1] The origins of this title page ascription are a mystery. Al-Ṭūsī's name is not mentioned in any of the surviving manuscripts, and certainly not in the manuscript from which the Rome edition was typeset.

[2] Modern scholarship accepts the attribution of book XIV to Hypsicles (1st century BCE), perhaps based on an earlier lost treatise by Apollonius. Book XV is now considered to be a compilation from several sources, one of which appears to be Isidorus of Miletus (6th century CE), who is reputed to have written commentaries on the *Elements* that no longer survive (Vitrac & Djebbar 2011, 31-32).

[3] The majority of these notes were borrowed from the *Kitāb fī Ḥall Shukūk Kitāb Uqlīdis* [On the resolution of doubts raised by Euclid's treatise] written by noted mathematician Ibn al-Haytham (died 432 / 1031) although Ibn al-Haytham's name is not mentioned (De Young 2009).

Such extensive variations in formulation, not to mention the many differences in diagram patterns make it more difficult to assume that the Rome edition is merely a re-editing of al-Ṭūsī's treatise.

Although we can now be sure that the author of the Rome redaction was not al-Ṭūsī, we do not know who to assign as author. Some scholars (Rosenfeld & Ihsanoğlu 2003, 211–219)have argued that it must be the son of al-Ṭūsī, Ṣadr al-Dīn, who took over his father's position as head of the research institute at Marāgha. But until now we have no contemporary documentary evidence concerning the author's identity, so many scholars prefer to designate him as Pseudo-Ṭūī. I shall follow this designation as well.

3 Typesetting the Rome Euclid

It had long been known that there were two manuscript copies of the Pseudo-Ṭūsī in the Biblioteca Medicea Laurenziana in Florence. These manuscripts were misidentified, however, in one of the most widely used reference works in the history of science — the *Dictionary of Scientific Biography*. In his article outlining the complex transmission of the *Elements*, John Murdoch, identified these manuscripts as Bibl. Laur. or. 2 and or. 51. Neither number seems to be correct if one checks the online catalog from the library. Only after expending more than a thousand euros of my university's research budget to purchase scans of every work identified with al-Ṭūsī in the orientali collection did I discover that the correct identification (or. 20 and or. 50) had been published years earlier in a footnote to one of Sabra's studies (1968, 15) of parallel lines in the Arabic tradition. Needless to say, I quietly glossed over this minor point when making my report to the university administration.

In my investigation of this well-known impostor and based on these expensive scans, I can now state with confidence that the Rome edition was typeset from manuscript orientali 20. Typesetter notes in Italian and other markings in the margin of the manuscript make this conclusion certain. I can also state with certainty that the text of orientali 20 was copied from orientali 50. The copying is evident when we examine passages that were canceled out in the text of Or. 20 and rewritten in different form in the margins. In one case, the copyist accidentally turned two leaves. When he discovered the error, he canceled the entire section and rewrote it correctly in the margins of Or. 20. We can see that the lacuna left by the copyist extended from the last word on folio 46a until the first word on folio 47b of Or. 50 (De Young 2012).

It is curious that the diagrams of manuscript Or. 20 were not copied from Or. 50. This unusual situation is clearly evident when we look at diagrams for books VII–IX, Euclid's discussion of numbers and their characteristics. Traditionally, diagrams found in these book used line segments to represent numbers. Or. 50 follows this convention, but Or. 20 adopts a new technique – using columns of dots to represent numbers. Although this new style may be more consistent with the spirit of Euclid's work, it leaves a mystery — where did this innovation come from? So far, the technique has not been observed in any earlier manuscripts, so perhaps it was the copyist himself who made this change in the diagrams.

4 How did the press get its Arabic manuscripts?

Although the story is still somewhat murky, it appears highly probable that the manuscripts used by the press in its publishing efforts were brought to Italy by Ignatius Niʿmatallāh, former Patriarch of the Syrian Orthodox communion (1557-1576), who had been forced into exile through another kind of mistaken identity. Ignatius, a typical medieval polymath, was also a practicing physician. Because of his skills, he had become personal physician to the local Muslim governor. The local Muslim intelligentia were not happy to have a high-profile Christian in such a powerful position and frequently stirred up trouble at the court opposing the Patriarch. Perhaps to diffuse some of this tension, the governor, during one of his evening salons, "honored" the Patriarch by placing his own turban on his head and declaring him a "convert" to Islam. Although there is no indication that the Patriarch wished or intended to renounce Christianity, the governor's "honor" forced Niʿmatallāh (through a kind of mistaken identity — his fellow Christians now regarded him as a traitor to his faith) to abdicate his position. He fled to Italy, taking with him his collection of manuscripts. In Italy, he continued to pass himself off as "Patriarch" (perpetuating yet another mistaken identity) and was generally received with great honor in the halls of power. While traveling from Venice to Rome, he was apparently introduced to Ferdinand de Medici, who was contemplating a new business venture — a publishing house to produce Arabic texts. Ferdinand had money but needed Arabic manuscripts while Niʿmatallāh had Arabic manuscripts and needed money. A deal was finally struck and the "Patriarch" joined the board of the infant publishing house, putting his manuscripts at its disposal and setting in motion one of the great mistaken identities in the history of mathematics.[4]

5 Influence of the Pseudo-Ṭūsī edition in Arabic

In Arabic, the Pseudo-Ṭūsī redaction was far less influential than the treatise actually authored by Naṣīr al-Dīn — at least if we consider number of surviving manuscripts to be any indication of popularity and influence. Nevertheless, it is now possible to trace instances of influence from the Rome 1594 edition for several centuries in the Arabic-speaking world. For example, a century after the publication of the treatise, a manuscript copy (Tehran, Sipahsalar 540) was written out from the typescript. Since manuscripts do not have title pages, the treatise is not explicitly assigned to al-Ṭūsī. Nevertheless, this manuscript has also been the victim of a mistaken identity. Someone — almost certainly not the copyist himself — wrote a note on the flyleaf identifying the manuscript as the *Iṣlāḥ* [*Uṣūl Uqlīdis*] [Correction of Euclid's Elements] by Athīr al-Dīn al-Abharī (died 663 / 1265). The text makes clear, however, that it is actually a copy of the Pseudo-Ṭūsī *Taḥrīr* and not al-Abharī's *Iṣlāḥ*.

Initially, we might be surprised that someone would want to copy a printed book by hand. But the conclusion that this is what happened follows from a small note inserted in the margin beside the demonstration of proposition VI, 1: "Apparently a diagram should exist here." The note makes sense when we look at the Rome edition, page 134. There is a blank space, apparently left for a diagram that was never inserted. Moreover, the diagrams in Sipahsalar 540 preserve the same distinctive features shown in the Rome edition. Although we don't know where it was copied, the manuscript is dated 1101 / 1670.

Some two centuries later, the Pseudo-Ṭūsī version surfaced once again, this time in the form a two-volume lithograph edition published in Fez (1293 / 1876). Like the earlier Rome edition, the title page proudly proclaims it the work of Naṣīr al-Dīn al-Ṭūsī. So even three centuries after the first print edition, the mistaken identity of the author continued to live on. This new lithograph version

[4] Saliba (2008, 199–212) gives one of the most complete summaries of the colorful Patriarch's life. Toomer (1996, 22–24) also provides a short biographical sketch of his role in transmitting Arabic learning to Europe.

has not yet been fully studied, but preliminary surveys indicate that its diagrams exhibit distinctive features exactly like those found in the Rome edition. The text is identical to that of the Rome edition, although written now in Maghribi Arabic script and many of the obvious typographical errors in the Rome edition have been corrected. The observation that proposition III, 11 is incorrectly numbered III, 12 in each printed version is very strong evidence for a direct genetic connection between the two. The probability that the incorrect number should appear independently seems incredibly small.

It seems clear that even though there are few surviving manuscript copies, the text continued to be read and copied from time to time. But at the same time, there is no evidence of its influence in the broader tradition. No commentary on the text has yet been identified, and the surviving copies have little or no marginalia apart from corrections to scribal errors. The lack of influence is somewhat puzzling because the treatise seems ideal for mathematical education, especially for younger students just beginning their journey in mathematics. Al-Ṭūsī's treatise was aimed at more mature students who already had acquired some basic knowledge of philosophy and logic. The Pseudo-Ṭūsī treatise assumes very little from the learner. The extensive inclusion of references to earlier propositions, similar to the system one finds in many modern textbooks of Euclidean geometry help novices to find their way through the logic of the arguments. Only the existence of the Fez lithograph, though, hints toward any educational use of the treatise. And this hint is indirect and incomplete at the moment because there seem to be only a few copies in existence and those that have been digitized on line show no evidence of use by students. Still, why would anyone print a treatise on Euclidean geometry unless it was expected that there would be some market to repay the investment. And the most logical market to assume is from the madrasa. More study will be needed on the educational system in North Africa in the 19th century in order to resolve some of these puzzles.

6 Influence of the Pseudo-Ṭūsī in Europe

From the time of its publication, the treatise had been of considerable interest to European mathematicians. Their focus had been almost entirely on one small section of the treatise, though — the demonstration of Euclid's parallel lines postulate. This demonstration follows proposition I, 28.[5] It is built on three lemmas:

1. Any two straight lines [being] placed in a plane, if there fall upon them lines, each one of which is perpendicular to one of the two and cutting the other in acute and obtuse angles such that the all the acute angles are toward one end of the lines and the obtuse angles toward the other end, I say that the lines are getting shorter the closer one moves toward the side facing the acute angles and that they are getting longer the closer one moves toward the side facing the obtuse angles.
2. Two straight lines extended perpendicularly from the endpoints of a straight line being equal to one another and we connect their two endpoints with a straight line, each of the angles formed between the two perpendiculars and the straight line connecting their endpoints is right.
3. For any triangle whose sides are straight lines, its three angles are equal to two right angles.

[5] A modern French translation of the entire demonstration has been given by Jaouiche (1986, 233-241).

Having established these lemmas, the author turns to Euclid's statement of the parallel lines postulate: "If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles (Heath 1956, I, 155)." There are several possibilities to consider. The angles formed by the line falling on the two lines will be either (a) one right and the other acute, (b) both acute, or (c) one obtuse and the other acute. In each case, the postulate claims that the two lines, if extended indefinitely, will meet on that side. The author gives a demonstration for each of these possible cases, thus "demonstrating" Euclid's postulate.

The existence of this "demonstration" had apparently been traveling as a rumor among mathematicians for some time before the Rome edition was published. The Jesuit mathematician, Christoph Clavius (1538-1612) wrote in the introduction of his 1589 edition of Euclid (Knobloch 2002, 419): "We learned long ago that the Arabs demonstrated the same principle. Though I diligently looked for the demonstration for a long time, I could not see it, because it is not yet translated from the Arabic into Latin." Clearly, some rumors were spreading concerning the existence of an Arabic demonstration of Euclid's postulate during the years before the Rome edition was printed. And in the last edition of his work, Clavius lamented (Knobloch 2002, 419): "I never got the permission to read it though I continually asked for it [from] the owner of the Arabic Euclid."

Who was this rather selfish owner of Arabic manuscripts who would not allow Clavius to view his sources? Saliba (2004, 211-212) argues that it was most probably the Patriarch himself. He had, at considerable difficulty, carried along from Antioch a considerable number of manuscripts. He probably had a good idea of their worth in the European market, and, as an exile, he knew he would need to support himself in a foreign country. Eventually, he negotiated an agreement with Ferdinand to donate his manuscripts to the Press. And in return, he received an annual stipend and was guaranteed access to his books as long as he lived. But during the period that Clavius was requesting permission to see the manuscripts, the negotiations were still taking place, and the Patriarch was probably unwilling to jeopardize their outcome by allowing access to his collection. As a result, Clavius had to figure out the mathematics alone. His results, however, are in some ways remarkably similar to those in the printed edition (Clavius 1612, 48-53).

Clavius had complained that the Arabic text of the demonstration was not available in Latin. So far as can be ascertained at present, the treatise was never translated in its entirety.[6] The first Latin translation of the demonstration of the parallel lines postulate was made by Edward Pococke (1604-1691). It seems not to have been published at that time, but was quoted by John Wallis (1616-1703), Savillian Professor of geometry at Oxford, in a lecture on the parallel lines postulate (11 July 1663). Wallis (1693, II, 665–678) included the translation in his magnum opus. He critiqued the Pseudo-Ṭūsī demonstration in order to set the stage for his own approach to the problematic postulate.

Wallis's critique was studied by Saccheri (1667–1733), one of the founders of non-Euclidean geometry. In Scholion III of proposition XXI in his classic treatise, *Euclides ab omni nævo vindicatus* [Euclid vindicated from every blemish] (1733 / 1920, 100-109) he discussed the approach of Pseudo-Ṭūsī, citing Wallis as his source. His own investigation of the "Saccheri quadrilateral" has many similarities

[6]Youschkevitch (1976, 183) claimed the existence of a complete translation, published in Rome in 1657. No one has been able to verify the existence of this translation (Mercier 1994, 213).

with the second lemma developed in the earlier Arabic proof.[7] And so an obscure and still unnamed Arabic mathematician came to leave his mark on the history of non-Euclidean geometry, while the much more famous mathematician, Naṣīr al-Dīn, who also explored the parallel lines postulate in his own writings, left almost no direct impression on the development of the subject.

7 What can we learn?

We now know much more about the history of this particular Arabic discussion of Euclid's *Elements*. We can trace both the antecedents of the printed Rome 1594 edition and its influence over the next three centuries in both the Islamic and European civilizations. Given the paucity of documentary evidence at our disposal, this is a rather surprising result.

In some respects, it seems surprising as well that this particular treatise seems to have been so completely overshadowed by the genuine redaction of Euclid written by al-Ṭūsī. The Pseudo-Ṭūsī version seems in some ways better adapted to serve as an introductory textbook. It makes the logical arguments of geometry more explicit, and it provides copious references to earlier propositions and premises that support Euclid's arguments and conclusions. At the same time, we might speculate that it was precisely these pedagogical conveniences that might have helped to keep the treatise alive over the centuries although there is at present no documentary evidence to offer support for the hypothesis. Aware of the many mistaken identities that have developed over the centuries, we can always hope that as we examine more manuscripts we may find additional copies that of the Pseudo-Ṭūsī treatise that might help to answer some of our questions.

8 Summing up

Trying to sort out the intricacies of this curious history of errors for the past year has been great fun for me personally. I have always liked jigsaw puzzles. For me as a historian, collecting pieces of information and relating them together to make a coherent historical picture is not unlike working a jigsaw puzzle. And when the puzzle is complete, there is a kind of personal satisfaction and a sense of accomplishment.

But, as my colleague, Glen van Brummelen (2010, 2) wrote recently, we historians are caught up in an important practical dilemma: "On the one hand the proper scholarly treatment of history, especially in the past few decades, increasingly demands good contextual awareness and a resistance to glib answers. On the other hand, our main clients, school teachers and educational associations, demand easily digested 'sound bites' that may be inserted with little fuss into an existing curriculum."

I think very few of us would take refuge behind the oft-quoted sentiment of G. H. Hardy (2005, 49): "I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world." We believe mathematics is useful and many of us believe that history of mathematics is also useful. But useful in what specific ways? I can only suggest a few general ideas.

On the most mundane level, whether we produce or use history of mathematics we should obviously make every effort to insure that our facts and interpretations are historically correct when

[7] Although Saccheri's work languished in obscurity for almost a century, it was revived in the nineteenth century and is now recognized as a classic.

we bring historical vignettes into our classrooms. Unfortunately, many writers of recent histories of mathematics geared toward non-professional audiences, have often simply repeat what older sources have said and so perpetuate the myths that have crept into our discipline. Our students also need to be aware of the multi-cultural fabric from which modern mathematics has been cut. Few discussions of Saccheri, for example, mention the importance that concepts developed in the remote mountains of medieval Persia centuries earlier played in the development of his ideas. This despite the fact that Saccheri explicitly cites Wallis who had explicitly quoted the demonstration of the supposed Naṣīr al-Dīn which he acquired in Latin translation from Pococke. A great deal of work has been done over the past fifty years to explore the multi-cultural dimensions of mathematics. If we can begin to incorporate some of that work into our classrooms, we will certainly enrich our students understanding of mathematics as a human enterprise as well as increase their sensitivity to and appreciation for other cultures.

Although I must leave development of lesson plans or learning modules to mathematics educators, I suggest that one possible use for this historical episode, might be to introduce the subject of non-Euclidean geometry. One might do so by providing students with a translation of the lemmas and demonstrations of the Pseudo-Ṭūsī and ask them to critique the arguments from a Euclidean standpoint — can students find where the proof assumes an equivalent of the parallel lines postulate? (Even though no English translation from the Arabic is now available in print, there is a high-quality French translation by Jaouiche (1986, 233-241) that could be used to generate an English text. Jaouiche also gives an informative discussion of the mathematical and philosophical implications of the argumentation, both by the Pseudo-Ṭūsī and other Islamic mathematicians which might serve as useful background for the teacher who feels unsure of his abilities.) Those researching the application of ethno-mathematics in mathematics education can no doubt suggest other fruitful ways to use this material.

I would only say in conclusion that I believe we educators should resist the attempt to integrate history of mathematics into mathematics education. Although this may initially seem an odd statement, allow me to explain. Integration implies the forced merger of two essentially disparate entities — Americans might think of think of integrating the segregated school systems of the 1960s civil rights movement. The result can be a half-hearted token representation of history of mathematics. Rather than force a few bits or "sound-bites" from history of mathematics into an alien mathematics curriculum in the name of integration, I think we should aim to make history of mathematics integral to mathematics education. Rather than continue to perpetuate the long-standing disciplinary compartmentalization of mathematics and history and literature, our educational systems and our students can be enriched by interdisciplinary initiatives that emphasize the web of inter-connections among the traditional disciplines. Such interdisciplinary perspectives on mathematics can be especially useful to students who are not focusing on mathematical disciplines. Of course, as individual teachers we can often be overwhelmed by the demands of our day-to-day activities. We may feel we have no time or energy to develop interdisciplinary approaches. But organizations like HPM that regularly bring together historians and mathematics educators in fruitful exchanges can play a key role in supporting and furthering our efforts.

REFERENCES

– Cassinet, R., 1993, L'aventure de l'édition des Éléments d'Euclide en arabe par la Société Typographique

Médicis vers 1594, *Revue française d'histoire du livre*, 78–79, 5–51. An earlier version appeared under the same title in *Cahiers d'histoire des mathématiques de Toulouse* 9 (1986).

- Clavius, C., 1999, Opera Mathematica, V. tomis: Commentaria in Euclidis Elementa Geometrica. Facsimile reprint Hildesheim: Olms-Wiedmann. All references are to the reprint edition.
- De Young, G., 2012, Further adventures of the Rome 1594 Arabic Redaction of Euclid's *Elements*. *Archive for History of Exact Sciences*, in press. [DOI: 10.1007/s00407-012-0094-9]
- De Young, G., 2009, The *Taḥrīr Kitāb Uṣūl Uqlīdis* of Naṣīr al-Dīn al-Ṭūsī: Its Sources. *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften*, 18, 1–71.
- Hardy, G. H., 2005, *A Mathematician's Apology*. Regina: University of Alberta Mathematical Society. Originally published Cambridge: Cambridge University Press, 1940. All references are to the 2005 electronic edition.
- Heath, T., 1956, *The Thirteen books of Euclid's Elements*. New York: Dover. Three volumes.
- Jaouiche, Kh., 1986, *La Théorie des parallèles en pays d'Islam*. Paris: Vrin.
- Jones, R., 1994, The Medici Oriental Press (Rome 1584–1614) and the impact of its Arabic publications on northern Europe. In G. Russell (ed.), *The 'Arabick' Interests of the Natural Philosophers in Seventeenth Century England*, pp. 88–108. Leiden: Brill.
- Knobloch, E., 2002, Christoph Clavius (1538-1612) and his knowledge of Arabic sources. In G. P. Brizzi and R. Greci (eds), *Gesuiti e università in Europa (secoli XVI–XVIII)*, pp. 403–420. Bologna: CLUEB.
- Rosenfeld, B. A. and Ihsanoğlu, E., 2003, *Mathematicians, Astronomers, and Other Scholars of Islamic Civilization and Their Works (7th–19th c.)*. Istanbul: IRCICA.
- Mercier, R., 1994, English Orientalists and Mathematical Astronomy. In G. Russell (ed.), *The 'Arabick' Interests of the Natural Philosophers in Seventeenth Century England*, pp. 158–223. Leiden: Brill.
- Sabra, A. I., 1968, Thābit ibn Qurra on Euclid's parallels postulate. *Journal of the Warburg and Courtauld Institutes*, 31, 12–32.
- Sabra, A. I., 1969, Simplicius's proof of Euclid's parallels postulate. *Journal of the Warburg and Courtauld Institutes*, 32, 1–24.
- Saccheri, G., 1733, *Euclides ab omni nævo vindicatus* Mediolani: Ex typographia Pauli Antonii Montani. All references are to the English translation: G. Halsted (ed. and trans.), *Girolamo Saccheri's euclides vindicatus*. Chicago: Open Court (1920).
- Saliba, G., 2008, Embedding scientific ideas as a mode of science transmission. In: E. Calvo, M. Comes, R. Puig, M. Rius (eds.) *A Shared Legacy: Islamic Science East and West*, pp. 193–213. Barcelona: University of Barcelona Press.
- Toomer, G. J., 1996, *Eastern Wisedome and Learning: The Study of Arabic in Seventeenth-Century England*. Oxford: Oxford University Press.
- Van Brummelen, G., 2010, Filling in the sort blanks: Musings on bringing the historiography of mathematics to the classroom. *BSHM Bulletin* 25, 2–9. [DOI: 10.1080/17498430903321125]
- Vitrac, B. and Drebbar, A., 2011, Le livre XIV des Éléments d'Euclide: versions grecque et arabes (première partie). *SCIAMVS* 12, 29–158.
- Wallis, J., 1693, De postulato quinto et definitione quinta lib. 6 Euclidis: disceptatio geometrica. In his *Opera mathematica, volumen secundum* [*De algebra tractatus : cum variis appendicibus*]. Oxford: E Theatro Sheldoniano (1693), pp. 665–678. Reprinted Hildesheim: Olms (1972).
- Youschkevitch, A. P., 1976, *Les mathématiques arabes: VIIIe–XVe siècles* (M. Cazenave and Kh. Jaouiche, trans.). Paris: Vrin. Originally published in Russian under the title *Istoriya matematiki v srednie veka*. Moscow, 1961.

DUTCH ARITHMETIC, SAMURAI AND WORSHIP

Teaching Western Mathematics in Pre-Meiji Japan

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ABSTRACT

This paper discusses the scarce occasions in which Japan came into contact with Western arithmetic and algebra before the Meiji restoration of 1868. It concentrates on the reception of Dutch works during the last decades of the Tokugawa *shogunate* and the motivations to study and translate these books. While some studies based on Japanese sources have already been published on this period, this paper draws from Dutch sources and in particular on witness accounts from Dutch officers at the Nagasaki naval school, responsible for the instruction of mathematics to selected samurai and *rangakusha*.

1 Motivation

The history of traditional Japanese mathematics (*wasan*) is a gratifying subject for study as it confronts us with basic questions on the development of mathematics. The relative isolation of Japanese intellectual culture during the Edo period provides us with almost experimental conditions for the question if mathematics evolves in some necessary order or pattern. If it does, we should observe the same developments in the mathematics of an isolated Japan as they appeared in the West during the early-modern period. The answer to this question is not straightforward. *Wasan* practitioners achieved important results independently from the West.[1] The most salient ones are the development of *bōshohō* ("side-writing", a kind of symbolism in algebra), the foundations of infinitesimal calculus and Seki's work on infinite series which baffles Smith and Mikami (1914, 14). On the other hand, the complete lack of mathematics in the study of the physical world, the specific esthetics of *sangaku* problems and the modes of proof and demonstration are some of the idiosyncrasies which can best be explained within the context of the Edo society.[2]

The end of the seclusion period (*sakoku*, 1635–1868) gave rise to a situation of an equal interest. The confrontation with Dutch mathematics at the end of this period raises the issue how foreign knowledge can and should be incorporated within existing traditions. Of the possible strategies of adaption, integration or replacement the Meiji regime chose drastically for the latter one, abandoning

[1] The most comprehensive overview of *wasan* in a Western language is by Horiuchi (1994). The book is recently translated into English: Horiuchi (2010).

[2] For one of the few studies on the relation of *wasan* and physics see Ravina (1993).

its own rich and flourishing *wasan* tradition. The possible choices are exemplified by the first two Japanese works on Western mathematics.

I will situate the appearance of these two works within the context of foreign military threat and the subsequent development of a Japanese naval force. Previous studies have been published which approach this historical overturn based on Japanese sources (Sasaki 1994, 2002). This account is mostly based on witness accounts of Dutch teachers at the Nagasaki naval training program. The Dutch played a major role in placing education in Western mathematics as a condition for their support to the Japanese enterprise in naval warfare.

2 Samurai and warships

Before the second half of the nineteenth century there was probably no influence of Dutch mathematics in Japan, and if there would have been any, it was very limited. Mikami (1913, 177–8) summarizes the situation as follows:

> Some of the mathematicians in the first part of the 19th century were able to read Dutch works, though their knowledge of the language was of an exceedingly limited kind. A certain number of Dutch astronomical works were possessed by the Astronomical Board of the Shogunate, but we know practically nothing of what were the mathematical treatises brought from Holland to Japan in those days. Nor are we able to find traces of the Dutch influence upon the writings belonging to this epoch. No quotations, no references are found. The relation of the Dutch science and the mathematics cultivated in Japan still remains unexplored. It is almost the whole of the Dutch influence, of which we know, that some of the writings of Kawai [Kyotoku, Kaishiki Shinpō, 開式新法], Shiraishi, Ichino [Mokyo] and others contain some deformed Roman characters as symbols. Reflecting on the incorrectness with which the names of the authors are spelled, their knowledge learned from Dutch works appears to have been very limited if any. We have no knowledge of any Occidentalist, who was at the same time a mathematician.

The situation changed dramatically on July 1853 with the arrival of commodore Matthew C. Perry and a flotilla of four warships at Uraga Harbor. His insistence on forcing the opening of Japan had a dramatic threatening effect on the shogunate. Several actions were taken as a response including the fortification of coastal areas such as Tokyo bay. On 7 August a messenger was sent to the Dutch *opperhoofd* (chief merchant) at Deshima with two questions: what is the cost of a frigate and steam warship and can the Dutch deliver these?[3] An approximate answer was given to the first question and he was told that an answer to the second required consultation with the Dutch government. They were told that it could take up to three years before such ships would be delivered to Japan. On 15 October the governors of Nagasaki communicated the decision by the Edo government to found a naval force according to Western principles and asked the Dutch for help. The Dutch responded with a list of demands in order to assist them in that. However, they made clear from the beginning that it makes no sense to deliver Western frigates without the proper training of Japanese naval officers. The proposal by the last *opperhoofd* Donker Curtius to send Japanese youths to Holland for an intensive training

[3] We here closely follow the Dutch account by Van der Chijs (1867, 414–498), including the Dutch translation of the official Japanese documents.

program was declined with an unambiguous warning not to raise this delicate topic again. Instead, it was chosen to train Japanese naval officers near Deshima. The Dutch delivered a list of subjects that needed to be taught to the new officers by Dutch military teachers. They listed 14 disciplines, putting the mathematical courses on top:[4]

1. Geography and navigation according to Western principles using European maps
2. Astronomy
3. Arithmetic according to the Western method
4. Algebra *(stelkunst)*
5. Geometry
6. . . .

continuing with crafts specific to naval and military expertise. The Dutch response stressed the fact that "war ships carrying officers and personnel without experience in these disciplines would lead to a disadvantage rather than be of any use". A clearer statement of the utility of Western mathematics to modern warfare will be hard to find. The motivation to establish the *Bansho shirabe-sho* (Institute for the investigation of barbarian books) in 1856 should also be seen in direct relevance to the military threat.[5] The Dutch were considered instrumental in the acquisition of Western knowledge necessary for a defense. Thomas C. Smith (1948, 131) quotes one Mito official who stated in 1854 that "the necessity of defense against the barbarians requires that we know them and know ourselves; there is no other way to know them than through Dutch learning".

After months of negotiation the Dutch government sent the paddle steamers *Soembing* and *Gedeh* to Japan, lead by captain Gerhardus Fabius. The *Soembing* arrived at Nagasaki on 22 August 1854 under the command of Gerhard Christiaan Coenraad Pels Rijcken, the *Gedeh* on 21 July 1855. The *Soembing* was equipped with six big guns. It functioned as the training ship for the newly established Nagasaki *Kaigun Denshu-sho* (Nagasaki naval school). The *Soembing* was handed over to Japanese authorities on 5 October 1855. Dozens of young samurai were sent to Nagasaki by the shogunate to be trained as naval officers. Several had already some experience with Dutch as *rangakusha*. Some of the students we know by name and became important to the Japanese naval command. Kaishu Katsu (勝 海舟, 1823–1899) had studied European military science from Dutch books. He was assigned the command of the *Kanrin-maru* in 1860 and became a statesman. Utida Tsunejirō became a naval commander. Horie Kuwajirō (堀江 鍬次郎, 1831–1866) had studied chemistry from Dutch text books and became one of Japan's earliest photographers. Other students include Matsumoto Ryojun who studied Dutch medicine. Some student of the naval school were sent to the Netherlands from 1862 such as Enomoto Takeaki (榎本武揚, 1836–1908) Uchida Masao (Kojirō, 1839-1876), who worked at

[4]Translated from the official notes of 15 Oct 1853, van der Chijs (1867, 415–416). For a discussion of the other courses, related to navigation and military science, see Arima (1964).

[5]Any Dutch book that could be found on one of the ships at Deshima became a precious item and circulated within a small group of Japanese scholars called *rangakusha*. Most of the Dutch books ended up in *Bansho shirabe-sho* together with the astronomical observatory. In 1863 it was renamed *kaiseijo*. In 1869 these books moved to the newly established Imperial University, now The University of Tokyo.

the *Kaisejo* after his return in 1867, and Akamatsu Noriyoshi (Daizaburo, 1841-1920) who became Vice Admiral.[6]

Classes commenced on board of the *Soembing* in the bay of Nagasaki on 24 Oct 1855 and lasted until May 1859 when the institute was moved to Tsukiji.[7] The Dutch officers were lodged on Deshima but could freely move around Nagasaki by the end of 1855. Pels Rijcken was responsible for the training program until another screw-driven steam warship, the *Kanrin maru*, was delivered by a second detachment on 21 Sept 1857. Its commander Willem Johan Cornelis Huyssen van Kattendycke took over on 1 Nov for the following two years. Also both these commanders were involved in teaching. Pels Rijcken taught navigation, cannonry and mensuration. Other instructors included Cornelis Hendrikus Parker de Jonge, the physician dr.Jan Karel Van den Broek, H. O. Wichers (2nd class officer) teaching geometry, algebra, trigonometry, navigation, and from 1858 also descriptive geometry. C.J. Umbgrove (3rd class officer) was teaching "arithmetic in whole and broken numbers, proportions and root extraction". Several witness accounts of the training program by Dutch officers have been preserved. W.J.C. Huyssen van Kattendycke kept a diary from which extracts have been published in 1860. Van den Broek (1862) wrote a flaming response, revealing much of the sensitivities and rivalries of the first detachment. The naval doctor J.C.L. Pompe van Meerdervoort (1867–8) kept notes during his five year stay in Japan and as such provides the most extensive witness account of life at Deshima during that period. He taught medicine to 150 students of which 61 graduated as medical doctors in 1862. His account of Japan, based on his notes, was published in a two-volume book.[8] H.O. Wichers kept a diary from 1857 to 1859, which remains unpublished but is preserved at the museum of the Dutch Navy in The Hague.[9] He became a Dutch marine minister in 1877. Van Kattendycke (1860, 23) lists some details on the mathematics courses. Teaching was done through the aid of interpreters. However, the Dutch complained that the Japanese translators did not understand the Dutch language enough and that they had problems with translating the technical terms. Geometry and algebra were taught 5 hours per week by a 2nd class lieutenant. Arithmetic was taught for 9 hours per week by an administration officer.

Van Kattendycke (1860, 73) was at first skeptical about the assignment: "Pompe's teaching could be brought to direct use, by the treatment of diseases in Nagasaki, but algebra and navigation, or put differently, everything related to mathematics, what could be expected from that?" We also should take into account the social context of Edo intellectual culture. Mathematics exemplified the divide between *samurai* and *chonin* (merchant) cultures. Smith and Mikami (1914, 46) note that "samurai despised the plebeian soroban, and the guild of learning sympathized with this attitude of mind". As abbaco arithmetic and algebra were associated with the merchant class in sixteenth-century Europe, so it was during the Edo period in Japan. Rikitaro Fujisawa (1912, 22), in a comprehensive report on mathematics education in Japan writes that "The intellectual education was essentially classical in nature, and profoundly influenced by the Confucianism tinged with the chivalrous spirit of feudal

[6]These names appear on a roll kept at the National Diet Library, Japan, ms. 93 in the Katsu Kaishu manuscript collection.

[7]Some sources, such as Arima (1964) give Feb 1859 as the end of the Nagasaki naval school. This does not fit with the Dutch records.

[8]The book is translated into English and Japanese. He also published some medical studies related to his stay in Japan: Pompe van Meerdervoort (1859), on the cholera epidemics in Japan: Pompe van Meerdervoort (1998). *His Korte beschouwing der pokziekte en hare wijzigingen* was translated in Japanese by Mitsukuri Genpo but never printed.

[9]Instituut Voor Maritieme Historie, "Dagboek verblijf in Japan als instructeur Japanse marine", 1857–1859, Inventaris van de losse stukken, archive nr.266.

times. To do such things as calculation was thought condescending beneath the dignity of the samurai rank. They even went to the length of boasting of their ignorance in the art of computation practiced by trades-folk, who were looked upon as an inferior class of people". This does not mean that the samurai class detested mathematics—as there were samurai mathematicians—but the esthetics of *sangaku* appealed more to their intellectual aspirations than the practicalities of merchant arithmetic.[10] So it must have been a culture shock for students from the samurai class to get confronted with down-to-earth calculations which they associated with merchant arithmetic. Its direct relevance to naval warfare was not always apparent and posed problems of motivation.

The first year was difficult as we read from a report by Pels Rijcken (Van der Chijs 1867, 462): "Without a soroban they were not able to make the most elementary calculations". Students were afraid to ask questions, even when they did not fully understood what was being taught. Only when the Japanese students met with the teaching officers outside of the classroom—a practice which was encouraged—did they find the courage to ask questions.[11] After several weeks, on 8 Dec 1855 the courses were reorganized and the conditions improved. More attention was now given to the teaching of Dutch language and basic arithmetic. After a year most of the students mastered the basic of mathematics in Western style (Van der Chijs 1867, 465). They could operate on numbers, solve the more difficult arithmetical problems and use algebra. They could solve linear problem with several unknowns, solve quadratic equations and use logarithms with ease. Those who had done algebra were also taught solid geometry and from Oct 1856 onwards plane and spherical trigonometry. All these students could read Dutch texts and consulted Dutch text books. In the early morning of 29 March 1857 the *Soembing*, in Japanese renamed as *Kanko Maru* (観光丸) was put under steam and sailed off to Tokyo manned by a 105 Japanese crew. All of them had followed 15 months of training by the Dutch. In April lessons were resumed for the new arrivals and the students who were left behind. In March 1859 the Dutch were surprised and the students disappointed to hear that the operations of the naval school would be moved to Edo (Van der Chijs 1867, 489-490). The move took place in May and all teaching on Nagasaki officially ended. However, some students still followed lessons at Deshima until the second detachment left on 4 Nov 1859.

The first Japanese works on Western mathematics

The *Seisan Sokuchi* (*A short course on Western Arithmetic*) by Riken Fukuda published in 1856 and the *Yōsan Yōhō* (*The method of Western arithmetic*) by Yanagawa Shunsan in 1857, listed by Mikami (1909) were the first Japanese works describing Western style arithmetic. It is not clear to what degree the appearance of these books is related to two new established institutes. However, several facts point into such direction. First note that the publication dates coincide with the establishment of the *Bansho shirabe-sho* institute. The idea of such institute was suggested by Katsu Kaishu to Abe Masahiro (Jansen, 1958). Katsu, one of the students of the naval school, was involved later with the selection of subjects for study and translation by the institute (Hommes 1993, 33). The overlap in subjects taught by at the *Kaigun Denshu-sho* is remarkable. Also, the 575 books used by the Dutch ended up in the library of the *Bansho shirabe-sho* (Hommes 1993, 76–7).

[10] *Sangaku* are intricate geometrical problems from the Edo period which were often displayed at the entrance shrines and temples. See Fukagawa, H. and Rothman (2008) for an excellent work in English on this tradition.

[11] Private teaching by the Dutch officers became more the rule than an exception in 1858 as reported in Van der Chijs (1867, 488).

Not only did the Dutch arithmetic books spread, the very teaching of arithmetic went beyond the direct needs of the naval training school. Van Kattendycke reports that following a suggestion by Donker Curtius, a class of 25 to 30 children of translators between 8 and 15 were taught the basics of arithmetic. However, this project was abandoned after a fire on Deshima (on 7–8 March 1858).[12] The interest of the Dutch in Japanese mathematics and science also went beyond the practical needs for organizing training. Donker Curtius collected about hundred thirty books authored by *rangakusha* during his stay in Japan.[13] He owned a copy of *Yōsan Yōhō* (nr. 60 in the Catalogue by Serrurier, 1879) as well as a copy of *Seisan Sokuchi* (catalogue nr. 62a). These are the only works on European mathematics in his collection and probably the only works in this category at that time. In 1861 he sold his books to the Dutch government and the collection is now kept at Leiden University Library (Kerlen, 1996). One medical instructor of the first detachment, Van den Broek, collected 27 rare manuscripts and books by *rangakusha* which were only recently rediscovered by Herman J.Moeshart (2003) at the Library of Arnhem in Holland. Both had a reasonable knowledge of the Japanese language. Donker Curtius dispatched a manuscript on Japanese grammar already in 1855 to Hoffmann, who was then a professor of Japanese at Leiden. Van den Broek (1862, 133), who felt sabotaged in his plans to publish his own dictionary, complained about the manuscript that "words were spelled so poorly that it was of no use at all". Hoffmann spent considerable effort to improve it to a publishable work (Donker Curtius, 1857). Apparently there was a demand for Japanese dictionaries and grammars as it was soon adapted to French (Pages, 1861). So, the interest of the Dutch and Japanese in each other's language and knowledge was at least reciprocated.

Before I look at the two works in detail let us remind us the unique historical conditions under which the books appeared. As Sasaki has pointed out at several occasions (Sasaki 1994a, 1994b, 2002), evolutions in mathematics are best understood from the social history in which they occur. The *sakoku* policy already provided us with almost experimental conditions for the study on the contingency of mathematics. The Kanagawa treaty of 1854 effectively ended two centuries of seclusion policy and is an interesting experiment as was the closure of Japan: What is the best way to introduce *yōzan* or Western science and mathematics to Japan? As a comparison, the transition from Roman numerals and calculation with the Gerbertian abacus to the use of Hindu-Arabic numerals took several centuries in Europe. In Japan, the transition from *wasan* to *yōzan* was an even more drastic one and took only some decades to accomplish. The choice for such drastic reform was undoubtedly inspired by the military threat and was part of the transformation of the *bakufu* and *samurai* culture to a modern Japan.

Let us now have a closer look at the two Japanese works and how they relate to Dutch works on arithmetic.[14] It seems that there were two options for a Japanese author of a book on Western mathematics in 1855. Either they approach it from the *wasan* tradition and explain how Western mathematics functions different and relates to existing methods and practices. Or they approach it as *rangakusha* with respect for the original language, terminology and presentation of the Western works. Both these books represent one of these options. From a first comparison it becomes clear that the *Seisan Sokuchi*

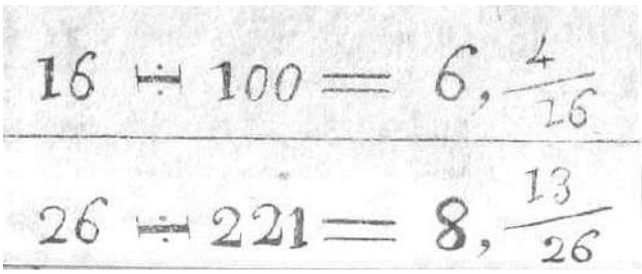
[12]Huyssen van Kattendijke (1860, 74). The story is confirmed by Van der Chijs, (1867, 483).

[13]In this he followed the German physician Philipp Franz von Siebold (1769–1866) who stayed at Deshima from 1823 to 1830, who already brought a large collection of Japanese books and manuscripts (together with paintings, instruments etc.) to Europe. The items are described by J. Hoffmann (1845).

[14]For a modern facsimile edition see Shin'ichi (1979).

is more integrated with *wasan* than the *Yōsan Yōhō* is. The author of the first, Riken Fukuda (1815–1889) of Osaka, was a *wasan* scholar and he made a considerable effort to transliterate Western methods of arithmetic within existing *wasan* knowledge of mathematics. Western numerals and signs for operations are completely absent from the book. Multiplication tables, to be found in every elementary work on arithmetic since Fibonacci's *Liber abbaci*, are presented in the *Seisan Sokuchi* with Japanese numerals. In the *Yōsan Yōhō*, on the other hand, Hindu-Arabic numerals feature prominently in the first section on numeration.[15] Yanagawa Shunsan (柳川春三, 1832–1870) was from a very different background.[16] He was the son of a tool maker in Nagoya, who studied literary Chinese and Dutch at a young age and was reputed for his calligraphy. He moved to Edo and Nagasaki where he worked as a *rangakusha*. In 1861 he moved to the *Kaiseijo* translation bureau in Kanda. Like Horie Kuwajirō, he was interested in photography. In 1864 he was appointed teacher at the *Kaiseijo* and became publisher of the *Shimbun Kaisō*. In 1867 he founded *Seiyō Zasshi*, the first Japanese periodical. With the Meiji restoration in 1868 he became the head of the *Kaiseijo*. Yanagawa also published a Japanese edition of a Chinese translation of a work on Western learning *Zhihuan qimeng* (Elementary lessons in the circle of knowledge) in 1862.[17]

A salient aspect of Yanagawa's book is that throughout the Dutch pronunciation of operations is added in katakana. Such a practice has no function at all for learning Western arithmetic unless ... the teaching would be in Dutch, or is intended for *rangakusha* with some knowledge of Dutch. After numeration it is explained how whole and broken are to be pronounced (*Yōsan Yōhō*, 1857, 11a). This approach runs easily into problems as language conventions do not follow the rules of arithmetic. For example, the number 321 is pronounced in Dutch 'driehonderdéénentwintig' and is in an order different from English or Japanese. It would sound as "three hundred one and twenty". As this would present difficulties, in the book this example is rendered as 3 hundreds, 2 tens and 1 ones (*Yōsan Yōhō*, 1857, 11b).



division in the *Yōsan Yōhō*

Also differences in writing conventions between European and Japanese languages pose problems. As shown in Figure 1, this representation of division would be correct if you read it from right to left: 6, 4/26 is the result of dividing 100 by 16. However, if it is read from the left to the right (or top to bottom as it appears in the book) the statement $16 \div 100 = 6, 4/16$ would be wrong (Yōsan Yōhō, 1857, 12a). The use of a comma is also puzzling. While the comma is used as a decimal point,

[15]Yanagawa Shunsan, *Yōsan Yōhō*, 1857, p. 9a. All references refer to the original edition counting from the title page as 1a, 1b, 2a. . .

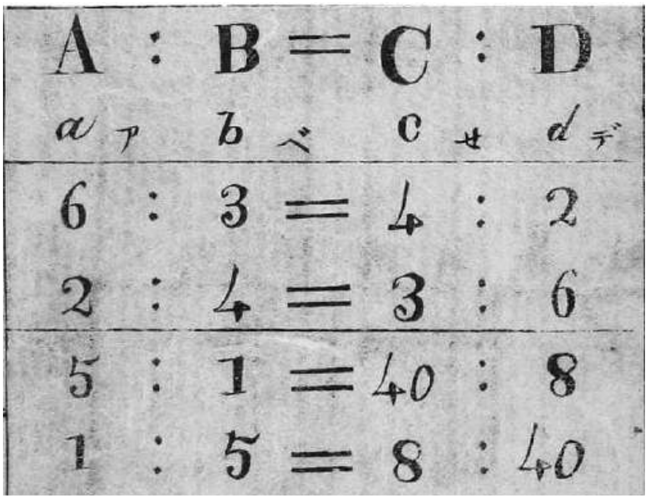
[16]These biographical data are based on Munson (2005). There also exists a Japanese biography of Yanagawa by (1940) of which I did not have the opportunity to consult.

[17]Liu Jianhui (2000), translated by Fogel (2009).

the combination of the comma and fraction does never appear in Dutch arithmetic books.

Not only are the Dutch pronunciations listed for numbers up to a million, also the multiplication tables are transliterated as if they were drilled in Dutch. We find in the book the katakana for “één maal één is één”, “twee maal twee is vier”, “twee maal drie is zes”, “drie maal drie is negen”, “acht maal negen is tweeënzeventig” and “negen maal negen is éénentachtig” (respectively the products 1×1, 2×2, 2×3, 3×3, 8×9, 9×9). Again, the Dutch pronunciation of such tables is irrelevant for learning Western arithmetic unless it is intended to be taught in Dutch. The Dutch names for basic operations of arithmetic and terms such as ‘fractions’, ‘equality’ and so on are given in katakana as well as.[18] One specific term in the long list is peculiar. ‘Eigenlijdigheid’ as such is not a Dutch word. It is derived from the uncommon term ‘eigenlijdig’ which was used only in the medical sense, meaning the local illness of body part which does not affect the other parts.[19] Its use in a book on arithmetic is surprising and may reveal some familiarity of the author with Dutch medical literature.

The faithful rendering of Dutch arithmetic in the *Yōsan Yōhō* had the advantage that Western symbolism was introduced at this early stage. Figure 2 shows the notation for proportions with the use of letters $A : B = C : D$, which are transliterated in katakana (*Yōsan Yōhō*, 1857, 27b). The book is probably the first Japanese work to use the symbol x for the unknown, in relation to proportions: $a : b = c : x$. (*Yōsan Yōhō, 1857, 15a*). The choice for using Western symbols was a deliberate one as we read in the introduction. The author complains about the lack of systematization in *wasan* works: “People find [occidental calculation] difficult to learn because they do not retain the Dutch ciphers But in fact it is much easier to understand than to operate the *soroban* and saves us a lot the trouble of memorization”.[20]



proportions in the *Yōsan Yōhō*

He is possibly referring to the *Seisan Sokuchi* which did not retain the symbolism of Dutch arithmetic. Also interesting is that Yanagawa compares the symbols for the unknown x, y, z with ideograms as shown in Figure 3 (*Yōsan Yōhō*, 1857, p. 8a).

A comparison of the *Yōsan Yōhō* with several Dutch arithmetic books from Hayashi’s list does not

[18]For a discussion of these terms and their use for the official “Government course guidelines for elementary school, Arithmetic” issued later see Kiyosi (1996).

[19]According to the historical dictionary *Woordenboek der Nederlandse Taal*.

[20]Translated from the French from Horiuchi (1996, 260).

reveal any direct source.[21] In all probability, Yanagawa's book is not a translation of a Dutch book on arithmetic but rather a commentary or a collection of notes on Dutch arithmetic. The notes may have been collected while consulting Dutch arithmetic books but it might also possible that these were based on the lessons that were taught at the Nagasaki naval school. I have already situated the book in a context where knowledge of Dutch terms for operations is important and the use of Dutch phrases to memorize multiplication tables. Another indication is the kind of problems we find treated in the book:[22]

> One battle ship in Germany has 80 guns and 2000 crew members. One frigate has 36 guns and 600 crews. How many guns and crew members do five battle ships and 10 frigates have?

With Huyssen van Kattendyke complaining that his Japanese students could not always appreciate the relevance of arithmetic and algebra for the art of navigation, it makes sense to set arithmetical problems within the practical context of naval warfare.[23]

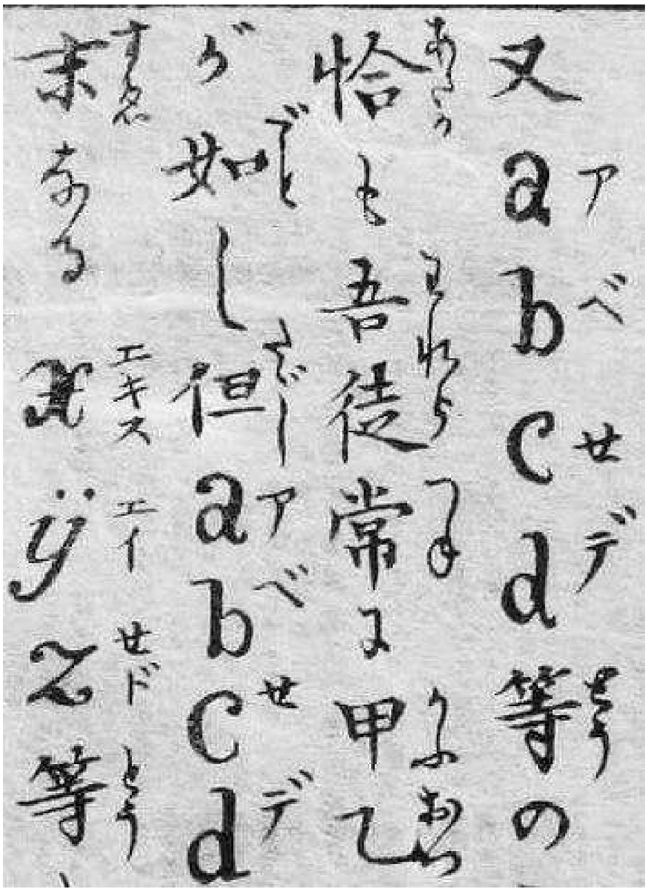
It comes as no surprise that there is a strong parallel between the educational philosophy of the Nagasaki naval school—and consequently early Meiji education—with what Danny Beckers (2003) has called the 'propaedeutic function of mathematics' in the Netherlands during the first half of the nineteenth century. Despite the strong emphasis on mathematics in the Nagasaki naval training program, the intention was not to create able mathematicians but to use mathematics as a basis to transform the samurai mind to modern Western thinking and learning. The main proponents of the new educational program in Holland at that time were the Leyden professor Jacob de Gelder, I.R.Schmidt and Rehuel Lobatto from the Delft polytechnic school and J. Badon Ghijben en H. Strootman from the Breda military academy. These mathematicians produced the textbooks we find used in Japan during the 1850's and trained the teachers of the Nagasaki naval school.[24] Beckers (2003, 238) describes the rise of the propaedeutic function of mathematics as "the successful mix of belief in progress, educational ideas and the rise of modern nations" which thus provides the best fit for the needs of the new Meiji era of the Japanese society. If indeed we may attribute such important influence of the Dutch to education in Japan of the early Meiji period, the relevance of Holland, the Dutch language and *rangaku* rapidly evaporated. While some students of the naval school were sent to Holland for further study, together with instructors of the *Bansho shirabe-sho* such as Nishi Amane and Tsuda Mamichi, they soon found out that the Dutch language was of minor importance in the intellectual, cultural and political centers of the West. Jansen (1958, 595) reports that Matsuki Kōan of the institute complained that he could hardly find any Dutch books and that the "Hollanders themselves all read their books in French and German". With the opening of Japan, two centuries of *rangaku* came to an end and English, French and German books soon replaced the Dutch works as sources of Western learning.

[21] Hayashi (1905) with some additions in Hayashi (1909a, 1909b). In particular I looked at the arithmetical works: de Gelder, J. *Allereerste gronden der cijferkunst*, s'Gravenhage en Amsterdam: Gebr. van Cleef, 1824; Wiskundig Genootschap, *Verzameling van nieuwe wiskundige voorstellen*, Amsterdam: Weijtingh en Van der Haart, vol. 1: 1820, vol. 2: 1846; and the elementary algebra by J. Badon Ghijben, H. Strootman, *Beginselen der Stelkunst, bevattende: De behandeling der stelkunstige vormen , de oplossing der vergelijkingen van den eersten en tweeden graad, De treorien der gewone logarithmen, reken –en meetkunstige reeksen en kogelstapels*. Breda: Koninklijke Militaire Academie, 1840 (2nd ed.).

[22] *Yōsan Yōhō* (1857, 34b). The problem is quoted by Shigeru (2000, 446).

[23] Huyssen van Kattendijke (1860). I have no evidence that this problem is derived from the *Kaigun Denshu-sho* but as the available Dutch books do not contain such examples and its first appearance in a book of 1857 while the lessons took place in Nagasaki makes a connection very likely.

[24] Pels Rijcken, the main instructor in mathematics and officer in charge of instruction of the first detachment, was trained at the Breda military academy.



the introduction of literal symbolism in the *Yōsan Yōhō*

3 Conclusion

The confrontation with Dutch mathematics at the end of the Tokugawa period raises the issue how foreign knowledge can and should be incorporated within existing traditions. Of the possible strategies of adaption, integration or replacement the Meiji regime chose drastically for the latter one, abandoning its own rich and flourishing *wasan* tradition. The possible choices are exemplified by the first two Japanese works on Western mathematics. The *Seisan Sokuchi* tried to adapt Western procedures to *wasan*. Yanagawa criticized such approach and intended his *Yōsan Yōhō* to be a faithful rendering of Dutch methods and procedures but also terms and symbolic notations. I have situated this revolution in Japanese mathematics within the context of foreign threat and the development of a Japanese naval force. The Dutch played a major role in placing education in Western mathematics as a condition for their support to this enterprise. I have pointed out several connecting lines between the Nagasaki naval training program and the newly established *bansho shirabe-sho* institute. The educational approach taken in the Dutch books used for the training program fitted very well the ambitions of the Meiji regime. The influence of Dutch learning in the latter years of the Edo period may have been greater than believed.

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REFERENCES

– Arima, S., 1964, "The Western Influence on Japanese Military Science, Shipbuilding, and Navigation", *Monumenta Nipponica*, **19**, (3–4), 1964, p. 352-379.

– Beckers, D., 2003, "*Het despotisme der mathesis*", *Opkomst van de propaedeutische functie van de wiskunde in Nederland 1750–1850*, (Dutch) Ph.D. Dissertation, Nijmegen University.

– Donker Curtius J. H., 1857, *Proeve eener Japansche spraakkunst, toegelicht, verbeterd en met uitgebreide bijvoegselen vermeerderd door dr. J. Hoffmann*, Leiden: A. W. Sythoff.

– Fogel, J. A., 2009, "Demon Capital Shanghai: The "Modern" Experience of Japanese Intellectuals", *Sino-Japanese Studies*, 16, pp. 126–145.

– Fujisawa R., 1912, *Summary Report on the Teaching of Mathematics in Japan*, Tokyo: International Commission on the Teaching of Mathematics.

– Fukagawa, H. and Rothman, T., 2008, *Sacred mathematics: Japanese temple geometry*, Princeton NJ.: Princeton University Press.

– Hayashi, T., 1905, "A list of Dutch books on mathematical sciences imported from Holland to Japan before the restoration in 1868", *Nieuw Archief voor Wiskunde*, serie **2**, 7, pp. 232–238.

– Hayashi, T., 1909a, "Some Dutch books on mathematical and physical sciences imported from Holland to Japan before the restoration in 1868", *Nieuw Archief voor Wiskunde* serie **2**, 9, pp. 39–41.

– Hayashi, T., 1909b, "How have the Japanese used the Dutch books imported from Holland?", *Nieuw Archief voor Wiskunde*, serie **2**, 9, pp. 42–48.

– Hoffmann, J., 1845, *Catalogus librorum et manuscriptorum Japonicorum a eo collect. annexa enumeration illorum, qui in Museo Regio Hagano servantur*. Leiden, 1845

– Hommes, J. M., 1993, The *Bansho shirabesho: A Transitional Institution in Bakumatsu Japan*. Bachelor thesis. University of Pittsburg.

– Horiuchi, A., 1994, *Les Mathematiques Japonaises a L'Epoque d'Edo (1600–1868): Une Etude des Travaux de Seki Takakazu (?–1708) et de Takebe Katahiro (1664–1739)*. Paris: Librairie Philosophique J. Vrin.

– Horiuchi, A, 1996, "Sur la recomposition du passage mathématique japonais au début de l'epoque Meiji", in Goldstein, C., Gray, J ; and Ritter J. *L'Europe mathématique: histoires, mythes, identités*, Paris : Fondation Maison des sciences de l'homme, 1996, pp. 247–270.

– Horiuchi, A., 2010, *Japanese Mathematics in the Edo Period (1600–1868. A Study of the Works of Seki Takakazu (?–1708) and Takebe Katahiro (1664–1739)*, Science Networks. Historical Studies, Vol. 40, Basel: Birkhauser.

– Huyssen van Kattendijke, W.J.C., 1860, *Uittreksel uit het dagboek van W.J.C. Ridder Huyssen van Kattendijke, gedurende zijn verblijf in Japan in 1857, 1858 en 1859*, 'S Gravenhage: W.P. van Stockum.

- Jansen, M. B., 1958, "New Materials for the Intellectual Study of Nineteenth-Century Japan", *Harvard Journal of Asiatic Studies*, **20** (3–4), p. 567–597.
- Kerlen H., 1996, *Catalogue of pre-Meiji Japanese books and maps in public collections in the Netherlands*, Japan Neerlandica 6, Amsterdam: J.C. Gieben.
- Kiyosi, Y, 1996, 洋算用法"における蘭和数学用語記号について (On Mathematical Terms and Notations in Dutch and Japanese in the "Yosan Yoho" or Usage of Western Arithmetic) (1857) Shunsan Yanagawa) *Journal of the Faculty of International Studies of Culture, Kyushu Sangyo University*, 5, pp. 137–152.
- Liu Jianhui, 2009, *Mato Shanhai: Nihon chishikijin no "kindai" taiken* (Tokyo: Kōdansha, 2000), pp. 94–127. Translated by
- Mikami, Y., 1909, "Remarks on T. Hayashi's 'Brief History of Japanese Mathematics' " *Nieuw Archief voor Wiskunde* Serie **2**, 9, pp. 373–386.
- Mikami, Y., 1913, *The development of mathematics in China and Japan*, Leipzig: Teubner.
- Moeshart, H. J., 2003, *Guide & Concordance To Nineteenth-century Dutch/Japanese Japanese/Dutch Manuscript Dictionaries and Related Documents: The J .K. Van Den Broek Collection*, Arnhem: Arnhem City Library, Gelderland Documentation Centre.
- Munson, T.S., 2005, "The Procurement of Rarities is a Sign of Peace: Ideological Resistance to Imperialism in Yanagawa Shunsan's *Yokohama hanjoki*," in *Early Modern Japan: An Interdisciplinary Journal*, 13, pp. 1–10.
- Osatake Tateki, 1940, *Shinbun zasshi no soshisha Yanagawa Shunsan* (*Yanagawa Shunsan, Founder of a Newspaper and a Magazine*), Tokyo: Takayama shoten.
- Pages, L. (ed. tr.), 1861, *Essai de grammaire japonaise composé par M.J.H. Donker Curtius, enrichi d'éclaircissements et d'additions nombreuses par M.J. Hoffman*, Paris: Remquet, Coupy et Cie.
- Pompe van Meerdervoort, J.C.L., 1867–1868, *Vijf Jaren in Japan (1857–1863) Bijdragen tot de kennis van het Japansche Keizerrijk en zijne bevolking*. Van den Heuvel en Van Santen, Leiden (2 vols.).
- Pompe van Meerdervoort, J.C.L. "Verslag over de gouvernements geneeskundige dienst op het eiland desima en in Japan, over 1857 en 1858", *Geneeskundig tijdschrift voor Nederlandsch Indie*, Lange en Co., Batavia,1859, pp. 495–572.
- Pompe van Meerdervoort, J.C.L., Matsumoto R., Westendorp, R. 1998, *3 cholera epidemics in Japan, 1858–1859–1862: a description*, Leiden: De Baaierd.
- Ravina. M. "Wasan and the Physics that Wasn't. Mathematics in the Tokugawa Period", *Monumenta Nipponica*, 48, 2, (Summer, 1993), pp. 205–224
- Sasaki, C., 1994a, "The history of Japanese mathematics with the Adoption of Western Mathematics in Meiji Japan, 1853–1905", in C. Sasaki, M. Sigiura, J. W. Dauben (eds.) The *Intersection of history and mathematics*, Basel: Birkhauser, pp. 165–186.
- Sasaki, C., 1994b, "Asian Mathematics from Traditional to Modern", *Historia Scientiarum*, **4**, (2), pp. 69–77.
- Sasaki, C., 2002, "The Emergence of the Japanese Mathematical Community in the Modern Western Style, 1855–1945" in K. H. Parshall, A. C. Rice (eds.), *Mathematics Unbound: The Evolution of an International Mathematical Community, 1800–1945*, Providence, RI.: American Mathematical Society, pp. 229–252.
- Serrurier, L., 1882, *Verzameling van Japanse boekwerken door mr. J.H. Donker Curtius, Nederlands Commissaris in Japan, op zijne reid naar Yedo in 1858 voor het rijk ingekocht. Beschreven door wijlen dr. J.J. Hoffmann*, Leiden: Leiden University Library.
- Shigeru, J., 2000, "The Dawn of Wasan (Japanese Mathematics", in H. Selin, U. d'Ambrosio, *Mathematics Across Cultures: the History of Non-Western Mathematics*, Dordrecht: Kluwer Academic, pp. 423–454.

– Shin'ichi, Ōya, 1979, (ed.) *Seisan sokuchi, Yōsan Yōhō*, Tokyo: Kochi Shuppan.

– Smith, D. E. and Mikami Y. *A History of Japanese Mathematics*. Chicago: The Open Court Publishing, 1914. (New York: Dover books, 2004).

– Smith, T. C. 1948, "The Introduction of Western Industry to Japan during the Last Years of the Tokugawa Period", *Harvard Journal of Asiatic Studies*, **11** (1/2), pp. 130–152.

– Van den Broek, K.J., 1862, "Nederland en Japan. Kantteekeningen bij officieusen tekst", *De Tijdspiegel*, 1862, pp. 126–135.

– Van der Chijs, J.A., 1867, *Neêrlands streven tot openstelling van Japan voor den wereldhandel: Uit officieele, grootendeels onuitgegeven bescheiden toegelicht*, Amsterdam: Frederik Muller.

REMARK ON THE NOTION OF GOLDEN RATIO*
—concerning "Divine Proportion" in the Renaissance period—

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ABSTRACT

In this paper, we try to show diverse aspects of the notion of Golden Ratio seen in its historical development. This notion should be discussed not only from a mathematical viewpoint, but also from various viewpoints based on human cultural activities and natural phenomena. Through the discussion we try to suggest that this notion could be understood in wider and deeper context and even to stimulate mathematics teachers' interests in the notion. And especially, we try to point out the fact that this notion was called "divine proportion" in the Renaissance period and that this name (term) would be derived from the collaborative study of Leonardo da Vinci and Luca Pacioli. Leonardo da Vinci might be the first to introduce this terminology in the "*Paragone*" of his book "*Libro di Pittura*" (on Painting). This fact is considered to be an aspect in which mathematical notions and human activities have a close relation to each other. Mathematics is a product of human wisdom, to be sure, and therefore mathematical notions should be the product of human activities. But nowadays, we are apt to perceive mathematics to be a conceptual discipline, which has been formed by cutting off many concrete human-cultural parts and by rearranging the remaining conceptual parts into a logical and concise system. Thinking so, Leonardo da Vinci's "divine proportion", as well as the notion of Golden Ratio, could be considered to be one of the most appropriate examples to mathematics education, in order to understand the original features of mathematics related to human life and culture. This kind of fact might be profitable for teachers who want to encourage the students in the mathematics classroom.

1 Introduction

The Golden Ratio might be one of the most fecund notions in the history of mathematics, and therefore, it should be also one of the most applicable notions to mathematics education in classroom. In fact, we could find various aspects concerning the Golden Ratio in historical steps of human activities; in the development of mathematics, especially geometry, in human cultural activities, for example the fine art of the Renaissance period, and even in the human efforts to explain some principles seen in Nature. The various factors of these kinds might give us lots of topics, which are interesting and even sometimes mystic, for mathematics education.

*This paper has been read on the occasion of the 4th Conference on History and Pedagogy of Mathematics in Shanghai, China in 2011, and was ameliorated with more discussion.

Nevertheless, despite of these possibilities to apply this notion into mathematics classrooms, many mathematics teachers seem to be apt to depend on fragmentary topics concerning this notion. Actually, so many books and articles are published, to report and discuss diverse aspects of the Golden Ratio, on the one hand. Much of them should be written with the quite precise and steadfast intention of discussing various factors related to this notion as mentioned above. But, on the other hand, there is no denying that many topics can be taken up without full context but separately and fragmentarily in mathematics classroom. It is because the fragmentary topics on the Golden Ratio might be susceptible for students' understanding and also enable teachers to encourage easily the students in their interests and curiosity. This can be true to a certain extent, and however, it is better and even effective for mathematics education that teachers should try to understand this notion as well as possible; for, from a historical viewpoint, the notion of Golden Ratio should be deeply related to human thoughts and activities for the objects in the world. This is also related essentially to the aim and the significance of mathematics education.

In this paper, we try to show diverse aspects of the Golden Ratio seen in its historical development, to suggest that this notion should be discussed in wider and deeper context and even to stimulate mathematics teachers' interests in the notion. And especially, we try to point out the fact that this notion was called "divine proportion" in the Renaissance period and that this name (term) would be derived from Leonardo da Vinci's "*Paragone*" of his book "*Libro di Pittura*" (on Painting).

2 Brief history of the Golden Ratio

The Table 1 shows an example of the brief history of the Golden Ratio. Especially, this notion became related to Fibonacci number; it originated in Fibonacci's "Liber Abaci" (The Book of Abacus) written at the beginning of the 13^{th} century, but the close linkage between the two notions had been found and discussed from the 18^{th} century onward.

Considering the history of human efforts to investigate the Golden Ratio, we can find, from a viewpoint of harmony or mystery possessed in the notion, the three categories of human understanding, as follows:

(a) - harmony and mystery in mathematics

(b) - harmony and mystery in human cultural activity(art, etc.)

(c) - harmony and mystery seen in Nature.

(a) Harmony and mystery in mathematics The Golden Ratio is, of course, a mathematical notion, and many studies have been done over its historical development. It is because this notion possesses various properties, which are mathematically interesting and even mystic. For example, Pythagorean School is said to take notice of a regular pentagon and to adopt a pentagram as their symbol. It is well-known that the ratio between the diagonal and the side of the regular pentagon is equal to the

[1]Cf. Sculptures: "*Diadumenus*"(B.C. 430, "*Man Tying on a Fillet*"), and "*Doryphorus*" (c. B.C. 450, "Spear Bearer")

[2]The book contained the following recreational problem: "How many pairs of rabbits can beproduced from a single pair in one year if it is assumed that every month each pair begets a new pair which from the second month becomes productive?"

[3]For two consecutive Fibonacci numbers, f_n and f_{n+1}, the ratio f_{n+1}/f_n approaches $\phi(\phi = 1.6180\ldots$, the golden ratio) as n becomes larger.

B.C. 6C.–5C.	- Pythagorean School: Study on Pentagon (Pentagram) and on the notion of incommensurability. - Polykleitos : "*Canon of human proportion*," which is considered as a theoretical work that discusses ideal mathematical proportions for the parts of the human body[1].
c. B.C. 300?	Euclid: "*Elements*." Introduction of the notion of "*extreme and mean ratio*," which signifies the now called Golden Ratio.
B.C. 1C.–1C.	Vitruvius: "*De architectura*" (On Architecture), in which Vitruvius discussed the human proportion by employing a square and a circle as a frame of an ideal human body.
1202	Fibonacci: "*Liber Abaci*" (The Book of Abacus). Introduction of the Fibonacci numbers[2].
c. 1482	Piero della Francesca: "*De quinque corporibus regularibus*" (On the Five Regular Bodies), a manuscript which was not published at the time.
c. 1490	Leonardo da Vinci: Sketch of "*Vitruvian human proportion*." Leonardo da Vinci: "*Libro di Pittura*" (On painting).
1498	Luca Pacioli: "*De Divina proportione*" (Divine proportion), a manuscript. But the version published in 1509 was in the form of printing.
1525	Albrecht Durer: "*Underweying der Messung mit dem Zirckel und Richtschyt*" (Treatise on measurement with compass and ruler). Dürer introduced implicitly the notion of the logarithmic spiral.
1638	René Descartes: Study on the logarithmic spiral which Descartes called the "*equiangular spiral*."
1753	Robert Simson: Study on the relation between the Golden Ratio (ϕ) and Fibonacci numbers[3].
1835	Martin Ohm: "*Die Reine Elementar-Mathematik*" (The pure elementary mathematics), in which the term of "Golden" Section was first used.
1830s	Alexander Braun, Auguste Bravais & his brother Louis Bravais: Discussion on the Fibonacci sequence seen in diverse botanic Phenomena (seeds of a sunflower, a pinecone, a kind of leaf arrangement, etc).
1843	J. P. M. Binet: Introduction (reintroduction) of the formula to gain any Fibonacci number; the formula is given by using ϕ.
1914	Thomas Cook: "*The Curve of Life*." Discussion many types of spirals in Nature, including the spiral of Nautilus shell.
1942	D'Arcy Thompson: "*On Growth and Form*" (first published in 1917). Introduction of the term "homogeneity" (self-similarity), which concerns with the Fibonacci numbers and the Golden Ratio.

Brief History of the Golden Ratio

Golden Ratio ($\phi = 1.6180\ldots$). In addition, as to all the segments seen in a pentagram, the ratio of every two segments taken in order of length is equal to this ratio. And, when the five diagonals are drawn in a regular pentagon, they will form another regular pentagon in the center; this operation can be continued without end. Therefore, the Golden Ratio can be represented with only the unit "1" (in the form of a continued fraction, $\phi = \frac{1}{1+\frac{1}{1+\frac{1}{\cdots}}}$). These properties of this ratio indicate that it possesses a kind of harmony in mathematics. According to this fact, the Golden Ratio also creates us some kind of mystic atmosphere.

(b) Harmony and mystery in human cultural activity In fact, the Golden Ratio has affected human mind. It stimulated the various artists to pursue their ideal harmony and beauty. Such was the case especially in the Renaissance period. For example, lots of studies on Leonardo da Vinci' drawings have been done from a viewpoint of this notion. Considering Leonardo's *dessin* of human proportion (*Vitruvian human proportion*), it is often said that the position of navel might indicate the Golden Section of the human height (between the top of a head and the sole of a foot). This idea has been still discussed even in the 19^{th} century, by the German psychologist Zeising[4]. We can find other example of this kind.

(c) Harmony and mystery seen in Nature The Golden Ratio often appears with various forms in Nature. For example, by investigating the features of the spiral that the *Nautilus* shell creates during its growth, we can find that the spiral is almost logarithmic and that the coefficient of the growth is related to the Golden Ratio. Another example is seen in the arrangement of sunflower seeds. The seeds of the sunflower spiral spread outward in both clockwise and counterclockwise directions from the center of the flower. The numbers of clockwise and counterclockwise spirals are considered to be two consecutive numbers in the Fibonacci sequence. Nowadays, we can find many cases related to the notion of Golden Ratio in the morphologic study on the nature. This fact impresses us the harmony and the beauty of the nature.

3 The name of "Divina Proportione" written by Luca Pacioli

Although the notion of the Golden Ratio is quite fecund and popular for many teachers and students, it is possible that an essential question will arise in human mind; it concerns the name of the notion. From a viewpoint of harmony and mystery, "Golden Ratio" should be just the right name, i.e. ideal and even perfect for naming. Then, who named it first? It is not so easy to clarify, because we should be confronted at least with the three different names called to each period in its history: "*extreme and mean ratio*," "*divine proportion*" and "*golden ratio*."

It is perhaps Euclid (c. B.C. 300) who was the first to define explicitly the notion. In the Book VI of his "*Elements*," he gives the definition, as follows:

Definition 3 A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less[5].

[4] About Zeising's discussion, for example see below: Adolf Zeising, *Neue Lehre von den Proportionen des menschlichen Körpers, aus einem bisher unerkannt gebliebenen, die ganze Natur und Kunst durchdringenden morphologischen Grundgesetze*, Leipzig, Rudolph Weigel, 1854

And in Proposition 30, he also shows the geometrical construction of this ratio[6]. This definition means the golden section. It is considered as a quite mathematical definition which formally depends on geometry. In addition, in the Book XIII, he discusses the five regular polyhedrons, especially a regular dodecahedron with the notion.

On the other hand, the name of "Golden Ratio" appeared much later, in the 19^{th} century. Johannes Tropfke, a German mathematician, suggested this matter in his book "*Geschichte der Elementar-Mathematik*" (History of elementary mathematics). According to Tropfke, it was the German mathematician Martin Ohm who first adopted the name of "Golden Section" (not precisely "Golden Ratio") in his second edition of "*Die Reine Elementar-Mathematik*" (The pure elementary mathematics) published in 1835. Ohm said that "one also customarily calls this division of an arbitrary line in two such parts the golden section[7]." Tropfke also noted there were other mathematicians who used this name at that time. So it might be certain that the name of "Golden Ratio" was used and established around in the 1830s and in the 1840s.

Then, what about the name of "divine proportion?" It is known that this naming appeared in the Renaissance period. It was perhaps Luca Pacioli, an Italian friar and mathematician, the first one to use it, when he published a book with the name of "divine proportion" as a title. In fact, he wrote "*De Divina Proportione*" (Divine Proportion)[8] in 1498, and presented the manuscript to the grand Duke of Milan. And in 1509, he published the book of the same title in the form of the letterpress printing.

In his "*De Divina Proportione*", Pacioli treats various kinds of polyhedrons, but the discussion centers round on the five regular polyhedrons. In addition, the figures of various kinds of polyhedron are attached at the end of this book, and Leonardo da Vinci drew them. We should notice, here, that a regular dodecahedron is formed with twelve regular pentagons as its faces and that the Golden Ratio can work as an important factor in the relation between the regular polyhedrons: for example the relation between an octahedron and an icosahedron[9].

At the starting point, Pacioli, by himself, adopted Plato's idea on the five regular polyhedrons. In "*Timaeus*"[10] Plato had considered that the five regular polyhedrons correspond to the four elements of the world and the fifth essential element; Earth corresponds to a cube, Water to an icosahedron, Air to an octahedron, Fire to a tetrahedron, and finally, the fifth essential element, i.e. the sacred Cosmos, to a dodecahedron. On the basis of this idea, Pacioli developed his argument that the ratio appearing in a pentagon was considered as an important an even noble factor to discuss polyhedrons. Therefore, for Pacioli, such a ratio should be called "divine proportion".

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[5] Thomas L. Heath, "*The Thirteen Books of Euclid's Elements*," vol. 2, Dover, 1956, p. 188.

[6] Ibid., pp. 267–268. "To cut a given finite straight line in extreme and mean ratio."

[7] Johannes Tropfke, *Geschichte der Elementar Mathematik in systematischer Darstellung*, Berlin und Leipzig, 1923, Band 4, p. 187.

[8] Luca Pacioli, *De Divina Proportione*, Introduzione di Augusto Marinoni, Silvana Editoriale, Milano, 1986. See also French translation, Divine proportion, (Traduction de G. Duchesne et M. Giraud avec la collaboration de M.-T. Sarrade), Librairie du Compagnonnage, 1988.

[9] For example, see H. S. M. Coxeter, *Introduction to Geometry*, Second Edition, John Wiley & Sons, Inc., 1989.

Coxeter writes: "The faces surrounding a vertex of the icosahedron belong to a pyramid whose base is a regular pentagon. Any two opposite edges of the icosahedron belong to a rectangle whose longer sides are diagonals of such pentagons......this rectangle is a golden rectangle...". Therefore, "the twelve vertices of the icosahedron are the twelve vertices of three golden rectangles in mutually perpendicular planes". Considering that a golden rectangle can be inscribed in a square so that each vertex of the rectangle divides a side of the square in the golden ratio, an octahedron can be also constructed with these three rectangle mentioned above (pp. 162–163).

[10] For example, see Plato, *Timeaeus and Critias*, Penguin Books, 1977, pp. 73–77.

4 Concerning the name of "Divine Proportion"

As mentioned above, Pacioli's "*De Divina Proportione*" seems to be the first book with this name as a title. But Pacioli seems not to be the first to call "divine proportion" to the ratio in question. To investigate the problem, we should look around the situation at that time.

At first, we should pay attention to the fact that Pacioli's "*De Divina Proportione*" is said to be a kind of plagiarism. Pacioli seemed to plagiarize Piero della Francesca's writing on the same subject, "*De quinque corporibus regularibus*" (On the Five Regular Bodies)[11]. But in the latter book, although Piero treated the five regular polyhedrons, he did not use the term "Divine Proportion". Secondly, we should also take notice of Pacioli's book "*Summa*"[12] written in 1494. In this book, Pacioli discusses geometry by presenting certain of the books of Euclid's "*Elements*." But in the part correspondent to the Book VI of the "*Elements*," Pacioli did not use the term "divine proportion," but the same term as Euclid. Consequently, it is suitable to think that Pacioli and Piero have not used yet the term "divine proportion" before 1498.

On the other hand, we can find the term "divina proportione" (divine proportion) in the writing of Leonardo da Vinci, precisely in the "Paragone" of his "*Libro di Pittura*" (On painting)[13]. This is Leonardo's posthumous work rearranged from the manuscripts by his disciple Francesco Melzi[14]. But, since Pacioli mentioned Leonardo's work on painting in his book "*De Divina Proportione*," Leonardo had already written some parts of "Libro" in 1509. In it, the term "divina proportione" appears three times, and all of them correspond with harmony and beauty which are kept within the sublime deity. Some studies suggest that Leonardo's term "divina proportione" signifies, in itself, the mathematical meaning of the Golden Ratio[15], and however, this insistence might be still disputable. Anyway, it is certain that Leonardo used the term.

Moreover, we should also notice that Pacioli and Leonardo had a close acquaintance with each other. Since 1481, Leonardo has resided in Milan, where Pacioli was assigned in 1496. The two deepened their friendship and perhaps they might exchange their discipline. Therefore, it is possible to consider that Pacioli's book "*De Divina Proportione*" might be the result of their collaboration.

Considering such a situation, we could frame a hypothesis. It would be Leonardo who first introduced the term "divine proportion" in his "*Libro*" around in the 1490s. Although Leonardo's term would be based on his philosophical idea, it might not be appreciable whether it contains the mathematical meaning. Pacioli might be influenced by Leonardo's idea through their collaboration, and finally the former would adopt the term "divine proportion" including its philosophical meaning.

In consequence, Pacioli achieved his "*De Divina Proportione* as a mathematical treatise on the regular polyhedrons with the notion of "Golden Ratio" and it should be supported with Leonardo's idea and terminology (see Fig. 1).

[11] Piero della Francesca, *Libellus de quinque corporibus regularibus*, Giunti Gruppo Editoriale, Firenze, 1995.

[12] Luca Pacioli, *Summa de arithmetica, geometria, proportioni et proportionalita*, Reprint (Originally published,Venice, 1494), Kyoto, Daigakudo Books, 1973.

[13] Leonardo da Vinci, *Libro di Pittura*, Giunti Gruppo Editoriale, Firenze, 1995.

[14] Concerning Melzi's effort to compile Leonardo's manuscripts on Painting, see for example: *Leonardo on Painting, edited by Martin Kemp*, Yale Nota Bene, 2001.

[15] For example, Soichi Mukogawa, a Japanese historian of Art, discusses this matter in his article as follows: Soichi Mukogawa, On the Meaning of Leonardo da Vinci's 'divina proportione' in His 'paragone', *Bijutushi*, 51(2), 2002, pp. 282–296 (in Japanese).

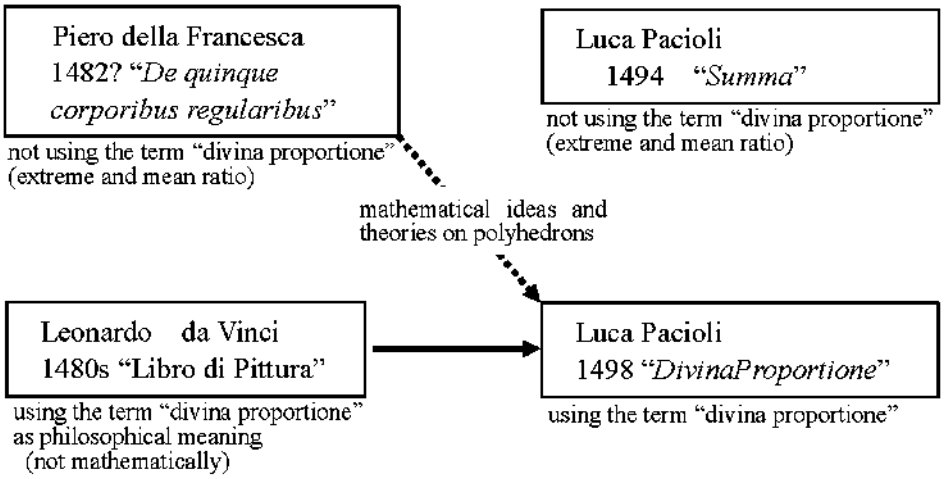

Fig. 1

5 Conclusion

The "Golden Ratio" is certainly a mathematical notion, but it has also influenced various aspects of human activities through its history; i.e. development of mathematics in itself, human artistic activities and human understanding of Nature. The name of this notion was fixed at least in the 19th century after many twists and turns. And the name of "divine proportion" introduced in the Renaissance period seems to symbolize the significance of the notion; for it has provided us for some conceptual factors of harmony and mystery.

In this paper, we tried to suggest that it would be Leonardo da Vinci who first introduced the term "divine proportion." It might be the first hypothesis in the discussion on the development of the Golden Ratio. The fact seems also to symbolize the period, because in the Renaissance period, mathematics was considered to be much linked with human activities and to developed practically rather than theoretically. J. V. Field says that "the name Renaissance is really a label for a style not for a period"[16] In fact, the Renaissance mind pushed up the significance of the notion to the level of harmony, beauty, mystery and even divinity.

It is certain that historical topics are applicable for mathematics classroom, because they can give us lots of interesting and attractive contents. But it is also true that we should remember keep in our mind that the history of mathematics contains various kinds of factors, because it is also the history of human thoughts and activities.

[16] J. V. Field, *The invention of infinity*, Oxford University Press, 1997, p. 1.

EUCLID'S PROPOSITION II.5: A VIEW THROUGH THE CENTURIES

Geometry, Algebra and Teaching

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ABSTRACT

Book II of Euclid's *Elements* has played an important role in the historiography of Greek mathematics. The main reason for this is that its propositions express geometrical results that are easily reformulated in modern algebraic symbolism. This has given rise to a well-known historiographical debate on whether Euclid's original conception behind Book II was purely geometrical (and hence interpreting it in algebraic terms is anachronistic) or if, rather, it was algebra written in the language of geometry (and hence it can be seen as be characterized as "geometric algebra").

Beyond the historiographical debate, a look at the ways in which Book II was presented in the various editions of the *Elements* starting from the Middle Ages and up to the late 19th century shows that not only historians, but also mathematicians trying to come to terms with its contents, looked at this book in different ways, concerning the roles of algebra and geometry in the results presented in it. Thus, Book II of the *Elements* offers a unique point of view from which to consider in historical perspective the changing relationship between geometry and algebra.

More specifically, one illuminating way to understand this process is to focus on a specific result of Book II, Proposition II.5, and to analyze the metamorphoses underwent by it, since the time of classical Greek mathematics to the early twentieth century. In this talk we take a guided tour that highlights the ways in which changing views about the interrelations between algebra and geometry in different mathematical cultures may influence the multifarious interpretation given to one and the same result. Particular attention is paid to the ways in which the uses of the *Elements* as a textbook for the study of elementary mathematics affected this issue. By analyzing selected texts produced in changing historical contexts, it is shown that, while symbolic manipulation and other mathematical ideas that we typically associate with algebra were incorporated in various ways to proofs of II.5 already beginning with the Greek commentators of Euclid, none of these additions or their combination did ever imply a definite change of orientation that all subsequent authors felt compelled to follow. At various times and up until the nineteenth century, one can still find mathematicians who preferred, for different reasons and in changing circumstances, to move back and forth from a purely geometrical to a more algebraically-oriented approach to Book II of the *Elements*, and particularly to II.5.

Among the mathematicians whose versions of II.5 we explore are Heron, Ibn-Qurra, Gersonides, Clavius, Barrow and Wallis.

THE NEW *CURRICULUM STANDARD* AND THE NEW MATHEMATICS

the Union of History of Mathematics and Mathematics Education

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ABSTRACT

On the background of the history of mathematics with mathematics education, this paper explains the history of mathematics and mathematical cultural that was increased in the new "Curriculum Standard". It also discusses the way for the integration of the history of mathematics and mathematics education. It also points the integration can be regarded from mathematics cultural perspective and mathematics quality education. As the Pythagorean theorem in the East-West formulation and application of mathematics for example, this paper elaborates the related teaching mentality. We should change the traditional mathematics values while hand down a tradition culture essence, so that we can effectively merges the history of mathematics and mathematics education into one organic whole. For this, we can achieve the purpose of understanding mathematics the long and short of the story, the widespread application as well as the prospects for development.

Keywords: mathematics curriculum standards; history of mathematics; mathematics education; teaching reform

At the beginning of the new century, Chinese Ministry of Education has promulgated two new "Math Curriculum Standards": "full-time compulsory education in mathematics curriculum standards (trial version)" and "high school mathematics curriculum standards (trial version)" (this paper referred to as "Curriculum Standards"). One of the significant changes is to update new teaching thought and teaching content, especially the history of mathematics and mathematics education will be closely integrated to form a new model of mathematical education. This is a landmark in the history of mathematics education in China.

1 The history of HPM in China

HPM is a shorter form of "International Study Group on the Relations between the History and Pedagogy of Mathematics". [Ou Shifu, 2003]HPM has its corresponding organizations, conferences and journals.April 1998 by the International Commission on Mathematics Education (ICMI) was launched, HPM hosted "The Role of History of Mathematics in Mathematics Education" International Symposium in Marseilles, France. The delegates from China Mainland, Hong Kong and Taiwan attended the meeting. [Zhang Dianzhou, 2009]Hong Wansheng, Taiwan Normal University Professor founded the Chinese-language publications "HPM Newsletter" in October 1998, published with the history of mathematics and mathematics education-related news and papers, 2010, 13 volumes have been published more than 100 period.

The history of the HPM in Chinese mainland can be traced back in 1996. “The 3th International Symposium On the History of Mathematics and Mathematical Education Using Chinese Characters” held in Hohhot, Inner Mongolia Normal University. This is the first academic conference for the history of mathematics and mathematics education opened together. However, it is the only form of connection, the contents of the vast majority papers presented at the meeting were pure history of mathematics. In 2002 ”The 5th International Symposium on the History of Mathematics and Mathematics Education Using Chinese Characters” held in Tianjin Normal University. Although most of the papers’ content in this conference were the history of mathematics, but more than 10 articles involved combination of history of mathematics and mathematics education. In May 2005 “The 1st National Conference on History and Pedagogy of Mathematics” held in Xi’an, Northwestern University. It opened officially the curtain of HPM professional meetings. This conference theme is to explore how to combine the history of mathematics and mathematics education, and how to use the history of mathematics in mathematics teaching. Many papers presented at the meeting is to explore the combination and application for history of mathematics and mathematics education, which involving higher mathematics, secondary mathematics, and elementary mathematics education. This conference has produced a great response in the history of mathematics and mathematics education community.

In April 2007, “The 2nd National Conference on History and Pedagogy of Mathematics” was held in Hebei Normal University. It shows a clear sign of the HPM. 180 delegates attended the meeting were continue to build a friendship bridge between the history of mathematics and mathematics education. They created a mutual benefit and win-win favorable situation. In May 2009, “The 3rd International Conference on History and Pedagogy of Mathematics” held in Beijing Normal University, it extended conference to the international scope. The total number of participants and the papers has increased. The important part of the papers is to study history of mathematics effective use in mathematics education, and interpretation of HPM in the actual case in the theory and methods. The integration of history of mathematics and mathematics education has been recognized by mathematicians and mathematics educators. It becomes as a new development model.

In May 2011, “The 4th National Conference on History & Pedagogy of Mathematics” held in Shanghai East China Normal University, participants include representatives from Canada, Germany, Korea, Japan, India and other international experts in history of mathematics and mathematics education. A total of 224 domestic experts and scholars and more than 130 articles in Chinese and English was presented in the meeting. The meeting focused on the theory and practice of HPM, colleges curriculum for history of mathematics and mathematical cultural topics. It shows the fruitful results and future prospects of HPM.

2 Improvements of new “curriculum standards”

In 2001, the China Ministry of Education issued “full-time compulsory education in mathematics curriculum standards (trial version)”. Then they made a revision by the opinions during practice in 2007. By the year of 2003, a new “ high school mathematics curriculum standards (trial version)” [The people’s Republic of China Ministry of Education, 2003] was promulgated, the basic philosophy emphasizes: offers a variety of courses to meet individual choice; advocate actively for the exploration of learning styles; focus on improving students' mathematical thinking; to develop students awareness of the mathematics application; keeping pace with the times to understand the "double base"; stressed nature, attention to appropriate formal; reflect the cultural values of mathematics; focus on the integration of information technology and mathematics curriculum.

This corresponds with the new "curriculum standards" of course a significant adjustment: to reduce the required courses, a significant increase in elective courses. Some of the traditional elementary math content, such as "conic and equation", "number system expansion and the introduction of plural" are included in the elective course. The hyighlighted parts in a required course are "preliminary algorithms, statistics, probability", It put "counting principles, statistics cases, probability" in elective courses again, embodies the concept of advanced mathematics course times.

New "curriculum standards" arranged "the history of mathematics election lecture" in the "Elective Series 3". In fact, after the required courses and elective Series 1 and Series 2 is basically the knowledge of the original schools. From the series 3 is the new part. Series 3 and Series 4 contain 16 topics, the history of mathematics ranked first. In the the third part of "curriculum standards" also arranged the "mathematical culture" (do not teach separately). "Curriculum standards" stressed the need to "reflect the knowledge of the occurrence and development process, promote student self-exploration" (The people's Republic of China Ministry of Education, 2003, p117) and the "penetration of mathematical culture, embody the human spirit" (The people's Republic of China Ministry of Education, 2003, p119) in the "materials for the proposal".

Compared to the previous "teaching program" the course objectives (the purpose of teaching) have been greatly changed: add "the development of mathematical application consciousness and sense of innovation, and strive for the real world contains some of the mathematical model of thinking and making judgments"; "has a certain mathematical perspective, the gradual understanding of the scientific value of mathematics, applications and cultural values, to form the habit of thinking critically, the rational spirit of respect for mathematics, experience the aesthetic significance of mathematics," and so on.

In fact, the new "curriculum standards" of the purpose of teaching can be attributed to three sentences:

To know how mathematics came out, that requires students to understand the causes of mathematics, to know the development of mathematical context.

To know how mathematics is used, that is, to understand mathematics in the production life of the application, know the value of mathematics.

To know how mathematics is changing, that is, to understand the background of developments in mathematics and its development prospects.

In mathematics teaching, to achieve these three goals, knowledge of history of mathematics into mathematics teaching is not an option, but essential.

3 The ways of integrating history of mathematics and mathematics education

In the international scope, the history of mathematics into mathematics teaching has been a long time exploration of practice and successful models, such as two-cycle mode of hermeneutics, patterns of resources integration contact, history-psychology of epistemological mode, the model of integration logic-history-cognition, "why-how to" mix mode, and so on. [Zhu Fengqin, Xu Bohua, 2010] Now the problem need to be explored is how to localized.

In recent years, at domestic-related meetings, various artists discuss freely, lays out the current trends in mathematics education. Their opinions for practicing of HPM in China had a very good inspiration. It can be concluded as the following points:

First, China and the West have differences in cultural traditions of the fundamental starting point and the basic way of thinking. The result not only affect the science and technology, but even affect culture, correct understanding and development of innovation. Thus it has an impact on the general quality of the whole nation. As a result, we should let mathematics into Chinese fine cultural traditions, then we can get lessons from the development of mathematics in comparison with the West.

Second, we need to create an artistic conception of the cultural for history of mathematics, and to emphasize the role of education of mathematics culture. For example, Zhuang Zi's famous quote "the wood stick of a length, get rid of half every day, eternally inexhaustible". It was used as an example of infinitesimal in ancient Chinese.(Liang Zongju, 1981) In fact, Zhuang Zi's intention is to "eternally inexhaustible", not to say that "this is the limit of the process tends to 0." It is its artistic conception can helps to understand the limit. Through the process of taking half of all, people felt the state of "a wood stick close to zero but not zero". As a result, mathematicians use this sentence as mathematics historical datas application. We will strive to reveal the cultural connotation of the history of mathematics knowledge, and further make the history of mathematics into mathematics education.

Third, combination with the demands of current new "curriculum standards" for mathematics competence education, we must not only insist on the history of mathematics based on mathematics education should put the history form into an education form, but also insist on mathematics education based on the history of mathematics should to find new the growing point in history of mathematics. American famous mathematician, mathematics historians and mathematics educators M.Klein (Morris Kline, 1908 $\sim$ 1992) showed us in the "Mathematics in Western Culture" preface:"The historcal order happens to be most convenient for the logical presentation of the subject and is the natural way of examining how the ideas arose, what the motivations for investigating these ideas were, and how these ideas influenced the course of other activities."[M. Klein, Zhang Zugui translation, 2005] Klein believes that the history of mathematics can provides an overview of the whole curriculum, so that course content with each other, and connect with the main part of mathematical thoughts; The history of mathematics allows students to see how real mathematicians create history-how to fall down, how to find their way in the fog, to muster the courage to study; To tell them about mathematics from a historical perspective, is one of the best ways to make students understand mathematics content and mathematics attractive appreciation(M. Klein, Zhang Zugui translation, 2005, xv pages). History of mathematics is an effective way to achieve the objectives of the new mathematics curriculum.

4 Train of thoughts on teaching

We can see from the ancient Chinese classic "Jiu Zhang Suan Shu(Nine Chapters on Arithmetic)" that, the Chinese mathematical culture originated in the actual needs, such as measurement of land, measuring volume and so on. Chinese mathematics made the actual production of civil life for object of study, and solved practical problems for the target. It start around of establishing and improving computing technology and algorithms, emphasising on the results should be in the analysis and induce based on observation and experimentation. It made the mathematical truth in calculation, putting mathematics in a few self-evident, on the principles of the visual image. However, at present, the content and methods of Chinese mathematics education has been westernized. Chinese mathematics education's form using the Western model, but in cultural psychology it also use unconsciously the Chinese traditional mathematics and culture. It lead to many confused problems in mathematics education in realistic. They are, how to deal with the relationship between training thinking and guidance practice, emphasizing the pursuit of mathematical visual effect and its practical effect, or its rational debate and logical deduction. We will take the cultural background of the birth and development of Gou Gu Ding Li (Pythagorean theorem) in China and the West as an example, by use of history of mathematics to explor the relationship between cultural traditional and modernization of mathematics education.

Gou Gu Ding Li is the root of Chinese geometry, and the essence of Chinese mathematics. The birth and development of Chinese mathematics metheds, such as Kai Fang Shu(square root methed), Fang Cheng Shu(equation methed), Tian Yuan Shu(establish eqution methed) and other techniques, have a close relationship with Gou Gu Ding Li. [Qian Baocong, 1964]Combinating Gou Gu Xing(Right triangle) and ratio algorithm has been constituted various measurements (such as Liu Hui's "Chong Cha Shu"). Ancient mathematicians often used Gou Gu Xing triangle instead of general triangle, which can avoid to discuss the nature of angle. By this way, Chinese ancient mathematicians did not touch the complex theory of parallel, making geometric system is concise, problem solution more practical. Investigating the birth and development of Chinese Gou Gu Ding Li, we can see that ancient Chinese cultural tradition of mathematics obvious emphasis had obvious characteristics, such as pay attention to application, focusing on linking theory with practice, combinating with number and shape, using calculate mainly, being good at establishing a set of algorithms system. However, China's traditional culture pay attention to "pragmatism", the way of thinking which believed in speaking instead of doing. This research methods makes Gou Gu Ding Li has not been beyond the visual experience and the specific operations from the beginning of its born. It is always as an art in the dissemination and application instead of a complete set of deductive reasoning. Its mode of development is to solve the practical problems. This value orientation of the technique application influence our understanding of mathematics, and our mathematics teaching. In our secondary school textbooks, we directly express that Gou Gu Ding Li in the form of number. This is a reflection of this value orientation. [Wang Qingjian, 2004]

In the West, from began with Pythagoreans discovered "irrational number with the rational incommensurable", the Pythagorean theorem as a metric benchmarking of Euclidean space, through deductive reasoning, giving the new chapter for the improvement and development of geometric axiom systems. While proofing the Pythagorean theorem, Euclidean combined with graphical analysis, deductive reasoning to obtain a series of theorems and corollaries. Since then, Western mathematicians extended the Pythagorean theorem to seek a positive integer of indefinite equation from its number meaning, leading to the famous Fermat's conjecture and Mordell conjecture. They extended the Pythagorean theorem from its graphics meaning, leading to the relationships for the area of plane figure, and the surface area of three-dimensional graphics. By this way continue, through the process of pursuit of rigorous logic and mathematical beauty, modern mathematics access to development. This mathematics tradition for advocates rational and emphasize deductive reasoning contained western religious and philosophical values, which for we learn mathematics and understand the formation of the architecture of modern mathematics has important implications.

The Pythagorean theorem that was born in the different cultures of China and the West and its development path gives us the inspiration, which is that we must inherit the essence of traditional culture as the same times to change the traditional view of mathematics value, in order to learn the western mathematical axiomatic system, embarked on mathematics education modernization. For this purpose, we should be designed to meet our own cultural traditions and customs of classroom teaching. The Pythagorean theorem teaching, for example, can be use the following links to instructional design.

First, we start from the cultural traditions, by using of modern teaching methods for math test. Such as students draw a right triangle calculations themselves, or use the Geometer's Sketchpad software to automatically measure the length of three sides for verify. Second, we can use the comparison of China Liu Hui's proof of the Pythagorean theorem and the Euclidean method in "Element", to tap the cultural connotations.

Liu Hui's proof is based on a self-evident, intuitive image "Chu Ru Xiang Bu" principle, that process can be operated with in-kind.[Wu Wenjun, 1998] So that practical problem can be into mathematics, and ultimately achieve the construction of meaning for mathematical theorems. The proof of Euclid's method is completely divorced from material object. It shows the pursuit of beauty of mathematics and mathematical rational. In this way, students' think can be trained.The presentation of latter method can be used math "re-creation" theory. Throught analysis its exploration process, making the proof idea is gradually revealed. So students can finish the deep understanding for the deductive system structure of the axiomatic.

In summary, we can start from the cultural traditions, using modern means of education to inherit and carry forward the traditional culture, mining the meaning of traditional culture, to achieve the modernization of mathematics education.

From the current situation in the world, the further integration for the history of mathematics and mathematics education is the general trend. The cultural of mathematics education becomes priorities. During the years of research and exploration about the new "Curriculum Standards", history of mathematics teaching practice in China has become possible. The integration of history of mathematics and mathematics education will have a more promising future.

REFERENCES

- Klein M., Zhang Zugui translation, 2005, *Mathematics in western culture*, Shanghai: Fudan University Press, vii.
- Liang Zongju, 1981, *Compendium of History of World Mathematics*,Shengyang: Liaoning People Press,59.
- Qian Baocong, 1964, *Chinese Mathematics History*, Beijing: Science Press.
- Ou Shifu, 2003, "The history of HPM :1976–2000", *HPM Newsletter*, **6**(10), 2–4.
- The people's Republic of China Ministry of Education, 2003, *Ordinary high school mathematics curriculum standards (experimental)*, Beijing: People's Education Press.
- Wang Qingjian, 2004, *Compendium of History of Mathematics*, Beijing: Science Press, 251–263.

– Wu Wenjun, 1998，*The Large System of Chinese Mathematics History*, vol. Ⅲ，Beijing: Beijing Normal University Press, 215–217.

– Zhang Dianzhou, 2009, *My Experience in Mathematics Education*, Nanjing: Jiangsu Education Press, 341.

– Zhu Fengqin, Xu Bohua, 2010, “The international research and enlightenment for the pattern of mathematical history into mathematics teaching”, *Journal of Mathematics Education*, **19** (3), 22–25.

SOCIAL STRUCTURES IN MATHEMATICS EDUCATION
Researching the History of Mathematics Education with Theories and Methods from Sociology of Education.

Johan PRYTZ

Introduction

This talk is divided in two main parts. In the first part, I discuss external motives for studying the history of mathematics education. They are external in the sense that they do not primarily concern the production of knowledge to the field of history of mathematics education itself. In my view there are two external motives. The first concerns how research in the history of mathematics education can make contributions to the field of history of education in general. The second motive is to do with how research in mathematics education may benefit from research in history of mathematics education. In the discussions of both these motives, I touch upon a need to study the people and the groups of people behind concepts, arguments and stories connected to mathematics education. In the second part of the talk, I discuss in more detail why and how I have included a sociological perspective. This second part takes off in a discussion on contemporary research about the history of Swedish mathematics instruction. My criticism is, however, of rather general nature and I am inclined to believe that also an international audience can appreciate the discussion. The second part also contains a section where examples of my results are presented; this section serves as a demonstration of how theories and methods from sociology of education can be applied. The second part, but also the talk, ends with an outline of my future research.

Part I – Motives for studying the history of mathematics education

I believe that research like mine, i.e. projects that include comprehensive studies of the goals, contents and methods of mathematics education, can make significant contributions to educational history in general; significant in the sense that historians in other specialties (e.g. politics, economics, ideas, science or education in general) cannot attain the same result. Lövheim (2006) and Prytz (2007) constitute a good example for this discussion. They treat the same type of school (secondary schools), the same time period (about 1900-1960) and the same country (Sweden), but with different focus and quite different sources. Lövheim, with a background in history of ideas, focus on more public curriculum debates about science and mathematics, while Prytz makes a comprehensive study of the contents and methods of mathematics education. To some extent, they do treat the same sources: curriculum documents and teacher periodicals. But were as Lövheim studies also debates in for instance the parliament and the public press, Prytz turns to textbooks, exams and literature specialized in mathematics education. Thus, two types of educational debates are studied: public and professional.

In Lövheim's treatise, two main ideals, with accompanying lines of argumentation, stand out: on one side the classical or humanistic ideal and on the other the realistic ideal. To keep it short, the most important goal of the humanists was to cultivate the students' knowledge about man and his culture. Important school subjects in that perspective were languages[1] , history and religion. The main goal of the realists' was to promote knowledge about things and nature. Important school subjects in that perspective were for instance science, economy and geography. Lövheim analyses how the arguments changed over time, in different debates and in different media, and to what extent they had impact on curriculum documents. However, if we turn to Prytz study of the professional debate on geometry instruction[2] in the lower secondary schools, it was not a reflection of the debates studied by Lövheim. First of all, geometry instruction of that time contained a large proportion of theoretical geometry with focus on propositions, proofs and purely geometrical problems. This applies in particular to the later years of the secondary schools: the axiomatic method was introduced in the equivalent of year 7 in the Swedish school system of today. Applications in science, work life or everyday life were not prominent in the geometry textbooks. However, this orientation of geometry instruction, despite the ongoing battle between realists and humanists in the public debate, was never really put into question in the professional debate; the need to include more applications was not an issue. The more or less unquestioned overarching goal of geometry instruction of lower secondary school was to foster a general formal education, in this case logical thinking and a critical attitude regarding language and spatial intuition. Rather, the main disagreements concerned the teaching of the subject and the design of textbooks, e.g. the order of propositions and the design of proofs. Moreover, Prytz shows that this by no means was a superficial debate; textbooks were produced in accordance with the different viewpoints.

The apparent focus on the theoretical side of mathematics rather than its applications, but also the explicit goal about general formal education, implies that the debaters in the professional debate advocated a classical ideal. During the 19th century, the argument about general formal training was often used by classicists as they defended the position of classical languages. However, these debaters, all of them secondary school teachers and textbook authors by the way, cannot have been negative to a realistic education. All of them were science teachers and taught physics in secondary schools, i.e. they had a solid background in science. Hence, the public debate and the professional debate were different in character in the sense that the former concerned general goals, while the latter mainly concerned methodological issues. This is perhaps not that surprising. The interesting thing is that the disagreements regarding goals in the public debate did not turn up in the professional debate. Moreover, the conflict between proponents of classic and realistic ideals, so fundamental in the public debates if we follow Lövheim, was not a major element in the professional debate.

The point I want to set forth is that these differences are possible to discern only by a thorough investigation of the goals, contents and methods of school mathematics that were debated in teacher periodicals and realized in textbooks. And it is interesting to know about such differences; they tell us something about not only school mathematics, but also something about how the Swedish school

[1]In the beginning of the period 1900-1960, the position of classical languages was still an issue. But the focus of the debate was shifting towards just modern languages.

[2]This debate took place in the leading teacher periodical on mathematics and science instruction of this time. To a reader of today, major debates about geometry instruction might seem a bit odd; geometry is a rather small topic in the Swedish syllabus. However, during the period 1900-19600 geometry constituted a much greater part of school mathematics than it does today. The big change in this respect took place in the early 1960's.

system functioned. That may in turn provide a better understanding of how and why school mathematics changed in a certain way.

However, the differences in the debates just accounted for might perhaps be accidental or a result of my selection of sources. Moreover, how do we know that the professional debate mattered? The debaters might have been a group of people without influence? These questions touch upon a more general problem. My impression is that much of the research on the history of mathematics education is based on textual analysis. The problem is that purely textual analysis cannot answer questions about the importance or influence of the investigated texts. In the next part, I develop my thoughts about this problem and how sociological analysis can help us overcome it.

Before that, I turn to my second motive for studying the history of mathematics education. This motive is in linked to the fact that educational researchers in Sweden quite often make statements about the history of mathematics education. This applies for researchers from mathematics education as well as education. These statements are not part of the empirical investigations, but occur in some sort of background chapters. As I see it, these historical accounts serve as a motivation for posing the actual research questions; a motivation alongside arguments regarding scientific and practical needs.[3]

The main problem with these historical backgrounds is not the accuracy of factual claims, but their bias. A common feature is that the researchers pay considerable attention to the great school reforms of the 1960's, i.e. the introduction of Grundskolan[4] (a new school type that includes the former primary and lower secondary schools) and Gymnasieskolan (a new school type that comprises the former upper secondary school and different higher vocational schools). In these backgrounds, the authors point out that the reforms were accompanied by radical changes in mathematics education, e.g. the abandonment of classical geometry, new teaching methods, and later on the introduction of the New Math. Another common feature of these backgrounds is that they contain explanations of why mathematics instruction was reformed during the 1950's and the 60's. Three types of reformist arguments regarding mathematics are put forward.

- School mathematics was considered old-fashioned from a scientific point of view. Therefore, the courses were changed.
- School mathematics was considered old fashioned in relation to the needs of a modern society. These needs could be in science or technology, but also so-called everyday situations. Therefore, in order to provide knowledge more suitable for such areas, the courses in mathematics were changed.
- Pedagogical and psychological research had brought a new awareness about mathematics instruction and learning, which resulted in a reformation of the courses and new teaching methods.

By putting forward these arguments as a source of change, the authors of the historical backgrounds links the changes in school mathematics to changes in society and science. This relationship is underscored with even greater emphasis by Selander (2001) in an essay on Swedish school mathematics of the 18th century:

[3]A more comprehensive account of the backgrounds together with references to the works is given in Prytz (2007).
[4]Note that Grundskolan (year 1-9) and Gymnasieskolan (year 10-12) are the basic types of schools in Sweden today.

> For centuries school mathematics has evolved from a formal education and a deductive method to today's functional orientation with elements of an inductive method. Early on, one considered mathematics, along with Latin, important for the students' ability to think and make judgments, and mathematics provided useful training in systematic working procedures and clarity of thought. ···During the 18th century, a long mathematical tradition based on Euclid's geometry was complemented by arithmetic; thereby, mathematics was more adapted to the new needs of the era that followed upon the expansion of shipping and trade. ···During the late 19th century, with the breakthrough of industrialism and the formation of modern schools, with industrial chemistry and new sources of energy, with railroads and national time standards, with the organization of labor and capital and the formation of the modern national state, school mathematics was renewed once again, as the practical relevance of school mathematics was accentuated even more (Selander 2001), pp. 41-42)[5]

In the historical backgrounds, the first half of the 20th century is treated very briefly in comparison with the concern about the 1950's and 60's, most times not at all. In cases where mathematics instruction of this period is given a slightly longer treatment, it is often described in terms of its traditionalism, isolation, and stagnation. In a report, on theories on common education[6] and mathematics instruction in Sweden, Magne (1986) claims that mathematics instruction at the common level was completely unaffected by the reform movements in other Western countries before 1950. However, he does suggest that mathematics instruction at secondary level was influenced by international movements.[7] In the same report, though, Magne points out that the international debates on geometry instruction around 1900 never reached Sweden, nor did the international debates on algebra instruction during the 1920's. Håstad (1978) takes it even further in his doctoral thesis on mathematics education:

> If we must mention the force that has played the leading role in the development of mathematics instruction, the answer is simple: *tradition*. However, there are a number of other "persons in power" that have possessed important supporting roles. The study of their influence is crucial. And how important is tradition? In order to come to grip with its role [the role of tradition], I make the following simplification. *The mathematics instruction that took place up to the 1950's should be considered tradition*. The plausibility of such an assumption is compellingly vindicated by the fact that mathematics instruction has been relatively static during a long period of time and that only minor modifications have taken place during the previous decades (Håstad 1978, p. 134).[8]

In an essay on school mathematics, Unenge (1999) summarizes his experiences of Swedish mathematics instruction before 1960 in a similar way.

> Well until the late 1950's, mathematics instruction was more or less unchanged. The way I was taught as a student in Realskolan was the way I taught my students in Realskolan 15 years later (Unenge 1999, pp. 24-25).

[5]The quote is originally in Swedish. The translation is done by Prytz (2007).

[6]Common education is a translation of the Swedish word folkundervisning. In Sweden this concept concerns education for the masses or the commons. Other translations of folkundervisning would be public education or peoples' education.

[7]Magne (1986), p. 6

[8]The underlining as well as quotation marks are original.

These historical accounts are by no means entirely wrong. I do not question the correctness of the factual claims about the 1950's and 60's, but we have to be aware of what aspects of the situation they describe. I have no doubts about the descriptions of the reformist arguments and the changes in the curricula; undeniably, these arguments were a part of the debate and the changes in the curriculum did take place. The problematic aspect is that only the arguments of the 'winning team' are mentioned – the arguments of those whose wishes came true in the reforms of the 1960's. The critics of the reforms and their counter arguments are not considered at all. In fact, from the historical backgrounds you cannot tell whether there were any critics. It is almost as if the changes that took place followed some kind of natural order related to the industrialization of Swedish society.

Another problematic aspect is the descriptions of the time before 1950 as being traditional and static. Prytz (2007) shows that changes in mathematics education did take place, in this case geometry. In the leading teacher periodical, teaching methods and contents were discussed and textbooks, e.g. Euclid's *Elements*, were criticized. New alternative textbooks were indeed produced; for instance, some authors began to use the symmetry concept and propositions about symmetry replaced traditional propositions about congruence. The latter are essential in traditional editions of Euclid's *Elements* and should not be considered a "minor modification" as Håstad puts it. Therefore, I think that Magne's and Håstad's characterizations of the time before 1950 as being static and traditional are misleading: mathematics education did change. However, attempts to reform mathematics education in other countries might have been more radical than in Sweden. Moreover, since the changes of Swedish school mathematics in the late 60's were so fundamental, e.g. the introduction of New Math, less fundamental changes in the past may stand out as insignificant, especially to people like Magne and Håstad, who in the 60's belonged to various expert groups that worked for the reformation of mathematics education.[9] Håstad was for instance one of the experts on New Math in Sweden. Thus, if we consider the historical accounts of Magne and Håstad, it is fair to say that the view of the 'winning team' has been transformed to historical facts.

I think it is important to detect this type of biases, not just because historical accounts in educational research ought to be unbiased, but also because researchers, teacher educators or administrators may benefit from a less biased history as they pose questions about mathematics education. In order to pose innovative questions it may be fruitful to question things that we today take for granted. If we consider the historical backgrounds mentioned above, they all include an idea of rapid and fundamental changes of mathematics education during the 60's and early 70's. However, if we compare these changes with the previous changes during the period 1900-1960, which indeed took place, we find people in leading position whose arguments about how to develop mathematics education that appears to be similar (Prytz 2007, 2012). Even though they considered different concepts, e.g. symmetry in the 1920's and sets in the 1960's, they relied on the same educational assumption: mathematical concepts and the teaching ought to match the students' spatial intuition.[10] Thus, the choice of concepts and representations of concepts were considered important in the development of mathematics education. It would be interesting to investigate further what happened with this assumption after 1970. My impression though, after having taught at teacher education for the last five years, is that it

[9] Kilborn et al (1977), *Hej Läroplan. Hur man bestämmer vad våra barn ska lära sig i matematik.*

[10] Before 1950, the debaters used the Swedish word "åskådning". The concept is quite similar to the German concept anschauung. In the 1960' s, the debaters used the word "konkretisering", a word that was used also in the methodological part of the curriculum documents. I usually translate konkretisering by visualization. Despite the different word, if we compare the reasoning about "åskådning" and "konkretisering", there are considerable similarities.

still is an important assumption. For example, textbooks about mathematics education often contain a chapter about how concepts may be represented and explained by different types of materials or pictures (cf Löwing & Kilborn 2002). These books put an emphasis on why different materials and pictures ought to be used and they provide examples of how materials and pictures can be used to explain different concepts, e.g. division and percentage. Yet there is a lack of concepts and general principles about how to design materials and pictures for different mathematical topics. Moreover, in most cases there are no scientific references attached to central propositions and examples regarding these matters. In what respect these missing parts reflect a shortage of research results or an inability to include such research results in teacher education I cannot tell. Nor do I pass any judgments about this missing part. I just want to draw attention to that these missing parts in teacher education, and possibly also in research, do not match some of the historical accounts mentioned above; the latter establish that pedagogical and psychological research became a part of Swedish mathematics education after 1960. My point is that biased historical accounts may conceal continuities or discontinuities in the development of mathematics education that might deserve serious questioning.

So as to detect biases in historical accounts, I think it is helpful to know about the authors' involvement in reformations of mathematics education. In that perspective, the relation between position and standpoints becomes interesting. Knowledge about a person's positions may reveal motives as well as influence. Håstad and Magne, mentioned above, is a good example of that. In the next part of this paper I discuss how this relation may be studied by means of a sociological analysis.

Part II - On the use of a sociological perspective in research on the history of mathematics education

Why a sociological perspective?

If we by the phrase 'using a sociological perspective' mean that we consider relations between people and groups of people as we study the history of mathematics education, the question might seem a bit trivial. However, it is not, at least if we consider contemporary research about the history of mathematics education in Sweden (cf Kilpatrick & Johansson 1994, Bjerneby-Häll 2002, Hatami 2007, Prytz 2007, Lundin 2008). The main material in these works is educational texts, e.g. textbooks, policy documents and teacher periodicals and the main method is pure textual analysis. The latter makes it difficult to explain why content and methods of school mathematics changes, even though the researchers put changes in school mathematics in connection with general scientific, economic, political and social changes. My main objection is that these researchers cannot explain why some arguments, textbooks, authors and debaters were more influential than others. General societal changes (economic, political or social) do not, per se, produce a certain kinds of school mathematics. Nor do changes in mathematics as a science. You may understand such general changes as possible incentives for change in a school system, but you cannot derive from them, in more detail, the contents and methods of school mathematics during a specific time period. You would then ignore that an educational system most likely comprise different groups of people who interpret, not necessarily in the same way, what societal change is, but most importantly, what its consequences for school mathematics are. Moreover, these groups may have had different amounts and types of influence over school mathematics. The previous comparison between public and professional debates about mathematics education indicates that this was the case during the period 1900-1960. My point is that if we want to understand why mathematics education change in a certain direction it is necessary to integrate textual analysis with a sociological analysis that focuses on groups with influence over school mathematics.[11]

My approach in this matter has been to think of different arenas of decisions. There is a central political arena (including for instance a parliament and a government) and a central administrative arena (e.g. a national school board) where groups of people make decisions about education. Another type of arena comprises all the teachers that make decisions as they plan and carry out the teaching. A question that interests me is what other types of arenas of decisions are there and who inhabits these arenas? An important research task in this perspective is to discover such arenas, i.e. to identify and describe groups of people and positions that had influence over mathematics education during certain periods of time. If we know about these arenas it is possible to investigate influence and why mathematics educations change in a certain direction. More precisely, to follow the transmission and reception of ideas about school mathematics between arenas in both directions: which ideas that were recognized as important and which that were ignored; which ideas became influential, which did not.

For some years I have grappled with questions about the influence of ideas, arguments or textbooks in mathematics education. Obviously, these questions concern power in a school system. My underlying idea has been to study course plans, textbooks, teacher periodicals and exams; educational texts that have affected teachers as they plan, carry out and evaluate their teaching. Consequently, the people in control over the design and production of these educational texts have power over the content and methods of teaching. However, I am by no means the first to stress the significance of educational texts in studies of how school systems work. For example Lindensjö & Lundgren (2000) explain why the production of educational texts should be an integral part in studies of how educational reforms are received in different contexts. The important aspect from my point of view is that if we can determine groups of people involved in the production of educational texts, we can also determine one or more arenas in the decision-making process. Moreover, by studying relationships between people within a group, but also relationships between groups, we may say something about influence.

How to study social structures of mathematics education—some general notes

In my studies of the social structures I have applied a prosopographic method combined with Bourdieu's theory of field and capital (cf Bourdieu 2000, Broady 2002a). A prosopography can be considered a collective biography (cf Broady 2002b). The collective in this case are the authors or editors of educational texts. An important part of the analysis is the identification and description of these peoples' properties (or capital or assets) and how people and properties are linked to each other. Examples of properties are teacher experience, educations and authorship in different areas or genres, but also positions in school administration or science and positions such as editor of teacher periodicals.

[11] For a thorough discussion about social and institutional aspects regarding the formation of school mathematics, see for instance Popkewitz (1988). Regarding the need to study social contexts in research about the history of mathematics education it has been emphasized by for instance Schubring (2006) and Howson, Keitel & Kilpatrick (1981). But, also researchers who focus on school governance emphasize this need; for example Lindensjö & Lundgren (2000) underscore that social relationship and power relationships should be considered in studies of educational processes.

Here, standpoints about mathematics and education are also seen as properties. Another important part of the analysis is to identify those assets that were valued by people in the investigated collective. This can be done by investigations of explicit statements, grading systems or symbols of recognition, but also by comparing assets of people in certain positions with the assets of those who was not in these positions. Note that position may function as an asset. For instance, headmaster of a school is a position, which can be an asset if you want to make a career in the school administration. By identifying patterns in peoples' properties and positions, it is possible to discern value and reward systems (systems of recognition) and how they are linked to such things as standpoints regarding education and mathematics, but also the production of educational texts. Hence, an important part of the analysis is to discern the logic between assets, positions and standpoints. For that end there are specific concepts and techniques, see Broady (2002a). This is, however, not the place to go into the details.

Regarding the choice of sources, my basic criterion has been that the texts have been produced for teachers and used by teachers in connection to teaching, including for instance curriculum documents, textbooks, books about teaching, teacher periodicals and exams. But, teacher periodicals are particularly interesting since they contain debate articles in which it is possible to discern demarcations between different groups of people as contradictory opinions become visible. However, the method outlined above has its limitations in historical studies; limitations related to the existence of educational texts and to what extent practices within an educational system (national, regional or local) relied on such texts. So far, I have mainly studied the history of mathematics education in Sweden from about 1860 and onwards and for that period the approach works. But for periods further back in time, I think that the supply of educational texts was different. This is to do with the professionalization of teachers and the occurrence of professional periodicals and specialized literature on teaching, but also changing systems of degrees and national exams and less detailed central policy documents. Yet, these reservations should not immediately imply that the supply of educational texts was smaller and the approach should be abandoned. A first step should perhaps be to rethink the idea of educational texts. It might happen that types of texts that we today do not associate with usage in schools, may have been used for educational purposes in past times. The Bible or why not Euclid's *Elements* are examples of that.

Some results

In this section I outline some of the main findings and results from my research in the history of Swedish mathematics education during the 20th century.[12] In connection to this I discuss two sets of questions that have been touched upon earlier on: 1) How do we know that differences in arguments, debates or textbooks are not accidental? Or are they a result of the selection of sources rather than a reflection of actual differences of opinion? 2) How do we know that a certain educational texts mattered? Where some authors more influential than others?

As mentioned before, the public debate and the professional debate about mathematics education

[12]Thorough descriptions and analyses of debates and arguments about content and methods of geometry instruction during the period 1905-1962 are accounted for in Prytz (2007). In the same treatise you find also thorough descriptions and analyses of textbooks together with statistics on textbooks. Facts about the debaters during the period 1905-1962 and the sociological analysis are accounted for in Prytz (2009). The analysis of changes connected to the major school reforms in the 1960's is about to be published in 2012. The paper was presented at the Second International Conference on the History of Mathematics Education held in Lisbon, Portugal, in October 2011.

during the period 1900-1960 were different. One difference not yet discussed is the small number of people engaged in the professional debates. If we consider the professional debate on geometry instruction, it involved a total of seven peoples during the period 1905-1962. All of them were authors or editors of textbooks on geometry and other topics. Apart from these seven, there were about five more authors of geometry textbooks during this period of time. You might think that the small number constitutes a historiographical nuisance as the standpoints of these people cannot be considered representative for all teachers. This is indeed true, but we must remember that we are considering the production of educational texts. In that perspective 7 people out of 12 is a rather large proportion. Moreover, an interesting fact is that the production of educational texts about geometry was concentrated to a small group of people. I will return to this fact later on in this chapter.

First, let us look at the differences in contents of the public and professional debates. These differences appear not to have been a coincidence if we consider the professional debate in a sociological perspective. There is evidence of that the group of people involved in the production of educational texts about geometry instruction was independent in relation to other institutions in society. Prytz (2009) shows that the professional debate on geometry instruction, during this period, functioned as a field. This means that there was a certain pattern in the debaters' background, their way of reasoning and their positions. One important aspect of a field is that the people in it tend to value, but also fight about, standpoints, actions, objects and symbols that are common to this group of people. A second important aspect is that the same people tend to disregard other types of standpoints, actions, objects and symbols that are valued by other groups of people or institutions. The people who debated geometry instruction fit into this pattern.

The peoples who reached leading positions in school mathematics, i.e. editor of the major teacher periodical on mathematics and science education and advisor at the central school board regarding mathematics education, shared some properties. These properties separate them from people who did not reach these positions. If we consider the standpoints of the peoples who reached leading positions in school mathematics, they all made a clear distinction between scientific and educational standards. School geometry had to apply to both, but in cases in which a conflict arose, the latter should prevail. This distinction was not made by people that did not occupy this type of positions. Instead, they meant that scientific standards had an educational value. In fact, much of the debates were about the relevance of various scientific and educational standards, but also which of the two types of standards that would be superior to the other.

This difference in standpoints divides the debaters in the same way as differences in the debaters' backgrounds do, which is also typical of a field. One part of the analysis of backgrounds has been to consider different types of recognized skills. Here we can see that skills related to being a prominent research mathematician was not a common property of the people in leading positions. Indeed, all the debaters had a background in science since they had a Ph D in mathematics, but only one of them made some kind of scientific career in mathematics. However, he did not reach a leading position in connection to school mathematics. Instead, the property of the people in leading positions that was common to them and only to them was success as textbook author. They were among the most successful of their time, not only in geometry but in other mathematical topics as well. Moreover, I would say that a symbol of recognition was linked to textbooks, a recognition of the skill to write and edit textbooks. I think it is reasonable to say that recognition was given by teachers as they chose to purchase a textbook. But, we cannot understand recognition in the sense that teachers chose the textbook that was optimal in relation to their idea of teaching. However, it is fair to say that if a teacher[13] chose a textbook, he or she also valued that same textbook on the basis of some, at least, elementary standards; especially when there were always at least two different textbooks to choose from that were regularly republished during the whole period 1900-1960. In this sense, I think it is reasonable to talk about teachers giving recognition of skills to author and edit textbooks. Another argument to why a symbol of recognition was linked to textbooks is that textbook design was something the debaters fought about: it was a central subject in the professional debate about geometry instruction. Moreover, the discussions about textbook design were initiated by non-established textbook authors, who happened to be the same people that did not reach leading positions. The critical articles were also comprehensive and so were the replies of the criticized authors. My point is that textbooks and textbook design triggered action and reaction. When the contenders in the professional debate attacked the established, they focused on things with high symbolic value among professionals. Issues that were important in the public debate about mathematics education came in the background.

Here, I want to pick up the question about whether the differences between the public and the professional debate should be considered accidental or not. I would say no. As just mentioned, Prytz (2009) has showed that the people involved in the professional debate about geometry instruction functioned as a field. An important aspect of a field is independence. The differences between the public and professional debates reflect the independence of people involved in mathematics education vis-à-vis other groups in society, e.g. scientists and politicians.

I also want to return to the questions about the importance of the professional debate and influence of debaters and the texts. The sociological analysis puts us in the position to answer such questions. I think it is fair to say that some of the debaters did have more influence or power than others. The group of people who reached leading positions in school mathematics not only had these positions; they also controlled a large part of the supply of educational texts regarding geometry as they were editors of the major periodical that treated the subject and they authored the most used textbooks. On this point it is important to know that central curriculum documents during the period 1900-1950 were very brief and general on the contents and methods of school mathematics and all other school subjects. Moreover, there were no specific books about mathematics instruction in secondary school until the early 1950's. There were however national exams at the end of lower secondary level (Realskolan) and the end of upper secondary level (Gymnasiet), which may have influenced the teachers. Still, this influence was restricted to a small number of exercises on a couple of occasions every year. Moreover, not all students took the exam at lower secondary level; it was not a prerequisite for entrance to upper secondary level. My point is that textbooks and articles in the leading teacher periodical on mathematics and science education constituted the bulk of educational texts that treated content and methods of mathematics instruction in a comprehensive way. Hence, the production of the main part of the educational texts regarding geometry was controlled by half a dozen people. So I think it is fair to say that these people had influence and that the professional debate mattered. Moreover, if we want to understand why mathematics education changes in a certain way, why some ideas and arguments gain influence or not, this is an important group to study.

[13]Or the teaching staff at a school.

Further research

In the coming five years, I will be researching to what extent general reforms of the Swedish school system, but also specific attempts to change school mathematics, affected the content and methods of school mathematics. I will also try to answer questions about why some attempts to change were successful and others were not. The theories and methods that have been previously described will then be applied. A first step is to study the period 1950-1975. This was a period when the Swedish school system went through great changes; a system with parallel school types was replaced by a system with one compulsory school type (year 1-9) and a voluntary upper school type (year 10-12). Also mathematics education changed in connection to this. One of the major specific attempts to change school mathematics was the new course plan in mathematics, influenced by New Math, which was introduced in 1969. In order to discern changes, comparisons with the first half of the 20th century will be necessary. On that point, it will be possible to use the results from my previous studies (Prytz 2007, 2009) on geometry instruction during the period 1905-1962.

One aspect that interests me in particular is how changes of structures that are to do with the organization of the school system affect the content and methods of school mathematics. That is, I am also interested in causes other than those that are related to explicit attempts to changes the contents and methods of school mathematics.

A structural change of this kind is the forms of course development. In the 1960's, the state tried to get greater control over the courses. One means was to issue comprehensive course plans. In the secondary school statutes of 1933, descriptions of the contents of courses are brief. For instance, the contents of the mathematics courses of Realskolan (lower secondary school, year 5 to 9) were described in approximately 240 words. Moreover, the secondary school statute as a whole did not contain sections on teaching methods, neither general ones nor specific ones for each subject. In the curriculum of 1962, the description of the purely mathematical content of the corresponding courses of Grundskolan (year 5-9)[14] comprises about 1000 words. In addition to that there are comments and directives regarding planning and teaching methods in mathematics that comprises approximately 1550 words. And in addition to that there is a final section on how, in what order and to what extent each mathematical topic ought to be treated. This final section comprises about 3500 words (Prytz 2012). Apart from that, there was a general section on teaching methods. Considering this difference in the national curriculum documents, the textbook authors[15] before 1950 had greater freedom, but also more power over the content of the mathematics courses. Thus, the forms for the development of the mathematics were different before and after 1950.

A preliminary study (Prytz 2012) indicates that the professional discourse in the leading professional periodical about mathematics education changed in connection to this; more precisely the leading peoples' attitudes towards teachers and authorities. Before 1950, the leading person presented important assumptions and research results as possible to discuss; their certitude was not presented as evident. In the 1960's the attitude was different. The leading person presented important assump-

[14]Year 7-9 of Grundskolan comprised two types of courses in mathematics: general and special. The latter corresponded to the mathematics courses of Realskolan. The chapter of the 1962 curriculum that treats the mathematics courses contains common sections for both types of courses and specific sections for each type of course. I have counted the words of both the common sections and the sections that treat the special course specifically.

[15]Unless otherwise indicated, the term textbook author also refers to the publishers involved in the production of the textbook.

tions and research results more as non-discussable facts. There were also other changes in the investigated articles that reflect this change in attitude. The argumentation got shorter. The meaning of the propositions became less distinct and explicit references to scientific treatises disappeared. Moreover, this change in attitude can be tied to changes of the forms of course development. It is about how dependencies between leading people and the teachers changed as course plan authors got more power on the behalf of textbook authors. Textbook authors are dependent on the teachers, the buyers of textbooks, in a way that the course plan authors are not.

A second structural change that interests me is the abandonment of a parallel school system and the introduction of one mandatory school type. This change also included the decision to have a heterogeneous classes in the whole of Grundskolan rather than specialized classes during the last years, for instance academic and vocational classes. Thus, the conditions for differentiation became quite different. In Sweden, the concept individualization was central in the discussion about how to deal with heterogeneous classes. The Central school board even initiated a project that was supposed to develop material and teaching methods for school mathematics so that the teaching could be individualized (Prytz 2012).

In order to discern to what extent organizational changes affected the contents and methods of school mathematics, the investigations will be repeated in Germany. Germany is interesting since Swedish and German school systems, on a general level, were organized in similar ways before World War II. In the 60's, however, the Swedish school system changed in a more comprehensive way. In West Germany they kept, for instance, a parallel school system. By using the German case as a kind of reference point, it is possible to study to what extent the Swedish organizational model with heterogeneous classes affected the contents and methods of school mathematics. Or rather, it is then possible to avoid pure counterfactual speculations if you want to answer such questions.

Of course, this type of comparative studies can be applied on other countries. The more countries and school systems that are investigated, the more comparisons can be made. As I see it, more countries also mean better opportunities to discern global similarities and local characteristics. So, I want to end this talk with an offer. If you are interested to carry out the kind of studies that is described in this talk, please let me know and I will provide relevant references. I also want to invite you to the Third International conference on the History of Mathematics education in Uppsala next summer. In connection, to this conference it will be possible to arrange a short introductory course about prosopography, Bourdieu's theory on field and capital and how they can be applied in historical research.

References

- Bjerneby Häll, Maria (2002). *Varför undervisning i matematik?: argument för matematik i grundskolan - i läroplaner, läroplansdebatt och hos blivande lärare*. Lic.-avh. Linköping : Univ.
- Bourdieu, P. (2000). *Konstens regler. Det litterära fältets uppkomst och struktur*. Stockholm: B Östling publishing house. Translation of: Bourdieu, P. (1992), Les règles de l'art. Genèse et structure du champ littéraire. Paris: Seuil.
- Broady, Donald (2002a), Nätverk och fält, in Sociala *nätverk och fält*, ed. Håkan Gunneriusson, Opuscula Historica Upsaliensia 28, Uppsala universitet

- Broady, D. (2002b). French prosopography. Definition and suggested readings. *Poetics*, 30, 381-385.
- Hatami, Reza (2007). *Reguladetri: en retorisk räknemetod speglad i svenska läromedel från 1600-talet till början av 1900-talet. Lic.-avh. Växjö* : Univ., 2007
- Howson, A.G., Keitel, C. & Kilpatrick, J. (1981). *Curriculum development in mathematics*. Cambridge: Cambridge University Press.
- Håstad, M. (1978). *Matematikutbildningen från grundskola till teknisk högskola i går - idag - i morgon: [Training in mathematics from grade school to technical university yesterday - today - tomorrow]*. Diss. Uppsala : Univ.. Stockholm.
- Kilpatrick, J. & Johansson, B. (1994) Standardized mathematics testing in Sweden: The legacy of Fritz Wigforss. *Nordisk Matematikdidaktik (Nordic Studies in Mathematics Education) 2 (1)*, 6-30.
- Lindensjö, Bo; Lundgren, Ulf P. (2000), *Utbildningsreformer och politisk styrning, Stockholm*
- *Lundin, Sverker (2008), Skolans matematik: en kritisk analys av den svenska skolmatematikens förhistoria, uppkomst och utveckling*, Diss. Uppsala : Uppsala universitet, 2008
- Lövheim, D. (2006). *Att inteckna framtiden: läroplansdebatter gällande naturvetenskap, matematik och teknik i svenska allmänna läroverk 1900-1965*. Diss. Uppsala : Uppsala universitet, 2006. Uppsala.
- Löwing, M. & Kilborn, W. (2002). Baskunskaper i matematik: *för skola, hem och samhälle*. Lund: Studentlitteratur.
- Magne, Olof (1986), *Teorier för folkundervisning i matematik*. Malmö: Institutionen för pedagogik
- Popkewitz, Thomas S. (1988), Institutional Issues in the Study of School Mathematics: Curriculum Research, *Educational Studies in Mathematics*, Vol. 19, No. 2, Mathematics Education and Culture (May 1988), pp. 221-249
- Prytz, J. (2007). *Speaking of Geometry: a study of geometry textbooks and literature on geometry instruction for elementary and lower secondary levels in Sweden, 1905-1962, with a special focus on professional debates*. Diss. Uppsala: Uppsala universitet, 2007. Uppsala.
- Prytz, Johan (2009), Professional debate and social structure in Swedish mathematics education, 1905-1962. The case of geometry instruction at the lower secondary level. In *"Dig where you stand". Proceedings of a conference on On-going research in the History of Mathematics Education"*, eds Kirstín Bjarnadóttir, Fulvia Furinghetti & Gert Schubring (Reykjavík 2009)
- Prytz, Johan (2012, forthcoming), Changes in the exercise of power over school mathematics. The case of mathematics education (year 1-9) in Sweden, 1930-1970, in *Proceeding of the second international conference on the history of mathematics education*,
- Schubring, Gert (2006), Researching into the History of Teaching and Learning Mathematics: the State of the Art. *Paedagogica Historica*, Volume 42, Number 4 (2006), pp. 665-677
- Selander, Staffan (2001), Från geometri till aritmetik: 1700-talets skolmatematik, in *Vänbok till Wiggo Kilborn*, ed Löwing, Madeleine, Göyeborg

– Unenge, J. (1999). *Skolmatematiken i går, i dag och i morgon: -med mina ögon sett*. (1. uppl.) Stockholm: Natur och kultur.

THE ROLE OF THE FRENCH ASSOCIATION OF MATHEMATICS TEACHERS APMEP IN THE INTRODUCTION OF MODERN MATHEMATICS IN FRANCE (1956–1972)

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ABSTRACT

The French Association APMEP (Association of Professors of Mathematics in Public Education) played a major role in the implementation of the Reform of Modern Mathematics. As soon as 1956, in an APMEP meeting, the mathematician Gustave Choquet compared the mathematics teachers to members of a Museum, who show dusty and useless objects. In the years 1958–1960, teachers began to introduce some elements of Modern Mathematics in classrooms, like sets and relations. From 1960, there had many papers in the *Bulletin* of the Association to explain Modern mathematics to teachers and to exhibit experiments in classrooms with these mathematics. In 1964, it was the beginning of local courses and national sequences for the Television organized by the Association.

We propose to come back to this rich period with numerous books and papers to understand the interest of the Association APMEP for the Modern Mathematics and the interrelations between teachers, mathematicians and others. For the teachers, Modern mathematics will permit to develop mathematical activities, to democratize teaching and to adapt the mathematical teaching to the modernity of the society. But it appears clear rapidly that the new notions present difficulties in regards of the ancient contents of teaching, for instance in geometry. We will focus on this aspect to analyze the first failures of the Reform. We will also examine how the teaching of mathematics became quickly a teaching of a complicate language. In its "Charter of Chambéry" of 1967, the Association APMEP asked for a Reform of mathematics teaching, but also for the creation of IREM (Institute of Researches on Mathematical Education). At this period, the IREM had the task to teach Modern Mathematics to all the French teachers, but some years after, many of their members strongly criticized the Reform.

1 The French context of the Reform

The reading of the Official Instructions of Ministry of Education of January 1957 gives an idea of the French context of the Reform of modern Mathematics. It was emphasized on the serious danger that France meets, on the intellectual level and on the economical level, because of the lack of increasing number of engineers, researchers and technicians and also on the urgent necessity to turn a increasing number of young people towards the scientific careers.

1.1 The economical and scholar context

The period corresponds to the "Cold War" with a major event in 1957, which is the launching of the Spoutnik by the URSS. The European Organization of Economical Cooperation (OECE created in 1948 and became OCDE in 1961) created in 1958 a Committee to make more efficient the teaching of sciences and of technics and organised a Colloquium in 1959 in Royaumont (France) to promote a reform of the contains and of the methods of the mathematical teaching (Barbin 1989).

On this occasion the French mathematician Gustave Choquet presented a program of teaching for secondary schools and the French Bourbakist Jean Dieudonné exclaimed "A bas Euclide!" (Down with Euclid!).

Many reforms of the scholar system occurred in the years 1960. The Reform Berthoin extended the schooling of the pupils until sixteen years old (1959). Last year, the examination to enter in the Lycées (upper secondary schools for sixteen to eighteen pupils old) was deleted. In 1963, the important Reform Fouchet created the "Collèges d'Enseignement Secondaire" (CES), they replaced "little classrooms of Lycées" to welcome all the pupils in the same institutions.

1.2 The social context and the modern mathematics

In the end of the years 1960, some famous works of social researchers are edited in France. In his paper "Teaching systems and Thought systems" (1967), Pierre Bourdieu opposed "schools of thought" and "culture of class". He explained that as the School is put in charge to communicate the social system, it has been organized to answer to this function. For him, to realize this programme, named "culture", the School has "to programm a culture" in such a way to facilitate a methodological learning. Four years after, the sociologists Christian Baudelot and Roger Establet wrote *L'école capitaliste en France* (1971), which showed in detail the links between the social system and the scholar system.

The idea that the modern mathematics was the mathematics needed for a democratic School was expressed often in the years 1960–1970. For instance, the French mathematician André Lichnerowicz, which was in charge to organize the Reform, wrote in a union journal of January 1973 that the new mathematical teaching was necessary, not to form professional mathematicians, but above all and at first to form the future citizens. For him, the future citizens had not to be passively subject to the various frames, which will be imposed or proposed to them. They must have the power to say no to the too much clever manipulators of computers and they don't have to capitulate in front of a pseudo-scientific terrorism.

There was also the idea that the pupils of all the social class will be in an equal situation because all the parents will be in an equal position towards of a new mathematic that they did not know. The French mathematician Jean Frenkel wrote in a *Bulletin of APMEP* in 1972 that for some years, parents of all the social class would be equally disarmed to help their children to learn. It was naïve, because special books for parents were edited and private schools were created at soon as the Reform was in the air.

2 The "leaven" of the Reform : the French Association APMEP (1956 – 1968)

In a meeting in Melun in 1952 (a town in the South of Paris), the mathematicians Dieudonné, Choquet and Lichnérowicz met the Swiss psychologist Jean Piaget. Four years later, in a meeting of APMEP in Sèvres near Paris, Choquet compared the teachers of mathematics to guards of Museum, which show dusty objects without interest for most of them (Bareil, 1992). From the years 1958–1960, teachers and schoolbooks began to introduce basic notions of modern mathematics, like sets and relations. The French Association des Professeurs de Mathématiques de l'Enseignement Public (APMEP) is an Association created in 1909, which gathers all the mathematical teachers from the Nursery School to the Universities.

2.1 New mathematics in the *Bulletin* of APMEP (1960–1963)

First studies on the theory of sets and on experiments in classrooms appeared in the *Bulletin* of APMEP in the years 1960–1961. In particular, two important papers appeared on the theory of the modern mathematics: Choquet wrote a paper entitled "Research on a easy axiomatic for the first teaching of geometry" and an other French mathematician Jean Colmez wrote a paper "The structure of modern mathematics". In this paper of 1961, Colmez explained that this structure of mathematics can contain every rational theory relative to any human activity. For him, the Mathematics are the science of reasoning and propose plans of reasoning to the other sciences, which are in a way prefabricated.

Claude Pair was a young mathematics teacher of the Lycée Poincaré in Nancy when he wrote a paper on "The affine geometry in the first grade of the Lycées", (1961) that means for pupils aged fifteen or sixteen. Some months later, he related "An experience of teaching of modern notions", where he emphasized that we have to keep the great geometrical ideas for later to serve as examples, because they will enter in wider theories.

The *Bulletin* of May-June of 1963 contained papers about the relations between mathematics and the reality. Inside, Gilbert Walusinsky, a famous leader of APMEP, wrote a paper entitled "The rule of three will not held". The title is a joke using the two sentences: "La Guerre de Troie n'aura pas lieu" (a drama of Jean Giraudoux) and "La règle de trois n'aura pas lieu". He considered that the automatism of the mathematical "rule of three" is not without danger and he showed to avoid it by using modern mathematics in the lower grade of Collèges, with pupils eleven or twelve aged.

Always in this *Bulletin*, Walusinsky wrote a paper on "Spatial studies and mathematical teaching", where Spoutnik I and Exporer I are taken as examples. While, Jean Kuntzmann wrote about the needs of men working on applied mathematics and on automatisms. But he made clear that these careers were also opened to the young women. Remember, that in this period, French young women began to consider that it is normal for them to have a work.

In the annual commentaries of votes in 1963, we can read that some teachers of the Association APMEP judged that it was necessary to adapt the curricula to the "possibilities" of the pupils and to take in account the "pedagogical realities of now". Indeed, in the beginning of the years 1960, a new population of pupils acceded to the Collèges (see above) and it seems that the teachers thought that the difficulties met by these pupils could be solved by the introduction of a modern teaching.

2.2 The "Chantiers" of APMEP (1964) and the Charter of Chambéry (1968)

On Walunsky's initiative, the Association APMEP created in the years 1964 "Les chantiers mathématiques", which were a television serie on modern mathematics and prepared the "Grande Commission".The "Commission Lichnérowicz" began to work three years later, and many of its 18 members belong to APMEP.

In January 1968, the Association APMEP organized a Colloquium with around forty participants, where a first text on "the steps and the perspectives of a Reform of mathematical teaching" was elaborated. This text will be completed and adopted by the General Assembly under the name of "Charte de Chambéry" The text answered to three questions: Why the mathematical teaching has to be reformed from the Nursery Schools to the Universities? Why the reform is possible? How to realize the Reform? The answers were accompanied of many concrete proposals, in particular to determine the different steps of the Reform.

The first main goal was to teach to all pupils a mathematic useful for the modern world. Indeed, there was a true democratic willingness to render an abstract level more accessible, and not reserved to only some privileged persons as in the past. The purpose is to teach a contemporary mathematic, which had to be a part of the culture for everybody, in the period of computers and automatisms.

The second goal was to teach the mathematic with an active pedagogy. The Charter emphasized that the introduction of a new contain in the mathematical teaching will be ineffective, and may be harmful, if it is not accompanied by an appropriate pedagogy: active, open, the less dogmatic as possible, organising the work of the pupils by groups and appealing to their imagination.

The Charter of Chambéry stressed on the necessity to create Academic Institutes on Research on Mathematical Education (IREM), because the effectiveness of the Reforms had to begin by a serious experimentation and by an increased effort to develop the in-service teacher training.

3 The great lines of the Reform and the special case of Geometry

The official new curricula appeared steps by steps from July 1968 to June 1971. The first IREM began to be created in Main text in 1969, which means that the Reform will be installed before the works in the IREM really was always effective. It is possible that the social movement of May 1968 explains the rapidity of the decisions.

3.1 The introduction of modern mathematical notions

From the Nursery School to the Universities, the new teaching was symbolized by the notion of set represented by the drawing of a "patate" (potato) with the drawings of union and intersection of sets. The relation of equivalence, with the properties of reflexivity, etc., and the properties of the operations, like commutativity, associativity, etc. were considered as basics notions. They were presented in a naïve manner from the lower grades.

In the two first years of Collèges, with pupils from eleven to thirteen aged old, there was an introduction of the "language" of sets and relations. In the two next years, the fields Q of the rational numbers and the field R of real numbers were "constructed". The initiation to the geometrical proof disappeared of the Curricula in the same time than the three "cases of equalities of triangles" of the "old" geometry. In place, Curricula offered an affine geometry turned towards the linear algebra.

In the Lycées, the "vocabulary" of the sets and the relations was linked with a teaching of logic. The affine and Euclidean geometries were based on linear algebra. The Curricula gave a good place to calculus. The differential calculus contained the notions of continuity of a function, limits, derivability of numerical functions of one real variable in the upper classroom. The integral calculus was based on the sums of Riemann, on differential equations. A part concerned probabilities on a finite set.

3.2 The special case offered by the geometrical teaching

The geometry offered a special case for the members of APMEP. In the years 1960, it seems that they could not imagine to not teach geometry. Indeed, the geometry was considered as the field for initiation of the reasoning from many centuries. But, what kind of geometry?

The members of APMEP wrote in the *Chantiers mathématiques* of 1964–1965 that, as it was necessary to teach geometry, the true problem was to improve its presentation (Daubelcour 2004, p.174). Moreover, they considered that it was more or less obvious that the study of vector spaces would permit to give to the geometry his place in modern mathematical teaching, in such a way the geometry would be adapted to the nowadays conditions of the science and of the world.

Here it is interesting to mention the two books of Choquet and Dieudonné of 1964, which were intended to teachers and to students. Choquet wrote in his *The teaching of Geometry* that we have to prefer the methods leading to the fundamental notions that twenty centuries of mathematics finished to bring out: the notions of set, the relations of order and of equivalence, the algebraic laws, the vector spaces, symmetry, and transformations. Always in 1964, Dieudonné explained in his *Linear Algebra and Elementary Geometry* that, in the time of a great proliferation in all the sciences, all the things that can condense and lead to unification has a virtue that we could not over-estimate.

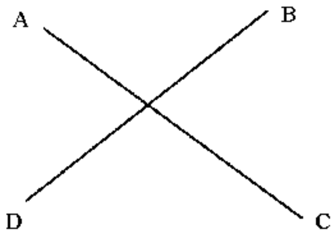
Frenkel edited a book entitled *Geometry for future teacher* (1973), where he developed the ideas expressed ten years before by Dieudonné. He considered that, mathematically speaking, elementary geometry has not to be distinguished from linear algebra, but only by artificial boundaries on the dimension and sometimes by the field concerned. As consequence, we find in the schoolbook for the last grade of the Lycées this definition: in an affine space E associated to a vector space e, we named affine variety through a point A and with direction e', a subspace of e, the set of the points M of E, such that the vector AM is an element of e'. We noted (A, e') this affine variety (Daubelcour 2004, p.184).

3.3 The geometry in the "retraining" of teachers in the IREM

The first IREM created in France, Paris, Strasbourg, Bordeaux and then Rouen, etc. were generally managed by militants of the Reform, and many trainers are teachers which belonged to APMEP, because they had study modern mathematics already. As a young mathematical assistant of the University, I entered in the IREM of Rouen in 1974 because the director, which was also my teacher in the University, initiated me to the new geometry.

To explain that the difficulties with the Reform were reinforced by the contain of the schoolbooks, I would like to tell an experience that I met in a training session in 1975–1976. A group of teachers of Collèges came to show to me an exercise of a schoolbook, because they were not able to solve it. It was a book for the pupils thirteen or fourteen aged, where the geometry was presented from the axioms of connection, in the same manner that Hilbert in his *Foundations of the Geometry* of 1899. For instance, "two distinct points determine a straight line" is an axiom.

The exercise shown by the teachers considered a finite geometry with four points A, B, C, D. The lines were noted $\{A, B\}$, $\{A, C\}$, etc. The first question asked to prove that A, B, C, D are two parallel straight lines. The teachers explained to me that it was not true and they point out this drawing, where two straight lines cut.



That means, that the idea that two points not only define, but that they "are" a straight line was too difficult to understand, because of the vividness of the old geometry. But $\{A, B\}$, $\{C, D\}$ has not a common point, so they are parallels.

4 APMEP and the reactions against the Reform of Modern Mathematics

The Association APMEP decided to issue a statement of the Reform in 1971 and to publish a new charter, entitled *Charter of Caen* (1972). The purpose was to make concrete proposals in favour of the Reform, about the necessity to create an IREM in each Academy and the manner to organize in-service teachers training, but also to examine some criticisms, which are pronounced from 1971.

4.1 Against the new teaching of Geometry in Collèges

From June 1971, the APMP expressed a doubt about the conditions in which the new Curricula has to be applied in the two last grades of Collèges. The modern mathematics already applied in the two first grades met a success, but the next step seemed compromised. In a letter published in the Bulletin of APMEP of the Autumn of 1970, some members of the Institutes of IREM asked that the Curricula of the two last years of Collèges did not contain a complete deductive theory of geometry and a systematic construction numerical structure.

In despite of that, new Curricula appeared in 1971 for the last two classrooms of the Collèges. One mathematical definition crystallized the critics: the one of an affine straight line, which was intended to pupils thirteen or fourteen aged.

By definition, an affine straight line D is a set E equipped with a family F of bijections from E to R such that:

a) For every element f of F and for every element (a, b) of $R^* \times R$, the application defined by $g(M) = af(M) + b$ also behold to F

b) Reciprocally, if f_1 and f_2 are two any elements of F, there exists (a, b) belonging to $R^* \times R$ such that $f_2(M) = af_1(M) + b$

The set F is called the support of the affine straight line D, an element M of E is called a point of the affine straight line D.

This definition provoked a little scandal. Jean Leray, French mathematician and member of the French Academy of Sciences, red it to two successive Ministers of Education of France, and they were not able to understand it. A funny paper appeared in the famous French satirical journal *Le Canard Enchaîné* (Bareil, 1992). In this period, the two promoters of the Reform, Choquet and Dieudonné, also protested. The first one wrote that, like it is putted, the Reform was an attack against the Geometry and against the recourse to the intuition. While the second one explained that it was a new scholastic, a more aggressive and more stupid scholastic placed under the banner of the modernism.

4.2 Against an elitist and dogmatic mathematical teaching

From 1975, there were more general critics, which considered that the generous ideas of the promoters of the Reform had not been reached, and moreover, that they completely failed.

In a paper entitled "Retro mathematics or modern mathematics" (1975), the French Physicist and thinker François Lurçat wrote that the formalism is tolerable for the Youth coming from the Upper Class, in despite it is bored and it discourage a part of them. But, for him, it constituted an almost insupportable obstacle for a young pupil coming from the Low Class or the Popular Class of the Society.

In the Institutes IREM, some researchers presented alternative proposals opposed to the Curricula of the Reform, like the teaching by themes, the teaching by projects, etc. In the middle of the year 1975, a National Committee "Epistemology and History of Mathematics" was created by the IREM, where the idea emerged rapidly that History of Mathematics could be a therapeutic against dogmatism (IREM, 1982). The Institutes IREM of the North and the West of France organised three National Colloquiums on "Mathematics and Society". They emphasized the elitist character of the Curricula of the Reform, which replaced the Latin by the Mathematics as a scholar discipline to select pupils.

5 Conclusion: on the "pernicious effects" of the Reform

By "pernicious effects" of a Reform, we mean effects which are not foreseen by the promoters, but which finally go against their ideas (Barbin, 1989). For any Reform, there are "pernicious effects" often linked to the change of a pedagogical innovation into an institutional renovation. But there are also particular effects, which are interesting to analyze in the purpose to understand more about the conditions of changing in Curricula, specially for scientific teaching.

Here we will focus on the epistemological conceptions, which was under the Reform of modern mathematics, because we would like to emphasize that a Reform not only needs mathematical trainings for teachers, but also epistemological trainings which can render more clear the purposes and the possibilities.

5.1 The mathematic as a study of structures

One of the main idea of the Reform of modern mathematics was the necessity to teach the notions of sets and relations, the fundamental structures of the algebra, the basic notions of topology. But the question was: how it could be possible to reconcile this mathematical contains to an active pedagogy?

For instance, in the third grade of the Collèges, the pupils are asked to start with objects or drawings to "bring out" the axioms of Geometry. That means that the mathematical foundations and structures could be spontaneously go out from the heads of the pupils, which are putted in situation to "recognize" them. But the mathematical theories had been constructed by the humanity in a long history where the problems and situations were more complicated than those proposed to the pupils.

This led "to manipulate" the pupils: after to have "bring out" the axioms they were asked to prove the theorems, which concern formal objects, by axiomatic. In particular, pupils had to prove theorems without using figures, which could help their reasoning. At this period, the situation of the pupils, which are supposed to construct themselves all the notions, appeared in all the scholar disciplines. It was one of the subjects of "Alert the babies!", a film produced by Jean-Michel Carré (1978).

5.2 The mathematic as an universal and abstract language

This conception led to a formal discourse, a "manipulation" of symbols, an excess of vocabulary and of sophisticated notions. The idea that "the mathematic is a language" came from the conceptions of some Bourbakists, like Dieudonné, and it was widely adopted by mathematicians, politicians and educationalists. It is clear that such a conception isfavourable to select pupils who are accustomed to handle language without references to any problem, which could bring a meaning.

Moreover, the notion of "abstract language" reinforced the opinion that there exist two kinds of students: "abstract pupils" and "concrete pupils". Indeed, it was the only manner to explicate the failure of many pupils, and it was also a good way to explain that it is not possible to make something with the "concrete pupils".

The idea of a universality of the mathematical language was also a good manner to justify that the mathematics was the discipline of selection "par excellence". Indeed, this selection did not depend of the social origin of the pupils, but only expressed the necessity to learn the modern rationality.

REFERENCES

- APMEP, 1967, "Charte de Chambéry", *Bulletin de l'APMEP*, **257**, 138–144.
- APMEP, 1972, *Charte de Caen*, Paris: Bibliothèque d'information sur l'enseignement mathématique.
- Barbazo E., 2010, "Les années des mathématiques modernes", Bulletin de l'APMEP, **490**, 543–553.
- Barbin, E., 1989, "Les effets pervers de la Réforme des "mathématiques modernes", *Société Française*, **33**, 26–28.
- Barbin, E., 2008, "Perennial notions and their teaching", in E. Barbin, N. Stehlikova, C. Tzanakis (eds), *History and Epistemology in Mathematics Education: Proceedings of the fifth European Summer University*, Prague : Vydavadelsky Press, 157–161.
- Barbin, E., to appear, "Le genre 'ouvrage d'initiation' : l'Exposé moderne des mathématiques élémentaires de Lucienne Félix (1959)", in E. Barbin, M. Moyon, *Les ouvrages de mathématiques dans l'histoire.. Entre recherche, enseignement et culture*. Limoges: Presses Universitaires.
- Bareil, H., 1992, "La réforme des mathématiques modernes vue par un enseignant du terrain". Bulletin de la SMF, **54**, 13–24.
- Baudelot C., Establet R., 1971, *L'école capitaliste en France, Paris*: Maspero.
- Bourdieu, P., 1967, "Systèmes d'enseignement et systèmes de pensée", *Revue Internationale de sciences sociales*, **XIX**, 367–388.
- Carré, J.-M., 1978, *Alertez les bébés* !, Paris: Les films grain de Sable.
  http://www.filmsdocumentaires.com/films/392-alertez-les-bebes

- Choquet, G., 1964, *L'enseignement de la géométrie*, Paris: Hermann.
- Colmez, J., 1961, "La structure des mathématiques modernes", *Bulletin de l'APMEP*, **217**, 3–10.
- Daubelcour, J.-P., 2004, "Les mathématiques modernes en 1970", *Évolution des programmes d'analyse et de géométrie au XXe siècle en Terminale scientifique*, IV, 149–214 http://irem-old.univ-lille1.fr/publications/fascicules.htm
- D'Enfert, R., Kahn, P., 2011, *Le temps des réformes—Disciplines scolaires et politiques éducatives sous la Ve République. Les Années 1960*, Grenoble: Presses universitaires.
- D'Enfert, R., Gispert, H., 2011, "L'élaboration d'une réforme à l'épreuve des réalités : le cas des mathématiques modernes au tournant des années 1970", *Histoire de l'éducation*, **38**, 59–119.
- Dieudonné, J., 1964, *Algèbre linéaire et géométrie élémentaire*, Paris: Hermann.
- Frenkel, J., 1973, *Géométrie pour l'élève-professeur*, Paris: Hermann.
- IREM, 1982, *La rigueur et le calcul*, Paris: CEDIC.
- Lurçat, F., 1976, "Mathématiques 'rétro' ou mathématiques modernes", *Orientations*, **55**, 331–369.
- Pair, C., 1961, "Une expérience d'enseignement des notions modernes" *Bulletin de l'APMEP*, **217**, 29–34.
- Walusinsky, G. W., 1963, "La règle de trois n'aura pas lieu", Bulletin de l'APMEP, **231**, 386–388.
- Walusinsky, G. W., 1963, "Études spatiales et enseignement des mathématiques", *Bulletin de l'APMEP*, **231**, 411–416.

BOOK XIII OF THE *ELEMENTS* :
Its Role in the World's Most Famous Mathematics Textbooks

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ABSTRACT

Euclid's *Elements* is the prototype geometry textbook. Yet, what this means for us and what it may have meant for a student in antiquity are not necessarily the same. A sign of this difference is Proclus' claim that Book XIII of the *Elements* concerning the construction of the regular solids and their mutual relations, which is usually left out of geometry textbooks—even those closely based on the *Elements*—is one of the two main goals of Euclid's work. It is argued here that what makes the *Elements* an ideal textbook for students, in Proclus' view, a view fitting the ancient idea of *paideia*, is its ability to turn students towards the kind of ordered cosmos that Plato sets out in the *Timaeus*. Conversely, the manner in which wholes are constructed in Book XIII, both the individual solids and their *taxis*, also turns one towards the wholeness of the entire treatment of geometry in the *Elements*.

Keywords: Euclid's *Elements*, Book XIII, Proclus, textbook

Introduction

Euclid's *Elements* is the prototype geometry textbook. The format of Euclid's *Elements* has certainly served as the inspiration for geometry textbooks in the past, and, in fact, at least until the end of the 19thcentury geometry textbooks tended to be directly based on the *Elements* itself, or a version of it, or a reaction to it.[1] One might even go so far as to say that what it meant to write a new geometry textbook then was to repackage Euclid in a more appealing, more pedagogically sound, or more modern or rigorous form.

In the case of Simson's *The Elements of Euclid* (1781/1804) and Playfair's *Elements of Geometry* (1795/1866), this meant keeping the development bound to Euclid's text. Both introduce modifications to be sure: Playfair, for example, points out the usefulness of algebra, adding though that it is in a simple form and does not depart from the nature of the reasoning (Playfair, 1795/1866, preface, page numbers absent). Still, the first sentence of Playfair's preface provides the main message: "It is a remarkable fact in the history of science, that the oldest book of *Elementary Geometry* is still considered as the best, and that the writings of Euclid, at the distance of two thousand years, continue to form the most approved introduction to the mathematical sciences." And Playfair goes on to give the reasons for this: "This remarkable distinction the Greek Geometer owes not only to the elegance and

[1] The overwhelming presence of Euclid's *Elements*, either as a model or as an *anti*-model (!) for mathematics textbooks, can be seen, for example, in Roméra-Lebret (2009), Giacardi (2006), Sinclair (2008), and Carson & Rowlands (2006).

correctness of his demonstrations, but to an arrangement most happily contrived for the purpose of instruction...” Even Legendre, whose Elements de Géométrie departs significantly from Euclid, does not fail to acknowledge Euclid, saying that while his goal is to improve on the rigor of Euclid he stays fairly close to the method of the Elements of Euclid as well as that of Archimedes’ *On the Sphere and the Cylinder* (Legendre, 1794, p.v). And it is significant that the epigraph for the book is, “Si quid novisti rectius istis, candidus imperti,” “If you know better than this, tell me candidly”(Horace, *Epistles* I:6): whether it is Legendre boasting about his own improvements of Euclid or about Euclid himself, it is a challenge that almost certainly one way or the other is directed towards Euclid.

As textbooks of geometry, all of these textbooks based on Euclid—Legendre’s as well—emphasized principally the subjects contained in Books I–VI, XI, XII of the *Elements*, that is, the basic propositions about triangles, rectangles, and parallelograms; circles and inscribed and circumscribed figures; ratio, proportion, and similar figures; and elementary solid geometry. This was indeed the core of the school course. In the case of higher education at the start of the 19thcentury, Nathalie Sinclair (2008) summarizes that situation, saying, “Typically, the geometry curriculum [she is specifically referring to that] would include the first six books of Euclid, and perhaps...Books XI and XII” (p.15).

None of this is shocking: whether or not we use Euclid as a geometry textbook, we recognize the contents of Books I–VI to constitute the *sin qua non* material for geometry instruction. In fact, if we think of the *Elements* as a textbook, we are likely to think mostly of the *first book only* of the *Elements*, the book where the definitions, postulates, and basic theorems of geometry are laid out. Legendre’s *Elements* does treat the regular solids, the content of the last book of the Elements; however, he does so in an appendix. Thus it is with considerable surprise that we discover when we read Proclus’ *Commentary on the First Book of the Elements* that the real point of the whole work, in his view, was in fact the last book, Book XIII on the five regular, or cosmic, solids, precisely the book that for the most part failed to find its way into the geometry curriculum, and still does not. What is this book about? Why might Proclus think that it is at the heart of the *Elements*? And since, as we shall see in a moment, Proclus’ interests were deeply educational, what does this say about the *Elements* as an educational text and how Proclus, at least, understood the goals of mathematics education? These are the questions that I would like to consider in this paper. I am not sure I shall succeed in answering them definitively; however, I will be pleased if they bring out the great difference that may exist between how we, since the eighteenth century, conceive the intent of a geometry text and how that was conceived in antiquity.

1 Proclus’ View of Book XIII

Before giving a brief sketch of Book XIII of the *Elements*, I want to amplify Proclus’ estimate of its importance to the whole of the *Elements* and to make it clear that Proclus’ views here are held with an eye to the *Elements* as a textbook, that is, as a book for learners.[2]

Now, if there is any doubt whether Proclus meant what he said about Book XIII, it is enough to remark that Proclus tells us *no less than four times*[3] that of the two aims possessed by the entire work one is the investigation of the cosmic figures, the five regular solids. This claim flies in the face not only of

[2]Much of what follows in this and the remaining sections appear in an expanded form in Fried (in press).

[3]*In Eucl.* (Friedlein). pp. 68, 70, 71,74 (see also pp. 82–83). The English translations are taken from Morrow (1970); however, I will always use Friedlein’s page numbers, which, being included also in Morrow’s text, will allow the reader to

what modern mathematicians, among whom I include Simson, Playfair, and Legendre, consider the core of the *Elements*, but also of what historians of Greek mathematics, raised on this more modern Euclidean tradition, often have thought. When Thomas Heath comes to this claim by Proclus, for example, he says confidently that it is "obviously incorrect," expanding thus: "It is true that Euclid's *Elements* end with the construction of the five regular solids; but the planimetrical portion has no direct relation to them, and the arithmetical no relation at all; the propositions about them are merely the conclusion of the stereometrical division of the work" (Heath, 1956, I, p.2). Heath's skepticism[4]of course is not out of place. It is indeed difficult to see how Proclus can put aside the immense wealth of other mathematical work in the *Elements* and single out so pointedly one of its shortest books, Book XIII on the regular or Platonic solids—Book XIII, in this view, must take precedence over Book X with its 115 propositions on incommensurables, or Book V which sets out the theory of proportion, or Books VII–IX on the theory of numbers, not to speak of Books I, III, IV, and VI, which make up every modern student's own elements of geometry. Nor does the deductive structure of the book give credence to Proclus' claim, for while the propositions in Book XIII do rely widely on propositions from other parts of the *Elements*, they do not draw from all parts of the *Elements*, as Heath points out, and, when they do, it is not always in a very striking way.[5]

As for the Elements being a kind of textbook, the second of the two aims of the *Elements*, as Proclus sees them, refers explicitly to learners, to *manthanontes*: it is that with Euclid's work at hand learners will have a treatment of the elements before them and the means to perfect their understanding of the whole of geometry (*In Eucl*. p.71). On the face of it, this seems to be a different kind of aim than the first. Heath unsurprisingly takes this aim more seriously than that concerning the Platonic solids, which he attributes only to Proclus' Platonic loyalties. Proclus himself, according to Heath, had difficulty reconciling the two aims and that, "To get out of the difficulty. . ." (p.2) Proclus delegated one aim to learners and the other to the subject. But, if this really is a difficulty for Proclus, it is hard to see how it is resolved by referring one aim to the learner and the other to the subject; can learners perfect their knowledge about the whole of geometry and yet exclude its subject? It is more likely that not only did Proclus not see any difficulty here but that he also saw these two aims as essentially complementary.

As for Proclus' Platonic loyalties, that is not to be denied—Heath is certainly right on that count. Moreover, Proclus does associate Euclid with Plato; indeed, he does so explicitly just when he first tells us that Book XIII was the aim of the *Elements*.[6] Proclus was in fact one of the last heads of the Platonic academy, and that should be kept in mind not only because the academy was a place of learning but also that education itself was at the center of Plato's thought. In that light, Proclus' assertion that "Euclid belonged to the persuasion of Plato. . .," meant, in effect, that Euclid's Elements had a place among the studies of the academy, that it had educational value, or, to use the far more subtle Greek term, that it contributed to *paideia*. To understand how Book XIII can be conceived as the aim of the *Elements*, then, one must view it in terms of *paideia*—not so much what facts or skills it affords the learner as what view of the world it gives the learner and what it does to the learner. This, in a nutshell, is the thesis of my entire paper. But to see how Book XIII serves this purpose, we do need to

follow the Greek text or Morrow's translation.

[4]The skepticism is also shared by Morrow (1970, p. l)

[5]Mueller is less dismissive of Proclus' claim, even though Mueller, like Heath, places great weight on deductive structure in judging the relative importance of parts of the *Elements* (see Mueller, 1981, p. 303).

[6]*In Eucl*. p. 68: "Euclid belonged to the persuasion of Plato and was at home in this philosophy; and this is why he thought the goal of the *Elements* as a whole to be the construction of the so-called Platonic figures".

take at least a brief look at the contents of the book.

2 A Brief Sketch of Elements, Book XIII

Book XIII can be divided into four parts. The first part spans propositions 1 to 6; the second, propositions 7 to 12; the third, propositions 13 to 17; while the last part contains just the single proposition, proposition 18.

Propositions 1–6, making up the first part of Book XIII, contain aspects of the extreme and mean ratio (*akron kai meson logon*), what is popularly called the "golden section." With one exception, XIII.2, these propositions serve as lemmas used in the rest of the book[7], particularly in propositions 16 and 17 in which the icosahedron and dodecahedron are constructed. The fact that there is an exception at all, however, makes it clear that these are not only a set of tools but also a subject for the book. Viewed this way, Euclid would be underlining that the regular solids are born in the realm of proportion; the proportion determined by the extreme and mean ratio, moreover, has a special place, being a continuous proportion (*sunexēs analogia*) whose three terms are the segments of a line and the whole line.[8] And it is at least suggestive that Plato calls a continued proportion, which ". . .makes itself and the terms it connects a unity in the fullest sense. . .," the "most beautiful of bonds" (*desmōn kallistos*).[9]

The way in which the segments of a line divided according to the extreme and mean ratio are classified and compared in XIII.6 and the other propositions in the first part of Book XIII hints, perhaps, at the way the sides of the cosmic figures will be compared and classified later in the constructions and especially in the final part, proposition 18, where all of the sides of the various figures are brought together and described in one diagram. But in propositions XIII.1–6 that hint is at best vague and visible at all only on hindsight. In the next set of propositions, propositions XIII.7–12, the prefigurement of the constructions, however, is more cogent.

Propositions XIII.7–11 all concern, one way or another, properties of regular pentagons or of the decagons derived from them, while proposition 12 concerns an equilateral triangle inscribed in a circle, telling us, in particular, that the square on the side of the triangle is triple the square on the radius of the circle. Indeed, except for XIII.7, all of these propositions consider the regular figures inscribed in circles, just as cosmic figures, the regular solids, are inscribed in spheres.

Thus, taken together, propositions XIII.1–12, what I have framed as the first two parts of Book XIII, prepare us for the constructions of the regular solids not only by providing necessary lemmas, that is, prepare us in a deductive sense, but also it prepares us, one might say, in a thematic sense. With that, let us move on to the third part of Book XIII, the constructions themselves.

To start, it is worth noting that, except for the tetrahedron, the definitions of the regular solids are not given in Book XIII, but in Book XI.[10] One arrives to Book XIII, therefore, already knowing what the solids are: the job of Book XIII, rather, is the construction and ordering of the solids. The construc-

[7] Noted also by Heath (1956) and Mueller (1981).

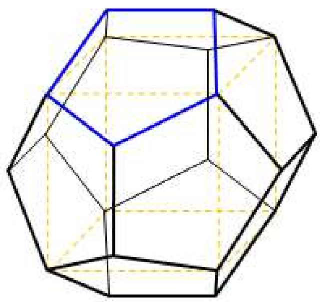
[8] Just as a reminder regarding the meaning of these terms. The quantities A, B, C are in a *continuous proportion* whenever, A:B::B:C. A segment S is divided into the *extreme and mean ratio* whenever the two parts of the segment and the whole segment are in a continuous proportion, that is, if S is divided into two segments A and B (where B is the greater segment), then A:B will be an *extreme and mean ratio* if A:B::B:S, that is, A:B::B:(A+B).

[9] *Timaeus*, 31c, translation by M. Cornford (1975), with some minor modification by myself.

[10] The definitions are precise; yet they do not allow one to visualize the solids easily. For example, the dodecahedron is defined as a "solid figure contained by twelve equal equilateral, and equiangular pentagons." The constructions in Book XIII also leave to the reader much of the work of visualizing the solids, but enough guidance is given to make that possible.

tions appear in the book as follows. The tetrahedron composed of four equilateral triangles, three triangles meeting at each vertex, is the first solid constructed in XIII.13. Next, the octahedron composed of eight equilateral triangles, four triangles meeting at each vertex, is constructed in XIII.14. The cube is constructed in the next proposition. The icosahedron composed of twenty equilateral triangles, five triangles meeting at each vertex, is constructed in XIII.16. Finally, in XIII.17 the dodecahedron composed of twelve regular pentagons is constructed. The form of each of these propositions follows the same general pattern: each requires the figure be constructed, inscribed in a sphere, andits side compared to the diameter of the sphere. The constructions themselves, however, are completely distinct—there is no master scheme for the constructions; there are five procedures for the five solids.

Thus for example, in XIII.17 where Euclid describes the construction of the dodecahedron, he begins with another previously constructed figure, the cube, and then literally builds the dodecahedron around the cube (see figure 1), showing in the course of that that the ratio of the side of the dodecahedron to the side of the cube is the extreme and mean ratio. It is worth noting in this connection that what Euclid actually constructs in XIII.17 is just one face of the dodecahedron (see fig. 2) and simply tells the reader that, "Therefore, if we make the same construction in the case of each of the twelve sides (*pleurai*) of the cube[11], a solid figure will have been constructed which is contained by twelve equilateral and equiangular pentagons, and which is called a dodecahedron." In other words, the reader, the learner, is not only asked to complete the details of how the construction is to be completed for the other sides of the cube, but also to imagine the shape of the finished figure.[12]



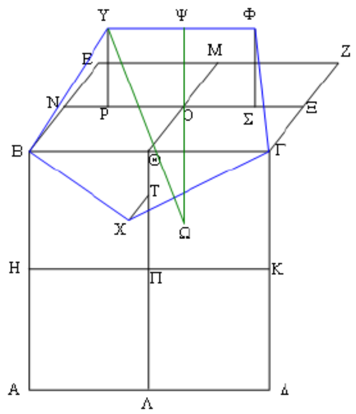
XIII.17: The completed dodecahedron

While this task is not impossibly difficult given the head start Euclid has already provided, it is not effortless. It requires an act of the imagination more than a concrete execution with, say, compass and straight edge; the construction is distinguished in this way from constructions such as that of the equilateral triangle, the first regular figure, which opens the very first book of the *Elements*.

The next parts of the proposition are 1) to show how the dodecahedron can be inscribed in a sphere and 2) that the side of the dodecahedron is an "apotome" (*apotomē*). This is a term from Book X of the *Elements*: an apotome is the difference between two rational lines that are commensurate in square only, that is two lines a and b such that sq.a: sq.b::m:n where m and n are whole numbers, for

[11] That is, the edges of the cube.

[12] A comparison might be made here to the constructions ending Book I of Apollonius' *Conica* (for example, Conica, I.52). Having provided the vertex and base circle, Apollonius asks us there to "imagine a cone"(*noesthō kōnos*), that is, to complete the picture.

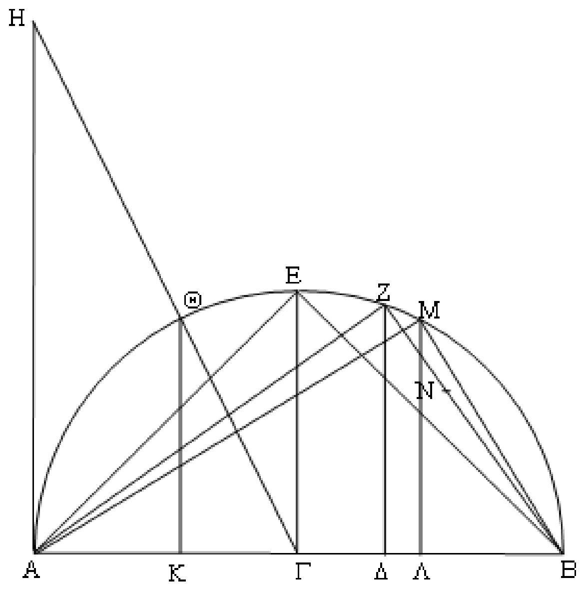


Proposition XIII.17: Construction of the face of the dodecahedron and finding the circumscribing sphere

example, the difference between the diagonal of a face of a cube and the diagonal of the cube itself. It is fair to ask, why is it important to show that the side of the dodecahedron is an apotome? It is not hard to see why one should want to know that the ratio of the side of the dodecahedron to the side of the cube is the extreme and mean ratio, for, together with the fact that the square on the diameter of the sphere circumscribing the cube is three times the square on the side of the cube, this fact allows one to construct the side of the dodecahedron given the side of its circumscribing sphere. But why must we know that it is an *apotome*? I think the only reasonable answer is that Euclid wanted to show how the ordering of the solids fits with the other great taxonomic book in the *Elements*, namely, Book X.[13]

Indeed, reading proposition 18, the closing proposition of the book, one sees that a taxonomy of the solids and their sides was truly on Euclid's mind. Proposition XIII.18, which I have deemed a distinct part of Book XIII, comprises two completely independent parts. The first part of this proposition is a synopsis of the relations between the sides of the various solids and the diameter of the sphere circumscribing them. It is unique in the *Elements* in that while it contains geometrical arguments and there are claims proved along the way, its main goal is not to prove a particular claim or solve a particular problem; what it does is "to set out and bring into comparison with one another" (*ekthesthai kai sunkrinai pros allēlas*) various lines that have already been determined. The sides of the five constructed figures are laid out all together in a half circle whose diameter is the diameter of the sphere circumscribing them. Thus, in figure 3, AB is the diameter of the sphere, AZ is the side of the tetrahedon, BZ is the side of the cube, BE is the side of the octahedron, MB is the side of the icosahedron, and BN is the side of the dodecahedron. The point Γ divides AB into half; Δ divides AB such that AΔ is twice ΔB or two thirds AB; Λ is such that $\Gamma\Lambda = \Lambda K$ where K is determined by the intersection of the circle with HΓ drawn from the extremity of AH, which is equal to AB and at right angles to it; and N divides BZ, the side of the cube, in the extreme and mean ratio. Little new is proven in this first part of XIII.18: all have been established in the constructions themselves. As the

[13] The point remains even if one fully accepts that Theaetetus was behind both Book X and Book XIII. Euclid was an editor, if not more than that: he had to concur that the connection between the theory of irrational lines and the regular solids that might have been in Theaetetus' own work (assuming that a written work existed) was essential for the flow of the work.



The setting out and comparison of the sides of the five regular solids

diagram itself suggests, what Euclid does here is to bring the cosmic figures together in the text and compare them as parts of a cosmos: one is brought together with the other like the clicking of glasses at a meeting of friends.

The second part of XIII.18 shows that no other regular solids exist besides the five constructed in the previous propositions. This Euclid shows quite simply using the fact proved in Book XI (prop. 21) that a solid angle is contained by (three or more) plane angles which together must be less than four right angles. Therefore, the solid angles of a regular solid can be contained by three, four, or five equilateral triangles, three squares, or three pentagons. The second part of XIII.18 is, in a way, a limit of possibility, a *diorismos*, but it is not a typical diorismos. Rather than bounding an unlimited number of possibilities, it shows that the five figures constructed in propositions 13–17 cannot be exceeded, that is, they form a totality. The first part of XIII.18 shows the relations between the cosmic figures with respect to a single sphere that circumscribes them: now we know that that sphere circumscribes a true cosmos, an ordered whole (as in the sense of the Greek, *kosmos*). Obviously there is more one can say about the regular solids, but, unlike any other topic in the *Elements*, this concerns a set of *objects* that cannot be further extended. There is no other book in the *Elements* that presents a *whole* in this sense and a true ending.

This display of wholeness, I believe, is the key to understanding the special status Proclus attaches to Book XIII as a *telos* and to understanding its special educative value. But to complete that argument, we need to consider first what being educative might have meant for a thinker like Proclus, and that means, among other things, considering what place mathematics occupied in education in Classical times.

3 Book XIII as an Educational Text

We have grown used to a mathematics education that begins with elementary school mathematics and continues by stages to university level mathematics. One expects to find a kind of educational program in the classical world that led to mathematical work such as Archimedes' *On the Sphere and Cylinder*, or Apollonius' *Conica*, or Euclid's *Elements*. But one's expectations here are frustrated: there is no clear path from very rudimentary mathematical training to studies in higher mathematics at institutions of advanced learning, such as the *Museum* in Alexandria and the *Academy* in Athens; there appears to be a gap.[14] Part of the problem may well be that we think of mathematics today as an entire somehow self-contained domain. The steps towards mathematics are, therefore, also mathematical in character: numbers lead to arithmetic; arithmetic leads to algebra; algebra leads to analysis, etc. There is good reason to think, however, that mathematical inquiry in classical times was not so far from philosophical inquiry (see Fried, in press). What steps lead towards that kind of inquiry? What kind of training is required? What kind of textbook? We may be looking in the wrong direction in trying to see why Euclid might serve as a textbook. Perhaps we should look more in the direction of philosophy or rhetoric, which was a central locus of educational practice in the classical world (see Bernard, 2003a, 2003b).

What I am suggesting is that the propaedeutic for higher mathematics may not necessarily have been a similar but simpler type of inquiry. If one thinks about the way to philosophy as being akin to the way to mathematics, one might consider a more general education. The idea of a general preparation that forms the constant foundation of one's thoughts, actions, and works is close to the classical idea of *paideia*. As an educational ideal, *paideia* was hardly the monopoly of philosophy; the rhetorical tradition claimed that for itself quite as much, if not more, but both aspired to genuine knowledge and a perspective on how one should live. In the philosophical tradition as represented by Plato, possessing *paideia* meant leading a philosophical way of life. As Jaeger puts it in his great three volume work (Jaeger, 1945) entirely dedicated to defining the notion of *paideia*, "[Plato's] *philosophos* is not a professor of philosophy, nor indeed any member of the philosophical 'faculty', arrogating that title to himself because of his special branch of knowledge (*texnudion*)... Although...he uses the word to imply a great deal of specialized dialectical training, its root meaning is 'lover of culture', a description of the most highly educated or cultured type of personality...[Plato's *philosophos*] is averse to all petty details; he is always anxious to see things as a whole; he looks down on time and existence from a great height" (vol. II, p.267).

Seen in the context of a philosophical education, one would have to view mathematics in terms of its potential to make one "see things as a whole" and point one to a higher life; this gives mathematics a place in one's thinking life, in one's *paideia*. The true mathematician, that is, one for whom mathematics is part of *paideia*, is the kind of mathematical learner that Proclus must have in mind when he claims that one of two principal aims of the *Elements* is Book XIII.

What then is it that this philosophico-mathematical learner gains from Book XIII? The main thing a learner gains is what we have already discussed at the end of the last section, namely, that one is

[14] Morgan (1999) writes: "Among non-literary elements of education, our ignorance of mathematics and mathematical education in the Classical period constitutes a problem in a class of its own" (p.52) (See also Mueller, 1991, p.88). For example, one of the only texts referring to older students' studying geometry is a remark by Teles, quoted by Stobeus to the effect that teachers of arithmetic and geometry (and riding!) are among chief plagues of students' life (see Marrou, 1982, p.176; Freeman, 1912, p.160).

given an image of a whole in a way that is more clear than in any other book of the *Elements*. This is both because the five solids are only five and, therefore, form a totality, and because each figure is itself a kind of totality, whose likeness to a sphere is underlined by each being circumscribed within a sphere. This is the time to remind the reader that these five figures are also called the Platonic solids because of their role in Plato's *Timaeus*. The *Timaeus* is very much a dialogue about wholeness: words referring to the whole (to *holon* or the adjective *holos*) or the all occur throughout it with striking frequency. Proclus wrote a commentary on the *Timaeus*, and it is clear that the *Timaeus* was not far from his thoughts when he wrote his commentary on the *Elements*. He refers to the dialogue often. For example, in describing the contribution of mathematics to the theory of nature (*phusikēn theōrian*) he writes:

> It reveals the orderliness (*eutaxian*) of the ratios according to which the universe is constructed (*dedēmiourgētai to pan*) and the proportion that binds things together in the cosmos, making, as the *Timaeus* somewhere says, divergent warring factors into friends and sympathetic companions. It exhibits the simple and primal causal elements as everywhere clinging fast to one another in symmetry and equality, the properties through which the whole heaven was perfected (*ho pas ouranos eteleōthē*) when it took upon itself the figures appropriate to its particular region; and it discovers, furthermore, the numbers applicable to all generated things and to their periods of activity and of return to their starting-points, by which it is possible to calculate the times of fruitfulness or the reverse for each of them. All these I believe the *Timaeus* sets forth, using mathematical language throughout in expounding its theory of the nature of the universe [literally, the "whole"] (*peri tēs phuseōs tōn holōn theōrian*). (*In Eucl*. pp.22–23)

The connection between the regular solids and the construction of the elements described in the *Timaeus* can also be seen in a scholium to Book XIII in which the identification of the tetrahedron, octahedron, cube, and icosahedron with fire, air, earth, and water, and the dodecahedron with the all (*tōi panti*) is repeated.[15] It is not altogether clear when that scholium was written or by whom, but there is good reason to believe it was written prior to Proclus' time;[16]nevertheless, it shows that not only Proclus related the material in Book XIII to what is spoken of in the *Timaeus*. Recalling the *Timaeus* is important also in the way it presents wholes coming to be, namely, as, in some way, being constructed (the nature of which is, of course, one of the points where Neo-Platonism is not clearly in line with Platonism). In the *Timaeus* itself the four solids corresponding to the elements are constructed in a heuristic way, as if only to show that the elements can be constructed rationally the way the *demiourgos* constructs the entire universe, the whole itself. The dodecahedron is not constructed in the *Timaeus*, though it is, as we have seen, in the *Elements*. What one sees then in Book XIII is the precise completion of those constructions, how they might be realized in detail. It ought to be recalled, however, that that detail is meant only to be enough to allow the learner to imagine the completion: the geometrical constructions, as we described in the last section, are a kind of prompt for them to open the eyes of their imaginations.[17]

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[15]See Euclid (1969–77), vol. V., p. 309.

[16]The particular scholium cited here is from Heiberg's *Schol Vat. series P*, which means there is a good chance it was written before Theon's time in the 4thcentury (see Heath, 1956, I, pp. 64–69).

[17]One might make a comparison here to how Socrates, at the start of the *Timaeus*, asks to see the image created in the previous day's talk to come to life (*Tim*. 19b): learners, reading Book XIII, must bring the cosmic figures to life.

But construction in the *Timaeus* is central not only to the conception of the individual cosmic figures, but also of cosmos considered as a whole, whether or not Plato took that construction literally or metaphorically. Be that as it may, what construction shows is how parts fit together, or, to use the more suggestive Greek word, how they demonstrate *harmonia*. Indeed it is the fitting together into a whole that makes a cosmos a cosmos. Its order is one of organization and place, of taxis, rather than, as Jacob Klein (1985, esp. pp. 30–34) has pointed out, one of *lex*, of law, which marks the modern sense of an ordered universe. It is this kind of order that is exemplified so well in proposition XIII.18 setting out the relationships of the five regular solids. As suggested in that discussion, this is not the only proposition which relates one kind of mathematical object to another—what proposition does not?—but, with the final part of XIII.18 showing that there are no other regular solids, that the five form a totality, the first part becomes a *taxis* indeed of a whole.

Interestingly enough, this brings us to Proclus' second claim regarding the aim of the *Elements*, namely, that the work presents learners with a thorough treatment of the elements of geometry and the means to perfect their understanding of the whole of geometry (*In Eucl*. p.71). Now, we have already addressed the point that learners are those who want to cultivate their *paideia* and therefore are not easily separated from other mathematicians. But what ought to be said in addition is that learners' perfecting their understanding of the whole of geometry may not be very far from their contemplating a cosmos, like that suggested by the material of Book XIII. There is no fine line between the taxis of the material of geometry and its discursive means. Thus Proclus' description of the formal aspects of geometry, of synthesis and analysis, of definitions, hypotheses, and demonstrations, flows seamlessly from his description of the material of geometry moving from the simple to the complex, from points to bodies, and from the complex back to the simple (*In Eucl*. pp.57–58). In this, he may be betraying his Neo-Platonic rather than strictly Platonic outlook. For in struggling with the question of wholeness of The Soul with respect to individual souls, Plotinus seriously considers the image of geometrical science, how, "Each theorem contains accordingly the total science in potency and the total science does not exist one whit the less" (*Enneads*, IV, 3, in O'Brien, 1964), though, it must be added that Plotinus, for his own reasons, ultimately rejects the image.

In any case, what is clear is that to the extent consideration of the five regular figures is consideration of a cosmos, an organized whole, this seemingly special topic can be related to the organized whole of the *Elements* as a science. Proclus' two aims of the *Elements* thus really become one: seeing the whole portrayed by the science becomes at once seeing the whole of the science. As a textbook for a Greek mathematician, then, we begin to see the *Elements* in a different light. Besides providing tools and specific facts, it also turns viewers toward a view of the world, or rather shows them what a world can be. This makes it a very different kind of textbook than the kind Simson or Legendre imagined, but, for one like Proclus, a book very much for students nonetheless.

REFERENCES

- Bernard, A. (2003a). Sophistic Aspects of Pappus's Collection. *Archive for History of Exact Sciences*, 57(2), 93–150.
- Bernard, A. (2003b). Ancient Rhetoric and Greek Mathematics: A Response to a Modern Historiographical Dilemma. *Science in Context*, 16(3), 291–412
- Carson, R. N., & Rowlands, S.: 2000. *A Synopsis of the Collapse of the Geometry Standard in the UK*. Unpublished report of the Centre for Teaching Mathematics, University of Plymouth, UK

- Cornford, M. (1975). *Plato's Cosmology*. Indianapolis, IN.: Bobbs-Merrill Company, Inc.
- Euclid (1969–77.) *Euclidis Elementa*, post Heiberg edidit E. Stamatis, 5 vols. Stuttgart: B. G. Teubner.
- Freeman, K. (1912). *Schools of Hellas*. London: MacMillan and Co. Limited.
- Fried, Michael N. & Unguru, Sabetai (2001). *Apollonius of Perga's Conica: Text, Context, Subtext*. Leiden, The Netherlands: Brill Academic Publishers.
- Fried, Michael N. (in press). Euclid's Book on the Regular Solids: Its Place in the Elements and Its Educational Value. In B. Sriraman (Ed.), *Crossroads in the History of Mathematics and Mathematics Education*. Charlotte, NC: Information Age Publishing.
- Giacardi, L. (2006). From Euclid as Textbook to the Giovanni Gentile Reform (1867–1923): Problems, Methods and Debates in Mathematics Teaching in Italy. *Paedagogica Historica,. 42*(4&5), 587–613.
- Heath, T. H. (1981). *A History of Greek Mathematics*, 2 volumes. New York: Dover Publications, Inc.
- Heath, T. L. (1956). *The Thirteen Books of Euclid's Elements. Translated from the text of Heiberg with introduction and commentary*, 3 vols. New York: Dover Publications, Inc., 1956)
- Jaeger, W. ,1945, *Paidea: The Ideals of Greek Culture*, 3 vols., Gilbert Highet (trans.), New York: Oxford University Press.
- Klein, J. (1985). The World of Physics and the "Natural" World. In R. B. Williamson and E. Zuckerman (Eds.), *Jacob Klein: Lectures and Essays* (pp. 1–34) Annapolis, MD.: St. John's College Press.
- Legendre, A. M. (1794). *Elements de Géométrie*. Paris: Didot.
- Legendre, A. M. (1825). *Elements of Geometry*, 2ndEdition. Translation by John Farrar. Cambridge: At the University Press.
- Marrou, H. I. (1982). *A History of Education in Antiquity*. George Lamb (trans.). Madison, Wisconsin: University of Wisconsin Press.
- Morgan, T. J. (1999). Literate Education in Classical Athens. *Classical Quarterly*, 49(1), 46–61.
- Morrow, G. R. (1970). *Proclus: A Commentary on the First Book of Euclid's Elements*. Princeton: Princeton University Press.
- Mueller, I. (1981). *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. Cambridge, MA: MIT Press.
- Mueller, I. (1991). Mathematics and Education: Some Notes on the Platonic Program. In ΠΕΡΙ ΤΩΝ ΜΑΘΗΜΑΤΩΝ, *special edition of Apeiron*, *XXIV*(4), 85–104.
- O'Brien, E. (1964). *The Essential Plotinus*. New York: New American Library.
- Playfair, J. (1795/1866). *Elements of Geometry: Containing the First Six Books of Euclid*. Philadelphia: J. B. Lippincott & Co.
- Proclus (1967). *Procli Diadochi in primum Euclidis Elementorum librum commentarii*, (G. Friedlein). Hildesheim.
- Roméra-Lebret, P. (2009). Treaching New Geometric Methods with an Ancienct Figure in the Nineteenth and Twentieth Centuries: The New Triangle Geometry in Textbooks in Europe and USA (1888–1952). In K. Bjarnodóttir, F. Furinghetti, G. Schubring (eds.), *"Dig Where You Stand" Proceedings of a Conference on On-Going Research in the History of Mathematics Education.* Reykjavík: University of Iceland.
- Simson, R. (1781/1804). *The Elements of Euclid; viz. The First Six Books, Together with the Eleventh and Twelfth*, 12thEdition . London: F. Wingrave.
- Sinclair, N. (2008). *The History of the Geometry Curriculum in the United States*. Charlotte, NC: Information Age Publishing.

THE MATHEMATICS DEVELOPMENT OF THE BOOK SEA MIRROR OF CIRCLE MEASUREMENTS (CEYUAN HAIJING)

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ABSTRACT

The book "Sea mirror of Circle measurements" (Ceyuan Haijing) is completed in 1248 by Li Zhi. The mathematics of the book is to investigate cases of circles inscribed in a right angle triangle. The circle in the book represented a circular fort, with a diameter set at 240. Different information on the sections of the triangles are given in order to calculate the diameter using Tian Yuan Algebra (setting up equations). If a, b, c are respectively the three sides the right angle triangle and the diameter of the inscribed circle is d, then we have the following two conditions: (i) $a^2 + b^2 = c^2$ (ii) $a + b = c + D$ (D as diameter of the inscribed circle). There are 12 chapters in the book Ceyuan Haijing, and each questions in the book is deal with individually in setting up equation to find the diameter. In this study, the mathematics contents of chapter 3 to chapters 6 could be summarised into a general type of mathematics problem based on the above two conditions. For example, questions in chapter 3 can be summarised as question satisfying the following two conditions (i) $\langle c + b\rangle(b) = p$, and (ii) $\alpha_0\langle ab\rangle + \alpha_1\langle ca\rangle + \alpha_2\langle cb\rangle + \alpha_3\langle a^2\rangle + \alpha_4\langle b^2\rangle + \alpha_5\langle c^2\rangle = q$, where α_0, α_1, α_2, α_3, α_4, α_5, p, q are constant (here $\langle a\rangle$ denotes a ratio representation, for example $r = \frac{ab}{a+b+c}$ is denoted by $r = \langle a\rangle(b) = \langle b\rangle(a) = \langle ab\rangle$), As questions in chapter 3 and chapter 4 are symmetric, the process on setting up equation in chapter 3 and 4 is very similar. The same process applies to questions in chapter 5 and chapter 6, as questions in these two chapters are also symmetric. The exploration of this general method will help to relate the work in mathematics in each chapterof the book and discover the possible relation among the thinking of the problems.

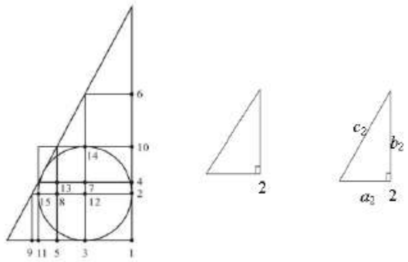
1 Introduction

In the book of sea mirror, there are a total of 12 chapters and 170 mathematical questions. All questions required to find the diameter of a circular fort inscribed in a right angle triangle, and in the following question format.

[Assume there is a circular fort of unknown diameter and circumference.]
One person walks out of the south gate 135 steps and another person walks out of the east gate 16 steps, and then they see each other.
[What is the diameter of the circular fort?]

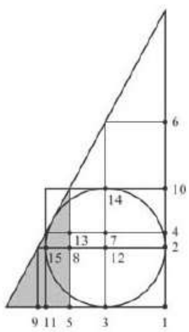
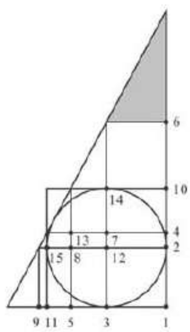
The statement in the squared bracket is the condition and question tag of the problem.

The condition of the question could be described by the following diagram. The point 2, 14, 15, and 3 in the diagram represent the four gates according to the following 2 (west gate), 14 (south gate), 15 (east gate), 3 (north gate). And in the diagram, the numbered point is the vertex of a right angle triangle. The three sides are denoted by a (gou, the base), b (gu, the height), c (xian, the hypotenuse). There are 15 triangles in the diagram. The triangle are numbered, for example, triangle 2 is the triangle with the right base vertex be numbered 2, and then the three sides corresponding are a_2, b_2, and c_2. The book aimed to establish, for each question, equation that will lead to the answer of the diameter of the fort (which is always 240). The unknown variable of the equations studied is given the name Tian-yuan (celestial element).



However, though there are some general formula mentioned in chapter 1 for solving the equation, there is no one general method of solving the equation in each chapter. The equation in each chapter is deal with individually. For example the process of setting equations of the 17 quetsions are quite different.

This paper aimed to give a general solution for each of the chapter 3, 4 5, and 6. For example, in chapter 3 of the book. Two conditions are given (uaually the length of two segments of the triangle in the 17 questions. For chapter 3, one condition is always the height b_2 of the triangle, and for chapter 4, one variable is always the base a_3.

Chapter 3	Chapter 4
	

The following is the list of the 17 questions in chapter 3 and chapter 4.

The following is the explanation of solving of the problem in the book from chapter 3 to chapter 6. We will use chapter 3 as an example to illustrate the process of setting up the equations. And provide the basic formula and proof for the remaining chapters, while the details of the all the examples (70 in total) are not included here.

b_2**(chapter 3)**																
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
c_4	a_{11}	b_{11}	a_{15}	a_{14}	c_{10}	c_2	c_1	c_6	b_{14}	a_{10}	c_{15}	c_{14}	c_6	c_8	$b_{14} + c_{14}$	$a_{15} + c_{15}$
b_2**(chapter 3)**																
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
c_5	b_{10}	a_{10}	b_{14}	b_{15}	c_{11}	c_{13}	c_1	c_9	a_{15}	b_{11}	c_{14}	c_{15}	c_9	c_7	$a_{15} + c_{15}$	$b_{14} + c_{14}$

2 The setting up of the generalised method

We will start with Chapter 3 (the 17 questions for $\mathbf{b_2}$). Problems in chapter can be generalised into the following format, finding the diameter of the circle with the following two conditions.

i $\langle c + b\rangle(b) = p = b_2$, the value of p equal to the length of leg (gu) b_2,

ii $\alpha_0\langle ab\rangle + \alpha_1\langle ca\rangle + \alpha_2\langle cb\rangle + \alpha_3\langle a^2\rangle + \alpha_4\langle b^2\rangle + \alpha_5\langle c^2\rangle = q$

where α_0, α_1, α_2, α_3, α_4, α_5, $p = (b_2)$, q are constants.

In order to do so, we set up the following proof.

Formula	Proof (by i. it could be proved)
❶ $\langle b^2\rangle = \frac{p^2-r^2}{2p}$	By i. $\langle c+b\rangle(b) = p$, $\langle cb\rangle = p - \langle b^2\rangle$ $\langle cb\rangle^2 = (p - \langle b^2\rangle)^2$ $\langle c^2\rangle\langle b^2\rangle = (\langle a^2\rangle + \langle b^2\rangle)\langle b^2\rangle$ $= p^2 - 2p\langle b^2\rangle + \langle b^2\rangle^2$ $\Rightarrow \langle a^2\rangle\langle b^2\rangle = r^2 = p^2 - 2p\langle b^2\rangle$ $\Rightarrow \langle b^2\rangle = \frac{p^2-r^2}{2p}$ ❶
❷ $\langle cb\rangle = \frac{p^2+r^2}{2p}$	By i. $\langle c+b\rangle(b) = p$, then $\langle cb\rangle = p - \langle b^2\rangle$ Using ❶ and substitute into RHS : $\langle cb\rangle = p - \frac{p^2-r^2}{2p}$ $\Rightarrow \langle cb\rangle = \frac{p^2+r^2}{2p}$
❸ $\langle a^2\rangle = \frac{2pr^2}{p^2-r^2}$	As $\langle a^2\rangle\langle b^2\rangle = r^2$, that is $\langle a^2\rangle = r^2/\langle b^2\rangle$ Using ❶ and substitute into RHS : $\langle a^2\rangle = \frac{2pr^2}{p^2-r^2}$
❹ $\langle ca\rangle = \frac{p^2+r^2}{p^2-r^2}r$	$\langle ca\rangle = \langle ab\rangle\langle cb\rangle/\langle b^2\rangle = r\langle cb\rangle/\langle b^2\rangle$ Using ❷ and ❶ and substitute into RHS : $\langle ca\rangle = \frac{p^2+r^2}{p^2-r^2}r$
❺ $\langle c^2\rangle = \frac{(p^2+r^2)^2}{2p(p^2-r^2)}$	$\langle c^2\rangle = \langle a^2\rangle + \langle b^2\rangle$ Using ❸ and ❶ and substitute into RHS : $\langle c^2\rangle = \frac{(p^2+r^2)^2}{2p(p^2-r^2)}$

Putting ❷, ❹, ❸, ❶ and ❺ into ii. We will get the equation of the diameter of the circle.

With the above 5 formula, the general procedural could be explained through examples in the problems of chapter 3.

Example 1: (problem 1 of chapter 3) Find the diameter with the information of the length b_2, c_4. Known i. $\langle c+b\rangle(b) = p = b_2$ ii. $\langle 2b\rangle(c) = q = c_4$
Original question and its method One person walks south out of the east gate without knowing the number of steps and stand still. And another person walks south from the west gate 480 steps, and then they see each other. Then he walk 510 steps to meet the first person. Method in the book To find D with length of b_2 and c_4. : $c_4 - b_2 = b_{15}$, and so $\sqrt{2(c_4 - b_2)\cdot 2b_2} = D$

In this question, the working of the generalised method is as follow:
By ii. $\langle cb\rangle = q/2$, using ❷ and substitute, we have $\frac{q}{2} = \frac{p^2+r^2}{2p}$, the equation $r^2 = pq - p^2$ is obtained after simplification, which is $r^2 = b_2c_4$–$(b_2)^2$, the same as $\sqrt{2(c_4 - b_2)\cdot 2b_2}$ in the book.

Example 2: (problem 2 of chapter 3) Find the diameter based on the information of the length b_2, a_{11}. Known: i. $\langle c+b\rangle(b)=p=b_2$, ii. $\langle c-b+a\rangle(a)=q$
The original question A person leave the west gate and walk south 480 steps. Another person start from point 11, and walk 80 steps eastwards. They then see each other. Method of the Book: Let $x=D$, $2b_2-x=2b_{10}$ and $a_{11}(2b_2-x)=D^2$ With $x^2=D^2$, then $x^2=a_{11}(2b_2-x)$ and $-x^2-a_{11}x+2a_{11}b_2=0$

Using our format, the process of setting up equation can be described as follow: From ii. $\langle c-b+a\rangle(a)=q$, i.e. $\langle ca\rangle - r + \langle a^2\rangle = q$

Using the formula and substitute, we have $\frac{p^2+r^2}{p^2-r^2}r - r + \frac{2pr^2}{p^2-r^2} = q$,

simplify the equation and the equation of the diameter of the circle is $2r^2+qr-qp=0$

As $p=b_2$, and $q=a_{11}$, we have $2r^2+a_{11}r-a_{11}b_2=0$. which is the same equation $-x^2-a_{11}x+2a_{11}b_2=0$ set up by the method of the book.

Example 3: (problem 3 of chapter 3) Find the diameter based on the information of the length b_2, b_{11}. Known: i. $\langle c+b\rangle(b)=p=b_2$, ii. $\langle c-b+a\rangle(a)=q=b_{11}$
The original question A person leave the west gate and walk south 480 steps. A person walk south from point 11 for 150 steps. They then see each other. Method(in the Book): Let $x=r$, $b_{11}-x=b_{15}$(area of trapezium) $b_2(b_{11}-x)=r^2$ and $x^2=r^2$ Which implies $-x^2-b_2x+b_2\cdot b_{11}=0$

Base on the general method here, the process of setting up the equation is as follow: From ii. $\langle c-b+a\rangle(b)=q$, i.e. $\langle cb\rangle - \langle b^2\rangle + r = q$.

Through substition, we have $\frac{p^2+r^2}{2p} - \frac{p^2-r^2}{2p} + r = q$, simplify the equation and we obtain the equation of the diameter of the circle: $r^2+pr-pq=0$. With $p=b_2$, $q=b_{11}$, the equation becomes $r^2+b_2.r-b_2b_{11}=0$ which is the same equation obtained in the book.

Through the above examples, it can be proved that other similar problems in chapter 3 can be solved by using the above method.

3 Chapter 4(17 questions frome on a_3, the base of the triangle 3)

Problems in chapter 4 can be generalised into the following format.

Known : i. $\langle c+a\rangle(a)=p$

ii.$\alpha_0\langle ab\rangle+\alpha_1\langle ca\rangle+\alpha_2\langle cb\rangle+\alpha_3\langle a^2\rangle+\alpha_4\langle b^2\rangle+\alpha_5\langle c^2\rangle=q$, where α_0, α_1, α_2, α_3, α_4, α_5, p, q are constants

As the process of proof is similar to those in chapter 3, we only list the results of the formula for comparsiion purpose.

Formula on Chapter 3	Formula of chapter 4
$\langle a^2\rangle = \frac{2pr^2}{p^2-r^2}$	$\langle a^2\rangle = \frac{p^2-r^2}{2p}$
$\langle cb\rangle = \frac{p^2+r^2}{2p}$	$\langle cb\rangle = \frac{p^2+r^2}{p^2-r^2}r$
$\langle b^2\rangle = \frac{p^2-r^2}{2p}$	$\langle b^2\rangle = \frac{2pr^2}{p^2-r^2}$
$\langle ca\rangle = \frac{p^2+r^2}{p^2-r^2}r$	$\langle ca\rangle = \frac{p^2+r^2}{2p}$
$\langle c^2\rangle = \frac{(p^2+r^2)^2}{2p(p^2-r^2)}$	$\langle c^2\rangle = \frac{(p^2+r^2)^2}{2p(p^2-r^2)}$

4 Generalised process in chapter 5 and chapter 6

In chapter 5, there are 18 questions based on the information of the length of b_1. And in chapter 6, there are 18 questions based on the information of the length of a_1. We list the conditions and formula below, and provide the proof of formula for chpater 5. One example for this method is included.

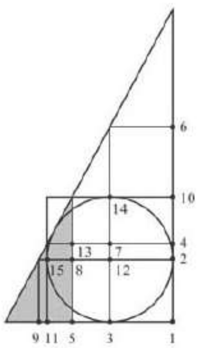
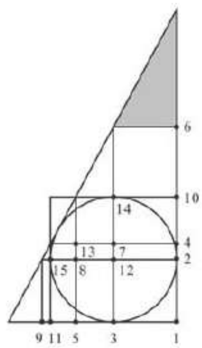
Problems in chapter 5 can be generalised into the following format.

Known : i. $\langle c+b+a\rangle(a) = p$

ii.$\alpha_0\langle ab\rangle + \alpha_1\langle ca\rangle + \alpha_2\langle cb\rangle + \alpha_3\langle a^2\rangle + \alpha_4\langle b^2\rangle + \alpha_5\langle c^2\rangle = q$, where $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, p, q$ are constants and problems in chapter 6 can be generalised into the following format.

Known : i. $\langle c+b+a\rangle(a) = p$

ii.$\alpha_0\langle ab\rangle + \alpha_1\langle ca\rangle + \alpha_2\langle cb\rangle + \alpha_3\langle a^2\rangle + \alpha_4\langle b^2\rangle + \alpha_5\langle c^2\rangle = q$, where $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, p, q$ are constants

Formula on Chapter 5	Formula of chapter 6
	
A. $\langle a^2\rangle = \frac{2r^2(p-r)}{p(p-2r)}$	A. $\langle a^2\rangle = \frac{p(p-2r)}{2(p-r)}$
B. $\langle cb\rangle = \frac{p^2-2pr+2r^2}{2(p-r)}$	B. $\langle cb\rangle = \frac{p^2-2rp+2r^2}{p(p-2r)}r$
C. $\langle b^2\rangle = \frac{p(p-2r)}{2(p-r)}$	C. $\langle b^2\rangle = \frac{2r^2(p-r)}{p(p-2r)}$
D. $\langle ca\rangle = \frac{p^2-2rp+2r^2}{p(p-2r)}r$	D. $\langle ca\rangle = \frac{p^2-2rp+2r^2}{2(p-r)}$
E. $\langle c^2\rangle = \frac{4r^4-8pr^3+8p^2r^2-4p^3r+p^4}{2p(p-r)(p-2r)}$	E. $\langle c^2\rangle = \frac{4r^4-8pr^3+8p^2r^2-4p^3r+p^4}{2p(p-r)(p-2r)}$

5 The following is the proof for fomule in chapter 5(questions with length b_1)

Formula on Chapter 5	Proof(by i it could be proved)
A. $\langle b^2\rangle = \frac{p(p-2r)}{2(p-r)}$	From i. $\langle c+b+a\rangle(b) = p$, we have $\langle cb\rangle = p - \langle ab\rangle - \langle b^2\rangle = p - r - \langle b^2\rangle$ $\langle cb\rangle^2 = (p - r - \langle b^2\rangle)^2$ $\Rightarrow \langle c^2\rangle\langle b^2\rangle = (\langle a^2\rangle + \langle b^2\rangle)\langle b^2\rangle = (p-r)^2 - 2(p-r)\langle b^2\rangle + \langle b^2\rangle^2$ $\Rightarrow \langle a^2\rangle\langle b^2\rangle = r^2 = (p-r)^2 - 2(p-r)\langle b^2\rangle$ $\Rightarrow \langle b^2\rangle = \frac{(p-r)^2-r^2}{2(p-r)} = \frac{p(p-2r)}{2(p-r)}$
B. $\langle cb\rangle = \frac{p^2-2pr+2r^2}{2(p-r)}$	From i. $\langle c+b+a\rangle(b) = p$, we have $\langle cb\rangle = (p-r) - \langle b^2\rangle$ using A and substitute into RHS: $\langle cb\rangle = (p-r) - \frac{p(p-2r)}{2(p-r)}$ $\Rightarrow \langle cb\rangle = \frac{p^2-2pr+2r^2}{2(p-r)}$
C. $\langle a^2\rangle = \frac{2r^2(p-r)}{p(p-2r)}$	$\because \langle a^2\rangle\langle b^2\rangle = r^2$ $\langle a^2\rangle = r^2/\langle b^2\rangle$ using A and substitute into RHS: $\langle a^2\rangle = \frac{2r^2(p-r)}{p(p-2r)}$
D. $\langle ca\rangle = \frac{p^2-2rp+2r^2}{p(p-2r)}r$	$\langle ca\rangle = \langle ab\rangle\langle cb\rangle/\langle b^2\rangle = r\langle cb\rangle/\langle a^2\rangle$ using both A and D and substitute into RHS: $\langle ca\rangle = \frac{p^2-2rp+2r^2}{p(p-2r)}r$
E. $\langle c^2\rangle = \frac{4r^4-8pr^3+8p^2r^2-4p^3r+p^4}{2p(p-r)(p-2r)}$	$\langle c^2\rangle = \langle a^2\rangle + \langle b^2\rangle$ using both A and B and substitute into RHS: $\langle c^2\rangle = \frac{4r^4-8pr^3+8p^2r^2-4p^3r+p^4}{2p(p-r)(p-2r)}$ $= \frac{(p^2-2rp+2r^2)^2}{2p(p-r)(p-2r)}$

We provide an example in chapter 6, and putting A, B, C, D and $\langle ab\rangle = r$ into ii. We will get the equation of the diameter of the circle after simplification.

Example 1: (problem 1 of chapter 6) Known: i. $\langle c+b+a\rangle(a) = p = a_1$, ii. $\langle c-b\rangle(a) = q = a_{15}$
Question A person leave from east gate and walk straight for 16 steps. Another leave from vertex 1, walk east for 320 steps. They then see each other (the sight line is tangent to the fort).

Using the representation, and from ii. $\langle ca\rangle - r = q$, Using Equation A and substitute $\frac{p^2-2pr+2r^2}{2(p-r)} - r = q$, the equation of diameter: $-4r^2 + (4p-2q)r + 2pq - p^2 = 0$ is obtained after simplification.

6 Conclusion

The generalised method for solving the work of setting up the equations show that Li Zhi may notice the symmetric property of the triangle and this properties can bed used to solve symmetric problem. This may be the reasons that the 17 questions in chapter 3 and the 17 questions in chapter are linked. So does the 18 problems in chapter in chapter 5 and 6. In working out with all possible equations, we can understand the hard work and the difficult point of the mathematics problem at that time. Also, the working of a generalised method is only possible through using good symbols.

The possible extension of using the generalise formula may also help us to understand how the degree of the equations obtained be mininized. In most of the equation provided by Li Zhi, the degree of the equations are not more than 4 and this is not easy at that time without such symbols that we used today.

REFERENCES

- Guo Shuchun 2010, “history of science and technology in China, volume in mathematics”, Beijing, Science Press.
- Li Yan 1931-1932 “Ceyuan Haijing yanjiu licheng kao” (Reserach into the Ceyuan Haijing) in Zhong suan shi luncong, in the edited work by Guo Shuchun .“completed work of Li Yan and Qian Baocong”

HPM AND PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS

—On Mathematics Teachers' Quality and Its' Advance in Mathematical History

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ABSTRACT

History of mathematics is very important for mathematics education, but mathematics teachers make rarely use of mathematical history in mathematics class. For history of mathematics coming into mathematics classes，it is crucial that researching and advancing mathematics teachers' quality in mathematical history. The quality includes three aspects: the recognition on mathematical history, the knowledge about mathematical history and the ability to use mathematical history in teaching. According to the Evaluation Theory of SOLO Classification, we can classify teachers' quality in mathematical history into 5 levels. Through adopting appropriate measures, we can advance teachers' quality in mathematical history, and further promote mathematics teachers' professional development.

Keywords: mathematics teacher; quality in mathematical history; level of quality; advance; professional development of teacher.

Remark: The name marked an asterisk (*) is presenter.

Full Text in Chinese:

与数学教师专业发展

—谈数学教师的数学史素养及其提升

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摘要： 数学史在数学教育中陷入“高评价、低运用”误区。研究和提升数学教师的数学史素养，对推进数学史走进数学课堂至关重要。数学教师的数学史素养包含教师对数学史的认识、数学史知识、运用数学史教学的能力三个要素。根据 SOLO 理论，数学教师的数学史素养可划分为 5 个水平。教师在自身努力下，通过课题带动、专项培训、创建 HPM 资料库、开展反思性教学、发挥群体优势等措施，可提升数学史素养水平，促进专业发展。

关键词： 数学教师；数学史素养；素养水平；提升；教师专业发展

“数学史与数学教育（HPM）”是国际数学教育研究的热点论题。数学史的教育价值日益凸显。我国数学课程标准关注数学史，旨在把数学史引入数学课堂。然而，数学史在教学中“高评价、低运用”的现象普遍存在。调查显示，当前我国数学教师的数学史素养普遍偏低是其关键因素。[1] 然而，何谓数学教师的数学史素养？教师的数学史素养应如何提升？却是少有人问津。为此，本文首先探讨数学教师数学史素养的内涵，并在此基础上提出数学教师数学史素养的提升策略。

一、数学教师数学史素养的内涵

数学史素养是数学教师专业素养的重要组成部分，是数学教师全面了解数学、优化数学知识结构、提高数学教学能力的重要基础，也是搞好数学教学的重要保证。数学教师的数学史素养不同于数学史学家的数学史素养，它是为数学教学服务，并非纯粹研究数学史。该素养涉及三个要素：对数学史的认识、数学史知识、运用数学史教学的能力。本文主要论述中学数学教师的数学史素养。

数学教师对数学史的认识，是数学教师数学史素养的重要组成部分，它对教师教学中数学史的运用状况起到指挥和调控作用。例如，有教师认为，学生学习数学的目的是掌握数学知识，提高数学思维能力和解决实际问题的能力，而不是学习数学史。教学中掺入数学史只能侵占宝贵的课堂时间，增加学生学习负担。这样，他们在教学中将尽量少用或不用数学史；但是，如果教师认识到数学史对数学教育的重要作用：激发学生学习兴趣、深刻领会数学教育价值、指导教学实践（历史上数学家曾经遇到过的困难，课堂上，学生同样会遇到）等，他们设计教学方案时，就会有意识地思考数学史，教学中也主动运用数学史。香港大学的萧文强教授在总结教师教学中不愿用数学史的一些常见理由时，就指出数学教师对数学史的认识程度影响着数学教师学习和运用数学史的积极性 [2]。

数学史知识的广博程度，是衡量数学教师数学史素养的重要指标。若一位数学教师不知道祖冲之、刘徽为何许人，对阿基米德、牛顿、高斯闻所未闻，《几何原本》、《九章算术》不知为何物，很难说他具有较高的数学史素养。中学数学教师需要了解的数学史知识，主要指与教学内容相关的和“数学史选讲”中涉及的数学史知识。如数学概念及重要成果的产生背景、重要数学思想的诞生历程及著名数学家的感人故事及趣闻轶事等。

运用数学史教学的能力，是数学教师借助数学史提高教学效果所必需的技能。Fulvia Furinghetti 指出“不同作者对数学史作用得出的不同结论，并不是数学史自身作用的问题，而缘于不同数学教师对数学史的不同运用方式”。[3] 这显示了数学教师运用数学史教学能力的重要性。近几年，很多学者赞成“数学史融入数学教学”，其主要形式是教师结合数学史对教学内容重新设计和加工，制作适用于教学的“历史套装”，让数学史料“随风潜入夜，润物细无声”，学生于潜移默化之中便领悟到数学史上的数学思想、思维方式等。洪万生教授领导下的台湾 HPM 团队中，有很多教师开发了一些供课堂教学使用的“学习单”，并应用于教学实践 [4][5]。这些具体例证对提高教师运用数学史教学的能力上值得借鉴。

二、数学教师数学史素养的水平划分

不难发现，不同教师的数学史素养存在较大差异。例如，对于勾股定理，有的教师仅把它看作一个抽象枯燥的数学结论；而有的教师更了解历史上勾股定理的来龙去脉和生动有趣的奇闻轶事；还有的教师不仅能把勾股定理置于古今东西方文化之中，还能把相关史料有机地融入数学教学，既提高了学习

效果，又让学生感受到多元文化的魅力和学习数学的价值。对于不同水平的数学史素养，应该如何划分呢？根据 SOLO 理论 [6]，结合数学教师数学史素养的组成要素，我认为数学教师的数学史素养也可划分为与 SOLO 对应的五个水平。综合 Hans Niels Jahnke 诠释学理论与台湾苏意雯的相关研究，[7] 对这五个水平分别介绍如下：

第 0 水平 (前结构层次): 教师缺乏数学史意识，对数学史知识及数学史在数学教学中的作用毫无知晓，教学中从不利用数学史。此水平的教师可以说全无数学史素养。教师的教学主要借助数学知识逻辑关系（这里不排除做游戏、做实验等与数学史无关的手段）促进学生的数学认知，根本想不到运用数学史。（如图 1）

第 1 水平（单一结构层次）：教师已具备少量数学史知识 (图中用虚线表示)，但认为数学史与数学教学无关，在教学中不使用数学史。如图，教师虽对数学史有所了解，但仍是借助数学知识逻辑关系促进学生的数学认知，不触及数学史范畴。数学史对于数学教学来说，仍是“天外之物”。（如图 2）

第 2 水平（多点结构层次）：教师已具备局部的数学史知识。对数学史在教学中作用的认识还比较肤浅，仅停留于提高学生的学习兴趣的低层次上（图中用虚线表示。在某些章节教学时偶尔运用数学史，其方法或是直接提供史料文本，或是讲数学史故事、作数学史报告等，数学逻辑与数学史料未能有机整合，数学史的讲授与数学知识的教学可谓泾渭分明。由于不能较好地两者兼顾，经常是顾此失彼。（如图 3）

第 3 水平（关联结构层次）：教师拥有局部的数学史知识，对教学中数学史的作用已有较深刻认识，认识到数学史可以促进学生对数学知识的理解、有利于形成良好的人格和精神等（图中虚线加粗）。对于特定单元（如勾股定理等），了解较丰富的数学史知识，教学设计时能从数学知识逻辑与数学史料两个面向促进学生数学认知，能较好地把数学史融入数学教学（图中虚线加粗）。但由于教师数学史视野所限和经验不足，在教学中难免有所欠缺。（如图 4）

第 4 水平（抽象拓展层次）：教师拥有较广博的数学史知识，对数学史的作用已有深刻认识。HPM 教学不再局限于特定章节，整个教学都从数学知识逻辑、数学史料两个层面考虑促进学生的数学认知，对于课程内容与 HPM 的适切性已有全面了解，教学中能够高效地发挥数学史的作用。HPM 教学已成为一种教学观念，一种教学意识。如下图所示，此图与图 4 的不同之处在于左右两圈的扩大，表明教师实施 HPM 教学的章节不断拓展，对 HPM 教学的理解与实践能力也有很大提高。这一水平应是数学教师的数学史素养的理想层次，也是数学教师努力的目标。（如图 5）

若从动态的角度来看，上述五个水平可看做 HPM 视角下数学教师专业发展过程中的五个阶段。数学教师从数学史素养的 0 水平提升到第 4 水平，实际上是数学教师 HPM 概念形成的过程。在此过程中，教师的数学史知识不断增多，对数学史的认识逐步加深，教学中由想不到数学史，发展到尝试运用数学史，进而在任何教学单元都能进行有效的运用数学史，形成 HPM 教学观念。

三、数学教师数学史素养的提升策略

根据前文所述，数学教师的数学史素养是决定数学史能否有效融入数学教学的关键。通过对中学数学教师数学史素养的调查发现，当前中学数学教师的数学史素养大多停留在第 1 水平或第 2 水平。为探寻数学教师数学史素养的提升策略，我们开展了“基于教师专业发展的数学史与数学教学整合的研究”实验课题。结合课题实践，参照有关资料，笔者认为，要提高数学教师的数学史素养可从以下几个方面努力：

（一）利用课题带动或开展切实有效的专项培训

我们实施为期两年的“基于教师专业发展的数学史与数学教学整合的研究”实验课题后发现，参与教师的数学史素养都得到不同程度的提升。我们认为，开展课题研究是实现 HPM 概念形成的有效社会互动方式，是提升教师专业素养的重要途径。

继续教育研究显示，专项培训是提高教师专业水平的有效策略。要提升数学教师数学史素养，培训中应要切实好以下几项工作：第一，为教师推荐部分实用的、经典的数学史学习材料，如汪晓勤与韩祥临合写的《中学数学中的数学史》、克莱因撰写的《古今数学思想》等。同时通过听专家报告，撰写论文等形式提高教师对数学史的认识，增加教师的数学史知识；第二，培训工作应与教学实践相结合。培训结束前，每人至少准备一节“数学史融入数学教学”的观摩课，或课堂录像资料，以此促使教师提高运用数学史教学的能力；第三，制定科学、严格的考核制度，把数学史素养的三个要素均作为考核内容，走出只注重数学史知识的误区。把考核结果与教师利益挂钩，真正起到培训的效果。

（二）教师自身应提高认识，加强学习，自觉提升数学史素养

虽然外在环境是影响教师成长和发展的重要因素, 但就教师个体而言, 发展和成长的关键还在于自身的努力。

由前面分析可知，教师数学史素养的提升从心理学上讲，一般要经历堆积、复合思维、抽象化三个阶段。而每一阶段的跨越都需要教师艰辛的智力操作。数学史素养的三要素紧密相连，相互影响，教师必须统筹兼顾，才有可能实现数学史素养的整体提升。因此，教师平时要多读一些反映数学史价值、介绍数学史知识的书籍、期刊，经常浏览数学史网站，积极参加有关的报告会、讨论会，撰写学习心得。同时，在设计教学方案时，既考虑数学知识逻辑、学生认知水平，又要查阅相关数学史料，挖掘与教学有关的历史素材，提炼有价值的数学思想方法，制作出关照三维度的“历史套装”，实践于数学课堂。所有这些，没有教师的高度关切和全身心的投入，提升教师的数学史素养只能是天方夜谭

（三）创建 **HPM** 教学资源库，为教师运用数学史教学提供参考

让数学教师完整地经历数学史融入数学教学的设计、实施全过程，对数学教师数学史素养的提升很有益处。但是，由于广大数学教师毕竟不可能像数学史学家那样对数学史知识了如指掌，再加上平时工作繁忙，教学压力较重，有时试图运用数学史，却由于数学史料搜寻困难，设计教学需要花费很多时间，便望而却步；部分教师对 HPM 概念了解甚少，不知道在教学中如何运用数学史，对数学史只能敬而远之；也有教师根据自己的设想在教学中运用数学史后，发现数学史对数学教学没有作用，于是决定不再涉猎数学史。在课题实施中，与中学教师座谈时，很多教师反映“如果能提供一些数学史融入数学教学的范例，让我们有个参照就好啦”。可以看出，从事研究 HPM 的专家学者当务之急就是设计出一线教师在教学中切实可行的数学史融入数学教学的范例，或者对善于运用数学史的教师的教学设计进行加工提炼，创建对中学教师真正有用的 HPM 教学资源库，既可吸引更多的教师关注 HPM，又能为尝试运用数学史的教师提供参考，这必将扩大 HPM 的影响，推动数学史融入数学教学的开展。

（四）提倡反思性教学，发挥群体优势，提高教师运用数学史的能力

研究发现，教学反思是推动教师成长的核心因素，也是促进教师专业成长的必由之路。例如，教师运用数学史教学后，发现效果并不理想，可反思教学设计理念、教学过程，探寻改进方案；教师对教学效果满意时，可反思成功的原因，提炼成功的经验。经过不断的反思，教师可做到扬长避短，逐步提高运用数学史教学的能力。实验课题也证实了反思性教学的重要作用。

因为每位教师都有自己长期形成的教学信念、教学风格，仅凭借自身的努力可能很难超越自我，教师数学史素养的提升更需要群体的智慧。学校要充分利用教研组这一平台，开展“我对 HPM 的理解”、“数学史在数学教学中的定位与作用”等专题讨论，集思广益、寻求共识；开展“数学史融入数学教学”讲课比赛，让每位教师经历教学过程；请优秀执教者上示范课，提供教学样板。它山之石，可以攻玉。相信通过采取有效措施，教师的数学史素养一定会得到提升。

（五）改革高师数学史教育模式

当前，多数高师院校把数学史作为选修课，普遍重视不够、要求不严。有研究者对即将走向工作岗位的高师毕业生做过调查 [8],，结果显示，有相当一部分学生对数学史知识缺少基本的了解。因此，要提高未来数学教师的数学史素养，就必须改革目前的数学史教育模式。

首先，明确课程定位。高师院校数学教育专业的培养目标是为中学输送合格的数学教师。数学史对数学教学必不可少。因此，数学史应定位于必修课，作为学生必须具备的一种素养，在课程考评上要和专业基础课放到同等重要的位置。

其次，改革教学方式。教学中不应把数学史当成真正的“历史”来讲授，要把它作为一种研究活动过程、方法、技术和能力，让学生在学“史”中促进自己专业素养的提高。为此，建议大三、大四都应开设数学史课程。教学过程可分为两个阶段：第一阶段，安排在大学三年级。让学生获得中学数学教学必备的数学史知识，了解历史上数学思想方法的演变，体会数学与数学文化的联系；第二阶段，安排在大学四年级，请 HPM 教学经验丰富的教师做指导，通过开辟“数学教育实验室”开展教学实验，重点培养学生运用数学史教学的能力。

最后，加强专业师资队伍建设。数学史是一门博大精深的学科。任课教师需要有较深厚的数学文化底蕴，进行过系统的数学史学习，对数学的概貌有正确的理解和认识。而目前绝大部分高校缺少高水平的数学史教师，致使开设的课程没有发挥应有的作用。笔者认为，要提高数学史教育效果必须加强专业师资队伍建设，引进数学史方面的专业人才，为培养学生的数学史素养提供保障。

总之，数学史素养，是数学教师专业素养的重要组成部分，也是数学教师专业化的重要体现，更是能否实现数学史教育价值的关键因素。面对当前中学数学教师数学史素养普遍不高的现实，重视数学教师数学史素养理论及提升策略的研究，对适应课程改革，提高教学质量，更好地落实 HPM 的研究成果，推进数学教师专业发展都有着现实和深远的意义。

Bibliography

[1] 李红婷. 课改新视域：数学史走进新课程 [J]. 课程. 教材. 教法，2005（9）：51–54

[2] M.K. Siu, “No, I don’t use history of mathematics in my class. Why? ”, in Proceedings of HPM 2004 & ESU4 (ICME 10 Satellite Meeting of the HPM Group & Fourth European Summer Univer-

sity on History And Epistemology in Mathematics Education) at Uppsala, July 2004, edited by F. Furinghetti et al, Uppsala Universitet, 2006, 268–277.

[3] Fulvia Furinghetti. Teacher education through the history of mathematics. *Educational Studies in Mathematics*, 2007,66,131–143.

[4] 许志昌. 数学史融入数学教学—以概率单元为例 [J].HPM 通讯，2006（8）：10–19

[5] 苏惠玉. 无限与微积分学习单组模设计Ⅱ. 微积分的概念学习单 [J].HPM 通讯，2008（9）3–11

[6] SOLO 分类理论http://baike.baidu.com/view/1554511.htm

[7] 苏意雯. 数学教师专业发展的一个面向：数学史融入数学教学之实作与研究 [D]. 台湾：台湾师范大学数学系，2005.14-6-158

[8] 李开慧. 高师数学教育专业毕业生数学史知识存在的问题、原因及对策 [J]. 重庆师范大学学报 (自然科学版).2006(6):88–92.

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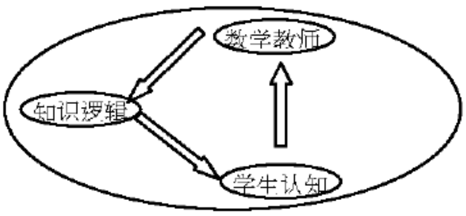

图 1 数学教师数学史素养第 0 水平

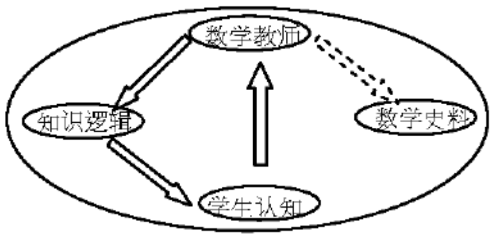

图 2 数学教师数学史素养第 1 水平

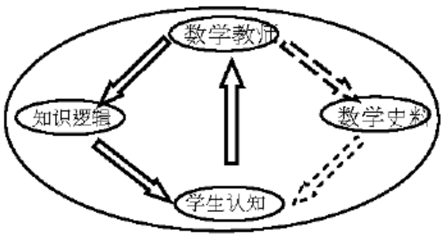

图 3 数学教师数学史素养第 2 水平

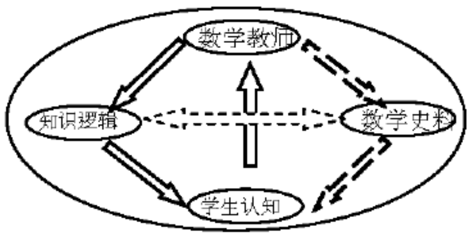

图 4 数学教师数学史素养第 3 水平

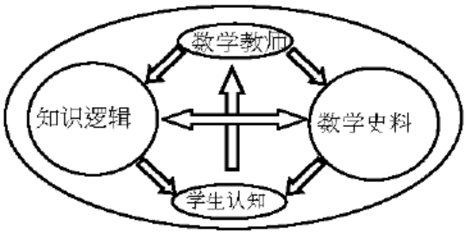

图 5 数学教师数学史素养第 4 水平

COMMON CORE STATE STANDARDS MOVEMENT IN U.S MATHEMATICS CURRICULUM

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ABSTRACT

On June 1, 2010, the Common Core State Standards (CCSS) in mathematics was released. The adoption of CCSS is voluntary, but the U.S. Department of Education will provide some financial incentives for those states that accept it. In the beginning, 48 states initially signed on, and 46 states have officially adopted the CCSS for their state standards so far. This paper looks back the national standards movement in U.S mathematics curriculum and discusses an opportunity that CCSS become national standards in U.S mathematics Curriculum.

1 National Standards Movement

The national standards movement began with the publication of *A Nation at Risk* in 1983. The state governors and business leaders began to increase their involvement in the formation of educational policy that would ultimately become the standards-based reform movement (Puma et al, 2000). In 1989, President Bush proposed the America 2000 Act, which called for mandated national testing. However, this act failed to pass through congress. In 1993, Goals 2000 proposed by President Clinton was passed by Congress. The significance of the Goals 2000 was that it took an important step in requiring states to have education standards in order to receive Title I funds (Miller, 2000). With the enactment of the No Child Left Behind (NCLB) Act of 2001, the Federal Government expanded its role significantly. This act required states to test more frequently with the National Assessment of Education Progress (NAEP) and set more ambitious and uniform improvement goals for their schools, and the Federal Government took action on schools that failed to meet those goals (Fuhrman, 2004). Although no national curriculum was proposed during the initial phase of the NCLB Act, there was a national system of standards-based accountability imposed with the understanding that every school should do well in mathematics.

2 Standards-Based Reform Movement

The standards-based reform movement emerged in the late 1980s and 1990s through the work of a group of education leaders, governors, researchers, curriculum development companies, and professional organizations. The National Council of Teachers of Mathematics (NCTM) has developed standards that address students' learning goals, assessment, and instruction (NCTM, 1989, 1991, 1995, 2000, 2006). NCTM's publications in 1989, 2000, and 2006 were for curriculum, the 1991 publication was for appropriate teaching, and the 1995 publication was for assessment. Falling NAEP scores and a proposed solution from NCTM led to the development of a nationally recognized set of content standards in mathematics (McClure, 2003).

Mathematics education has long been divided by contentious debates about curriculum and instruction. By the Mathematical Association of America, two mathematicians and three mathematics educators gathered to seek common ground in their efforts to improve K – 12 mathematics teaching and learning with two pilot meetings in December 2004 and June 2005. They agreed upon a set of understandings in seven issues and terms: Automatic recall of basic facts, calculators, learning algorithms, fractions, teaching mathematics in "real world" contexts, instructional methods, and teacher knowledge (Ball et al, 2005). The following year, NCTM published the *Focal Point*, which provided a set of core ideas for mathematics in K – 8.

3 Common Core State Standards Movement

In April 2006, President George W. Bush created the National Mathematics Advisory Panel to examine and summarize the scientific evidence related to the teaching and learning mathematics. In 2008, the final report was published with 45 findings and recommendations on key topics, curricula content, learning processes, teachers and teacher education, instructional practices, instructional materials, assessments, and research policies and mechanisms (NMAP, 2008a). In addition to the final report was a set of task group reports on (a) standards of evidence, (b) conceptual knowledge and skills, (c) learning processes, (d) teachers and teacher education, (e) instructional practices, (f) instructional materials, and (g) assessment, as well as (h) a national survey of Algebra I teachers (NMAP, 2008b).

On June 1, 2010, the Common Core State Standards (CCSS) in mathematics, a state-led effort, was released and written through a joint effort by the National Governors Association and the Council of Chief State School Office to develop common K – 12 and college and career ready mathematics standards (CCSSI, 2010). The adoption of CCSS is voluntary, but the U.S. Department of Education will provide some financial incentives for those states that accept it. In the beginning, 48 states initially signed on except for Texas and Alaska, and 46 states have officially adopted the CCSS for their state standards so far. Each state could adopt the CCSS either directly or by fully aligning the state standards with the CCSS. States may also add additional standards.

CCSS set grade-specific standards but do not dictate teachers on what and how to teach. The standards do not define the intervention methods or materials necessary to support students. CCSS focus on understanding mathematics in such ways: 1) conceptual understanding and procedural skills are equally stressed, 2) key ideas, understandings, and skills are identified, 3) deep learning of concepts is emphasized, and 4) being able to apply concepts and skills to new situations is expected (Hunt, 2010). The K – 5 standards provide that students build a strong and a solid foundation in whole numbers, addition, subtraction, multiplication, division, fractions, and decimals to enable student hands on learning in geometry, algebra, and probability and statistics. The middle school standards provide a coherent and rich preparation for high school mathematics, whereas the high school standards provide practice in applying mathematical knowledge to real world issues and challenges as well as emphasize mathematical modeling.

4 Common Core State Standards and Other Standards

CCSS is a state-led effort that is not part of the Federal Government development. It is, however, possible that the CCSS had a chance to create the national standards in mathematics by looking at the number of states that initially signed on and by the support provided from the Federal Government to states as they began adopting the standards. What would be the reasons for this? Also, what are some counter arguments?

The first reason is the benefits for states and local districts that adopt CCSS as their standards. The benefits are as follows: 1) CCSS allows collaborative professional development to be based on best practices, 2) CCSS allows the development of common assessments and other tools, 3) CCSS enables comparison of policies and achievement across states and districts, and 4) CCSS creates potential for collaborative groups to gain more economical efficiency for content standards, assessments, professional development, and pre-service teacher education. Each state does not have to develop its own curriculum guide, assessment, and content standards. However, Usiskin (2007) argued that a single set of national standards does not promise that schools speed up to change the curriculum, that students' performance in mathematics will not be necessarily improved with national standards, and that national standards may not be better than local and state curriculum. Another argument against national standards is such that one size does not fit all, since local districts, not the Federal Government, know what is best for their students.

The second reason is that CCSS present different characteristics from other states and professional organization standards. The differences are the followings:

- Fewer, more rigorous, and clearer goal
- Aligned with college and career expectations
- Internationally benchmarked
- Includes rigorous content and application of higher-order skills (mathematical modeling)
- Builds upon strengths and lessons of current state standards
- Research-based
- A stronger emphasis on mastery of basic arithmetic and fractions in elementary school
- A focus on more memorization and automaticity with mathematics facts over elimination and use of calculator (putting the calculators away)
- Pushing for completion of algebra by the end of eight grade, although not mandatory
- Pushing for all students to complete at least algebra II-level mathematics in high school (the minimum expectations for high school mathematics are likely to increase)

Porter, McMaken, and Yang (2010) reported the alignment with three levels – low, moderate, and high – between CCSS and state standards/assessment, CCSS and NCTM standards, and CCSS and NAEP. They found low to moderate alignment between CCSS and state standards and between CCSS and NCTM standards. For assessment, they found that CCSS has low alignment with state assessments and NAEP, but NAEP is more aligned with CCSS than state assessments. The study concluded that CCSS is considerably different from state standards and assessments and NCTM standards. We do not know if CCSS will bring positive or negative impact in mathematics education because the more change CCSS present, the harder it will be to fulfill the change. On the other hand, the more change CCSS present, perhaps the more positive effects CCSS will bring.

5 Conclusion

Many researchers and educators addressed that the traditional U.S. mathematics curriculum is "a mile wide and an inch deep." Quay (2010) reported that state standards are confusing and inconsistent across states, often holding low expectations for students in their mastery and rigor, not adequately aligned with demands of college and career, and do not pass muster with international competitors. TIMSS (2008) showed that the average mathematics score of U.S. fourth- and eighth-graders was higher than TIMSS scale average of 500, but lower than five Asian countries that have national mathematics standards. Moreover, the PISA (2010) result showed that the U.S. average score in mathematics literacy was lower than the OECD average score.

Considering these facts, the U.S. mathematics standards must become substantially more coherent and focused in order to improve student achievement. Based on the experience with current state standards and professional organization standards, national standards must present concerns about the quality and equity of elementary and secondary education (Goertz, 2008). It also makes some difference in what is taught and what is learned. Can CCSS bring coherence to a highly decentralized and fragmented mathematics education in the U.S. as a national standard?

REFERENCES

- Ball, D.L., Ferrini-Mundy, J., Kilpatrick, J., Milgram, J., Schmid, Wilfried, & Schaar, R. (2005). Reaching for common ground in K-12 mathematics education. *Notices of the American Mathematical Society*, 52 (9), 1055–1058.
- Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Author
- Fleischman, H.L., Hopstock, P.J., Pelczar, M.P., and Shelley, B.E. (2010). *Highlights From PISA 2009: Performance of U.S. 15-Year-Old Students in Reading, Mathematics, and Science Literacy in an International Context* (NCES 2011–004). U.S. Department of Education, National Center for Education Statistics. Washington, DC: U.S. Government Printing Office.
- Fuhrman, S. (2004). *Less Than Meets the Eye: Standards, Teasing, and Fear of Federal Control*. In Who's in Charge Here? The Tangled Web of School Governance and Policy, edited by Noah Epstein, 131–163. Washington, D.C.: Brookings Institution.
- Goertz, M, (2008). *Standards-Based Reform: Lessons from the Past, Directions for the Future*. Edited by Kenneth Wong and Robert Rothman. New York: Peter Lang Publishing.
- Gonzales, P., Williams, T., Jocelyn, L., Roey, S., Kastberg, D., and Brenwald, S. (2008). *Highlights From TIMSS 2007: Mathematics and Science Achievement of U.S. Fourth- and Eighth-Grade Students in an International Context* (NCES 2009-–001 Revised). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education. Washington, DC.
- Hunt, J. (2010). *Common Core State Standards: General Brief*. Chapel, NC: JR Institute for Educational Leadership and Policy.
- McClure, P. (2003). Where standards come from. *Book excerpt from Theory into Practice*, 44(1), 4–10.
- Miller, R. (2000, June). *A summary of efforts in school reform since 1983*. Washington, DC: Association of American Colleges and Universities.
- National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author

–– National Council of Teachers of Mathematics. (1991). *Professional standards*. Reston, VA: Author

–– National Council of Teachers of Mathematics. (1995). *Assessment standards*. Reston, VA: Author

–– National Council of Teachers of Mathematics. (2000). *Principles and standards for School Mathematic*s. Reston, VA: Author

–– National Council of Teachers of Mathematics. (2006). *Curriculum Focal Points*. Reston, VA: Author

–– National Mathematics Advisory Panel. (2008a). *Foundations for success: Findings and recommendations from the National Mathematics Advisory Panel [brochure]*. Retrieved January 8, 2011,

from http://www2.ed.gov/about/bdscomm/list/mathpanel/index.html

–– National Mathematics Advisory Panel. (2008b). *Reports of the task groups and subcommittees*. Washington, DC: U.S. Department of Education. Retrieved January 8, 2011,

from http://www2.ed.gov/about/bdscomm/list/mathpanel/index.html

–– No Child Left Behind Act of 2001 (2002). Public Law 107–110-Jan. 8, 2002. 107^{th} Congress of the United States of America.

–– Puma, M., Raphael, J., Olson, K., & Hannaway, J. (2000). *Putting standards to the test: a design for evaluating the systematic reform of education*. Washington, DC: Urban Institute Education Policy Center.

–– Quay, L. (2010). *Higher Standards for All: Implications of the Common Core for Equity in Education*. Berkeley, CA: UC Berkeley School of Law, Warren Institute on Race, Ethnicity and Diversity.

–– Schmidt, W., McKnight, C., Houang, R., Wang, H., Wiley, D., Cogan, L., Wolfe, R. (2001). *Why schools matter: A cross-national comparison of curriculum and learning*. San Francisco: Jossey-Bass.

–– Usiskin, Z. (2007). Do We Need National Standards with Teeth? *Educational Leadership*, 65(3), 38–42.

SOCIAL CONSTRUCTION OF THE ALGEBRAIC STRUCTURES

A Model for its Analysis

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ABSTRACT

It is written the first part of the results of an investigation that characterizes the Social Construction of the Algebraic Structures using the methodology of an historical analysis that allowed to give account of the social context and as well as of the circumstances in which diverse books are published and articles that were written between 1870 and 1945 around the Algebraic Structures. A conceptual system was constructed to characterize the constituent elements of the social construction of the algebraic structures within the framework of some theoretical constructors on the social construction of the knowledge of P. Berger and. Chevallard; internalization, externalization, diffusion, representation and reproduction of the knowledge. Under this constructed frame we described three processes that comprise of the epistemological construction of the algebraic structures that we described by contextual phases they locate in the time. For each phase we emphasized the importance of action that certain agents that, in the history of algebra, they are marked by his incidence in the consideration of the importance of the formal scheme that prevails in the education of abstract algebra in the level superior.

Keywords: Keywords: Social Construction, internalization, externalization, validity, legitimation and institutionalization of knowledge

1 Introduction

The mathematical knowledge that is found proposed in the programs of study of the different educational levels have its origin, in principle, in the mathematical knowledge developed as knowledge used. This knowledge generally suffers a process of functional transformation to convert in teachable knowledge, processes that Yves Chevallard it has called, didactic transposition. However, the mathematical knowledge that will be transposed has passed already for at least three processes previous to those of the transposition; it has been validated by the academic community that produces, it has been institutionalized for this same community or a great that contains them and has been legitimated as a didactically significant knowledge.

The validity of the knowledge is tied to who to produces and it happens in group of agents that are integrant of one determined academic community. The legitimation as a didactically significant knowledge, on the other hand, is tied to whom, without having produced the knowledge, it qualify it in order that this one is incorporate to the system of teaching, by distinguishing it and by separating those of its history and of the community that validated it to put it to disposition of who transposes it (Chevallard, 1998)

Chevallard consider that the social construction of the knowledge begins when is diffused, moment to the one that arrive after that this knowledge already has passed for a process of to clear the personality of its producer and begins to fulfill totally different functions; reproduction and representation of the knowledge. In the sociological ideas of Berger (1969), we can locate this moment with the one that he calls externalization through which the knowledge is label as human product. According to this author the externalization is the permanent overturning of the human being towards the world, so much of the physical activity as mental and mark the moment by means of which the material objects and the non-material created for the man reaches the objective reality. This material and symbolic objects enrich the totality of the existent objects and impose its logic to the individuals that they employ them. The cultural permanence of these objects (its epistemological valence, according to Chevallard), it will depend of a specific community and of the use that this give you.

The symbolic objects and material understood as product of the physical and intellectual activity of the man acquires meant for other through the internalization process, causing that the man becomes converted in product of the society according to Berger. The actions of the other also is internalize for the man and they begin to acquire for it a significant character by converting in a typical conduct it arrives even to normed their own actions. In the measure in which is given this internalization and in the measure of the intersubjective that they acquire the objects and actions in a community of individuals, it is constituted a process of institutionalization of the knowledge, reaching the maximum level as social construction, that is to say, once the actions is identified as one form to act and they are habit, that form to make the things in the frame of meanings associated to the objects symbolic and material they constitute a program that regulates the interaction of the integrant of the community and that they put into practice as prescribed execution manners that it enjoy an acceptance generalized and unconditional (Berger and Luckman, 1997). To other communities to correspond to transform the knowledge institutionalized as object of knowledge in object to teach by means of the process of legitimation, eliminating the context significant to put it in hands of the community that finally will convert it in object of teaching through the process of didactic transposition.

The diffused knowledge, either for oral media and/or writings it is put within reach of other individuals that are part of certain communities and for every one of these communities this knowledge acquires a different meaning; for the academic community, for example, this knowledge is a susceptible knowledge, in principle, of being questioned and afterwards validated. It is the academic community who finally will produce it as scientific knowledge, as product of the activity of the mathematicians. According to Chevallard is the community that transformed it in object of knowledge.

According to the model described so far, a study of social construction of knowledge accounts for the moments of externalization, internalization, the process of validation, legitimization and institutionalization, the communities where such moments and processes occurred, of the agents that make up these communities, the role they have in them, the actions performed and cultural conditions that frame these actions over time. Thus, each of these elements are considered constituents of the social construction of knowledge.

This article describes the framework that occurred the axiomatization, classification and systematization of particular algebraic structures as processes that are part of the development of tools and creation of a language that shape and support the emergence of the concept of algebraic structure that we located as products of moments of externalization. We place the productions written and reported by some authors (Corrry, 2002 (d), Katz, 2007, Hernandez, 2009) as important by its incidence about the exposition of a way new to generate mathematical knowledge, as well as the influence that they had in the emergency, of the call, structural vision of the mathematical one that it consolidates as well conceptually to the algebraic structures.

2 Antecedents

In the educational mathematics a study of social construction of the knowledge can be given in two lines; 1) The context of the school, with the end of understanding how the students and teachers build the school mathematical knowledge and the factors that, of one or otherwise, they influence in this construction and 2) The social construction of the mathematical knowledge as knowledge produced in the academic community and what happen until becoming in teachable knowledge, for understand or give sense to its presence in the escolar system.

The search that we did and whose results we present immediately, they allowed, by a side, to notice the scarce investigation that there is in the plane of our interest (the second) and by other, identify the difficulties of the students in the apprenticeship of the algebraic structures for, in our investigation, follow the track that what is tried get the apprentice, taking those subjects to the classroom.

They are few the studies that, as those of Katz (2007), they describe the happen historical of the algebraic structures as mathematical knowledge that after passing by the processes of didactic validity, legitimation, institutionalization and transposition they are imprisoned of individual or collective ideological and philosophical conditioners that do it little or more attainable in a education-learning process. The article of Katz, present the history of the divided algebra in four stages that he calls "conceptual stages"; the geometric, the static thing of solution of equations, those of function dynamics and the abstract. Describes how is what, the century XX, algebra had become converted in the search of common structures of diverse mathematical objects for establishment of an axiomatic system that it would systematize them. Katz asserts that before teaching the algebraic structures of groups, ring and fields it is necessary that the student know sufficient examples of these to give you a chance to consider them as a useful and necessary generalization.

In the investigations about the problematic of the apprenticeship of the abstract algebra, the area of the mathematics in which is located principally the study of the algebraic structures (groups, rings, modules, between others), we identify two aspects that they are related directly with the interest of our investigation. One of them is the emphasis that it makes in the problem they have the students to formalize concepts that if they obtain to construct with respect to different mathematical objects, they cannot symbolize because a language and a system of appropriate symbols lack that allow them, then, to manipulate them (Dubinsky, Dauthermann, Leron and Zazkis, 1994) and by other side, the emphasis that becomes in the incapacity which they present the students who have identified conducts or characteristics common in sets of mathematical objects but that they cannot organize nor systematize so that, he is useful for the construction of new mathematical knowledge (Simpson and Stehlílková, 2006).

Works as those of Hazzan (1999), they raise that the students of abstract algebra tend to reduce the level of abstraction of different forms to achieve understand the concepts and properties of this subjects, converting them unconsciously in mentally more accessible concepts.

The structural vision; conception of the generated mathematics by David Hilbert (1862–1943) where the form to represent and to reproduce the mathematical knowledge, occurs as an abstract way that it unifies several existent theories until before 1930, principally in the theory of algebraic of the numbers and the theory of the polynomials; it seems to be the origin of the aspects problematic that interested us and that are described in the mentioned investigations, this agrees with some initial conclusions at that we arrived after a first closeness to the historical-epistemological evidence of the social construction of the algebraic structures. This preliminary analysis, carried out as part of the methodology that we employ, it showed these they emerge once particular structures, that it spring up in diverse fields of the mathematics, in moments and different stages, its axiomatic is constitutes, is classified and is systematized afterwards of its construction. More still the genesis of the actions that is organized in a systemic and that generates the emergency of the structural vision, occurs to satisfy a philosophical and ideological perspective that it proposes new forms in the construction of mathematical knowledge, actions that they are regulated by intentions also of philosophical and ideological character and propose new forms of construction of mathematical knowledge.

3 A Model for the Characterization of the Social Construction of the Algebraic Structures

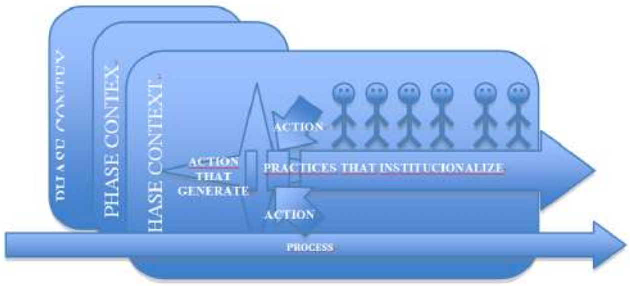
We leave from the principle of that the reality is constructed socially, by understanding this reality, *"as quality of the phenomena that we recognize as independent of our volition"* (Berger y Luckmann, 2001, p. 13); this quality dominates our actions and our ideas in the measure in which we recognize. Each of the phenomena they compose to the reality have specific characteristics and each individual accumulates knowledge of this characteristics that they give meaning and they go constructing the reality at the same time as it regulates the actions that in it carries out. It establishes a *dialectic process* between the reality that its imposes as pre-existent and the actions that in her they are realized by means of which the society constitutes itself and consequently the social contexts in that those phenomena finish immersed.

According to Berger (1969), the dialectic process mentioned happens for three moments; externalization, objectivation and internalization. The externalization moment it gives through actions driven by the accumulation of knowledge. Here they interest particularly; the diffusion; that it is understood as the moment in which a built knowledge in the individuality to give know other by some oral or written means, the representation; understood as the whole of material and symbolic objects next to the meaning that is associated to them and that give sense to certain knowledge produced for the individual, and the knowledge reproduction, that refers to the employment of a certain built knowledge by other in a specific context. The products of these actions that it can be articles, books and all kind of documents by means of which is reported results of an investigation, as well as, languages, symbols and any tools that does possible the objectivation of the knowledge.

The actions by means of which externalization is possible, they are oriented by an intentionality of double character: an individual character that obeys the need of the social being that identifies it or puts as part of certain environments and communities; and a social character that obeys the need, to next to others, producing the reality (Berger 1969). These actions they diversify while the reality becomes more complex and it is grouped, according to the social character of the intentionality with which it is realized until comprising of processes by means of which a body of knowledge it is established as social reality. These processes are part of the validation, institutionalization and legitimation of the knowledge, that is to say, by means of them who the externalize knowledge that an individual, objectivized by its products, it is internalized by the integrant of the community to the one that belongs, community that prints you a level of acceptance that gives you the necessary solidity in order that, as knowledge validated, as knowledge validated, they was useful for the construction of new knowledge in the same community or puts it to disposition of other communities orienting and sustaining its development. The validation of knowledge, mark the begin process of institutionalization.

As the validity, the institutionalization and the legitimation of the knowledge occur in communities whose members internalizes the symbolic objects that are product of the actions, we give off that these occur by phases, that is to say, they occur in a community located in the time and they are realized by certain agents, for which, the products of the diffusion, representation and reproduction of the knowledge acquire a significant character by converting to the actions and its intention in a typical conduct that it gets to normed its own actions. To this typical conduct as system of actions constructed in the frame of social and individual intentions that it is constituted in a regulator of the interaction of the members of a community and that they carry out as execution manners prescribed that it enjoy an acceptance generalized and unconditional we will call it *practice that institutionalizes*.

We will call *context phase* to a system of practices that institutionalize that it is originated by an action, we will call, *action that generates* and that was carried out an individual that we will call *agent* of the context phase.



A schematic representation of the characterization of the social construction of the mathematical knowledge.

4 Methodology

We carry out an investigation of historical-epistemological character by following the methodology of a historical analysis considered in Sierra and González (2003) and consists in 5 stages: 1) initial statement of the investigation, 2) heuristic criticism stage, 3) analysis of the documentation, 4) hermeneutics stage, and 5) exposition stage.

In the stage of initials statements of the investigation, in addition of delimiting the subject of investigation, it is necessary to define the line that will follow the investigation and the general frame below which will be developed. In base to a first revision of the bibliographical material, one draws up a possible trajectory that it allows to decide the feasibility of the investigation according to the material whereupon it is counted and initial hypotheses are formulated.

In our case we delimit the subject of investigation and the general line of development by locating it as a study of social construction of the mathematical knowledge that fundamentally we will guide below the sociological ideas of Berger and Luckmann and from the anthropological perspective of the outlined education by Chevallard.

In the first bibliographical revision we find the book, Modern Algebra And The Rise Mathematical Structures of, written by Leo Corry that speak about the origin of the algebraic structures from a historical perspective. In this one, Corry explains the origin and development of the ideas of structure and the dialectic relation that exists between this and the structural vision. This text was determinant for the location of dates, actions, ideas and its agents that are part of the social construction of the algebraic structures. Also it relate about the influence of ideological cut that they occurred between these agents and that were fundamental in the publication of books, that Corry indicate, as determinants in the inclusion of diverse themes of the abstract algebra in the education.

In the heuristic-critic stage it is realize the search and selection of documentary sources that contribute with some element about the subject in discussion. It necessary to locate and classifies the sources of forms such that do not produce documentary vacuums. It elaborated a historical critic for verifying the authenticity of the source and that it does for reach of the objectives of the investigation. In this stage we select four publications that are part of our material of analysis; the Treatise Algebra on of George Peacock published in 1845, *Lehrbuch der Algebra* of Heinrich Martin Weber published in 1895, Modern Algebra of B. L. Van der Waerden, published in 1930 and the Elements of the Mathematics of Nicolas Bourbaki published in 1935. We established the context phases of the social construction of the algebraic structures and that will describe as part of the results at present article.

The stage of analysis of content consists in studying the material under historical criterions, design some instruments that will permit approach the objective that one has considered, same that the interpretation of the data via the analysis of data that it realized. We design tow instruments for analysis of contend; the first, permitted us, without carrying out properly the analysis of content, locate the documents as product of processes of externalization and internalization of certain agents. This information we found it in the prologue and/or introduction of the text as well as in the biography of the authors. The second instrument consisted in a second card in which, by each work analyzed, it locate some aspects that permitted us identify the other constituent elements of the social construction of the knowledge.

The hermeneutics stage the which that it give appropriate answer to the planted questions in the investigation and indicate the possible causes by those who it is produced the analyzed historical facts, This stage we are realizing it following the context phases that we established in the critical heuristic stage. In present article it corresponding to the first context phase; the phase of incipient formalizations of the algebra.

5 Results

5.1 Process and Context Phases

The algebraic structures in its actual form, it can say that it characterized fundamentally by three aspects; the formal treatment, its function that systematize and abstraction level. These aspects are those who finally generated the emergency as a treatment for the mathematics in general and of the algebra in particularly.

To speak of the formal aspect of the algebraic structures it is necessary to speak of theories and of the language. The formal language is built with words of the common language or of the common sense, undressing them of the meanings that have in these for using them explicitly as says the formal language constructed to speak of a theory. The taken words of the ordinary language are devoid of meaning, that eliminates, in the measure of the possible, double interpretations.

To formalize a mathematical theory this must be provided of an axiomatic system; the axioms are considerate as basic points, independents and consistent to each other and on which, a mathematical theory without internal contradictions can be constructed (Rañada, 2003).

The axiomatic method it fixed as initial posture in the writing of mathematical theories (in texts or articles) of form such that its formalization is easy to conceive. Thus, to provide with an axiomatic system to a theory, represents important advantages as for development, giving multiple contents to the undefined terms (elements of a group, for example) or dissociate the diverse aspects of a mathematical situation to study them with greater depth. For to formalize an theory that it has a axiomatic system, it select a first-rate language appropriate L for the theory it serves to speak about the elements of set and that have the function to avoid the different interpretations; this language has a precise syntax, in such a way that does it moreover, susceptible to study as a mathematical object. Through of the formalization of mathematical theories the basic processes of the thought are optimized.

By the described it in the previous paragraphs, we divide the social construction of the algebraic structures in three processes: The process of Formalization, The Process of Systematization and the Process of Didactic Transposition of the Algebra. Next we describe the elements that they influenced the determination of the context phases they integrate every one of these processes.

In the Lehrbuch der Algebra of Heinrich Martin Weber, published in 1895, it can be identified a faithful image of the algebraic knowledge and of the form in which the algebra was conceived in its time. Weber makes an abstract presentation of many concepts that until the moment had been treated in a particular way (Corry, 2002b). The work that Heinrich Weber and Dirichlet made together in some writings about the algebraic functions it seems to have been an important influence in order that Weber directed his efforts to the study of the algebra and the numbers theory. Dirichlet in addition influenced a strongly Richard (1831–1916) when this one carried out a stay in the university of Göttingen where Dirichlet was professor. Dedekind published a complement of the *Vorlesungen über Zahlentheorie* of Dirichlet as class notes (Reck, 2008). We consider the work of Weber as a product of externalization based on the internalization of the ideas and knowledge of Dirichlet and other mathematicians of his time. We locate to Weber as agent of the *context phase of incipient formalization of the algebra*, first phase of the *Formalization process*.

Heinrich Weber went professor of Hilbert to his entrance to the university of Königsberg, under its influence, of Lindermann and of Dedekind went that Hilbert was interested in the theory of invariants, which represented his first area of investigation (Rodríguez, 2000).

Hilbert was professor at the university of Gotinga since 1893, his work and personality it positioned as the center of mathematics and its teaching, more important of the time. He was protagonist of one of the more important discussions in the history of the mathematics, in which it is considered emerges, the structural vision of the mathematics. As a result of these discussions they arise, those that are known as philosophical currents the mathematics; formalism, logicism and intuitionism that they occurs between 1870 and 1910. The origin of these discussions happened by the propose the distinguish between the that exists by "construction," it' s to say, that it exists because it establishes an explicit algorithm for construct, and that it exists because it is "fact", as Hilbert said, "*It can demonstrate that the attributes contain a notion, cannot conduct a contradiction, by the application of a finite number of logical deductions. I will say that it is demonstrated the mathematical existence of the notion under discussion*" (Ferreiros, 1999, p. 5). This proposition divided the options of the mathematicians, the way to represent and to reproduce mathematics that they were known it and its development since then. Meanwhile, the axiomatic method that was created by Euclid can be considered axiomatic concretely, and his geometry a constructive mathematics—given us two different points, A and B, it can be constructed a segment that unites them—. The axiomatic geometry that was formalized by Hilbert is considered as a formal and abstract mathematics, proposes an mathematics of the "fact";—given two points, A and B, exists a segment that the unites them— The contemporary mathematicians of Hilbert' s time, who was without a doubt one of the central personalities in mathematics in the eighteenth century, it distinguished between those who were with his ideas; Emil Artin (1898–1962) and Emmy Noether (1882–1935), for example, and those who were against his ideas; Luitzen Egbertus Jan Brouwer (1881–1966). Or those who simply demonstrated the impossibility of reaching important key goals proposals by Hilbert, as in the case of Kurt Gödel (1906–1978).

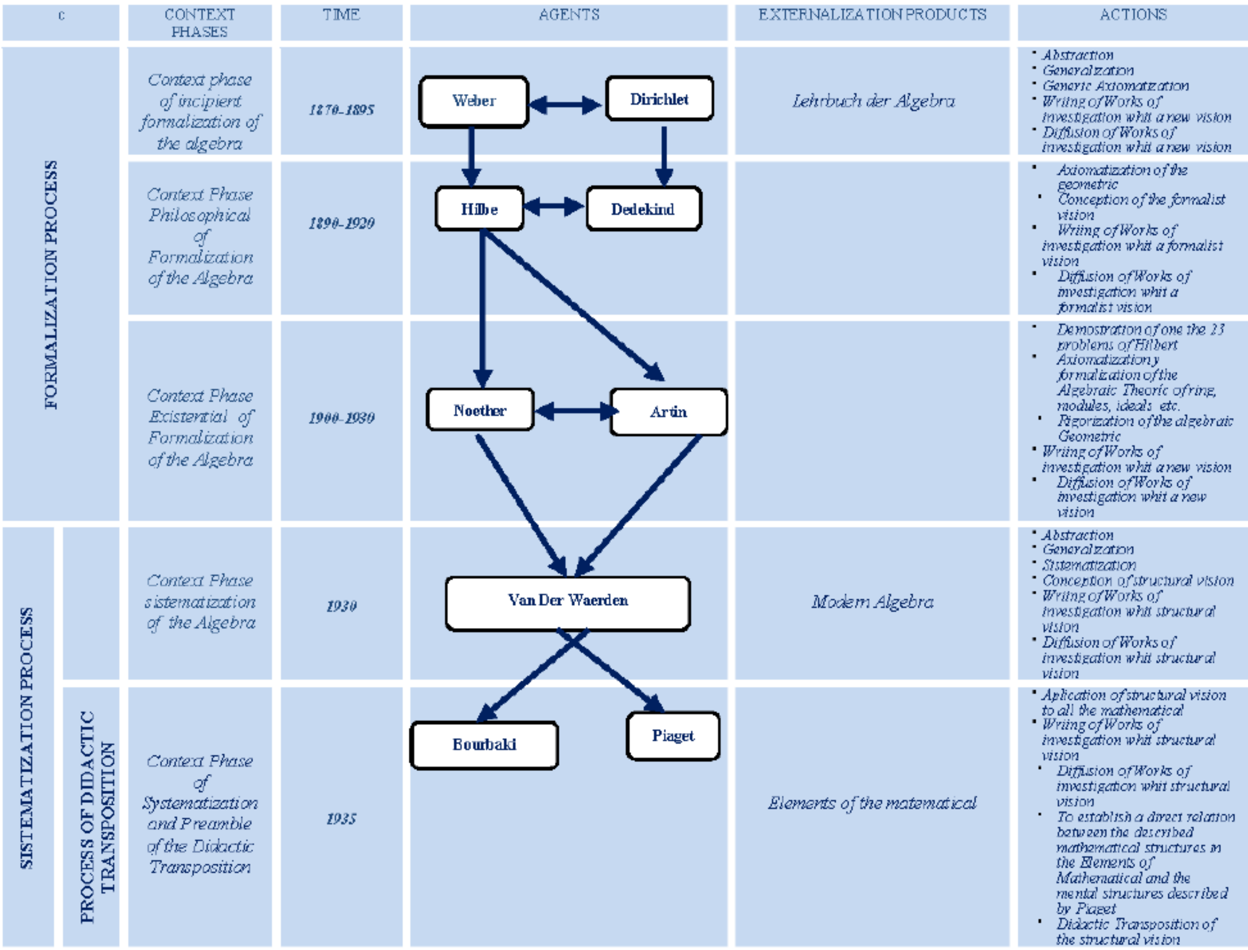
Hilbert conceived the idea of the formalism as the facts to reduce the mathematics to one finite game, with a finite number of formulates, defined of finite form. According to these ideas, the mathematics they became for Hilbert, in a general theory of the formalism and the axiomatic in the method of mathematics investigation. The mathematicians of the era (1870–1910) lived the preoccupation of called *"rigorization of the mathematics"*, that it consisted justly in establishing an axiomatic system for the mathematical theories existent, not by considering the axioms as absolute and evident truths, but as starting points that refer the fulfillment of certain properties for the elements of a set, that can include some elements no defined, from which they could be deduced one long list of theorems that will shape the theory.

By all the described, we consider to Hilbert as agent of *Contextual Phase of Formalization of Algebra*, whose products of externalization represented a strong influence for the destiny of the mathematics.

The internalization that did Emmy Noether and Emil Artin, disciples of Hilbert, of their ideas, they were translated in products of externalization representing, reproducing and divulging his built knowledge. Van der Waerden declares in the *Modern Algebra; "Based in part on lectures by E. Artin and E. Noether"*, it find that go der Waerden went disciple of Noether during seven months in the university of Göttingen and knew to Artin in the university of Hamburg in where participated in a course and they programmed to write a book together. It says that after an initial revision of advance that present Waerden, Emil suggests you that continue writing and publish it but in an independent way (Rodríguez, 2000), the result of this work went, finally, the two volumes of *Modern Algebra.* This permitted us identify to Emmy and Emil as agents of the *Phase Existential of Formalization of the Algebra*, third context phase of the *Process of Formalization of the Algebra*. *Modern Algebra* is a product of the process of externalization of Van der Waerden through which diffused the built knowledge in base to the internalization of the ideas and knowledge of Noether and Artin. We locate to Waerden as agent of the Context Phase of Systematization of the Algebra.

In the Modern Algebra and in the axiomatic method of Hilbert is that a community of French mathematicians finds the inspiration for, as a product of the externalization of his ideas, written in the *Elements of the Mathematics* of Nicolas Bourbaki, the image that of the mathematics that it had internalization and that it clarified the fundamental goal of their publications; endow to the mathematics of a theoretical unification in the presence of one apparent dispersion that lived during the first third of the century XX (Hernández, 2009).

The Bourbaki text resulted from great impact in the insertion of diverse topics of the algebra in its structural vision in the escolar environment as product, between other things, to the declarations of Jean Piaget in those who it referred as structures naturals to the mother structures of Bourbaki. This insertion we considers it as a product of the internalization of those ideas in the escolar environment, and to the Bourbaki group as the agent of the context *Phase of Systematization and Preamble of the Didactic Transposition* of the corresponding algebra simultaneously to *the Process of Systematization and to the Process of Didactic Transposition of the Algebra*.



The processes and the phases of the social construction of the algebraic structures.

c	CONTEXT PHASES	TIME	AGENTS	EXTERNALIZATION PRODUCTS	ACTIONS

5.2 Formalization Incipient Context Phase

The beginning of the process of formalization of the algebra went marked in the history of the mathematics for the publication of the Heinrich Weber text for two reasons fundamental. From that Niels Abel demonstrate the insolubility by radicals of the equation of degree five in 1824, and until 1845 that it is known the works of Galois, the investigations that it is considered in the frame of the algebra were fundamentally those who it is dedicated to the study of the forms and conditions to find the roots of polynomials. It sprang up however, of 1854 to 1880, diverse investigations in different theories of the mathematics in those who introduced, by a side, the language gives the set theory and for the other, the application of results of the Galois theory, making evident its value as unifying factor. As examples we can refer to: the theory of algebraic curves of Dedekind and Weber, investigations on the theory of matrix and quaternions realized by Cayley, the theory of the regular polyhedrons worked by Hamilton in 1856 in the Treaty of Substitutions, the Camille Jordan works published in 1870, investigations about the classification of the geometries made by Kline in 1872 and in the algebraic theory of the numbers promoted by Dedekind in 1877. These investigations sustain the institutionalization of the ideas and forms to approach subject of the generated algebra for Galois, that was validated until after its death.

Heinrich Weber (1842–1913), conscious of the last advances in the algebra, molds the possibility to formulate new algebraic concepts in purely abstract terms. The action we consider as generating of the context phase of incipient formalization it happens in 1893; Weber went the first that gave abstract definitions of group and field in the frame of a single article named; *"Die allgemeinen Grundlagen der Galois"* where says *"Ein system von dingen irgend welcher Art in endlicher oder unendlicher Anzahl wird zur Gruppe, wenn folgende Voraussetzungen erfüllt sinde...* [A system of a finite or infinite number of things of any type are a group if the following conditions fulfill]. At once it enunciates three of the four properties of group know now as, associative and existence of the inverse and neuter element, it enunciates through its equivalent, called cancel laws by the right and by the left. Clear up moreover that the composition of two elements not always is commutative, a few paragraph after defines field "*Eine gruppe wird zum körper, wenn in ihr zwei Arten der Composition möglich sind, von denen die erste Addition, die zweite Multiplication genannt wird*" [A group is converted in a field, if possible, in its two types of composition, of those which the first his name is addition and the second multiplication]

The diffusion as generating action is product of alternate actions of observance of similitudes or common aspects, between two or more, theoretical subjects and its objects of study, that we considers as an abstraction action.

In the mentioned article of Weber observes us clearly an action of generalization and axiomatization, in which he present the conditions for consider that a set of " things of any type " below a specific operation, it is a group. These actions; axiomatization, abstraction and generalization, have the intention, that finally this described in the mentioned article and in *Lehenbuch der Algebra*; integrating the knowledge that during a period of time had developed in subject of the algebra. In your Weber writings it incorporates a complete body of new individual ideas and the developed techniques along the century XIX that are part of the necessary base for the formalization of the algebra. However the study of the groups appears subsidiary to the study of the polynomials equations and your solutions, even in the *Lehrbuch der Algebra* the general concept of group appears after 475 pages.

The community of mathematicians of which it comprised Weber (Dedekind, Cayley, Jordan and Kline) shared already some elements of the formalistic current proposed by Hilbert from 1870, in the investigations generated by their integrant molded already the incipient stage a new tendency to investigate in mathematics and particularly in algebra.

The declaratory thing of Weber in the prologue leaves clear his intentionality "*Es war meine Absicht, ein Lehrbuch zu geben, das, ohne viel Vorkenntnisse vorauszusetzen, den Leser in die moderne Algebra einführen* *und auch zu den höheren und schwierigeren*" [It was my intention gives a textbook that not presupposes you previous knowledge to the reader and made a presentation of the modern algebra of the easy thing to the difficult thing]. It stands out the importance of the development of the algebra and the fundamental influence that has had the theory of groups above all in the numbers theory. "*die neueste Entwicklung der Algebra ganz besonders von Bedeutung geworden sind; das ist auf der einen Seite die immer mehr zur Herrschaft gelangend Gruppentheorie, deren ordnender und klärender Einfluss überall zu spüren ist, und sodann das Eingreifen der Zahlentheorie"* [the most recent development of Algebra has reached a very special importance; this is on the one hand the group theory that every time obtains more dominion and whose organizer influence are perceived by all sides, particularly in the theory of numbers]

Also it emphasizes the influence and participation of some of his colleagues so much in the writing of the book as in the development of his ideas that also deposes explicitly in the prologue:

"*Zuerst gilt dieser Dank meinem Freunde Dedekind für seine treue Hülfe bei der Correctur, und wenn er auch auf den Plan und die Ausführung meines Werkes keinen Einfiuss ausgeübt hat, so möchte ich doch nicht unerwähnt lassen*" (Weber, 1895, p.7) [This gratefulness firstly for my Dedekind friend, by its faithful aid in the correction, and even though has not exerted influence in the plan and the accomplishment of my work, would not want to omit to mention it].

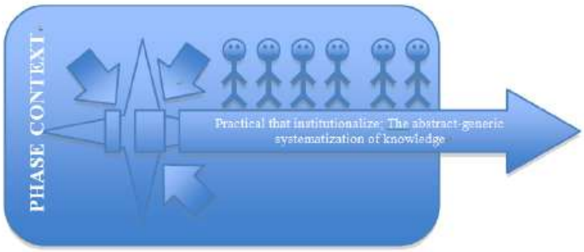
"*Auch der mannigfachen Anregung und Belehrung habe ich hier zu gedenken, die ich meinem Freund und-Collegen F. Klein verdanke, der das Fortschreiten der Arbeit mit regstem Interesse begleitet hat und dessen sachkundiger, stets bereitwilligst gegebener Rath in manchen Theilen des Buches von grossem Einfiuss gewesen ist*" (Weber, 1895, p. 7), [Here also I have to remember the multiple suggestions and instructions, that I must thank a friend and colleague F. Klein, who accompanied the advance the work with much interest and whose competent advice, gave always with good will, has had a great influence in some parts of the book].

For the context phase of the incipient formalization of the algebra we locate a practical that institutionalize; The abstract-generic systematization of knowledge integrated by actions of abstraction, generalization and generic axiomatization and the generating action of oriented diffusion below diverse intentions; the integration of generated knowledge until the moment, presentation of knowledge below a same general idea, it facilitates the application of these new ideas to other mathematical theories; with Dedekind and Weber as agents.

The scientific production of the era was not regulated absolutely, this depended of the relations and recognition that the authors had generated in a community of individuals that it would constitute as the group of specialists that they would administer the accumulated knowledge that it generated in the algebra under relevance and structure of meaning of them. This community it ended establishing the mediations for the recognition of the scientific results that were susceptible writings of diffusion.

6 Conclusions

The concept of algebraic structure emerges once particulars structures, that arise in diverse fields of the mathematics, in moments and different stages, they are equipped with an axiomatic system, classified and systematized afterwards to its construction. The structural vision as conception of the mathematics, it implies, represents and reproduces the mathematical knowledge, unifying several existent theories until before 1930 anchored initially to the algebraic theory of the numbers and the



Scheme of the Phase of Incipient Formalization of Algebra.

theory of the polynomials, has its roots in the formalistic current, proposal by Hilbert, that it prevailed in the community of mathematicians in front of the intuitionist currents and logicist.

Until before 1845 the representation and reproduction of the molded mathematical knowledge in objects of oriented diffusion, it made evident a null presence of symbolic material associated to ideas of abstract-generic representation of concepts of now called, Abstract Algebra. It until the publication of the *Lehrbuch der Algebra* of Henrich Weber it that appears, of way objectivized, an incipient abstract-generic reproduction of the concept of group and excel as product of the externalization of his ideas that as well represent the internalization of the externalizates concepts by Galois in his writings know in a posthumous way.

The phase of incipient formalization of the algebra represents for the structural vision the start-up of actions of systematization and generalization of the ideas of Galois, that together the ideas of Hilber, it profiles the axiomatization of the theory of the today known as abstract algebra and of the mathematics in general.

The practical that institutionalize of abstract-generic systematization as system of actions driven by the accumulation of knowledge and individual and social intentions, it is established in effect, as regulating of the interaction of the integrant of the community of mathematicians of the era, generating manners of representation and reproduction prescribed of the mathematical knowledge they begin to enjoy an acceptance generalized and unconditional originated by the publication of the *Lehrbuch der Algebra* of Henrich Weber.

The oriented diffusion in the scientific communication and the divulging of the works of investigation proposal, of incipient way, in the *Lehrbuch der Algebra* of Henrich Weber as a new form of making algebra and is more profitable given its Generality, offers the possibility of explaining diverse theoretical elements and give origin to other new. It profiles as action that it generates it shows one's profile as generating, non-single of the phase in this article, but of the other context phases of construction of the algebraic knowledge.

The community of mathematicians they identified as agents of the context phase of formalization incipient, impel, in an implicit way, new forms of representation and reproduction of the mathematical knowledge. Forms that are sustained and consolidated by the agents of the subsequent context phases (philosophical and existential of formalization), by complementing of this way the process of formalization of the Algebra.

REFERENCES

– Berger, P. (1969). El Dosel Sagrado. Elementos para una Sociología de la Religión. Buenos Aires Argentina:

Amorrortu Editores.

- Berger, P. & Luckmann, T. (1997). Modernidad, Pluralismo y crisis de sentido: la orientación del hombre moderno, España: Paidós Studio.
- Berger, P. & Luckmann, T. (2001). La Construcción Social de la Realidad. Buenos Aires Argentina: Amorrortu Editores.
- Chevallard Y. (1998). La Transposición didáctica. Del saber sabio al saber enseñado. AIQUE grupo Editor. Argentina
- Corry L. (2002 (a)), Heinrich Weber: Lehrbuch der Algebra (1895–96), Cohn Institute for History and Philosophy of Science, Tel Aviv University Ramat Aviv, 69978, December 6.
- Corry L. (2002 (b)), David Hilbert y su Filosofía Empiricista de la Geometría, Boletin de la Asociación Matemática Venezolana. IX(1).
- Dubinsky, E., Dautermann, J., Leron, U. & Zazkis, R.(1994), On learning fundamental concepts of group theory, Educational Studies in Mathematics Springer Netherlands, 27 (3), 267–305.
- Ferreiros José, (1999), Matemáticas y Platonismo(s), La Gaceta de la Real Sociedad Española de Matemáticas 2, pp. 446–473.
- Hazzan, O. (1999), Reducing abstraction level when learning abstract algebra concepts, Educational Studies in Mathematics 40(1), 71–90
- Hernández J. (2009), Las Estructuras Matemáticas y Nicolas Bourbaki. Universidad Autónoma de Madrid, Seminario “Orotava” de Historia de la Ciencia, Año IV.
- Katz Víctor J. (2007). Stages in the History of Algebra with Implications For Teaching, Educational Studies in Mathematics, Springer 185–201
- Rañada Manuel F. (2003). David Hilbert, Hermann Minkowski, la Axiomatización de la Física y el Problema número seis, La Gaceta de la RSME, Vol. 6.3, pp. 641–664
- Reck Erich (2008), Dedekind, Contribution to the Fundations of Mathematics, Stanforrd Encyclopedia of Philosophy
- Rodríguez Salinas Baltasar (2000), Hilbert, Números Dialet, Las matemáticas del siglo XX: una mirada en 101 artículos, pp. 43–44.
- Sierra Vázquez M., González Astudillo M.T. (2003), El Método de Investigación Histórica en la Didáctica del Análisis Matemático, Investigación Matemática: Séptimo Simposio de la Sociedad Española de Investigación en Educación Matemática, 109–130, Granada.
- Simpson Adrian, StehlíIková Nada, (2006), Apprehending Mathematical Structure: A Case Study Of Coming To Understand A Commutative Ring, Educational Studies in Mathematics., Springer, 61 pp. 347–371
- Weber Henrich, (1895), Lehrbuch Der Algebra, Friedrich Vieweg Und Sohn, Braunschweig

HISTORY OF ARITHMETIC TEXTBOOK AND COMPOSITION OF CONTENT BASED ON COUNT PRINCIPLE METHOD

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ABSTRACT

本稿においては，藤沢利喜太郎の数え主義とクニルリングの数え主義について考察し，日本の第一期国定教科書の内容を構成する方法としての数え主義を明らかにし，現在の日本の小学校における算数科の数教育の指標の一つとして，数え主義の視点から整理し論究した

緑字（赤字は削除）··· 修正が望ましい箇所[赤字] ··· コメントです

1 はじめに

現在，日本で小学校の算数科の授業で使用されている教科書は，いわゆる緑表紙教科書や現代化の影響があり，第一期国定教科書は塩野直道や生活算術によりその影響は希薄であるとの見方がある．一般的に，現在の日本における算数科教育がどのような必然性や偶然性により実施されているのか，その事実関係を明確にすることにより，理由そのものの現時点における有効性について具体的に検討し，今後の日本及び世界の算数科教育の可能性について考察することは，極めて重要なことである．

現在の日本における算数教育における数教育を見る指標の一つとして，数え主義を考察することにより，日本における第一期国定教科書における数え主義について論究する．本稿においては，具体的な分析方法として，当時の日本の教科書の教材の内容を構成する方法について全数に制限して論理展開する．

通常，日本における第一期国定教科書に関する指摘は，内容が藤沢利喜太郎の影響を受け，数え主義に立脚しているという点にある．この数え主義が，そもそもどの様な数え主義なのかを徹底的に議論するする必要がある．藤沢利喜太郎は，1985 年の「算術条目及教授法」や 1900 年の「数学教授法」等により中等学校の数学教育に影響力を持った．数学者である藤沢利喜太郎の数え主義は，数学論を基軸とした心理学的な考察は希薄な数え主義であり，初等教育ではなく中等教育を意識していたと考えるのが妥当である．そのため，直観主義であり実際の指導事例も現存しないので，どの様な観点から当時の日本の小学校の第一期国定教科書を作成したのかを知ることは極めて難しい．

しかし，算数の第一義の目的として，日常生活を意識したことは認められる．例えば，「加減乗除，普通の度量衡，貨幣，日用適切ノ雑題，簡易ナル分数，小数，比例，利息算ヲ教ユルモノ」といった観点から考察を加える必要がある．他の数え主義には，ドイツのクニルリングの心理学的考察から発信されている数え主義があり，直接的影響の例を算数教育の実践家である佐藤武にみることができる．

2 藤沢利喜太郎の数え主義

藤沢利喜太郎の数え主義について，4 つの視点から述べることとする．

(1) 数系列の獲得の周辺としての視点　藤沢利喜太郎の数学教育への関心は,「数学教授法講義筆記」を著した時期を境に，中等教育から初等教育へと変化した．また，小学校の教授法に触れてはいるが，具体的な教授細目は作っていない.「算術教科書」(師範学校用，中学校用教科書) に具体的な内容を記載しているだけで，その特徴は，数概念の指導から量を排除し，命数法　記数法において整数の加減乗除を，数えることで体系化した点にある．数指導は，分離量への対応でなく数詞を覚えることが先決となっている．日本語の数詞は，十進構造を内包しているが，英語表現で目にする十が 3, 十が 12 という十を単位として数える方式を含んでいないために「いち」も「じゅういち」も数系列では同等の存在となる.

(2) 四則演算の数体系の視点　数系列上の移動により加法を定義し,「数え上げ」操作の省略としての足し算が，また，数え下がりの省略が引き算を含め，数系列の移動として説明する以上，大きい数は引かれる側の数であり，数系列上右に存在する．数学者である藤沢利喜太郎は，その点をよく理解していたからこそ，敢えて大小の概念を何ら説明しなかったと推察される．累加で乗法を定義し，乗数は，数系列上の移動ではなく移動の回数を示す数となっている．メートル法度量衡等の十進諸等数における単位換算のため，特殊な数の説明の後に基数，一般整数という順序をとっている．除法の計算説明は，乗法の時とほぼ同じで，(名数)÷(名数)=(不名数) である包含除と (名数)÷(不名数)=(名数) である等分除の二種類の定義を与えている.

(3) 筆算の生成の視点　数系列上の移動により，四則演算の体系を作ったが，数が大きくなると数系列上の移動が困難となり筆算が出てくる．加法の筆算の前段階として，暗算の習熟が必要になる．その後，筆算の加法を説明するが繰り上がりの説明には頓着しない．減法は，一般の整数の減法であって，繰り下がりの無い場合，繰り下がりのある場合，小数を含む場合に分けて説明している．基数と基数の乗数は，九九の呼び声を使い，それ以外は，乗数が基数の場合，乗数が有効数字の右にいくつかの 0 がつく場合，一般の数の場合の順序で説明がなされている．除法は，法が基数の場合と 2 桁以上の場合に分けて説明している.

(4) 目指した方向性と発想の限界からの視点　藤沢利喜太郎は，寺尾壽を中心とした理論算術や明治以降の輸入算術教授，さらに，競争試験の難問である三千題流の算術を批判した．藤沢利喜太郎は，順序数によって体系付けられた数と計算の体系を天下り的な量の当てはめの排除を試みるが，そもそも日常生活においては，量を排除できない．しかし，もしも，藤沢利喜太郎を弁護すると，当時は，数と量の関係を天下り的でない方法で説明できる方法の成功例は存在しなかった.
　以上のことから，藤沢利喜太郎の数え主義は，十進構造の欠如という限界がある.

3　ドイツのタンクとクニルリングの数え主義

ドイツのタンクとクニルリングの数え主義について数図を認めて以降の数え主義を中心に述べる．そもそも，数え主義は，藤沢利喜太郎の指摘の通り，ドイツ心理学の影響を受けたタンクとクニルリングの著書が発端となっている．やがて，クニルリングは，考え方を変更して数図を積極的に認め，鈴木筆太郎に「直観主義的数え主義」と名付けられるようになった.

3.1　算術教授法

「1 の増減」「2 の増減」という考え方は，いわゆる緑表紙教科書の編集者たちの指導形態である．いわゆる緑表紙教科書における「数え主義」は，クニルリングや第一期国定教科書とは明らかに違った段階に入った証拠になる．現在においても，この考え方を採用している教科書もある．いわゆる緑表紙教科書は，一気に加減法へと進まないで数の分解を扱う．数えるから出発して，数図も扱い，数の分解へと進む．このストーリーは，第一期国定教科書では見られない現象で，いわゆる緑表紙教科書において，はじ

めて見られる事実である．そこでは，数図は，その配列では数をとらえさせようとはしていない．数はあくまでも数える対象である．したがって，ベーツ等の様な〇の配列によって，数を認識させることを目標にはしていない．したがって，日本においてグルーベ主義は採用されたが，直観主義そのものは実践された事実を確認されていない．

一方，クニルリングの数え主義は，数図を積極的に認めるという意味では直観主義である．算術教授の目的は，従来の規則計算主義，形式的淘汰主義，そして事物計算主義を折衷した立場である．教材の選択と配列は，計算の熟達を主目的とする技術計算においては純粋無名の数を取り扱う．実用計算においては，実用的な内容を有する問題を扱い，十分な生活の準備を与える助けとなっている．また，科学的な計算においては，可能な限り児童に多方面的な興味を持たせなければならないとし，ペスタロッチの四則順進主義やグルーベ主義の多方的処分に対して，自然停止点到達まで一則を単位としている．教授方法は，規則計算主義の注入方法，ペスタロッチの直観的方便の数図の導入やヘルバルトの 5 段階に対抗して 3 段階説を採用している．

3.2　数概念に対する心理学的考察

数とは何かに関する従来からの見解を以下の様に分類する．「数は物の外に立ち，物を超越するものである．数は，不生不滅不変なものであり，物の根元である．」「数は，色形などと同じ物の属性の一つである．つまり，数は現実に存在するが，それ自身独立して存在するものではない．むしろ，物に付着する性質である．」「数は，独立に存在するものではなく，また、他に付着する性質でもない．むしろ，人が物を知覚し，説得する形式に過ぎない．したがって，純主観的なものであり，この主観的形式に相当する実在する物は，外界にあることはない．」「数は物と同一であり，物の多さである」．

この分類に従えば，クロネッカーや藤沢利喜太郎の考え方は，第 3 番目に相当するが，クニルリングの考え方は第 3 番目と第 4 番目を融合させた客観的概念として表現される．両概念は別々に存在しえないことを，現実性により両者が結合されることで証明し，数概念の心理学的考察を加えている．数概念の発生　成立には，少なくとも 2 つの精神活動が必要であるとしている．はじめに，客観概念としての数は，「感覚および知覚の複合より，いわば抽き離し抜き出す．」といった分解がどうしても必要になる．一方，関係概念としての数は，「総括し結合する精神活動」である結合が必要になる．そして，この 2 つの精神活動によって，数概念は形成されると結論づけている．

このようにして，数概念を定義　分類した．従来の分類は，具体数と抽象数の二通りの見方になっていて，異なる数の種類を示すものではない．また，定数と不定数という分類は，数それ自身が定まっていることから，この分類は正しくなく，定数しか存在しないとする．この様な考え方から，定数は，主数と副数に分類される．主数は，「自然単位の数」「測定単位の数」「数学　哲学上の数」に，副数は「順序数」「論理的雑多を示す数」「運算数」に分類される．このようにして，クニルリングが現実の量を認めていることは明らかであり，数と量の関係を天下り的に適用した藤沢利喜太郎の数え主義とは異なっている．

3.3　数概念を生成する構成要素

今度は，数概念がどの様にして得られるかが重要な課題となる．そこで，クニルリングは，数概念を獲得するために，「数直観」「数観念」「数え方」「数系統」「計算」を必要とした．まず，数直観は，ペスタロッが述べる直観と同一の概念で直観を認めている．しかし，直観主義で利用される数図を，量と数の中間に位置づけ，直観を支援するための目的で利用する．数観念は，自然的数観念と人為的数観念とに分類され，前者は，単に感覚的な知覚から直観から生じた概念で，観念においても正確に思考し，記憶することはできないとし，表面的な観念だけでは満足できず，そこに算術教授の難しさを指摘する．後者は，数えることを基礎として生じたもので，自然数観念とは本質的に異なり，否定的であった数図を積極的に認める．

つぎに，数を発見する手順としての数え方は，以下の 3 段階を経て発達したと主張する．まずは，目測による数え方としては，プライエルの「児童の心」の中にある実験を紹介している．2 歳 5 ヶ月の小児が 9 個のケーゲルを「一ツ．一ツ，一ツ，一ツ，一ツ，モーツ，モーツ，モーツ，モーツ」と言いながら一つ一つ並べたものであり，物体について物体によって行う動きである．しかし，これでは不十分なために，次の段階である指による数え方が出現する．目測による数え方は，4 以上の数は不明瞭であり，5, 6, 7 などをはっきりと識別するために，両手の指に一対一対応させるための根拠として，数詞の語源をあげる．今度は，言葉による数え方であるが，これは，指で数えていたものに従属する関係でいたものが主役になったものであるとする．また，数えるだけでは数概念は得られないために，十進的数が必要となる．数系統は指によって数えることから発達してきたが，数系統は，大きな数を確実かつ短時間に算定可能なものである．数系統は，数概念を創造する方法であるとともに，すべての十進数を十個のアラビア数字で表現できる数の名前を作る方法でもある．

藤沢利喜太郎の数え主義は，計算は，数系列上の操作であり，2 重の考え方が含まれている．前者は，数系統を無視し，後者は，実際に合わせて一から数え始めているとして批判している．以上の様に，数概念は，数直線，数観念，数え方，数系統及び計算によって獲得されると述べている．しかし，数直線と数え方，数観念と計算の関係のいずれをとっても不明瞭である．

数え主義は，時間系列を，直観主義は，空間を背景にしている．教材教具としての数範囲については，算術教授全般にわたって，有効な心理学的法則によって定めなければならないとしている．例えば，手で数える場合には，1〜5, 両手で数える場合には，10 が自然で数えたときの限界である．すなわち，10 と基数の複合により得られるものが，11 から 20 である．十位で数えて，十の十倍の 100 まで，百位で数えて 100 まで，千位で数えて 1000×10000 ＝ 1,000,000 までは，定めることは可能である．また，1〜20 までの数範囲における数および計算運算を直観させるために計算盤を用いている．

結局，クニルリングの数え主義は，藤沢利喜太郎の数え主義とは異なる．しかし，数図の限界を指摘しながらも，数概念に対する見解が実際の教授とも必ずしも一致しないために未完成のものである．

4　日本における第一期国定教科書と数え主義

第一期国定教科書は，藤沢利喜太郎の影響が強くあらわれて，理念，方法，内容を指向したと言われる．しかし，その内容は，児童の心理学的な考察も含まれているため，藤沢利喜太郎の意向がすべてではない．数範囲の区分に関しては，クニルリングの数え主義に似ている．ただ，算術書とは別に，実際の教授は，個々の教師に任せられているために，実際に，学校現場でどのような意図を含んでいたのかを判断するのは難しい．授業においては，指導書の記述通りに，まず数詞を唱え，その後，数えることによって，集合の大きさを数として捉えさせるように指導し，これが数え主義の特徴とされている．数え方の順序は，具体から抽象への流れではあるが，使う実物の順序も抽象へ流れる媒体となっている．そのために，藤沢利喜太郎の方便とは違っていて実物利用である．その意味においては，日本の第一期国定教科書は，加法では教えることを基本としたが，減法においては，加法の逆算としての位置付けから，数え下がりだけによっていた分けではない．

数え主義の典型的な指導法は，佐藤武に散見される．唱えることから始まり，数えることに移って行く．対象は，はじめは無名数で数を唱えるが，やがて「個」「本」「匹」「人」などの名数を付けた数え方とへと移って行く．ここで，数図は使われないが，数える対象に○や●を使用している．しかし，それに触れているのは，安藤寿郎のみである．日本における第一期国定教科書においては，「4+3」も「4+2」も同じ手法で扱っており，直観主義が入る余地はない．数え主義は，数の系列を利用した加減算であり，最終的には，唱える数詞が，数の大きさを表現することに特徴をもつと指摘されている．

5 まとめ

数学者の藤原利喜太郎の数え主義は，数学論から出発して，順序数により有理数の体系をつくった．数え方は，数系列上の操作であり，十進構造の説明は呼び方のみで，心理学的考察は希薄である．タンクやクニルリングの数え主義とは，明らかに異なった様相を呈している．クニルニングの数え主義は，当時萌芽したドイツ心理学的考察から出発した．直観主義を乗り越え，数えることが数概念の獲得手段で，，これに数えた結果の暗記及び十進構造の理解により，数計算へと発展する．心理的数え主義の様相を呈しており,「自然に適へる教授法」の教授学的原則の基に数範囲の限定や教材教具の開発に関しては，まさに特筆すべきことがあった．

日本における第一期国定教科書は，その完成度や意図から，児童の生活と結びついたものである．日常生活という生活単元学習の観点から評価することが妥当な教科書である．例えば，佐藤武の様に，計算術について積極的に数え主義を用いた者もいた．生活算術の立場の者たちの批判は，日本における第一期国定教科書で採用した数から量へという観点についてである．この量重視の姿勢が，両九九を生み日本の第三期国定教科書に掲載されるに至っている．日本の第一期国定教科書においては，江戸時代の和算の影響もあって，片九九になっている．また，藤原安次郎は，グルーベ主義の加減算の原理としての数の分解を採用している．数の分解は，児童にとって必ずしも易しくはないが，日本における現在使われている教科書を用いた授業実践に繋がる算数科教育の大切な意義が，日本の明治時代の数学史の立場から存在すると考える．

REFERENCES

- 藤沢利喜太郎，1895, 算術条目及教授法，三省堂書店
- 藤沢利喜太郎，1907, 算術教科書，上巻
- 藤沢利喜太郎，1900, 数学教授法講義筆記，大日本図書
- 藤沢利喜太郎，1906,「算術科教授細目案の比較」，東京物理学校雑誌　第 180 号
- クニルリング，1905、数え主義　算術教授法真髄，上巻
- クニルリング，1906、数え主義　算術教授法真髄，下巻
- Tanc,1884,Das Rechner auf der Unterstufe
- Knilling,1884-1886, Zur Reform der Rechenuterrichts
- 鈴木筆太郎，1911, 算術教授法に関する新研究，宝文館
- 鈴木筆太郎，1927, 低学年算術教授案，モナス
- 佐藤武，1919, 算術新教授法の原理及実際，同文館
- 佐藤武，1919, 算術教授革新論，同文館
- 佐藤武，1924, 発生的算術新教育法，文教書院
- 佐藤武，1924, 尋四算術教授毎配当実際，同文館
- 佐藤武，1938, 数え方の指導，モナス
- 寺尾寿，1888,「中等教育算術教科書」上巻，P.11
- 塩野直道，1947, 数学教育論，河出書房
- 塩野直道，1961, 算数　数学教育論，啓林館
- 文部省編，1905, 尋常高等小学算術書編集趣意書
- 文部省編，1951, 学習指導要領
- 文部省編，1958, 学習指導要領
- 文部省編，1963, 近代日本教科書教授法資料集成，第十三巻　編集趣意書 2

TEACHING AND LEARNING OF FUNCTIONS IN MODERN JAPAN

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ABSTRACT

The concept of functions was introduced into Japan around 1860, in learning differential and integral calculus. At first, a function was considered as a synonym of expression. At the Imperial College of Engineering in Tokyo, Professors Ayrton and Perry taught students engineering and mathematics unified together. They used squared papers extensively from 1876. Functions were treated extensively there.

The ideas of the movement for reform of mathematical education, advocated by Perry and Klein, were introduced into Japan early in the twentieth century. Functions and graphs were gradually introduced into school mathematics. In 1942, a drastic change of the curricula of the mathematics and natural sciences for secondary schools was made. Functions played a central role in the new syllabus of mathematics. It was a reception of the ideas of reform of mathematics education as well as the adoption of the traditional way of learning in Japan. Though the syllabus was not carried out completely due to the war, functions have played an important role in school mathematics since then.

1 Introduction

This paper deals mainly with teaching and learning of functions in Japan from 1860s to the middle of the twentieth century. In the following, names of Japanese and Chinese are written in the order of surname and given name, according to the customs in Japan and China.

First we mention briefly about learning of Western mathematics in Japan up to 1870s. Japan closed the country to outsiders from the seventeenth century to the middle of the nineteenth century. Only very limited trade with China and Holland was permitted. Dutch and Chinese vessels were allowed to trade at Nagasaki. Christianity was strictly forbidden. Even Chinese translations of Western books on natural science, unrelated to Christianity, if these books were translated by missionaries, were prohibited to import to Japan until 1720. As a result, only a very small number of Japanese had some knowledge of Western mathematics at the period of national isolation. Most of them had only some knowledge of Western elementary mathematics: for instance, arithmetic only, or arithmetic and elementary algebra. Only a few of them learned some topics of higher mathematics.

Until early 1870s, the ways of learning Western mathematics were as follows:

(1) learning mathematics under a Western teacher,

(2) learning mathematics under a Japanese teacher having some knowledge of Western mathematics,
(3) learning mathematics from Western books,
(4) learning mathematics from Chinese translation of Western books,
(5) learning mathematics from books on Western mathematics written in Japanese.

Among these, (1) was realizable only for very limited ones. Others learned Western mathematics either under a Japanese teacher at first, then by themselves from books, or by themselves from books from the beginning. Western mathematics was learned mainly from Western books, written in Dutch in the early days, later in English or in French, sometimes in German. Western mathematics was also learned from Chinese translations. For Japanese at that time, books written in Chinese were easier to read than those written in Western language. Publication of books on Western mathematics written in Japanese began in 1857, only elementary mathematics at first. For learning higher mathematics, (5) is excluded, as there were no books on higher mathematics written in Japanese at that time.

After the Meiji Restoration (1867–1868), Japanese Government intended to modernize Japan by introducing Western civilization, especially Western technology, into Japan. Modern educational system was introduced in 1872. As to mathematics, the Department of Education decided to teach only Western mathematics at all school levels—teaching of traditional Japanese mathematics was abolished. It was too radical to be carried out, however. Many teachers were unfamiliar with Western mathematics. So the curriculum of arithmetic of elementary schools was revised within a few months. Since then, mathematics has been taught in Japan in Western style, with some consideration on the traditional way of calculation, the use of *soroban* in elementary arithmetic.

2 The concept of functions introduced into Japan in the nineteenth century

The concept of functions was introduced into Japan around 1860 in learning differential and integral calculus. Until 1870s, the number of Japanese who learned elements of differential and integral calculus was very small, and most of them learned calculus from books, written in Western language or by Chinese translation of Western books. Among these, the following books played important roles in learning differential and integral calculus at that time.

(1) “Elements of the Differential and Integral Calculus” by Albert Ensign Church, revised edition,
(2) “A Treatise on the Differential Calculus”, “A Treatise on the Integral Calculus” by Isaac Todhunter,
(3) “Daiweiji Shiji” by Alexander Wylie and Li Shanlan (Chinese translation of “Elements of Analytical Geometry and of the Differential and Integral Calculus” by Elias Loomis); Japanese call this book “Daibiseki Shūkyū”,
(4) “Weiji Suyuan” by John Fryer and Hua Hengfang (Chinese translation of an article by Wallace); Japanese call this book “Biseki Sogen”.

In most books on differential and integral calculus which were introduced into Japan in the nineteenth century, a function is defined as a variable quantity connected with another variable quantity (or other variable quantities). For instance, Church defines a function as follows (Church 1876, p. 1):

> One variable quantity is a function of another, when it is so connected with it, that any change of value in the latter necessarily produces a corresponding change in the former.

We quote also here from Todhunter's Differential Calculus. Todhunter gives some examples of functions (Todhunter 1871, p. 1):

> Suppose two quantities which are susceptible of change so connected that if we alter one of them there is a consequent alteration in the other, this second quantity is called a *function* of the first. Thus if x be a symbol to which we can assign different numerical values, such expressions as x^2, 3^x , $\log x$, and $\sin x$, are all *functions* of x. If a function of x is supposed equal to another quantity, as for example $\sin x = y$, then both quantities are called *variables*, one of them being the *independent variable* and the other *dependent variable*. An *independent variable* is a quantity to which we may suppose any value arbitrarily assigned; a *dependent* variable is a quantity the value of which is determined as soon as that of some independent variable is known.

In "Daiweiji Shiji", a function is defined as an algebraic or analytic expression of another variable quantity (or other variable quantities).

In "Weiji Suyuan", a function is defined like in Church, but actually a function is regarded as an algebraic or analytic expression of the independent variable(s).

Simple examples of functions and most functions treated in the textbooks of calculus are those represented by algebraic or analytic expressions. At that time, an algebraic function was defined as a function represented by an algebraic expression of the independent variable. This is the old usage of the term "algebraic function". Therefore, even if the students regarded functions as algebraic or analytic expressions of independent variables, setting aside the definition of a function in the textbook—just like the case of "Weiji Suyuan", they wouldn't have much difficulty and trouble in learning elements of calculus. In this way, "functions" were actually considered as a synonym of "expressions" at that time.

Graphs of functions are treated simply in the textbooks of differential calculus at that time. Textbooks at that time laid emphasis on geometric properties of various plane curves and tracing of curves as applications of differential calculus to geometry, and not on the graph of a function as a geometric representation of the function.

3 Functions taught at Tokyo Kaisei Gakkō

We mention in this and the next sections briefly about educational institutions in Japan where elements of calculus (and consequently the concept of functions) were taught in 1870s and in the early 1880s.

Tokyo Kaisei Gakkō, a predecessor of the University of Tokyo, was established in 1873 as an institution of higher education giving professional education in various fields, and the origin of this institution was "Yōgakusho" (Institution of Western Studies), established by Tokugawa Government in 1856. In 1877, the University of Tokyo was established by amalgamating Tokyo Kaisei Gakkō and Tokyo Igakkō (Tokyo Medical School). At Tokyo Kaisei Gakkō, five departments of special and technical learning were intended at first: Law, Chemical Technology, Engineering, Polytechnical Science, and Mining. Among these five departments, the first three were planned to be taught in English, the fourth in French, and the fifth in German. Professors were invited from Western countries. Mainly for financial reasons, however, Mining was abolished, and Polytechnical Science was reduced to Physics (in French) at first, and then was also abolished in 1880.

Engineering course started in 1874 by appointing Robert Henry Smith, a graduate of the University of Edinburgh, to Professor of Mechanical Engineering. He taught mechanical engineering and higher mathematics in the academic year 1874–1875, and in the subject "Higher Mathematics" he taught quaternions and differential and integral calculus. From examination papers of Differential Calculus and those of Integral Calculus recorded in the "Calendar" of Tokyo Kaisei Gakkō, we find that an outline of differential and integral calculus was taught in that academic year. Textbook of differential and integral calculus was unknown, but, judging from the examination papers, the textbook was easier one than Todhunter's.

In the next academic year, "Higher Mathematics" was taught by James R. Wasson, Professor of Civil Engineering. He was a graduate of West Point Military Academy in the United States, and he taught calculus using Church as the textbook. Church's books were used as textbooks at West Point for a long time, and Wasson taught calculus based on the method at West Point.

Mathematics and Physics taught in Physics in French course were, until its end in 1880, of the highest level in Japan at that time. According to the curriculum of this course, the final topic in mathematics was the mathematical theory of heat, but the details of the actual teaching of each topic are unknown. The concept of functions and an introduction to differential calculus were taught in a subject "Complementary Algebra" before learning "Differential and Integral Calculus", following the curriculum of mathematics in lycée in France. Judging from the outline of the contents and order of "Complementary Algebra" cited in (Ogura 1948, p. 189), the author considers that the main textbook of this subject was Briot's one (Briot 1874).

In the University of Tokyo, the curriculum of mathematics at the Department of Science was drawn up in 1880. It was modeled after that of Great Britain. Textbooks of Differential Calculus and Integral Calculus were Todhunter's ones. As to functions, the first teaching of the theory of functions of a complex variable and elliptic functions was carried out in the academic year 1883–1884 by Terao Hisashi (1855–1923), Professor of Astronomy. Though the details are unknown, the author considers that the topics were treated in French style. For, Terao was a graduate in Physics (in French) of the University of Tokyo, and he studied astronomy in Paris.

4 Teaching of functions and graphs at Kōbu Daigakkō

Kōbu Daigakkō, or the Imperial College of Engineering, is a predecessor of the College of Engineering of the University of Tokyo. The college was planned in 1871 by Kōbushō, the Department of Public Works of the Government, as a college to train students to be engineers serving as government officials in that Department. Main factories in Japan were under government management at that time. The actual start of the college under the name of Kōgakuryō was in 1873. All professors were invited from the United Kingdom, most of them from Scotland. The principal of the college was Henry Dyer (1848–1918), Professor of Civil and Mechanical Engineering. He was a graduate of the University of Glasgow. He stayed at this College until 1882. Professor of Mathematics (from 1873 to 1878) was David H. Marshall, a graduate of the University of Edinburgh. Among the professors were William Edward Ayrton (1847–1908), Professor of Natural Philosophy and Telegraphic Engineering from 1873 to 1878 and John Perry (1850–1920), Professor of Civil and Mechanical Engineering from 1875 to 1879.

It was a six-year college of technical education. The whole course was divided into three: (1) the general and scientific course, the first two years, (2) the technical course, the next two years, and (3) the practical course, the final two years. Theory and applications, teaching and learning in school and practical training outside school were unified together. As to mathematics, a subject "Elementary Mathematics" was taught firmly in the general and scientific course. It was a standard course of elementary mathematics with some applications to practical problems, and not a merely application-oriented one. The contents were geometry, algebra, plane trigonometry, mathematical tables, spherical trigonometry and geometrical conics. Coordinate geometry and calculus were treated in "Higher Mathematics" in the technical course.

In the technical course, Ayrton and Perry taught subjects of engineering by unifying technology and mathematics together. They taught science and engineering by applying mathematics, and also topics of higher mathematics through solving practical problems in science and engineering. They used squared papers extensively since 1876. Functions were treated extensively, not only functions represented by analytic expressions but also functions represented by tables or graphs. They taught also finding of functional relation between two kinds of quantities from experimental data by plotting these on a squared paper. It was an epoch-making event in the history of the teaching of the concept of functions. Perry's experiences in Japan and United Kingdom resulted in his idea of "Practical Mathematics" and that for reform of mathematical education setting "utility" as the core.

In 1886, this College merged with the University of Tokyo, and the Imperial University was established. At the College of Engineering of the Imperial University, mathematics was taught by developing the methods in Kōbu Daigakkō. In 1890s, mathematics was taught by Inokuty Ariya (1856–1923), Professor of Mechanical Engineering. He was a graduate of Kōbu Daigakkō and was taught from Perry there. Inokuty taught calculus emphasizing graphical methods.

5 A step to the spread of the concept of the functions

First we consider books on calculus written in Japanese in the early Meiji era. Publication of Japanese translation of books on higher mathematics began 1872, and the number of publications treating some topics of higher mathematics began to increase from the middle of 1880s, though slowly.

Fukuda Han, who was a mathematician and a military engineer captain, intended to translate "Daiweiji Shiji" into Japanese. The Japanese translation "Daibiseki Shūkyū Yakukai" was planned to be published in ten volumes altogether, but actually only the first volume was published in 1872, after reviewed by his father Fukuda Riken (Fukuda Izumi), a mathematician who studied Wasan at first and wrote books on Western elementary mathematics in 1870s. The contents of "Daibiseki Shūkyū Yakukai" were an abridged translation of the first four books of "Daiweiji Shiji", that is, an introduction to coordinate geometry in plane, and no functions in this volume. In Japanese translation, formulas were written as in English original, and not in Chinese style.

Eight years after, Fukuda Han edited and, after reviewing by Fukuda Riken, published a book on differential and integral calculus, "Hissan Biseki Nyūmon" (Introduction to Differential and Integral Calculus by Written Calculation) in two volumes in 1880. This is the first book on differential and integral calculus written in Japanese. This book was edited by referring several books on calculus written in English and those written in Chinese. Among these, "Daiweiji Shiji" and Loomis' original text, and "Weiji Suyuan" played important roles. Formulas are written in Western style. A function is defined as in "Weiji Suyuan". "Hissan Biseki Nyūmon" was a book on differential and integral calculus in the transition period from Wasan to Western mathematics. Fukuda Riken and Han comprehended differential and integral calculus relating with Wasan, especially "Enri" (literal translation: theory of circles, a branch of higher mathematics in Wasan). For instance, we see in the exercise problems of "Hissan Biseki Nyūmon" mensuration of various figures as are seen in Wasan books.

Japanese translation of Todhunter's "Differential Calculus" was published in 1881, and that of his "Integral calculus" in 1882. The translator was Nagasawa Kamenosuke (1860–1927).

Okamoto Norifumi intended to translate Church's book on calculus into Japanese. Only the first part, differential calculus was published in 1883. In translating into Japanese, he made some enlargement to the original text; he considered it better to Japanese audience at that time.

In this way, textbooks on differential and integral calculus were translated into Japanese. At higher educational institutions, however, the textbooks of calculus were Western books written in English such as Todhunter of Williamson and not the ones written in Japanese until 1920s. For, the styles of Japanese translations was rather stiff, and, for a person who had a fair knowledge of English, reading original English text would be better for learning calculus.

As mentioned in previous sections, Japanese in the 1870s and in the early 1880s learned the concept of functions in learning calculus, a branch of higher mathematics. Therefore, only a very small number of Japanese had some knowledge of functions. In Western countries at that time, textbooks of elementary algebra which treated functions and graphs and books on mathematics for general audience containing some topics from higher mathematics were already published.

For instance, "Common Sense of Exact Sciences", a posthumous publication of William Kingdon Clifford (1845–1879), edited by Karl Pearson, treated several topics from "higher mathematics". It was a new departure in a book on mathematics for general audience. Functions and graphs were treated in this book. This book was translated into Japanese by Kikuchi Dairoku (1855–1917), Professor of Mathematics at the Imperial University, and published as "Sūri Shakugi" (it means: Explanation of Mathematical Sciences) in 1886.

Japanese translations of the textbooks of algebra which treated functions and graphs were published from the late 1890s: for instance, H. Bos's textbook of algebra in French and George Chrystal's "Textbook of Algebra" in 2 volumes and "Introduction to Algerbra". Some books on algebra written by Japanese authors also treated functions and graphs. In this way, the concept of functions were, though very slowly at first, spreading in Japan since the last decade of the nineteenth century.

6 Introduction of functions and graphs into school mathematics

In 1902, the syllabus of middle schools (in Japanese: *chugakkō*), boys' five-year schools for general education, was officially announced. As to mathematics, mathematics is divided into four subjects: arithmetic, algebra, geometry and trigonometry, and each subject should be taught rigorously by its own method. There are no functions except the term "circular functions" in the syllabus of trigonometry.

The movement for reform of mathematics education from the beginning of the twentieth century, advocated by John Perry and Felix Klein (1849–1925), was intended to reform school mathematics into modern and practical one by introducing the concept of functions into school mathematics. Perry stressed radical reform of mathematics education by setting "utility" as the core, and by getting rid of Euclid. Klein regarded the concept of functions in geometric form as the central idea in school mathematics.

The idea of the movement for reform was introduced into Japan early in the twentieth century, first by Hayashi Tsuruichi (1873–1935), Professor of Mathematics at Tohoku Imperial University, Kuroda Minoru (1878–1922), Professor of Mathematics at Tokyo Higher Normal School, and Inokuty Ariya, Professor of Mechanical Engineering at the Imperial University of Tokyo. Kuroda studied mathematics education under Klein at Göttingen, and Inokuty was greatly influenced by Perry at Kōbu Daigakkō. Hayashi and Kuroda worked eagerly for the spread of the new trends of mathematics education, especially Klein's idea for reform. Inokuty introduced Perry's idea, especially that of "Practical Mathematics", and worked actively for the spread of the idea of practical mathematics and Perry's idea for reform of mathematics education into Japan. He was the first advocate of Perry's idea in Japan. But his activity was limited within the technical schools and not to teachers of mathematics at schools for general education. Meanwhile, some of the ideas for reform, for instance, the use of graphs as the visualization of functions, were gradually known to mathematics educators and teachers in the first two decades of the twentieth century.

The syllabus was revised in 1911. The new syllabus of mathematics begins with a remark:

> Though the syllabus of mathematics is written by dividing into four subjects: arithmetic, algebra, geometry and trigonometry, mathematics should be taught always considering mutual relations among (contents of each year and) subjects.

The syllabus had no mention of functions. Responding the remark cited above, however, "teaching of mathematics with consideration for the connection of algebra and geometry" became a topic to be considered in teaching of mathematics in middle schools in Japan, and it resulted in the introduction of the concept of functions into mathematics in secondary education.

Publication of textbooks of algebra for middle schools treating functions and graphs began in 1913. Namely, a textbook of algebra by Hayashi and that by Kuniyeda Motoji (1873–1954), Professor of Tokyo Higher Normal School, both published in 1913, treated functions and graphs. These textbooks were in use since 1914.

In Kuniyeda's book a function is introduced in the fourth year class, in the last chapter after learning ratio and proportion.Coordinates, graphs and functions are introduced starting from proportional relations

By arranging a chapter on functions and graphs in the end, previously learned topics may be summarized by using the idea of functions, and also pupils can see a new view of mathematics from the standpoint of "functions". Moreover, even omission of the chapter is possible: for instance, in case of shortage of school hours, and for a teacher who is passive to introduce functions in school mathematics.

On the other hand, Hayashi treats functions from the beginning. Functions are introduced in the second year class, the first year of learning algebra. After learning an algebraic expression and its values, graphical representation of the variation of the values of an algebraic expression is introduced. Then, coordinates, functions and graphs are introduced. Linear equations and linear functions are treated simultaneously. Hayashi's plan is based on Klein's idea: "The concept of functions in geometric form should be the central idea in school mathematics".

In this way, the idea of “introduction of the concept of functions into school mathematics” had been gradually spreading among teachers of mathematics in Japan. But, in the early stage, teachers of mathematics often understood “teaching of functions in geometric form” merely as “teaching of graphs” or as “teaching elements of coordinate geometry as a topic of connection between algebra and geometry”.

By the way, the year 1913 was also the year of the first publication of a book on the theory of functions by Japanese author. Namely, “Kansūron” (Theory of Functions) by Yoshikawa Jitsuo (1878–1915), Professor of Mathematics at Kyoto Imperial University, was published in that year. Yoshikawa was a graduate of the Imperial University of Tokyo and studied analysis under David Hilbert in Göttingen. In this book, elements of complex analysis and some advanced topics were explained clearly, and the idea of Riemann surfaces were introduced early.

The Mathematical Association of Japan for Secondary Education, the predecessor of the present Japan Society of Mathematical Education, was established in 1919 to improve mathematical education in Japan. In 1924, both Ogura Kinnosuke (1885–1962) and Sato Ryoitiro (1891–1992) stressed reform of mathematics education in middle schools in Japan by taking the concept of functions as the central idea, and they also stressed that the idea of differential and integral calculus should be introduced as a final step of teaching the concept of functions. Ogura argued the necessity of radical reform of mathematics education by getting rid of Euclid and following the idea of Perry. According Ogura, “the essence of mathematics education is development of scientific mind of pupils” and “the core of mathematics education is cultivation of the concept of functions”. Ogura’s book (Ogura 1924) had influence on teachers and educators, especially those of elementary schools, and it resulted in a sweeping revision of the textbook of arithmetic in elementary schools in the thirties. Sato explained in his book of 1929 teaching of mathematics in middle schools more in detail by giving teaching plans that he had already practiced at the Middle School Attached to Tokyo Higher Normal School.

The curriculum of middle schools was revised in 1931 to cope with the spread of secondary education and to meet the demand of the times. The new syllabus of mathematics was very simple and indicated only main contents of the each year, without dividing into subjects such as algebra, geometry and so on. The syllabus also indicated that,

> In teaching mathematics, cultivation of the concept of functions should always be kept in mind.

This was the first mention of functions in the syllabus of mathematics of middle schools. Functions were not the central topics in the curriculum, however.

The national textbook of arithmetic for elementary schools was renewed entirely in the thirties. The new textbook, “Jinjō Shōgaku Sanjutsu” (Arithmetic for Elementary Schools), was edited and published by the Ministry of Education every year for one grade, beginning 1934, and the new textbook came into use from the first grade children of the school year 1935. It was edited following the ideas for reform of mathematics education from the beginning of the twentieth century and intended to develop mathematical thinking of school children through their various activities. Graphs and topics related to the concept of functions were treated extensively.

7 A drastic change

In 1942, the curricula of mathematics and science for middle schools and those for girls' schools were revised thoroughly and drastically. The revision was to cope with a time of crisis for the nation and to meet the pressing demand for the improvement of science and mathematics education at that time. Until that time, teaching of science in secondary education was rather old-fashioned. The new curricula were intended to cultivate scientific minds and creativity of pupils so as to cope with various difficult situations which they will encounter and to meet the national demand of the times.

The aim of the new syllabus of mathematics was the reorganization of school mathematics by getting rid of the "traditional system" of mathematics education, to cope with the national movement for the new order at that time. The new syllabus was intended to develop pupils' mathematical thinking and creativity through their various activities. Functions played a central role in the new syllabus of mathematics. Utility and applications of mathematics were emphasized. Many new topics were introduced in mathematics of middle schools: elements of analytic geometry, nomography, descriptive geometry, elementary probability and statistics, and the idea of differential and integral calculus. On the other hand, Euclidean geometry was not treated in the traditional way.

The syllabus indicated in an item of the remarks that

> Thoughout the syllabus, attention should be paid to cultivate the concept of relations.

The concept of relations is far more broad than that of the functions. The author considers that it was a step forward to introduce fundamental ideas in modern mathematics into school mathematics.

Emphasis was laid on learning by doing and heuristic methods: to let pupils discover mathematical facts through their various activities such as observations, experiments and considerations, and to let them synthesize and systematize these mathematical facts with appropriate suggestions and advices of the teacher. To acquire knowledge, doing was required. This was just the traditional way of learning in Japan: learning through training and self-study—acquisition of knowledge through training by themselves.

Emphasis was laid also on utility and applications. Knowledge without accompanied by practice was regarded as worthless at that time, and "integration of knowledge and practice" was advocated. This was influenced by learning through training, the traditional way of learning in Japan and the doctrine of inseparability of knowledge and practice by Wann Yann Ming (1472–1528), a Chinese philosopher of early sixteenth century. By the emphasis on utility and applications, many ideas which Perry had proposed and had stressed from the late nineteenth century were realized in the curriculum. It was a reception of the ideas of reform of mathematics education since the beginning of the twentieth century as well as the adoption of the traditional way of learning in Japan.

This curriculum was not carried out completely due to the World War II. This curriculum is essentially a curriculum in the peace time and not the one in the wartime, and careful preparations were essential for the carrying out the curriculum successfully, as Ogura and Nabeshima wrote in their book (Ogura & Nabeshima 1957). Though the curriculum was not carried out completely, functions have played an important role in school mathematics since then.

8 After the World War II

In this section we mention briefly about the teaching of functions after the World War II. After the war, the educational system in Japan was reformed entirely. The new system came into operation in 1947. Since then the curricula for elementary, lower-secondary and upper-secondary schools were revised several times, about every ten years.

Mathematics in upper-secondary schools has been taught systematically. Functions have been playing important roles. Elements of differential and integral calculus were taught systematically, not like the one in the syllabus of 1942, in an elective subject of mathematics, from the start of upper-secondary schools.

The revision of the Course of Study in 1960 was intended to develop students' basic knowledge and skills and to improve scientific and technological education so as to meet the demands of the society. Calculus and analytic geometry were enriched, and some new topics such as vectors and the idea of sets were introduced. On the other hand, to cope with the spread of upper secondary education, subjects whose contents were only basic ones were prepared.

Introduction of calculus into school mathematics was, at first, to introduce the ideas of calculus into school mathematics as a final topic of teaching the concept of functions. Since the 1960s, calculus in upper secondary schools has been changed to lay stress on students' acquisition of skills to cope with the rapid development of science and technology.

The New Math Movement, influenced by the works of Bourbaki group, spread worldwide from 1960s to 1970s. As a result, upper secondary mathematics leaned towards formal and abstract one and not practical one. Mathematics education from 1960s to 1970s was going in the direction opposite to the direction in which it had proceeded since the beginning of the twentieth century.

Functions are playing an important role in school mathematics since 1940s, and the introduction of elements of differential and integral calculus into secondary education has been made. In this way, popularization of calculus has been made.

The spread of upper-secondary education, emphasis on skills in school mathematics, introduction of new topics of abstract nature such as the idea of sets, and popularization of calculus caused new problems. For instance, students learn calculus of simple polynomial functions at first. Polynomial functions, however, can differentiated and integrated formally without using the concept of limits. Therefore, students can solve routine problems concerning differentiation and integration of polynomial functions just as solving problems of calculations in algebra, without understanding the concepts of functions and limits. We should make efforts to improve the situation and to develop teaching of mathematics.

REFERENCES

- Bertrand, J. L. F., 1870, *Traité d'Algèbre, Deixième Partie, à l'usage des classes de mathématiques spéciales*, New ed., Paris.
- Briot, C. A., 1874, *Leçons d'Algèbre, conformés aux programmes officiels de l'enseignement des lycées, Deuxième Partie, à l'usage des élèves de la classe de mathématiques spéciales*, 7th ed., Paris.
- Church, A. E., 1876, *Elements of the Differential and Integral Calculus, Revised Edition, Containing the Elements of the Calculus of Variations*, New York: A. S. Barnes and Co.

- Editorial Committee (ed.), 1983–1984, *Nippon no Sūgaku Hyakunenshi* (A History of Mathematics in Japan in Recent One Hundred Years), 2 vols., Tokyo: Iwanami Shoten.
- Fryer, J. & Hua, H. (trs.), 1874, *Weiji Suyuan* (Chinese translation of an article on "fluxions" in the 8th edition of the Encyclopaedia Britanica), Shanghai. (This book is called "Biseki Sogen" in Japan).
- Fukuda, H. (tr.), 1872, *Daibiseki Shūkyū Yakukai* (Japanese translation with commentaries of "Daiweiji Shiji", Chinese translation of "Elements of Analytical Geometry and of the Differential and Integral Calculus" by E. Loomis), revised and annotated by Fukuda Riken, Tokyo.
- Fukuda, H., 1880, *Hissan Biseki Nyūmon* (Introduction to Differential and Integral Calculus by Written Calculation), revised and annotated by Fukuda Riken, 2 vols., Tokyo.
- Japan Academy (ed.), 1954–1960, *Meijizen Nippon Sūgakushi* (A History of Japanese Mathematics before Meiji Era), 5 vols., Tokyo: Iwanami Shoten.
- Kota, O., 1994, "Bisekibun Kyōiku no Hanseiki" (Fifty Years of Calculus Education in Japan), *St. Paul's Review of Science* **34**, 1–18.
- Kota, O., 2005a, "Meiji Shoki no Kōbu Daigakkō ni okeru Sūgaku Kyōiku" (Mathematics Education at the Imperial College of Engineering in the Early Meiji Era), *Sūrikaisekikenkyūsho Kōkyūroku* **1444**, 43–58.
- Kota, O., 2005b, "Meiji Shoki no Kōgakuryō, Kōbu Daigakkō ni okeru Sūgaku Kyōiku" (Mathematics Education at the Imperial College of Engineering in Tokyo in the Early Meiji Era), *Journal for Historical Studies in Mathematics Education* **5**, 26–37.
- Kota, O., 2007, "Meiji Zenki ni okeru 'Seiyō Kōtō Sūgaku' no Kyōiku" (Teaching of 'Western Higher Mathematics' in the First Half of the Meiji Era), *Sūrikaisekikenkyūsho Kōkyūroku* **1546**, 230–246.
- Kota, O., 2010, "Meiji Jidai ni Manabareta Furansu-ryū Sūgaku" (Mathematics in the French Style Learned in the Meiji Era), *Sūrikaisekikenkyūsho Kōkyūroku* **1677**, 230–242.
- Kota, O., 2012, "Kindai Nippon ni okeru, Kansū no Gainen to soreni Kanrenshita Kotogara no Juyō to Fukyū" (The Reception and Spread of the Concept of Functions and Related Subjects in Modern Japan), *Sūrikaisekikenkyūsho Kōkyūroku* , forthcoming.
- Matsubara, G., 1982–1987, *Nippon Sūgaku Kyōikushi* (A History of Mathematics Education in Japan), 4 vols., Tokyo: Kazama Shobo.
- Nagasawa, K. (tr.), 1881, *Bibungaku* (Japanese translation of Todhunter's Differential Calculus), Tokyo: Sūri Shoin.
- Nagasawa, K. (tr.), 1882, *Sekibunngaku* (Japanese translation of Todhunter's Integral Calculus), Tokyo: Sūri Shoin.
- Ogura, K., 1924, *Sūgakukyōiku no Konpon Mondai* (Fundamental Problems of Mathematical Education), Tokyo: Idea Shoin.
- Ogura, K., 1932, *Sūgaku Kyōikushi* (A History of Mathematics Education), Tokyo: Iwanami Shoten.
- Ogura, K., 1935, 1948, *Sūgakushi Kenkyū* (Researches in History of Mathematics), 2 vols., Tokyo: Iwanami Shoten.
- Ogura, K., 1973–1975, *Ogura Kinnosuke Chosakushū* (A Collection of Writings of Ogura Kinnosuke), 8 vols., Tokyo: Keisō Shobo.
- Ogura, K. & Nabeshima, N., 1957, *Gendai Sūgaku Kyōikushi* (A History of Mathematics Education in Modern Times), Tokyo: Dainippon Tosho.
- Okamoto, N. (tr.), 1883, *Sasi Bibunsekibunngaku*, vol.1 (Church's Differential and Integral Calculus), Tokyo: Ministry of Education.

- Perry, J., 1899, *Practical Mathematics, Summary of Six Lectures Delivered to Working Men*, London: HMSO.
- Perry, J., 1900, *England's Neglect of Science*, London: T. Fisher Unwin.
- Perry, J. (ed.), 1902, *British Association Meeting at Glasgow, 1901. Discussion on the Teaching of Mathematics*, 2nd ed., London: Macmillan and Co.
- Research Group for the History of the University of Tokyo (ed.), 1993–1994, *Tokyo Daigaku Nenpō* (University of Tokyo Annual Report), 6 vols., Tokyo: University of Tokyo Press.
- Sato R., 1924, *Shotō Sūgakukyōiku no Konponteki Kōsatsu* (Fundamental Considerations to Elementary Mathematical Education), Tokyo: Meguro Shoten.
- Sato R., 1929, *Sūgakukyōiku Kakuron* (A Detailed Explanation of the Teaching of Mathematics), Tokyo: Tōyō Tosho.
- Todhunter, I., 1871, *A Treatise on the Differential Calculus with Numerous Examples*, 5th ed., London: Macmillan and Co.
- Todhunter, I., 1878, *A Treatise on the Integral Calculus and its Applications with Numerous Examples*, 5th ed., London: Macmillan and Co.
- Wylie, A. & Li, S. (trs.), 1859, *Daiweiji Shiji* (Chinese translation of "Elements of Analytical Geometry and of the Differential and Integral Calculus" by E. Loomis, 1852), Shanghai. (This book is called "Daibiseki Shūkyū" in Japan).
- Yoshikawa J., 1913, *Kansūron* (Theory of Functions), Tokyo: Fuzanbō.

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18th CENTURY MATHEMATICS EDUCATION
Effects of Enlightenment in Iceland

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ABSTRACT

Living conditions in Iceland worsened in the period 1600–1800, and the greatest lava flow on earth in historical times in 1783–84 was accompanied with severe earthquakes and famine. Concurrently, the Enlightenment movement, channelled from Germany through Denmark, had considerable influence in Iceland from 1770 onwards. People, interested in progress in Iceland, established a society which advanced the Enlightenment by publishing a journal and books on various practical matters. Among them were philosopher Ó. Olavius and lawyer Ó. Stephensen, later Governor of Iceland, both educated in Copenhagen. The Enlightenment movement produced in the 1780s substantial arithmetic textbooks deliberately intended to raise the educational standards of Icelanders. One of them was by Olavius in 1780, modelled on Danish and German textbooks. The other arithmetic textbook published in 1785 by Stephensen, was accompanied with introduction to algebra, modelled on lectures at the University of Copenhagen. It was immediately sanctioned as a textbook at the two Latin schools, where, however, mathematics education was at its nadir until 1822. These two books, in addition to a book of arithmetic tables, remained the only arithmetics textbooks available in Icelandic until the 1840s. Both were good representatives of the typical European arithmetic textbook of the *practica* type. Their influence in the aftermath of the disastrous events will be explored as well as their roots in German arithmetic education tradition in the Enlightenment period.

1 Introduction

The Enlightenment was an educational movement which had considerable effect in Denmark whence it arrived from Germany. The Enlightenment left a marked impact on many Icelanders who studied in Copenhagen, some of whom subsequently became leading officials in Iceland, both secular and religious, and could thus use their influence (Sigurðsson, 1990, p. 293).Arithmetic textbooks constituted an important part of the efforts of behalf of the members of the Enlightenment movement to raise the educational standards of Icelanders.

We shall study three printed arithmetic textbooks, published in the period 1780–1785. Considering that only one 14 page long arithmetic textbook in the Icelandic language had been printed earlier, these were important enterprises and worth exploring. Several questions come up:

Which political aims led to the publications of the textbooks?

Who funded the publications?

From where did the authors seek their models?

Which didactical aims did the authors have?

We shall elaborate on each of the three books, one by one, and finally draw some conclusions about the above questions. But first we shall set the scene of the Icelandic society in the 18^{th}century.

2 Iceland in the 18^{th}century and the Enlightenment

In the Age of Enlightenment, Iceland was a part of the Danish state and cultural and other influences from abroad reached the country mostly from Denmark. At the end of the 18^{th} century the population numbered only 47,000 and was almost exclusively rural (Sigurðsson, 1990). The 18^{th} century had been marked by periods of famine and misery. Consequently, the state of general education worsened in Iceland while it was rising in Europe.

From the 12^{th} century there had been cathedral schools at the two Episcopal Sees in Skálholt and Hólar where the syllabus was preparation of the clergy and Latin was the main subject. In 1743, an ordinance was issued which required knowledge in the four arithmetical operations in whole numbers and fractions (Janus Jónsson, 1893, pp. 38–41). Geometry was not mentioned. There were no schools for the general public, while regulations on knowledge in reading and Christendom were set in 1743 as requirement for confirmation at the age of 15. The instruction was the responsibility of the families under the supervision of the clergy.

Records are available on population of year 1703: 50,358, and year 1769: 46,271. Repeated period of famine had led to population decrease. The census of 1703 provides information about occupations, see Table 1. As the structure of society did not change markedly during the 18^{th} century one may assume that the division of the population into classes was similar around 1780.

Karlar *Males*	
Lögmenn *Prefects*	3
Landsskrifari *Secretary of the public court of law*	1
Lögréttumenn *Members of the public court of law*	43
Sýslumenn og lögsagnarar *Sheriffs*	21
Hreppstjórar *Municipal administrators*	670
Umboðs- og klausturhaldarar *Administrators of public estates*	5
Biskupar *Bishops*	2
Prestar *Clergymen*	245
Djáknar *Deacons*	4
Skólameistarar og kennarar *Headmasters and teachers*	7
Skólalærðir *School graduates*	26
Skólapiltar *Students*	76
Fyrrverandi embættismenn *Retired officials*	6
Kaupmannsfullmektugir og eftirlegumenn *Commercial managers*	5
Fálkafangarar *Falcon catchers*	6
Smiðir *Carpenters/artisans*	108
Bókbindarar *Bookbinders*	2
vBrytar *Stewards*	4
Total	**1224**

Occupations in 1703 (Hagskinna, 1997).

Public administrators, public employees, academics, lawyers and other professional occupations numbered about 350, among them 245 clergymen many of whom took young boys for learning. Another group of people, who must have had to know some counting, were the municipal administrators who numbered 670. Adding their family members and skilled workers, one can assume that at least

1500 persons in the country could make use of arithmetic textbook. Many skills deteriorated during the misery of the 18th century while the ordinance about reading skills in the 1740s counteracted that.

In spite of many calamities that befell the population of Iceland in the last quarter of the 18th century, it was also a dawn to new era. The policy of the Danish authorities towards Iceland was in many ways influenced by the Enlightenment movement. After a period of famine by the mid-1700s the Danes were shocked to realize that the population in 1767 was less than 50,000 instead of the 80,000 which had earlier been assumed. Various bodies in the Danish state supported efforts to enhance progress in Iceland. Not all of them were well founded, however, but a number of new legislation acts were enacted in 1772–1787 with the aim of modernization (Björnsson, 1990). Research reports were made on the state and trends in the colony, and e.g. in the period 1770–1780 a great number of informative printed texts were distributed free in Iceland (Magnússon, 1990).

The Danish authorities had run monopoly trade from 1602 where the inhabitants were obliged to trade within certain trading districts, each allotted to a particular merchant. A terrible famine following volcanic eruptions and earthquakes in 1783–84 led to sudden abolition of the monopoly trade in 1787. The earthquakes destroyed the Episcopal See of Skálholt and its cathedral school which consequently was moved to the new capital Reykjavík and became one of the seeds to growing urbanisation in the 19th century.

The Icelandic society had stayed intact for centuries, and things were not easily changed, least by external force, however good intentions and resources were put into the effort. The catastrophic eruption and earthquakes of 1783–84 and financial difficulties in Denmark caused by the Napoleonic wars in the early 1800s were finally to reduce to results to next to nothing (Björnsson, 1990).

3 The textbooks

Three books on arithmetic were published in a five year period, 1780–1785. They may be considered as a result of the wide-reaching effort of the Danish authorities and the proponents of the Enlightenment to raise the level of education in the country. All the authors were active members of societies of the movement and agents of the Danish authorities in one way or another.

The textbooks largely adhered to the European tradition of practical arithmetic, *practica*, originating in the late Middle Ages. They began by an explanation of the concepts of a number and of numerical place value, followed by the arithmetic operations in whole numbers and fractions: addition, subtraction, multiplication, division and extracting square roots. The remaining content concerned mathematical techniques for business use: use of the Rule of Three, monetary exchange, problems of partnership and barter (Swetz, 1992). However, the nature of the three textbooks was different and they may have complemented each other.

3.1 Olavius: Clear Guide

One of the most prolific writers of the Icelandic Enlightenment movement was Ó. Olavius (c. 1741–1788), who wrote a number of books about nature science and economic matters. One of his greatest feats, while also his fallacy, was to establish the first print shop in the country to print secular books in 1773. The print shop printed a total of 83 books. Olavius had, however, to leave the enterprise in 1784 for the sake of disagreements on its financial matters. The following years he travelled around

Iceland to write a research report, *Økonomisk Rejse / Economic Travel* on utilization of harbours, drift wood and abandoned farms, to finally move to Denmark in 1779.

In 1780 Olavius published in Copenhagen an arithmetic textbook, called *Greinilig Vegleidsla til Talnalistarinnar / A Clear Guide to the Number Art* (374 pp. + foreword, xxviii pp.). The book seems to have had support from influential Danish parties. The book is dedicated to a Mr. Schach Rathlau who may have funded the publication. In his foreword, Olavius recounts that there no instruction in the indispensable knowledge of mathematics had existed in the county's vernacular. This could have harmful consequences when trade was made with foreign merchants and experienced calculators who sometimes wanted to earn more than they should. He intended the book for use at the Latin Schools which were two at that time but also for "of other children of the country, who might find an urge to exercise in computing." (Olavius, 1780, p. vii–viii).

While one can sense a sincere wish to improve education in Iceland with this book, an expression like the one later in this address: "... who should believe that the country, deprived of this knowledge amongst others, should be able to stand up so long, as it nonetheless has, even if brought into the shape it presently has, and over which one cannot be surprised . . ." (ibid), disturbed many a good inhabitant of Iceland.

The writer next addresses the benevolent reader for 20 pages. The author is concerned that no one teaches the general public anything about arithmetic, which he counts as most other arrangements in the non-country and every common person who want to learn something must be his/her own teacher. So he concludes that he must explain with a great number of examples.

Unfortunately for the book and the young author, an influential minister of the church, Gunnar Pálsson, a former schoolmaster, and a member of the group who kept the Hrappsey print shop running, gave the book bad review in a letter to a colleague. He seized the phrase "non-country" and others similar, and deemed the book lacking *dexteritas didactica*, didactical adroitness, which it certainly is not in modern understanding. Further inspection reveals that Pálsson wrote the letter before he had seen any more than the forewords. The disagreements and disappointments in connection with the print shop probably caused Pálsson's expressions, which have coloured the reputation of the book until present (Bjarnadóttir, 2006, p. 76).

The textbook was never used at the Latin Schools, but it survived at least for half a century. Olavius recounts in his *Economic Travel*, published in 1780, the same year as the *Clear Guide*, that the *Clear Guide* was distributed for free in Iceland in 1300 copies. Assuming 7000 homes, leads to the conclusion that the book was available in nearly every fifth home in the country. We shall now look at its model and didactical content.

The author informs the reader that among his models were textbooks by the Dane Chr. Cramer, adherent to the Enlightenment, and the German Christlieb von Clausberg (1732, 1748, 1762), the author of *Der Demonstrative Rechenkunst*, to which the *Clear Guide* bears resemblance. Clausberg's book was published in four volumes, a total of over 1400 pages. The *Clear Guide* contains traditional arithmetic of the *practica* type: The number concept, numeration, the four arithmetic operations in whole numbers, "multiple" numbers (with units) and fractions, monetary and measuring conversions, and the Rule of Three, a method of calculating proportions, which was indispensable in unit conversions and price computations. What is different from many other textbooks, ancient and modern, is that the author stressed that calculation methods are only procedures that may be altered and he gives a number of alternatives (Olavius, 1780, pp. 32–33, 41–42, 65–89).

A great part of the book is devoted to "number tricks", translated from the Clausberg's text, named *Rechungsvorteile*, calculating advantages. The number tricks are based on understanding of compositions of numbers. Some of them are simple aid to mental arithmetic, such as multiplying by 100 and dividing by 4 instead of multiplying by 25, or multiplying by 8 and dividing by 1000, instead of dividing by 125. Others suggest e.g. that when multiplying 96 by 39, one may multiply by 40 and subtract 96 once, see Fig. 1:

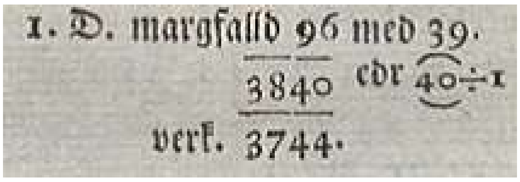


Figure 1

Multiplying by 96 was very useful as 1 rixdollar was 96 shillings. That could be done in various ways, such as multiplying by 6, and then 16; or by 4, 4, and 6, or even 6, 8 and 2, see Fig. 2. The question is: How many shillings are 484 rixdollars?

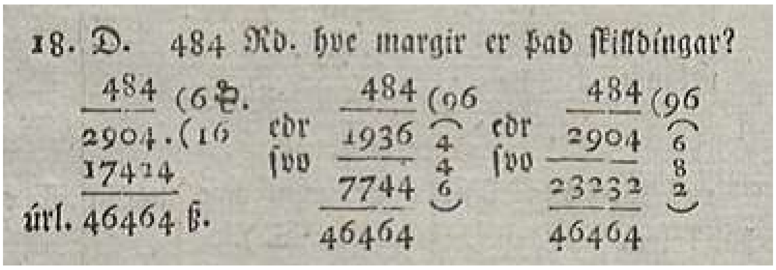


Figure 2

The reverse processes could be applied when converting shillings to rixdollars, see Fig. 3:

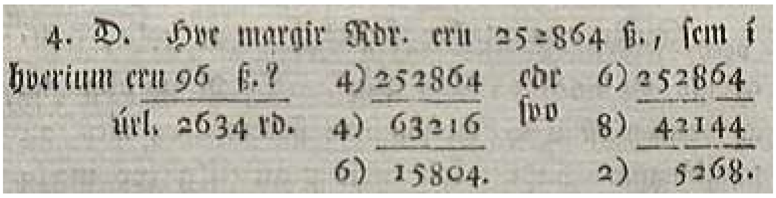


Figure 3

Division was still cumbersome in the 18th century, but number tricks could be good aids. Fig. 4 and 5 demonstrate division by 99 and 97.

678345 : 99 is solved by repeated subtraction:

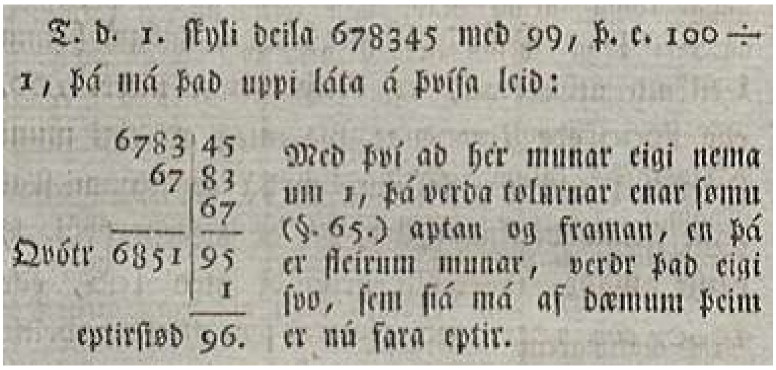
$678345 = \mathbf{6783}(100 - 1) + 45 + \mathbf{67}(100 - 1) + 83 + 67$

The quotient is $6783 + 67 = 6851$ and the remainder is $45 + 83 + 67 = 195$, which raises the quotient to 6852 and the remainder to 96.

Dividing by 97 by this method is a little trickier, see Fig. 5:

$379681 = \mathbf{3796}(100 - 3) + 81 + \mathbf{113}(100 - 3) + 88 + \mathbf{3}(100 - 3) + 39 + 9$

The quotient is 3796 + 113 + 3 = 3912, remainder 81 + 88 + 39 + 9 = 217,


T. d. 1. skyli deila 678345 med 99, þ. e. 100 ÷ 1, þá má það uppi láta á þvísa leid:

6783 | 45
67 | 83
| 67
Qvótr 6851 | 95
| 1
eptirstöd 96.

Med því ad hér munar eigi nema um 1, þá verda tölurnar enar sömu (§. 65.) aptan og framan, en þá er fleirum munar, verdr það eigi svo, sem siá má af dæmum þeim er nú fara eptir.

Figure 4

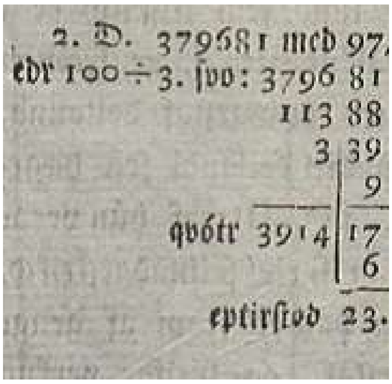

2. D. 379681 med 97, edr 100 ÷ 3. svo: 3796 81
113 88
3 39
9
qvótr 3914 17
6
eptirstöd 23.

Figure 5

raising the quotient to 3914 with remainder 23.

The examples above demonstrate the author's intentions; to enable people to play with the numbers to achieve skills in the art of computing. How fertile the soil was for such methods were in Olavius's "non-country" with no schools for the general public is questionable. However, from the modern point of view, the examples witness a deep understanding of arithmetic and a wish to share it. The model is also clear; a number of the examples in the *Clear Guide* are also found in Clausberg's *Der Demonstrativen Rechenkunst*. The pedagogical aims are declared in the address to the benevolent reader:

> I say of myself, that I learn more from one solved problem than 10 unsolved ones, as there are always some differences, and then more can be learnt from 3 or 4 than from one, how elsewhere shall be done. When beginners will be handed these problems for solving, or they take themselves the same and solve wrongly, then they may see from the doings of the book, wherein their error is hidden, and thus bring their number to the right track, but it must be exercised on a question and an answer. For this reason I call my little work Clear Guidance to the Number Art (Olavius, 1780, p. xv).

The books by Olavius and Clausberg resemble other products of protestant educational efforts (Bjarnadóttir, 2011). They contain examples deduced from the Bible, such as: "The number of years since the creation of the Earth is 5746 years but from the birth of Christ 1779 year. How many years from the creation of the Earth were gone when he came into being?" (Olavius, 1780, p. 39). This problem is found in Clausberg's textbook where the creation of the Earth was 5678 years and Christ born 1729 years earlier (Clausberg, 1732, p. 66), so the two authors do not quite agree upon the age of the Earth. Other examples count the time since the flood of Noah and the time since the Lutheran Reformation. In his examples Olavius does not refer much to Icelandic environment or occupations but primarily to didactical arithmetic concerns.

There is a section about ratio and proportions, and arithmetic and geometric progressions are mentioned. In the section about fractions, numbering 112 pages, the greatest common divisor is found by Euclid's algorithm in a somewhat advanced version. Number tricks with fractions cover 90 pages. The final chapter on *regula de tri* numbers 84 pages.

On the whole, the book is a good introduction to arithmetic, and much of its content could still be of use today as an aid to mental arithmetic. The good intentions to explain and be of assistance shine from the pages, even if the author may have managed to offend his intended readers.

3.2 Johnsonius: A Pocket Book for Farmers

In 1782, the next arithmetic book was published. A learned man, Jón Jónsson Johnsonius (1749–1826), published a pocket-book with tables, handy for the exchange of goods. It was called *Vasa-qver fyrir bændur og einfalldlínga á Islandi: edr ein audvelld Reiknings-List / A Pocket-Book for Farmers and Simple-minded People or an Easy Computing Art* (249 pages). A further elaboration of the title recounts that it contains all kinds of computations in purchases and sales, both by domestic and foreign price levels, and furthermore an extract of the royal Icelandic ordinance on buying rates and mailings.

Johnsonius studied in Copenhagen and received from 1779 a stipend to serve at the Arnamagnaean Manuscript Collection. A. Magnússon's collection, made in the early 18th century, was cherished by the Danes as a Nordic heritage but only Icelanders could read and edit it. Johnsonius became county magistrate in 1797.

In its foreword the author says that even if farmers and the bourgeoisie in Denmark are much better informed in the computing art than the common people in Iceland, a number-booklet such as this one has been published in Copenhagen long ago, and many times. They are better informed, as there they at least have some experience of it in the primary schools [which did not exist in Iceland]. And even if Iceland has already had printed *Number Art* [*The Clear Guide to the Number Art*], this present booklet might nonetheless be of good use to many, who do not have the opportunity to soak their head in arithmetic studies. The other book, even if good and well composed, might not be of good use for the common people.

The author hoped this present book to be of use for those who were not skilled in arithmetic, for their inspection at home and by the merchant so that they were not depending on the merchant or other customers how to calculate the prices which they could read and find themselves in the booklet. But then they would have to study the booklet thoroughly in beforehand. The book and in particular the interest tables, were, according to the author, modelled after the Danish booklet, which is attributed to Søren Matthisen (1680) (Johnsonius, p. 5–7).

The content of the *Pocket-Book* is mainly tables. Firstly, there are multiplication tables up to 100 times 100 on 25 pages, then multiplication of the present currency on 102 pages, and interest-tables on 25 pages. The remainder of the book contains conversion tables between different currencies and scales, and an extract of the charge-table published as a royal ordinance, dated in 1776, about the Danish trade monopoly. These were basically Danish measures, in addition to the ancient *landaurar*, the *hundrað*/hundred equivalent to 120 *alin*/ells (an ell was approx. 60 cm of woollen cloth), 240 fishes or one cow.[1]

The *Pocket-Book* can therefore not be considered as a textbook but a collection of tables. Its contents are in most respects similar to the first textbook, published in 1746, *Lijted Agrip*, an extract of Hatton's *Tradesman's Treasury* (1712), translated by the Rev. Halldór Brynjólfsson, later bishop. The "fish" trading unit is related to other units, such as rix-dollars and shillings, as well as the *Cronas* and *Specias*. Also included is a table of the *tíund* or tithe, which was computed as a 1% property tax, 1 sheep of every 100 sheep, and 1 cow of every 100 cows.

Possibly this book had more influence on the mathematics education of the public than the two good textbooks published by the proponents of the Enlightenment in that same decade.

3.3 Stephensen: Short teaching

In 1785 Ó. Stephensen published at his own cost a textbook (Stefánsson, 1785), *Stutt Undirvísun í Reikníngslistinni og Algebra / A Short Teaching on the Computing Art and Algebra* (248 pages), which became a required reading at the two Latin schools (*Lovsamling for Island 5*, 1855: p. 244). In his preface Stephensen recounts that in 1758 he had written down what he learned in Copenhagen in order not to forget it. As many copies of that work were still in distribution he decided to have it printed. In his address to the reader the author feels necessary to safeguard himself against those already knowledgeable:

> ... this booklet is not composed in the understanding that there are not many in this country that well can compute, and can, without it, teach it to others, especially officials of the religious and secular classes; rather it is intended for use to youngsters and adolescents ... (Stefánsson, 1785: To the reader).

Stephensen was a student of the Rev. Brynjólfsson, later bishop, who translated the *Little Compendium*, and knew undoubtedly his arithmetic well. An inspection of Stephensen's manuscript (Lbs. 409, 8vo), originating in 1758, reveals however a concise text, different from the textbook. The autobiography of Stephensen's son, M. Stephensen recounts that he revised his father's manuscript before printing according to a Professor Geuss's lecture notes (Lbs. 408, 8vo) in 1781–82. The textbook is therefore at university level of the times. The son had added chapters on decimal fractions, which were a novelty, ratios, proportions and sequences, algebra and linear and quadratic equations, to the traditional content of his father concise *practica* textbook manuscript (Bjarnadóttir, in print).

The *Short Teaching* was immediately authorized as a textbook for the two Latin schools. The Latin Schools deserved to have an arithmetic textbook to assist the teaching, but the authorization of the book by Stephensen, which may have been linked to his high administrative position, excluded Olavius's book from school use. Earthquakes, which destroyed one of the schools in 1784, may, however, have reduced the educational impact of Stephensen's textbook. In 1802, the two Latin Schools were

[1] *Hagskinna* (1997). Measures, weights and currency: A hundred = 20 *aurar* (pennies) = 120 *álnir* [ells, originally a measure of woollen cloth]. In terms of fish-value, one hundred = 6 *vættir* = 240 (valid) fishes. This Icelandic currency system existed from medieval times up to the 20th century, called *landaurar* (land-pennies). A hundred was the equivalent of a cow, i.e. a middle-aged, faultless cow in spring, or six sheep, woolly and carrying lambs, in spring. The monetary value of *landaurar* was variable, and up to and beyond the 18th century there were differences of opinion on how to compute it. . . . Farms were also measured by hundreds. An average farm was valued at 20 hundreds, and it was supposed to support livestock of 20 cows or 120 sheep.

merged into one where mathematics education was marginal. According to a memoir: "Everyone who reached the upper grade was given Governor Ólafur's Arithmetic, but it was up to the pupils whether they ever opened the book or not." (Helgason, 1907–1915: pp. 85–86). From 1822 onwards, when B. Gunnlaugsson (1788–1876) had become mathematics teacher at the Latin School, it adhered to Danish regulations and Danish textbooks were used.

As the two other textbook authors, Stephensen mentioned the trade, but first he flattered the Danish authorities which he said that if the inhabitants themselves had most graciously been left to choose, they would have wished themselves nothing else in this respect: that the country's trade would be entrusted to such enlightened, so righteous and so much for its affluence and prosperity, meticulous gentlemen's management (Stefánsson, 1785, p. *3). One should recall that Stephensen was the second highest official in the royal Danish administration and had to be loyal to his superiors. It is, however, hard to credit the sincerity of these words, considering the situation of trade monopoly, which was abolished in the following year. It seems that the author was aware of the forthcoming abolition, as he hopes that the book will be of aid to the up-growing youth; either to be able to themselves, in due time, to participate to some degree in the trade, or as servants to further the same. For this purpose he dedicated the book to these honourable Excellencies and good gentlemen (Stefánsson, 1785, p. *4).

The content of the examples in Stephensen's book concerns farming in the general sense. One example reflects class division of society, exercised since medieval times:

> 70 guests were invited to a wedding reception. They are to sit at three tables. Available amount of money is 200 rixdollars. The elite are to have food for 4 rd., guests at the middle table for 3 rd., and at the lowest table for 2 rd. How many could sit at each table? (Stephensen, p. 170).

The problems otherwise tell about fish catch, sheep, horses, mowing fields, credits with the merchant, alms to paupers, cost of food for students and servants, and problems of age, time, distance etc. Generally, the examples reflect the world of farmers, people who run farms and boats and deal with merchants, while they would hardly be considered rich elite in the modern sense.

4 Distribution of the 18th Century Textbooks

Remarkable as it was to have a choice of two good printed arithmetic textbooks in the last decades of the 18th century, no proper textbook was published again until 1841. The two books and the *Pocket-Book* were therefore the basis for mathematical knowledge for over half a century, at least for those who did not attend a learned school.

A survey exists of inventories of books in seven out of nine benefices in the county of Austur–Húnavatnssýsla in northwest Iceland from the first three decades of the 19th century (Jensdóttir, 1969). The sources were on one hand annual church censuses, and on the other hand probate records and records of administration of estates at death, available at the National Archives of Iceland. The inventories were not all compiled in the same year; it differs from place to place in which year the best list was taken. The first one was made in 1809, another in 1823, while the other five were made in 1826–1830. Books were counted on 159 farms, a total of 2,490 religious books; the average number on each farm being 16, not counting four larger libraries. Secular books were not counted on the regular farms, only when the estate of a deceased person was evaluated. For the larger libraries complete, but not all equally accurate, lists exist. From them and from estates of deceased persons one can deduce that secular books comprised about one-eighth of the total number of books, approximately 350 books, which gives an estimated total of 2,840 in addition to 562 books in private libraries, a grand total of about 3,400 books (Jensdóttir, 1969, p. 164).

In 129 estates of deceased persons in the period 1800–1830, there were a total of 8 arithmetic textbooks out of 189 secular books. In the four private libraries there were 6 arithmetic books out of 562 total books and an estimated 70 secular books.

Out of the approximately 260 secular books found in estates of deceased persons and in private libraries, four copies were found of the *Pocket-Book*, three copies of the *Clear Guide* and seven copies of the *Short Teaching*. All these books were at that time between 25 and 50 years old. The *Short Teaching* was found in three out of the four private collections, all owned by learned persons. Of the three copies of the *Clear Guide* two existed in private collections.

One may estimate approximately 160 secular books not counted at the regular homes and not included in the estates of deceased persons, so one could expect 5–7 arithmetic books not counted.

The population in the region in question may roughly be estimated as 1/30 of the whole population. Assuming 20 arithmetics textbooks in the region, gives 600 arithmetic textbooks in the country. This number for the 25–50 years old book is not so little, taken into consideration that each copy may have had many users and that Icelandic farmhouses were not well suited for conserving book collections for many decades, in the badly heated buildings made of turf and stone, where things quickly moulded and rotted.

5 Conclusions

We have seen that three useful books on arithmetic for the general public, a total of close to 1000 pages were distributed in Iceland in a period of great difficulties, probably all of them more or less for free. What effect did it have?

The textbooks in concern were deliberately aimed at teaching farmers and young people basic elements of arithmetic in their dealing with foreign merchants. The political interests of the textbook authors, Olavius, Johnsonius and Stephensen, all proponents of the Enlightenment movement and all of them employed by the Danish administration, can be read in their forewords to their arithmetic books. They all state clearly that their intentions are to provide individuals with prerequisites to cope with trade and skilled merchants. Their main aim was to contribute to the technological and socio-economic development of Icelandic society.

We do not have many sources of information about their effects. Certainly they were better than nothing. Young people who had the endurance for self-study had access to sources of knowledge, but how accessible were they? Olavius tells us that there were no teachers in 1779, when the *Clear Guide* was written. The Rev. Helgason tells us that everyone was given the book, which reveals that the *Short Teaching* was distributed for free, but no one cared if the pupils studied it or not. As there were no mathematical requirements for students entering the University of Copenhagen, the cathedral Latin schools did not have to take that aspect into consideration until 1822.

What good was it then? Considering that the Clear Guide was published in 1300 copies and the other two hardly in less than 600 copies each leaves 2500 copies in Icelandic homes, and at least 600 copies seem to have existed 25–50 years later. Even if people favoured poetry and stories more, and was obliged to read a certain amount of religious literature, it is not unlikely that a couple of hundred books were studied thoroughly and another 500 inspected irregularly. Home- and self-study was the only acknowledged way of studying until late 19th century. The only Icelandic mathematician until 20th century, B. Gunnlaugsson, did not attend school until the University of Copenhagen, where he earned a gold medal for solving a mathematical problem before he even enrolled in the university. Gunnlaugsson mentions both books, by Olavius and Stephensen, in his textbook (Gunnlaugsson, 1865). These books were without doubt bases for Gunnlaugsson's early mathematical studies with his father, a poor tenant farmer who had earned several prizes from the King, on the initiative of the proponents of the Enlightenment, for invention of utilities, such as a loom and a harpoon.

The early printing of handbooks for farmers with conversion tables of trading units, the *Lijted Agrip / Little Compendium Intended for Farmers* (Hatton, 1746) and *Pocket-book / A Pocket-Book for Farmers and Simpleminded People or an Easy Computing Art* (Johnsonius, 1782) reveal a need for handbooks as an aid in trade. They may have served more people than the comprehensive arithmetic textbooks, good though they were.

The political aims that led to the publications of the textbooks were to enhance education and subsequent progress in Iceland, the Danish colony, and they were merely funded by Danish actors. The authors were educated Icelanders who were employed by the Danes. They were therefore subjected to suspicion, to impose information and instructions upon the inhabitants which they themselves had not asked for. This is an old and a new story of the relations between the colonial masters and the subordinated. The masters believe that they know what is best for their subjects. Some of it may be ill-founded which awakes suspicions about whatever the masters may have to offer.

Each textbook author had his own didactical aims. From the modern point of view, Olavius's book is an excellent textbook in arithmetic versatility and Stephenson's book brings the latest technique in arithmetic, such as decimal fractions. Both authors modelled their books after acknowledged authors, Olavius on Clausberg's book, which adhered to the fashion structured by the Lutheran Reformation, and Stephensen on the latest university syllabus. They wanted to share the latest and the best with their countrymen and some of their seed may have hit a fertile ground like Gunnlaugsson.

REFERENCES

- Bjarnadóttir, K., 2006, *Mathematical education in Iceland in historical context-—Socio-economic demands and influences*. Reykjavík: Háskólaútgáfan. Available at `http://rudar.ruc.dk/bitstream/¬¬1800/2914/1/Chapter0_IMFUFA.pdf`.
- Bjarnadóttir, K., 2011, "17th and 18th century European arithmetic in an 18th century Icelandic manuscript", in *Proceedings of the 6th European Summer University. History and Epistemology in Mathematics Education*, Vienna University of Technology, 19–23 July, 2010.
- Bjarnadóttir, K., in print, "Values and virtues of a rural society as reflected in 18th and 19th century arithmetic textbooks in Iceland", in *Proceeding of the second conference on on-going research on the history of mathematics education*, Lisbon, October 2–6, 2011.
- Björnsson, L., 1990, "Atvinnumál", in *Upplýsingin á Íslandi. Tíu ritgerðir*, I. Sigurðsson, (ed.), Reykjavík: Hið Íslenska bókmenntafélag.
- Clausberg, C. v., 1732, 1748, 1762, *Der Demonstrativen Rechenkunst*. Leipzig: Breitkopf.
- Gunnlaugsson, B., 1865, *Tölvísi*. Reykjavík: Hið íslenska bókmenntafélag.
- *Hagskinna. Sögulegar hagtölur um Ísland / Icelandic Historical Statistics*, 1997. Reykjavík: Hagstofa Íslands / Statistics Iceland.

- Hatton, E., 1712, *The Merchant's Magazine or Trades Man's Treasury*, 6th ed., London.
- Hatton, E., 1746, *Lijted Agrip Vmm þær Fioorar Species I Reiknings Konstenne*. Hólum: Hólaprent.
- Helgason, Á., 1907–1915, "Frásagnir um skólalíf á Íslandi um aldamót 18. og 19. aldar. 1. Skólahættir í Skálholti og í Reykjavíkurskóla hinum forna", in *Safn til sögu Íslands og íslenskra bókmennta að fornu og nýju*, IV, 74–98. Copenhagen and Reykjavík: Hið íslenska bókmenntafélag.
- Jensdóttir, S. B., 1969, "Bókaeign Austur-Húnvetninga 1800–1830". *Landsbókasafn Íslands. Árbók* 25, 142–166.
- Johnsonius, J.J., 1782, *Vasa-qver fyrir bændur og einfalldlínga á Íslandi eðr ein auðvelld Reiknings-List*. Copenhagen: The author.
- Lbs. 408, 8vo. *Professor J. M. Geuss Collegium, over Arithmetiken, holdet Anno C.N. 1781 & 1782*. National and University Library of Iceland, manuscript department.
- s. 409, 8vo. *Stutt Undervisan umm Arithmeticam Vulgarem eða Almenneliga Reikningslist*. National and University Library of Iceland, manuscript department.
- *Lovsamling for Island 5. 1784–1791, 1855*, "Reskript til Stiftsbefalingsmand Levetzow og Biskopperne Hannes Finnsson og Arni Thorarensen, ang. Indförelse i Skolerne af Stephensens Lærebog i Regning. Christiansborg 10. Februar 1786". Copenhagen.
- Magnússon, H., 1990, "Fræðafélög og bókaútgáfa", in *Upplýsingin á Íslandi. Tíu ritgerðir*, I. Sigurðsson, (ed.), Reykjavík: Hið Íslenska bókmenntafélag.
- Matthisen, S., 1680, *Compendium Arithmeticum eller Vejviser*. Copenhagen.
- Olavius, Ó., 1780, *Greinilig Vegleidsla til Talnalistarinnar með fjórum höfudgreinum hennar og þriggja lida Reglu skipud eptir Landsvísu og Kauplögum Íslendínga*. Copenhagen: The author. Available at http://baekur.is/bok/000302066/.
- Sigurðsson, I., 1990, "The Icelandic Enlightenment. A Brief outline", in *Upplýsingin á Íslandi. Tíu ritgerðir*, I. Sigurðsson, (ed.), Reykjavík: Hið íslenska bókmenntafélag.
- Stefánsson (Stephensen), Ó., 1785, *Stutt Undirvísun í Reikníngslistinni og Algebra. Samantekin og útgefin handa Skóla-lærisveinum og ødrum ýnglíngum á Íslandi*. Copenhagen: The author. Available at http://baekur.is/is/bok/000302457/.
- Stephensen, M., 1888, "Autobiographia Drs. Magnúsar Stephensens (Brot)." *Tímarit hins íslenska bókmentafélags*, 9, 197–272. Available at http://timarit.is/view_page_init.jsp?pageId=2317831.
- Swetz, F., 1992, "Fifteenth and sixteenth century arithmetic texts: What can we learn from them?" *Science and Education*, 1, 365–378.

GEOMETRY TEXTBOOKS IN NORWAY IN THE FIRST HALF OF THE 19TH CENTURY

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ABSTRACT

Bernt Michael Holmboe(1795–1850) wrote the textbooks in mathematics that were predominantly used in the learned schools in Norway between 1825 and 1860, and he was one of the most influential persons in the development of school mathematics in this period. His way of presenting the subject matter was, however, challenged by his colleague and former mentor, Christopher Hansteen (1784–1873). Holmboe's textbook in geometry came in four editions, 1827, 1833, 1851 and 1856, and they were—with one exception—used in all the learned schools in Norway in this period. Holmboe's presentation of the subject matter was in many ways traditional and Euclidean. Hansteen wrote a geometry textbook in 1835 where he challenged this way of presenting the subject matter, and where he let utilitarian considerations overrule logical deduction and theoretical thinking. Holmboe's textbooks where the first Norwegian textbooks that were used in the learned schools, but we know that textbooks by the Danish mathematics teacher, Hans Christian Linderup (1763–1809) was used in Christiania Learned School prior to Holmboe's.

I will give a presentation of the geometry textbooks by Linderup, Holmboe and Hansteen, and their ways of presenting the subject matter with special focus on fundamental concepts like *point*, *line* and *parallel lines*. I will also try to present these textbooks way to present the *parallel axiom* from the Elements by Euclid. By this I will try to describe how geometry teaching traditionally was done in this period, and how it was challenged.

Keywords: history of mathematics education, textbooks, geometry

1 Introduction

Towards the end of the 18th century, a great effort was done to establish mathematics as a school subject in the higher education in Norway, and a school reform that was introduced around year 1800 lead to proper teaching in mathematics. The mathematical community in Norway at that time was small, and all the participants did necessarily become significant members of the society. This was a time of considerable development in the subject of mathematics, and this also influenced the debate about mathematics education.

Cathedral schools[1] were schools from the medieval time that were connected to cathedrals, and they were meant to give a theological education to future priests. All cathedral schools were turned into *Latin schools*, or *grammar schools*,[2] when the reformation was introduced in Norway in 1539, and it

[1] Katedralskoler
[2] latinskoler

was mandatory for every town to have a one. The new Latin schools, together with the old cathedral schools, constituted the so-called *learned schools*. Most of these Latin schools were, however, of a very poor quality, so in reality, the higher education—preceding the university—in 1814, was only four cathedral schools with a total of 200 pupils, in addition to some that had private tuition. The schools were referred to either by their names like *Stavanger Latin School* or *Christiania Cathedral School*, or by for instance *Christiania Learned School*[3]—Christiania was the name of Oslo approximately 1600–1925. By a governmental decree in 1809, the pupils started at the learned schools at the age of 9–10 years, and the duration was normally eight years consisting of four two-year grades, and each day at school was seven hours—four before noon and three after. The university qualifying examination[4] was arranged by the university. The learned schools gave a classic education, and a higher education in scientific subjects at the same standard as the learned schools could be achieved at the Military Academy. This school admitted pupils from the age of 12–14. Several intermediate schools[5] were established in smaller towns after 1814, and they were learned schools without the upper two-year grade. [Andersen, 1914]

Christopher Hansteen describes the situation of the learned schools in the beginning of the 19th century in a footnote in his textbook in plane geometry [Hansteen, 1835, XVI–XVII]. When Christiania Learned School was reformed in 1800, there were in the upper classes hardly any pupil[6] that with skill could do arithmetical operations with whole numbers. Geometry and other mathematical subjects were probably not even known by their names. When Hansteen, together with six other pupils were discharged from Christiania Learned School in 1802 to start at the university, they knew geometry, plane trigonometry, stereometry,[7] conic sections and equations of some other curved lines, the elementary arithmetics, higher-order equations and their solutions, the main statements of combinatorial and permutational analysis, the sum of series, and some astronomy, mechanics and physics—all this due to a very able teacher.[8] Hansteen concludes this footnote by stating that not only is mathematics a subject for cultural formation in the same manner as other sciences, but it is also an indispensable tool in the study of the nature.

2 The textbook authors

Hans Christian Linderup (1763–1809), a Danish mathematics teacher that wrote a textbook in two volumes in 1799–1803, *«The first grounds for the pure mathematics»*,[9] the second edition was published in 1807 [Dansk biografisk Lexicon, 1899, Linderup, 1807]. Linderup writes in the preface that there are several not insignificant modifications, corrections and amendments from the first to the second edition, and the second edition from 1807 will be used in this presentation. Linderup's textbook was used as textbook in geometry in Christiania Learned School *before* Holmboe's books were published.[10]

[3] Christiania Lærde Skole
[4] examen artium
[5] middelskoler
[6] Discipel
[7] space geometry
[8] Hansteen dedicates his textbook in plane geometry to his former teacher, professor Søren Rasmusen, whose clear and thorough lectures arouse Hansteen's lust for the mathematical sciences.
[9] De første Grunde af den rene Mathematik
[10] Documentation found in the archives of Oslo Cathedral School; teaching reports signed Søren Rasmusen, dated 1810–1812.

Christopher Hansteen (1784–1873) was born in Christiania in Norway. He was first a law student in Copenhagen, but became interested in the natural sciences when he met the physicist H. C. Ørsted. He became a teacher in applied mathematics at the university in Christiania in June 1814, and he was professor from March 1816 till he retired of medical reasons in 1861. Hansteen was very productive, and wrote about terrestrial magnetism, northern light, meteorology, astronomy, mechanics, etc. He received international recognition after an expedition to Siberia in 1828–30 to study the geomagnetism. Hansteen moved with his family and servants into the new Observatory in Christiania in 1833.

In 1835, Hansteen wrote a textbook in geometry where he challenged the traditional Euclidean geometry, and he introduced the subject matter in a very «un-Euclidean» way. He refers in the preface to Abraham Kästner (1719–1800), a German who published textbooks in geometry in 1758, –63, –74, –88, and they where translated into Danish. Kästner writes that «*any of the countless number of geometry textbooks possesses so much less value and clarity of geometry, the more they deviate from Euclid's Elements*» and Hansteen continues to quote Kästner—«*one should never publish a textbook before one has used it for several years in lectures, and learned to know its shortcomings*». Hansteen then admits that in his book, he has in some sections deviated considerably not alone from Euclid, but also from all the other textbooks he knows, *and* he has not had time to follow the other rule by Kästner, as he has only been working on this book for about 6 months.

Bernt Michael Holmboe (1795–1850) was born in southern Norway as a son of a vicar of the Church. He is mostly known as being the teacher of Niels Henrik Abel, when he was at Christiania Cathedral School. Bernt Michael Holmboe became a student in 1814, and in 1815 he became Christopher Hansteen's assistant at his astronomical observatory. After completing his exams he worked from 1818 till 1826 as teacher at Christiania Cathedral School, then as a lecturer at the university from 1826 till 1834, and after that as a professor. Holmboe was an influential person in the development of mathematics education in the first half of the 19th century. He was the third person to be appointed professor in mathematics at the new university in Christiania. Holmboe was a mathematician at heart, and still a young man when he started teaching in 1818. It is said that his teaching was more «lively» and enthusiastic than what his students were used to. He gave them exercises and assignments out of the ordinary, and he caught their attention [Christiansen, 2009]. Holmboe wrote—with a few exceptions—all the textbooks in mathematics that were used in the learned schools in Norway between 1825 and approximately 1860.

Holmboe wrote textbooks in Arithmetics, Geometry, Stereometry, Trigonometry and Higher Mathematics. These were the textbooks in mathematics that were predominantly used in the learned schools in Norway between 1825 and 1860, a decade after Holmboe's death. He was probably one of the most influential persons in the development of school mathematics in the first half of the 19th century in Norway. His way of presenting the subject matter was in many ways very traditional, and they were challenged by his colleague and former mentor, Christopher Hansteen.

The table 1 shows an overview over Holmboe's textbooks, their Norwegian titles and their various editions.

An Overview Over Holmboe's Textbooks

TITLE	Edition	Year	Edited by	Publisher
Lærebog i Mathematiken	1st	1825		Jacob Lehmann
Første Deel, Inneholdende Indledning	2nd	1844		J. Lehmanns Enke
til Mathematiken samt Begyndelses-	3rd	1850		J. Chr. Abelsted
grundenetil Arithmetiken	4th	1855		R. Hviids Enke
	5th	1860		R. Hviids Enke
Lærebog i Mathematiken	1st	1827		Jacob Lehmann
Anden Deel, Inneholdende	2nd	1833		Jacob C. Abelsted
Begyndelsesgrundene til Geometrien	3rd	1851	Jens Odén	R. Hviids Enke
	4th	1857	Jens Odén	J. W. Cappelen
Stereometrie	1st	1833		C. L. Rosbaum
	2nd	1859	C. A. Bjerknes	J. Chr. Abelsted
Plan og sphærisk Trigonometrie	1st	1834		C. L. Rosbaum
Lærebog i den høiere Mathematik Første Deel	1st	1849		Chr. Grøndahl

3 The textbooks

3.1 [Linderup, 1807]

[Linderup, 1807] emphasizes a platonic view on the being of mathematics when he in the first section explains the fundamental concepts of *body*, *face*, *line* and *point*. A body has three dimensions, a face has two, a line has one, and a point has no dimensions. The boundary of a body is faces, the boundary of a face is lines and the boundary of a lines is points. The boundary of a «thing» is not a part of the «thing», but on the contrary its cessation—the face is not a part of the body, the line is not a part of the face, and the point is not a part of the line. Innumerable points constitute therefore no line, innumerable lines constitute no faces, and innumerable faces constitute no body. From this one may conclude that it is not possible to draw geometric points, lines and faces, as all drawn objects are real physical bodies, whose physical extension in space are geometric bodies. We must, in other words, *abstract* the concept from drawn bodies. [Linderup, 1807, 145–146]

Linderup then gives an explanation of *lines* by imagining a point—which is without any extension—set in motion. This point runs through a trajectory,[11] or a length, without breadth and thickness, and that is a *line*.[12] If the direction[13] of the point is unchanged during the motion, it describes a *straight line*,[14] or *recta*, and the point moves from one place to another following the shortest trajectory. If, on the other hand, the direction of the point is changed during the motion, then it describes a *curved line*,[15] or *curva*. Linderup admits that these explanations are the simplest ideas we may have on straight and curved lines—«*The concepts of straight and curved lines are so plain that they are not possible to define in an understandable manner*».[16] That is why Linderup turns to this circular explanation by explaining «line» by using «direction», and explaining «direction» by using «line». [Linderup, 1807, 146–147]

Linderup's definition of parallel lines is two straight lines in a plane that, no matter how far they

[11] Vey
[12] Linie
[13] Retning
[14] ret Linie
[15] krum Linie
[16] Begreberne om rette og krumme Linier ere saa enkelte at de ikke lade sig definere eller gjøre forstaaelige ved Forklaring

are prolonged, never will meet [Linderup, 1807, 168–169]. In a following theorem, he proves that two straight lines are parallel if, when intersected by a transversal, either

1. the sum of consecutive interior angles equals two right angles
2. corresponding angles are equal
3. alternate angles are equal

Linderup gives some descriptions on how to use compass, ruler and protractor in spite of his platonic view on geometric objects.

3.2 [Holmboe, 1827, Holmboe, 1833]

Holmboe's textbook in geometry came in a total of four editions, but only the two first were published in Holmboe's lifetime. There are some differences from the first edition to the second, but not concerning the fundamental concepts discussed in this paper.

The textbook in basic geometry [Holmboe, 1827] starts with several definitions of basic concepts. The very first definition describes *geometry* as a science about the *coherent magnitudes*. Coherent magnitudes are the space with all available dimensions and time. According to [Solvang, 2001], Holmboe's way of organizing the subject matter was influenced by Adrien-Marie Legendre's (1752–1833) introduction to geometry [Legendre, 1819]. The geometry of Legendre is constructed mainly the same way as Euclid, and starts with a long list of what he calls *explanations*, similar to what Euclid calls *definitions* [Euclid, 1956].

The first definition in [Legendre, 1819] defines *geometry* as a *science which has for its objects the measure of extension. Extension has three dimensions, length, breadth, and thickness*. With reference to classification of coherent magnitudes in space and time, Holmboe classifies geometry in two parts:

1. *The real geometry* defined by the relations between the various magnitudes in space, without considering their changes in time.
2. *Mechanics*, defined by the changes the magnitudes goes through in time. All changes on a magnitude through time are called motion, and it is conditioned by force.

It is postulated that the space stretches indefinitely.[17] Holmboe advises the teacher to show moderation in the review of proofs, and to show examples using numbers before the examination of the proof. This practical advice contradicts the structure of his textbooks, which is strictly Euclidean. There are few exercises and numerical examples, and the notion of construction means to elucidate the concept, and not to use compass and ruler. Holmboe does not give any detailed instructions on how to use ruler and compass in this book, nor does he mention geometric locus, but writes about elucidative[18] objects, magnitudes and concepts. His idea may be that the mathematics teaching shall educate the students with respect to formal logic, by encouraging them to think and conclude.

[17]Geometrie er en Videnskab om de sammenhængende Størrelser. Sammenhengende Størrelser er ere Rummet med enhver deri forekommende Udstrækning og Tiden. Med Hensyn til de sammenhængende Størrelsers Inddeling i Rum og Tid, inddeles Geometrien i 2 Dele. (1) Den egentlige Geometri, der bestemmer de i Rummet forekommende Størrelsers Forhold til hinanden uden Hensyn til deres Forandring i Tiden. (2) Mekanik, der bestemmer de Forandringer, som Størrelserne undergaae i Tiden. Anm. Enhver Forandring, som en Størrelse i Tiden undergaaer, kalles Bevægelse, hvis betingelse kaldes Kraft. Fordringssætning. Rummet maa tænkes udstrakt i det Uendelige.

[18]anskueliggjørende

Holmboe's first definition is a description of a classification—any bounded part of the space is called a *body*,[19] the boundary of the body is called a *face*,[20] the boundary of a face is called a *line*,[21] and the boundary of a line is called a *point*.[22] A body has three extensions, called *length*,[23] *breadth*[24] and *height*,[25] a face has two extensions, called *length* and *breadth*, a line has one extension, called *length*, and a point has no extensions. Holmboe then states without any explanation, but with an illustration,

> «*Fundamental concept. A straight line.*»[26]

A *curved line*[27] is a line of which no part is a straight line, and a *straight plane*[28] is a plane where one, between two arbitrary points may draw a straight line. The fundamental statements of the straight line is that a straight line may be prolonged infinitely, one may always draw one straight line between two points, and one may never draw more than one straight line between two points. The part of the straight line that lays between the two points is the shortest of all lines drawn between the points, and it is called the *distance*[29] between the points. [Holmboe, 1827, 2–4]

Holmboe has in his textbook two subsections called «*About two straight lines intersected by a transversal*»,[30] and «*About parallel lines*».[31] The first subsection gives a thorough description of all pairs of angles this situation produces. This is followed by the consequences of two corresponding angles being equal, and vice versa, the situations which have the consequence that the corresponding angles are equal [Holmboe, 1827, 11–16]. The latter of the two subsections then has a theorem with proof which states that when two straight lines are intersected by a transversal, such that an outside angle is equal to its corresponding interiour angle, then the two straight lines can not intersect no matter how far they are prolonged in both directions. This is followed by Holmboe's definition of *parallel lines*.

> «*Two straight lines in the same plane that does not intersect when prolonged indefinitely to both sides, are* parallel *to each other, or the one is parallel to the other*»[32] [Holmboe, 1827, 46]

In two following theorems, Holmboe demonstrates, using the same situation of two straight lines intersected by a transversal, that when the two straight lines are parallel, then the corresponding angles are equal, and if one of the angles in a pair of corresponding angles is greater than the other, then the two straight lines are *not* parallel. [Holmboe, 1827, 50–53]

Holmboe is in this textbook very true to the *Elements* [Euclid, 1956], but without ever referring to Euclid.

[19]Legeme
[20]Flade
[21]Linie
[22]Punkt
[23]Længde
[24]Brede
[25]Høide
[26]Grundbegreb. En ret Linie.
[27]krum Linie
[28]ret Flade
[29]Affstanden
[30]Om to rette Linier, som overskjæres af en tredie
[31]Om parallele Linier
[32]To rette Linier i samme Plan, som til begge Sider forlængede i det Uendelige ikke skjære hinanden, siges at være *parallele* med hinanden, eller den ene at være parallel med den anden

3.3 [Hansteen, 1835]

In 1835, Christopher Hansteen published a textbook in basic geometry [Hansteen, 1835], which in many ways challenged Holmboe's textbooks. Hansteen's book was 278 pages, which is a lot more than what is expected of a textbook in elementary geometry. The author is intentionally trying to tear down the walls that existed between the classical geometry on one side, and the newer analytical geometry and the infinitesimal geometry on the other. The basis of the textbook is real life, with references to artifacts like corkscrews, stove pipes and hourglasses. The presentation of the subject matter is very unlike Euclid's Elements. The style is narrative and written in the first person, sometimes very lengthy, and there are many numerical examples. Hansteen tried to expand Euclid's definition of straight lines and of parallel lines, and Euclid's parallel postulate [Euclid, 1956].

Hansteen's textbook contains a comprehensive preface which also contains definitions of fundamental concepts. The first concept to be defined is the straight line [Hansteen, 1835, III–IV], which is also, according to Hansteen, «*the foundation of geometry*».[33] It is of great importance that this concept is clearly defined, especially in a science that demands a consistent and logic practice. Hansteen presents five different ways a straight line may be defined

- «*A straight line is a line which lies evenly with the points on itself*» from [Euclid, 1956]. Close to this is also Baron Wolff's definition stating that «*a line is straight when a part is similar to the whole*».[34]
- Archimedes, and most French geometers after him, defined the straight line as «*the shortest trajectory between two points*».
- Some geometers regard the straight line as a hereditary concept that only needs to be mentioned to be understood, and defines a straight line as «*those things which is known to be a straight line*».[35]
- Abraham Kästner says that «*a straight line is that whose points all bear against one trace*»,[36] and he adds that «*no one will learn the straight line to know from an explanation, and no one needs to; but one may say something about it, that guides the attention to a closer attention to what makes it a straight line*».
- Finally, others say that «*when a point moves continually in the same direction, then its trajectory is a straight line*».

According to Hansteen, after such definitions, all geometers introduces a postulate which states that «*one may create a straight line between two given points, and prolong such a given straight line in any direction in both directions as one pleases*». Hansteen makes noteworthy objections to such a postulate by asking with what tool such a prolonging shall be made, and how to make sure that the line made by such a tool is homogeneous, or that is satisfies the demands made in the various definitions of a straight line [Hansteen, 1835, VI–V].

Hansteen elaborates towards a definition where he let lines be produced by the movement of a point, and there are two kinds. One kind has the quality that when two points of a part of the line are placed on two arbitrary points on the whole line, then all points of the part of the line will coincide with points in the whole line—analogous, if we let a part of a line move along the whole line, and the part always fits with the whole line. Such lines are called *homogeneous lines*,[37] and there are two types—*straight* and *curved* lines. A homogeneous line has all over the same curvature, and all perpendiculars of any plane homogeneous line will, when duly prolonged, either intersect in one

[33] Geometriens Grundvold
[34] Linae recta est, cujus pars quæcunque est toti similis
[35] Qvæ linea recta dicatur notum est
[36] en ret Linie er den, hvis Punkter alle ligge hen mod een Egn
[37] eensartede Linier

point, or not intersect at all. There are in other words only two types of homogeneous lines in a plane—the straight line and the circle. When a point moves from one place to another in a space, then it describes a line. If this line is straight, it is called the direction of the motion [Hansteen, 1835, IV–V,7,9,35–38] From the concept of the straight line we may derive the concept of the plane, and from these definitions we may prove that a line is straight when all the points in the line remains unchanged in the same position is the line is rotated around two arbitrary points on the line [Hansteen, 1835, 9], and that a straight line between two points is shorter than any curved or broken lines between these two points [Hansteen, 1835, 40].These two statements are not axioms, but theorems.

It is more proper that a «mechanical artist» derives rules for his practice from the definitions and theorems of the geometry, than that the theoretical geometer shall direct his concepts and definitions towards this practice. The carpenter's planer and the metalworker's file are tools that are suitable for producing homogeneous planes and lines, and the geometer should not neglect to acquire the theoretical principles on which these methods are based. A *ruler* is described as a tool—made by wood or metal—by which one may produce straight lines in a plane. [Hansteen, 1835, VIII,13]

The cause for the much discussed controversy Hansteen's textbook made was the handling of parallel lines. Hansteen states very clearly that the Euclidean definition of parallel straight lines, embrace by nearly all geometers, has all the logic errors a definition may have. He states correctly that parallel lines are defined, according to Euclid, by a *negative* quality, and not a *positive*. He continues by stating that the quality by which the parallel lines are defined, is *outside all experience and test*, as it points towards the infinite. Euclid's definition may also not be used on curved lines, which may also be parallel—according to Hansteen. «*No one will hesitate in declaring to two concentric circles reciprocal parallel*». There is a definition stating that if two lines in a plane never intersect, no matter how far they are prolonged in any direction, does not make an angle [Hansteen, 1835, 28]. There is, however, no mentioning that these lines are parallel.

Hansteen argues for an understanding of parallelism where one let a perpendicular to any kind of line move along this line, in such a way that it always is a perpendicular. Any point on this perpendicular then describes a line, where any point's smallest distance to the original line all over is the same [Hansteen, 1835, IX]. Consequently, Hansteen has a definition of parallel lines

> «*Any line that is being described by a point on the perpendicular to a given line, when it moves along the same with an unaltered angle, is said to be parallel to the directrix*» [Hansteen, 1835, 59]

and the characteristics of a line, parallel to another, is therefore

- it always cuts off equal parts of all its perpendiculars
- any perpendicular to one of these lines is also a perpendicular to the other

and parallel of a straight line has in addition the following characteristics

- the parallel is also a straight line
- as these straight lines never intersect, they form no angle with each other
- if the parallel lines are intersected by a transversal, then the alternate interior angles are equal, the corresponding angles are equal, and the consecutive interior angles equals 2R

By following these properties of parallelism, Hansteen transform Euclid's disputed axiom into a theorem which he proves [Hansteen, 1835, 70].

Hansteen does all over let lines and planes be produced by the motion of points and lines, because this method gives the clearest conception of a line's direction in any point. One may easy imagine that a point in motion has a certain bearing in any place of its trajectory. Some geometers object to this method since motion involved *time* and *power*, two concepts that are irrelevant to geometry, but belongs in the mechanics. Hansteen states that the motion of an immaterial point requires now power, and that we are only elucidating a motion in our minds. [Hansteen, 1835, XII]

Hansteen's textbook was published in one edition only, but one reason may be that it contained much subject matter outside the school curriculum. He explains that because of a limited production of textbooks in Norway, he has added subject matter beyond the curriculum of the learned schools, but should be of interest for students that want to prepare themselves for a study of the higher mathematics [Hansteen, 1835, XVIII]. It is also worth while to mention as a curiosity that Hansteen in his textbook introduces and describes *Metre* as a the new unit of length [Hansteen, 1835, 81].

4 A comparison, and some concluding remarks

Hansteen's textbook is a textbook that stands out, and is different from the contemporary textbooks. Both Holmboe's and Linderup's books were firmly in the Euclidean tradition that was typical for geometry textbooks in the 18th and 19th century. There was a present debate about the use of Euclidean ideas in textbooks in geometry, and when Christopher Hansteen published his textbook in geometry [Hansteen, 1835], it was evidently a controversial issue, and an attack on the Euclidean textbooks [Piene, 1937, Solvang, 2001].

It is an noteworthy difference that Hansteen tries to give a thorough definition of *lines*, both straight, curved and broken, whilst both Linderup and Holmboe accept the straight line as a self-evident concept that is understood, and does not need to be explained.

The most interesting difference between Holmboe and Hansteen is their understanding of parallelism. Holmboe's definition of parallel lines is similar to Euclid's, and he was also firmly in agreement with Legendre's understanding of parallelism. Hansteen's definition of parallel lines covers all types of lines—straight and curved, homogeneous and not homogeneous. A bitter controversy broke out between Holmboe and Hansteen, and the polemics that followed in the newspapers has later been called the dispute about parallelism. Both Holmboe and Hansteen published pamphlets where they justified their own views.

Holmboe's textbooks were more or less controlling the market regarding textbooks in mathematics in the first half of the 19th century.

Bibliography

[Andersen, 1914] Andersen, O. (1914). *Norges høiere skolevæsen 1814–1914. En oversigt*. Kirke- og undervisningsdepartementets jubilæumsskrifter 1914. J.M. Stenersens forlag, Kristiania.

[Christiansen, 2009] Christiansen, A. (2009). Bernt Michael Holmboe (1795–1850) and his mathematics textbooks. *BSHM Bulletin: Journal of the British Society for the History of Mathematics*, 24:105–113.

[Dansk biografisk Lexicon, 1899] Dansk biografisk Lexicon (1899). A digital facsimile from Project Runeberg. Accessible from `http://runeberg.org/dbl/`.

[Euclid, 1956] Euclid (1956). *The Thirteen Books of the Elements*, volume I–III. Dover Publications, Inc., New York, Second Unabridged edition. Translated with Introduction and Commentary by Sir Thomas L. Heath.

[Hansteen, 1835] Hansteen, C. (1835). *Lærebog i Plangeometrie*. Johan Dahl, Christiania.

[Holmboe, 1827] Holmboe, B. M. (1827). *Lærebog i Mathematiken. Anden Deel, Inneholdende Begyndelsesgrundene til Geometrien*. Jacob Lehman, Christiania, First edition.

[Holmboe, 1833] Holmboe, B. M. (1833). *Lærebog i Mathematiken. Anden Deel, Inneholdende Begyndelsesgrundene til Geometrien*. Jacob Abelsted, Christiania, Second edition.

[Legendre, 1819] Legendre, A.-M. (1819). *Elements of Geometry*. Hilliard and Metcalf. Translated from the French for the use of the students of the University at Cambridge, New-England.

[Linderup, 1807] Linderup, H. C. (1807). *De første Grunde af den rene Mathematik. Forsøg til en Lærebog for Skoler*. J. F. Schults, Kiøbenhavn, Second edition.

[Piene, 1937] Piene, K. (1937). Matematikkens stilling i den høiere skole i Norge efter 1800. *Norsk Matematisk Tidsskrift*, 19(2):52–68.

[Solvang, 2001] Solvang, R. (2001). Skolematematikken i Norge på Abels tid. *Nordisk Matematisk Tidsskrift*, 49(3):111–138.

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THE ISSUE OF MATHEMATICS TEXTBOOKS IN THE CORRESPONDENCE OF GIOVANNI NOVI TO ENRICO BETTI DURING THE UNIFICATION OF ITALY

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ABSTRACT

In the *Archivio Betti*, at the *Scuola Normale Superiore* in Pisa, there are 48 letters that the mathematician Giovanni Novi (1826–1866) sent his friend Enrico Betti (1823–1892) from Dicember of 1850 to October of 1864. Novi traslated important treatises on Arithmetic and Geometry, Betti asked him to publish the *Trattato di algebra superiore*, based on his lectures at university. In their correspondence, Novi and Betti talk about the issue of mathematics textbooks at high schools and new organization of universities. The Unification of Italy represented a political, educational and mathematical turning point. Among leading researchers were Betti, Francesco Brioschi, Felice Casorati, Luigi Cremona, who worked to raise both the mathematical and political standard in Italy through their active role in the reform of the Italian educational system at the secondary as well as at the university levels. One of the aims was to adapt the elementary books of mathematics to the progress of mathematics through the translation of new works.

1 The correspondence between Giovanni Novi and Enrico Betti

Enrico Betti was born in Pistoia, Tuscany, in 1823; he studied there and graduated from the University of Pisa in 1846. After working as assistant at the University of Pisa, Betti returned to Pistoia where he became a teacher of mathematics at secondary school in 1849. In 1854 he moved to Florence where again he taught in secondary school. He was appointed as professor of Higher algebra at the University of Pisa in 1857. Betti joined the battalion led by Ottaviano Fabrizio Mossotti and he fought in two battles, being those of Curtatone and Montanara in 1848. In 1864 Betti succeeded Mossotti when he was appointed to the chair of Mathematical physics and became director of its teacher's college, the Scuola Normale Superiore, in 1864 holding this post until his death, in 1892. He served as an undersecretary of state for education for a few months in 1874 and as a senator in the Italian Parliament in 1884.[1]

[1]See *Dizionario Biografico degli Italiani* For Betti's researches, see also Mercurio A.M., Palladino N., "Intorno alla risoluzione delle equazioni algebriche di quinto grado per funzioni ellittiche in Betti e Brioschi", Palladino F., Mercurio A.M., Palladino N., *Per la costruzione dell'Unità d'Italia. Le corrispondenze epistolari Brioschi—Cremona e Betti—Genocchi*, Bottazzini U., *Va' pensiero. Immagini della Matematica nell'Italia dell'Ottocento.*

Giovanni Novi was born in Naples in 1826; he studied at *Accademia Militare della Nunziatella* and the worked at the geodesic department of Officio Topografico. In 1848, because of the Neapolitans revolutions, he moved to Florence where he started to teach *Meccanica e Artiglieria* at *Liceo Militare*; after the deposition of The House of Habsburg-Lorraine, he moved to Pisa where he was appointed as professor of Higher algebra at university. In Tuscany, Novi married Teresa Pozzolini.[2]

In the *Archivio Betti*, at the *Scuola Normale Superiore* in Pisa, there are 48 letters that Giovanni Novi sent his friend Enrico Betti from Dicember of 1850 to October of 1864. Italian scholars paid modest attention to Giovanni Novi, in the field of research concerning mathematicians during the period of Italian Unification. The Unification of Italy represented a political, educational and mathematical turning point. Among leading researchers were Betti, Francesco Brioschi (1824–1897), Felice Casorati (1835–1890), Luigi Cremona (1830–1903), who worked to raise both the mathematical and political standard in Italy through their active role in the reform of the Italian educational system at the secondary as well as at the university levels. One of the aims was to adapt the elementary books of mathematics to the progress of mathematics through the translation of new works.

The reading of the recently published correspondences Novi-Betti [3] and Betti–Tardy [4] allows us to say that Giovanni Novi played an important role in the Italian mathematics life. He was in close friendship with Betti, Placido Tardy and in close touch with Giuseppe Battaglini, Barnaba Tortolini, Angelo Genocchi, Ottaviano Fabrizio Mossotti, Eugenio Beltrami, Luigi Cremona, Bernhard Riemann.[5]

Novi contribution is found in the work of translation of high-level foreign works and composition of treaties for teaching mathematics at high school. His productions includes: *Elementi d'aritmetica*; the translation *Memoria sopra le trasformazioni generali di date funzioni. Estratto di un opuscolo del sig. O. Schlömilch, Sul moto dei proietti nell'anima delle bocche da fuoco*, the translation *Elogio di Carlo Gustavo Jacob Jacobi letto nell'Accademia delle Scienze di Berlino il 1° di luglio 1852 da Lejeune Dirichlet*, the review *Lezioni di Meccanica razionale di O. F. Mossotti, La Statica dei sistemi di forma invariabile di F. Brioschi, Milano 1859, Elementi di Meccanica razionale di D. Chelini delle Scuole Pie* published in *Annali di Scienze matematiche e fisiche* [6] *; Riduzione in serie delle facoltà analitiche* and *Sugl'invarianti e i covarianti delle forme binarie* published in 1864 in "Giornale di Matematiche"[7] by Battaglini; the unpublished *Lezioni di Artiglieria*.

Betti and Novi collaborated in the drafting of *Trattato di algebra superiore – Parte Prima, Analisi Algebrica*. In 1854 Betti moved to Florence where he began teaching Algebra superiore at *Liceo*; there he began to translate another school text, namely Bertrand's *Algebra elementare* (published in 1856). In *Avvertimento del traduttore*, Betti announces his intention to publish a *Corso di Algebra;*

On the 23rd February 1858, from Genoa, Tardy wrote to Betti:

[2] A Novi's brief biography is held in Valerio V., *Società Uomini e Istituzioni Cartografiche nel Mezzogiorno d'Italia*.

[3] Palladino N., Mercurio A.M., "La corrispondenza Giovanni Novi-Enrico Betti"

[4] Cerroni C., Martini L., *Il carteggio Betty-Tardy (1850–1891)*.

[5] Bernhard Riemann (1826–1866) studied in G ttingen with Gauss, then in Berlin. In 1851 he held his lecture about a general theory of functions of a complex variables and made lasting contributions to mathematical theories; in 1859 he started teaching at the same Gauss department. From October 1863 to July 1865, he moved to Pisa because of health problems.

[6] The *Annali*, edited by Tortolini and published in Rome from 1850 to 1865, were known as "Annali di Tortolini", but their full name was "Annali di Scienze matematiche e fisiche". From 1865, Enrico Betti, Francesco Brioschi and Angelo Genocchi started to collaborate with Tortolini and the journal name became "Annali di matematica pura e applicata". In 1865 all publications stopped and started again in Milan from July 1867 directed by Francesco Brioschi and Luigi Cremona.

[7] In 1863, Giuseppe Battaglini founded "Giornale di matematica ad uso degli studenti delle Università italiane" with Vincenzo Janni (1819–1891) and Nicola Trudi (1811–1884). From 1866 to 1893, Battaglini edited the journal and for this reason it was generally knonw as "Giornale di Battaglini"

"Scrivimi a lungo e dimmi come ti trovi in Pisa e come va la nuova scuola. Hai cominciato a lavorare per il corso di algebra che devi pubblicare? Ricordati che è un lavoro necessario e tu ormai l'hai promesso. Davvero che un libro di algebra il quale introducesse alle teorie moderne sarebbe un gran benefizio. I trattati che abbiamo lasciano la scienza almeno 30 anni indietro";[8]

Betti replied on the 26th February 1858:

"Io ora sono tutto ed esclusivamente occupato dal Corso di Algebra Superiore. E' un impresa ardua a cui non so se basteranno le mie forze. Non bisogna contentarsi di por giù tutti questi nuovi acquisti della Scienza fatti da 30 anni in quà, uno accanto all'altro; ma bisogna farne un corpo, ben organizzato dove tutto sia al suo posto; e poi qui più che in una memoria è necessaria chiarezza, e rigore e semplicità. Ho fatto un Piano un po' nuovo, e mi pare che su questo vengano bene e naturalmente a disporsi le parti nuove e vecchie della Scienza. Intanto su questo Piano fo le mie lezioni per l'Università; dove però bisogna che esponga molto di quello che andrà nel Corso. Ci h qualche giovane capace e che studia con passione. Ho molto piacere di recarmi a Genova, anche per potere parlare insieme con Te di questo Corso. Speriamo che allora avrò potuto portarlo molto più avanti".[9]

Betti's purpose was actualized by Novi in 1863: he collected Betti's Lessons at lyceum and university and published *Trattato di algebra superiore*; in the preface, Novi thanks Betti and declares the purposes that they wonted to achieve:

"Un'opera di tal fatta era stata promessa dal professor Betti, corrono già varii anni [...]. Ma nel 1859 il professor Betti fu chiamato a dettar lezioni di analisi superiore, ed il nuovo ufficio obbligandolo ad altri studi, lo distolse di mandare ad effetto il suo primo pensiero [...].

Succeduti noi al professor Betti nella cattedra di algebra superiore dell'Università di Pisa, fummo dallo stesso esortati ad effettuare la promessa che le nuove circostanze impedivano a lui di adempire. Per agevolarci questo compito, ingombro di non lievi difficoltà, egli pose gentilmente a nostra disposizione i manoscritti delle lezioni fatte negli anni 1858 e 59, e ci offrì i suoi consigli e i suoi aiuti con quella schietta benevolenza che rende sì preziosa la sua amicizia a noi ed a quanti lo conoscono".[10]

Their close friendly relationship is manifest in every letter of the correspondence. Betti wrote to Tardy from Pisa on the 26th February 1858[11]:

"Abbiamo quasi fissato con Brioschi di trovarsi a Genova nelle vacanze di Pasqua, dove forse verrà anche Genocchi. Vogliamo parlare un poco anche insieme con Te del nostro Giornale; e stabilire bene tutto ciò che è necessario per il migliore andamento dello stesso. Io ho accettato l'invito di Brioschi tanto più volentieri, perché mi darà occasione di stare un poco con Te. Nelle vacanze di Carnevale fui

[8]"Please, write me and let me know about your new experience in Pisa and at the new school. Have you started to think about your course of algebra? Remeber: this is a necessary work and you promised it to me. A book about algebra with an introduction on modern theories would be a great benefit. All treatis we have leave science behind at least 30 years".

[9]"I am really busy with this course of higher algebra. It is hard and I don't know if I have enought strenght for this. It is necessary to not be content with these gains of the Science started 30 years ago and placed side by side; but it is necessary to create one body, well organized in which everything is in the right place; moreover we need clarity, and rigour and semplicity rathen then memory. I have organized a quite new plane and I think old and new parts of the Science have the right conditions to coexist. In the meantime, my lessons at the University are based on this new Plan. There, I have some smart and passionate students. It is a pleasure for my to go to Genova and discuss with you about this course. I hope I'll develop it for that moment"

[10]"Professor Betti promised this work years ago. [...] But he moved in 1859 to the chair of higher analysis and this commitment has distracted him from his promise, because of other studies. [...] I succeeded him on the chair of higher algebra at the University of Pisa and Betti invited me to perform his promise. He provided his lessons of the years 1858 and 1859, in order to help me in this aim and he offered advices and assistance with his kindness".

[11]In Cerroni C., Martini L., *Il carteggio Betty-Tardy (1850–1891).*

a Firenze, e mi provai a persuadere il Novi a venire anche Lui; ma per ora non sono riuscito".[12]

On the 23rd March 1858, in his letter to Betti, Novi wrote:

"Ma Tardy mi scrive che sei sano e che certamente avrà il piacere di abbracciarti a Genova; lo che mi piace moltissimo. Io, come ben puoi immaginare, appena pronunzio Genova mi attiro una tempesta dalla mia dolcissima Gegia; quindi per ora non bisogna pensarci. Ti rivedrò prima della tua gita? Lo spero, e intanto desidero più dirette notizie di te e della tua casa".[13]

These letters deal with an important desicion immediatly after Italian unification: the creation of a real Italian mathematical journal, able to confront itself with the most important European journals, founded in the first half of the century, as *Journal di Crelle*[14] or *the Annales de Mathématiques*[15]. In 1857, Brioschi, with Cremona, Betti and Genocchi, decided to carry out a "progetto intorno ad un giornale di matematica Italiano da surrogarsi agli Annali del Tortolini, quando non potesse essere una continuazione di questi",[16] as he wrote in a letter to Genocchi in May 1857.[17] Novi himself was invited to Tardy's house during Easter holidays in 1858 for the important meeting[18] between Brioschi, Tardy, Betti and Genocchi in which they discussed about the *Annali di Matematica pura ed applicata* and also to organize the famous trip of Brioschi, Betti and Felice Casorati[19] around different German and French cities.[20]

Some letters between Novi and Betti are dedicated to the riorganization of the mathematical studies and, as consequence, the engineer carrer.[21] In particular, these themes were examinated in letters dated 20th February, 22nd February and 4th March 1861, in occasion of the purporse of the Minister Mamiani, at the beginning of 1861, to reorganize the University and adapt the Legge Casati to the new political situation[22]. As known, all proposals didn't become law. In 1861 Betti participated (with Tardy) at a ministerial commission for the reform, as he wrote in a letter[23]:

"Ricevetti il giorno avanti che arrivasse la tua una lettera del Mamiani colla quale m'invitava a far parte della Commissione per la legge sull'istruzione pubblica. Io ho accettato. Non ti nascondo però

[12]"I and Brioschi determined that we will meet in Genova, in the Easter holidays, where Genocchi maybe will be to. We are talking with you about our Giornale; and we are deciding what is necessary for its best performance. I accepted Brioschi's call also because I will see You. In the carnival holisays I was in Florence to to persuade Novi to come with us; but at the moment I can't do it".

[13]"But Tardy wrote me that you are good and surerly he will have the pleasure to meet you in Genova. As you can imagine, everytime I mention it my sweet Gegia get annoyed, so let's forget this for now. Will I see you before your trip? I hope so and look forward hearing news from you".

[14]The *Journal für die reine und angewandte Mathematik*, published by Walter de Gruiter, founded in 1826 in Berlin by Leopold Crelle (1780–1855), who edited the journal until his death. For this reason it was known as Giornale di Crelle. Similary, from 1856 to 1880, Carl Wilhem Borchardt (1871–1880) edited the journal and started to be named "Giornale di Borchardt".

[15]Nouvelles annales de mathématique founded in France in 1842 by Olry Terquem (1782–1862) and Camille Gerono (1799–1892).

[16]"Project for a new Italian mathematical journal to susbtitute the Annali del Totolini or create just a continuation".

[17]In Carbone L., Mercurio A.M., Palladino F., Palladino N., La corrispondenza epistolare Brioschi-Genocchi, pp. 273–274.

[18]A report of the meeting is present in the Betti obituary written by Broschi and present in Cerroni C., Martini L., Il carteggio Betty-Tardy (1850–1891).

[19]Felice Casorati (1835–1890) graduated in engineering in Pavia in 1856; in 1859 he was appointed professor of Algebra and Analitical Geometry at the University of Pavia. He also started to teach Infinitesimal analysis and Higher analysis; (cfr. *Dizionario Biografico degli Italiani*).

[20]See Bottazzini U., Va' pensiero. Immagini della Matematica nell'Italia dell'Ottocento, Carbone L., Mercurio A.M., Palladino F., Palladino N., La corrispondenza epistolare Brioschi-Genocchi.

[21]Cfr. Soldani S., "Ingegneri e studi di ingegneria nella Firenze di metà Ottocento" and references in it.

[22]Cfr. Polenghi S., La politica universitaria italiana nell'età della Destra storica, chapter II, L'istruzione superiore e l'unificazione nazionale, pp. 91–112.

[23]In Cerroni C., Martini L., Il carteggio Betty-Tardy (1850–1891), letter of Genuary of 1861.

che trovo questo incarico assai grave per le difficoltà che ci saranno a potere far bene"[24].

A letter written by Tardy in Cremona, on the 11th June 1861, testifies that everywhere discussions were strong. Moreover, in this letter Tardy underlined the fact that it wasn't found a concrete solution:[25]

"I giovani vengono alle università unicamente perché la laurea è necessaria per esercitare una professione. Per gl'ingegneri poi si vorrebbe che fossero assai limitati gli studi matematici, e si vorrebbero anzi fare due classi d'ingegneri, una delle quali non uscisse nemmeno dalle università, e però vedete quanti pochi accorrerebbero alla nostra scuola. Ma tutti questi discorsi sono rimasti senza risultato".[26]

2 The problem of mathematical books

In 1858 *Trattato di geometria elementare di A. Amiot* was edited. Cremona[27] wrote an extensive review on this book: "noi non abbiamo buoni libri elementari che siano originali italiani e giungano al livello de' progressi odierni della scienza. Forse ne hanno i Napoletani che furono sempre e sono egregi cultori delle matematiche; ma come può aversi certa notizia se quel paese è più diviso da noi che se fosse la China?".[28] This is held in *Considerazioni di storia della geometria in occasione di un libro di geometria elementare pubblicato a Firenze*, «Il Politecnico», IX (1860). The original Amiot's treatise was *Leçons nouvelles de géométrie élémentaire*, Parigi, Dezobry et E. Magdeleine libraires-éditeurs, 1850. In 1862 Novi wrote *Trattato d'aritmetica; prima traduzione italiana con note ed aggiunte di Giovanni Novi*. Firenze, Le Monnier, traslated from a work of J. Bertrand. Even by Bertrand, Betti had written *Trattato d'algebra elementare*, Firenze, Le Monnier, 1859. The paper focuses on Novi's contributions in educational and scientific venues, based on his letters to Betti, his works, his additions to the original treatises of Bertrand and Amiot.

Some great Italian mathematicians, among these Cremona, Betti, Brioschi, had an important role in the Italian Public Instruction. One of the problems was to give a uniform instruction to citiziens, updating all elementary and high school books with recent advances of mathematics, using the translations of new foreign works.

In 1856, Giuseppe Bertrand published the *Trattato di aritmetica*, translated by Giovanni Novi and then the *Trattato d'algebra elementare*, translated by Enrico Betti; Alfredo Serret published the Trattato di trigonometria translated by Antonio Ferrucci. A year later, Novi's *Elementi d'aritmetica* were published. In 1858, *the Trattato di geometria elementare, written by A.Amiot, prima traduzione italiana con note*

[24]"The day before he got your letter, I received a letter by Mamiani; he invited me to join the Commission for Public Education. I accepted. But I can not deny that I find this job very seriously because of the difficulties to do well".

[25]In Cerroni C., Fenaroli G., Il Carteggio Cremona-Tardy (1860–1886).

[26]"Young people are graduating at university because it is necessary to practice a profession. For engineers, then it would that mathematical studies were very limited, and it would indeed make two classes of engineers, one of which does not even come out of universities, and yet you see how few would be at our school. But all these speeches have been without result".

[27]Luigi Cremona (1830–1930) taught at Cremona high school and, from 1859 in Milan at Beccaria high school. In 1860s he became professor of higher Geometry at the University of Bologna. In 1867 he was appointed professor of higher Geometry and graphical statistics at the higher technical college of Milan. In 1873 he was called to Rome by the Minister of Public Instruction to organize the college of engineering. There he was also appointed professor of graphical statistics and higher Geometry. He was nominated Minister of Public Instruction and Senator (biography written by U. Bottazzini and L. Rossi in *Dizionario Biografico degli Italiani)*.

[28]"We have no good elementary books wich are originally Italian ones and wich correspond to the recent progress of sciences. Perhaps Neapolitans, who were and are lovers of mathematics; but how can we have some news if that country is more divided from us that if it was China?"

aggiunte, by Giovanni Novi, was published and Luigi Cremona wrote an extensive text titled *Considerazioni di storia della geometria, in occasione di un libro di geometria elementare pubblicato recentemente a Firenze in Il Politecnico*.[29] The efforts of great mathematicians towards the production of books didactically effective come from these first attempts, started before Italian Unification. For example, it is necessary to take in account the book written by Cremona on projective geometry, those on elementary geometry by Veronese, D'Ovidio and others, and the works written by Enriques and Amaldi destined to be reproduced for a whole century.

From 1853, Novi started to dedicate his time to treaties translation, as Novi himself wrote to Betti on 25th July 1853:[30]

"Quando ci vedremo, o se tarderai quando ti scriverò nuovamente, ti dirò le ragioni per le quali debbo dimenticare ogni cosa per varii mesi e mettermi a scrivere trattati, la fatica più esimia di questo mondo. Ci vuol pazienza".[31]

About the difficult issue of the books at high school and university, there is a letter written by Novi to Betti on the 21st of April 1858, in which he wrote:

"In Firenze nulla di notabile, eccetto un curioso fatterello avvenuto al tuo Merlo in occasione della Geometria di Amiot.[32] Merlo credendo che tu avessi disposto favorevolmente Carloni alla nuova Geometria, si era creduto in diritto di adottare l'Amiot senza tenerne parola all'Abate. Un bel giorno, com'era da aspettarsela, il rubicondo si presentò al Merlo domandandogli ragione dell'inatteso cangiamento. Qui non ti starò a ripetere i dialoghi corsi fra i due: ti dirò solo che dopo vari va e vieni, il reverendo consultò l'oracolo, il quale rispose che finché l'Università non mutava Trattato, non gli pareva conveniente avvenissero mutazioni nei Licei: parlato che ha l'oracolo tutto è finito. Il povero Merlo si trova nella durissima necessità di dire ai suoi scolari che si è ingannato, e che ora bisogna prendere il Legendre[33] e qui nuovo e curiosissimo dialogo fra gli scolari e il maestro. Per chiudere finalmente in modo degno questa commedia, eccoti un'ultima scena; Le Monnier si vede arrivare gli scolari che domandano restituzione dei quattrini; e Le Monnier restituisce!! Ciò posto, mio carissimo Betti, tu vedi bene che la quistione dipende interamente dall'illustre professore di Geometria dell'Università.[34] Le Monnier ti mandò, per passarli allo Sbragia, una copia della Geometria piana,[35] te ne ha detto nulla? Interroga, e con prudenza spingilo a parlare. Ho saputo da Le Monnier che l'affare con Matteucci è combinato; il Sig. Felice,[36] con quel suo sorriso che dice tante cose, mi faceva sapere che Matteucci aveva detto che non poteva accettare meno di quattromila lire[37], e poi aveva preso duecento scudi![38] Son cose che mi fanno male ai nervi; gli scienziati dovrebbero rispettarsi un

[29]It was founded in Milan in 1839 by Carlo Cattaneo (1801–1869).

[30]Cfr. letter of the 25th July 1853 in Palladino N., Mercurio A.M., "La corrispondenza Giovanni Novi-Enrico Betti".

[31]"I will explain all the reasons I devote all my time to write treaties when I see you or in my next letter. It must be patient".

[32]Novi G., *Trattato di Geometria elementare di A. Amiot, prima traduzione italiana con note ed aggiunte di Giovanni Novi, Professore di Meccanica nell'I e R. Liceo militare di Firenze, con un atlante di 59 tavole.*

[33]Adrien Marie Legendre (1752–1833) wrote *Éléments de géométrie*, Paris, Firmin Didot, 1794; his treaty was used for years in the most part of European schools and more than once translated and published in Italy.

[34]He maybe refers to Fabio Sbragia.

[35]He refers to the first part of the Trattato by Amiot, translated by Novi.

[36]Felice Le Monnier.

[37]During Italian Unification, Gran Duchy of Tuscany currency was *Lira toscana* or *fiorentina*, corresponding to 84 cent of *Italian Lira.*

[38]One *Tuscan Scudo*, or *Piastra*, corresponds to 6.66 *Tuscan Liras*; so two hundred Scudi of the letter correspond to 1332 Tuscan Liras.

poco in più".[39]

This is one of the scientific initiatives carried out by Italian scientists after the known "Decennio di preparazione". The aim of these works is expressed in the preface of *Trattato di Geometria elementare di A. Amiot, prima traduzione italiana con note aggiunte di Giovanni Novi:*

"Le opere destinate ad iniziare i giovani nello studio di una scienza, debbono contenere le basi teoriche che per la loro generalità e fecondità vogliono esser considerate come fondamentali. E ciò rende ragione perché di tratto in tratto sia necessario mutare i libri elementari".[40]

It was important the attention that Luigi Cremona dedicated to the question; in his text "Considerazioni", Cremona praised the personal work of Novi, who enhanced the original work written in 1850 by Amiot *Leçons nouvelles de géométrie élémentaire*; Creomona also agreed with the initiative of the publisher to update the book with all the last progress of Mathematics; he wrote:

"Il merito di queste interessanti publicazioni non può essere ritratto in brevi parole, né può appieno sentirsi se non da chi le abbia avute in mano, e con diligenza studiate. Non solo sono state scelte le migliori opere originali fra le recentissime, ma anche furono arricchite ed ampliate con preziose note ed aggiunte, che ne accrescono singolarmente il pregio. Così, per le utili fatiche de'chiari uomini nominati, noi possediamo attualmente ottimi trattati d'aritmetica, d' algebra, di trigonometria e di geometria. Facciamo voti che sì eccellenti principj siano seguiti da cose maggiori".[41]

Cremona ended the "Considerazioni" inviting all to turn the attention to Novi's notes "destinate quasi esclusivamente allo sviluppo delle teorie recenti soltanto abbozzate nel testo"[42] and to "brevi note, poste dal traduttore, allo scopo di indicare nuove conseguenze de' teoremi esposti dall' autore, o più semplici dimostrazioni, o maniere più generali di considerare certi argomenti".[43]

In 1863 Novi wrote the *Trattato di algebra superiore – Parte Prima, Analisi Algebrica;* in preface, he wrote:

"Abbiamo divisa l'opera in tre parti. Nella prima si espone quanto si riferisce alla teoria delle serie, dei prodotti infiniti e delle frazioni continue; nella seconda, che volge intorno alle frazioni algebriche razionali, si troveranno le teorie dei determinanti, delle funzioni simmetriche, dell'eliminazione e quella importantissima e tutta moderna delle trasformazioni lineari per le funzioni omogenee; nella terza finalmente si tratta delle funzioni algebriche irrazionali, e vengono dichiarati i risultati a cui sono

[39] "I have no news from Florence, except for a curious episode happened to Merlo and related to the work Geometria by Amiot. Merlo thought you suggested the new Geometria to Carloni, so he decided to use Amiot's work without Abbot authorization. As you can imagine, the ruddy man asked to Merlo about the unexpected change. I don't write you all the conversation: all you need to know is that the clergyman, after speaking with oracle, declared that it was impossible to make some changes at high school until University didn't change its Trattato. So, the poor Merlo was obliged to communicate to his scholars he was wrong and they had to use the Legendre: you can imagine the curious conversation between professor and students. To end this comedy, here the final scene: students went to Le Monnier asking the money back; and Le Monnier repaid them!! So, my dear Betti, it is clear that the distinguished Professor of Geometry decides everything. Le Monnier sent you a copy of Plane Geometry to give it to Sbragia, do you know? Speak with him and lead him to say everything. Le Monnier said me that the business with Matteucci is done; Mr Felice, with his eloquent smile, let me know that Matteucci said that he couldn't accept less than four thousands liras but at the end he took two hundred scudi! This thing annoys me; scientists should respect each other".

[40] "All works with the aim to initiate young people into science have to include all theorical basis, considered fundamental. This is the reason why it is necessary change elementary books following new treaties".

[41] "Only people who read them can express the merit of these publications. It has been chosen the best works among the most recent and enriched ones by precious notes. So thanks to the work of those men mentioned before, we have the best works on arithmetic, algebra and geometry. We hope this is only the beginning".

[42] "Designed for the development of recent theories".

[43] "Short notes, written by the translator, with the aim to indicate the new consequences of the theory expressed by the author, or simply demostrations, or general way to considerer some themes".

giunti alcuni geometri moderni, quali Betti, Hermite, Kronecker, Weierstrass ec., circa alla risoluzione algebrica dell'equazioni.

Oggi pubblichiamo la sola prima parte, che distinguiamo col nome di analisi algebrica, per seguire l'esempio di altri geometri".[44]

In *Trattato di Algebra Superiore's* index, these arguments are written: *Nozioni sulla teoria delle combinazioni, Numeri complessi, Limiti e continuità delle funzioni, Serie, Convergenza delle serie, Serie doppie, Serie potenziale, Serie esponenziale e logaritmica, Serie circolari ed iperboliche, Ulteriori ricerche sulle serie, Prodotti infiniti, Facoltà analitiche, Frazioni continue, Riduzione delle frazioni continue in serie e in prodotti infiniti e viceversa.*

Novi discussed about his treaty with Angelo Genocchi (1817–1889), as it is possible to observe reading the only letter left that Novi sent to Genocchi from Pisa the 28th of December 1863:[45]

"La ringrazio della sua lettera e delle espressioni gentili che s'è pregiato avere riguardo la mia *Analisi Algebrica*. Ho la coscienza di non aver risparmiato fatiche per fare un lavoro il meno imperfetto che mi fosse possibile, malgrado una lunga ed istimata malattia reumatica che mi tormenta da quattro anni. Tuttavia non mi dissimulo, che in Opera sì ampia e nello svolgimento di tante e sì diverse dottrine, qualche cosa non mi sia sfuggita; è perciò che prego gli uomini autorevoli a volermi essere cortesi delle loro osservazioni, per migliorare il già fatto. Così l'insigne geometra Riemann[46], che passa l'inverno a Pisa, e al quale offrii una copia del mio lavoro, mi ha fatto avvertito di una inesattezza sfuggitami a pag. 56 ove dico che se $\lim u_n = 0$, la serie non può essere indeterminata. Si vede bene che se indico con S_n ed S_m la somma dei primi n ed m termini della serie, e se suppongo che m sia una tale funzione di n (per es. una potenza di n), che cresca più rapidamente di n, può bene accadere che la differenza $S_m - S_n$, *comunque composta di termini convergenti a zero, tende nondimeno verso un limite* finito diverso da zero. Questa inesattezza ne ha portato dietro qualche altra, che farò corregere a LeMonnier. Il mio errore è pervenuto dal Trattato sulle serie di Catalan, libro che bisogna leggere con gran cautela, perché contiene più di una svista; e poi dal perché tutti gli autori che parlano di serie indeterminate, danno esempi ai quali si applica rigorosamente la dimostrazione della pag. 56. Riemann mi ha promesso un esempio.

Io la sarò gratissimo se leggendo la mia Opera più minutamente, Ella mi farà qualche osservazione che possa essermi utile. Come pure se avesse a premunirmi contro qualche possibile sbaglio nella 2a parte, mi farà un vero regalo ad avvertirmene.

Così Betti mi parlò di qualche errore che Ella aveva trovato nel Complemento di Rubini; ma non rammentava bene in che consistessero; abbia la gentilezza di dirmelo, affinché io verifichi la cosa e stia in guardia. Parimente se Ella avesse qualche comunicazione a farmi sopra i soggetti che debbo esporre nella 2a parte, io mi reputerò fortunatissimo di poterla inserire (sotto il suo nome) nel mio Trattato. Indotto a fare questa opera per ragioni di utilità e non per ambizione, sono lieto quando mi

[44]"We diveded the work in three parts. In the first we explain the theory of the series, of infinitive products and continued fractions; in the second part, based on the rational algebric fractions, there are the theory of determiners, of symmetrical functions, of the elimination and the most important and modern of the linear transformation for homogeneous functions; in the third part, we expose the irrational algebric functions and the results of some modern scientists, as Betti, Hermite, Knonecker, Weierstrass, ecc. about the algebric resolution of the equation.
Now, we only publish the fist part, named Analisi algebrica, following the example of other scientists".

[45]In FONDO GENOCCHI, BIBLIOTECA PASSERINI LANDI in Piacenza.

[46]From the letter, it is possible to understand that Novi met Bernhard Riemann (1826–1866) for the first time during Rieman stay in Pisa.

1858. *Trattato di Geometria elementare di A. Amiot, prima traduzione italiana con note ed aggiunte di Giovanni Novi, Professore di Meccanica nel R. Liceo militare di Firenze, con un atlante di 59 tavole.* Firenze, Le Monnier.	**1856.** *Trattato d'aritmetica di Giuseppe Bertrand; prima traduzione italiana con note ed aggiunte di Giovanni Novi.* Firenze, Le Monnier.
1881. *Trattato di geometria elementare di A. Amiot; prima traduzione italiana con note ed aggiunte di Giovanni Novi.* Firenze, successori Le Monnier.	**1862.** *Trattato d'aritmetica di Giuseppe Bertrand; prima traduzione italiana con note e aggiunte di Giovanni Novi.* Firenze, Le Monnier.
1899. *Trattato di geometria elementare di A. Amiot; prima traduzione italiana, con note ed aggiunte di Giovanni Novi.* Firenze, Successori Le Monnier.	**1886.** *Trattato d'aritmetica di Giuseppe Bertrand; prima traduzione italiana con note ed aggiunte di Giovanni Novi.* Firenze, F. Le Monnier.
	1894. *Trattato d'aritmetica di Giuseppe Bertrand; traduzione italiana con note ed aggiunte di Giovanni Novi.* Nuova edizione con modificazioni ed aggiunte per cura di Antonio Socci. Firenze, Successori Le Monnier.
	1920. *Trattato d'aritmetica di Giuseppe Bertrand; traduzione italiana con note ed aggiunte di Giovanni Novi.* Nuova edizione con modificazioni ed aggiunte per cura del dott. Antonio Socci. Firenze, Le Monnier.

Novi translations of Amiot and Bertrans treaties

è dato attingere a fonti italiane a preferenza delle straniere".[47]

In 1857, Novi published Elementi d'aritmetica, Le Monnier, and in the "Avvertenza" declaired that they were only the preliminaries of *Trattato d'aritmetica di Giuseppe Bertrand; prima traduzione italiana con note ed aggiunte di Giovanni Novi*, traslated from *Traité d'arithmétique* in 1849 edited by Le Monnier. *Elementi d'aritmetica* is constituited of 150 pages on: *addizione e sottrazione, moltiplicazione, divisione, divisori di numeri—numeri primi, frazioni, numeri decimali, numeri complessi, rapporti e proporzioni, regole del tre, problemi, sistema metrico*.

Novi translations of Amiot and Bertrans treaties were republished, as the chart 2 shows.

[47]"I thank you for your letter and your kind expression towards my Analisi Algebrica. I work hard to write the less imperferct treaty, despite my long rheumatic ill started four years ago. However I don't hide that in this big and long work I miss something; for this reason, I ask to influential people to give their observations on it, in order to improve my work. So the distinguished Riemann who stays here in Pisa and has read my work, informed me about an inaccuracy at pag.56 where I wrote that if, the series can't be indeterminate. If I indicate Sn and Sm as the addition of the first n and m elements of the series, and if I suppose that m is a n's function (for example a power of n), that it increases faster than n, it could be possible that the difference Sm– Sn, in any case formed by elements converging at zero, tents to finite limit different from zero. This inaccuracy cause others and I will coommunicate to Le Monnier to correct them. My mistake comes from the Trattato sulla serie by Catalan, a work we need to read with great attention because it contains more than one inaccuracy; moreover Rieman promised to give me some examples for the demostration at pag. 56. I would be grateful if, reading carefully my work, you make useful observations and warn me against possible mistake in the 2nd part.

Betti confirmed that you found some mistake in the Complimenti di Rubini; but he didn't remember them; please, let me know these mistakes in order to avoid them. In the same manner, if you have comments on the themes for the 2nd part, it would be an hounour for me to mention you in my traty. Usefulness and not ambition persuades me to write this traty, and I'm happy to use Italian and not foreign sources".

REFERENCES

- Amiot A., *Leçons nouvelles de géométrie élémentaire*, Parigi, Dezobry et E. Magdeleine libraires-éditeurs, 1850.
- Amodeo F., *Vita matematica napoletana*, Napoli, vol.I, Tip. Giannini e Figli, 1905; vol.II, Tipografia dell'Accademia Pontaniana, 1924.
- Baccini I., *La mia vita. Ricordi autobiografici*, Roma-Milano, Società editrice Dante Alighieri di Albrighi, Segati e C., 1904.
- Bertrand J., *Traité d'arithmétique*, Paris, Hachette, 1849.
- Bertrand J., *Traité élémentaire d'algèbre*, Paris, Hachette, 1850.
- Betti E., *Trattato di algebra elementare. Prima traduzione italiana con note ed aggiunte di Enrico Betti*, Firenze, Le Monnier, 1856.
- *Opere matematiche di Enrico Betti*, Milano, pubblicate per cura della R. Accademia de' Lincei, Hoepli, 2 tomi, 1903–1913.
- Bollati E., *Fasti legislativi e parlamentari delle rivoluzioni italiane nel secolo XIX*, Vol.II 1859–61 Toscana, Milano, Civelli, 1866.
- Bosi P., *Dizionario storico, biografico, topografico d'Italia*, Torino, Tipografia Candeletti, 1870.
- Bottazzini U., *Va' pensiero. Immagini della Matematica nell'Italia dell'Ottocento, Bologna*, Il Mulino, 1994.
- Cambi F., Staccioli G., *Il gioco in Occidente. Storia, teorie, pratiche*, Roma, Armando, 2008.
- Carbone L., Mercurio A.M., Palladino F., Palladino N., *La corrispondenza epistolare Brioschi-Genocchi*, «Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche di Napoli», s. IV, vol. LXXIII, 2006.
- Catenacci G., *Francesco de Sanctis. Discorso ai giovani. Napoli, 18 febbraio 1848*, Scuola militare "Nunziatella", Istituto italiano per gli Studi filosofici, Napoli,, Associazione nazionale ex allievi Nunziatella, 2008.
- Cerroni C., Fenaroli G., *Il Carteggio Cremona-Tardy (1860–1886)*, Milano, Mimesis, 2007.
- Cerroni C., Martini L., *Il carteggio Betty-Tardy (1850–1891)*, Milano, Mimesis, 2010.
- Cremona L., "Considerazioni di storia della geometria, in occasione di un libro di geometria elementare pubblicato recentemente a Firenze", *Il Politecnico*, vol. IX, Milano, Editori del Politecnico, 1860, pp. 286–322.
- *Centro Ricerche Prato*, www.crprato.it.
- *Dizionario Biografico degli Italiani*, www.treccani.it.
- De Feo F., *Itinerario spirituale di Cesare Guasti*, Roma, Edizioni di storia e letteratura, 1989.
- De Sanctis F., *Lettere dall'esilio (1853–1860)*, raccolte e annotate da Benedetto Croce, Bari, Laterza, 1938.
- De Sanctis F., *Lettere a Pasquale Villari*, con introduzione e note di Felice Battaglia, Torino, Einaudi, 1955.
- *Enciclopedia Britannica*, www.britannica.com.
- Gioberti V., *Del rinnovamento civile d'Italia*, Parigi e Torino, a spese di Giuseppe Bocca, 2 voll., 1851.
- Guizot F., *Mémoires pour servir à l'histoire de mon temps*, Paris, Michel Lévy frères, 8 voll., 1858–1867.
- *Sito web Istituto e Museo di Storia della Scienza*, brunelleschi.imss.fi.it.
- *Liceo ginnasio "Dante" di Firenze*, http://www.liceoclassicodante.fi.it.
- *Lessico Universale Italiano*, Istituto della Enciclopedia Italiana fondata da Giovanni Treccani, Roma, Arti grafiche Marchesi per i tipi dell'Istituto Poligrafico e Zecca dello Stato, www.treccani.it.

- Mercurio A.M., Palladino N., “Intorno alla risoluzione delle equazioni algebriche di quinto grado per funzioni ellittiche in Betti e Brioschi”, *Memorie scientifiche, giuridiche e letterarie. Accademia Nazionale di Scienze Lettere e Arti, Modena*, serie VIII. vol. X, Fasc. II (2007), pp. 391–441.
- Novi G., “Memoria sopra le trasformazioni generali di date funzioni. Estratto di un opuscolo del sig. O. Schlömilch”, *Annali di Scienze matematiche e fisiche*, Roma, t. I (1850), pp. 517–527.
- Novi G., “Sul moto dei proietti nell’anima delle bocche da fuoco. Considerazioni del Sig. Giovanni Novi, Professore di Artiglieria nel Liceo militare di Firenze”, *Annali di Scienze matematiche e fisiche pubblicati* in Roma Ottobre e Novembre 1854, Roma, Tipografia delle belle arti, 1854.
- Novi G., “Elogio di Carlo Gustavo Jacob Jacobi letto nell’Accademia delle Scienze di Berlino il 1° di luglio 1852 da Lejeune Dirichlet. Traduzione dal tedesco di Giovanni Novi, Professore di Meccanica, e di Artiglieria nel Liceo militare di Firenze”, *Annali di Scienze matematiche e fisiche*, t. VII (1856), Roma, Tipografia delle belle arti, pp. 342–372.
- Novi G., *Trattato d’aritmetica. Prima traduzione italiana con note ed aggiunte di Giovanni Novi*, Firenze, Le Monnier, 1856.
- Novi G., *Elementi d’aritmetica*, Firenze, Le Monnier, 1859.
- Novi G., *Trattato di Geometria elementare di A. Amiot, prima traduzione italiana con note ed aggiunte di Giovanni Novi, Professore di Meccanica nell’I e R. Liceo militare di Firenze, con un atlante di 59 tavole*, Firenze, Felice Le Monnier, 1858.
- Novi G., “Lezioni di Meccanica razionale di O. F. Mossotti, La Statica dei sistemi di forma invariabile di F. Brioschi, Milano 1859, Elementi di Meccanica razionale di D. Chelini delle Scuole Pie, Bologna 1860”, *Annali di Matematica pura e applicata*, Roma, t. 3 (1860), pp. 245–251.
- Novi G., *Trattato di Algebra superiore, Parte I. Analisi algebrica*, Firenze, Le Monnier, 1863.
- Novi G., “Riduzione in serie delle facoltà analitiche”, *Giornale di Matematiche*, vol. II, Napoli, Benedetto Pellerano Editore, 1864, pp. 1–46.
- Novi G., “Sugl’invarianti e i covarianti delle forme binarie”, *Giornale di Matematiche*, vol. II, Napoli, Benedetto Pellerano Editore, 1864, pp. 306–330.
- Palladino F., Mercurio A.M., Palladino N., *Per la costruzione dell’Unità d’Italia. Le corrispondenze epistolari Brioschi—Cremona e Betti—Genocchi*, Firenze, Olschki, 2009.
- Palladino N., Mercurio A.M., “La corrispondenza epistolare di Battaglini a Cremona”, *Rendiconto dell’Accademia delle Scienze Fisiche e Matematiche di Napoli*, 2011.
- Palladino N., Mercurio A.M., “La corrispondenza Giovanni Novi-Enrico Betti”, *Rendiconto dell’Accademia delle Scienze Fisiche e Matematiche di Napoli*, 2011.
- Parshall K.H., *James Joseph Sylvester: Jewish mathematician in a Victorian world*, The Johns Hopkins University Press, 2006.
- Piromalli A., *La letteratura calabrese*, Cosenza, Pellegrini Editore, 1996.
- Polenghi S., *La politica universitaria italiana nell’età della Destra storica*, Brescia, Editrice La Scuola, 1993.
- Pozzolini A., “Lettere e frammenti inediti di Antonietta Pozzolini”, *Letture di Famiglia*, Tip. di M. cellini e C., 1875.
- Pozzolini Siciliani C., *Lettere da Parigi*, Firenze, Tipografia Barbera, 1904.
- *Sito PRISTEM*, Università Bocconi, matematica-old.unibocconi.it.
- *Portale per la storia di Firenze*, www.dssg.unifi.it.

- *Relazione del Ministro di Agricoltura, Industria e Commercio sopra gli Istituti tecnici, le Scuole di Arti e Mestieri, le Scuole di Nautica... Presentata alla Camera dei Deputati nella tornata del 4 luglio 1862*, Torino, Tipografia della Camera dei Deputati, 1862.
- Romano M., *Storia di una famiglia borghese: i Vallone di Galatina (secc. XVII-XX)*, Milano, FrancoAngeli Editore, 2003.
- Russo G., *La Scuola d'Ingegneria in Napoli. 1811–1967*, Napoli, E.S.I, 1967.
- *Sistema bibliotecario di ateneo dell'Università di Pisa*, biblio.unipi.it.
- *Sistema Informativo Unificato per le Soprintendenze Archivistiche*, Archivi di personalità, siusa.archivi.beniculturali.it.
- Signorini M.A., Visconti A., "Il salotto di Gesualda e Cesira Pozzolini nella Firenze del 1859", in *Salotti e ruolo femminile in Italia*, Venezia, Marsilio, 2004, pp.381–403.
- Soldani S., "Ingegneri e studi di ingegneria nella Firenze di metà Ottocento", in *Alle radici della moderna ingegneria. Competenze e opportunità nella Firenze dell'Ottocento*, Firenze, Firenze University Press, Firenze, 2010.
- *Sito web del Senato della Repubblica*, http://www.senato.it/.
- Tricomi F.G., "Matematici Italiani del Primo Secolo dello Stato Unitario", in *Memorie dell'Accademia delle Scienze di Torino – Classe di Scienze Fisiche Matematiche e Naturali*, Serie 4a, n. 1, 1962.
- Valerio V., *Società Uomini e Istituzioni Cartografiche nel Mezzogiorno d'Italia*, Firenze, Istituto Geografico Militare, 1993.
- Vapereau L.G., *Dictionnaire universel des contemporains*, Paris, Librairie de L. Hachette, 1865.
- Venturoli G., *Elementi di meccanica e d'idraulica*, Bologna, Tip. F.lli Masi e C., 2 voll., 1806–1807.

AFTER THE GÖSTA MITTAG-LEFFLER (1846-1927) AND JULES HOUËL (1823–86) CORRESPONDENCE : THEIR GENERAL AND PARTICULAR THOUGHTS ABOUT MATHEMATICS TEACHING

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ABSTRACT

Jules Houël (1823–86) taught real and complex analysis at the Bordeaux' faculty of science from 1859 to 1884. In 1872, the young Gösta Mittag-Leffler (1846–1927) wrote him about complex analysis. Then they corresponded until 1883 about the theory of functions and complex analysis, how to teach it and also the mathematics teaching organization in Europe (especially France and Germany).

INTRODUCTION

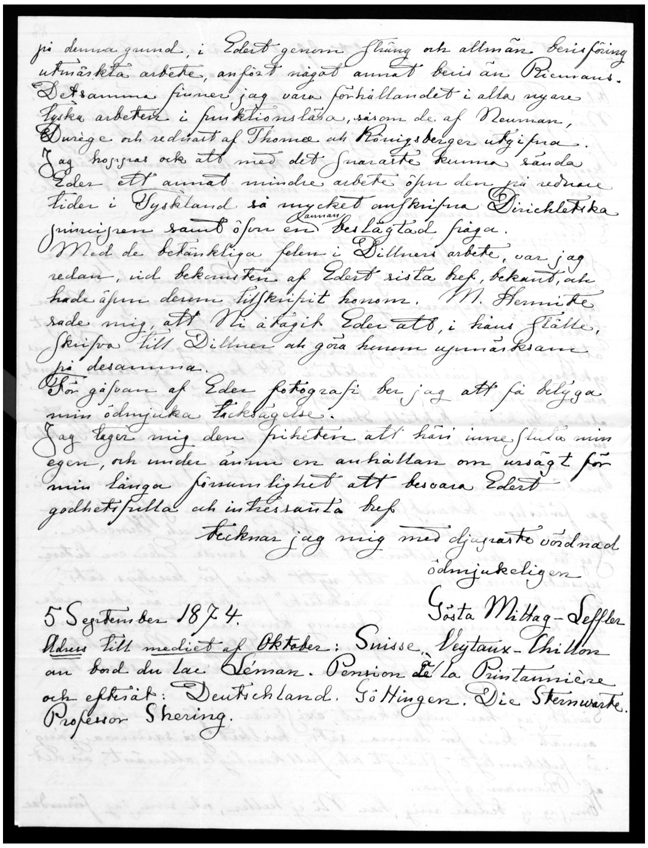
Nowadays, the name of Mittag-Leffler is well-known in the mathematics studies because of the theorem of factorization of a meromorphic function[1], although Houël's name is anonymous. In 1872, Mittag-Leffler was 26 years old and docent at Uppsala University. At that period, Houël was 49 years old and was well recognized by many European mathematicians. In his first letter, Mittag-Leffler wrote to Houël on the 15th July 1872 :

> Mister distinguished Professor,
>
> As I know you are one of the few French mathematicians that master my mother language, I took the liberty, encouraged by Mister the Professor Dillner, of sending you my work titled «Separation of roots of synectic functions of one variable» I just published. First of all, please allow me to offer my humble and respectful thanks for the knowledge of the complex quantities that I have gained from reading your great and exhaustive handbook «Théorie élémentaire des quantités complexes».

That was the beginning of the correspondence between the two mathematicians; it lasted until May 1883. Their correspondence is made of seventy letters: thirty-five written in Swedish by Mittag-

[1] See (Rudin, 1975) for instance.

Leffler and thirty-five written by Houël in French.[2] One can find the Mittag-Leffler's letters[3] in the Caen-la-mer library in Caen, France and the Houël's letters[4] are in the Kungliga Vetenskaps Akademien in Stockholm.[5] Eric Lehman, professor emeritus of mathematics at the Caen university translated the Mittag-Leffler's letters into French.[6] The correspondence between Mittag-Leffler and Houël is really interesting because Houël is the first non Scandinavian mathematician whom Mittag-Leffler wrote to so we can follow the genesis of his ideas and the many issues are covered. Obviously, the starting point is mathematics and more precisely complex analysis. Repeatedly, Mittag-Leffler and Houël discuss the theory of functions of one complex variable and elliptic functions; they discussed also the ways of teaching them. The organization of mathematics teaching in Europe and especially in France and Germany is a recurrent topic. Finally, the mathematics journals are omnipresent in the correspondence: Houël's *Bulletin*, Dillner's Tidskrift, Weyr's Archiv, Mittag-Leffler's Acta, . . . In our paper we'll present the exchanged ideas on complex analysis teaching in connection with the state of the theory of functions at that period, the organization of mathematics teaching in France and Germany according to Houël and Mittag-Leffler. But first of all, we have to present both Mittag-Leffler and Houël then the historical, political and mathematical contexts necessary to appreciate their points of view.


på denna grund, i Edert genom sträng och allmän bevisföring
utmärkta arbete, anfört något annat bevis än Riemanns.
Detsamma finner jag vara förhållandet i alla nyare
tyska arbeten i funktionsläran, såsom de af Neuman,
Durège och redigerat af Thomae och Königsberger utgifna.
Jag hoppas och att med det snaraste kunna sända
Eder ett annat mindre arbete öfver den på senare
tider i Tyskland så mycket omskrifna Dirichletska
principen samt öfver en (annan) beslägtad fråga.
Med de betänkliga felen i Dillners arbete, var jag
redan, vid bekomsten af Edert sista bref, bekant, och
hade öfver denna tillskrifvit honom. M. Hermite
sade mig, att Ni åtagit Eder att, i hans ställe,
skrifva till Dillner och göra honom uppmärksam
på desamma.
För gåfvan af Eder fotografi ber jag att få betyga
min ödmjuka tacksägelse.
Jag tager mig den friheten att häri innesluta min
egen, och under ännu en anhållan om ursägt för
min långa försumlighet att besvara Edert
godhetsfulla och intressanta bref
tecknar jag mig med djupaste vördnad
ödmjukeligen
Gösta Mittag-Leffler
5 September 1874.
Adress till medlet af Oktober: Suisse. Veytaux-Chillon
au bord du lac Léman. Pension de la Printannière
och efteråt: Deutschland. Göttingen. Die Sternwarte.
Professor Schering.

The last page of the letter dated of 5th September 1874 written by Mittag-Leffler to Houël

[2]Mittag-Leffler learned French in school and university and studied in Paris six months in 1873. So he understood well French.

[3]Certain letters are slightly damaged because the paper used was quite thin.

[4]They are in good condition.

[5]There are copies in the Mittag-Leffler Institute, Djursholm, Sweden.

[6]Let me thank him and his mother.

1 ABOUT GÖSTA MITTAG-LEFFLER

Gösta Mittag-Leffler[7] (1846–1927) was issued of an old Swedish family with German origins and was made of artists, pastors, politicians, entrepreneurs. He studied French, German, Latin, mathematics, physics in Uppsala university and supported his Ph.D.'s thesis in 1872—dir. Göran Dillner—in complex analysis on «applications of the argument principle». Mittag-Leffler wanted to specialize in elliptic functions ; at that period, Dillner was connected to French mathematicians as Hermite, Briot, Bouquet, Houël. Mittag-Leffler, docent in 1872 at Uppsala university, decided to study in Paris during a year. He spent six months in 1873 in Paris studying with the most well-known French mathematicians ; but Hermite advised[8] him to study in Germany—Göttingen or Berlin. Finally, Mittag-Leffler studied a couple a months in Göttingen with Schering and then in Berlin with Kronecker and Weierstrass. He was one of the best Weierstrass' students and became his friend.

In 1876, Mittag-Leffler taught at Helsingfors[9] university for five years and then came back to Stockholm to teach in Stockholm college, founded by the new king Oskar II of Sweden in order to promote Scandinavian mathematics. In 1882, he married Signe af Lindfors issued of a rich and powerful Finnish family. Mittag-Leffler became an entrepreneur too. In Stocholm college in 1884, he recruited the brillant S. Kowaleskaja recommended by Weierstrass. The Sthockholm college became an important centre of mathematics, competing with the best French and German universities : the school of Scandinavian mathematics was born.

Gösta Mittag-Leffler about 1880

[7] See (Stubhaug, 1999) for a deailed biography of Mittag-Leffler.
[8] «Weierstrass : he is a master of all» would have said Hermite to Mittag-Leffler.
[9] Helsinki now

Mittag-Leffler founded in 1882 the mathematical journal Acta Mathematica from a Sophus Lie's idea ; still in order to promote Scandinavian mathematical school. The Acta Mathematica is still an important mathematical journal edited by the Mittag-Leffler Institute in Djursholm.[10] He was connected to many mathematicians : Hermite, Poincaré, Picard, Painlevé, Weierstrass, Houël, Markov, Moore, . . . Some say that his correspondences total about 12 000 letters. He continued his mathematical research on complex analysis and elliptic functions throughout his life. His most famous theorem is about factorization of meromorphic functions ; his masters Weierstrass and Hermite gave another proof of that theorem. He was connected to many important European politicians. So we can assume that he was a key figure of the mathematical world about 1880. He received many honours from the mathematical community.

2 ABOUT JULES HOUËL

Jules Houël (1823–86), issued of an old protestant Normand family, studied mathematics in Ecole Normale Supérieure from 1843 to 1846 ; he failed the « agrégation» in 1846—which he passed in 1847—, Houël began with teaching in different high schools. He worked in the high schools in Bourges (1847), in Pau (1848–49) , in Bordeaux (1850) and Alençon (1851–52). In 1852, he left teaching for mathematics and astronomy research, which led in 1855, to the defence of two theses—one in mechanics and one in astronomy (Houël 1855)—. Cauchy, member of the jury thesis, showed a real interest about them. Houël wanted to work on astronomy but couldn't enter[11]the Observatoire de Paris. After all, Houël took over V.A. Le Besgue at the «chaire de mathématiques pures» in Faculté des Sciences de Bordeaux in 1859 where he taught until 1884—when he retired because of health troubles.

Jules Houël about 1880

[10]Mittag-Leffler had no child and bequeathed his beatiful house in order to create a mathematical Scandinavian center.

[11]After Houël's descendants, U. Le Verrier director of the «Observatoire de Paris» would have presented him from entering it. That information is still to control.

He published the *Théorie élémentaire des quantités complexes* (four volumes) and the *Cours de calcul infinitésimal* (four volumes), fruits of his teachings from 1860 s until 1880 s. Those courses are really interesting because they are quite exhaustive, rigorous[12]; some pedagogic examples can help the reader and a history of the theories taught too. The *Théorie élémentaire* and the *Cours de calcul infinitésimal* had a certain success in France and Europe.

Houël enlivened the Société des sciences physiques et naturelles de Bordeaux. He promoted it so much that in 1867[13], «almost all the Bordeaux' mathematicians were enrolled in it». Houël published many mathematics papers, historic papers or translations in the *Mémoires* of the «Société», especially about complex analysis and non Euclidean geometries.[14] From 1864 to 1872, Houël was archivist of the « | Société» and developed substantially the activity and the connections. In 1872, the «Société des sciences physiques et naturelles» de Bordeaux counted more than a hundred connections among the learned societies in the whole world.

Houël was a polyglot : after the French mathematician P. Barbarin (Barbarin 1927), he knew at the end of his life «all the European languages», although he did never travel out of France. He translated, for instance : from German, papers written by Lejeune-Dirichlet, Riemann, Balzer, Lobatchevski[15], Lipschitz ; from Swedish, papers written by Mittag-Leffler about elliptic functions ; from Hungarian, the opuscul written by J. Boliay about non euclidean geometry; from Russian, books written by Lobatchevski and papers by Imschenetski and Bougaïev; from Norwegian, *La vie d'Abel* by C.A. Bjerknes, from Italian la *Théorie des équipollences* by Bellavitis.

Houël corresponded with many European mathematicians from the 50 s until the 80 s. here are additional examples : in France, Darboux, Laisant, Lefoy ; in Italy, Cremona, Forti ; in Germany, Balzer, Lejeune-Dirichlet, Borchard, Klein, Ohrtmann, ; in Scandinavy, Bjerknes, Dilllner, Lie ; in Bohemia, Durege, Emil and Eduard Weyr.

Houël was from 1870 to 1882, co-editor with Darboux of the *Bulletin des sciences mathématiques et astronomiques*, journal founded under the direction of the «Commission des hautes études»—presided by Chasles—with the support of the «Ministère de l'instruction publique» in order to diffuse[16] the new mathematics from Germany into France.[17] The polyglottism and opening mathematical mind of Houël were principal motivations of that choice. The *Bulletin* lasted until about World War II.

[12]See (Zerner 2008).

[13]As Houël wrote it to Charles Berger, in his letter dated of 12th January 1867 almost summoned in the introduction.

[14]On the life and work of N. Lobatchevski, for instance.

[15]The most of Lobatchevski's papers were written in German.

[16]After the 1870 French-Prussian war and the success of the French analytical School in the beginning of XIXth century, the French mathematicians didn't use to study German sciences, which were really important during the second part of the XIXth

[17]See (Gispert 1987).

3 ELEMENTS OF CONTEXTUALIZAION: HISTORIC LANDMARKS, TEACHING POLICY IN FRANCE AND MATHEMATICS (ANALYSIS) TOPICS DURING THE XIXTH CENTURY

3.1 Historic landmarks in the period of the correspondence between Mittag-Leffler and Houël in Europe

The period of correspondence (1872–83) is really interesting in Europe: it stands just after the 1870 French-Prussian war which ended by the foundation of the Third Republic in France, the Second Reich in Germany led by the conservative and nationalist Otto von Bismarck.

The fact that victorious Germany took Alsace and Lorraine—almost—fanned the hatred of the French against the Germans. The beginnings of Third Republic in France are marked by an largely royalist National Assembly—legitimist and pro Orléans, until 1876, when Republicans became majoritary because the Chambord's Count[18] alias Henri V, refused the Crown of France for ideological[19] reasons. Sweden had a new king since 1872, Oskar II who was enlightened conservative and created in 1878 Stockholm College. Finland was under Russian rule since Napoleonian wars after seven centuries of Swedish occupation.

3.2 Teaching policy in France during the XIXth century

Before the French Revolution, most of teachings in France were the fact of religious people—for instance, Jesuits. In Year II of the French Revolution, the Convention voted a text[20] telling that teaching would be layman, free and necessary and the Daunou[21] law. That text claimed actually teaching was not necessary, organized an primary and secondary teachings. According the French Revolution, public teaching had to give culture to citizens—Enlightment influence—and legitimate/assure the survival of democracy. Universities had been replaced by professional schools in medicine and law at that time and «grandes écoles» such as Ecole Polytechnique and Arts et Métiers.

Under the «Consulat», the first high schools were created in 1802. In 1806-8, Napoleon Ist founded the «baccalauréat»—exam at the end of high schools, the imperial university[22] and forced the teaching[23] of philosophy in high schools. Actually, teaching monopoly by the state was organized by those laws: the different degrees of teaching would be the faculties—medicine, law, literature, sciences, the high schools, the colleges, the institutions, boarding schools and primary schools. Under the «Restauration», it was decided[24]that all communes should propose primary teaching to all children without any resources condition. In 1833, the Guizot law encouraged the foundation of primary superior schools for poor children, who could access neither college nor high school.

The Falloux[25]law of 1850 authorized the creation of catholic schools and confered to Catholic Church the control of their programms and teachers. There was no more important change before the

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[18]The last Bourbon.

[19]One important point is that he wanted the white flag for France back, which was impossible

[20]5thNivôse, Year II.

[21]3rd Brumaire, Year IV.

[22]10thMay, 1806.

[23]17thMarch 1808.

[24]29thFebruary 1816.

[25]15thMarch.

His Grace Dupanloup about 1870

beginnings of the Third Republic. In the 1870's, the power of the clerical party, leaded by His Grace Dupanloup, was clear : the «Assemblée nationale» was in majority made of Royalists—Orleanists and legitimists. The Comte Jaubert proposed, in 31th July 1871, a new text in order to finish the state monopoly on superior teaching : there was three votes deliberations[26]about it, which went to the freedom on superior teaching law, in 12th July 1875—316 votes for and 266 against. At the end of 1870's, the Republicans were majoritary in the «Assemblée» and were capable of decide of a new policy in public instruction, which leaded to the famous Ferry laws, in 1880-2. They limited the power of Catholic Church in superior teaching, made the teaching layman, necessary and free for children between 6 and 13 years old. The Goblet law, in 1886, forbid to religious people to teach in public schools. Public instruction became the key of Third Republic.

3.3 Mathematical topics (analysis) in Europe during the first half of XIXth century

In the first half of XIXth century, some mathematicians tried to found rigorously the notions of real analysis[27], new analytical theories appeared and some ancient theories took a new turn.

In his *Cours d'analyse de l'Ecole Polytechnique*, Cauchy defined rigorously about 1820 the notion of convergence of a series, worked on continuity and derivability too. Bolzano worked also on continuity and prooved the intermediate values theorem in 1817 But continuity was not precisely defined enough. Weierstrass gave the modern way of its definition at the end of the century.

The theory of complex functions began with Cauchy's works; from 1815 to 1845, he worked on «Cauchy-Riemann equation», integration of a complex function along a contour, the residues, complex derivation. In the 1850 s, Riemann exposed the theory of multiform functions and the applica-

[26] 3-4-5thdecember 1874, beginning June 1875, 8-12 July 1875.

[27] Infinitesimal calculus.

tions to abelian functions.

The theory of elliptic functions comes from the idea of rectification of ellipses: it began from the Greek antiquity and took another form with the analytical geometry of Descartes in XVIIth century. The elliptic curves were «transcendental» curves: they were studied first by the Bernoullis and Leibniz. Then Mac Laurin, D'Alembert, Euler studied those curves and at the end of the XVIIIth century, Gauss expressed certain elliptic integrals with the lemniscatic sinus and cosinus. In 1825, Legendre classified the elliptic integrals in three types and in 1826, Abel had the genial idea of inversing the elliptic integrals and considering them as complex functions. That was the rebirth of the elliptic functions theory : Jacobi, Gudermann[28], Liouville, Riemann, Hermite, Briot, Bouquet, Clebsch, Kronecker, Weierstrass, Mittag-Leffler ... continued the Abel's work. All along the XIXth century, the elliptic functions theory was a main topic especially in Germany and France.

4 ABOUT TEACHING COMPLEX FUNCTIONS ACCORDING TO HOUËL AND MITTAG-LEFFLER

The beginning of the correspondence between Mittag-Leffler and Houël deals with complex analysis and the way of teaching it. In 1872, Mittag-Leffler was docent in Uppsala university and taught already complex analysis ; Houël just published the first three parts of the *Théorie élémentaire des quantités complexes* and taught since several years the complex analysis in «licence ès sciences mathématiques». In his letter answering to the Mittag-Leffler's first letter, Houël wrote in July 1872 :

> Here's how I understand that we must expose this theory in elementary education $f(x)$. We start by taking a function f(x), well know for real values of the variable. One replaces x by a complex variable z and one verifies that—except for singular cases that I do not throw away for, and which I reserve for later discussion—the ratio $[f(z+dz)-f(z)]/dz$ approaches certain value, function of z, and independent of the argument of dz.

Mittag-Leffler answered in August 1872 :

> Define a function of complex quantities as a function obtained by replacing in the form of a real function $z = f(x)$ the real variable by a complex variable, has, in my opinion, several disadvantages.
>
> The first drawback is that you build then the theory of complex quantities on a previous theory of real quantities, while , in my opinion, the correct approach should be the opposite. The theory of real quantities should be a special case of the theory of complex quantities.
>
> A second disadvantage is that it is not sure at all that the fundamental relation $df/dx+idf/dy = 0$ is a consequence of what we have for x real : lim $f(x+h) - f(x)/h = limf(x-h) - f(x)/h$. It will happen to the forms of elementary functions, but this can not be a property of functions in full generality .
>
> Moreover, if one relies on the definition of a function of a real variable , one must develop it in detail, and might one not then be facing the same problem as when you want directly define what is a complex function ?

[28] Weierstrass' master.

Finally, the problem of teaching complex functions lies in the inadequate state of the theory; in 1870, the notion of a general function is not clear neither the notion of a complex function, as Houël wrote in July 1872 :

> If we call any function of x quantity whose value depends on that of x in a certain way, then we must abandon establish a general theory of functions, because there is not, i believe, a single proposal for which one can not imagine a function that puts it in default. Some functions, such that $f(n) = 1 + 1/2 + \cdots + 1/n$, are essentially discontinuous. Others, though continue, have no differential, even for the case of real variables. Hankel[29]has shown than can be expressed by the signs of analysis—definite integrals or infinite series—functions with a finite range of singularities that make inapplicable to those functions all the rules of analysis. After showing none of the definitions proposed so far for the idea of function, take in its generality was satisfactory, he concludes that we must adopt to the function definition has given Riemann , which is nothing else that the definition of a monogenic function by Cauchy. This means that a non continuous and non monogenic function can not give rise to any general theory , and that in a theory that wants to be comprehensive , it must dismiss the functions that would put all the theorems in default.

In the first letter written by Mittag-Leffler, it is explicit that the different properties of complex functions are not well connected :

> As you see, I started my work by specifying the definitions of continuity, monogenity, monodromy. I wrote the definitions a little differently from what is usually done. In particular, the definition of monogeneity is of a different nature of this concept as it has over recent times. I do not think one is entitled, in an overview of the theory of synectic functions to evacuate the monogeneity from the building of synecticity.

And Houël in his first letter to Mittag-Leffler :

> Sometimes a function is synectic for the generality of the points in the plan and it ceases to be in some singular points ; and it can cease to be continue without ceasing being monogenic, as is the case for the points I have called, according to Neumann, the first kind of infinite, or it may lose both continuity and monogenity, as in the case of $e^{1/z}$ for $z = 0$ (second kind of infinite).

Both points of view on complex analysis teaching are interesting : Houël proposing a inductive way to make the students «feel» the principle although Mittag-Leffler would think of a deductive way in order to avoid to build «complex intuition» from «real intuition». Houël's point of view is the one of a pedagog but the Mittag-Leffler's one is entirely theoretical. In both points of view, it appears that the notion of a general function and a complex function is not clarified and the different properties - continuity, monodromy, synecticity, monogenity - of complex functions are not in the early 1870s well connected.

Another topic in the correspondence between Mittag-Leffler and Houël connected to complex analysis teaching is elliptic functions teaching. As soon as 1877, Houël asked Mittag-leffler in order to

[29]See (Hankel, 1870).

build a course and a chapter of his *Cours de calcul infinitésimal* reachable by his students of «licence ès sciences mathématiques» : that exchange of letters on that topic is quite technical, so we can't include it in our paper. We present the asking of Houël about it, in his letter dated march 1877:

> I am currently in a very great embarrassment, about a chapter on elliptic functions that I would like to insert in my *Cours de calcul infinitésimal*, and that was not in the original plan of the book. But a ministerial decree which introduced the elliptic functions in the program of the «licence ès sciences mathématiques» will require that I devote some pages. Secondly, the program of the exam is already very busy, I can hardly think of anything else to give an overview presentation was with rigor and at the same time in a manner quite elementary. There is a considerable number of approaches to this theory ; but the old methods of Jacobi, Gudermann, etc do not seem quite rigorously establish the double periodicity. There is the second method of Jacobi , adopted by Schellbach, based on the properties of the functions. But this method does not seem very easy for the beginners. You would do me the biggest favor by giving me some pointers on how I should follow. Is it possible, in thirty or forty pages, to share a simple an rigorous way to a beginner, possessing the first principles of the theory of functions of a complex variable, until the notion of the integral taken around a point—the residue of Cauchy ? Is it, I say, possible to teach seriously periodicity of elliptic functions, notions of integrals of second and third species and especially the expressions of these functions from functions θ?

That part of the correspondence about elliptic functions teaching for beginners is contained in eight letters.

5 THE ORGANIZATION OF MATHEMATICS TEACHING IN FRANCE, GERMANY ACCORDING HOUËL AND MITTAG-LEFFLER

The French mathematician Justin Bourget—father of the novelist Paul Bourget—and friend of Jules Houël thought that Houël was probably the person who knew the most about teaching organization in Europe, in the letter dated 17$^{\text{th}}$ April 1876 :

> What are we going to do for universities? You have to take advantage of it and study this topic thoroughly in a newspaper like *Le temps* ou *La république française*. You have spent a lot of time thinking about the subject, you are familiar with the way things are organized in other countries, you are able to say a million of intelligent things on the topic.

Jules Houël was interested for a long time by the organization of teaching generally and mathematics teaching ; he used to ask his correspondents how it worked in their homelands, as for instance to Mittag-Leffler, in his letter dated 13$^{\text{th}}$ September 1874,

> If you can spare the time I would appreciate if you could give me details about the way universities in Sweden, Finland and Russia are organized.

At that period of the 1870 s, which is the beginning of the Third Republic in France, Houël was very critical about the organization of teaching: according to him, there were too many vacations, too many exams, not enough work from both students and professors. The following quotations are in the letter dated 13th September 1874 written by Houël:

> We nearly spend half the year doing exams or being in holyday. With the time we have left we only teach 2 lessons a week (I teach 5 of them as I can't accept the title of professor for nothing). Are there still professors who think teaching is tiring !
>
> You must know that there is a lot to say about further education in Paris. So what are you going to say once you have seen how it works in the rest of France.
>
> We don't have universities but only faculties which are part of universities but are not linked together. Divide ut imperes, the motto of dictators, to which the famous emperor has conformed.
>
> In France, public instruction is not better organized than in Sweden. I even think we should copy your institutions, especially those of your important universities.

Houël pointed out the fact that French faculties were too small and had too few students ; Houël had probably between 2 and 8 students[30]each year in his courses. The superior mathematics teaching was centralized in Paris where the «Ecole Normale Supérieure», the «Ecole Polytechnique» and the «Sorbonne University» drained the majority of the students. Many professors worked in both of those schools.

On the contrary, the organization of mathematics teaching in Germany and Russia appeared as an example to Houël :

> I don't know when France has universities as Germany or Russia (who have 7 without Helsingfors and Dorprat).

According Houël—Mittag-Leffler agreed—the origins of bad oragnization of education in France came from domination of catholicism and from Napoleon Ith' organization.

> I am entirely of your opinion: times have changed since the days when Latin et Greek in Europe have produced this great intellectual revolution as the Renaissance ! [...] We suffer the fate of all countries where Catholicism dominates. If governments of François Ist and Louis XIV had not paralyzed by their persecutions, as absurd as cruel, the beneficent influence of the Reformation, France would not have experienced all these setbacks,, and would have continued to hold its rank among the nations that lead the progress. [...] But here it is the bishops who interfere with that, and hardly wise minister had the time to abolish the absurd practice of Latin verse in high schools, His Grace Dupanloup restored it by his credit.

And

> . . . Void where our higher languishes, with the absurd organization bequeathed to us by evil genius of France, this evil being that insists on naming the great Napoleon! This scourge of God harmed in any possible way, but I think he did more fatal is its organization of public education, where he demonstrated both ignorance of a corporal and a Jesuit obscurantism.

[30]See (Zerner, 2008).

CONCLUSION

We have now to resume the information given about the correspondence between Mittag-Leffler and Houël in regard with their context. First, the mathematical considerations, which are the starting point of their epistolary exchange, are totally in the era of time: complex functions and elliptic functions. It is true that Houël had more a pedagogic position although Mittag-Leffler was a theorist and specialist of complex analysis—Houël was an astronomer. They felt that those theories were not achieved but Mittag-Leffler was more modern in his points of view in particular in the idea of making a theory of continuous functions.

About public instruction and its organization, they agreed—that's why we present only the Houël's point of view: about it, Houël's ideas were revolutionary in the 1870 s. Houël promoted the model of German universities against the French organization. Houël wasn't content of remarking it : it analyzed the failure on the one hand by the centralized organization and the baccalaureate weight decided by Napoleon Ist and on the other hand by the influence of the clerical Party in France. Mittag-Leffler who studied in France and Germany confirmed the views of Houël.

To conclude, we would like to compare these topics with those of other correspondences of Houël or Mittag-Leffler to show its originality. The main correspondences with Mittag-Leffler already studied are with Poincaré and Hermite. The correspondence between Mittag-Leffler and Poincaré deals with the participation of Poincaré to Acta Mathematica; the one between Mittag-Leffler and Hermite is more diverse: it deals with different mathematicians, new mathematics—set theory of Cantor for instance—and mathematics in general.

The main correspondences with Houël already studied are with Beltrami and De Tilly. Both have as a great topic the non Euclidian geometries; with Beltrami, it is more about differential geometry and geodesy and with De Tilly about generalities and foundations of those geometries. In regard of those different correspondences, the correspondence between Mittag-Leffler and Houël is really interesting because it is in tune with the mathematical times but really personal too in their analyses on public instruction, which is a great topic of the Third Republic in France.

REFERENCES

- Azéma J.-P. & Winock M., 1986, *La Troisième République. Paris : Hachette*, collection "Pluriel".
- Barbarin P., 1926, La correspondance entre Houël et De Tilly. *Bulletin des sciences mathématiques et astronomiques* (2) 50, 50-62 et 74-88.
- Bertrand J., 1867, *Rapport sur les progrès les plus récents de l'analyse mathématique*. Paris : imprimerie impériale.
- Baubérot J. & Zuber V. , 2000, *Une Haine oubliée. L'Anti protestantisme avant le "Pacte laïque" (1870-1905)*. Paris: Albin Michel.
- Belhoste B. (Dir.), 1995, *La France des X, deux siècles d'histoire*. Paris.
- Belhoste B., 1985, *Cauchy, un mathématicien au XIXe siècle*. Paris : Belin.
- Beltrami E., 1998, *La découverte de la géométrie non euclidienne sur la pseudosphère. Les lettres d'Eugenio Beltrami à Jules Houël (1868-81).*
  Introduction, notes et commentaires critiques par Luciano Boi, Livia Giacardi, Rossana Tazzioli. Préface Ch. Houzel et E. Knobloch. Paris : Albert Blanchard.
- Chevalier J.-J., 1986, *Histoire des Institutions et des Régimes politiques de la France, de 1789 à nos jours. Paris :* Dalloz.

– Darboux G., 1873, *Sur une classe remarquable de courbes et de surfaces algébriques et sur la théorie des imaginaires.* Paris : Gauthiers-Villars.

– Gerbod P., 1965, *La Condition universitaire en France au XIXème Siècle (1842-1880).* Paris : P.U.F.

– Gispert H., 1987, *La correspondance de G. Darboux avec J. Houël. Chronique d'un rédacteur (déc. 1869 - nov. 1871).* Cahiers du séminaire d'histoire des mathématiques 8 , 67-207.

– Hankel H., 1870, *Untersuchungen über die unendlich oft oscillirenden und unstetigen Functionen.* Tübingen : Teubner.

– Hermite Ch., 1989, *Lettres de Charles Hermite à Gösta Mittag-Leffler (1892-1900).* Cahiers du séminaire d'histoire, tome 10, p. 1-82.

– Houël J. , 1855, *Sur l'Intégration des équations différentielles dans les problèmes de mécanique. Paris* : Mallet- Bachelier.

– Houël J., 1855, *Sur le développement en fonctions périodiques de la fonction pertubatrice de Jupiter*. Paris : Mallet-Bachelier.

– Houël J., 1867, *Essai critique sur les principes fondamentaux de la géométrie élémentaire ou commentaire sur les XXXII premières propositions d'Euclide*. Paris : Gauthier-Villars.

– Houël J., 1869, *Sur le calcul des équipollences (Méthode d'analyse géométrique de M. Bellavitis)*. Paris : Gauthiers-Villars.

– Houël J., 1871, *Sur l'impossibilité de démontrer, par une construction plane, le principe de la théorie des parallèles du postulatum d'Euclide*. Paris : Gauthiers-Villars.

– Houël J., 1878-79-80-81, *Cours de calcul infinitésimal*. Quatre Tomes. Paris : Gauthier-Villars.

– Houël J., 1867-68-69-73, *Théorie élémentaire des quantités complexes*. Quatre Tomes. Paris : Gauthiers-Villars.

– Launay M., 1988, *L'Église et l'École en France, XIXème-XXème Siècles*. Paris : Desclée.

– Lespiault G., 1887, *Notice sur Guillaume-Jules Houël*. Extrait du Mémorial de l'Association des anciens élèves de l'Ecole Normale. Archives de l'Académie des Sciences.

– Milza P. & Berstein S., 1994, *Histoire du XIXème Siècle*. Paris : Hatier.

– Mittag-Leffler G., 1882, *Sur la théorie des fonctions uniformes d'une variable*. Paris :Comptes rendus de séance de l'Académie des sciences 94.

– Neuenschwander E., 1984, *Die Edition Mathematischer Zeitschriften im 19. Jahrhundert und ihr Beitrag zum Wissenschaflichen Austausch zwischen Frankreich und Deutschland*. Göttingen : Mathematisches Institut der Universität Göttingen.

– Poidevin R. & Bariéty J., 1977, *Les relations franco-allemandes, 1815-1975*. Paris : A. Colin.

– Poincaré H. & Mittag-Leffler G., Nabonnand Ph., 1999, *La correspondance entre Henri Poincaré et Gösta Mittag-Leffler (1881-1911)*. Basel : Birkhaüser.

– Prost A., 1979, *Histoire de l'enseignement en France, 1800-1967*. Paris : Armand Colin.

– Rémond R., 1974, *Introduction à l'histoire de notre temps, T2, Le XIXème siècle (1815-1914). Paris* : Seuil, coll. "Points".

– Rudin W., 1975, *Analyse réelle et complexe.* Traduction de N. Dhombres et F. Hoffman. Paris : Masson.

– Stubhaug A., 2006, *Gösta Mittag-Leffler : a man of conviction*. Traduction de T. Nunnaly. Berlin-Heidelberg : Springer-Verlag.

– Zerner M., 2008, *La transformation des traités d'analyse (1870-1914)*. Archives ouvertes de l'Université de Nice : hal-00347740.

THEORY OF EQUATIONS IN THE HISTORY OF CHOSUN MATHEMATICS
朝鮮의 方程式論 歷史

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ABSTRACT

Theory of equations has been one of the most important subjects in the history of algebra. In eastern mathematics, geometry was somehow contained in the theory of equations and its number system was restricted to the field of rational numbers and hence its theory of equations has a completely different history from the western counterpart. Chosun mathematics was strongly influenced by Chinese mathematics, but there were very few Chinese mathematics books available that Chosun mathematicians could study. Despite such poor conditions, Chosun mathematicians succeeded in establishing a perfect theory of equations in the context of eastern mathematics, namely the theory of polynomial equations and that of solving them.

This paper intends to illustrate this achievement of the theory of equations in Chosun dynasty and its implications to the school mathematics.

Keywords: Chosun Mathematics, Theory of equations, TianYuanShu(天元術), ZengCheng-KaiFangFa(增乘開方法), Park Yul(박율, 朴繘, 1621-1668), Hong Jung Ha(홍정하, 洪正夏, 1684-?), Lee Sang Hyuk(이상혁, 李尙爀, 1810-?), Nam Byung Gil(남병길, 南秉吉, 1820-1869).

1 Introduction and Chosun mathematics

In eastern mathematics, geometry for practical mensuration was carried out by the theory of right triangles and areas of triangles so that it was studied in the setting of algebra. Unlike geometrical algebra in ancient western mathematics, algebra got its position as a main subject in eastern mathematics. Further, the theory of equations was a main area of the study of algebra. The other subjects like mensuration including a good approximation of π, problems of ratios, extractions of square and cube roots and systems of linear equations were completely settled in JiuZhang SuanShu(九章算術), which was commented thoroughly by Liu Hui(劉徽) in the third century. Thus subsequent mathematicians took interest mainly in the theory of equations. Theory of equations divides into two parts. One is how to construct and represent equations from given conditions and the other is how to solve them. Both problems were resolved by mathematicians of Song(960-1279) and Yuan(1271-1368) dynasties. For the former, they introduced TianYuanShu(天元術) for representing and operating polynomials and then extended it to SiYuanShu(四元術) for polynomials with four indeterminates and hence they could deal with polynomial equations together with systems of equations of higher degrees. For the latter, the eastern mathematics has a completely different approach from the western counterpart, because JiuZhang SuanShu set the tradition that the study of mathematics was restricted to the field of rational numbers so that irrational solutions of equations were not allowed. Thus they tried to find a method to get approximations to the solutions. We note that eastern mathematics regards radical roots $\sqrt[n]{a}$ as their approximations and hence eastern mathematics is not interested in the formulas of solving equations by radicals. Furthermore, using the synthetic divisions, they could solve polynomial equations once and for all in the twelfth century. The method is called ZengChengKaiFangFa(增乘開方法) which is known by Ruffini(1765-1822)-Horner(1786-1837)'s method. These results are presented in the books by Li Ye(李治, 1192～1279), Qin Jiu Shao(秦九韶, 1202～1261), Yang Hui(楊輝) and Zhu Shi Jie(朱世傑). Chinese mathematicians were quite satisfied by these results, and Chinese theory of equations saw no more development and hence its history is somewhat suspended. But the western one gave a great impetus to the modern mathematics. Furthermore, the mathematics of Song and Yuan eras were not fully appreciated by the next generations so that it was almost forgotten until the end of the 18th century.

Mathematics in Chosun dynasty(1392-1910) as in previous dynasties in Korean peninsular was strongly influenced by Chinese mathematics. Although China is Chosun's neighboring country, their capital cities were very far apart. As the clause of June 16, 1460 shows in the annals of the Chosun dynasty(朝鮮王朝實錄) of King SeJo(世祖, 1455 ～1468), King SeJong(世宗, 1418 ～1450) regretted the scarcity of mathematics books for the study of astronomy, in particular ShouShiLi(授時曆) and mathematics as well because of the distance between the two countries, and then succeeded in importing Zhu Shi Jie's SuanXue QiMeng(算學啓蒙, 1299), Yang Hui SuanFa(楊輝算法, 1274-1275) and JieYong JiuZhang(捷用九章) along with other books on astronomy([26]). We note that the first two books together with An Zhi Zhai's XiangMing SuanFa(詳明算法, 1373) were designated as the required subjects for the national examinations for the selection of mathematical officials(籌學取才) from the beginning of Chosun dynasty. There were two more notable occasions of importation of mathematics books. One was in the middle of the 17th century when ShiXianLi(時憲曆) initiated by the western astronomy was introduced into Chosun so that books with the western mathematics were also brought into Chosun. Chosun mathematicians in the 18th century paid a great deal of attentions to these books, in particular ShuLi JingYun(數理精蘊, 1723). The other was in the first half of the 19th century when they brought in books such as Qin Jiu Shao's ShuShu JiuZhang(數書九章, 1247), Li Ye's CeYuan HaiJing(測圓海鏡, 1248) and YiGu YanDuan(益古演段, 1259), Zhu Shi Jie's SiYuan YuJian(四元玉鑑, 1303) with the commentary by Luo Shi Lin(羅士琳, 1774-1853) and appendices by Yi Zhi Han(易之瀚). They also imported old classics included in ShiBu SuanJing(十部算經) and books written by major mathematicians of Qing dynasty like Wang Xi Chan(王錫闡, 1628-1682), Mei Wen Ding(梅文鼎, 1633-1721), Jang Yong(江永, 1681-1762), Jiao Xun(焦循), Zhang Dun Ren(張敦仁), Li Rui(李銳, 1768-1817) and Song Jing Chang(宋景昌). Thus Chosun mathematicians could have an almost full list of Chinese mathematical works at long last in the middle of the 19th century. By these, we figure out that the first two introductions of Chinese mathematics were motivated for the sake of systems of calendars but eventually effected the development of Chosun mathematics and that the last one was made for the study of mathematics per se. We also note the relatively poor environment of Chosun mathematics and its relation with Chinese mathematics.

Any mathematical work in Chosun dynasty before the 17th century was not handed down except SanHak BalMong (산학발몽, 算學發蒙) which is an appendix to GyoSik ChuBoBub(교식추보법, 交食推步法, 1458) compiled by Lee Sun Ji(이순지, 李純之, ?-1465) and Kim Suk Je(김석제, 金石梯). It consists of just 23 pages and deals with methods of multiplications, divisions and extractions of square roots. The same clause of the Annals of the Chosun dynasty mentioned above reveals that Chosun mathematicians could carry out only the above three subjects and that Chosun mathematics went into a decline after King SeJong except that of basic calculations and mensurations for the government affairs.

In 1660, Kim Si Jin(김시진, 金始振, 1618-1667) republished SuanXue QiMeng, which rekindled Chosun mathematics. In his preface, Kim said that Chosun mathematics in the period could handle mathematics of XiangMing SuanFa and that he could barely get SuanXue QiMeng(算學啓蒙, 1299) and Yang Hui SuanFa(楊輝算法, 1274-1275), although the latter had been reprinted in 1433 and distributed to various government offices. In particular, he mentioned the importance of TianYuanShu for the study of mathematics, and its study was followed by many mathematicians, notably Park Yul(박율, 朴繘, 1621-1668) and Hong Jung Ha(홍정하, 洪正夏, 1684-?). We also note that Kim's reprint was exported to China in the 19th century and that it was republished in 1839 by Luo Shi Lin. Contrary to the fact that TianYuanShu was rather neglected in China, it was a major subject in Chosun mathematics, so that Chosun mathematics, in particular theory of equations came to have a different history from its Chinese counterpart.

These two facts namely different approaches to the theory of equations by eastern and western mathematics and those by Chosun and Chinese mathematics, motivate us to investigate the history of theory of equations in Chosun dynasty.

Investigating the theory of equations in Chosun dynasty, we conclude that Chosun mathematicians established a perfect theory on equations although they had only a few rudimentary informations.

Because of the limit of the presentation, we will deal with the main history on the theory of equations in Chosun dynasty. The paper divides into two sections. In the first half, we investigate the earlier results obtained in Park Yul's SanHak WonBon(산학원본, 算學原本, 1700, [7]) and Hong Jung Ha's GuIlJib(구일집, 九一集, 1724, [2]). The second half is devoted to the results obtained in the 19th century.

For the Chosun mathematics, we refer to HanGuk GwaHak GiSulSa JaRyo DaeGye(한국과학기술사자료대계, 韓國科學技術史資料大系, [3]) and for the Chinese mathematics to ZhongGuo KeXue JiShu DianJi TongHui ShuXueJuan(中國科學技術典籍通彙, 數學卷, [10]) and ZhongGuo LiDai SuanXue JiCheng(中國歷代算學集成, [11]). Those books compiled in them will not be numbered as an individual reference except a few important ones. We also include our papers ([12 - 25]) related to theory of equations which are written in Korean. Among them, English versions of [21] and [22] will be appeared in the present proceedings. For the history of Chinese mathematics, we refer to [8, 9].

2 Park Yul, Hong Jung Ha and their theory of equations

Despite the mid 17th century's meaningful contact with Chinese mathematics including western mathematics, it was first studied mainly by the officials of the state observatory(관상감, 觀象監). Chosun mathematics revived after the republication of SuanXue QiMeng in 1660 and its renaissance was prompted by the study of the classical SuanXue QiMeng, Yang Hui SuanFa, XiangMing SuanFa together with SuanFa TongZong(算法統宗, 1592) written by Cheng Da Wei(程大位, 1533-1606).

Gyung Sun Jing(경선징, 慶善徵, 1616-?) published MukSaJib SanBub(묵사집산법, 默思集算法, [4]) which is presumably the first full mathematics book in the history of Korean mathematics that we now have. But it does not deal with TianYuanShu that was introduced in the last book of SuanXue QiMeng and ZengChengKaiFangFa. For ZengChengKaiFangFa, SuanXue QiMeng explains it by the process of extracting radicals although it indicates FanFa(飜法) in ZengChengKaiFangFa in the same chapter KaiFangShiSuo(開方釋鎖). Yang Hui SuanFa includes ZengChengKaiFangFa for quadratic equations as JianZongKaiFangShu(減縱開方術). Since Jia Xian(賈憲) and his ZengChengKaiFangFa were mentioned only in Yang Hui's XiangJie JiuZhang SuanFa(詳解九章算法, 1261), they were not known to Chosun for the book was not brought into Chosun.

Park Yul's SanHak WonBon is the second full book of mathematics whose main subject is the theory of equations. It consists of three books. The first book deals with simple equations $ax^2 = b$ related to right triangles. Park was interested in the interpolation of roots and its reverse. In order to compare solutions with those obtained by the interpolation, he indicated both solutions for those problems with rational solutions. Even for extracting the square roots, he indicated the exact equations. The second book is devoted to equations $ax^n = b$ related to circles, spheres and regular polyhedrons. In this case, he used three well known values of π, i.e., 3, $\frac{157}{50} = 3.14$ and $\frac{22}{7}$, and the basic rules $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{ab^{n-1}}}{b}$. The author included these two books as a preliminary for the last one in which he expounded TianYuanShu. In order to build equations by TianYuanShu, one has to calculate $(x-a)^n$ and hence Jia Xian's triangle also known as Pascal's triangle was introduced. We note that Jia Xian's triangle is not included in SuanXue QiMeng and Yang Hui SuanFa, but in SuanFa TongZong. For the triangle, Park quoted Yang Hui's quotation of the interpolation of BianGu TongYuan(辨古通源) in XuGu ZhaiQi SuanFa(續古摘奇算法, 1275) and then obtained it by the synthetic division used in ZengChengKaiFangFa. This shows that Park had only SuanXue QiMeng and Yang Hui SuanFa when he wrote SanHak WonBon. After this, he included 18 problems of polynomial equations and 5 problems of linear equations. The first part presents TianYuanShu. We just quote Problem 2 to compare his problems with those in SuanXue QiMeng.

> 今有平圓徑 平方面 立方面 三乘方面 四乘方面 五段共積 一萬六千七十七尺 只云 平方面爲 四乘方二分之一 三乘方三分之一 立方面四分之一 而圓徑少平方面一尺 問各幾何

The author explained how to get the equation(개방식, 開方式) by TianYuanShu in detail and then presented its solution simply by the statement "solving it(開之)". The second part deals with a linear equation and 4 systems of linear equations. It is notable that he obtained the linear equation by TianYuanShu. SanHak WonBon is the first book that dealt with TianYuanShu in eastern mathematics after Zhu Shi Jie's SiYuan YuJian(1303) whose existence had not been known to Chosun until the 19th century. It shows that Park had a thorough insight into the subject. The preface of the book mentions that the author was indebted to Im Jun(任濬) for his mathematics, who is also mentioned in the preface of Kim's republication of SuanXue QiMeng. But we now do not have any of Im's own publications. SanHak WonBon was quoted by various mathematicians and made a great contribution to the development of theory of equations in Chosun.

The next major work for theory of equations was written by a great mathematician Hong Jung Ha. While Park Yul passed a national examination for the civil services(文科 科擧) in 1654 and became a governor of a prefecture (현감, 縣監), Hong was from Jung In(中人) family who were not allowed to take such examination as Park passed but only those for minor officials for administrative affairs and technology including medicine and music(雜科, 取才). See [21, 24] for the detail of Hong's personal history. He wrote GuIlJib(구일집, 九一集, 1724) consisting of 9 books. The first eight books contain 20 chapters with 493 problems and were written before 1713 when Hong met He Guo Zhu(何國柱) to discuss mathematics. Book 9 was added in 1724 as an appendix. Book 9 shows that Hong did not have any information on western mathematics although Choe Suk Jung(최석정, 崔錫鼎, 1646-1715) quoted extensively TongWen SuanZhi(同文算指, 1613) in his book GuSuRyak(구수략, 九數略) and Cho Tae Gu(조태구, 趙泰耉, 1660-1723) revealed his extensive study of western mathematics in his book JuSeo GwanGyun(주서관견, 籌書管見, 1718). Contrary to Hong, Choe and Cho were both the highest government officials(領議政). Three chapters in Book 4 of GuIlJib deal with theory of equations related to TianYuanShu. It is well known that when dealing with equations related to finite series(堆垛術), Zhu Shi Jie did not use TianYuanShu in SuanXue QiMeng but he used it extensively in SiYuan YuJian so that the latter book is known to contain extremely developed theory of finite series as well. But the book received almost no attention until the 19th century. Hong Jung Ha, however used TianYuanShu when he constructed equations related to finite series. We also note that the theory of equations in JiGu SuanJing(緝古算經) written by Wang Xiao Tong(王孝通, ca. 7th C.) was not fully understood by Chinese mathematicians except those who used TianYuanShu in Song-Yuan eras. Hong also deals with the equations in JiGu SuanJing in Book 4 and 5. Thus dealing with the usual mathematics except theory of equations in the first three books and one chapter in Book 4, and dealing with theory of equations including those related to right triangles(句股術) in the remaining four books, Hong gives his exclusive attention to theory of equations in GuIlJib.

Quoting some results in GuIlJib, we now investigate the excellence of Hong's theory of equations. In MyungSeungBangSik(明乘方式) of the introductory remark(범례, 凡例), he shows that the coefficients of $a(x+b)^n$ can be obtained by the synthetic divisions, which was hinted at in SanHak WonBon for the case $(x+b)^n$. We refer to [22] for the process for $(x+b)^n$ given by Yang Hui which may be explained as follows: One can find $a_0, a_1, \ldots, a_n$ of $y^n = a_n(y-b)^n + a_{n-1}(y-b)^{n-1} + \cdots + a_1(y-b) + a_0$ by synthetic divisions as those in ZengChengKaiFangFa and then by $x = y - b$, or $y = x + b$, one has coefficients of the binomial expansion. Hong extends this to ay^n. Furthermore he includes Jia Xian's triangle up to the degree 10 under the same title GaeBangGuRyumYulJakBubBonWonDo(개방구렴율작법본원도, 開方求廉率作法本源圖) in SuanFa TongZong, which was transmitted through Wu Jing(吳敬)'s JiuZhang SuanFa BiLei DaQuan(九章算法比類大全, 1450) from Yang Hui. He also includes the triangle for $(x-1)^n$ up to degree 12 under the title GaeBangGuRyumYulJungBuJiDo(개방구렴율정부지도, 開方求廉率正負之圖). For this, Park Yul gave just the signs of coefficients of the binomial expansion.

A general quadratic equation $ax^2 + bx = c$ was first introduced once only in the chapter GouGu(句股) of JiuZhang SuanShu. Although there are some other problems in the chapter related to general equations, their solutions were obtained without constructing equations. We call this process traditional approach to right triangles. JiGu SuanJing is known to be the first book that constructed equations of higher degrees. Zhu Shi Jie introduced TianYuanShu in problems related to right triangles and a system of linear equations in SuanXue QiMeng and he compared his construction of equation by TianYuanShu with the traditional approach. General quadratic equations not related to right triangle were introduced to find two sides of a rectangle with its area and the sum or difference of the two sides, which may be also understood by the relation between roots and coefficients. Zhu also deals with them by TianYuanShu. Hong Jung Ha deals with right triangles in the chapter GuGoHoEunMun(구고호은문, 句股互隱門) in Book 5 altogether with 78 problems. He thoroughly comprehends mathematical structures of the traditional approach, TianYuanShu and quadratic equations with the relation between roots and coefficients, and he obtains the most advanced theory of GouGoShu in the beginning of the 18th century. He also detects the symmetry between Gou(句) and Gu(股). We note that TianYuanShu should be divided into two types. One is the method to represent just polynomials $\sum_{i=0}^{n} a_i x^i$, which is used in SuanXue QiMeng. The other is one to represent those with negative powers $\sum_{k=1}^{m} b_k x^{-k} + \sum_{i=0}^{n} a_i x^i$, which is used in CeYuan HaiJing. Although Hong did not come into contact with the latter book, we strongly gather that he would have used the latter TianYuanShu because there were many cases where he should use it to construct equations (see [19] for the detail). We also note that Mei Jue Cheng(梅瑴成, 1681 ~ 1763) missed the latter TianYuanShu in his ChiShui YiZhen(赤水遺珍, 1759) so that he wrongly claimed that TianYuanShu is nothing but JieGenFang(借根方) method to represent polynomials introduced in ShuLi JingYun. In three Books 6-8 of GuIlJIb consisting of three parts of chapter GaeBangGagSulMun(개방각술문, 開方各術門), Hong used TianYuanShu in the sense of the first one. Like the equations in SuanXue QiMeng, Hong's equations assume the form of $p(x) = 0$ and hence for a traditional form $ax^n = b$, he represents them as $ax^n - b = 0$. We note that the equal sign was not yet introduced, and hence the equation is denoted by "GaeBangSik(개방식, 開方式) $ax^n - b$". His problems of TianYuanShu deal with conditions regarding right triangles, rectangles, circles, spheres together with regular polyhedrons of dimensions up to 10.

Regarding Hong Jung Ha's method of solving equations, we have already discussed it in detail in [22]. As mentioned earlier, Hong just studied the method ZengChengKaiFangFa merely in SuanXue QiMeng and TianMu BiLei ChengChu JieFa(田畝比類乘除捷法, 1275) which is a part of Yang Hui SuanFa. But he grasps its mathematical structure and extends it to the arbitrary equations. Following the method of extraction of roots in JiuZhang SuanShu, they tried to get digits of solutions, so that they must use the process of retreats(一退, 二退, ⋯) which is rather awkward for the synthetic divisions. As we quoted in [22], Hong's ZengChengKaiFangFa appears among others in Problem 48 in Book 6 where he solves the equation $x^4 + 2,256x^2 - 4,264,225 = 0$. Excluding the paragraph of retreats, his process to get the equation for the second estimate and eventually have the solution, is exactly ZengChengKaiFangFa which can be used now. Furthermore, Hong referred to coefficients of x, x^2, x^3, x^4 as GabJong(甲從), EulJong(乙從), ByungJong(丙從) and JungJong(丁從), respectively, where Gab, Eul, Byung and Jung were chosen from the ten heavenly stems(天干). His notions are much more adaptable than those related to Lian(廉).

In all, theory of equations in Chosun mathematics was completed by Hong Jung Ha except that of systems of equations of higher degree which will be discussed in the next section. Unfortunately, Hong's GuIlJib has been read only by a small number of his descendants and their relatives so that it did not influence the development of Chosun mathematics although it is the greatest work produced in Chosun dynasty. GuIlJib was very much appreciated by Lee Sang Hyuk(이상혁, 李尙爀, 1810-?) and Nam Byung Gil(남병길, 南秉吉, 1820-1869), which will be discussed in the next section.

3 Theory of equations in the 19th century

Because of ShuLi JingYun and Mei's erroneous claim mentioned above, TianYuanShu was almost forgotten in Chosun after Hong Jung Ha. But at the end of the 18th century, Song-Yuan mathematics revived in China and their commentaries began to be published by various mathematicians. And thus, TianYuanShu was widely recognized as a better tool for theory of equations than JieGenFang. Although Mei used TianYuanShu in SiYuan YuJian to compare it with JieGenFang, he does not seem to have fully appreciate the book. But from the beginning of the 19th century, various mathematicians, notably Luo Shi Lin and Shen Qin Pei(沈欽裵), studied SiYuan YuJian and wrote commentaries on it. Furthermore, Kim's republication(1660) of SuanXue QiMeng was republished in China by Luo Shi Lin in 1839. Consequently, Song-Yuan mathematics were completely recovered in China and their results in turn were brought into Chosun mostly by Nam Byung Gil so that Chosun mathematicians also began to study Song-Yuan mathematics, in particular its theory of equations again. Nam passed the national examination for the civil services in 1848 and became a minister(禮曹判書) and the head of the state observatory(觀象監提調). He wrote many books on astronomy and mathematics. His brother Nam Byung Chul(남병철, 南秉喆, 1817-1863) also served as the head of the state observatory and ministers of various administrative departments and wrote books on the same subjects as his brother. The state observatory had its own library with a large collection of books on astronomy and mathematics. Thus they could import books from China and hence have a full collection of books on Song-Yuan mathematics. Nam Byung Gil had an exceptional collaborator, Lee Sang Hyuk.

Lee Sang Hyuk is also from JungIn family as Hong Jung Ha. He passed the national examination for astronomy(陰陽科) in 1831 and then the examination for mathematics(籌學取才) in 1832. He got appointed as UnGwaJung(운과정, 雲科正, 正三品) at the state observatory. A granduncle of Lee's father(李秉喆) is Hong Jung Ha so that Lee could read Hong's GuIlJib. His maternal grandfather was a medical doctor. In 1850, Lee published a book GyuIlGo(규일고, 揆日考) on astronomy and Nam Byung Gil wrote its preface. Since then, Lee and Nam exchange prefaces and corrections in their publications, where they show their great respect to each other. Because of their affinity and deep expertise in mathematics and astronomy, they don't have any barrier arising from their different social standing. Notwithstanding his low status of JungIn, Lee could have all the information available on contemporary mathematics and astronomy, so that his environment is much more different from Hong Jung Ha's.

Since ShuLi JingYun was added as the subject for the national examination for astronomy, Lee had to study it. JieGenFang was first introduced in Book 31-37 which belong to the last part(末部) of ShuLi JingYun but equations were greatly discussed in its earlier books. This means that its approach to theory of equations before JieGenFang is rather traditional like the chapter GouGu in JiuZhang SuanShu. He wrote ChaGeunBangMongGu(차근방몽구, 借根方蒙求, 1854) to redress this deficiency of ShuLi JingYun. Indeed, he took problems from Book 3 - 25 of ShuLi JingYun together with 3 problems in the last part and his own problems, and then constructed equations using JieGenFang method. As we noted, JieGenFang is a method to represent only polynomials and therefore Lee meets some difficulties when the given conditions relate to the rational polynomials. Presumably, one of the great Song-Yuan mathematical works, CeYuan HaiJing were brought into Chosun around 1850's and Lee certainly studied the book. Incidentally Nam Byung Chul wrote a commentary (해경세초해, 海鏡細艸解, 1861) on CeYuan HaiJing. Furthermore, Lee and Nam Byung Gil take examples extensively from

CeYuan HaiJing in their works. Thus they ought to have known TianYuanShu in the book and hence the difference between TianYuanShu and JieGenFang.

Lee does not use JieGenFang since his publication SanSul GwanGyun(산술관견, 算術管見, 1855), using TianYuanShu instead even when he solves the problems in ShuLi JingYun. Moreover, in the chapter WonYongSamBangHoGu(원용삼방호구, 圓容三方互求) Lee deals with three congruent squares in a circle(品字) which was in ChaGeunBangMongGu but he solves the same problem using TianYuan-Shu this time. Lee takes the series representations for sine(正弦) and versine(正矢) *in a circle* which were introduced into China by Jartoux(1668-1720) and quoted in ChiShui YiZhen. Using the partial sums and their values in the chapter, HoSiGuHoDo(弧矢求弧度), he solves polynomial equation to get the angle and the length of an arc, i.e., value of arcsine. It is very much notable that in these problems, Lee mentioned that the number of terms for the partial sum can be determined by the absolute value of the first neglected term. Both are alternating series. We don't know how he obtained such an important theory for alternating series and their errors.

Lee is an actual coauthor of Nam Byung Gil's SanHak JungEui(산학정의, 算學正義, 1867, [6]) which is a comprehensive work on the subjects dealt in ShuLi JingYun and Song-Yuan mathemtics including ShuShu JiuZhang, CeYuan HaiJing, YiGu YanDuan and SiYuan YuJian. Nam and Lee left notes of their study on SiYuan YuJian XiChao(四元玉鑑細艸, 1834) compiled by Luo Shi Lin with Yi Zhi Han's appendices. Besides these, Nam published five more books, JibGo YunDan(집고연단, 緝古演段), GuJang SulHae(구장술해, 九章術解), YuSi GuGo SulYo DoHae(유씨구고술요도해, 劉氏句股術要圖解), MuIHae(무이해, 無異解, 1855), CheukRyang DoHae(측량도해, 測量圖解, 1858). GuJang SulHae is a unique full commentary on JiuZhang SuanShu in Chosun. JibGo YunDan is a commentary on JiGu SuanJing based on JieGenFang, a counterpart of Zhang Dun Ren's JiGu SuanJing XiCao(1801) based on TianYuanShu. MuIHae is a commentary on the Li Rui's commentary on Li Ye's CeYuan HaiJing and YiGu YanDuan. This shows that Nam and Lee studied for the first time Song-Yuan mathematics around 1850's.

Lee Sang Hyuk was much interested in SiYuan YuJin XiCao, in particular, the theory of finite series in the book along with TianYuanShu through SiYuanShu(四元術). Furthermore, he analyzed thoroughly the theory of equations in Yi Zhi Han's appendices. Since he studied ShuLi JingYun, he fully understood identities and their properties.

In fact, traditional linear equations were implicitly of the form $ax = b$ and systems of linear equations were explicitly represented by the present form of matrices. Moreover, negative numbers were introduced in the chapter of JiuZhang SuanShu, FangCheng(方程). Thus the systems(方程) have been always associated with positive and negative numbers(正負). Since equations in Chinese mathematics were derived from word problems involving ordinary affairs, they should have positive solutions. For such an equation $p(x) = 0$, both positive and negative coefficients of $p(x)$ should be presented. Let $m(x)$ be the polynomial composed of positive terms in $p(x)$ and let $n(x)$ the polynomial made of absolute values of negative terms in $p(x)$, then $p(x) = 0$ is equivalent to $m(x) = n(x)$ as linear equations $ax = b(a, b > 0)$ with positive solutions. This fact was called ZhengFu XiangDang(正負相當). Furthermore, ZengChengKaiFangFa was explained by ZhengFuLun(正負論) in SuanXue QiMeng. It was also called ZhengFuKaiSanChengFangTu(正負開三乘方圖) for an equation of degree 4 in ShuShu JiuZhang. Thus solving polynomial equations is also related to ZhengFuLun. In all, two main frames of theory of equations are both unified by ZhengFuLun.

Lee Sang Hyuk took this aspect for his theory of equations and wrote the first part JungBuRon(정부론, 正負論) of IkSan(익산, 翼算, 1868, [5]). Iksan, together with GuIlJib, is the most important work in the history of Chosun mathematics. He took relevant examples from JiuZhang SuanShu, ShuShu JiuZhang, CeYuan HaiJing, YiGu YanDuan, SuanXue QiMeng, SiYuan YuJian XiCao with appendices by Yi Zhi Han, Mei Wen Ding's FangChengLun(方程論, 1674) and ShaoGuang ShiYi(少廣拾遺, 1692), ShuLi JingYun and SanHak JungEui and then built his theory of equations based on ZhengFuLun. As one can easily deduce, Lee studied almost all Chinese and Chosun works which dealt with theory of equations, whose range was really extensive, from linear equations to equations of any degree including systems of equations of higher degrees. He mixes the traditional and modern style of presentations. Indeed, he gives basic definitions and properties(= theorem) of a subject and then using examples, he explains and deduces its mathematical structures. Sometimes his application of ZhengFuLun is a bit excessive but that is because he tried to keep the consistency of the whole theory. It is strange that he didn't quote any works by Yang Hui and hence didn't use the name ZengChengKaiFangFa. Presumably, XiangJie JiuZhang SuanFa was never brought into Chosun. Since Lee's aim in JungBuRon was to present mathematical structures of theory of equations, he systematized the theory up to Yi Zhi Han and hence most of the results in the book are known. But we should mention that he obtained sufficient conditions for quadratic equations to have FanJi(飜積) and YiJi(益積), also called HuanGu(換骨) and TouTai(投胎), respectively in ShuShu JiuZhang. Following KaiFangFa in Yi Zhi Han's appendix, he explains the process of ZengChengKaiFangFa in which he does not obtain digits of solutions but any approximations of solutions. In due course, there appear synthetic divisions by $x+a(a>0)$. He is the first author to ues them in Chosun. In the end of the book, he appended a short history of theory of equations where he added ShouShiLi and Jiao Xun's KaiFang TongShi(開方通釋, 1801) for the reference together with books mentioned above.

Park Yul and Hong Jung Ha use freely n-dimensional regular polyhedrons for x^n when they deal with equations of higher degrees. Zhu Shi Jie did not use them except for a problem to find $\sqrt[4]{a}$ but for equations of higher degrees, he used conditions given by rational and irrational equations together with finite series. Zhu has extended the theory of finite series(堆垛術) far more than the usual arithmetic series, SanJiaoDuo(三角垛) $\sum_{k=1}^{n}(\sum_{m=1}^{k} m)$ and SiJiaoDuo(四角垛) $\sum_{k=1}^{n} k^2$ and then using this new theory, Zhu obtains equations of higher degrees. We note that finite series are much more tangibly related to ordinary affairs than higher dimensional polyhedrons. But Zhu simply uses finite series to construct equations by TianYuanShu and his presentations are not systematic.

Lee Sang Hyuk observed this and hence wrote ToeTaSeol(퇴타설, 堆垛說) as the second part of IkSan. Since Lee was an astronomer, he studied theory of progressions of differences in ShuShiLi which is related to SanJiaoDuo and its extensions so that he included the theory in his book. He first defines $\sum_{k=1}^{n} k$, discusses its properties, and then applies it to the traditional arithmetic series. He then inductively defines SanJiaoDuo and its natural extensions up to the fifth stage which we call the class of SanJiaoDuo(三角垛系列). Using the class, he added the theory of progressions of differences. As Zhu Shi Jie introduced, Lee studies the series $\sum_{k=1}^{n} ka_k$ for a series $\sum_{k=1}^{n} a_k$, which is called LanFeng(嵐峰) of the given series as in SiYuan YuJin. Lee's most interesting contribution to finite series is JulJuk(절적, 截積) which calculates $\sum_{k=m}^{m+n-1} a_k$ instead of the whole sum(전적, 全積) like $\sum_{k=1}^{n} a_k$. Using m, n, he represents sums of JulJuk. To do so, he introduces a method called BunJukBub(분적법, 分積法) which may be understood as follows:

$$\sum_{k=m}^{m+n-1} k = \sum_{k=1}^{n} k + (m-1)n$$

.

The above is also geometrically explained by the fact that the area of a trapezoid can be calculated by its division into a triangle and parallelogram.

Applying the above processes to SiJiaoDuo, Lee built the class of SiJiaoDuo(四角垛系列) and obtained their sums and sums of JulJuk. He also introduced $\sum_{k=1}^{n} k^3$ as a LanFeng of SiJiaoDuo and extended it to $\sum_{k=1}^{n}(\sum_{m=1}^{k} m^3)$ as before. Both of them were not introduced in SiYuan YuJin. He added a summary which reveals mathematical structures of the series and their relations between various series. Finally he applied his theory to the construction of equations in 12 problems made for himself as those in chapter RuXiang ZhaoShu(如像招數) but his method is much more improved than those in SiYuan YuJin XiCao.

As we have already mentioned, Lee Sang Hyuk studied Hong Jung Ha's GuIlJib. Although the author of SuanXue QiMeng JuHae(산학계몽주해, 算學啓蒙註解) is not known, it includes a part of theory of equations in GuIlJib, and commentaries of SuanXue QiMeng mainly on chapter DuiJi HuanYuanMen(堆積還源門) and chapter KaiFangGeShuMen(開方各術門) in detail. In the former, the author used TianYuanShu as Hong Jung Ha did. In the latter, the author included the Yi Zhi Han's method containing the synthetic division by $x + a(a > 0)$([17]). Although the name of the book is a commentary of SuanSue QiMeng, its main object is theory of equations and it may be actually a study note made by Lee Sang Hyuk. Thus the book ought to have been written before IkSan. After IkSan, there is a study note SanHak SeubYu(산학습유, 算學拾遺, 1869, [1]) written by Cho Hee Sun(조희순, 趙羲純). He made 14 problems for himself in the chapter, SaJi SanRyak(사지산략, 四之算略) and using ErYuanShu, SanYuanShu and SiYuanShu, he constructed systems of equations of higher degrees and solved the systems. Nam Byung Gil wrote a preface for the book and Cho's book is the last significant work on theory of equations in Chosun dynasty.

4 Conclusion and Educational implications

Because of absence of mathematical works in Chosun before the middle of the 17th century, it is impossible to study history of Chosun mathematics during the period except informations related to government affairs, national examinations and educations. For the education of mathematics, basic mathematics involving daily affairs was taught in SuDang(서당, 書堂) which served as present-day elementary school. There was an institute called SanHak(산학, 算學) that trained minor official candidates who would deal with governmental affairs related to mathematics. Since the quota of the institute was restricted by the government law and SuDang was attended only by the upper-class people, neither institute had any significant effect to the general education of mathematics. Indeed, general education system in Chosun was introduced in the very end of the 19th century and implemented in the 20th century. For SanHak and the national examination for mathematicians, there should have been various required textbooks. But according to Kim Si Jin's preface of the republication of SuanXue QiMeng(1660), they were almost reduced to XiangMing SuanFa and the other books like SuanXue QiMeng and Yang Hui SuanFa were almost lost.

However, as soon as SuanXue QiMeng was republished in Chosun, its mathemtics revived greatly. Most of traditional mathematical theories except theory of equations were well compiled in the books mentioned above and they are well quoted and extended in Chosun. Studying only rudimentary results of theory of equations from SuanXue QiMeng and Yang Hui SuanFa, Chosun mathematicians, especially Park Yul and Hong Jung Ha, perfected theory of equations which can be used even today. Chosun mathematics in the latter half of 18th century was strongly influenced by ShuLi JingYun and TianYuanShu ceased almost to exist in Chosun. Beginning at the end of the 18th century, Song-Yuan mathematics thrived again in China. Their mathematical works were brought into Chosun in the middle of the 19th century. Mathematicans like Lee Sang Hyuk and Nam Byung Gil who were already familiar with ShuLi JingYun, could combine Song-Yuan mathematics with the western approach introduced in ShuLi JingYun. Thus they succeeded in revealing a theoretical aspect of equations. Lacking the theory of real numbers let alone that of complex numbers, theory of equations in eastern mathemtics had a critical limitation in itself and hence it could not make any significant contribution to the development of modern algebra.

Although basic number system in mathematics is either the field of real numbers or that of complex numbers, we don't use irrational numbers but their rational approximations in everyday life so that we can be said to live in the field of rational numbers. It is well known that numerical data in computer are all rational approximations. Thus it would be desirable for students to understand its mathematical structures. For elementary operations of polynomials, we only need coefficients with degrees of powers in a polynomial indicated as with digits in the expression of numbers $\sum_{i=0}^{n} a_i 10^i$, which is exactly TianYuanShu. To represent polynomial functions, we need x as $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ but for algebraic nature of polynomials it is enough to represent it by the sequence $a_0, a_1, \ldots, a_{n-1}, a_n$ or a vector $(a_0, a_1, \ldots, a_{n-1}, a_n)$ of $\mathbb{R}^{n+1}$. We note that the operations of the ring of polynomials can be easily taught by those of numbers via sequence expressions and that the structure of vector space of polynomials can be exposed directly by the expressions without x^k's. Thus TianYuanShu should be taught in school mathematics.

ZengChengKaiFangFa was originated by geometrical natures but carried out by synthetic divisions([22]). We recall that its algebraic nature is explained by the following representation of a polynomial $p(x)$:

$$\begin{aligned} p(x) &= b_n(x-\alpha)^n + b_{n-1}(x-\alpha)^{n-1} + \cdots + b_1(x-\alpha) + b_0 \\ &= \left\{ b_n(x-\alpha)^{n-1} + \cdots + b_1 \right\}(x-\alpha) + b_0 \\ &= \left[\left\{ b_n(x-\alpha)^{n-2} + \cdots + b_2 \right\}(x-\alpha) + b_1 \right](x-\alpha) + b_0. \end{aligned}$$

Clearly it involves the remainder theorem and the repeated synthetic divisions and offer excellent motivation for us to teach it in school mathematics. Furthermore, it can be used in the introductory part of differential calculus course. Indeed, if x is sufficiently near α, then every power $(x-\alpha)^k$ can be neglected and hence the polynomial function around $x = \alpha$ can be approximated by the constant function b_0. Furthermore, if powers $(x-\alpha)^k (2 \leq k)$ are neglected, then the function around $x = \alpha$ can be approximated by the linear function $b_1(x-\alpha) + b_0$. These facts also give a good motivation for continuous or differentiable functions together with derivatives and tangent lines, and eventually for analytic functions and their Taylor expansions. We also note that polynomial functions are familiar to secondary school students . Thus ZengChengKaiFangFa is definitely not obsolete but should be treated as an important subject in school mathematics.

Bibliography

[1] 趙羲純, 算學拾遺, 韓國科學技術史資料大系, 數學編 卷八, 驪江出版社, 1985.

[2] 洪正夏, 九一集, 韓國科學技術史資料大系, 數學編 卷二, 驪江出版社, 1985.

[3] 金容雲 編, 韓國科學技術史資料大系, 數學編 卷一 –卷 十, 驪江出版社, 1985.

[4] 慶善徵, 默思集算法, 韓國科學技術史資料大系, 數學編 卷一, 驪江出版社, 1985.

[5] 李尙爀, 翼算, 韓國科學技術史資料大系, 數學編 卷四, 驪江出版社, 1985.

[6] 南秉吉, 算學正義, 韓國科學技術史資料大系, 數學編 卷七, 驪江出版社, 1985.

[7] 朴繘, 算學原本, 國立中央圖書館, 1700.

[8] K. Shen, J. N. Crossley, A. W.-C. Lun, *The Nine Chapters on the mathematical Art*, Oxford Univ. Press, 1999.

[9] 吳文俊 主編, 中國數學史大系, 第一卷 – 第八卷, 副卷, 北京師范大學出版社, 1998.

[10] 中國科學技術典籍通彙, 數學卷, 全五卷, 河南教育出版社, 1993.

[11] 中國歷代算學集成, 上, 中, 下, 山東人民出版社, 1994.

[12] Hong Sung Sa, Finite series in Chusun Dyntasty Mathematics(朝鮮 算學의 堆垜術), The Korean Journal for History of Mathematics(한국수학사학회지) 19(2006), No. 2, 1–24.

[13] Hong Sung Sa, Hong Young Hee, Chosun mathematician Lee Sang Hyuk's theory of equations(朝鮮 算學者 李尙爀의 方程式論), The Korean Journal for History of Mathematics(한국수학사학회지) 17(2004), No. 1, 1–14.

[14] Hong Sung Sa, Hong Young Hee, Mathematics in Chosun Dynasty and SiYuan YuJian(朝鮮 算學과 四元玉鑑), The Korean Journal for History of Mathematics(한국수학사학회지) 20(2007), No. 1, 1–16.

[15] Hong Sung Sa, Hong Young Hee, Nam Byung Gil and his Theory of Equations(南秉吉의 方程式論, The Korean Journal for History of Mathematics(한국수학사학회지) 20(2007), No. 2, 1–18.

[16] Hong Sung Sa, Hong Young Hee, Lee Sang Hyuk's ChaGeunBangMongGu and ShuLi JingYun(李尙爀의 借根方蒙求와 數理精蘊), The Korean Journal for History of Mathematics(한국수학사학회지) 21(2008), No. 4, 11–18.

[17] Hong Sung Sa, Hong Young Hee, Chosun Mathematics Book SuanXue QiMeng JuHae(朝鮮 算書 算學啓蒙註解), The Korean Journal for History of Mathematics(한국수학사학회지) 22(2009), No. 2, 1–12.

[18] Hong Sung Sa, Hong Young Hee, Chang Hye Won, History of FanJi and YiJi(適과 益積의 歷史), The Korean Journal for History of Mathematics(한국수학사학회지) 18(2005), No. 3, 39–54.

[19] Hong Sung Sa, Hong Young Hee, Kim Chang Il, Gou Gu Shu in the 18th century Chosun(18世紀朝鮮의 句股術), The Korean Journal for History of Mathematics(한국수학사학회지) 20(2007), No. 4, 1–22.

[20] Hong Sung Sa, Hong Young Hee, Kim Chang Il, Gou Gu Shu in the 19th century Chosun(19世紀朝鮮의 句股術), The Korean Journal for History of Mathematics(한국수학사학회지) 21(2008), No. 2, 1–18.

[21] Hong Sung Sa, Hong Young Hee, Kim Chang Il, Hong Jung Ha's Number Theory(洪正夏의 數論), The Korean Journal for History of Mathematics(한국수학사학회지) 24(2011), No. 4, 1–6.

[22] Hong Sung Sa, Hong Young Hee, Kim Young Wook, Liu Yi and Hong Jung Ha's Kai Fang Shu(劉益과 洪正夏의 開方術), The Korean Journal for History of Mathematics(한국수학사학회지) 24(2011), No. 1, 1–13.

[23] Hong Young Hee, Mathematics of Chosun Dynasty and ShuLi JingYun(朝鮮 算學과 數理精蘊), The Korean Journal for History of Mathematics(한국수학사학회지) 19(2006), No. 2, 25–46.

[24] Kim Chang Il, Hong Sung Sa, Hong Young Hee, Chosun Mahtematician Hong Jung Ha's Genealogy(朝鮮 算學者 洪正夏의 系譜), The Korean Journal for History of Mathematics(한국수학사학회지), 23(2010), No. 3, 1–20.

[25] Kim Young Wook, Hong Sung Sa, Hong Young Hee, Park Yul's SanHak WonBon(朴繘의 算學原本), The Korean Journal for History of Mathematics(한국수학사학회지), 18(2005), No. 4, 1–16.

[26] 朝鮮王朝實錄, http://sillok.history.go.kr.

CHOSUN MATHEMATICIAN HONG JUNG HA'S LEAST COMMON MULTIPLES
朝鮮 算學者 洪正夏의 最小公倍數

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ABSTRACT

Though greatest common divisors were introduced in the first chapter fang tian(方田) of JiuZhang SuanShu(九章算術), least common multiples were rather neglected in mathematics of eastern Asia. We investigate a method of finding least common multiples that the greatest mathematician Hong Jung Ha(洪正夏, 1684～?) in Chosun dynasty introduced in his book GuIlJib(九一集, 1724). He first noticed that for the greatest common divisor m and the least common multiple n of two natural numbers a, b, $n = a\frac{b}{m} = b\frac{a}{m}$ and $\frac{a}{m}, \frac{b}{m}$ are relatively prime. He then showed that for natural numbers $a_1, a_2, \ldots, a_n$, their greatest common divisor d and least common multiple l, $\frac{a_i}{d}(1 \le i \le n)$ are relatively prime and there are relatively prime numbers $c_i(1 \le i \le n)$ with $l = a_i c_i (1 \le i \le n)$. This is one of the most prominent mathematical results on Number Theory in Chosun dynasty. The purpose of this paper is to show the process for Hong Jung Ha to capture and reveal the mathematical structure related to greatest common divisors and least common multiples. We also discuss Hong Jung Ha's pedagogical attitude.

Keywords: Hong Jung Ha(洪正夏, 1684～?), GuIlJib(九一集, 1724), Number Theory, greatest common divisors, least common multiples.

1 Introduction

JiuZhang SuanShu(九章算術, [4, 5]) established an approach to mathematics in China and eastern countries including Chosun, laying out mathematical subjects and presentations. In the first chapter fang tian(方田), it restricts the study of mathematics to the field $\mathbb{Q}$ of rational numbers rather than to the field of real numbers. The theory is described by relevant word problems that start with jin you(今有), their answers and processes to get them. But one can easily find mathematical structures involved in sequences of related problems. The field $\mathbb{Q}$ of rational numbers is given by fractions, and for reduction of fractions greatest common divisors and the Euclidean algorithm to compute them are introduced in the chapter. But least common multiples are rather neglected in the eastern mathematics except in a few cases, e.g. JiuZhang SuanShu and Zhang Qiu Jian SuanJing(張丘建算經,

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[4, 5]) to get common denominators([3]). Besides these, least common multiples were mostly restricted to relatively prime cases and thus obtained by multiplying the given numbers. General cases were dealt in Yang Hui(楊輝)'s XuGu ZhaiQi SuanFa(續古摘奇算法, 1275, [4, 5]) but Yang Hui didn't notice that for the greatest common divisor m and the least common multiple n of two natural numbers a, b, $ab = mn$ and he obtained common multiples instead of the least one. Eastern mathematicians did not show any interest in factorizations and hence prime factorizations except Yang Hui, who did use factorizations but only for short methods of multiplications and divisions in ChengChu TongBian SuanBao(乘除通變算寶, 1274, [4, 5]). Thus Number Theory of the eastern mathematics came to have a completely different history from the western counterpart.

The purpose of this paper is to show that Hong Jung Ha(홍정하, 洪正夏, 1684 ∼ ?) characterizes least common multiples in his mathematics book GuIlJib(구일집, 九一集, 1724, [2]), making a great contribution to Number Theory among others. Hong Jung Ha was from a Jung In(中人) family, passed the national examination for mathematicians(籌學取才) in 1706 and became Hoe Sa(會士, 從九品) in that year, Hun Do(訓導, 正九品) in 1718 and finally Gyo Su(教授, 從六品) in 1720([9]). He wrote GuIlJib consisting of nine books. The first eight books contain 20 chapters with 493 problems, where a chapter(句股互隱門) in Book 5 deals with the right triangles with 78 problems and the last three books(開方各術門) are devoted to the theory of equations with 166 problems. Thus it may be assumed that his main interest in the book is the theory of equations. He studied Yang Hui SuanFa(楊輝算法 1274-1275, [4, 5]), Zhu Shi Jie(朱世傑)'s SuanXue QiMeng(算學啓蒙, 1299, [4, 5]), An Zhi Zhai(安止齋)'s XiangMing SuanFa(詳明算法, 1373, [4, 5]), Cheng Da Wei(程大位)'s SuanFa TongZong(算法統宗, 1592, [4, 5]) and Chosun mathematician Gyung Sun Jing(경선징, 慶善徵, 1616 ∼ ?)'s MukSaJib SanBup(묵사집산법, 默思集算法, [1]). Gyung is Hong's great grand uncle([9]). Using the TianYuanShu(天元術) in SuanXue QiMeng and the method of solving equations(開方法) in Yang Hui's TianMu BiLei ChengChu JieFa(田畝比類乘除捷法, 1275), Hong perfected the theory of equations. The first eight books were completed before 1713 and the final Book 9(雜錄) was added later in 1724. The final book is more of an appendix in which he discussed rudiments of astronomy, five notes(五音) and twelve pitch-pipes(十二律), and mathematical discussions with He Guo Zhu(何國柱) in 1713. He Guo Zhu was surprised at the excellence of Hong's mathematics, in particular at the TianYuanShu of which he presumably informed Mei Jue Cheng(梅瑴成, 1681 ∼ 1763).

Least common multiples are dealt in the first chapter GuiChun ChaBunMun(귀천차분문, 貴賤差分門) in Book 2. Begining with the least common multiple of two numbers, he arrives at the conclusion that for natural numbers $a_1, a_2, \ldots, a_n$, their greatest common divisor d and least common multiple l, $\frac{a_i}{d}(1 \le i \le n)$ are relatively prime and there are relatively prime numbers $c_i(1 \le i \le n)$ with $l = a_i c_i (1 \le i \le n)$. Hong Jung Ha's result reveals the mathematical structure of greatest common divisors and least common multiples.

2 Hong Jung Ha's least common multiples

We first quote the problem 7 in the chapter GuiChun ChaBunMun in Book 2.

> 今有甲乙二人同起程 只云甲日行八十五里 乙日行六十五里
>
> 問甲乙所行各幾日 以行里適等

答曰 甲一十三日 行一千一百五里, 乙一十七日 行一千一百五里

法曰 置甲日行八十五里 乙日行六十五里 約之得五爲法

另列八十五里以法五除之 得十七日此乙之日也

又列六十五里以法五除之 得十三日此甲之日也

乃列各日以其日行乘之 各得其里數一千一百五里 合問

一法 置甲日行八十五里 乙日行六十五里 乘之得五千五百二十五里爲實

另甲乙日行約之得五爲法 除之亦得各其行一千一百五里

The problem deals with the least common multiple of 85 and 65. Indeed, if two persons, A and B travel daily 85 Li(里) and 65 Li respectively, how many days do A and B take to travel the same Li of distance? First he finds the greatest common divisor 5 of 85 and 65 and by $\frac{85}{5} = 17$, $\frac{65}{5} = 13$ he concludes that A, B take 13 and 17 days respectively to cover the same distance of 1105 Li, i.e., the least common multiple is $85 \times 13 = 65 \times 17 = 1105$. He gives another method to get the least common multiple by $\frac{85 \times 65}{5} = 1105$, which means $5 \times 1105 = 85 \times 65$.

In the above problem, Hong precisely reveals the relation $dl = ab$ for the greatest common divisor d and the least common multiple l of a, b. Further using this in Book 9, he corrects Yang Hui's solution of a problem in XuGu ZhaiQi SuanFa. Indeed, he finds the least common multiple 84 of 12 and 28 and presents a common multiple 168 given by Yang Hui as another solution. He adds another problem to get the least common multiple 420 of 60 and 28 by the same relation in Book 9.

The next problem 8 is as follows:

甲乙丙三人行步不等 甲日行四十五里 乙日行四十里 丙日行三十五里

問幾日 以所行里相等

答曰 甲五十六日 行二千五百二十里 乙六十三日 行二千五百二十里

丙七十二日 行二千五百二十里

法曰列三人日行數互相約之得五 以除四十五里得九爲甲分母

以除四十里得八爲乙分母 以除三十五里得七爲丙分母

乃列於左行 又列各人日行於右行 以左行互乘右行三位 各得二千五百二十里 此相等之數 各以日行除之得各日

The problem 8 is exactly same as the above except that three persons, not two, travel daily 45 Li, 40 Li and 35 Li respectively. As above, he finds the greatest common divisor 5 and divides 45, 40 and 35 by 5 to get 9, 8, 7 which are called each person's denominator. He now applies the method HoSeung(호승, 互乘) to reduce the common denominator and then gets the least common multiple 2520 by $45 \times 8 \times 7 = 40 \times 9 \times 7 = 35 \times 9 \times 8 = 2520$. Further dividing 2520 by daily traveling Li, he obtains the number of days for each person to travel 2520 Li.

The last problem 9 dealing with least common multiple is as follows:

今有欲買牛馬騾驢 而其牛隻價十八兩 馬隻價十二兩 騾隻價九兩

驢隻價六兩 問四色幾隻之價適等

答曰 牛二 馬三 騾四 驢六 各價皆三十六兩

法曰各列四色價互相約之得三爲法 以除各價得牛六馬四騾三驢二

乃列於左行 又列各價於右行 以左行互乘右行四位

各得四百三十二兩爲實 各以其價除之得 牛二十四馬三十六

騾四十八驢七十二此亦等數

又四位互相約之–不能約者置之–得十二乃爲法

以除牛二十四得牛二 以除馬三十六得馬三 以除騾四十八得騾四

以除驢七十二得驢六

以除四百三十二兩得四色等價三十六兩 合問

A person wants to buy cows, horses, mules and donkeys priced at 18, 12, 9 and 6 coins respectively but to pay the same amount of coins for each animal. Thus the problem is about finding the least common multiple of 18, 12, 9 and 6. As in the problem 8, Hong gets 6, 4, 3 and 2 by dividing 18, 12, 9 and 6 by their greatest common divisor 3 and then using HoSeung(互乘) $18 \times (4 \times 3 \times 2) = 12 \times (6 \times 3 \times 2) = 9 \times (6 \times 4 \times 2) = 6 \times (6 \times 4 \times 3) = 432$, he obtains a common multiple 432 of the four numbers. Further he calculates the greatest common divisor 12 of the four numbers $4 \times 3 \times 2$, $6 \times 3 \times 2$, $6 \times 4 \times 2$, $6 \times 4 \times 3$ and gets 36, 2, 3, 4, 6 by dividing 432 and the above four numbers by 12. Clearly 2, 3, 4, 6 are relatively prime and Hong obtains the least common multiple $36 = 18 \times 2 = 12 \times 3 = 9 \times 4 = 6 \times 6$ of 18, 12, 9, 6, where 2, 3, 4, and 6 are the numbers of cows, horses, mules and donkeys respectively. In the process of the division by the greatest common divisor 12, he adds a comment that one doesn't need the process if the numbers are relatively prime as in the previous two problems.

Putting together the process of solving the above three problems, one can conclude the following:
the least common multiple of 85, 65 is given by $85 \times 13 = 65 \times 17 = 1105$,
the least common multiple of 45, 40, 35 is $45 \times 56 = 40 \times 63 = 35 \times 72 = 2520$, and
the least common multiple of 18, 12, 9, 6 is $18 \times 2 = 12 \times 3 = 9 \times 4 = 6 \times 6 = 36$, where the two numbers 13, 17, three numbers 56, 63, 72, and four numbers 2, 3, 4, 6 are all relatively prime.

In all, using the greatest common divisor and HoSeung(互乘), he finds a common multiple of given numbers. He divides the common multiple by the given numbers and then concludes that the common multiple is the least common multiple of given numbers when the quotients are relatively prime as in the first two problems. Otherwise, dividing the common multiple by the greatest common divisor of the quotients, Hong obtains the least common multiple as in the last problem.

Generalizing the above process, we have the following theorem.

Theorem(홍정하(Hong Jung Ha)) Let $a_1, a_2, \ldots, a_n$ be natural numbers and d, l their greatest common divisor and least common multiple respectively. Then one has the following.

i) $\frac{a_1}{d}, \frac{a_2}{d} \cdots \frac{a_n}{d}$ are relatively prime.

ii) There are relatively prime $c_1, c_2, \ldots, c_n$ with $l = a_i c_i (1 \leq i \leq n)$ and the converse also holds.

First we denote the set of natural numbers by $\mathbb{N}$ and define an order $\ll$ on $\mathbb{N}$ as follows: $x \ll y$ iff x is a divisor of y, i.e., $x \mid y$. Then it is obvious that $(\mathbb{N}, \ll)$ is a partially ordered set. It is also well known that for $a, b \in \mathbb{N}$, the greatest common divisor d(the least common multiple l, resp.) is precisely the meet $a \wedge b$(the join $a \vee b$, resp.) in the ordered set $(\mathbb{N}, \ll)$ and hence it is a lattice. We now go back to the proof of the theorem.

Proof i) is trivial by the definition.

Let's prove ii). Suppose that l is the least common multiple of $a_1, a_2, \ldots, a_n$, then there are $c_i(1 \leq i \leq n)$ with $l = a_i c_i (1 \leq i \leq n)$. Let p be the greatest common divisor of $c_1, c_2, \ldots, c_n$, then there are $x_i \in \mathbb{N}$ with $c_i = p x_i (1 \leq i \leq n)$. Thus $\dfrac{l}{p} = a_i x_i (1 \leq i \leq n)$ imply that $\dfrac{l}{p}$ is a common multiple of $a_1, a_2, \ldots, a_n$, so that $p = 1$. In all, $c_1, c_2, \ldots, c_n$ are relatively prime.

For the converse, assume that there are relatively prime $c_1, c_2, \ldots, c_n$ with $l = a_i c_i (1 \leq i \leq n)$. Let m be the least common multiple of $a_1, a_2, \ldots, a_n$. Then by the above proof, there are relatively prime $p_1, p_2, \ldots, p_n$ with $m = a_i p_i (1 \leq i \leq n)$. Since l is a common multiple of $a_1, a_2, \ldots, a_n$, there is q with $l = mq$, which implies $a_i c_i = a_i p_i q$. Thus $c_i = p_i q (1 \leq i \leq n)$ and therefore q is a divisor of relatively prime $c_1, c_2, \ldots, c_n$ and hence $q = 1$. This shows that $l = m$ is the least common multiple of $a_1, a_2, \ldots, a_n$.

3 Conclusion

In many occasions in GuIlJib, we can find that Hong Jung Ha did not simply follow the old methods of other mathematicians but tried to disclose mathematical structures in the problems and then established his own mathematics([6, 7, 8]). For example, in the introductory remark(凡例) of the book, he showed that Jia Xian(賈憲)'s triangle can be obtained by the synthetic division process used in ZengChengKaiFangFa(增乘開方法). He didn't have any information on Yang Hui's comment on the process which was quoted in YongLei DaDian(永樂大典, 1407) and Wu Jing(吳敬)'s JiuZhang SuanFa BiLei DaQuan(九章算法比類大全, 1450) but he did figure out the process for himself. Further, he extended the process to get the binomial coefficients of $a(x+b)^n$ for any a, b and also included the binomial coefficients of $(x-1)^n$ up to $n = 12$. We have not been able to confirm that the latter was obtained by the synthetic division process. But if it is the case, then Hong might be the first eastern mathematician to use the division for negative numbers. It is notable that Hong derived the above results from just a few examples of extractions of the nth roots and Liu Yi(劉益)'s method to solve quadratic equations in those books mentioned in the introduction.

Hong Jung Ha's result on Number Theory discussed in this paper also shows his attitude to mathematics. He could obtain the least common multiple of two numbers a, b by $ab = dl$, where d, l are the greatest common divisor and the least common multiple of a, b respectively. But he further noticed that $l = a\dfrac{b}{d} = b\dfrac{a}{d}$ and that $\dfrac{a}{d}, \dfrac{b}{d}$ are relatively prime and this fact can be extended to the case of any number of natural numbers so that he captures the mathematical structure involving greatest common divisors and least common multiples.

The process to reach the results also indicates Hong's pedagogical attitude. Indeed the first problem deals with a case of two numbers, the second one with the exactly same case with three numbers and the last one with general cases where he notes that the first two problems are simply special cases of the general theory.

REFERENCES

– 慶善徵, 默思集算法, 韓國科學技術史資料大系, 數學編 卷一, 驪江出版社, 1985.

- 洪正夏, 九一集, 韓國科學技術史資料大系, 數學編 卷二, 驪江出版社, 1985.
- K. Shen, J. N. Crossley, A. W.-C. Lun, *The Nine Chapters on the mathematical Art*, Oxford Univ. Press, 1999.
- 中國科學技術典籍通彙, 數學卷, 全五卷, 河南教育出版社, 1993.
- 中國歷代算學集成, 上, 中, 下, 山東人民出版社, 1994.
- Hong Sung Sa, Hong Young Hee, Kim Chang Il, Gou Gu Shu in the 18th century Chosun(18世紀 朝鮮의 句股術), The Korean Journal for History of Mathematics(한국수학사학회지) 20(2007), No. 4, 1–22.
- Hong Sung Sa, Hong Young Hee, Kim Chang Il, Hong Jung Ha's Number Theory(洪正夏의 數論), The Korean Journal for History of Mathematics(한국수학사학회지) 24(2011), No. 4, 1–6.
- Hong Sung Sa, Hong Young Hee, Kim Young Wook, Liu Yi and Hong Jung Ha's Kai Fang Shu(劉益과 洪正夏의 開方術), The Korean Journal for History of Mathematics(한국수학사학회지) 24(2011), No. 1, 1–13.
- Kim Chang Il, Hong Sung Sa, Hong Young Hee, Chosun Mahtematician Hong Jung Ha's Genealogy(朝鮮算學者 洪正夏의 系譜), The Korean Journal for History of Mathematics(한국수학사학회지), 23(2010), No. 3, 1–20.

LIU YI AND HONG JUNG HA'S KAIFANGSHU

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ABSTRACT

In *TianMu BiLei ChengChu JieFa*(田畝比類乘除捷法) of *Yang Hui SuanFa*(楊輝算法), Yang Hui annotated detailed comments on the method to find roots of quadratic equations given by Liu Yi in his *YiGu GenYuan*(議古根源) which gave a great influence on Chosun Mathematics. In this paper, we show that *ZengChengKaiFangFa*(增乘開方法) evolved from a process of binomial expansions of $(y + \alpha)^n$ which is independent from the synthetic divisions.

We also show that, extending the results given by Liu Yi - Yang Hui and those in *SuanXue QiMeng*(算學啓蒙), Chosun mathematican Hong Jung Ha(洪正夏) elucidated perfectly the *ZengChengKaiFangFa* as the present synthetic divisions in his *GuIlJib*(九一集).

Keywords: Solutions of polynomial equations, Liu Yi(劉益), Yang Hui(楊輝), *XiangJie JiuZhang SuanFa*(詳解九章算法), Yang Hui SuanFa(楊輝算法), TianMu BiLei ChengChu JieFa(田畝比類乘除捷法), YongLe DaDian(永樂大典), SuanXue QiMeng(算學啓蒙), ZengChengKaiFangFa(增乘開方法), Hong Jung Ha(洪正夏), GuIlJib(九一集).

0 Introduction

Chinese mathematics which started with *JiuZhang SuanShu*(九章算術) was mathematized in the Liu Hui's *Commentary*(九章算術注). It is well known that this book together with *HaiDao SuanJing*(海島算經) written shortly after laid the foundation of Chinese mathematics. Later Wang Xiao Tong(王孝通) generalized the theory of polynomial equations up to and including 4th degree equations in his book *JiGu SuanJing*(緝古算經), but didn't have much effect on Chinese mathematics until the publication in 1801 of *JiGu SuanJing XiCao*(緝古算經細艸) by Zhang Dun Ren(張敦仁).

However in the Song and Yuan period, the development of *TianYuanShu*(天元術) continued to *SiYuanShu*(四元術) completing the construction method of equations. This with the contemporary development of calculations of the roots in the theory of polynomial equations marks as the greatest accomplishment of Chinese mathematics. But most of the early works containing these researches of the pre-thirteenth century are lost, and the only survivals are in collected forms in the works like, *CeYuan HaiJing*(測圓海鏡, completed in 1248, published in 1282) and *YiGu YanDuan*(益古演段, 1259) by Li Ye(李冶, 1192–1297), *ShuShu JiuZhang*(數書九章, 1247) by Qin Jiu Shao(秦九韶, 1202–1261),

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XiangJie JiuZhang SuanFa(詳解九章算法, 1261)[13] and *Yang Hui SuanFa*(楊輝算法, 1274–1275) by Yang Hui(楊輝), and *SuanXue QiMeng*(算學啓蒙, 1299) and *SiYuan YuJian*(四元玉鑑, 1303) by Zhu Shi Jie(朱世傑).

In Korea, among these surviving works the only books studied were *Yang Hui SuanFa*, *SuanXue QiMeng* and *XiangMing SuanFa*(詳明算法, 1373) by An Zhi Zhai(安止齋). In the 17th century these books also suffered the near loss due to continued foreign intrusions. In 1660 Kim Si Jin(金始振, 1618–1667) reprinted *SuanXue QiMeng* and also found a copy of *Yang Hui SuanFa* so that the study continued. [7] Unlike Ming, Chosun mathematics followed the tradition of Song and Yuan and studied *SuanXue QiMeng* so that the *TianYuanShu* continued in the works like *SanHak WonBon*(算學原本, 1700) by Park Yul(朴繘, 1621–1668) and eventually further improved in the work of Hong Jung Ha(洪正夏, 1684–?)'s work *GuIlJib*(九一集).

The methods of solving polynomial equations are dealt with in the previously mentioned books, *ShuShu JiuZhang*, *XiangJie JiuZhang SuanFa*, *Yang Hui SuanFa* and *SuanXue QiMeng*. Jia Xian(賈憲)'s methods of computing square roots and cube roots, called the ShiSuoKaiFangFa(釋鎖開方法) and the *ZengChengKaiFangFa* are commented in Yang Hui's *XiangJie JiuZhang SuanFa* and thus survives to today. On the other hand the Shao Guang Chapter of the *XiangJie JiuZhang SuanFa* is lost but some of its contents the *ZengChengKaiFangFa*, the Jia Xian's triangle (Chinese equivalent of Pascal's triangle) and a problem calculating the 4th root using the *ZengChengKaiFangFa* which are all explained in the book are commented in *YongLe DaDian*(永樂大典) and the name *DiChengKaiFangFa*(遞乘開方法) is used instead. *SuanXue QiMeng* also contains a calculations of square root and cube root using *ZengChengKaiFangFa* but no explicit names were given to the method. In *ShuShu JiuZhang*, Qin Jiu Shao used *ZengChengKaiFangFa* to solve general polynomial equations. The name he used for the method was *ZhengFuKaiSanChengFangTu*(正負開三乘方圖). On the other hand, in the 2nd volume of the *TianMu BiLei ChengChu JieFa* of *Yang Hui SuanFa*, Yang Hui constructed the *YuanDan*(演段) method for quadratic equations following Liu Yi's now lost *YiGu GenYuan*(議古根源) and commented on it. It starts as follows [15, 14]:

> 中山劉先生序謂 算之術入則諸門出則直田 議古根源故立演段百問
> 蓋欲演算之片段也 知片段則能窮根源 既知根源以於心無懞昧矣
> 今姑摘數問詳註圖章 以明後學其餘自可 引而伸之觸類而長 不待盡術也

It is probable that the forementioned *ShaoGuang* Chapter of the *XiangJie JiuZhang SuanFa* had a diagram explaining the *ZengChengKaiFangFa*. But the earliest surviving diagram for the method is the one for general quadratic equations in *TianMu BiLei ChengChu JieFa*. We do not know whether the words 故立演段百問 mean that *YiGu GenYuan* included the method above. But, what Yang Hui quoted from *ShaoGuang* Chapter tells us that he studied fully the *ZengChengKaiFangFa* of Jia Xian and Liu Yi, and it is also probable that the diagram for the solution was added by Yang Hui. From now on we call the diagram Liu-Yang diagram.

The purpose of this paper is to analyze the Liu-Yang diagram to uncover the process in the evolution of *ZengChengKaiFangFa*, and to study the effect of Yang's method on the mathematics of Chosun.

This paper is split into two sections. In the first section, we explain that the synthetic divisions method of *ZengChengKaiFangFa* actually evolved out of the expansion of the binomials $(y + \alpha)^n$ in the calculations of square roots and cube roots, and we conclude that Liu-Yang diagram was introduced in the process. The only imported mathematical books present in 17th century Korea were

Yang Hui SuanFa, *SuanXue QiMeng*, and Cheng Da Wei's *SuanFa TongCong*(1592). In the second section, we figure out that, from the study of only these three books, Hong Jung Ha developed the *ZengChengKaiFangFa* into the modern form of synthetic divisions and also used the perfect *TianYuan-Shu* in the construction of higher order polynomial equations in his book *GuIlJib*, thereby showing that his mathematics is the best mathematical achievement in the 18th century Asian mathematics.

The sources used in this paper for the Chosun mathematics is *The Collections of the Historical Sources in the Science and Technology of Korea*(韓國科學技術史資料大系). And the sources for the Chinese mathematics is *ZhongGuo KeXue JiShu DianJi TongHui*(中國科學技術典籍通彙) and *ZhongGuo LiDai SuanXue JiCheng*(中國歷代算學集成). The quotes in this paper without references are from one of these references.

1 ZengChengKaiFangFa of Liu Yi and Yang Hui

It is well known that the method of solving polynomial equations in China began with the computations of square roots and cube roots in *JiuZhang SuanShu*. The method computes each digit of the solutions one by one downward. The guess for the highest digit is called the *first quotient*(初商). The rest of the solution is called the *second quotient*(次商). The *JiuZhang SuanShu*'s method is the one for computing the next quotients. This method goes as follows.

If the equation $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ is given, we guess the first quotient as α and if we let the second quotient $y = x - \alpha$, then since $x = y + \alpha$, the equation

$$a_n(y+\alpha)^n + a_{n-1}(y+\alpha)^{n-1} + \cdots + a_1(y+\alpha) + a_0 = 0$$

can be solved to get the second quotient y. The process can be repeated and this is called the method of *JiuZhang* or the *ShiSuoFa*(釋鎖法) of Jia Xian.

Another method is to put $y = x - \alpha$ in the equation. Then we have

$$\begin{aligned} p(x) &= b_n(x-\alpha)^n + b_{n-1}(x-\alpha)^{n-1} + \cdots + b_1(x-\alpha) + b_0 \\ &= \left\{b_n(x-\alpha)^{n-1} + \cdots + b_1\right\}(x-\alpha) + b_0 \\ &= \left[\left\{b_n(x-\alpha)^{n-2} + \cdots + b_2\right\}(x-\alpha) + b_1\right](x-\alpha) + b_0. \end{aligned}$$

Now we divide $p(x)$ by $(x-\alpha)$ in the last expression. Using the synthetic division, we get the remainder b_0, and then we divide the quotient by $(x-\alpha)$ again to get the remainder b_1, and so on. Once done, $b_0, \ldots, b_n$ form the coefficients of the equation for the second quotient y:

$$b_n y^n + b_{n-1}y^{n-1} + \cdots + b_1 y + b_0 = 0.$$

From this equation one guesses the second quotient and then determine the equation for the next quotient and so on. This method is called the *ZengChengKaiFangFa*(增乘開方法). [14, 6] The kind of notations in the explanation above was not available.

In the computations of square roots and cube roots, it is probable that the area of squares or the volume of cubes were used in expanding $(y+\alpha)^n$ and in their generalizations. But the synthetic division method was never conceived until the emergence of *ZengChengKaiFangFa*.

In the *TianYuanShu* expressions, the polynomials could be divided by the powers x^n of the *TianYuan* x, but it was impossible to divide them by $(x-\alpha)$.

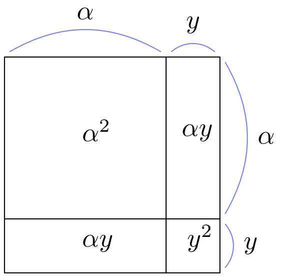


Figure 1

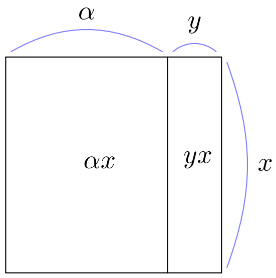


Figure 2

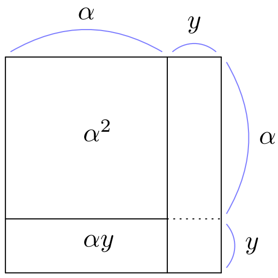


Figure 3

Now we try to show that, in fact, the synthetic division of *ZengChengKaiFangFa* evolved from the Liu-Yang diagram. We begin with the simple equation of $x^2 = A$. This problem is to find the side of a square with its area A.

First we look at the method of *JiuZhang* or *ShiSuoFa*. Using a square diagram as in the Figure 1, from the first quotient α, one can expand the left hand side of the equation $(y+\alpha)^2 = A$ geometrically and get $y^2 + 2\alpha y = A - \alpha^2$, which can be solved for the second quotient y.

Now the process can be divided into smaller steps. In the Figure 2 one divides the width x into two segments as $(y + \alpha)$ and the area is divided as $(y + \alpha)x = yx + \alpha x$. Then as in the Figure 3, the height of the left hand square is divided into $(y+\alpha)$ and then the height of the right hand square into $(y + \alpha)$, which produces

$$yx + (y + \alpha)\alpha = yx + \alpha y + \alpha^2 \tag{1}$$

and then (1) becomes

$$(y + \alpha)y + \alpha y + \alpha^2 = (y^2 + \alpha y) + \alpha y + \alpha^2. \tag{2}$$

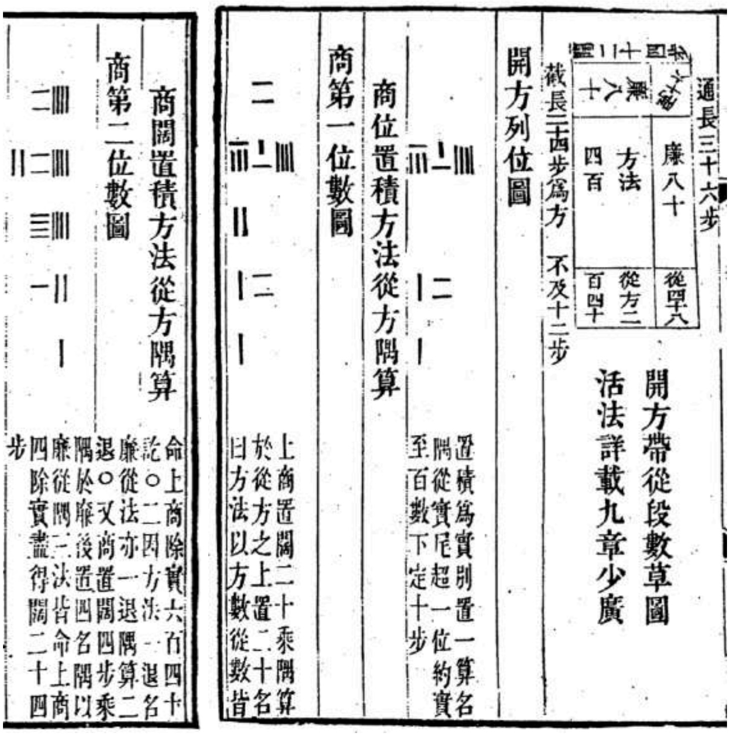
This process can be summarized as the synthetic division (in modern notation) as follows [13]:

	1	0	0	(α
(1)		α	α^2	
	1	α		
		α		
(2)	1	2α		

In fact in all the early works, the *ZengChengKaiFangFa* was used as in the form $x^2 = A$ rather than in the modern form $x^2 - A = 0$ and the positive constant term(實) changes from A to $A - \alpha^2$.

In the case of cubic equation $x^3 = A$, the method of *JuiZhang* or *ShiSuoFa* uses the expansion of $(y+\alpha)^3$ to get the equation $y^3+3\alpha y^2+3\alpha^2 y = A-\alpha^3$ for the second quotient. ([4]) On the other hand the *ZengChengKaiFangFa*, similarly to the quadratic case, is believed to divide the length, width and height (all these edges equal to x) one by one into $y + \alpha$ to get the equation for the second quotient as follows:

$$\begin{aligned} x^3 &= (y+\alpha)x^2 = yx^2 + \alpha x^2 = yx^2 + \alpha(y+\alpha)x \\ &= yx^2 + \alpha yx + \alpha^2 x = yx^2 + \alpha yx + \alpha^2(y+\alpha) \\ &= yx^2 + \alpha yx + \alpha^2 y + \alpha^3. \end{aligned} \tag{1}$$



TianMu BiLei ChengChu JieFa in *Yang Hui SuanFa*

Now this step (1) is continued as follows:

$$\begin{aligned} & y(y+\alpha)x + \alpha y(y+\alpha) + \alpha^2 y + \alpha^3 \\ & \quad = y^2x + \alpha yx + \alpha y^2 + \alpha^2 y + \alpha^2 y + \alpha^3 \\ & \quad = y^2x + \alpha y(y+\alpha) + \alpha y^2 + \alpha y^2 + \alpha^2 y + \alpha^2 y + \alpha^3 \\ & \quad = y^2(y+\alpha) + 2\alpha y^2 + 3\alpha^2 y + \alpha^3 \end{aligned} \tag{2}$$

And then the last step of (2) produces the equation $y^3+3\alpha y^2+3\alpha^2 y+\alpha^3$, which gives us the coefficients of the desired equation and the constant term $A - \alpha^3$. This probably is also the method used in the construction of the triangle of Jia Xian as shown in the *YongLe DaDian*. ([15])

The Liu-Yang diagram which shows the previous argument is contained in the second book of *TianMu BiLei ChengChu JieFa* as the 6th problem with the following explanations:

直田積八百六十四步 只云闊不及長一十二步 問闊幾步 答曰 二十四步

術曰 置積爲實 以不及步爲從方 開平方除之

This problem is to find the shorter side x in the rectangle of area 864 with the difference of the two sides 12, and the answer 24 is given. The method of solution is the division of the given rectangle into a square and another rectangle of which one side is the same as the difference of the two sides of the original rectangle. This way the equation $x^2 + 12x = 864$ was obtained and solved as follows: in the Figure 1, below the rectangle diagram is written "開方帶從段數草圖 活法詳載九章少廣". From this we can tell that Yang Hui had already explained in detail the solution of quadratic equations in the *ShaoGuang* chapter of his book *XiangJie JiuZhang SuanFa*.

The way the rectangle was divided and the computation method above of square root are shown in the *KaiFang LieWeiTo*(開方列位圖). Also the *CongFang*(從方 $= 12$) was added below the diragram. Then the width is divided into the first quotient ($= 20$) and the second quotient. Following the computation of square root, the 20 was put below the constant term and was called the *FangFa*(方法) so that it is distinguishable from the *CongFang*. Now compute $20^2 + 12 \times 20$ and subtract it from the constant term to get the new one ($= 224$) for the second quotient and then double the *FangFa* to get 40.

Now the second quotient is guessed as 4 and then, following the solution method for quadratic equations, *FangFa* becomes 44 and the computation $(44 \times 4) + 12 \times 4 = 224$ shows that the solution is 24.

Problem 7 is to find the longer side in the same problem. The same method as the previous one is presented in the book and was named *YiJiKaiFangShu*(益積開方術) and then another solution which is the same as the *ZengChengKaiFangFa* is given with the following words in the name of *JianCongKaiFangShu*(減從開方術).

> 直田積八百六十四步 只云闊少長一十二步 問長幾步 答曰 三十六步
> (益積開方)術曰 置積爲實 以不及步爲負隅 開平方除之得長

The problem 7 has several errors in its solution. Following the diagram inserted with the solution, the equation $x^2 - 12x = 864$ is obtained from a square whose side coincides with the height of the rectangle of the given problem as follows: Divide the width of this square into the rectangle of the problem and one with the width the same as the difference of the two sides of the rectangle. Considering this, the *FuYu*(負隅) in the solution above should have been written *FuCongFang*(負從方) instead. The following supplement follows the solution.

> 草曰 置積八百六十四於第二給 置差十二步於第四給爲負從
> 置負隅一算於第五給 於第一給上商置長三十步
> 以乘負隅於第三給置方法三十 以上商三十乘負從十二 添積三百六十
> 卻除積九百餘積三百二十四步 二因方法共六十改名
> 廉法一退 負從一退 負隅二退 又於實上商置長六步
> 以乘負隅一置六於廉次名隅以上商六命負從
> 添積七十二共積三百九十六 以廉隅之數命上商除實積盡 得三十六步合問

But in this supplement the *FuYu* also should have been *Yu*(隅). The computation method is exactly the same as the problem 6 except that the *CongFang* term is negative here. Thus the new constant term for the second quotient is obtained by subtracting $30^2+(-12)\times 30$ from the original term 864. The solution considers this computation as $864 - (30^2 - 12 \times 30) = (864 + 12 \times 30) - 900$ and saids "add 360"(添積三百六十). In case of second quotient the only difference is the computation $324 - (66 \times 6 - 12 \times 6) = (324 + 72) - 396$.

The name *YiJiKaiFangShu*(益積開方術) should be interpreted that, since *CongFang* is negative, the term 12×30 should be subtracted in the computation of $30^2 + (-12) \times 30$. In fact, in problem 9, the same area is given with the sum of the two sides (instead of the difference) and it asks to find the width. This time the equation becomes $-x^2 + 60x = 864$ and the solution method is named *YiJiShu*(益積術). Using the same method above, the constant term for the second quotient is computed as $864 - (60 \times 20 - 20^2) = (864 + 20^2) - 60 \times 12$ and the term -20^2 is named *YiJi*(益積). The solution method of the problems 6, 7 and 9 are basically identical except that, in the problems 7 and 9, areas are subtracted. These all seem to be due to the lack of understanding of negative numbers.

Back in the problem 7, once the *YiJiShu* solution is given, another solution *JianCongKaiFangShu* is added as follows:

> 草曰 依五給資次布 置商積方法負從隅算 置積爲實 於實上商置長三十
> 以乘隅算置三十於實數之下 名曰方法 以負從十二減三十餘一十八
> 命上商除實五百四十餘積三百二十四 復以上商三十乘隅得三十
> 併入方法共四十八 退位廉其隅算再退 又於實上商長六步
> 以乘隅算得六 併入廉法共五十四 命上商六步除實盡 得長三十六步合問

This method is the same computation as the *YiJiShu* above except the computation goes through as $30^2-12\times30 = (30-12)\times30$ so that, in the computation of the second quotient, it is replaced by $[\{(30-12)+30\}+6]\times 6$. And this exactly the *ZengChengKaiFangFa*. Also in the problem 9, after *YiYuShu*(益隅術) solution is given, *ZengChengKaiFangFa* solution is added with the name *JianCongKaiFangShu*. The only flaw is that the words "retreat *Lian*(廉) and *Yu*"(退位廉其隅算再退) is unnecessary because they did not use the decimal position method in the computations. The problem 10 is to compute the height in the same problem as 9. This is an *YiYuShu* problem but this name is not mentioned in the book. Instead, since it is *ZengChengKaiFangFa* with *FanJi*(飜積) appearing in its process, the name *FanJiShu*(飜積術) was given. The equation of the problem 12 is $-5x^2 + 228x = 2592$ and the method should have been named *YiYuShu* instead of the name *YiJiShu* given in the book. The method used was *ZengChengKaiFangFa* and this problem is the most general form of quadratic equation whose 2nd degree term(隅) is not 1. The problem 23 is a general form of quartic equation but computation for the second quotient is unnecessary and the *ZengChengKaiFangFa* is used as well. Begin prior to the introduction of *JianCongKaiFangShu*, the solutions of the problems 6, 7 and 9 have identical structures, which utilize diagrams and it is not hard to be recognized as a primitive form of *ZengChengKaiFangFa*.

As was explained above, Yang Hui must have explained in detail the methods of solving polynomial equations in the *ShaoGuang* Chapter of his book *XiangJie JiuZhang SuanFa* but, since the part is lost, we cannot make out the history of the development of the *ZengChengKaiFangFa* exactly. Nevertheless, through the analysis of the problems 6–23 above, we can be sure that the *ZengChengKaiFangFa*, independent of the synthetic division method, was developed in the process of the expansion of the equation $p(y+\alpha) = A$ diagramatically.

2 Kai fang shu of Hong Jung Ha

In this section we investigate the KaiFangShu of Hong Jung Ha, one of the greatest mathematician of ChoSun Dynasty.

As explained above, Pak Yul composed polynomial equations using *TianYuanShu* in his book *San-Hak Wonbon*. But we cannot find the proof that Hong Jung Ha studied the book. *Yang Hui SuanFa*, *SuanXue QiMeng* and *SuanFa TongCong* are the only books Hong is known to have consulted in constructing his theory of equations in writing *GuIlJib*. In the preliminary chapter, he generalized the GaeBangGuRyumYul JakBub BonWonDo(개방구렴율작법본원도, 開方求廉率作法本源圖) of *SuanFa TongCong*, that is the JiaXian's triangle, to the degree 10(구승방, *JiuChengFang*) which is 10th degree. And then he proceeds to show the 12th degree coefficients of $(x - y)^n$ as JiaXian's triangle or GaeBangGuRyumYul JeongBuJiDo(개방구렴율정부지도, 開方求廉率正負之圖).

In *GuIlJib*, representations of polynomial equations using *TianYuanShu* already appear in the Book 4. ([11]) But constructions of equations by setting *TianYuan* appears in the Books 6∼8 GaeBangGakSulMun(개방각술문, 開方各術門). In the previous 3 books are dealt with the problem of 4th roots but Hong dealt with the 5th roots. Moreover *SuanXue QiMeng* dealt with the 4th roots but not as equations. On the other hand Hong used *TianYuanShu* and constructed all problems in the form $x^n - A = 0$. The solution of the problem is also in the Book 4. In fact, in the Book 4 section 2 *GuChukHaeEunMun*(구척해은문, 毬隻解隱門), the 8th problem is the first to use *TianYuanShu* to represent *GaeBangSik*(개방식, 開方式), and gave the solution with the word "using *FanFa*, solve it"(以飜法平方開之). *FanFa* is only possible in *ZengChengKaiFangFa*.

ZengChengKaiFangFa appears first in *GuIlJib* in the Book 5 *GuGoHoEunMun*(구고호은문, 句股互隱門) problem 64. The Book 5 deals with the theory of equations involving the Pythagorean theorem ([11]), and the problem 64 is to solve the equation $0.5625x^2 - 18x - 81 = 0$ for its root 36. In the computation, the decimal places were not used but modern synthetic division was used for the first quotient 30. There the small lettered comments "*CongFang* moves forward"(從方進一位 隅法進二位) and "*CongFang* and *YuFa* retreats"(從方一退 隅法二退) is not appropriate. But in the computation method the name *YiJiFa*(益積法) is mentioned. Here he also mentions that the constant term for the second quotient becomes $-81 + (-33.75) = -114.75$ and that they are all negative, which shows that he understood the *YiJiFa* properly. Therefore Hong's *ZengChengKaiFangFa* is different from Liu-Yang's method and follows closely the modern method of synthetic division. He also mentioned correctly the *BeonGam*(번감, 飜減). ([10]) But he called 114.75 "as the next constant term"(仍爲實) and did not show the sign.

In the problem 65, the same method is used to solve $0.36x^2 - 18x + 81 = 0$ and in this case appears *BeonJeokBeop*(飜積法). Hong used the same method to get the constant term $81 + (-114) = -63$ for the second quotient. Solved problems are all quadratic equations. Equations of degree higher than 2 is also dealt in *GuGoHoEunMun* but no computations were given and only comments like "solve the equation of degree 4"(三乘方開之) and "solve the equation with *FanFa*"(飜法開之) are given.

The solution of these appear for the first time in the Books 6∼8 *GaeBangGakSulMun* of *GuIlJib*. As we have seen he solved the n-th roots for n up to and including 5. In this computation, he uses *ZengChengKaiFangFa* on the equations $x^n - A = 0$, but once using the *ZengChengKaiFangFa* for the first quotient α, he subtracted α^n from A and the went back to the Liu-Yang's method or the method of *SuanXue QiMeng*. Here instead of using *TianYuanShu* as Qin Jiu Shao did, he denoted the coefficients as *GabJong*(갑종, 甲從), *EulJong*(을종, 乙從), etc and then added the products of these coefficients to the lower degree terms. On the other hand, the aforementioned problems 6 and 7 of *TianMu BiLei ChengChu JieFa* are inserted in the *GuIlJib* as the problem 24 of the Book 6. The original solution in the *TianMu BiLei ChengChu JieFa* used the primitive version of *ZengChengKaiFangFa*, but Hong did not mention this and used directly the *ZengChengKaiFangFa*. Then, in place of the original *JianCongKaiFangShu* solution of the problem 7, he used the method in the *GuGoHoEunMun* and the he listed the old method(고법, 古法) and the *YiJiShu* as follows:

益積術曰 置積於第二層爲實 置差於第四層爲從方 置一於第五層爲遇法
以平方益積法開之得長 依圖布算 (생략) 從方進一位 遇法進二位
置上商三十於第一層 以隅法一與上商三十相呼得三十 乃入於第三層

名曰方法 又以從方十二與上商三十相呼得三百六十 乃加入於實積 此益積法
共得一千二百二十四尺 則以方法三十與上商三十相呼得九百
以減於所得益積九百 餘積三百二十四 又方法倍之得六十
乃方法一退 從方亦一退 隅二退 次商六步於上商三十之次
以隅法一與次商六步相呼得六 加入於第三層方法 共得六十六
又以從方十二與次商六步相呼得七十二 加入於所得餘積三百二十四
共得三百九十六 此亦益積也 則以方法六十六與次商六步相呼得三百九十六
除所得益積三百九十六 恰盡 亦得長

Previously we pointed out errors in the *YiJiShu* solution of the problem 7 of *GuIlJib*. Here Hong Jung Ha said about putting *CongFang* in the 4th floor and the *CongFang* in his *EuiDoPoSan*(依圖布算) is written as positive numbers. His *YiJi* so completely ignores this that he could not show the difference from the *YiJiFa* above very well. Hong used the same term *SangHo*(상호, 相呼) for all the processes of multiplying the quotients and the numerators.

Now the problems 9 and 10 of *TianMu BiLei ChengChu JieFa* is included in the *GuIlJib* as the problem 25, and unlike Yang Hui's equation $-x^2 + 60x = 864$ he set up his equation $x^2 - 60x + 864 = 0$ which solves for the width and the height. Hong used the term *JianCongKaiFangShu* for the *ZengChengKaiFangFa* in *YangHui SuanFa* and probably he termed it after *JianCong*. He gave solutions for the same problem using *ZengChengKaiFangFa* and then with the old method he gave the *YiYuShu* of the problem 9 of *TianMu BiLei ChengChu JieFa* under the name *YiJiShu* as in the problem 24. This solution sets up the equation $-x^2 + 60x = 864$ as Yang Hui did, but he neglects the sign as in the problem 4, which shows his misunderstanding of Yang Hui. ([12])

The following quotation from the problem 48, Book 6 of *GuIlJib* best shows the *ZengChengKaiFangFa* of Hong Jung Ha.

今有直積二千六十五步 只云較乘和得二千二百五十六步 問長平各若干 答曰長五十九步

For this he sets up the equation $x^4 + 2,256x^2 - 4,264,225 = 0$ and then solves as follows:

初商三十於上 以丁從一與初商三十相呼得三十 乃入於丙位名曰丙從
卽與初商三十相呼得九百 加入於乙從共得三千一百五十六
就與初商三十相呼得九萬四千六百八十 乃以於甲位名曰甲從
就與初商三十相呼得二百八十四萬四百 除實二百八十四萬四百
餘實一百四十二萬三千八百二十五步 另以丁從一與初商三十相呼得三十
加入於丙從共得六十 就與初商三十相呼得一千八百
加入於乙從共得四千九百五十六 就與初商三十相呼得一十四萬八千六百八十
加入於甲從共得二十四萬三千三百六十
又以丁從一與初商三十相呼得三十 又加於丙從共得九十
就與初商三十相呼得二千七百 又加於乙從共得七千六百五十六
又以丁從一與初商三十相呼得三十 又加於丙從共得一百二十
乃甲從一退乙從二退丙從三退丁從四退*) 次商五步於初商三十之次
以丁從一與次商五步相呼得五 加入於丙從共得一百二十五
就與次商五步相呼得五 加入於丙從共得一百二十五
就與次商五步相呼得六百二十五 加入於乙從共得八千二百八十一
就與次商五步相呼得四萬一千四百0五 加入於甲從共得二十八萬四千七百六十五

就與次商五步相呼得一百四十二萬三千八百二十五
除實一百四十二萬三千八百二十五步恰盡得平三十五步 以平除積得長五十九步 合問

The change of decimal places at $*)$ is not necessary as was explained before. Except for the neglect of the sign in the remaining constant term, Hong's *ZengChengKaiFangFa* is perfect. Consulting only *YangHui SuanFa*, *SuanXue QiMeng* and even inferior *SuanFa TongCong*, Hong's achievements in reinventing such *KaiFangShu* is the best among 18th century oriental mathematics.[1]

3 Conclusion

Theory of equations in the 12th century Chinese mathematics was completed by the introduction of the *TianYuanShu* and the *ZengChengKaiFangFa*, and diverged away from the theory in the west. Now the contemporary works of that time are lost. The *TianYuanShu* was passed on in its complete form until 14th century, however the *KaiFangShu* was all independently written in *ShuShu JiuZhang*, Yang Hui's books, *SuanXue QiMeng* and even its names were different, so that neither the origin nor the structure was clear. Nevertheless Yang Hui gave Jia Xian's theory in his *XiangJie JiuZhang SuanFa* and Liu Yi's theory in his *TianMu BiLei ChengChu JieFa*, which made us possible to figure out a rough picture of the 12th century Chinese KaiFangShu.

From the diagram of the Liu Yi's solution of general quadratic equations which was in the lost book of *YiGu GenYuan* which is passed on to us by Yang Hui and presumed to be the primitive form of *ZengChengKaiFangFa*, and from the *JianCongKaiFangShu* method which is almost identical to the *ZengChengKaiFangFa*, we confirmed that the *ZengChengKaiFangFa* had emerged from the process of expanding the binomial term $(y+\alpha)^n$ as did the Jia Xian's *ShiSuoFa*. Hence we could establish the process of the evolution of *ZengChengKaiFangFa* independently of the synthetic division method, without the help either of the divisions algorithm of polynomials or of the remainder theorem.

On the other hand the *TianYuanShu* which was lost in the Ming dynasty of China, is well passed on in Chosun and blossomed by Pak Yul and Hong Jung Ha. Also in KaiFangShu, Hong Jung Ha developed the perfect *ZengChengKaiFangFa* knowing merely the primitive form of *ZengChengKaiFangFa* of Liu-Yang. Since the *XiangJie JiuZhang SuanFa* was never introduced in Chosun, Hong did not have a chance to use the term *ZengChengKaiFangFa* but we confirmed that his *KaiFangShu* coincides perfectly with the modern interpretations of *ZengChengKaiFangFa*.

Hong was a very well-known mathematician in the beginning of 18th century Chosun, but his great accomplishments did not have much influence on the Chosun mathematics until it was rediscovered by the 19th century mathematicians Yi Sang Hyuk(李尚爀, 1810～?) and Nam Byung Gil(南秉吉, 1820～1869).

Bibliography

[1] *ZhongGuo KeXue JiShu DianJi TongHui* (中國科學技術典籍通彙) Mathematics Part (5 volumes), 河南教育出版社, 1993.

[1] Korean mathematicians of 17th and 18th century had the only opportunity to consult *YangHui SuanFa*, *SuanXue QiMeng* and *SuanFa TongCong*.

[2] *ZhongGuo LiDai SuanXue JiCheng* (中國歷代算學集成) (3 volumes), 山東人民出版社, 1994.

[3] *The Collections of the Historical Sources in the Science and Technology of Korea* (韓國科學技術史資料大系), Mathematics Part (10 volumes), 驪江出版社, 1985.

[4] J. N. Crossley, A. W.-C. Lun, K. Shen, *The Nine Chapters on the Mathematical Arts*, Oxford University Press, 1999.

[5] S. S. Hong, Y. H. Hong and Y. W. Kim, *SanHak WonBon of Pak Yul* (朴繘의 算學原本), 한국수학사학회지 18(2005), No. 4, 1–16.

[6] S. S. Hong, *The Solutions of Polynomial Equations of Chosun* (朝鮮의 方程式 解法), 대한수학회 소식지, 제131호(2010), 11–14.

[7] S. S. Hong,*The Environment of Chosun Mathematics* (朝鮮 算學의 環境), 대한수학회 소식지, 제125호(2009), 32–35.

[8] S. S. Hong and Y. H. Hong, *SanHak KyeMong JuHae, a Chosun Mathematics Book* (朝鮮 算書 算學啓蒙註解), 한국수학사학회지 22(2009), No. 2, 1–12.

[9] S. S. Hong and Y. H. Hong, *The Theory of Equations of Nam Byung Gil* (南秉吉의 方程式論), 한국수학사학회지 20(2007), No. 2, 1–18.

[10] S. S. Hong, Y. H. Hong, H. W. Chang, *The History of FanJi and YiJi* (飜積과 益積의 歷史), 한국수학사학회지 18(2005), No. 3, 39–54.

[11] S. S. Hong, Y. H. Hong, C. I. Kim, *The 18th Century GouGuShu of Chosun* (18世紀 朝鮮의 句股術), 한국수학사학회지 20(2007), No. 4, 1–22.

[12] Y. H. Hong, *The Theory of Equations of Chosun* (조선시대의 방정식론), 한국수학사학회지 17(2004), No. 4, 1–16.

[13] Y. W. Kim, *ZengChengKaiFangFa*(增乘開方法), preprint, 2009.

[14] B. C. Qian(錢寶琮), 增乘開方法的歷史發展, 宋元數學史論文集, 36–59, 科學出版社, 1985.

[15] W. J. Wu(吳文俊) Ed., 中國數學史大系, 第一卷–第八卷, 副卷, 北京師範大學出版社, 1998.

YANG HUI'S *NAYIN QILI*
楊輝의 納音 起例

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ABSTRACT

The sexagesimal cycle(干支) has been playing a very important role in ordinary human affairs including astrology and almanacs as well as the arts of divination(術數) since the second millennium B.C. Yin-Yang(陰陽) school related the cycle with the sixty four hexagrams, and the system of five notes(五音) and twelve pitch-pipes(十二律). The processes to relate them are called NaJia(納甲) and NaYin(納音) respectively and quoted in Shen Kuo(沈括, 1031-1095)'s MengQi BiTan(夢溪筆談, 1095). Yang Hui(楊輝) mathematicized the latter process in his book Xugu Zhaiqi Suanfa(續古摘奇算法, 1275). In this paper we show that Yang Hui introduced the earliest concept and notion of functions, in particular those with finite domains and codomains and then using the theory of congruences and composite of functions, he represented the NaYin process as a 1-1 onto function on the set of sexagesimal cycle into the set of pairs given by the five notes and twelve pitch-pipes.

Keywords: five notes, twelve pitch-pipes, sexagesimal cycle, NaYin(納音), Shen Kuo(沈括), MengQi BiTan(夢溪筆談, 1095), Yang Hui(楊輝), Xugu Zhaiqi Suanfa(續古摘奇算法, 1275), functions, congruences.

1 Introduction and the traditional NaYin process

The sexagesimal cycle(干支) has been introduced as early as the Shang(商 = 殷) dynasty in the second millennium B.C. for the dating of days. It is made of combinations of ten heavenly stems(天干) and twelve earthly branches(地支). The heavenly stems consist of Jia(甲, 1), Yi(乙, 2), Bing(丙, 3), Ding(丁, 4), Wu(戊, 5), Ji(己, 6), Geng(庚, 7), Xin(辛, 8), Ren(壬, 9), and Gui(癸, 10), and the earthly branches of Zi(子, 1), Chou(丑, 2), Yin(寅, 3), Mao(卯, 4), Chen(辰, 5), Si(巳, 6), Wu(午, 7), Wei(未, 8), Shen(申, 9), You(酉, 10), Xu(戌, 11), and Hai(亥, 12), where numbers indicate the order in the system. The set of heavenly stems(earthly branches, resp.) will be denoted by H(E, resp.) which is equipotent to $\{1, 2, \ldots, 10\}$ ($\{1, 2, \ldots, 12\}$, resp.). Using these notations, one can figure out the hexagesimal cycle as

$$\{(m, n) \in \{1, 2, \ldots, 10\} \times \{1, 2, \ldots, 12\} \mid m \equiv n \mod 2\}.$$

*First Author

Further its elements are called by juxtaposing two elements as JiaZi(甲子), YiChou(乙丑), . . ., GuiHai(癸亥), or (1, 1), (2, 2), ⋯, (10, 10), (1, 11), (2, 12), (3, 1), ⋯, (10, 12) (see Table 1). The set of the sexagesimal cycle will be denoted by S, which is clearly equipotent to $\{1, 2, 3, \ldots, 58, 59, 60\}$.

The sexagesimal cycle has played a very important role in human affairs such as astrology and almanacs, and the arts of divination(術數) up to present days. It is used to represent year(歲次), month(月建), day(日辰) and hour(時) and to relate hexagrams in YiJing(易經) whose process is called NaJia(納甲).

Two basic frames of Chinese scales were introduced in the third century B.C. One is five notes(五音) consisting of Gong(宮), Shang(商), Jue(角), Zhi(徵) and Yu(羽) and the other is twelve pitch-pipes(十二律) which are divided into two parts, namely six Lu(六律) and six Lu(六呂). The former consists of Huang Zhong(黃鐘), Da Cu(大簇), Gu Xian(姑洗), Rui Bin(蕤賓), Yi Ze(夷則), Wu Yi(無射) and the latter of Da Lu(大呂), Jia Zhong(夾鐘), Zhong Lu(仲呂), Lin Zhong(林鐘), Na Lu(南呂), Ying Zhong(應鐘)([2, 6]).

Yin-Yang(陰陽) school was formed in the Former Han dynasty(206 B.C. - 24 A.D.). Yin and Yang are two mutually complementary principles whose ceaseless interplay gives rise to all natural phenomena. Yang and Yin are represented by odd and even numbers respectively. Five Elements(五行) consist of wood(木), fire(火), earth(土), metal(金), water(水) which also correspond to Jupiter, Mars, Earth, Venus, and Mercury respectively and circulate in a fixed circulating order. The Yin-Yang school is based on correlation made between the Five Elements, the four compass points with the center(中央, 東, 西, 南, 北), the four seasons, the five notes, the twelve months, the twelve pitch-pipes and the sexagesimal cycle. Indeed, water, fire, wood, metal and earth were represented by 1(north, winter), 2(south, summer), 3(east, spring), 4(west, autumn) and 5(center, earth) but the fixed circulating order of their force(氣) is given by wood(3), fire(2), earth(5), metal(4) and water(1)([2]).

The five notes Gong(宮), Shang(商), Jue(角), Zhi(徵) and Yu(羽) were corresponded to earth(5), metal(4), wood(3), fire(2) and water(1), respectively. The six Lu(六律) Huang Zhong(黃鐘), Da Cu(大簇), Gu Xian(姑洗), Rui Bin(蕤賓), Yi Ze(夷則), Wu Yi(無射) are conceived of as Yang and hence represented by 1, 3, 5, 7, 9 and 11 respectively and the six Lu(六呂) Da Lu(大呂), Jia Zhong(夾鐘), Zhong Lu(仲呂), Lin Zhong(林鐘), Na Lu(南呂), Ying Zhong(應鐘) as Yin and hence represented by 2, 4, 6, 8, 10 and 12 respectively, which determine the circulatory order of twelve pitch-pipes.

In the following, the sets of five notes and twelve pitch-pipes will be denoted by N and P respectively and then the set $N \times P$ has 60 elements and hence is equipotent to the set S of the sexagesimal cycle. Yin-Yang school made a 1-1 correspondence between two sets which is called NaYin(納音). Shen Kuo(沈括, 1031-1095) quoted the process in his book MengQi BiTan(夢溪筆談, [6]). Since the set E of earthly branches and P of twelve pitch-pipes have twelve elements, one has a 1-1 correspondence which may be recognized by the identity function of $\{1, 2, \ldots, 11, 12\}$, where the elements of E and P are as above represented by the elements of $\{1, 2, \ldots, 11, 12\}$. In other words, Zi $\mapsto$ Huang Zhong, Chou $\mapsto$ Da Lu, Yin $\mapsto$ Da Cu, ⋯, Xu $\mapsto$ Wu Yi, Hai $\mapsto$ Ying Zhong. Thus it remains to construct a function on N to the sexagesimal cycle which leads to a 1-1 correspondence between $N \times P$ and S of the sexagesimal cycle. The Na Yin process was designed by four rules, namely counterclockwise direction(左旋), pairing of yin and yang(娶妻), interval given by eight(隔八生子) and jump by three

甲子, 乙丑	丙寅, 丁卯	戊辰, 己巳	庚午, 辛未	壬申, 癸酉
商	徵	角	宮	商
甲戌, 乙亥	丙子, 丁丑	戊寅, 己卯	庚辰, 辛巳	壬午, 癸未
徵	羽	宮	商	角
甲申, 乙酉	丙戌, 丁亥	戊子, 己丑	庚寅, 辛卯	壬辰, 癸巳
羽	宮	徵	角	羽
甲午, 乙未	丙申, 丁酉	戊戌, 己亥	庚子, 辛丑	壬寅, 癸卯
商	徵	角	宮	商
甲辰, 乙巳	丙午, 丁未	戊申, 己酉	庚戌, 辛亥	壬子, 癸丑
徵	羽	宮	商	角
甲寅, 乙卯	丙辰, 丁巳	戊午, 己未	庚申, 辛酉	壬戌, 癸亥
羽	宮	徵	角	羽

Na Yin

intervals(遁甲三元). Indeed the law of counterclockwise direction means that the circulating order is given by metal(4, Shang), fire(2, Zhi), wood(3, Jue), water(1, Yu) and earth(5, Gong) contrary to the order 3, 2, 5, 4, 1 of the force. The law of pairing of yin and yang is to associate the same note to each pair of consecutive heavenly stems starting from Jia. The law of interval given by eight is that two elements of the sexagesimal cycle with the difference 8 in their order correspond to the same note. Finally the law of jump by three intervals means that the last two laws are applied only to three intervals and then the next cycle begins anew by the law of counterclockwise direction. Thus JiaZi(甲子), YiChou(乙丑) $\mapsto$ Shang, i.e., (1, 1), (2,2) $\mapsto$ Shang by the first two laws and RenShen(壬申, (9, 9)), GuiYou(癸酉, (10, 10)), GengChen(庚辰, (7, 5)), XinSi(辛巳, (8, 6)) $\mapsto$ Shang as well by the law of interval given by eight. Furthermore, WuZi(戊子, (5, 1)), JiChou(己丑, (6, 2)) $\mapsto$ Zhi by the law of jump and then continue the same process to the rest of the sexagesimal cycle. These processes assign five notes to the half of the sexagesimal cycle. For the remaining half, one applies again the four rules from JiaWu(甲午, (1, 7)) and then one has a function $f : S \to N$. Values of f are indicated in Table 1.

In order to indicate the circulating order in the sexagesimal cycle, we use the quotient functions given by congruences as follows. The set of integers will be denoted by $\mathbb{Z}$. For $a, b, m \in \mathbb{Z}$, we say that a is congruent to b modulo m if m divides $a - b$ and write $a \equiv b \mod m$. It is well known that the relation given by the above relation is a congruence relation on the ring $\mathbb{Z}$ ([1]), the quotient ring is denoted by $\mathbb{Z}/[m]$ and the quotient function by $q : \mathbb{Z} \to \mathbb{Z}/[m]$.

Using these notions, the sexagesimal cycle is equipotent to $\mathbb{Z}/[60]$ so that these sets will be also considered as the set S of the sexagesimal cycle. Similarly the set E of twelve earthly branches and the set P of twelve pitch-pipes are identified with the quotient set $\mathbb{Z}/[12]$ and the set N of the five notes with $\mathbb{Z}/[5]$.

Let $g : \{1, 2, \ldots, 59, 60\} \to \mathbb{Z}/[12]$ denote the restriction of the quotient function $q : \mathbb{Z} \to \mathbb{Z}/[12]$. We now define a function $h = f \sqcap g : \{1, 2, \ldots, 59, 60\} \to N \times P$, i.e., $h(n) = (f(n), g(n))$ for $n \in \{1, 2, \ldots, 59, 60\}$, where the function f is given in the above. One can prove that the function h is indeed 1 - 1 onto and hence gives rise to the NaYin process. The detail of proof can be found in [11].

In all, the NaYin process by YinYang school has been derived by the above four rules. There are numerous explanations for the motivation of the above four rules and they are quite remarkable but not self-evident.

The purpose of this paper is to illustrate that Yang Hui(楊輝) mathematicized the traditional NaYin process in his book Xugu Zhaiqi Suanfa(續古摘奇算法, 1275). To do so, he introduced the earliest concept of functions, in particular those with finite domains and codomains, and that of composite of functions.

2 Yang Hui's NaYin QiLi(納音 起例)

Yang Hui is one of the four great mathematicians in Song(宋, 960-1279) and Yuan(元, 1271-1368) dynasties. The others are Li Ye(李冶, 1192-1279), Qin Jiu Shao(秦九韶, 1202-1261) and Zhu Shi Jie(朱世傑)([5, 7]). Yang Hui wrote XiangJie JiuZhang SuanFa(詳解九章算法, 1261) and Yang Hui SuanFa(楊輝算法, 1274-1275). The latter consists of three books ChengChu TongBian SuanBao(乘除通變算寶, 1274), TianMu BiLei ChengChu JieFa(田畝比類乘除捷法, 1275) and Xugu Zhaiqi Suanfa(續古摘奇算法, 1275)([9, 10]). In these books, Yang Hui included his own mathematical results together with the history of the theory of equations. The theory of equations, in particular the theory of solving polynomial equations formulated in the 11 ~ 12th century, is one of the most important results in the history of Chinese mathematics. But the original books with the results are all lost and hence we can know about their history only through Yang Hui's works. Yang Hui is also an innovator. He didn't just follow traditional mathematics in various subjects but improved them by mathematical structures. It is well known that the diagrams HeTu(河圖) and LuoShu(洛書) played a very important role for Chinese philosophy, in particular in YiJing and Yin-Yang school. Traditionally LuoShu is known to be a magic square of order 3 and HeTu deals with numbers from 1 to 10. The labeling of these diagrams were strongly defended by Shao Yong(邵雍, 1011-1077) and Zhu Xi(朱熹, 1130-1200) who formed a mainstream of philosophy in Song dynasty([8]). Dissenting from the mainstream, Yang Hui labeled both diagrams in Xugu Zhaiqi Suanfa following Liu Mu(劉牧, 1011-1064)'s opinion that HeTu is the magic square. However, in this paper we will follow the traditional labeling, so that LuoShu is a magic square. Illustrating how to construct LuoShu, Yang Hui mathematicized it. Further, he added magic squares of order $n(4 \leq n \leq 10)$ and some variations of magic squares. His magic square of order 10 is not exact and corrected by Chosun mathematician Hong Jung Ha(洪正夏, 1684-?) in his GuIlJib(九一集, 1724, [4]).

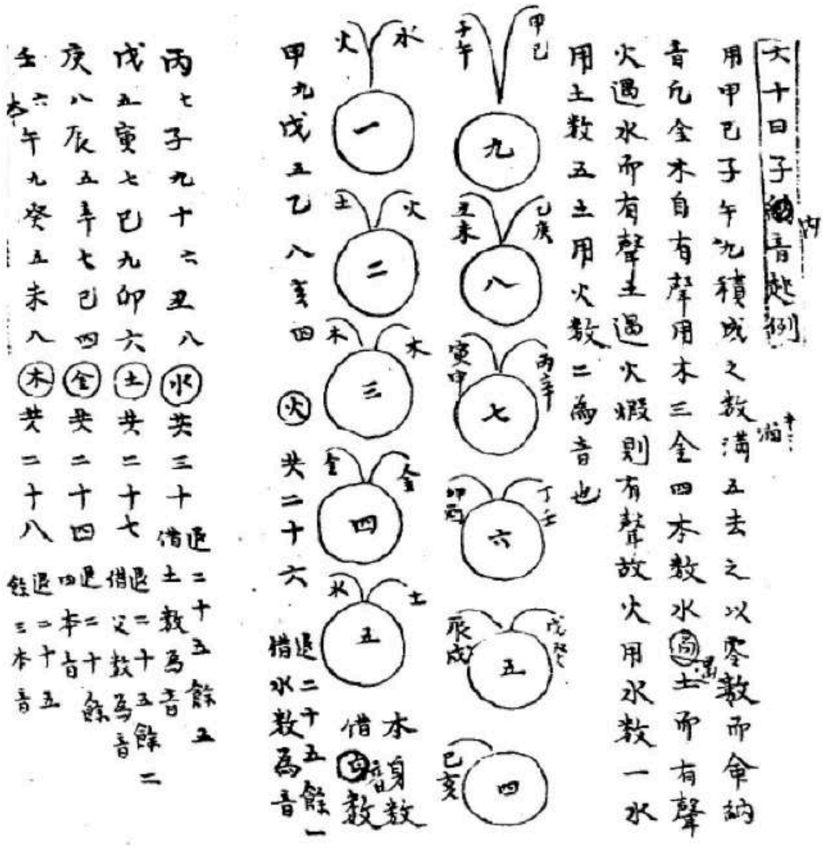
In this section, we investigate Yang Hui's mathematicized NaYin process. For his theory, we use the following notions: The set H of heavenly stems means $\mathbb{Z}/[10]$, the set E of earthly branches $\mathbb{Z}/[12]$ and the equivalence class $[n]$ of n is simply denoted by n. Thus $\mathbb{Z}/[10] = \{1, 2, \ldots, 9, 10\}$ and $\mathbb{Z}/[12] = \{1, 2, \ldots, 11, 12\}$. But operations in these sets mean those in the quotient rings. For example, for $5, 8$, $5 + 8 = 3$ when they are elements of H, but $5 + 8 = 1$ for elements of E. The set S of sexagesimal cycle means the subset $\{(m, n) \mid 1 \leq m \leq 10, 1 \leq n \leq 12, m \equiv n \mod 2\}$ of the product set $H \times E = \mathbb{Z}/[10] \times \mathbb{Z}/[12]$.

Yang Hui defines a function $c : \{1, 2, \ldots, 10\} \to \{9, 8, 7, 6, 5\}$ for heavenly stems as follows:

$$c(1) = c(6) = 9, \quad c(2) = c(7) = 8, \quad c(3) = c(8) = 7,$$
$$c(4) = c(9) = 6, \quad c(5) = c(10) = 5. \quad \text{(See Figure 1)}$$

Proposition A. The function c for heavenly stems satisfies the following.

i) If $m \equiv n \mod 5$, $c(n) = c(m)$.



NaYin

ii) $c(n) + n \equiv 0 \mod 5$.

iii) $c(n+8) - c(n) \equiv -3 \equiv 2 \mod 5$.

The statements i) and ii) are immediate from the definiton of c. Since $c(n) \equiv -n \mod 5$ by ii), one has iii), for

$$c(n+8) - c(n) \equiv c(n+3) - c(n) \equiv -n - 3 - (-n) \equiv -3 \mod 5.$$

Yang Hui also introduced a function $e : \{1, 2, \dots, 12\} \to \{9, 8, 7, 6, 5, 4\}$ for earthly branches as follows:

$$e(1) = e(7) = 9, \quad e(2) = e(8) = 8, \quad e(3) = e(9) = 7,$$
$$e(4) = e(10) = 6, \quad e(5) = e(11) = 5, \quad e(6) = e(12) = 4. \quad \text{(see Figure 1)}$$

Proposition B. The function e for earthly branches has the following properties:

i) If $m \equiv n \mod 6$, $e(n) = e(m)$.

ii) $e(n) + n \equiv 10 \equiv 4 \equiv -2 \mod 6$.

iii) $e(n+8) - e(n) \equiv -2 \equiv 4 \mod 6$.

The statements i), ii) are clear by the definition of e. Since $e(n) \equiv -n + 4 \mod 6$, one has

$$e(n+8) - e(n) \equiv e(n+2) - e(n) \equiv -n - 2 + 4 - (-n+4) \equiv -2 \mod 6.$$

We note that the above two functions have finite domains and codomains and that Yang Hui's concept of such functions are the earliest one in the history of mathemtics. He also introduced a very much descriptive method to represent the functions as in Figure 1, which may be used even now.

The function c for heavenly stems relates to a doctrine of YinYang school and extending this, Yang Hui defines mathematically the function e for earthly branches.

We now discuss values of functions c, e according to the two laws of interval given by eight and jump by three intervals and show that Yang Hui's functions reveal mathematical structures of two laws. Beginning at JiaZi(甲子), we calculate the values of the function c for heavenly stems and e for earthly branches by the above Proposition A, B which are as follows:

$$c(n)\ :\ 9,6,8;\quad 5,7,9;\quad 6,8,5;\quad 7,9,6;\quad 8,5,7.$$
$$e(n)\ :\ 9,7,5;\quad 9,7,5;\quad 9,7,5;\quad 9,7,5;\quad 9,7,5.$$

Similarly, beginning at YiChou(乙丑), one has the values of c, e for the law of pairing of yin and yang as follows:

$$c(n)\ :\ 8,5,7;\quad 9,6,8;\quad 5,7,9;\quad 6,8,5;\quad 7,9,6.$$
$$e(n)\ :\ 8,6,4;\quad 8,6,4;\quad 8,6,4;\quad 8,6,4;\quad 8,6,4.$$

We recall that the NaYin process in the previous section starts at JiaZi($(1,1)$) and JiaWu($(1,7)$). Since their heavenly stems are both 1, the values of c in the above table do not change when we begin with JiaWu instead of JiaZi. Furthermore noting that $1 \equiv 7 \mod 6$ for Zi(1) and Wu(7), the values of e in the above table are those of e when we begin at JiaWu.

Similarly, starting at the YiWei instead of YiChou, one has the exactly same table as above for values of c, e.

We note that the NaYin process is determined by a function on S onto the set $\mathbb{Z}/[5]$ of the five notes and hence the values of the function are devised modulo 5. Contrary to the fact that the values of c follow modulo 5 by Proposition A, the values of e follow modulo 6 by Proposition B. Thus the values of e in one cycle determined by the law of jump by three intervals, decrease by 2 but the next cycle begins increased by 4. Although $-2 \equiv 4 \mod 6$, they are not congruent modulo 5.

Yang Hui noted that the values of c increase by 2 and those of e decrease by 2 *in one cycle* and hence the sum of values of two functions c, e are fixed *in one cycle*. Indeed, the sums of values of two functions are 3, 4, 5, 1, 2 in the five cycles beginning at JiaZi. Similarly, the sums of values of two functions are 1, 2, 3, 4, 5 in the five cycles beginning at YiChou which is the pair of JiaZi. In all, in each cycle of the five cycles, the sum of sums of values of two functions at the pairs by the law of pairing of yin and yang is still fixed and the sums are 4, 1, 3, 5, 2 modulo 5 in the five cycles. We denote five cycles by I(JiaZi, RenShen, GengChen), II(WuZi, BingShen, JiaChen), III(RenZi, GengShen, WuChen), IV(BingZi, JiaShen, RenChen), V(GengZi, WuShen, BingChen) and the cycles determined by the pairs of the above cycles by I′, II′, III′, IV′, V′. Using these notions and collecting the above facts, we now have the following theorem.

Theorem. The functions c, e satisfy the following.

i) For the cycles I, II, III, IV, V determined by the laws of the interval given by eight(隔八生子) and jump by three intervals(遁甲三元), $c(m) + e(n)$ is fixed modulo 5 for any member (m, n) in the same cycle of the five cycles. Furthermore, the sums are 3, 4, 5, 1, 2 for the cycles I, II, III, IV, V, respectively.

ii) For the cycles I′, II′, III′, IV′, V′, $c(m) + e(n)$ is fixed modulo 5 for any member (m, n) in the same cycle of the five cycles. Furthermore, the sums are 1, 2, 3, 4, 5 for the cycles I′, II′, III′, IV′, V′, respectively.

iii) For the cycles I, II, III, IV, V, $\{(c(m) + e(n)) + (c(m + 1) + e(n + 1))\}$ is fixed modulo 5 for any member (m, n) in the same cycle of the five cycles and its pair $(m + 1, n + 1)$. The sums are 4, 1, 3, 5, 2 for the cycles I, II, III, IV, V, respectively.

iv) The above i), ii) and iii) hold for the five cycles determined by the two laws beginning at JiaWu(甲午).

Using the above theorem, Yang Hui has obtained the NaYin process. Indeed take any pair (m, n) which appears at an odd place in the ordered set S of sexagesimal cycles endowed with the usual order and its next pair $(m + 1, n + 1)$ and then calculate the sum

$$(c(m) + e(n)) + (c(m + 1) + e(n + 1)).$$

He then has a function $s : S \to N$ which corresponds one of the five notes Gong(宫, 5), Shang(商, 4), Jue(角, 3), Zhi(徵, 2) and Yu(羽, 1) by the sum to (m, n) and $(m + 1, n + 1)$ and then a 1-1 correspondence $y = s \sqcap g : S \to N \times P$, where g is the function introduced in the previous section. One can easily figure out $g(m, n) =$ the nth pitch-pipe. The detail of the proof for y being 1-1 and onto can be found in [11]. In all y is a new NaYin process which Yang Hui called a temporary process(借音).

For the traditional NaYin process, Yang Hui introduced a function $t : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ as follows:

$$t(1) = 2,\ t(2) = 5,\ t(3) = 3,\ t(4) = 4,\ t(5) = 1.$$

For a motivation of the function t, Yang Hui stated the following:

"金木自有聲 用木三金四本數 水遇土而有聲 火遇水而有聲
土遇火煆則有聲 故火用水數一 水用土數五 土用火數二"

We recall that one has a 1-1 correspondence between the five elements and the five notes by water(1, Yu), fire(2, Zhi), wood(3, Jue), metal(4, Shang), earth(5, Gong). Yang Hui's comment states that the permutation t of $\{1, 2, 3, 4, 5\}$ is determined by properties of sounds of the five elements.

Yang Hui shows that the function $f : S \to N$ for the traditional NaYin process is precisely $t \circ s : S \to N$ and hence he can have the traditional NaYin process h.

The tradtitional NaYin process is constructed by the order structure of the sexagesimal cycle but Yang Hui's process by the algebraic structure of the pairs of heavenly stems and earthly branches. Furthermore, the traditional one needs to be applied twice starting from JiaZi and JiaWu. But Yang Hui's process enables us to directly get the NaYin for any member of the sexagesimal cycle. Indeed for any member of the sexagesimal cycle with the odd earthly branch, one has its NaYin by iii) of the above theorem and the permutation t and for one with the even earthly branch, one has its NaYin by that of the sexagesimal cycle which precedes the given one.

Although Yang Hui uses iii) of the above theorem for his NaYin process, one may also use i) of the theorem. In this case, the permutation t should be replaced by $t(1) = 1$, $t(2) = 5$, $t(3) = 4$, $t(4) = 2$, $t(5) = 3$ for the traditional process. The property ii) of the theorem also determines a NaYin process. They provide three different NaYin processes. These new processes might not have been accepted because the traditional one was strongly tied with the doctrine of Yin-Yang school. We note that the new ones are precisely determined by the mathematical structure of the sexagesimal cycle and that the doctrine of Yin-Yang school may contain ambiguous claims.

3 Conclusion

Chinese mathematics was developed by JiuZhang SuanShu(九章算術) as the western mathematics by Euclid's Elements. JiuZhang SuanShu was completed as a mathematics book by Liu Hui's commentary ([3, 9, 10]). Liu Hui's preface for the book begins by the following statement.

> 昔在庖犧氏始畫八卦 以通神明之德 以類萬物之精 作九九之術 以合六爻之變
>
> 暨于黃帝神而化之 引而伸之 于是建歷紀 協律呂 用稽道原
>
> 然後兩儀四象精微之氣可得而效焉 記稱隷首作數 其詳未之聞也
>
> 按周公制禮而有九數 九數之流 則 九章是矣.

In ancient China, Fu Xi(庖犧=伏羲) has introduced the eight trigrams(八卦) and mathematics(九九之術) and extending these, they developed astrology(曆法) and music(律呂) and then Zhou Gong(周公) propagated six ethical codes for the education of young generations. One of those codes is JiuShu(九數) which is an origin of JiuZhang(九章). This claim of Liu Hui's has become the most important motivation for the study of mathematics throughout the history of oriental mathematics.

Yang Hui of Southern Song not only transmitted very important mathematical results obtained in the 11～12th century but also improved the traditional Chinese mathematics including JiuZhang SuanShu based on the mathematics of Song era. His improvement is prominent especially in his Xugu Zhaiqi Suanfa(續古摘奇算法, 1275). He didn't just follow the mainstream advocated by Shao Yong and Zhu Xi and showed that LuoShu, which was believed as one of bases for ShangShuXue(象數學), is simply a magic square of order 3. Further he claimed that the NaYin process arranged by the Yin-Yang school and its structure are simply mathematical consequences. To do so, he introduced a concept of functions including their composites and using this together with the theory of congruences, Yang Hui constructed the NaYin process and revealed the mathematical structure involved in the process.

But his ingenious achievements were neither appreciated nor understood by the next generations so that the earliest concept of functions in the history of mathematics has been completely forgotten in the eastern mathematics.

Bibliography

[1] D. M. Burton, *Elementary Number Theory*, fifth ed. McGraw-Hill, 2002.

[2] Fung Yu-Lan(馮友蘭), *A History of Chinese Philosophy*(中國哲學史), Vol. I, II, tr. D. Bodde, Princeton University Press, 1952.

[3] 郭書春, 九章算術 譯注, 上海古籍出版社, 2007.

[4] 洪正夏, 九一集, 韓國科學技術史資料大系, 數學編 卷2, 驪江出版社, 1985.

[5] 阮元, 疇人傳, 臺灣商務印書館, 1968.

[6] 沈括, 夢溪筆談, 臺灣商務印書館, 1967.

[7] 吳文俊 主編, 中國數學史大系, 第一劵 – 第八卷, 副卷, 北京師范大學出版社, 1998.

[8] 朱熹, 易學啓蒙, 金珍根 옮김, 청계출판사, 2008.

[9] 中國科學技術典籍通彙 數學卷 全五卷, 河南教育出版社, 1993.

[10] 中國歷代算學集成, 上, 中, 下, 山東人民出版社, 1994.

[11] Hong Sung Sa, Hong Young Hee, Lee Seung On, Yang Hui's NaYinFa(楊輝의 納音法), The Korean Journal for History of Mathematics(한국수학사학회지), 24(2011), No. 3, 1–12.

AN EXPERIMENT ON TEACHING THE NORMAL APPROXIMATION TO THE SYMMETRIC BINOMIAL USING DE MOIVRE & NICHOLAS BERNOULLI'S APPROACHES

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ABSTRACT

De Moivre (1730, 1733, 1738, 1756) and N. Bernoulli's (1713) approaches on the approximation of the binomial distribution allows us to identify simple but fundamental conceptual elements that capture well the essential characteristics of the binomial distribution for large number of trials. Using these conceptual elements substantially facilitates the understanding of the binomial distribution and its normal approximation. An experimental teaching work that we have designed and implemented based on De Moivre and Bernoulli's approaches made accessible to our (department of education) students, these nontrivial and complex issues.

1 Introduction

The normal distribution (ND) and the Central Limit Theorem (CLT) are key concepts of Statistics, which present important difficulties for the students' learning. Didactical research points out that usual introductory statistics courses addressed to students considered as potential users of statistics (e.g. students of social sciences, medicine, biology, ...) have very poor learning outcomes concerning these subjects (Batanero et al. 2004, Chance et al. 2004, Mathews & Clark 1997, Clark et al. 2003, Garfield & Ben-Zvi 2007). For example, Mathews, Clark and their colleagues (Mathews & Clark 1997, Clark et al. 2003, Garfield & Ben-Zvi 2007, p.377) examined students in four tertiary USA institutions, shortly after they had completed their introductory statistics course with *grade A*; the large majority of these students could not understand and compose the basic constitutive elements of the CLT, thus most of the examined students have only fragmentary recall of the CLT and very few have a viable understanding of the theorem.

Lack of understanding and misunderstandings concerning the CLT and the ND are frequent also among students and scientists that have received statistics education beyond an introductory one (e.g. see Cummins 1991, Wilensky 1997, Crack & Ledoit 2010). This often has important negative consequences even in their professional practice (Barbieri et al. 2009, Brockett 1983, Cummins 1991,

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Crack & Ledoit 2010). Concerning these concepts, two important defects of the usual statistics courses addressed to statistics' users are: (a) The empirical work to be done by the students is insufficient and often inadequate; (b) few (if any) elements of explanations and proofs are given on why the CLT holds and why the ND is an adequate model of the relevant real phenomena mentioned in the course, thus seriously limiting students' understanding and giving rise to misunderstandings and misuses (Brockett 1983, Cummins 1991, Crack & Ledoit 2010)[1].

Didactical research has addressed (a) and underlined the importance of overcoming this defect (e.g. see Batanero et al 2004, Chance et al. 2004, Blanco & Ginovart 2010, Lunsford et al. 2006), but has paid little attention concerning (b) (but see Wilensky 1997, 2003 and Crack & Ledoit 2010). This paper aims to contribute to this.

2 The normal approximation to the binomial distribution

The normal approximation to the binomial is the simplest case related to the CLT and thus more adequate to be discussed in an introductory statistics course. Searching in the rich reservoir of the history of statistics, we identified an approach on the subject based on De Moivre's (1730, 1733, 1738, 1756) and Nicholas Bernoulli's (1713) works, which may be used in an introductory statistics course[2]. Our a priori analysis pointed out that this approach offers an important possibility for better understanding the binomial and the ND, and allows to explain why the normal approximation to the binomial holds and thus to understand better the fundamental link between these distributions. Following this analysis we designed and implemented a relevant experimental teaching in an introductory statistics and probability course to 30 students of the Department of Education of the University of Crete (prospective primary school teachers)[3]. Students worked in pairs and performed *guided research work* (cf. Freudenthal 1991, Legrand 1993, Goos 2004, Stonewater 2005) in which emphasis is given in students investigation work. Moreover, students had not only to work out problems given by the teacher, but also to get involved in forming the research questions, and gradually pose their own research questions and problems (closed and open questions, conjectures etc).

Below we present the approach used concerning the normal approximation to the symmetric binomial, as well as, some main results of this teaching.[4]

Prior to the teaching of the normal approximation to the symmetric binomial the teacher had

[1]Moreover, Wilensky examined social and other scientists and found that the lack in their education of explanations and legitimization concerning the use of ND in the modeling of real phenomena created important feelings of confusion, discomfort and insecurity to them concerning the ND and its use in the related modeling (Wilensky 1977 and references therein).

[2]For De Moivre and Bernoulli's works see Montmort 1713 pp. 388–394, De Moivre , 1730, 1738, 1756, Hald 2003, ch16, 17.3, ch24, 2007, ch3, Stigler 1986, ch2.

[3]23 had followed a "science" or "technology" orientation in high school and 7 had followed the "human sciences" orientation.

[4]The presentation is restricted here to the work done on the approximation of the symmetric binomial, on the one hand, because of space limitations and, on the other hand, because almost all main ideas and methods necessary for the normal approximation to the binomial distribution are already introduced and treated in the case of the symmetric binomial (see also section 6). Concerning this, it is interesting to note that most of De Moivre's work on the normal approximation to the binomial distribution concerned the case of the symmetric binomial with even number of trials. In *The Doctrine of Chances* (1756) he devoted only a little more than one page (pp. 249–250) in which he stated the extension of his results for the general case of the binomial, considering that they are an easy extension of the results obtained for the symmetric case (De Moivre, 1756 pp.242-254, Stigler 1986, ch2).

discussed with students the formula of the binomial distribution[5]

$$P(N,k,p) = \frac{N!}{k! \cdot (N-k)!} \cdot p^k \cdot q^{N-k} \tag{1}$$

Additionally, the mode, the expected value, the variance and the standard deviation of the binomial distribution were examined.[6]

Furthermore, examples of applications involving the binomial distribution were discussed (e.g. chance games, newborns sex, success in examinations, simple insurance models).

The interest of studying the binomial distribution was discussed with the students not only in connection with practical applications, but also with reference to the relevant argumentation developed by J. Bernoulli (1713) and De Moivre (1730, 1738, 1756) (Hald 2003 ch15, ch16, ch24, Hald 2007, ch2, ch3, Sheynin 1968, Sheynin 2005, Stigler 1986, ch2). In particular, the importance and the historical difficulty of calculating probabilities of binomial distributions for large number of trials were discussed; in particular, De Moivre's comment (*corollary*) on pp.234-235 of *The Doctrine of Chances*, 1738, (1756 edition, p.242) where he clearly states that this problem is connected with the evaluation of conclusions that may be drawn from empirical evidences (see also Sheynin 1968).

Then, students with teacher's guidance searched empirically how characteristic values of the symmetric binomial, with large even number of trials (N), vary following the change of the number of trials; students used a spreadsheet with incorporated binomial function (Excel) to obtain more easily the needed numerical examples.[7]

- At first, they found that for large, even numbers of trials, the probability of the modal value of the symmetric binomial is approximately proportional to the inverse of the square root of the number of trials[8]

$$P(mN, mN/2, \frac{1}{2}) \approx \frac{1}{\sqrt{m}} P(N, N/2, \frac{1}{2}) \tag{2}$$

- They observed that for large even N, the same inverse proportionality relation holds approximately for the probability of values, deviating the same number of standard deviations from the middle value. Given this and the previous relation, it was derived and then confirmed experimentally that the ratio

$$R(N,a) = \frac{P(N, N/2 + a \cdot \frac{1}{2} \cdot \sqrt{N}, \frac{1}{2})}{P(N, N/2, \frac{1}{2})} \text{ (with } a \cdot \frac{1}{2} \cdot \sqrt{N} \text{ integer)} \tag{3}$$

remains approximately constant when a remains constant; i.e. that it is almost independent of N, for large N. - Moreover, they found that for large even N, the sum of probability of values between the middle value and $a \cdot \frac{1}{2} \cdot \sqrt{N}$ is approximately independent of N. Then, upon teacher's suggestion,

[5] $P(N,k,p)$ is the probability of having exactly k successes in N trials, p been the probability of success in a single trial and $q = 1 - p$. Eq(1) was first derived for the symmetric binomial ($p = q = 1/2$) and then it was generalized.

[6] Measures of central tendency (mode, mean and median) and variation (range, interquartile range, mean absolute deviation, variance and standard deviation) were discussed in the part of the course on descriptive statistics, which preceded the discussion on the binomial.

[7] The binomial with $p = q = 1/2$ and N even, besides of been symmetrical, has one modal value which is equal to its expected value and its median. That the values of its centers are equal, facilitates the investigation of its properties; hence, the teacher suggested to the students that it was a good starting point for their investigations.

[8] In the empirical investigations that led students to this and the two following properties, they used examples where the N varies from 50 to 160000; though interesting, this is not presented here, for brevity.

students examined the symmetric binomial with large odd N and found that the three properties above, with adequate adaptation, hold also in this case.

3 The Ratio

3.1 An important element in De Moivre's work on the binomial distribution is that he considered the ratio of the probability at distance d from the center of the distribution to the probability at the center ($R_1(N,d,p) = \frac{P(N,Np+d,p)}{P(N,Np,p)}$).[9] Using the ratio, he reconsidered the probability at distance d from the center as $P(N, Np+d, p) = R_1(N,d,p) \cdot P(N, Np, p)$[10] and this analysis was of key importance for he finally achieved the normal approximation to the binomial (Hald 2003, ch24).

De Moivre was not the only one who worked on this ratio; Nicholas Bernoulli (1713), Daniel Bernoulli (1770) and later, in different circumstances, Karl Pearson (1895) are among those who did important work on this ratio and the approximation of the binomial distribution (Hald 1984, 2003 ch16, 17.3, ch24, Sheynin 1970, Pearson 1895, Stigler 1985, ch10). Moreover, Laplace followed De Moivre and considered the probability at distance d from the center of the distribution, in a similar way, as the product of the probability at the center and the ratio of these two probabilities, but for a class of distribution much larger than the binomial, and he used this analysis in the work (1810) where he derived his Central Limit Theorem, a theorem that Stigler (1986 pp136,137) qualifies as a major generalization of De Moivre's limit theorem.

The ratio $R_1(N,d,p)$ is a conceptual object of key importance for understanding the binomial distribution and its normal approximation; however it is also a complex object, whose characteristics and behavior is difficult to be understood by the students, especially for large value of N. In this case the study of the history of probability, in particular of relevant works of De Moivre and of Nicholas Bernoulli, was of great help since it permitted us to identify conceptual elements that are simple, or at least accessible to the students, and capture the essential characteristics and the structure of $R_1(N,d,p)$; thus, their didactical use may substantially facilitate its understanding. We did a quite detailed teaching work on $R_1(N,d,p)$ using these simple conceptual elements to achieve students' better understanding of the subject and to obtain elements of answer to relevant questions of our didactical research. We present, in some length, this teaching work in subsections 3.2-3.4. However, it is useful to keep in mind that the aforementioned conceptual elements can be presented to the students in more or less details than ours, according to the teaching approach and depth of examination of the subject sought, which depend on the level of the course, the time available and the mathematical background of the students.

3.2 After the aforementioned empirical investigation, students had found some basic characteristics and properties that the symmetric binomial acquires when N is large. However, students found these properties only through empirical investigation; hence many of them asked for explanations on why these properties hold. The vivid interest of students on this issue created an adequate teaching environment to discuss explanations and proofs of these properties.

[9] In his work, De Moivre assumed explicitly or implicitly, that in the cases of the binomial distribution that he examined, Np is integer; thus the center of the distribution is both the mode and the expected value.

[10] This consideration of $P(N, Np+d, p)$ leads to consider the whole distribution as organized around its center, the ratio $R_1(N,d,p)$ determines the structure of the distribution, while $P(N, Np, p)$ plays the role of a scale parameter.

The teacher proposed them to work first on explaining and proving the second of the three aforementioned properties. He remarked, that in order to do so, they had first to carefully examine the ratio $R_1(2m,d) = \frac{P(2m,m+d,1/2)}{P(2m,m,1/2)}$[11]. During the discussion on this subject, the ratio was presented in three forms:

(a) $R_1(2m,d) = \frac{m!\cdot m!}{(m+d)!\cdot(m-d)!}$

(b) $R_1(2m,d) = \frac{m\cdot(m-1)\cdots(m-(d-1))}{(m+1)\cdot(m+2)\cdots(m+d)}$

(c) $R_1(2m,d) = \frac{1\cdot(1-1/m)\cdots(1-(d-m))/m}{(1+1/m)\cdot(1+2/m)\cdots(1+d/m)}$[12]

Then the teacher said that De Moivre found that, when $2m$ is large and d is small compared to $2m$, the ratio $R_1(2m,d)$ can be approximated by a much simpler fraction in which the sequences of factors involved in $R_1(2m,d)$ are substituted by a few factors raised to adequate powers, and asked students if they could guess such fractions.

Students discussed on this issue and a common idea that emerged from their discussion was that the factors to be used could be middle values of the involved sequences of factors in $R_1(2m,d)$, raised to powers equal to the number of factors in the sequences.

Based on this idea, students worked in pairs to elaborate approximate ratios. They proposed different such ratios and after empirical investigation they found several of them that approximated well $R_1(2m,d)$ for d small compares to $2m$.[13]
Among them are:

$$R_{c_1}(2m,d)\frac{(1-(d/2)/m)^{d-1}}{(1+(d/2)/m)^{d-1}\cdot(1+d/m)} \text{ (proposed by one pair),}$$
$$R_{c_2}(2m,d)\frac{(1-(d-1)/2/m)^{d}}{(1+(d+1)/2/m)^{d}}\text{(proposed by two pairs),}$$
$$R_{b_1}(2m,d)\frac{(m-(d-1)/2)^{d}}{(m+(d+1)/2)^{d}}\text{(proposed by two pairs),}$$
$$R_{a_1}(2m,d)\frac{(m/2+1/2)^{2m}}{(m/2+(d+1)/2)^{m+d}\cdot(m/2-(d-1)/2)^{m-d}}\text{(proposed by one pair)}^{14}.$$

The empirical tests also gave other interesting information. For example, students observed that in all the examined examples $R_{c_1}(2m,d)$, $R_{c_2}(2m,d)$, $R_{b_1}(2m,d)$ were greater or equal to $R_1(2m,d)$, while $R_{a_1}(2m,d)$ was smaller than $R_1(2m,d)$.

Remark 1

De Moivre used the following ratio to approximate $R_1(2m,d)$

$$R_1(2m,d) \approx \frac{(m)^{2m}}{(m+d-1)^{m+d-1/2}\cdot(m-d+1)^{m-d+1/2}\cdot(m+d)/m}$$

[11] $2m$ being the number of trials, m a positive integer and d a positive integer not larger than m; for simplicity $R_1(2m,d,1/2)$ is denoted as $R_1(2m,d)$.

[12] For (b) and (c) obviously the initial factors m, $(m-1)$, $(m+1)$, $(m+2)$, $(1-1/m)$, $(1+1/m)$, $(1+2/m)$ exist for $d \geq 2$; (b) and (c) were presented in this analytical way in order to be better understood by the students. Additionally, the use of concrete examples clarified further the meaning of the three formulas, as well as, the cases when d equals 0, 1 and 2.

[13] Upon teacher's suggestion in the empirical check, students examined values of d up to five standard deviations.

(Stigler 1984 ch2, Hald 2003, ch24)[15]. However, De Moivre was not the first with the idea to approximate $R_1(2m, d)$ with a product of a few factors raised to adequate powers. Nicholas Bernoulli had worked a hypothesis test on Arbuthnott's data concerning newborns' sex ratio and communicated his results to Montmort in a letter that he published in his own book with Bernoulli's permission (Montmort 1713, pp388-394, Hald 1984, 2003 ch16, 17.3). In the context of this work Bernoulli wanted to find a convenient approximation to the ratio

$$\frac{P(N, Np, p)}{P(N, Np-d, p)} = \frac{(Nq+d)(Nq+d-1)\cdots(Nq+1)}{(Np-d+1)(Np-d+2)\cdots Np} \cdot \frac{p^d}{q^d}$$

.[16] He remarked that when d is small compared to Np and Nq, the quantities $f_i = \frac{Nq+d+1-i}{Np-d+i}$ $(1 \leq i \leq d,\ i \in N)$ involved in the ratio closely approximate the terms of a geometrical progression and their logarithms the terms of an arithmetic one. Based on this remark, for such d, Np and Nq, he approximates the ratio with $(\frac{(Nq+d)(Nq+1)}{(Np-d+1)Np} \cdot \frac{p^2}{q^2})^{d/2}$.[17,18].

It is worth noting that De Moivre and Bernoulli's approximating ratios constitute, in exchange with some loss of accuracy, an important simplification of the complex ratio $R_1(2m, d)$ both conceptually and computationally.

That De Moivre and Bernoulli conceived and treated the same basic idea (though using different methods), namely to approximate the ratio $R_1(2m, d)$(or its inverse) by the product of a few factors raised to adequate powers, is already a remarkable fact.

After informing students on this idea and despite their limited mathematical background, they succeeded to find simple such ratios that approximate efficiently $R_1(2m, d)$[19].

These results constitute a strong indication that the exploration of this idea of De Moivre and Bernoulli in the classroom, allows for a conceptually natural and didactically efficient approach to the subject.

3.2.1 Approximately equal factors

Two questions that students posed after their findings in 3.2 were how it can be explained that these ratios approximate $R_1(2m, d)$ and why the observed inequalities hold. The teacher proposed to examine these questions first for $R_{c_2}(2m, d)$.

To this end, he proposed to reconsider $R_1(2m, d)$ in form (c) above, by rearranging the factors of the numerator in increasing order, and get $R_1(2m, d) = \frac{(1-(d-1)/m)\cdots(1(d-i)/m)\cdots 1}{(1+1/m)\cdots(1+i/m)\cdots(1+d/m)}$. The ratios $F_i =$

[15] In *Miscellanea Analytica*(1730), De Moivre worked with the inverse ratio which approximates $1/R_1(2m, d)$; in 1733 he inverted the ratio and derived its approximation to $e^{-d^2/m}$(Hald 2003 ch24).

[16] In this ratio he assumed that Np, Nq are integers.

[17] Hald (2003, pp266,267) remarks that Bernoulli having this approximation, he could very easily find that it converges to $e^{-d^2/(2Npq)}$ if only d is equal $O(\sqrt{N})$ and $N \to \infty$. However, since this was not necessary to his work, he did not investigate the approximation further.

[18] De Moivre's ratio approximates $R_1(2m, d)$ better than Bernoulli's approximating ratio. Nevertheless Bernoulli's approximation is an efficient one that allows for the normal approximation to the binomial (Hald 2003, ch16, 24) Moreover, to obtain his ratio, De Moivre used the polynomial expansion of the logarithm of $R_1(2m, d)$ and (James) Bernoulli's formula for the sum of powers of integers, hence, a more advanced and complex conceptual apparatus than that needed to derive Bernoulli's approximation.

[19] Students approximating ratios are conceptually close to Bernoulli's; both Bernoulli and the students derived their approximating ratios based on compensation reasoning. Of course Bernoulli's rationale is more elaborated than students' simple reasoning. Still, their simple reasoning allowed them to find approximating ratios of about the same accuracy as Bernoulli's (see also §3.2.2).

$\frac{1-(d-i)/m}{1+i/m}$ ($i = 1, 2, \ldots, d$), whose product is $R_1(2m,d)$, also equals $F_i = \frac{m-d+i}{m+i} = 1 - \frac{d}{m+i}$. The teacher remarked that the last form makes obvious that if d, and thus i, is small compared to mF_i increases as i increases, but the change of F_i is small. To better appreciate the variation of F_i, he asked students to calculate $F_{i+1} - F_i$ and $F_d - F_1$. Students found that $F_{i+1} - F_i = \frac{d}{(m+i)\cdot(m+i+1)}$ and $F_d - F_1 = \frac{d\cdot(d-1)}{(m+1)\cdot(m+d)}$. These results, combined with some adequate examples, made more obvious to the students that the change of F_i as i increases, is small compared to the magnitude of F_i, when d is small compared to m.[20]

Next, he explained to the students, using also examples, that if d is of the order of some standard deviations ($d = a \cdot \sqrt{m/2}$) and m is large enough, then the F_i become approximately equal; thus in this case, $R_1(2m,d)$ is the product of d approximately equal fractions. Therefore, it is reasonable that $R_1(2m,d)$ could be approximately equal to one of the middle value fractions among the F_i -or some close value- raised to a power equal to the number of factors that it substitutes. But this was precisely the case of their $R_{c_2}(2m,d)$, $R_{c3}(2m,d)$ and $R_{b_1}(2m,d)$.

A student remarked that if the F_i are approximately equal, not only the aforementioned ratios, but also powers of the other fractions F_i close to the middle could approximate $R_1(2m,d)$, an idea that the teacher confirmed. After this remark, another student asked if even the extremes F_1^d, F_d^d approximate $R_1(2m,d)$. Other students responded that it was not reasonable to use the extreme fractions F_i to approximate $R_1(2m,d)$. However, the teacher remarked that when $2m$ is very large and d is of the order of a few standard deviations, the F_i are so close that even the extremes F_1^d, F_d^d approximate $R_1(2m,d)$, but F_i^d with i having a middle value do this much better. Then he urges students, using spreadsheets, to do some relevant numerical examples in order to acquire some experience on the accuracy of these approximations[21].

3.2.2Factors approximately of geometric progression

Then the teacher remarked that the variation of the fractions F_i although small, has practical importance since, because of it, F_i^d with i having a middle value permits a decent approximation of $R_1(2m,d)$ even for $2m$ not very large, while, if we use the extremes, F_1^d, F_d^d, the approximation is poorer and to achieve equally accurate approximations, much larger value of $2m$ are needed. Thus it is worthy to examine closer the F_i and their variation. After that, he told students that in 1713, Nicholas Bernoulli remarked that the factors involved in $R_1(2m,d)$ were approximately terms of a geometric progression and that this is an important property to understand, because it permits a better understanding of the sequence F_i and of $R_1(2m,d)$.

Then, considering $R_1(2m,d) = \frac{(1-(d-1)/m)\cdots(1-(d-i)/m)\cdot 1}{(1+1/m)\cdots(1+i/m)\cdots(1+d/m)}$ he proposed to the students to examine initially if the factors of the denominator are approximately terms of geometrical progression, by considering the ratio of two successive such factors: $l_{den,i} = \frac{1+(i+1)/m}{1+i/m} = \frac{m+(i+1=)}{m+i}$, which also equals

[20]The teacher also remarked that when d is small compared to m, $F_{i+1} - F_i \approx d/m^2$ and $F_d - F_1 \approx d \cdot (d-1)/m^2$.

[21]For example, for $2m = 10000$ and $d = 100$ they found $R_1(2m,d) = 0,135344\ldots$, $F_1^d = 0,132673\ldots$, $F_d^d = 0,138032\ldots$, $R_{c_2}(2m,d) = 0,13535332\ldots$, $R_{c_1} = 0,13535306\ldots$ Thus, students saw that in such cases F_1^d and F_d^d indeed approximate $R_1(2m,d)$, but their approximating ratios do this much better. The reason for this is that for $2m$ large enough and $d = a \cdot \sqrt{m/2}$, considering a constant, the ratio of the approximation error to the exact value ($Rer = (apr. - exact)/exact$) is approximately proportional to $\sqrt{1/(2m)}$ for F_1^d and F_d^d, while it is approximately proportional $1/(2m)$ for $R_{c_2}(2m,d)$ and $R_{c_3}(2m,d)$). However, they also saw that for $2m = 100$ and $d = 10$, $R_1(2m,d) = 0,136247\ldots$, $F_1^d = 0,112755\ldots$, $F_d^d = 0,161505\ldots$, $R_{c_2}(2m,d) = 0,137146\ldots$, $R_{c_1}(2m,d) = 0,136920\ldots$, so they remarked that in such cases the approximation of F_1^d and F_d^d to $R_1(2m,d)$ becomes very poor, while their approximating ratios still provide acceptable approximations, if the accuracy asked is not too high.

$l_{den,i}1 + \frac{1}{m+i}$.
The teacher remarked that the last form makes obvious that if d, and thus i, is small compared to m: (a) $l_{den,i}$ is close to 1 and (b) $l_{den,i}$ changes a little; more precisely it decreases a little, as i increases.

However for better appreciating the variation of $l_{den,i}$ he asked students to calculate $l_{den,i+1} - l_{den,i}$ and $l_{den,d-1} - l_{den,1}$. Students found that $l_{den,i+1} - l_{den,i} = -\frac{1}{(m+i+1)\cdot(m+i)}$ and $l_{den,d-1} - l_{den,1} = -\frac{d-2}{(m+d-1)\cdot(m+1)}$. These results and some adequate examples made even more obvious to the students that the change of $l_{den,i}$ as i increases is very small compared to the magnitude of $l_{den,i}$, when d is small compared to m.[22] Students did a similar examination for the sequence of factors in the numerator, and then with teacher's help they examined in the same way the ratio $l_{F,i} = \frac{F_{i+1}}{F_i} = \frac{1-(d-i-1)/m}{1+(i+1)/m} \cdot \frac{1+i/m}{1-(d-i)/m} = 1 + \frac{d}{(m+i-d)(m+i+1)}$. This examination work pointed out to the students that also the sequence of factors in the numerator and the sequence of fractions F_i are approximately terms of geometrical progressions. In fact they found that the F_i approximate even better the terms of a geometrical progression than the factors of the denominator or of the numerator, since $l_{F,i}$ change less than $l_{den,i}$ and $l_{num,i}$[23] as i increases[24].

Additionally the teacher remarked that F_i are approximately terms of a geometrical progression with ratio very close to 1 and because of this closeness the F_i are also approximately equal as discussed previously. However, considering that the F_i are approximately terms of such a geometric progression, takes better in to account their small differences than simply considering that they are approximately equal.

Then the teacher discussed with the students that for d terms of a geometrical progression, $a_1, \cdots, a_d$, it holds that $a_1 \cdot a_d = a_2 \cdot a_{d-1} = a_i \cdot a_{d-i+1}$, with $i \in N$, $1 \leq i \leq [d/2]$. He also remarked that for d odd, $a_i \cdot a_{d-i+1} = a^2_{(d+1)/2}$.

Then he told students that, since F_i, whose product constitutes $R_1(2m, d)$, are approximately terms of a geometric progression for $2m$ large and d small compared to $2m$, $F_i \cdot F_{d-i+1}$ should be approximately stable for d fixed. Thus it was interesting to consider $R_1(2m, d)$ as constituted by such couples of factors by rearranging F_i. Students did this and found that for d even, $R_1(2m, d) = (F_1 \cdot F_d) \cdots (F_i \cdot F_{d-i+1}) \cdots (F_{d/2} \cdot F_{d/2+1})$, whereas for d odd (and $d > 1$) $R_1(2m, d) = (F_1 \cdot F_d) \cdots (F_i \cdot F_{d-i+1}) \cdots (F_{(d-1)/2} \cdot F_{(d+3)/2}) \cdot F_{(d+1)/2}$.

The teacher remarked that, following the aforementioned property, for d odd $F_i \cdot F_{d-i+1}$ should be also approximately equal to $F^2_{(d+1)/2}$ and thus $F^d_{(d+1)/2}$ should be approximately equal to $R_1(2m, d)$. But $F^d_{(d+1)/2}$ equals their approximating ratios $R_{c_2}(2m, d)$ and $R_{b_1}(2m, d)$, and this was an additional explanatory element on why these ratios approximate well $R_1(2m, d)$. Moreover, the teacher explained that when d is even, although $F_{(d+1)/2}$ is not among the factors of $R_1(2m, d)$, it is a value between $F_{d/2}$ and $F_{d/2+1}$. Additionally, $F_{d/2}$, $F_{d/2+1}$ are very close when d is small compared to $2m$ and therefore $F_{d/2} \cdot F_{d/2+1} \approx F^2_{(d+1)/2}$. Thus $R_{c_2}(2m, d)$ and $R_{b_1}(2m, d)$ can be used to approximate $R_1(2m, d)$ also in case d is even.

However the discussion on the products of pairs $F_i \cdot F_{d-i+1}$ had an informal character and the closeness of the values of $F_i \cdot F_{d-i+1}$ was not evaluated quantitatively. Thus the teacher proposed to

[22] The teacher also remarked that when d is small compared to m, $l_{den,i+1} - l_{den,i} \approx -1/m^2$ and $l_{den,d-1} - l_{den,1} \approx -(d-2)/m^2$

[23] $l_{num,i}$ been the ratio of two successive factors of the numerator of $R_1(2m, d)$

[24] Indeed, students found that when m is large and d is small compared to m, $l_{F,i+1} - l_{F,i} \approx -2d/m^3$ and $l_{F,d-1} - l_{F,1} \approx -2d \cdot (d-2)/m^3$, which are substantially smaller than the corresponding differences of the $l_{den,i}$ or the $l_{num,i}$.

the students to examine more closely $F_i \cdot F_{d-i+1} = \frac{1-(d-i)/m}{1+i/m} \cdot \frac{1-(i-1)/m}{1+(d-i+1)/m}$. Students with teacher's help transformed this into $F_i \cdot F_{d-i+1} = 1 - \frac{d\cdot(2m+1)}{(m+(d+1)/2)^2-((d+1)/2-i)^2}$ In this form it was easy for the students to see that, for the values of i which are no greater than $(d+1)/2$ (see above), $F_i \cdot F_{d-i+1}$ increases as i increases. This also implies that $F_i \cdot F_{d-i+1} < F^2_{(d+1)/2}$, for $i < (d+1)/2$. Therefore, for $d \geq 2$, $R_1(2m, d) < F^d_{(d+1)/2} = R_{c_2}(2m, d) = R_{b_1}(2m, d)$; this answered the relevant students' question mentioned earlier.

Moreover having found the monotonicity of $F_i \cdot F_{d-i+1}$, students answered easily teacher's question, which is the smallest $F_i \cdot F_{d-i+1}$? From this answer it was easily derived that for $d \geq 2$, $(F_1 \cdot F_d)^{d/2} \leq R_1(2m, d)$[25].

To appreciate the change of $F_i \cdot F_{d-i+1}$ as i increases, students worked in the same way as previously and calculated $D_{i+1,i} = F_{i+1} \cdot F_{d-i} - F_i \cdot F_{d-i+1}$ and $D_{(d+1)/2,1} = F^2_{(d+1)/2} - F_1 \cdot F_d$. They found that for $2m$ large and d small compared to $2m$, $D_{i+1,i}$ approximately equals $\frac{4d(d/2-i)}{m^3}$ and $D_{(d+1)/2,1}$ approximately equals $\frac{d\cdot(d-1)^2}{2\cdot m^3}$. These results, combined with some adequate numerical examples, pointed out to the students that when d is small compared to $2m$, both $D_{i+1,i}$ and $D_{(d+1)/2,1}$ are very small compared to $F_i \cdot F_{d-i+1}$, and in this sense it is reasonable to characterize $F_i \cdot F_{d-i+1}$ involved in $R_1(2m, d)$ as approximately stable.

Then he discussed with students that when $d = a \cdot \sqrt{m/2}$ and m is large enough, the distance between $(F_1 \cdot F_d)^{d/2}$ and $R_{c_2}(2m, d)$ is very small, and, both these (lower and upper) bounds of $R_1(2m, d)$ can be used as its approximations[26].

3.3 The following step was to look for the limit of the upper bound, $F^d_{(d+1)/2} = R_{c_2}(2m, d)$, and the lower bound $(F_1 \cdot F_d)^{d/2}$ (that we call now on $R_{c_5}(2m, d)$) when m tends to infinity and $d = a \cdot \sqrt{m/2}$.[27]

Initially the teacher reminded students that when $n \to +\infty$, $\lim(1 + \frac{1}{n})^n = e$, a property that students had been taught in high school. Then he explained that more generally for any sequence x_n of real numbers such that $x_n \to +\infty$, $\lim(1 + \frac{1}{x_n}^{x_n}) = e$. He also explained that, if $x_n \to +\infty$, $\lim(1 + \frac{c}{x_n})^{x_n} = e^c$ and $\lim(1 + \frac{c+b/x_n}{x_n})^{x_n} = e^c$.

Since $d = a \cdot \sqrt{m/2}$, substituting to the numerator of $R_{c_2}(2m, d)$ this one equals $(1 - \frac{a\sqrt{m/2}-1}{2m})^{a\sqrt{m/2}} = ((1 - \frac{a/4-(1/4)/\sqrt{m/2}}{\sqrt{m/2}})^{\sqrt{m/2}})^a$, so its limit for $m \to +\infty$ equals $(\lim(1 - \frac{a/4-(1/4)/\sqrt{m/2}}{\sqrt{m/2}})^{\sqrt{m/2}})^a$. Putting $x_m = \sqrt{m/2}$ and applying the last of the aforementioned properties, it was obtained that this limit is $e^{-a^2/4}$. Working similarly with the denominator of $R_{c_2}(2m, d)$ it was obtained that its limit is $e^{a^2/4}$, so the limit of $R_{c_2}(2m, d)$ is $e^{-a^2/2}$.

Working similarly with $R_{c_5}(2m, d)$[28] it was obtained that its limit is also $e^{-a^2/2}$.

Then, the teacher explained that since $R_{c_2}(2m, d)$ and $R_{c_5}(2m, d)$ have this same limit, for an ε, even if it is very small, there is N_0, depending on ε, such that for each $N > N_0$ the distances of both,

[25]So, the lower bound of $R_1(2m, d)$ thus derived, is equal to the inverse of the approximating ratio of Nicholas Bernoulli for the case $p = q = 1/2$ (recall that he approximated $1/R_1(N, d, p)$, see "Remark 1" above).

[26]During this discussion $D_{i+1,i}$ and $D_{(d+1)/2,1}$ were compared with $F_{i+1} - F_i$ and $F_d - F_1$, and the comparison pointed out to the students that although the F_i are approximately equal for d small compared to m, $F_i \cdot F_{d-i+1}$ are even less variant. Thus using $(F_i \cdot F_{d-i+1})^{d/2}$ and $F^d_{(d+1)/2} = R_{c_2}(2m, d)$ permits to bound $R_1(2m, d)$ in a much smaller interval than that defined by F^d_1, F^d_d.

[27]The teacher reminded students that $d = a \cdot \sqrt{m/2}$ means that d is a fixed multiple of the standard deviation.He also explained that in order to look for the limits of $R_{c_3}(2m, d)$ and $R_{c_5}(2m, d)$ it was not necessary to be restricted in the cases that d is integer.

[28]Recall that $R_{c_5}(2m, d) = (\frac{1-(d-1)/m}{1+1/m} \cdot \frac{1}{1+d/m})^{d/2}$.

$R_{c_2}(2m, d)$ and $R_{c_5}(2m, d)$ from $e^{-a^2/2}$ are smaller than ε. Therefore, for each value of N greater than N_0 and so that $d = a \cdot \sqrt{m/2}$ is an integer, $R_1(2m, d)$, which is between $R_{c_2}(2m, d)$ and $R_{c_5}(2m, d)$, has a distance less than ε from $e^{-a^2/2}$. So, for all such values of N, $R_1(2m, d)$ closely approximates $e^{-a^2/2}$. The teacher also remarked that since $d = a \cdot \sqrt{m/2}$ we can also say that $R_1(2m, d)$ approximates $e^{-d^2/m}$.

3.4 Students posed some interesting questions whose treatment permitted to elaborate further on the subject:

- Can it be proved that the other approximating ratios they have found (see §3.2) have the same limit when $m \to +\infty$?
  The proofs that worked out in the treatment of this question were similar, or close, to the previous proof and offered students the occasion to better understand the involved proof process.
- In cases $d = a \cdot \sqrt{m/2}$ is not an integer, can we replace it with an integer close to it (e.g. $d = [a \cdot \sqrt{m/2}]$)? And if we do so, does the limit of $R_1(2m, d)$ remain the same? Working on this question with the students it was found that when $m \to +\infty$, the limit of $R_1(2m, [a \cdot \sqrt{m/2}])$ is $e^{-a^2/2}$ as well.
- If we consider d to be constant ($d = c$) instead of being $d = a \cdot \sqrt{m/2}$, what is the limit of $R_1(2m, d)$? If we consider d to be a constant fraction of the number of trials ($d = a \cdot 2m$), what is the limit of $R_1(2m, d)$?

Working on the first question, students found that when $m \to +\infty$ and $d = c$, the limit of $R_1(2m, d)$ is 1. The teacher explained that this also means that if m is large enough and d is constant, then the difference of $P(2m, m, 1/2)$ and $P(2m, m + d, 1/2)$ becomes very small compared to their magnitude.

Working on the second question, students found that when $m \to +\infty$ and $d = a \cdot 2m$, the limit of $R_1(2m, d)$ is 0. They also found that in this case the limit of $e^{-d^2/m} = 0$. However, upon teacher's suggestion, the limit of the ratio $R_1(2m, d)/e^{-d^2/m}$ when $m \to +\infty$ was examined with the students, and it was found that in this case($d = a \cdot 2m$) this limit is 0, while in all the previously examined cases this limit is 1. This result and the associated discussion helped students significantly to understand an aspect of the normal approximation to the binomial which is subject to frequent and important misunderstandings[29].

Remark 2

As already mentioned, the ratio $R_1(N, d)$ is an essential conceptual object for understanding the binomial distribution and the normal approximation to the binomial; however, for the students it is also complex and difficult to understand. In this case the study of the relevant history of probability and statistics is of great help since it permits to identify simple conceptual elements that capture the essential characteristics and the structure of $R_1(N, d)$ and, thus, their didactical use substantially facilitates its understanding. The first such conceptual element is that, for d small compared to N, $R_1(N, d)$ is the product of d approximately equal fractions (F_i). Since the small differences between these fractions still have practical importance (see 3.2.1, 3.2.2), a second conceptual element is needed for a deeper understanding of these differences and of the structure of the ratio, namely that the F_i are approximately terms of a geometric progression, where the ratio of this progression is very close to 1.

[29]On these misunderstandings see e.g. Brockett 1983, Cummins 1991.

Because these two conceptual elements capture the essential characteristics of $R_1(N, d)$, even if they are presented to the students in a less detailed way than the one presented here, still, important results can be easily derived. For example, powers of simple ratios can be found that approximate well $R_1(N, d)$, some of which are lower or upper bounds of $R_1(N, d)$. Moreover, it can be proved that they are such bounds, provided that the monotonicity of the factors of $R_1(N, d)$ is also considered. Then, to obtain the limit values of $R_1(N, d)$, for $N \to +\infty$ and d equal $O(\sqrt{N})$, is matter of a few simple exercises on limit of sequences.

A more detailed examination of these two conceptual elements, as the one presented here, offers further quantitative information on the smallness of the differences of the factors of $R_1(N, d)$ compared to their size and thus leads to a better understanding of them.

Further examination involving these two conceptual elements naturally leads to the quantification of the approximation errors of $R_1(N, d)$ and to the determination of its rate of convergence. These important issues were only touched upon, at the empirical level, in the teaching work examined here, since it was an introductory teaching on the subject. However, the treatment of these issues can be based on, and is a natural prolongation of, the teaching work presented here. In fact, analysis of the ratio $R_1(N, d)$ based on the two aforementioned conceptual elements can be used in different teaching approaches whose depth of examination of the subject depends on the level of the course, on the time available and on the mathematical background of the students.

4 Approximating the middle term of the Symmetric Binomial

De Moivre, in his *Miscellanea analytica* (1730 pp. 173–174) gives a short and elegant proof on the approximation of the middle term of the symmetric binomial, based on Wallis theorem on the approximation of π[30](Hald 2003, ch24).

Of course, the full understanding of this approach requires the understanding of the proof of Wallis theorem, which we considered to be too hard for our students. Nevertheless, using Wallis product, it is simple to prove an essential element of this approximation, namely that for large numbers of trials ($2m$), $P(2m, m, 1/2)$ is approximately inversely proportional to $\sqrt{2m}$ (which is precisely one of the properties that students have found earlier empirically -see section 2).

This proof was discussed with the students in the following way:
Initially the teacher explained that, after adequate simplifications, $P(2m, m, \frac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdot 6 \cdots 2m}$,
so , $P^2(2m, m, \frac{1}{2}) = \frac{1 \cdot 3^2 \cdot 5^2 \cdots (2m-1)^2}{2^2 \cdot 4^2 \cdots (2m)^2}$, which can be also written as $\frac{1}{2} \cdot \frac{3^2}{2 \cdot 4} \cdot \frac{5^2}{4 \cdot 6} \cdot \frac{6^2}{6 \cdot 8} \cdots \frac{(2m-1)^2}{(2m-2) \cdot (2m)} \cdot \frac{1}{2m}$.
Thus $B_m = (2m) \cdot P^2(2m, m, \frac{1}{2}) = \frac{1}{2} \cdot \frac{3^2}{2 \cdot 4} \cdot \frac{5^2}{4 \cdot 6} \cdot \frac{6^2}{6 \cdot 8} \cdots \frac{(2m-1)^2}{(2m-2) \cdot (2m)}$.
Since $\frac{B_{m+1}}{B_m} = \frac{(2m+1)^2}{(2m) \cdot (2m+2)} > 1$, B_m increases when m increase.
We can also write that $B_m = \frac{1 \cdot 3}{2^2} \cdot \frac{3 \cdot 5}{4^2} \cdot \frac{5 \cdot 7}{6^2} \cdots \frac{(2m-3) \cdot (2m-1)}{(2m-2)^2} \cdot \frac{(2m-1)}{(2m)}$, in this form all the involved fractions are smaller than 1, so it is obvious that $B_m < 1$. Since B_m is an increasing and bounded sequence, its

[30]Earlier, in 1721, De Moivre had realized a different, more involved approach on the subject and succeeded to find the approximation sought. However, he determined the constant involved in the approximation only approximately. Around 1725 he informed Stirling on this problem. Stirling's answer resolved this issue and, later, De Moivre (1738, p. 236) remarked that his answer "has spread a singular Elegancy in the Solution" (of the approximation) (Hald 2003, ch24, Stigler 1986 ch2).

limit for $m \to +\infty$ is a constant C. So, when $m \to +\infty$, $\lim \sqrt{B_m} = \lim \sqrt{2m} \cdot P(2m, m, 1/2) = \sqrt{C}$. This means that when m is large $\sqrt{2m} \cdot P(2m, m, 1/2)$ is very close to $\sqrt{C}$ and thus it remains approximately constant. This justifies that for large value of m, $P(2m, m, 1/2)$ is approximately inversely proportional to $\sqrt{2m}$.

At this stage, the demanding problem of determining the value of $\sqrt{C}$ remained; in the context of the aforementioned simple proof, a few further steps on this were discussed with the students: $E_m = \frac{2m}{2m-1} B_m$ is a decreasing sequence and its limit, when $m \to +\infty$, is equal to the limit of B_m, thus $E_m > C > B_m$. So $\sqrt{\frac{(2m)^2}{2m-1}} \cdot P(2m, m, 1/2) > \sqrt{C} > \sqrt{2m} \cdot P(2m, m, 1/2)$. Students based on this result and using the function of the binomial distribution of Excel achieved to find the first four decimal digits of $\sqrt{C}$ $(0, 7978)$.

Then the teacher informed students that Wallis, in the mid-seventeenth century, had found that the limit of $1/E_m$ equals $\pi/2$, so $\sqrt{C} = \sqrt{2/\pi}$. This surprised students, who associated π with the cycle and could not see how it was involved in the examined probability approximation. However this question was not answered in the context of this course[31].

5 Sum of probabilities

Later, the approximation of $\sum P(2m, m+d, 1/2)$, with $d \in N$ and $0 \leq d \leq b \cdot \sqrt{m/2} = D$ was discussed. This sum was transformed in the form $\sum P(2m, m, 1/2) \cdot R_1(2m, d) = P(2m, m, 1/2) \cdot \sum R_1(2m, d)$. Then, using the result of section 3.3 and the graphical representations of the involved sums, the teacher explained that the ratio $\frac{\sum R_1(2m,d)}{\sum e^{-d^2/m}}$ approximates 1 as $m \to +\infty$. Then, using graphical representations, he explained that $\frac{\sum e^{-d^2/m}}{\int_0^D e^{-x^2/m} dx}$ approximates 1 as $m \to +\infty$. Thus $\frac{\sum R_1(2m,d)}{\int_0^D e^{-x^2/m} dx}$ approximates also 1 as $m \to +\infty$. Although simple and understood by the students, this explanation was not rigorous, since for the proof, properties of uniform convergence are needed, otherwise it becomes quite complicated. However this concept was unknown to our students.

Then using the result of section 4 it was derived that $\frac{P(2m,m,1/2) \sum R_1(2m,d)}{P(2m,m,1/2) \int_0^D e^{-x^2/m} dx}$ has the same limit with $\frac{P(2m,m,1/2) \sum R_1(2m,d)}{\frac{2}{\sqrt{2\pi \cdot 2m}} \int_0^D e^{-x^2/m} dx}$, which is 1. This means that for large enough m, $\sum P(2m, m+d, 1/2)$ is approximately equal to $\frac{2}{\sqrt{2\pi \cdot 2m}} \int_0^D e^{-x^2/m} dx$.

6 Subsequent work

Subsequently, the teacher discussed with the students the normal approximation to the binomial distribution for the case of (i) the symmetric binomial with an odd number of trials (ii) the non-symmetric binomial (probability of success $p \neq 1/2$) with Np integer and (iii) the non symmetric binomial with Np non-integer.

Case (i) was a simple extension of what was discussed previously.

For (ii), the approximation of the ratio $R_1(N, d, p)$ was based on simple variations of the ideas and methods already discussed for the case of $R_1(2m, d, 1/2)$. So, the subject was treated as a sequence

[31] Some students expressed a vivid interest on this question, and looked for the proof in mathematics textbooks and the internet. The teacher helped them by explaining the proofs they found, however even with this help most of them found these proofs hard to understand. The issue of a more accessible approach to this subject remains; an interesting question for further didactical investigation.

of exercises, which students formulated, discussing with the teacher and then worked out, with his help whenever needed. This approach permitted them to better assimilate the conceptual elements introduced for the case of $R_1(2m, d, 1/2)$ and extended the relevant methods. For the approximation of $P(N, Np, p)$ Stirling's formula was needed and a relevant proof was discussed with the students. This proof is significantly simpler than those usually presented in mathematics textbooks, since in this proof it is used as a lemma the approximation of $P(2m, m, 1/2)$ discussed in section 4.[32]

Case (iii) was discussed, in a non-rigorous way for reasons of saving teaching time, and it was derived that the normal approximation holds also in this case for N large enough.

Then applications using the normal approximation to the binomial were worked out with the students. Among these applications were problems derived from the problem discussed by Nicholas Bernoulli in his letter to Montmort (see Remark 1 and references there in). However, for these problems actual data, and not Arbuthnott's data, were used.

7 Final Remarks

The normal approximation to the binomial distribution is the easiest case related to the Central Limit Theorem; still it is a complex subject posing important difficulties for the students. In this case the didactically oriented study of the relevant history of probabilities and statistics was of great help since it permitted us to identify simple conceptual elements that capture well essential characteristics of the binomial distribution for large number of trials. The didactical use of these conceptual elements not only facilitates students' understanding of final results of the normal approximation to the binomial, but also makes accessible to them explanations, justifications and even proofs of properties and results concerning this approximation, otherwise difficult within the usual approaches.

Moreover, history offers vivid material on the questions and problems that leaded to the emergence and posing of the problem of the approximation of the binomial distribution, as well as, interesting application problems. The discussion of such material stimulates students' interest and permits the understanding of elements concerning the importance of the subject for scientific and practical life.

REFERENCES

- Barbieri,A., Dubikovsky,V., Gladkevich,A., Goldberg,L.R. et Hayes,M.Y., (2009) "Central Limits and Financial Risk" (available at `http://ssrn.com/`) SSRN-id1404089.pdf
- Batanero,C., Tauber,L.M., Sanchez,V. (2004) "Students' Reasoning about the Normal Distribution", in D.Ben-Zvi & G.Garfield (eds), *The Challenge of Developing Statistical Literacy, Reasoning and Thinking* (pp.257–276), Dordrecht: Kluwer.
- Blanco,M. & Ginovart,M. (2010) "How to introduce historically the normal distribution in engineering education: a classroom experiment", IJMEST 41(1), 19-–30.
- Brockett,P. (1983) "On the Misuse of the Central Limit Theorem in Some Risk Calculations" *The Journal of Risk and Insurance*, 50(4), 727–731
- Chance,B., delMas,R., Garfield,J. (2004) "Reasoning about Sampling Distributions ", in D.Ben-Zvi & G.Garfield (eds), *The Challenge of Developing Statistical Literacy, Reasoning and Thinking* (pp. 295–318), Dordrecht: Kluwer

[32] This proof and the relevant discussion with the students are not presented here for the sake of brevity.

- Clark,J., Karuat,G., Mathews,D. & Wimbish,J. (2003) *The Fundamental Theorem of Statistics: Classifying Student understanding of basic statistical concepts*, unpublished paper, `http://www1.hollins.edu/faculty/clarkjm/stat2c.pdf`
- Cummins,J.D. (1991) "Statistical and Financial Models of Insurance Pricing and the Insurance Firm", *The Journal of Risk and Insurance* 58(2), 261–302
- Crack,F. T. & Ledoit,O. (2010) "Central Limit Theorems when data are dependent: addressing the pedagogical gaps", *Journal of Financial Education*, 36(spring/summer), 38–60
- De Moivre,A., (1730) *Miscellanea analytica de seriebus et quadraturis* London : J. Tonson & J. Watts. (Reprinted: Gale Ecco, Print Editions, 2010)
- De Moivre,A., 1738, 1756, *The Doctrine of Chances*, 2^{nd} ed London: Woodfall, 3^{nd} ed London: Millar, Reprinted 1967, NY Chelsea
- Garfield,J. & Ben-Zvi,D. (2007) "How Students Learn Statistics Revisited", *International Statistical Review* 75(3), 372-–396
- Goos,M. (2004) "Learning Mathematics in a Classroom Community of Inquiry", *JRME* 35(4), 258-–291
- Freudenthal,H., 1991, *Revisiting mathematics education – China lectures*, Dordrecht: Kluwer
- Hald,A. (1984), "Nicholas Bernoulli's theorem" International Statistical Revue, 52, 93–99
- Hald,A. (2003) *"A History of Probability and Statistics and their applications before 1750"* NJ: Wiley
- Hald,A. (2007) *A History of parametric statistical inference from Bernoulli to Fisher 1713–1935*, Springer
- Legrand,M (1993) "Débat scientifique en cours de mathématiques et spécificité de l'analyse", *Repères*, 10, 123–158
- Lunsford,M., Rowell,G., Goodson-Espy,T. (2006) "Classroom Research: Assessment of Student Understanding of Sampling Distributions of Means and the Central Limit Theorem in Post-Calculus Probability and Statistics Classes", *Journal of Statistics Education* 14(3) www.amstat.org/publications/jse/v14n3/lunsford.html
- Mathews,D. & Clark,J. (1997) "Successful Students' Conceptions of Mean, Standard Deviation and the Central Limit Theorem", *Paper presented at the Midwest conference on teaching statistics*, WI Oshkosh, (`http://www1.hollins.edu/faculty/clarkjm/stats1.pdf`)
- Montmort,P.R. de (1713) *Essay d'Analyse sur les Jeux de Hazard*, 2^{nd} ed, Paris Quillau,. Reprinted 1980, NY Chelsea
- Pearson,K. (1895) "Contributions to the Mathematical Theory of Evolution. II. Skew Variation in Homogeneous Material", *Philosophical Transactions of the Royal Society of London* (A), Vol. 186, pp. 343–414
- Sheynin,O.B. (1968) "On the early history of the Law of Large Numbers", *Biometrika*, 55(3), 459–467
- Sheynin,O.B. (1970) "Daniel Bernoulli on the normal law", *Biometrika*, 57(1), 199–202
- Stigler,S.M (1986) *The history of statistics: the measurement of uncertainty before 1900*, Harvard University Press
- Stonewater,J. (2005) "Inquiry Teaching and Learning: The Best Math Class Study", *School Science and Mathematics*, 105(1) 36–47
- Wilensky,U., (1997) "What is normal anyway? Therapy for epistemological anxiety" *Educational Studies in Mathematics*, 33, 171–202
- Wilensky,U. (2003) "Statistical mechanics for secondary school: The GasLab multi-agent modeling toolkit" *IJCML* 8, 1-41

EDUCATIONAL MEANING OF THE THEORY OF RECTANGULAR ARRAY IN NINE CHAPTERS ON THE MATHEMATICAL ART

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ABSTRACT

We first propose a principle of comprehensive cognitive development process by reviewing and synthesizing Piaget's theory of cognitive development, learning theory such as Dubinsky's APOS theory, and the epistemology of ancient Chinese philosopher Xun Zi(荀子). Our principle of cognitive ability development process comprises 4 stages: adaptation stage(familiarization of external stimulation); receptiveness stage(acceptance of information); completion stage(completion of conceptualization); utilization stage(application of knowledge). We think that a person experiences these 4 stages of cognitive process internally when he or she gets to acquire some knowledge.

We investigate the 8th chapter ⟨Rectangular Arrays(方程)⟩ of the ancient Chinese mathematics book ⟨Nine Chapters⟩[1] on the Mathematical Art(九章算術)⟩. ⟨Rectangular Arrays(方程)⟩ deals with the theory of systems of linear equations. We discuss an educational meaning and the value of ⟨Rectangular Arrays⟩ from the viewpoint of our principle of cognitive ability development.

1 Introduction

Euclid's ⟨Elements⟩ is regarded as the origin of systematically organized western mathematics. Around the time when ⟨Elements⟩ was written, an anonymous ancient Chinese mathematical book, ⟨Nine Chapters on the Mathematical Art(九章算術)⟩ was compiled during the Former Han(前漢) Dynasty (206 BC–8 AD). Some contents of the book date back to before Qin(秦) Dynasty (221–207 BC)[3]. Since the book had been compiled, several annotations had been added to the book, among which the commentary made by the first annotator Liu Hui(劉徽) in AD 264 is highly evaluated in that he supplemented the book using his own approach based on novel theories and ideas.

⟨Nine Chapters⟩ is considered one of the oldest and the most influential ancient Asian mathematical books. It had constituted the basis of eastern mathematics until the Western mathematics was introduced. The book is comprised of 246 practical problems and solutions which are categorized into nine chapters, namely, 1. rectangular fields(方田), 2. millet and rice(粟米), 3. distribution by proportion(衰分), 4. short width(少廣), 5. construction consultation(商功), 6. fair levies(均輸), 7. excess and deficit(盈不足), 8. rectangular arrays(方程), 9. right triangles(句股).

[1] ⟨Nine Chapters on the Mathematical Art⟩ is shortened to ⟨Nine Chapters⟩.

⟨Nine chapters⟩ is well organized in the pedagogical point of view. The problems in each chapter in the book are arranged from easy to more complicated ones to solve. We here are particularly interested in chapter 8 ⟨Rectangular Arrays(方程)⟩. It deals with a system of linear equations and the solution method given there is essentially the same as the Gaussian elimination method for a matrix.

In this paper, we investigate the structure of the chapter 8 in terms of the learning process or cognitive development process of human mind. To do this, we first propose a principle of comprehensive cognitive developmental process by reviewing and synthesizing Piaget's theory of cognitive development, learning theory such as Dubinsky's APOS theory, and the epistemology of ancient Chinese philosopher Xun Zi(荀子).

2 Cognitive ability development process

Human's cognitive ability is cultivated and developed from education. Education is the process which provides human with resources and experiences so that he or she may acquire knowledge and various skills to carry out any task while living as a human. In that sense education is very important and indispensible for human life. The more knowledge and information one has, the faster one's cognitive ability expands. The further one's cognitive ability grows, the faster one absorbs knowledge and skills. It is not hard to believe that education is very closely related to human cognition ability and so education should be designed in accordance with human cognition level. So when a book is written for the purpose of teaching, reader's cognitive ability development process should be considered.

In this section, we propose a principle of cognitive ability development, which has much in common with each of Piaget's cognitive developmental theory, Dubinsky's mathematics learning theory and the epistemology of Xun Zi(荀子).

2.1 Piaget's theory of cognitive development

Piaget ascertained some patterns and strategies of thought in each of the four cognitive developmental stages: sensorimotor stage, preoperational stage, concrete operational stage, formal operational stage. These stages are divided according to children's age.

He also suggested a developmental process that is consisted of a cycle: The child performs an action which has an effect on or organizes objects, and the child is able to note the characteristics of the action and its effects(*action*); Through repeated actions, perhaps with variations or in different contexts or on different kinds of objects, the child is able to differentiate and integrate its elements and effects(*process of reflecting abstraction*) and the child is able to identify the properties of objects by the way different kinds of action affect them(*process of empirical abstraction*); By repeating this process across a wide range of objects and actions, the child establishes a new level of knowledge and insight(*process of forming a new cognitive stage*); The child use them to create more complex objects and to carry out more complex actions(*creation*); As a result, the child recognize still more complex patterns and construct still more complex objects and new stage begins [5].

Piaget's cognitive development theory forms the foundation of the constructivism, which is a psychological theory on cognition. It says that knowledge and meaning are constructed by individual experiences and so learning and thinking require participation of the learner. Piaget believed that knowledge is constructed and re-constructed by the learner by interacting with objects [7].

2.2 Dubinsky's APOS theory

Dubinsky made a constructivist theory of learning in undergraduate mathematics by extending to college level the reflective abstraction in children's learning in Piaget's developmental process. His theory is based on the idea that an individual acquires mathematical knowledge by constructing mental actions, processes, and objects and organize them in schemas to make sense of situations and solve problems.

An *action* is a transformation of individual's externally perceived object by step by step instructions on how to perform the operation. When an action is repeated and the individual reflects on it, he or she make an internal mental construction called a *process*. When the individual becomes aware of the process as a totality and realize that transformations can act on it, an *object* is constructed. A *schema* for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked to form a framework in the individual's mind [1].

2.3 The epistemology of Xun Zi(荀子)

While the logical account for the cognitive process started to be given from Piaget's work in the west, there were attempts to make a logical interpretation of the human cognitive process in the east way long before Piaget.

A Chinese Confucian philosopher Xun Zi(荀子, 312–230 BC) pointed out that human mind is working through 4 stages: nature of things(性), human feeling of things(情), human thinking of things(慮), human action(僞) [2]

That is to say, when one perceives a thing, he or she has a feeling about it and then he or she think about it and then finally takes an action according to the thought. This means that after obtained by sensory perception, knowledge or information is organized and readjusted in accordance with one's recognition, and then it becomes utilized.

2.4 A principle of cognitive ability development

Education is the process through which knowledges are accumulated and cognitive ability is cultivated. Education process helps each individual build a thinking system. From well established thinking system comes the ability to concretely operate concepts according to each individual's cognitive level. By enhancing the cognitive ability, intellectual capacity is expanded and a series of actions leading to the development of intellectual capacity is understood as a main factor of education.

We propose that the cognitive ability development process in individual's mind is comprised of 4 stages, namely, *adaptation stage*(familiarization of external stimulus), *receptiveness stage*(acceptance of information), *completion stage*(completion of conceptualization), and *utilization stage*(application of knowledge).

In the adaptation stage, an individual shows rather passive attitudes to the external stimuli such as new concepts, new information, or new methods. He or she just senses the stimuli, not showing much interest. When one is getting familiarized with the repeating stimuli and gets interested in them, he or she moves on to the next stage.

In the receptiveness stage, one takes more active part to accept the new concepts and methods. He or she takes time and efforts to reflect and practice in order to accept them. As the external stimuli are maturing internally by one's reflections and practices, he or she gets to be more and more appreciating and understanding them. This concept-maturing-process is indispensable for knowledge establishment.

In the completion stage, a concept is acquired and fully understood so it becomes established knowledge in one's mind, ready to be utilized whenever needed. In the utilization stage, one can apply the established knowledge in order to interpret various phenomena and to solve real world problems. In this stage, intellectual capacity is further expanded and the cognitive ability is more promoted than before.

One notable feature of the principle of cognitive ability development is what distinguishes adaptation stage from receptiveness stage. In the adaptation stage the cognitive level allows no more than rote or mechanical learning, while in the receptiveness stage self-directed learning based on constructivism takes place. Individual's action and experience play very important roles in concept maturing which brings knowledge establishment.

Our principle of cognitive ability development has much in common with each of Piaget's cognitive developmental process, Dubinsky's APOS theory, and Xun Zi's epistemology. Following table shows how these theories can be compared.

	adaptation	receptiveness	completion	utilization
Piaget	action	reflective & empirical abstraction	forming a new cognitive stage	creation
Dubinsky	action	process / object	schema	
Xun Zi	nature	feeling	thinking	thoughtful action

3 Rectangular Arrays(方程) and the cognitive ability development process

The 8th chapter ⟨Rectangular Arrays⟩ of ⟨Nine Chapters⟩ deals with systems of linear equations. The solving method used here is ⟨rectangular array method(方程術)⟩, which is essentially the same as the Gaussian elimination method. The Gaussian elimination method was named after Carl Friedrich Gauss, but it was not invented by him. The method stems from the notes of Isaac Newton in 17th century [4]. This fact indicates that the theory of a system of linear equations was developed in the east much earlier than it appeared in the west.

In this chapter, a rule for calculating negative numbers, called the ⟨sign rule(正負術)⟩, is also introduced. Shen mentioned that it is surprising to find that Liu Hui summed up the ⟨sign rule⟩ 1700 years ago in the same way we understand and apply now. Such a treatment noted in the west is not found until after the Renaissance [3].

We briefly discuss the characteristics of each problem in light of our cognitive ability development process below. We present the original text and the corresponding English translation of some of the problems to help readers grasp what the chapter is all about.

第一問. 今有 上禾三秉 中禾二秉 下禾一秉 實三十九斗. 上禾二秉 中禾三秉 下禾一秉 實三十四斗. 上禾一秉 中禾二秉 下禾三秉 實二十六斗. 問上中下禾一秉各幾何?

Problem 1. Now there are 3 sheaves of top grade rice, 2 sheaves of middle grade rice, 1 sheaf of low grade rice. Total volume is 39 dou[2]of rice; 2 sheaves of top grade rice, 3 sheaves of middle grade rice, 1 sheaf of low grade rice. Total is 34 dou; 1 sheaf of top grade rice, 2 sheaves of middle grade rice, 3 sheaves of low grade rice. Total is 26 dou. What is the volume of 1 sheaf of each grade rice?

The ⟨rectangular arrays(方程)⟩ and the ⟨rectangular array method(方程術)⟩ are introduced as a way of solving problem 1, which is represented by a system of linear equations with 3 unknowns in modern algebra. The solving method suggested in ⟨Nine Chapters⟩ can be briefly explained as follows: First of all, write down the given numbers of things in each column, making a ⟨rectangular array⟩, or a 4×3 matrix. Then eliminate small top entries by performing a series of column operations until the resulting array becomes a lower triangular matrix, from which final answers can be obtained by division and substitution. This method is called the ⟨rectangular array method⟩, which is the same as the Gaussian elimination method. Below shown are the starting array and the ending array for this problem.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{pmatrix}$$

第 二 問. 今有 上禾七秉 損實一斗 益之下禾二秉 而實一十斗. 下禾八秉 益實一斗與上禾二秉 而實一十斗. 問上下禾實一秉各幾何?
Problem 2. Now there are 7 sheaves of top grade rice, subtracting 1 dou from them and adding 2 sheaves of low grade rice makes 10 dou; 8 sheaves of low grade rice, adding 1 dou and adding 2 sheaves of top grade rice makes 10 dou. What is the volume of 1 sheaf of each grade rice?

Problem 2 can be expressed using modern notation by linear equations with 2 unknowns

$$7x - 1 + 2y = 10 \text{ and } 8y + 1 + 2x = 10.$$

Problem 2 explains transposing constant terms with the passage that "subtracting is adding (to another)[3]and adding is subtracting (to another) ". So the columns of the array are $(7, 2, 11)$ and $(2, 8, 9)$.

第 三 問. 今有 上禾二秉 中禾三秉 下禾四秉 實皆不滿斗. 上取中 中取下 下取上 各一秉 而實滿斗. 問上中下禾實一秉各幾何?
Problem 3. Now there are 2 sheaves of top grade rice, 3 sheaves of middle grade rice, 4 sheaves of low grade rice, the volume of each is less than 1 dou. Adding (1 sheaf of) middle grade rice to the top grade rice, (1 sheaf of) low grade rice to the middle grade rice, (1 sheaf of) top grade rice to the low grade rice makes 1 dou each. What is the volume of 1 sheaf of rice of each grade?

Problem 3 is to solve the system

$$2x < 1, \quad 3y < 1, \quad 4z < 1, \quad 2x + y = 1, \quad 3y + z = 1, \quad 4z + x = 1.$$

[2]Dou(斗) is a unit of volume.
[3]"損之曰益 益之曰損"

Here, negative numbers inevitably appear in the matrix reducing process. A method of addition and subtraction of negative numbers is explained, which is called the ⟨sign rule(正負術)⟩, where ⟨正⟩ and ⟨負⟩ refer to positive and negative, respectively. The ⟨sign rule⟩ is summarized as saying that "like signs subtract, opposite signs add (when subtracting); opposite signs subtract, like signs add (when adding)."

Problems 1, 2, and 3 provide a certain kind of problem situations, for which a new mathematics is required. These problems introduce a new concept such as ⟨rectangular array⟩, a new method such as ⟨rectangular array methods⟩, and a new rule such as ⟨sign rule⟩.

Exposing learners to such problem situations and introducing new concepts and new methods related with the problem is like providing a stimulus. This is the adaptation stage for learners. Providing a stimulus is in turn like sowing a seed. Like a big tree starts from a seed, knowledge is formed from a stimulus. A seed can hardly be seen at first but soon with a good care it will grow to show a little leaf and then it will be noticed. When a stimulus is repeatedly given to learners, who may not have recognized it at first, they will sense it stronger and stronger and adapt to it without deep understanding of it. As learners try intentionally to understand and get used to it, they are gradually accepting such stimulus as meaningful information or knowledge.

> 第 四 問. 今有 上禾五秉 損實一斗一升 當下禾七秉. 上禾七秉 損實二斗五升 當下禾五秉. 問上下禾實一秉各幾何?
> Problem 4. Now there are 5 sheaves of top grade rice. If 1 dou 1 sheng[4] is subtracted from them, the rest is equivalent to 7 sheaves of low grade rice; 7 sheaves of top grade rice subtracted 2 dou 5 sheng is equivalent to 5 sheaves of low grade rice. What is the volume of 1 sheaf of rice of each grade?

In modern notation, the system $5x-11 = 7y$, $7x-25 = 5y$ represents the problem 4. The suggested solution is to make an array consisting of two columns $(5, -7, 11)$ and $(7, -5, 25)$. We notice here that y terms are transposed as well as the constant terms and negative numbers appear as some entries of the array.

Problems 5 and 6, like problem 4, deal with linear equations with 2 unknowns involving negative coefficients, and they are solved by applying the concepts and rules introduced before. All the problems 1 through 6 are asking what the volume of 1 sheaf of rice of each grade is.

Problems 4, 5, and 6 intend to help learners get more familiarized to the problem situations and solving methods newly introduced before. These problems are simple variations of the problems in the adaptation stage. With these problems learners can practice applying new concepts and methods. While learners are taking time and efforts to figure out what the new stimulus or information means and to reflect on it, they become more receptive and experience the newly adapted concepts and methods maturing in their minds. This is the receptiveness stage of the cognitive ability development process. This is the period for which a very little plant is taken very good care of to grow continuously.

> 第 七 問. 今有 牛五 羊二 直金十兩. 牛二 羊五 直今八兩. 問牛羊直金其何?
> Problem 7. Now there are 5 cows and 2 sheep, the total price is 10; 2 cows and 5 sheep, the total price is 8. What is the price of a cow and a sheep each?

[4]Sheng(升) is a unit of volume. 1 dou = 10 sheng.Sheng(升) is a unit of volume. 1 dou = 10 sheng.

Problem 7 deals with the price of domestic animals. According to the suggested solution given in ⟨Nine Chapters⟩, the resulting matrix is different from the one given in problem 1, where the first (from right) column remains the same throughout the whole process, while in problem 7 the first column (from right) has been multiplied as shown below. When a varied column operation is introduced, learners' computation skills will be improved.

$$\begin{pmatrix} 2 & 5 \\ 5 & 2 \\ 8 & 10 \end{pmatrix} \Rightarrow \begin{pmatrix} 10 & 10 \\ 25 & 4 \\ 40 & 20 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 10 \\ 21 & 4 \\ 20 & 20 \end{pmatrix}$$

第 八 問. 今有 賣牛二 羊五 以買十三豕, 有餘錢一千. 賣牛三 豕三 以買九羊 錢適足.
賣羊六 豕八 以買五牛 錢不足六百. 問牛 羊 豕價各幾何?
Problem 8. Now sell 2 cows, 5 sheep, and buy 13 pigs, then there left 1000 coins; sell 3 cows, 3 pigs, and buy 9 sheep, then no coin is left nor deficient; sell 6 sheep, 8 pigs, and buy 5 cows, then there is 600 coins shortage. What is the price of a cow, a sheep, and a pig each?

Problem 8 involves the situation with zero constant term and negative coefficients as well. It has 3 unknowns and asks about the price of domestic animals. Problem 9 is an application problem with 2 unknowns asking the weight of birds.

第 十 問. 今有 甲乙二人持錢不知其數. 甲得乙半而錢五十. 乙得甲太半而亦錢五十.
問甲乙持錢各幾何?
Problem 10. Now there are two persons A and B, each of whom has unknown amount of coins. If A gets half of what B has then A will be having 50 coins. If B gets two thirds of what A has, then B will also be having 50 coins. What is the amount of coins A and B each has?

Problem 10, asking the amount of coins with 2 unknowns, shows how to deal with arrays including fractions. It suggests to multiply an appropriate number to the column including fractions to get integer entries. Problem 11, together with problem 10, extends the theory to more general situations to handle fractions. It asks about the price of domestic animals with 2 unknowns.

Problems from 7 to 11 require mixed calculations involving negative integers, fractions, and 0. They look the type of problems that have the intention to improve learners' calculation skills. While problems from 1 to 6 ask only about the volume of 1 sheaf of rice, various objects such as the price of domestic animals, the weight of birds, the amount of money are dealt with in these problem situations. These problems provide more complicated situations than the previous ones so the learners can establish knowledge completely which has been maturing all along. By facing a variety of problem situations and solving them on their own, learners get confident with the new concepts and methods, and new knowledge becomes fully established. Their cognitive levels are upgraded in this completion stage. The seed has completely grown to be a big tree ready to produce fruits.

Problem 12 deals with horse power with 3 unknowns. Problem 13 asks the length of ropes in the situation expressed with 5 equations and 6 unknowns. Problem 14 asks about the yield of each field with 4 unknowns. Problem 15 asks the weight of rice with 3 unknowns and negative coefficients. Problem 16 asks the number of chickens eaten up with 3 unknowns. Problem 17 asks the prices of domestic animals with 4 unknowns. Problem 18 asks the prices of grains with 5 unknowns.

The problem situations given in problems from 12 to 18 are still more complicated than before: the numbers of unknowns are getting larger; an indeterminate system is included; a variety of objects including abstract ones such as horse power should be dealt with. These various real world problems can be solved with careful application of established knowledge. Learners who have fully established knowledge can utilize it to solve application problems. So their cognitive ability is enhanced. This happens in the utilization stage. We can enjoy the fruits of the big tree which will produce many more fruits over the years.

4 The educational meaning of the ⟨Rectangular Arrays⟩

As we saw in the above section, the 18 problems in the chapter ⟨Rectangular Arrays⟩ are arranged from easy to complicated ones with respect to problem situations and solving method. More specifically, mathematical levels are upgraded from problems involving only positive numbers to ones with integers or fractions; the concepts included in problem situations extend from concrete ones such as volume and length to abstract ones such as horse power and price; the number of unknowns increases from 2 to 6 so the matrix operations are getting more complicated. One of the problems deals with an indeterminate system of linear equations.

In this section, we summarize what we analyzed the ⟨Rectangular Arrays⟩ chapter of ⟨Nine Chapters⟩ from the standpoint of cognitive ability development.

Problems 1, 2, and 3 are designed for introducing a certain kind of problem situations and solving methods so that learners can familiarize and adapt themselves to them. These are the problems for the adaptation stage.

Learners can practice solving problems by applying the rules instructed in the previous problems to similar situations given in problems 4, 5 and 6. These problems are meant to help learners become receptive to new concepts and methods and understand them. By facing and solving various problem situations repeatedly, learners can make new concepts and methods mature internally. This happens in the receptiveness stage and these problems are for the receptiveness stage.

Learners get the chance to face varied problem situations involving different objects starting from problem 7. By extending the methods to even more complicated problem situations as exemplified in problems from 8 to 11, learners can deepen their understanding and get confident with the newly developed concept. So they finally can reach to the stage of completion of knowledge.

Learners can apply their established knowledge on the subject in order to solve more complicated and practical real world problems such as problems from 12 to 18, and especially, to solve an indeterminate system of linear equations like problem 13. So these problems are for the utilization stage.

There are some other notable features of this chapter in educational viewpoint. The problems used to explain the methods ⟨rectangular array method(方程術)⟩ and ⟨sign rule(正負術)⟩ have 3 unknowns, whereas 2 unknowns are used when a rather sophisticated computation skill is needed. Application problems have mainly 3 or 4 unknowns with one problem with 5 and 6 unknowns each. These features coincide with our thoughts. When we teach a system of linear equations or matrices to students, we typically introduce the topic with problems with 3 unknowns and we say that the same method applies to more general situations involving more than 3 unknowns. If we have to deal with more complicated situations where a special computation technique is needed, then it will be best to start with 2 unknowns and then generalize. In that sense, ancient Chinese mathematical book seems to reflect our way of thinking in terms of education.

5 Final Remark

⟨Nine Chapters⟩ was published way long before there appeared a theory of human cognitive development. Even so, the problems in each chapter are organized and arranged as if human cognitive development process is considered. The fact that psychology in the west was developed and established in 20th century does not mean that there had been no recognition of human mind or psychology before. When there was no theory on learning, cognition, or psychology, the compilers of ⟨Nine Chapters⟩ seemed to understand how human cognitive ability was developing and how to teach people efficiently.

Finally, the comment made by Liu Hui below problem 18 is worth noticing. Problem 18 is a rather complicated one involving 5 unknowns and in the commentary, Liu Hui encouraged readers to solve problems not by mechanically mimicking the solutions of previous problems but by applying a better and simpler method developed from one's own reflection and thought. Liu Hui meant that mathematical technique is developed by practicing simple operations and then combining them in more complicated ways to get to the solution of more difficult problems. And this is what we mean by completion of knowledge and utilization of it. His comment indicates that he understood mathematics as a subject for cultivating creative thinking, which would be what mathematics education intends to achieve now.

REFERENCES

[1] Dubinsky, E., McDonald, M., "APOS: A constructivist theory of learning in undergraduate mathematics education research", http://www.math.kent.edu/~edd/ICMIPaper.pdf.

[2] Kim, K. and Lee, H., 2009, *Essays on the Eastern Philosophy*, 2nd ed. Dongnyuk Publishing.

[3] Shen, K., Crossley, J., and Lun, A., 1999, *Nine Chapters on the Mathematical Art*, Oxford university press,.

[4] http://en.wikipedia.org/wiki/Gaussian_elimination

[5] http://en.wikipedia.org/wiki/Jean_Piaget

[6] http://en.wikipedia.org/wiki/Xun_Zi

[7] http://www.learningandteaching.info/learning/piaget.htm

MEANING OF THE METHOD OF EXCESS AND DEFICIT

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ABSTRACT

Young Bujok(盈不足, Ying Buzu in Chinese) is the title of the 7th chapter of The Nine Chapters on the Mathematical Art(九章算術), one of the oldest and the most influential ancient mathematical books. Young Bujok mathematically means the method of excess(盈) and deficit(不足). It is well-known that this method was transmitted from China to Europe via Arabia, becoming the rule of double false position.

We investigate the method of excess and deficit to conclude that it was a mathematical product of the creativity of the ancient Asians. It can be considered as a trial for construction of mathematical theories by Chinese, with the following chapters : Retangular Arrays and Right-angled Triangles. Even though the method has not been developed as a modern mathematical theory, we still possibly compare the method to the division algorithm and/or the modulo arithmetic via the concepts of division described by 實如法而一 and cross multiplication(維乘), from the point of view to deal with the remainders of divisions.

Keywords: Young Bujok(盈不足, Ying Buzu in Chinese, the excess and deficit), The Nine Chapters on the Mathematical Art(九章算術), the rule of double false position, cross multiplication(維乘)

1 Introduction

The Nine Chapters on the Mathematical Art(九章算術, Gu Jang San Sool in Korean, Jiu Zhang Suan Shu in Chinese) is one of the oldest and the most influential ancient mathematical books in East Asia, as like as Elements by Euclid in the West. Young Bujok(盈不足, Ying Buzu in Chinese) is the title of the 7th chapter of The Nine Chapters on the Mathematical Art (the Nine Chapters, in short). While we let Young Bujok itself to mean the chapter, it mathematically means the method of excess(盈) and deficit(不足). It is well-known that this method was transmitted from China to Europe via Arabia, becoming the rule of double false position [1, 5].

Chang Hyewon raised a question in [1] about the origin of such mathematical knowledges as the method of excess and deficit. She said that it was relatively easy to clarify the validity of the algorithms depicted in ancient mathematical books, but hard to figure out the origins of such mathematical knowledges, and only some conjectures about the origins were possible.

However, the method of excess and deficit looks very natural if we take for granted that the ancient Asians were clever and accustomed to division. The method, even though novel, ingenious in theory and powerful in applications, looks very alike to elementary number theory, for example, division algorithm.

With this thought, we here look for a mathematical and historical meaning and value of Young Bujok, the chapter of the method of excess and deficit. We first set up a viewpoint and/or an attitude for analysing Young Bujok. We then take a close look for the problems and the method of Young Bujok. We finally conclude that the method of excess and deficit is alike division algorithms, and it is very novel and ingenious mathematical product of the creativity of the ancient Asians.

2 Viewpoints and attitudes for the analysis on Young Bujok

We are going to adopt threefold viewpoints/attitudes for the analysis of the mathematical and historical meaning and value of the chapter Young Bujok.

We first adopt the attitude of Liu Hui, an ancient Chinese mathematician who made commentaries on the Nine Chapters. He himself told his attitude to look at the mathematics of the Nine Chapters in his preface of it [2, p. 56]:

> 徽幼習九章 長再詳覽 觀陰陽之割裂 總算術之根源 探賾之暇 遂悟其意.
> 是以敢竭頑魯, 采其所見, 爲之作注. 事類相推, 各有攸歸.
> When he was young, Lui Hui studied the Nine Chapters. Getting old, he closely investigate it again, and got to be aware of the change of Um Yang(陰陽, Yin Yang in Chinese) and the roots of arithmetic(mathematics). And researching the relationships(gaps) among them, he at last got the very meanings.
> Inspite of his dullness, he gathered his opinions and made commentaries. After his investigations and classification of things, (he concluded that) there are principles (of their own).

The above translation here is somewhat rough. However, since the ancient Chinese and English are a lot different form each other, it probably better to translate so as to figure out the meaning clearer. For the word by word or more detailed translation, refer to [5].

We adopt such attitude of Liu Hui so that we closely investigate each problem in Young Bujok to be aware of the root principles. So we minimize the references rather than widen, to analyse the mathematical and historical meaning and values of Young Bujok. We are rather going to only research on Young Bujok itself.

Secondly, we adopt Hawking's *model-dependent realism*, which can be clearly explained in the following quotation from *The Grand Design* [3] by S. Hawking and L. Mlodinow:

> Until the advent of modern physics it was generally thought that all knowledge of the world could be obtained through direct observation, . . . (omitted) . . . that is not the case. The naive view of reality therefore is not compatible with modern physics. To deal with such paradoxes we shall adopt an approach that we call *model-dependent realism*. . . . (ommitted) . . . But there may be different ways in which one could model the same physical situation, with each employing different fundamental elements and concepts. If two such physical theories or models accurately predict the same events, one cannot be said to be more real than the other; rather, we are free to use whichever model is most convenient.

In this quotation, we specially focus on the last statement: If there are several models which predict the same events, we are free to use the most convenient model. In this sense, we make a kind of model under which we understand the given situations, we mean here, the problems of Young Bujok.

Lastly, we look at the functions of mathematical problems. We know that there are problems simply for joy or time consumption, with no specific purposes, for example, some puzzles like cross-word puzzle and *sudoku*. However, problems in mathematical books are not like that. They have their own meaning and purposes. Such meaning and purposes are possibly roughly divided into two parts: one is the introduction/explanation of a theory, and the other is for practices/applications of the theory.

Since the Nine Chapters, and so the chapter Young Bujok also, consist of problems, we are going to think about the purposes of the problems of Young Bujok.

3 Problems and the method of Young Bujok

The Nine Chapters on Mathematical Art consist of nine chapters(books) with total 246 problems: chapter 1 方田 (Field Measurement) of 38 problems, chapter 2 粟米 (Millet and Rice) of 46 problems, chapter 3 衰分 (Distribution by Proportion) of 20 problems, chapter 4 少廣 (Short Width) of 24 problems, chapter 5 商功 (Construction Consultations) of 28 problems, chapter 6 均輸 (Fair Levies) of 28 problems, chapter 7 Young Bujok(盈不足, Excess and Deficit) of 20 problems, chapter 8 方程 (Rectangular Arrays) of 18 problems, and chapter 9 句股 (Right-angled Triangles) of 24 problems. For the details about the Nine Chapters, refer to [2], and for its English translation, refer to [5].

The chapter Young Bujok of the Nine Chapters consists of 20 problems as mentioned above. The first eight problems explain the method of excess and deficit, and the remaining twelve problems are the use of the the method in various kinds of situations, that is, the applications of the method of excess and deficit.

We closely look at the first eight problems, but are not going to discuss about the remaining twelve problems for now. We might have a chance to discuss about them later. The first 8 problems are:

1. 今有 共買物. 人出八, 盈三. 人出七, 不足四. 問人數, 物價各幾何?
   Here are something to buy together. If each person pays 8, then the amount exceeds the price by 3. If pay 7, then the deficit is 4 to the price. How many people are there? How much is the price?

2. 今有 共買鷄. 人出九, 盈十一. 人出六, 不足十六. 問人數, 鷄價各幾何?

3. 今有 共買璡. 人出半, 盈四. 人出少半, 不足三. 問人數, 璡價各幾何?
   To buy a jade together, if each pays a half, then the excess is 4, and if each pays a third, then the deficit is 3 to the price of the jade. How many people are there? How much is the price?

4. 今有 共買牛. 七家共出一百九十, 不足三百三十. 九家共出二百七十, 盈三十. 問家數, 牛價各幾何?

5. 今有 共買金. 人出四百, 盈三千四百. 人出三百, 盈一百. 問人數, 金價各幾何?

6. 今有 共買羊. 人出五, 不足四十五. 人出七, 不足三. 問人數, 羊價各幾何?

7. 今有 共買豕. 人出一百, 盈一百. 人出九十, 適足. 問人數, 豕價各幾何?

8. 今有 共買犬. 人出五, 不足九十. 人出五十, 適足. 問人數, 犬價各幾何?

We translate only two problems. But it is enough to see that these statements are in a certain pattern, and what problems mean. The problems are to say that: There is something for some people to buy together(今有 共買物). If each person pays(出) some money, then there be a remainder, an excess(盈) or a deficit(不足) to the price of that *something*. In case of problem 7 and 8, there is no remainder in one condition. Given two conditions, the number of people and the price are questioned.

After problem 4, the general method of excess and deficit is explained as

> 置所出率, 盈, 不足各居其下. 令維乘所出率, 并以為實. 并盈, 不足為法.
> 實如法而一. 有分者, 通之. 盈不足相與同 其買物者.
> 置所出率. 以少減多, 餘, 以約法, 實. 實為物價, 法為人數.[1]
> Put each of payments and each remainder(excess or deficit) under each of them. Make a cross multiplication(維乘) among the payments and the remainders. Let the sum of the multiplied payments be sil(實), and the sum of excess and deficit bub(法).
> Divide sil by bub(實如法而一). If there are fractions, reduce them to a common denominator.
> If the remainders are sumed up to zero, then it means the purchase together.
> Make a difference(餘) of the payments. Divide bub(法) by the difference(餘) to find the number of people. Divide sil(實) by the difference(餘) to get the price.

There are more explanations on problems 5 and 6, and 7 and 8, respectively. But these method are special cases of the above so that we omit them.

4 Analysis of Academic/Historical Meanning and Value of Young Bujok

We first think of the phrase 實如法而一 which comes out of everywhere in the Nine Chapters. Koh and Ree, in [4], looked for logic or mathematical formalism underneath the ancient Asian mathematics. They focused on sil(實) and bub(法). Sil(實, shi in Chinese) means a reality, that is, what one has in real, meanwhile bub(法, fa in Chinese) means a basic standard and/or a unit. So the phrase 實如法而一 is understood as that if what a person has is the same as the unit, then it becomes one. This, therefore, means the division, even though it does not mention about the remainder.

As we know, the determination of a unit motivates the inventions of the natural numbers, and then naturally additions of them. With some cleverness and creativity that human beings have, the multiplication naturally comes out. So, in some sense, division is a starting point of arithmetic or mathematics. Therefore, setting up 實 and 法, or the phrase 實如法而一, is considered as the very beginning of the ancient Asian mathematics.

We now look at the chapter Young Bujok. It starts with the explanation of its title [2, p. 119] as:

> 盈不足, 以御隱雜互見.

This means that the method of excess and deficit deals with complex and complicated situations to clear up and simplify them. This is a sort of fundamental and basic meaning and value of mathematics. In mathematics, fundamentally, we simplify complicated situations using given clues or conditions to figure out the situations(the present). With such anaysis of the situations, we can predict our near

[1]In [2] the statements of the method is a little bit differently given: 盈不足相與同 共買物者. 置所出率, 盈, 不足各居其下. 令維乘所出率, 并以為實. 并盈, 不足為法. 有分者, 通之. 副置所出率. 以少減多, 餘, 以約法, 實. 實為物價, 法為人數.

future, or invert the situation to find the causes(our near past). This is a value of doing mathematics. Young Bujok affords such privilege.

It is also important to notice that there is a mathematical pattern in the statements of the first eight problems in the chapter Young Bujok. For example, let a denote the number of people, and b the price of the thing to buy together in the problems. In problem 1, the first condition tells

$$8a = b + 3 \quad \text{or} \quad b = 8a - 3,$$

and the second condition tells

$$7a = b - 4 \quad \text{or} \quad b = 7a + 4.$$

Similarly, in problem 3, the the first and second conditions tell

$$\frac{1}{2}a = b + 4 \quad \text{or} \quad b = \frac{1}{2}a - 4,$$

$$\frac{1}{3}a = b - 3 \quad \text{or} \quad b = \frac{1}{3}a + 3.$$

These conditions look like the division algorithm: Given natural numbers $a \leq b$, there exist natural numbers q and r such that

$$b = aq + r \quad \text{with } 0 \leq r < a.$$

Combining 實如法而一 and the pattern of the given statements in Young Bujok, we see a similarity between the representation of problems in Young Bujok and division algorithm.

It is sure that there are big differences between them. For example, in division algorithm, only natural numbers are treated and the remainder r should be in the range of $0 \leq r < a$, a the divisor. Meanwhile, in Young Bujok, the dividend or 實 b and the divisor or 法 a, even the quotient q and the remainder r are not necessarily the natural numbers. They can be rational numbers, and even the remainder(excess or deficit) can be any number. However, using 實如法而一, we can divide 實 b by 法 a as much as we want, and then we get 盈 or 不足, which are considered as the remainder. In this way, Young Bujok combined with 實如法而一 is considered as a sort of generalized form of division algorithm.

There is one more thing worth noticing. Division algorithm is a sort of deductive computation, which means the quotient and the remainder are uniquely determined from the given dividend and divisor. However, Young Bujok problems are given in the form of inversion. We are given the quotient and the remainder, and asked to find the dividend and the divisor. In this sense, Young Bujok is much more *mathematical* rather than *arithmetical*.[2]

We then consider the meaning of the cross multiplication(維乘) as the management tool of the remainders(excess or deficit). For example, in problem 1, when a denotes the number of people and b the price, if each person pays 8, then there is an excess 3 of the price so that $8a = b + 3$. And if each person pays 7, then there is an deficit 4 of the price so that $7a = b - 4$. Now if we make a cross multiplication which means the multiplication of 4 and 3 to the former and the latter equations, respectively, to get

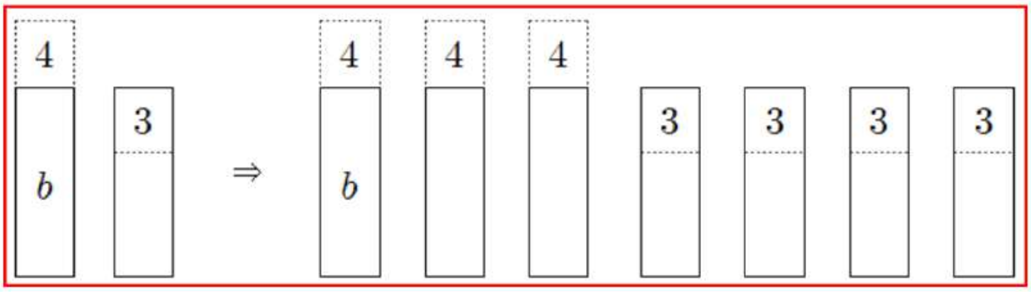
$$32a = 4b + 12 \qquad \text{and} \qquad 21a = 3b - 12,$$

[2]Needham insisted that the West owes to Chinese mathematics for the use of indeterminates. - in Martzloff, J. C., 1997, *A History of Chinese Mathematics*, Springer, Berlin.

and then add the equations up to get

$$(32+21)a = (4+3)b,$$

in which there is no remainder, that means, the cross multiplication is a method to find a multiple of a certain number, in this case, the price of the thing to buy together. In this sense, the cross multiplication is compared to modulo arithmetic in number theory. In the explanation of the method of excess and deficit, it is said 盈不足相與同 其買物者, which means that to give and take 盈 and 不足 to get rid of each other is just the purchase together of the thing (in multiple). This is depicted in the figure 1.



⟨ Fig. 1 ⟩ cross multiplication

The figure shows that the utility of cross multiplication. Even though we want to buy a thing, we duplicate the situation of common purchase to get rid of remainders(excess or deficit). The number of duplications are given by the excess and the deficit. This is the ingeneous, even though simple, creative mathematical product of the ancient Asians.

We finally consider the nature of the problems in Young Bujok. As mentioned above, the first eight problems have the purpose of the introduction and explanation of the mathematical theory, the method of excess and deficit. In fact, the first four problems provide a prototype and the other four give some variations. But if we write these variations in modern mathematical terminology, then it is easy to see that they are surely contained in the prototype.

Even though we haven't touched on the application problems which come after the first eight problems, the application is easily derived from the the method of excess and deficit, which are now considered as the rule of double false position.

If we look at the title of the Nine Chapters, then we can feel a kind of practicality from the first six titles. It seems that the mathematics in the first six chapters would rather be used in governmental offices, for example, the office of taxes. But the latter three chapters, Young Bujok(Excess and Deficit), Retangular Arrays and Right-angled Triangles seem rather mathematical(theoretical). In fact, in the sense of providing a theoretical prototype in the first few problems, they are considered as an effort or a trial to make theories by the ancient Chinese mathematicians.

5 Conclusion

We investigate the method of excess and deficit to conclude that it was a mathematical product of the creativity of the ancient Asians. It can be considered as a trial for construction of mathematical theories by Chinese, with the following chapters : Retangular Arrays and Right-angled Triangles. Even though the method has not been developed as a modern mathematical theory, we still possibly compare the method to the division algorithm and/or the modulo arithmetic via the concepts of division described by 實如法而一 and cross multiplication(維乘), from the point of view to deal with the remainders of divisions.

Bibliography

[1] Chang Hyewon, 2005, The Excess and Deficit Rule and the Rule of False Position, *Korean Journal for History of Mathematics* Vol. 18, No. 1, 33–48.

[2] 靖玉樹編勘, 中國歷代算學集成, 山東人民出版社, 1994.

[3] Stephen Hawking & Leonard Mlodinow, 2010, *The Grand Design*, translated version in Korean, ckachi Publisher, ISBN-13 9788972914921.

[4] Koh Youngmee & Ree Sangwook, 2010, Logic of Ancient Mathematics of East Asia: Epistemology by Xun zi, Logic by Mozi, *The Korean Jouranl for History of Mathematics* Vol. 23, No. 3, 33–44.

[5] Shen Kangshen, John N. Crossley & Anthony W.-C. Lun, 1999, *The Nine Chapters on the Mathematical Art: Companion and Commentary*, Oxford University Press.

RESEARCH ON THE MUK SA JIB SAN BEOB

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ABSTRACT

Included in the first chapter of the Mathematics Section of the Collection of the Historical Sources in the Science and Technology of Korea [1], *Muk Sa Jib San Beob* by Kyong Sun-Jing(1616–?) is one of the very few existing mathematical work of 17th century Korea, and is also an important primer of mathematics in that time of Korea.

In this paper we claim that Kyong Sun-Jing first mastered the contents of the *Xiang Ming Suan Fa* and then combined *Suan Xue Qi Meng* and *Yang Hui Suan Fa* together to write his book.

Keywords: Chosun mathematical primer, *Muk Sa Jib San Beob*, *Yang Hui Suan Fa*

1 Introduction

Included in the first chapter of the Mathematics Section of the Collection of the Historical Sources in the Science and Technology of Korea [], *Muk Sa Jib San Beob* by Kyong Sun-Jing(1616–?) is one of the very few existing mathematical work of 17th century Korea, and is also an important primer of mathematics in that time of Korea. According to the record by the Collection of the Historical Sources in the Science and Technology of Korea, this book was originally preserved in the library of Peking University and the one in Korea is a copy of it. [1] Our paper is on the basis of this version.

Being one of the earliest remaining mathematical works of Korea, this book has been studied by many scholars. Hur Min translated this book into Korean with a thorough study of the book included in the translation. He and several others wrote papers on the book, namely Pythagorean Theorem in 18th Century Korea[2], Pythagoras Theorem in 19th Century Korea and Theory of Equations in the Choseon Period was written by Professor Hong Younghee, Comparisons between *Suan Xue Qi Meng* and *Muk Sa Jib San Beob*[3], was written by Hur Min, Kyong Sun-Jing and a Mathematics Workshop in 17th Century Choson Korea written by Horng Wann-Sheng and Li Chien-Tsung [4], etc.. From the papers mentioned above, we see that all of them make extensive comparison with the book *Suan Xue Qi Meng* . [5] In this paper we make a comparative study of *Muk Sa Jib San Beob* with the Chinese books *Xiang Ming Suan Fa* , *Suan Xue Qi Meng* and *Yang Hui Suan Fa*, which were two of the three only books available in Korea at that time.

*First Author

Through careful study of the distinct aspects of its formation, advantages, disadvantages, influences on the later generations and so on, we claim that Kyong Sun-Jing first mastered the contents of the *Xiang Ming Suan Fa* and then combined *Suan Xue Qi Meng* and *Yang Hui Suan Fa* together to write his book.

2 Background

2.1 The Background Information of *Muk Sa Jib San Beob*

Kyong Sun-Jing, courtesy name YeoHyeu汝休，was born in Cheongju, Korea. In 1640(庚辰) Kyong passed the government exam and became an official. In 1669(己丑) he held the post of mathematics teacher. In 1674 (甲寅) he got a higher position as a professor. Then he was promoted to be a governor of the Living Department which is a branch of Ministry of Rites In 1676(丙辰). [6]

Kyong Sun-Jing left two works which are *Sang Myeong Su Gyeol*（详明数诀）and *Muk Sa Jib San Beob*. It is a pity that *Sang Myeong Su Gyeol* has not been handed down to today. When Hong Daeyong wrote his book *Requirement for the Understanding of Mathematics*(筹解需用), he quoted that "*Sang Myeong Su Gyeol* was written by Kyong Sun-Jing." When noble Cho Taegu wrote his book *JuSeoGwanGyeon*(籌書管見), he membered Kyong Sun-Jing as one of the numerous contemporary scholars. Moreover, Kyong has been praised as a top-rated mathematician by Choi Seok-Jeong. In addition, in the algorithm of Stacking problem in *Sanjiaoguo* included in *Gu Su Ryak*, Choi had clearly written, "in the west we have Matteo Ricci and Johann Adam Schall von Bel and in east we have the famous ··· Kyong Sun-Jing." We can also realize Kyong's proficient mathematic skill in the forewords of *Suan Xue Qi Meng* written by Kim Shi-Jin. Kim speaks highly of Kyong. From all these evaluations we recognize Kyong's status in the noble Confucian mathematical society of Chosun during the 17th and 18th century.

From the historical sources, we can discover that three Chinese mathematical books had been introduced to Choseon before 15th century AD. Those books were *Xiang Ming Suan Fa* (1373) written by An Zhiqi, *Suan Xue Qi Meng* (1299) written by Zhu Shijie of Yuan Dynasty and *Yang Hui Suan Fa*(1275) written by Yang Hui of Song Dynasty. [7] However only a handful of people could understand these books. It can be seen from the book *SeJong Shillok* that there were scholars who had learned *Qi Meng Suan* and also had the knowledge of it in 1430. However in the same SeJong Shillok is recorded that, in 1460 no one can understand *Qi Meng Suan*. According to the volume six of *YiJe's Posthumous Papers*, it is written of the book "Because the contents of the book are from shallow to deep and no one could have explained it, the scholars do not cherish or value the book as it is still simple and crude." it is clear that people have not paid enough attention to *Suan Xue Qi Meng* and few people could really understand it. Once one reads Kyong Sun-Jing's *Muk Sa Jib San Beob*, one can easily discover that Kyong's mathematical thoughts have been strongly influenced by the three books mentioned above. During the procedure of writing, he consulted the contents of these three books. *Muk Sa Jib San Beob* has three volumes named heaven (upper), earth (middle) and men (lower). The book is divided into 25 subjects and 400 questions altogether. Among them, there are 293 questions having one solution, 99 questions having two solutions and 8 questions having three solutions. The main frame of catalogue is based on *Xiang Ming Suan Fa* and has some similarity with *Suan Xue Qi Meng* . The questions are expressed in the form—"question—answer" which includes "question—answer—solution", "question—answer—first solution—second solution", "question—answer—first solution—second solution—third solution" and so on. The book pays clear attention to the problems with many solutions. The style of giving solutions is new and original.

2.2 The Three Chinese Mathematical Works Available in Korea at That Time

Xiang Ming Suan Fa's Contents

Xiang Ming Suan Fa is divided into two parts. Volume 1 contains 16 parts, which includes:

1. Nine chapters concrete number	2. Big or small concrete number
3. Composite number table	4. Measure with bucket and hu (dry measures)
5. Scales and acreage	6. Arithmetical table
7. Multiply divide	8. Factorization
9. Addition	10. Multiplication
11. Division on the abacus with a divisor of one digit	12. Subtraction
13. Division on the abacus with a divisor of two or more digits	14. Demand one
15. Quotient division	16. Reduction of a fraction

Details: the question of "factorization" is that one factor has one digit; the question of "addition" is that one of the factor's first number must use digit. In "multiplication", multiplier has no distinction; "Subtraction" problem is just the opposite to the addiction. Subtrahend's first number must use digit 1; "Demand one" and "quotient division" are all division problem; "Reduction" is the problem for a fraction.

Volume 2 contains 11 parts, which includes:

17. Multiply the difference and divide the same	18.Expense from the goods
19. Cui Fen	20.Harmonious Cui Fen
21. Cloth piece	22. Scales
23. Stacking	24. Calculate storage
25. Measure acreage subject	26. Food of land
27. Construction	

Details: "Multiply the difference and divide" the same is about the ratio. "Expense from the goods" includes two questions. One is the grain transportation, the other is about dyeing the silk gauze, which is about the expense is part of the goods. "Cui Fen" means to calculate part of the sums from the already known amount and the whole price by using the price difference of each unit. "Harmonious Cui Fen" only has two questions. The last few questions are about the authority dividing the grain, silver and silk. "Cloth piece" and "Scales" are not named after the definition of maths, the questions of which belong to Division by multiply subject. Product from the already known Stacks base angle includes four questions as the name says. "Calculate storage" includes eight questions meaning to get the storage rices through the already known shape and size of the cellars. "Measure acreage subject" includes 13 divisions about the acreage of different fields. Not named after the definition of maths, the questions of "food of land" belong to division by multiply subject. "Construction" is about nine questions which deal with the volume of wall, platform, dyke, weir, and canal.

The feature of *Xiang Ming Suan Fa* Every question has its own solution. The content is rather simple and obvious. The classifying styles of each part of the contents are not based on the mathematical operation's pattern. Resaonably enough *Muk Sa Jib San Beob* does not adopt some of those names, such as cloth piece端疋，scales斤秤 and land and food田亩纽粮.

Suan Xue Qi Meng's Contents [8]

Suan Xue Qi Meng is really a good introductory book. The whole book has three volumes named heaven, earth and men, including 20 subjects. It contains 259 mathematic problems, and arranged them in the style of 'from shallow to deep', starting from four arithmetic operations to evolution and *tianyuan* method and so on. It contains all the frequently used mathematics problems of that time. At the beginning of *Suan Xue Qi Meng*, the author gives 18 mathematics formulas in lyric form, conversion tables, frequently used mathematical constants, methods of positive and negative numbers, *Kai Fang Shu*, and then some applications. It's a good prerequisite for further reading the *Suan Xue Qi Meng*. This style of writing is rarely seen in the previous books. It gives vivid feelings of mathematics. The first parts in *Xiang Ming Suan Fa* are familiar with this part. But it is less vivid than *Suan Xue Qi Meng* . The main body of the *Suan Xue Qi Meng* is divided into three parts: upper, middle, and lower.

Upper Volume It includes 8 subjects which has 113 mathematic problems. It contains multiplier which has one digit in multiply, multiplier's first place are digit 1 in multiply, long numbers in multiply, divisor's first place is digit 1 in division, long numbers in division, all kinds of ratio problems(including calculating interests, taxes, etc.) and so on.

Middle Volume It includes 7 subjects which has 71 mathematic problems. It contains all kind of acreage, volume, earthwork, complex ratio problems and so on.

Lower Volume It includes 5 subjects which has 75 mathematic problems. It contains many fractional number calculations, stacking problem, round-off method, first- order equation sloving method, tianyuan method and so on.

Yang Hui Suan Fa's Content

Yang Hui has many works. Nine Chapters on Mathematical Procedures in Detail(1261)(详解九章算法), Daily Algorithm(1262)(日用算法)and *Yang Hui Suan Fa*(1274–1275)(杨辉算法) are handed down from the ancient time. *Yang Hui Suan Fa* includes Cheng Chu Bian Tong Ben Mo, Tian Mu Bi Li Cheng Chu Jie Fa and Xu Gu Zhai Qi Suan Fa. He has made it his mission to inherit and carry forward mathematical achievements of Nine Chapters on Mathematical Procedures. He has made outstanding achievements in solution of equation of higher degree, high-order arithmetic progression summation, and magic square multiply and division fast arithmetic. Guo Shuchun acclaimed him as the last mathematician who made progress of mathematics through making explanatory notes of Nine Chapters on Mathematical Procedures during the Chinese mathematic history.

3 Comparative Analysis

3.1 Analysis of the framework and the contents

Only a few people have analyzed the beginning part of the book. Professor Hur Min was the only one who has mentioned it in his translated books and research papers. He has mentioned that "Composite number table" in *Muk Sa Jib San Beob* begins from that nine multiple nine is eighty-one. But in *Xiang Ming Suan Fa* , *Suan Xue Qi Meng* and *Yang Hui Suan Fa* they all begin with one multiple one is one. This article makes more meticulous comparison on "*Po San Seon Seub Mun*布算先习门".

"*Po San Seon Seub Mun*" in Muk Sa Jib San Beob	*Xiang Ming Suan Fa*	*Summary* of *Suan Xue Qi Meng*
九九合数	九九合数	释九数法
大小名数	大小名数	大数之类　小数之类
斗斛法	斗斛	斗斛法起率
丈尺法	丈尺	段匹起率
斤秤法	斤秤	斤秤起率
田亩法	田亩	田亩起率

Among them, *Muk Sa Jib San Beob*, *Xiang Ming Suan Fa* , and *Suan Xue Qi Meng* have 6 names which is named for its contents. But in *Muk Sa Jib San Beob* adopts 2 names of *Xiang Ming Suan Fa* and only transforms the other 4 names:

Some parts which *Xiang Ming Suan Fa* does not has but *Suan Xue Qi Meng* has. for example: Nine chapters concrete number, Arithmetical table and Multiply divide these three parts. *Muk Sa Jib San Beob* does not absorb these three parts in it.

The parts which *Suan Xue Qi Meng* has but *Xiang Ming Suan Fa* does not have:

"*Po San Seon Seub Mun*" in *Muk Sa Jib San Beob*	*Summary* of *Suan Xue Qi Meng*
九归法	九归除法
纵横诀	明纵横诀
斤下留两法	斤下留两
铢两相求法	求诸率类
古圆径法	古法圆率
刘徽新术	刘徽新术
冲之密率	冲之密率
开方法	明开方法
约分作名法	明异名诀

The parts which *Suan Xue Qi Meng* has but *Xiang Ming Suan Fa* does not have and also not been adopted by *Muk Sa Jib San Beob*: signed and unsigned number, multiply and division .

The parts which only *Muk Sa Jib San Beob* has:

Localization method, Fang Xie method, punch a hole to firm soil method, area of circle method, area of spherome method, Gou gu method, plane geometry method and solid geometry method.

At this part, Kyong simplifies the formula, especially that for the π for which he uses number 3 instead. So the listed areas, volumes, etc. all uses the number 3 instead. The other noteworthy place is that *Gou Gu* is construed as the form of $3:4:5$ or the form of general condition. Although the

In Muk Sa Jib San Beob	In *Xiang Ming Suan Fa*	Remarks
纵横因法门	因法	Similar
身外加法门	加法	Similar, only 3 questions without corresponding
列位乘法门	乘法	Similar
单位归除门	归法	Similar
身外减法门	减法	Similar
随身归除门	归除	Similar
随身归除门	求一；商除	Similar
身外减法门	商除	Similar
异乘同除门	异乘同除	Similar
归除乘实门	异乘同除	Similar
就物抽分门	就物抽分	Similar
归除乘实门	差分	Similar
就物抽分门	就物抽分	Similar, only 3 questions without corresponding
差等均配门	差分；和合差分	Similar
堆垛开积门	堆垛	Similar, only 5 questions without corresponding
仓囤积粟门	盘量仓窖	Similar
田亩形段门	丈量田亩	Included
商功修筑门	修筑	Similar , 4 questions without corresponding

Comparison of the contents in *Muk Sa Jib San Beob* and *Xiang Ming Suan Fa*

narration is not wrong, he uses 3 : 4 : 5 as a known conditions in all the right-angle side of right angled triangle in the following examples. It states that he does not deeply understand all the *Gou Gu* method. This paper will further analyze this condition in tetails later in the paper.

Except the “scales” and “cloth piece”, subjects in *Xiang Ming Suan Fa* do not have corresponding subjects in *Muk Sa Jib San Beob*, other parts all have corresponding ones.

But “scales” and “cloth piece” are contained in “*Gui Je Sung Shil Mun*” and “Multiply the difference and divide the same” and these parts are better concluded in *Muk Sa Jib San Beob* than in *Xiang Ming Suan Fa*. “Reduction of a fraction” in *Muk Sa Jib San Beob* does not have corresponding parts in other books. “Reduction of a fraction” has 4 questions which make the fraction into the most simple type of fraction. “Land and Food” in *Xiang Ming Suan Fa* are not used in *Muk Sa Jib San Beob*. It can be seen that *Muk Sa Jib San Beob* has absorbed all the good parts. The problems are better accepted, rejected and concluded by Kyong.

From above, we can conclude that the content of *Xiang Ming Suan Fa* has been covered. The content which is not concluded into *Xiang Ming Suan Fa* , Muk Sa Jib has been summed up and made its choice in Muk Sa Jib San Beob.

Form of comparison of the contents in *Muk Sa Jib San Beob* and *Xiang Ming Suan Fa* is as in the Table 3.1:

3.2 Analysis of Several Distinctive Aspects

In *Muk Sa Jib San Beob*, “*Hwa Chui Ho Hae Mun*和取互该门” 10 questions has special part. The contents of it correspond with *Suan Xue Qi Meng*’s “equation plus-minus subject”. In *Muk Sa Jib San Beob*, there are several original questions. The supreme feature is that Kyong did not use the original solving steps. Kyong solves the question on the basis of the feature of unknown number’s quotient. It is really good that solving problem has his only skill. But from avoiding using negative number, it reflects the author's determination not to use the negative numbers. However, this kind of refusal of the new ideas must have its own limitations. From other perspectives, it is reflected that his cognition of mathematical theory is still weak. Furthermore, from the angle of solving the *Gou Gu* subject problem, it also can reflect it. There are four problems which are converted from 2 problems in "equation plus-minus subject". It loses Zhu Shijie's punchline or its profound theory. At the evolution part, the solving capacity does not reach the level of power of four. Moreover, it does not use *tianyuan* method to solve the problem. It shows that Kyong did not recognize the advanced technic of tianyuan method in depth. Nonetheless, from the solving capacity of Question 40 to Question 43 in "evolution to solve the secret subject", it seems that algebraic thought has sprouted in it. However, there is no manifestation of unknown number and it only is some particular example. It can not cover all the equation problems, so it can not be expanded. It is a pity. We can only regard it as his capacity in solving method.

4 Educational Significance

Kyong Sun-Jing wrote the book from the view of an educator. He spared no effort to write an advanced mathematics book which most of the people cannot understand. *Muk Sa Jib San Beob* aims at the division into details. That is to say, the problems are easy and simple for people to understand with several solutions. In addition, the author can analyze and solve the problem from specific to general. It can be said that this book was very rare and estimable in 17th century Chosun Korea.

5 Conclusion

To sum up all the factors, after comparing the *Muk Sa Jib San Beob* with the main Chinese works introduced in Korea and after analyzing the existing historical materials, we can conclude that Kyong Sun-Jing's *Muk Sa Jib San Beob* is written on the basis of the book *Xiang Ming Suan Fa* and consulted *Suan Xue Qi Meng* and *Yang Hui Suan Fa*. Kyong does not fully understand the book *Suan Xue Qi Meng* and *Yang Hui Suan Fa*. It is known that in the Chosun Period the books *Xiang Ming Suan Fa*, *Suan Xue Qi Meng* and *Yang Hui Suan Fa* were used for selection exam for officers. But in our opinion, because Kyong Sun-Jing is a dignitary and a main instructor of mathematics, mathematics level of *Xiang Ming Suan Fa* is more universal in 17th century Chosun and few people could understand *tianyan* method. There are hardly any people who could understand and make good use of *Suan Xue Qi Meng*.

Bibliography

[1] Collection of the Historical Sources in the Science and Technology of Korea: Mathematics. Ed. Kim Yong-Woon. 1 vol. Soul: Lijiang P, October 1985.

[2] Horng Xingshi, Horng Yingen, and Kim Changyi."Pythagoras Theorem in 18th Century Korea" in the Journal of Korean society for the history of mathematics 20.4 (2007):1–22.

[3] Hur Min. "Comparisons between *Suan Xue Qi Meng* and *Muk Sa Jib San Beob*." in the Journal of Korean society for the history of mathematics 20.1 (2008):1–22.

[4] Horng Wann-Sheng, and Li Chien-Tsung. “Kyong Sun-Jing and a Mathematics Workshop in 17th Century Choson Korea” in Sino-logical Study. Taiwan. 25.1 (June 2007):313-3-14.

[5] Jing Yushu. Chinese Mathematics Collection though the Ages. 1 vol. Jinan: Shandong People’s P, March 1994.

[6] Collection of the Historical Sources in the Science and Technology of Korea: Mathematics. Ed. Kim Yong-Woon. 10 vol. Seoul: Ryeogang P., October 1985.

[7] Yu Feng. History of china: Science History in Yuan Dynasty. Ed. Shi Zhongwen and Hu Xianglin. Beijing: People’s P. April 1994.

[8] Li Yan. The History of Chinese Mathematics. Shanghai: The Commercial P., 1935.

A STUDY ON “GUSURYAK” OF CHOI SEOK JUNG
최석정의『구수략』에 대한 연구

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ABSTRACT

We study Choi Suk Jung’s『GuSuRyak』on perspective of philosophy of mathematics. Choi Suk Jung wanted to explain systems of mathematics on the basis of『Book of Changes』in『GuSuRyak』. He especially tried to embody Shao Yung’s ‘Sasang Theory’ in『GuSuRyak』. He eventually redefined『Nine Chapters on the Mathematical art』in order to account for rule of mathematics as ‘Sasang Theory’. This is the unique view on mathematics in history of Chosun mathematics. We conjecture that Choi Seok Jung tries to establish the mathematical principle based on Shao Yung’s ‘Sasang Theory’ in『GuSuRyak』.

Keywords: Choi Seok Jung(崔錫鼎), GuSuRyak(九數略), Nine Chapters on the mathematical art(九章算術), Shao Yung(邵雍), Book of Changes(周易), Sasang Theory(四象論), Chosun mathematics

1 들어가는 말

조선시대 산학서 번역 작업이 본격적으로 시작되기 전 우리에게 가장 많이 알려진 산학서는 다양한 형태의 마방진이 실려 있는 최석정(崔錫鼎)의『구수략』이다. 아마도 이는 다른 산학서의 저자들과 달리 최석정의 높은 사회적 신분과 일본에서도『구수략』에 대한 연구가 이뤄졌다는 입소문에 기인할 것이다(김용국, 김용운, 2009).

조선시대 산학서 번역 작업이 어느 정도 진척된 이후 다른 산학서와『구수략』의 내용을 상세히 살펴보면, 그 명성만큼 조선시대 산학서 중 수학적 아이디어가 뛰어난 저작에 해당되지는 않는다는 것을 확인할 수 있다. 실제로 산학 번역 작업 이전의 명성이 과대평가였다는 논지의 글을 확인할 수 있다(장혜원, 2006; 홍성사, 2006; 홍영희, 2006).

사실, 시종일관 역학의 관점에서 산학을 정리하고『구장산술』을 재진술하고 있는『구수략』을 단순히 산학서로만 받아들여 평가한다면 온전한 모습을 보기 힘들 것이다.『구수략』은 역학자의 입장에서 보면 독특한 역학서로, 수학자의 입장에서 보면『구장산술』의 특이한 주석서로 읽힐 수 있을 것이다. 두 입장 중 어느 한 쪽만을 취한다는 것은 반쪽짜리『구수략』을 대면하고 있는 형상이라 할 수 있을 것이다. 결국 수학자와 역학자의 교류가 거의 없는 현 연구 풍토에서 최석정의『구수략』은 자신의 자리를 제대로 잡지 못하고 부유하고 있다고 볼 수 있다.

최석정은 역학 이론 중 소옹(邵雍)의 사상론(四象論)을 바탕으로 산학의 법칙을 설명하고 있다. 사상론은 모든 존재의 이치를 사상, 즉 태양(太陽), 태음(太陰), 소양(少陽), 소음(少陰)으로 설명하는 이론이므로 산법(算法) 역시 이 이치를 벗어나지 않는다는 것이 최석정의 기본 생각이다. 이것은 산법을 하나의 술(術)로 해석하지 않고 그것들이 작동하는 원리의 기본 철학을 제공하려는 것이 최석정의 새로운 기획이라 할 수 있다. 이러한 관점은 넓은 의미의 수학철학을 제공하려는 시도로 해석할 수 있을 것이다

2 최석정의 삶

『구수략』의 저자 최석정(1646 1715)의 본관은 전주, 초명은 석만(錫萬), 자는 여시(汝時)·여화(汝和), 호는 존와(存窩)·명곡(明谷), 시호는 문정(文貞)이다. 할아버지는 영의정 명길(鳴吉)이며 아버지는 한성좌윤 후량이고, 아우 석항(錫恒)과 함께 후상(後尙)에게 입양되었다. 또한, 최석정은 남구만(南九萬)과 박세채(朴世采)의 문인이다(이경구, 2006).

아홉 살 때 『시경』과 『서경』을 외웠고, 열두 살에 『주역』을 풀어 신동으로 알려진 최석정은 1666년에 진사시에 합격했고, 1671년에 정시문과에 급제하여 검열, 설서, 봉교, 교리 등을 역임했다. 1685년, 부제학으로 있을 때 당시 소론의 영수이던 윤증(尹拯)을 옹호하고 영의정 김수항(金壽恒)을 논척하여 한때 파직되었다. 그 뒤 이조참판, 한성부판윤, 이조판서를 지내고, 1697년 우의정이 되어 왕세자 책봉을 위해 주청사로 청나라에 다녀왔다. 1699년 좌의정에 올라 대제학을 겸하면서 『국조보감』의 속편을 편찬하고 『여지승람』을 증보하도록 했다. 1701년 영의정이 되었으나, 왕세자의 보호를 위해 장희빈의 처형에 반대하다가 진천에 부처(付處)가 되었다. 이듬해 풀려나 판중추부사를 거쳐 다시 영의정이 되었으며, 이후 노론과 소론의 격렬한 당쟁 속에서 소론을 이끌며 모두 여덟 차례 영의정을 지냈다.

최석정은 조부 최명길의 학문을 계승하고 일찍이 장유(張維)의 『계곡만필』에 자극받아 왕수인(王守仁)의 『전습록』에 관심을 가지게 되면서 양명학을 공부했다. 그는 조부가 양명학자가 아니라고 역설하면서도 왕수인을 양명자(陽明子)로 부르는 등 겉으로는 주자학을 추종하면서도 속으로는 양명학을 긍정하는 학문 태도를 지니고 있었다. 따라서 그는 이 시기 노론의 경직된 학문 풍토에서 벗어나 주희의 설에 구애받지 않는 자유로운 사고를 전개시킬 수 있었다. 일찍이 그가 찬집한 『예기유편』도 그 내용에 있어서 정주(程朱)가 정한 것과 다른 점이 많아 노론의 거센 공격을 받았다. 1709년 홍문관에서 이 책을 간행하기로 했으나 승지 이관명(李觀命), 성균관 유생 이병정(李秉鼎) 등으로부터 주자를 배반했다는 비난을 받아 결국 판본이 소각되었다. 문장과 글씨에 뛰어났으며, 음운학에도 정통하여 『경세정운도설』 등을 찬집했다. 저서로 『명곡집』 36권이 있으며, 편서로는 『좌씨집선』, 『운회전요』, 『전록통고』 등이 있다. 숙종 묘정에 배향되었으며, 진천 지산 서원에 제향되었다.

3 『구수략』의 구성과 내용

『구수략』은 건과 곤, 두 권으로 나뉘어 있고 건과 곤은 각각 갑과 을, 병과 정, 모두 네 편으로 이루어져 있다. 본편인 갑, 을, 병의 내용은 '수원(數源)', '수명(數名)', '수위(數位)', '수상(數象)', '수기(數器)', '수법(數法)'으로 구성되어 있고, 부록인 정의 내용은 '문산(文筭)', '주산(珠筭)', '주산(籌筭)', '하락변수(河洛變數)'다

갑편이 시작되기 전에 목차에 해당되는 '목록'과 긴 '인용서적'이 나오는데, 대부분의 조선 산학서와 달리 다양한 참고문헌을 '인용서적'에 모아 놓고 있다. '인용서적'에는 사서삼경을 비롯해 『주례』, 『예기』 등을 포함하는 경서(經書) 12권, 『장자』, 『순자』, 『『손자』 등을 포함하는 제자(諸子) 6권, 『사기』, 『한서』,

『강목』 등 제사(諸史) 3권, 『소자전서』, 『주자대전』, 『역학계몽』 등을 포함하는 제집(諸集) 5권, 『구장산경』, 『칠정산』, 『산학계몽』, 『산학통종』, 『승제산』, 『적산기법』, 『전묘비류』, 『천학초함』, 『주산』, 『상명산법』. 『묵사집』 등 산서로 이루어진 제서(諸書) 11권, 총 37권이 포함되어 있다. 산서는 원나라, 송나라, 명나라 등의 곽수경, 주세걸, 정대위, 양휘의 저서뿐만 아니라 서양 수학의 내용을 번역한 『천학초함』과 『주산』이 포함되어 있고, 조선 산학서로는 유일하게 경선징의 『묵사집』을 언급하고 있다(최석정, 2006a).

3.1 갑편

『구수략』은 책 전반에 대한 별도의 서문은 없지만, 중요한 장에 짧은 서문을 덧붙이는 형식이다. 전체적으로 살펴보면 갑의 '수명', '수위', '수상', '수기'와 '수법'의 앞머리가 실질적인 전체 서문을 대신하고 있다고 볼 수 있다. 이 책의 첫 대목인 '수원'은 수의 근원에 대해 설명하고 있는데, 수의 발생은 도(道)에서 시작된다고 보고 있다. 또한 1, 3, 5, 7, 9에 해당하는 천수(天數)와 2, 4, 6, 8, 10을 지칭하는 지수(地數)가 하도낙서(河圖洛書)에서 비롯됨을 <역전>의 '계사전'을 들어 설명하고 있다. 다음으로 '수명'은 도량형 각각에 대한 의미와 사용되는 단위명과 그것들의 관계에 대해 설명하고 있다. 또한, 정수(正數, 一 萬), 대수(大數, 億 載), 소수(小數, 分 漠)를 나타내는 명칭을 간략하게 소개하고 있다. 세번째, '수위'에서는 수의 자리에 대해 소수, 정수, 대수로 나눠 설명한다. 정수와 대수는 10000을 기준으로 설명하고, 소수는 주로 0과 1사이의 자릿값을 나타내지만 기본적으로 기준 단위 아래의 수를 지칭한다는 것을 실제 도량형에서 사용되는 단위의 예를 들어 설명하고 있다. 그 다음, '수상'에서는 수의 형상을 나타내는 것을 산대로 보고, 산대의 형태와 산대를 놓는 방법을 설명하고 있다. 마지막에 산대를 놓는 방법에 대한 포산구결(布筭口訣)을 덧붙이고 있다. '수기'에서는 수의 기(器)를 수의 사물(物)로 풀이하며, 이것을 율, 도, 량, 형에 해당된다고 보고 있다. 또한 규구준승(規矩準繩)이 그것의 법칙이라고 설명한다. 본격적으로 산술의 법칙을 설명하는 '수법'의 앞머리에서 수의 법칙은 용(用)에 해당되고 이것이 변화를 통해 수를 완성시킨다고 설명한다. 이 법칙의 신묘함을 밝히는 것은 인간의 몫이라고 주장하며 서론에 해당되는 부분을 마치고 있다.

'수법'은 크게 '통론사법', '통론팔법', '통론사상' 등 세 부분으로 나눌 수 있다. '통론사법'과 '통론팔법'은 기본적인 사칙 연산에 대해 설명하고 있고, '통론사상'은 앞의 내용과 사상론을 기초로 『구장산술』에서 다루는 내용을 최석정만의 관점으로 설명하고 있다. 따라서 『구수략』의 핵심 내용은 을과 병편에 해당되는 '통론사상'이라고 볼 수 있다.

'통론사법'에서는 4가지 법칙, 현재의 사칙연산에 해당되는 가(加), 감(感), 승(乘), 제(除)를 설명하고 있다. 가법(加法)은 첨가되어 자라나는 성질이 있다는 측면에서 양(陽)에 대응시키고, 감법(減法)은 덜어서 줄어드는 성격이라는 점에서 음(陰)에 대응시키고 있다. 또한 이 두 법칙의 환원관계, 즉 역연산이라는 측면을 강조하고 있다. 마찬가지로 승법(乘法)은 새로운 수를 낳는다는 측면에서 양, 제법(除法)은 약(約)해서 나눈다는 측면에서 음에 대응시킨 다음 이 두 법칙의 환원관계를 설명한다. 이 4가지 법칙이 산술의 권여(權輿), 즉 시작)이고 형범(型範, 모범)이 됨을 강조하며 뒤이어 이 4가지 법칙의 구구를 자세히 기술한다. 그리고 뒷 부분에 '구구도사(九九圖四)'와 '구구합수구결(九九合數口訣)'을 싣고 있다. '구구도사'에서 곱셈 구구에 대한 4가지 그림을 제시하고 있는데, 2가지는 한문으로 채워진 도표이고 나머지는 산대 형태의 그림으로 채워진 도표이다. 특히 산대 그림 중 구구모수상도는 채원정의 '범수방도'를 인용하고 있다. 이 4가지 그림은 결국 곱셈 구구를 수(數)와 상(象)으로 형상화한 것으로 볼 수 있다. '구구합수구결'은 지금의 곱셈 구구단에 해당되는데, 1단부터 시작되는 것이 지금과 다른 점이다. 맨 마지막에 가감승제, 즉 사칙연산 모두에 익숙해지기 위해서는 곱셈 구구가 가장 중요하다고 덧붙이고 있다.

‘통론팔법’에서는 승법과 제법을 8가지로 분류해서 산대셈을 기본 원리로 설명하고 있다. 정수(正數)와 변수(變數)로 나눠 정수에 2가지, 변수에 6가지 법칙을 대응시킨다. 여기서 정수와 변수는『구수략』에서 자주 나오는 범주인데, 이것은『주역』에서 시초점을 칠 때 괘를 뽑는 과정을 빌려 온 것으로 보인다. 괘를 뽑고 나서 변하지 않는 효(爻)가 있고 변하는 효과 있는데, 변하는 효를 변효(變爻)라 부른다. 이것과 유사하게 변하지 않는 법칙을 정수, 기본 법칙에서 변형되는 법칙을 변수라 명명한 것으로 보인다. 실제로 ‘음양정수이법’ 기본적인 승법(양)인 보승법(步乘法)과 기본적인 제법(음)인 상제(商除)를 지칭한다. ‘음양변수육법’에서 승법에는 인법(因法), 가법(加法), 승법(乘法), 제법에는 귀법(歸法), 감법(減法), 제법(除法)이 해당된다. 자세한 분류는 [표 1]과 같다.

	제법 (나눗셈) - **음**			승법 (곱셈) - **양**		
정수 正數	**상제법**			**보승법** (상승, 영산)		
변수 變數	**귀법** (구귀법)	**감법** (신외감법) (경신제)	**제법** (귀제법)	**인법**	**가법** (신외가법)	**승법** (유두승법)

[표1: 팔법의 분류]

우리가 보기에는 동일한 곱셈과 나눗셈을 굳이 분류하는 것이 다소 억지스러워 보일 수 있지만, 이런 분류는 경선징의『묵사집산법』을 비롯해 다른 산학서에서도 나타난다(경선징, 2006; 황윤석, 2006). 여기서 우리가 주목해야 될 것은 분류된 법칙들이 수학적으로 어떤 차이점을 갖느냐가 아니라 8가지로 분류된 방식이다. 최석정이 8가지 법칙을 “건과 곤이 여섯 자식을 거느리는 것과 같다(如乾坤之統六子)”(최석정, 2006, p.41)라고 표현한 것은 이것을 팔괘와 대응시키려는 의도를 나타낸 것으로 볼 수 있다. 8가지로 분류해 나가는 법칙을 ‘문왕팔괘순서지도’([표2])와 비교해보면 좀 더 분명해진다(주희, 1998). 정수의 보승법과 상제법이 각각 건괘와 곤괘에 해당되고 변수의 인법, 가법, 승법이 간괘, 감괘, 진괘에 해당되고, 변수의 귀법, 감법, 제법은 태괘, 리괘, 손괘에 해당된다(최석정, 2006a).

곤모 坤母			**건**부 乾父		
☷			☰		
태 兌 (소녀 少女)	**리** 離 (중녀 中女)	**손** 巽 (장녀 長女)	**간** 艮 (소남 少男)	**감** 坎 (중남 中男)	**진** 震 (장남 長男)
☱	☲	☴	☶	☵	☳
得坤上爻	得坤中爻	得坤下爻	得乾上爻	得乾中爻	得乾下爻

[표 2 : 문왕팔괘순서지도 文王八卦順序之圖]

‘승제원류’에서는 덧셈과 곱셈의 관계, 뺄셈과 나눗셈의 관계에 대해 설명하고 있다. 기본적으로 곱셈

은 누가에서 비롯되는 것이고 나눗셈은 누감에서 시작된다고 보고 있다. 이러한 관계의 원리를 '손익구결(損益口訣)', '구일구결(求一口訣)', '쌍일구결(雙一口訣)'을 통해 다시 설명하고 있다. 갑편의 마지막인 '지분약법'은 분수 계산에 대해 다루고 있다. 분수 계산의 유형을 명분(命分), 약분(約分), 과분(課分), 합분(合分), 석분(石分), 통분(通分), 감분(減分), 승분(乘分), 제분(除分) 등 10가지로 나눠 설명한 다음 다시 10가지 예문을 제시한다. 마지막에 『천학초함』의 14문제를 제시하여 앞의 분수 계산법을 활용하게 한다.

3.2 을편과 병편

최석정은 '통론사상'에서 수법(數法)에 대한 자신의 사상론을 본격적으로 전개하고 있다. 앞에서 가법과 승법은 양에, 감법과 제법은 음에 대응시켰는데, 통론사상에서는 양의(兩儀)에서 사상(四象)으로 나아가 가감승제를 각각 태양, 태음, 소양, 소음으로 세분화시켜 대응시킨다. 최석정은 세상의 모든 이치가 사상에서 벗어날 수 없으므로 수의 이치 역시 사상론에 종속된다고 주장하며, 이를 바탕으로 『구장산술』의 모든 법칙을 설명하고자 한다. 통론사상의 서문은 다음과 같다.

> 『전』에서 말하길 사물이 생긴 후에 상이 있고, 상 이후에 만물이 번성하고, 만물이 번성한 이후에 수가 있다. 수는 1에서 근원하는데, 1이 태극이 된다. 1이 2를 낳으니 양의고, 2가 4를 낳으니 사상이다. 총괄해서 말하면 가감승제는 산법의 사상이고, 나누어서 말하면 사상은 각각 사수를 갖춘 것이다. 양의 양은 태양이 되고, 음의 음은 태음이 되고, 음의 양은 소양이 되고, 양의 음은 소음이 된다. 태양은 일이고 태음은 월이고 소양은 성이며 소음은 신이다. 하늘과 땅 사이에 오직 사상만이 있을 따름이다. 수의 이치가 비록 지극히 심오하다 할지라도 어찌 이것에서 벗어나겠는가? 이제 사상에 새로운 뜻을 밝혀서 『구장산술』의 모든 법칙들을 풀고자 하니, 보는 사람은 새로운 설을 만든다고 말하면서 홀대하지 말라. (최석정, 2006a, p.105)

최석정은 '통론사상'을 사상정수(四象正數)와 사상변수(四象變數)로 나눈 다음, 사상정수를 다시 정(正)과 변(變)으로 나눠 정에는 누가, 누감, 상승, 상제, 변에는 총승, 총제, 준승, 준제로 분류한다. 여기서 을편이 끝난다. 병편으로 넘어가 사상변수를 사상정수와 마찬가지로 정과 변으로 나눠 정에는 방승, 방제, 준승, 준제, 변에는 체승, 체제, 교승, 교제로 분류한다. 이렇게 분류한 법칙 중 누가와 누감을 제외한 14가지 법칙으로 『구장산술』에 나오는 구장의 내용을 각각 설명하고 있다.

'방정구결'은 음수와 양수가 섞인 '방정(方程)'장과 관련된 문제에서 부호 처리를 하는 방법을 소개하는 구결이다. 그리고 최석정은 방정은 영육(盈朒, 영부족)이 변한 것이라며 이 두 가지 관계에 대해 설명을 덧붙인다. 그 뒤 '구장명의'는 『구장산술』의 구장 각각 이름의 뜻을 밝히고 있다.

마지막으로 '통론사상'에서 자신이 전개해 온 내용을 토대로 '구장분배사상'에서 구장을 사상에 대응시키고 다시 '사상제법분배구장'에서 사상의 16가지 법칙을 구장에 대응시킨다. 구체적인 대응 내용은 [표3]과 [표4]와 같다(최석정, 2006b).

'통론사상'에서 자신의 논지를 정리한 다음, 최석정은 '고금산학'에서 역대 산학자를 정리한 내용으로 병편을 끝내고 있다. 예수가 산(筭)과 수(數)를 만든 것에서부터 시작해서 주공이 구장을 만들어 육예(六藝)로 만민을 교화시키고, 공자는 예에 능하고 공자의 제자 72인은 육예에 능통하다고 기술한다. 또한, 한, 당, 송, 명, 청의 유명한 유학자와 산학자, 선교사까지 예를 들고 있다. 우리나라 산학자는 최치원부터 남호, 황희, 이황, 이이, 김시진, 박율, 경선징 등을 거론하고 있다.

태양 太陽	일日	일一	방전 方田
태음 太陰	월月	이二	속미 粟米 소광 少廣
소양 少陽	성星	삼三	상공 商功 쇠분 衰分 영육 盈朒
소음 少陰	신辰	사四	균수 均輸 구고 句股 방정 方程

[표3 : 구장분배사상(九章分配四象)]

정수 팔법 正數 八法	태양	일	일	누가	총승 (방전)
	태음	월	이	누감	총제 (속미)
	소양	성	삼	상승 (방전)	준승 (상공)
	소음	신	사	상제 (속미)	준제 (균수)
변수 팔법 變數 八法	태양	일	일	방승 (방전)	체승 (방전)
	태음	월	이	방제 (소광)	체제 (소광)
	소양	성	삼	자모준승 (쇠분)	영허교승 (영육)
	소음	신	사	구고준승 (구고)	정부교제 (방정)

[표4 : 사상제법분배구장(四象諸法分配九章)]

3.3 정편

정편은 부록에 해당되고 '문산', '주산(珠筭)', '주산(籌筭)', '하락변수' 등 네 부분으로 구성되어 있다. '문산'은 사칙연산을 포지금(鋪地錦)으로 설명한 다음, '문필구결(文筆口訣)'을 덧붙인다. 마지막으로 칸을 만들어 가며 사칙 연산을 하는 방법에 대해 전체적인 설명을 한 다음, 이 모양새와 사각형 그림에서 연산을 해 나가는 방향이 <주역>의 몇몇 괘의 형태와 유사한 점을 들어 이 두 가지가 같은 류라고 설명한다. 또한 최석정은 정대위(程大依)의『산법통종』은 중요한 책이지만 승법만 설명하고 있어 이것이 근심스러워 양과 음의 짝을 맞추기 위해 자신이 제법을 첨가하고, 이 두 가지 법칙의 원류를 보여주기 위해 가법과 감법을 드러낸다고 덧붙이고 있다(최석정, 2006b). 이것은 황윤석(黃胤錫)이『산학입문』에서『산법통종』을 인용하며 곱셈만 설명하고 있는 것과는 다르다(황윤석, 2006). 이러한 차이는 수학적 견해보다는 산법을 음양과 사상의 틀에 넣어 설명하려는 최석정의 독특한 시각에 기인한다고 볼 수 있다.

'주산(珠筭)'에서는 중국의 주판셈을 소개하고 있다. 자세한 계산 방법은『산법통종』에 있으니 반복하지 않겠다고 말하며 간단히 넘어가고 있는데, 이는 주판셈보다는 산대셈에 대한 최석정의 선호도를 반영하는 것처럼 보인다. 이는 앞의 '문산'에서의 태도와 사뭇 다르다. 또한 최석정은 근세 중국의 관사가 모두 주산(珠筭)을 사용하고 산대를 폐하고 일본 역시 이 주산을 사용하고 있는 것은 모두 깨달음이 부족한 것이라고 직접적으로 비판하고 있다. 이것은 조선이 중국이나 일본에 비해 산대셈을 오랫동안 고집한

시대적 분위기에 기인하는 것이라고 가늠해 볼 수도 있겠지만, 이보다는 『주역』에 나오는 시초의 형태와 산대의 유사성에 기인한다고 볼 수 있다.

'주산(籌筭)'에서는 선교사 쟈크로의 『주산』을 토대로 승법, 제법, 개평방법(開平方法)을 설명하고 있다. 마지막에 최석정은 서양 산법인 문산과 주산(籌筭)을 비교하면서 두 가지가 목적이 같으나 방법이 다르다고 지적한다. 문산은 보편적이고 정교하나 주산((籌筭)은 지루하며 졸렬하다고 비판하면서 본인이 글로 남기는 이유는 새로운 것에 관심 있는 사람들이 취할 수 있도록 할 따름이라고 말한다.

마지막 '하낙변수'는 하도와 낙서로 시작하여 이들의 다양한 변형인 마방진 46개에 대해 다루고 있다. 마방진 일부는 『양휘산법』과 『산법통종』에 나와 있는 것이고, '낙서오구도', '낙서육구도', '낙서칠구도', '낙서팔구도', '낙서구구도', '범수용오도', '기책팔구도', '중상용구도', '구구모수변궁용도' 등은 다른 산학서에서 찾아볼 수 없는 형태이나 이것들에 대한 자세한 설명은 생략되어 있다

4 『구수략』과 『주역』

'주역'이라는 이름은 주대(周代)의 역(易)이라는 뜻이다. 『주역』은 서주(西周) 시대에 형성되었고 시초의 변화로서 변역(變易)을 예측하는 것이다. 『주역』은 폭넓게 전해졌으나 후대 사람들이 이해하기 어려워 전수 과정에서 설명과 해석이 나타나기 시작했다. 원전과 해석을 구분하기 위해 『주역』을 원본인 『역경』과 해설서인 『역전』으로 구분하기 시작했다. 『역경』은 64괘로 구성되어 있고, 매 괘마다 괘명, 괘상, 괘사, 효사 등 4부분으로 이뤄진 간단한 기술을 포함한다. 이러한 간략한 내용에 의미를 부여하고 설명을 덧붙이는 것이 『역전』이다. 『역전』은 『역경』 점서(占書)를 넘어서서 하나의 철학서로 자리매김을 할 수 있는 토대를 마련한 것으로 볼 수 있다. 『역전』 중 '단사상전', '단사하전', '상사상전', '상사하전', '문언전', '계사상전', '계사하전', '설괘전', '서괘전', '잡괘전' 등 10가지를 '십익(十翼)'이라고 부르며 중요하게 생각하여 한대(漢代) 이후에는 『주역』에 『역경』과 '십익'을 모두 포함하고 있다(주백곤, 1999).

『역경』은 중국 고대의 가장 중요한 전적(典籍) 가운데 하나로 2000년에 달하는 경학(經學) 시대 동안 가장 으뜸가는 위치를 차지해 왔다(고지마 스요시, 2004). 한대부터 『역경』을 주석하고 연구한 저작이 쏟아져 나옴으로써 중국 고대 문화 발전의 중요한 자리를 차지하게 되었다. 이러한 연구 흐름은 크게 두 학파, 즉 상수학파(象數學派)와 의리학파(義理學派)를 형성하였다. 음양기우(陰陽奇遇)의 수와 괘효상 및 8괘가 상징하는 물상(物象)으로 『주역』을 설명하는 데 중점을 둔 역학을 상수학이라 한다. 반면에 괘명의 의의와 괘의 성질로 『주역』을 해석하여 괘효상과 괘효사의 의리를 밝혀 드러내는 데 중점을 둔 역학이 의리학이라 한다(임병학, 2008). 상수학은 한대 역학의 주류였고 위진 수당 시기에는 의리학파가 우위를 점하였다. 송명 시대에는 상수학파와 의리학파가 병행하면서 도학(道學)과 상호 적응하여 상수학파는 수학파(數學派) 상학파(象學派)로 나뉘고, 의리학파는 이학파(理學派), 기학파(氣學派), 심학파(心學派), 공리학파(功利學派) 등 여러 갈래로 나뉘었다. 청대에 이르러 다시 한대 역학의 부흥이 이루어지며 다양한 방면으로 역학이 발전했다. 이러한 역학(易學)의 발전은 현대까지 이어지며, 동아시아 사상사에서 중요한 역할을 차지한다(료명춘 외, 1994).

최석정의 『구수략』은 내용과 구성에 있어서 역학자 중 소옹의 영향을 가장 많이 받은 것으로 보인다. 소옹은 수의 설명을 중시하는 선천학(先天學)을 제시하였고 역학의 유파 가운데 수학파를 건립하였다. 이 유파는 기우수(奇偶數)의 기초 하에 괘효상의 변화를 설명하여 '수(數)가 상(象)을 생(生)한다'고 주장하였기 때문에 수학파로 불린다(주백곤, 1999). 소강절은 이 이론을 토대로 일년 사시 음양이기의 소장 과정을 설명하고 나아가 인류의 역사와 우주 역사의 변천 과정을 추산하였다. 이러한 소옹의 사상은 주희에 의해 도학의 전통에서 다시 부활하였다. 주희는 주돈이, 소옹, 장재, 정호, 정이를 등을 '북송오자

(北宋五子)'라 부르며, 이들이 공자와 맹자의 도통을 잇는다고 주장하였다. 주희는 크게 의리학파 입장에서 적극적으로 상수학파의 주장을 포섭하는데, 이는 『역학계몽』에 잘 드러나 있다(주희, 2008). 이렇듯 성리학의 성립 배경 속에도 소옹의 사상은 하나의 흐름을 형성하고 있다.

최석정이 『구수략』의 첫 장, 첫 문장에서 수의 근원은 도(道)라고 일컫는 것을 이러한 흐름을 반영한다고 볼 수 있다. 즉, 이는 도학의 전통에서 『구수략』을 저술하려는 의도로 읽힌다. 조선 시대 선비들이 세상만물을 보는 기본 철학은 『주역』이다. 최석정 역시 이것을 바탕으로 산학의 경전인 『구장산술』의 모든 산술 법칙을 설명하고자 했던 것으로 보인다. 따라서 '고금산학'에서 우리나라의 유명 산학자를 신라시대 유학자인 최치원부터 이황, 이이까지 예를 드는 것은 우리에게는 낯설게 보이지만 최석정에게는 당연한 이치일 것이다. 사실 이황은 주희의 『역학계몽』의 체제를 따른 자신의 저서 『계몽전의』에서 주희의 하도낙서와 선후천역에 대한 탐구를 통해 천하만물의 당연법칙과 소이연의 법칙인 리(理)의 철학 체계를 밝히고 있다(임병학, 2008). 또한 이이는 이황처럼 체계적인 저서는 없지만 『천도책』과 『역수책』에서 『주역』을 원용하며, 이기론의 확실한 논거로 역학 이론을 사용하고 있다(임병학, 2008). 이렇듯 조선 시대 대학자인 이황과 이이 역시 자신들의 이론의 근거로 『주역』에 대한 자신의 해석을 제시하고 있다. 따라서 조선시대 학자로서 어떤 주장을 피력하기 위해 그 근거로 『주역』에 대한 자신의 견해나 선진유학자의 이론을 제시하는 것은 드믄 일이 아니라 할 수 있다.

최석정의 독특한 점은 『주역』에 대한 이론을 인간사의 법칙도 아니고, 만물의 법칙도 아니고, 점을 치는 수의 법칙도 아닌 산학에서 사용되는 법칙을 대상으로 적용하고 있다는 점이다. 또한 역학의 많은 이론 중 상대적으로 조선 학자들에게 그리 큰 영향을 미치지 않은 소옹의 이론을 바탕으로 한다는 점이 특이하다 할 수 있다. 이러한 특이점이 가능했던 것은 양명학적 소양을 갖춘 최명길과 유·불·도를 자유롭게 넘나들었던 장유와 남구만의 영향에 부분적으로 기인한다고 볼 수 있을 것이다(이경구, 2004).

5 나가는 말

『주역』은 현대의 일상사에서 점치는 책으로 경시될 수도 있지만, 공자의 위편삼절(韋編三絶)이라는 말에서 보듯 『주역』은 도학에서 가장 중요한 경전이요, 도학자들이 세상을 보는 기본 철학이라는 것을 알 수 있다. 이러한 역학의 관점에서 산학의 법칙들을 설명한다는 것은 최석정의 입장에서 산학을 하나의 잡학으로 보지 않고 도학의 한 계통으로 설명하고 싶은 의도가 숨어 있다고 볼 수 있다. 즉, 조선시대 수학의 철학적 기초를 다지려는 시도로 해석할 수 있을 것이다.

한편, 최석정의 『구수략』은 소옹의 기획을 확대하려는 시도로도 읽힐 수 있다. 소옹이 『황극경세서』에서 '사상체용지수도'의 번호를 붙여 분류하는 형태와 최석정이 산술의 법칙을 분류하는 방식이 매우 유사함을 확인할 수 있다(소옹, 2002). 소옹은 이 책에서 자신의 사상론에 기초해 음운학에 대해 설명하고 있는데, 최석정의 『경세정운도설』 또한 훈민정음을 사상론에 기초해 음운학적으로 도해한 저서이다. 또한 소옹과 최석정 모두 당대에 시에 능했다는 평가를 받았고 문집을 남기고 있다. 이러한 측면에서 본다면 최석정은 소옹의 학문적 삶을 따라가기 위해 부단히 노력했던 것으로 보인다. 소옹의 『황극경세서』는 역사, 천문학, 음운학 등 다방면을 다루고 있지만 수의 법칙에 대해서는 다루고 있지 않다. 바로 비워 있는 이 지점이 최석정으로 하여금 『구수략』을 기획하게 된 배경으로 추정해 볼 수 있을 것이다. 그렇지만 최석정이 『구수략』을 저술한 진정한 의도에 근접해가기 위해서는 좀 더 큰 틀에서 본격적인 연구들이 필요하다.

REFERENCES

- 경선징, 2006, 묵사집산법 천, 유인영, 허민(역), 서울: 교우사.
- 고지마 스요시, 2004, 사대부의 시대, 신현성(역), 서울: 동아시아.
- 김용국, 김용운, 2009, 한국 수학사, 서울: 살림.
- 료명춘, 강학위, 양위홍, 1994, 주역철학사, 심경호(역), 서울: 예문서원.
- 이경구, 2009, 17세기 조선 지식인 지도, 서울: 푸른역사.
- 임병학, 2008, 易學과 河圖洛書, 서울: 한국학술정보(주).
- 소옹, 2002, 황극경세서, 노영균(역), 서울: 대원출판.
- 장혜원, 2006, 산학서로 보는 조선 수학, 서울: 경문사.
- 주백곤, 1999, 주역산책, 김학권(역), 서울: 예문서원..
- 주희, 1998, 주자전의 상, 성백효(역), 서울: 전통문화연구회
- 주희, 2008, 역학계몽, 김진근(역), 서울: 청계.
- 최석정, 2006a, 구수략 건, 정해남, 허민(역), 서울: 교우사.
- 최석정, 2006b, 구수략 곤, 정해남, 허민(역), 서울: 교우사.
- 황윤석, 2006, 산학입문, 강신원, 장혜원(역), 교우사.
- 홍성사, 2006, 朝鮮 算學의 堆垜術, 한국수학사학회지 19, No 2, 1-24.
- 홍영희, 2006, 朝鮮 算學과 數理精蘊, 한국수학사학회지 19, No 2, 25-46.

- 홍영희, 2006, 朝鮮 算學과 數理精蘊, 한국수학사학회지 19, No 2, 25-46.

HISTORIC INVESTIGATION OF LEGENDRE'S PROOF ABOUT THE 5TH POSTULATE OF "ELEMENTS"

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ABSTRACT

本稿は，高等学校数学科の教材としての可能性を検討するために，「原論」の第 5 公準の証明の誤りの歴史を考察した．その結果，証明は間違いではあるが，数多くの知恵を絞って挑戦してきた証明の数々には，間違いだらけだったと評価することだけでは済まされない教育的価値が内包されていると考える．

1　はじめに

本稿においては，高等学校数学科の教材開発を目的に，「原論」の第 5 公準の証明の試みの歴史に関する考察を行う．そもそも，「原論」の議論そのものは極めて明確であるが，第 5 公準に関してだけは，他の 4 つの公準と比較しても明らかに複雑すぎる．原論」の出版以来，実に多くの数学者がその証明を試みてきた．具体的には，第 1 公準から第 4 公準を前提として，第 5 公準を証明しようという試みである．「原論」の第 1 巻の注釈を書いたプロクレスは，プトレマイオスの試みを執筆した後で，自分自身の証明を記載している．

□一方で，Euclid(330?-270?B.C.) は，どこまでいっても交わらないものを平行と述べている．アラビアにおいて，なぜ，「原論」に第 5 公準が必要であったのかという問題と同時に，「原論」第 5 公準を証明しようとする努力がアラビアの数学者に引き継がれた．その中の証明の一部は，ヨーロッパに輸入された．ヨーロッパにおいても，数学の進展にともない多くの数学者によって，その挑戦がなされたのである．その中においては，Girolamo Saccheri（1667－1733）と Johann Heinrich Lambert（1728－1777）による研究は非常に有名であり，この挑戦から，やがて公準が成立しないとする非ユークリッド幾何の誕生に至るのである．

□この様な「原論」第 5 公準の証明の歴史的変遷の中で，高等学校数学科の教材開発をすることを究極の目的に，そのはじめとして，あまりにも有名なユークリッドの「原論」の第 5 公準の証明について，その歴史的変遷に焦点をあて，その特徴について歴史経緯を数学教育に生かす方向性や具体例を示すことが目的である．

2　ある証明

第 5 公準を証明するためにある命題から始めることとしたものがあるので，その命題から考察する．

命題　　　ある直線が平行な二直線の一方向に交われば他方にも交わる．

証明

半直線BFとFGは点 F から引かれた半直線だから，両者の距離はいくらでも大きくなる．すなわち，点Pが半直線FB上を点B方向に動く時，点 P と半直線 FG との距離 (点Pから半直線 FG におろした垂線の長さ) は点 P が点 F から遠ざかるにつれていくらでも大きくなる．したがって，ある時点で平行線の距離を超えてしまう．これは，FGとCDが交わることを意味する．

実は，この証明には 2 つの問題点がある．まず，最初の問題点は，「点Pが半直線FB上を点Bの方向に動く時，点Pと半直線FGとの距離 (点Pから半直線 FG におろした垂線の長さ) は，点Pが点Fから遠ざかるにつれていくらでも大きくなる．」という点である．点Pから半直線FGにおろした垂線の長さが，増大して行くことは確かであるが，それが無限大に発散することは自明ではなく，証明が必要である．さらにもう一つの問題点は，「平行線の間の距離が一定である．」とするが，ここで証明しなければならない命題である第 5 公準を利用して証明している点である．いわゆるトートロジーである点にある．当然，平行線の定義には，二直線の距離が一定であることは含まれていない．距離が一定であることを証明するためには，どうしても第 5 公準が必要となる．

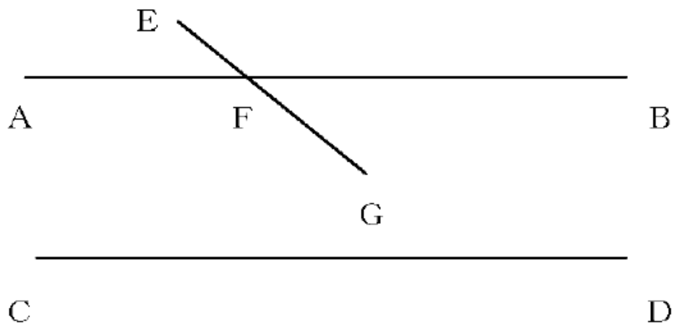

Figure 1: *

図 1 AB // CD の時，直線 EG が直線 AB と交われば直線 CD と交わる．

ただ，一方で，このプロクレスの命題が証明された場合，第 5 公準を証明することは極めて簡単となる．そこで，以下にその証明をする．

証明

$\angle BEF + \angle DFE < 2R$と仮定する．点$E$を通り，$\angle HFE + \angle DFF = 2R$が成り立つ様に$E$を通る直線 KH を引くと，$KH \parallel CD$となる．直線$AB$は，$KH$と交わるので$KH$に平行な直線$CD$とも交わることがプロクレスの命題を利用すれば容易に証明可能である．

3　ルジャンドルによる証明の歴史的変遷

本章においては，19 世紀初頭におけるラグランジェによる第 5 公準の証明

について考察してみる．Adrien Marie Legendre1(752～1833) は，幾何学の教科書 Éléments de géométrie(幾何学原論) を 1794 年に出版し，版を重ねて 15 版は，1866 年に出版された．ルジャンドルの教科書は，フランスの幾何学の教科書の標準となった．このルジャンドルの本は，ガロアが読んで大きな影響を受けたことで知られている．また，ルジャンドルの教科書は，アメリカでも翻訳されて，幾何学の教科書として出版されている．日本においても，明治期の初期に幾何学の教科書として使用された．ルジャンドルのこの本は，本当によく売れた様である．ルジャンドルのこの幾何学の教科書は，各版において第 5 公準の証明を発表している．しかし，この本が非常に売れたにもかかわらず，自らの証明の誤りに気付いては，次の版で新しい証明に挑戦するということを繰り返した．これは何を意味するかと言えば，既に，ガウスが非ユークリッド幾何の可能性に気付いていた時に，ルジャンドル自身は，まだまだ，証明に挑戦していたことになる．ルジャンドルは，1823 年に最後の 12 版が出版されるまで，自身の証明を発表している．以下に，に掲載されたその証明の内の一つについて考察する．また，第 5 公準を仮定すれば，三角形の内角の和は二直角であることが証明可能である．ルジャンドルは，この逆が成り立つこと，言いかえると三角形の内角の和が二直角であることと第 5 公準とは同値であることからその証明をスタートする．

4 ルジャンドルによる証明

命題すべての三角形の内角の和が 2 直角であれば，第 5 公準が成り立つ．

証明$\angle BEF + \angle DFE < 2R$ と仮定する．下記の様に，点F から，$\angle BEF = \angle EFG = 2R$ が成り立つ様に直線FG をひく．さらに，F からAB 上の任意の点 N へ線分を引く．

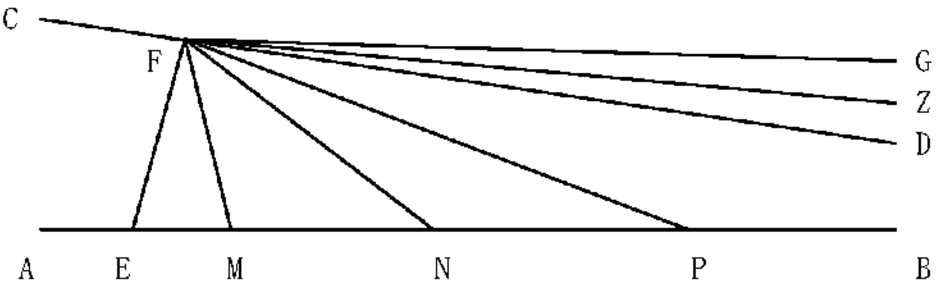

Figure 2: *

図 2

次に，直線AB 上に$MN = FM$ となる様に点M を選ぶ．そうすると，三角形FEM は二等辺三角形となり，その底角$\angle MFN = \angle MNF$ は，$\angle FME$ の1/2 である．$\angle MFN = (1/2)\angle FME \cdots *$, ここで，三角形の内角の和は 2 直角であることを利用した．すなわち，$\angle MFN + \angle MNF + \angle EMN = 2R$ であったので，

$\angle MFG = \angle EFG - \angle EFM = \angle EFG - (2R - \angle FEM - \angle FME) = (2R - \angle FEM) - (2R - \angle FEM - \angle FME) = \angle FME$ となる．したがって，$\star$ より, $\angle NFG = \angle EFG - \angle MFN = \angle FME - \angle MFN = (1/2)\angle FME = (1/2)\angle MFG$ となる．次に直線AB 上に点P を$FN = NP$ が成り立つ様にとると，上の議論と同様にして，$\angle PEG = (1/2)\angle NFG = \angle(1/4)\angle MFG$ を得ることができる．以下，このご議論によってAB 上の点R を$\angle RFG < \angle DFG$ であるように選ぶことが可能である．そうする

と，半直線FDは三角形FERの内部を通ることになり，したがって，三角形の辺EFと交わることになる．よって，直線FDは，ABと交わる．

以上のことから，第 5 公準を証明するためには，三角形の内角の和が 2 直角であることを証明すれば良い事になる．そこで，ルジャンドルは，先ず，はじめに，次の命題を証明することにしたのである．

命題　すべての三角形の内角の和は 2 直角以下である．

証明　まずはじめに，三角形の内角の和は 2 直角よりも真に大きいと仮定する．辺ACを延長して三角形CDEを三角形$ABC\equiv$三角形CDEとなる様に作図する．以下，この様な作業を n 回繰り返して，図 1 の様に，三角形ABCと合同な三角形を作図する．

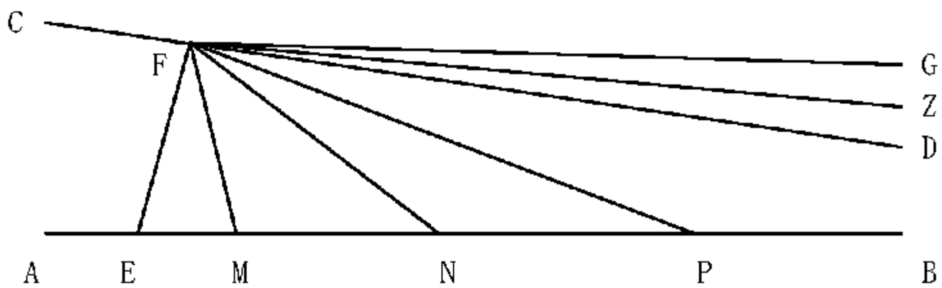

Figure 3: 図 1

つまり，三角形$ABC\equiv$三角形$CDE\equiv$三角形$EFG\equiv\cdots\equiv$三角形NOP. さらに，点Bと点Dを線分で結び，同様にして点Dと点F, 点Fと点Hと次々に線分で結ぶ．$BD\cdots MO$は折れ線となり，一直線にあるとは限らない．

三角形$ABC\equiv$三角形CDEより, $\angle BAC=\angle DCE$であり，ACEは，線分であるので

$2R=\angle BCA+\angle BCD+\angle DCE=\angle BCA+\angle BCD+\angle BAC$が成り立つ．一方，仮定より，$2R<\angle BCA+\angle ABC+\angle BAC$が成り立つ．上の等式を比較して，$\angle ABC>\angle BCD$が成り立つことが理解できる．そうすると，三角形$ABC$と三角形$DCB$において, $AB=DC$, $BC=CB$, $\angle ABC>\angle DCB$が成り立つので，ユークリッドの「原論」の第 1 巻の命題 24 より，大きい角に対応する辺の方が大きい．

すなわち，$AC>BD$となることが理解される．そこで，$AC-BD=\delta>0$とおくと，$AP=nAC$, 折れ線$BD\cdots MO$の長さは，$(n-1)BD$となる．点Aと点Pを結ぶ折れ線$ABD\cdots MOP$の長さは，

$$
\begin{aligned}
&AB+(n-1)BD+OP=AB+(n-1)BD+BC\\
=&AB+BC-BD+nBD=AB+BC-BD+nAC-n\delta\\
=&AP+(AB+BC-BD-n\delta)
\end{aligned}
$$

と書くことができる．そこで，nを十分大きくとり，

$AB+BC-BD-n\delta<0$となる様にすると，点Aと点Pを結ぶ折れ線$ABD\cdots MOP$の長さ$AP+(AB+BC-BD-n\delta)$は線分APの長さより短いことになる．2 点間を結ぶ折れ線で最短のものは，その 2 点間を結ぶ線分であるので矛盾である．この矛盾は，三角形ABCの内角の和が 2

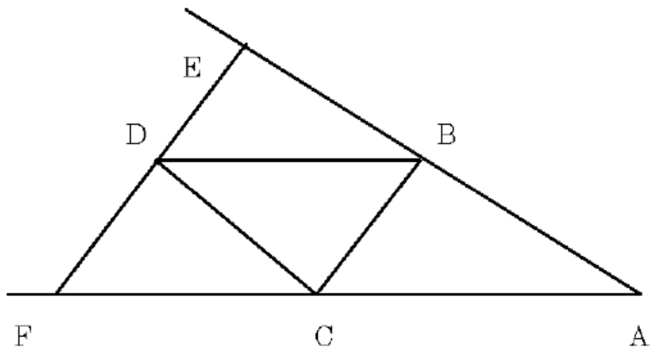

Figure 4: 図 2

直角より真に大きいと仮定したことにより生じたものである．したがって，三角形の内角の和は，2 直角以下でなければならないことになるが証明された．

最後に，ルジャンドルは以下の命題を証明しようとするが，証明に間違いがある．以下，ルジャンドルのした証明を示す．

命題　すべての三角形の内角の和は，2 直角である．

証明　三角形ABC の内角の和は 2 直角より真に小さいと仮定し，三角形ABC の内角の和$=$ $2R-\delta,$, $\delta > 0$ と書く．三角形ABC の辺BC を共有して，A と反対側に三角形DBC を三角形$DBC\equiv$ 三角形ABC であるようにとる．図 2 の様に，点 D を通る直線を引き，AB の延長と$E,$ AC の延長とF で交わるとする．この時，図 2 における 4 つの小さな三角形$DBE,$ 三角形$ABC,$ 三角形BCD 及び三角形CDF の内角の和を全て加える．$ABE,$ ACF 及びEDF が直線であることを利用すると，$\angle BAC+\angle ABC+\angle BCA+\angle BCD+\angle CBD+\angle CDB+\angle DCF+\angle CFD+\angle FDC+\angle BDE+\angle DBE+\angle BED=\angle BAC+\angle CFD+\angle DEB+(\angle BCA+\angle BCD+\angle DCF)+(\angle ABC+\angle CBD+\angle DBE)+(\angle FDC+\angle CDB+\angle BDE)=\angle BAC+\angle CFD+\angle DEB+6R$

となる．

一方で，三角形の内角の和は 2 直角以下なので，三角形$ABC\equiv$ 三角形DCB の内角の和は$2R-\delta\delta$ であったので，$\angle BAC+\angle CFD+\angle DEB+\angle 6R\leq 2(2R-\delta)+2\times 2R$ が成り立つ．以上のことから，$\angle BAC+\angle CFD+\angle DEB\leq 2R-2\delta$

となることが理解できる．これは，三角形AEF の内角の和に他ならない．すなわち，三角形AEF の内角の和は$2R-2\delta$ である．さらに，三角形AEF に対して，上と同様にすると内角の和が，$2R-4\delta$ の三角形が得られる．したがって，n 回同様にすると，内角の和が$2R-2n\delta$ の三角形が得られる．しかし，$\delta>0$ であるので，n を十分大きくとると，$2R-2n\delta<0$ とすることが可能である．ところが，三角形の内角の和なので，その値は正でなければならないので，矛盾である．これは，そもそも，三角形のABC の内角の和が 2 直角より真に小さいと仮定したことによって生じたものである．したがって，全ての三角形の内角の和は 2 直角である．

以上で証明終わりとしているが，実は証明になっていないことは明らかである．即ち，点D を通る直線が$AB,$ AC の延長した 2 本の半直線と交わることが自明ではないことは明らかである．ここに，決定的なこの証明の誤りがあるのである．

5 日本における学習指導要領の数学的活動との関係性

日本の学習指導要領において，数学的活動がその目玉の一つになっている．そこでは，スパイラルに授業展開が示されているが，図形領域について考察する時，彌永昌吉氏の指摘する幾何の 4 段階が有効かもしれない．幾何の 4 段階とは，次の様なものである．①直観的段階··· 小学校教育の段階，②局所的段階 (ターレスの段階)··· 中学校教育の段階，③体系的論証の段階 (ユークリッドの段階)··· 高等学校教育の段階，④公理的段階 (ヒルベルトの段階)··· 大学教育の段階．本稿で示したユークリッドの「原論」は，彌永昌吉氏の指摘するまさに高等学校教育の段階に符合する．③の高等学校での体験が経験に変容することによって，④の大学段階での公理的段階になって行くのである．その意味においても，本稿で述べたユークリッドの「原論」の第 5 公準の証明の歴史は，高等学校における一つの有効な教材になり得ると考える．

6 まとめ

本稿においては，第 5 公準の証明の誤りの歴史を見てきたが，実は結果的に証明は間違いではあるが，数多くの知恵を絞って挑戦してきた証明の数々は，結果的に間違いだらけだったと評価することだけでは済まされない教育的価値が内包されている．教育的に見た時，その証明の考え方そのものは非常に興味深いものがあり，高校の数学教育に応用可能であろうと思われる．その間違った証明そのものの中に極めて興味深い事柄が多数あることにこそ高校教員は注目すべきである．高校生にユークリッド幾何を学習させるということではもちろんない．しかし，数学の学習においては，厳密な証明は極めて重要である．ただ，同時に直観も大切である．この両者のからみで考えることが教育的には重要であると思われる．他には，フィッシャー等を利用して，日本の学習指導要領の目玉の一つである数学的活動に取り入れることも十分可能であると思われる．今後の高校現場での実践に期待したい．

REFERENCES

- T.L.Heath,Euclid’s Elements,Vol.1,Dover.
- B.A.Rosenfeld,1988, A History of Non-Euclidean Geometry, Springer.
- Legendre, A.M., 1794,“Éléments de géoérie, avec des notes, . . .”, Paris.
- Legendre, A.M., 1824, “Elements of Geometry”, Edinburgh.
- Hartshorne, R.,1977,” Companion to Euclid” Amer.Math.Soc.
- Katz, V.J.,1988, “A History of Mathematics” Second Edition Addison-Wesley

THE TRENDS ON MATHEMATICS IN NOVELS

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ABSTRACT

Ordinary people think that the works of literature amuse people and are helpful to their mind. But, it is easy for them to have an implicit bias regarding mathematics, that mathematics is a difficult subject and is hard to understand. If mathematics is fused in novels, the contents of novels become rich and people can learn knowledge of mathematics naturally.

Historically, only a few mathematical items appeared in the novels for a limited case, usually in the mystery novels. For example, in *Chasing Vermeer* (2004), a novel for children by Blue Balliett, Pentomino code was used in many places. In *The Da Vinci Code* (2003), a novel by Dan Brown, Fibonacci sequence and anagram are important keys to solve a murder. In *The Sudoku murder* (2007) , a novel by Shelly Freydont, a Sudoku was found on the scene of murder. In *Deep-rooted Tree* (2006), a novel by Jeongmyung Lee, a 3x3 magic square of Hangeul characters has an important meaning.

In The Ants (1991), a novel by Bernard Werber, a sequence of numbers gives a fun to readers. *The Dwarf* (1978), a linked novel by Se-hui Cho, contains short stories with titles 'Möbius Strip' and 'the Bottle of Mr. Klein'. Mathematical titles were used as a metaphor to represent author's intention. In *1Q84* (2009-2010), a novel by Haruki Murakami, a writer teaching mathematics explains what mathematics is.

Nowadays, in some novels mathematics plays a significant role in the storyline. In *The Professor's Beloved Equation* (2003), a novel by Yoko Ogawa, a mathematics professor appears as a main character. In *the Last Pythagorean* (2010), a faction by Sunyoung Lee, an exciting event in the history of mathematics that is the discovery of an irrational number is an important ingredient in the storyline. In *Uncle Petros and Goldbach's conjecture* (1992), a faction by Apostolos Doxiadis, readers can learn Goldbach's conjecture that is a big unsolved problem. In *Logicomix: An Epic Search for Truth* (2009), a graphic novel by Apostolos Doxiadis and Christos Papadimitriou, its story is based on the search for foundation in mathematics.

In summary, we investigate the elements of mathematical motivation in some novels and analyze the trends how mathematics is involved in novels.

Keywords: mathematics, novel, mathematical motivation, mystery novel, faction

1 Mystery Novels

1.1 *Chasing Vermeer* (2004): a mystery novel by Blue Balliett

Mathematical clue for mystery : Pentomino

i) Pentomino code
ii) message of the number 1212:
12 names with 12 letters
iii) the name of the painting with 12 letters, *A Lady Writing* (a painting of Johannes Vermeer (1632~1675))

1.2 *The Da Vinci Code* (2003): a mystery novel by Dan Brown

Mathematical clue for murder :
Fibonacci sequence and ciphers

i) 13-3-2-21-1-1-8-5 → Fibonacci seq.

ii) Atbash: fold-over using Hebrew alphabet

A	B	G	D	H	V	Z	Ch	T	Y	K
Th	Sh	R	Q	Tz	P	O	S	N	M	L

B-P-V-M-Th → Sh-V-P-Y-A
Sh Sh K → B B L

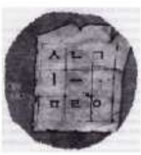
iii) Anagram:
O, Draconian devil! → Leonardo da Vinci!
Oh, lame saint → The Mona Lisa!

1.3 *The Sudoku murder* (2007): a mystery novel by Shelly Freydont

Mathematical clue for murder :
Sudoku

1.4 *Deep-rooted Tree*(2006):a mystery novel by Jeongmyung Lee

Mathematical clue for mystery:
Magic square



Magic square of Hangeul (Korean language) characters

2 Novels with mathematical items

2.1 *The Ants* (1991): a novel by Bernard Werber

Mathematical items:
a sequence: 1, 11, 12, 1121, 122111, 112213, 12221131, 1123123111, ⋯
Puzzles: Make 4 (and 6) equilateral triangles with 6 matchsticks :

2.2 *The Dwarf* (1978) : a linked novel by Se-hui Cho

Mathematical titles 'Möbius Strip' and 'the Bottle of Mr. Klein': metaphor

2.3 *1Q84* (2009-2010): a novel by Haruki Murakami

An unpublshed novelist who works as a math tutor says what math needs.

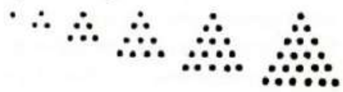
2.4 More list of Fictions including mystery, see the last 4 references:

3 Novels with significant mathematical storyline

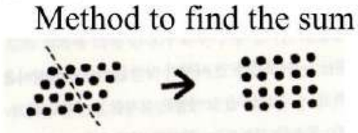
3.1 *The Professor's Beloved Equation* (2003): a novel by Yoko Ogawa

A mathematics professor connects everything with numbers.

i) Amicable number:
220&284, 1184&1210
ii) Perfect number: 6, 28, 496, 8128, 33550336, 8589869056
Property:
6=1+2+3, 28=1+2+3+4+5+6+7
iii) Abundant / Deficient number
iv)Triangular number:1,3,6,10,15,21,…



Method to find the sum 1+2+3+4 :



v) Fermat's Last Theorem

vi) Mersenne prime : $2^{3217}-1$

vii) The professor's beloved equation :

$e^{\pi i}+1=0$

viii) Every prime numbers $p \neq 2$ can be expressed by $4p+1$ or $4p-1$.

13=4x3+1, 19=4x5-1

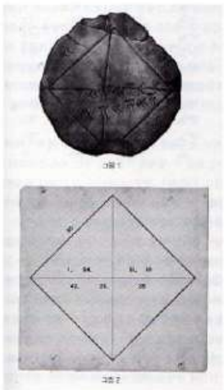
3.2 *The Last Pythagorean* (2010): a Faction by Sunyoung Lee

A sentence:*Hippasus who discovered an irrational number was thrown into a well by Pythagorean school.*

→ Faction (Fact+Fiction)

Fact: Pythagorean school (Pythagoras, Hippasus, Ninon, Damo), Pythagorean theorem,

Fiction: Story about Pythagorean school related to irrational number



$\alpha = 30$,
1, 24, 51, 10= β,
42, 25, 35= γ

42, 25, 35

$= 42+\frac{25}{60}+\frac{35}{60^2}$

$= 42 + 0.416 + \cdots + 0.00972$

$= 42.42638\cdots$

$\alpha = 30,\ \ \beta = ?,\ \ \gamma = 42.42638\cdots$

$\rightarrow\ \gamma^2 = 2\alpha^2$

→

1, 24, 51, 10

$= 1\text{x}60^0 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$

$= 1 + 0.4 + 0.0141 + 0.00004629629\cdots$

= 1.41421296296···

3.3 *Uncle Petros and Goldbach's conjecture* (1992): a Faction by Apostolos Doxiadis

Fact: Goldbach's conjecture

Fiction: Life of a mathematician who tried to solve it

i) Euclid showed by indirect proof that there are infinitely many primes.

ii) Goldbach's conjecture : Every even integer > 2 can be expressed as the sum of 2 primes.
4=2+2, 6=3+3, 8=3+5, 10=3+7, 12=5+7, 14=7+7, …

iii) Goldbach's 2nd conjecture : Every integer > 5 can be expressed as the sum of 3 primes.

iv) Sieve of Eratostenes: a method to find a prime number by Eratostenes

v) Distribution of primes:
There are many twin prime pairs:
(11, 13), (41, 43), (9857, 9859), …
$(835335^{39014}-1,\ 835335^{39014}+1)$

vi) For positive integer k, successive k composite numbers can be found.
(k+2)!+2, (k+2)!+3, … (k+2)!+(k+1), (k+2)!+(k+2)

vii) **Prime Number Theorem (PNT)** :
If a random integer is selected near to some large integer *N*, the probability that the selected integer is prime is about 1 / ln(*N*).

3.4 *Logicomix: An Epic Search for Truth* (2009): a graphic novel by Apostolos Doxiadis and Christos Papadimitriou

Contents: Bertrand Russell's life story to pursue the foundational quest in mathematics

<Great mathematicians of the 20th century in this book>

i) B. Russell (1872-1970):

The Foundations of Arithmetic (1884, with Frege), *Basic Laws of Arithmetic* (Vol.1(1893), Vol.2 (1903) with Frege), *The Principles of Mathematics* (1903), *Principia Mathematica* (3 Vols., 1910–1913, with Whitehead)

Russell's Paradox (There doesn't exist a set of all sets.)

ii) G. Cantor (1845-1918): Set Theory

iii) L. Wittgenstein(1889-1951): *Tractatus Logico-Philosophicus* (1922)

iv) G. E. Moore (1873-1958): Teaching Leibniz's usage related to conjuction and etc. to Russel

v) A. N. Whitehead (1861-1947): Professor of Russell

vi) G. Frege (1848-1925): a student of Russell

vii) H. Poincaré (1854-1912): belief on Intuition

viii) D. Hilbert (1862-1943): belief on Exactness of Logical proof

ix) K. Gödel (1906-1978): Incompleteness Theorems

x) A. Turing (1912-1954): designing Turing Machine to define the proof

Conclusion

In old days, Mathematics didn't play a leading role in novels. A few mathematical items can be found in mystery novels and etc.

But, nowadays, Novels can have significant mathematical storyline. Faction and Graphic novel are new trends on mathematics in novels to inform the importance of mathematics to public with interest and without difficulty.

References

- Balliett, B., 2004, *Chasing Vermeer*, New York, Scholastic Press.
- Brown, D., 2003, *The Da Vinci Code*, New York, Doubleday.
- Doxiadis, A., translated by Jeong, H., 2000, *Uncle Petros and Goldbach's conjecture*, Seoul, Thinking Tree Publishing Co. Ltd
- Doxiadis, A. and Papadimitriou, C., translated by Jun, D., 2011, *Logicomix: An Epic Search for Truth*, Seoul, Random House Korea Inc.
- Freydont, S., translated by Cho, Y., 2011, *The Sudoku Murder*, Seoul, Millionhouse Publishing Inc.
- Lee, J., 2006, *Deep-rooted Tree*, Seoul, Millionhouse Publishing Inc.
- Lee, S., 2010, *The Last Pythagorean*, Seoul, Gimmyoung Publishers Inc.
- Murakami, H., translated by Yang, Y., 2009-2010, *IQ84*, Seoul, Munhakdongne Publishing Corp.
- Ogawa, Y., translated by Kim, N., 2004, *The Professor's Beloved Equation*, Seoul, Ire Inc.
- Werber, B., translated by Lee, S., 1993, *The Ants*, The Open Books Co.
- http://news.dongascience.com/PHP/NewsView.php?kisaid=20100218100000000206&classcode=0113.
- http://divisbyzero.com/2009/11/16/ math-in-literature.
- http://divisbyzero.com/2010/05/24/mathematics-in-novels-and- martin-gardner-rip.
- http://kasmana.people.cofc.edu/MATHFICT/mfview.php?callnumber=mf798.
- http://www.monmouthcountylib.org/FictionaboutMath.doc.

ORTHOGONAL LATIN SQUARES OF CHOI SEOK-JEONG

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ABSTRACT

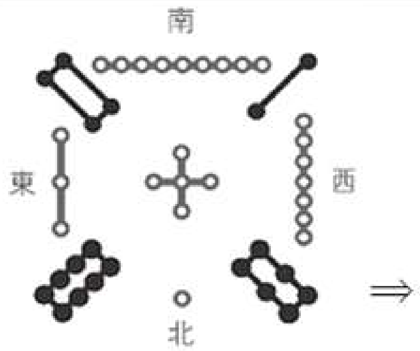
A latin square of order n is an $n \times n$ array with entries from a set of numbers arranged in such a way that each number occurs exactly once in each row and exactly once in each column. Two latin squares of the same order are orthogonal latin square if the two latin squares are superimposed, then the n^2 cells contain each pair consisting of a number from the first square and a number from the second. In Europe, Orthogonal Latin squares are the mathematical concepts attributed to Euler. However, an Euler square of order nine was already in existence prior to Euler in Korea. It appeared in the monograph Koo-Soo-Ryak written by Choi Seok-Jeong(1646–1715). He construct a magic square by using two orthogonal latin squares for the first time in the world. In this posterr, we explain Choi's orthogonal latin squares and the history of the Orthogonal Latin squares.

1 Definition of magic square.

A **magic square** of order n is an arrangement of n^2 numbers, usually distinct integers in a square such that the n numbers in all rows, all columns, and both diagonals sum to the same.

2 History of magic square.

Chinese literature dating from as early as 650 BC tells the legend of Lo Shu or "scroll of the river Lo". In ancient China there was a huge flood. The great king Yu (禹) tried to channel the water out to sea where then emerged from the water a turtle with a curious figure/pattern on its shell; circular dots of numbers which were arranged in a three by three grid pattern such that the sum of the numbers in each row, column and diagonal was the same: 15, which is also the number of days in each of the 24 cycles of the Chinese solar year. This pattern, in a certain way, was used by the people in controlling the river.



4	9	2
3	5	7
8	1	6

3 Latin square.

A Latin square is an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column. Here is an example:

2	3	1
1	2	3
3	1	2

1	3	2
3	2	1
2	1	3

4 Orthogonal Latin square.

Orthogonal Latin squares of order n over two sets S and T, each consisting of n symbols, is an $n \times n$ arrangement of cells, each cell containing an ordered pair (s, t), where s is in S and t is in T, such that every row and every column contains each element of S and each element of T exactly once, and that no two cells contain the same ordered pair.

2	3	1
1	2	3
3	1	2

+

1	3	2
3	2	1
2	1	3

=

2, 1	3, 3	1, 2
1, 3	2, 2	3, 1
3, 2	1, 1	2, 3

5 Euler square & Conjecture..

Orthogonal Latin squares were studied in detail by Leonhard Euler, who took the two sets to be $S = \{A, B, C, \ldots\}$, the first n upper-case letters from the Latin alphabet, and $T = \{\alpha, \beta, \gamma, \ldots\}$, the first n lower-case letters from the Greek alphabet—hence the name orthogonal Latin square.

In the 1780s Euler demonstrated methods for constructing orthogona-Latin squares where n is odd or a multiple of 4. Observing that no order-2 square exists and unable to construct an order-6 square (see thirty-six officers problem), he conjectured that none exist for any oddly even number $n \equiv 2(\mod 4)$. Indeed, the non-existence of order-6 squares was definitely confirmed in 1901 by Gaston Tarry through exhaustive enumeration of all possible arrangements of symbols. However, Euler's conjecture resisted solution for a very long time.

In 1959, R.C. Bose and S. S. Shrikhande constructed some counterexamples (dubbed the Euler spoilers) of order 22 using mathematical insights. Then E. T. Parker found a counterexample of order 10 through computer search on UNIVAC (this was one of the earliest combinatorics problems solved on a digital computer).

In 1960, Parker, Bose, and Shrikhande showed Euler's conjecture to be false for all $n \geq 10$. Thus, Graeco-Latin squares exist for all orders $n \geq 3$ except $n = 6$.

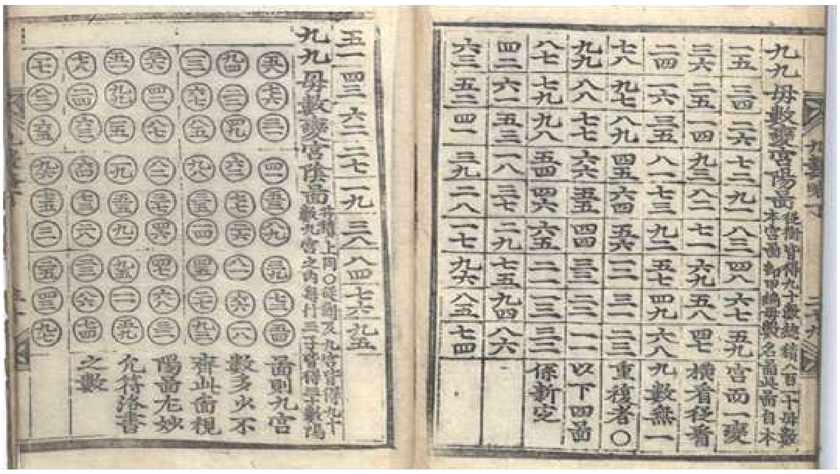
6 Orthogonal Latin square Choi Seok-Jeong

5,1 6,3 4,2	8,7 9,9 7,8	2,4 3,6 1,5
4,3 5,2 6,1	7,9 8,8 9,7	1,6 2,5 3,4
6,2 4,1 5,3	9,8 7,7 8,9	3,5 1,4 2,6
2,7 3,9 1,8	5,4 6,6 4,5	8,1 9,3 7,2
1,9 2,8 3,7	4,6 5,5 6,4	7,3 8,2 9,1
3,8 1,7 2,9	6,5 4,4 5,6	9,2 7,1 8,3
8,4 9,6 7,5	2,1 3,3 1,2	5,7 6,9 4,8
7,6 8,5 9,4	1,3 2,2 3,1	4,9 5,8 6,7
9,5 7,4 8,6	3,2 1,1 2,3	6,8 4,7 5,9

$x(i-1)+j$

37 48 27	70 81 62	13 24 5
30 38 46	63 71 79	6 14 22
47 28 39	80 61 72	23 4 15
16 27 8	40 51 32	64 75 56
9 17 25	33 41 49	57 65 73
26 7 18	50 31 42	74 55 66
67 78 59	10 21 2	43 54 35
60 68 76	3 11 19	36 44 52
77 58 69	20 1 12	53 34 45

Koo-Soo-Ryak (Printed by woodblock)



7 Handbook of Combinatorial Design.

On the page 12 of Handbook of Combinatorial Design, the following statement was included.

> The literature on latin squares goes back at least 300 years to the monograph Koo-Soo-Ryak by Choi Seok-Jeong (1646–1715); he uses orthogonal latin squares of order 9 to construct a magic square and notes that he cannot find orthogonal latin squares of order 10.

This implies that the orthogonal latin squares of Choi Seok-Jeong is at least 67 years earlier than Euler's

REFERENCES

- Academy of Korean studies(1991), Encyclopedia of Korean Heritages and culture.
- Allan Adler, *Magic N-Cubes Form a Free Monoid*, the elctronic journal of combinatorics 4 (1997), #R15
- Charles J. Colbourn , Jeffrey H. Dinitz(Editor)(2006), *Handbook of Combinatorial Designs*, 2nd edition, page 12, Chapman and Hall.
- Choi Seok-Jeong(1646–1715), *Koo-Soo-Ryak*, (M. Hur & H. N. Chung, Trans.), Kyowoosa, 2006.
- Hahn Sang-Geun and Oh Yoon-Yong(1993), Choi Seok-Jeong and his magic squares, *Journal of the Korea Society of Mathematical Education* Series A, Vol. 32, No3, 205–219.
- Kim Y. W. and Kim Y. K. (1982), *History of math in Korea*, Yeol Hwa Dang.
- Song, Hong-Yeop, *Choi's orthogonal latin squares is at least 67 years earlier than Euler's*, A presentation to the 2008 Global KMS Conference, Jeju, Korea.
- Lih. Ko-Wei, (2010), A *remarkable Euler square before Euler*, Mathematics Magazine, 83, 163–167.

TEACHING HISTORY OF MATHEMATICS BY CREATING YOUR OWN DECK OF CARD AND POSTER OF WORLD MATHEMATICIANS
including your fellow countrymen

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ABSTRACT

While we are teaching the history of mathematics, we found that most of our students lost their interest after half of the course. We examined reasons for that. What we found was there are too many old names of western mathematicians to remember. Our students told that they like to know more about mathematicians of 20th century and from their neighboring countries. Attempting to change the students' attitude we started to implement a different teaching strategy. Specifically, we propose to start teaching the class by discussing the contribution of mathematicians from the home country. Next, we proceed to discuss contribution of mathematicians from neighboring countries. We had a limit of 54 great mathematicians which is the number of a deck of poker cards which the first author learned from Prof. Wann-Sheng Horng of Taiwan. So we made a rule to find half of great mathematicians from Asia and other half from the rest of the world.

This process should take up half of the course. For example, we asked our students to discover the mathematician that is the first bachelor/master/doctoral degree holder from their country, neighboring countries, and their area of mathematical interest etc. Then we had discussed whose contributions are bigger than others in order to give a numbering. This approach motivated students to think and discuss each other even after the class. In the process, students become a company of mathematical figures and propel them to do their best.

In the process, we tried to make the class more fun by making a deck of card in real. Effectively, we created a deck of cards on world mathematicians from the view point of students. There are many ways this can be done. For example, we could select and classify mathematicians as Korean mathematicians, Asian mathematicians, European mathematicians and Anglo-American mathematicians. After we created a Korean version of the playing cards, a couple of students got involved to make the English version of it. Our family members like to see all mathematicians in big paper. So we made a poster from the card set.

Certainly, someone from a different country could implement a different classification scheme. For example, a Brazilian student can think of his own card set with Brazilian mathematicians (1/4), South and North American mathematicians (1/4), European mathematicians (1/4) and other important mathematicians from all other countries (1/4). Every student may like to have their own set of card and poster while leaning the history of mathematics, not only from other countries but also from their neighboring countries.

In this poster presentation, we introduce our process of selecting and classifying mathematicians as well as designing and arranging the cards with the outputs which can be used as a learning tool. We liked to share some of our experience and to learn your ideas from the history of mathematics classes.

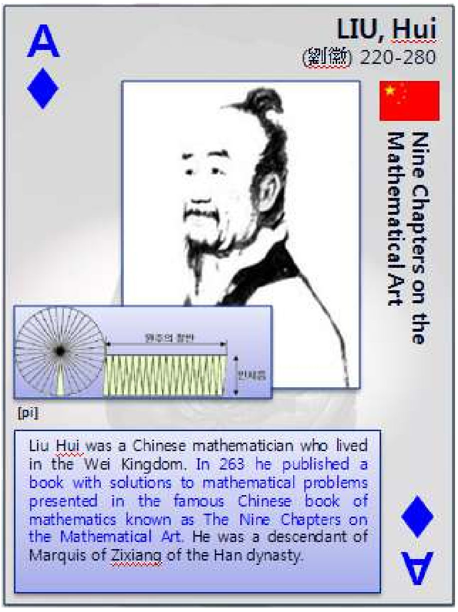
1 Mathematicians Cards

1.1 Korean Mathematicians (Homeland)

Cards	Name
♥ A	HONG, Jung-Ha; 洪正夏
♥ K	LEE, Sang-Hyuk; 李尙爀
♥ Q	LEE, Sang-Seol; 李相卨
♥ J	REE, Rimhak; 李林學
♥ 2	KYONG, Seong-Jing; 慶善徵
♥ 3	CHOI, Seok-Jung; 崔錫鼎
♥ 4	HONG, Dae-Yong; 洪大容
♥ 5	HWANG, Yoon-Seok; 黃胤錫
♥ 6	NAM, Byung-Gil; 南秉吉
♥ 7	CHOI, Kyu-Dong; 崔奎東
♥ 8	LEE, Chun-Ho; 李春昊
♥ 9	CHOI, Yoon-Sik; 崔允植
♥ 10	CHANG, Ki-Won; 張起元

1.2 Asian Mathematicians (Neighboring Country)

Cards	Name
◆ A	LIU, Hui; 劉徽
◆ K	Al-Khwarizmi
◆ Q	Seki KOWA; 関孝和
◆ J	ZHU, Shijie; 朱世傑
◆ 2	ZU, Chongzhi; 祖沖之
◆ 3	Bhāskara II
◆ 4	LI, Ye; 李冶
◆ 5	QIN, Jiushao; 秦九韶
◆ 6	YANG, Hui; 楊輝
◆ 7	LI, Shanlan; 李善蘭
◆ 8	Teiji TAKAGI; 高木 貞治
◆ 9	Srinivasa Ramanujan
◆ 10	Shiing Shen CHERN; 陳省身

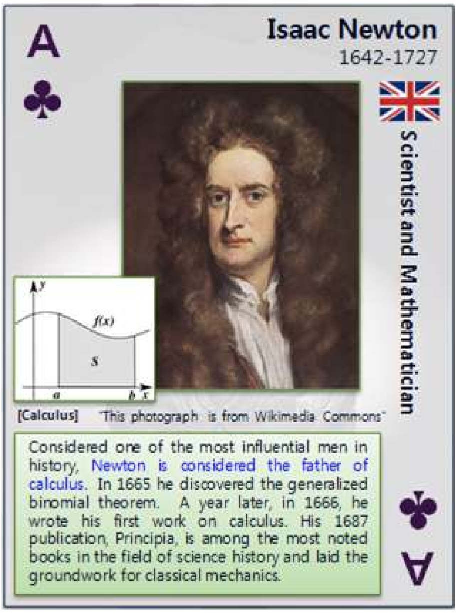


1.3 European Mathematicians (Big figures in World Math. History-Spade)

Cards	Name
♠ A	Euclid
♠ K	Leonhard Euler
♠ Q	Pythagoras of Samos
♠ J	Nicolas Bourbaki
♠ 2	Pierre de Fermat
♠ 3	Gottfried Leibniz
♠ 4	Jean Baptiste Joseph Fourier
♠ 5	Augustin Louis Cauchy
♠ 6	Niels Henrik Abel
♠ 7	Georg Friedrich Riemann
♠ 8	Georg Cantor
♠ 9	Felix Christian Klein
♠ 10	Kurt Gödel

1.4 USA and UK Mathematicians (Some Modern Mathematicians)

Cards	Name
♣ A	Isaac Newton
♣ K	Bertrand Russell
♣ Q	Eliakim Hastings Moore
♣ J	John von Neumann
♣ 2	Benjamin Peirce
♣ 3	James Joseph Sylvester
♣ 4	George Boole
♣ 5	Arthur Cayley
♣ 6	George David Birkhoff
♣ 7	Leonard Eugene Dickson
♣ 8	Oswald Veblen
♣ 9	Robert Lee Moore
♣ 10	Alan Turing



2 Conclusion

The above idea will be presented in a poster. This subject is relevant to HPM theme 5, Cultures and Mathematic. It gives some ideas of how to produce material (deck cards) that will provide some concise information on great mathematical figures. What is the key point of this approach is that one has only a limited number (54) of mathematicians to select among a much greater one and therefore one has to specify appropriate criteria that could be country/culture-dependent, historical period-dependent, mathematical subject-dependent etc. In addition, to provide a very short, but still concise and informative text that could be included in a deck card is not straightforward. This could mean that in producing this material, one has to study much more on the achievements of the mathematicians to be selected.

In our poster, we did use our criteria of selecting and classifying mathematicians as well as designing and arranging the cards with many of our colleagues. We expect to share some of our experience and output.

REFERENCES

- Sang-Gu Lee, J. Noh, S. Song, 2009, Obstacles in the Development of Korean Mathematics in the late 19th and early 20th Centuries, The Korean Journal for History of Mathematics.
- Sang-Gu Lee and Y. Ham, 2006, J. J. Sylvester , F. Klein and and American Mathematics in 19th Century, The Korean Journal for History of Mathematics, **19**(2), pp.77-88
- Sang-Gu Lee, Y, Ham, 2009, Educational policy and curriculums of Korean school mathematics in the late 19th and early 20th century, Korean tertiary mathematics and curriculum in early 20th century, **22**(3), pp.207-254.
- Sang-Gu Lee and Y. Ham, 2009, Contribution of Oswald Veblen to AMS and its meaning in Korea.The Korean Journal for History of Mathematics **22**(2).
- Sang-Gu Lee, Sangwook Ree and Duk-Sun Kim, 2008, R. L. Moore's Moore Method and its meaning in Korea, The Korean Journal for History of Mathematics, **21**(1), pp.77-88
- `http://www.nomadbook.co.kr/`
- `http://people.naver.com/DetailView.nhn?frompage=nx_people&id=9345`
- `http://news.khan.co.kr/section/khan_art_view.html?mode=view&artid=200708070937422&code=900314`
- `http://people.naver.com/DetailView.nhn?frompage=nx_people&id=49052`
- `http://krdic.naver.com/detail.nhn?kind=korean&docid=2207900`
- `http://100.naver.com/100.nhn?docid=147889`
- `http://100.naver.com/100.nhn?docid=172058`
- `http://100.naver.com/100.nhn?docid=173843`
- `http://100.naver.com/100.nhn?docid=35590`
- `http://100.naver.com/100.nhn?docid=147740`
- `http://blog.naver.com/onisl3k?Redirect=Log&logNo=20009055215`
- `http://100.naver.com/100.nhn?docid=148044`
- `http://100.naver.com/100.nhn?docid=775414`

– `http://www-history.mcs.st-and.ac.uk/PictDisplay/Cantor.html`

– `http://www-history.mcs.st-and.ac.uk/PictDisplay/Von_Neumann.html`

– `http://planetmath.org/encyclopedia/NicolasBourbaki.html`

– `http://en.wikipedia.org/wiki`

ARITHMETICAL AND GEOMETRICAL PROGRESSIONS, AND NUMERICAL SERIES IN CHINA BEFORE 14TH CENTURY

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ABSTRACT

Chinese mathematicians did many advances in different subjects but most of their discoveries were hidden for the rest of world mathematicians. We searched for specific developments on arithmetic and geometric sequences, and finally infinite numerical series. All this bibliographic research was done with the main objective of get our students involved in mathematics history.

1 Introduction

Ancient Chinese mathematicians had many advances in Mathematics but mathematicians outside China did not know about these discoveries. The use of numerical series started with arithmetical and geometrical progressions many centuries before we imagined.

In this poster we show some of the advances in the Chinese mathematics on the use of arithmetical and geometrical progressions to solve problems. Also the sums of some finite series were calculated. We list only four mathematicians who worked in this area.

2 Chinese mathematicians

The book Jiuzhang Suanshu or Nine Chapters on the Mathematical Art (circa 200 B.C.) involve the use of arithmetic and geometrical progressions to solve some problems in chapter 3. Xu Yue wrote three different systems of powers of 10, a lower system 10, 10^2, 10^3, 10^4, 10^5,⋯a middle system of powers of 10^4 like 10^4, 10^8, 10^{12}, 10^{16},⋯and a upper system based on the squares of powers of 10 like 10^4, 10^8, 10^{16}, 10^{32},⋯

Zhang Qiujian wrote the book Zhang Qiujian suanjing (Mathematical Manual of Zhang Qiujian) around the year 486 A.D. where he included some problems to be solved using arithmetical progressions. He summed some arithmetical progressions. Yang Hui (13^{th} century) summed some finite series like the squares of the natural numbers in the interval m^2 to $(m+n)^2$ and also gave $1+3+6+\ldots+n(n+1)/2 = n(n+1)(n+2)/6$

Zhu Shijie gave sums for finite numerical series like the first n naturals, the sum of the cubes of the first n naturals, sums of arithmetical progressions, and the sum of finite series. He also established some sums for infinite series like $1 + 4 + 9 + 16 + 25 + \ldots$. $1 + 5 + 14 + 30 + 55 + 91 + \ldots$

These advances were part of some lectures used in a calculus class and let students know that numerical series can be developed using calculus but also geometry and algebra. This bibliographic search was part of a doctoral dissertation on numerical series.

REFERENCES

– Mactutor History of Mathematics. (n.d.). Web page retrieve from http://www-history.mcs.st-and.ac.uk/

– Rosas, A. (2007). *Transposición didáctica de las series numéricas infinitas. Una caracterización del discurso escolar actual en el nivel superior* [Didactic transposition of infinite numerical series. A characterization of the present school discourse in university] (Unpublished doctoral dissertation). CICATA-IPN, Mexico

CHOSUN MATHEMATICIAN LEE SANG HYUK'S GENEALOGY
조선 산학자 이상혁 (李尚㷜)의 계보

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ABSTRACT

Lee Sang Hyuk(李尚㷜, 1810–?) is one of the most prominent ChungIn(中人) mathematicians in Chosun dynasty and wrote a book GyuIlGo(揆日考,1850) on astronomy and three mathematics books ChaGeunBangMongGu(借根方蒙求, 1854), SanSulGwanGyun(算術管見, 1855) and IkSan(翼算,1868) and left a few mathematical notes. He worked in HoJo(戶曹) and GwanSangGam(觀象監).

Investigating his colleagues and kinship relations including the affinity and consanguinity, we show that Lee Sang Hyuk is related to the other prominent ChungIn mathematicians Gyung Sun Jing(慶善徵, 1616–?) and Hong Jung Ha(洪正夏, 1684–?)mathematically through marriage and belongs to one of preeminent clans of ChungIn mathematicians in Chosun dynasty.

1 Introduction

Lee Sang Hyuk(李尚㷜), Gyung Sun Jing(慶善徵) and Hong Jung Ha(洪正夏) are three ChungIn mathematicians who publilshed mathematical books. Investigating their family lineages we find that there are genealogical connections between them. In figure 1 and 2, vertical lines represent a direct family line, slant lines for fathers-in-law, and parentheses for the clan names.

2 Lee Sang Hyuk's Genealogy

In figure 1, Lee Chung Il(李忠一) is Gyung Sun Jing(慶善徵)'s father-in-law. Lee Chung Il(李忠一)'s another son-in-law, Lee Paeng No(李彭老) is Hong Yu Won(洪有源)'s father-in-law, whose father is Hong Seo Ju(洪敘疇). Kyung Yeon(慶演), Hong Jae Won(洪載源)'s father-in-law, is Lee Sin Il(李信一)'s son-in-law who is Lee Chung Il(李忠一)'s younger brother.

In figure 2, Lee Yeong Hyeon(李英顯) is Hong Si Won(洪始源)'s son-in-law who is Hong Jung Ha(洪正夏)'s uncle, and his son, Lee Tae Yun(李泰胤), is Hong Jung Ha(洪正夏)'s brother-in-law. Lee Sang Hyuk(李尚㷜)'s father, Lee Byeong Cheol(李秉喆) is Hong Jung Ha(洪正夏)'s nephew-in-law.

3 Conclusion

Based on the genealogical lineages, mathematics of three ChungIn mathematicians which started from Gyung Sun Jing(慶善徵) eventually could be handed down to Lee Sang Hyuk(李尙爀). Gyung Sun Jing(慶善徵) published MukSaJibSanBub(黙思集散法) which is the first full mathematics book which we now have. Hong Jung Ha(洪正夏) wrote GuIlJib(九一集) which is the greatest mathematical writing in Chosun dynasty. These three mathematicians contributed a lot to development of Chosun mathematics and they are all related by marriage.

REFERENCES

- 雜科榜目, `http://www.aks.ac.kr`
- 韓國科學技術史資料大系, 數學編, 1卷–10卷, 驪江出版社, 1985.

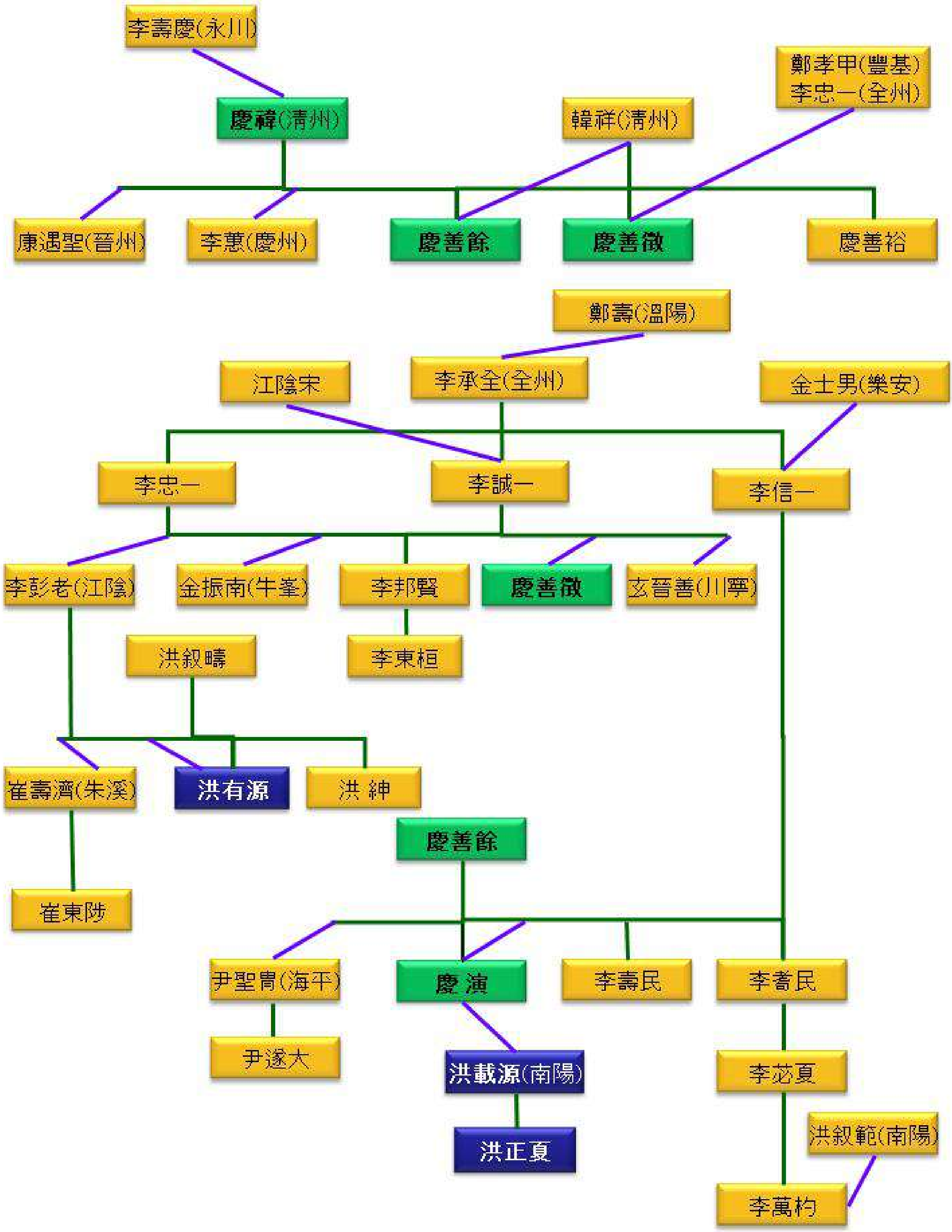
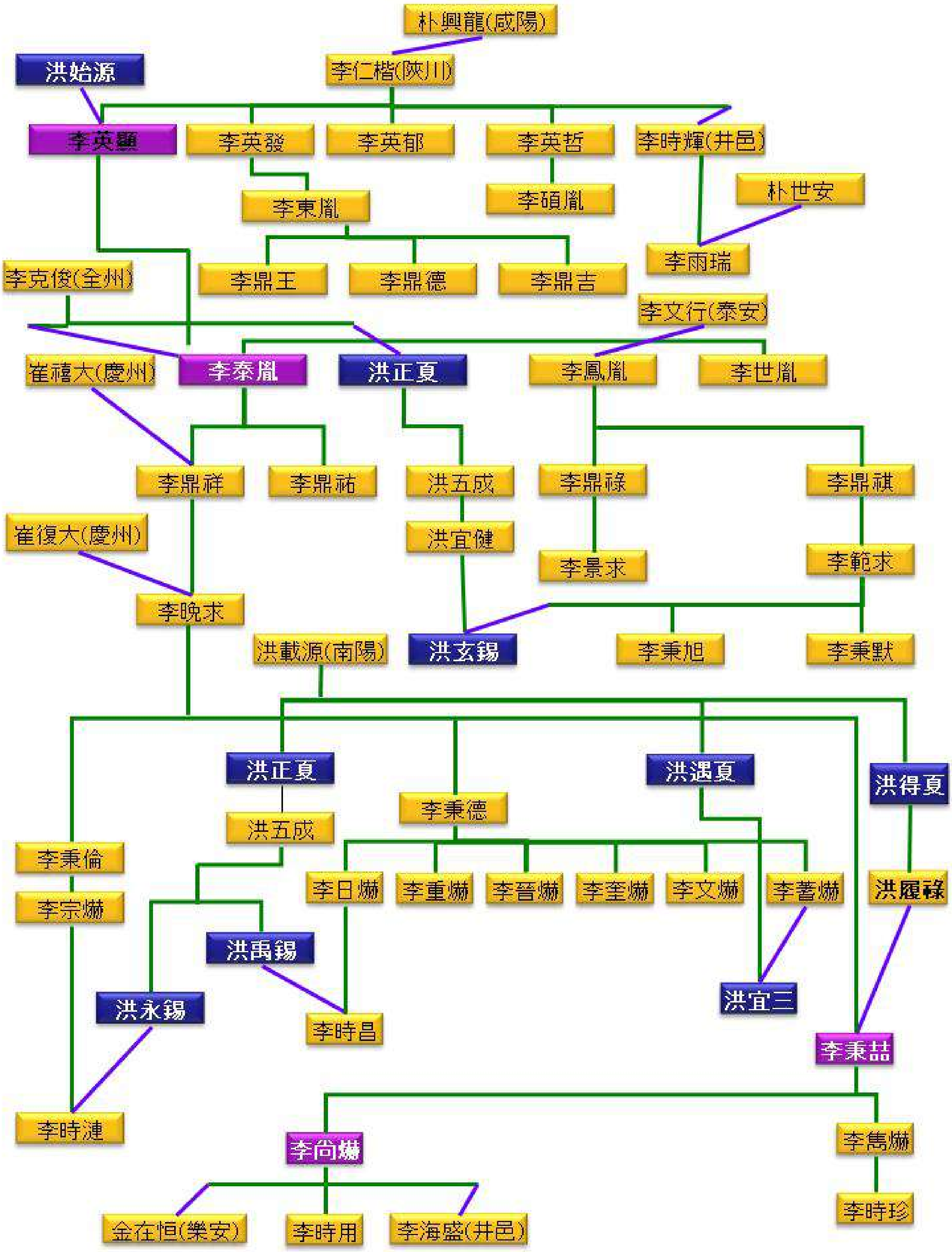

李壽慶(永川)
鄭孝甲(豐基)
李忠一(全州)
慶禕(淸州)
韓祥(淸州)
康遇聖(晉州)
李蕙(慶州)
慶善餘
慶善徵
慶善裕
鄭壽(溫陽)
江陰宋
李承全(全州)
金士男(樂安)
李忠一
李誠一
李信一
李彭老(江陰)
金振南(牛峯)
李邦賢
慶善徵
玄晉善(川寧)
洪叙疇
李東桓
崔壽濟(朱溪)
洪有源
洪 紳
慶善餘
崔東陟
尹聖胄(海平)
慶 演
李壽民
李耆民
尹遂大
洪載源(南陽)
李苾夏
洪正夏
洪叙範(南陽)
李萬枃

Figure 1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李時輝(井邑)
李東胤
李碩胤
朴世安
李克俊(全州)
李鼎王
李鼎德
李鼎吉
李雨瑞
李文行(泰安)
崔禧大(慶州)
李泰胤
洪正夏
李鳳胤
李世胤
李鼎祥
李鼎祐
洪五成
李鼎祿
李鼎祺
崔復大(慶州)
洪宜健
李景求
李範求
李晩求
洪載源(南陽)
洪玄錫
李秉旭
李秉默
洪正夏
洪遇夏
洪得夏
李秉德
洪五成
李秉倫
李宗爀
李日爀
李重爀
李晉爀
李奎爀
李文爀
李蓍爀
洪履祿
洪禹錫
洪永錫
洪宜三
李時昌
李秉喆
李時漣
李尙爀
李雋爀
金在恒(樂安)
李時用
李海盛(井邑)
李時珍

Figure 2

FIRST STUDY ON A JOSEON MATHEMATICS BOOK SUHAKJEOLYO (數學節要, 수학절요), WHICH WAS WRITTEN BY JONG-HWA AN IN 1882

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ABSTRACT

In 2007, a Taiwanese mathematics historian Wann-Sheng HORNG made a visit to Kyujanggak(the royal library of Joseon Dynasty) in Seoul, Korea. During this visit, he found the Korean math book SuHak-JeolYo (<數學節要>), which was written by Jong-Hwa AN(1860-1924) in 1882. Then he mentioned the possible importance of AN' s book in his article in the Journal Kyujanggak(vol. 32, June 2008).

Jong-Hwa AN is a Korean scholar, activist of patriotism and enlightenment in the latter era of Joseon Dynasty. He passed the last examination of Joseon Dynasty to become a high government officer in 1894. The father of the modern mathematics education in Korea, Sang-Seol LEE(1870-1917) also passed the same examination with him. It is interesting that government high officer AN and LEE both wrote mathematics books in 19th century. In this talk, we now analyze this mathematics book of Joseon written in 1882.

1 안종화에 대하여

한말 국학자이며 애국계몽운동가인 안종화(安鍾和, 1860-1924)는 충청남도 당진(唐津) 출신으로 1894년(고종 31) 치러진 조선의 마지막 과거시험인 식년시에 3등으로 진사가 되었다. 이어서 문과전시(殿試) 병과(丙科)에 독립운동가이자 한국 근대수학교육의 아버지로 평가되는 이상설(李相卨, 1870-1917)과 함께 급제하여, 궁내부낭관(宮內府郎官), 법부서관(法部書官), 홍문관시독(弘文館侍讀) 등을 거쳐 종2품 가선대부(嘉善大夫)를 역임했다. 안종화는 1882년 전통수학책 <수학절요(數學節要)>를 저술하였다.

2 <수학절요> 탐구

대만수학사학자 洪萬生은1882년에 쓰여진 <수학절요>를 처음 발견하고, 그 가치에 대하여 주목한다고 지적하였다(洪萬生, 2008). 본 원고에서는 안종화가 1882년 저술한 <수학절요>에 대하여 최초로 그 내용을 분석한다. 구한말 안종화가 <수학절요>를 저술한 사실이 그를 다룬 많은 글에서 전혀 언급되지 않은 것은 특이한 일이다. 안종화의 <수학절요>는 전체 세 권(卷之一, 二, 三)으로 되어 있으며 각각 천(天), 지(地), 인(人)으로 표시하였다.

서문에 따르면, 명(明)대 숭정(崇禎)년간(1628-1644)에 태서(泰西)인 나아곡(羅雅谷, Jacques Rho, 1593-1638)이 주산사선법(籌算斜線法)을 만들고 후에 매문정(梅文鼎, 1633-1721)이 주산을 수직으로 고쳐서 그 집록(輯錄)한 것을 더 채웠는데 매우 자세하여 사람들이 이것을 최고로 삼았으며 이 책이 기록한 바는 문산(文算), 포지금(鋪地錦), 사산(斜算)이라는 여러 이름으로 불리기도 했다. 그러나 초학자들이 배우기에 어려워

서 안종화가 부족한 부분을 채워 저술하였음을 밝히고 있다. 그러나 안종화는 나아곡의 주산사선법과 문산의 사선법을 구별하지 못하고 있는 것으로 판단된다. 그 이유는 나아곡이 언급한 주산은 본래 일종의 계산기로 그 기판은 사선으로 되어있고 후에 매문정이 이를 다시 수직으로 만들어 사용했던 반면, 문산은 사실 필산법이기 때문이다. 문산은 이미 오경(吳敬, 약 15세기경)의 <구장산법비류대전(九章算法比類大全)>(1450)에 사산(寫算)이라는 이름으로 들어있다(李儼 & 杜石然, 1964). <수학절요>의 통론에 나오는 서명응(徐命膺, 1716-1787)은 나아곡의 주산법과 문산을 제대로 구별하고 있는데 안종화는 서명응의 고사신서(攷事新書) 8권을 잘못 인용한 것으로 추정된다(박권수, 2010).

다음으로 목록을 보면 하도(河圖), 낙서(洛書)와 도량형(度量衡), 그리고 가감승제 후에 방전(方田), 속포(粟布), 쇠분(衰分), 소광(少廣), 상공(商功), 균수(均輸), 영뉵(盈朒), 방정(方程), 구고(句股)의 순서로 편성이 되어 있어 이 책이 주로 <구장산술(九章算術)>의 내용을 다루고 있음을 알 수 있다. 그밖에 3권에서는 차근방(借根方), 체류(體類), 난제(難題), 제가설(諸家說)도 다루고 있음을 알 수 있다.

이 책에서 저자는 본문 상단에 작은 글씨로 본문에서 언급한 개념에 대하여 보충설명을 주고 있으며, 특히 승법(乘法, 곱셈) 중국 전통수학에서 포지금(鋪地錦)이라고 부른 필산법을 사용하였다. 한국수학사학자들이 <수학절요>에 대한 언급을 한 기록이 전혀 없어 외국의 수학사학자들은 그 존재조차 몰랐다. 대만수학사학자 洪萬生이 규장각에 직접 방문하여 <수학절요>라는 책이 충남대학교 도서관에 있다는 것을 알고, 구하여 읽은 후 이 책의 가치를 높이 평가하였다. 이 책은 조선산학을 다룬 현존하는 마지막 책으로 그 이후의 수학책이 바로 이상설의 <수리>이다. <수리>는 조선산학에서 벗어나 근대수학을 다루는 첫 번째 책이 된다.

3 결론

<수학절요>는 한말 국학자이자 애국계몽운동가이며 이상설(1870년생)과 함께 1894년 대과에 동시에 급제한 안종화(1860년생)가 대과에 합격하기 12년전에 저술한 조선산학책이다. 본 연구에서는 <수학절요>의 가치를 파악하기 위하여 최초로 그 내용을 분석하였다. 목록을 살펴보면 기본적으로 <구장산술>의 내용을 중심으로 다루고 있음을 알 수 있으며 특히 승법은 포지금이라 불리는 방법으로 계산되어 있다. 또한 <수학절요>가 포함하고 있는 차근방, 체류, 난제, 제가설 등의 내용으로부터 이 책이 <수리정온>(聖祖 (淸), 1722)의 영향을 크게 받은 책임을 알 수 있었다. <수학절요>는 기존에 잘 알려진 조선산학의 내용을 주로 다루었기 때문에 본 발표에서는 최근에 학계에 처음으로 공개된 조선산학을 다룬 현존하는 마지막 책인 <수학절요>에 대하여 특징을 보여주는 일부만 분석하였다. 후속연구에서는 완역이 되어 조선산학과 근대수학을 잇는 과정에 대한 추가 연구가 있기를 기대한다.

REFERENCES

– 김남석, 2003, "한말 국학자 안종화 연보", 내포문화, 15, pp. 200-215.
– 南秉吉, 1867, 算學正義.
– 박권수, 2010, "규장각 소장 <攷事新書>에 대하여", 奎章閣, 36, pp. 1-25.
– 安鍾和, 1882, 數學節要.
– 최기영, 1994, "안종화", 한국의 역사가와 역사학 (하) (창작과 비평사), pp. 23-34.
– 聖祖 (淸), 1722, 數理精蘊 (國學基本叢書 四百種 : 王雲五 主編, 臺灣商務印書館, 1968).
– 李儼 & 杜石然, 1964, 中國古代數學簡史 下册, 北京 : 中華書局.
– 洪萬生, 2008, 奎章閣收藏算書初訪, 奎章閣, 32, pp. 283-293.

Chinese Mathematics in Chosun

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ABSTRACT

It is well known that the Chinese mathematics has had a great influence on mathematics of eastern Asian countries including Chosun. Although Chosun is China's neighbouring country, their capital cities were very far apart and hence mathematical exchanges between two countries have been materialized only through mathematics books.

The purpose of this talk is to investigate Chinese mathematics books which had influence on the development of Chosun mathematics. We also investigate Chosun's efforts to import those books including King SeJong(世宗, 1397–1450)'s effort and exchanges of mathematics books in eastern Asian countries.

Three Authors of the *Taisei Sankei*

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ABSTRACT

The *Taisei Sankei* (大成算経 in Japanese) or the *Dacheng Suanjing* (in Chinese) is a book of mathematics written by Seki Takakazu 関孝和, Takebe Kataakira 建部賢明 and Takebe Katahiro 建部賢弘. The title can be rendered into English as the *Great Accomplishment of Mathematics*. This book can be considered as one of the main achievements of the Japanese traditional mathematics, *wasan*, of the early 18th century.

The compilation took 28 years, started in 1683 and completed in 1711. The aim of the book was to expose systematically all the mathematics known to them together with their own mathematics. It is a monumental book of *wasan* of the Edo Period (1603–1868).

The book is of 20 volumes with front matter called Introduction and altogether has about 900 sheets. It was written in classical Chinese, which was a formal and academic language in feudal Japan.

In this lecture we would like to introduce the *wasan* as expressed in the *Taisei Sankei* and three authors of the book.

The plan of the paper is as follows: first, the Japanese mathematics in the Edo Period was stemmed from Chinese mathematics, e. g., the *Introduction to Mathematics* (1299); second, three eminent mathematicians were named as the authors of the *Taisei Sankei* according to the *Biography of the Takebe Family*; third, contents of the book showed the variety of mathematics which they considered important; fourth, the book was not printed but several manuscripts have been made and conserved in Japanese libraries; and finally, we show a tentative translation of parts of the text into English to show the organization of the encyclopedic book.

1 Introduction to Mathematics

One Chinese book of mathematics called the *Suanxue Qimeng* 算学啓蒙 (*Sangaku Keimō* in Japanese, *Introduction to Mathematics* in English) was written by Zhu Shijie 朱世傑 in 1299 during the Yuan dynasty. This book is a systematic textbook of traditional elementary mathematics and explains the theory of celestial element, *tianyuanshu* 天元術, which amounts to the theory of polynomials and algebraic equations of one variable in modern mathematics. It disappeared in China during the Ming dynasty but conserved and reprinted several times in years of King Sejong the Great in Korea. After the Japanese invations to Korea between 1592 and 1598, a copy of the *Suanxue Qimeng* was transfered

*A part of this paper was read at the second International Conference on the History of Modern Mahtematics, Xi'an China, May 17 ∼ 20, 2012.

to Japan (see for example [Kawahara2010]). The book was reprinted in 1658 by Hisada Gentetsu, and in 1672 by Hoshino Sanenobu with simple commentary (see [Morimoto2008]).

In 1690, Takebe Katahiro published the *Accomplished Vernacular Commentary* on the *Suanxue Qimeng* 算学啓蒙諺解大成. Takebe's commentary was written in colloquial Japanese and explained every detail of Chinese traditional mathematics including the *tianyuanshu*, and became one of the fundamental books for Japanese mathematicians. Takebe's master Seki Takakazu improved the *tianyuanshu* to be able to handle polynomials of several variables and invented the theory of elimination of variables in a system of algebraic equations. Seki's theory of elimination can be found in Volume 17 of the *Taisei Sankei*.

One of the common interests of Japanese mathematicians were the calculation of the circular rate π in several digits, which was done with a method of acceleration of a sequence as shown in Volume 12 of the *Taisei Sankei*. This investigation eventually leaded to the discovery of the Taylor expansion formula of an inverse trigonometric function by Takebe Katahiro (see [Mor-Oga2007]).

2 Three authors

No name of authors is written in the *Taisei Sankei*. We know the authors according to the Biography of the Takebe (see [Meiji1956, p.270]) written by Takebe Kataakira. We present here a tentative English translation :

> ···[1] Although there have been plenty of books of mathematics in Japan and China, it is regrettable that the explanation of series of operations has not been done thoroughly. Thus three gentlemen, under the leadership of Katahiro, consulted and started in summer of the third year of Ten'a Period (1683) the compilation of a book to describe all the details of the exquisite theory newly invented, and expose all the theories descended from antiquity. The compilation lasted until the middle of Genroku Period (around 1695)[2]. A book of 12 volumes in total was named the *Sanpō Taisei*, the *Accomplished Mathematics*. When it was almost copied, he [Katahiro] became a busy officer of the government and could not engage in the research of delicate mathematics, and Takakazu, being senile and sick for many years, could not think, examine and do research seriously[3].
>
> Therefore, from winter of the 14th year of the same period (1701), while I engaged in the govenmental duties, I, Kataakira, worked on the book concentrating my own thinking for ten years, generalized ideas and wrote them in details into 20 volumes, and named the book the *Taisei Sankei*, of which I myself made a fair copy for completion[4]. (The compilation

[1] 少年（十六歳）より其の弟賢弘と相共に数学に参し、甚だ此の芸に志し有りて異国本朝の算書を披きて、其の旨を暁にすといえども、解難の理会を以て得る事無し。于時関新助孝和（甲府相公綱重卿の家臣）が算数世に傑出せりと聞きて、兄弟各々是を師として学ぶに、暦法天文同じく心を留めて、昼夜寝食を忘れて巧夫をなし、共に術理貫通の道を深く発明す。蓋し、孝和が数に於いて稟（う）くる処生知安行（せいちあんこう）なり。賢弘も又太（はなはだ）叡智にして是に亜（つ）げり。

[2] 凡そ、倭漢の数学、其の書最も多しといえども、未だ釈鎖の奥妙を尽くさざることを嘆き、[関孝和、建部賢明、建部賢弘の]三士相議して天和三年[1683]の夏より賢弘其の首領と成りて各々新に考え得る所の妙旨を悉く著し、就て古今の遺法を尽くして、元禄の中年[1695年頃]に至りて編集す。

[3] 総十二巻、算法大成と号して、粗是を書写せしに、[賢弘は]事務の繁き吏と成され、自ら其の微を窮することを得ず。孝和も又老年の上、爾歳病患に逼られて考検熟思すること能わず。

[4] 是に於て同十四年[1701]の冬より賢明官吏の暇に躬ら其の思いを精すること一十年、広く考え詳に註して二十巻と作し、更に大成算経と号して、手親ら草書し畢れり。

> of this book started in Ten'a Period and completed at the end of Hōei Period (1711). Each volume was revised several ten times. Because of this elaboration, all of the compilation took 28 years to complete.)[5] Nevertheless, as I have tendency to indulge in seclusion, I do not like to become famous. As I have intention to remain incognito and hide my achievements, I handed over all my merits to Katahiro and professed to be a lunatic[6].

According to this information, the *Taisei Sankei* was edited by three gentlemen: Seki Takakazu 関孝和 (1642?–1708), Takebe Kataakira 建部賢明 (1661–1716) and Takebe Katahiro 建部賢弘 (1664–1739). Seki was a founder of Japanese traditional mathematics, *wasan*, and the Takebe brothers (Kataakira and Katahiro) were students of Seki.

The compilation started in 1683 on the initiative of Takebe Katahiro. The authors intended to present systematically all Japanese and Chinese mathematics known to them, and furthermore new theories inaugurated by Seki Takakazu, e.g., theory of resultants and determinants.

An interim version of twelve volumes was prepared around 1695, which was called the *Sanpō Taisei* (算法大成 *Accomplished Mathematics*), which was lost. At the final stage of compilation of this interim version, Seki Takakazu was senile and Katahiro was busy as government officer, Katahiro's elder brother Kataakira took care of the task, generalized description of mathematics, enlarged the book into 20 volumes and completed the compilation alone in 1711. The compilation took 28 years in total.

3 Contents

The *Taisei Sankei* has an independent front matter called Introduction and 20 volumes are divided into three parts: Introduction 首篇, Part A 前篇 (Volumes 1–3), Part B 中篇 (Volumes 4–15), and Part C 後篇 (Volumes 16–20).

Some manuscripts contain the General Catalogue, which lists up all volumes and chapters.

Introduction It includes Discussion on Mathematics and Numbers 算数論, Basic Numbers 基数, Large Numbers 大数, Small Numbers 小数, Degree 度数, Quantity 量数, Weight 衡数, Time 鈔数, Counting Board 縦横, Red and Black Counting Rods 正負, Operation on Counting Board 上退, and Terminologies 用字例.

Part A It treats elementary arithmetic ending with the introduction of determinants.

Volume 1 is entitled Five Techniques 五技 and treats Addition 加, Subtraction 減, Multiplication 因乗, Division 帰徐, and Extraction of Root 開方;

Volume 2 is entitled Miscellaneous Techniques 雑技 and treats Addition and Subtraction 加減, Multiplication and Division 乗除, and Extraction of Root 開方;

Volume 3 is entitled Various Techniques 変技 and treats advanced aspects of the content of the previous volumes, e.g., Discriminant of algebraic equation of degree 2, 3, 4, and 5.

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[5] (此の書、天和の季に創りて宝永の末[宝永八、1711]に終わる。毎一篇校訂すること数十度なり。此の功を積むに因て総て二十八年の星霜を経畢んぬ。)

[6] 然れ共、元来隠逸独楽の機ある故、吾が身の世に鳴ることを好まず。名を包み徳を隠すを以て本意とする者なれば、吾が功悉く賢弘に譲りて自ら癡人と称す。

Part B Volume 4 serves as an introduction to Part B, which are further divided into two parts: Methods of Symbols 象法 (volumes 5–10) and Methods of Figures 形法 (volumes 11–15). The former treats traditional mathematics and games, and the latter treats various problems on geometry and measurement.

Volume 4 is named Three Essentials 三要 and includes Symbol and Figure 象形, Flow and Ebb 満干, and Numbers 数；

Symbol and Figure are the classification of mathematical objects and hence, problems. Volume 4 defines the structure of Part B.

Methods of Symbols Volume 5 treats Mutual Multiplication 互乗, Repeated Multiplication 畳乗, and Pile sums 垜積；

Volume 6 treats Fractions 之分, Several Methods of fractions 諸約, and Art of Cutting Bamboo 翦管；

Volume 7 treats Magic Squares, Magic Circles 聚数, Josephus Problems 計子, 算脱, and Coding Problems 験符；

Volumes 8 and 9 treat Daily Mathematics 日用術.

Methods of Figures Volume 10 treats Regular Squares 方, Rectangles 直, Regular Triangles 勾股, and Polygons 斜(三斜、四斜、五斜);

Volume 11 discusses Regular Polygons 角法；

Volume 12 is concerned with Rates of Figure 形率, i.e., Circle Theory 円理, and treats the length of the circular circumstance 円率, the length of an arc 弧率, the volume of a ball 立円率, and the volume of spherical figures 球闕率.

Volume 13 is the same as Seki's monograph the Measurement 求積.

Volumes 14 and 15 are concerned with Techniques of Figure 形巧.

Part C It is composed of the last 5 volumes and treats Seki's theory of equations.

Volume 16 is named Discussion on Problems and Procedures 題術辨 and the same as Seki's Critical Studies of Problems 題術辨議之法.

Volume 17 is named Solutions of Well-posed Problems 全題解 and similar to Seki's Trilogy 三部抄, which contains Explicit Problems (i. e., direct calculation) 見題, Implicit Problems (i. e., equation of one variable) 隠題, Concealed Problems (i. e., equation of several variables) 伏題, and Submerged Problems (i. e., non algebraic equations) 潜題.

Volume 18 is similar to Seki's Restoration of Defective Problems 病題擬；

Volumes 19 and 20 are named Examples of Operations 演段例 and contain 23 examples of algebraic equations.

4 More than 20 Manuscripts

The *Taisei Sankei* was never published but at least 20 manuscripts have been kept in Japanese libraries. See [Komatsu2007]. We cite here some of them:

1. MSS Kashū: University Tokyo Library, T20/29, 34, 61 $\sim$ 73, 75.
2. MSS Kyoto A: Kyoto University, 219316.

3. MSS Kyoto B: Kyoto University, 102021.
4. MSS Kanō A: Kanō Collection, 7-31453.
5. MSS Kanō B: Kanō Collection, 7-20820.
6. MSS Okamoto: Okamoto Collection, 41-16964.
7. MSS Fujiwara: Fujiwara Collection, 450.
8. MSS Daté: Daté Collection.

1. MSS Kashū was copied by a Confucian scholar Sakakibara Kashū while Takebe Katahiro was alive, and kept at his hand. There is no Introduction, and all volumes lack the name *Taisei Sankei*. It looks like a collection of independent booklets. Many scholars assume that MSS Kashū represents an early stage of the *Taisei Sankei*.

2 and 3. MSS Kyoto A and B are at Department of Mathematics. Both are available in color on the web page of the Kyoto University Library Network. As the red counting rods means positive numbers and the black negative, we cannot distinguish them in black and white picture. MSS Kyoto B was copied in 1851 and revised in 1853.

4, 5, 6, and 7. Four MSS 4–7 are conserved at Tohoku University Library and available in black and white on the web page of Tohoku University Library.

8. MSS Daté belonged to the Daté family. It is now conserved at Miyagi Prefectural Library and has been published in Volume 4 of the book [Seki2010]. The MSS Daté was copied by a mathematician Toita Yasusuke in 1780 and lacks Volume 20.

5 Structure of the *Taisei Sankei*

Along with the tradition of Chinese mathematics, Takebe Katahiro recognized mathematics as a bunch of mathematical problems. He tried to classify mathematics (i.e., mathematical problems) and to organize the Complete Book of Mathematics.

The general framework of the *Taisei Sankei* is stated at "Discussion on Mathematics and Numbers" in Introduction.

Volume 4 "Three Essentials" can be considered as the classification of all mathematical objects, thus giving a framework of known problems of mathematics treated in Part B.

Part C treats Seki's theory of algebraic equations, which was outlined in the "discussion on Problems and Procedures" (Volume 16). Note that not only well-posed problems but defect or incomplete problems are under consideration.

5.1 Discussion on Mathematics and Numbers

Here is a tentative translation of the text into English:

> Mathematics is the Number. The Number describes the essence of the original property of all things; Mathematics describes everything already clarified and is employed for application. Certainly Chaos is originally "No Ultimate" but "Great Ultimate", which is the commencement of all Reasons. It moves and does generate One. [7]

[7] 算者数也。数言万物本具之体。算言已顕而相為之用也。蓋混沌本無極而太極，是衆理之肇，動而生一焉。

One is *yang* and odd. Here the emergence of the number is recognized as Increase, and as Flow. If discussed by Reason, it is the positive; if named by Substance, it is the symbol; if said by Technique, it is the addition. Number One becomes stable and generates Number Two.[8]

Two is *yin*, and even. Here the generation of the number is recognized as Decrease, and as Ebb. If discussed by Reason, it is the negative; if named by Substance, it is the figure; if said by Technique, it is the subtraction.[9]

In this way, two kinds of number, odd and even, are generated; the increase and the decrease, the Flow and the Ebb are named; two Reasons, positive and negative, are equipped; two Substances, the symbol and the figure, are distinguished; two Techniques, the addition and the subtraction, are prepared. They influence each other and accumulate numbers, where the names of large and small numbers are classified.[10]

Here the repeated addition and subtraction are not fast in calculation. Therefore, create the multiplication table "nine times nine", and establish the method of ordering and shifting numbers. By these methods two techniques [i.e., addition and subtraction] directly attain the number and quickly determine its order.[11]

Repeated addition is called multiplication; repeated subtraction is called division, so-called the quotient-division method, the operation of which is difficult to understand easily. Therefore, compose the chant of division to replace it. The repeated division is called the root extraction. These [addition, subtraction, multiplication, division, and root extraction] are called "Five Techniques."[12]

Further, classify the symbol and the figure into four forms, establish three levels of the flow and ebb, and distinguish numbers in four kinds. Apply them to each other. These [symbol and figure, flow and ebb, and number] are so called "Three Essentials."[13]

Afterwards, as the condition becomes elaborated and the techniques are applied in varied way, problems are established and procedures of solution developed. These [problem and procedure] are called the "Two Significances."[14]

In this way, [the book] is equipped with the formula of five techniques, explains the methods of symbols and figures, analyze problems precise or not, and distinguish procedures good or bad. These altogether, with their variations, exhaust the way of mathematics fully. Therefore, I edit an article of summary, place it at the beginning of the book of 20 volumes to write the outline of the organization of the book for the reference of the reader.[15]

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8 一者陽也、奇也。是数所始為増、為満。由理論之, 則為正；由物名之, 則為象；由技言之, 則為加也。一数静而生二焉。

9 二者陰也、偶也。是数所成為損、為干。由理論之, 則為負；由物名之, 則為形；由技言之, 則[為]減也。

10 既而奇偶両数相生, 増損満干名立, 正負二理相具, 象形二物相分, 加減両技相備, 則自是交感積数而分大小之名義焉。

11 於是累加減者, 以功不速, 故創造九九合数, 立定位渉降法焉。両技由此, 直得其数, 速定其位矣。

12 乃累加者, 号乗；累減者, 号除；是所謂商除也。然以其所為輙難暁, 故制帰除句訣而代之。累商如者号開法。謂之五技。

13 又分象形四体, 立満干三科, 別数四等, 而互相為用。謂之三要。

14 然後成云為之巧, 致変化之技, 題問自斯而立, 法術自斯而起。謂之両義。Some MSS 両義 is written as 両儀.

15 於是備五技之式, 积象形之法, 解題之精粗, 分術之邪正, 悉挙其変而以尽数学始終之道也。是故綴総括一篇, 弁諸巻首, 誌編次之大意, 以迪来者云。

5.2 Three Essentials

Volume 4 was named Three Essentials 三要, in which Takebe Katahiro's philosophy on mathematics was exposed. Volume 4 is divided into three sections: Symbols and Figures 象形; Flow and Ebb 満干; and Numbers 数. Each section starts with a general statement followed by problems (67 in total) which serve as examples for the general statement.

In 2010, we already discussed Volume 4 in an international conference in Xi'an (see [Morimoto2010]) and we refrain from discussing it again.

5.3 Discussion on Problems and Procedures

Part C treats Two Evidences, i.e., problem and procedure.

Volume 16 starts with the following statement

> The problem is to write the condition on symbol and figure and to question the number hidden in it. Therefore, the condition is for the problem and can be changed properly.[16]
>
> If the number originally equipped is clarified and stated, the condition is right; If it is stated with many techniques, the condition is changeable. If the statement is complicated and the condition is elaborated, then frequently its reason is covered. By this problem is classified into four categories: visible, hidden, concealed, and submerged.[17]
>
> That being so, numbers can be used to express the condition and to give proof to the answer. If the numbers are not appropriate and the condition is not stated [properly], then there are some troubles in the reason of procedure and in answer numbers. [18]
>
> The procedure is to describe the technique of finding [answer] numbers. In its description, always there are formula, some of which cannot be changed and some can. The rule and the procedure are to find the numbers which are not stated in the problem to solve. Therefore, whenever we state a problem and seek the answer, we should observe the reason of action and perform its technique.[19]
>
> That is, if the difference is given, apply the addition; if the sum is given, apply the subtraction; if the product is given, apply the extraction of the root; if the quotient is given, apply the multiplication. In every case, we recover the original status. If the sum is questioned, apply the addition; if the difference is questioned, apply the subtraction; if the product is questioned, apply the multiplication; if the quotient is questioned, apply the division. In every case, we are convenient in finding the answer. These two situations are all based on the natural reason. [20]
>
> Certainly, the reason is hidden, clarified, general, or stuffed; the technique is slow, fast, former, or later. Hence, we classify all procedure involved into four categories: real, forcible, biased, and wrong. Therefore, we raise below 87 problems and completely dis-

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[16] 題者, 署象形云為之辞, 而問所蔵之数也。是故, 辞者, 題之用而有正変之義矣。

[17] 顕本具数而言者, 正辞也；致衆技而言者, 変辞也。若所言混雑而巧辞, 則屡蔽其理。是以見、隠、伏、潜之四題分焉。

[18] 然而数能成辞之用, 亦為答之証。若数不応, 辞不称, 則有術理答数之煩, 是以八条之病生焉。

[19] 術者、述求数之技也。其所為, 常以有定式, 応変之異者, 曰法, 曰術, 皆求題中不言之数。故每設題下問, 必察当為之理而成其技矣。

[20] 乃言差者, 加之；言和者, 減之；言積者, 開除之；言商者, 因乗之。各復于其旧也。若問和者, 加之；問差者, 減之；問積者, 因乗之；問商者, 開除之。各応于其求也。是皆自然之理也。

> cuss the meaning of loss and gain to demonstrate the norm to state the problem and to apply the procedure.[21]

REFERENCES

[Kawahara2010] Kawahara Hideki: History of Mathematics in Korea (in Japanese), University of Tokyo Press, 2010.

[Komatsu2007] Komatsu Hikosaburo: Report of the present status of a revision of the *Taisei Sankei* (in Japanese), RIMS Kokyuroku 1546(2007), 140–156.

[Meiji1956] Japan Academy: History of Mathematics before the Meiji Restoration (in Japanese), Volume 2, Iwanami Shoten, 1956

[Mor-Oga2007] Morimoto Mitsuo and Ogawa Tsukane: The Mathematics of Takebe Katahiro: His Three Formulas of an Inverse Trigonometric Function, SUGAKU EXPOSITIONS, Volume 20 (2) (2007), 237–252. (original paper in Japanese, Sugaku 56(3) (2004), 308-319.)

[Morimoto2008] Morimoto Mitsuo: The *Suanxue Qimeng* and its Influence on Japanese Mathematics, To appear in the Proceedings of the International Conference on History of Mathematics in Memory of Seki Takakazu (1642?–1708), 2008.

[Morimoto2010] Morimoto Mitsuo: Mathematical Philosophy of Takebe Katahiro, International Conference on the History of Modern Mathematics, August 11–17, 2010, Xi'an, China. This paper will appear in RIMS Kokyuroku.

[Seki2010] Study Group for History of Mathematics in East Asian: Great Collection of *wasan* books in Seki school– Seki San Shidensho–(in Japanese), Bensei Shuppan, 2010.

[21] 蓋理有隠、顕、通、塞。技有遅、速、先、後。是以諸術之所起, 有実、有権、有偏、有邪。是故, 今下八十七問, 悉論得失取捨之義而以為設題施術之軌範矣。

HPM in China

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ABSTRACT

We will describe the short history of HPM in China and its relation with the Chinese Society for the History of Mathematics.

TBA

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Highlights of Chosun Mathematics
조선 수학의 명장면들

Jangjoo LEE

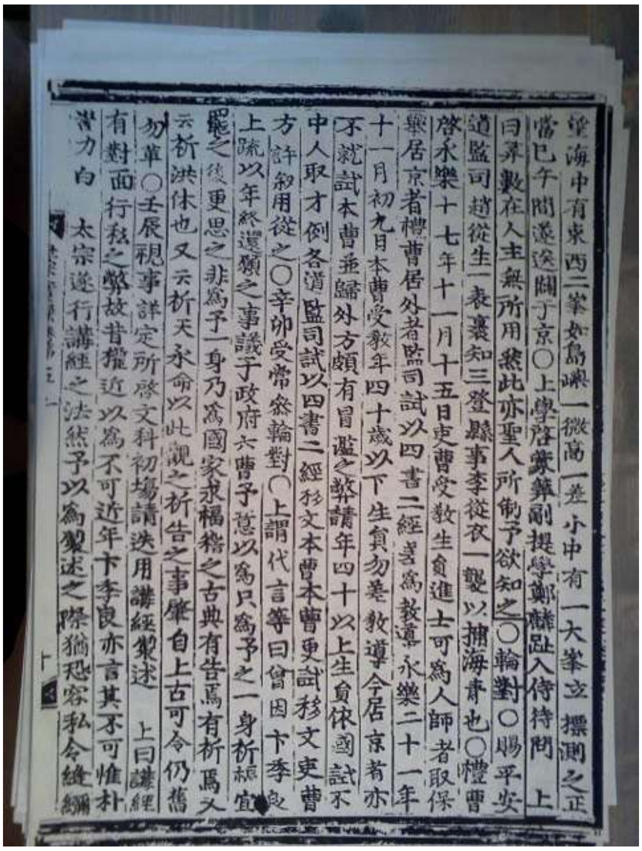
■ 전시 내용 **Contents of Display**

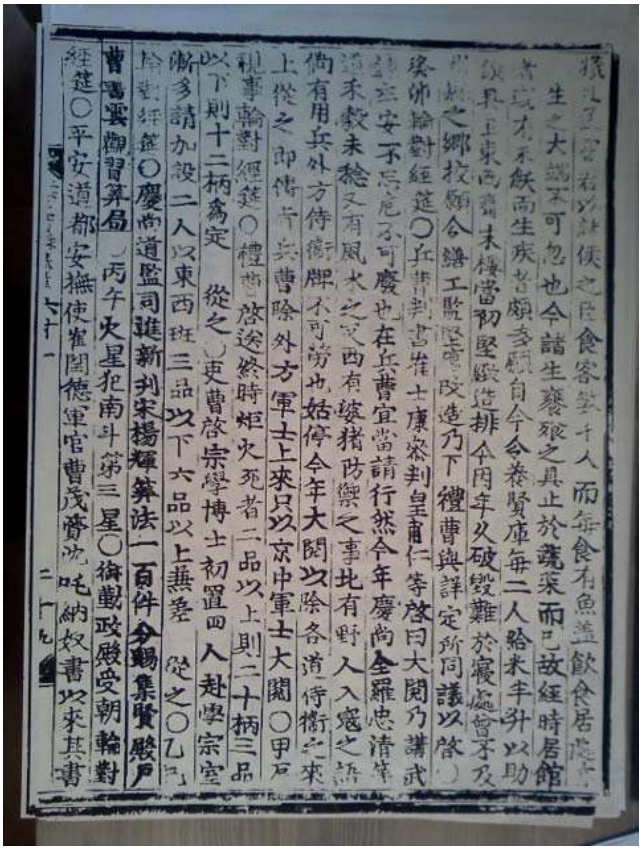
조선 산학의 명장면 24개를 이용한 8폭 병풍 (한국과학창의재단에서 후원)
Twenty four highlights of Chosun mathematics in an eight-fold folding screens

■ 내용 요약 **Summary**

We select indigenous mathematics in Chosun dynasty
in HanGook GwaHak GiSulSa JaRyo DaeGye(韓國科學技術史資料大系) and
the Annals of the Chosun dynasty(朝鮮王朝實錄).
There are 3 scenes in each fold screen which totaled the 24 scenes in eight fold screens.
The folding screens are to be seen starting from the right in the chronological order. We add short explanations which are numbered 1 to 24 according to the order.

1 **2**





1. 세종실록 12년(1430년) 10월 23일 임금이 계몽산(啓蒙算)을 연구하는데, 부제학 정인지(鄭麟趾)가 들어와서 모시고 질문을 기다리고 있으니, 임금이 말하기를, "산수(算數)를 배우는 것이 임금에게는 필요가 없을 듯하나, 이것도 성인이 제정한 것이므로 나는 이것을 알고자 한다."하였다. 여기서 성인이란 공자를 말하는데 신하들의 다른 의견을 이 한마디로 막은 세종의 재치를 엿볼 수 있다. 실록의 오른쪽 두 번째줄부터 세 번째줄까지의 내용이다.

Sejong(世宗,1397 1450) records October 23, 1430—The king was studying SuanXue QiMeng(算學啓蒙, 1299) when BuJeHak Jung InJi(副提學 鄭麟趾, 1396 1478) came in and waited for the king's order. The king said that it seems that there is no need for a king to study mathematics, but since it is also established by saints, I desire to know it. This paragraph can be found on the right of the records from the second to third line.

2. 세종실록 15 년(1433 년) 8 월 25 일 경상도 감사가 새로 인쇄한 송나라의 양휘산법(楊輝算法, 1274-1275) 100권을 진상하므로, 집현전과 호조와 서운관의 습산국(習算局)에 나누어 하사하였다.

Sejong(世宗,1397 1450) records August 25, 1433— GyungSang province governor(慶尙道監司) republished YangHui SuanFa(楊輝算法, 1274-1275) and his 100 books were presented to the King SeJong. The King distributed them to JipHyunJon(集賢殿), HoJo(戶曹) and SeoUnGwan SubSanGuk(書雲觀習算局)

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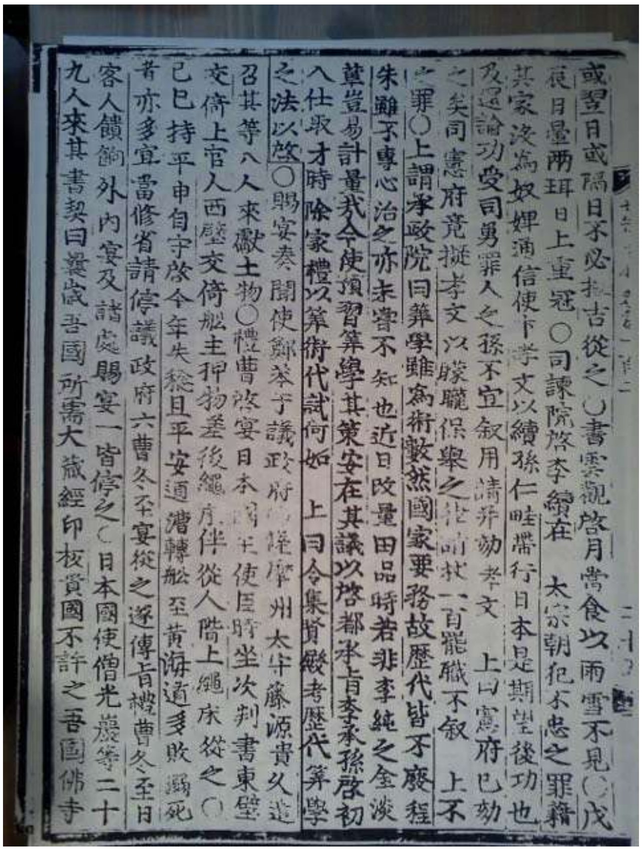

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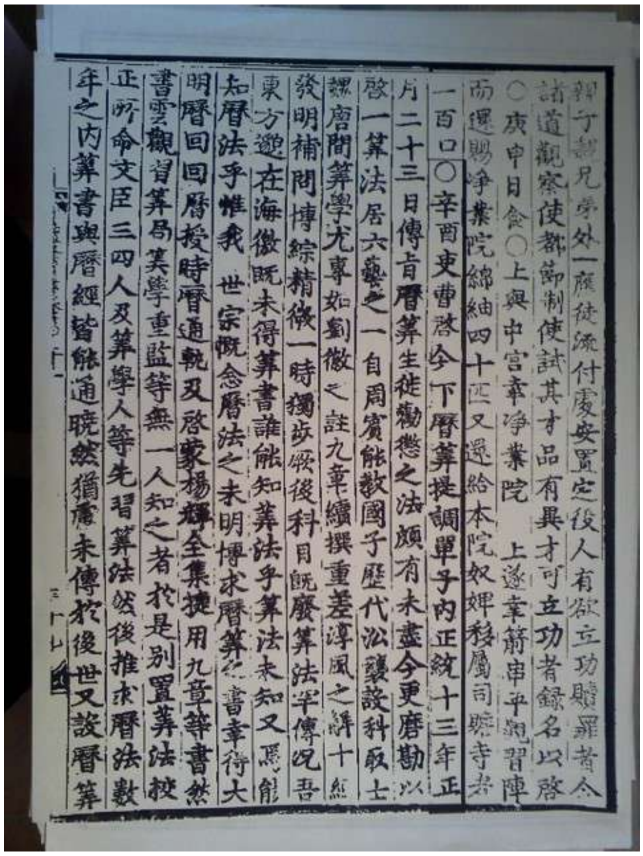

3. 세종실록 25 년(1443 년) 11 월 17 일 임금이 승정원(承政院)에 이르기를, "산학(算學)은 비록 술수(術數)라 하겠지만 국가의 긴요한 사무이므로, 역대로 내려오면서 모두 폐하지 않았다. 정자(程子) • 주자(朱子)도 비록 이를 전심하지 않았다 하더라도 알았을 것이요, 근일에 전품을 고쳐 측량할 때에 만일 이순지(李純之) • 김담(金淡)의 무리가 아니었다면 어떻게 쉽게 계량(計量)하였겠는가. 지금 산학을 예습(預習)하게 하려면 그 방책이 어디에 있는지 의논하여 아뢰라." 하니, 도승지 이승손(李承孫)이 아뢰기를, "처음에 입사(入仕)하여 취재할 때에 가례(家禮)를 빼고 산술(算術)로 대신 시험하는 것이 어떻겠습니까." 하매, 임금이 말하기를, "집현전(集賢殿)으로 하여금 역대 산학의 법을 상고하여 아뢰게 하라." 하였다.

Sejong(世宗,1397 1450) records November 17, 1443—The king told SeongJeonWon(承政院), "Although mathematics might be considered as a scheme, nobody has abolished it because it deals with nation's important affairs. Even though ChengZi(程頤, 1033-1107) ZhuZi(朱熹, 1130-1200) had not devoted themselves to mathematics, they must have known it. If it had not been for Lee SoonJi(李純之, 1406 1465) and Kim Dam(金淡, 1416 1464), the mensuration for the new tax system would not have been so easily completed. Discuss how to study mathematics and report back to me." So DoSeungJi Lee SeungSon(1394 1463) proposed, "How would it be we test mathematics instead of GaRye(家禮) when low level officials firstly take the position and take the test called ChuiJae (取才)?" The king replied, "Order JibHyunJun to study the past laws on mathematics and then report back to me."

4. 세조실록 6 년(1460 년) 6 월 16 일 처음 쪽 조선에서 역산 생도를 권려하고 징계하는 일의 개선책에 대해 논하였다. 산학계몽의 서문의 일부를 인용하고 조선에서 산서를 구하는 일이 매우 어려웠음을 볼 수 있다. 세종 대에는 산학이 어느 정도 발전이 있었음을 잘 나타내어 주고 있다. 세조가 수학을 아주 잘 이해하고 있음도 볼 수 있다.

SeJo(世祖,1417 1468) records June 16, 1460—There was a discussion on reform measures on advising and taking disciplinary actions on YokSanSaengDo. It begins with a quotation of a part of SuanXue QiMeng(算學啓蒙, 1299)'s preface. One can gather that it is quite difficult for Chosun to import books on mathematics and astronomy. It shows mathematics had been developed to a certain degree during Sejong's era. Also, we can see that Sejo(世祖, 1417 1468) understood mathematics very well.

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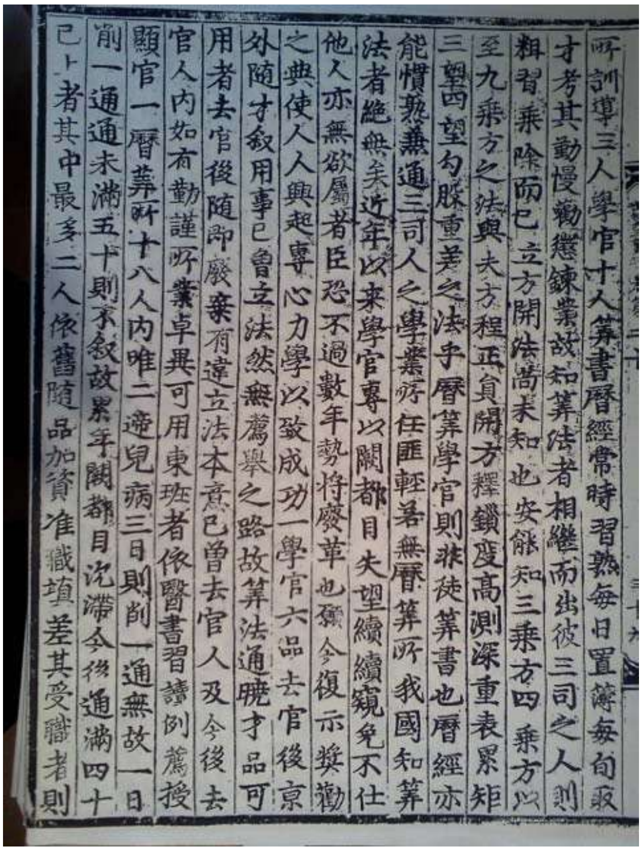

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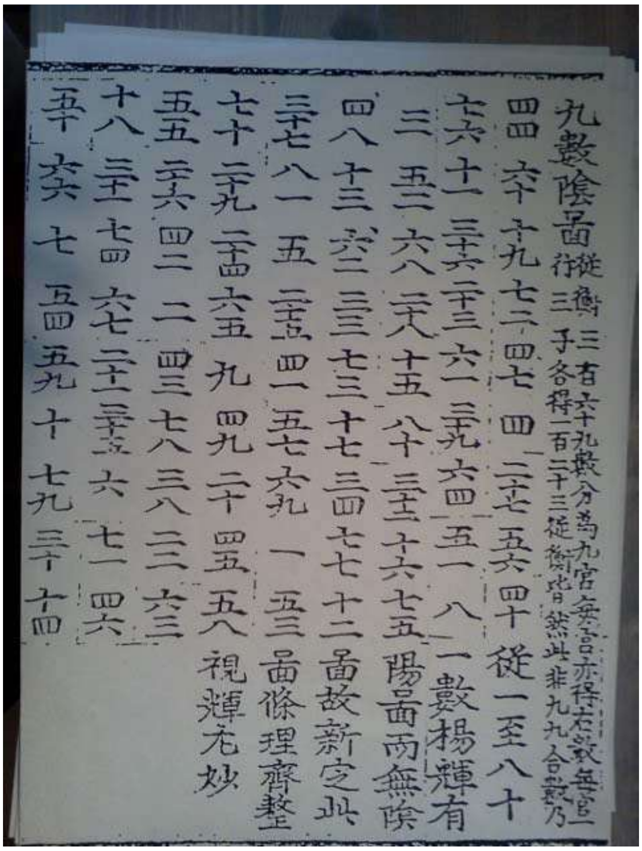


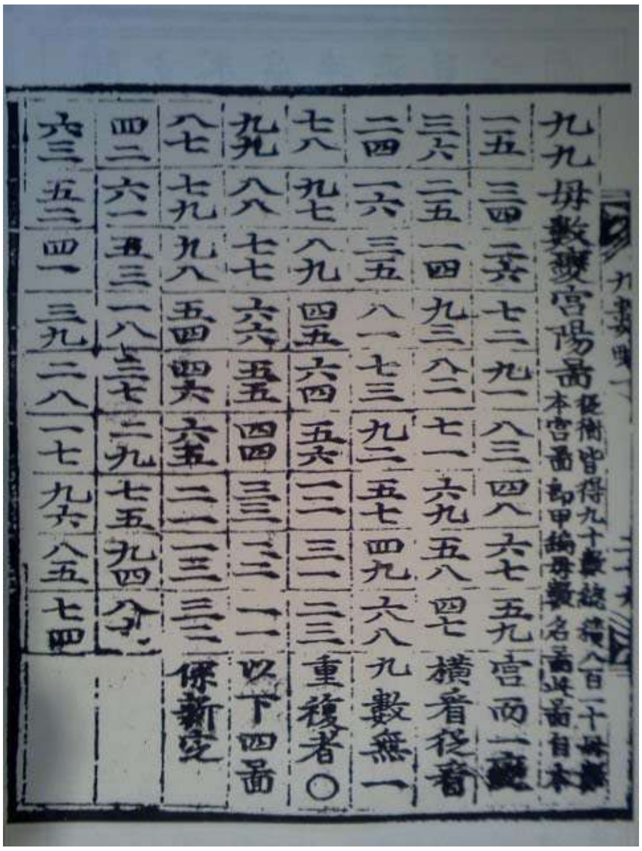
5. 세조실록 6 년 6 월 16 일 두 번째 쪽 세조 대에 와서 산학이 퇴보하여 겨우 계산법과 제곱근만 구할 줄 알아서 연립 1 차방정식 (方程正負) 과 방정식의 구성과 개방법 (開方釋鎖) 을 제대로 이해 못하고 있음을 보여주고 있다. 수학 실력을 높이기위해서 채찍질 하고 있음을 알 수 있다. 正負가 正員으로 잘못 표기 되어있다.

Sejo(世祖,1417 1468) records June 16, 1460—This shows that mathematics had regressed so far back that mathematicians barely knew methods of multiplications and divisions, and how to find square roots during Sejo's era. Furthermore, it shows they did not understand systems of linear equations, construction of equations, and their solving method Gaebangbup(開方法). We can also see how hard they were trying to upgrade their mathematics.

6. 최석정 (1646-1715) 의 구수략 (갑을병정 네 편으로 이루어 졌는데, 갑은 가감승제, 을은 응용, 병은 개방, 입방, 방정, 정은 산법및 마방진의 연구의 내용으로 구성되었다.) 에 들어있는 9 차 마방진으로 최석정이 독창적으로 만들었다. 나머지 마방진은 모두 양휘산법에서 인용했음을 볼 수 있다.

In Choi SukJeong 's Gusuryark(九數略), Choi(1646-1715) quoted magic squares in Yang Hui Suan Fa and added many more new magic squares. One of these is Gusuumdo(九數陰圖) which is a magic square of order 9.

7



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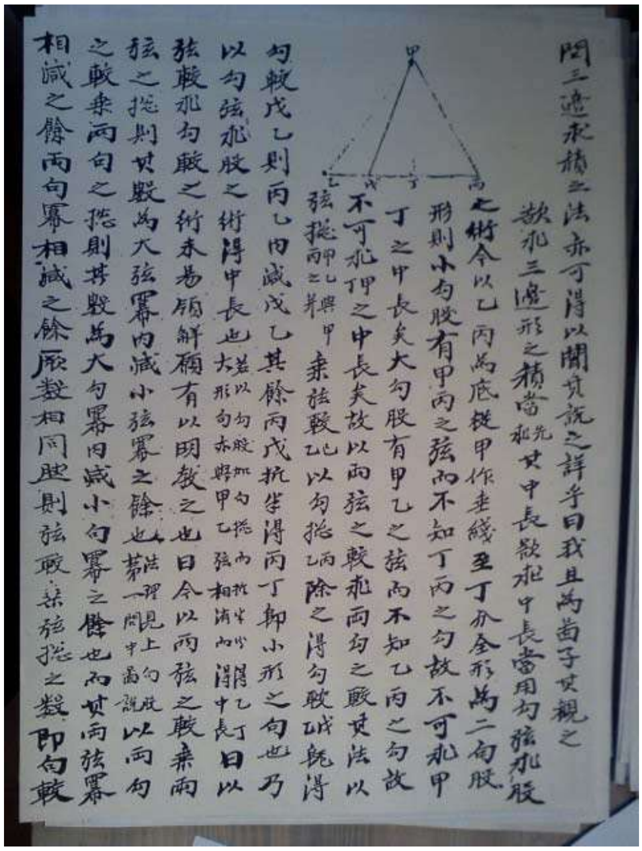


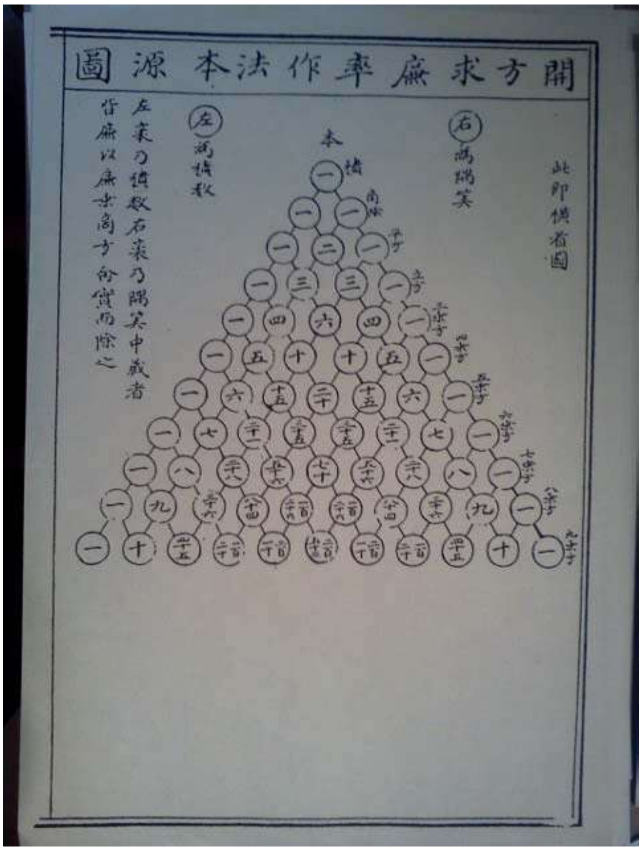
7. 최석정의 구구모수변궁양도 (orthogonal Latin square = Euler square) 이다. 서양에서는 Euler 의 방진으로 잘 알려져 있다. 우리나라가 앞선 것인데 오히려 외국에서 최석정의 학문적 업적을 먼저 인정하였다. Euler(1707–1783) 는 18 세기의 가장 위대한 수학자로 orthogonal Latin square 의 개념을 도입하였다.

Choi SukJeong(崔錫鼎, 1646-1715) also includes an orthogonal Latin square of order 9 as GuGuMoSuByunGungYangDo(九九母數變宮陽圖). Euler(1707-1783) introduced the concept of orthogonal Latin square and hence orthogonal Latin square is also called Euler's square.

8. 조태구 (1660-1723) 의 산학서 '주서관견' 속의 내용이다. 삼각형의 세 변을 주고 수선의 길이를 구하는 법에 대한 증명인데 중요한 사실은 조선산서에 들어있는 최초의 증명이라는 점이다. 수선의 길이를 구하는데 예각, 둔각 삼각형으로 나누어서 증명하였다.

This is an excerpt of Cho Tae Goo(趙泰耉, 1660-1723)'s mathematics book, Juseogwangyeon(籌書管見, 1718). This is a part of finding the height of a triangle with its three sides where it also includes the proof. This is the very first proof in Chosun(朝鮮) mathematics book.

9



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9. 산학자 홍정하(1684-?)의 구일집(1724)의 내용이다. 가헌(12 세기)의 삼각형, 일명 Pascal's triangle 으로 알려져 있다. Pascal(1623-1662)과 마치 오래 전부터 교류하고 있었다는 착각을 불러 일으킨다.

Guiljib(九一集) is written by one of the greatest mathematician, Hong Jung Ha(1684-?). Jia Xian's(賈憲) triangle, also known as the Pascal's triangle were included in the book.

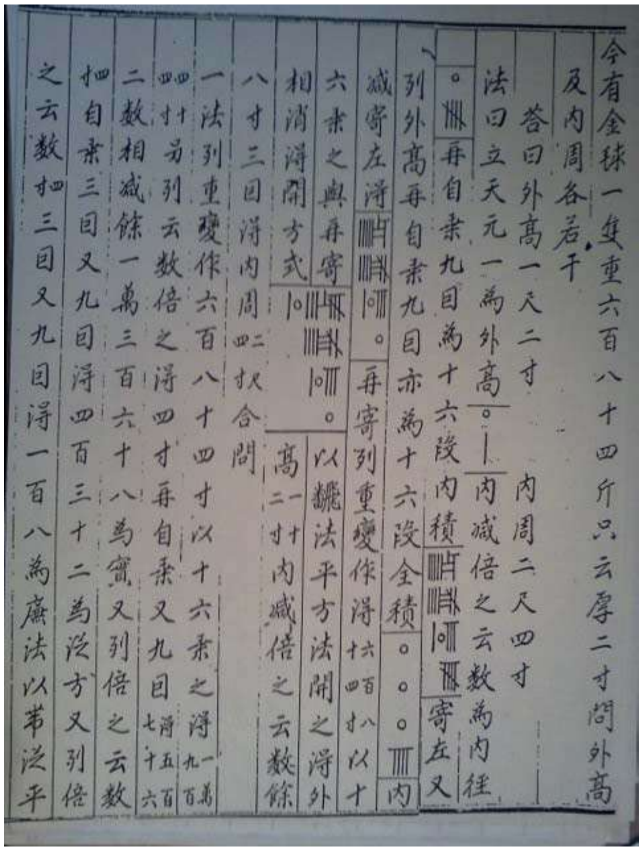
10. 홍정하의 구일집에 나온 내용이다. 현재 고등학교 수학에서 중요하게 배우는 부분이다. $(x-1)^n$ 의 2항계수를 표현하고 있음을 볼 수 있다.

Unlike other mathematics books in Eastern Asia, he also includes the triangle for $(x-1)^n$.

11


百子圖

12



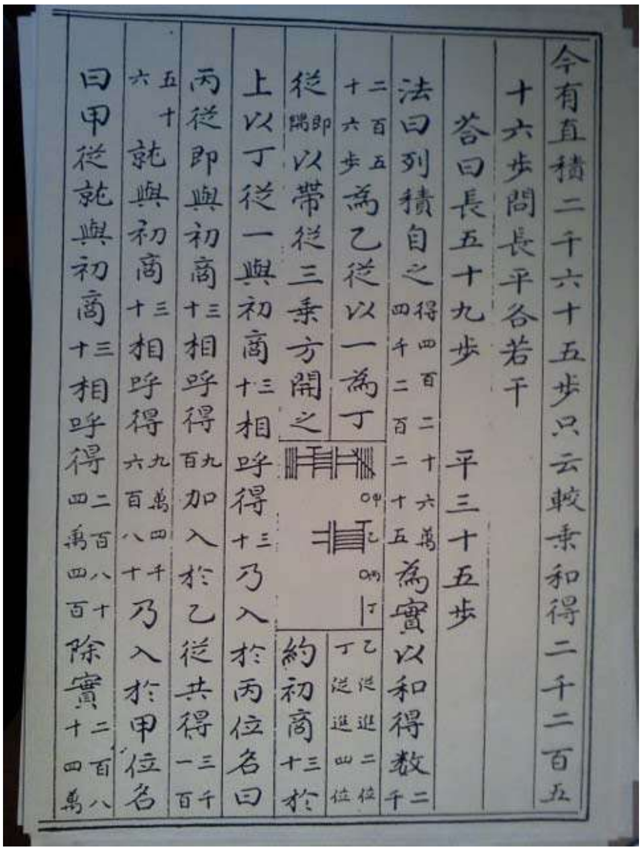
11. 홍정하의 구일집에 실린 백자도이다. 양휘의 잘못된 10 차 마방진을 정대위, 최석정은 자기의 저서에 그대로 인용하였는데 홍정하는 잘못된 10 차 마방진을 고쳤다. 산학의 대가들의 계산을 바로잡아 수학의 정확성을 보여주고 있는데 홍정하의 수학자다운, 오류를 바로잡는 힘과 정직성 그리고 치밀함을 엿볼 수 있다.

The Backjado(百子圖), the magic square of order 10 is found in Hong Jung Ha(1684-?)'s Guiljib(九一集). Yang Hui included an incorrect magic square of order 10 in YangHui SuanFa(楊輝算法, 1274-1275) which subsequently quoted by Cheng Da Wei(程大位, 1533-1606) and Choi Suk Jeong(崔錫鼎, 1646-1715) . Hong corrected it with comments.

12. 홍정하가 천원술을 이용하여 방정식을 구성한 것으로 중국 산학서인 산학계몽에 들어있지 않은 우리 산학의 독창적인 형태의 문제로 의미가 있다.

Using TianYuanShu(天元術), Hong Jung Ha constructs an equation.

13



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答曰長五十九步　平三十五步
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二百五十六步)爲乙從以一爲丁(乙從進二位 丁從進四位)
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上以丁從一與初商(三十)相呼得(三十)乃入於丙位名曰
丙從即與初商(三十)相呼得(九百)加入於乙從共得(三千一百
五十六)就與初商(三十)相呼得(九萬四千六百八十)乃入於甲位名
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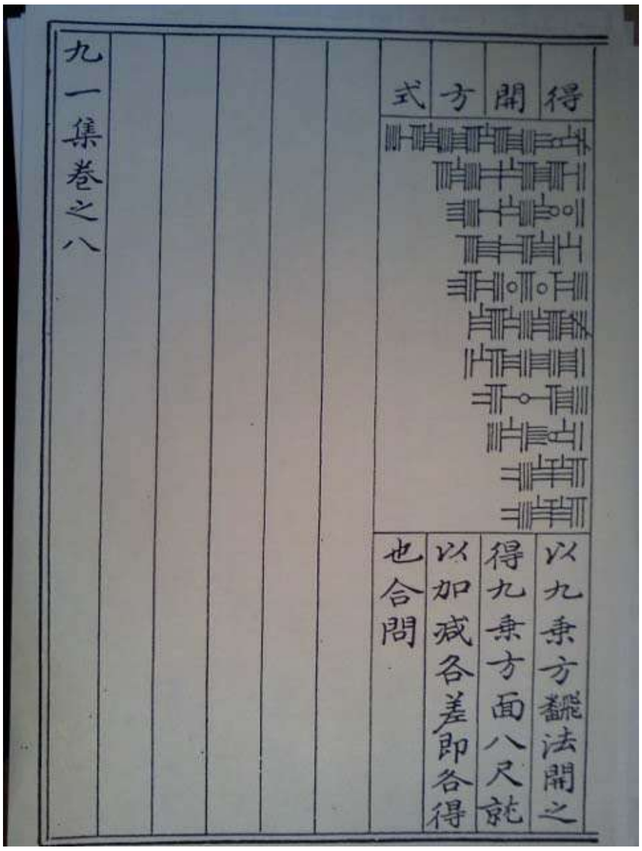

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13. 홍정하의 구일집의 내용으로 전형적인 방정식 이론인 증승개방법을 보이고 있다. 위의 천원술과 방정식의 풀이법으로 드디어 홍정하는 방정식론을 완성하였다.

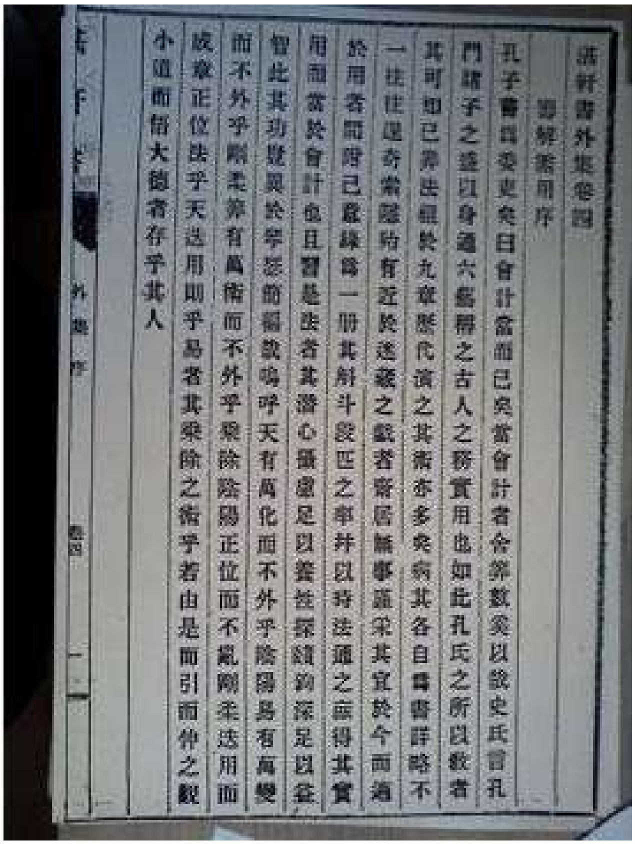
This is a ZengChengKaiFangFa(增乘開方法) found in Hong Jung Ha's Guiljib(九一集). With this together with TianYuanShu(天元術). Hong's equation theory was finally completed.

14. 홍정하의 구일집에 있는 10 차 방정식의 천원술 표시이다. 산대의 조형적인 아름다움이 돋보인다. 훌륭한 디자인적인 요소가 잘 드러나 있어서 현대의 여러 제품의 무늬로 활용해도 참 좋겠다는 생각이 든다. 복잡한 것이 아름답다는 역설이 묘하게 들어맞는다.

Using TianYuanShu(天元術), Hong constructed an equation of degree 10.

15

16



15. ‘주학 팔세보’라는 산학자 집안 가계도에 있는 홍정하의 족보이다. 명문 산학자 집안(남양 홍씨)의 가계를 잘 보여주고 있다. 산학자 집안 가계도는 이외에도 ‘주학 입격안’, ‘주학 선생안’과 같은 종류의 책에 전해진다.

In the family tree called “Juhak Palsebo(籌學 八世譜)” we can find Hong Jung Ha’s genealogy. This shows the genealogy of a prestigious ChungIn mathematician family, Hong of the clan name NamYang. There are two books “Juhak Ipgyuckahn(籌學 入格案)” and “Juhak Seonshangahn(籌學 先生案)” in which we find informations on ChungIn mathematicians who passed the national exam(取才) for mathematical officials in HoJo.

16. 담헌 홍대용(1731-1783)의 수학책 ‘주해수용’의 서문인데 수학의 실용성을 강조하고 있다. 번역은 다음과 같다.

This is the preface of a mathematics book “Juhaesuyong(籌解需用)” by Hong Dae Yong (1731-1783), which emphasizes the practicality of mathematics. The translation is as follows:

공자가 일찍이 ‘위리(곡식의 창고를 관장하는 벼슬)’라는 벼슬을 한 적이 있다. 일명 ‘회계’를 말하는 것인데, ‘회계’라는 것이 수학을 버리고 어찌 설명할 수 있겠는가? 역사가들이 말하길 공자의 제자들이 집대성 하여 몸소 육예에 능통했다고 그것을 칭한다. 고인들이 실용에 힘썼다는 뜻과 같은 개념일 것이다.

Confucius(孔子, BC 551 BC 479) was a “WeiLi(委吏),” a government position in charge of the storage room for crops, earlier on in his life. In other words, he was the accountant, but how can we explain accounting without mathematics? According to historians, Confucius's students made a comprehensive survey and were proficient at the six arts (etiquette, music, archery, horsemanship, calligraphy, and mathematics). Hence, it is the same conception of their being strived for the practicality of mathematics.

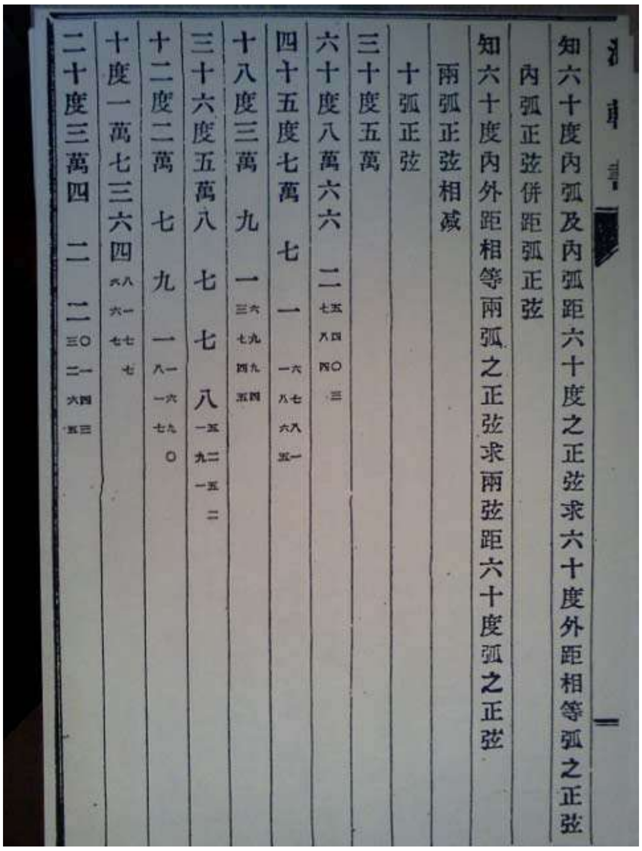
산법은 <구장산술>에 기초하는데, 대대로 내려오는 방법 또한 여러 가지가 있다. 그것들은 자세한 것도 있고 간략한 것도 있고, 들쭉날쭉하여 한결같지 않다. 풀어놓은 것을 보면 대개 특이한 부분이나 숨겨진 방법을 찾는 것이 거의 숨바꼭질과 가깝다. (아마도 그 이유는 저자들이 여흥이나 유희 식으로 소일거리 삼아 계통 없이 수학을 대했던 태도에 기인하지 않나 생각한다.) 나는 지금의 실정에 맞게 실용적으로 수학을 다룬 내용을 찾아서 나의 뜻에 부합된 것을 부쳐 한 권의 책으로 꾸며보았다. 언제든지 용량과 길이의 비율, 상황에 맞는 실용성을 활용하여 회계를 처리할 수 있게 하였다. 또 이 법을 익히는 자는 마음을 가라앉혀 깊이 생각하면, 족히 본성을 기를 수 있고, 깊이 탐구하고 깊이 찾으면 족히 지혜에 도움이 될 수 있다. 이 공이 어찌 좋은 악기를 얻고 좋은 책을 얻는 것과 무엇이 다르겠는가?

Mathematics has its foundation on Gujangsansul(九章算術), but there are many ways on how it came to be. There are detailed and simple methods among them, so they are inconsistent. If we take a look at the steps in solving them, finding any peculiar parts or the hidden methods are almost like hide-and-seek. I have made this book by compiling the practicality of mathematics in the current sate along with my will. I have made it so that you can apply the ratio of volume and length to any practical situation to do accounting. Also, if anybody who learns this method think deeply, study deeply, and search deeply enough, they will find it useful to the wisdom. How can it be different from earning a good instrument or book?

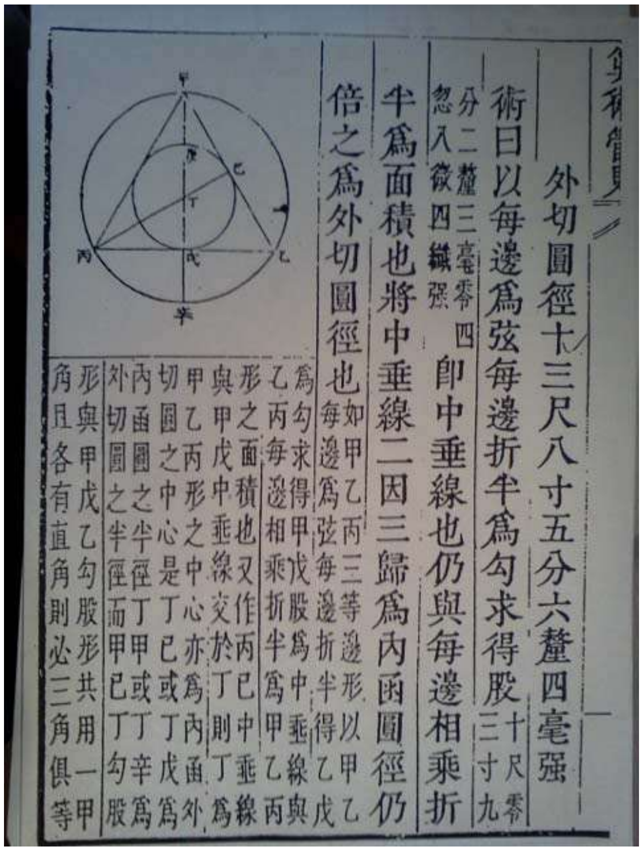
하늘은 만물의 변화가 있어 음양의 이치에 벗어나지 않고, 주역은 만물의 변화가 있어 강하고 부드러운 것에 벗어나지 않고, 수학은 만물에 있어 승제에 벗어나지 않는다. 음양이 바른 자리에 서면 어지럽지 않고, 강하고 부드러움이 질서에 맞게 잘 교차하면 성장과 조화를 이룬다. 바른 자리는 하늘의 법도가 되고, 교차하여 쓰이면 주역에서 법도가 되니 어찌 수학에 있어서 승제의 기술이 아니겠는가? 만약 이러한 논리를 바탕으로 이 논리를 넓히고 잘 펼치어 작은 도리를 보고 큰 법을 깨닫는 것은, 이 책을 읽는 사람의 몫일 것이다.

All creation on earth does not stray away from the yin yang logic, the driving force does not stray away from the wild and gentle, and mathematics does not stray away from multiplication and division. If the yin and yang are at the right position, nothing goes out of order. If the wild and gentle cross each other in order, they create harmony. The right position becomes the law of the heavens and, if used interlockingly, it becomes the law of driving force. Then how can it not be a technique of multiplication and division in mathematics? It is up to the reader to realize a bigger law from a small duty by analogy and extension

17



18



17. 담헌 홍대용(1731-1783)의 수학책 '주해수용'에서 보이는 특수각의 $r \sin x$ 값($r = 100,000$ 인 경우)이다. 이로써 서양수학인 삼각함수표를 중국으로부터 얻어서 사용하였음을 볼 수 있다. 즉, 서양수학의 본격적인 유입이 시작되었다는 사실을 알 수 있다.

Table for rsin x (r=100,000) for special angles shown in "Juhaesuyong(籌解需用)." The book is the first mathematics book in Chosun which deals with plane trigonometry.

18.~19. 조선 제일의 천재 수학자 이상혁(1810–?)의 산학서인 산술관견(1855)에 들어있는 문제로 원에 내접하고 외접하는 여러 정다각형 문제들이다. 현재 되살려서 고등학교 교과서의 문제로 활용해도 손색이 없을 정도로 뛰어나다. 붓으로 어떤 도구를 가지고 원과 다각형을 그렸는지 궁금하다.

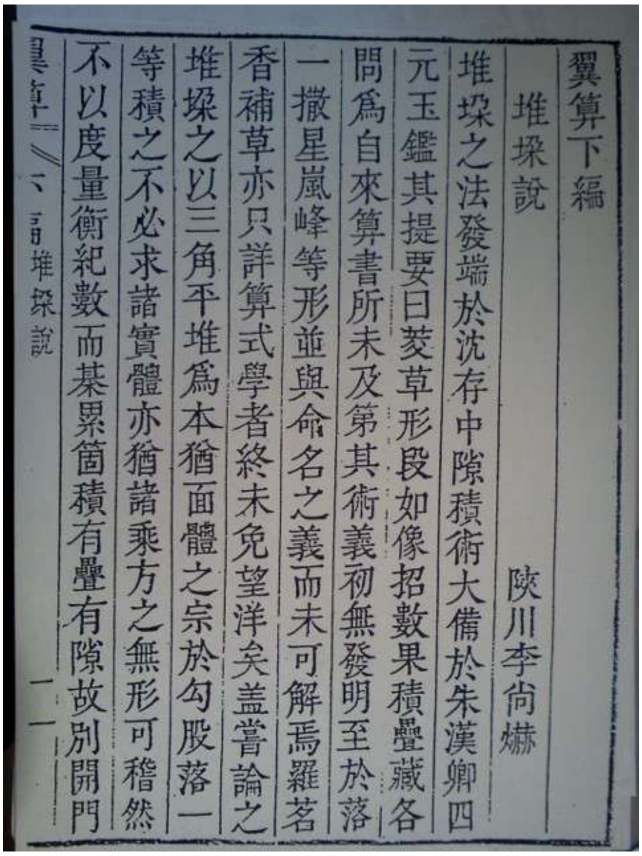
These problems dealing with inscribed and circumscribed regular polygons, are found in Sansulgwangyeon(算術管見, 1855) written by Joseon Dynasty's greatest mathematician, Lee Sang Hyuk (1810–?).

19

濶較卽中率倍之數也 卽八等邊形之內
函圓徑也以爲股每邊爲勾求
得弦卽外切圓徑也且以內函
圓半徑與每邊相乘折半得數
八因之爲面積也 如甲乙丙丁戊已庚辛八
等邊形作甲已辛戊二線則俱
爲內函圓徑又作乙已丙庚二
線則俱爲外切圓徑又作乙辛
線則乙甲已勾股形與乙甲壬
勾股形共用一甲直角且甲已
乙角所對甲乙弧與甲乙壬角
所對甲辛弧等而角度亦等則
三角必俱等而爲同式形故甲
已股與甲乙勾之比同於甲乙
股與甲壬勾之比而爲連比例

各等邊形拾遺 八

20



翼算下編
堆垜說 陜川李尚爀
堆垜之法發端於沈存中隙積術大備於朱漢卿四
元玉鑑其提要曰茭草形段如像招數果積疊藏各
問爲自來算書所未及第其術義初無發明至於落
一撒星嵐峰等形並與命名之義而未可解焉羅茗
香補草亦只詳算式學者終未免望洋矣盖嘗論之
堆垜之以三角平堆爲本猶面體之宗於勾股落一
等積之不必求諸實體亦猶諸乘方之無形可稽然
不以度量衡紀數而棊累箇積有疊有隙故別開門

堆垜說 一

20. 이상혁의 익산(1868) 에 있는 내용이다. 퇴타술(유한급수론) 에 대한 역사와 동기에서 저자가 얻어낸 중요한 업적중의 하나인 기본 철학을 나타낸 부분이다. 세계 어느 곳에도 없는 그의 독창적인 아이디어가 돋보인다. 아마도 같은 부호와 현대 수학의 여러 급수를 이 분이 알고 있었다면 세계에서 가장 뛰어난 급수이론의 종결자가 되지 않았을까?

Lee Sang Hyuk (1810 - ?)'s last book Ik-San(翼算, 1868) is consisting of two books, JungBuRon(正負論) and ToeTaSul(堆垜說) The former deals with theory of equations and the latter with theory of finite series. In this preface, Lee shows his motivation for ToeTaSul(堆垜說), mathematical and historical motivations together.

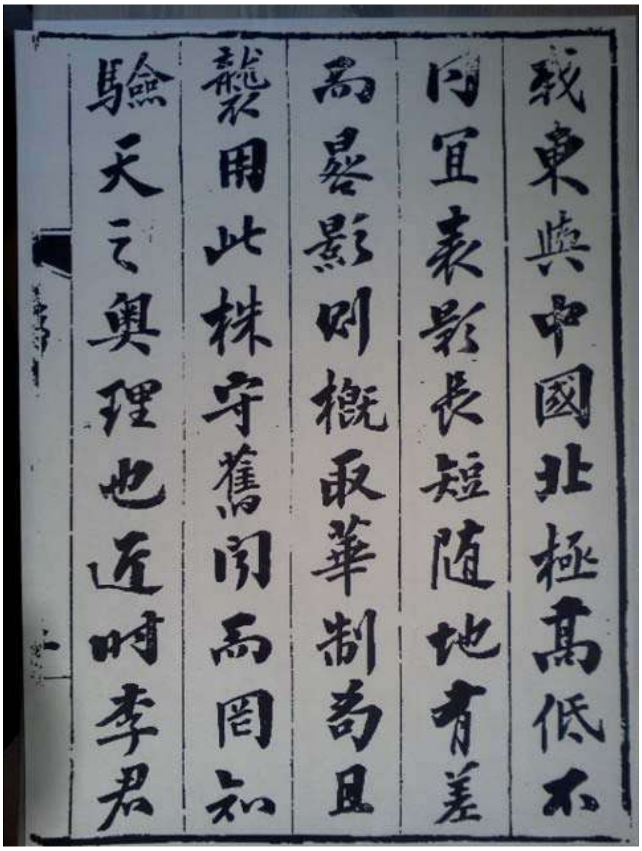
21.~24. 남병길이 이상혁의 천문학에 관한 저서 <규일고(1850)>에 써 준 친필 서문이다. 당대의 정승인 남병길이 중인 산학자 이상혁에게 써준 서문인데 친구라고 일컬은 것으로 보아 학문적인 교류와 함께 계급적인 관념을 초월한 수학의 세계를 엿볼 수 있다. 그리고 거리낌 없는 필체의 멋도 느낄 수 있어서, 병풍으로 위의 내용을 만들고 보니 가장 멋진 부분이라고 많은 분들이 즐거워하였다.

Nam Byung Gil(南秉吉, 1820 1869) wrote a preface to Lee Sang Hyuk (1810 - ?)'s book <Gyu Il Go (揆日考, 1850)> on the Chosun(朝鮮) astronomy. Nam is from Yang Ban(兩班) family and gets appointed a minister of YeJo(禮曹) and he is a well established mathematician and astronomer who published many books by himself. By Nam, an elitist of the noble group, coining the term "friendship" with his relationship with Lee, we can see the scholarly collaboration along with the world of mathematics, which could eliminate their different social standing. This also shows the beauty of Nam's calligraphy.

21

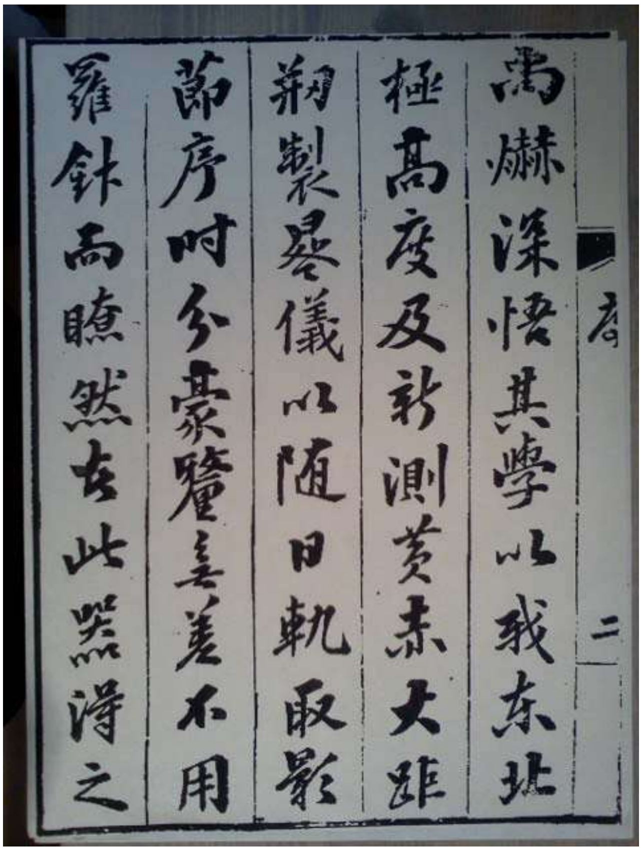
揆日考序
六執之教數居其一曆象
之學是也余考察疇人諸
書始悉寢食未嘗一日而
廢焉粤自黄帝迎日推策

22



我東與中國北極高低不
同冝表影長短隨地有差
而晷影則概取華制而且
襲用此株守舊聞而罔知
驗天之與理也近時李君

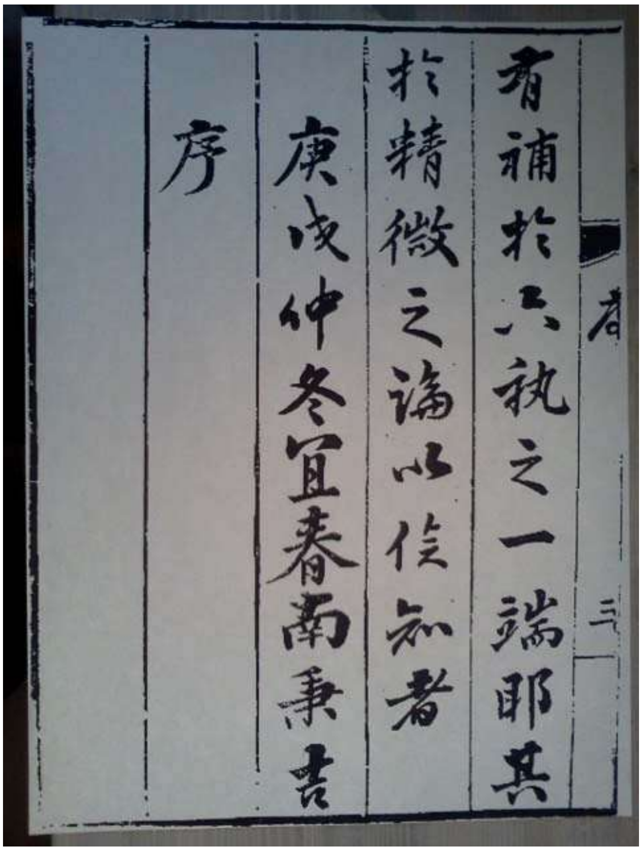
23



為爀深悟其學以我東北
極高度及新測黃赤大距
新製晷儀以隨日軌取影
節序時分毫釐之差不用
羅針而瞭然在此器得之

序 二

24



有補於六執之一端耶其
於精微之論以俟知者
庚戌仲冬宜春南秉吉

序

序 三

MODERN MATHEMATIC BOOKS BY KOREAN AUTHORS DURING 1884–1910

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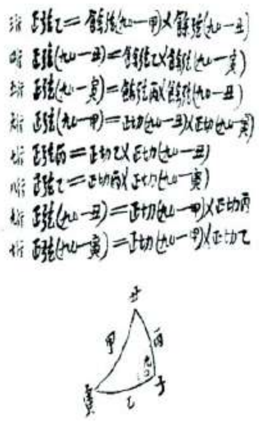
ABSTRACT

19세기말 한국의 근대 수학은 일본과 중국을 통해서만 도입되었다고 일제에 의하여 소개되어, 그것이 정설로 여겨졌으나 사실은 우리의 자주적인 노력도 함께 있어왔다. 21세기 한국 수학의 도약을 위하여 근대수학 도입과 발전과정에 대한 정확한 이해는 역사를 바르게 보고 미래를 준비하는데 필수적이다.

조선의 과학은 수학과 천문학이 중심이었으며, 개화기에도 갑오경장을 전후하여 1885년부터 수학책은 꾸준히 소개되었다. 그러나 1884년에서 1910년 사이의 한국 수학에 대한 연구는 그간 황무지였다. 중요한 것은 개화기 과학책은 1900년 이전의 사료는 거의 존재하지 않지만, 개화기 근대수학책은 현재 발굴한 李相卨의 <數理>, 학부가 급히 발간한 <簡易四則問題集>(1895), <近易算術書>(1895), 학부의 의뢰로 이상설이 편찬한 <算術新書>(1900) 등 근대과학보다 수학의 소개가 훨씬 먼저이며 더 중요하고 연구가 필요한 추가 사료들이 망라되어 있다. 저자가 현재까지 파악한 한국 근대수학책은 이미 약 70 종이 넘는다. 따라서 본 연구진이 추가 발굴한 개화기 조선인들이 쓴 <數理>, <數理學雜誌>를 포함한 120여 년 전부터의 수학사료들을 전시하여 제공하는 것은 우리에게 새로운 시야를 갖게 해줄 것이다. 따라서 본 전시에서는 저자가 그간 발굴한 1884 년에서 1910 년 사이의 한국 근대수학책을 소개한다. 이 내용은 세계에 전혀 알려져 있지 않아서 세계 수학자들도 많은 관심을 보이는 분야이므로 본 국제학회를 통해 한국 수학의 발전 과정에 대하여 공유하는 계기로 삼고자 한다.






<List of Modern mathematic books by Korean authors during 1884–1910>

We will try to display some of the following Korean Mathematics books that were published during 1884–1910 by Korean authors. Many books were written with Korean and Chinese characters.

Year	Title of book	author	(발행년)	publisher	현재소장처	기확보여부
1887~1898	수리	보재 이상설-성균관장	1887~1898	친필	이문원 개인 소장	이상구 확보
1895	간이사칙문제집	학부편집국 편	1896 / 1895	학부	국립중앙도서관	이상구 확보
1900	산술신서 상(1, 2권) 1판, 2판	보재 이상설-학부의뢰	1900	학부?	국립중앙도서관	이상구 확보
1900	정선산학(상)	남순희- 학부편집국 검정	1900	탑인사	국사편찬위원회	이상구 확보
1901	신정산술 新訂算術(一, 二, 三)	남원 양재?	1901 광무 5년 / 光武 10년-1906년		국립중앙도서관	이상구 확보
1902	산술신편	FIELD 필하와	1902		연세대학교 도서관 5층 국학자료실	이상구 확보
1905	FIRST BOOK IN ARITHMETIC	영 문. 도 산 안창 호 (安 昌 浩 :1878~1938) 가계소장서적으로.	1905	정리사	1905년 11월호부터 1906년까지 통권8호	이상구확보
1905	수리학잡지 數理學雜誌	유일선	1905	정리사	1905년 11월호부터 1906년 까지 통권 8호	이상구 확보
1907	초등 산학신편(국문)	E. H. Miller, 오천경(편역)(국문)	1907/ 광무 11년/ 융희 1년	대한예수교 서회간인	독립기념관	이상구 확보
1907	중등산술교과서상	이원조?	1907		연세대학교 도서관	확보
1907 1907	중등산술교과서상 중등산술교과서상	이원조? 이원조?	1907 1907		/5층 국학자료실	확보 확보
1902	산술신편	FIELD 필하와	1902		연세대학교 도서관	이상구 확보
1907 1906/1907	중등산술교과서하 算術教科書	이원조? 이원조	1907 1906	大同報社	/5층 국학자료실	확보 보안확보
1905	수리학잡지 數理學雜誌	유일선	1905	정리사	이화여대 중앙도서관	이상구 확보
1907	중등산술교과서 상, 하	玄公廉發行	1907		연세대학교 도서관 확인중	
1907	중등산학, 하	이원조	1907		연세대학교 도서관	부분확보/연대
1907 1908	중정 산학통편 산학 신편(상, 하) 산학계몽	이명칠 필하와 (Eva Field) 저, 신해영 술	1907 1908	대한아소 교서회	독립기념관 소장	이상구 확보 이상구 확보
1907	중등산학, 하	이원조	1907		국립중앙도서관	부분확보/연대

1908	대수학교과서	김준봉 저. 유일선 교열	1908	정상환	연세대학교 도서관	이상구 확보 / 이대보안확보
1908	보통학교 산술교과서	홍병선(洪 秉 琁, 1888. 12. 7~1967. 7. 19)	1908		보통학교 산술보충교과서(普 通 學校算術補充敎科書)1927	홍병선의 보통학교 산술교과서 *(1908),-홍성사교수님?
1908 1908	(新式) 算術敎科書	이상익 홍종욱洪鍾旭	1908 1908	大東書市 普文社	이화여대 중앙도서관	보안 확보 이지희확보
	(新撰) 算術通義 (上)				이화여대 중앙도서관	
1908	신식산술교과서(전)	이상익저, 안종화(安鍾和)의서(序)	1908년 6월	일한인쇄주식회사, 日韓圖書印刷株式會社	?	
1909	산술교과서(상)	이교승 저, 이면우 공열	1908	이면우법률사무소	연세대학교 도서관	파손으로 대출불가표지사진확보
1909	산술교과서(하)	이교승 저, 이면우 공열	1908	이면우법률사무소	연세대학교 도서관 / 한결 김윤경 문고	이상구 확보
1908	초등근세산술(전)	이상익	1908	휘문관(徽文館)	연세대학교 도서관	확보
1908	초등산술교과서(상권)	유일선	1908	정리사	연세대학교 도서관/독립기념관	이상구 확보
1908	최신산술상 · 하2권.	김하정(金夏鼎)/오영근(吳榮根) 교열	1908	서울 일신사(日新社)에서 간행	한밭교육박물관	대전/복사예정
1908	중등용기서법	吳榮根	1908		연세대학교 도서관	부분확보/연대
1908 1908	중등용기서법 융희신산술상편(1908)	吳榮根 정운복	1908	일한서방	국학자료실	부분확보/연대 확보
1909	산술교과서(하)	이교승 저, 이면우 공열	1908	이면우법률사무소	연세대학교 도서관/국학 자료실	이상구 확보
1909	(학부인가)산술지남상권 算術指南上卷	유석태	1909년5.	발행기관 : 휘문소	조흥금융박물관	이상구 일부확보
1910	산술지남하권 算術指南	유석태	1910년	휘문소	독립기념관	이상구 확보
1909	학부검정-산술교과서(하권)	이교승- 사립학교 고등교육 수학과학(교)원용	1909/ 융희3년1월28일	휘문관(徽文館)에서 간행	?	확보
1909	근세대수 (상)	이상익	1909	유일서관	연대/ 고려대학교 도서관	부분확보/연대

1909 1910	평면기하학 중정 대수학교과서	이명구 유일선	1909 1910	광 동 서 국, 중앙 서관 한성신창 서림	서원대학교 박 물관	이상구자료일 부확보/복사 금지하고있음 확보
1909	근세대수 (상)	이상익	1909	유일서관	연세대학교 도서관	부분확보/연대
1910	고등 산학신편 3판	필하와 저. 신해영 술 (3차 출판)	1910/ 융희4년	대한아소 교서회	국립중앙도서관	이상구확보

基于“数学史融入数学课程”的教科书编写

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ABSTRACT

在考察教科书中数学史的呈现形式、影响教师实施“数学史融入教学”的因素和影响学生数学史学习的因素的基础上，提出基于“数学史融入数学课程”的教科书编写策略：史学形态转化为教育形态、关注数学思想方法、呈现多元文化数学内容、建设资源库和教师的发展。

Keywords: 数学史；数学课程；教科书

数学史是数学的一个分支，数学史教育则是数学教育的一个部分；“数学史与数学教育”这个研究领域被数学家、数学史工作者以及数学教育研究者共同关注。HPM 已经成为国际数学教育的新思潮之一。[1] 在我国，无论义务教育阶段还是普通高中阶段的《数学课程标准》[2][3]，都有与数学史相关的论述。两年一次的数学史与数学教育会议迄今已经召开过四次。人们认识到，数学是人类文化的重要组成部分，数学教育是数学文化的教育；而数学史是数学文化的一种载体，数学史融入数学课程有助于学生认识数学、理解数学，感受数学文化。

数学课程是数学教育改革的关键。如何将数学史融入数学课程，使数学史成为数学课程的有机组成部分，是广大数学教育工作者和数学史家共同关注的问题。本文主要论述基于“数学史融入数学课程”的教科书编写问题。

一、教科书中数学史的分类和作用

出现在数学教科书中的数学史料，粗略地分，有显性和隐性两种形式。

（一）显性数学史内容

显性的数学史内容，主要是历史上的人物（主要是数学家）及其贡献、重要的历史事件、发生的时间等等。它们出现在教科书的正文、注解或阅读等栏目中。一些重要的知识点，需要学生熟知和掌握的内容，出现在正文中；而略显次要的，或者文字较多的内容则出现在注解和阅读中。至于形式，显形的数学史以文字、图片和表格等出现。文字有助于学生细致了解数学史的内容；而生动的图片则有助于学生直观、形象地感受人物与事件。

（二）隐性数学史内容

隐性的数学史内容，主要是古人的思想方法。数学的思想方法可以出现在正文中，主要是例题的解法和证法。同时，也可以安排在习题中，让学生去解决古人的问题，或使用古人的方法去解决现在的问题（前者相对比较显性，而后者就显得非常隐性）。

在教科书中，隐性内容相对难做，也比较容易受忽视。事实上，隐性的内容也许才是数学史融入数学教科书、提高学生数学素养的主战场。因为，我们给学生介绍数学史，应该立足过去，把握现在，

指向未来。让学生感受历史、感受数学、感受方法。方法可以训练学生的数学思维，提高学生解决问题的能力，并提高学生的数学素养。

（三）教科书中数学史内容的作用

不管是显性还是隐性的内容，数学史出现在教科书中，并进而出现在课堂教学中，它的作用可以归纳为激趣、激情、激志、激智等等。激趣，激发兴趣，激发学生的学习兴趣；激情，激发情绪，激发学生投入学习的情绪，一种淡定、坦然以至兴奋、激昂的情绪；激志，激发斗志，虽然遇到困难和挫折，但依然坚持不懈；激智，激发智慧，从古人那里得到启发，闪现灵感。不管怎么样，数学史融入数学教科书的最终目的还是为了促进学生的数学学习。如何来判断教科书中数学史内容的多寡、优劣，就看它们是否有效地通过教师的教促进了学生的数学学习。

二、影响数学史教学的因素

影响教科书中数学史内容发挥其作用的因素有很多，本文仅从教师和学生两方面来考察。

（一）影响教师数学史教学的因素

数学史融入教科书后，需要教师加以实施。教师如何在教学中使用数学史，也受到诸多因素的影响。我们认为影响教师实施“数学史融入教学”的因素主要有知识储备和观念这两个。

(1) 知识储备
教师的知识储备会影响教师的教学。如果教师对数学史的知识了解得很少，那么在教学中他（她）只可能将教科书和教学参考书中的内容干巴巴地传递给学生；如果他（她）了解得比较多，则可以旁征博引、生动有趣、富有启发地进行教学。

(2) 观念
教师的观念会影响教师的教学，与数学史有关的观念会影响教师数学史的教学。

首先是教师对数学史重要性和必要性的认识。如果教师认识到这一点，那么他（她）在教学中就会非常重视数学史内容，在课前会非常认真地准备相关素材，在教学中会花费一定的时间在上面。相反，如果没有认识到这一点，纵使教科书中有数学史内容，教师在教学中也会作淡化处理，甚至忽略。那么教科书中的数学史内容仅仅成了摆设，而没能起到应有的作用。

在实际教学中，我们发现教师有一观念，即“重算轻史”。对于教科书中作为例题、习题出现的史料，教师仅仅是把它作为一个数学问题。教师的任务是和学生一起，或者让学生自己，解决这个数学问题。教师重视这个问题是什么，它的解法是什么；而对于它的历史，它的背景，它在数学史中的地位、意义等都是不关心的。这样，数学史在数学教学中的地位和作用被大大降低了。

当然，除去知识储备和观念这两个主要因素外，教师自身的素质、教学基本功、教科书的编写质量等也会影响教师数学史的教学。反过来，关注教师的教学，在教科书编写时关注这些因素，可以促进教科书的编写。

（二）影响学生数学史学习的因素

数学史出现在数学教科书中，其目的是为了促进学生的学习。在现实中，考试的导向作用、内容的呈现形式、教师的讲解方式等因素会影响数学史对学生学习的促进作用。在教科书编写时考虑到这些因素，也有助于教科书编得更好。

就本文主题而言，教科书中，数学史内容的呈现方式是否有吸引力，会影响学生的学习。文字的叙述、插图的安排、活动的设计、问题的设置，这些都需要慎重考虑和精心安排。如果把一段原始的古文文献直接呈现给学生，学生就会感到沉闷、乏味、枯燥，就会远离它。但是用浅显易懂的现代文表示出来，适当地配上插图，学生就会比较容易接受。如果再安排有趣味并有一定挑战性的问题，设计成数学活动，这样就让数学史亲近学生。

三、教科书编写理念：数学史融入数学课程

数学史如何融入数学课程？对这一理念的具体阐述，我们从“定位 —— 挖掘 —— 转化 —— 呈现”这四方面着手：首先确立价值，随后根据价值选择内容，之后将这些内容转化形态呈现在数学课程中。[4]

（一）数学史的核心教育价值是培养创新

数学史可以为学生的数学学习提供有力的支持和支撑。数学课程中多介绍一些数学产生和发展的故事，数学家的生平、成果和贡献，都有助于学生感受数学，感受数学文化。数学家是如何发现问题、发明方法、创造思想并解决问题的，他们的数学思想方法是如何推动数学的发展的，对于这些问题的思考和解决有助于加深学生对数学的认识和理解。在确立数学史的核心教育价值为培养学生的创新精神和创造能力之后，我们又如何来培养这种创新精神和创造能力呢？可以说，数学史更多的是数学思想方法史，那么，向学生介绍数学史应以介绍历史上的数学思想方法为重点。事实上，“不管他们从事什么业务工作，唯有深深地铭刻于脑际的数学的精神、数学的思维方法、研究方法、推理方法以及着眼点，却随时随地发生作用，并使他们终生受益。……这样的教育，才是最好的教育。”[5] 那么，在设计课程和编写教科书时（尤其是在高中阶段），应重视选取那些有助于发展学生数学思维和培养创新能力的数学史料，而这样的内容应包含丰富的数学思想方法。让学生自然而然地重演（这种重演并非完全遵照古人的做法）古人对某一问题的发现、探索和解决过程并体验其中蕴涵的思想方法则有助于实现我们的目标。通过感受思想方法和解决问题来培养学生的创新意识和创造能力，使他们可以更好地学好数学、用好数学、发展数学。

（二）数学史的重点内容是数学思想方法

数学史可以为学生的数学学习提供有力的支持和支撑。数学课程中多介绍一些数学产生和发展的故事，数学家的生平、成果和贡献，都有助于学生感受数学，感受数学文化。数学家是如何发现问题、发明方法、创造思想并解决问题的，他们的数学思想方法是如何推动数学的发展的，对于这些问题的思考和解决有助于加深学生对数学的认识和理解。

在确立数学史的核心教育价值为培养学生的创新精神和创造能力之后，我们又如何来培养这种创新精神和创造能力呢？可以说，数学史更多的是数学思想方法史，那么，向学生介绍数学史应以介绍历史上的数学思想方法为重点。事实上，“不管他们从事什么业务工作，唯有深深地铭刻于脑际的数学的精神、数学的思维方法、研究方法、推理方法以及着眼点，却随时随地发生作用，并使他们终生受益。……这样的教育，才是最好的教育。”[5] 那么，在设计课程和编写教科书时（尤其是在高中阶段），应重视选取那些有助于发展学生数学思维和培养创新能力的数学史料，而这样的内容应包含丰富的数学思想方法。让学生自然而然地重演（这种重演并非完全遵照古人的做法）古人对某一问题的发现、探索和解决过程并体验其中蕴涵的思想方法则有助于实现我们的目标。通过感受思想方法和解决问题来培养学生的创新意识和创造能力，使他们可以更好地学好数学、用好数学、发展数学。

（三）数学的史学形态转化为教育形态

宋乃庆教授在首届全国数学史与数学教育大会上做大会报告时提出一个观点：“数学的史学形态转化为教育形态”。[6] 数学史具有丰富的教育价值，由此也决定它除了自身的史学形态外，还有应用于数学教育的教育形态。

事实上，在数学史中，数学概念的形成、数学思想的来历、数学方法的应用、数学定理的审美内涵、数学家的思维方法和科学与人文精神、数学的哲学基础、数学发展的社会背景等都是极富价值的素材，对于学生全面深刻地认识数学及数学的发展非常重要，也更富有教育意义。数学教育应充分发掘数学史的教育功能，有效地发挥数学史的教育价值，需要把数学史的史学形态转化为教育形态，充分发挥数学发展过程中人类思维的本质性、思想性、连续性和完整性以及数学所蕴涵的理性精神对学生学习数学的启发意义和感召作用。[7]

数学课程中引入数学史，绝非简单的嫁接、拼凑和移植，而需要对其进行深入地挖掘、改造、提炼和升华。数学史作为数学思想的发展史，蕴涵了丰富的思想方法。有些思想方法在当今已经沉寂，有些依然活跃在数学和科学领域。不管是沉寂的还是依然活跃的，对学生的思维都具有一定的启发意义。不过，作为动态演化的历史凝聚成静态的数字、公式和文字之后，在某种程度上却掩盖了深层次的，同时又是作为数学核心部分的思想方法。所以，我们面对的是作为学术形态的数学史。如何把学术形态的数学史料转化为教育形态的教学材料，需要我们对古代数学的概念、原理、思想、方法做出认真的思考和梳理，进行加工和创造，深入挖掘材料背后隐含的价值，使之适合当今学生的认识水平、心理特征以及中小学数学课堂的特点，并探索如何展现在具体的课程、教科书，以至教学中。

（四）数学史在数学课程中的呈现方式

数学课程中的数学史可以以数学课本、数学读本、选修课程和专题研究等形式呈现。[8]

(1) 数学课本。数学课本可以把故事、历史名题作为问题情境，比如丢番图的年龄问题引出一元一次方程；也可以简单介绍数学家的事迹来鼓励学生热爱数学、勤奋学习、刻苦钻研，例如阿基米德在死神降临之时仍醉心于数学研究。而且，我们更应注重将数学史料（尤其是数学的思想方法）有机地渗透融合到课本中。比如，在“等比数列求和公式”这一节中，除了课本中的“错位相消法”之外，不妨介绍《原本》第九卷中给出的方法，从定义出发进行推导。

(2) 数学读本。数学课本应当体现数学的特色，简洁而明确，只需给出问题、方法和结论，并配以一定的练习。至于这些问题是如何提出的，这些方法是如何想到的，有没有其他的方法，这些问题的历史背景如何，它们的来源以及其后的发展又是如何，各个数学分支、数学内容之间的联系如何，数学与其他学科之间的联系又如何，等等，这些内容不妨详细地写入数学读本，让有兴趣的学生课外去阅读。

(3) 选修课程。《高中课标》已将“数学史选讲”作为选修课程的一个专题。选修课程可以把系统学习和专题讲座结合起来。系统学习指利用教科书（选修）有计划、有条理地介绍数学史；专题讲座则是指不定期地邀请一些教师、专家开设讲座，或者组织学生观看与数学史有关的视频。

(4) 专题研究。中小学数学课程应当给学生留出一定的思考空间。数学课程不仅仅是介绍数学的发展，而且还应创造机会让学生重演这种过程。比如，在“正四棱台体积公式”这一节中，给出一定的背景和必要的提示之后，留出空白，让学生重演古埃及人、古巴比伦人以及中国古人对这一公式的推导过程。[9] 此外，还可以通过“成果综述”和“问题拓展”来进行专题研究。比如，在“勾股定理”中，让学生通过各种途径查阅资料，对不同时期不同地区的证明方法进行综述，并体验这些方法的巧妙和优美，同时对它们进行比较、分析。在研究“正多面体和多面体欧拉公式”之

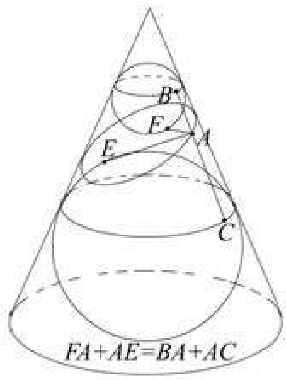

Figure 1: *

图 1 Dandelin 双球

后，不妨提问：正多面体只有 5 个，那么类似足球的“半正多面体”[10] 又有多少个，你能把它们描述（计算）出来吗？这就拓展了问题。我们认为，“专题研究 — 问题拓展”这一形式最能培养学生的创新意识和创造能力。这里的问题可以是课程设计者或者教师给出的，也可以是学生自己提出的。

四、基于“数学史融入数学课程”的教科书编写策略

根据上述理念，我们提出以下基于“数学史融入数学课程”的教科书编写策略。我们提出这些策略的直接目的是提高教科书的编写质量，最终目的还是为了促进教师的教和学生的学。

（一）前提：史学形态转化为教育形态

把数学史由史学形态转化为教育形态，让学生亲近和喜欢。这是前提，同时也是关键的一步。教科书不能把原始的数学史料直接拿来，放置其中。要让学生亲近、接受和喜欢，必须改变其枯燥乏味、晦涩难懂的原始面貌，以学生喜闻乐见和易于理解的形式呈现出来。这样有助于发挥出数学史应有的价值和作用，一方面有助于教师的使用，另一方面也有助于学生提高学习兴趣、进入历史世界，从而更好地理解数学史、理解数学。

例如，在“圆锥曲线”这一单元，教科书会介绍圆锥曲线的简单历史，比如梅内赫莫斯为解决倍立方体问题而发现圆锥曲线，阿波罗尼斯在前人工作的基础上创立了相当完美的圆锥曲线理论，他的《圆锥曲线论》就是这方面的系统总结，等等。除了这些，也会介绍 Germinal Dandelin 的工作和 Dandelin 双球。如图 1 所示，用一个平面去截圆锥，这个平面与圆锥的交线是一个椭圆。在圆锥内做大小两个球分别与圆锥和截面相切，切于点 E，F，在截口曲线上任取一点 A，过点 A 作圆锥的母线，分别与两个球相切与点 C，B。由球和圆的几何性质，可以知道 FA+AE=BA+AC=BC。而且，截面与两个球的切点恰是椭圆的两个焦点。

对这一原始素材，其实有很多工作要做。首先，可以用圆柱体取代圆锥，依然可以得到一个椭圆，依然有 FA+AE=BA+AC，但问题已经简单化了。（在研究双球问题之前，可以先研究单球问题。）把简单的圆柱体里的 Dandelin 双球问题研究透后再来研究圆锥里的 Dandelin 双球问题，这就为学生的学习做了铺垫。其次，如何向学生说明 E、F 就是椭圆的焦点，并且为什么被称为焦点，也需要做进一步的梳理。再次，教科书为什么将坐标系的原点选在椭圆的中心，椭圆的标准方程与阿波罗尼斯的方程之间有什么联系，都需要做一些思考和加工。

Figure 2: *
图 2 长方体房间图

那么，将数学史的史学形态转化为教育形态，会不会因此淡化数学史？或者，会不会因为强调数学味，而使教育形态呈现不足？其实这些问题的本质涉及数学教育的基本矛盾。“数学方面”与“教育方面”的对立统一事实上构成了数学教育的基本矛盾。前者是指数学教育应当正确地体现数学的本质，后者则是指数学教育应当充分体现教育的社会目标并符合教育的规律。能否处理好这一矛盾（或者说，搞好这两个方面的均衡取得平衡）也正是搞好数学教育的关键所在。[11] 数学教育具有一般教育过程的性质，又具有自身特殊过程的性质，那么数学教学问题的研究就得以沿着“教与学对应的原理”和“教与数学对应的原理”双重轨道进行。[12] 这两条原理的构建也是符合上述数学教育的基本矛盾。那么，数学史融入数学课程，也应该在史学形态（数学方面）和教育形态（教育方面）取得适当的平衡。

（二）重点：关注数学思想方法

教科书中应有一定的数学史内容；数学史应该以合适的方式适量地呈现在教科书中。教科书中的数学史内容，应方便教师在教学中使用，方便学生直接接触。对于具体内容的选择，在小学阶段，可以通过生动有趣的故事，绚丽多彩的插图来呈现，主要是为了吸引学生的注意力，激发他们学习数学的兴趣，感受数学文化。但随着年级的提高，图片应该逐渐减少，而文字逐渐增加。尤其是到了高中阶段，我们应把目光聚焦于让学生“进入”数学史，古为今用、推陈出新；文字更要关注涉及数学本质的内容，关注数学的思想方法，而不仅仅是外在的故事和奇闻趣事。因为学习数学史，不仅仅是知道、了解一些史实，更多的是感受数学思想方法。所以，教科书中应该呈现较多的显性数学史料，但同时也设置一定的隐性内容。基于“数学史融入数学课程”的教科书重点是关注数学数学思想，具体做法是将数学史显性和隐性内容相结合，尤其注重通过隐性的数学史内容让学生感受数学思想方法。在不知不觉中，重演古人对某一问题的解决过程，感受他们的解题方法，尝试用他们的方法去解决今天碰到的数学问题，由他们的问题出发提出一系列新的问题。这样有助于提高学生的数学素养，进一步培养学生的创新意识和创造能力。这既是重点，同时也是手段。

例如，我们在我国几套现行初中《数学》教科书《勾股定理》单元中可以看到类似“已知圆柱的底面半径为 6cm，高为 10cm，蚂蚁从 A 点爬到 B 点的最短路程是多少（精确到 0.1cm）（图略）?”这样的问题。在此题之后，不妨安排以下一道历史名题，作为进一步的拓展：“如图 2，在一个长、宽、高分别为 30、12、12 英尺的长方体房间里，一只蜘蛛在一面墙的中间离天花板 1 英尺的 A 处，苍蝇则在对面墙的中间离地面 1 英尺的 B 处，苍蝇是如此地害怕，以至于无法动弹。试问，蜘蛛为了捉住苍蝇需要爬行的最短距离是多少？（提示：它少于 42 英尺）”这一“蜘蛛与苍蝇”问题最早出现在 1903 年的英国报纸上，它是杜登尼最有名的谜题之一。杜登尼是 19 世纪英国著名的谜题创作者，他创作的这一问题对全世界难题爱好者的挑战，长达四分之三个世纪。[13]

这一问题看似简单，但要正确地解决它也并非易事。解决这一问题的基本工具是勾股定理，同时，它还需要用到以下数学思想方法 —— 转化思想：把立体图形转化为平面图形；变换思想：正方形的旋转变换；分类讨论思想：分三种情况分别求出蜘蛛的爬行距离。这一个“蜘蛛问题”，问题本身充满趣味性，而且在解决它的时候需要转化和分类讨论，对促进学生思维的灵活性和严谨性有一定的作

用。总得来说，这是一道非常不错的问题，通过充满趣味的表面可以直达数学的本质，而且可以促进学生“数学地思维”。

（三）突破点：呈现多元文化数学内容

呈现多元文化数学内容，让学生尊重、分享、欣赏和理解不同文化背景下的数学成果。这是“数学史融入数学课程”和“数学史融入教科书”的突破点。

数学通过数学教育得以传播和发展，所以数学教育必须置身于人类文化，尤其是数学文化之中。在多元文化的视野中，为认识数学教育的过去、面对数学教育的现在、展望数学教育的未来，从不同文化背景深入探究不同数学教育体系的特点与规律，这既是人类认识的必然，又是人类发展的需要；既要体现不同文化的特色差异，又形成不同文化交融的基础。[14]

在不同的历史时期，在各个文化背景下，数学的发生发展出现了许多不同的数学成果，东西方的数学思想方法更是有很大差异而各有其特征和优势。通过对不同时期、不同地区数学成果以及思想方法的比较，可以使学生认识到数学并不是只属于某个民族、某种文化，让学生用一种更宽广的胸怀和视野去看待数学及身边的世界。数学教科书和数学教学引导学生尊重、欣赏、分享、理解其他文化下的数学，由此拓宽学生的视野，深化对数学知识的认识和理解，培养开放的心灵。以往我们过份强调某项数学成果我国比西方早多少年，这其实滋长了狭隘民族主义的思想；那么本着一种尊重、理解和支持的态度向学生介绍多元文化的数学，重在对所有数学成果的欣赏和分享上，就可以让学生用一种“泛爱万物”的胸怀去了解不同时期、不同文化背景下的思考方式。

以“勾股定理”为例，对于其证明，一方面，可以训练学生缜密的数学思维；另一方面，其方法据说超过 400 种，而且不同的方法与不同的文化、不同种族的思维方式紧紧联系在一起。所以可以说，它是体现多元文化数学的极好题材。那么，在《勾股定理》单元中，教科书应以适当的方式呈现若干种经典证法。比如欧几里得《原本》中的证明方法就很值得向学生介绍，与赵爽的方法作一对比，学生能体会到古希腊人对理性的追求。具体说来，赵爽的弦图证法（几何观点）充分运用了直角三角形易于移补的特点，给出了简洁、直观的证法，其相应的几何思想是图形经移、补、凑、合而面积不变，不仅反映了我国传统文化中追求直观、实用的倾向，而且其展示的割补原理和数形结合思想让我们看到我们传统文化中的精髓，对我们继承和发扬传统文化起着潜移默化的熏陶作用。而欧几里得证法给我们展示的是西方数学文化传统的另一侧面，即严谨的逻辑和理性的推理。[15] 如果再对相关背景做介绍，学生可以意识到不同的文明产生了不同的数学。欧几里得方法可能对学生而言比较难，不是那么容易理解，教师可以做适当的处理，比如借助计算机做动态演示，一般学生还是可以接受的。

（四）保障：资源库的建设和教师的发展

教科书中不可能出现很多的数学史内容，编者只能精选其中最重要、最有价值的部分。除了精心编写教科书外，还需大力建设配套的课程资源。这里的资源可以包括相关文献、参考书目、网络链接、视频资源等等。课程资源库的建设，这是“数学史融入数学课程”“数学史融入教科书”的保障，方便教师查找资料、设计教学，促进学生自主学习、拓展视野，加深对数学的认识和理解。

前文已经述及，教师的数学史知识储备会影响教师的教学；那么，教师增加自己数学史知识储备可以通过阅读、培训、交流等途径进行。首先，教师可以阅读数学史书籍，还可以阅读期刊杂志。不少中学数学杂志上会刊登一些数学史以及有关数学史与数学教学的联系的论文，数学史应用于数学教学的案例。这些教学案例对中学教学更有启发意义，教师可以借鉴，在自己的课堂中加以应用。其次，培训可以分职前和职中两种。职前培训其实就是在大学里接受系统的数学史课程学习。其是师范院校，将数学史与数学教育结合起来，不仅有助于全面认识数学，而且还有助于未来的教师合理应用数学史于课堂教学。从事教师职业后，教师还可以参加各级各类的培训，在培训中也会有数学史相关

的讲座，教师可以从中获得相关知识。再次，参加学术会议，与专家、同行交流也是获取数学史知识的一种途径。两年一次的数学史与数学教育会议迄今已经召开过四次。其他的数学教育会议，也会有涉及数学史与数学教育的议题。在会议中可以听取专家的报告，也可以就自己关心的问题与专家、同行交流讨论。在这过程中，可以增长自己的数学史知识。

REFERENCES

- 汪晓勤, 欧阳跃. HPM 的历史渊源 [J]. 数学教育学报, 2003,12(3):24-27.
- 中华人民共和国教育部. 义务教育数学课程标准 (实验稿) [S]. 北京：北京师范大学出版社. 2001.
- 中华人民共和国教育部. 普通高中数学课程标准 (实验)[S]. 北京：人民教育出版社, 2003.
- 朱哲, 宋乃庆. 数学史融入数学课程 [J]. 数学教育学报, 2008,17(4):11-14.
- （日）米山国藏. 数学的精神、思想和方法 [M]. 毛正中, 吴素华译. 成都：四川教育出版社, 1986.
- 冯振举, 杨宝珊. 发掘数学史教育功能, 促进数学教育发展 —— 第一届全国数学史与数学教育会议综述 [J]. 自然辩证法通讯, 2005,27(4):108-109.
- 李铁安. 基于笛卡尔数学思想的高中解析几何教学策略研究 [D]. 重庆：西南大学. 2007.8.
- 朱哲, 张维忠. 中小学数学课程中的数学史 [J]. 浙江师范大学学报 (自然科学版),2004,27(4):422-425.
- 朱哲, 张维忠. 一节基于数学史的教学案例：正四棱台的体积公式 [J]. 中学数学教学参考, 2004,3:8-11.
- 沈康身. 历史数学名题赏析 [M]. 上海：上海教育出版社, 2002.736.
- 郑毓信. 数学教育哲学 [M]. 成都：四川教育出版社, 2001.228-229.
- 涂荣豹. 数学教学认识论 [M]. 南京：南京师范大学出版社, 2006. 前言 4.
- 徐鸿斌, 张维忠. 从一道历史数学名题谈研究性学习 [J]. 数学通报, 2004,(7):31-32.
- 汪秉彝, 吕传汉. 再论跨文化数学教育 [J]. 数学教育学报, 1999,8(5):16-19,25.
- 张维忠, 汪晓勤. 文化传统与数学教育现代化 [M]. 北京：北京大学出版社, 2006.267.

A Study on‘Mathematization’ of Obstacles and Solving Appoaches

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ABSTRACT

Since‘mathematization’ was first proposed by Hans Freudenthal, many theories and experiments regarding the‘mathematization’ were interpreted continuously, and the concept of ‘mathematization’becomes more and more copious. Our research is going to analyze‘mathematization’ from the perspective of the mathematical culture, and try to find out the detail reasons of the following problems about teachers and students. Fist problem is that teachers are much lacking at having a deep understanding on the procedure of mathematization, and they also can not fully grasp the developing process of the content of mathematics, which is the second problem. And next problem about students is that emotional factors lead students to a passive mathematics learning. What’s more, there is no a clear connection between mathematization and the live of students. Based on solving these problems, our paper also gives some approaches to convert the education theory of“mathematization” into practice.

数学化自从弗赖登塔尔（Hans Freudenthal）提出以后，在理论和实践中不断得到诠释，内涵也不断得到丰富。本文力求从数学文化的视角对“数学化”进行分析，并对目前数学化过程中教师对数学的理解不够深刻、对数学内容的发生发展过程把握不够全面，学生由于情感态度等因素导致被动学习数学、数学教学过程中数学化与生活化关系不明确等方面的问题进行了剖析。在此基础上，提出了一些使“数学化”从教育理念转化为教育实践的对策思考。

Keywords: mathematization; problems analyzing; solving appoaches; 数学化；障碍分析；对策思考

数学教育的目的是数学化（Mathematising），数学化的思想最早由荷兰数学教育家弗赖登塔尔（Hans Freudenthal）提出，弗赖登塔尔认为，所谓数学化就是人们在观察、认识和改造世界的过程中运用数学的思想和方法来分析和研究客观世界的种种现象并加以整理和组织的过程。

一. 数学文化的视角认识数学化

要理解好数学化，应该先对数学文化进行认识。“化”，本义为改易、生成、造化，如《庄子· 逍遥游》：“化而为鸟，其名曰鹏”；《黄帝内经· 素问》：“化不可代，时不可违”；《礼记· 中庸》：“可以赞天地之化育”等，归纳各种说法，“化”指事物形态或性质的改变，并由此引申为行迁善之义 [1]. “文化即人文，即人的精神.”[2] 数学文化乃通过数学对人的精神进行改变.

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围绕数学化的概念，有研究者给出了一些界定，如“数学化应该是数学认知的形成和改进过程，是指数学地组织现实世界的过程和方式即抽象化、符号化、模式化、形式化和系统化这是数学思想的基本原则.”[3] 再如“是指人们在观察现实世界时, 应用数学的方法分析研究各种具体现象并加以整理组织的过程. 其中包括把现实问题转化为数学问题及使数学更完整、更系统和更抽象这样两个过程”[4].

从上述各种对数学化的理解我们可以看出，数学化最主要的问题还是需要解决观念的问题，形成数学的思想，数学化在一定意义上应该是思想数学化，“数学思想需要满足两个条件：一是数学产生、发展过程中所必须依赖的那些思想，二是学习过数学的人所具有的思维特征.”[5] 因而数学化就应该是和数学学习结合起来才有意义.

从文化的角度来审视数学，来讨论数学教育是一个很不轻松的话题，一方面数学的特点决定了这是一项艰苦的工作，另一方面从文化视角来理解和认识数学教育，使数学教育的责任更大. 承载着提升人，因而也就提升社会的基本素质的责任.

然而，现代数学发展朝着越来越深的专业化方向迈进，而且由于现代数学艰涩程度高而难以被一般人理解. 这就形成了一个主要问题，就是数学化不能脱离数学，数学又不是一般人能理解. 因而，数学化问题由于数学本身的特点而显得有较大的难度.

除了数学本身的特点给数学化带来障碍外，在数学教育过程中，一些基本问题往往也成为数学化的困难所在. 目前数学教育在数学的途径中遇到了教师不够数学化、学生不愿数学化等问题. 本文力求对数学化的相关问题进行梳理，并提出一些教学思考.

二. 教师不够数学化

由于教师对数学本身的理解没有达到一定的深度，所以，在数学教学中，很难将数学思想的教学渗透到数学教学中.

1. 对数学知识理解不透

“数学文化观念下的数学价值和功能的基本定位是反对关于数学的任何片面的、固定的和狭义的理解.”[6] 因而在数学教育中，需要既揭示数学的科学性的一面，又揭示其人文性的一面.

教师数学化的程度在很大程度上决定了学生数学化的程度. 然而目前数学教师的数学化程度应该还处于一个有待提高的阶段. 对于一些数学知识所涉及的数学思想往往理解不到位，导致在教学时不能将数学思想渗透到数学教学中.

如在介绍乘法和加法运算时，教师关注的比较多的是，加法运算要对齐数位，乘法运算在第一步（乘）的过程中对数位对齐就不提要求，（只在乘了以后再对齐）. 由于对思想方法不理解，就让学生记住“加法要对齐”，而学生却经常弄错. 如果教师能在介绍运算时渗透分类和分步运算的思想，则学生犯错的概率就会大大减小.

有老师在教学中语言叙述也有一些不严密或对概念理解不准的特点，如“算术平方根等于本身的数是 1 或 0”（应是 1 和 0）；“写成q/p形式的数叫有理数”（没有说明p, q应该是整数）；“由π 和$\sqrt{2}$在数轴上的表示，我们可以看出每一个无理数都可以在数轴上表示”（这种推理前提不是结论的充分条件）；“无理数就是开方开不尽的数”；“这个几何体的投影是什么？”（将物体说成几何体）；“圆心的代数化就是坐标，半径的代数化就是数”（将对应说成代数化）；倾斜角的取值范围是$[0°, 180°)$（区间是对实数而言的）.

再如顾沛先生所举之例，小学等于的思想实际上是从自然界中的“一样”抽象出来的，而等于的本质就是事物之间的“反身性”、“对称性”和“传递性”的一种关系，它的推广就是“等价关系”. 教师在介绍等于时，一般不易将思维延伸，认识到其“等价”特点. 教师在介绍“一笔画”问题时，

虽然介绍欧拉“七桥问题”，但对“七桥问题”所含的数学推理模式、抽象模型过程却理解不透，难以把握．

2．对数学知识的关系不明确

对数学知识的关系把握不好主要表现在两个方面，一个方面是对知识本身的联系关系把握不够，如有的教师在介绍算术平方根是说：“算术平方根的定义是一个正数x的平方等于a，……”给后期学习平方根带来负迁移；教师在教学中说“没有一个数的平方为负数．”为后期学习复数带来不利影响．这也就说明教师对后续知识和前期学生所学知识了解不够，关注比较多的是自己所教得那部分知识．

对数学知识的关系把握不够的另一个方面是对数学与实际的关系把握不够，如在数学与实际的关系处理上所设计的实际问题或解决方案在实际中难以行得通．如“有 8 个展览点，为方便游客参观，组织者设计了两条游览线路，每一条都由折现构成的折线图”首先这里路和“折线”的关系没有理清，另外，实际路都不会是笔直的线；再如“用木桩固定旗杆，使旗和桩都在一个三角形面内．这样的固定显然如果垂直三角形的风一吹，旗杆就很难保持平衡了”；“A，B，C，D 为四个村庄，现在这四个村打算建一个学校，为了使学校到四个村庄的距离之和最小，问校址选在哪里?”，这里的问题是，办学校不可能将校址选在一个前不着村，后不着店的地方．

高中教师在介绍函数概念时，不明确初中的函数概念与高中的函数概念的区别与联系，因而不能理解函数的本质特点，也不是特别明确为什么要介绍两个函数概念．

三．学生不愿意数学化

数学化在文化的视角下就应该是使学生通过数学学习，在科学思想、人文意识、审美情趣等方面得到完善，因而，从数学文化的视角理解数学化，就是使学生通过数学学习，基本素养得到全面提升．然而，目前学生学习数学的情况是令人担忧的，数学化的对象（学生）存在主观拒绝或不愿意被数学化的情况，内因是变化的依据，离开了内因谈数学化是困难的．

我们一次初中骨干教师培训班上的调查显示，调查人数为 80 人，有 25.93% 的教师认为自己所教班级的学生不喜欢数学的比例占 60%，38.89% 的教师认为自己所教班级的学生不喜欢数学的比例占 50%，31.48% 的教师认为自己所教班级的学生不喜欢数学的比例占 40%，这些数据可以从一个侧面反映学生对数学的喜欢程度．66.10% 的被调查教师选择的原因是数学太难导致学生不喜欢数学．陈熙仁先生的研究 [7] 说明，初中学生对数学学习的自信心不足，内在动机不明确．有些学生对数学产生了焦虑、厌学情绪，从而可能会走向自我放弃数学学习的歧途．

由于高考很多题目都远离教材，所以，教师对教材的挖掘也没有关注，学生将教材作为最基本的练习册，先完成教材中的练习，再完成其他各种复习资料的练习，学生学习数学的过程实际成了不断完成各类题型训练的过程．对数学的其他意义和价值就无暇顾及了．高考复习基本采用“覆盖”方式，也就是，考试有 22 题，我就做 220 道题，甚至 2200 道、22000 道题，力争在高考中“碰到”所做的题目的类型．这样的大运动量，给学生带来的是负担不断加重、难度不断增加，苦不堪言，谁还有心思“数学化”．

综上所述，目前在数学化的过程中无论从数学本身的艰涩性，还是数学教师的状况，还是学生的主观愿望都还有很多障碍，这些问题产生的主要原因是数学教学比较普遍的现象是关注了知识的科学价值，而相对忽视知识的人文价值．对数学与生活的关系认识还存在偏差．

四．教学思考

上述分析我们可以看出，在数学化的途径中存在三个方面的障碍，而主要障碍还是数学本身的特点及教师的状况，学生的状况在很大程度上由这两个方面的状况决定．数学本身的特点启示我们要将数学化分层次，不同的人在不同的层次上实现数学化，而教师的状况启示我们要关注高师的数学教育．

1．数学化的层次性

“数学是关于数的世界、形的世界、逻辑关系的乃至有关的世界．它基于实践，但能远远超越实践，不断自我超越．” [8] “教育的主题是唤醒人的超越性” [9]，数学教育的过程是一个不断超越自我的过程，所以数学化是一个过程，也是一个结果，而一个人的数学化是在不断提升的．

确定数学化的层次要求是重要的．对不同层次的人确定不同层次的数学化程度要求，也就是在不同的层次上形成不同的数学观，不同的对数学的理解和认识．我们可以从数学的知识、思想、能力三个角度来理解数学化的层次性．

（1）数学知识的层次性

对于数学知识，不同的人掌握的情况是不同的．小学、中学、大学等对数学知识的理解也就会处于不同的层次．

如对无理数的认识，在小学阶段认识了π这样的无理数，在初中给出了无理数的直观定义：无限不循环的小数，到了高中发现要用初中定义去证明一个数是无理数不是一件容易的事情，于是，考虑到无理数是不能写成p/q $(p, q \in \mathbb{Z})$ 的形式的数，到了大学学习了戴德金分割和康托的有理数基本列后，对无理数有更深的认识．这一不同层次对数的认识，也就使得人们对数的发展过程有了理解，也就是对数学的理解也就不断深刻．

再如，对函数的认识，初中认识到函数是研究变量的数学模型，在函数的定义中体现了“变”的特点，到了高中，觉得有的函数是不变的，于是觉得需要对函数概念进一步抽象，发现其跟本质的特点，于是从对应的视角发现，函数最本质的特点是两个集合之间具有的特殊对应，因而，抓住了函数的本质“三要素”，到了大学，又从关系的角度来理解，发现函数就是满足一些要求的笛卡尔积的子集．

这样不断的对函数的理解的加深，形成了不断抽象的知识状况．

（2）数学思想的层次性

数学思想随着数学知识的加深而不断深化．如对无穷的思想，人们开始可能从自然数的特点发现数学中具有无穷的思想方法．但对无穷的认识还出于感性的阶段，也不能区分各种无穷，不知道怎样思考无穷的问题．对无穷的理解处于通常所说的“潜无穷”的状态．当学习了微积分后，具有了极限的思想，可以在一定程度上区分无穷大与无穷小，学习了集合理论，对无穷的认识进一步加深，认识到无穷的不同层次性，而以自然数为代表的无穷是“最小”的无穷，无穷是一个发展的过程，具有不同的层次，可以“无穷”的发展，这样就在更高的层次上把握了无穷，形成了具有“实无穷”观的思想特点．

（3）数学能力的层次性

数学能力主要是抽象思维能力、逻辑思维能力、解决问题的能力等．抽象思维能力主要体现在对问题的抽象化上．小学阶段可以从名数抽象为数字、可以从车轮、苹果等抽象出圆、点等，从黑板、桌子抽象出矩形等，也就是形成了一般的抽象思维．到了中学就可以从数字抽象为一般的符号和字母、从

圆、点、线等抽象出多边形、圆形等，也就是形成了科学的抽象特点．到了高中就可以将一些问题转化为完全不考虑“实际”的演算和推理了，把握事物的本质，把繁杂问题简单化、条理化，能够清晰地表达，能够去掉具体的内容，利用概念、图形、符号、关系表述事物．能通过假设和推理建立法则、模式或者模型，并能够在一般意义上解释具体事物．

逻辑思维能力，不同的阶段具有不同的状态，在小学阶段，可能是具有有条理的说话，为一个现象提供说理．到了中学就要进行一些证明了，到了高中证明的问题越来越复杂，需要从一部看到若干步的发展．

解决问题的能力，在小学阶段可能是将一些问题转化为简单的数学模型，如加法、乘法等，到了初中可能就要转化为较为复杂的数学模型，高中所需要的模型就更复杂．具有了转化思想后，就可以将问题数学化，接下来就是运用数学解决问题了。不同的层次所能解决的问题的复杂程度也就不一样．

实际上，知识到思想再到能力本身就是数学化的不同层次，在这个层次的最基础部分是数学的知识层次，所以，离开了数学知识就不能数学化了，但如果知识不经由思想发展为能力，也就在很大程度上失去了知识的作用了．

2．高师数学教育中要关注数学化

提高教学水平，关键是提高教师的水平．这不是一句空话，需要扎扎实实地工作 [10]．高师是数学教师的摇篮，关注数学化要从高师的数学教育入手．

（**1**）从数学化的角度开展教学

高师数学教学应充分挖掘数学知识内容中所包涵的数学思想和方法，包涵的数学发展史，从数学发展背景提炼数学内容的特点，使高师学生完整地掌握数学．

现代数学教育的立体化目标，极需要教师的教学对学生产生探寻科学真理的感召力，熏陶良好品质的感化力，认清事物本质的洞察力，揭示数学理论的穿透力，激发学习兴趣的策动力 [11]．而要实现这一目标，需要高师数学教育体现“数学化”．

如高师的数学系的基础课程《数学分析》从知识的角度来说，包括函数研究、导数和微分、积分、实数性质、级数理论等内容，是高师学生高中阶段函数研究的继续，又是进一步学习数学的基础，在整个数学学习及未来的数学教育中具有极其重要的地位．

数学分析又是一种重要的思想方法，其核心是极限思想，极限思想是解决无穷问题的重要的思想和方法，有了极限的思想，人们在解决无穷问题时便具有了一种基本的方法．

数学分析还是人类数学史上最辉煌的成就，数学分析的基本内容是微积分，微积分的基础是变量，而将变量引进数学是 17 世纪数学最重大的思想变革，这一变革带来了整个数学的变革．

微积分的理论基础是微积分基本定理，这一定理揭示了微分和积分之间的内在联系，只有这一关系的发现，才说明了微积分的诞生．因而在介绍微积分基本定理的教学时，结合微积分的产生过程的介绍，学生收获的就不只是一个定理了．因而高师《数学分析》课程的教学，不仅要关注学生能解决各种积分或微分等问题，也要关注其中的数学思想的特点和数学知识背景后面的数学发展历程，只有这样，学生才能在掌握基本知识的同时，理解数学的价值和思想的特点，受到数学美的熏陶，完善对数学的认识和理解．

（**2**）强化对中小学的引领作用

伍鸿熙教授指出“数学家应该致力于师资培训，但如要有收获，得需要对中小学数学又深切的认识．” [12] 高师的数学教师应该对中小学数学熟悉，中小学数学具有“数学化”的理念，但据笔者对部分高校数学教师的相关调查 80% 的教师没有听过“数学化”这一词，这不能不让人感到问题严重．

高师数学要引领基础教育的数学，关键是要能够站在高等数学这一观点下经常对基础数学进行反思，揭示初等数学中一些重要问题的本质．如中学一个重要的概念函数，函数是研究变量的关系的数学模型，数学研究变量是 17 世纪开始的，恩格斯在《反杜林论》中说道：“初等数学，即常数的数学，是在形式逻辑的范围内活动的，至少总的说来是这样，而变数的数学 —— 其中最重要的部分是微积分 —— 本质上不外是辩证法在数学方面的运用．”[13]

（**3**）从文化的视角分析数学教育

文化是数学的最本质的特征．从文化的角度来理解数学教育，是对影响数学发展的各种因素进行分析、对数学知识中蕴含的数学思想进行挖掘、对数学方法给予关注、对数学与实际的关系厘清，因而从文化的视角来分析数学，对我们完整的理解数学教育的本质、目的、内容、方法等都具有重要的意义．

从文化的视角理解数学，可以从数学的知识、结构、发展背景、所涉及的数学家的生平和个性、思想特点、方法特点、对人的个性品质的影响等多个方面来理解数学．这样看到的数学就是立体的，也就是在数学教育过程中就能对人的各个方面进行影响．

REFERENCES

[1] 张岱年，方克立．中国文化概论【M】．北京师范大学出版社，2004. 1:1．

[2] 张楚廷．数学文化【M】．高等教育出版社，2000. 7:1．

[3] 张国栋，数学思想与数学教育【J】，北京教育学院学报，1998（2）：8–10．

[4] 张荣淼．中外数学家对培养学生数学学习能力的研究概述【J】．中小学教材教学，2005（6）：20–22）．

[5] 史宁中．漫谈数学的基本思想【J】．天津：数学教育学报，2011.20（4）：8．

[6] 黄泰安，关于数学文化的若干重要相关研究领域【J】，数学教育学报，2007．16（2）：4–7．

[7] 陈熙仁等．初中生数学学习情感的调查研究【J】，天津：数学教育学报．2011. 20（4）：32–36．

[8] 杨叔子．文理交融，打造“数学文化”特色课程【J】．数学教育学报，2011. 20（4）：7．

[9] 李铁安等．高中解析几何教学策略 —— 数学史的视角．数学教育学报，2007． 16（2）：90–94．

[10] 张英伯．傅种孙 —— 中国现代数学教育的先驱．数学通报，2008. 47（1）：8–10．

[11] 黄安成．谈数学教师的“创作激情”与“创作冲动”，北京：数学通报，2008. 47（1）49–52．

[12] 伍鸿熙．一个数学家在数学教育界的经验．数学通报，2008.47（1）：6．

[13] 张楚廷．数学文化．高等教育出版社，2000，第 82 页．

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