HPM 2012

The HPM Satellite Meeting of ICME-12

DCC, Daejeon, Korea
July 16~20, 2012

PROCEEDING

BOOK 1

HISTORY and PEDAGOGY of MATHEMATICS * HPM

International Study Group on the Relations Between the HISTORY and PEDAGOGY of MATHEMATICS
An Affiliate of the International Commission on Mathematical Instruction

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Workshop

4. Mathematics and its relation to science, technology and the arts: historical issues and educational implications.

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Schedule

[Day 1] MONDAY, JULY 16

08:00–19:00	Registration & Information	
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10:00–11:00	Plenary Lecture**(Theme 2) [Man-Keung Siu]**	
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11:00–11:30	Break	
11:30–12:30	Oral Presentation**(Theme 2)**	
	Session 1 [Evelyne Barbin] Uffe Thomas Jankvist: *A Historical Teaching Module on "The Unreasonable Effectiveness of Mathematics"—the Case of Boolean Algebra and Shannon Circuits.* Jerry Lodder: Historical Projects in Discrete Mathematics.	**R101**
	Session 2 [Sunwook Hwang] Jin Ho Kim* & In Kyung Kim: *Future Research Topics in the Field of Mathematical Problem Solving: Using Delphi Method.* Kyunghee Shin: *Harriot's Algebraic Symbols and the Roots of Equations.*	**R102**
12:30–14:30	Lunch Break	
	Official Photographing at the entrance of DCC on 12:30	
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	Session 1 [Kristin Bjarnadottir] Shu Chun Guo: *A Discussion on the Meaning of the Discovery of Mathemetics in the Worriers and the Han Dinasty.* Mustafa Alpaslan*, Mine Isiksal & Cigdem Haser: *Relationship Between Pre-service Mathematics Teachers' Knowledge of History of Mathematics and Their Attitudes and Beliefs towards the Use of History of Mathematics in Mathematics Education.* Kathleen M. Clark: *The Influence of Solving Historical Problems on Mathematical Knowledge for Teaching.*	**R101**

	Session 2 [Michael Fried] Yoichi Hirano & Yoshihiro Goto: *An Essay on an Experiment in Mathematics Classroom – the Golden Ratio Related in the Form of the Nautilus Shell.* Man-Keung Siu & Yip-Cheung Chan*: *On Alexander Wylie's Jottings on the Science of the Chinese Arithmetic.* Jeyanthi Subramanian: *Indian Pedagogy and Problem Solving in Ancient Thamizhakam.*	**R102**
	Session 3 [Young Wook Kim] Guo-qiang Li & Li-hua Xu: *Analysis of Mathematics Teaching in the View of the History of Mathematics.* Toshimitsu Miyamoto: *Mathematics Education and Teaching Practice to Bring up History of Mathematics Culture Richly.* Nobuki Watanabe: *Sundial and Mathematics: Analysis of Oldest Horizontal Sundials in Japan by Mathematics.*	**R103**
16:00–16:30	Break	
16:30–17:30	Workshop**(Theme 2)**	
	Janet Heine Barnett, Jerry Lodder & David Pengelley: *Projects for Students of Discrete Mathematics via Primary Historical Sources: Euclid on His Algorithm.*	**R101**
	Renaud Chorlay: *The Journey to a Proof: If f' is Positive, then f is an Increasing Function.*	**R102**
	Frederic Metin: *Using History for Mathematics in EFL Teaching.*	**R103**
	Peter Ransom: *Fortifying France: Les Villes de Vauban.*	**R104**
17:30–18:30	Poster Session	**Lobby**
	Moonja Jeong: *The Trends on Mathematics in Novels.*	
	Sung Sook Kim: *Orthogonal Latin Squares of Choi Seok-Jeong.*	
	Sang-Gu Lee, Kyung-Won Kim & Jyoung jenny Lee: *Teaching History of Mathematics by Creating Your Own Deck of Card and Poster of World Mathematicians (Including Your Fellow Countrymen).*	
	Alejandro Rosas & Leticia Pardo: *Arithmetical and Geometrical Progressions, and Numerical Series in China before 14th Century*	
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	Sang-Gu Lee, Jae Hwa Lee* & Hyungwoo Byun: *First Study on a Joseon Mathematics Book SUHAKJEOLYO(數學節要, 수학절요), which was Written by Jong-Hwa An in 1882.*	
18:00–19:00	Preparatory Session 1 for Asian HPM **[Sung Sa Hong]**	**R107**
	Opening Remark: Sung Sa Hong(Korea)	
	Chang-Koo Lee (Korea): *Chinese Mathematics in Chosun*	
	Morimoto (Japan): *Three Authors of the Taisei Sankei.*	
19:10–20:30	Welcoming Banquet	
9:00–19:00	Exhibitions	**Lobby** (Continues during the conference.)
	Jangjoo Lee: *An Eight-Fold Folding Screen Using the 24 Finest Scenes of Mathematics of Joseon Dynasty.*	
	Sang-Gu Lee & Yoonmee Ham: *Modern Mathematic Books by Korean Authors during 1884–1910.*	
	Zhe Zhu: *Textbook Preparation Based on the Conception of "Integrating the History of Mathematics into Mathematics Curriculum".*	
	Yongmei Liu & Tingting Liu: *A Study on 'Mathematization' of Obstacles and Solving Appoaches*	

[Day 2] TUESDAY, JULY 17

8:00–19:00	Regisration & Information	
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	Janet Barnett (USA): *Bottled at the Source: The Design and Implementation of Classroom Projects for Learning Mathematics via Primary Historical Sources.*	
10:00–11:00	Plenary Lecture**(Theme 1) [Sung Sook Kim]**	
	Tinne Hoff Kjeldsen (Denmark): *Uses of History for the Learning of and about Mathematics: Towards a Theoretical Framework for Integrating History of Mathematics in Mathematics Education.*	**R107**
11:00–11:30	Break	
11:30–12:30	Oral Presentation**(Theme 1)**	

	Session 1 [Yoichi Hirano] Mi Kyung Ju, Jong Eun Moon & Ryoon Jin Song: *Ethnomathematics and its Educational Meaning: A Comparative Analysis of Academic Discourse and Educational Practice of Mathematics History in Korea.* Immaculate K. Namukasa: *History of Mathematics Implemented in Mathematics* Education Programs: *The Development, Implementation and Evolution of a University Course.*	**R101**
	Session 2 [Uffe Jankvist] Toshimitsu Miyamoto: *Theoretical Framework Concerning Drawing Method in Mathematics Education.* Kyeonghye Han: *The Historico-Genetic Principle and the Hermeneutical Methode as the Theoretical Background of Using History of Mathematics in Lesson.*	**R102**
	Session 3 [Frédéric Métin] David Guillemette: *Bridging Theoretical and Empirical Account of the Use of History in Mathematics Education? A Case Study.* René Guitart: *Misuses of Statistics in a Historical Perspective: Reflexions for a Course on Probability and Statistics.*	**R103**
	Session 4 [Maria del Carmen Bonilla] Patricia Baggett* & Andrzej Ehrenfeucht: *The "Ladder and Box" Problem: From Curves to Calculators.* David Pengelley: *Teaching Number Theory from Sophie Germain's Manuscripts: A Guided Discovery Pedagogy.*	**R104**
12:30–14:30	Lunch Break	
	Korean Traditional Music Performance: (12:30–13:00) Five musics performed by Daejeon Korean Traditional Orchestra	**R107**
14:30–16:00	Oral Presentation**(Theme 4)**	
	Session 1 [Mustafa Alpaslan] Oscar João Abdounur: *The Division of the Tone and the Introduction of Geometry in Theoretical Music in the Renaissance: a Historic-Didactical Approach.* Sunam Cho: *The Formation of Mathematics Curriculum Characteristics by Augustus de Morgan in University College, London: On the Boundary between Mathematics and Natural Philosophy.* George Heine: *The Flattening of the Earth: Its Effect on Eighteenth Century Mathematics.*	**R101**

	Session 2 [Youngmee Koh] Woosik Hyun: *Mathematical Foundations of Cognitive Science.* Taewan Kim & H. K. Pak: *Analysis on Lines and Circles in Secondary School Mathematics Textbooks according to the Types of Concept.* Ho-Joong Lee: *On the Eigenvalues of Three Body Problem in the Early 20th Century.*	**R102**
	Session 3 [Renaud Chorlay] Leonardo Venegas: *The Topological Intuition of Leonardo da Vinci* María del Carmen Bonilla: *Visualization of the Mechanical Demonstration to Find the Volume of the Sphere Using Dynamic Geometry.* Shuping Pu* & Xiao-qin Wang: *How to Integrate History of Mathematics into Mathematics Textbooks: Case Study of Junior High School Textbooks in China and France*	**R103**
16:00–16:30	Break	
16:30–17:30	Workshop**(Theme 3)**	
	Bjørn Smestad: *Teaching History of Mathematics to Teacher Students: Examples from a Short Intervention.*	**R101**
	Man-Keung Siu* & Yip-Cheung Chan: *Chinese Arithmetic in the Eyes of a British Missionary and Calculus in the Eyes of a Chinese Mathematician: Collaboration between Alexander Wylie (1815–1887) and LI Shan-Lan (1811–1882).*	**R102**
	Anne Michel-Pajus: *Historical Algorithms in the Classroom and in Teacher-Training.*	**R103**
17:30–18:30	Poster Session	**Lobby**
	Moonja Jeong: *The Trends on Mathematics in Novels.*	
	Sung Sook Kim: *Orthogonal Latin Squares of Choi Seok-Jeong*	
	Sang-Gu Lee, Kyung-Won Kim & Jyoung jenny Lee: *Teaching History of Mathematics by Creating Your Own Deck of Card and Poster of World Mathematicians (Including Your Fellow Countrymen).*	
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	Sang-Gu Lee, Jae Hwa Lee* & Hyungwoo Byun: *First Study on a Joseon Mathematics Book SUHAKJEOLYO(數學節要, 수학절요), which was Written by Jong-Hwa An in 1882.*	
18:00–19:00	Preparatory Session 2 for Asian HPM **[Chang Kyoon Park]**	**R107**
	Anjing Qu (China): HPM in China.	
	Evelyne Barbin (Chair of HPM): TBA.	

[Day 3] WEDNESDAY, JULY 18

8:00–12:00	Regisration & Information	
9:00–10:00	Plenary Lecture**(Theme 4) [Janet Barnett]**	**R107**
	Dominique Tournè's(France): *Mathematics of the 19th Century Engineers: Methods and Instruments.*	
10:00–10:30	Break	
10:30–12:00	Panel Discussion 1 **[Kathleen Clark]**	**R107**
	Theme 1: Why Do We Require a "History of Mathematics" Course for Mathematics Teacher Candidates? (And What Might Such a Course Look Like?).	
	Panelists: Mustafa Alpaslan (Turkey), Sang Sook Choi-Koh (Korea), Kathleen Clark (USA), Frédéric Métin (France).	
12:00–18:30	Excursion	

[Day 4] THURSDAY, JULY 19

8:00–18:30	Regisration & Information	
9:00–10:00	Plenary Lecture**(Theme 5) [Tinne Hoff Kjeldsen]**	**R107**
	Anne Michel-Pajus (France): *A Voyage into the Litterary-Mathematical Universe*	
10:00–11:00	Plenary Lecture**(Theme 6) [Tinne Hoff Kjeldsen]**	**R107**
	Johan Prytz (Sweden): *Social Structures in Mathematics Education. Researching the History of Mathematics Education with Theories and Methods from Sociology of Education*	
11:00–11:30	Break	
11:30–12:30	Oral Presentation**(Theme 6)**	
	Session 1[David Guillemette] Evelyne Barbin: *The Role of the French Association of Mathematics Teachers APMEP in the Introduction of Modern Mathematics in France (1956–1972).* Michael N. Fried: *Book XIII of the Elements: Its Role in the World's Most Famous Mathematics Textbook.*	**R101**

Session 2[Sanwook Ree] **R102**
Chun Chor Litwin Cheng:
The Mathematics Development of the Book Sea Mirror of Circle Measurements (Ceyuan Haijing).
Guo-qiang Li* & Li-hua Xu:
On Mathematics Teachers' Quality and Its' Advance in Mathematical History.

Session 3 [Dominique Tournés] **R103**
Hoyun Cho:
Common Core State Standards Movement in U.S Mathematics Curriculum.
Lorena Jimenez Sandoval & Gustavo Martinez Sierra:
Social Construction of the Algebraic Structures. A Model for Its Analysis.

12:30–14:30 Lunch Break

14:30–16:00 Panel Discussion 2 **[Uffe Thomas Jankvist]** **R107**

Theme 2: Empirical Research on History in Mathematics Education: Current and Future Challenges for Our Field.

Panelists: Uffe Thomas Jankvist (Denmark), Yi-Wen Su (Taiwan), Isoda Masami (Japan), David Pengelley (USA).

16:00–16:30 Break

16:30–17:30 Oral Presentation**(Theme 6)**

Session 1 [Chun Chor Litwin Cheng] **R101**
Toshimitsu Miyamoto:
History of Arithmetic Textbook and Composition of Content Based on Count Principle Method.
Osamu Kota:
Teaching and Learning of Functions in Modern Japan.

Session 2 [Kathleen Clark] **R102**
Kristín Bjarnadottir:
18th Century Mathematics Education: Effects of Enlightenment in Iceland.
Andreas Christiansen:
Geometry Textbooks in Norway in the First Half of the 19th Century.

Session 3 [Bjørn Smestad] **R103**
Nicla Palladino:
The Issue of Mathematics Textbooks in the Correspondence of Giovanni Novi to Enrico Betti during the Unification of Italy.
François Plantade:
After the Gösta Mittag-Leffler & Jules Houël Correspondence, Their General and Particular Thoughts on Mathematical Teaching.

17:30–18:30 HPM Session **[Evelyne Barbin]** **R107**

Opening Remarks: Evelyne Barbin (Chair of the HPM Group).
Introducing New Chair of the HPM Group.
Announcement of HPM 2016.

[Day 5] FRIDAY, JULY 20

8:00–15:00	Regisration & Information	
9:00–10:00	Plenary Lecture**(Theme 7) [Tsang-Yi Lin]**	**R107**
	Sung Sa Hong (Korea): *Theory of Equations in the History of Chosun Mathematics.*	
10:00–10:30	Break	
10:30–12:30	Oral Presentation**(Theme 5)**	
	Session 1 [Jinho Kim] Chun-yue Stanley Lee & Mei-yue Christine Tang: *A Comparative Study on Finding Volume of Spheres by LIU Hui(劉徽) and Archimedes: An Educational Perspective to Secondary School Students.* Chang K. Park & Sun Bok Bae: *Cultural Prime Numbers: 2, 3 and 5.* Young Hee Kye: *Math and Art in View Point of Perspective Drawing of the West and East.*	**R101**
	Session 2 [Anne Michel-Pajus] André Cauty: *Invitation to Revisit the Mesoamerican Calendars. The Count That is Called Real Calendar.* Gregg De Young: *A Colorful Case of Mistaken Identities.* Albrecht Heeffer: *Dutch Arithmetic, Samurai and Warships: The Teaching of Western Mathematics in Pre-Meiji Japan.*	**R102**
	Session 3 [Evelyne Barbin] Yoichi Hirano: *Remark on the Notion of Golden Ratio—Concerning "Divine Proportion" in the Renaissance.* Leo Corry: *Euclid's Proposition II.5: A View through the Centuries-Geometry, Algebra and Teaching.* Qing-jian Wang: *The New "Curriculum Standard" and the New Mathematics—the Union of History of Mathematics and Mathematics Education*	**R103**
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14:30–16:00	Oral Presentation**(Theme 7)**	
	Session 1 [David Pengelley] Sung Sa Hong, Young Hee Hong & Chang Il Kim: *Chosun Mathematician Hong Jung Ha's Least Common Multiples.* Sung Sa Hong, Young Hee Hong & Young Wook Kim: *Liu Yi and Hong Jung Ha's KaiFangShu.* Sung Sa Hong, Young Hee Hong & Seung On Lee: *Yang Hui's NaYin QiLi.*	**R101**

	Session 2 [Sang-Gu Lee] Michael Kourkoulos* & Constantinos Tzanakis: *An Experiment on Teaching the Normal Approximation to the Symmetric Binomial Using De Moivre & Nicholas Bernoulli's Approaches.* Youngmee Koh: *Educational Meaning of the Theory of Rectangular Array in the Nine Chapters on the Mathematical Art.* Sangwook Ree: *Meaning of the Method of Excess and Deficit.*	**R102**
	Session 3 [Sangki Choi] Yuzi Jin & Young Wook Kim: *Research on the Muk Sa Jib San Beob.* Hae Nam Jung: *A Study on "GuSuRyak" of Choi Seok Jung.* Toshimitsu Miyamoto: *Historic Investigation of Legendre's Proof about the 5th Postulate of "Elements".*	**R103**
16:00–16:30	Closing Ceremony **[Jinho Kim]**	**R107**
	Closing Remarks: Evelyne Barbin(Chair of the HPM Group), Luis Radford(New Chair of the HPM Group), Sunwook Hwang(Chair of HPM 2012 LOC)	

USES OF HISTORY FOR THE LEARNING OF AND ABOUT MATHEMATICS

Towards a theoretical framework for integrating history of mathematics in mathematics education

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ABSTRACT

The purpose of the present paper is to present a theoretical framework for analyzing, criticizing and orienting designs and implementations of history of mathematics in mathematics education in order to address the questions of how integrating history of mathematics benefits students' learning of mathematics and how uses of historical elements to support students' learning of mathematics develop students' historical awareness. To address the second question, a multiple perspective approach to history of practices of mathematics is introduced together with a set of concepts that can be used to identify and articulate different forms of people's uses of history. To address the first question uses of history and history of mathematics are linked to a competence based conception of mathematics education and Sfard's theory of mathematics as a discourse. To illustrate the framework and how it can be used, two examples, one from a university master's program and one from a Danish high school, of integrating history into mathematics education are presented and analyzed.

1 introduction

Despite a range of well known arguments[1] for integrating history in mathematics classrooms, and the inclusion of history in the national mathematics curriculum in some countries,[2] history does not play a significant role in general mathematics education. This might seem strange for someone from the outside, considering that mathematics has a history that goes more than 5000 years back, so the past provides a huge reservoir of authentic mathematical texts and activities, and why not learn from the masters?[3] However, as every one knows who has tried it, it is not so straightforward to integrate history in mathematics teaching and learning.[4] On one hand there is the question of how integrating history of mathematics benefits students' learning of mathematics, and on the other hand when

[1]See e.g. Beckman (2009), Fauvel (1991a, 1991b), Fauvel and van Maanen (2000). See also the review article by Jankvist (2009).

[2]Examples are Denmark and France.

[3]David Pengelley and Reinhard Laubenbacher have developed several courses where they teach with original historical sources. These are described on their homepage http://sofia.nmsu.edu/ history/. They have published several papers and books explaining their approach, see e.g. Laubenbacher and Pengelley (1996).

[4]See e.g. Siu (2007).

historical elements are used to support students' learning of mathematical concepts, theories or techniques, or to humanize mathematics, there is the question of in what sense such implementations develop students' historical awareness.[5]

In the present paper I will focus on these two issues. To deal with the second issue, I will in section 2) introduce notions from research in people's uses of history and from the academic discipline of history of mathematics. Recent research has shown that people use history in many different contexts, with different approaches and for different purposes, i.e. we attach several, partly different meanings to history. The task is not to announce one approach as the right one and discard the others, but to unfold the differences between the various ways in which history is being used and understood. The challenge is not to dissolve the complexity but to explore it; to clarify how history is (can be) understood and for what purposes it is or can be used, in order to capture some of the multifaceted ways in which history can benefit students' learning of and about mathematics.

To deal with the first issue, uses of the past and history of mathematics needs to be linked to theories from didactics that connects to conceptions of mathematics education and to learning of mathematics. This will be done in section 3) and section 4), respectively. Ideas from didactics of mathematics are introduced to discuss and analyze how and in what sense different approaches to history can benefit teaching and learning of mathematics. In section 3) Mogens Niss' proposal for a competence based understanding of mathematics education that addresses the question of what it means to master mathematics is introduced.[6] In section 4) Anna Sfard's theory of mathematics as a discourse is presented and I will argue, that history of mathematics can function at the core of what it means to learn mathematics.[7]

Together, these theories and notions will span a theoretical framework that can be used to place, analyze and criticize implementations of historical elements in mathematics classrooms to understand how history is used and in what sense it can benefit students' learning of mathematics, as well as orient designs of implementations such that learning goals and teaching intentions can be made clearer and targeted – whether these goals are directed towards the learning of mathematics or of history of mathematics. The framework allows teachers to make informed and reflected choices about how and for what purpose(s) historical elements can enter mathematics classrooms.

To illustrate the theoretical framework and how it can be used, two examples of integrating history into mathematics education will be presented and analyzed in section 5). The first example is a report from a project work carried out by a group of students in a university master's programme in mathematics in Denmark. The second example is an experimental teaching course on implementing problem oriented project work in history of mathematics that was carried out in a Danish high school.

2 A multiple perspective approach to history of mathematics and different uses of history

The notion of a "multiple perspective" approach to history is borrowed from the Danish historian Eric Bernard Jensen's (2003, 16–17) writings about historiography. The multiple perspectives enter, because the underlying premise is that people are understood as being shaped by history and being

[5] For discussions of these two issues, see also Fried (2001, 2007).
[6] See Niss (2004) and Niss and Højgaard (2011).
[7] See Sfard (2008).

shapers of history. History is studied from perspective(s) of the historical actors, paying attention to these actors' intentions and motivations, as well as to intended and unintended consequences of their actions. It is an action-oriented conception of history were people, their projects and their actions are taken as point of departure for historical investigations to achieve a historical-social understanding of how people have thought and acted at different times and in different cultures.

Such an approach to history can be adapted to history of mathematics, if we think of mathematics as a cultural and historical product of knowledge that is produced by human intellectual activities. The knowledge that is produced by a mathematician, or group of mathematicians, at a certain time in history depends on the knowledge and mathematical culture available for these mathematicians and it (might) shape or define guidelines for further developments of mathematical knowledge. In this sense, such an action-oriented perception of history of mathematics can be pursued, where the historian study the history of mathematics from perspectives of past mathematicians, their projects and motivations, situated in certain contexts, at specific places, at certain times, and under particular historical circumstances in order to understand and explain historical processes in the development of mathematics. In the academic profession of history of mathematics, contextualized historical investigations of this kind are undertaken. One approach it to study concrete episodes of production of mathematical knowledge within the "work place" of the involved (past) mathematicians, studying the development of these mathematicians' production of mathematics from their practice(s) of mathematics, trying to follow the development of these mathematicians' ideas and techniques.[8]

Besides the perspectives of the historical actors, the perspective of the historian also needs to be taken into account. A historian's research inquiry is always guided by some questions, problems, or wonderings that she/he wants to answer, solve or understand. Hence, the choice of perspective(s) is determined in a dialectic process between the historian's perspective(s), i.e. what she/he wants to understand regarding the historical episode in question, and the historical actors' perspectives as they unfold during the research process.

The strength of such a multiple perspective approach where the development of mathematics is studied from different points of observation is that the historical analyzes are attached to concrete episodes of mathematical research and research practices from where relations and connections can be unfolded and explored. The perspectives can be of different kinds. In some instances the historian might be interested in e.g. how other disciplines influenced the development of pieces of mathematics, or how and why techniques of proofs changed, or if and how applications of mathematics influenced its developments etc., asking questions such as why mathematicians introduced specific definitions and concepts, which particular problems did they work on, what techniques did they use and why, how did mathematical objects emerge and develop.[9] These kinds of perspectives and historical questions regarding mathematical research practices relate to the content and inner core of mathematics, and consequently, such a multiple perspective approach to history of mathematics studied from practices of mathematics has the potential to play a significant role for the learning of mathematics.

This approach to history of mathematics comes from the academic discipline i.e. how professional historians of mathematics think and practice history. Research into people's uses of history has shown

[8]See e.g. Leo Corry's introduction (Corry, 2004) as well as the rest of the papers published in *Science in Context*, 17(½), 2004. See also Epple (2000), Kjeldsen (2004), Kjeldsen and Carter (in press) to name just a few where also further references can be found.

[9]See Epple (2004).

that such an academic approach to history is jut one of many approaches. It has shown that people's historical awareness is formed in many different contexts, that people use history in many different connections and for many different purposes, e.g. in movies, when we travel, in family histories, in computer games, in school subjects, in museums, in memorial places and landmarks.[10]

Jensen has written about people's conception and uses of history. In this context, he defines history as follows: "when a person or a group of people is interested in something from the past and uses their knowledge about it for some purpose" (Jensen 2010, 39). When history is viewed in this broader perspective it becomes a complex concept—an umbrella term for a collection of related forms of knowledge and practices that people uses in their life. The task is not to identify one form of history as the right one, but to reveal similarities and differences in approaches and ways in which history is understood and used.

For this purpose, Jensen (2010, 40–57; 141; 145) has introduced four pairs of concepts, which he uses to identify and articulate different forms of people's uses of history. They are: (1) lay history and professional history; (2) pragmatic history and scholarly history; (3) actor history and observer history; (4) identity concrete and identity neutral history.[11] These concepts can be used as guideposts when we want to understand and analyze our own and other's conception and uses of history. They address different aspects: methodological aspects of research in history, history as an academic field of research and an 'every day use' of history, and the intentions of specific uses of history. Hence, they do not mutually exclude one another, they overlap, and they can be present in various degrees in concrete uses of history.

Lay history and *professional history* distinguishes between every-day (life world) uses and professional uses of history i.e. it is about differences in the context in which history is used. Lay people's (i.e. non-professional historians') uses of history have become an object of research within the last decades. According to Schörken (1981), professional historians consider lay history to be naïve and lay people think of professional history as lifeless and distant from the real world. Many mathematicians including mathematics teachers read and use past mathematical texts for research, in teaching and out of sheer interests for its history. History of mathematics is also an academic discipline with its own research programmes, educational programmes, academic degrees and prices, journals, international conferences etc. Hence, the distinction between lay history and professional history makes sense when it comes to conceptions and uses of history of mathematics and some of the historiographical debates exhibit this distinction. To give just one example, the historian of mathematics Grattan-Guinness (2004, 163) complains that mathematicians are not sympathetic to history (as professional historians of mathematics conceive of it) because "their normal attention to history is concerned with heritage: that is, how did we get there? Old results are modernized in order to show their current place; but the historical context is ignored and thereby often distorted. By contrast, the historian is concerned with what happened in the past, whatever the modern situation is." What Grattan-Guiness draws attention to in this quote is a difference that is one of the characteristics between lay persons' and professional historians' concerns of and with the past.

Pragmatic history is history studied from a kind of utility perspective. This is the case when history is conceived of as "the master of life" so to speak when we think, we can learn from history's mistakes that history can teach us better ways to live our lives, the *historia magistra vitae* conception of

[10]See e.g. Ashton and Kean (2009), Eriksen and Sigurdsson (2009), Jensen (2010).
[11]My translation into English.

history. A pragmatic historian will try to make history relevant in a contemporary context. Many professional historians now a day dissociate themselves from a pragmatic conception of history which they think inhibits our understanding of history epistemologically. They favour a *scholarly approach to history*[12] where they maintain a critical distance to the past and emphasise differences between now and then. History is about gaining insights into and understanding the past on its own terms. The multiple perspective approach to history of mathematics as described above would be characterized as a scholarly approach to history. The distinction between pragmatic and scholarly history overlaps with the distinction between lay history and professional history in the sense that lay people often have a pragmatic conception of history, whereas many professional historians now a days have a scholarly approach, but there are situations were lay history is guided by scholarly interests and the pragmatic approach to history has been a tradition within academic, professional history with the scholarly approach being the dominant one from the mid 19^{th} century (Jensen 2010, 48–52).

The notions *Actor history* and *observer history* are used to distinguish between whether people look at a past episode retrospectively or in a forward-looking perspective. It is about people's position regarding a past episode. The term actor history is used to characterize approaches to history where the past is used to orient one self and/or act in a present context. Jensen calls this an intervening use of history. In contrast, history can be used in a retrospective perspective with an enlightening purpose, in such cases Jensen (2010, 41) talks about *observer history*. As mentioned above, these concepts do not exclude one another e.g. an observer history can be contained in an actor history. Jensen (2011, 8) gives the example of a professional historian who uses a scholarly approach to understand something from the past (e.g. a war) in order to spread information and enlighten people in the present (about the relation to the country of the war).

History can be used in an intervening sense to form people's identity and in such cases Jensen talks about an identity concrete presentation of history. What is considered to be identity concrete or identity neutral history writing depends on culture and time – a history writing that is considered to be identity neutral in one culture might not be considered to be neutral by another culture, and what is considered to be an identity neutral history writing at one point in time might be considered to be identity concrete at another point in time (Jensen 2010, 52-57).

Besides the approaches to history covered in these four pairs of concepts, Jensen also includes the so-called '*living history*' concept as a playful approach to history. This form of history, where people actively participate in historical scenes and experience life from reconstructed specific historical periods and settings (e.g. a late 14^{th} century market town) is a way of using history to help participants develop historical awareness. According to Jensen (2010, 145), many people find the living history approach appealing, because the playful approach with its focus on developments of skills requires other learning strategies than the more intellectual approach that is used in much school teaching, where students learn from books.

These notions provide a set of glasses—a lens—through which we can identify, articulate and distinguish between different understandings and uses of history. Together with the multiple perspective approach to history of mathematics outlined above, they provide a theoretical framework that can be used to characterize, analyze and criticize uses and practices of history and implementations of history in mathematics classrooms. They can also be used to orient designs and future

[12] This is my translation of the Danish word "lærd"—which mean to be a scholarly person.

implementations of history to clarify and target learning goals and teaching intentions.

In the next sections, history of mathematics will be linked to theories from didactics that connect to conceptions of mathematics education and to learning of mathematics, in order to discuss aspects of how and in what sense history can function in mathematics teaching and learning within these theories.

3 A competence based mathematics education—and the role of history

By a competence based view of mathematics education, I refer to the understanding of mathematics education as it has been developed in the Danish KOM-project (Niss 2004, Niss and Højgaard 2011). The project was initiated by the Danish Ministry of Education, and its understanding of mathematical competence form the basis for curriculum developments and descriptions in general mathematics education in Denmark. The objective of the project was to identify purposes and learning outcomes of mathematics education when the goal is to educate people to master mathematics.[13] Instead of a traditional curriculum of lists of concepts, subjects, techniques, results, etc., the project group identified eight main competencies and three kinds of second order competencies that were argued to span mathematical competence.[14]

The eight main competencies are divided into two groups: Four competencies that have to do with abilities to ask and answer questions in and with mathematics, and four competencies that regard abilities and familiarities with language and tools in mathematics, see Figure 1.

1. Thinking competency	5. Representing competency
2. Problem tackling competency	6. Symbol and formalism competency
3. Modeling competency	7. Communicationg competency
4. Reasoning competency	8. Aids and tools competency

Figure 1: The two groups of main competencies.

According to the Danish KOM-project, mathematics education should also provide students with philosophical and historical insights of mathematics to achieve a balanced picture of mathematics. The three second order competencies take care of that. They are (1) meta-issues of actual applications of mathematics in other subject and practice areas, (2) the historical development of mathematics in cultures and societies, its internal and external driving forces and interactions with other fields, and (3) the nature of mathematics as a discipline (Niss 2004).

The intentions of the KOM-project behind the second order competency of historical awareness and insights are in accordance with the multiple perspective approach to history of mathematics studied from its practice, as it is described above. Hence, the historical awareness that the KOM-project wants to develop in students corresponds to a scholarly perception of history. Such an approach to history also has potential to train and develop (some) of students' main mathematical competencies.

[13]Some find the word "competence" non-appropriate because it labels students as being non-competent if they do not succeed in mathematics. The word competence was chosen because it focuses on abilities to cope with situations where mathematics plays or can play a role (Niss 2004, 182).

[14]A mathematical competency are developed and trained in relation to subject matters of mathematics. Ten subject areas are identified in the KOM-report as subject matters in which to develop and train mathematical competence in general education: The number domain, arithmetic, algebra, geometry, functions, calculus, probability theory, statistics, discrete mathematics and optimization (Niss and Højgaard 2011, 126-128).

This can be achieved by using the multiple perspective approach to history of mathematical practices in mathematics education on a small scale, by focusing on a limited amount of carefully chosen perspectives that address issues in concrete pieces of past mathematical activities, in order to have students become aware of and reflect upon e.g. research strategies or the function and nature of specific mathematical concepts, arguments, problems, methods, results etc. from the historical episode in question. In section 5) we will see an example where students, through such an approach to history, developed historical awareness in the sense of the KOM-project in a way that also invoked and trained (some) of the students' main mathematical competencies. This is an example where observer history is contained in action history.

4 Can history function at the core of what it means to learn mathematics?

In the competence based understanding of mathematics education, history as such is part of mathematics education through the second order competencies. Students' mathematical competencies can be invoked and trained in the process of developing their second order competency of historical awareness, but history is not essential for developing students' first order mathematical competencies. Therefore, in this section I will address the question whether history can function at the core of what it means to learn mathematics?

Here I will draw on Sfard's (2008) theory of *Thinking as Communicating*. The framework of mathematical competence presented above deals with how we can think of mathematics education in terms of what should come out of mathematics education, namely people who posses mathematical competence to some degree.[15] Sfard (2008) is concerned with human thinking in general and mathematical thinking in particular.

Sfard views mathematics as a discourse where discourse "refers to the totality of communicative activities, as practiced by a given community" (Sfard 2000, 160). Learning mathematics then means to become a participant in the discourse. Discourse denotes human activity, and the discursive interpretation of learning emphasizes the social nature of intellectual activities. The activity of communicating is regulated by rules. Sfard distinguishes between two types of rules: object-level rules and metalevel rules. Object-level rules concern the content of the discourse. They are narratives about properties of mathematical objects. Metalevel rules are rules about the discourse itself. They are implicitly present and they govern "when to do what and how to do it." (Sfard 2008, 201–202). They

> "manifest their presence ⋯in our ability to decide whether a given description can count as a proper mathematical definition, whether a given solution can be regarded as complete and satisfactory from a mathematical point of view, and whether the given argument can count as a final and definite confirmation of what is being claimed." (Sfard 2000, 167)

To become a participant in mathematics discourse, to learn mathematics, it is necessary to develop not only object-level rules but also proper metalevel rules. Hence, creating situations where metalevel

[15] In the KOM-report it is suggested that progression in an individual's mathematical competence is realised through its growth in three dimensions: degree of coverage, radius of action and technical level, (Niss and Højgaard 2011, 30).

rules are exhibited and made explicit objects of students' reflection is an essential aspect of mathematics teaching and learning – and it is with regard to this history of mathematics can function at the core of what it means to learn mathematics, because metalevel rules are contingent. These rules are not necessary. They develop and change over time. This means that these rules can be investigated at the object-level of history discourse.

According to Sfard, because of the contingency of metalevel rules, students are not likely to begin a metalevel change by themselves. This is most likely to happen if the learner becomes confronted with another discourse governed by metalevel rules that are different from the ones she or he has been acting in accordance with so far. Sfard have termed such an experience a *commognitive* conflict, and she defines it as "a situation in which different discursants are acting according to different metarules." (Sfard 2008, 256).

The scholarly multiple perspective approach to history of mathematics studied from its practice as outlined in section 2) views mathematics as something that develops because of human intellectual activities. Hence, there is no contradiction between this approach to history and a discursive view of mathematics. As we have argued in Kjeldsen and Blomhøj (2012), and as will be illustrated below, metalevel rules can be exhibited as explicit objects of reflection for students, by having students work with historians' tools on historical questions about the practice of mathematics in concrete mathematical episodes from the past. History provides a huge reservoir of authentic mathematical texts either published mathematical articles, correspondences between mathematicians, manuscripts for talks, and notes etc. Such sources can play the role as an "interlocutor". Students can examine them in their historical context and analyze the work of former mathematicians with respect to the way they formulated mathematical statements, the way they argued for their claims, their views on mathematics and so on. Hereby students can experience differences in meta-discursive rules between interlocutors (the historical text, themselves, their textbooks and/or their instructor). In this way metalevel rules can be revealed and made the object of students' reflections. Whether commognitive conflicts occur, depends on the chosen sources, and the metalevel rules that govern the students' own discourse. It is of course essential to use a scholarly approach to history, i.e. gaining insights into and understanding the past on its own terms. The so-called whig (or present-centredness) interpretation of history, where the readings and interpretations of historical sources are constrained by our modern conception of mathematics must be avoided.[16] If past mathematics is translated into modern mathematics and "old results are modernized", as was quoted from Grattan-Guiness in the description of lay history above, differences in discourse between the past and the present will (partly) disappear.

In the first example of the next section, some instances will be given, where metalevel rules were made into explicit objects of reflection for students through the students' work with sources from the past and historical investigations of an episode from the history of differential equations.[17]

[16]The term whig history comes from Butterfield (1931). See also Wilson and Ashplant (1988) and Schubring (2008).

[17]In Kjeldsen and Petersen (forthcoming) another example is presented where learning and teaching situations were designed using parts of the framework, with intention to make meta-discursive rules in mathematics objects of students' reflection and to detect students' metarules. Here the past was used with a deliberate intention of intervening. i.e. it is an example where observer history is contained in actor history.

5 Two examples: analysis of a student project and an experimental teaching course

In this section, two examples from teaching practice will be analysed and discussed within the framework developed and presented in section 2), 3) and 4). The first one is a project work conducted by a group of students in a university programme in mathematics and the second one is an experimental teaching course that was implemented and studied in a mathematics classroom in upper secondary school. Both of them function as examples of how concrete implementations of history can be analyzed to understand how history was used and in what sense it benefitted (or had the potential to benefit) students learning of mathematics.

5.1 Analysis of a student project work from a university master's programme in mathematics

Roskilde University is a reform university that was founded in 1972 in Roskilde, Denmark. The university implemented student centred, problem oriented and group organized project work as one of its main pedagogical principles. All students of the university, no matter which study programme they follow, participate in a group organized project work in every semester. The project work runs throughout the entire semester. At the end of the semester each group hands in a report of 50–100 pages in which they answer the problem formulation that has guided their project work. Besides the project work, students also follow regular courses. Course work and project work run in parallel each semester, and they each take up half a student's study load. A student has participated in 10 such projects when he or she receives his/her master's degree.[18] The project work analyzed below belongs to the first semester of the master's programme in mathematics.[19]The theme of the project is "mathematics as a discipline", and the requirement is that the students should work with a problem through which they will gain insights into the nature of mathematics and its "architecture" as a scientific discipline in a way that illustrates the historical developments of mathematics, its status and/or its place in society.

The project report in question was written by five students. It has the title *Physics' Influence on the Development of Differential Equations*. From their courses and project work in mathematical modelling in their bachelor studies, the students had experienced that differential equations play a central role in applications of mathematics in other sciences. They wanted to investigate how differential equations were developed and what motivated that development. They knew that during the last part of the 17^{th} century, mathematicians had begun to use infinitesimals to solve problems that were difficult to solve with classic geometry. Many of these problems were physical problems, and, as the students wrote in their report, physics is often mentioned in history of mathematics literature as an influential factor in the development of differential equations. The students were curious to find out how and in what sense physics had influenced the development of differential equations in the 17^{th} and 18^{th} century. Their historical investigations were guided by questions such as: "How did physics influ-

[18]Readers interested in the special problem oriented project work are referred to Kjeldsen and Blomhøj (2009), Blomhøj and Kjeldsen (2009), Salling Olesen and Højgaard Jensen (1999). See Niss (2001) for further information and discussions about the experiences with problem oriented student projects at Roskilde University.

[19]Before entering into the master's programme in mathematics, students have completed a three year interdisciplinary science of bachelor's programme where they have specialized in mathematics and one other subject.

ence the development of differential equations? Was it as problem generator? Did physics play a role in the formulation of differential equations as solutions to given problems? To what extent can the influence from physics be traced in the first systematization of the theory of differential equations? ” (Paraphrased from Nielsen, Nørby, Mosegaard, Skjoldager and Zacho (2005, 8))

To answer these questions, the students studied this episode in history of mathematics from its practice from the perspective of how problems from another discipline (physics) influenced the development of mathematics,[20] how they entered into mathematicians' formulation of problems and the techniques they used to solve the problems. The students chose two cases: the catenary problem and the brachistochrone problem. The catenary problem is to find the shape of the curve formed by a flexible string that hangs freely between two fixed points. The brachistochrone problem is to find the path of fastest descend for a point that moves from one fixed point to another only influenced by gravity. They read and analyzed three selected original sources from the 1690s that dealt with the two cases: Johann Bernoulli's solution of the catenary problem and of the brachistochrone problem and Jakob Bernoulli's solution of the brachistochrone problem.

The students studied and interpreted the three sources within the mathematical discourse of the time, discussing them within the broader social and cultural context of the contemporary mathematical community. To mention just a few points: (1) the students discussed what was to be understood by a mathematician at that time, (2) they explained that the borders between disciplines were much looser than today and that mathematics and natural philosophy were much more intertwined, (3) they outlined how mathematical results were circulated (or not) within the mathematical community at the time, and emphasized the importance of competition which they linked to how mathematicians functioned in society, (4) they took into account the perspective of the actors, by discussing the content of the sources with respect to the Bernoulli brothers' intentions e.g. whether the purpose of the brothers' work was to solve the problems of the catenary and the brachistochrone or rather to investigate the effectiveness of infinitesimals as a new technique in mathematics in the 17^{th} century.

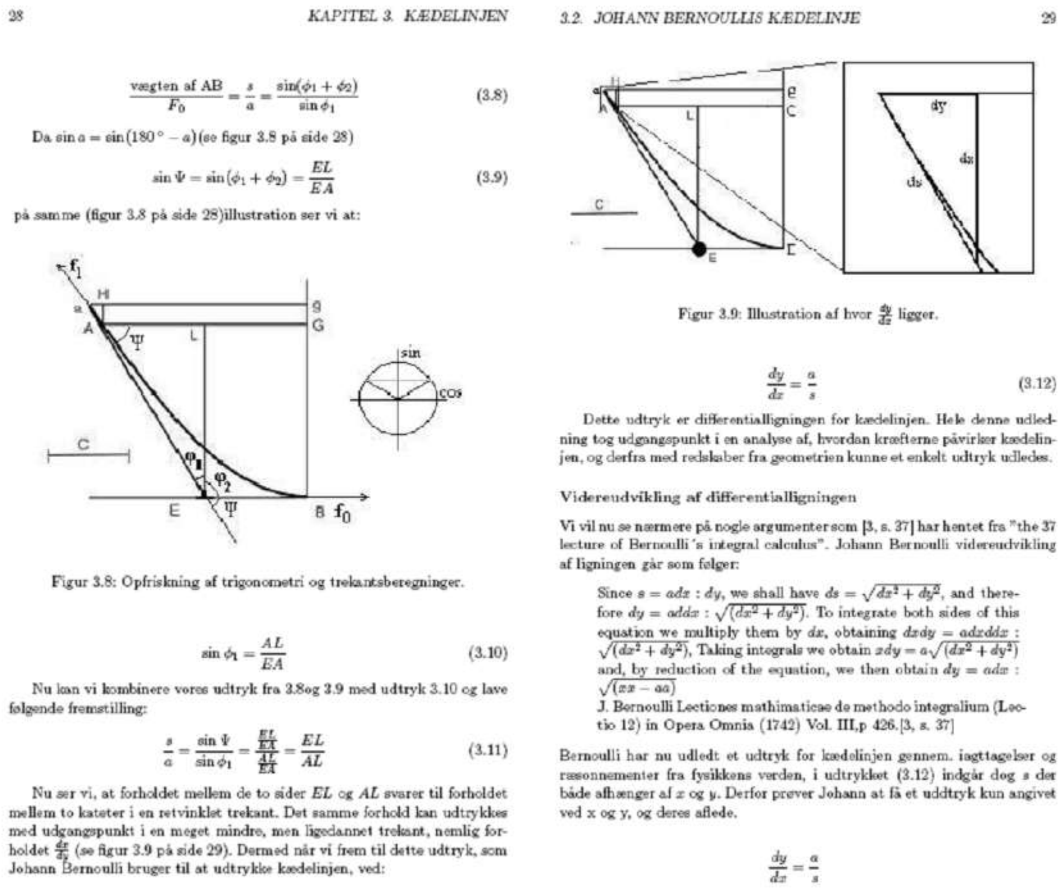
If we use the framework presented in section 2) to analyze this particular implementation of history into mathematics education to answer the second question that was raised in the introduction i.e. in what sense such an implementation develops students' historical awareness, we can conclude that for this particular implementation, the students had a scholarly approach to history with an enlightening purpose, i.e. observer history in Jensen's terminology.

In dealing with the mathematical content of the sources, the students made a detailed analysis of how the Bernoulli brothers derived the differential equations for the problems, how they formulated the equations and why they formulated them the way they did, how and with which methods they solved the equations. The students analyzed the sources with respect to what objects the Bernoulli brothers were investigating and which techniques they used to produce knowledge about the objects. All these issues are not only relevant for answering the students' historical questions, they are also relevant for the learning and understanding of differential equations. In the following I will analyze parts of the students' work within the framework presented in section 3) and 4) to answer the first question that was raised in the introduction, i.e. how integrating history of mathematics can benefit students' learning of mathematics. I will not go into all the details of the students' project work. In-

[20] The students realized in the course of the project work that physics and mathematics were not separated disciplines in the 17th century in the sense of how we consider them today, and that natural philosophy (as it was called) and mathematics were much more intertwined.

terested readers are referred to Kjeldsen (2011) for a discussion of mathematical competence, and to Kjeldsen and Blomhøj (2012) for a comprehensive analysis of the project work with respect to possibilities for meta-level learning.

In the catenary problem, Johann Bernoulli used five hypotheses from statics. In studying his treatment of the problem, the students had to mathematize these five hypotheses and to understand how Johann Bernoulli used them to describe the catenary. In working out this part of Bernoulli's text, the students' problem tackling competency, reasoning competency, representing competency, parts of their modelling competency[21] and their competency to handle symbols and formalism in mathematics were evoked and trained. In order to understand Bernoulli's mathematical representation of the catenary, they had a) to fill out many gaps themselves and derive intermediate results using arguments with similar triangles and from trigonometry, b) to introduce and understand the use of symbols, c) to mathematize the hypotheses. Figure 2 displays a couple of pages from the students' final report where they explain, how Bernoulli mathematized the hypotheses from statics and described the catenary.


28 KAPITEL 3. KÆDELINJEN

$$\frac{\text{vægten af AB}}{F_0} = \frac{s}{a} = \frac{\sin(\phi_1 + \phi_2)}{\sin \phi_1} \quad (3.8)$$

Da $\sin a = \sin(180° - a)$ (se figur 3.8 på side 28)

$$\sin \Psi = \sin(\phi_1 + \phi_2) = \frac{EL}{EA} \quad (3.9)$$

på samme (figur 3.8 på side 28)illustration ser vi at:

Figur 3.8: Opfriskning af trigonometri og trekantsberegninger.

$$\sin \phi_1 = \frac{AL}{EA} \quad (3.10)$$

Nu kan vi kombinere vores udtryk fra 3.8og 3.9 med udtryk 3.10 og lave følgende fremstilling:

$$\frac{s}{a} = \frac{\sin \Psi}{\sin \phi_1} = \frac{\frac{EL}{EA}}{\frac{AL}{EA}} = \frac{EL}{AL} \quad (3.11)$$

Nu ser vi, at forholdet mellem de to sider EL og AL svarer til forholdet mellem to kateter i en retvinklet trekant. Det samme forhold kan udtrykkes med udgangspunkt i en meget mindre, men ligedannet trekant, nemlig forholdet $\frac{dx}{dy}$ (se figur 3.9 på side 29). Dermed når vi frem til dette udtryk, som Johann Bernoulli bruger til at udtrykke kædelinjen, ved:

3.2. JOHANN BERNOULLIS KÆDELINJE 29

Figur 3.9: Illustration af hvor $\frac{dy}{dx}$ ligger.

$$\frac{dy}{dx} = \frac{a}{s} \quad (3.12)$$

Dette udtryk er differentialligningen for kædelinjen. Hele denne udledning tog udgangspunkt i en analyse af, hvordan kræfterne påvirker kædelinjen, og derfra med redskaber fra geometrien kunne et enkelt udtryk udledes.

Videreudvikling af differentialligningen

Vi vil nu se nærmere på nogle argumenter som [3, s. 37] har hentet fra "the 37 lecture of Bernoulli's integral calculus". Johann Bernoulli videreudvikling af ligningen går som følger:

> Since $s = adx : dy$, we shall have $ds = \sqrt{dx^2 + dy^2}$, and therefore $dy = addx : \sqrt{(dx^2 + dy^2)}$. To integrate both sides of this equation we multiply them by dx, obtaining $dxdy = adxddx : \sqrt{(dx^2 + dy^2)}$, Taking integrals we obtain $xdy = a\sqrt{(dx^2 + dy^2)}$ and, by reduction of the equation, we then obtain $dy = adx : \sqrt{(xx - aa)}$
> J. Bernoulli Lectiones mathimaticae de methodo integralium (Lectio 12) in Opera Omnia (1742) Vol. III,p 426.[3, s. 37]

Bernoulli har nu udledt et udtryk for kædelinjen gennem. iagttagelser og ræsonnementer fra fysikkens verden, i udtrykket (3.12) indgår dog s der både afhænger af x og y. Derfor prøver Johann at få et uddtryk kun angivet ved x og y, og deres aflede.

$$\frac{dy}{dx} = \frac{a}{s}$$

Figure 2: Page 28 and 29 of the students' report. The text is in Danish except from the quote from the source on page 29.

Bernoullli described the infinitesimals dx and dy of the curve geometrically and he used the so-called infinitesimal triangle to derive an equation between dx and dy. This part of Bernoulli's text presented cognitive obstacles for the students. In order to understand Bernoulli's arguments, the students

[21]Modelling competency is understood in the sense of Blomhøj and Kjeldsen (2010).

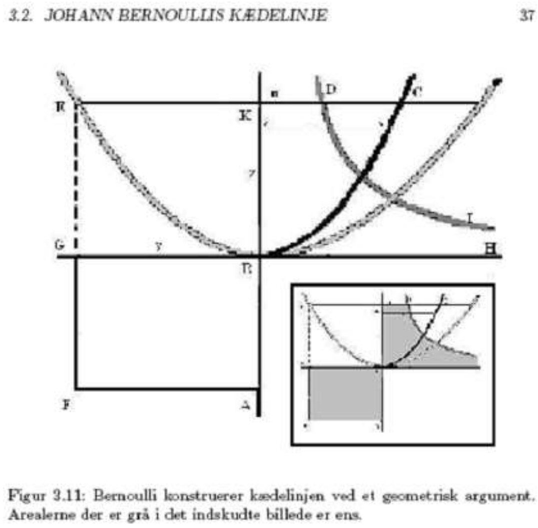
had to read and understand the text within the mathematical discourse of the time which is difficult, because the point of departure for them was their own mathematical discourse, which was different from Bernoulli's. As explained in section 4) it is precisely the contingency of metalevel rules of mathematical discourse that is the reason, why history can serve as a means to reveal meta-discursive rules and make them objects of students' reflections. The way Bernoulli used geometry, statics and infinitesimals to derive an equation for the infinitesimals of the catenary is very different from the way it was introduced in the textbooks from which the students had learned about differential equations. Especially Bernoulli's use of the infinitesimal triangle went fundamentally against the $\epsilon - \delta -$conception of rigour that the students' had been brought up with in their first year analysis course. The right page (page 29) of figure 2 shows the students' explanation and discussion about the infinitesimal triangle.

In this part of the project work, the students' competencies to think and reason mathematically were trained and challenged in a new context provided by the history. The thoughts and reasoning presented in the historical sources were very different from the way in which issues of infinitesimals calculus are thought about and reasoned with in their analysis textbooks. For the students, the historical context provided an authentic piece of mathematics that could not be dealt with and understood on its own terms by using standard methods from their analysis textbooks. This is an instance where connections were created in the learning situation between the students' experiences with the involved mathematics from their textbook and their historical experience—an instance that challenged the students to use other aspects of their mathematical conceptions of infinitesimal calculus in new situations provided by the historical context. To be more concrete, the students' problems with understanding Bernoulli's way of reasoning with infinitesimals provoked situations where the students examined why Bernoulli's method worked in this particular case, and it initiated discussions between the students and the professor who supervised their project work about criteria for rigour and how such criteria are determined. These were instances where the students experienced that standards for rigour have changed over time, which is an example of the contingency of metalevel rules in mathematics discourse.

Another instance for metalevel learning occurred when the students had to understand Johann Bernoulli's solution of the catenary differential equation. Figure 3 is a picture of a page from the students' report that belongs to the section in the report where they explained and discussed Bernoulli's method.

Bernoulli constructed the solution geometrically. This is not how the students were used to solve differential equations and this part of their work initiated discussions between the group and their supervisor about conceptual aspects of what a solution to a differential equation means. They became aware that this, too, changes over time. It is a metalevel rule of mathematics discourse. The learning and teaching situations were these kind of discussions emerged were created in the process of the project work, guided by the students' efforts to read and understand the historical sources in order to find answers to their guiding questions. To illustrate how the students' "dialog" with the original source made them reflect upon these metalevel rules, I have translated the following two paragraphs from the students' report. In the first paragraph, the students investigate and discuss what counts as a valid argument, and in the second they reflect upon style of argumentation and generality in mathematics:

> "On page 26 we can follow how Johann Bernoulli transformed his physical knowledge into mathematics. Approximately half of Johann Bernoulli's account for the derivation of the differential


3.2. JOHANN BERNOULLIS KÆDELINJE 37

Figur 3.11: Bernoulli konstruerer kædelinjen ved et geometrisk argument. Arealerne der er grå i det indskudte billede er ens.

Figure 3: From page 37 of the students' report. Bernoulli's geometrical construction of the catenary. The figure with the two shadowed areas was constructed by the students to illustrate a comparison of areas that are involved in Bernoulli's construction.

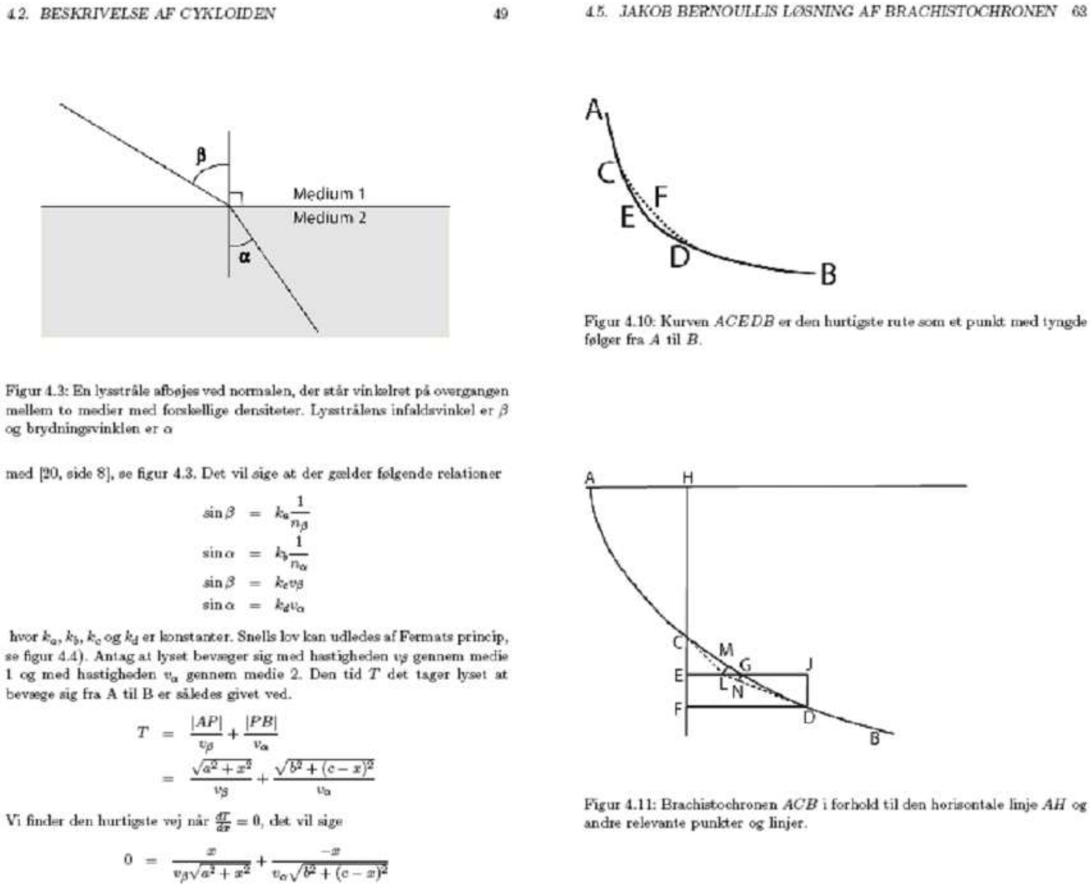
> equation of the catenary was paraphrased in that section. Whereas he later became much briefer in his derivations, he was very particular in this derivation. [$\cdots$]. We interpret this as if Johann Bernoulli felt a need to document that precisely this transformation from physics to differential equations was well founded. This was done with geometry which had a high degree of validity in this period." (Nielsen et al. 2005, 41)

> "Bernoulli did not know the logarithmic function, so he could not describe the curve [of the catenary] analytically. Even though he showed an incredible geometrical and mathematical intuition, his construction [of the solution] did not lead to a general solution of similar problems." (Nielsen et al. 2005, 42)

I will give one last example from the project work that illustrates how the students' mathematical competencies were trained. It is taken from the students' comparison of the Bernoulli brothers' different ways of solving the brachistochrone problem. Johann interpreted the moving point as a light particle that moves between two points. He used Fermat's principle of refraction and derived an equation involving the infinitesimals dx and dy. Jakob used a different strategy. He considered the problem as an extremum problem, using that an infinitesimal change in the curve would not increase time, due to the minimum property of the brachistochrone. Figure 4) shows two pages from the students' report, illustrating the two different approaches. The page to the left is from the students' treatment of Johann's solution and the page to the right is from their investigation of Jakob's solution.

In this part of their project work, the students' mathematical thinking competency was evoked and trained. The students experienced the characteristics of the nature of mathematics that makes it possible to generalize solution methods beyond particular, concrete problems. They wrote the following about the differences between Johann's and Jakob's approaches:

> " [$\cdots$] makes it [Johann's solution] weak from a mathematical point of view. His solution cannot be generalized because it was based on the physical situation. [$\cdots$] As indicated in section 4.5 Jakob's solution gave rise to the mathematical discipline of calculus of variations. His method


4.2. BESKRIVELSE AF CYKLOIDEN 49

Figur 4.3: En lysstråle afbøjes ved normalen, der står vinkelret på overgangen mellem to medier med forskellige densiteter. Lysstrålens infaldsvinkel er β og brydningsvinklen er α

med [20, side 8], se figur 4.3. Det vil sige at der gælder følgende relationer

$$\sin\beta = k_a\frac{1}{n_\beta}$$
$$\sin\alpha = k_b\frac{1}{n_\alpha}$$
$$\sin\beta = k_c v_\beta$$
$$\sin\alpha = k_d v_\alpha$$

hvor k_a, k_b, k_c og k_d er konstanter. Snells lov kan udledes af Fermats princip, se figur 4.4). Antag at lyset bevæger sig med hastigheden v_β gennem medie 1 og med hastigheden v_α gennem medie 2. Den tid T det tager lyset at bevæge sig fra A til B er således givet ved.

$$T = \frac{|AP|}{v_\beta} + \frac{|PB|}{v_\alpha} = \frac{\sqrt{a^2+x^2}}{v_\beta} + \frac{\sqrt{b^2+(c-x)^2}}{v_\alpha}$$

Vi finder den hurtigste vej når $\frac{dT}{dx} = 0$, det vil sige

$$0 = \frac{x}{v_\beta\sqrt{a^2+x^2}} + \frac{-x}{v_\alpha\sqrt{b^2+(c-x)^2}}$$

4.5. JAKOB BERNOULLIS LØSNING AF BRACHISTOCHRONEN 63

Figur 4.10: Kurven $ACEDB$ er den hurtigste rute som et punkt med tyngde følger fra A til B.

Figur 4.11: Brachistochronen ACB i forhold til den horisontale linje AH og andre relevante punkter og linjer.

Figure 4: Page 49 and 63 of the students' report. Johann's (page 49) and Jakob's (page 63) approaches to the brachistochrone problem. The text is in Danish

> could be used to solve other such kinds of optimization problems. [⋯] in contrast to his brother, Jakob was able to abstract from the mechanical framework of the problem of the brachistochrone. He separated the mathematics from the physics." (Nielsen et al. 2005, 75)

As the students explained, the technique Johann Bernoulli used to solve the problem was tied to the physical situation. It could not be generalized. Its scope was limited to the actual situation. In contrast, Jakob Bernoulli's solution method was independent of the concrete situation and could be used beyond that on other kinds of extremum problems—eventually leading to a new sub discipline in mathematics, called the calculus of variations.

As we have seen, this project work created complex teaching and learning situations where a scholarly approach to history was used that 1) developed the students' second order competency of having historical insights and possess historical awareness, 2) invoked and trained the students' mathematical competencies, 3) exhibited metalevel rules of mathematical discourse and made them objects of students' reflections. The students wrote their report using LaTeX and constructed the figures in their report using MatLab. If we include such skills in the competency to handle tools and aids of mathematics, the students were trained in all eight main competencies, as well as in the second order competency of developing historical awareness and insights into the history of mathematics.

The historical context provided situations where the students came to reflect upon differences between the sources' and their textbook's way of conceptualizing differential equations and their solutions, differences of argumentation, style and rigour. On the object level of mathematics discourse, this project work benefited the students' learning of mathematics especially through their discussions of why the Bernoulli brothers' use of infinitesimals as actual quantities gave "correct" answers despite the lack of rigour as we understand it today, where the concept of a function and of limit are crucial, concepts the Bernoulli brothers' did not have at their disposal. Through these discussions—dialogues—with the original sources, the students were forced into reflections upon their own understandings of the involved concepts on a structural level that went far beyond their initial operational dominated conception of differential equations.

5.2 Analysis of an experimental course in problem oriented project work: Egyptian mathematics.

The subject matter of the experimental course that will be analyzed in this section was Egyptian mathematics. The experimental course was developed in 2004 as part of an in-service course for upper secondary mathematics teachers in the Danish high school. The objective of the in-service course was to support mathematics teachers in developing, implementing and documenting problem oriented, group organised project work in history of mathematics in upper secondary mathematics education.[22] The in-service course began with a three day seminar during which the teachers in groups designed and developed a problem oriented project work of their own choice with specific learning objectives, course materials and products. Afterwards the teachers implemented their experimental course, i.e. their problem oriented project work, in one of their classrooms. During the experimental course, the teachers observed their students' group work in class. When the experimental course was finished, the teachers wrote a report documenting the implementation of their experimental course. The teachers' experiences with developing and implementing the problem oriented project work in history of mathematics in their classrooms where then discussed on the basis of these reports during a two-day seminar at the end of the in-service course.

The teacher, who developed and implemented the experimental course that will be analyzed in the following, chose Egyptian mathematics for several reasons. First of all, there is a textbook on Egyptian mathematics with sources translated into Danish.[23] And second, the course was meant to be interdisciplinary with history, and Egypt was suitable as a common theme that the mathematics teacher and the history teacher could agree upon.[24] The teacher's design of the experimental course was guided by his formulation of seven objectives for the students' learning. Four of these dealt with issues relating to independent study skills, and three of them concerned the history of mathematics as it was required in the new (2005) curriculum for mathematics in Danish high schools. In the following I will concentrate on his learning objectives regarding history and mathematics, which were the following: He wanted to

1. "have the students appreciate that mathematics has been different from what it is today

[22] History of mathematics and design and implementation of group and project organized teaching was part of a new curriculum in the Danish upper secondary school system (gymnasium) which was implemented in 2005.

[23] There exists only very little materials in Danish with sources from episodes in the history of mathematics. Lack of suitable resources is a major obstacle to introducing history for the learning and teaching of mathematics on a broader scope.

[24] Only the mathematics teacher participated in the in-service course.

2. develop the students' awareness that mathematical results have evolved, that mathematics is not static, which is contrary to the way it is often presented
3. develop the students' awareness that mathematics develops in an interplay with culture and society." (Wulff 2004, 2–3; my translation)

The teacher designed the project work in three phases: First he gave an introduction to Egyptian mathematics. He used two 45-minute lessons where he introduced the class to the Egyptian number symbols, showed them how the Egyptians multiplied numbers by repeating doubling and how they formulated problems. Second, the students were divided into groups of four. Each group worked with a chapter from the textbook on Egyptian mathematics: fractions, Pesu (bread and beer) exercises; first degree equations; two equations with two unknowns and second degree equations; the circle and approximations of π; the volume of a truncated pyramid; and computations of areas. Each group's work was guided by the problem formulation, chosen by the teacher: *How and why did the Egyptians calculate?* Eight lessons of 45 minutes were used in class for the independent group work (and an unknown amount of homework). During the group work, the teacher functioned as a consultant the students could call on for advice. Third, each group had to share the knowledge they had acquired in the group work. This was done in the form of a seminar where each group presented their work and their answers to the problem formulation supported by a power point presentation.

I will not go into further details about the work done in the groups,[25] but concentrate on using the theoretical framework to analyze the implementation of the experimental course and the teacher's evaluation. In his report, the teacher wrote about objectives 1) and 2) that "they were all about gaining insight into current mathematics precisely by studying the mathematics of another time" (Wulff 2004, 3). Hence, we are dealing with a use of the past from a utility perspective. In this part of the project work, the teacher took a pragmatic approach to history. This is also consistent with the teacher's attention in the classroom as he revealed in his report where he wrote: "Already during the first module [the first two lessons] came the classical question, why are we going to learn this? And we had a nice talk about the intended learning issues [1), 2) and 3) above], during which the class apparently accepted that historical mathematics, besides being interesting as such, could contribute to a more nuanced view on current mathematics." (Wulff 2004, 5). In the teacher's evaluation of to what degree the leaning objectives were realized, he thought that the students did not experience that mathematics develops over time, since the historical process of change was not dealt with in the project work. The students compared Egyptian and modern mathematics, through which they became aware that there were fundamental differences, i.e. the students experienced that mathematics has changed, but they did not study the actual process of change and how such processes come about. For the third learning objective, the teacher left the utility perspective and took a scholarly approach to history, as he wrote: "here is where the subject of history can be involved. From a general knowledge about Ancient Egypt and its society, students can discuss how society and culture have been driving forces for the mathematics of that time. At the same time the historians' method of source criticism is an essential tool for interpreting ambiguous and defective papyri" (Wulff 2004, 4).

The activities that guided the students' work, i.e. reading the sources and working with exercises presented in their respective chapters of the textbook on Egyptian mathematics can be interpreted as

[25] Interested readers are referred to Kjeldsen (2012) where some examples from the topics the students worked with are presented.

a kind of "living history" approach. The students had to learn and simulate how ancient Egyptians calculated, how they worked with geometry, posed mathematical problems etc. In doing so the students used different learning strategies and came to reflect upon mathematics on a structural level as can be seen from the following passage in the teacher's report: "Many students wondered about how "stupid" the Egyptians were. Why did they only use unit fractions? Why should a number be expressed as a sum of different unit fractions? On the other hand their methods were very difficult to understand; that is rather advanced, so in that respect they weren't stupid at all. I think that many of the students realized that current mathematics is not "just" like today, but is a result of a long development, during which many things have been simplified. [⋯] This [that mathematics had made progress] became especially obvious when the students constantly rewrote the Egyptian notation to current notation with x's, formulas, etc. After they had finished an Egyptian calculation they would say: 'but that just corresponds to ⋯' followed by a solution of an equation in our way. It was very inspiring to see how students, who normally were a bit alienated towards x's and equations now had taken those to themselves as their own, and all of a sudden perceived equations as an easy way to solve problems. The students became aware that modern notation makes the calculations much easier than they would have been otherwise" (Wulff 2004, 7).

The teacher evaluated the experimental teaching course as partly successful. All learning goals except the last one (the third one above) were fulfilled. The last objective was suppose to have been reached through the students' work with answering the "why" part of the problem formulation – that is, why did the Egyptians calculate. The teacher had hoped that through interdisciplinary work with the students' school subject of history and their history teacher, the students would have experienced concrete examples of developments of mathematical ideas driven by needs of society. That didn't happen because the history teacher focused on other issues. The teacher reported that afterwards the students seemed to posses a more mature and reflective attitude towards mathematics.[26]

In this problem oriented project work, the teacher used past mathematics in different ways, from different perspectives and for different purposes. He had the students deal with history of mathematics from its practice, having the students work with the content matter of past mathematical text from the perspective of which techniques the Egyptians used and the kind of problems they worked with. For parts of the learning objectives the teacher used a pragmatic approach to history and for other parts he used a scholarly approach. The above quotes from the teacher's report show that several of the students' mathematical competencies were invoked and trained through this project work on Egyptian mathematics, especially their problem tackling skills and their competencies to deal with different representations of mathematical entities and to handle symbols and formalisms.

6 Discussion and concluding remarks

The theoretical framework presented in section 2), 3) and 4) draws on theories from mathematics education and from historiography adapted to history of mathematics and to history of mathematics in mathematics education. It is composed in such a way that it can deal with two central questions in research in integrating history of mathematics in mathematics education: 1) how integrating history

[26]The teacher finished the report approximately three months after his experimental teaching course had finished. Unfortunately, the teacher did not give examples of how and in what sense the students had a more mature and reflective attitude towards mathematics.

of mathematics benefits students' learning of mathematics and 2) how uses of historical elements to support students' learning of mathematics develop students' historical awareness. The framework is broad enough to accommodate the richness of the spectrum of implementations of history in mathematics education, and it is narrow enough to function as a tool for analyzing, criticizing and orienting designs and implementations of history for the teaching and learning of and about mathematics. The framework captures (some) of the multifaceted ways in which history can benefit students' learning of and about mathematics.

The part of the framework that concerns historiography and different forms of history provides a set of concepts that can be used to explore and identify how history is or can be understood. By linking that part of the framework with the second part that concerns thinking about purposes of mathematics education and mathematics learning it becomes possible to clarify and distinguish between different purposes for integrating history in mathematics education in relation to the two central issues presented above.

The framework was used to analyse the design and implementation of the experimental teaching course in Egyptian mathematics. The analysis revealed that the teacher used different approaches to history for different purposes targeted towards different learning goals – some directed towards mathematics and some towards history of mathematics. He created a complex and rich learning situation were the students developed new learning strategies, enhanced several of their mathematical competencies, and gained insights into the history of practices of mathematics. The analysis of the project work on physics' influence on the development of differential equations that was carried out by a group of five students in a university master's programme in mathematics showed that the students' used a scholarly approach to history for enlightening purposes. Their approach can be characterized as a multiple perspective approach to history of mathematics from its practice, adapted to mathematics education by focusing on the perspective of whether, how, for what purposes and to what degree the historical actors were inspired by problems from physics. The analysis revealed that in the process of exploring the three original sources, chosen by the students', on their own terms, the students identified, discussed and reflected upon differences between the historical actors' mathematical practices and the ones presented in their textbooks. Hereby, connections were created in the learning situation between the students' experiences with the involved mathematics from their textbook and their historical experience. These connections challenged the students to use other aspects of their mathematical conceptions in new situations provided by the historical context. The multifaceted ways in which history can benefit students' learning of mathematics became visible by employing the second part of the theoretical framework. It showed that the students were trained in all the main mathematical competencies, that they gained insight into history of mathematics in the sense of the second order competency of the KOM-project, and that they came to reflect upon meta-discursive rules in mathematics.

Finally, the combination of a multiple perspective approach to history of mathematics studied from practices of mathematics and Sfard's theory of thinking constitutes a foundation, from which it can be argued that history can function at the core of the learning of mathematics. Since meta-discursive rules in mathematics are contingent, they can be objects of historical investigations. The analysis of the students' project in history of differential equations showed that, by having original sources play the role as "interlocutors", differences in metalevel rules in the discourse of the sources, the students' textbooks, and/or their instructor and themselves, can be revealed. Hereby metalevel rules are exhibited and can be made the object of students' reflections. This indicates that history might be an obvious strategy for detecting students' meta-discusive rules and for students to develop proper metarules. The first part is explored in (Kjeldsen and Petersen, forthcoming) where it was possible to detect (some) improper meta-rules in students' mathematical discourse through a teaching module in history of the concept of a function that was implemented by the use of a matrix-organization that provided the teacher with a window into students' meta-rules. The second part, whether this caused a change in these students' metarules, is another question. To answer this question, more research is needed. However, knowledge about students' improper metarules can be used by a teacher to target further teaching goals in ways that focus students' attention towards developing proper meta-discursive rules.

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ETHNOMATHEMATICS AND ITS EDUCATIONAL MEANING

A comparative analysis of academic discourse and educational use of mathematics history in Korea

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ABSTRACT

In this research, we comparatively analysed theoretical discourse in the research of mathematics history and educational use of mathematics history in mathematics textbooks to discuss implication for mathematics education in culturally diverse school. Our analysis focused on two kinds of documents: research documents of mathematics history published in Korea and Korean mathematics textbooks. Our analysis identified the tendency in which increasing numbers of Korean researchers adapt the perspective of mathematics as cultural knowledge. Parallel to the emergence of sociocultural discourse in the research of mathematics history, Korean mathematics textbooks introduce materials from the history of mathematics produced by diverse cultural groups. However, the Korean mathematics textbooks did not fully exploit the potential of mathematics history to bring up students' multicultural sensitivity. In Korean mathematics textbooks, use of mathematics history is still largely framed by the Eurocentric perspective on mathematics. This research has highlighted the significant discrepancy between the theoretical discourse of mathematics history and its educational use. This implies that it is essential to seek for ways of how to incorporate the issue of diversity and difference in educational use of mathematics history in textbooks.

1 Introduction

Recently, due to the increasing influx of immigrants, Korea undergoes a rapid transformation into ethnically and culturally diversified society. This cultural diversification demands fundamental restructuration of school education in Korea. In particular, nowadays an increasing number of immigrant kids enter Korean school and, as a consequence, school need to be prepared for students with various cultural backgrounds and to provide quality education that guarantees the equity in accessibility to all students. In this context, it is necessary to examine whether Korean mathematics textbooks are organized to be well-adapted into culturally diverse school. From the perspective, in this research, we comparatively analysed theoretical discourse in the research of mathematics history and educational use of mathematics history in mathematics textbooks to discuss implication for mathematics education in culturally diverse school.

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Research of mathematics history has shown that mathematics is a sociocultural and historical product and that mathematics has developed based on the dialogical relationship between diverse cultural systems of mathematics. We analysed mathematics textbook since they are regarded as documents that embody the educational visions of the national curriculum. Also, textbooks are essential teaching aid for teachers to plan and conduct their lessons in class. This implies that textbooks exert considerable influence on the teaching and learning of mathematics in school. Thus, it is of significance to investigate whether Korean mathematics textbooks are written to effectively exploit the potentials of mathematics history to address the educational needs raised by the cultural diversification in Korean school.

In addition to mathematics textbooks, we also analysed the theoretical discourse that has been raised in the research documents concerning history of mathematics. This analysis focused on the question of how the research of mathematics history in Korea has been changed in terms of its issues and theoretical discourse. Then, we compared the results of the textbook analysis and those of the research document analysis in order to investigate whether there are any connections or discrepancies between the perspectives about history of mathematics taken by the textbooks and by the research documents, respectively. Based on that, this research is to identify educational implication for the educational use of mathematics history in culturally diverse school.

2 Use of Mathematics History in Culturally Diverse Classroom

In the community of mathematics education, it has been argued that history of mathematics can play a valuable role in teaching and learning of the discipline (Fauvel, 1991). In fact, it is possible to locate historical quotations on the use of mathematics history in mathematics teaching and learning (Fasanelli, 2000). For instance, in his inaugural address as first president of the London Mathematical Society in 1865, de Morgan said:

> "I say that no art or science is a liberal art or a liberal science unless it be studied in connection with the mind of man in past times⋯The mathematician needs to know what the course of invention has been in the different branches of Mathematics; he wants to see Newton bringing out and evolving the Binomial Theorem by suggestion of the higher theorem which Wallis had already given. If he be to have his own researches guided in the way which will best lead him to success, he must have seen the curious ways in which the lower proposition has constantly been evolved from the higher" (recited from Fasanelli, 2000, p.35).

As this quotation suggests, mathematics educators have recommended that mathematics teachers creatively use history of mathematics in a variety of ways to teach mathematics. Especially, in the Korean national mathematics curriculum, educational use of mathematics history has been emphasized as one of methods for teaching and learning mathematics. Concomitant to this emphasis in the national mathematics curriculum, there has been a corpus of studies concerning the educational use of mathematics history.

In those documents such as national curriculum and educational theses, it is argued that history of mathematics reveals the connection between mathematics and human life and civilization. Thus, educational use of mathematics history can contribute to student motivation, the improvement of students' affective attitude and beliefs about mathematics by highlighting the human aspect of mathematics. Moreover, it is considered that history of mathematics can be a good resource for mathematics teachers to plan their teaching because history of mathematics shows how mathematics has been evolved in terms of curiosities, insights, tasks, methods, difficulties, and achievements that mathematicians has come up.

Another educational significance of mathematics history lies in the fact that it brings up the issue of cultural diversity in mathematics class. In fact, mathematics historians have revealed that the mathematics is a cultural hybrid which has had a nomadic life across diverse civilizations (Joseph, 1993). This perspective adapts a fact that mathematics is totally integrated with other manifestations of a culture. Culture as a strategy for societal action manifests itself through jargons, codes, myths, symbols, utopias, and ways of reasoning and inferring. A community has historically developed practices such as ciphering and counting, measuring, classifying, ordering, inferring, modelling, and so on, which leads to a unique system of an ethnomathematics (D'Ambrosio, 2010).

This relationship between culture and mathematics extends our understanding of the educational significance of mathematics history. For instance, Horng (2000) comparatively analysed the methods used by Euclid and of Liu Hui to find the greatest common divisor of two natural numbers. The analysis has shown that ancient mathematics of the West and the East have approached common problems differently and made different contributions to the development of mathematics. Horng (ibid.) concluded that history of mathematics can be useful resource for teaching mathematics to students in a meaningful way. That is, by introducing other possible forms of doing things through history of mathematics, mathematics class can provide a context for students to approach mathematics from diverse epistemological and methodological perspectives. Mathematics teachers may help students appreciate the multi-dimensional splendour of the discipline and its relationship to other cultural endeavours (Siu, 2000).

From this perspective, ethnomathematics such as Egyptian multiplication, Polynesian way of calculating distance, or methods to measure land productivity used in traditional New Guinea are regarded neither as primitive nor as uncivilized. Rather, it is considered that these ethnomathematics is the culmination of the communal consciousness that have historically developed in the context of a community. Therefore, as students explore the history of an ethnomathematics, they learn the uniqueness of each cultural system of mathematics and the relationship between ways of living and ways of knowing. Grugnetti and Rogers (2000) argue that this multicultural approach to history of mathematics would help students escape from ethnocentrism and extend their mathematical perspectives beyond their own cultural backgrounds. The sociocultural perspective of mathematics challenges the hierarchical relation between European academic mathematics and other cultural mathematics systems and then reconstructs an egalitarian power structure among all these ethnomathematics. Thus, it is considered that research of mathematics history highlights cultural facets of the discipline and its use may create a context for students to appreciate different ways of doing mathematics and to learn how to communicate over the difference (D'Ambrosio, 2010).

So far, the discussion shows various potentials that history of mathematics can offer to school mathematics. It is necessary whether our school mathematics exploits all the potential of mathematics history for teaching mathematics to empower our students mathematically. From this perspective, we have analysed research documents in order to identify theoretical discourse that provides implication for educational use of mathematics history. Also, we analyzed Korean mathematics textbook in order to investigate how effectively Korean mathematics textbooks are organized to address the issue of diversity. Based on the results of the comparative analysis, we will discuss educational implication.

3 Research Methods

3.1 Analysis of Theoretical Discourse in the Research of Mathematics History

In order to examine the research trends of thesis and articles about mathematics history published in Korea, we used representative searching engine provided by KERIS(Korea Education and Research Information Service). The reason why we chose the database of KERIS is that we focused on thesis and articles about mathematics history pulished in Korea and KERIS, as a governmental organization of Korea, has one of the most comprehensive database concerning research and publication of educational issues. Via the webside of KERIS, we searched thesis and articles by various keywords involving 'mathematics history', for example, 'Oriental history of mathematics', 'mathematics history of Chosun Dynasty', and 'Western history of matheamtics' as well as 'mathematics history' up to 2011. As a result of the search, we could locate 648 theses and articles published between 1970–2011.

We first classified data according to publication date in order to grasp the chronological changes in the issues of research. Then we classified them again into four categories according to research subject: General mathematics history (GMH), Western mathematics history (WMH), Eastern mathematics history involving Chinese and Japanese (EMH) and Korean mathematics history (KMH). If thesis and articles do not limit mathematics history regarding its origins, we sorted them into GMH. We counted materials dealing with Western mathematician, theories and approaches mainly to explain some mathematical theorem and principles produced by European mathematics to WMH. If thesis and articles deal with eastern mathematics history including Chinese, Japanese, or Korean mathematics history, we classified them as EMH. When as research document compared Western mathematics with Eastern or introduced Eastern mathematical development, solving problem process and literature, we categorized them into EMH. Thesis and articles concerning to Korean mathematics were categorized into KMH. Based on the categories, we analysed our data both qualitatively and quantitatively and described the tendency of how the theoretical discourse in the research of mathematics history has changed over the past five decades in Korea.

3.2 Analysis of Educational Use of Mathematics History

In order to investigate how mathematics history is used in Korean mathematics textbooks, we chose three mathematics textbooks for the 7th graders. These textbooks were selected because they were known as top ranked with respect to their portion in Korean textbook market. In Korea, a workbook accompanies a main textbook. So we analyzed both main textbooks and their workbooks. When we say "a textbook", it usually refers to a textbook with its workbook together, unless we specify which.

Based on theoretical literature, we constructed a preliminary frame for analysis and then applied it to the textbooks. Through the first pass of analysis, the preliminary frame was elaborated to fit better for our analysis. The final frame for analysis consists of factors of two dimensions, of which descriptions are following.

A. Origins of mathematics

The first dimension of the analytic frame is concerned whose history of mathematics is dealt with in textbook. We adapted the sociocultural perspective of mathematics so as to consider that European mathematics is not the only kind of mathematics. Based on the literature of ethnomathematics, we began with two categories that are concerned with European mathematics and non-European mathematics. After the preliminary analysis, we added the third category about the multicultural connection among ethnomathematics developed in many other nations, ethnics, racial groups. The multicultural connection refers to the pattern of use where the textbooks introduce materials from the history of culturally diverse mathematics to make a comparison or to integrate them to create a new of solving a problem. Therefore, this dimension consists of three categories: use of European mathematics history(EM), use of non-European mathematics history(NEM), and use of diverse ethnomathematics history for multicultural connection(MC).

B. Contents of mathematics history use

We also identified categories based on what kind of content is adapted from the history of mathematics. If textbooks introduced episodes of famous mathematicians, important mathematic problems, or historical anecdotes about mathematics, then we categorized them into C1. If textbooks present historical solutions or strategies about mathematical problems, we categorized them into C2. If a material from history of mathematics was used to provide a context for students to explore mathematical ideas and to develop their own mathematical thinking, then we categorize it into C3. Lastly, if a material from mathematics history is used to facilitate students to compare critically or to think creatively from perspectives of various ethnomathematics, then we categorized it into C4. C4 provides a context for students to approach mathematics from a multicultural perspective by comparing, deducing, creating based on the perspectives of diverse mathematics (Horng, 2000; Kim, 1999; Grugnetti and Rogers, 2000).

4 Findings

4.1 Theoretical *Discourse* in the Research of Mathematics History

We classified the collected research documents concerning history of mathematics and identified the frequency of WMH, EMH and KMH. Table 4.1 presents the results.

TABLE 4.1: The results of the frequency analysis

Period	Type								Total	
	WMH		OMH		WMH		GMH			
1970s	**1**	(0.1%)	**1**	(0.1%)	**2**	(0.3%)	-	-	**4**	(0.5%)
1980s	**4**	(0.6%)	**6**	(0.9%)	**5**	(0.8%)	**3**	(0.5%)	**18**	(2.8%)
1990s	**87**	(13.5%)	**5**	(0.8%)	**6**	(0.9%)	**10**	(1.5%)	**108**	(16.7%)
2000–2004	**176**	(27.2%)	**5**	(0.8%)	**11**	(1.7%)	-	-	**192**	(29.7%)
2005–2009	**236**	(36.4%)	**1**	(0.1%)	**16**	(2.0%)	-	-	**250**	(38.5%)
2010–2011	**68**	(10.5%)	**3**	(0.8%)	**3**	(0.5%)	-	-	**76**	(11.8%)
Total	**572**	(88.3%)	**23**	(3.5%)	**40**	(6.2%)	**3**	(2.0%)	**648**	(100%)

When we examine the above table more carefully, the number of the research documents on mathematics history increased rapidly over time. 88.3% of the research documents dealt with the history of Western European mathematics. Those documents introduced Western mathematicians as founder or developer of mathematical formula or algorithm. It can be said that those research adapts the Eurocentric perspective regarding that Western mathematics is the only legitimate kind of mathematics.

However, we found somewhat different position in Park(1977)'s thesis, which contained study on the history of Korean mathematics. She first compared the natural environment, industry, political and social structure, and ideology of Korea with those of China because she thought that all of those factors brought about different patterns of mathematical thinking and then examined the course of Korean mathematics. She argued that it needs more to establish formation process than accomplishments to understand truly mathematics of Koreans. This thesis shows that researchers of mathematics history began to recognize the relationship between mathematics and culture at that time using the concept of absolute mathematics. Especially, Park(ibid.) presented the necessity of research about Korean mathematics and explained that every nation has its own unique mathematics because of different climate, politics, economy, etc.

This point of view became extended in the 1980s. Researchers of the 1980s addressed viewpoints theoretically that they should get away from Western European mathematics traditions and regard mathematics as cultural heritage which belongs to every nation and ethnics by introducing the notion of ethnomathematics. With reference to the notion of ethnomathematics, Kim(1986) mentioned that every mathematics had contributed to the development of modern mathematics and that research trends was shifting from the history of European mathematics history to the history of the national and regional ethnomathematics. Hu(1997) also stated the differences around 1970 as follows: there were not only increasing number of literature on mathematics history but tendencies toward pluralist approaches reflecting a broad spectrum of mathematical concerns. Thus he referred lightly to literatures on the mathematics development in diverse societies like ancient China, India and medieval Muslim countries.

In the 1990s, theoretical researchers tried to establish the role and position of mathematics history in the mathematics education and emphasized the cultural value of mathematics. This effort had a great influence on the emergence of research on Korean mathematics history(Kim, 1999; Park, 2001). In the 1980s and the 1990s, in order to identify the status of Oriental mathematics, researchers tried to compare the Oriental mathematics and the Western mathematics by tracking their historical development. Researchers also sought to find the characteristics in the history of Korean mathematics that distinguish it from other Oriental mathematics, especially Chinese mathematics history. They were concerned to the fact that school mathematics included the knowledge of modern Western mathematics as only truth. Even mathematicians regard 'mathematics history' as Western European. This may lead students to think that there is no traditional mathematics in Korea. In fact, many people regarded Korean traditional mathematics as copies of Chinese mathematics. However, there are unique mathematics in Korea, so researchers investigated mathematical activities during the Chosun Dynasty in which many of historical artefacts remain.

At first researchers investigated the *Nine Chapters on the Mathematical Art*, one of the oldest ancient Asian mathematical books, which is a valuable material indicating that mathematics has existed with us for long time. Although it is from China, it can be said as significant part of Korean mathematics history in the regard that it influenced the formation of Korean mathematics. The Korean historical mathematical books, for example, *Gu-il-jip, Iksan, Chugryangdohae*, etc. emerged around 1700 (Koh & Ree, 2009). The history showed how traditional Korean lived to reveal the existence of mathematical culture. This prompted researchers to investigate Korean mathematics history. Since mathematics of the Chosun dynasty accounts for a great part of Korean mathematics history, researchers explored mathematicians, mathematical terms and literatures of Chosun Dynasty. In the beginning, they ended up introducing anecdotes or life stories of mathematicians, and various problems. They extended the boundary of research to include the inquiry of the historic-genetic principle and solutions of specific problems.

In summary, research topics of mathematics history in Korea could be categorized into Western European mathematics history, Chinese mathematics history and Korean mathematics history. Here, most researchers of Chinese and Korean mathematics history have adapted the sociocultural perspectives of mathematics and emphasized issues about the cultural aspect of mathematical thinking and reasoning. They argue that every ethnic group and nation has its own traditional mathematics and that each group has contributed to the development of the world mathematics. In particular, researchers of Korean mathematics history tried to distinguish Korean mathematics from Chinese mathematics to emphasize the uniqueness of Korean mathematics.

4.2 Educational Use of Mathematics History

In order to examine the educational use of mathematics history in mathematics textbooks, we counted the frequency of each code as the mathematical contents C1, C2, C3, C4 which are categorized into European, non-European, and multicultural connection. In the following, we describe the salient features in the use of mathematics history in Korean mathematics textbooks

TABLE 4.2: The frequency of each code in the use of mathematics history

EM 67 (48.9%)				NEM 31 (22.6%)				MC 39 (28.5%)				Total 137 (100%)
C1	C2	C3	C4	C1	C2	C3	C4	C1	C2	C3	C4	
46 (33.6%)	12 (8.8%)	7 (5.1%)	2 (1.6%)	15 (10.9%)	4 (2.9%)	11 (8.0%)	1 (0.7%)	28 (20.4%)	2 (1.5%)	4 (2.9%)	5 (3.6%)	

A. Use of anecdotes in mathematics history

In the analyis, we found that Korean mathematics textbooks use mathematics history in various ways. Among those ways of use, the most prevalent pattern of use was to introduce episodes and annnecdotes of famous mathematicians. As shown in TABLE 4.2, C1(64.9%) is the most prevalent throughout all the categories of the origins. In Korean textbook, mathematics history was most often used by introducing historical mathematicians and their mathematical achievements, artefacts, or events. They appeared on the first page of each chapter or as side readings in a chapter. This pattern of use simply list information without extending to the development of related mathematical ideas. In addition, TABLE 4.2 shows that most cases came from the history of European mathematics in this use of mathematics history.

Thus, the use of C1 may mislead students to think mathematics as an discipline that a few of genius mathematicians, especially, European male mathematicians, have produced. C1 in non-European mathematics and muticultural connection introduced students about mathematics history of non-European societies. For instance, one of the textbooks introduced non-European mathematicians and female mathematicians as well as European mathematicians on the chronology of mathematics histroy. Although C1 in combination with the history of diverse ethnomathematics may contribute to revealing students that mathematics is universal knowledge. However, mathematics educators should be cautious of whether this use of mathematics history successfully highlight the cultural facet of each ethnomathematics.

Even though the textbooks attemped to connect ethnomathematics of diverse groups, they failed to create a sound mathematical connection. For example, one of the textbooks presented coins of diverse countries and posed a problems asking the relationship between the size of the angles. This question is rarely relevant to the purpose of expanding students' mathematical understanding of geometric shapes. Also, while the textbooks introduced annecdotes and artefacts from mathematics of diverse groups, they did not successfully highlight the diversity in the way of developing mathematical concepts.

B. Prevalence of European mathematics history

The analysis has shown that European mathematics history was prevalent in mathematics textbooks. The percentage of its use is 48.9%. It means that Korean mathematics textbooks were heavily oriented toward the European mathematics knowledge. For instance, while the mathematics textbooks introduced many mathematicians, they were mostly European male mathematicians like Euclid, Pythagoras, Decartes, etc. It is important to point out that the prevalence does not simply have a quantitative meaning. Instead, it is concerned with a postion whose mathematics is representative and legitimate. For instance, when the textbooks presented tasks to require students to explore mathematical ideas in connection to history of mathematics, the tasks mostly adapted history of European mathematics to guide students' exploration. So Korean mathematics textbooks represent European mathematics as a normative of students' mathematical development. This monocultural tendency contrasts to the theoretical discourse that Korean researchers of mathematics history has emphasized the understanding of cultural identity in the research of mathematics history.

C. Use of multicultural connection in mathematics history

Although the use of European mathematics history was prevalent, Korean mathematics textbooks tried to include historical materials from diverse cultural groups. As TABLE 4.2 shows, the percentage of non-European cateogory is 22.6% and the percentage of multicultural connection is 28.5%. However, although the textbooks tried multicultural approach by introducing historical anecdotes and artefacts from ethnomathematics of various groups, the mathematical achievement of non-European groups was hardly acknowledged.

For example, some tasks presented problem solving strategies taken from the history of western mathematics and from non-European mathematics and then asked students to compare them and to write the strength of the western style strategy compared to those of non-western strategies. In this way, most tasks in multicultural connection underestimate the ethnomathematics of non-European groups and mislead students to a monocultural view of mathematics which assumes rationality and efficiency of European mathematics as standard.

On the contrary, it is necessary to note that there were good examples of multicultural connection in the use of mathematics history. For instance, there was a task that introduced different solutions of the linear equation traditionally used in China and in Korea. The task asked students to discuss what the advantages and disadvantages of the different ways of solving the problem are. The strength of this task lies in the fact that it encourages students to faily compare mathematics of diverse cultural groups and to explore their mathematical values, benefits and contributions.

In multicultural connection, the mathematics textbooks did not effectively exploit the potential of educational use of mathematics history for bringing up students' understanding of difference and diversity. The tasks presented the mathematics history of diverse cultural groups in ways that implicitly or explicitly penetrate the taken-for-granted hierarchical relation between European mathematics as superior and non-European mathematics as inferior. Thus, it is of essence to seek for ways of how to introduce history of diverse ethnomathematics as highlighting their own cultural values and help students extend their mathematical perspectives as experiencing different ways of doing mathematics.

5 Conclusion

In this research, we comparatively analysed theoretical discourse in the Korean research documents of mathematics history and educational use of mathematics history in Korean mathematics textbooks to discuss implication for mathematics education in culturally diverse school. Our analysis identified the tendency in which increasing numbers of Korean researchers adapt the perspective of mathematics as cultural knowledge. In their research of mathematics history, they have developed a position that each cultural group possesses its own unique system of mathematics and a cultural system of mathematics grows through dialogical relation with other cultural systems of mathematics. This implies that Korea has developed a unique and distinct mathematics and that Korean mathematicians contributed to the development of world mathematics. The researchers have recommended to introduce Korean mathematics history into school mathematics in order to help our students acknowledge our traditional mathematics and develop high self-esteem of their cultural heritage.

Parallel to the emergence of sociocultural discourse in the research of mathematics history, the analysis of the Korean mathematics textbooks shows that mathematics textbooks introduce materials from the history of ethnomathematics produced by diverse cultural groups. However, the analysis has revealed that there was a significant limitation in the way of using history of mathematics. Specifically, Korean mathematics history was used as a tool to provide students with mere excitement, interest and motivation. Its use rarely extended to the development of mathematical meaning. More importantly, in Korean mathematics textbooks, use of mathematics history is still largely framed by the Eurocentric perspective on mathematics. So the Korean mathematics textbooks did not fully exploit the potential of mathematics history to bring up students' multicultural sensitivity.

The analysis of this research has highlighted the significant discrepancy between the theoretical discourse of mathematics history and its educational use. This implies that it is essential to seek for ways of how to incorporate the issue of diversity and difference in educational use of mathematics history in textbooks. When history of mathematics is fairly used, it will help students appreciate the cultural facet of mathematics and acknoweldge the unique strength and weakness of ethnomathematics created by a certain cultural group. Through this kind of cultural exposure to different ways of doing mathematics, students may be encouraged to cross the boundary drawn by their own cultural background. As crossing the boundary, they may be encouraged to deconstruct and reconstruct knowledge hegemony taken-for-granted in society, and ultimately pursue freedom, equity, and peace via learning mathematics, which is the most valuable contribution that history of mathematics can make to mathematics education.

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HISTORY OF MATHEMATICS IN MATHEMATICS TEACHER EDUCATION PROGRAMS:
The development, implementation and evolution of a course

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ABSTRACT

Several countries including England, Scotland and Denmark have recently incorporated the history of mathematics in their school curriculum. This raises questions on the nature of teacher education and professional development programs in the history of mathematics courses and workshops. What are the examples of recent courses in the history for mathematics for teachers? In this paper, I present a case history of a mathematics course for teacher candidates at a Canadian University. I specifically studied the modifications which the course has undergone since its inception five years ago. The question at the center of this analysis is, in what ways has the course developed? And what are the reasons for the changes? I carried out a content analysis of the course's outlines, plans, lists of resources and teaching and assessment materials in order to study the changes and the possible reasons for the changes over the past five years. Interpretations of the results are framed by Furinghetti's framework, the role of history of mathematics in schools. I adapt the discipline of noticing as a method of inquiry. The discipline of noticing encourages teachers and educators to reflect on their practices. Results display major changes in course content, assignments and materials; while the course goals showed few changes. For example, new topics, such as, historical numbers and historical mathematics problems were introduced. Further, broader topics, such as, the evolution of specific concepts, vis-à-vis the narrower topic of history of number development were introduced. The nature of the course participants and the increased availability of multimedia texts appear to be the major reasons underlying the course changes.

Keywords: History of mathematics, teacher education, course implementation, history of mathematics-for-teachers tasks, practitioner research.

1 Introduction and Background

Several countries including England, Scotland and Denmark have recently incorporated the history of mathematics in their school curriculum in order to motivate and develop students' mathematical thinking (Marshall & Rich, 2000; Pope, 2010; Pritchard, 2010). This raises questions on the nature of history of mathematics courses and workshops offered in teacher education and professional development programs. What are the examples of recent courses in the history for mathematics for teachers? In this paper, I study modifications undertaken on a history of mathematics course for teacher candidates since its inception at a Canadian University. This paper relates to four themes of HPM 2012 meeting: Theoretical and/or conceptual frameworks for integrating history in mathematics education; History and epistemology implemented in mathematics education; Classroom experiments and teaching materials; and Topics in the history of mathematics education.

I developed a course the Historical-Conceptual Development of Mathematics, 18 hours, for elementary and senior teacher candidates in 2007 at a University in Canada. Since then, I have taught the course for three terms to an average of 20 teacher candidates who elect to take the course a term; increasingly a majority of the teacher candidates are from the secondary division. This elective course was designed to draw from the history and traditions of mathematics to inform the understanding and teaching of school mathematics. The course integrates the connections (and disconnections) between the historical and conceptual development of mathematics topics. It aims to: relate the history of mathematical concepts to sequencing topics in the school curriculum; assist teacher candidates to appreciate why concepts, such as integers, are better understood by learners in specific ways and not in other ways (say, integer multiplication in abstract ways and not in concrete ways); and illustrate the multiplicity of meanings and representations—including concrete, graphical and analytical—of several concepts. The classes are designed in the format of short lectures, problem solving activities of historical mathematics, acting out skits and plays, exploration of selected texts and documentaries, anecdotal displays on both conventional and non conventional mathematicians, researching the evolution of mathematics concepts in selected articles, and preparing creative teaching activities based on the history of mathematical concepts.

The format of the course in 2011 consisted of short lectures, problem solving of historical mathematics, acting out skits and plays, exploration of selected texts and documentaries, anecdotal displays on both convention and non conventional mathematicians, researching the evolution of mathematics concepts in selected articles, and preparing creative teaching activities based on the history of mathematics concepts.

In this paper, I study the course modifications that have occurred since the course's inception. I carry out a content analysis of the topical outlines, daily plans, readings lists and assignments in order to study how these areas have evolved over the past five years. I also share some classroom tasks. The question at the center of this analysis is, in what ways has the course evolved? And what might be the main reasons for the changes?

2 Framework

A plethora of professional and scholarly publications exist on incorporating the history of mathematics into school and university mathematics. Questions; such as why, how and what of teaching history of mathematics are addressed in the literature. Liu (2003), Bidwell (1993), Ernest (1998), Freudenthal (1981), Furinghetti (1997) outline roles of learning history of mathematics in school teaching. The roles range from students' learning outcomes such as improvement in attitudes toward mathematics; to pedagogical outcomes, such as opening up a window into studying aspects of students' mathematical thinking; through to broader humanistic roles that the history of mathematics plays in showing the ways mathematics has been practiced over time, and even further to the argument that the history of mathematics is part of mathematical learning. Further, several forms of teaching the history of mathematics are explored in the literature including biographies of mathematicians, anecdotal and visual displays of mathematicians, their works and facts about their lives, and the exploration of original works and ancient textbooks (Bidwell; Fauvel, 1991; Kaye, 2010; Pritchard, 2010; Wilson & Chauvot, 2000). Several researchers recommend incorporating the history of mathematics in the teaching of the subject matter; say as mini-lectures on any topic of interest. Despite the plethora of literature on the importance of having history in a mathematics class, Marshall and Rich (2000) identified a lack of "empirical studies that discuss the use of history to teach mathematics" (p.704). Swetz (2003) notes an increased interest in offering history of mathematics in university mathematics departments. There is need for empirical studies on classes, courses and workshop where history of mathematics forms an integral part of learning.

Swetz (2003) observes that it is important to consider the goal of exploring the history of mathematics when deciding which resources to be used say in university courses on the history of mathematics. A course instructor might choose to focus beyond the Western origins of mathematics, whereas another instructors might choose to focus on doing historical mathematics problems from varies mathematics traditions. Furinghetti (1997) offers a conceptual framework for integrating history in mathematics education. Furinghetti describes four approaches to teach the history of mathematics: a) a history of mathematics for promoting the image of mathematics; b) a history of mathematics as a source of mathematical problems; c) a history of mathematics as a different approach to concepts; and d) a history of mathematics as a different approach to mathematical concepts. Furinghetti thus links the role of the history of mathematics to how it is taught or incorporated in school mathematics and in teacher education. Furinghetti further identifies two streams of intervention of the "history of mathematics into mathematics teaching ⋯one stream is aimed at promoting mathematics, the other [is aimed] at reflecting on mathematics; the first is linked with the 'social' role of the discipline and its image, the second mainly concerns aspects interior to the discipline, such as development and the understanding of it" (p. 59). The study conducted by McBride and Rollins (1977), for instance, focuses on the changes in students' attitudes as a result of incorporating history into mathematics teaching. This study by McBride and Rollins belongs to the former category of the *social role*. D'Ambrosio (1997), on the other hand, promotes a cultural focus to the history of mathematics; he argues a cultural focus broadens students' understanding of mathematics. D'Ambrosio's focus aligns more with the latter category of the *mathematical role*, specifically the *ethno-mathematics* role. Swetz (1995) and Siu (1995) observe that exploring historical mathematical problems contributes to students' understanding of the processes, origins and development of mathematics; these researchers illustrate the latter category of the interior role to mathematics. Liu encourages incorporating the history of mathematics into teacher education because it expands teachers' mathematical knowledge. Exploring the history of mathematics in the context of the time, place, cultures and civilizations in which it was developed reveals much about the nature of how mathematics knowledge was develops and is learned (Bidwell, 1993).

3 Methods

Mason (2002) defines the *discipline of noticing*, what at other times he refers to as, *researching-from-the-inside*, as a method of inquiry for studying teaching and other practitioner practices. The *discipline of noticing* involves working towards improving one's ability to notice, mark, record, reflect, and analyze specific aspects of their professional practice. Mason encourages making a record of brief descriptions of a phenomenon while recording what one has noticed. These descriptions should be vivid and stripped off of any theory, judgment and personal views. He refers to such descriptions as *brief-but-vivid descriptions, accounts-of* an incident. These brief-but-vivid descriptions form the data to be analyzed. The data is then interpreted; the interpretations, explanations and application of theory follow after and remain separate from the descriptions. Data analysis, akin to several qualitative methods, involves searching for common themes among a collection of accounts. Mason outlines several aspects of noticing; which includes systematic reflection and the validating with others. Mason labels the reflection on the changes the *accounts*-for. To Mason and Spence (1999), systematic "reflection-on-action" promotes "reflection-in-action" (p.153). The discipline of noticing is useful at exploring more possibilities for acting when similar situations arise in the future. The discipline of noticing supports learning from experience and increases the possibilities to notice specific categories of a phenomenon.

In this paper, to study modifications which the course has have undergone since its inception, I study the documents for the course, the topical outlines, daily plans, reading lists and assignments specifications as the accounts of the course plans. I carry out the content analysis of both the 2007 and 2011 course plans to identify changes. Results of the analysis are summarized in tables. I share written plans of the classroom activities as brief-but-vivid descriptions of the course activities. The question at the center of this analysis is: in what ways have the course goals, content, teaching and assessment materials changed over the past five years? And what are the reasons for the changes?

Mason (1994) observes that noticing is what teachers and educators do all the time. But to engage in a conversation with and about what teachers and educators notice is what requires a deliberate choice on the side of the teacher-or educator-researcher. Mason developed the discipline of noticing as a practice for working with noticing. The techniques from the discipline of noticing adapted here involves marking the changes over time by comparing and contrasting documents, offering a description of what the changes are, in Mason's (2002) terms, the *accounts-of* the difference, and then reflecting on the changes. The reflection on the changes is intended to offer justifications of what I—as the educator who is researching the development of her own designed and own taught course—thought were the impetus for the changes, the *accounts-for*. It is the accounts-of and the accounts-for that are at the center of the results and their interpretations. Presenting my reflections on the course at the conference is a way of validating with others what I notice about a history of mathematic course for teachers.

4 Analysis and Results

In tables 1 to 4, I offer analyses of the content in the course's goals, content, teaching and assessment activities, and daily class plans for the first year the course was taught, 2007; as well as the most recent year the course was taught, 2011. Text that is *italicized* in the tables marks the differences between the two years. Where no differences exist I have note, —same as in 2007—.

Table 1, where course goals for the two years are compared, shows one difference between course goals in 2007 and 2011: a new goal was introduced in 2011 which emphasized the use of "understanding and reflection" from the course to "inform professional judgment in practice". All other five goals remained the same as in 2011.

Table 1: *Comparing and Contrasting the Goals of the Course*

2011 Aims and Goals	**2007 Aims and Goals**
The course aims to 1. —Same goal as in 2007— 2. —Same goal as in 2007— 3. —Same goal as in 2007— 4. —Same goal as in 2007— 5. —Same goal as in 2007— 6. *Encourage understanding and reflection on student development, learning theory, pedagogy, curriculum and research on the history of mathematics, and to use such understanding and reflection to inform professional judgment in practice.*	The course aims to 1. Develop understanding of the evolution of mathematical ideas historically. 2. Explore the implications of this evolution on conceptual development and across grades. 3. Explore the implications of this evolution on sequencing of topics, selecting teaching models, and representations for teaching mathematics. 4. Introduce a range of strategies for teaching mathematics in ways connected to history, culture and new discoveries in mathematics. 5. Demonstrate how history of school mathematics is usefully integrated in school mathematics.

Table 2 shows a comparison of course content of the two years. Four differences are evident: (a) an introduction of an overarching topic, Topic 1 on *Introduction, overview and role of history of mathematics* in 2011; (b) a broadening of Topic 1 (2007 column) to include the *history* ⋯*of geometry* as well; (c) an amalgamation of topics 2, 3, 4 and 5 (2007 column) on specific concepts into two general topics, *Historical mathematics problems* and *Evolutions of specific concepts*; (d) the separation of the 2007 Topic 8 *Mathematicians in School Mathematics* into *Selected mathematicians' stories, lives and works* and *Women, prodigies and other interesting mathematicians*; as well as a shift from focusing specifically on, say, *Fractal Geometry* ⋯*Euclidean and non-Euclidean Geometries* to describing a general focus on *Selected new mathematics discoveries* . Further, a new topic was introduced in 2011, Topic *8, a brief history of school mathematics*.

Table 2: *Comparing and Contrasting the Course Content*

2011 Course Content	**2007 Course Content**
1. *Introduction, overview and role of history of mathematics* 2. History of Number and Geometry in teaching and learning 3. Western and non Western roots of the school mathematics 4. *Historical mathematics problems* 5. *Selected mathematicians: stories, lives and works in school mathematics* 6. *Evolutions of specific concepts e.g. History of zero, evolution of functions, and times of multiplication* 7. *Women, prodigies and other interesting mathematicians* 8. *Selected new mathematics discoveries and a brief history of school mathematics*	1. History of Number developments 2. Western and Non western roots of Mathematics 3. *Zero and Integers—Chinese rods and relation to use of colored chips* 4. *Multiplication—development across grades and variety of algorithms* 5. *Fractions—Egyptians fractions and other Fractions* 6. *Babylonian Algebra—polynomial, Geometry and the Polykit and Algebra tiles* 7. *Fractal Geometry, landscapes as Fractals, Euclidean and non-Euclidean Geometries* 8. *Mathematicians in School Mathematics — Gauss and number theory; Pythagoras theorem, its origin, proofs and the internet* 9. *Patterning: Pascal; Fibonacci and number sequences*

Table 3: *Comparing and Contrasting the Course Materials*

2011 Course Materials	**2011 Course Materials**
The course texts will include: A. Selected chapters from history of mathematics books, textbooks and magazines. Examples include: a. Joseph, Ghever Ghese G. (1991). *The crest of the Peacock: Non-European roots of mathematics*. London: Penguin Books. b. Burton, David M. (2002). *The history of mathematics: An introduction*. McGraw-Hill Math. B. Selected scholarly and professional/teacher journals articles. Examples include: a. Using history in mathematics Education. *For the Learning of Mathematics, 11*, 3–6. b. Liu, P. H. (2003). Do teachers need to incorporate the history of mathematics in their teaching? *The Mathematics Teacher, 96*, 416–421. C. *Selected web pages on the history of mathematics. Examples include:* a. *The History of mathematics—BBC documentaries* b. *History of Mathematics Wikipedia page*	The course texts will include: A. Selected chapters from books such as Chapter 1, Joseph, Ghever Ghese G. (1991). *The crest of the Peacock: Non-European roots of mathematics*. London: Penguin Books. B. Selected chapters from textbooks such as Howard, Eves (1990). An introduction to the history of mathematics, 6th Edition C. Selected articles from Professional/teacher journals such as Jarvis, D. (2007). Mathematics and the visual Arts: Exploring the golden ratio. *Mathematics Teaching in the Middle School, 12*, 467–471. D. Selected sections of Scholarly articles such as Section 1 and 6 of Furinghetti, F. & Radford, L. (2002). Historical conceptual developments and the teaching of mathematics: From phylogenies and ontogenesis theory to classroom practice. In: L. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 631–654). New Jersey: Lawrence Erlbaum. Available online E. Selected websites such as The MacTutor History of Mathematics archive `http://www-groups.dcs.st-and.ac.uk/\~{}history/` Accessed January 4, 2007 F. Curriculum guidelines G. Curriculum Textbooks

The major change evinced in Table 3 is a move towards an increased use of nontraditional texts, especially to include visual-audio texts and texts published online. There was a resulting change of a reduction in the number of textual course materials listed.

Table 4: *Comparing and Contrasting the Course Assignments*

2011 Comparing and Contrasting the Course Assignments	**2007 Assignments and Other Course Requirements**
I. Attendance, classroom contribution and individual participation including small group work. *This will contribute 30% towards the final grade.* II. *One major assignment. You have a choice among A, B and C. This major assignment contributes 50% towards the final grade.* A. Mathematician assignment—Select a mathematician; research and write about one of his/her works and its relevance to school mathematics. Up to THREE pages double spaced. OR, B. Topical assignment—Select a mathematics topic or concept (e.g. integers, functions etc.); research and design a poster display that presents some of its historical, conceptual or curriculum (across grades) development. *Electronically designed posters are preferred but the instructor is open to other forms.* OR, C. *Small group performance project—in a group of 2 or 3 please plan a performance project that can be performed in class. The project may take the form of a lesson idea, a skit, a short video clip or any other agreed upon form. Samples will be shown in class.* III. *Presentations of the major assignment in seventh, eighth and ninth class. The presentation will contribute 20% towards the final grade.*	I. Classroom contribution and participation including homework, reading notes and assigned readings 25% II. Two written assignments, 2 pages double spaced each. • Topical assignment 25%—Select a mathematics concept (e.g. integers, fractions, angles etc.), research and write about some of its historical, conceptual or curriculum (across grades) development. • Mathematician assignment 25%—Select a mathematician and research and write about one of his works and it's the relevance of to school mathematics. III. III. A take home examination—25%

Three differences are seen in Table 4 which compares the assignment sections of the course: a) an introduction of a performance project; b) the provision of choice among the assignments on a mathematician, on a topic and a performance project; c) the inclusion of classroom presentation of the major assignment; d) the replacement of the take home assignment by classroom presentations based on a students' major assignment in the 2011 column.

The daily plans changed due to the changes in topical outline and due to the change in nature of participants. A majority course participants were from the IS division in 2011. This made it difficult to compare the class plans of the two years. I only carried out a holistic comparison of daily plans in 2007 as compared to those in 2011. In summary, the analyses of the class plans show more course readings are done in the context of in-class activities in 2011. Much of the teaching content and materials are still the same in 2011 as in 2007, but some content (say, evolution of specific concepts) is covered more in detail than other content (say, history of number) in 2011. Several more online, audio, and video texts are used during class in 2011, and more time in class is planned for students to present their work and assignments.

To exemplify the teaching materials, I share these in the context of selected tasks used in class. These are presented in Figures 1 to 5. These task descriptions, showing the topic, materials, supplemental materials, school classroom relevance and specific tasks, are adapted from the daily class plans. Figure 1 is an example of a task in which I use online video clips from a 4 hour documentary, *The story of maths*. Students listen to an archived radio program on the story of two mathematicians, one from the European mathematics tradition and the other from the Indian mathematics tradition, G. H. Hardy and S. Ramanujan in the task shown in Figure 2. Figure 3 displays a task in which part of the video recorded lectures on *Historical numbers, The Story of Euler's e* is utilized. Figure 4 describes the task in which students, using a play script and additional materials, prepare to act a play based on a mathematician, Évariste Galois. Figure 5, is an example of a classroom activity which involves reviewing selected articles to explore the life and times of a concept; in this case the multiple meanings and representations of the concept, in this case functions. It is worth noting that course participants submit exemplary assignments. These are also presented in class. Here I only mention assignment projects. These include but are not limited to:

- A short 5 minute video on conversations between Robert Hooke, Gottfried Wilhelm Leibniz and Sir Isaac Newton. This video centers on the Feud between Leibniz and Newton
- A comic strip based on selected mathematicians,
- A manuscript on Zero and another on Paul Erdös,
- A script of a skit centered on Emilie Du Chatelet, Hooke, Leibniz and Newton, and
- A brochure on Fractals, a song performance based on selected mathematician and mathematics concepts, and a poster on prime numbers.

Topic: Western and non Western roots of math

Materials: History of Math documentary by Prof. Marcus D Sautoy, which is organized by mathematics tradition. Sautoy, P. M. (Director). (2009). The story of maths [Motion Picture]. Also available at `http://topdocumentaryfilms.com/the-story-of-maths/`

Task: In groups of 3 we are going to watch the first 14 parts covering 7 early mathematics traditions. This is going to take us up to 25 minutes per group

Group 1: Part 3, 4 (stop at 7 minutes)—Babylonia –17 minutes
Group 2: Part 4 (at 7 minutes), 5, 6 Greece –25 minutes
Group 3: Part 7, 8—China—18 minutes
Group 4: Part 8 (start at 8 minutes), 9, 10 –Indian –22 minutes
Group 5: Part 10 (starts at 8 minutes), 11 —Islam –12 minutes
Group 6: Part 11 (Start at 7 minutes), 12, please skip 13, we viewed it.
during the previous class, 14—West and Europe – 23 minutes

(You may view parts 15 onwards in your spare time.)

Please view the video part indicated for your group with an intention to list on the chart paper provided

- some stories, ___________________________
- pictures or video clips, ___________________________
- math problems, topic or examples, ___________________________
- mathematicians ___________________________
- or general themes ___________________________

Supplementary materials: You may supplement the information provided in the videos with published literature from the selected chapters and chronological charts and maps provided from books such as Joseph, Ghever Ghese G. (1991).

Classroom Relevance: These stories may be narrated, shown, connected to, mentioned or simply utilized when teaching mathematics in your future classroom.

Figure 1. Use of a documentary in a history of mathematics-for-teachers task.

Topic: Selected mathematicians: stories, lives and works in school mathematics: G. H. Hardy and S. Ramanujan, A mathematical romance

Materials: Archived Radio Program,

Sautoy, P. M. (Composer). (2011). A brief history of mathematics: Hardy and Ramanujan. [BBC Radio, Performer, & P. M. Sautoy, Conductor] London, UK. Accessed Feb 2011 at `http://www.bbc.co.uk/iplayer/episode/b00ss1j4/A\_Brief\_History\_of\_Mathematics\_Hardy\_and\_Ramanujan/` (14 minutes)

Supplementary materials: Selected sources include Google timelines and Google images; as well as selected pages from books such as Pickover, C. A. (2009). *The Math Book, from Pythagoras to the 57th dimension 250 milestones in the history of mathematics*. New York: Sterling.

Figure 2. Use of an archived radio program in a history of mathematics-for-teachers task.

Topic: Historical numbers: The Story of Euler's e

Materials: A Video Lecture, Burger, B. B. (2007). Science and mathematics, Part 2, history of numbers. (Taught by Prof. Edward B. Burger). Chantilly, VA: Teaching Company.

Task: We are going to view the lecture by Burger and review the pages provided in order to write a brief one paragraph story about the evolution of the number e

Figure 3. Use of a video recorded lecture in a history of mathematics-for-teachers task.

Topic: Selected mathematicians: stories, lives and works in school mathematics: Évariste Galois—A play

Materials: (1998).The life and times of Évariste Galois: A play in four scenes by the second-year class of Mariono Moreno School, Argentina. *Mathematics in Schools* (September), 12–13.

Task: Half of the class is going to prepare to act out a play about the life of Évariste Galoise Please review the script provided and prepare to act.

In groups of 4, the other half of the class is going to review the additional materials on Évariste Galoise provided online, at the BBC I player website, and selected pages provided.

Figure 4. Use of plays and drama in a history of mathematics-for-teachers task.

The Life and Times of Functions: f, $f(x)$

Materials: 4 readings that give the time line and mathematicians involved in the life and times of, $f(x)$ are:
(a) Earliest uses of function symbols `http://jeff560.tripod.com/functions.html` ; and
(b)The history of the concept of functions and some education implications published in The Mathematics volume 3 Number 2. Also Available at PROQUEST database (c) Historical and pedagogical aspects of the definition of a Function by M A. Malik published in the *International Journal of mathematics education, science and technology, volume 11 number 4, pp. 489–492* also available at PROQUEST Database; (d) Functions (mathematics) at Wikipedia.
[In your groups (in 2011) or at home (in 2007)], please read one of the above articles assigned to you. You may review and complete the chart. We shall take up the chart together as a whole class.

Different meanings Today
What do functions mean for you?

1. ____________
2. ____________
3. ____________

Where is the function in the sequence 2, 4, 6, 8, ⋯⋯?

Functions were not explored by Ancient Math Traditions
But some of their mathematics areas might look like functions. Name three such areas

- ____________
- ____________
- *Cubics and cubic roots*

The **Greeks** studied distances and time but not speed, a relation between these two varying quantities.

Arabs and other earlier traditions did not study motion.

Euclid studied geometry of points, lines and planes by construction, devoid of any motion and formulae

Evolution of functions
The evolution of functions in mathematics is a recent concept dating to the end of the **17th century**

Early notion of Function 17th and 18th century

Galileo (1564-1642)—(in his quantitative study of Nature with **Kepler (1571–1639)** measured quantities and sought to identify patterns and regularities. The study of falling bodies, motions of planets, motion along curves lead to the rigorous study of proportions, polynomial and trigonometric equations

Descartes (1596–1650) and Fermat (1601–1665) introduced analytic geometry—curves in a plane described by equations.

Descartes stated an equation in two variables & geometrically represented it by a curve. He used this to show dependence between variable quantities (x, y). He described curves by motion/locus and formulae.

Newton (1642–1727) —showed functions in infinite series. He devised some terminology: Fluent for independent variable, *relata quantitas* for dependent variables and *genita* for constants

USE-Explore-Define-Apply

D' Alembert (1717–1783)—vibrating string theory; Fourier (1768–1830)—heat flow in material bodies.

Dirichlet (1805–1859)—studied **Fourier series** defined function as a <u>unique correspondence</u> between variables representing numerical sets thus separated it from analytic representation by formulae.

This fact is contested by George Hardy who reviewed Dirichlet's work and did not find any mention of this by Dirichlet.

Dirichlet also worked with functions that could be <u>discontinuous</u> at some points. As well this definition was for a long time rejected for being too broad.

Cantor (1845–1918)—his work on set theory contributed to the definition by correspondences between sets, <u>numerical or non numerical</u>

Caratherdory in 1917 defined function as <u>a rule</u> of correspondence from set A to real numbers.

Functions have origins in intuitive geometry and intuitive **calculus**

To begin with functions were used to designate correspondence between geometric objects e.g. curve and its quantities, the slope, tangent, gradient, the area under a curve, limits, etc.

What other contexts or topics are functions used in

____

Viete (1540–1603) with his influence in the creation of **Algebraic symbolisms** increased expressive possibilities in mathematics.

Graphing calculators and computers might be very helpful when studying functions.

Discuss the assertion in light of the evolution of functions

____

Origins of function terms:
Set theory—Domain-range;
Computing—algorithm, input, output; function machine;
Cartesian geometry— formulae, plots, gradient and graphs
Calculus—______________
Applied mathematics—tables of values; t-tables; differences;

PTO

Modern function Definitions

The **word function** is traced back to **Gottfried Leibniz (1647–1716) in 1673 (others say 1694)**. He used it in Calculus to describe quantities related to curves (graphs with no corners), specifically the gradient/ slope of a curve at a point.

Bernoulli (1718) defined function as a quantity composed from variable and constant.

More study of curves by the use of algebra and the use of analytical expressions became necessary thus fueling the need for function as a tool.

Leibniz and Jean Bernoulli, in their correspondences, adopted function to further study curves by using algebra

In mid 18^{th} century, the word function would later be used by **Leohard Euler (1707–1793), a former student of Bernoulli** to describe an expression or formulae,

PTO

Hardy in (1908) defined a function in the modern terms as "**a relation between two variable x and y such that to every value of x and any rate correspond values of y.**"

Bourbaki in 1939 defined function as a rule of correspondence between two sets.

Later in computation theory a function came to be understood as a computation.

In computing science as an Algorithm—including recursive functions/ spread sheet functions or non-analytic functions such as the one that maps natural numbers onto 0, 1, 1, 2, 3, 5, 8, ⋯.

Partial functions defined as functions for which some x-values have no y-values defined.

How is a function defined in the curriculum?
Narrowly or broadly?

____

Clearly functions are now formal mathematical objects and physics tools. The function concept moved away from its origin in calculus.

PTO

Algebraic— equations, expressions, or generalizations; **Motion Geometry**— Transformation, mappings; object and image? *I recall learning functions in the context of domain and range, mappings in grade 8, what earliest meaning of functions do you recall learning?* __________________________	**Euler later** proposed an alternative definition to broaden the definition—but this was ignored in favor of the analytic expression definition for the whole of the 18^{th} century. Beginning the 19^{th} century onwards definitions would be elaborated especially by **scientist/mathematics who sought its use in other theories**. **As well** the impetus to formalize of mathematics using set theory during the 19^{th} century helped. It is claimed; **Dirichlet and Lobachevsky** broadened and formalized the definition of a function. Now functions are used in many areas including Dynamical systems—partial differential equations, topology, and probability functions in stochastic models.	*What other Mathematics topics and symbols is* ***Leibniz*** *famous for* __________________________ *How many years did it take between early inventions and more formalized acceptance of functions?* __________________________

Lessons for teaching (generated from classroom discussion)
Mathematics concepts evolve over time.
Meanings and representations of concepts vary—some are more elementary than others.
Studying several contexts for functions (e.g., sets, machines, speed and velocity, graphing/analytic geometry, series) might guide us in not mixing meanings of a concept when teaching it.
In mathematics, multiple representations might help get to several meanings of a concept.
Rigor and generalizability were valued during later traditions of mathematics.
Need/or relevance to use a concept influences recognition or acceptance of a concept.
Practical uses of mathematics are important as mathematical and scientific uses of mathematics.

Figure 5. An example of a task to illustrate the origin of multiplicity of meaning of functions.

5 Interpretations

In this section I account for, to the extent that it is possible, the possible catalysts for the differences shown among the course aims, content, teaching and assessment materials over the course of the 5 years. My reflections are first and foremost informed by the literature reviewed and theoretical framework identified. But they are also informed by my experiences teaching the class and by the survey and written feedback solicited from the students at the beginning of the term, half-way through the term and at the end of the term. At the beginning of the course, in 2007 and 2011 I requested students to complete a questionnaire about themselves and their expectations from the course. Half-way through the term I solicit for written feedback on the course, and at the end of the course the students complete university wide course and instructor evaluation. Teacher candidates' abilities and expectations as well as the availability of a variety of useful and captivating resources appear to have been the major impetus for the changes.

The difference between goals was primarily in order to match the program requirements which were introduced since 2011. The goal in 2011 focused on professional judgment in practice. The reduction in course readings and the coverage of more readings in class were another difference accounted for in the 2011 program: Elective courses were to reduce the amount of readings and assignments offered beginning 2010.

Several difference show changes to do with the course itself. The difference in course content reflects a move toward the amalgamation of topics on specific mathematics concepts such as evolution of functions toward general topics such as on the evolution of specific topics. On the other hand, the separation of a topic on mathematicians into mainstream mathematics and other mathematicians was evoked by the need to explore mathematicians that represented various groups of course participants including but not limited to gender, race and age. Also, the introduction of women, mathematicians from other cultures and prodigy mathematicians was intended to introduce the cultural and social justice aspects into the course. The introduction of the new topic historical mathematics problems such as those on prime numbers, equations, matrices, and the golden ratio, was evoked by the possibility of exploring mathematics in historical texts that are of direct relevance to school mathematics. Along these lines of relevance to school mathematics, a new topic on the history of school mathematics was introduced. This topic was intended to make connections between the history of mathematics and pedagogical and curriculum reforms supported by the Ministry of Education and School Boards in the province.

The experiences teaching the course for two years meant that I increasingly, through my explorations as well as my interactions with students in the course and with history instructors, became aware of nontraditional texts and non-traditional approaches in the area. Also, some new texts were published after the first year of the course. For instance, the story of mathematics documentary was produced in 2009. Further, students' ongoing informal feedback during and after classes, and formal written feedback half-way during the course and at the end of the course indicated that teacher candidates, given their overall course load, found less traditional course materials to be more captivating. The inclusion of multimedia and online texts is at the same time accounted for by the increase in non-traditional texts readily available online as well as improvement in classroom technologies available at the university over the past three years.

The differences seen in the assignment materials of the course are mainly due to the three reasons: to reduce on the course load for students, to give choice for the students to explore topics of interest, and to leave room for the students to use formats of interest, especially those that were in line with the non-traditional texts explored in this class and in other mathematics methods classes.

6 Conclusions

My sense is that the changes implemented, since the course was introduced, have made the course more interactive for the students and has given them more choice to explore their topics of interest in formats that are more appealing to them. Several of the students who still find course readings and assignments that are in the format of manuscripts appealing still get the opportunity to use these formats. Most importantly, the results show that the course is evolving from the social role of using history for the purposes of promoting mathematics and improving on its image towards the mathematical role of encouraging teacher candidates to reflect more an understanding of mathematics, its origin and history more deeply (Furinghetti, 1997). The mathematical role of teaching the history of mathematics appears more appropriate for a course in which a majority of the members who elect to take the course are increasingly those that have interest, comfort and strengths in mathematics. Should the course have consisted of majority class members who profess discomfort and lack of strength in mathematics, the course's focus would have been interventions, using the history of mathematics, as described by Kaye (2010). Further, the inclusion of more diverse texts and the amalgamation of topics is supported by the possibility, as noted by Pritchard (2010), of the availability of thoughtful radio, television, and other multimedia texts on the history of mathematics. A major lesson for me from this analysis is that although the course goals remained the same, an instructor's interaction with the available resources and with the students was the key at forming modifications in the written plans of the course.

For further inquiry, it would be interesting to assess the course on the basis of the activities, resources, approaches, foci that have been noted in the literature on the use of history of mathematics in education. Further reflections on this, in addition to a content analysis of the course as planned, could also focus on the course as taught and experienced by the students.

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THEORETICAL FRAMEWORK CONCERNING DRAWING METHOD IN MATHEMATICS EDUCATION

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ABSTRACT

本稿においては，西洋数学及び日本の和算における正五角形の作図法に関しての考察とそれが作図可能であることについてのガロア理論に基づく証明及びそれに基づく作図のアルゴリズムについて述べる．具体的には，西洋数学としてトレミーによる作図法，日本の和算として平野喜房，藤田貞資及び安島直円による作図を示す．さらに，正五角形の作図可能性の証明については，円分体の手法による証明を与える．

1 はじめに

定規は線分を描きとコンパスは円を画くことのみに制限した作図は，古代ギリシャにおいて既に考えらえていたが，いくつかの作図不可能な問題について，事実として実際に作図不可能であることが分からなかったために難問とされた問題がいくつかある．その中で有名なものとしては，一般角の三等分の作図と円周率の作図及び 2 の作図の 3 つである．一方，作図可能なものについても古い時代から色々知られていた．例えば，角の 2 等分線の作図や線分の垂直 2 等分線の作図及び与えられた長方形と同じ面積を持つ正方形の作図があげられる．これ以外にも，正五角形の作図があげられる．角の 2 等分線が作図可能なために，正2^n×3 角形，正2^n×5 角形，正2^n 角形の作図も当時から分かっていた．学校数学においても色々な作図の指導がなされ，その中でも正五角形の作図は興味深いものがある．

実際に，子どもたちの中にはすべての正多角形が作図可能なのであろうかと興味を持つ子どもたちもいる．ギリシャ時代には，上で示した以外の正多角形の作図はできにと思われていたと思われる．実際，ガウスは，その時代にガロア理論は出来ていなかったために，1 の原始 p 乗根のみたす相反方程式の根の様子を調べてその結論を得ている．現在では，ガロア理論に基づいて，正多角形の作図可能性の証明が可能となっている．そこで，本稿においては，正五角形の作図可能性の証明をすることとする．

2 和算に対する評価

和算を現在の日本の学校の数学の授業で扱うことは，決して，不可能なことではない．ただ，それほど単純で簡単なものでもない．和算について，一度でも学習した学校教員であれば，その量の多さと質の高さに驚くのが普通である．学校数学においては，小学校の算数から高等学校の数学まで，どの単元であっても論理的には，利用可能である．当然のことではあるが，現在の日本の学校教員の多くは，明治時代と違って，残念ながら和算家ではない．したがって，和算の中のほんの一部の知識しか知らないために，現在の学校数学に和算を導入する際に必要以上に多くの苦労をせざるを得ない．それでも日本の小学校の算数科のすべての教科書会社の教科書に，和算の内容が一部取り入れられている．実際に，和算が流行した江戸時代には，数学嫌いの子どもは本当に少なかったと言われているこ

とは誰でもが知っている．現在の日本の子どもたちの数学嫌いと比較しても和算には，圧倒的な魅力を感じる．和算を現在の子どもたちのために，編集し直して，改めて現在を生きる子どもたちのための本をつくることができれば，和算は現在の日本の学校数学においても，有効であると評価することが可能であると思われる．

3　西洋数学と日本の和算家による正五角形の作図

本稿においては，正五角形の作図方法に関して，西洋の数学による方法としてトレミーによる作図法と日本の和算家によるその作図法として平野喜房と藤田貞資及び安島直円によるものを紹介する．また，正五角形が作図可能であることの証明として，現代数学による手法であるガロア理論に基づいた証明方法を紹介する．具体的には，円分体を利用した証明である．これは，幾何学の問題を代数学の手法によって証明したことにおいても意味がある．また，この証明方法に基づいた正五角形の証明は，数学としては珍しく計算そのものが興味深いと言う数学的意味もある．そこで，歴史的に古いが興味深い西洋及び日本の江戸時代に発達した和算による正五角形の作図法を示すとともに，その証明を幾何学の手法ではなくて代数学であり現代数学であるガロア理論を用いて証明した．

4　西洋数学による正五角形の作図の実際

4.1　トレミーによる正五角形の作図法

トレミーによる作図法は，「アルマゲスト」の中で「原論」第XIII巻命題10を引用して説明している．

以下，トレミーによる正五角形の作図法について述べる．

図1の様に，線分AB，線分CDは，円Oの直径である．また，線分ABと線分CDは，その円の中心Oで，互いに直行しているとする．さらに，円Oの半径CDの中点をEとする．線分EAを半径とする円と円Oの半径ODとの交点をFとする．この時，線分AFは，この円Oに内接する正五角形の辺の長さになる．実際，図2において，中心Aで半径AFの円がもとの円Oとの交点を点G，点Hとする．次に，点Gを中心とし半径GAの円が円Oと点Jで交わるとする．さらに，点Hを中心とし半径HAの円と円Oとの交点を点Kとする．5つの点，点A，点G，点J，点K，点Hを図3の様に結ぶと正五角形が得られる．

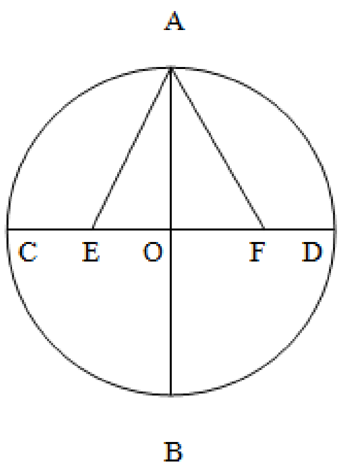

図 1

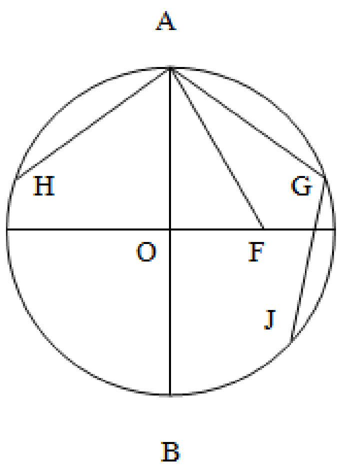

図 2

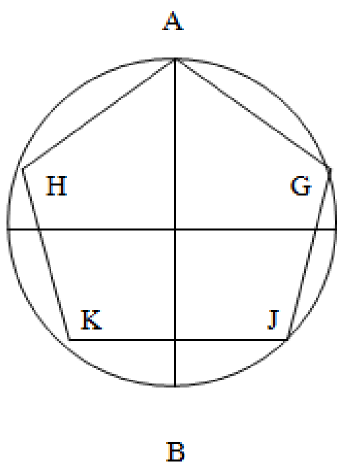

図 3

4.2 「原論」の正五角形の作図法

「原論」第Ⅳ巻命題 11 は，「与えられた円に正五角形を内接させること」という作図命題である．命題 10 は，「底辺における角の双方が残りの角の 2 倍である二等辺三角形をつくること」となっていて，正五角形の作図命題の準備となっている．命題 11 は，命題 10 の作図を行うようになっていて，正五角形の対角線と辺でつくられる三角形は，頂角が 36° の二等辺三角形になるので，これと相似な三角形を円に内接するように作図するとそれを基に正五角形を内接させることが可能であるという内容である．

正五角形は，辺と対角線で決まる．したがって，円がなくても辺と対角線があれば作図可能である．正五角形の辺a と対角線b との間には，$b(b-a)=a^2$ という式が成り立つので，これをb の 2 次方程式として解くと，
$b=\frac{\sqrt{5}}{2}a+\frac{1}{2}a$ となる．
したがって，与えられた辺a に対して，2 等分の線分$c=\frac{1}{2}a$ の$\sqrt{5}$ 倍の線分$\sqrt{5}c=\frac{\sqrt{5}}{2}a$ を作図すれば，対角線$\sqrt{5}+c$ の長さが求められるので，正五角形の作図が可能となる．

5 日本の和算家による正五角形の作図法の実際

本章においては，日本の 3 人の和算家平野喜房，藤田貞資，安島直円による正五角形の作図法について紹介する．

5.1 日本の和算家平野喜房による正五角形の作図法

平野喜房による正五角形の作図法は，次の通りである．

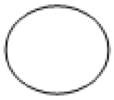

図 4

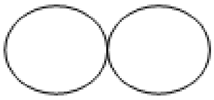

図 5

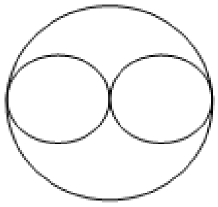

図 6

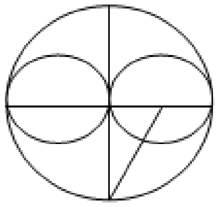

図 7

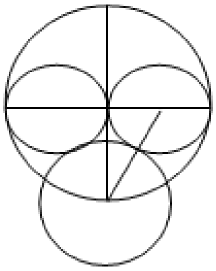

図 8

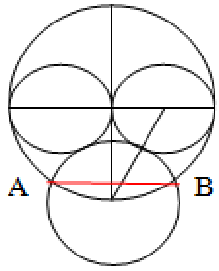

図 9

① 図 4 の様に，任意の大きさの円を画く．

② 図 5 の様に，図 1 で描いた円と合同なもう 1 つ円を画く．
この時，2 つの合同な円は，外接する様に描く．

③ 図 6 の様に，大きな円を画く．
この時に，図 2 で画いた 2 つの外接する円に外接する様に描く．

④ 図 7 の様に，3 つの円の直径となる様な線分を画く．
さらに，2 つの合同な円の共通外接線を大きい円の直径となる様に線分を画く
さらに，合同な 2 つの円の一方の中心と 2 つの合同な円の共通外接線と大きな円との交点を線分で結ぶ．

⑤ 図 8 の様に，もう一つ円を画く．
この時，外接している 2 つの合同な円の共通外接円となる様に画く．

⑥ 図 9 の様に，線分 AB を引く．
この時，図 5 で画いた円と外接している 2 つの合同な円の共通外接円との 2 つの交点を結ぶ線分になっている．そして，2 つの交点を A，B とし，画いた線分を線分 AB とする．そうすると，線分 AB が正五角形の 1 辺となっている．この様にして，正五角形を作図することができる．

ここで，以下にこの平野喜房による正五角形の作図法が正しいことを証明する．

証明

図 10 における∠DBA が 18° となることを示せば良い．図 10 にける最も大きい円の半径を 1 としても一般性を失わない．

そうすると，$DC = \sqrt{5}/2$，$DB = DC - 1/2 = (-1 + \sqrt{5})/2$

$$\sin(\angle DEB) = DB/DE = (-1 + \sqrt{5})/4$$

よって，$\angle DEB = 18°$

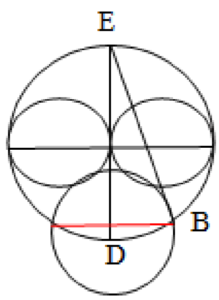

図 10

5.2 日本の和算家安島直円による正五角形の作図法

安島直円による正五角形の作図法は，次の図 11 の通りである．

① 2 本の線分をお互い垂直になる様に画き，その交点を A とする．
② 交点 A を中心として，半径が AB となる様な円を画き，①で横に引いた線分との交点の一方を点 C とする．
③ 点 B を中心にして，半径 BA の円を画く．
円 B と縦の線との交点を点 D とする．
④ 点 D を中心にして，半径 DC の円を画く．
円 D と縦の線との交点を点 E とする．
⑤ 点 B を中心にして，半径 BE の円を画く．
⑥ 円 B と横の線との交点を点 F，点 G とする．

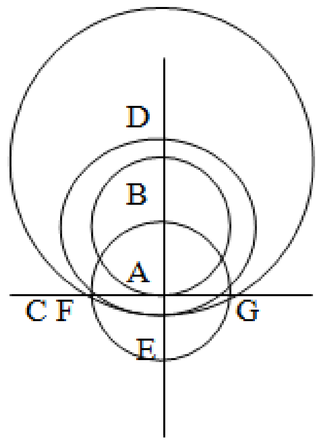

図 11

6 円分体による正五角形の作図可能性の証明

本章においては，ガロア理論に基づいて正五角形の作図可能性に関して証明する．
正五角形の作図と円周の 5 等分の問題は同値なので，円周の 5 等分が作図可能であることを証明する．

$360° \div 5 = 72°$ より，$\omega = \cos 72° + i \sin 72°$ とおく．
また，拡大体を L=K(ω) とおく．
ここで，ω の最小多項式を $f(x) = x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$ となる．
この時， L は，$f(x)$ の最小分解体となる．
ここで， L を K 上のベクトル空間とすると，
$\{\omega, \omega^2, \omega^3, \omega^4\}$ は，ベクトル空間の基底となる．
L 上の任意の要素を b とすると，$b = a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4$ となる．
ただし，$a_i \in$ K.
1 の n 乗根の性質より，$\omega + \omega^2 + \omega^3 + \omega^4 = -1$ となる．
拡大次数は，$[\mathrm{L} : \mathrm{K}] = 4$ となるので，ガロアの位数は 4 となり，
$\{\omega, \omega^2, \omega^3, \omega^4\}$ を置換する要素から構成される．
ここで，$\sigma(\omega) = \omega^2$ と置くと，
$\mathrm{G(L/K)} = \{1, \omega, \omega^2, \omega^3\}$ と表現され巡回群となる．

G(L/K) は，正規部分群$\{1, \sigma^2\}$ を持ち，正規真部分群は，これしか存在しない．
ただし，自明な部分群は除く．
ガロアの基本定理より，K と L に中間体が存在する．
それを M と置くと，中間体 M は，群 H の固定体となる．
そこで，係数の条件を以下の通り求める．

$$\begin{aligned}\sigma^2(b) &= \sigma^2(a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4)\\ &= a_1\sigma^2(\omega) + a_2\sigma 2(\omega^2) + a_3\sigma^2(\omega^3) + a_4\sigma^2(\omega^4)\\ &= a_1\sigma(\omega^2) + a_2\sigma(\omega 4) + a_3\sigma(\omega^6) + a_4\sigma(\omega^8)\\ &= a_1(\omega^4) + a_2(\omega^8) + a_3(\omega^{12}) + a_4(\omega^{16})\\ &= a_1\omega^4 + a_2\omega^3 + a_3\omega^2 + a_4\omega\end{aligned}$$

したがって，$\sigma(b) = b$ となるのは，$a_1 = a_4$ の時である．
また，ω の指数が 5 より大きい時，$\mod 5$ で$\{1, \omega, \omega^2, \omega^3, \omega^4\}$ にもどる．
よって，中間体 M の要素 α は，$a_1 = a_4 = b_1$，$a_2 = a_3 = b_2$ とおくと，以下の通りに書ける．
$b = b_1(\omega + \omega^4) + b_2(\omega^2 + \omega^3)$ とおく．さらに，$\omega + \omega^4 = \alpha$，$\omega^2 + \omega^3 = \beta$ とおくと，

$$\begin{aligned}\alpha + \beta &= \omega + \omega^2 + \omega^3 + \omega^4\\ &= -1\\ \alpha\beta &= (\omega + \omega^4)(\omega^2 + \omega^3)\\ &= \omega^3 + \omega^4 + \omega + \omega^2\\ &= \omega + \omega^2 + \omega^3 + \omega^4\\ &= -1\end{aligned}$$

ここで，2 次方程式の解と係数の関係より，α とβ は，2 次方程式$x^2 + x - 1 = 0$ の解となる．したがって，

$$\alpha = \frac{(-1 + \sqrt{5})}{2}, \quad \beta = \frac{(-1 - \sqrt{5})}{2}$$

となる．ただし，α とβ の符号は，下の図 12 の様に，$\{\omega, \omega^2, \omega^3, \omega^4\}$ の関係から決定する．
以上のことから，$M = K(5)$ となる．
したがって，M 上の代数方程式を考えると，ω, ω^4 は，方程式$x^2 - \alpha x + 1 = 0$ の解となり，
ω^2, ω^3 は，方程式$x^2 - \beta x - 1 = 0$ の解となる．α が作図できると，円の中心 O とα との中点から垂線を引くと，頂点ω, ω^4 が求められるので，円周を 5 等分することができる．

7　まとめ

本稿においては，正五角形の作図方法と現代数学に基づいたその証明の可能性について述べてきた．西洋の数学としては，ユークリッド原論やトレミーによる正五角形の作図方法について述べてきた．また，日本の 3 人の和算家平野喜房，藤田貞資，安島直円による正五角形の作図法について述べてきた．日本の和算家による，正五角形の作図方法は，その正当性を証明するとなると西洋数学には無い独自の文化によって形成されてきたものであるために，西洋数学の様な証明はなされてこなかった．しかし，その作図方法は，独創性があり見事と言うべきものであることが分かった．
最後に，現代数学の手法として，ガロア理論とりわけ円分体の理論に基づいて正五角形の作図可能性について考察してきた．通常は，この様な計算そのものは，興味の持たれないものではあるが，正五角形の作図可能性の証明に限っては，計算そのものが興味深い珍しい例であることが分かった

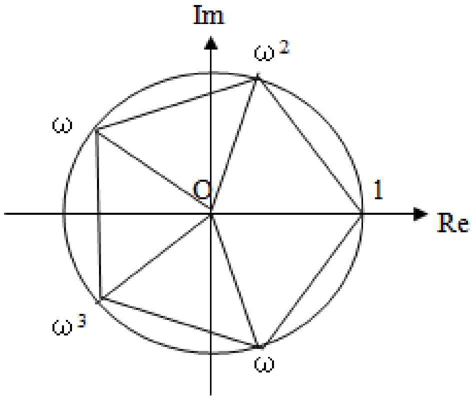

図 12

学校数学といえども，最終的には，どうしても数学プロパーが効いてくる．たとえば，図形の対称性を指導するには，どうしても，指導者には群論の素養が必要である．また，今回証明で利用したガロア理論は，方程式を指導する時には，当然必要な素養である．もちろん，子どもたちにそれを直接指導するということではない．現役の学校教員や教員を目指す大学生や大学院生には，少なくともこの様な数学そのものの素養が必要である．特に，学校数学と直接結びつきそうな教材に対応した数学プロパーの素養を身に付けることは必要であると思われる．質の高い授業実践を展開するためには，数学プロパーの素養を現役の教師にも学生にもできるだけ身に付けることが必要であると確信する．

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THE HISTORICO-GENETIC PRINCIPLE AND THE HERMENEUTICAL METHODEAS THE THEORETICAL BACKGROUND OF USING HISTORY OF MATHEMATICS IN LESSON
수학사 도입의 이론적 근거－역사 발생 원리와 해석적 방법론

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ABSTRACT

본 논문에서는 수학 수업에서의 수학사 활용의 이론적 근거를 비판적으로 고찰하도록 한다. 우선 수학적 인식에 관한 개인적인 발달과 역사적 발달 사이의 관계를 토대로 수학사 활용의 교육적 유용성을 가장 강력하게 이론적으로 뒷받침하는 역사 발생 원리의 생성, 전개 과정 및 그 의의와 한계 등을 논한다. 또한 인간 정신의 소산을 정확히 이해하려는 기술론인 해석학적 방법이 수학사 활용과 관련하여 어떻게 적용되는지를 살펴보도록 한다. 그리고 실제로 많은 논의에도 불구하고 수학 수업에 수학사적 내용이 광범하게 도입되지 않는 현실적 이유를 위 두 이론이 지닌 현실적 한계와 관련하여 논의하도록 한다.

1 논의의 전개

수학사를 교육적 측면에서 고찰하여 이를 수업에 활용하고자 하는 경향이 강해진 것은 1970년대 중반부터였는데 논의는 크게 세 가지 측면에서 진행되었다.

첫째, 학문 이론적 측면에서 보자면 수학교육의 근본적인 목표는 수학에 관한 균형 잡힌 상(像)에 도달하는 것이다. 기존의 수학 교육에서는 즉 공리, 결론, 증명 등으로 이루어진 형식적 체계와 동일시되는 순전히 형식적인 상을 그리도록 하게 되므로 이를 바로잡아 균형을 이루도록 해야 한다. 이를 위하여 수학을 이론의 발전이나 변화라는 인간 간의 관계를 토대로 한 역사적 현상으로 받아들여야 한다는 주장이 대두되었다. 이러한 학문 이론적 측면은 무엇보다도 교사들에게도 중요하다고 할 수 있다(Klein, 1924). 왜냐하면 수학에 대하여 적절한 상을 가지지 않은 상태에서 그것을 알린다는 것은 어려운 일이기 때문이다.

둘째로 교육 이론적 측면에서 보자면 우선 수학사에 대한 고찰은 수학뿐만이 아니라 문화적 배경에 대한 교육에도 일조한다고 볼 수 있다. Flegg는 “문화적인 배경은 왜 특정시대에 특정한 방향으로 수학적 개념이 발달했는지를 보여준다. 역으로 수학사에 대한 연구는 일반 역사가들이 통상 잊어버리는 문화적 배경이 되는 사실을 조명한다”(Flegg,1976, p.68) 면서 “수학은 우리 문화를 구성하는 데서 빠뜨릴 수 없는 부분이며 수학사를 가르치지 않으면 수학이 무엇인지에 대하여 부정확한 개념을 심어줄 수” 있으므로 “수학사는 정치사, 과학사, 예술사 등처럼 반드시 가르쳐야 한다” 고 주장하였다(Flegg, 1976 p.307).
그런 한편 수학사는 역사 수업의 긍정적 기능을 담당하게 될 수도 있다. Heymann에 따르면 “문화적 연관성/ 연속성”의 견지에서 역사적 지식을 얻게 된다고 한다(Heymann, 1996). 말하자면 수학과 “나머지 세상” 과의 다양한 관계를 통찰할 수 있게 한다는 것이다. 이러한 점에서 수학사는 좋은 역사 수업과 비슷한

교육 목표를 따르기도 한다. 또한 수학은 우리 문화의 구성 요소이며, 역사적 고찰 없이는 파악할 수 없는 문화적 현상으로 Scriba는 다음과 같이 주장하였다.

> “수학사를 수학 수업–중등 혹은 고등 교육–을 위해 활용할 수 있는지를 묻는 것은 잘못되었다. 오히려 ‘수학사 없는 수학을 생각할 수 있는가’라고 질문을 던져야 한다. 모든 것은 다음의 대답에서 비롯된다. ‘역사 없는 수학은 결코 있을 수 없다.’” (Scriba, 1983, p.113)

셋째로 교수 학습 방법론적 측면에서 보자면 수업에 역사적 요소를 도입함으로써 여러 가지 긍정적인 효과를 거둘 수 있다. 무엇보다 역사적 고찰은 수학의 이해를 돕는 수단으로 작용한다. 말하자면 “보편화된 개념에 대해 구체적이고도 세세하게 그 역사적 과정을 추적하는 것은 일반화와 추상화의 본질과 역할을 이해하도록 가르치는 최선의 길” 이라 할 수 있다. (Jones, 1978, p.61) 그리고 통상적인 수업과는 달리 수학을 역동적 과정으로서 인식할 수 있게 한다. “전통적인 수학 교수는 역사적 과정이 끝나는 지점에서 시작되기 때문에 그 분야를 이해하는 게 어려워”지는 것이다(Riedl, 1980, p.306). Boyer는 Cicero를 인용하면서 “네가 태어나기 전에 어떤 일이 일어났는지를 알 수 없다는 것은 영원히 어린아이로 남아있음을 뜻한다”(Boyer, 1964, p.24) 고 갈파하였다.

2 장애와 한계

수학사의 활용에 대하여 긍정적인 시각만 존재하지는 않는다. 우선 수학사 활용에 긍정적인 입장을 표명한 Freudenthal 역시 수학사에 대한 지식을 모든 학생들에게 요구할 수는 없다는 견해를 나타내어 하나의 선택 가능한 방안으로 간주했다.

이와는 약간 다른 맥락에서 Spalt는 우려를 표명했다. 수학의 역사를 다룬 다양한 출판물에서 제시된 목표는 어느 정도는 타당하지만 그로 인해서 수학의 역사를 과소평가하거나 왜곡 또는 학문적 의미에서 가볍게 다루게 되지 않을까 생각했던 것이다.(Spalt, 1987) 이러한 맥락에서 다음과 같은 주장이 제기되었다.

> “어떤 특정 분야에서 개인의 지적 발달은 이 분야의 역사적 전개과정과 비슷하게 진행된다는 전제 위에서 만들어지는 교육학적 개념을 설정하려는 시도는 좌초되었다고 간주해야 한다.”(Richenhagen, 1990 p.174)

또 아래처럼 말한 학자도 있다.

> “역사적 발전 과정을 배우고 난 후에 학생들은 수학 자체에 더욱 소원해질 수 있다” (Nagaoka, 1989p, 176)

이밖에 이론적으로는 타당하다고 인정하면서도 그 실질적 적용가능성에 대해서는 다소 회의적인 시각도 존재한다. 전통적인 연역적 전개 양식은 우아하고 체계적이며 명확하고 경제적인 교재 구성 방식을 가지는데 반해 수학사를 활용한 전개 과정은 자칫 지루해지기 쉬우며, 정확성과 우아함이 결여되기 쉽고 일반적인 이론적 체계화에 이르기가 어려울 뿐만 아니라 교재구성이 곤란한 경우가 많다는 것이다.(우정호, 1998, p.66)
하지만 많은 교사들이 수학사 도입에 긍정적이면서도 실제 수업에 활용하지 않는 이유로는 무엇보다도 하나의 주제를 다루는데도 시간이 상당히 많이 소요된다는 사실이다. 또 다른 이유로 수학사를 수업에 통합하여 다룰 수 있을 정도로 배경 지식을 쌓기가 힘들다는 현실도 무시할 수 없다. 추가적인 수업 자료의 도입으로 시간을 허비할지도 모른다는 우려도 아울러 제기된다. 게다가 전통적인 수학 수업은 그 목표까지

포함하여 수학사와 간극이 너무 커서 내용을 수업에 적합한 형태로 조직할 수 있는 교사가 극히 적다는 난점도 존재한다.

그렇지만 Jahnke에 따르면 수학사야말로 연관이 없어 보이는 연산 더미에 의미를 불어넣어줄 수 있으므로 수업의 보조 자료에 그쳐서는 안 되며 오히려 중요한 수업 내용이 되어야 한다고 주장하였다. 즉 인류가 직면했던 문제에 대한 해결책으로서 개념과 기술이 생겨났던 당대의 의미를 되새겨줄 수 있다는 것이다. (Jahnke 1991, p.6)

한편 어떤 학자들은 오늘날의 수학 교재가 지나치게 문제 위주로 구성되어 있는 사실을 지적하기도 한다.

> “많은 수의 연습문제가 인위적 성격을 띤다. 이들은 오로지 교실에서만 쓰일 용도로 구성되고 고안되었다. 학문으로서의 수학이나 어떤 종류의 전문적 수학과도 유사성을 띠지 않는다.”(Doerlfer, 1986, p.63)

수업 계획을 연산의 수준으로 작성하지 못하면 아예 고려 대상에서 제외해 버린다는 지적도 있다.

> “충분한 연습 기회를 줄 수 없는 주제는 학교 수학으로는 적합하지 않다. 심지어 다양한 연습 문제를 가지고 학교 수학을 분류할 수도 있다”(Doerlfer, 1986, p.63)

이러한 방향으로 진행되는 수업은 사실 수학을 세련된 계산술의 총화로 여기는 입장에 근거한 것이라 해도 과언이 아니다. 그 결과 학생들은 수학이란 개별적 과제 또는 문제를 모아놓은 것에 불과하다고 보게 되며, 수업 계획에 따른 단거리의 행보는 가능할지 모르나 수학 전체의 상을 그린다거나 그 가치를 인식하는 데는 도달하지 못하게 된다. (Kronfellner, 1998, p.18) 이에 대해 바겐샤인(Wagenschein)은 이의를 제기한다. (Wagenschein, 1962, p.67)

> “이렇게 수업 계획을 좌우하는 재고학문(언젠가 무엇을 위해선가 사용하게 되니 주의해 두어라!)으로서의 관점은 논리적으로 잘못되었다고 할 수 없으나 교육적으로 결코 옳다고 할 수 없다. 재고를 쌓기 위해서 늘 서두르게 되며 이는 학생들을 고무시키는 게 하니라 지루하다고 느끼게 만들 따름이다. 학생들은 얼마 지나지 않아 거의 대부분을 잊어버리고 만다.”

한편 평가와 관련해서도 수학사 도입의 난점이 존재한다. 교사는 학생, 학부모 모두 쉽게 수긍하는, 될 수 있는 대로 공정한 평가 기준을 가지기를 원한다. 이는 점수를 매기고 그에 근거한 성적을 내야 하는 현실 때문에 불가피하다. 이로 인해 시험을 대비한 연습 문제를 될 수 있는 대로 많이 풀어봐야 한다고 생각하게 되며, 이들 문제는 시험 문제와 유사하면서도 지나치게 유사해서는 안 된다. 수업에서 수학사가 차지하는 비중이 높아지면 성취도 평가에도 포함시켜야 하지 않는가라는 문제가 자연스럽게 제기된다.

> “수학 수업이 목표가 어떤 방향으로 설정되는지 상관없이 시험이나 성취도 평가에 통합되어야 한다. 시험 문제들은 수업의 중심이 무엇이었는지를 반영한다.”(Reichel, , H-Ch. : 1991, p.158)

수학사 관련 내용을 시험에 포함시키지 않는다는 전제 위에 수업에 도입한다면 여러 가지 문제가 발생한다. 특히 수학성적이 낮은 학생이나 그 부모들은 ‘중요한 것에서 벗어난다든지’ ‘시험과 관련 있는 매우 중요한 문제 연습을 소홀히 한다든지’ 하는 이유로 격렬하게 저항할 것이다.

이러한 상황이 수업에 수학사 도입을 꺼리도록 하는 주된 요인이다. 따라서 절충적 입장에서 교사나 일부 학자들 역시 수학사를 수업 분위기 형성이나 수업 내용을 점 더 잘 받아들일 수 있도록 주의를 환기시키기 위한 방안 정도로 수학사를 수업에 도입하기를 원한다.

그렇다면 수학 수업이 역사적 지식을 습득하도록 하는 목표를 가져야 하는가? 이는 어떤 종류의 학교 유형에도 걸맞지 않는다. 실업계 학교에서는 오히려 역사와 무관한 수업 방식을 선호할 것이다. 하지만 이 학교에서도 수학을 수단시 하면서 일반 소양 교육 방향으로 갈 수 있는 기회를 전적으로 쓸모없다고 치부

하지는 않는다. 이는 수학의 다양한 측면 가운데 어떤 점을 중시하는가 또는 교사가 어떤 의도를 가지고 있는가에 따라 달라진다.(Kronfellner, 1998, p.19–20)
이러한 가운데 수학사 도입의 가장 강력한 이론적 근거로 자리잡아 온 것은 바로 역사 발생 원리이다.

3 역사 발생 원리

3.1 발생 원리의 생성과 발전

수학교육에서 발생적 원리란 발달의 개념을 수학교육학의 중심에 놓고 수학의 학습-지도의 문제를 다루는 것으로 생성 배경으로는 무엇보다도 Euclid 기하학의 종합적 연역적 방법에 대한 강력한 비판을 들 수 있다.

구체적으로 보자면 Arnauld의 <새 기하학원론>*Nouveaux elements de geometrie*(1667) 에서 구체물의 참조를 허용하고 초등기하의 개념과 정리의 순서를 자연스럽게 재조정한 새로운 접근법과 증명을 제시한 것이 그 원조라 할 수 있다. 즉 "스콜라적이고 형식적인 수업 방식이 아닌 자연스러운 수업 방법으로서" 제기되었던 것이다(Schubring, 1978, p.18). 그밖에 Ramus, Bacon, Descartes 등도 Euclid 기하학의 접근법을 강력히 비판, 대안적인 해석과 접근법으로 발생적 관점을 주장하기도 하였다(우정호, 1998).
그 후 조금 더 진전된 이론적 틀에 입각한 일반적인 교수학적 구상으로 '역사 발생 원리'가 등장한다. 즉 19세기에 이르러 Lindner는 "소재를 그 자연스러운 순서에 따라 다루어 간단한 것으로부터 합성된 것으로, 원인으로부터 결과에, 보다 작은 것으로부터 보다 큰 것으로, 쉬운 것에서 어려운 것으로 나아가되, 하나하나의 동인을 아주 주의해서 서로 결합하는 것"을 발생적 방법이라 정의하고 발생된 순서에 따라 수학을 제일 먼저 지도해야 한다고 주장하였다(Schubring, 1978, p.62).

이처럼 인식의 발달과정에 근거하여 제기된 발생 원리는 19세기 후반에 들어서 생물학적 발달이론과 결합하면서 변화를 일으키게 된다. 진화론을 주창한 Darwin의 영향을 받은 E. Haeckel이 형식화한 '재현의 법칙', 즉 모든 개체는 발달 과정에서 역사적 과정에서 보였던 계통 발생을 재현한다는 것을 인식의 발생 원리에도 적용하였으며, 이를 이른바 역사 발생 원리라 일컫게 된 것이다. 발생적 원리는 19세기 후반에 형성된 일반적인 교수학의 기본원리로서 널리 인정되기는 하였으나 실제로 당시 중등교육의 현장에서는 광범하게 채택되지 않았다. 그러다가 Felix Klein(1849–1925)이 주창한 수학 교육 개혁의 움직임과 더불어 새로운 자각이 일어나게 되었다. Klein은 역사 발생 원리에 근거하여 수학사 활용을 한층 더 강화된 형태로 요구하였다.

> "물론 우리는 수학교육학적 입장에서 아동에게 너무 일찍 그처럼 추상적이고 어려운 것을 제시한다는 인상을 주지 않도록 해야 한다. 이 점에 대한 나의 견해를 정확히 규정하기 위하여 생물학적 기본 법칙을 인용하고자 한다. 그에 따르면 개체는 종족의 전 발달 단계를 단축된 순서로 거치면서 발달한다. 이러한 생각은 오늘날 모든 사람들의 교양이 되어 있다. 나는 이 법칙은 모든 수업과 마찬가지로 수학 수업 역시 반드시 따라야 할 일반 법칙이라고 본다. 즉 전 인류가 원시적 상태에서 높은 인식 수준으로 비약하게 되는 것과 똑같이 수학 수업 역시 소년기의 자연스러운 요구에 부응하여 점차 더 높은 내용으로 나아가서 결국 추상적인 공식화에까지 이르도록 해야 하는 것이다."(Klein, 1924, p.289)

다시 또 다른 저명한 수학자 Poincaré 역시 역사 발생 원리를 지지하였다.

> "어떤 동물의 태아 발달은 지질학적 시대의 그의 선조의 전체 역사를 매우 짧은 기간 동안에 경과한다고 동물학자들은 주장했다. 인간의 정신 발달에서도 마찬가지인 듯하다. 교육자는 아동을 그의 선조가 통과한 모든 단계를 매우 빨리 그렇지만 어떤 단계도 소실되지 않게 인도해야 한다. 이러한 이유에서 학문의 역사는 우리의 으뜸가는 안내자이어야 한다."(우정호, 1998, p.280, 재인용)

역사발생 원리의 또 다른 주창자 Toeplitz 는 대학 강의에서도 역사 발생 원리를 적용하려고 시도하였다. Toeplitz 는 사후에 발간된 “미적분개론”의 서문에서도 다음과 같이 밝히고 있다.

> “오늘날 우리가 규범적인 필수 요목으로 간주하여 가르치고 있는 미적분 계산의 이 모든 기본적인 내용; […] 어디서도 다음과 같은 의문은 제기되지 않는다: 왜 그렇게 하는가? 어떻게 해서 그렇게 되었는가? 이 모든 규범이 한때는 절실한 탐구의 목표였으며 창안되던 당시에는 아주 흥미로운 취급 대상이었음에 틀림없다. 만일 우리가 이들 개념의 근원으로 다시 돌아간다면 시간이 흐르면서 쌓인 먼지가 사라지고 다시 생명력 넘치는 모습으로 우리 앞에 나타날 것이다.”(Toeplitz, 1963 p. v)

Klein과 Poincaré가 언급한 역사 발생 원리는 하나의 가설로 제기된 것이며 이를 실제로 수업에 적용하기 위해서는 구체적인 방법론에 대한 모색이 필요했다. 수학교육에 대하여 수학사 도입의 두 가지 방향에 대해서는 Toeplitz 가 다음과 같이 지적하였다.

> “학생들로 하여금 희곡화를 통하여 실연해봄으로써 문제설정, 개념 및 사실을 발견하게 하거나ー나는 이것을 직접적인 발생적 방법이라 부른다ー교사 자신이 역사적인 분석을 통하여 각 개념이 원래 의미와 실질적인 핵심이 무엇인지를 알아내고 그로부터 이 개념을 가르치기 위한 결론을 끌어내는 것이다. 이는 역사와 그 자체로 관계를 가지는 것은 아니며 간접적인 발생적 방법인 것이다.”(Toeplitz, 1927, p.92)

그렇지만 당시 생물학적 발생의 기본 원리를 개인의 학습과정에 직접 적용하는 데 대해서는 회의적인 반응 역시 적지 않았다. 실제로 학생으로 하여금 학문의 역사에서 보이는 모든 우회도로와 막힌 길을 답습해 보게 하는 것은 어리석어 보이기 때문이다.

3.2 역사 발생 원리의 적용

Klein과 Toeplitz가 역사 발생 원리의 적용을 그토록 강력하게 주장한 배경에 대한 의문이 제기된다. 사실상 중등교육과정의 기하 영역 중 많은 내용이 Euclid의 <원론> 을 고스란히 담고 있는데도 말이다. 여기서 역사적 원전 자체에 관한 연구와 발생 원리의 배경이 되는 생각의 차이를 발견할 수 있다. 곧 역사적 원전은 그 자체가 수학적 인식이 어떻게 성립했는지에 관한 통찰을 주지는 않는다는 점으로 Töplitz 역시 발생 원리의 적용을 다음과 같이 주장했다.

> “역사 자체가 중심이 아니라 문제, 사실, 증명의 발생 그리고 이러한 발생 과정에서 결정적인 전환점이 무엇인지가 중심인 것이다.” (Toeplitz, 1927, p.94)

Wagenschein은 Toeplitz에 찬동하여 “발생은 역사가 아니다!”라는 기치를 내걸고 역사 발생 원리가 수학사를 다루면서도 역사 자체를 위한 것이 아니라 수학적 문제, 개념, 이론의 성립에 관한 통찰을 말하는 것임을 분명히 하였다(Wagenschein, 1989). 이러한 성립 과정이 밝혀질 때 비로소 역사적 원전을 수업에 성공적으로 투입할 수 있다는 것이다.
즉 수학적 인식에 이르는 유사한 상황에 처해 있는 것으로 가정하여 수학적 개념 등을 찾아내는 시도를 하는 것으로 Alexis-Claude Clairaut는 교과서 <기하학 원론> 에서 그러한 생각을 가지고 있었음을 보여준다.

> “나는 이 학문 역시 다른 모든 분야와 마찬가지로 점차적으로 세워진 것이라 생각했다[…] 그리고 이러한 최초의 과정은 초보자의 이해가 없이는 불가능했을 것이다. 왜냐하면 그렇게 발전시킨 이 역시 초보자였으므로.”(Vohns, 2003, p.8, 재인용)

Clairaut는 발견자의 인식이 어떻게 변화하는지를 찾아내려고 했다. 역사가의 눈으로 보면 이러한 사변적인 추정의 과정이 반드시 믿을 만한 것은 아니지만 교육자의 관점에서는 발생적 특성을 특별히 강조할 수 있는 흥미로운 가능성을 열어주는 것이다.
발생적 수학 수업의 주요 주창자 중 한 사람인 Wittenberg는 자신의 저서 <교육과 수학>*Bildung und Mathematik*에서 자신의 방법을 기하학에서 구체화시켰다. 그는 공리적이고 연역적인 방법에 대하여 명백히 반대하는 입장을 설파하였다(Wittenberg, 1990).

> "기하학에서는 무엇을 다루는가? 자와 컴퍼스로 공책이나 칠판 위에 그릴 수 있는 도형과 우리 주위의 세상, 들판이나 건물, 사용하는 대상에서 발견할 수 있는 도형일 것이다. 주의해 보면 공리나 증명에 관한 것도 아니고 크기가 없는 점이거나 두께가 없는 선에 관한 것도 아닌 것이다. 이 둘은 모두 내적인 요구가 없이 학생들에게 강제할 수 있는 동기가 없을 뿐 아니라 적절하지도 않다."(Vohns, 2003, p.9)

그의 제안에 따르면 역사적인 사실성을 고스란히 보여줌으로써 성과를 올릴 수 있는 것은 아니지만 흥미로운 방법을 찾을 수는 있다. 즉 그는 역사 발생 원리를 사실의 전개라는 측면보다 아동의 발달이라는 측면에서 구현하고자 했다는 것이 명백하다.

3.3 아동의 발달과 (역사적) 사실의 발달: 심리 발생 원리와의 통일

Klein의 동료인 Pringsheim은 1898년에 이미 역사 발생 원리의 고정된 한 측면에 대하여 의문을 제기하였다.

> "우리는 학문의 발달사에서 이전 세대가 범했던 결정적인 오류나 결함을 피하는 것을 배워야 한다 […] 더 나은 길이 보이지 않는 한 각자는 학문 자체의 발전과정과 근본적으로 같은 길을 거쳐야 한다. 그러나 더 나은 길이 있다면 그 길을 가리켜 줄 뿐만 아니라 그 길을 가도록 해야 하는 게 교사의 의무이자 과제이다."(Vohns, 2003, p.12)

한편 많은 교육학자들도 역사 발생 원리에 대하여 회의적인 입장을 취하기도 했다. Lutz Führer는 역사발생 원리를 절대시해서는 안 된다며 그 이유로 첫째, 개인의 발달과 인류의 문화 단계의 전개 사이에 보이는 소위 평행성이라는 것은 단지 표면적인 관찰을 근거로 해서 유효한 것이며, 둘째, 역사적인 길은 그다지 정확히 알려져 있지 않을 뿐 아니라 도달하고자 하는 지식으로 가는 더 쉽고 짧고 분명한 다른 길이 있을 수도 있기 때문이라고 주장하였다(Fuehrer, 1986). 그래서 더욱 포괄적으로 이해되는 심리 발생 원리를 제기하였다.

역사 발생 원리가 자료의 발달을 중심에 놓고 제기된 것이라면 심리 발생 원리는 아동의 발달을 중심에 놓는 것이다. 이러한 관점에서 학생은 고유한 인식을 지어나가는 목수로 이해된다. 심리 발생 원리는 역사 발생 원리와 비슷하게 20세기 초반에 당시 주류를 이루던 Herbart학파의 주장에 따라 정해진 대로 진행되는 수업 방법에 반대하는 것을 기초로 한 수학교육 개혁의 커다란 흐름과 더불어 상승기류를 탔다. 교육개혁가인 Johannes Kühnel(1869–1928)은 심리 발생 원리에 기초한 수업방법을 다음과 같이 규정하였다.

> "가르치고 제시하고 전달하는 것은 […] 과거의 수업기술이며 오늘날에 와서는 그 가치가 적어졌다 […]. 학생들 역시 지식을 습득해야 하겠지만 […] 우리는 학생들에게 가르치지 않고 스스로 깨우치도록 해야 할 것이다.[…] 학생들의 활동은 더 이상 수용이 아니라 획득으로 이해해야 한다. 미래의 학습과정은 지도와 수용이 아니라 조직과 활동으로 특징지어져야 할 것이다." (Vohns, 2003, p.13 재인용)

그런데 아동 중심의 수업과 교과 중심의 수업이 절충 가능할 것인가라는 문제에 대하여 교육가 John Dewey(1859–1952)는 확신을 가지고 긍정적인 답변을 하고 있다.

> “아동과 교과 내용은 하나의 과정을 정의하는 양 극이다. 두 점이 한 선분을 결정하듯이 당면한 아동의 발달 상태와 교과의 내용이 수업을 결정한다.”(Vohns, 2003, p.13)

심리 발생적인 사고방식에서 중심적인 명제는 교사의 임무가 학습내용을 아동에게 전달해주는 데 있지 않고 학습내용과 아동 사이의 작용을 중재해 준다는 데 있다는 것이다.
이 두 가지 방향의 결합은 ‘새 수학’의 물결에 밀려 빛을 보지 못하다 다시 새 수학의 좌초 후에 새로이 부상해 왔다.

> “수학의 기본적 관점을 좀 더 명확하게 규정함으로써 여러 입장에 적합한 함수론을 통일적으로 표현할 수 있게 되었다. 이러한 표현을 기초로 역사적으로 함수개념과 결합되어 있는 다양한 직관적 표상에 적합한 개념을 규정할 수 있게 되었다. 그렇지만 역사적 설명은 종종 함수개념의 제대로 이해하는 것을 어렵게 한다. 이처럼 이해를 가로막는 것을 단호하게 막기 위해서 Bourbaki는 ‘함수’라는 용어를 자신들의 방대한 저서 “수학원론”에서 아예 사용하지 말 것인가를 심각하게 고민하기도 하였다. 오늘날에도 (대학이나 학교의) 해석학 교과서에서 여전히 헷갈리게 하는 어법과 함수 개념의 도입 방식이 얼마나 문제가 많은지를 확인할 수 있다.”(Steiner, 1989, p.36)

이처럼 수학에 대한 정적인 관점과는 달리 발생 원리는 역동적인 관점을 요구한다.

> “수학 수업은 발생적 방법에 따라 조직되어야 한다.”(Wittmann, 1975 p.120)

라는 강력한 주장도 한편에서 제기되었다. Wittmann은 수학이라는 분야를 발생적으로 재현하는 것을 다음과 같이 언급하였다.

> “· 수업이 수학의 생성과 응용의 과정에 관한 자연스러운 인식론적 과정일 때이다. 정밀과학의 이론도 원시적인 초기 형태를 정제시키면서 문제를 탐구하여 이론을 발전시킨다는 사실에 상응하여 발생적 재현은 다음의 표식을 통하여 발생적으로 특징지어진다.
> 학생의 사전 이해와 결부되도록 할 것.
> 수학 안팎의 더 넓은 통일적인 문제의 맥락에서 고찰하도록 할 것.
> 전후 맥락에서 개념들을 비공식적으로 도입하도록 할 것.
> 직관적이고 발견적인 전제에 대하여 엄밀하게 숙고해보도록 지도할 것.
> 끊임없는 동기유발과 일관성.
> 진행될수록 안목이 넓어짐으로써 그에 상응하여 관점의 변화가 일어남.”(Wittmann, 1975, p.106)

한편 다음과 같은 견해가 강하게 제기되기도 하였다.

> “인간의 두뇌에서 전개되는 인식의 메커니즘은 역사를 관철하는 학문적 방법론의 발달로 표출되기 때문이다. ··· 인간 사고의 역사적 발달 단계는 어린이의 개체 발생의 정신적, 심리적 발달에서 재현된다.”(Oeser, 1991, p.84)

요컨대 생물학적 발생의 법칙을 근거로 두 갈래로 뻗어나간 심리 발생 원리와 역사 발생 원리가 수학 수업을 위한 하나의 법칙으로 다시 통일 되어야 한다는 것이다.

4 해석학적 방법

4.1 해석학적 방법의 의미

해석학적 방법이란 인간 정신의 소산을 정확하게 이해하려는 기술론으로 무엇보다도 원전의 해석을 기본으로 한다. 이는 본래 역사학자들이 즐겨 사용하는 방법이다. 유클리드의 <원론> 이 교과서로 사용되던 시대에는 이 방법이야말로 전 과정이었다고 해도 과언이 아니다. 학생들은 교사의 도움으로 원전의 번역본이나 해설본을 가지고 공부를 했다.

이 방법의 주창자인 Jahnke는 한 편의 학생들의 관점이나 사고방식과 다른 한 편 수학사의 맥락에서 이를 다루는 학자들 간에 생활 조건이나 문화적 또는 전공 체계에 따른 거리가 분명히 존재한다는 점을 주목하였다. 하지만 이러한 상황이 수학사의 일화를 수업에 도입하는 게 장애로 작용하지는 않음을 분명히 했다. 오히려 목적의식적으로 이러한 방식의 수업을 진행하여 소기의 목적을 이룰 수 있다고 보았다.(Jahnke, 1995, p.30–31)

Jahnke는 해석학적 방법을 Toeplitz의 발생 원리로부터 강하게 선을 그었다. 그는 전문적 이론의 전개 과정에 학생들을 참여시킴으로써 학생들로 하여금 그 발생으로부터 수학적 기능이나 숙련도, 개념 등을 익힐 수 있을 것이라는 견해에 부정적이었다. 수학의 역사를 수업 과정에 공식적으로 도입하라는 요구는 전문적 어려움 때문에 충족되기가 쉽지 않을 것으로 보았다.(Glaubitz 2003, p.71) 그렇다고 수업에서 수학사를 배제하는 것이 결코 옳다고는 보지 않았다.

Jahnke는 발생 원리를 적용하여 새로운 개념을 도입할 때 역사적 요소를 개입시키는 수업 상황을 변화시켰다. 학생들이 어느 정도 학습 주제에 대한 지식을 가지고 있을 때는 역사를 도입할 수도 있다. 전문적인 어려움은 상당히 줄어들 것이다. Jahnke는 적절한 시점에서 역사를 도입하는 근거로서 발생 원리보다는 해석학적 방법이 더 유용함을 주장하였다. 이를 위해서 학생들은 역사적 자료를 해석하고 해당 저자가 처했던 개인적인 또는 문화적, 학술적 상황을 살펴볼 필요성에 직면하게 된다. 이러한 요구에서 비롯되는 기대는 각각의 수학적 대상만이 아니라 수학 전체에 대한 나름의 시각을 익히도록 한다. Jahnke는 이러한 의미에서 수학사 원전 강독이 흥미롭고도 의미 있는 학습 경험을 축적하도록 이끌어낸다고 보았다. 그밖에 이러한 분석 과정 중에 자연스럽게 요구가 충족되기도 한다고 보았다. 이렇게 해서 대다수는 개방적이면서도 소통 중심의 논쟁적 수학에 대한 관점을 가지게 된다는 것이다. 그래서 원전 강독의 이점 세 가지를 다음과 같이 들었다. 첫째, 순전히 연산만으로 특징지어지는 내용은 의미를 잃게 된다. 고성능 컴퓨터, 계산기 등이 광범하게 사용되기 때문이다. 둘째, 공동체 안에서 교환 가능하고 사회적인 능력이 더욱 중요해지는데 이는 수학적 표현 능력을 익힘으로써 축적된다. 셋째, 학문은 미래 지향적이지 절대적인 진리는 아니다. 이러한 관점은 당면한 수학 원전을 천착해 나감으로써 개선되어 간다. 사고방식, 표현방식, 연구 방식과 관련하여 획득된 강조점의 변화에 따라 많은 학생들은 자기 나름의 방식으로 표현하고 관찰해 나가면서 수학에 접근하는 새로운 길을 열어가게 된다고 Jahnke는 주장하였다. 그는 여타의 수학 수업 역시 이로부터 이점을 취하게 된다고 본다.(Glaubitz, 2003, p.71–72)

학생 개개인이 지닌 언어 능력의 계발이라는 관점에서 언어와 수학의 상호관련성 역시 중요한 사항이다. 그런데 통상 원전의 강독에는 세 가지 언어가 필요하다. 우선 학술적 전문 언어와 일상 언어가 섞인 수업용 언어이다. 다음으로 자료에 사용된 역사적 언어가 요구되며, 수학을 대하는 개인 고유의 방식이 필요하다. 성공적인 수업에서는 이들 세 가지 언어가 상호 관계를 맺고 서로 뒤섞이면서 나타난다.(Glaubitz, 2003, p.72–73)

Jahnke를 비롯하여 해석학적 방법을 선호하는 여러 명의 학자들의 견해를 종합하자면 다음과 같다.

첫째, 학생들이 활동의 대상으로 삼을 수 있는 주제를 눈으로 볼 수 있게 나타낼 수 있다는 강점이 있다.

둘째, 원전을 다루는 것은 근본적으로 수업의 삽화적 성격에 적합하다. 많은 교과서에서 사료에 나오는 문제가 등장하지만 주로 이해를 돕거나 동기를 부여하기보다는 재미있게 다루려는 의도가 엿보인다.

셋째, 원전의 신뢰성에 따라 전혀 의도하지 않았던 요소가 수업 시간에 드러난다. 학생들은 원전을 강독

하면서 전혀 다른 사안을 연상하기도 한다.
Jahnke에 따르면 이러한 특성으로 인해 새로운 개념을 도입할 때 학생들이 응용하고 연습할 수 있는 역사적 원전을 투입하면 문제 상황의 적용에 따라 개념의 이해 정도가 달라질 것으로 보았다. 이들은 통상적으로 문제를 취급하는 방식과는 다르다. 각 사료의 맥락이나 저자의 전기적인 내용에 많은 비중이 할당된다. 각각의 원전은 고유의 역사적 특성을 지닌다. 무리하게 오늘날의 수학을 대입시켜 해석하는 것은 바람직하지 않다. 학생들은 해석 과정에서 자유로운 연상을 할 수 있으며, 논리적 입증이 목표는 아니다. 그러고 나서 역사적 원전에 들어있는 개념, 아이디어, 연산 기술과 오늘날의 이해 수준을 비교할 수 있다.

4.2 해석학적 방법의 적용

Rasfeld가 10단계 심화 과정의 확률론 수업에 해석학적 방법을 적용한 예를 소개하도록 한다. 일반적으로 1654년을 통계학 성립의 원년으로 간주한다. 이 해에 파스칼과 Fermat가 주고받은 편지에서 확률 문제를 다루었기 때문이다. 그 내용은 원래 Chevalier de Mere가 제기한 주사위 놀이의 기대값 문제를 해결을 구하는 것이었다.

> "도합 60점 10두카텐(Dukaten)이 걸려있다고 한다. 한 편이 50점, 다른 한 편은 20점을 땄을 때 사정이 생겨 게임을 중단하게 되었다면 각 편에 할당되어야 할 판돈의 몫은 얼마인가?" (Schneider, I.(ed.) 1988, p.11)

Pascal과 Fermat 이전에 이러한 문제에 대한 합리적인 해결책을 제시한 이는 없었다. 오늘날 대부분의 교과서에는 이와 유사한 분배 문제가 실려 있다. 하지만 역사적인 진행과정을 염두에 두지 않은 채 다루어진다. 그리고 토막토막 나뉜 채 배우게 되므로 어떠한 교육학적 가능성도 실현되지 않는 것으로 보인다.

이 지점에서 해석학적 방법으로, 즉 역사적 자료나 편지를 다양하게 해석하거나 학생들의 시각에 맞게 맞닥뜨려져야 한다는 주장이 제기된다. 이처럼 다양한 출발점에서 나아가면 찬성이나 반대와는 다른 맥락에서 여러 가지 해석이 공존하게 될 여지가 남게 된다. 그리고 르네상스 시대 이탈리아 수학자들의 초기 해법도 살펴 볼 수 있다. 이로부터 분명히 알 수 있는 사실은 수학자들 역시 이들 문제를 어려워했으며 단일한 해결책을 유도해 내지는 못했다는 사실이다. 그러고 나서 페르마와 파스칼의 편지로부터 오늘날 수용되는 해법을 배울 수 있게 된다. 이는 확률론적 기초 지식을 전제로 하며, 이러한 기초 지식에는 라플라스의 확률론과 간단한 확률 계산에 흔히 사용되는 수형도 등이 포함된다. 역사적 자료를 다룰 때 유명한 통계적 개념이 적용되어야 함은 분명하다. 학생들이 이미 가지고 있는 지식이 문제 해결에도 사용되며 한층 심화 확장되어야 하는 것이다. 이항계수의 적용과 같은 새로운 지식의 습득도 물론 가능하다.

Rasfeld의 예에서는 오래 걸리는 계산을 할 때 컴퓨터를 사용하게 하였다. 뿐만 아니라 부분 과제로서 각 시대에 활동했던 수학자나 그들의 업적을 인터넷을 활용하여 조사할 수 있도록 하였다. 이러한 과정을 거쳐 역사적 자료를 온전히 이해할 수 있게 된다. (Rasfeld, 2007,p.263–266)
이와 관련한 일련의 수업 과정은 모두 13차시로 이루어진다.(Rasfeld, 2007, p.267)

차시	내용
1 차시	문제 제기와 학생들의 최초 해법
2, 3, 4 차시	이탈리아 학자들의 초기 해법
5, 6, 7 차시	학생들의 새로운 해법과 파스칼의 해법
8, 9 차시	페르마의 해법
10, 11 차시	파스칼의 삼각형을 사용한 해법
12 차시	해의 공식 유도와 일반화
13 차시	검토와 무기명 질문

학생들은 처음에 다음 과제를 마주하게 된다.

> "알렉스와 베른트는 동전던지기 놀이를 하면서 전차를 기다리고 있다. 각자 60센트씩 걸고 먼저 4번을 이기는 사람이 건 돈을 모두 가지기로 하였다. 2:1로 알렉스가 이긴 상황에서 전차가 진입하였다. 그래서 놀이를 중단하고 건 돈을 공평하게 나누어 가지기로 하였다."(Rasfeld, 2007, p.267)

학생들은 해법을 각자 내놓아야 한다. 수학수업에서 확률론이 차지하는 비중은 여전히 작다. 사실 현상을 수학화하는 과정에서 대수적 모형을 구축하는 일은 확률론에 대해서 분명히 우위를 점한다. 그러므로 학생들이 승률을 간과하게 되어 똑같은 분배 비율 1:1을 제안하게 되기도 한다. 학생들이 토론 과정을 거치는 게 무리도 아니다. 이러한 보기는 기상 사정으로 중단된 테니스 경기에도 적용된다.
교사는 이 지점에서 아주 오래된 문제를 다룰 수 있음을 제안한다. 즉 학생들로 하여금 문제해결에 대한 동기를 부여하게 된다. Rasfeld는 여기서 이탈리아 수학자 Pacioloi의 해법을 제시하는데 이는 학생들이 내놓은 첫 번째 풀이법과 유사하다. 게임 참가자 A, B가 각각 도달 점수 n에서 a, b를 땄다고 한다면 ($b \le a < n$ 이며 각 판에서 승률이 같다고 가정한다) 풀이는 $a : b$ 이다. 이는 중세 후기 도시 간의 통상적인 거래에 참가한 쪽의 승패의 비에 따른 분배와 같다. 학생들은 1:0의 상황에서 100번의 게임을 협상하는 것은 옳지 않다고 비판하였다.(Rasfeld, 2007, p.268–269)
Cadano는 1939년 그의 저서 <수학과 측량의 실제>*practica Arithmeticae et Mensurandi Singularis*에서 Pacioli의 해법을 비판하였다. 학생들은 이제 이긴 게임의 수보다 이겨야 할 게임의 수에 주목하게 된다. 이 해법은 $(n-b) : (n-a)$ 라는 식으로 반영된다. 학생들 $n = 5$ 인 경우를 엑셀 표에서 시행해본다. 그러면 파촐리 해법보다 낫긴 하지만 여전히 그 결과에 흡족하게 되지는 않는다. 비로소 승률을 고려해야 한다는 인식에 도달하게 된다. 카르다노의 저서에서는 다음과 같은 식으로 제시되어 있다.(Rasfeld, 2007, p.269)

$$(1 + 2 + \cdots + (n - b)) : (1 + 2 + \cdots + (n - a))$$

Cardano의 합의 식은 설명이 제대로 되어있질 않아서 학생들이 느끼기에 만족스럽지 않다. 인터넷 검색을 통해서 학생들은 Cardano의 맞수 Tartaglia의 저서 <수(數)와 계측(計測)에 대한 일반론>*General trattato di numeri e misure*(1550)에 아래와 같은 비례식 설명으로 제시되어 있음을 발견한다.(Rasfeld, 2007, p.269–270)

$$(n + a - b) : (n + b - a)$$

Pacioli, Cardano, Tartaglia의 해법은 학생들에게 바로 받아들여졌다. 학생은 결정적인 사고 과정의 변화를 겪은 후에 위 수학자들과 마찬가지로 승률을 고려하기 시작했다. 경로 법칙을 고려하여 계산한 결과 알렉스에 대해서는 11/16, 베른트에 대해서는 5/16의 승률을 산출한다. 그리하여 게임이 중단된 상황에서 11:5의 비율로 판돈을 배분할 수 있다. 이로써 학생들은 그 결과에 만족하게 된다. 하지만 게임수가 커지면 수형도가 상당히 복잡해진다는 인식에 도달하면서 학생들은 불만스러워 하게 된다. 여기서 문제의 수학적 일반화가 시작된다. 학생의 입장에서 해결의 실마리가 떠오르지 않는다면 다시 역사에 등장하는 문제들을 다루도록 한다. 즉 수업 중에 Pascal이 Fermat에게 보내는 편지를 인용할 수 있다.

> "두 명의 게임 참가자가 예컨대 세 번의 승부를 가르는 게임을 벌였으며 각자는 32피스톨(pistol)을 걸었다고 한다.
> 첫 참가자가 두 판을 이기고 두 번째 참가자가 한 판 이겼다고 가정하면 결과는 다음과 같다. 첫 참가자가 이긴다면 모든 게임에서 이기게 되는 것이고 따라서 64피스톨을 가지게 된다. 다른 편이 이긴다면 이 대 이가 되므로 헤어질 때 각자 건 돈만큼 즉 32피스톨씩 가지게 된다. 근데 여기서 첫 번째 참가자가 이기면 64피스톨 지면 32피스톨을 가지게 된다. 그래서 더 이상 게임을 하지 않고 헤어지려 할 경우에 첫 번 째 참가자는 이렇게 주장하게 될 것이다. '32 피스톨은 명백하게 내가 가져야 한다. 내가 지더라도 가지게 되므로. 나머지 32피스톨은 내가 딸 수도 당신이 딸 수도 있으므로 그 가능성은 똑같다고 본다. 따라서 이 32피스톨을 이등분하여 나에게 더 주어야 할 것이다.' 이렇게 해서 그는 48피스톨을 가지게 되고 다른 참가자는 16피스톨을 가지게 된다.[⋯]"(Scheneider, 1988, 재인용)

자료에서는 이전과 마찬가지로 수형도로 나타내기가 용이하지 않아서 엑셀표 작성을 중단하게 되고 마는 두 가지의 게임 상황과 승률이 더 제시된다. 이로써 도표의 각 란을 채울 수 있게 된다.
다음 차시에 학생들은 Fermat의 답장을 접하게 된다. Fermas는 파스칼의 삼각형을 써서 조합론적인 방법으로 문제를 해결하고 결국 일반적인 공식으로 다음을 유도하게 된다.

$$\left[\binom{r}{n-a}+\binom{r}{n-a+1}+\cdots+\binom{r}{r}\right]:\left[\binom{r}{0}+\binom{r}{1}+\cdots+\binom{r}{n-a-1}\right]$$

학생들로 하여금 익명으로 자신의 의견을 나타내도록 하였더니 인터넷 검색이나 자료 강독 등이 긍정적이었다고 평가되었다. 컴퓨터 프로그램을 이용한 해법은 오히려 성가신 것으로 받아들였다. Rasfeld는 왜 학생들이 이러한 표 계산을 꺼려하는지를 해명하지는 않았다. (Rasfeld, 2007, p.282–283) 아마도 미리 연습을 하지 못해서 수업시간에 컴퓨터로 작업하는 것이 싫증이 나서 그럴 수도 있을 것이다.
Rasfeld는 자신이 제안한 일련의 수업 과정을 단축시킬 수도 있을 것이라고 강조한다. 학생들은 반응은 다음과 같았다고 한다.

> “나는 선생님께서 우리들 각자가 그 문제를 해결할 수 있는지를 물어 보신 것이 좋았다. 그리고 각 명제에 관한 토론은 아주 흥미로웠다. 마찬가지로 먼저 인터넷을 통하여 카르다노와 타르탈리아라는 인물이 어떠한지를 알아보는 일도 재미있었다. 하지만 카르다노와 타르탈리아의 해법을 PC에서 번역해놓은 것은 아주 무미건조했을 뿐만 아니라 길고도 지루했다.”
> “똑같은 문제를 여러 수학자들이 서로 다르게 해결했다는 것이 아주 흥미로웠다. 주제는 오래 지속해서 다루기에 좀 지루했다. 파스칼과 페르마 사이에 주고받은 내용은 좋았다. 수형도를 이용한 확률 계산은 좋았다.”(Rasfeld, 2007, p.284)

5 결어

수학사 도입의 타당성을 아무리 역설해도 역시 결국 남는 문제는 “수학사를 수학 수업에 성공적으로 도입할 수 있을까”이다. 많은 이들이 Jahnke의 한 제자가 다음과 같이 피력한 데 동감을 표한다.

> “수학사에 관하여 얘기하는 것은 과도하다고 생각한다. 일반적으로 우리는 무엇인가를 배우거나 현재 상황을 개선시키기 위하여 역사를 탐구한다. 그러나 모든 수학의 기본 정리나 발견은 결코 잊힌 적이 없으며, 실질적인 유용성도 알려져 있다. 아마도 내가 알지 못하는 예외가 있다면 그다지 중요하지 않거나 별로 사용되지 않는 발견일 것이다. 따라서 이러한 내용을 학교 수학에 도입하는 것은 과도하다.
> 원래 주제가 흥미로울지는 모르나 수학의 역사 안에서 헤매 다니느라 미래로 나아가지 못하고 과거로 돌아가는 대신, 오래된 발견에 기초해서 수학을 발전시켜야 한다.”(Jahnke, 1998, p.3)

수학에 정리나 계산술 말고는 알아야 할 것이 더 이상 없다고 가정한다면 위 인용문도 옳은 말이다. 그렇다면 사실상 수학은 죽은 학문에 불과하다. 그러나 사실과 대상 사이의 관계를 파악한다든지 인물과 동기를 이해한다든지 문화와 철학 및 전공 분야와의 관계, 즉 수학과 수학자 또는 이용자 또는 단순한 관찰자 가운데 어떠한 관계를 설정할 수 있는지는 밝히는 등 당면한 상황을 제대로 이해하기 위해서는 역사가 중요하다. (Jahnke 1998, p.3)
그렇지만 수학사를 수업에 도입하는 것이 필요한지에 관한 결론을 내리기 위해서는 실제 수업을 대상을 한 비교 연구가 필요하다. 지금까지 이루어진 약간의 비교 연구 결과를 참고할 수 있다. Glaubitz는 2007년 해석적 방법을 도입한 수업이 기존의 수업에 비해 평가 결과 등이 성공적이었음을 확인하는 연구 결과를 제시했다. 설문 결과에 따르면 이전에 해석적 방법에 가장 비판적이었던 학급에서 그 성과가 가장 컸다고

한다. 그리고 파지 효과도 실험 학급에서 훨씬 높았다고 한다.(Glaubitz 2007, p.262–263)
하지만 홍콩의 Lit/Siu/Wnog팀이 수행한 연구에 따르면 심지어 더 좋지 않은 결과를 산출하기도 했다.

> "결과에 따르면 실험 그룹이 전후 모두 통제 그룹보다 낮았으며, 그 차이는 테스트2와 3에서 통계적으로 유의하다. 통제 그룹의 점수가 3.22 오른 반면 실험 그룹의 점수는 4.56점 낮아졌다."(Lit, 2001, p.23)

이탈리아의 Bagni도 부정적인 결과를 제시하였다. 복소수 계산을 주제로 Rafael Bombeli의 자료를 도입한 실험 그룹의 점수가 통제 그룹과 비교해서 테스트 결과가 더 낮았다고 한다.(Bagni 2000, p.9)
이들 적은 양의 결과만으로 수학사를 도입한 수업의 질이 더욱 개선되었다거나 학생들의 능력이 향상되었다고 쉽게 결론내릴 수는 없다. 하지만 일련의 수업 말미에 기록한 학생들의 의견을 기초로 삼는다면 대체로 긍정적이다. 이러한 수업을 통해서 전혀 알지 못했던 가능성을 발견한 교사들에게서도 이러한 긍정적인 답변을 얻게 된다. 수업 자료를 연구하는 교육학자들 역시 긍정적인 입장이다.
수학사를 도입한 수학 수업은 원칙적으로 대부분 긍정적이라 할 수 있다. 하지만 구체적으로 어떤 주제의 수업에 어떤 방법을 적용하는 게 바람직한지에 관해서는 한가지로 답하기가 어렵다.
앞서 언급한 방법들(해석적, 역사 발생적, 역사적 확정) 가운데 한 가지를 선택하기로 결정하든지 아니면 이를 참조하여 독창적인 방법을 고안해내어 적용할 지는 전적으로 교사 자신의 몫이다. 이들 세 가지 방법을 이상적인 것으로 간주한다면 이 셋을 혼합한 방법을 생각해 볼 수 있다. 개별적인 경우마다 목표와 방법과 수단을 감안하여 결정해야 할 것이다. 물론 수학사 도입과 관련하여 제기되는 언어 문제의 해결 등도 몹시 중요하다. 총론적 관점에서 수학사 도입의 당위성에 관한 논의를 넘어서 더욱 진전된 적용의 결과를 토대로 다양하고 창의적인 수업이 이루어질 수 있도록 해야 할 것이다.

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BRIDGING THEORICAL AND EMPIRICAL ACCOUNT OF THE USE OF HISTORY IN MATHEMATICS EDUCATION? A CASE STUDY

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ABSTRACT

In this paper, I propose a study in which I take on this challenge and explore how *cultural understanding, replacement* and *reorientation* occur in the course of an experimentation in which students read and discussed a text by Pierre de Fermat as part of their calculus class. The data were collected within a study concerned with the metamathematical reflections (historicity of the concepts, rigor, intrinsic and extrinsic driving forces, etc.) students may develop when taking part in such activity. Two groups of students took part in the study (audio-recorded), and twenty-one interviews were conducted and transcribed. Using this set of data, I now confront them with Barbin and Janhke's three arguments regarding the use of history in the mathematics classroom and, in the same movement, confront the theoretical framework with the these data so to see how they could actually enrich one another.

Keywords: empirical studies, cultural understanding, replacement, reorientation, use of primary sources, learning and teaching calculus

1 Introduction

For several decades, many thinkers, researchers and teachers focused on the "how" and "why" of using the history of mathematics in mathematics education. Early in the 20th century, educators (Barwell, 1913), philosophers (Bachelard, 1938) and mathematicians (Poincaré, 1889; Klein, 1908; Toeplitz, 1927; Pólya, 1962) became interested. Until recently, it seemed like everyone, teachers and researchers, agreed that history was good and saw it as a motivating and effective tool for the learning of mathematics (Charbonneau, 2006). This enthusiasm has led to numerous studies concerning the use of the history of mathematics.

However, for 10 years now, the field of research around the use of history is being restructured. New and serious questions are born following the publication of the *History in mathematics education—The ICMI Study* (Fauvel & van Maanen, 2000). True health check of this area of research, the book brings together the beliefs, questions and concerns of researchers. This book invites researchers to take a step back from their beliefs. For example, the efficiency and relevance of many examples of application of history in class were questioned (Siu, 2000; Bakker, 2004). Some researchers stress the importance of being cautious when studying the historical aspects of mathematical concepts, they go as far as doubting the students and teachers ability to study these concepts from a historical point of view. (Fried, 2001; Charbonneau, 2002; Jankvist, 2009a). Others also question the "transferability" of positive experiences reported by practitioners from different academic levels (Tzanakis, 2000; Schubring, 2007). More broadly, we question the general way of conducting research in this field, in order to try to go beyond "the stories of practices". Thus, the lack of serious and systematic empirical studies questioning the possible contribution of history of mathematics in mathematics education is still around the table (Lederman, 2003; Siu & Tzanakis, 2004; Siu, 2007; Jankvist, 2009b).

2 Questions and research problem

Even today, if several studies inform us about positive experiences around specific activities involving the history of mathematics in the classroom (see Greenwald, 2005; Arcavi & Isoda, 2007; Hoyrup, 2007; Blomhøj & Kjeldsen, 2009), few of them, according to several (eg. Furinghetti, 2007; Charalambous, Panaoura & Philippou, 2008; Jankvist, 2009b; 2010; Guillemette, 2011), are making real analysis of how history is used, and how learners benefit from it.

2.1 Two types of studies

Looking through the scientific literature since the 1990s, one can classify the studies on the use of history in the mathematics classroom into two categories. There are those that usually take the form of stories of practices analyzed. These generally include initiatives of mathematics teachers of different academic levels who tried try to introduce history in various ways in their courses. However, most remain unsatisfactory from an experimental standpoint, because few of them present a framework for analyzing their data. They offer interesting reflections on the phenomenon, but they don't drive a precise analysis following a data collection methodologically established. In this sense, Gulikers and Blom (2001) observe that those cases are often isolated, creating a gap between "practical experiences" reported by some studies and theorical considerations reported by speculative reasearch. This brings me to the second type of studies, those who deal mainly with theoretical considerations. Their contribution is important because they provide ways of seeing and distinguishing that help us to look deeper on the arguments and methods regarding the use of history (Jahnke & al. 2000; Fried, 2007; 2008; Jankvist, 2009c). Having said this, despite the important contributions of these two types of studies, there is a need for empirical systematic studies on the mathematical learning process that unfold within the introduction of history of mathematics.

Moreover, these "practical experience" or theoretical considerations are rare. In his research, Jankvist (2007; 2009a) attempted to identify, from 1998 to 2009, all empirical studies published in English-language journals (Educational Studies in Mathematics, For the Learning of Mathematics, Mediterranean Journal of Research in Mathematics Education and Zentralblatt für Mathematik der Didaktik, as well as master's and doctoral theses and conference proceedings). In his work, he refers to empirical researches as "large scale quantitative studies to small scale qualitative studies, from experimental investigations to a teacher testing out a course using methods of questionnaires and interviews" (Jankvist, 2009a, p. 38). The studies he selected showed, in one way or another, empirical data on which the authors based their observations and conclusions. He found a total of 81 studies. On the other hand, he emphasized that throughout the 78 studies appearing in the HPM2004 & ESU4 (Furinghetti, Tzanakis & Kaijser, 2008) conference proceedings on the history of mathematics in mathematics education, only about 10% of them derive from empirical studies. That being said, he noticed an upsurge in the number of these studies since the middle of the decade, showing a change initiated in this field of study concerning this type of research.

Trying to bridge theorical framework and empirical account, my works is placing itself in this movement where researchers are looking to describe and explore what happens when using history for the learning of mathematics based on a conprehensive perspective.

The theoretical studies on the use of the history of mathematics in the class, discuss, among other things, the question of "why" history is used. From this central question emerge many arguments to justify the presence of history in the mathematics classroom. Two classifications stand out, on the one hand, Barbin (1994) and Janhke (2000) and, on the other hand, Jankvist (2009c). In the following, I present how each researcher dissected the question in their own way.

2.2 Jankvist and the "whys" of using history

Jankvist (2009c) divides the arguments for the use of history in two categories.

First, history can be seen as an effective and motivating cognitive tool that can assists and supports the teaching and learning of mathematics. Motivational factors, the humanization of mathematics, cognitive support for the student, the deepening of epistemological refexion for teaching, access to various problems and enriching the didactic reflexion around epistemological obstacles are clear arguments associated with this history perceived as a tool.

Second, a certain type of discourse that proclaims the teaching of the history of mathematics "as such" is a contribution to learning mathematics, in the sense that it teaches us what is mathematics. Jankvist does not hesitate to speak of the learning of "the sake" of mathematics through the history of mathematics (Jankvist, 2009c, p. 239). In this sense, the history of mathematics is seen as a goal in itself. It shows that mathematics evolves in time and space, and is "not something that has arisen out of thin air" (*ibid.*). Mathematics are a human activity wearing multiple facets trough cultures and societies. Its evolution is the result of intrinsic and extrinsic motivations animating the mathematicians in their day, are arguments that are part of a vision of history seen as a goal. These are arguments that are part of a vision of history seen as a goal.

Jankvist categorized the arguments that support the presence of history of mathematics in the classroom by observing the teachers' intention. If the intention relates more specifically to enrich the conceptual understanding of mathematical objects, the arguments are associated with the history seen as a tool. If the intention is mainly metamathematicals reflections (that is to say, the reflections that affect the historicity of the concepts presented, the historicity of the notation and the rigor associated, the mechanisms underlying the discovery of concepts, the intrinsic and extrinsic forces that drive mathematicians discoverers or the links between the development of these concepts and the development of societies and cultures) the arguments are then associated with the history seen as a goal.

2.3 Barbin and Jahnke and the "whys" of using history

For its part, Jahnke (2000) also questioned the arguments concerning the use of history, specifically concerning the use of primary sources. He highlights three main hypotheses. A first assumption is that history can provide a cultural understanding of mathematics. As he says: "The integration of the history of mathematics invite us to place the developement of mathematics in the scientific and technological context of a particular time and in the history of ideas and societies" (*id.*, p. 292). Thus, the history of mathematics is a way to place mathematicals objects studied in an historical continuum and an historical and social context, this enables us to see their progress and identify issues that have generated their development. Also, in this perspective, the links between the objects studied take a different shape from the simple sequence of concepts within a curriculum or ways in which the concepts of the discipline are typically organized.

The second assumption is that the integration of history leads to a replacement of mathematics, that is to say that "it allows mathematics to be seen as an intellectual activity rather than as just a corpus of knowledge or a set of techniques" (*ibid.*). These first two assumptions are related to the need to humanize mathematics, to emphasize its historicity and to emphasize the evolutionary aspect. It is a dimension frequently mentioned in literature (see Furinghetti, 2004; Tang, 2007; Blomhøj & Kjeldsen, 2009; Guillemette, 2009; Jankvist, 2010).

The final assumption is that history of mathematics brings a reorientation. In this sense, "the history of mathematics challenges one's perceptions through making the familiar unfamiliar. Getting to grips with a historical text can cause a reorientation of our view" (Jahnke & al., 2000, p. 292). From this perspective, history allows students to question their own assumptions and experiences related to mathematical objects by the encounter and comparison of another mathematical comprehension, one from another era. These concepts of reorientation, cultural understanding and replacement were first developed by Barbin (1994). Concerning reorientation, she stressed that "the history of mathematics, and this is perhaps the main attraction, has the virtue of allowing us to wonder what is obvious" (Barbin, 1997, p. 21, my translation).

With regard to research, if more fuel the discussion of these various arguments, they have seldom faced experimentation. The research, in terms of understanding how to operate this cultural understanding, this replacement and this reorientation in the learner and how it is articulated with the presence of historical elements in the mathematics classroom, is still in its infancy (Furinghetti, 2007; Siu, 2007; Charalambous, Panaoura and Philippou, 2008; Jankvist, 2009b). There is a real need to better understand these phenomena and what can potentially happen there. In a broader perspective, we must close this gap, raised by Guliker and Blom (2001) between the empirical studies and theoretical one.

Jankvist categorization between history seen as a tool and history seen as goal allows, firstly, avoiding a widespread confusion between arguments and methods and, secondly, facilitates the observation and the analysis of the relations between these two aspects of research. The categorization of Jankvist therefore aims to facilitate and guide the work of the researcher. On the other hand, the three arguments brought by Barbin and Janhke give a broad perspective of the potential benefits for the individual's learning and for the mathematics classroom.

Overall, this feed deep thought in the discussion about the "whys" of the use of history in the mathematics' classroom. From these theoretical considerations and assumptions, questions emerge: in wich particular manners the use the history in mathematics education can develops the components of cultural comprehension, replacement and reorientation for the learners? In particular, how are those components articulated with the use of primary sources in the mathematics classroom?

2.4 The "hows" of using history

Some researchers have considered the more specific "how" of the use of history. This is the case of Fried (2001; 2007; 2008), who highlights the difficulty of properly addressing the history of mathematics in class. He wants the history to be taken seriously and argues that its study should be very attentive. Otherwise, to Fried (2001), there is high risk of distortion of history, because history could be contaminated with a modern vision of mathematics that crushes the historicity of the concepts and sterilizes its exploration. The risk of anachronism and false readings of a progressive history are very high. Too often, historical aspects take the shape, as he said, of anecdotes and historical vignettes.

Concerning the "how", Jankvist (2009c) show three categories: the illumination approaches, the modules approaches and the history-based approaches. The first is the introduction of isolated facts of historical vignettes or anecdotes. A good exemple, described by Jankvist (*ibid.*), is the case of Lindstrøm (1995) who, at the end of each chapter of his book, added a small section on the development of the history of the concepts covered. The modules approaches offer problem-learning situations or sequences of lessons, varying in duration, based on the history of a specific mathematical topic. It is clear opportunities in the history that are supported mathematically and didactically and which may include the use of primary and secondary sources like reading historical texts or developing research projects by students and others. For the history-based approaches, it is based on the historical development of the mathematical object studied for the developement of a complete sequence of lessons. Directly or indirectly, the history is found in the mathematics classroom through the strategies adopted by the teacher, his attitude towards the presentation of the studied subjects, the issues raised from the historical context or the sequence of concepts discussed. Essentially, this third category includes practices based on genetic approach following the work of Toeplitz (1963) or, otherwise, Freudenthal (1991).

Thus, Jankvist (2009c) suggests that these different approaches, which include several specific methods of use, do not have all the same goals and that their scope is different from one another. In this sense, Fried (2007) states that reading historical texts appear as a preferred method when it comes to use history in a rigorous and serious ways. However, this difficult reading activity would imply a double perspective. To illustrate it, he stressed that the aim of the historian is to delve into the era of the mathematician, to perceive his idiosyncrasies and to situate his work within a continuum of mathematics development. The look of the mathematician, meanwhile, attempts to decode the obsolete symbols, returning them to the modern language and grasp the essential mathematical sens. He calls diachronic the reading of the historian and synchronic the reading of the mathematician, terms borrowed from the linguist de Saussure (1967/2005). He said that synchronic reading is too often reinforced by teachers. Also, the teacher's role is precisely to tip the student constantly between these two visions. This continuous back and forth work helps the learner to be aware of his own conceptions of mathematics, his personal insights and his ability to confront it constructively with those of others (self-knowledge). That is why the reading of historical texts appears as the preferred approach from the perspective of the learner to generate the three components of Barbin and Jahnke & al.

From these theoretical considerations and assumptions, one could ask: in which particular manners the use of primary sources can develop the components of cultural connotations, repositioning and reorientation for the learners? In particular, how are those components articulated with the reading of both synchronic and diachronic ancient texts in a mathematics class?

3 Collecting data, preliminary analyses and research perspectivs

Some preliminary data from my master degres' work (Guillemette, 2009) can give some clues. As mentioned above, the data were collected within a study concerned with the metamathematical reflections (historicity of the concepts, rigor, intrinsic and extrinsic driving forces, etc.) students may develop when taking part in historical text reading activity. In the next section, I will describe the context of this study and how the data were collected. Using this set of data, I will thereafter confront them with Barbin and Janhke's three arguments.

3.1 The use of an historical text

In this study, an activity based on the reading of a text by Pierre de Fermat was built and lived in a classroom. The text concerned the method of maxima and minima of Fermat. It is a well-known text through which Fermat provides an elegant method of solving optimization problems using similar principles of calculus that emerged at the time. The following text was used in class: *Method for finding the maximum and minimum (on the method of adegalation, 1629/1637)* (IREM de Basse-Normandie, 1999). This text present high potential for getting into anecdotes and stories surrounding the character. Moreover, the mathematical elements and the approach used by Fermat can be easily articulated with the elements of the calculus course that was taking place.

Fermat offers in this text a general method for finding the minimum and maximum of polynomial expressions. First, he proposes to put the problem in equations: "We first express the minimum or maximum using terms that may be of any degree". Then, he substitutes $(a + e)$ to the "primitive unknown a". Here, Fermat employs infinitesimal objects intuitively and without justification. He then "adegalise" the two expressions, the one using a and the other using $(a+e)$. For Fermat, they are nearly equal since e will be treated as an infinitesimal value.

In this equation, we find terms that are in e or powers of e : "affected of e", as he puts it. He can then divide each member as many times as he wants by e to obtain at least one term without e. He then considers the value of e as 0 and thus eliminates the terms in e or e powers. Solving the remaining equation provides the maximum or minimum which sought.

In the piece chosen, Fermat gives an example of application of his method: "Lets divide the line AC with E, so that the rectangle AEC is maximum". He puts $m\overline{AC} = b$ and a the length of a segment generated by the point E ($\overline{AE}$ or $\overline{EC}$). The other segment will be $b - a$. He then seeks to maximise $ba - a^2$.

By following his method explained earlier, he replaces a by $a + e$ as the first segment length. The second segment becomes $b - a - e$ and $ba - a^2 + be - 2ae - e^2$ the new product. Thus, we have: $ba - a^2 \approx ba - a^2 + be - 2ae - e^2$ (Adegalation), $be \approx 2ae + e^2$, $b \approx 2a + e$ (Dividing each member by e), $b = 2a$ (Eliminating the terms in e). Thus, the area of the rectangle will be the maximum if $b = 2a$. This conclusion is not followed by any justification.

In summary, Fermat was looking to maximise the function $f(x) = x(b - x) = bx - x^2$, where $f(x)$ is the area of the rectangle posing x, the length of a side of the rectangle, and where b is half the perimeter of that rectangle. Today we would have $f'(x) = b-2x$ and $f'(x) = 0 \Leftrightarrow b-2x = 0 \Leftrightarrow x = \frac{b}{2}$. Of course, because $f''\left(\frac{b}{2}\right) < 0$, $\left(\frac{b}{2}, f\left(\frac{b}{2}\right)\right)$ is a maximum of f.

More precisely, by moving closer to Fermat's method, we have: $f'(x) = \lim_{e\to 0}\left(\frac{f(x+e)-f(x)}{e}\right)$

$$0 = \lim_{e\to 0}\left(\frac{f(x+e)-f(x)}{e}\right) \quad \text{(with } f'(x)=0\text{)}$$

$$\Leftrightarrow \quad 0 = \lim_{e\to 0}\left(\frac{(bx-x^2+be-2ex-e^2)-(bx-x^2)}{e}\right) \quad \Leftrightarrow \quad 0 = \lim_{e\to 0}\left(\frac{-2ex+be-e^2}{e}\right)$$

$$\Leftrightarrow \quad 0 = \lim_{e\to 0}(-2x-e+b) \quad \Leftrightarrow \quad 0 = -2x+b \quad \text{and} \quad \Leftrightarrow \quad x = \frac{b}{2}$$

3.2 The experiment

Based on Fermat's text, I built an activity that was conducted with preuniversity students in a cours of calculus. The activity consisted of three parts: a brief overview of socio-historical context and the mathematics at the time of Fermat, an individual reading of the text and a return in large group around that reading.

The socio-historical and mathematical context was presented from a PowerPoint document. It contained many evocative images of the socio-historical and scientific climate of the time of Fermat. This document presented several pictures of Fermat, various mathematicians of the time, the city of Paris and Toulouse and old documents. I have included various biographical elements of Fermat, concerning his correspondence with various scientifics of the time and references to his last theorem and the entire movement around it. The emergence of the Academy of Sciences and the tendency of scholars of the time for seeking global methods for solving a set of problems were mentioned. Subsequently, students were asked to read the extract individually. I circulated in the classroom to address different questions and guide students in reading. The final phase of the activity was a large pleanary. Students could then share and respond after reading the excerpt. I resumed the process of Fermat and tryed to conciliate it with modern methods, to highlight some idiosynccrasies of the mathematicien and to situate his work within a continuum of mathematics development.

The activity was experienced twice, first in a class of 30 students in the sector of natural sciences, and secondly in a class of 11 students in the sector of social sciences and humanities. These two phases of experimentation allowed me to confirm the feasibility and effectiveness of the reading activity and to be more comfortable conducting the activity and the individual interviews that followed.

3.3 Collecting data

Short interviews were conducted individually immediately after the workshop. Thus, among the 30 students in the first experiment, nine volunteered for interviews. For the second experiment, the 11 students were interviewed.

These were semistructured interwiews conducted by myself for about 10 minutes individualy. The discussion was around different questions: Overall, what struke you the most during this training workshop? Which elements from the presentation in the introduction hit you? What elements struck you during reading the text? And during the plenary phase? What did you learn about mathematics in general? What did you learn about calculus? What do you think this kind of reading can bring to a mathematics course? Do you think such an activity belongs in a mathematics class? All interviews, and the wokshops in class were audio recorded. All this was transcribed and constitued the data of the study.

3.4 Preleminary analysis and research perpectives

This research allowed me to observe metamathematical reflexions that emerged from pre-university students whom take part of such activity. Those metamathematical reflexions in question were those who, through a mathematical activity, concerned the historicity of the concepts presented, the historicity of the notation and the rigor associated, the mechanisms underlying the discovery of the concepts explored, the intrinsic and extrinsic forces that drive mathematicians and the links between the development of these concepts and the development of societies and cultures.

The transcripts of the interviews as well as the transcript of the activity helped to establish valuable data regarding the emergence of metamathematical reflections in connection with a reading of ancient texts. However, as I said above, it seems interesting, *a posteriori*, to try to highlight, through excerpts from the transcript, the three arguments concerning the use of history in the mathematics classroom: cultural understanding, replacement and reorientation. For example, the following two extracts which are the reactions of students who have experienced the reading activity:

> ″You say to yourself, ah! Math is going to be bad! However, you have inform us of the entourage of his discoveries, it seems that we know more from the inside. You know, you get the character and taste how his business is found. You know, math, just numbers, at least here you have something back [⋯] It is less abstract ″(*id.*, p. 67).

> ″Then that happened to him ⋯in the air as well and we finally we can do it with our limites and optimization to reach the same answer as him ⋯I do not understand, he still did it and now we can justify it ″(*id.*, p. 50).

It is possible to consider these reactions in the three components introduced by Barbin and Jahnke. Indeed, it is conceivable that the learner behind the first quote demonstrates a cultural understanding of mathematics by saying "we know more from inside". He seems, somehow, to have anchored the concepts discussed in a socio-historical and cultural context. His look changed as he seemed to perceive the "entourage of discovery", which allowed him to take a fresh look on mathematics. It is also conceivable that the activity of reading, for this same student, would have led to a replacement of the mathematical objects in question. He mentioned that with this kind of activity mathematics is not "just numbers". It appears less frozen in time, unchanging or reified. In this sense, mathematical activity appears to be a true human activity. Objects and concepts discussed don't come from heaven, but are developed by men in particular intrinsic and extrinsic motivations and they are the fruit of long and sometimes tortuous reflections. Finally, it is possible that a shift has taken place in the learner, considering the second quotation. Indeed, he was surprised by the intuitive approach of Fermat to such an extent that he felt the need to reclaim the concepts in question to better understand it. He asked and attemped to highlight the links between the two forms of understanding, his and Fermat.

This preliminary analysis of data from a particular research project, that are perceived by simple outlines, can provide clues about what each of the assumptions may mean. The fact remains that the contours of these theoretical considerations remain unclear and a refinement is needed. Many questions remain, for example: Does a cultural understanding implies necessarily a replacement of mathematics? Is there a form of gradation between each component? Is cultural understanding necessary for a shift to occur? More broadly, are these assumptions sufficient to interpret benefits for learners?

4 Conclusion

This first level of interpretation, on the one hand, provides a partial understanding of the phenomena in question and, secondly, clearly underlines the need to enter more deeply and more systematically in the analysis. In this sense, it seems necessary to build new experiments to investigate more precisely the impact of the introduction of history in the mathematics classroom to better illuminate the arguments in favor of this introduction. Thus, we must find effective ways to make the learners "talk": different kinds of interviews, written reflections, questionnaires, mathematical productions, etc. Through the systematic analysis of the experience of the class, it will be possible to fully grasp and understand the issues surrounding the introduction of history in the mathematics classroom. This understanding will provide tools to deal effectively with objects of study in this field of research.

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MISUSES OF STATISTICS IN AN HISTORICAL PERSPECTIVE
Reflections for a Course on Probability and Statistics

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ABSTRACT

This paper is a kind of guide for a historical introduction to the beginnings of probability and statistics (limited to the end of XIXth century), through the "Reading of Master", introductive to a mathematical course on these subjects. We provide the minimal historical bibliography. We claim that in many historical masterpieces everyone could recover the original meaning of notions and computations, understand the constitutive relations between probability and statistics, and so avoid the confusion of interpretations made very often today. Indeed, any given practice of statistics or probability is a tailpiece of a historical sequence of fixations, hesitations, remorse, errors, of "mathematical pulsation", and this sequence is their true meaning. To be aware of that it is necessary, as well for new future developments as for applications. Especially it is important to know the history of major misuses and difficulties of probability and statistics in past.

History teaches us that statistics lay on several notional difficulties, as: what are hazard and chance, their dynamics, what are stochastic process and random variable, what are the good processes for limits or infinite gluing of data, from where data do come, what are the good descriptions of populations, characters and samples? All these subjects could be elucidated from classical books. So when doing statistics we have to construct interpretations of results with a serious critical open eye.

We propose to emphasize this historico-epistemological point at the beginning of any course in statistics, especially for the benefit of future mathematics teachers.

1 Introduction

1.1 Reading the Masters

This paper provides an historical introduction to probability and statistics, via the "Reading of Masters". Today we can observe a lot of misuses and misunderstandings of statistics. At first of course these misuses are due to a general ineptitude induced by a bad understanding of the true nature of any given mathematical tool, and the false conception that such a tool could be used blindly. But they came also from specific difficulties of statistics, which can be understood by its history, and the reading of the decisive historical works. We propose a minimal list of such references to read or to consult. Our explanations here are directly related to these historical materials.

1.2 Starting with elementary problematics, and coming back to history

Clearly it could be difficult to read directly the original papers, even if we know that some good explanations are written there. A preparation for such a reading could be at first to study directly two or three very elementary and very good books, as (Granouillac, 1974), (Moroney, 1974), (Lévy, 1979), about very concrete problematics of today in statistics, and with (Boll, 1941,1942) on probability. Secondly we recommend to begin the study of an important historical material, the book (Bertrand, 1889), which is very intuitive, with a lot of significant exercises. Thirdly, we could read historical analysis as (Desrosières, 1993) and (Edwards, 2001) on statistics, and (Barbin & Lamarque, 2004) on both statistics and probability. This last book is synthesized in (Barbin, 2004) in a preface explaining that the type of problems is not the same in probability and in statistics. For example on the question of the relation between causes and events: for Laplace the problem of causes is a question of probability (of causes), whereas for Cournot the problem is to investigate statistically on causes by analysis of effects. Also there are differences between decision and prediction, extrapolations and results, modelling and exploring, etc.

1.3 Two sources, one pulsative mathematical subject

Two subjects seem to be different and not to come from the same source: on the one hand the question of probability or "geometry of the hazard" (Pascal, 1654) regarding uncertainty (Pascal), or likelihood (Leibniz), for applications to equitable judgements, and on the other hand the question of statistics, analysis of data for a state, concerning health, trade, taxes etc., and beneficial decisions in these areas.

But the exploration of the history shows that from a mathematical point of view the two subjects became much related and intertwined; comings and goings between the two subjects is vital in their historical developments. It is perfectly possible to teach separately probability and statistics; but our claim is that the true mathematical subject is in between, and a deep understanding of the correct use of both theories assume a main attention to this "pulsation" (For the idea of "mathematical pulsation" see (Guitart, 1999)).

2 Average, probability and expectation

Today the scientific subject of probability and statistics is the construction of the laws of chance and of distributions of data and their use for analysis in various other sciences, for presentation and analysis of phenomenon and data, for explanation and prediction. In such a scientific use, mainly as a 'logic' to guide experiments and observations, four implicit credits or beliefs are to be examined: the belief in the 'average' as meaningful and significant in reality, the belief in the existence of a real structural organisation for any experimental data, the belief that the average is the natural minimal summary of the organization, and the belief that the variation in statistical data is an effect of chance and so it is a matter of probability.

2.1 Addition, proportion, multiplicity and average, percentage

If a multiplicity of n weighted data (p_i, x_i) is given, then the pondered average or pondered mean value is (p, m) with

$$(p, m) = ((p_1 + \cdots + p_n), (p_1x_1 + \cdots + p_nx_n)/(p_1 + \cdots + p_n)).$$

The average or mean value is m. Almost all concrete applications of probabilities and statistics could be enough directly reduced to computations of such convenient averages. And a lot of misuses come from the oversight that the average is not so natural and obvious.

The fact to adopt an average in place of a multiplicity of values is not a mathematical principle, but an empirical decision, related to the fact that it amounts to compute proportions, or linear functions, and indeed it is the simplest of arithmetical rules. It could be justified also by the fact that the average minimizes the weighted sum of square distances to the given data. But simplicity does not imply adequacy. The basic objection to average m of the x_i is that if f is a function of x, with $y = f(x)$, why it is not better to compute an average M of the y_i, and then to take, in place of m, the value $f^{-1}(M)$? For example why not to compute the average of x_i^2 rather than the average of the x_i? So it is left to the user to choose the good average (on x or on $f(x)$, and, furthermore, with which weightings). A good discussion on this point is in (Bertrand, 1889).

It is clear that $2 + 3 = 5$ is a theorem, and that 2kg + 3kg $=$ 5kg is a scientific law in physics of pondered bodies, experimentally proved through the Law of Lever or by the determination of barycenters (Archimedes). But on the other hand 2\$ + 3\$ $=$ 5\$ is not a theorem or a scientific law: It is a convention for an exchange accepted today by everyone to enter in the game of finance and trade. So everybody accepts this rule at the root of his practise with money, estimation of salaries, debts etc. Looking at the history of money and bank, it is easy to understand how this rule did not always exist. In fact this rule imposed the transformation of objects of exchange as magnitudes, comparable and additive. Then the exchange could be performed just as balancing in calculus, and henceforth the calculus became the obvious medium of trade. A similar point is in the principle of vote (for decisions in justice, for election of representatives, to choose a product in a market, etc.): the votes are added, and the convention is that the decision is taken according to a numerical calculus (for instance the principle of majority). It is recommended to read (Condorcet, 1785).

In fact these reductions to arithmetical practices of liberal market and liberal democracy could be seen as application of a meta-rule of "reduction to average". So for the addition of dollars we have: (2\$ + 3\$)$/(2 + 3) =$ 1\$. A good reading here is (Cournot, 1835).

A first source of errors with an average is the ignorance of the concrete source of where it comes from (kg, \$, votes, etc.): then a true criticism of the result is impossible. And then the worst consequence is that different averages, and percentages, in different heterogeneous areas, are compared, and do conduct to fallacious correlations and illusory understanding.

2.2 Probability and expectation

Usually it is considered that probability theory starts with the work of Pascal on games (Pascal, 1654), its correspondence with Fermat in 1654, and, that the first treatise is the booklet of Huygens, in 1657: "du calcul dans les jeux de hasards" (Huygens, 1657). At this stage, we get the notion of expectation, while the notion of probability will be introduced explicitly later. The expectation models what is equitable, in presence of a true random situation; it is not what is good, what is beneficial. The analysis by Pascal could be formulated today, in a very anachronistic way, as the analysis of a stochastic process, as the description of conditional expectation (the introduction of the idea of "martingale"); none of these words is from Pascal' s work, but this interpretation is plausible. On the other hand we can insist on the fact that it is not a calculus of probability, a proportion, the quotient of number of favourable cases by the total number of cases [according to Laplace' s view (Laplace, 1812), (Bertrand, 1889)]. A fortiori it is not a calculus of probability via frequencies!

In modern terms, the expectation $E(X)$ of an alea X (a random variable concerning a stochastic process) could be defined as the average of the possible values x_i for X, weighted by the probability p_i that X takes the value x_i. It is the average of the possible values, but it is not the most probable value.

So, on the one hand, historically the expectation is a primary notion, preceding the notion of probability, and a fortiori preceding the notion of frequency; and on the other hand, in its modern expression, this notion seems to be a derived notion, constructed with the notions of probability (the p_i), of statistical distribution (the i), of average. This too analytical view could be a source of misunderstanding.

2.3 Relativity of probability during the time of the process, logical aspects

A difficulty, well illustrated in the Bertrand' s treatise (Bertrand, 1889), is the question of relativity of chance in time, a priori and a posteriori probability, reversions from probability of effects to probabilities of causes, in relation to the converse Bayesian calculus (Bayes, 1763) as reformulated by Laplace (Laplace, 1812). The relativity is also a question of dynamics in a stochastic model (question of martingales), and is related to incomplete information. The difficulty is that these relativities are not often announced explicitly in a concrete problem.

These relativities have also to be mixed with some logical questions, when we compute the probability of a logical combination of random events or variables. This logical point is treated in (Boole, 1854), and is at the basis of the Kolmogorov axiomatics in (Kolmogorov, 1933, p.2). So the modern axiomatic, with a set E of elementary events, a probability function P on a field F of subsets of E, with random variables seen as functions X on E, etc.) is able to support these aspects.

3 Statistics, combinatorics, asymptotic calculus, normal law

3.1 Statistics versus probability

Let us start with a warning: from a mathematical point of view, in principle, statistics need not to be constructed through a probabilistic interpretation. For instance the approach by Jean-Paul Benzecri in his factorial analysis of correspondences is based on a geometrical analysis of the shape of a cloud of experimental data. The observation of symmetries, of repetitions of a motive, of frequencies of a phenomenon, is not necessarily related to a causal interpretation in terms of chance.

Nevertheless, very often it is the case that the variation in a data set is a consequence of chance: in these cases (and in these cases only), the frequency (a notion in analysis of statistical data) could be related to the probability (a notion in doctrine of chance).

The key point is the following. Starting with an elementary reproducible stochastic process, we imagine as an experiment that this process works several times, and we look at the sequence of results. This sequence is a statistical data, with frequencies, etc., and of course, in such a sequence, the variation is due to the stochastic nature of the elementary process, and we can ask for a mathematical relation between the probability of the elementary process and the frequency in the sequence.

So, the probabilistic interpretation of statistics consists in the converse: given a sequence x_i of data, we pose the hypothesis that it is produced as iterated values of an unknown random process X, according to a probabilistic law which is to be revealed.

The difficulties with this point of view become very serious in several directions. The information (the x_i) could be incomplete (in fact a sample in a population). The true nature of the process underlying X could depend in fact on the possible modification in time of this process according to its repetition (the various X_i are not independent). And when we would like to correlate and to compare several sequences x_i, y_j, associated to random variables X and Y: the underlying stochastic processes associated to X and to Y are not necessarily the same, or at least easily correlated. In order to surmount these difficulties we could read analysis on the theory of measurement (cf. § 3.4).

3.2 Probability and frequency, weak law of large numbers of Bernoulli

A central difficulty comes from the confusion between *a priori* probability and frequencies. In fact, history shows that these two aspects are closely interconnected at the heart of the subject: this interconnection finds its mathematical expression in the law of large numbers. We learn this from Laplace and Bertrand' s books (Laplace, 1812), (Bertrand, 1889), where they discuss hazard and chance.

The weak Law of Large Numbers is a mathematical result expressing that the mathematical model of probability is consistent with the frequency interpretation of probability. Informally, and in a rather vague way, this law says that: when the number N of independent repetitions of an elementary stochastic process increases, then the observed frequency f_N of favourable issues "probably approaches" the probability p of the favourable issue, that is to say that the probability $P_N(e)$ that the difference $p - f_N$ exceeds a given positive number e converge to 0, as N increases indefinitely. It is a fundamental result by J. Bernoulli (Bernoulli, 1713).

This beautiful result constructs a relation between three terms: an unknown probability p, an observable frequency f_N, and another unknown variable probability $P_N(e)$; the beginner has to be careful to distinguish among these three terms.

3.3 Asymptotic calculus, de Moivre's Normal Law, Central Limit Theorem

After the treatise of Huygens, three decisive steps in probability where the books by (Bernoulli, 1713), (de Montmort, 1708), (de Moivre, 1718). There the combinatorics is well developed, with a thought towards the statistics of (Graunt, 1662) and (Halley, 1694), around the Binomial Law, and even is pursued towards asymptotic calculus (e.g with the formula of de Moivre-Stirling) and the normal law.

At first we reach the delicate question of the natural extension from finite combinatorics towards probabilities and statistics considered as potentially infinite combinatorics. Hence the asymptotic calculus and variations on limits which has to be defined and put forward. So to use the law of large numbers is not trivial.

The "Normal Law" (so named only in the XIXth century) was discovered by de Moivre, as a limit case of the Binomial Law (Freudenthal, 1957). It was also studied again by Laplace and Gauss, and so it is also known as the "Laplace-Gauss Law". Its graph is the so called "Bell Curve". It is related to the elaboration of so called "Central Limit Theorem", a deep improvement of the weak law of large numbers, resulting from works of de Moivre, Laplace, Gauss. This is to be read in (Laplace, 1812), (Bertrand, 1889). Today, this Central Limit Theorem is considered by probabilists as the central object of probability theory.

3.4 Measurement: Least Squares Method

The theory of measurement by the method of Least Squares could be studied through the works of (Mayer, 1750), (Legendre, 1806), (Gauss, 1855). In Mayer a very interesting empirical method of grouping observations is used, for the analysis of astronomical observations of the Moon. But the more decisive step was the emergence of the so called Least Square Method. The first justifications of this method (Laplace, Gauss) passed through the Normal Law and the Central Limit Theorem. But in fact in (Legendre, 1806) and in the second attempt of Gauss (Gauss, 1855), the justification is outside the scope of probability, even if related to the Bell Curve. It is very instructive to read the story of this subject (Bertrand, 1889), (Derosières, 1993).

3.5 The Average Man

In the XIXth century we get the theory of standard deviation with respect to the average, the development of various statistical or probability laws, motivated by various domains of applications. The development of the subject of probability and statistics is mainly about applications of the method of least squares and the Bell Curve. It is instructive to read various utilizations of this material.

In these applications, a basic difficulty is the confusion between data of several measurements of a given object and values of a given character in a given population. It is perhaps the root of the difficulties with probabilistic justification of the mean square method, and this confusion is excessively admitted by (Quételet, 1835). With his construction of the "Average Man", Quételet's basic assumption is the following: there exists an ideal man, and each concrete human is a measure of this "Average"; furthermore this measure is a random variable, made by chance, and according to a normal distribution.

4 Conclusion

In this brief approach, we stop at the end of the XIXth century, even omitting Galton, Pearson, and then Fisher, and so the true birth of modern statistics in the 1920's, through the theory of samples. There, the central question would be the construction of a good poll: how to construct a representative sample (Fisher), in such a way to get reasonable predictions? Clearly, on the question of the probability that a poll is a good one, the statistics get a new link with probability. We have also omitted the birth of the statistics of laws which are very different from the normal law, and the statistics of extremes (Levy, Gumbel). So, in some sense a theory of exceptions (improbable values) was created, and this is again a new link between statistics and probability.

Nevertheless, today the calculus of probability is in a "pulsation" in itself, between two presentations: on the one hand "à la Kolmogorov" with an E, F, P etc., and on the other hand as a direct manipulation of random variables and of their laws. This has been observed judiciously in (Mazliak, 2002). We think that this pulsation is easy to perceive in History.

And furthermore, on reading the classics, we realized here (although stopping in the XIXth century) that between probability and statistics, another real pulsative knot had been constructed by History. We think that it is important for future teachers to know that; especially this could help them not to reduce the idea of probability to the idea of frequency; otherwise it will be a real fault with respect to the true nature of the subject. Here is the fundamental misuse of statistics, to forget its link with probability as an a priori theory of chance.

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THE "LADDER AND BOX" PROBLEM FROM CURVES TO CALCULATORS

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ABSTRACT

The mathematical content of the "Ladder and Box" problem (e.g., Monte Zerger (1987), The "Ladder Problem"; David Wells (1992), *The Penguin Book of Curious and Interesting Puzzles*, p. 130) is to find the legs of a right triangle when its hypotenuse and the sides of an inscribed rectangle are known. This puzzle is of recent origin (early 20th century), but the underlying mathematical problem has been traced back to geometric constructions using Nicomedes' conchoid, ca. 180 BCE (Audun Holme (2010), *Geometry: Our Cultural Heritage*). It also occurs in Newton's 1720 *Universal Arithmetick*, translated from the Latin by Raphson (Problem XIV, p. 112, with Figure 27 between pp. 122 & 123), and in Thomas Simpson's 1745 *A Treatise of Algebra*, problem XV, p. 250. The problem was challenging even in the case of an inscribed square; and for an arbitrary rectangle, its algebraic description uses a fourth degree polynomial (e.g., Karlheinz Spindler (1994), *Abstract Algebra with Applications: Vector Spaces and Groups*). The methods that have been used to solve it have changed over time, from a geometric construction using special curves, to solving a biquadrate polynomial equation by an algebraic technique. At present it can be reduced to a simple trigonometric equation that can be solved by numerical methods implemented on a graphing calculator, which makes it an easy exercise. But besides the change in the technique of solving problems, the very concept of a "solution to a mathematical problem" has also changed over time; and at present it has several meanings that are used in different contexts. We will describe these changes in technique and in the meaning of a solution, and we will show that the "Ladder and Box" problem is still a very interesting problem that can be used either in algebra classes or in introductory calculus classes, because it shows a "practical" question that can be easily solved by rather sophisticated mathematical methods.

Outline

1. Introduction: A "new" problem with historical roots and its mathematical formulation
2. A modern solution using a simple trigonometric equation
3. How the problem has been solved historically, and what it meant at each time "to solve a problem"
4. Final remarks

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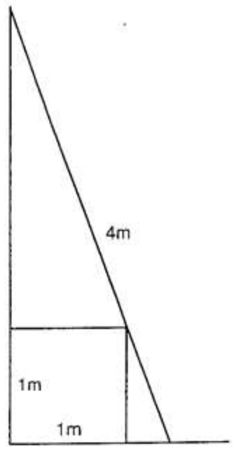
1 Introduction: A “new” problem with historical roots, and its mathematical formulation

399. The Ladder and the Box A ladder, 4 metres long, is leaning against a wall in such a way that it just touches a box, 1 metre by 1 metre, as in the figure. How high is the top of the ladder above the floor?



The ladder and box problem, from Wells, p. 130.

The “ladder and box problem” (the above example from Wells, 1992, p. 130) is relatively new; it first appeared in A. Cyril Pearson’s 1907 *20th Century Standard Puzzle Book* (London). But its mathematical underpinnings have been traced back to Nicomedes ($\sim$ 180 BCE), as well as to Isaac Newton (1720) and Thomas Simpson (1745). The problem is to build a right triangle, given a right angle, and an inscribed rectangle (above, the rectangle is a square), together with a segment s, representing the length of a hypotenuse.

2 A modern solution using a simple trigonometric equation

At present the lengths of the legs of the right triangle can be found analytically by solving a simple trigonometric equation that is easy to derive(Figure 2):

So we have:

$$s_1 * \sin(x) = b$$
$$s_2 * \cos(x) = a$$
$$s_1 + s_2 = s$$

By eliminating s_1 and s_2 we get

$$b/\sin(x) + a/\cos(x) = s$$

Both solutions to this equation can be easily found, for example, by using SOLVER on a graphing calculator. Then the required height h is

$$h = s * \sin(x)$$

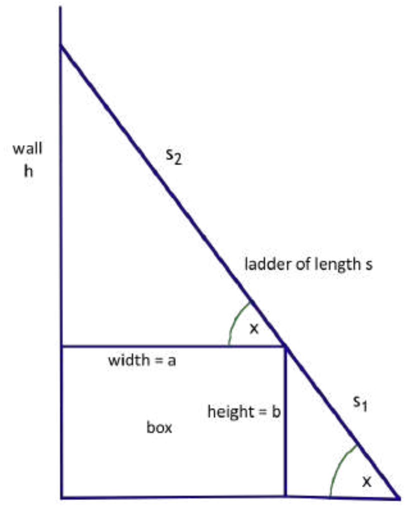


Diagram for a modern solution, from
http://sofia.nmsu.edu/~breakingaway/Lessons/LABP/LABP.html

Note that there are two solutions. The ladder can reach the wall either high or low. (http://sofia.nmsu.edu/~breakingaway/Lessons/LABP/LABP.html)

When $b = a$, this trigonometric equation can be reduced to two quadratic equations that can be solved using only square roots:
Let $y_1 = \sin(x)$ and $y_2 = \cos(x)$. So we have $\frac{a}{y_1} + \frac{a}{y_2} = c$.
Therefore we have:

$$y_1{}^2 + y_2{}^2 = 1$$
$$y_1 * y_2 = \frac{a}{c} * (y_1 + y_2)$$

Because $(y_1 + y_2)^2 = y_1{}^2 + y_2{}^2 + 2 * y_1 * y_2$, we have

$$(y_1 + y_2)^2 = \frac{2 * a}{c} * (y_1 + y_2) + 1$$

Let $z = y_1 + y_2$. We can find two values, z_1, z_2 of z by solving for z the following quadratic equation:

$$z^2 = \frac{2 * a}{c} * z + 1$$

Now for each z_i we have

$$y_1 + y_2 = z_i$$
$$y_1 * y_2 = \frac{a}{c} * z_i$$

So y_1 and y_2 can be found by solving for y the following quadratic equation:

$$y^2 - z_i * y + \frac{a}{c} * z_i = 0$$

For similar, but different, ways of solving this problem, see Simpson's 1745 solution in 3.b. below; and see also Uspensky (1948), p. 96, example 2; and Fisher (1972), pp. 97–98.

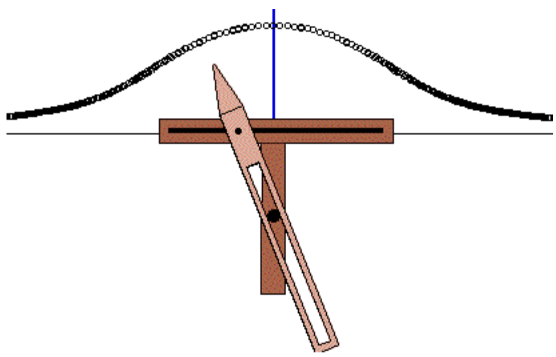
3 How the problem has been solved historically, and what it meant at each time "to solve a problem"

a. Geometric solutions
b. Algebraic solutions
c. Solutions as puzzles or as recreational mathematics

The problem is interesting from the point of view that its general case, finding a solution for an arbitrary inscribed rectangle with sides a and b, was considered to be very difficult. (It cannot be constructed with straight edge and compass, and the polynomial equation expressing the lengths of the legs in terms of a, b, and s, has degree four.) The main simpler version of the problem is when the rectangle is a square, which admits other techniques that don't work in the general case. The other special cases of the problem in which one can rather easily find the lengths of the legs of the required right triangle are when all the numbers involved, namely, the sides of the rectangle, the triangle's hypotenuse, and the two legs to be found, are rational. They often pop up in puzzle books and recreational mathematics, because they can be solved by "guess and check" methods.

a. Geometric solutions (at the time of the Greeks).

We credit this information to Audun Holme, *Geometry: Our cultural heritage* (2010, chapters 2, 3, and 4). For an arbitrary rectangle, the problem cannot be solved by a straight edge and compass construction. But it can be solved by using Nicomedes' conchoids, and tools related to it. Here is a diagram of such a tool:

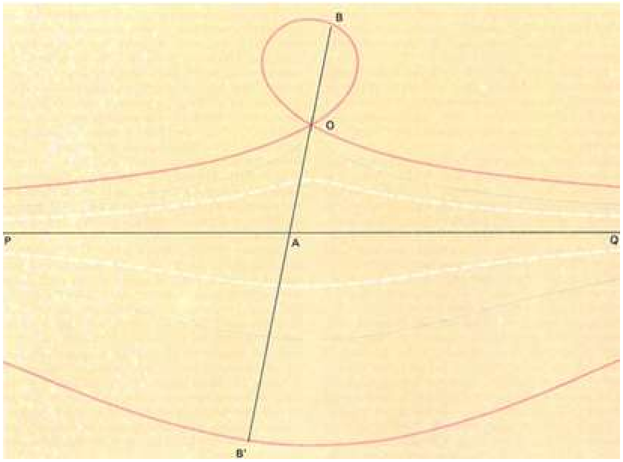


A tool for drawing the conchoids. from http://perseus.mpiwg-berlin.mpg.de/GreekScience/Students/Tim/Trisection.page.html

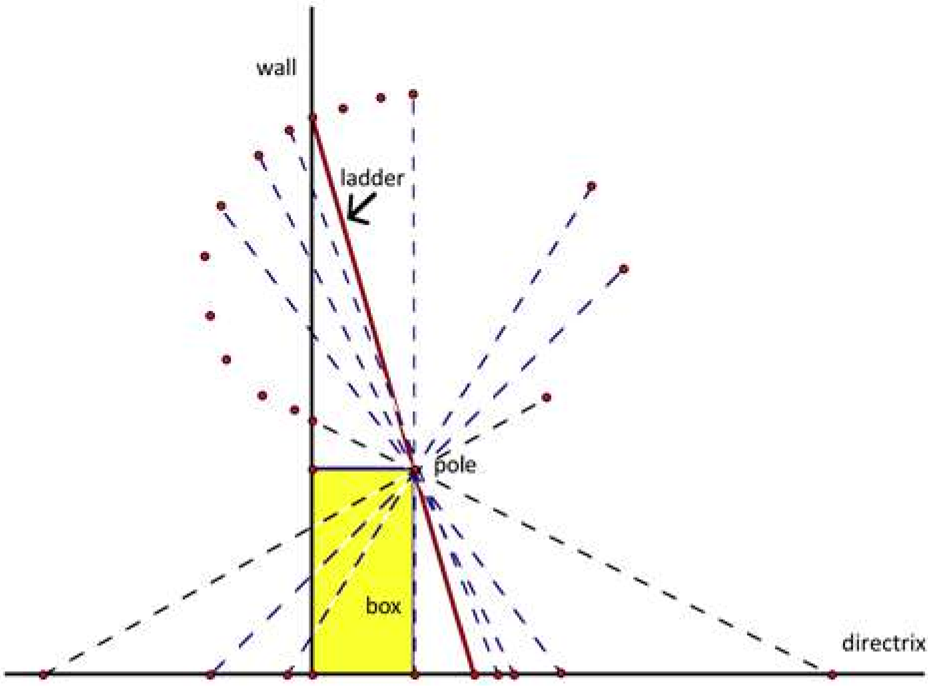
The tool above will draw both parts of the conchoid. Below we show both parts, with the "loop" above rather than below the directrix (the horizonal line PQ)(Figure 4).

But because for our problem we need only one point on a curve, a simple marked ruler suffices(Figure 5).

At that time the solution to this problem was a step-by-step description of a geometric construction using only specified "tools". But these "tools" seem to represent rather abstract operations. For example, using a "straight edge" meant that you can construct a unique straight line passing through any two given points, and using a "marked ruler" meant that you can put a point on a conchoid, given



The conchoid with the "loop" above rather than below the directrix, from http://www.daviddarling.info/encyclopedia/C/conchoid.html



A ladder against a wall (showing the conchoid's "loop"), drawn by authors.

a point on its directrix and its pole. No numbers were involved in any construction. The length of a segment was not a number, but the segment itself.

b. Algebraic solutions

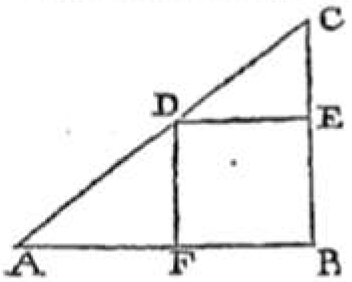
Both Isaac Newton in *Universal Arithmetick* (1720) and Thomas Simpson in *A Treatise of Algebra* (1745) analyzed a large number of geometric problems, showing how they can be solved using algebraic techniques. These solutions did not involve analytic geometry, because no coordinate systems were involved. Instead, the relationships among segments, such as proportions, were translated into equations involving lengths of those segments. Then, step-by-step procedures of solving these equations were shown. Both authors considered only the special case (where the rectangle is a square, $a = b$), because it leads to a biquadrate*[1] equation.

Below is the problem in Thomas Simpson's 1745 book.

250 *A Treatise of* ALGEBRA.

PROBLEM XV.

The Side of the inscribed Square BEDF, *and the Hypothenuse* AC *of a right-angled Triangle* ABC *being given; to determine the other two Sides of the Triangle* AB *and* BC.



Let DE or DF $= a$, AC $= b$, AB $= x$ and BC $= y$; then it will be as $x : y :: x - a$ (AF) $: a$ (FD) whence we have $ax = yx - ya$, and consequently $xy = ax + ay$. Also we have $AB^2 + BC^2 = AC^2$, or $x^2 + y^2 = b^2$ (by *Eu.* 47.1), to which Equation let the double of the former be added, and there arises $x^2 + 2xy + y^2 = b^2 + 2ax + 2ay$, that is $\overline{x+y}|^2 = b^2 + 2a \times \overline{x+y}$, and consequently $\overline{x+y}|^2 - 2a \times \overline{x+y} = b^2$; wherefore, by considering $x+y$ as one Quantity, and compleating the Square, we have $\overline{x+y}|^2 - 2a \times \overline{x+y} + a^2 = b^2 + a^2$; whence $\overline{x+y} - a = \sqrt{b^2+a^2}$, and $x+y = \sqrt{a^2+b^2}+a$; which put $= c$, and then, by substituting $c - x$ instead of its Equal (y) in the foregoing Equation, $xy = ax + ay$, there will arise $cx - x^2 = ac$; whence x will be found $= \frac{1}{2}c + \sqrt{\frac{1}{4}cc - ac}$ and $y = \frac{1}{2}c - \sqrt{\frac{1}{4}cc - ac}$.

The problem as shown in Simpson (1745), p. 250.

In all cases a solution was a step-by-step algebraic procedure for computing the required numbers. But the solution showed only the method of finding the numbers. No specific numbers were involved, either in the formulation of the problem or in the procedure. A method of showing alge-

[1] A biquadrate, or biquadratic, equation is a quartic equation that can be solved with only square roots (no cubic roots are needed).

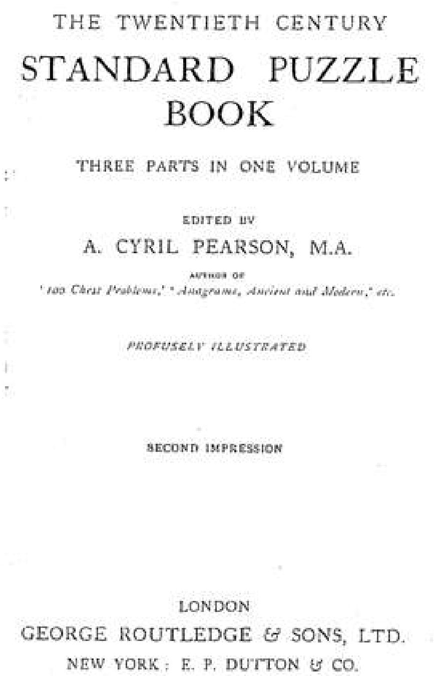
braic procedures by starting with a specific case with numerical coefficients, and only then showing a general procedure, which is common in modern algebra textbooks, was never used by Newton or by Simpson.

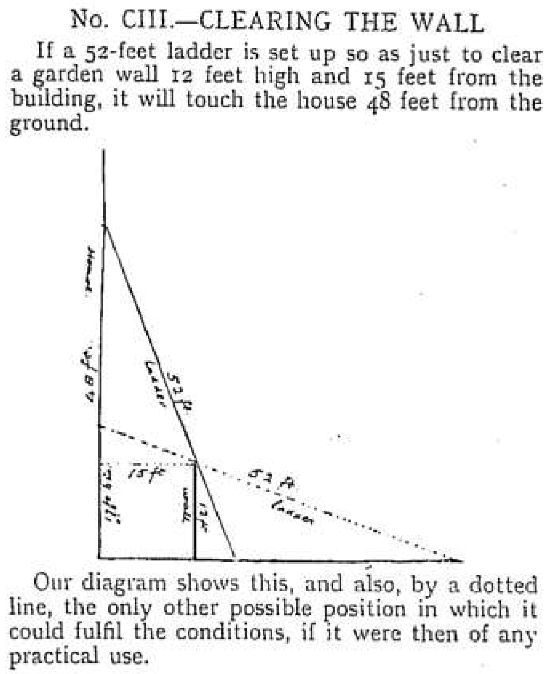
In applied mathematics, solutions to problems often include specific numbers. But they are almost never limited to numbers. Rather the opposite is true. In applied problems we usually want to know more about the solution, than we do in purely theoretical problems.

In a lecture she gave in 1969, Mary Cartwright talked about the fact that even when one looks just for a number, that is not all one wants to know, especially if one is an applied mathematician. She wrote about solutions to differential equations, " ⋯(one) really wants to know something about the solutions in general. Is there a periodic solution? Is it stable? Will it remain stable if I change a certain parameter? Will the period be longer or shorter?⋯" (Cartwright, cited in Ayoub, 2004.)

c. Puzzles and recreational mathematics

Here is the solution of the ladder and box problem in Pearson, *20th Century Standard Puzzle Book*, London (1907).


THE TWENTIETH CENTURY
STANDARD PUZZLE BOOK
THREE PARTS IN ONE VOLUME
EDITED BY
A. CYRIL PEARSON, M.A.
AUTHOR OF
'100 Chess Problems,' 'Anagrams, Ancient and Modern,' etc.
PROFUSELY ILLUSTRATED
SECOND IMPRESSION
LONDON
GEORGE ROUTLEDGE & SONS, LTD.
NEW YORK: E. P. DUTTON & CO.


No. CIII.—CLEARING THE WALL

If a 52-feet ladder is set up so as just to clear a garden wall 12 feet high and 15 feet from the building, it will touch the house 48 feet from the ground.

Our diagram shows this, and also, by a dotted line, the only other possible position in which it could fulfil the conditions, if it were then of any practical use.

Figures 7 and 8: from Pearson (1907), title page and p. 103.

(Pearson gives two solutions.)

Recreational mathematics is for amateurs. Problems are formulated, not in general, but in specific terms, and they are usually embedded in some kind of a story. The solution to the box and ladder problem is just a number, independent of the method used to find it. Also, in recreational problems, irrational numbers occur very rarely, and this explains why box and ladder problems have rational numbers as data and usually have rational solutions. All these examples fall into one category, which requires finding a rational root of a polynomial equation with rational coefficients. And this problem can be solved by Euler's method (which really is a "guess and check" method with a bounded number of guesses). But we did not find any author who discusses Euler's method in the context of this problem.

4 Final comments

In current mathematics, the modern meaning of solving a problem is really not much different from the meaning used by Newton and Simpson. We expect to see a general mathematical procedure to solve some reasonably large class of problems. And in applied problems we may also require some numerical values of the variables. The main difference is that the range of problems that can be solved is much larger and the use of technology is getting more and more prevalent.

But the use of technology brings some important changes. Now a person who solves a problem doesn't need to know *how* the problem is solved. So a student who solves the box and ladder problem by writing the equation,

$$b/\sin(x) + a/\cos(x) = s$$

and solves it on a graphing calculator for specific values a, b, and s, does not need to know anything about Newton's method, which provides solutions to these kinds of equations.

A comment about school mathematics

School mathematics is a special case. (By school math we mean all K-12 math and college math for non-math majors, that is not part of their professional training.) The concept of solving a problem in school mathematics is just like in recreational mathematics. Problems are embedded into some narratives (story problems) that are rarely realistic. What is required is usually just one or a few numbers, and the correctness of the answer is judged by their values. And also many educational researchers discourage teaching general procedures, and encourage improvisation as being more creative. This trend goes against the millennia-long trend in the development of mathematics that is going toward general solutions, which not only provide a numerical answer, but also explain how it can be done, and even why it should be done in this way and not in another way. But on the other hand, students are still drilled in very specific arithmetic procedures with very narrow ranges of application (for example, addition of common fractions with different denominators) that were designed a hundred years ago for accounting and other practical purposes.

The "box and ladder" problem in modern classrooms

The problem we have described is just one in a group of "ladder" problems (see http://www.mathematische-basteleien.de/ladder.htm#Sliding%20Ladder%20Problems). Others are the "Sliding ladder" as a geometric problem (e.g., Gutenmacher & Vasilyev, 2004, pp. 1–3, 113–114), the "Sliding ladder" as a dynamic calculus problem (e.g., Foerster, 2005, p. 178), and the "Two crossed ladders" problem (e.g., Gardner, 1979, pp. 62–64; Wells, 1992, p. 131).

With its rich history the "box and ladder" problem can be placed in several strands of high school mathematics.

In geometric constructions done either by hand or with computer software, the problem demonstrates the role of "basic tools", namely, the class of curves that can be drawn. In algebra its square box version is a very challenging problem that can be solved by the use of quadratic equations. Finally, the general problem can be solved easily with calculator technology (see 2. above). But this presents a dilemma. Traditionally, solving such problems in school is not a goal in itself. Instead, it is only done to teach students some techniques and to help them understand more general principles. Does this mean, for example, that we should not use the TI-84 SOLVER program unless we teach students Newton's method for solving equations, which underlies SOLVER's software?

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TEACHING NUMBER THEORY FROM SOPHIE GERMAIN'S MANUSCRIPTS: a guided discovery pedagogy

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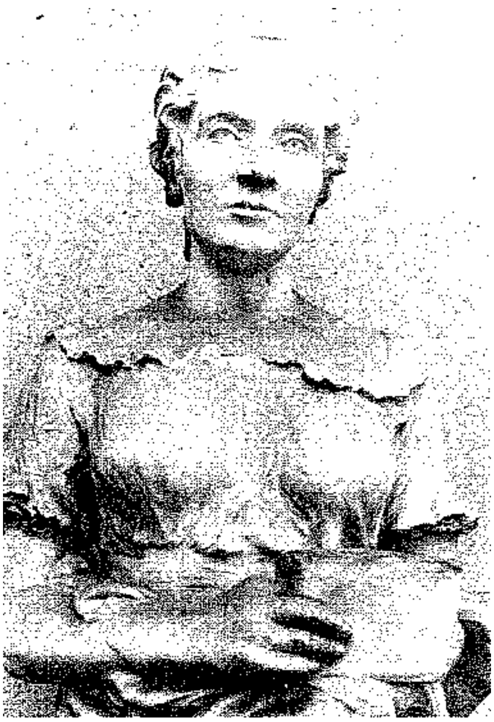
ABSTRACT

I will discuss a number theory course taught primarily through student study of Sophie Germain's research manuscripts and letters on Fermat's Last Theorem. Students learned everything in a first number theory course from original sources. I will discuss how original sources enhanced guided discovery and just-as-needed pedagogy.

Keywords: Number theory, Sophie Germain, Fermat's Last Theorem, manuscripts, primary sources, original sources, just-in-time, guided discovery, non-lecture, pedagogy



Sophie Germain (1776–1831)

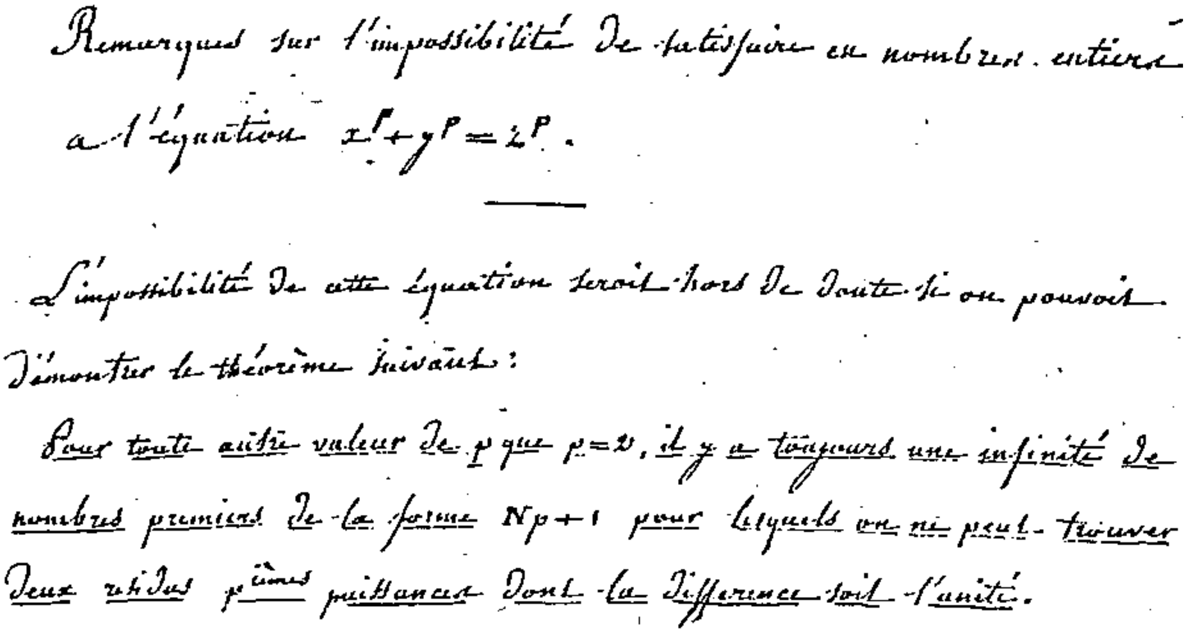
1 Introduction

I will describe the pedagogy of a number theory course taught entirely through studying original sources, primarily Sophie Germain's early nineteenth century research manuscripts and letters on

Fermat's Last Theorem. The course was taught as a mystery, with students as the detectives. A more detailed paper, with greater emphasis on the mathematical content of the manuscripts and the course, is in progress [10].

In all my courses, I aim to have my students study their mathematics directly from primary historical sources, and I have also moved away from lecturing to a classroom active with "guided inquiry" and "just-in-time" discovery. My goal is to dispense with textbooks presenting a purely modernized treatment, which often focuses on answers to questions not asked. I wish to base my courses on student discovery through primary sources aimed at answering meaningful questions [1, 2, 12]. Here I analyze how a method of "guided discovery" to learn mathematics "as needed" interacted with studying Germain's original manscripts in number theory, and discuss the student response.

2 A number theory course à la Sophie Germain


Remarques sur l'impossibilité de satisfaire en nombres entiers a l'équation $x^p + y^p = z^p$.

L'impossibilité de cette équation seroit hors de doute si on pouvoit démontrer le théorème suivant:

Pour toute autre valeur de p que $p=2$, il y a toujours une infinité de nombres premiers de la forme $Np+1$ pour lesquels on ne peut trouver deux résidus $p^{\text{ièmes}}$ puissances dont la différence soit l'unité.

Beginning of Sophie Germain's Manuscript A

Sophie Germain (1776–1831) was the first woman to do important original mathematical research [3]. In number theory she has been known only from a single unpublished result (today called "Germain's Theorem") toward proving Fermat's Last Theorem[1]. Recently, Germain's unpublished manuscripts and letters have revealed that she pursued an ambitious "grand plan" to prove Fermat's Last Theorem [3, 4, 6, 7]. I wondered if one could try teaching number theory largely with Germain's manuscripts.

Recently I was able to teach the standard beginning one semester number theory course at New Mexico State University. The course is at the advanced undergraduate and beginning graduate level. But as a first number theory course, the prerequisite is only a little abstract algebra, and I relied only on student facility at proving theorems.

[1]In the seventeenth century Fermat claimed that for a natural number $n > 2$, there are no natural number solutions to $x^n + y^n = z^n$. He was finally proven right by Andrew Wiles at the end of the twentieth century [6].

Let us compare the topics in a first course with Sophie Germain's research manuscripts. Germain was writing for experts, and was one of the first to utilize Gauss's congruence view. To understand her manuscripts and letters on Fermat's Last Theorem requires knowing unique factorization, Pythagorean triples, modular arithmetic, Fermat's Little Theorem, Lagrange's Theorem on the number of modular polynomial roots, modular roots of unity, and primitive roots modulo a prime. These are the topics of a first number theory course, along with the Quadratic Reciprocity Law. Conversely, if one understands the topics in a modern first course, one is equipped to understand Germain. Her research foundation is essentially what constitutes a first course today.

Germain was writing for readers such as Gauss and Legendre, so she did not develop any of the above topics in her manuscripts. Rather she assumed that her reader was already familiar with them, and used them freely in her writings. Thus it is more accurate to say that I taught "to" Germain's writings, not "from" them.

I told my students that their challenge was largely to understand Germain's progress towards proving Fermat's Last Theorem. This would be a detective story, because between any two Germain sentences there might be weeks of students learning what they needed in order to make the next leap. I planned to guide my students to learn all the topics in a first course by struggling to understand Germain's writings, with ancillary primary sources providing supplementary material on the Quadratic Reciprocity Law [5].

Next I provide just a small sample of Germain's writings, to give a sense of what students were challenged with, and what her grand plan was for proving Fermat's Last Theorem. A much more detailed picture of the source material and how it created the course content is provided in [10].

Already in the first few lines of Manuscript A of [7] (Figure 2) Germain gives away the big picture of her overall plan for proving Fermat's Last Theorem.

Sophie Germain, from Manuscript A

Remarks on the impossibility of satisfying in whole numbers the equation $x^p + y^p = z^p$.

The impossibility of this equation would follow without doubt if one could demonstrate the following theorem:

For every value of p other than $p = 2$, there is always an infinity of prime numbers of the form $Np+1$ for which one cannot find two p-th power residues whose difference is unity.

Just this much provokes the first several weeks of course study, including unique factorization, Pythagorean triples, residues, power residues, and inverses modulo primes. Students spend a lot of time deciphering and proving this claim.

In a nutshell, Germain's reason for her claim is what I call her

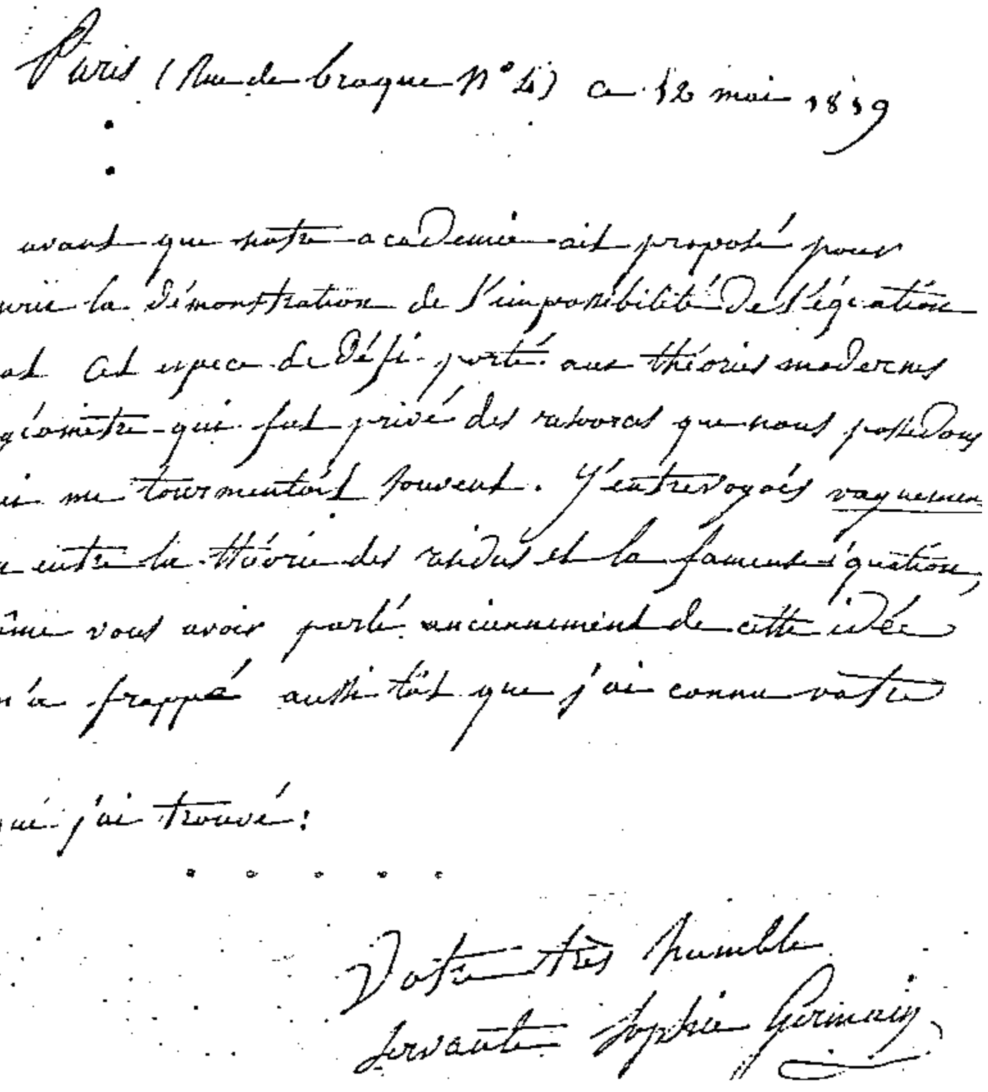
Basic Lemma. *Suppose $x^p + y^p = z^p$, and that θ is a prime satisfying*

Condition N-C. *There do **n**ot exist two nonzero **c**onsecutive p^{th} power residues, upon division by the prime θ.*

Then one of x, y, z is divisible by θ.

From this lemma, which students can prove, it is clear that if there are infinitely many primes $\theta = Np + 1$ satisfying Condition N-C, then since each of these would divide one of x, y, z, no such x, y, z can exist. Germain believed that infinitely many such primes θ existed, thus proving Fermat's Last Theorem. This was her grand plan, and most of her work was devoted to carrying it out.

Germain's extensive manuscripts, and her letters to Gauss (Figures 2, 2) and Legendre [7], lead to a multitude of student questions and investigations. These require Gauss' congruence viewpoint, and the existence of primitive roots for a prime modulus, which Germain uses heavily in her detailed analysis.


Paris (Rue de Braque N° 4) ce 12 mai 1819

Longtemps avant que notre académie ait proposé pour sujet de prix la démonstration de l'impossibilité de l'équation de Fermat cet espece de défi porté aux théories modernes par un géomètre qui fut privé des ressources que nous possédons aujourd'hui me tourmentait souvent. J'entrevoyais vaguement une liaison entre la théorie des résidus et la fameuse équation, je crois même vous avoir parlé anciennement de cette idée car elle m'a frappé aussitôt que j'ai connu votre livre.

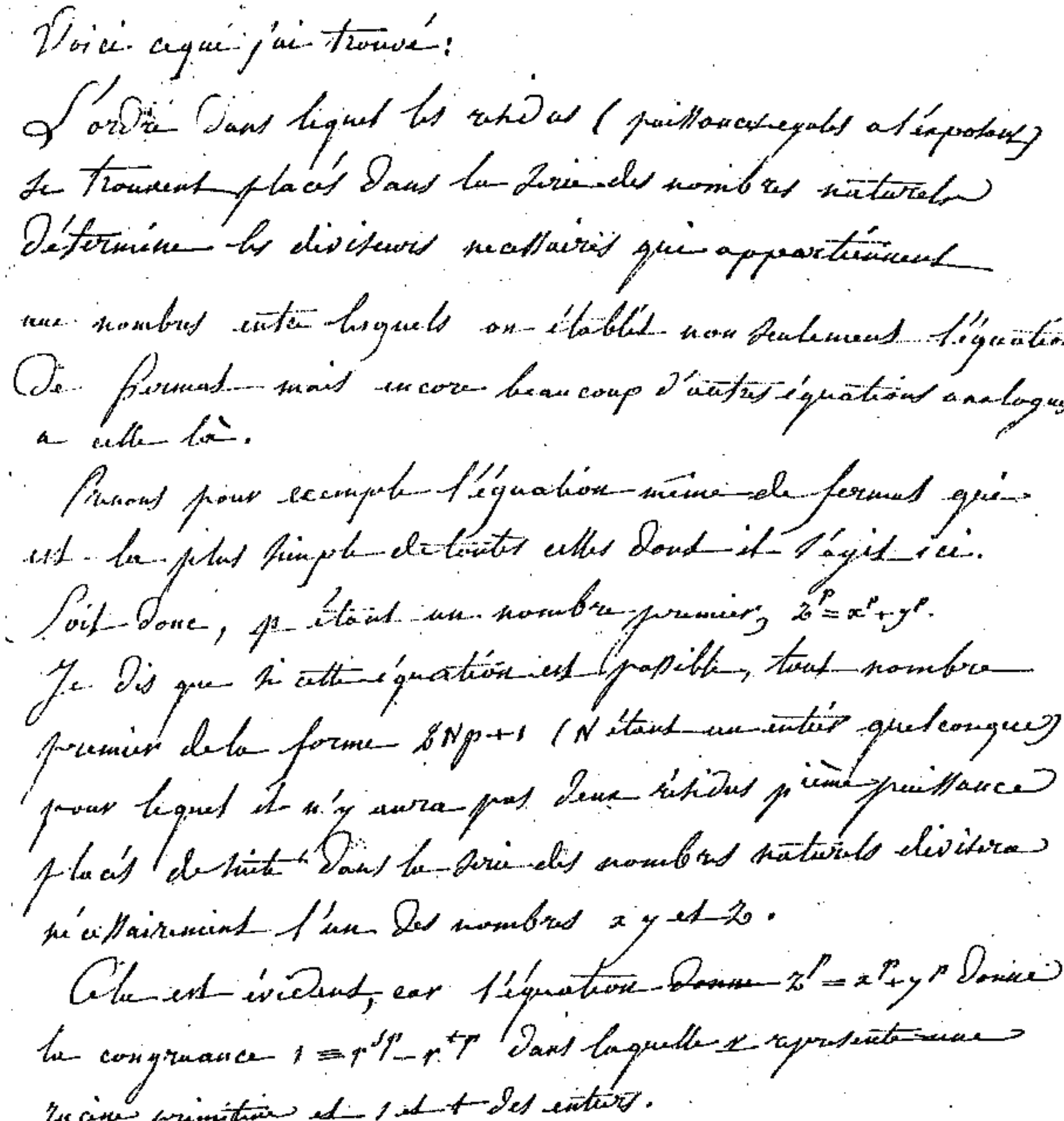
Voici ce que j'ai trouvé:

. . . . .

Votre très humble servante Sophie Germain

Germain's 1819 letter to Gauss

Students also study Germain's proof in her manuscripts of the unpublished "Germain's Theorem" on which her reputation formerly rested. She proves what we today call Case 1 of Fermat's Last Theorem under certain hypotheses, and also something more powerful, a "large size" theorem, "the necessity that the same numbers x, y, and z would be extremely large numbers". However, there is a flaw in Germain's proof of the large size of solutions, and one of the challenges to students is to find



Voici ce que j'ai trouvé :

L'ordre dans lequel les résidus (puissances égales à l'exposant) se trouvent placés dans la série des nombres naturels détermine les diviseurs nécessaires qui appartiennent aux nombres entre lesquels on établit non seulement l'équation de Fermat mais encore beaucoup d'autres équations analogues à celle là.

Prenons pour exemple l'équation même de Fermat qui est la plus simple de toutes celles dont il s'agit ici. Soit donc, p étant un nombre premier, $z^p = x^p + y^p$. Je dis que si cette équation est possible, tout nombre premier de la forme $2Np+1$ (N étant un entier quelconque) pour lequel il n'y aura pas deux résidus $p^{\text{ième}}$ puissance placés de suite dans la série des nombres naturels divisera nécessairement l'un des nombres x y et z.

Cela est évident, car l'équation $z^p = x^p + y^p$ donne la congruence $1 \equiv r^{sp} - r^{tp}$ dans laquelle r représente une racine primitive et s et t des entiers.

"Here is what I have found" from Germain's letter to Gauss and understand this.

In sum, understanding Germain's manuscripts ultimately entails almost all the topics of a first number theory course.

3 A pedagogy of just-in-time guided discovery

I integrated student study of Germain's manuscripts with non-lecture pedagogies [11]. The following pedagogies meld, and support each other. However, I will discuss them individually, including how they integrated with Germain's primary sources.

- *A question and inquiry based curriculum, with discovery guidance:*

  I wanted a curriculum driven by student investigation of meaningful questions, not by an instructor or book providing answers to unasked questions. Germain's manuscripts were perfect for this, with the larger question of Fermat's Last Theorem motivating understanding Germain's work, leading to numerous questions as she pursues her goal. Students are faced with numerous questions about how Germain knows the things she claims. I endeavored not to prove anything for students, but rather for them to learn everything through their own discoveries, with just the right tasks and guidance from me. I tried always to keep students in the driver's seat, with me charting the path.

  The experience indicates that a question based curriculum of guided discovery fits well with primary sources. Historical sources were usually written for experts of the period, not as teaching materials, and naturally evoke a wealth of questions. What could be more perfect? My challenge was to provide the right tasks and guidance.

- *Just-in-time (or just-as-needed):*

  I wanted motivation for new understanding always to come from investigation of Germain's writings. My goal was that students should always learn new things just-in-time, or even more strongly, just-as-needed. The drive for exploring something new should come from keeping one's eye firmly on the ball, namely Germain's research program. Only when students were stuck on something from Germain would I guide them to some new result that they needed to learn. My only exception would be sometimes to have students generalize something that had been discovered first for understanding Germain.

  I found that all desired course topics arose just-as-needed in understanding the manuscripts, though perhaps in different order than in a modern textbook. I forced myself to adhere to the just-as-needed maxim, and the result was highly satisfying. Students were always motivated by what was needed at each moment to advance with Germain.

  Primary sources are good for a just-as-needed pedagogy, since the barriers requiring new knowledge rise up naturally while studying a bigger picture.

- *No textbook in common:*

  I decided not to have a common textbook between me and my students, even as a supplement. There was a danger that I or my students would begin referring to the book in class interactions, and the focus would then shift away from Germain. By not having a common book, I would force myself to prepare assignments with all eyes always on Germain's manuscripts, and thereby reinforce both the guided discovery and just-as-needed pedagogies.

However, I did not want my students to feel too anxious at a lack of resources, so I asked each student to choose a book of their own to have as a comforting security blanket, and said I would be happy to discuss their book material with them at any time. This seemed to work very well.

I found that having no single textbook in common fit extremely well with primary sources, since it kept the focus on the primary sources, while providing a comforting sense of security for students.

- *Mystery detectives:*

  I designed the course with student as detective, learning mathematics as needed to follow Germain's trail to prove Fermat's Last Theorem. The primary sources were well suited to this, partly because they were research manuscripts, so one could see problems being solved firsthand, and one needed to learn all the background to keep up with the trail. I believe many courses could be designed around primary sources as a detective mystery for solving big, interesting questions.

- *Reading in advance, preparatory work for an active classroom, then work to complete at home:*

  I have a three part non-lecture pedagogy to obviate lecture, described in detail at [11]. Students have three types of work to prepare for each class day, staggered over three consecutive units of material. First (Part A), they read new material for two class days hence, write questions about it or respond to my questions, and I receive these and read them, to help me prepare for that future class; my direction of in-class activity will be guided by these reading responses. Second (Part B), students prepare a mathematical assignment of medium level "warm-up" exercises, on material already previously read, to bring to class. Class time is spent first discussing the responses I received earlier to the reading, and then mostly on the warm-up work they have prepared for that same unit. It is discussed and dissected in groups, and as a whole class, presented on the board, etc. Third (Part C), an assignment of higher level "final" exercises is completed at home on material already worked through in class, for careful marking by me, and indicates the level of understanding reached by the student. New Parts A, B, C then continue for the next class day, each on different units of material.

  I found that this worked every bit as well in this course, with the Germain sources, as in my other courses. It fit particularly well with wanting to put students in the driver's seat to decipher Germain.

Here is an example of homework assigned on a single day early in the semester. It includes reading and writing in advance (6A) on new material for two class days hence (unit 6), warm-up exercises (5B) for the next class day (unit 5 in-class work), and final work (4C) to complete at home after today's classroom activities (unit 4) focused on the previously assigned Part 4B.

Homework 6A/5B/4C

Please write these three assignments on separate sheets

6A: Hand in at beginning of class next time.

Read and write your questions on the next paragraph(s) of Germain's manuscript A: "If for proving that ..." through "even more satisfying results".

5B: Prepare these to discuss together and present in class next time.

1. Look up the classification of Pythagorean triples, i.e., natural number solutions to the equation $x^2 + y^2 = z^2$. Their complete classification has been known for a long time. Write down the details of classifying primitive Pythagorean triples, i.e., those where x, y, z, have no common divisor.

4C: Hand in at beginning of class next time.

1. State and prove a theorem to justify Germain's claim that

   Now since nothing prevents the successive assignment of an infinity of values to N, one can conclude from what precedes that there must exist an infinity of values of p for which the equation $x^p + y^p = z^p$ is impossible.

   In other words, state a hypothesis making explicit what she claims in the preceding, and show how it would prove Fermat's Last Theorem for infinitely many values of p, by showing how it would lead to her italicized theorem at the beginning of Manuscript A.

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4 Conclusion

I found that guided discovery to learn mathematics "as needed" interacted extremely well with studying original sources. What were the particular challenges for me?

First, I had to prepare just the right assignments, and guidance for students, and be flexible based on what happened in the classroom. From non-lecture teaching I was already used to adjusting as I go along, and I knew Germain's manuscripts well, so I was mathematically equipped for this pedagogical task. The Germain manuscripts were conducive materials, always providing questions to further challenge my students.

Second, I had to resist the temptation to introduce new phenomena before they actually arose in Germain's writing. I found that as the topics arose naturally in Germain's manuscripts, I gained the resolve to let "just-as-needed" perform its function. Students were constantly motivated by each new challenge towards the big goal.

Teaching number theory to Sophie Germain's manuscripts was the most exciting teaching experience I have ever had, and the students rose to the challenge, embracing the experience from the beginning. We also read and discussed the book *Sophie's Diary* [8, 9], a fictional diary by Sophie Germain during the ages 13–17, teaching herself the mathematics she will need to gain the serious attention of Lagrange at age 18 (as she did in real life), battling societal and familial pressure not to study mathematics because of her sex, and living in the middle of the French revolution outside her door in the heart of Paris.

I am writing a book for the course based on Germain's manuscripts, with tasks and guidance for student and instructor to follow Germain's path. I endeavor to leave almost all the proofs to the student, with ample guidance and optional further exercises.

Number theory cookies

My students should have the last word. But first, I explain the final exam period, where groups presented their work on proof and applications of the Quadratic Reciprocity Law. Students brought homebaked number theory cookies (Figures 4, 4). Try deciphering the icing on the cookies, which include ingredients of Germain's plan for proving Fermat's Last Theorem, quadratic reciprocity, Louis XVI at the guillotine, and an escargot.

Here are selected student comments from anonymous course evaluations:

"I have seen so many connections this semester between what we do in this course and the other math classes I've been taking, more than I have in any other course. I will definitely continue to study number theory on my own because of this class! "; "much more challenging than I expected"; "I will hold on to these papers forever"; "I loved this class"; "It was really cool to read and learn directly from primary sources like Germain's manuscripts and letters! "; "I really liked the high level of student participation"; "Dr. Pengelley's way of teaching will influence my way of teaching in future. What I learn is to be a good teacher, teacher need to work hard more than anyone else. To make student active learner, the role of teacher is very important."; "I truly enjoyed learning number theory in a historical context through Sophie Germain's manuscripts, while also learning the material of a normal number theory course. I also really liked reading 'Sophie's Diary' and thought it promoted some good discussions"; "the fact that every class was conducted in a way that invited open discussion meant that I was comfortable adding my thoughts or asking questions"; "I really like the way historical sources were incorporated into the course, and that Dr. Pengelley has personally translated manuscripts from French and used them to help us learn number theory".

More number theory cookies

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USING HISTORY OF MATHEMATICS IN HIGH SCHOOL CLASSROOM:
SOME EXPERIMENTS IN TAIWAN

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1 Introduction

When did Taiwan start to develop the HPM? The date is hard to be definite. However, 1996 must be a decisive year. In this year, Prof. Horng Wann-Sheng accepted the task to host the conference HPM 2000 Taipei and initiated *HPM Taipei Tongxun* in order to promote HPM in Taiwan. There are two vital impacts on mathematics education of Taiwan. First, *HPM Taipei Tongxun*, now renamed as *HPM Tongxun*, has become the most important platform for high school mathematics teachers to acquire as well as to share knowledge, information, teaching skills, teaching experience, and materials of the HPM.[1]

Secondly, the HPM 2000 Taipei attracted many mathematics educators of every level in Taiwan, so that they became more familiar with the HPM and started to be involved in the HPM. For instance, there are one doctoral and thirty-two master's dissertations on the HPM in high school in the period between 2001 and 2011. In contrast, there are only two master these before 2001. Furthermore, authors of these dissertations are all in-service or pre-service teachers of mathematics, and they did their research and experiments in actual classrooms. Besides, from 2000 onwards, there are many articles about HPM, especially concerning historical materials and using history of mathematics in classroom. Now HPM comes to be a meaningful and legitimate subject in Taiwan.

In what follows, I will introduce some experiments and examples of using history of mathematics in high school, and share my own experience as well. Before that, I have to clarify that by "experiment" I mean in this article does not necessarily refer to a formal or an academic one, it may be an example or experience of a teacher using history of mathematics in his or her class.

2 Experiments and examples of using history of mathematics

2.1 Research projects conducted by Prof. Wann-Sheng Horng

In Taiwan, Prof. Wann-Sheng Horng is the first scholar to investigate how history of mathematics integrated with education of mathematics in all possible aspects. He led his team completing several

[1] In Taiwan, high school includes junior high school and senior school. The former is for students aged from 13 to 15, and the latter from 16 to 18.

research projects, like "Ancient Mathematical Texts used in the Classroom", "Teacher's Professional Development in Terms of the HPM" and so on. Since most members of Prof. Horng's team are teachers of high school mathematics, these research projects combined practical experience and aimed at realistic applications.

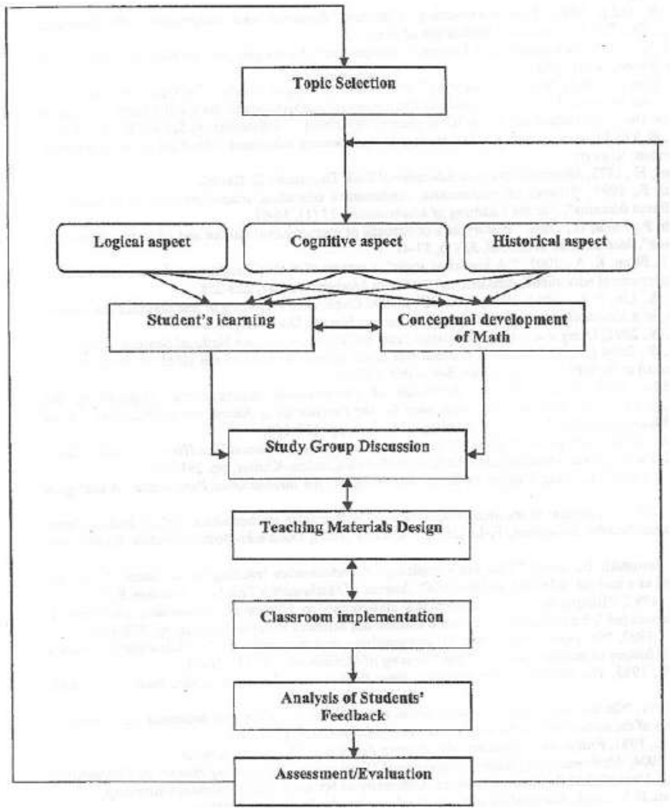
On the project "Ancient Mathematical Texts used in the Classroom", they have developed as many as twenty-nine teaching projects and worksheets/work-cards in terms of the HPM. "However, the participants were not aware that in this connection a subtle reconciliation of historical reflection with cognitive approach was necessary." (Horng, 2004) Jing-Ru Chiu and I took part in this project, and we designed three teaching projects and brought them into use in her classes. What we learned in the end of the project is that it is quite difficult to make a transition from history of mathematics to what students have learned or are going to learn. There are many interesting topics in history of mathematics, but what teachers think interesting is not necessarily suitable to students. For example, we developed a teaching project of Egyptian fractional numbers for 7th grade students. In the first class, students were all highly attracted by how ancient Egyptians wrote integers and used them to do addition, subtraction, multiplication, and division. Nevertheless, in the second class, students got confused with the reason why they needed to learn so complicated and useless Egyptian fractions. With these teaching experiences, we came to realize that Egyptian fractional numbers were fascinating to us, but they did not make much sense to some students. We appreciated the significance and the unique style of Egyptian fractions and their representations and arithmetical operations, but students did not. Moreover, did 7th graders need to know what Egyptian fractions were all about?

Through this research project we have learned that history of mathematics do bring to teachers as well as to students some benefit such as motivating learning of mathematics, appreciating cultural aspects of mathematics, seeing mathematics as human-being activities, and so on. However, history of mathematics is a double-edged sword, and it will do harm to math classes if teachers use it without second thought. How to use history of mathematics in teaching is a significant theme to which many papers have contributed. I will not go into the details. Instead, I am going to tell a story that has a significant impact on me for a long while.

When Jing-Ru Chiu and I participated in the research project, she was a junior high school teacher with practical experience of teaching mathematics. By contrast, I was a graduate student with lots passion in HPM but with little practical experience. When we cooperated to design teaching projects, we usually stood at the two ends of a balance. I wanted to put more and more history of mathematics into the teaching project while she concerned primarily not only about performance of teaching but about outcome of students' learning as well. Frankly speaking, at that moment I thought she was not capable enough to use history of mathematics in teaching. Nevertheless, in the end of this two-year research project, I had learned a lot from her about how to provide students proper and accessible materials related with history of mathematics.

2.2 Dr. Su Yi-Wen's doctoral dissertation

Dr. Su Yi-Wen's doctoral dissertation, *Mathematics Teachers' Professional Development: Integrating History of Mathematics into Teaching*, is one of the major outcomes of Prof. Horng's research project, "Teacher's Professional Development in Terms of the HPM". It is about her school-based research during a two-year period. There were four participants including Su herself in the research, and they were all mathematics teachers in the same senior high school. They developed a HPM model for designing teaching materials (see the diagram below, cited from Su, Yi-Wen, 2006) and finally completed teaching worksheets for eight topics: complex numbers, Heron's formula, circles, the mathematical expectation, metrics, transformations of translation and rotation, the concept of limit, and applications of limit. The overall process of completing teaching worksheets can divide into four parts.



First, after selecting a topic of senior high school mathematics, each participant needed to read relevant papers or books on history of mathematics and mathematics education. He or she not only analyzed the logical, cognitive, and historical aspects of the selected topic, but also took students' learning and conceptual development of mathematics into consideration. Then he or she discussed those with other participants and started to design teaching materials. The second part was implementing designed teaching worksheets/materials into class. Next, the designer shared his/her reflections on performance and responses from students with other participants, and looked for advices from others. Finally, the designer wrote a final report on his/her worksheets, and proposed suggestions for future users. Hence, teachers who are interested in these worksheets can easily get into practice through these reports.

After making a comprehensive survey of these reports, I discover several interesting things from designers' reflections and suggestions:

(1) What students are going to learn is mathematics, not history of mathematics

In each report, I see many positive responses from students. However, not every student approved of history of mathematics. Some thought learning from history of mathematics was inefficient, and some thought history of mathematics made mathematics even more difficult to learn. Therefore, the designers remind readers that the key to integrating history of mathematics into teaching is selecting proper historical materials and adapting them into accessible materials for students.

(2) History of mathematics inspires students not only in mathematics, but also in personality.

Prof. Wann-Sheng Horng has pointed out that telling historical stories can inspire students in personality. After finishing the research project, "Meta-Development of Teachers' Beliefs and Knowledge on History of Mathematics", Feng-Jui Hsieh concluded that one of the significances of integrating history of mathematics into instructions is to develop positive values in life. (Su, Yi-Wen, 2004) In the report on metrics, the author/designer told a story that he successfully helped a depressed student getting through his hard time by means of the story of Cayley and Sylvester. After listening to the story, the student actively fitted into his class and organized a study group. In a period, all members of the group progressed in studies.

(3) Responses from students helped the designers' professional development.

In all of the reports, authors/designers expressed their pleasure and satisfaction with most students approving of learning mathematics through the worksheets. Those feedbacks encouraged them to design worksheets for the other topics. This virtuous circle enhanced their professional expertise in terms of the HPM in an efficient way. Yi-Wen Su puts it that "by the end of the two-year project, it is obviously that the participants, in particular, T_1, have enhanced their professional expertise in terms of the HPM in following ways, namely, (i) they can begin to write popular mathematics articles; (ii) they are more reflective into their teaching than ever; (iii) they are able to integrate their mathematics knowledge into a broad picture; and (iv) he starts to care about the students' thinking. As a conclusion, the outcome of the project indicates that HPM approach can help the participants' professional development in an efficient way and can be another way for the in-service training." (Su, 2006)

2.3 Jun-Hong Su's Award Winning Teaching Projects

Jun-Hong Su is an experienced teacher of high school mathematics, and he wrote many articles about the HPM in varied publications for high school teachers in Taiwan. In addition, he designed several teaching projects in terms of the HPM, and won the first prize of the contest of teaching projects of high school science in 3 consecutive years, 2006, 2007, and 2008. The SpringSoft Education Foundation held this contest from 2005 to 2008. It asked participants to use the PowerPoint software to design and to present their projects. Su combined his experience of teaching mathematics and knowledge of mathematics history to win the prizes. The topics of these three teaching projects are the cosine formula, irrational numbers and conic sections.

The teaching project of the cosine formula has a distinctive feature that it deeply connects to the Pythagorean Theorem. It not only shows the connection between the cosine formula and the Pythagorean Theorem, but also illustrates that we can modify Euclid's proof of the Pythagorean Theorem in *Elements* to prove the cosine formula. See Figure 1. It is analogous to the diagram with which Euclid gives his proof except that the triangle ABC is not right-angled. We have rectangle AJ equal to rectangle AK, and rectangle BL equal to rectangle BK. Therefore, the sum of square AD and BH exceeds square AF by the sum of rectangle CJ and CL. Actually, rectangle CJ and CL are both equal to $\overline{AB} \cdot \overline{BC} \cdot \angle ABC$, and then we have the cosine formula:

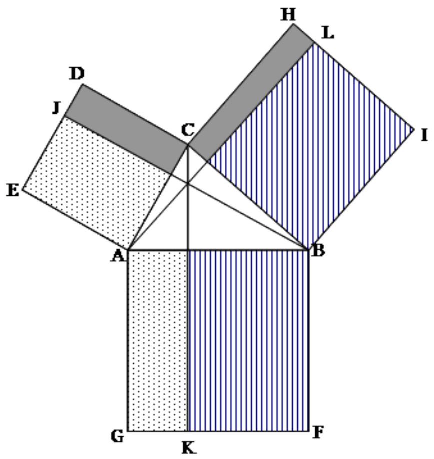


Figure 1

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 - 2 \cdot \overline{AB} \cdot \overline{BC} \cdot \angle ACB$$

Su expects that through this proof, students can regard the cosine formula as an inherent logical entailment of the Pythagorean Theorem.

Su's second teaching project aims to make students truly "perceive" irrational numbers. Students are told that irrational numbers are numbers that are not rational numbers. However, this kind of definition or explanation gives students almost nothing as to what is irrational number. In order to improve the situation, Su introduces the concepts of "commensurable" and "incommensurable" of Euclid's *Elements*. First, he shows the connection between rational numbers and commensurable magnitudes, and uses the Euclid algorithm to find the greatest common measure of two commensurable magnitudes. Second, he demonstrates the diagonal and the side of a square are incommensurable to explain the square root of 2 is irrational. (See Figure 2) Finally, he concludes that irrational numbers are those numbers that cannot be written to be fractional, ratios of two integers.

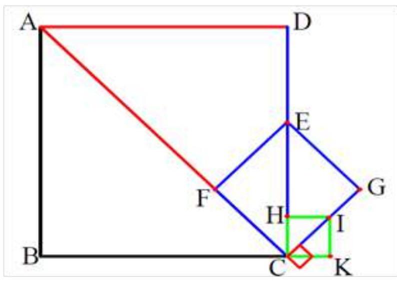


Figure 2

The third teaching project is about conic sections. The first part of it is Dandelin's theorem, and the most interesting thing is that Su uses the software, Cabri 3D, to display Dandelin spheres dynamically. The second part focuses on the meaning of the *latus rectum*, which Apollonius (ca. 262 BC 190 BC) called the upright side. In Taiwan, we have definitions of *latus rectum* for conic sections and formulas for calculating their lengths in senior high school mathematics textbook. However, that is all, nothing more. Therefore, students only memorize them without knowing how actually they meant. Hui-Yu Su, the editor of *HPM* Tongxun, wrote an excellent article to expound the original connotation of the latus rectum related to naming conic sections as *parabole*, *elleipsis*, and *hyperbole* in Apollonius' *Conics*.

Moreover, in the light of their original meaning, parabola, ellipse, and hyperbola may all have the same form of analytic expression, $y^2 = px \pm \frac{p}{d}x^2$. (See Hui-Yu Su, 2005) This article inspired Jun-Hong Su to integrate it into the second part of this teaching project (see Figure 3 below).

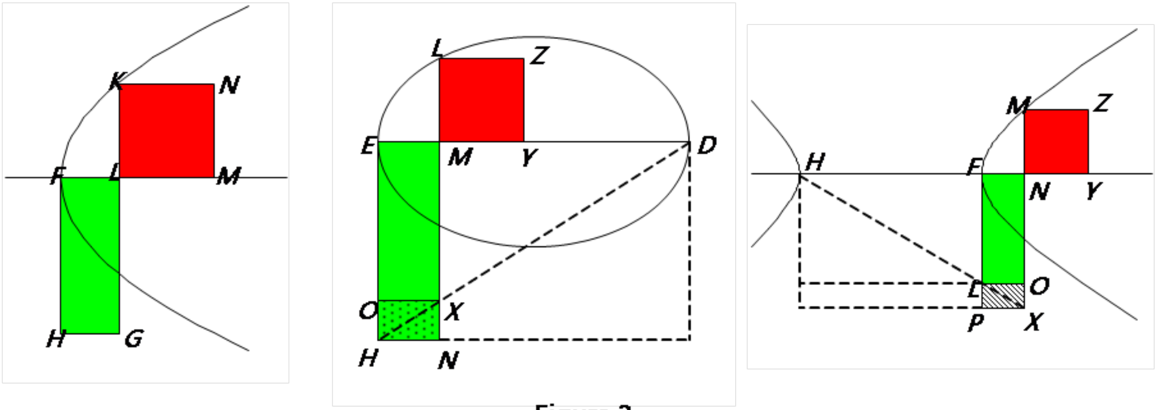


Figure 3

2.4 My Own Teaching Projects

Since 2007, I have designed several teaching projects, and two of them, tables of logarithm, and Cramer's rule, are illustrated in the following paragraphs. I used them in my classes of National Tainan First Senior High School.[2]

2.4.1 Tables of Logarithm

In Taiwan, every student has to learn logarithm in his or her first year of senior high school, and to memorize the number 0.301 as an approximation of log2. One day a question crossed my mind that if a student asked me about how to find out the approximation of log2, what my answer would be. The only method I remembered then was to approximate it by using Taylor's Formula, but Taylor's Formula is beyond what my students has learned. In addition, using Taylor's Formula to explain the "genetic" development of logarithms tables is anachronistic. Therefore, I looked into the history of logarithm and found something worth showing to students. Consequently, I designed a series of worksheets of logarithm tables and used them in my classes. Responses from my students were quite positive, so I wrote two articles about this topic, my worksheets and my students' feedbacks. Some teachers told me these articles not only inspired them but also intrigued them to take my worksheets into use. Those flatter me a lot indeed.

When I developed my worksheets of logarithm tables, I took some ideas and materials from two articles concerning using history of mathematics to teach logarithm in *HPM Tongxun*. They are "Shu Xue Shi Rong Ru Giao Xue: Yi Dui Shu Wei Li" (integrating history of mathematics into teaching: take logarithm for example) and "Dui Shu Sui Bi" (some things about logarithm) written by Jun-Hong Su and Zhi-Yang Horng respectively, who are both senior high school mathematics teachers. In his article, Su offers four worksheets of logarithm. His first worksheet is devoted to Nicholas Chuquet (1455-1488) while the others Napier's logarithm. My first worksheet essentially bases on his first

[2]Founded in 1922, National Tainan First Senior School is one of the most academically competitive senior high schools in Taiwan. It has 57 classes with around 2,300 students, and only around 10 students are girls. Roughly speaking, nearly each student's percentile ranke of academic performance in the Basic Competence Test for Junior High School Students is above 94%.

one. In Horng's article, he applies logarithm to find the relation between orbital periods and radii of planets, and in this way, the third Kepler's law of planetary becomes easier to follow. I adapted this application for my last worksheet.

	Indices	Numerus notarum	
1	0		
2	1		
4	2	1	
16	4	2	Tetras prima
256	8	3	
1024	10	4	
10,48576	20	7	
109,9511627776	40	13	Tetras secunda
12089,25819,61463	80	25	
12676,50600,22823	100	31	
16069,38044,25899	200	61	
25822,49878,08685	400	121	Tetras tertias
66680,14432,87940	800	241	
10715,08607,18618	1000	302	
11481,30695,27407	2000	603	
13182,04093,43051	4000	1205	Tetras quarta
17376,62031,93695	8000	2409	
19950,63116,87912	(1000)	3011	→ 10000

Except for the first and the last worksheet, the remaining included two parts. First, Henry Briggs (1561-1630) suggested John Napier (1560-1617) to change the logarithm into what we use today, and utilized his brilliant method to calculate approximations for the logarithm tables. The cited right tabulation is from Ian Bruce's translation of Briggs' *Arithmetica Logarithmica*, and there is a mark I made to show an error in the tabulation. This tabulation illustrates how Briggs approximated log2. The second column represents the degree of 2^{100}. Take 100 for example, the number 12676,50600,22823 in the first column is the first fifteen digits of , and the number 31 in the third column indicates there are 31 digits of 2^{100} . Then Briggs multiplied 12676,50600,22823 by itself to get the first fifteen digits and the total number of digits of 2^{200} . Briggs did not stop doing multiplication until he got the number 30,1029,9956,
6399, which is the total number of digits of $2^{10^{14}}$. Then Briggs obtained an approximation of log2 with high accuracy.

In what follows let me explain the procedure in today's notation:
$2^{10^{14}} = N \times 10^{30,1029,9956,6399-1}$,
$1 < N < 10 \Rightarrow 10^{14} \cdot \log 2 = (30, 1029, 9956, 6399 - 1) + \log N \Rightarrow \log 2 \doteq 0.30102999566398$

In my classes, I took $2^{10} = 1024$ to show my students how to find an approximation of log2:
$2^{10} = 1024 = 1.024 \times 10^3 \Rightarrow 10 \cdot \log 2 = \log 1.024 + 3 \Rightarrow \log 2 = \frac{1}{10} \cdot \log 1.024 + 0.3 \doteq 0.3$
Then I challenged them to get the approximation as accurate as possible by using electronic calculators. Actually, most calculators cannot show more than 13 digits which is the number of digits of 2^{40}. Moreover, the approximation of $\log 2$ coming from 2^{40} is 0.3 which is as same as the approximation coming from 2^{10}. This outcome somewhat depressed my students. Therefore, when they knew what Briggs had done without an electronic calculator, they all felt amazing and admired Briggs for his clever method and persistence as well.

Secondly, establishing logarithms tables was a slow, laborious job in the time without electronic calculators yet it brought much convenience for posterity though. I wanted my students to experience the process, so in the third worksheet of my teaching project, I asked them to calculate the approxi-

The first five digits of 7^n	n	The number of digits of 7^n	The first five digits of 7^n	n	The number of digits of 7^n
49	2	2		200	
2401	4	4		400	
	8	7		800	
	10	9		1000	
	20			2000	
	40			4000	
	80			8000	
	100			10000	

mation of log 7 by following Briggs' way, but used electronic calculators and wrote down the first five digits of 7^n (see the tabulation). Even using electronic calculators, they still spent some time to finish the task. Through this activity, they profoundly realized that to complete tables of logarithm was a huge task and people in that time definitely were in need of these tables for otherwise they would not need to do it. The following are my students' feedbacks:

* The method is amazing and unexpected!
* Briggs spent so many years on calculation and surprisingly, he could acquire so precise approximations without an electronic calculator. This shows that he was extremely persistent.
* It was very fortunate that Briggs altered Napier's logarithm with base $1 - 10^{-7}$. Briggs benefited later students. Thank you Briggs!!
* I finally realized that completing logarithm tables was a huge task, and surveys of astronomy and navigation would become easier with these tables. We should learn the predecessors' method well.
* I thought every concept of mathematics was easy to construct. Now I know that each concept we thought it as a matter of course came from hard work of mathematicians who even had devoted his whole life to it.
* It is more attractive to us to present mathematics in this way, and let us know development, application, and interesting things of mathematics.

2.4.2 Cramer's Rule

Hui-Yu Su, the editor of *HPM Tongxun*, selected 90 articles from volume 1 to volume 10 of *HPM Tongxun*, and sorted them by topics of senior high school mathematics. Actually, there are more than 120 articles relating to these topics in *HPM Tongxun* so far. Although the amount is huge, there are still some topics lacking research articles. Therefore, Su asked for them. As a deputy editor of *HPM Tongxun*, I chose the topic of Cramer's rule and designed a teaching project that I drew upon in two of my classes.

Nowadays, Cramer's rule is presented in the form of determinant. This, however, is completely different from the original version in Cramer's *Introduction à l'analyse des lignes courbes algébrique* (1750). In what follows, I take for example linear equations with three unknowns to explain Cramer's original

APPENDICE. 657

No. I.

Voyez pag. 59 & 60.

Soient plusieurs inconnues z, y, x, v, &c. & autant d'équations

$$A^1 = Z^1 z + Y^1 y + X^1 x + V^1 v + \&c.$$
$$A^2 = Z^2 z + Y^2 y + X^2 x + V^2 v + \&c.$$
$$A^3 = Z^3 z + Y^3 y + X^3 x + V^3 v + \&c.$$
$$A^4 = Z^4 z + Y^4 y + X^4 x + V^4 v + \&c.$$
&c.

où les lettres A^1, A^2, A^3, A^4, &c. ne marquent pas, comme à l'ordinaire, les puissances d'A, mais le prémier membre, supposé connu, de la prémiére, seconde, troisiéme, quatriéme &c. équation. De même Z^1, Z^2, &c. sont les coëficients de z; Y^1, Y^2, &c. ceux de y; X^1, X^2, &c. ceux de x; V^1, V^2, &c. ceux de v; &c. dans la prémiére, seconde, &c. équation.

Cette Notation supposée, s'il n'y a qu'une équation & qu'une inconnue z; on aura $z = \frac{A^1}{Z^1}$. S'il y a deux équations & deux inconnues z & y; on trouvera $z = \frac{A^1 Y^2 - A^2 Y^1}{Z^1 Y^2 - Z^2 Y^1}$, & $y = \frac{Z^1 A^2 - Z^2 A^1}{Z^1 Y^2 - Z^2 Y^1}$. S'il y a trois équations & trois inconnues z, y, & x; on trouvera

$$z = \frac{A^1Y^2X^3 - A^1Y^3X^2 - A^2Y^1X^3 + A^2Y^3X^1 + A^3Y^1X^2 - A^3Y^2X^1}{Z^1Y^2X^3 - Z^1Y^3X^2 - Z^2Y^1X^3 + Z^2Y^3X^1 + Z^3Y^1X^2 - Z^3Y^2X^1}$$
$$y = \frac{Z^1A^2X^3 - Z^1A^3X^2 - Z^2A^1X^3 + Z^2A^3X^1 + Z^3A^1X^2 - Z^3A^2X^1}{Z^1Y^2X^3 - Z^1Y^3X^2 - Z^2Y^1X^3 + Z^2Y^3X^1 + Z^3Y^1X^2 - Z^3Y^2X^1}$$
$$x = \frac{Z^1Y^2A^3 - Z^1Y^3A^2 - Z^2Y^1A^3 + Z^2Y^3A^1 + Z^3Y^1A^2 - Z^3Y^2A^1}{Z^1Y^2X^3 - Z^1Y^3X^2 - Z^2Y^1X^3 + Z^2Y^3X^1 + Z^3Y^1X^2 - Z^3Y^2X^1}$$

Introd. à l'Analyse des Lignes Courbes. Oooo L'é-

658 *APPENDICE.*

L'examen de ces Formules fournit cette Règle générale. Le nombre des équations & des inconnues étant n, on trouvera la valeur de chaque inconnue en formant n fractions dont le dénominateur commun a autant de termes qu'il y a de divers arrangements de n choses différentes. Chaque terme est composé des lettres $ZYXV$ &c. toujours écrites dans le même ordre, mais auxquelles on distribue, comme exposants, les n prémiers chiffres rangés en toutes les maniéres possibles. Ainsi, lorsqu'on a trois inconnues, le dénominateur a [$1 \times 2 \times 3 =$] 6 termes, composés des trois lettres ZYX, qui reçoivent successivement les exposants 123, 132, 213, 231, 312, 321. On donne à ces termes les signes + ou −, selon la Règle suivante. Quand un exposant est suivi dans le même terme, médiatement ou immédiatement, d'un exposant plus petit que lui, j'appellerai cela un *dérangement*. Qu'on compte, pour chaque terme, le nombre des dérangements: s'il est pair ou nul, le terme aura le signe +; s'il est impair, le terme aura le signe −. Par ex. dans le terme $Z^1Y^2V^3$ il n'y a aucun dérangement: ce terme aura donc le signe +. Le terme $Z^3Y^1X^2$ a aussi le signe +, parce qu'il a deux dérangements, 3 avant 1 & 3 avant 2. Mais le terme $Z^3Y^2X^1$, qui a trois dérangements, 3 avant 2, 3 avant 1, & 2 avant 1, aura le signe −.

Le dénominateur commun étant ainsi formé, on aura la valeur de z en donnant à ce dénominateur le numérateur qui se forme en changeant, dans tous ses termes, Z en A. Et la valeur d'y est la fraction qui a le même dénominateur & pour numérateur la quantité qui résulte quand on change Y en A, dans tous les termes du dénominateur. Et on trouve d'une manière semblable la valeur des autres inconnues.

Figure 1: *

Figure 4 Cramer's Rule of *Introduction à l'analyse des lignes courbes algébrique*

rule.

$$\begin{cases} A_1 = Z_1 \cdot Y_1 \cdot y + X_1 \cdot x \\ A_2 = Z_2 \cdot Y_2 \cdot y + X_2 \cdot x \\ A_3 = Z_3 \cdot Y_3 \cdot y + X_3 \cdot x \end{cases}$$

$A_1 \sim A_3$ are constants, $Z_1 \sim Z_3$, $Y_1 \sim Y_3$, $X_1 \sim X_3$ are coefficients of unknowns z, y, x respectively

$$z = \frac{A_1Y_2X_3 - A_1Y_3X_2 - A_2Y_1X_3 + A_2Y_3X_1 + A_3Y_1X_2 - A_3Y_2X_1}{Z_1Y_2X_3 - Z_1Y_3X_2 - Z_2Y_1X_3 + Z_2Y_3X_1 + Z_3Y_1X_2 - Z_3Y_2X_1}$$

Cramer gave a series of specific regulations to put down the values of unknowns:

(i) The values of unknowns have a common denominator. Each term of the denominator is in the form of $Z_aY_bX_c$, and a, b, and c are arrangements of number 1, 2, and 3: 123, 132, 213, 231, 312, 321. Therefore, the denominator has 6(=3!) terms.

(ii) We attach the sign "+" to the term with even number of "derangement", or we attach the sign "−". The "derangement" means infringing the condition $a < b < c$. For examples, the term $Z_3Y_2X_1$ has the sign "−" because there are two derangements: 3 before 1 and 3 before 2. The term has the sign " " because there are three derangements: 3 before 1, 3 before 2 and 2 before 1. After these, we have the denominator.

(iii) We change Z_1, Z_2, Z_3 into A_1, A_2, A_3 respectively, and then we find the numerator of the value of the unknown Z. We change Y_1, Y_2, Y_3 into A_1, A_2, A_3 respectively, and then we find the numerator of the value of the unknown Y. The value of the unknown X is obtained in a similar way.

In a word, Cramer used permutations to find the values of unknowns, rather than determinants. In my teaching project, I asked students trying to write down the Cramer's rule with four unknowns, and to imagine it with five unknowns. In addition to Cramer's original rule, I showed several pages of Colin Maclaurin's (1698 1746) *Treatise of Algebra* (1748) to my students, and asked them to explain what is the subject of these pages (see Figure 5). It was not difficult for them to identify the content of these pages as the so-called Cramer's rule. Moreover, after they knew Maclaurin already wrote the rule in 1729 before Cramer published his book in 1750, they answered the following three questions:

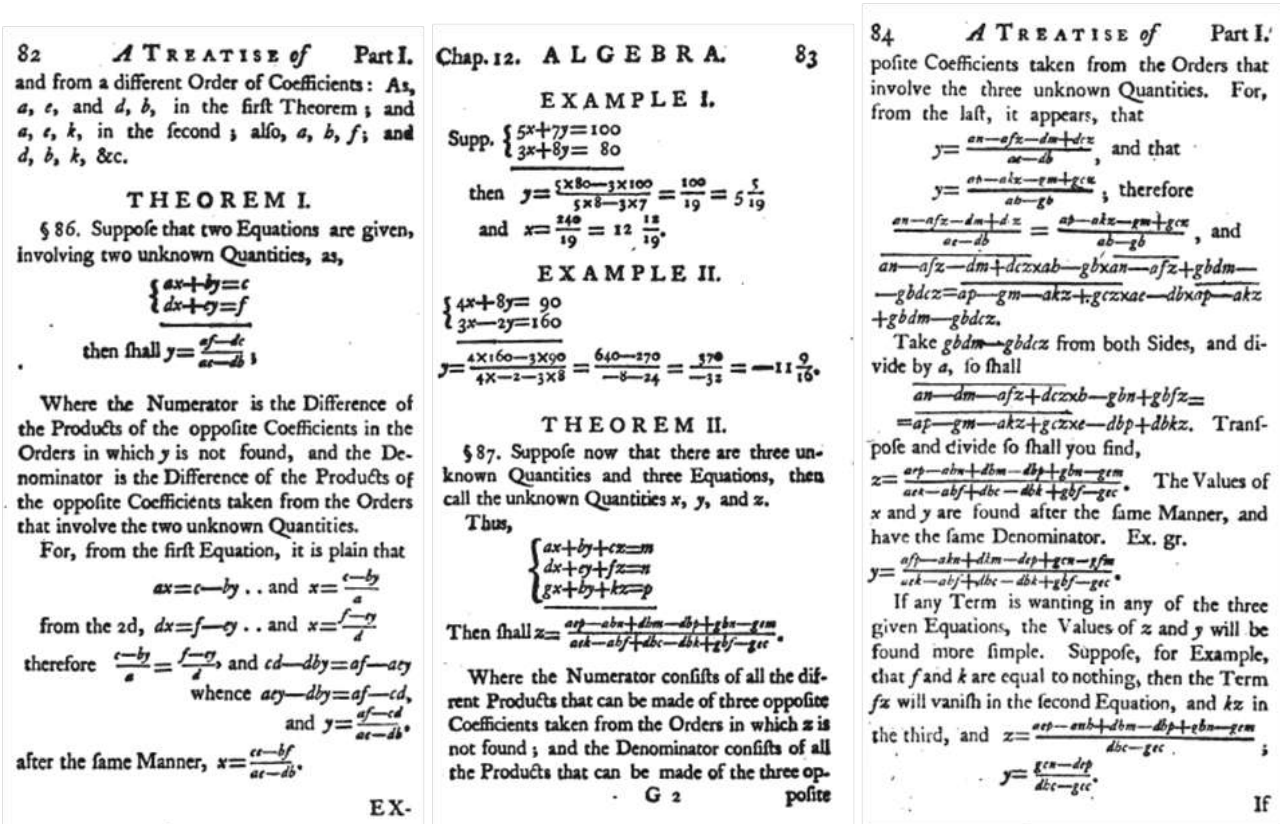
Question 1: Which one do you like? Cramer's or Maclaurin's rule?

Question 2: Should we call the name of the rule as Cramer's rule or Maclaurin's rule?

Question 3: What are the advantages of representing the rule in the form of determinant?

There are 85 students in my two classes, and in the case of question 1, only 2 students liked Cramer' original rule, 30 students liked Maclaurin's, and 33 students disliked both of them. Most students disfavored Cramer' original rule because of its complexity of expression, and they preferred Maclaurin's rule because they comprehended what Maclaurin had done. When it came to question 2, needless to say, most students approved of the name, Maclaurin's rule. However, one student defended for Cramer because Cramer told us how to find the values of unknowns no matter how many unknowns, but Maclaurin only told us the rule with three unknowns. After listening to his explanation, many students became hesitant. I was very glad that some one could perceive the deep difference between them and triggered off other students' second thought.

Although I have some experience in terms of the HPM, this teaching project was not successful



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and from a different Order of Coefficients: As, *a*, *e*, and *d*, *b*, in the firſt Theorem; and *a*, *e*, *k*, in the ſecond; alſo, *a*, *b*, *f*; and *d*, *b*, *k*, &c.

THEOREM I.

§ 86. Suppoſe that two Equations are given, involving two unknown Quantities, as,

$$\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$$

then ſhall $y=\frac{af-dc}{ae-db}$;

Where the Numerator is the Difference of the Products of the oppoſite Coefficients in the Orders in which *y* is not found, and the Denominator is the Difference of the Products of the oppoſite Coefficients taken from the Orders that involve the two unknown Quantities.

For, from the firſt Equation, it is plain that

$ax=c-by$ . . and $x=\frac{c-by}{a}$

from the 2d, $dx=f-ey$ . . and $x=\frac{f-ey}{d}$

therefore $\frac{c-by}{a}=\frac{f-ey}{d}$, and $cd-dby=af-aey$

whence $aey-dby=af-cd$,

and $y=\frac{af-cd}{ae-db}$.

after the ſame Manner, $x=\frac{ce-bf}{ae-db}$.

EX-

Chap. 12. ALGEBRA. 83

EXAMPLE I.

Supp. $\begin{cases} 5x+7y=100 \\ 3x+8y=80 \end{cases}$

then $y=\frac{5\times 80-3\times 100}{5\times 8-3\times 7}=\frac{100}{19}=5\frac{5}{19}$

and $x=\frac{240}{19}=12\frac{12}{19}$.

EXAMPLE II.

$$\begin{cases} 4x+8y=90 \\ 3x-2y=160 \end{cases}$$

$y=\frac{4\times 160-3\times 90}{4\times -2-3\times 8}=\frac{640-270}{-8-24}=\frac{370}{-32}=-11\frac{9}{16}$.

THEOREM II.

§ 87. Suppoſe now that there are three unknown Quantities and three Equations, then call the unknown Quantities *x*, *y*, and *z*. Thus,

$$\begin{cases} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p \end{cases}$$

Then ſhall $z=\frac{aep-ahn+dhm-dbp+gbn-gem}{aek-ahf+dhc-dbk+gbf-gec}$.

Where the Numerator conſiſts of all the different Products that can be made of three oppoſite Coefficients taken from the Orders in which *z* is not found; and the Denominator conſiſts of all the Products that can be made of the three oppoſite

G 2

84 *A* TREATISE *of* Part I.

poſite Coefficients taken from the Orders that involve the three unknown Quantities. For, from the laſt, it appears, that

$y=\frac{an-afz-dm+dcz}{ae-db}$, and that

$y=\frac{ap-akz-gm+gcz}{ah-gb}$; therefore

$\frac{an-afz-dm+dcz}{ae-db}=\frac{ap-akz-gm+gcz}{ah-gb}$, and

$\overline{an-afz-dm+dcz}\times\overline{ah-gb}\times\overline{an-afz}+gbdm-gbdcz=\overline{ap-gm-akz+gcz}\times\overline{ae-db}\times\overline{ap-akz}+gbdm-gbdcz.$

Take $gbdm-gbdcz$ from both Sides, and divide by *a*, ſo ſhall

$\overline{an-dm-afz+dcz}\times h-gbn+gbfz=$
$=\overline{ap-gm-akz+gcz}\times e-dbp+dbkz.$ Tranſpoſe and divide ſo ſhall you find,

$z=\frac{aep-ahn+dhm-dbp+gbn-gem}{aek-ahf+dhc-dbk+gbf-gec}$. The Values of *x* and *y* are found after the ſame Manner, and have the ſame Denominator. Ex. gr.

$y=\frac{afp-akn+dkm-dcp+gcn-gfm}{aek-ahf+dhc-dbk+gbf-gec}$.

If any Term is wanting in any of the three given Equations, the Values of *z* and *y* will be found more ſimple. Suppoſe, for Example, that *f* and *k* are equal to nothing, then the Term *fz* will vaniſh in the ſecond Equation, and *kz* in the third, and $z=\frac{aep-anh+dhm-dbp+gbn-gem}{dhc-gec}$;

$y=\frac{gcn-dcp}{dhc-gec}$.

If

Figure 2: *

Figure 5

enough in practice. I discussed it with a teacher of reaching-practice status who went to the classroom and made a recording on digital video. We both primarily agreed that this teaching project has many merits. However, we could make it better in two aspects. First, I nearly adopted all of Cramer's symbols and notations in order to bring the flavor of history to students. However, they brought cognitive obstacles to students at the same time. It was not easy for students to learn new things with an unfamiliar system of symbol. In addition, there were so many symbols that Cramer's rule became an abstract monster. Therefore, I should use symbols that are familiar to students instead of Cramer's, and start the rule on two unknowns.

Second, it may be better to use this teaching project before students know Cramer's rule. I expected that students could appreciate mathematicians' efforts on developing this rule as students knew its modern form. However, it did not work. On the one hand, the original rule is much more complicated and troublesome than the modern one, so students became impatient and unwilling to follow it. On the other hand, the time I brought this teaching project into classroom was six day earlier than the final examination of the semester, and it was no wonder that a few students were not in the mood to know something unrelated to the exam.

Besides these two flaws mentioned above, there was an unanticipated and surprising gift from Cramer's original rule. In Cramer's approach, it is easy to count the number of derangements of any given term. However, if we ask the inverse question: how many terms have the same number of derangements, then we enter the realm of 20th century discrete mathematics. Actually, this question turns to be a problem about permutations and inversions (derangements). For example, permutation 31524 has four inversions, namely (3,1), (3,2), (5,2), and (5,4), and how many permutations formed by 1, 2, 3, 4, and 5 have four inversions? This line of research can trace back to Eugen Netto's Lehrbuch der Combinatorik in 1901. (Bóna, 2004) On the last day of the semester, I got together six students who are good at mathematics, and proposed two interesting methods to them to get start the study of this problem. (See Appendix) Although they all highly attracted by these two methods, they did not go further in their winter vacation. I wrote an article to show the way to solve this problem in detail, and two of my colleagues are very interested in it. We will work together to modify my teaching project to become a six-class lesson, including Cramer's original rule and modern discrete mathematics as well, for students in mathematics and science talented classes.

3 Concluding Remarks

From these experiments mentioned above, we can see that the HPM has rooted in these teachers' PCK (pedagogical content knowledge). When they integrated history of mathematics into instructions, they combined their expertise of the HPM and of the PCK to prepare practicable and suitable teaching materials for their students. Prof. Wann-Sheng Horng developed the hermeneutic tetrahedron from Niels Hans Janhke's hermeneutic twofold cycle to illuminate how the HPM enhances teachers' PCK. (Horng, 2004b & 2005) Being a mathematics teacher in high school, I propose my suggestions for teachers and scholars who are interested in using history of mathematics in classroom.

To design such kind of teaching materials, obviously, teachers need to know the history of the chosen topic in deep sense. It even requires an overview of history of mathematics. Take for instance Jun-Hong Su's award winning teaching projects, he cannot create them without recognizing the significance of incommensurable magnitudes, the Pythagorean Theorem in Euclid's *Element*, and the *latus rectum* in Apollonius' *Conics*. However, few teachers in high school are as good as Jun-Hong Su expertise in history of mathematics. So, how can we help them? To offer them a serious course of history of mathematics might be an option, but it is not attainable for most teachers. I asked some teachers and myself a question that *being a mathematics teacher in high school*, what kind of resources of the HPM is most helpful or useful for us.

I got two answers. One is popular articles about the history of topics in high school math textbooks. I have to stress the word "popular". There are many articles and books on history of mathematic in Taiwan, but most of them are written for mathematicians and historians, not for teachers in high school. Take logarithm for example, Napier is the main character in many articles and books, but Briggs is rarely mentioned or referred to. However, Napier's logarithm is unintelligible not only for students, but also for most teachers. Fortunately, there gradually come out articles and books basically written for high school teachers. *Math through the Ages: A Gentle History for Teachers and Others* was translated into Chinese by team members of *HPM Tongxun* in 2008. My wife Jing-Ru Chiu who is now a senior high school mathematics teacher told me she likes this book very much and thinks it useful. In addition, Hui-Yu Su, the editor of *HPM Tongxun*, continues to write a series of articles, named "HPM Gao Zhong Jiao Shi" (HPM in senior high school classroom), for senior high school mathematics teachers since 2011. Up to now, she has written seven series articles, and all of them are devoted to history of math topics in textbooks. These articles provide teachers abundant materials for designing teaching projects.

The other answer is teaching projects with guidelines in detail. Teachers who are not capable of designing teaching projects of history of mathematics are capable of integrating history of mathematics into instruction, as long as there are some things like reports written by participants of Dr. Su Yi-Wen's research program, or articles about my own teaching projects. Through these reports and articles, teachers acquire not only guidelines and advices about implementation, but knowledge of history of mathematics. Moreover, teachers can easily adapt these teaching projects for their students. In other words, these teaching projects are prototypes that can produce many teaching projects. As a teacher adapts more and more teaching projects, he or she would acquire more and more expertise in terms of the HPM. One day he or she may be able to create a new prototype for others.

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Appendix

The symbol $f_m(n)$ denotes the number of permutations formed by 1, 2, 3…m with n inversions. For example, permutations formed by 1, 2, and 3 have three numbers of inversions:

Number of inversions	0	1	2	3
Permutation	123	213、132	231、312	321

We write $f_3(0)=1$, $f_3(1)=2$, $f_3(2)=2$, and $f_3(3)=1$. Let start the observation from $m=1$.

$m=1$：

1 $\Rightarrow$ number of inversions：0

n	0
$f_1(n)$	1

$m=2$：

1 2 $\Rightarrow$ number of inversions：0

2 1 $\Rightarrow$ number of inversions：1

n	0	1
$f_2(n)$	1	1

$m=3$：

□ □ 3 ⟨ 1 2 3 $\Rightarrow$ number of inversions：0
2 1 3 $\Rightarrow$ number of inversions：1

$\Rightarrow$ The numbers are as the same as $m=2$.

□ 3 □ ⟨ 1 3 2 $\Rightarrow$ number of inversions：1
2 3 1 $\Rightarrow$ number of inversions：2

$\Rightarrow$ Each number increases 1.

3 □ □ ⟨ 3 1 2 $\Rightarrow$ number of inversions：2
3 2 1 $\Rightarrow$ number of inversions：3
$\Rightarrow$ Each number increases 2.

n	0	1	2	3
	1	1		
		1	1	
			1	1
$f_3(n)$	1	2	2	1

$m=4$：

□ □ □ 4 $\Rightarrow$ The numbers are as the same as $m=3$

□ □ 4 □ $\Rightarrow$ Each number increases 1

□ 4 □ □ $\Rightarrow$ Each number increases 2.

4 □ □ □ $\Rightarrow$ Each number increases 3.

n	0	1	2	3	4	5	6
	1	2	2	1			
		1	2	2	1		
			1	2	2	1	
				1	2	2	1
$f_4(n)$	1	3	5	6	5	3	1

$\vdots$

$\vdots$

$$\{f_5(n)\}_{n=0}^{10} = \{1,\ 4,\ 9,\ 15,\ 20,\ 22,\ 20,\ 15,\ 9,\ 4,\ 1\}$$

$$\{f_6(n)\}_{n=0}^{15} = \{1,\ 5,\ 14,\ 29,\ 49,\ 71,\ 90,\ 101,\ 101,\ 90,\ 71,\ 49,\ 29,\ 14,\ 5,\ 1\}$$

$$\{f_7(n)\}_{n=0}^{21} = \begin{Bmatrix} 1,\ 6,\ 20,\ 49,\ 98,\ 169,\ 259,\ 359,\ 455,\ 531,\ 573, \\ 573,\ 531,\ 455,\ 359,\ 259,\ 169,\ 98,\ 49,\ 20,\ 6,\ 1 \end{Bmatrix}$$

These sequences constitute the Triangle of Mahonian Numbers (or Mahonian Triangle). The name shows the respect for Percy Alexander MacMahon (1854~1929):

1
1 1
1 2 2 1
1 3 5 6 5 3 1
1 4 9 15 20 22 20 15 9 4 1
1 5 14 29 49 71 90 101 101 90 71 49 29 14 5 1
1 6 20 49 98 169 259 359 455 531 573 573 531 455 359 259 169 98 49 20 6 1

Now, we introduce discrete mathematics. Rewrite the tabulation of $f_4(n)$ as follows:

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6
x^0	1	2	2	1			
x^1		1	2	2	1		
x^2			1	2	2	1	
x^3				1	2	2	1
$f_4(n)$	1	3	5	6	5	3	1

Then we see the procedure of finding $f_4(n)$ from $f_3(n)$ is multiplication of two polynomials:

$$(1+2x+2x^2+x^3)(1+x+x^2+x^3) = 1+3x+5x^2+6x^3+5x^4+3x^5+1\cdot x^6$$

In the similar way, we have $1+2x+2x^2+x^3 = (1+x)(1+x+x^2)$. Therefore, we find the generating function of $f_m(n)$:

$$F_1(x) = 1$$

$$F_2(x) = 1\cdot(1+x) = 1+1\cdot x$$

$$F_3(x) = 1\cdot(1+x)\cdot(1+x+x^2) = 1+2x+2x^2+1\cdot x^3$$

$$F_4(x) = 1\cdot(1+x)\cdot(1+x+x^2)(1+x+x^2+x^3) = 1+3x+5x^2+6x^3+5x^4+3x^5+1\cdot x^6$$

$$F_5(x) = 1 \cdot (1+x) \cdot (1+x+x^2)(1+x+x^2+x^3)(1+x+x^2+x^3+x^4)$$

$$= 1 + 4x + 9x^2 + 15x^3 + 20x^4 + 22x^5 + 20x^6 + 15x^7 + 9x^8 + 4x^9 + 1 \cdot x^{10}$$

$$F_6(x) = 1 \cdot (1+x) \cdot (1+x+x^2)(1+x+x^2+x^3)(1+x+x^2+x^3+x^4)(1+x+x^2+x^3+x^4+x^5)$$

$$= 1 + 5x + 14x^2 + 29x^3 + 49x^4 + 71x^5 + 90x^6 + 101x^7 + 101x^8 + 90x^9 + 71x^{10} + 49x^{11}$$

$$+29x^{12} + 14x^{13} + 5x^{14} + 1 \cdot x^{15}$$

$$\vdots$$

$$\vdots$$

$$F_m(x) = 1 \cdot (1+x) \cdot (1+x+x^2) \cdots\cdots (1+x+x^2+\cdots\cdots+x^{m-1}) = \prod_{k=0}^{m-1} (1+x+x^2+\cdots\cdots+x^k)$$

The coefficient of x^n of the expansion of $F_m(x)$ is $f_m(n)$. However, it still requires some work to write down the expression of $f_m(n)$.

A HISTORICAL TEACHING MODULE ON "THE UNREASONABLE EFFECTIVENESS OF MATHEMATICS"

The case of Boolean algebra and Shannon circuits*

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ABSTRACT

As a reaction to E. P. Wigner's paper on *The unreasonable effectiveness of mathematics in the natural sciences* from 1960, R. W. Hamming wrote a paper in 1980 called *The unreasonable of effectiveness of mathematics*, where he expanded Wigner's discussion by looking into the use of mathematics in computer science, being able to draw on his own 40 years of experiences in the area.

In the setting of the Danish upper secondary mathematics program this paper reports on the design and implementation of a so-called HAPh-module (History, Application, and Philosophy), where students were to read the original text by Hamming as well as two original texts by G. Boole and C. E. Shannon, respectively, illustrating the main point of Hamming's paper. More precisely the students worked with Boole's *An investigation of the laws of thought on which are founded the mathematical theories of logic and probabilities* (1854) and Shannon's *A symbolic analysis of relay and switching circuits* (1938). The implementation of the teaching module took place in a third year upper secondary mathematics class (students age 18–19) in the fall of 2011. Besides discussing the design of the module, a selection of data gathered during the implementation will be provided to illustrate outcomes (positive as well as negative) of the module.

1 Introduction

The motivation for carrying out the study described and discussed in this paper is threefold. First, in Denmark a reform, initiated in 2005, of the upper secondary school led to a more serious inclusion of elements of history in the mathematics program. Through "modules in the history of mathematics" students now must be able to "demonstrate knowledge about the development of mathematics and its interplay with the historical, scientific, and cultural development" (UVM, 2008, appendix 35, articles 2.3 and 2.1, my translation from Danish). Further, actual applications of mathematics play an important role in the new program, e.g. it says that students also must be able to "demonstrate knowledge about application of mathematics within selected areas, including knowledge about application in the treatment of a more complex problem" (ibid., article 2.1).

*The study presented in this paper is supported by the Danish Agency for Science, Technology, and Innovation.

Second, in 2002 a Danish report was published on *Competencies and Mathematical Learning* (recently it was translated into English: Niss & Højgaard, 2011), which besides listing eight mathematical competencies that students of mathematics are to develop and come to possess as part of their training, also lists three types of 2^{nd} order competencies or types of *overview and judgment* (OJ):

- o OJ1: the actual application of mathematics in other subject and practice areas;
- o OJ2: the historical evolvement of mathematics, both internally and from a social point of view; and
- o OJ3: the nature of mathematics as a subject.

Where mathematical (1^{st} order) competencies are a kind of "well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge", the three types of overview and judgment are "'active insights' into the nature and role of mathematics in the world" which "enable the person mastering them to have a set of views allowing him or her overview and judgement of the relations between mathematics and in conditions and chances in nature, society and culture" (Niss & Højgaard, 2011, pp. 49, 73).

Finally, the purpose of including history above, both in the Danish upper secondary mathematics program and as OJ2 in the report on competencies, has to do with a use of 'history as a goal', rather than one for instruction, i.e. 'history as a tool' (Jankvist, 2009a). Something similar is the case for including aspects of the actual application of mathematics, both in the upper secondary program and as OJ1. And also for OJ3, which we may associate with a use of philosophy in mathematics education, the purpose is rather one of 'goal' than of 'tool' (for further discussion, see Jankvist, forthcoming). Often when original sources play a role in mathematics education, it is mainly in the sense of a tool (e.g. Glaubitz, 2011; Barnett et al., 2011; Kjeldsen & Blomhøj, In press). But as we also know, e.g. from Jahnke (2000) and Fried (2001), a study of history through original sources introduces many aspects which are not only related to the actual learning of some mathematical concepts, theories, or methods. The study of original sources bears with it considerations of many aspects, e.g. in relation to the three types of overview and judgment mentioned above, which are not only a matter of understanding the mathematics treated; this being the third part of the motivation.

Thus, the overall motivation for the presented study is to do with if and how we may design upper secondary level teaching modules that through a use of original sources take into account all three types of overview and judgment simultaneously.[1] My way of trying to address this question in the present paper shall be mainly through students' own reactions and responses to such a teaching module. Since the three types of overview and judgment may be said to deal with the *H*istory (OJ2), the *A*pplications (OJ3), and the *Ph*ilosophy (OJ3) of mathematics, respectively, I shall refer to such teaching modules as *HAPh-modules*.

[1]See also Jankvist (2012).

2 HAPh-modules—design and implementation

What then is a HAPh-module? Following the concept of a *guided reading* of primary original sources, as developed by Pengelley, Lodder, Barnett, and others associated with the NMSU group,[2] where the reading of the original text(s) is 'interrupted' by explanatory comments, tasks, etc. (see in particular Barnett et. al, 2011), the idea is to have one original source representing the historical, the applicational, and the philosophical dimension, respectively. For the HAPh-module to be discussed here, the three original texts which the students studied – in Danish translation – were:

- o GEORGE BOOLE, 1854: *An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities*. (Boole, 1854)
- o CLAUDE E. SHANNON, 1938: *A Symbolic Analysis of Relay and Switching Circuits*. (Shannon, 1938b)
- o RICHARD W. HAMMING, 1980: *The Unreasonable Effectiveness of Mathematics*. (Hamming, 1980)

I shall explain in detail the contents of these three texts, their interrelations, and touch upon their exemplarity in relation to the three types of overview and judgment in the next section, but for now I shall focus on implementation and design.

During the three years of Danish upper secondary school, a class of 27 students were given two HAPh-modules, the first one on Euler's solution to the Königsberg bridge problem, Dijkstra's algorithm for finding shortest path, and Hilbert's 1900-lecture on mathematical problems (see Jankvist, 2011b; 2011c; forthcoming), and the one discussed in this paper. The first module was implemented in first year of upper secondary school (student age 16-17 years), and the second in their third and final year. After each implementation, the students were given a questionnaire containing both questions on the content of the modules and on their opinion about it. Half of the class was also interviewed about their questionnaire answers as well as their hand-in written material. During each implementation, I followed a focus group of 5 students in their work with the original texts and the associated tasks and assignments.[3] The duration of each teaching module was approximately ten 90-minutes lessons.

For each lesson, the students were to prepare in advance by reading a selection of the teaching material including the original texts. When meeting in class they then split into seven prefixed groups of approximately 3-4 students – the same groups during the entire implementation. Here they worked on the tasks given in relation to the texts as part of the guided reading approach. These tasks could be of various kinds, asking, for example: how we may be certain that Boole assumes associative and multiplicative properties for his classes; what Hamming means, when he says that science describes only 'how' and not 'why'; or in relation to the text by Shannon to have the students search the Internet to find out what a 'relay' is, a 'switch', a 'circuit', and to find examples of pictures of such. In general, the teaching material (Jankvist, 2011d) was designed for 'self-study', meaning that the students worked mainly by themselves in their groups, but with the possibility of asking their teacher

[2]NMSU is New Mexico State University. The group's teaching materials based on original sources may be found at: http://www.math.nmsu.edu/hist_projects/ and http://www.cs.nmsu.edu/historical-projects/ (Retrieved on February 15, 2012). In particular the projects by Janet Heine Barnett (2011a; 2011b) have served as a source of inspiration for the HAPh-module discussed in this paper.

[3]The focus group(s) were video filmed during the two implementations.

for help if needed. This also meant that the teacher would not give lectures at the blackboard. But by circulating among the groups during lessons, the teacher would still have an idea of how the students progressed with the material.

After having read and studied the three original texts and done the related tasks, the students were to do a collection of *essay assignments*. In a previous study I have found that this is a good way of bringing students to work with aspects of history as a goal (Jankvist, 2011a). For this reason, the same approach was taken to bring in the two dimensions of applications and philosophy. The particular setting creates a scene, where students near the end of the implementation of the teaching module are to discuss in their groups selected meta-perspective issues – or *meta-issues* – regarding the case. These meta-issues are chosen beforehand and included in the description of the essay assignments. More precisely, as part of the essay assignments, the more historical text by Boole and the application oriented text by Shannon were to be related to the philosophical discussion in Hamming's text, a task which to some degree also demands understanding of the inner mathematical issues – or *in-issues* – dealt with in the original texts. Furthermore, such discussions may force out some of the interplay between the three dimensions of history, application, and philosophy, although still exemplified by the concrete case. Once having outlined the content and context of the three original sources in the following section, I shall display an example of an essay assignment as well as an example of a student group 'essay', i.e. their answer.[4]

3 The historical case(s)

As mentioned, the philosophical theme for this module was Hamming's (1980) comment to a paper by the physicist Eugene Wigner from 1960, in which he discusses the "unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1960). Where Wigner's examples stem from the physical sciences, Hamming sets out to illustrate this unreasonable effectiveness of mathematics drawing on his own experiences from engineering – and aspects of what we today would consider to be computer science:

> During my thirty years of practicing mathematics in industry, I often worried about the predictions I made. From the mathematics that I did in my office I confidently (at least to others) predicted some future events – if you do so and so, you see such and such – and it usually turned out that I was right. How could the phenomena know what I had predicted (based on human-made mathematics) so that it could support my predictions? It is ridiculous to think that is the way things go. No, it is that mathematics provides, somehow, a reliable model for much of what happens in the universe. And since I am able to do only comparatively simple mathematics, how can it be that simple mathematics suffices to predict so much? (Hamming, 1980, p. 83)

As may be seen from the above quote, Hamming approaches his question from a rather constructivist point of view, which of course rules out some of the more Platonic explanations for the effectiveness of mathematics. This can also be seen from his statement that "Indeed it seems to me: The Postulates of Mathematics Were Not on the Stone Tablets that Moses Brought Down from Mt. Sinai" (Hamming, 1980, p. 86). Nevertheless, Hamming does point out that even though the standards of

[4]The two teaching materials may be found as texts 486 and 487 at http://milne.ruc.dk/ImfufaTekster/

rigor in mathematics may change over time, and with that definitions and proofs, the mathematical results often stay intact. After having discussed the effectiveness of mathematics and what mathematics is, Hamming (1980, pp. 88-89) goes on to provide some partial explanations for the unreasonable effectiveness of mathematics arranged under four headings, among these that "*We see what we look for*", meaning that we approach situations with an intellectual apparatus so that in many cases we can only find what we do – Hamming provides a parable by the physicist Arthur Eddington, saying "Some men went fishing in the sea with a net, and upon examining what they caught they concluded that there was a minimum size to the fish in the sea" – and that "*We select the kind of mathematics to use*", meaning that we select the mathematics to fit the situation, and that the same mathematics does not work in every place.

Boole's *The Laws of Thought*$\cdots$ from 1854 is an example of the latter, since he selects the elements from standard (arithmetic) algebra that applies to his logic system, the purpose of which he describes as follows:

> The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method; to make that method itself the basis of a general method for the application of the mathematical doctrine of Probabilities; and, finally, to collect from the various elements of truth brought to view in the course of these inquiries some probable intimations concerning the nature and constitution of the human mind. (Boole, 1854, p. 1)

In chapters II and III of his treatise, Boole considers the role of language in relation to the above and introduces a number of signs and laws to do so. More precisely, he introduces literal symbols x, y, etc. representing classes, and signs of operation $+$, $-$, $\times$ (times) and the sign of identity $=$ to be used on these classes. For example, if x stands for 'white things' and y for 'sheep', then the class xy stands for 'white sheep', similarly if z stands for 'horned things', then zyx stands for 'horned white things'. After associating the sign $+$ with the words 'and' and 'or', Boole deduces a number of laws which have their equivalent counterparts in standard arithmetic, e.g. the commutative law $x+y=y+x$, the distributive law $z(x+y)=zx+zy$, and the associative law (although this is not done as explicitly as for the others), and he deduces laws for the operation $\times$ (times) as well. The more interesting thing, however, is Boole's observation that in the context of his investigation we have that $xx=x$ (or $x^2=x$). If for example x stands for 'good', then saying 'good, good men' is the same as saying 'good men'. Boole then draws the consequence of comparing this to standard algebra:

> Now, of the symbols of Number there are but two, viz. 0 and 1, which are subject to the same formal law. We know that $0^2=0$, and that $1^2=1$; and the equation $x^2=x$, considered as algebraic, has no other roots than 0 and 1. Hence, instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity admitting only of the values 0 and 1. Let us conceive, then, of an Algebra in which the symbols x, y, z, etc. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. (Boole, 1854, pp. 26-27)

Boole's ideas went on to be adapted within mathematical logic and set theory, and the notion Boolean algebra was conceived. Some eighty years later, however, the ideas showed valuable in a very different setting than that of language and thought, namely design of electric circuits.

Shannon was a student at MIT when he got the idea for describing electric circuits by use of logic. With a set of postulates from Boolean algebra ($0 \cdot 0 = 0$; $1 + 1 = 1$; $1 + 0 = 0 + 1 = 1$; $0 \cdot 1 = 1 \cdot 0 = 0$; $0 + 0 = 0$; and $1 \cdot 1 = 1$) and their interpretations in terms of circuits (e.g. $0 \cdot 0 = 0$ meaning that a closed circuit in parallel with a closed circuit is a closed circuit; $1 + 1 = 1$ meaning that an open circuit in series with an open circuit is an open circuit), he was able to deduce a number of theorems which could be used to simplify electric circuits (see below). In an interview from 1987 in the magazine Omni, Shannon explained his use of Boolean algebra:

> It's not so much that a thing is 'open' or 'closed,' the 'yes' or 'no' that you mentioned. The real point is that two things in series are described by the word 'and' in logic, so you would say this 'and' this, while two things in parallel are described by the word 'or.' The word 'not' connects with the back contact of a relay rather than the front contact. There are contacts which close when you operate the relay, and there are other contacts which open, so the word 'not' is related to that aspect of relays. All of these things together form a more complex connection between Boolean algebra, if you like, or symbolic logic, and relay circuits.
>
> The people who had worked with relay circuits were, of course, aware of how to make these things. But they didn't have the mathematical apparatus of the Boolean algebra to work with them, and to do them efficiently. [⋯] They all knew the simple fact that if you had two contacts in series both had to be closed to make a connection through. Or if they are in parallel, if either one is closed the connection is made. They knew it in that sense, but they didn't write down equations with plus and times, where plus is like a parallel connection and times is like a series connection. (Shannon, 1987 in Sloane & Wyner; 1993, p. xxxvi)

For a given electric circuit $a - b$, Shannon defined the hindrance function X_{ab} to be 1 if $a - b$ is open and 0 if closed. For example, figure 2 (left) has the hindrance function $X_{ab} = W + W'(X + Y) + (X + Z) \cdot (S + W' + Z) \cdot (Z' + Y + S'V)$, where $+$ indicates series, $\cdot$ parallel and W' is the negation of W. Now, by means of manipulations according to his theorems of the expression for X_{ab}, Shannon is able to reduce this to $X_{ab} = W + X + Y + ZS'V$, the circuit of which is illustrated on figure 1 (right). (For exact reductions and theorems used, see Shannon, 1938b, p. 715 or Jankvist, 2011d, pp. 62-63.)

4 An example of an essay assignment and a student group essay

Having been introduced to the mathematical theorems and their proofs behind the reductions of the hindrance function above as well as the mathematics of Boole, the students were given the following essay assignment in order to relate the three original texts, and thus the three dimensions of history, application, and philosophy, to each other:

a. According to Hamming, what does it mean that a piece of mathematics is effective?

b. Do a comparison of the relative effectiveness of Boole's and Shannon's works (systems) distinguishing between effectiveness in terms of philosophy and effectiveness in terms of applications.

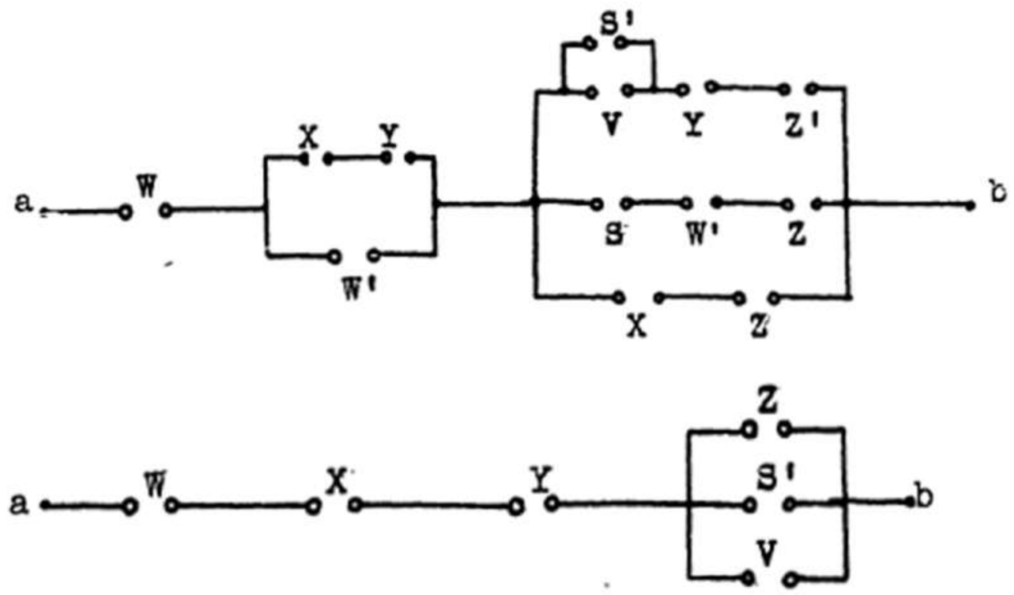

Figure 1 Left: The circuit to be simplified. Right: The simplified circuit after reductions on the hindrance function. (Shannon, 1938a).

c. Based on your answers to the above questions, discuss different types of 'the effectiveness of mathematics'. Recapitulate what Hamming means by the title of his paper *The Unreasonable Effectiveness of Mathematics*, and why it may be seen as 'unreasonable'.

d. Do you consider Boole's introduction and Shannon's application of the idea of an algebra only operating on the elements 0 and 1 along with the mathematical interpretation of 'and' and 'or' as an example of Hamming's 'the unreasonable effectiveness of mathematics'?

As an illustrative example of the students' work with this essay-assignment, I provide the following translated excerpt from one out of the seven groups:

> According to Hamming a piece of mathematics is effective when it can describe and predict natural phenomena. He finds mathematics puzzling in the sense that it can describe Nature using relatively simple formulas and expressions, practically without doing any experiments. [⋯]
>
> The philosophical effectiveness we see with Boole is connected to thoughts and the philosophy behind mathematics. It can say quite a bit about how we understand and confine mathematics with axioms [laws], for example when Boole makes it an axiom that x must be either 1 or 0. On the basis of philosophy he concludes something about mathematics' ways of thinking and methods. Shannon, however, on the basis of Boole's philosophical effectiveness, uses a method leading to his own effectiveness regarding application when transferring the [mathematical] theories to real life, where he at the same time tests them and thereby obtains an evaluation of the effectiveness. This is seen from the system [of electric circuits] on which his theorems are used.
>
> What Hamming believes is that mathematics is unreasonable because you are able to describe real events by simple mathematics and that we as human beings are finding it difficult to comprehend that apparently there are no limits to the range of mathematics regarding use in everyday life. By 'unreasonable' he means that it seems illogical that nature can be described by such simple

> operations, since nature for us seems complex, incomprehensible, and unpredictable – i.e. that we 'just' cannot plot them on a graph and get a result.
>
> Yes, we do consider Boole's introduction and Shannon's application as being examples of the unreasonable effectiveness of mathematics, since Boole shows that mathematics can [help] explain composition of language and that you can translate language directly into mathematics. In addition to this, Shannon shows, by means of Boole's introduction, that [the idea of using] the elements 0 and 1 can be applied on a circuit and hereby find the most simple [circuit]. These two examples fall outside what is usually considered the main field of mathematics and should, following Hamming, be described by more complex systems – but since this is not the case, it is natural, according to Hamming's line of thought, to call this mathematics unreasonable. [⋯] (Group 2, excerpt from hand-in)[5]

I'll return to the above excerpt in the discussion section of the paper, but first let us turn to the students' own reactions to module.

5 Students' reactions

It is difficult to draw a completely conclusive picture of which of the three dimensions the students' preferred and found to be fulfilled best in the module. When asked about this in the post-interviews, some would say the historical and some the philosophical, even though the majority of the students did claim that for them personally it was important to see an actual application of the mathematics they had to learn, i.e. the application of Boolean algebra to electric circuit design. Of course, the students' answers to this question are to some degree dependent on which of the three texts they personally preferred. When asked about this, as part of the essay assignments, many would lean towards Shannon's text because it was closets in presentation to what they were used to, for example one group wrote:

> In Shannon's text there was more mathematics than text, which made it easier to picture, more palpable. There were examples of what he said, which helped us to understand his conclusions, his intermediate results, and purpose. Of the three texts this was the most accessible, because it was more visual than the other two. Our favourite! (Group 6)

Not surprisingly, when studying original texts, language becomes a major factor, one to which students immediately refer:

> We find Hamming's text more relevant for teaching in the way that he manages to explain the limits of mathematics without drawing a conclusion, thus leading us to think further for ourselves. For that reason we perceive Hamming's text as more open and accessible. Besides, it is easier to read in terms of language, which makes it possible for us to focus on the mathematics, while working with it. [⋯] (Group 2)

This group, Group 2, on the other hand found Boole's text to be close to inaccessible for the same reason, i.e. language, which is completely in opposition to the evaluation of Group 4:

[5] All excerpts from student group's hand-ins or from student interviews have been translated from Danish.

> Boole's text from 1854 is the one you remember the best. He makes it simple, and then be builds on top and on top, so that we continuously and gradually become wiser. He understands how to create images in our heads that are easy to remember and with which you can easily identify. (Group 4)

Thus, it is important to remember that the experience a student has with reading different original sources is individual, and that this experience may have an effect on their preference of one of the three dimensions over the others. Actually, this may not be so surprising, because original sources often are discussed in their role of 'interlocutors' (e.g. Jahnke, 2000; Kjeldsen & Blomhøj, In press). So in the same way that we as individuals may communicate better with some persons than others, we may simply 'communicate' better with one original source than another. Perhaps because we can relate better to its author, its objective, or the language it is written in. Nevertheless, some students were able to look past the language barriers. For example, although Group 3 also found Boole's text difficult, they stated:

> Relatively strange, but good because it provides us with a new way of thinking. (Group 3)

The idea of the reading of original sources providing the students with something else than just a knowledge of the mathematical in-issues of the text, is, however, something that several students bring up themselves in the interviews. One of the focus group students (Group 7), Sophia, said:

> Well, it's been dry getting through it, it has, but it's also been very⋯Well, it has provided insight, I think, on how mathematics has been used before and how it has come into being, quite precisely. That was really cool, I think. Even though it wasn't mega exciting and even though it was enormously difficult to interpret, it also gave something in a sense. You got a lot of information about how mathematics was applied or about how some clever fellow formulated it back then and so. That was exciting, I think. Okay, maybe not necessarily exciting, but I think it was really cool to see that, how it worked. (Sophia, post-interview, November 3^{rd}, 2011)

Another focus group student, Nikita, provides a much more elaborated account:

> Well, first of all, I think that language wise it was very, very different from what we normally read. That is, what we normally read is much milder, academically speaking. It is described in basic words, or how you say it, it is almost 'baby talk', so you really can follow. Whereas this, not only did you have to understand what it was about, you also had like the language of it, and it has been a different way of thinking compared to the mathematics we are usually taught, where we have this formula and it works like this, this, and this. Here you got all the background knowledge, and how he arrived at it, etc. For me, I personally think that I get much more interested, when I see it all, than if I'm only told that now we are studying vectors and we must learn how to dot these vectors and then we must be able to calculate a length, right. That's all very good, but what am I to use it for? Whereas, when you know about the background, the development up till today, that I think was exciting. Because when we began with the first text [Boole's text], it was kind of like, yeah, that's alright, he can figure out this thing here, and this equals that, I can follow that, and 'white sheep' and so⋯That was good for starters. Then more is built on top, and all of a sudden we see: Why, it's a [electric] circuit we are doing! You could begin to relate it to

> your own reality; that is, something you knew already. So, the thing about starting from scratch, which I kind of felt I did, and suddenly seeing it form a whole, what it was used for today, and be able to relate it to something. Something you knew about. That, I think, was way cooler. (Nikita, post-interview, November 3^{rd}, 2011)

6 Discussion and conclusions

As evident from the above quotes, language is indeed a factor when using original sources in mathematics education. But despite the difficulty of reading old style language (even in translation), the students seem to find that the reading of the original sources in itself provides them with something, this 'something' ranging from: insights into a, for them, new way of thinking (Group 3); knowledge about the application of mathematics and original formulation of mathematical ideas (Sophia); historical background knowledge, the origin of mathematical ideas, and actual modern applications (Nikita). Results from previous studies also indicate that students may find a use of history more relevant if either there is an applicational side to it, or if the history is not too remote in time from themselves – because they feel that they can relate to this better than to something from "⋯before Christ was born⋯" (Jankvist, 2009b). And the student quotes above do seem to confirm this.

Now, regarding the module's philosophical dimension, to which the title of this paper refers, this was probably the one of the three dimensions which caused the students the most trouble. When asked about this particular dimension of the module, two students, Katharine from Group 4 and Sophia from the focus group, replied:

> Hamming was a little like being on the moon for me. Well, I understood it, but I had to read it twice before I could do the connection. [⋯] of course he [Hamming] taught me something, but for me it was on this high strange level, because I'm like; I just want the math, and then calculate and stuff, right. So, it was a bit high floating, I preferred the other two better [Boole and Shannon]. But it did provide an incredibly good connection between the parts, that there were the three dimensions. (Katharine, post-interview, November 3^{rd}, 2011)

> Well, I found it difficult. I found it *very* difficult. It was difficult to think philosophically like that. I don't think that I've been asked to think in that way before. [Usually] it is more like; that's the way it is, now try and work on it. I found it challenging. (Sophia, post-interview, November 3^{rd}, 2011)

When discussion falls on the use of original sources, it is often argued that although this may be one of the most ambitious, demanding, and time consuming ways of teaching mathematics/introducing history, it is also one of the most rewarding (e.g. Jahnke, 2000; Glaubitz, 2011). Something similar may be the case for the introduction of a philosophical dimension into the teaching of mathematics, because even though most students agreed to this being very challenging indeed, the essay assignments along with their questionnaire answers and post-interview utterances bear witness to this dimension having actually 'moved' something—not least in relation to the students' knowledge of meta-issues of mathematics, that is 'overview and judgment'. Focus group student Jean was even able to articulate some of his newly gained insights as well as pinpointing these to the presence of the philosophical dimension:

> I also think that the philosophy part was exciting, but rather abstract⋯It may sound a bit strange, but I do find it kind of cool to sit and think about mathematics [the subject], that it's not only a tool. Or yes, it is a tool, but you can kind of view it from different perspectives and, yes, view it more philosophically. (Jean, post-interview, November 3^{rd}, 2011)

When asked if this module (as well as the first one) had any impact on the way in which he perceived mathematics as a discipline, Jean replied:

> Yeah⋯definitely the view of mathematics has been altered because you've had the philosophical dimension as part of it [the modules], a fairly big part, I think. So, you gained a different insight into this than you had before. You kind of feel that you've reached a higher level⋯Yes, because you are able to see mathematics in a different way. And that has sort of surprised me; that you can view it in this way⋯that mathematics also has a philosophical side to it. That it is not only, as I said before, numbers. (Jean, post-interview, November 3^{rd}, 2011)

As promised, let us return to the excerpt from Group 2's essay assignment. Now, this essay may be considered a fairly deep answer for students at this particular level. In a few paragraphs, and particularly the last one, they are able to coin the essence of interplay between the three dimensions in the HAPh-module by providing their own sound argumentation for Boolean algebra and Shannon's use of it in electric circuit design being an example of Hamming's unreasonable effectiveness of mathematics. Admittedly, this essay answer is one of the better from the seven groups, but even so, it provides an *existence proof* of it being possible to have students reach the intended level of meta-issue abstraction, i.e. relating the three original texts to each other, and thus also the three dimensions of history, application, and philosophy.

That it is possible to introduce a historical dimension into mathematics teaching is well known from four decades of HPM research.[6] That it is also possible to introduce an applicational dimension into mathematics teaching is equally well known and documented. And that it is possible to introduce a historical dimension and an applicational dimension at the same time, has also been shown (e.g. Jankvist, 2009b; 2010; 2011a). But as of yet, only very few studies address the introduction of a philosophical dimension in mathematics teaching (see Jankvist, forthcoming, for a list), meaning that it is not a priori given how to do so. The above description of a concrete HAPh-module provides an example of this, one in connection with the dimensions of history and application where each of these three dimensions is introduced through a guided reading of an original source and the interplay of the dimensions (and sources) is dealt with in essay assignments. Judging from the student groups' essays and the student interviews, some of which were displayed above, a development of the three types of 'overview and judgment' does appear to be present. All in all, this points in direction of the laid out scheme of design indeed being 'marketable'.

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[6]HPM is International Study Group on the relations between History and Pedagogy of Mathematics.

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HISTORICAL PROJECTS IN DISCRETE MATHEMATICS

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ABSTRACT

Presented is a summary of two historical curricular modules for undergraduate discrete mathematics. The first "Deduction through the Ages" is a discussion of how modern mathematics arrived at the truth of an implication (an "if-then" statement) in propositional logic. The second "Networks and Spanning Trees" presents motivational material for the definition, enumeration, and application of trees in graph theory.

Keywords: Propositional logic, implication, graph theory, labeled trees, minimal spanning tree, Borůvka's algorithm.

1 Introduction

In this talk we discuss teaching discrete mathematics from primary historical sources and provide the results of a statistical study concerning the impact of this pedagogical technique on student learning attitudes. Over the past four years our interdisciplinary, intercollegiate team of seven faculty have developed 18 curricular modules that incorporate passages from primary sources to teach core content in finite mathematics, combinatorics, logic, abstract algebra, algorithm design, and computer science courses. This builds on a pilot study to teach from historical projects [3]. Each module is designed around one or several historical sources and develops a key concept (or several concepts) in the curriculum by examining the work of the pioneers and offering student exercises that illuminate and extrapolate from the source. Topics for the modules are often an examination of the ideas behind modern definitions, algorithms or lemmas that appear as opaque or unmotivated statements in today's textbooks, such as the truth table of an implication in propositional logic, the definition of tree in graph theory, or the formula for the summation of squares, $\sum_{i=1}^{n} i^2 = (n^3/3) + (n^2/2) + (n/6)$, and the unenlightening proof of this equality by formal mathematical induction. For the complete list of our curricular projects, along with the text of each one, see our web resource [2].

Why teach from historical sources when textbooks offer a concise, mathematically precise presentation of the subject? First, historical sources add context, with the original author keenly motivated to solve a particular problem or find a robust setting for previously fragmented solutions. We read what the problem was and witness a pioneering, often paradigm-setting approach. The primary source reveals the motivation for study of the subject or paradigm. Historical sources add direction to the subject matter. We observe where the author begins, how a problem is solved, and what subsequent work builds on the solution. Additionally we as readers are forced to grapple with the verbal meaning of a passage, consider non-standard formulations of ideas, and ask "What is an appropriate system of notation for this problem?" "What are the key properties to a solution to this problem?" We learn through cognitive dissonance. The thought process required to bridge the gap between the historical and the modern offers an invaluable learning experience. We gain insight into the process of discovery as well as an appreciation of the cultural and intellectual setting in which the author was writing. For further reasons to study from primary historical sources, see [1, 3]. For the results of a pilot study using this pedagogical technique, see [3]. To illustrate how the historical approach can be used to teach mathematical content, we examine two historical modules in detail: "Deduction through the Ages," and "Networks and Spanning Trees." The first is a study of the original work of several philosophers, logicians and mathematicians who have contributed to an understanding of the truth table of an implication (an "if-then" statement). The second examines the notion of tree and its applications before graph theory was an independent subject of study.

2 Deduction through the Ages

While today the truth of $p \to q$ (p implies q) is a matter of settled logic, the ancient Greeks debated at length when the following hypothetical proposition holds: "If a warrior is born at the rising of the Dog Star, then that warrior will not die at sea." The Greek philosopher Philo of Megara (ca. 4th century B.C.E.) maintained that a valid hypothetical proposition is "that which does not begin with a truth and end with a falsehood" [18, II. 110]. The on-line written project "Deduction through the Ages" [2] outlines five argument forms stated by Chrysippus (ca. 280–206 B.C.E.) [11, p. 189], and raises the question (for students and instructors) whether these five rules could be special cases of just one rule. This presentation focuses on the following three (of five) rules:

1. If the first, [then] the second. The first. Therefore, the second.

3. Not both the first and the second. The first. Therefore, not the second.

5. Either the first or the second. Not the first. Therefore, the second.

Verbal argument asserting the equivalence of these rules is difficult, and a more streamlined method for discussing their relation to each other is sought. An old point of view on logic is to reduce the subject to a system of calculation, whereby the rules of reasoning could be automated. The German philosopher, mathematician, and universalist Gottfried Wilhelm Leibniz (1646–1716) was one of the first to pursue this idea, and sought a *characteristica generalis* (general characteristic) or a *lingua generalis* (general language) that would serve as a universal symbolic language and reduce all debate to calculation. This in part served as motivation for Leibniz to introduce his symbols for differentiation and integration.

2.1 Boole's Algebra of Statements

In the modern era, an initial attempt at a symbolic and almost calculational form of elementary logic was introduced by the English mathematician George Boole (1815–1864). Author of *An Investigation of the Laws of Thought* [4, 5], Boole believed that he had reduced language and reasoning to a system of calculation involving the signs "$\times$", "$+$", "$-$", where "$\times$" denotes "and," "$+$" denotes "or," and "$-$" denotes "not." Boole writes [5]:

Proposition I.

All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz:

1st. Literal symbols, as x, y &c., representing things as subjects of our conceptions.

2nd. Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the elements.

3rd. The sign of identity, $=$.

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra. ...

If x represent any class of objects, then will $1 - x$ represent the contrary or supplementary class of objects, i.e. the class including all objects which are not comprehended in the class x.

The symbols "$\times$", "$+$", "$-$", however, lose their arithmetic meaning when applied to the logic of statements. For example, letting a denote the class of apples and b the class of red objects, then in Boole's notation the class of objects that are not red apples would be $1 - ab$. Objects that are either not apples or not red would be $(1-a) + (1-b)$. Thus, in Boole's notation

$$1 - ab = (1-a) + (1-b),$$

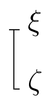
which reflects a statement in logic, not arithmetic. Also, Boole does not introduce a symbol for an "if-then" statement, so writing Chrysippus's first rule in this arithmetic notation is difficult.

2.2 Gottlob Frege Invents a Concept-Script

Let's now turn to the work of the German mathematician and philosopher Gottlob Frege (1848–1925) who sought a logical basis, not for language as Boole, but for mathematics. In *The Basic Laws of Arithmetic* [12], Frege introduces his own system of notation, called a *concept-script* or "Begriffsschrift"in the original German, which shows no kinship with the arithmetical symbols "$\times$", "$+$", "$-$". The centerpiece of Frege's notation is the condition stroke[1]. From *The Basic Laws of Arithmetic*, we read:

§12. **Condition-stroke, And, Neither-nor, Subcomponents, Main Component.**

In order to enable us to designate the subordination of a concept under a concept, and other important relations, I introduce the function of two arguments



[1] 

by stipulating that its value shall be the False if the True be taken as ζ-argument and any object other than the True be taken as ξ-argument, and that in all other cases, the value of the function shall be the True. . . . The vertical stroke I call the *condition-stroke*. . . . .

Thus, the symbol $\begin{array}{l}\ulcorner B\\ \llcorner A\end{array}$ is false only when A (the beginning proposition) is true and B (the ending proposition) is false. The reader is asked to compare the truth of Frege's condition stroke to Philo's verbal statement that a valid hypothetical proposition is "that which does not begin with a truth and end with a falsehood." The condition stroke is true when it is **not** the case that it begins (A) with a true statement and ends with a false statement (B). Thus, the condition stroke is Frege's symbol for an implication (a hypothetical proposition in ancient Greece). We use a few other symbols from the "Begriffsschrift." A horizontal line — denotes a "judgment stroke" that renders the value of either true or false when applied to a proposition. For example, $— 2^2 = 5$ returns the value "false," while $— 2^2 = 4$ returns "true." The symbol $\top \xi$ denotes the negation of $— \xi$, while $\vdash \zeta$ denotes that ζ is a true statement. These symbols may be combined in what Frege calls "amalgamation of horizontals," so that $\vdash (\top \Delta)$ becomes $\vdash\!\top \Delta$, meaning that the negation of Δ is true, i.e., Δ itself is false.

Let's now write Chrysippian rules (1), (3), and (5) above entirely in the concept-script. Frege himself states the "First Method of Inference" as from the propositions $\begin{array}{l}\ulcorner B\\ \llcorner A\end{array}$ and $\vdash A$ we may infer $\vdash B$. Letting A denote "the first" and B denote "the second," this "First Method of Inference" becomes verbally: "If the first, then the second. The first is true, therefore, the second is true." How can we write Chrysippus's third rule in Frege notation? Recall that the symbol

$$\begin{array}{l}\ulcorner B\\ \llcorner A\end{array}$$

is false only when A is true and B is false. Thus $\begin{array}{l}\ulcorner\!\top B\\ \llcorner A\end{array}$ is false only when A is true and B is true, which has the same truth value as "not both A and B." Again, letting A denote "the first" and B denote "the second," we see that "not both the first and the second, not the first, therefore, not the second" can be written as from $\begin{array}{l}\ulcorner\!\top B\\ \llcorner A\end{array}$ and $\vdash A$, it follows $\vdash\!\top B$. Finally, to write the fifth Chrysippian rule in the concept-script, note that the symbol $\begin{array}{l}\ulcorner B\\ \llcorner\!\top A\end{array}$ is false only when A is false and B is false, which has the same truth value as "either the first or the second," using the inclusive "or." Thus, "either the first or the second, not the first, therefore, the second" can be rendered as from $\begin{array}{l}\ulcorner B\\ \llcorner\!\top A\end{array}$ and $\top A$, it follows $\vdash B$.

Thus, rules (1) , (3) and (5) can all be written using the same root symbol, the condition stroke, and minor variations on negating or asserting its arguments. This demonstrates the interconnectedness of these rules, and offers insight into their possible equivalence.

2.3 Russell and Whitehead Find New Notation

While somewhat awkward in execution, Frege's condition stroke advances his philosophy that mathematical truths should follow from truths in logic, a point of view known today as logicism. Two later practitioners of logicism whose work set the stage for mathematical logic of the twentieth century were Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947). Russell was a prolific writer, contributing to the fields of education, history, religion, and political theory, not to mention philosophy and logic. Let's read a short excerpt from Russell and Whitehead's monumental collaboration *Principia Mathematica* [17], where an implication (an "if-then" statement) is formally defined. Note how the definition of "p implies q" reduces to the equivalent inclusive "or" statement in Frege's notation.

The fundamental functions of propositions. . . .

[T]here are four special cases which are of fundamental importance, since all the aggregations of subordinate propositions into one complex proposition which occur in the sequel are formed out of them step by step.

They are (1) The Contradictory Function, (2) the Logical Sum or Disjunctive Function, (3) the Logical Product, or Conjunctive Function, (4) the Implicative Function. . . .

The Contradictory Function with argument p, where p is any proposition, is the proposition which is the contradictory of p, that is, the proposition asserting that p is not true. This is denoted by $\sim p$. Thus, $\sim p$ . . . means the negation of the proposition p. It will also be referred to as the proposition not-p. . . .

The Logical Sum is a proposition with two arguments p and q, and is the proposition asserting p or q disjunctively, that is, asserting that at least one of the two p and q is true. This is denoted $p \vee q$. . . . Accordingly $p \vee q$ means that at least p or q is true, not excluding the case in which both are true.

The Logical Product is a propositional function with two arguments p and q, and is the proposition asserting p and q conjunctively, that is, asserting that both p and q are true. This is denoted by $p.q$ . . . . Accordingly $p.q$ means that both p and q are true. . . .

The Implicative Function is a propositional function with two arguments p and q, and is the proposition that either not-p or q is true, that is, it is the proposition $\sim p \vee q$. Thus, if p is true, $\sim p$ is false, and accordingly the only alternative left by the proposition $\sim p \vee q$ is that q is true. In other words if p and $\sim p \vee q$ are both true, then q is true. In this sense the proposition $\sim p \vee q$ will be quoted as stating that p implies q. The idea contained in this propositional function is so important that it requires a symbolism which with direct simplicity represents the proposition . . . . The symbol employed for "p implies q", i.e. for "$\sim p \vee q$" is "$p \supset q$." This symbol may also be read "if p, then q." . . .

But this . . . by no means determines whether anything, and if so what, is implied by a false proposition. What it does determine is that if p implies q, then it cannot be the case that p is true and q is false, ...

With these crisp definitions, Chrysippus's rules can be written as follows in the notation of *Principia Mathematica*:

$$1.\ \ p \supset q,\ p,\quad \therefore\ q$$
$$3.\ \ \sim (p.q),\ p,\quad \therefore \sim q$$
$$5.\ \ p \vee q,\ \sim p,\quad \therefore\ q.$$

To discuss the relation between rules (1) and (5), note that from [17] every implication is equivalent to a certain inclusive "or" statement and vice versa.

$$p \supset q \equiv \sim p \vee q, \qquad p \vee q \equiv \sim p \supset q.$$

The relation between rules (1) and (3) can be discovered from the equivalence between an implication and a negated "and" statement and vice versa.

$$p \supset q \equiv \sim p \vee q \equiv \sim (p.(\sim q)), \qquad \sim (p.q) \equiv \sim p \vee \sim q \equiv p \supset \sim q.$$

Thus, the major premise of rule (1), "if the first, then the second" is equivalent to a certain inclusive "or" statement, which in turn is equivalent to a certain negated "and" statement. Of course, the individual arguments of these "or" and "and" statements may themselves be negated, as we saw when discussing writing Chrysippus's rules in the "Begriffsscrift."

2.4 Post Develops Truth Tables

Emil Post (1897–1954) developed a highly efficient method to represent the truth values of compound statements involving the connectives "and," "or," "not," and "if-then." He dubbed these schematic representations "truth tables," a term which is in current use today. Emil was born in Poland, of Jewish parents, with whom he emigrated to New York in 1904. He received his doctorate from Columbia University, where he participated in a seminar devoted to the study of *Principia Mathematica*. In his dissertation of 1921, "Introduction to a General Theory of Propositional Functions" [14], he develops the notion of truth tables and clearly displays the table for an implication. With this in hand, the equivalence of the major premises in Chrysippus's rules is reduced to mere calculation of truth values.

INTRODUCTION TO A GENERAL THEORY OF ELEMENTARY PROPOSITIONS.

By Emil L. Post.

INTRODUCTION.

In the general theory of logic built up by Whitehead and Russell [17] to furnish a basis for all mathematics there is a certain subtheory . . . this subtheory uses . . . but one kind of entity which the authors have chosen to call elementary propositions. . . .

2. Truth-Table Development—Let us denote the truth-value of any proposition p by $+$ if it is true and by $-$ if it is false. This meaning of $+$ and $-$ is convenient to bear in mind as a guide to thought, ... Then if we attach these two primitive truth-tables to $\sim$ and $\vee$ we have a means of calculating the truth-values of $\sim p$ and $p \vee q$ from those of their arguments.

p	$\sim p$
$+$	$-$
$-$	$+$

$p,\ q$	$p \vee q$
$+\ +$	$+$
$+\ -$	$+$
$-\ +$	$+$
$-\ -$	$-$

... It will simplify the exposition to introduce ...

$$p \supset q \ . = . \ \sim p \vee q$$

read "p implies q," ... having the table

$p,\ q$	$p \supset q$
$+\ +$	$+$
$+\ -$	$-$
$-\ +$	$+$
$-\ -$	$+$

With the truth table of an implication we have arrived, after more than two millennia of deductive thought, where modern discrete mathematics textbooks begin a discussion of propositional logic. It is now a textbook exercise to verify, via truth tables, that the following logical equivalence holds:

$$p \supset q \equiv \sim p \vee q, \qquad p \vee q \equiv \sim p \supset q$$
$$p \supset q \equiv \sim (p.(\sim q)), \qquad \sim (p.q) \equiv p \supset \sim q.$$

3 Networks and Spanning Trees

In 1857 Arthur Cayley (1821–1895) published a paper [9] that introduces the term "tree" to describe the logical branching that occurs when iterating the fundamental process of (partial) differentiation. Of composing four symbols that involve derivatives, Cayley writes "But without a more convenient notation, it would be difficult to find [their] corresponding expressions . . . . This, however, can be at once effected by means of the analytical forms called trees . . . " [9]. Without defining the term "tree," Cayley has identified a certain structure that occurs today in quite different situations, from networks in computer science to representing efficient delivery routes for transportation. In a later paper "A Theorem on Trees" [10] published in 1889, Cayley counts trees in which every node (vertex) carries a fixed name or label, arriving at a result that today is known as "Cayley's formula" for the number of labeled trees on n vertices. His proof is a bit incomplete, and we discuss the work of Heinz Prüfer (1896–1934) on counting labeled trees via an enumeration of certain railway networks [15]. This is followed by a discussion of Otakar Borůvka's (1889–1995) work on finding a net of least total edge length, i.e., a minimal spanning tree, from all labeled trees on n fixed vertices [6].

3.1 Prüfer's Enumeration of Trees

The German mathematician Heinz Prüfer offers a quite clever and geometrically appealing method for counting what today are called labeled trees. He uses no modern terminology, not even the word "tree" in his work. Instead, the problem is introduced via an application [16]: Given a country with n-many towns, in how many ways can a railway network be constructed so that

1. the least number of railway segments is used; and

2. a person can travel from each town to any other town by some sequence of connected segments.

The ideas expressed here, that the least number of railway segments is used, yet travel remains possible between any two towns, are recognized today as properties that characterize such a railway network as a tree. Since the towns are fixed, their names (labels) are not interchangeable, and a labeled tree is an excellent model for this problem. Prüfer wishes to count all railway networks satisfying properties (1) and (2) above, and in doing so, he arrives at a result that agrees with Cayley's formula. Prüfer assigns to each tree a particular symbol based on the point labels (town names). Counting the resulting symbols is then much easier than counting trees. Of course, establishing a one-to-one correspondence between symbols and trees requires some work, which Prüfer writes "follows from an induction argument" (on the number of towns). Let's read a brief excerpt from "A New Proof of a Theorem about Permutations" [15, 16]:

[We] assign to each railway network, in a unique way, a symbol $\{a_1,\ a_2,\ \ldots\ ,\ a_{n-2}\}$, whose $n-2$ elements can be selected independently from any of the numbers 1, 2, . . ., n. There are n^{n-2} such symbols, and this fact, together with the one-to-one correspondence between networks and symbols, will complete the proof.

In the case $n = 2$, the empty symbol corresponds to the only possible network, consisting of just one single segment that connects both towns. If $n > 2$, we denote the towns by the numbers 1, 2, . . ., n and specify them in a fixed sequence. The towns at which only one segment terminates we call the endpoints. . . .

In order to define the symbol belonging to a given net for $n > 2$, we proceed as follows.

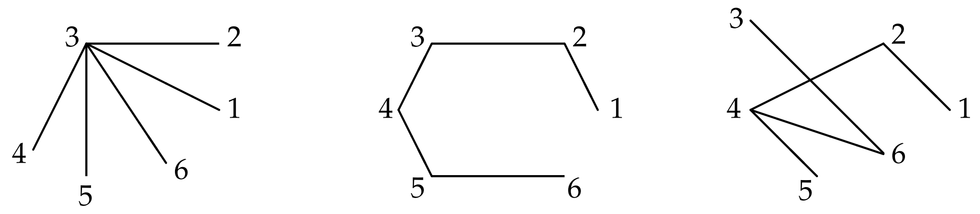
Let b_1 be the first town which is an endpoint of the net, and a_1 the town which is directly joined to b_1. Then a_1 is the first element of the symbol. We now strike out the town b_1 and the segment $b_1\,a_1$. There remains a net containing $n-2$ segments that connects $n-1$ towns in such a way that one can travel from each town to any other.

If $n-1 > 2$ also, then one determines the town a_2 with which the first endpoint b_2 of the new net is directly connected. We take a_2 as the next element of the symbol. Then we strike out the town b_2 and the segment $b_2\,a_2$. We obtain a net with $n-3$ segments and the same properties.

We continue this procedure until we finally obtain a net with only one segment joining 2 towns. Then nothing more is included in the symbol.

Examples:

Nets:



Symbols: $\{3,\ 3,\ 3,\ 3\}$ $\{2,\ 3,\ 4,\ 5\}$ $\{2,\ 4,\ 6,\ 4\}$

3.2 Borůvka's Solution to a Minimization Problem

In 1926 Otakar Borůvka (1899–1995) published [6, 7] the solution to an applied problem of immediate benefit for constructing an electrical power network in the Southern Moravia Region, now part of the Czech Republic. In recalling his own work, Borůvka writes [8, 13]:

> My studies at polytechnical schools made me feel very close to engineering sciences and made me fully appreciate technical and other applications of mathematics. Soon after the end of World War I, at the beginnings of the 1920s, the Electrical Power Company of Western Moravia, Brno, was engaged in rural electrification of Southern Moravia. In the framework of my friendly relations with some of their employees, I was asked to solve, from a mathematical standpoint, the question of the most economical construction of an electric power network. I succeeded in finding a construction ... which I published in 1926 ... .

Let's examine specifically how Borůvka phrased the problem [7]:

> There are n points in the plane (in space) whose mutual distances are all different. We wish to join them by a net such that:
>
> 1. Any two points are joined either directly or by means of some other points.
> 2. The total length of the net would be the shortest possible.

Thus, of all n^{n-2} labeled trees on n points (towns), which tree(s) has (have) the shortest possible total edge length. Borůvka proposes a simple algorithm to find such a net of minimal total length, based on the guiding principle "I shall join each of the given points with the point nearest to it" [7].

A Contribution to the Solution of a Problem on the Economical Construction of Power Networks

Dr. Otakar Borůvka

In my paper "On a Certain Minimal Problem," I proved a general theorem, which, as a special case solves the following problem:

There are n points in the plane (in space) whose mutual distances are all different. We wish to join them by a net such that:

1. *Any two points are joined either directly or by means of some other points.*

2. *The total length of the net would be the shortest possible.*

It is evident that a solution of this problem could have some importance in electrical power network designs; hence I present the solution briefly using an example. . . . .

I shall give the solution of the problem in the case of 40 points[2] given in Fig. 1.

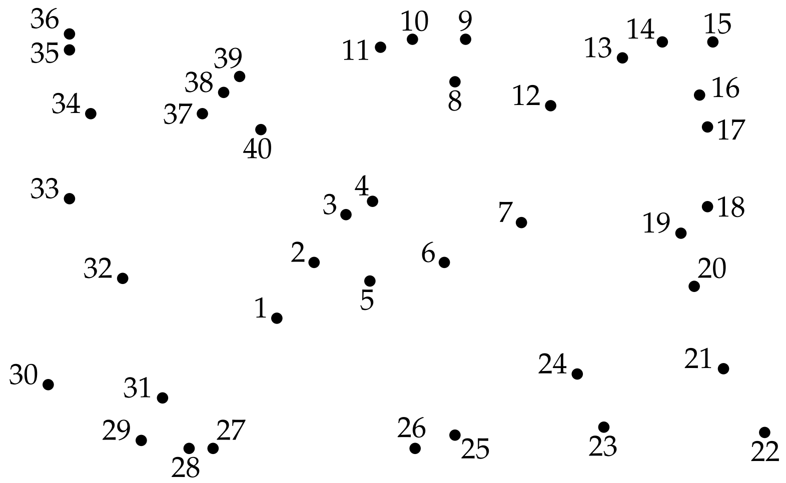


Fig. 1.

I shall join each of the given points with the point nearest to it. Thus, for example, point 1 with point 2, point 2 with point 3, point 3 with point 4 (point 4 with point 3), point 5 with point 2, point 6 with point 5, point 7 with point 6, point 8 with point 9 (point 9 with point 8), etc. I shall obtain a sequence of polygonal strokes 1, 2, . . . , 13 (Fig. 2).

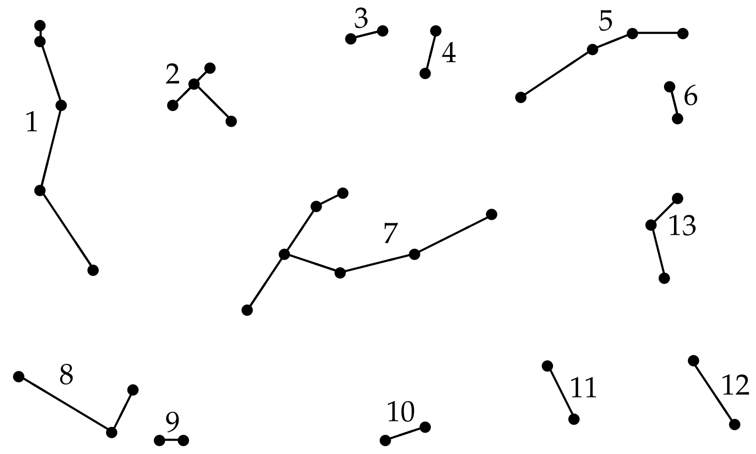


Fig. 2.

I shall join each of these strokes with the nearest stroke in the shortest possible way. Thus, for example, stroke 1 with stroke 2 (stroke 2 with stroke 1), stroke 3 with stroke 4 (stroke 4 with stroke 3), etc. I shall obtain a sequence of polygonal strokes 1, 2, 3, 4 (Fig.3).

I shall join each of these strokes in the shortest way with the nearest stroke. Thus stroke 1 with stoke 3, stroke 2 with stroke 3 (stroke 3 with stroke 1), stroke 4 with stroke 1. I shall finally obtain a single

[2]Borůvka only labeled the points 1 through 9 in his original paper. We have included labels of all points for later reference.

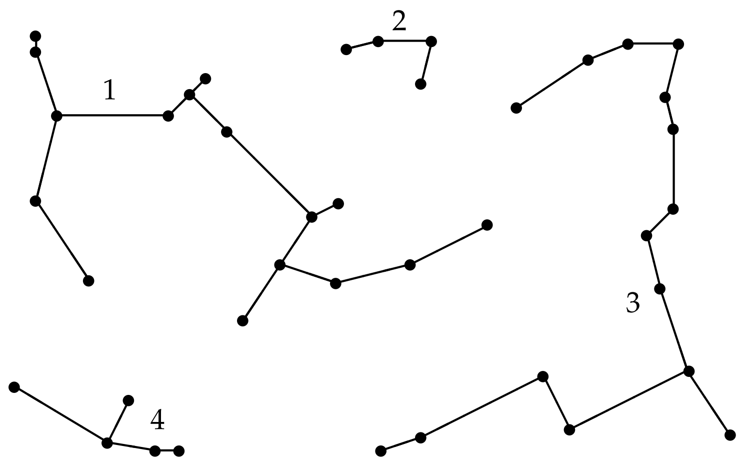


Fig. 3.

polygonal stroke (Fig. 4)[3] which solves the given problem.

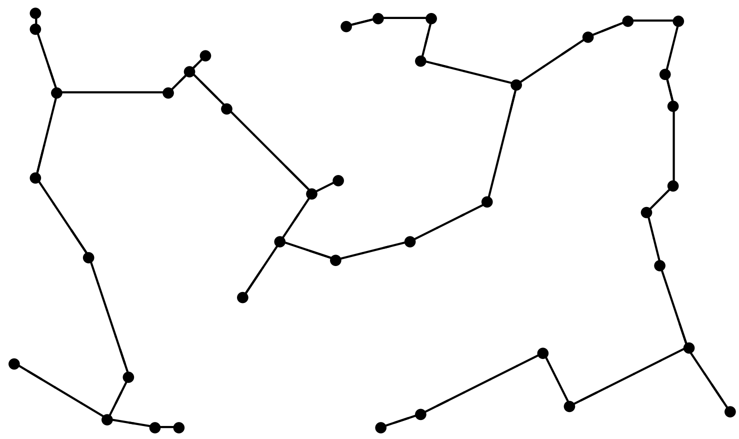


Fig. 4.

4 Impact on Student Leaning Attitudes

Over the past four years the impact of our historical projects on student learning and attitudes has been assessed by our statistical consultant, Dr. David Trafimow of the Department of Psychology, New Mexico State University. Students are asked to complete a pre- and post-course questionnaire which are matched by the use of anonymous codes. A sample question includes:

[3]In the original paper [7], Figure 4 is rotated 180°.

Which best describes you:
I am capable (Extremely) : (Quite) : (Slightly) : (Neutral) : (Slightly) : (Quite) : (Extremely) incapable of explaining Math/Computer Science concepts in writing.

Above, "extremely capable" is given the value of 3, "quite capable" the value of 2, ..., and "extremely incapable" the value -3, forming a scale from $+3$ to -3, with 0 being neutral. First the questionnaire was shown to be reliable by repeating similar questions, and Cronbach's alpha reliability factor was .89 on the pre- and .90 on the post-course questionnaires, where an alpha factor greater than .7 is considered reliable. On the scale from $+3$ to -3 above, the mean from pre- to post-course questionnaire increased from 1.13 to 1.47. Given the null hypothesis that there is no difference from pre- to post-course questionnaires, the paired T-test between the means of these two questionnaires yields $p < .001$, indicating that the probability of the difference occurring by chance is less than 1 in 1000. Our consultant reaches the conclusion that students' estimates of their Math/Computer Science understanding increased from pre- to post-test for courses using historical projects.

Acknowledgment

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FUTURE RESEARCH TOPICS IN THE FIELD OF MATHEMATICAL PROBLEM SOLVING: USING DELPHI MOTHOD
수학적 문제 해결 연구에 있어서 미래 연구 주제 : 델파이 기법

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ABSTRACT

Mathematical problem solving have placed as one of the important research topics which many researchers have been interested in from 1980's until now. A variety of topics have been researched: Characteristics of problem; Processes of how learners to solve them and their metacognition; Teaching and learning practices. Recently, the topics have been shifted to mathematical learning through problem solving and the connection of problem solving and modeling. In the field of mathematical problem solving where researcher have continuously been interested in, future research topics in this domain are investigated using delphi method.

Keywords: Mathematical problem solving, Delphi method, Research topics.

1 도입

인간사회는 수렵사회, 농경사회, 산업사회, 과학사회, 지식기반사회 등으로 끊임없이 변모해 오고 있다. 한편, 앞으로 다가올 인간 사회를 지식융합사회라고 한다. 이처럼 사회 형태가 변하면 변화된 사회에서 삶을 살아야 하는 아동들을 교육하는 목적도 변한다. 과거에는 많은 지식을 소유한 사람이 필요했었던 반면에, 현재 지식기반사회에서 요구되는 인간상은 지식을 창출할 수 있는 창의력을 갖춘 인간이다(교육부, 1997; 교육과학기술부, 2007, 2009; NCTM, 1989, 2000). 즉, 단순히 많은 지식을 알고 그 지식을 주어진 상황에 적용할 수 있는 능력을 갖춘 인간이 아니라, 자신이 알고 있는 지식과 지적능력을 활용해서 주어진 문제를 해결할 수 있는 능력을 갖춘 인간을 육성하는 것이 교육의 목적이 된 것이다. 이 둘을 동일한 것으로 보는 경향이 있지만, 학습상황에서 이 두 견해에 따라 실천되는 교실 풍경은 학습자, 교사, 학습내용 등에서 전

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적으로 달라진다 (김진호, 2010; Gainsburg, 2006; Hamilton, 2007; Zawojewski, Hjalmarson, Bowman, & Lesh, 2008; Zawojewski & McCarthy, 2007). 즉, 이전 사회 형태에서 요구되는 인간상을 육성하는데 적합한 것으로 인정되던 교육방법으로는 변한 사회에서 요구하는 인간을 육성해 낼 수 없다는 데에 있다 (구광조, 전평국, 강완, 1996; 김진호, 2010).

이런 맥락 속에서, 최근 들어 연구자들은 학습자들이 수학 수업을 통해서 스스로 지식을 구성할 수 있는 능력을 형성할 수 있도록 돕는 수학적 문제해결을 통한 수학 학습을 지지하고 있다 (Baroody, 1998; Lester, 2003). 이런 지지는 그 동안 Polya의 문제해결 전략, 발견술 등을 수학 수업의 실제에 적용하려는 시도들이 수학적 문제해결 능력 신장에 실패한 것을 통하여 알 수 있기 때문이기도 하다 (길양숙, 1991; English, & Sriraman, 2010; Schoenfeld, 2007). 이런 연구 결과는 우리나라 교육과정 및 초등수학교과서에도 반영될 필요가 있다 (보다 자세한 내용은 II장 2절 참고). 이런 관점에서, 세계 여러 나라의 교육당국 (류성림, 최창우, 남승인, 김상룡, 최재호, 김진호, 2011) 과 수학교육단체들 (NCTM, 1989, 2000) 은 자국의 교육과정에 수학적 문제해결을 주요 학습목표로 설정하고 있다.

한편, 수학적 문제해결에 관심을 가진 것은 오랜 역사를 갖고 있다. 수학적 문제해결을 무엇으로 보느냐에 따라서 일부 연구자 (Silver, 1985) 는 소크라테스 시대까지 거슬러 올라가는가 하면, 일부 연구자 (강옥기, 신성균, 강완, 류희찬, 정은실, 박교식, 우정호, 1985) 는 1930년대를 기점으로 삼기도 한다. 이처럼 오랜 역사를 갖고 있는 수학적 문제해결은 그 만큼 다양한 연구가 진행되어 온 연구영역임에 틀림없다. 그런데, 많은 연구자들의 수학적 문제해결 연구 동향을 분석한 결과, 공통적으로 언급되고 있는 흥미로운 사실은 바로 수학적 문제해결과 관련된 연구주제의 흥망성쇠를 볼 수 있다는 점이다 (황치홍, 2001; Hino, 2007; Lesh, 2010; Lester, 1994, 2003; Schoenfeld, 2007; Silver, 1994; Stacey, 2007). 수학적 문제해결 그 자체에 대한 연구가 수학 교육의 연구 중심에 서 있던 시기가 있는가 하면, 이 분야에 대한 연구가 시들해진 시기도 있다 (Hino, 2007; Lester, 1994; Schoenfeld, 2007). 또한 수학적 문제해결과 관련된 하위 연구 주제들도 같은 경향을 보인다. 예를 들어, 최근에 실생활 문제를 해결할 수 있는 학습자의 능력을 향상시키는 것과 관련된 연구는 감소되고 있는 반면에, 수학적 문제 해결을 통한 학습자의 수학 지식의 학습과 관련된 연구는 새롭게 부각되고 있는 연구주제이다 (NCTM, 1980a; Stacey, 2007). 한편, ERIC에서 "수학적 문제해결, 성차" 를 검색어로 1980년부터 2011년까지 수학적 문제 해결과 관련된 연구물을 검색하면 단 10편이 검색된다. 이 수치로 볼 때, 수학적 문제해결과 관련된 성차에 대한 연구는 거의 이루어지고 있지 않다고 볼 수 있다. 또 다른 한편으로, 공학의 발달로 인하여 공학을 이용한 수학적 문제 해결은 새로운 연구 주제로 주목을 받고 있다. 이와 같은 경향은 우리나라의 연구물들에서도 찾아 볼 수 있다 (권정은, 최재호, 2008; 류희찬, 권성룡, 김남균, 2005; 황치홍, 2001; 황혜정, 2007).

요약하면, 수학적 문제해결은 학교수학에서 기본 목적 중의 하나이며 다양한 주제들이 연구 되어 오고 있을 뿐만 아니라 미래에도 지속적으로 연구가 되어야 할 영역이라는데 연구자들은 동의하고 있다. 본 연구의 목적은 미래에 수학적 문제해결에서 어떤 주제들에 대한 연구가 이루어져야 할 것인지를 알아보는데 있다. 이를 알아보기 위해 델파이 기법을 이용하였다.

2 수학적 문제해결 연구 동향

2.1 미국의 문제 해결 연구 동향

수학교육 분야에서 수학적 문제해결에 대한 관심이 아주 오래 전부터 있었던 것은 사실이지만, 주요 이슈가 된 것은 NCTM(1980b) 이 An agenda for action: Recommendations for school mathematics of the

1980s을 출간한 이후이다. 1960년대와 1970년대의 수학교육을 비판하면서 NCTM이 낸 8가지 권고 중 첫 번째 권고가 "문제해결이 1980년대 학교 수학의 중심이 되어야 한다." 였으며, 이를 위해서 다음의 실천안을 내 놓았다(NCTM, 1980b, pp. 1-5)

① 수학과 교육과정은 문제해결을 중심으로 조직되어야만 한다.

② 충분한 수학적 응용 가능성이 있는 광범위한 전략, 과정, 표현 양식을 포함할 수 있는, 수학에서의 문제해결에 대한 정의와 용어들이 개발되어야 하고 확장되어야 한다.

③ 수학교사는 문제해결이 수업의 중심이 되는 교실 환경을 창출해야만 한다.

④ 문제해결의 교수학습에 필요한 적절한 교육과정 자료가 모든 학년급을 위해서 개발되어야 한다.

⑤ 수학 프로그램들은 학생들이 응용문제의 해결에 참여하도록 하여야 한다.

⑥ 연구자들과 연구지원 기관들은 문제해결의 본질과 문제해결자들을 발달시킬 수 있는 효과적인 방법을 연구하는데 우선권을 주어야 한다.

먼저, NCTM(1989)이 Standards를 출판 한 직후인 1990년대 초반까지의 연구물들을 토대로, 이와 같은 실천안들이 성공적으로 실천되었는지 살펴보고, 그 이후의 연구물들에 대해서 살펴보도록 한다. 수학교육자 및 교육학자들이 수학적 문제해결에 대해 관심을 갖게 된 획기적인 계기가 두 개 있었다. 하나는 인간의 사고 및 사고과정에 관심을 갖은 형태주의심리학과 Piaget를 비롯한 구성주의의 출현이고, 또 다른 하나는 수학자가 문제를 해결하는 중 일어나는 내면의 정신작용을 학습자들도 경험하여 학습자들이 성공적인 문제해결자가 될 수 있음을 안내한 Polya(1945, 1954, 1981)의 일련의 저술들의 출현이다.

인간의 사고과정에 대한 연구가 활발하게 이루어지기 시작한 것은 인간의 사고를 컴퓨터에 비유한 정보처리 심리학의 출현(Simon & Newell, 1972)과 행동주의를 바탕으로 하면서도 인간의 사고 과정에 관심을 갖는 Anderson(1976)과 같은 인지심리학자들의 출현이다. 하지만 이들을 중심으로 한 연구는 1990년대 들어서면서 점차 그 세력을 잃어간다. 그 이유는, 앞에서도 밝혔듯이, 1990년대 들어서면서 우리 사회는 스스로 지식을 창출할 수 있는 창의력을 갖춘 인간 육성을 교육의 사상으로 삼게 되는데, 이 두 이론에서는 창의성을 설명할 수 없었다. 대신에, 창의성을 설명할 수 있는 구성주의가 1990년대 이후로 각 연구자들로부터 각광을 받게 된다. 즉, 1990년대 이후 구성주의의 견해에서 인간의 사고과정을 탐구하는 경향을 보인다.

한편, 1970년대 이전의 수학적 문제해결을 다룬 연구들을 요약한 Kilpatrick(1969)은 이 연구들이 이론적이거나 체계적이지 못하고, 전형적인 교과서 형식의 문장제에만 관심을 갖고 있으며, 문제해결 행동에 대한 양적 연구에만 몰두하고 있다고 특징지었다. 이 당시의 연구에 사용된 문제들은 정형적인 것과 비정형적인 것이었는데, 학습자들이 비정형적인 문제를 해결하는데 어려움을 겪는 것으로 나타났기 때문에, 문제해결 발견술과 전략에 대한 개념을 소개한 Polya의 제안이 이 당시에 각광받게 되었다(English & Sriraman, 2010). 수학교육자들은 그가 제안한 것이 학습자들이 익숙하지 않은 문제를 해결할 때 발생하는 막힘 현상을 해결해 줄 수 있을 것으로 기대했다. 따라서, 1970년대와 1980년대 대부분의 연구들은 문제해결 과정과 전략적 사고에 초점을 두었다. 하지만, 학생들에게 발견술과 전략을 지도하는 것은 학생들의 문제해결력 향상에 영향을 미치지 못하는 것으로 나타났다(Lesh, & Zawojewski, 2007; Lester & Kehle, 2003; Schoenfeld, 1992, 2007; Silver, 1985).

수학적 문제해결에 대한 연구가 이루어지던 초기에는 “문제” 그 자체에 대한 이해를 위한 연구가 활발하였다(Lester, 1994; Lester & Kehle, 2003). 문제의 정의, 문제의 유형, 좋은 문제의 조건들, 문제의 난이도와 같은 하위 영역이 연구되었다. 이 중 문제의 난이도에 대한 연구의 집합체는 “task variables in mathematical problem solving”이다(Goldin & McClintock, 1979/1984). 문제 난이도에 영향을 미치는 변인은 내용과 맥락 변인, 구조 변인, 구문론 변인, 발견술 행동 변인의 네 가지 변인이다. 초기에는 이들 변인들에 대한 연구가 선형회귀모형으로, 나중에는 정보처리 기법을 이용해 실행되었다. 그 이후에, 이 계통의 연구는 과제 변인과 문제해결자의 특성 사이의 상호작용을 연구하는 것으로 대체되었다(Kilpatrick, 1985). 이는 당연한 연구 주제의 발전이다. 즉, 문제란 문제해결자의 특성을 고려하지 않은 채 다루어질 수 없는 대상이기 때문이다(김진숙, 1997; Krulik & Rudnick, 1995). 수학적 문제해결 연구가 1990년대 중반에 잠시 반성기를 겪었다(Hino, 2007; Lester, 1994). 최근 들어서 수학적 문제해결이 재조명되면서 문제 그 자체에 대한 관심이 일고 있는데, 개방형 문제, 실생활 문제, 실생활적 문제, 여러 수학 지식을 통합하는 문제 등이다(류성림 등, 2011).

한편, 1980년대 들어서면서, 학습자의 문제해결과정에 대한 연구는 Chi, Feltovich, Glaser(1981)의 연구를 필두로 전문가와 비전문가의 문제해결과정의 차이를 밝히는데 초점을 두게 된다. 이런 접근의 배경에는 전문가들의 문제해결과정을 비전문가 즉 학습자들에게 습득시킬 것을 목적으로 하고 있다. 문제해결에 있어서 전문가와 비전문가에 대한 연구물들의 결과를 요약하면 다음과 같다(Lester & Kehle, 2003). 첫째, 좋은 문제해결자는 나쁜 문제해결자보다 많은 지식을 알고 있다. 즉, 그들이 아는 것, 그들이 어려워하는 것을 안다. 또한, 그들의 지식은 풍부한 스키마와 잘 연결되어 형성되어 있다. 둘째, 좋은 문제해결자는 문제의 구조적 특징에 대해 그들의 주의를 기울이는 경향이 있고, 나쁜 문제해결자는 표면적 특징에 초점을 둔다. Kilpatrick(1985), Krutetskii(1976), Schoenfeld(1985), Silver(1985) 등은 숙련자들이 좀 더 많은 수학을 알고 있기 때문일 뿐만 아니라 초보자보다 수학의 어려움을 더 잘 알고 있기 때문에 더 잘 수행할 수 있다고 설명한다.

이처럼, 1990년대 초반까지 수학적 문제해결과 관련된 많은 연구들이 있었음에도 불구하고, 이들의 노력은 실패한 것으로 결론을 내리고 있다(Lester, 1994; Schoenfeld, 1992). 이와 같은 지적이 지난 20여년간 있었음에도 불구하고, 연구자들은 2000년대 들어서도 수학적 문제해결에 대한 연구들에 대해서 같은 결론을 얻고 있다(English & Sriraman, 2010; Lester & Kehle, 2003; Schoenfeld, 2007).

많은 연구들이 실패한 것은 Polya의 제안이 잘못된 것이 아니라, 당시의 연구자들의 접근에 문제가 있어 보인다. 첫 번째, 문제해결 중심의 수학 수업에서 중요한 것은 참된 문제(A genuine problem)를 제공했어야 하는데(NCTM, 1989, p. 10), 당시 미국의 수학교실에서 “문제해결”은 1단계 또는 2단계로 해결되는 문제와 같은 단순한 문장제 수준이었다. 이런 경향은 NCTM(1989)이 Standards를 출간한 후에 제작된 미국 및 여러 나라의 수학교과서에서도 볼 수 있었다(교육부, 1989; Baroody, 1993; Schoenfeld, 2007). 두 번째, 교수법적인 측면에서 이 시기의 연구들은 발견술적 기능의 개발을 지나치게 강조하였으며, 이로 인해 학생이 자신의 활동을 조정하는 관리적 기능을 무시하는데 실패의 원인이 있다(Garofalo & Lester, 1985). 즉 문제해결 전략은 가르친다고 해서 이해될 수 있는 것이 아니라, 문제해결자로서 학습자 스스로 전략들을 구성해 냈을 때 학습효과가 있는 것이다. 또한, 발견술적 기능의 개발을 지나치게 강조한 교수법을 적용하여 수학적 문제해결을 지도함으로써, 학생들의 활동은 발견술과 수학적 전략의 부재를 가져왔고 더 나아가 메타인지의 발달을 촉진시키지 못하였다. 그 결과, 기계적이며 생각 없는 풀이방법 및 주요어를 찾아 문제를 해결하는 경향을 보였다(Greer, 1997; Schoenfeld, 1992; Yoshida, Verschaffel, & De Corte, 1997). 심지어, 성공적인 수학 학습이 발생한 연구에서 조차 학습 전이는 일어나지 않았다(Silver, 1985).

이렇게 학습된 수학 지식은 학습자들이 일상생활 속에서 사용할 수 없다(Hiebert, Carpenter, Fennema, Fuson, et al. 1996) 즉, 학습자에게 문제해결전략을 지도할 목적으로 이루어진 수업의 결과는 학습 전이를 기대할 수 없다는 것이다(Skemp, 1987). 세 번째, Polya가 제시한 전략과 발견술이 기술적이었지 처방적이지 못한 것에 실패의 원인을 찾아 볼 수 있을 것이다(Levin, 1976; Schoenfeld, 1992). 네 번째, 문제해결이 많은 교실에서 전통적인 수업 방법으로 교수된 것이 학습자의 문제해결력을 향상시키는데 걸림돌이었다(English & Sriraman, 2010). 문제해결에 대한 예전의 견해나 현재의 견해는 문제해결을 하나의 독립된 주제인 양 다루고 있다는 점이다. NCTM (1980a, 1989)에서 알 수 있듯이, 문제해결은 그 자체로 수학교육이다. 그럼에도 불구하고, 이 때 시도한 연구들은 문제해결력이 기본 개념과 절차를 먼저 학습하고 "문장제"를 해결하면서 발달할 수 있는 것으로 가정되었었다. English와 Sriraman (2010)의 초기 수학적 문제해결을 위한 연구물들의 실패의 원인에 대한 지적은 그 핵심을 짚은 것이라 볼 수 있다. 학습자의 수학적 문제해결력 향상에 적합한 참된 문제를 제공하지도 못하였지만, 참된 문제가 제공했다하더라도 부적절한 수업 방법을 사용하면 그 결과는 부정적일 수 밖에 없다. 이는 개혁 기반 교육과정에 따라 개발된 수학교과서를 가지고 전통적인 방식으로 수업을 하면 그 수업효과는 의도했던 결과를 가져올 수 없는 것과 마찬가지이다(Senk & Thompson, 2003). 다섯 번째, 좀 더 근본적인 이유는 관련 정부 기관 및 전문 기관들이 이미 문제해결에 대해 분명히 이해하고 있고 현재 드러난 문제들은 어떤 것이 작용하는지를 보여주는 개별 연구들을 통해서 해결되어야 한다고 믿고 있다는데 있을 수 있다(Lesh, 2010).

1990년대 중반으로 접어들면서, 학습자들의 수학적 문제해결력을 향상시키려는 연구자들의 노력이 잠정적인 실패로 확인되고, 수학적 문제해결 연구는 점차 줄어들게 되어(JRME에 게재된 수학적 문제해결 관련 논문편수는 1980년대에는 31편, 1990년대는 22편, 2000년에서 2003년에는 4편이다. 자세한 것은 Lester & Kehle, 2003 참고) 이에 관한 지식의 축적은 뒤처지게 되었다. 그렇다고 해서, 이 기간의 연구들이 아무런 진전을 보이지 않았다는 것은 아니다. NCTM(1980b)이 제안한 권고들이 전면적으로 새로운 조명을 받게 되었으며, 더 나아가 분명하게 Standards에서 수학적 문제해결은 이것 자체로 독립적으로 다루어져서는 안 되고 다른 standards와 통합적인 관점에서 다루어져야 한다는 점이 부각되었으며, 수업 및 교육과정의 중심이 되어야 함이 확인되었다.

연구자들은 1990년대 중반 수학적 문제해결에 관한 연구가 숨고르기를 하는 원인으로, 수학적 문제해결을 이론화하는 작업이 쉽지 않으며(Schoenfeld, 2007), 수학적 문제해결과정을 밝혀내고 이를 수학교육에 적용하려 하였지만 자신들이 예상하였던 것 보다 훨씬 더 복잡하다는 것을 알게 되었다(Lester, 1994; Lester & Kehle, 2003). 이를 통해서 여러 연구자들은 수학적 문제해결의 연구가 다음과 같은 방향으로 나아가야 한다고 제언한다. 지금까지 연구 주제들이 원자화되어 가는 경향이 있었는데, 이제는 주제간 통합이 이루어져야 한다(Hino, 2007). 인지심리학에서 벗어나 구성주의를 바탕으로 한 수학적 문제해결에 대한 연구가 진행되어야 한다(Stacey, 2007). ICT를 활용한 모델링을 통한 수학적 문제해결이 중심이 되어야 한다(English & Sriraman, 2010). 전통적인 실험설계에서 벗어난 디자인 실험(Design Experiments)을 통한 교실 수업 연구로의 전환을 시도해야 한다(Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003).

2000년대로 접어들면서, 위와 같은 새로운 요구에 부응해서 수학적 문제해결에 대한 연구는 새로운 국면을 맞게 된다. 연구자들은 규준을 성취할 수 있는 수업을 설계하여 실험을 하기 시작하였다. 이런 실험을 디자인 실험이라고 하는데, 전형적으로 디자인 실험은 직접적으로 문제해결을 지도하는 것이 아니라 개념적 이해, 즉 학습할 수학 지식을 새로운 방법 그리고/또는 교수법으로 수학 지식을 지도할 것을 목적으로 한다. 성공적인 학습 환경이 이루어지기 위해서는 방법론적 그리고 개념적 진전이 있어야 한다. 2000년대 들어서 새롭게 대두된 모토가 문제해결을 통한 수학학습이다 (Lester, 2003; Schoen & Charles, 2003). 문

제해결을 통한 수학 교수는 이미 1980년대 꾸준하게 가능한 한 가지 접근으로 제기되었다(Baroody, 1993; Branca, 1980; Schroeder & Lester, 1989). 그렇지만, 문제해결전략 및 발견술 지도로서의 목적을 띄고 있었기 때문에, 이 접근은 수학교육과정 발달사적 입장에서 상대적으로 새로운 아이디어였지만(Lester, 1994), 문제해결을 통한 수학 학습이 학습자의 학습을 촉진시킬 수 있을 것이라는 데는 동의를 하고 있었다. 이 새로운 접근과 전형적으로 관련된 많은 아이디어들─아동관의 변화, 교사의 역할의 변화, 학습자가 학습할 수학 내용으로서의 지식관에 대한 변화, 수업을 위한 문제의 개발 및 선택, 모든 학습자가 학습하는 학습공동체 형성, 협력학습, 교육과정 및 수학교과서의 개발, 새로운 교수법 등─이 광범위하게 연구되었다. 현재 문제해결 수업을 통한 수학 학습과 관련해서 교사들이 던지는 많은 질문들에 대해서 연구결과물을 토대로 하는 답을 할 수 있다. (이에 대한 자세한 내용은 Cai(2003), Lindquist(1989), Schoenfeld (2006) 을 참고하기 바란다.)

수학적 문제해결과 관련된 모든 새로운 견해들에 대해 논의하는 것은 본고의 범위를 벗어나므로, 본고에서는 문제를 중심으로 다른 요소들과 관련지어가면서 논의하도록 한다. 왜냐하면, 학습자에게 제공되는 학습자료 중의 하나인 문제는 그 문제가 학습자에게 요구하는 것이 무엇이냐에 따라 이후 학습 장면에 지대한 영향을 미치기 때문이다. 주로 교사가 문장제를 제시하여 풀이과정을 설명하고 학습자는 경청하고 나서, 교사는 학습자들에게 유사문제를 내고 학습자들은 이들을 푼다. 이후 교사와 학습자가 학습자들의 반응을 평가하는 수업을 진행할 수 밖에 없는 것이다. 참문제는 상황을 제시하고, 학습자들이 이 상황으로부터 한 가지 또는 그 이상의 적절한 해를 개발해 낼 수 있는 상황을 의미한다(NCTM, 1989). 이를 위해서는 학습자들은 주어진 상황을 탐구하고, 그 상황이 제공하는 문제에 대한 답을 줄 수 있는 자료를 수집하고, 분석하고, 얻어진 해에 대하여 논의를 하고, 그 해가 주어진 문제 상황에 적합한지 반성해야 한다. 이에 적합한 소재는 바로 실생활에서 얻어지는 소재들이다. NSF(National Science Foundation)의 재정 지원을 받아 개발된 모든 초·중·고 개혁기반(또는 규준기반) 교육과정은 실생활 상황을 그 학습소재로 한다(Senk & Thompson, 2003). 또한, Burns와 동료연구자들도 실생활소재를 중심으로 한 수학 수업 자료를 활용하고 있다(Burns & Wickett, 2001; Wickett, Ohanian, & Burns, 2002).

학습자는 주어진 문제를 해결함으로써 단순히 자신이 알고 있는 지식을 적용하는 것이 아니라 수학적 지식을 학습할 수 있어야 한다. 이때, 관심을 가져야 할 것은 현재 교실 공간에 있는 모든 학습자들은 저마다 다른 지적 능력을 소유하고 있기 때문에 저마다 다른 이해를 취할 수 있다는 점이다(김진호, 2010; Cai, 2003). 그런 점에서 보면 참문제는 다양한 접근이 가능해야 할 뿐만 아니라 다양한 결과를 가져올 수 있어야 한다. 교사는 이런 학습자의 지적 능력을 인정해야지만 개혁기반 수업에서 성공할 수 있다. 교사가 이에 동의하지 않으면 학습목표라는 하나의 지식에 초점을 둔 수업을 하게 되고 학습자들의 사고의 다양성은 존중받지 못하게 된다(Kamii, 1994). 한편, 대부분의 교사들은 이런 문제로 진행되는 수업 관행에 익숙하지 않기 때문에 새로운 수업 관행에 익숙해질 필요가 있다. 이러한 교사들에게 익숙하지 않은 관행 중에 두 가지를 꼽는다면, 하나는 학습자들의 이해 정도에 따라서 수업자료를 선택해야 하는 것이고, 다른 하나는 수학적 의사소통이 활발한 교실 문화를 형성하는 것이다. 이런 관행에 익숙해 질 수 있는 효과적인 방법은 워크샵이나 각종 연수를 받는 것 보다 수학 학습 상황에서 수학적 아이디어를 분석하고 연결성을 발견하는 기회를 가질 필요가 있다(Ball, 1993; Bransford, Brown, & Cocking, 2000).

수학적 문제해결이란 주제는 끊임없이 새로운 연구주제가 생겨난다는 점에서 수학교육 분야에서 마르지 않는 샘과 같다고 할 수 있다. 현재의 이 연구가 미래의 연구에 긍정적인 영향을 미치기를 기대해 본다.

2.2 우리나라에서 수학적 문제해결 연구 동향

우리나라는 국가수준의 교육과정을 갖고 있는 국가이기 때문에, 수학적 문제해결에 대해 언급하면서 수학과교육과정 및 (초등) 수학교과서를 언급하지 않을 수 없다. 수학적 문제해결이 주요 목표로 제시되는 것은 제4차 수학과교육과정으로, 시기적으로 보아 미국에서 수학적 문제해결이 강조되던 시기와 거의 같은 시기임을 알 수 있다. 국가수준의 교육과정을 운영하기 때문에, 우리나라는 교육과정 개정과 동시에 2학년에서 6학년 초등수학교과서에 "여러가지 문제" 단원을 구성할 수 있었던 반면에, 국가 교육과정이 없는 미국에서는 NSF의 지원을 받아 개발된 개혁 수학교과서들이 제작되면서 비로서 문제해결이 수학교과서에 반영되기 시작하였다 (Schoenfeld, 2007). 1970년대 및 1980년대 초반에 수학적 문제해결과 관련된 연구물이 거의 없었을 뿐만 아니라 전문 연구자 또한 거의 없었다는 점에서 이런 과감한 시도는 획기적이었다고 하지 않을 수 없다. 하지만, 이런 시도는 학습자의 문제해결력을 신장시키기 위한 문제의 게재에 불과하다 (백석윤, 1993). 수학적 문제해결을 수학교육과정에 반영하려는 근본적인 철학을 이해하지 못한 채, 당시의 일부 수학교육관련 종사자들은 "기존의 수학교육이 바로 문제해결을 포함하고 있기 때문에 새삼스러울 것이 없다." 는 주장을 내세운다 (백석윤, 1993). 또한, 수학적 문제해결을 마치 "문제풀기" 또는 "문장제의 풀이" 로 오해하고, 이런 경향은 이후에도 지속적으로 유지되어 오고 있다 (박교식, 1996; 정인수, 2003).

여러 가지 문제점을 안고 시작한 교육과정상의 수학적 문제해결의 목표를 달성하기 위해 연구진들이 다양한 연구들을 진행하였다 (강옥기, 신성균, 강완, 류희찬, 정은실, 박교식, 우정호, 1985; 강옥기, 정은실, 박교식, 강문봉, 1989; 성인서, 1987; 신성균, 강문봉, 황혜정, 1993; 양인환, 1991).

수학적 문제해결의 강조는 제5차 및 제6차 수학과 교육과정에서도 계속되고 있다. 지도내용면에서 문제해결 전략이나 내용, 방법 등을 포함시킴으로써 보다 문제해결 교육에 적극성을 띈다. 교육과정상에서 수업 방법에 대해서 강조를 하고 있음에도 불구하고, 초등수학교과서는 학습자들에게 문제해결전략을 지도할 목적으로 여러 가지 문제 단원을 활용한다. Kim(2002) 이 지적하였듯이, 한 단원에 한 두 전략을 독립적으로 배정하는 방식보다 다양한 전략을 경험할 수 있도록 배정하는 방식이 채택해야 하지 않았나 싶다. 그러나 실제 교실수업에서는 창의적인 사고력과 문제해결력을 기르는 교육이 강조되지 않고 있었다 (조경원, 김경자, 노선숙, 2000).

이후, 제7차 수학과 교육과정에서는 문제해결전략 중심의 수학적 문제해결에서 벗어나 문제해결을 통하여 교수·학습할 것을 강조하고 있다. 이는 문제해결이 마치 독립된 내용지식인양 다루어지고 있는 현실에 대한 반성의 결과이었다. 이런 시도는 수학의 내용을 문제해결방식을 통하여 문제해결의 정신에 입각한 방식으로 교수·학습하고자 하는 것이다 (교육부, 1997). 수학적 문제해결에 있어서 그 강조점의 변화라는 점에서 매우 주목받아야 한다. 강조점의 변화는, 앞 절에서도 진술하였듯이, 세계적인 변화와 같이 한다는 점에서 의미있는 시도이다. 하지만, 다시 한 번 수학교과서에는 반영되지 못한다. 제7차 수학과 교육과정에 따른 초등수학교과서는 여전히 문제해결전략 중심으로 구성되어 있음을 알 수 있다 (박교식, 2001; 방정숙, 김상화, 2006). 이런 점은 다른 영역에서도 나타나고 있는 수학과교육과정과 수학교과서 사이의 괴리의 연장선상에서 이해할 수 있다 (김진락, 1992; 소경희, 2000; 이부다, 김진호, 2010). 더 심각한 괴리 문제는 교육과정내에서 발생하는 괴리이다. 제7차 수학과교육과정이 문제해결을 통한 수학 학습을 강조하고, 문제해결전략 중심의 교육에서 벗어나고자 한다고 진술하고 있음에도 불구하고, 교수·학습 방법에 대해 다음과 같이 진술하고 있다 (교육과학기술부, 2010).

> 문제해결력을 신장시키기 위하여 문제해결 과정 (문제의 이해 → 해결계획 수립 → 계획 실행 → 반성) 에서 구체적인 해결 전략 (그림 그리기, 예상과 확인, 표 만들기, 규칙성 찾기, 단순화 하기, 식 세우기,

> 거꾸로 풀기, 논리적 추론, 반례 들기 등)을 적절히 사용하며, 문제해결의 결과뿐만 아니라 해결과정과 그 방법도 중시하도록 한다. (p. 13)

문제해결전략 지도로서 수학적 문제해결이 아닌 문제해결을 통한 수학 학습으로서 수학적 문제해결로의 패러다임적 전환에 대한 고민이 더 필요하다고 볼 수 있다. 이러한 문제점은 최소한 2007 개정교육과정에서는 해소되었다고 볼 수 있다(교육과학기술부, 2007).

> 수학적 문제해결력 신장은 ... 개정 교육과정에서도 지속적으로 강조될 필요가 있다. 이를 위하여 개정교육과정에서는 교육목표 뿐만 아니라 내용, 교수·학습 방법, 평가에 걸쳐 일관되게 강조하고 있다. (p. 40)

개정교육과정에서 명시적으로 선언한 위와 같은 진술이 반영된 수학교과서 제작 및 이를 반영한 수업이 진행되고 있는지는 새로운 연구대상이다.

지금까지 우리나라의 수학과 교육과정에서 문제해결을 어떻게 다루고 있는지 살펴 본 결과로부터 내릴 수 있는 결론은 다음과 같다. 우리나라의 수학적 문제해결과 관련된 교육과정의 진술은 세계적인 동향과 그 맥락을 거의 같이 하고 있다고 볼 수 있지만, 이를 실천하기 위한 수업자료인 수학교과서에는 제대로 반영되지 못하고 있다. 앞으로는 이런 괴리를 줄이려는 노력이 필요하겠다.

이제, 우리나라의 수학교육에서 문제해결에 대한 연구는 어떻게 이루어졌는지 살펴보자. 문제해결에 대한 연구는 주로 1980년대 이후부터 나타나기 시작하였는데 1990년대 초반까지 그 연구의 경향을 살펴보면 다음과 같다(백석윤, 1993). 첫째, 수학교육에 문제해결의 소개를 겸한 기초 이론적 정리, 둘째, 수학적 문제해결력에 대한 분석, 셋째, 문제해결의 지도-학습 방법에 대한 고찰, 넷째, 문제해결 결과의 평가 방법에 대한 고찰, 다섯째, 문제해결 교육을 위한 문제의 탐색 및 개발 등이며, 이 중에 문제해결 학습-지도를 위한 방법의 모색이 주를 이루고 있다. 이러한 주제들을 바탕으로, 국내에서 1980～1995년에 78편, 1996～2000년에 89편의 문제해결력에 관한 연구가 수행 되었다(방승진, 이상원, 황동주, 2002).

2001년부터 2004년까지 우리나라에서 수학교육과 관련한 주된 학회지에 실린 논문 중 문제해결과 관련한 내용을 구체적으로 살펴보면(박교식, 2005 참조), 웹이나 공학을 활용한 문제해결, 수학적 의사소통과 문제해결과의 관계, 문제중심 수업 또는 이와 관련된 학습자료 개발, 문제해결 과정에서의 메타인지적 활동 또는 직관과 논리의 역할, 문제 만들기 활동과 문제해결력과의 관계, 문제해결 과정에서 학생들이 사용하는 전략 분석 또는 학생들이 경험하는 오류나 사고과정 분석, 수학적 문제해결력과 관련된 정의적 요소 분석, 교사의 문제해결 지도에 관한 접근 방식에 따라 학생에게 미치는 영향 등 다른 관련된 구성요소와 연계하여 다양한 연구가 진행되어 왔음을 알 수 있다(방정숙, 김상화, 2006).

한편, 황치홍(2001)은 1980년대부터 2000년까지의 국내 문제해결 관련 연구—국내 석박사 학위논문, 학회지 논문 등—에서 주로 다루어지고 있는 주제 및 그 동향에 대해서 분석하였다. 그에 따르면, 문제해결 학습지도, 문제해결 교수, 문제해결 전략 등에 대한 연구가 주를 이루고 특히 문제해결 학습지도에 집중되어 있는 경향이 있었다. 컴퓨터와 문제해결이라는 주제가 새로운 주제로 활발하게 연구되고 있었다. 하지만, 문제해결과 평가, 문제해결과 태도 등 일부 주제들에 대한 연구는 미미한 상태라서 이 주제에 대한 연구들이 더 있어야 할 것으로 보여진다.

우리나라의 문제해결에 대한 연구 동향을 살펴보면, 미국의 문제해결 연구 동향과 큰 차이점이 있다. 미국의 연구 동향은 문제해결내의 특정한 주제에서 다른 특정한 주제로 변화되어 왔음을 알 수 있다. 하지만 우리나라는 특정한 주제에 따른 흐름이 있는 것이 아니라 처음부터 폭넓게 문제해결내의 여러 분야가 동시에 연구가 이루어지고 있음을 알 수 있다. 물론, 초기에 문제해결과 문제의 정의, Polya의 이론에

관한 연구 등 문제해결 이론의 기초가 되는 부분이 집중적으로 연구가 이루어졌었다. 하지만, 그 이후에는 다양한 분야의 연구가 동시에 이루어지고 있음을 알 수 있다.

3 연구 방법

3.1 연구 방법

추세 외삽법 (Trend Extrapolation), 추세 영향 분석 (Trend Impact Analysis), 교차 영향 분석 (Cross Impact Analysis), 델파이기법 (Delphi Technique) 등 다양한 방법을 사용해서 미래예측을 할 수 있다 (Phi Delta Kappa, 1984) 이런 기법들 중 델파이기법은 추정하려는 문제에 관한 정확한 정보가 없을 때 "두 사람의 의견이 한 사람의 의견보다 정확하다."는 계량적 객관의 원리와 "다수의 판단이 소수의 판단보다 정확하다."는 민주적 의사결정의 원리에 논리적 근거를 두고 있다 (이종성, 2001). 델파이기법은 예측하려는 문제에 관하여 전문가들의 견해를 유도하고 종합하여 집단적 판단으로 정리하는 일련의 절차라고 할 수 있다.

델파이 과정은 이론적으로 합의가 이루어질 때까지 계속적으로 반복되어야 하지만 (Hsu & Sandford, 2007), 대부분의 경우에 3～4회에 걸친 질문으로 충분하다 (이종성, 2006; Custer, Scarcella & Stewart, 1999; Ludwig, 1997). 4회에 걸친 델파이 과정의 절차는 다음과 같다. (보다 자세한 과정은 Hsu & Sandford, 2007; Yousuf, 2007 참조)

제1회 : 추정하거나 해결하려는 문제에 해당하는 분야의 전문가 또는 이해집단 구성원을 선정 (패널이라고 함) 하여 이들로 하여금 상호접촉하지 않고 연구문제에 대한 개방형 질문에 응답하도록 하여 일련의 판단을 수집한다.

제2회 : 1회 개방형 설문으로 수집한 비체계적인 개방형 응답들을 편집하여 구조화된 폐쇄형 질문들을 만들어 다시 패널들로 하여금 질문의 각 항목내용의 중요성, 희망, 가능성 등에 대하여 동의하는 강도 (보통 likert형 척도) 를 평정하도록 한다.

제3회 : 2회에서 회수한 패널들의 반응에 대하여 집중 경향과 변산도 (중앙값과 사분점간 범위 (사분범위라고도 함) 또는 평균과 표준편차) 를 산출한다. 3회 설문의 각 패널들에게 각 질문의 집중 경향과 변산도 측정값 (통계적 집단 반응) 과 패널 본인의 제2회 반응을 피드백하여 질문에 대한 반응을 재고하고 수정할 수 있는 기회를 제공한다. 제3회 질문의 각 설문에 대한 반응란에는 동의의 강도뿐만 아니라 다수의 의견으로부터 벗어난 반응을 할 때에는 다수의 의견과 달리하는 이유를 적을 수 있는 란을 포함한다.

제4회 : 제3회에서 회수한 패널들의 반응에 대하여 집중경향과 변산도를 다시 산출하고 다수 의견으로부터 벗어난 소수의견을 수합한 보고서와 함께 질문을 반복한다. 패널들의 의견이 어느 정도 일치할 때까지 몇 차례 질문을 반복한다.

본 연구에서는 시간과 여러 가지 제약에 의해 3회에 걸쳐서 델파이 과정을 시행하였다.

3.2 연구 대상

델파이기법에서 연구 대상의 선정은 대단히 중요한 과정이다. 왜냐하면, 이것이 생산된 연구결과에 지대한 영향을 미치기 때문이다. 그럼에도 불구하고, "델파이기법에 관한 문헌에서, 연구 대상을 선정하는 준거에 대해서는 애매모호한 상태로 남아 있다"(Kaplan, 1971). 델파이 연구 대상의 선정은 수행 중인 연구 주제에서 요구하는 전문지식의 정도에 의존한다.

본 연구에서는 제1회 델파이 과정을 위해서 문제해결과 관련한 책을 저술하거나 학술지에 문제해결 관련 논문을 다수 게재하거나 문제해결과 관련한 프로젝트에 참여한 7명의 초등수학교육자를 선정하였다. 그러고 나서, 제2회 델파이 과정에 참여할 전문가는 초등수학교육자 (교육대학교수), 초등수학교육 박사학위자, 초등수학교육 박사과정생으로 총 67명을 선정하였다. 이메일을 통한 설문을 실시하기 위하여 이들의 이메일을 각 교육대학교 홈페이지, 대한수학교육학회 및 한국수학교육학회 인명록, 한국교원대학교 동문회 명부를 통해서 이들의 이메일 주소를 확보하였다. 확보한 이메일이 변경되어 설문에 참여하지 못한 연구자를 제외하고 7인의 패널을 포함하여 최종적으로 67명이 참여하게 되었다. 2회 델파이 설문에 실제적으로 참여한 전문가는 총 41명이었고, 이들을 대상으로 제3회 델파이 설문을 실시하였다. 이 중 제3회 델파이 설문에 응답한 전문가는 총 38명이었다.

3.3 연구 절차

제1회 델파이 과정을 위해서 7인의 패널을 선정한 후, 이들을 대상으로 제1차 델파이 설문을 실시하였다. 각 패널은 연구진이 제시한 개방형 질문(앞으로 우리나라에서 문제해결 중 중요하게 다루어져야 하는 주제를 적어주세요.) 에 대해서 4∼6개의 반응을 하였다.

연구진은 이들의 반응을 정리하고 주제별로 분류하여 세부 문항까지 총 23개의 설문 문항으로 제2회 델파이 과정을 위한 2차 델파이 설문지를 작성하였다. 23개의 문항은 문제해결에서 예비교사교육과 교사교육과 관련한 문항 5개(문항 1.(1) 에서 문항 3), 문제와 관련한 문항 7개 (문항 4에서 문항 10), 학생관련 문항 5개(문항 11에서 문항 15), 문제해결과 관련된 여러 가지 사항에 관련된 문항 6개(문항 16에서 문항 21) 로 구성되어있다.(<부록 1> 참고)

<표 1> 각 주제에 관한 문항

문제해결에서의 주제	문항
예비교사교육과 교사교육	1.(1) ∼ 3
문제	4 ∼ 10
학생	11 ∼ 15
문제해결과 관련된 여러 가지 주제	16 ∼ 21

초등수학교육자 35명, 초등수학교육 박사학위자 12명, 초등수학교육 박사과정생 20명, 총 67명을 대상으로 제2회 설문을 실시하였다. 실시 방법은 전체 대상자에게 전자메일을 사용하여 설문지를 보내고 답메일을 받았다. 1차로 전체 전자메일을 2월 22일에 보내어 응답하지 않은 사람을 대상으로 개별 메일을 보내거나 직접 설문을 하거나 전화를 통하여 응답하도록 요구하였다. 2차 설문에 응하지 않은 설문자들을 대상으로 설문에 반응해 줄 것을 재차 요청하였다. 그리하여 2011년 2월 22일부터 4월 8일까지 제2회 설문의 응답자는 초등수학교육자 20명, 초등수학교육 박사학위자 5명, 초등수학교육 박사과정생 16명으로 총 41명이다. 그래서 2회 설문의 응답률은 61.19%이다. 2회 설문지의 응답에 관하여 사분점간 범위를 구하였다.

그러고 나서, 제3회 설문을 제2회 설문에 응답한 설문자들을 대상으로 실시하였다. 제3회 설문은 각 설문자가 응답한 내용과 전체 응답자가 응답한 결과를 비교하여 다른 경우에 자신의 응답을 수정할 수 있도록 하기 위함이다. 델파이 방법에서 이런 과정은 합의를 위한 과정으로 생략해서는 안 되는 과정이다. 제3회 설문 또한 전자메일을 이용하여 5월 12일부터 28일까지 실시되었다. 제2회 설문과 마찬가지로 응답이 없는 경우는 직접설문, 전화 설문, 전자메일을 이용해 재차 삼차에 걸쳐 설문에 응답해 줄 것을

요청하였다. 연구대상자 중 개인 사항으로 인하여 빠진 3명을 제외하고 38명이 응답하여, 응답률 92.68%이다.

2회 응답반응과 3회 응답반응을 비교분석하여 2회 응답보다 3회 응답에서 의견이 좀 더 수렴되었는지, 어느 문항이 변화가능성이 큰지, 작은지 등을 분석하였으며, 연구대상을 전문연구가인 교수 집단(이하 전문연구 집단)과 초보연구가인 박사 및 박사과정생 집단(이하 초보연구 집단)으로 나누어 비교분석하였다.

3.4 자료 분석 방법

합의와 수렴에 대한 분석 방법

설문지의 각 항목의 변화가능성과 희망에 관한 설문 참여자의 합의는 표준편차로, 두 집단 간의 합의(희망)는 일원분산분석(F-prob)으로 분석하였다. 델파이 절차를 적용한 목적의 하나는 횟수를 거듭함에 따라 설문 참여자들의 응답이 수렴하도록 유도하기 위한 것이다. 제2회와 제3회의 각 항목에 대한 응답의 표준편차로 비교하고, 수렴에 대한 분석은 3회 결과를 바탕으로 하였다.

변화 가능성과 희망

제1차 델파이 설문으로 얻은 23개 주제에 대하여, 전문가들이 그 주제가 앞으로 연구되어야 할 것이라고 객관적으로 의견을 나타낸 것을 변화가능성이라고 보았다. 그리고 전문가들이 개인적으로 주관적으로 그 주제가 앞으로 연구되었으면 좋겠다고 의견을 나타낸 것을 희망이라고 보았다. 이는 모두 5점 척도(Likert-type의 응답척도)로 제시하게 하였다. 변화가능성의 정도는 Likert-type의 응답척도를 퍼센트로 전환하기 위하여 다음 선형 공식을 적용하여 퍼센트로 환산하였고, 그 환산값은 <표 2>와 같다(이종성, 2006).

<표 2> Likert-type척도의 퍼센트 환산값

Likert-type 척도	%
1	96.00
2	73.25
3	50.50
4	27.75
5	5.00

$Y = -22.75X + 118.5$ (단, X: Likert-type 척도상의 응답, Y: %)

이와 같이 퍼센트로 환산된 변화가능성 확률은 다시 정상적으로 분석하기 위하여 변화가능성 정도를 <표 3>과 같이 분류하였다(이종성, 2001). 희망가능성도 같은 척도를 사용하였다.

<표 3> 변화가능성 확률에 따른 변화가능성 정도

변화가능성 확률(%)	변화가능성 정도
67 이상	높다
51 $\sim$ 66	있다
50 이하	낮다

4 연구 결과

선정된 67명의 설문대상자 중 이들의 여러 가지 사정으로 인하여, 제2회 설문에는 41명, 제3회 설문에는 38명의 전문가들의 의견을 수렴하게 되었다. 이 전문가들의 제2회 응답과 제3회 응답을 비교하고, 어떤 주제를 중요시하는지 살펴보고, 전문가들을 전문연구자 집단과 초보연구자 집단으로 나누어 각 집단의 의견을 비교하였다.

4.1 합의와 수렴

델파이기법 중 반복해서 같은 문항으로 여러 번 설문 조사를 하는 이유는 응답자들의 반응들의 합의와 수렴 정도를 알아보기 위해서이다. 본 연구에서, 3회 설문의 합의 정도는 변화가능성에서 전척도대비 11.83%에서 20.30%까지 나타났으며, 희망에 대한 합의는 10.46%에서 19.69%로 나타났다. 이는 각 문항의 표준편차 값의 범위를 나타낸다.

연구 대상은 전문연구자 집단과 초보연구자 집단—모두 교사로 이루어져 있으므로 <표 4> 에는 교사로 제시하였다—으로 구성되어있다. 이 두 집단 간의 합의도 이루어졌는지도 살펴보았다. 변화가능성에서는 t-검정을 실시한 결과, 두 집단의 응답 중 유의미한 차를 나타내는 문항은 존재하지 않았다. 희망에서는 t-검정을 실시한 결과 두 집단의 응답 중 유의미한 차를 나타내는 문항은 총 7개가 존재했다. 문항 8, 9는 문제와 관련된 문항이고, 문항 13, 14, 15는 학생과 관련된 문항이며, 문항 17, 18은 문제해결과 관련된 여러 가지 사항에 관한 문항이다.

<표 4> 집단간 유의미한 차를 나타낸 항목

문항	집단	인원	평균	표준편차	t	p
8	교사	18	85.89	14.01	2.567	0.015
	교수	20	70.98	20.75		
9	교사	18	80.83	11.04	2.340	0.025
	교수	20	69.84	16.95		
13	교사	18	75.78	13.26	3.862	0.000
	교수	20	55.05	18.96		
14	교사	18	83.36	14.01	2.189	0.035
	교수	20	72.11	17.27		
15	교사	18	84.63	17.88	2.248	0.031
	교수	20	70.98	19.39		
17	교사	18	88.42	11.04	2.480	0.018
	교수	20	75.53	19.39		
18	교사	18	90.94	9.73	3.044	0.004
	교수	20	75.53	19.39		

$p < 0.05$

결론적으로, 변화가능성에 대한 교사와 교수의 응답이 통계적으로 유의미한 차를 나타낸 것이 존재하지 않으므로, 모두의 합의를 이끌었다고 말할 수 있다. 희망에서만 23문항 중 7개의 문항에서 유의미한 차를 나타내었다. 16개의 문항에서는 전문연구자 집단과 초보연구자 집단의 합의가 이루어졌다고 말할 수 있다.

전제적으로, 각 문항에 대해 다양하게 반응이 나온 것은 아니다. 3회 설문에서는 50%이하의 문항이 하나도 없었다는 것은 전문가들이 각 문항이 추후 연구될 필요가 있는 주제들이라고 판단하는 것이라고 할 수 있다. 박사과정의 교사 몇 명과 인터뷰해 본 결과, 그들은 제1회에서 전문가들이 주제를 잘 선정해 서인지 모두 변화되어야 하거나 희망한다고 생각한다고 말했다. (<표 8>, <표 10> 참고) 이는 제1회에서 전문가 패널이 미래의 연구가 이루어져야 할 주제를 잘 선정한 것이라고 볼 수 있을 것이다.

4.2 변화가능성과 희망

3회에 걸친 델파이 설문을 통하여 변화가능성과 변화에 대한 응답자의 주관적인 희망을 분리하도록 노력하였다. 이를 검증하기 위하여 제2회와 제3회의 변화가능성, 제2회와 제3회의 희망에 대한 관계를 분석하여 보았다. 이를 위하여 <표 5> 를 살펴보면, 제2회 설문 및 제3회 설문의 변화가능성의 평균의 차이가 거의 나지 않는다. 따라서, 제4회 설문을 실시하지 않아도 전문가들 간에 의견의 합의에 이르렀다고 볼 수 있다. 이는 희망에 대한 수치도 마찬가지이다.

<표 5> 각 부분의 평균과 표준편차

	문항	평균	표준편차
2회 변화가능성	23	62.84	5.37
3회 변화가능성	23	62.82	6.03
2회 희망	23	77.57	6.43
3회 희망	23	78.38	6.69
합계	92	70.40	9.73

제2회와 제3회의 변화가능성을 한 집단으로 두고, 제2회와 제3회의 희망을 한 집단으로 구성하여, 전문가 집단이 변화가능성과 희망을 따로 분리하여 생각하여 델파이 설문에 응하였는지를 살펴보았다 (<표 6> 참고). 집단간 비교를 위한 F값 46.544에 대한 유의확률은 0.000으로, $p < 0.05$가 성립한다. 따라서 변화가능성과 희망이 통계적으로 유의미한 차이가 있다. 이러한 경우 네 개의 영역 사이의 차이가 있는 지 검증을 위해 사후비교 (Post Hoc Tests) 를 추가로 시행하여야 한다.

<표 6> 각 부분에 따른 분산분석

	제곱합	자유도	평균제곱	F	유의확률
집단-간	5281.251	3	1760.417	46.544	0.000
집단-내	3328.388	88	37.823		
합계	8609.639	91			

$p < 0.05$

그리하여, 제2회와 제3회의 변화가능성과 희망이 분리되어 의견이 제시되었다는 것을 세분화하여 살펴보면 (<표 7> 참고), 제2회와 제3회 변화가능성의 차이와 제2회와 제3회의 희망에 대한 차이는 유의확률 $p < 0.05$ 에 해당되지 않으므로, 통계적으로 유의미한 차이는 없다. 즉, 제2회의 변화가능성과 제3회의 변화가능성에 대한 의견이 차이가 없다는 것을 말해준다. 또한, 제2회의 희망과 제3회의 희망에 대한 의견에도 차이가 없다는 것을 알 수 있다. 이 결과가 제4회의 델파이 설문을 실시하지 않은 이유이다. 제2회의 변화가능성과 희망, 제3회의 변화가능성과 희망, 제2회의 변화가능성과 제3회의 희망, 제3회의

변화가능성과 제2회의 희망사이에는 통계적으로 유의미한 차이가 있음을 알 수 있다. 이는 전문가들이 변화가능성과 희망을 분리하여 생각한 결과라고 볼 수 있다.

<표 7> 모든 통계치의 순위 상관계수

	제2회 변화 가능성	제3회 변화 가능성	제2회 희망	제3회 희망
제2회 변화가능성		1.000	0.000	0.000
제3회 변화가능성			0.000	0.000
제2회 희망				0.978
제3회 희망				

$p < 0.05$

이와 같은 결과를 바탕으로, 제3회의 변화가능성과 희망에 대한 응답을 분석하여 살펴보았다.

변화가능성

변화가능성에서 각 문항의 변화가능성의 정도는 아래 <표 8>와 같다. 변화가능성이 낮은 문항은 하나도 없었고, 변화가능성이 높은 문항은 5개 문항이며, 나머지 문항은 모두 변화가능성이 있음을 나타내었다.

<표 8> 각 문항의 변화가능성의 정도

변화가능성 정도	문항
높다	1.(1), 1.(2), 2, 3, 4
있다	1.(3), 5～21
낮다	없음

변화가능성에서 평균가능성이 가장 높은 문항을 순서대로 정리하면 <표 9>과 같다. 문제 3은 문제해결을 통한 수학학습을 실행하기 위해서는 교사는 반드시 학습자의 현재의 이해를 바탕으로 수업을 진행해가야 한다는 점과 현재 이루어지고 있는 수업을 통해서는 이 주제를 성취해 낼 수 없음을 전문가들이 인식하고 있음을 반영한 결과라고 할 수 있다. 문항 1.(1) 은 수학을 지도하는데 필요한 지식 (MKT: Mathematical knowledge for Teaching) 에 대한 연구가 활발한 가운데 이 중 한 분야로 문제해결을 지도하는데 필요한 수학지식 (MKPS: Mathematical knowledge for problem solving) 에 대한 연구의 필요성을 제기한 것으로 볼 수 있다. 문항 1.(2) 와 문항 2는 예비교사 교육에서 여전히 Polya의 문제해결 4단계 중심의 문제해결 교수법과 교사들 사이에서 문제해결과 문제풀기를 혼돈하고 있음에 대한 전문가들의 인식이 반영된 것이라고 볼 수 있다. 문항 4는 문제에 관한 문항이다. 문제해결을 통한 교육의 시작은 좋은 문제로부터 시작한다는 점에서 전문가들은 이런 인식은 시사하는 바가 크다고 볼 수 있다. 여러 차례에 걸쳐 교육과정이 개정되었고 현재도 개정에 대한 논의가 이루어지고 있는 시점에서 여전히 전문가들의 인식은 초등수학교과서에 제시된 문제들이 문제해결을 통한 수학학습에 적합하지 못하다고 판단하고 있음을 알 수 있다. 앞서 진술하였듯이, 문제해결을 통한 수학학습에서는 개방형 문제를 도입할 것을 주장하는데 반해, 현재의 초등수학교과서의 접근은 학습한 내용의 응용 또는 적용을 위해서 개방형 문제를 도입하고 있다.

희망

희망에서 각 문항의 평균희망지수의 정도는 다음 <표 10>과 같다. 연구를 희망하지 않는 문항은 하나도 없었고, 1개의 문항을 제외한 모든 문항이 모두 높은 희망지수를 나타내었다. 이 한 개의 문항은 '학년 또는

<표 9> 높은 변화가능성을 보인 항목

문항	문항 내용	평균가능성(%)
3	예비초등교사/초등교사의 학생들의 문제해결 과정에 대한 이해 증진	73.85
1.(1)	예비초등교사/초등교사의 문제해결이나 문제해결 교육에 대한 PCK	70.26
1.(2)	예비초등교사/초등교사의 문제해결이나 문제해결 교육에 대한 인식	69.66
2	문제해결 교수법(교사교육관련)	69.66
4	좋은 문제 개발	68.46

학년군별 문제해결능력의 세분화' 에 관한 연구가 이루어져야 한다는 문항 13으로, 64.87%의 희망지수를 나타내었다.

<표 10> 각 문항의 희망의 정도

희망 정도	문항
높다	1.(1) ~ 12, 14 ~ 21
있다	13
낮다	없음

희망에서 평균희망지수가 높은 항목을 정리해보면 <표 11>과 같다. 변화가능성과 순위가 다르기는 하지만 변화가능성과 마찬가지로 문항 4를 제외하고, 다른 문항들은 예비초등교사와 초등교사 교육에 관한 것이다. 이로 미루어, 전문가들은 앞으로 변화도 가능하지만 무엇보다 전문가들이 예비초등교사와 초등교사 교육에 관한 연구가 많이 이루어졌으면 하는 희망이 크다는 것을 알 수 있다.

<표 11> 높은 희망을 보인 항목

문항	문항 내용	평균희망지수(%)
1(1)	예비초등교사/초등교사의 문제해결이나 문제해결 교육에 대한 PCK	89.41
1(2)	예비초등교사/초등교사의 문제해결이나 문제해결 교육에 대한 인식	89.41
2	문제해결 교수법(교사교육관련)	85.82
4	좋은 문제 개발	85.82
1(3)	예비초등교사/초등교사의 문제해결이나 문제해결 교육에 대한 수업분석	84.63
3	예비초등교사/초등교사의 학생들의 문제해결 과정에 대한 이해 증진	84.63

5 맺음말

본 연구는 수학교육학 분야의 주요 연구 관심사인 문제해결 분야에서의 미래에도 연구되어야 할 연구 영역이 무엇인지를 알아보고자 하였다. 이를 알아보기 위해서 전문가들의 견해에 대한 합의를 추정해 볼 수 있는 델파이 기법을 사용하였다. 앞 절에서 한 논의들이 반드시 미래에 연구가 되어야 한다는 것을 의미하지는 않는다. 다만, 초등수학교육 전문가들은 이런 견해를 가지고 있음을 보이는 것이다.

델파이 제1회에서 전문가들이 제시한 23개 문항에 대해서 제2회와 제3회의 반응이 단 한 문항도 "변화가능성이 없다" 그리고 "희망하지 않는다"고 하지 않은 것은 앞으로 문제해결과 관련해서 연구해야 할 주제들이 많음을 입증하는 것이라고 할 수 있다. 문제해결이 수학교육 분야에서 지속적으로 연구되어야

할 분야임을 의미한다고 볼 수 있다. 전문가들이 선정한 문제해결에 관련된 주제의 변화가능성 및 희망에 대해서 높은 가능성을 보인 문항들이 모두 예비초등교사 교육과 초등교사 교육과 관련된 문항이었던 점은 특히 주목할 필요가 있다.

델파이방법에서는 제1회 설문에 참여하는 전문가들의 견해가 매우 중요하다. 왜냐하면, 제2회 및 제3회 델파이 설문은 이들이 생성해 낸 설문 문항에 대한 변화가능성과 희망 정도를 점검해 보는 것이기 때문이다. 따라서, 연구진이 선정한 제1회 델파이 설문 전문가가 아닌 다른 전문가를 선정한다면 다른 결과가 나올 수도 있음을 밝혀둔다.

또한, 본 연구는 국내의 초등수학교육자들만을 대상으로 하였기 때문에, 국외의 (초등) 수학교육자들을 대상으로 같은 연구를 실시해 볼 필요가 있다. 외국에서는 최근 들어, 문제해결과 모델링 같은 새로운 연구주제가 활발하게 연구되고 있는 반면에, 국내에서는 이 분야에 대한 연구는 부족한 편이다. 우리나라에서 관심을 갖고 있지 못한 연구주제들에 선호가 있을 수 있다.

본 연구에서 제1회 델파이 설문에서 전문가들에 의해서 생성된 연구주제들이 모든 연구자들에게 의해 재조명될 수 있기를 희망한다. 또한, 이들 연구결과들이 수학교육의 실제 및 실행에 영향을 미치기를 기대한다.

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<부록 1> 델파이 제2회 설문지(델파이 제3회 설문지도 동일)

문제해결에서 중요한 주제 : <u>델파이 제2회 설문</u>

* 다음 각 문항에 대하여 해당 부분에 √ 표시하세요.

		변화가능성 척도 본 응답척도에는 변화에 대한 동의 여하에 관계없이 변화가 가능한 정도를 추정 표시하여 주시기 바랍니다.					희망 척도 본 응답 척도에는 변화에 대한 주관적인 희망을 표시하여 주시기 바랍니다.				
주제		거의 확실하다 (96% 이상)	대단히 가능하다 (68~95%)	가능성이 있다 (36~65%)	가능성이 희박하다 (6~35%)	거의 불가능하다 (5% 이하)	꼭 필요하다	바람직하다	보통이다	바람직하지 않다	유해하다
1. 예비초등교사/초등교사의 문제해결이나 문제해결 교육에 대한	(1) PCK										
	(2) 인식										
	(3) 수업분석										
2. 문제해결 교수법(교사교육관련)											
3. 예비초등교사/초등교사의 학생들의 문제해결 과정에 대한 이해 증진											
4. 좋은 문제 개발											
5. 문제에 대한 정의와 문제의 유형화 작업											
6. 교과서에서 문장제를 모든 단원에서 체계적으로 다루는 문제											
7. 실생활 문제에 대하여 수학적 모델링을 활용하여 문제해결력을 높이는 문제											
8. 개방형 문제를 통한 창의성 신장방안에 대한 실제적인 연구											
9. 문제중심 학습 방법에 대한 방안											
10. 문제만들기의 체계적인 지도											
11. 학생들의 문제 해결 유형											
12. 문제를 잘 해결하는 학생의 사고 특성 추출											
13. 학년 또는 학년군별 문제해결능력의 세분화											
14. 문제해결 과정에 나타나는 사고 과정 분석											
15. 우리나라 학생들의 문제해결 능력과 기타 수학적 능력(추론능력, 의사소통능력, 연결능력, 표현능력 등)의 비교 연구											
16. 수학적 문제해결 활동의 메타인지적 측면에 대한 연구											
17. 문제해결 능력을 신장시킬 수 있는 구체적 방안 모색											
18. 수학적 문제해결 능력의 평가 방법											
19. 문제해결 접근 방법을 통한 수학 수업 구성 방안											
20. 학교수학 현장에서 문제해결 교육의 실천 방안											
21. 사회적 과정으로서의 문제해결 연구											

HARRIOT'S ALGEBRAIC SYMBOLS AND THE ROOTS OF EQUATIONS

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ABSTRACT

Thomas Harriot(1560 1621) introduced a simplified notation for algebra and his fundamental research on the theory of equations was far ahead of that time. He invented certain symbols which are used today. Harriot treated all answers to solve equations equally whether positive or negative, real or imaginary. He did outstanding work on the solution of equations, recognizing negative roots and complex roots in a way that makes his solutions look like a present day solution. Since he published no mathematical work in his lifetime, his achievements was not recognized in mathematical history and mathematics education. In this paper, by comparing his works with Vié ta and Descartes who were mathematicians in the same age, I will show his achievements in mathematics.

Keywords: Harriot, algebraic symbol, the roots of equation

MSC 2000 : 01A40, 01A45, 0803

ZDM Classification : A30

1 Introduction

Although there have been constant disputes among scholars about the mathematician who first invented the notation of the algebraical symbol between Thomas Harriot and Viéta, François, Thomas Harriot, a British scientist and mathematician in the late sixteenth and the early seventeenth centuries, took algebraical symbols in the equation and some parts of them have been used up to this day. It is very progressive that Thomas Harriot received the idea of complex roots and negative roots in the equations and endeavored to generalize the form of equation.

Notwithstanding the achievements, his works have been dealt with carelessly. That is because he does not have any mathematical work but a posthumous book.

First of all, this paper will look into the mathematical history when negative numbers were not received as solutions of equation. A brief description of Harriot's life in connection with mathematical history, his scientistic and mathematical works will be dealt in the second part. Harriot's algebraical symbols and solutions of equation will be compared with Viéta, François and Descartes, René for bring out the differences. This will be a meaningful work and help to shed light on one of the foremost mathematicians.

2 Solutions of Equations and History of Algebraic Symbol

2.1 History of Negative Numbers

It took long time that mathematicians receive negative numbers thoroughly. Mathematicians found it difficult to adopt negative numbers. There was possibly difficulties in detecting a visual and geometric meaning and operating[8, 9]. It had reached the peak of disputes on an approval of negative numbers in sixteenth and seventeenth centuries. Even in eighteenth century, there were a lot of scholars who did not receive negative numbers for reasons of irrationality. The history of negative numbers, inversely, has showed farseeing intelligence of Harriot. Arcavi[1] made a study of history of negative numbers, which particularly had difficulties to be received, in his paper on the methods of instruction of it.

Diophantus, Greek mathematician in the third century, did not receive negative numbers as solutions of a linear equation because he regarded it as an absurd one. He only adopted positive numbers and nothing else is possible to extract. Brahmagupta in the early seventh century looked into multiplication between signs but he did not receive negative numbers as solutions of a quadratic equation. Since then, multiplication between signs became known to the whole India. Though Al-Khowarizmi in the ninth century made notes of negative and positive roots of a quadratic equation, it is doubtful that he understood it completely without any comments on this.

Fibonacci in the early thirteenth century rejected negative roots, but took a step forward when he interpreted negative numbers in a problem concerning money as a loss instead of a gain[1, 14]. In the fifteenth century, though Pacioli used a minus sign in the equation such as $(7-4)(4-2) = 3 \times 2 = 6$, he did not understand the meaning of negative numbers. On the other hand, French mathematician Chuquet may have been the first mathematician to recognize negative numbers as exponents[9]. Stifel, a well known German mathematician in the mid sixteenth century, recognized negative numbers as absurd ones saying 'negative numbers are smaller than nothing'. Cardano notes that the product of multiplication between two negative numbers has a positive sign in his work in 1545 but he was doubtful of negative numbers as a fictitious one. Bombelli also understood it insufficiently by appending a term 'm and n are positive numbers' to $m-n$ in 1572[1, 9]. Viéta explained some laws of algebra in <In Artem Analyticem Isagoge> (1591) but he left out a specific explanation of negative numbers. He set limits on coefficient in the equation as positive numbers[1].

Hudde in Germany in 1659 did not make a distinction between positive and negative numbers. Harriot consented to his idea but literature disputes whether Harriot received negative numbers as solutions of an equation and understood the meaning of it[1, 2, 8, and 9]. Descartes partly received negative numbers. He called negative roots as a fake one because he thought that negative numbers are smaller than nothing. He inferred that positive numbers would be genuine roots of the equation.

It was the seventeenth century that mathematician received and applied freely. Practicalism was surged forth in the mid seventeenth century even in mathematics. It made an algebra design a consistent theory so that the application of negative numbers was unrestrained. However logical consideration about negative numbers was unsatisfactory. Since the notion and logic of negative numbers was unreliable, mathematicians evaded to comment or object to use of it. d'Alembert said 'if there was an equation having negative roots, there should be a misdirection' and 'the right answer was with a plus sign'[1, 8]. Maseres, a British mathematician, disregarded negative numbers as not understandable one in his book. Euler misjudged negative numbers as 'bigger one than infinity' in the letter to Wallis but corrected it later. This conclusion was deduced from the sequence — $1/4, 1/3, 1/2, 1/1, 1/0, 1/-1$, $1/-2, 1/-3$, — but later he changed position after the observation of the sequence — $1/9, 1/4, 1/1$, $1/0, 1/1, 1/4, 1/9$, — . When Pascal said 'subtracting 4 from 0 leaves 0', his friend Arnauld made an objection to it bringing $-1 : 1 = 1 : -1$, the ratio of smaller to bigger one was bigger to smaller one. De Morgan in the nineteenth century consented to d'Alembert saying 'there is no numbers smaller than 0'.

After adopting the logical formalism which excluded self-contradiction, mathematicians started to receive a negative number. Whitehead and Russel said that 'if receiving a symbol as an operator, there would be no restraint' in <Prinkipia>; as the solution of the equation $x + 1 = 3$ is $x = 3 - 1 = 2$, the solution of $x + 3 = 1$ is $x = 1 - 3 = -2$. That is, the solution of $x + a = b$ is $x = b - a$. They emphasized that an idea of $+$ and $-$ as an operator would remove absurd and irrational working. It was a consequence of mutual supplementation of Fibonacci's intuitive and unconscious point of view which regarded negative numbers as a meaningful one and the logical and consistent point of view which insisted an adoption of negative numbers or complex numbers as solutions of equation just like a Euclidean geometry. The former emphasized intuitive and comprehensive understanding and the latter stressed analytic and axiomatic formalism.

2.2 History of Algebraical Symbols

It was necessary to have mathematical signs for modern mathematics. Consequentially the development of mathematical notation brought about the development of mathematics. In the sixteenth century in Europe, mathematical notations were invented and applied. The sign of $+$ and $-$ were first appeared in <Mercantile> by Widmann in 1489. In this book, the 'excess' and 'shortage' were represented in equations instead of 'addition' and 'subtraction' or positive and negative numbers. Heocke was the first mathematician who used it as an algebraic symbol in 1514. Recorde, a British mathematician in 1557, first presented an equal sign '$=$' and he explained that equality means the parallel. At that time, the length of an equal sign was little longer. Or, two parallel vertical lines and '  ' also used as an equal sign. The multiplication sign '•' appeared Harriot's <Artis Analyticae Praxis> but Oughtred used '$\times$'[4] as the multiplication sign. Harriot, the initiator of an inequality sign, devised more convenient signs than Oughtred's one.  and  was appeared in Harriot's manuscript but we could find '$>$' and '$<$' in his book <Artis Analyticae Praxis>[11]. At that time, the present sign of division '$\div$' was appeared in Rahn's algebra book and a radical sign '$\sqrt{}$' was used in Rudolff's paper[4, 9].

3 Thomas Harriot

Thomas Harriot as the virtual first Britain algebraist introduced algebraic signs such as the interpretation of equation and inequality signs. An equal sign (=) generally known as the sign which a mathematician Robert Record had invented had become famous because Harriot aggressively used the sign.

Unfortunately, Harriot did not leave any of his books in mathematics in his entire life. After ten years since his death, his book <Artis Analyticae Praxis> was published by his colleagues in 1631. Furthermore, this fact had not been known to the world until Descartes's book referred to Harriot's thesis in 1637.

3.1 Harriot's Lifetime

Harriot had been born in Oxford, England in 1560 and graduated from Oxford University when he was twenty years old. There was not known about his private life when he was young. He had two tutors in his entire life and those were Minister Walter Ralegh and Count Henry Percy. After he had graduated from university, he became Walter's private mathematics teacher and took part in the lesson of the reclamation of new world and exploration to the new world in the North America with Walter's companies. As he brilliantly had worked as an adviser of the American expedition, for instance he had done the design and production of the ship and recruitment; 'a report about the new world, Virginia' which had been issued in 1589 paid the highest tribute of admiration[4, 6, and 9]. In addition, the report was filled with the native's language, religion, the method of the trade; therefore it became a famous thesis of settlers to the new world. This book is the only book he made in person. After Walter had warped up in political chaos, Henry began to help Harriot since 1598. Walter unstinted in his praise of Harriot's work and he described Harriot as 'Count magician'. Thanks to Henry, Harriot was able to focus on a stable study in a science lab with mathematicians Walter Warner and Thomas Hughes.

As Count Henry underwent the hardships of prison life for political reason in 1605, Harriot was also suspected but released soon. After that time, he devoted himself to the study of natural science such as mathematics, astronomy, mechanics and optical science and left outstanding academic achievements which great scholars paid little attention to. Though he attained materials for figuring the sun's rotation period by observing a solar spot in 1613, this period was his last moment to have a passion for his academic work. He was under continuous adverse circumstance of Henry's unfortunate death, his colleague's death and his disease. After five years of struggle against his disease, he passed away in 1621[11].

Although he made splendid achievements in mathematics and science, he did not leave any publications in his life. Confusing socio-political atmosphere and his meticulous characteristics shown in the report about Virginia had not gained an opportunity to publish his work.

3.2 Harriot's Natural Science

Most of Harriot's manuscripts were about natural science. Looking into his achievements in natural science would help to understand Harriot.

Harriot was endowed with practical and profound scientific knowledge and he achieved results in the fields of astronomy, optical science and dynamics. He was interested in astronomy after the discovery of a comet at that time and he observed the movement of the comet, later called Halley's Comet, from 1607. He observed the comet using a telescope in 1609 and it was ahead of Galilei. He first discovered sunspots while he observed Jupiter and after that time he had 199 times of record of sunspots observation from 1610 to 1613[9, 11].

Earlier in 1597, he discovered sine rule about refraction of lens and it was twenty years ahead of Willebrord Snell who was known as the originator of the theory[11]. A multi-color spectrum of light inspired Harriot and he developed the theory of a rainbow. Kepler got the news and sent a letter to Harriot but they never exchanged their theories. Harriot might feel discomfort to deliver his idea to Kepler directly or he might plan to publish when he regained his health [14]. In dynamics, he studied free fall of a parabola in the no resistance condition ahead of Galilei. Harriot discovered a track of a shell which described a parabola and could be divided into horizontal and vertical components. He also theorized various problems of sea navigation and his calculation was so accurate that he won praise[6].

His strong will and untiring observation and study brought him achievements of natural science.

3.3 Harriot's Mathematics

Since Harriot published nothing in his lifetime, his achievements are underestimated. In 1631, after ten years from his death, his colleagues published < Artis Analyticae Praxis> which contained his achievements. He in fact wanted Nathaniel Torporley to publish his works but Torporley's intimacy with Viéta made him to hesitate. Manuscripts which are kept in the library in Britain lack consistency and were out of order. Further, there are differences between manuscripts and <Artis Analyticae Praxis>, though the book was based on the manuscripts[11]. It seemed that some mathematicians revised arbitrarily, in the process of editing, when they concluded his mathematical results were wrong. Such being the case, estimations of Harriot in mathematics history varied. Some literature mentioned his unacceptance of negative and complex roots. A recent research tends to give priority to manuscripts when they compare manuscripts and the book. In 1883, 260 years after his death, Sylvester showed his respect to Harriot as 'a father of modern mathematics who introduced algebra to analytics' in a letter to Cayley[5, 11].

Algebraic Symbols and Roots of Equations

Even though most of literature mention Harriot as the originator of inequality signs, he used  and  instead of '<' and '>'. Moreover Robert Recorde's equal sign was used in his book and the equality sign spreaded among people.

Warner, the editor of <Artis Analyticae Praxis>, wrote down comments under the error Harriot had made. Harriot used algebraic symbols except an exponent in an excellent way. Viete used vowels for unknowns and consonants for knowns and Harriot adopted it in his solution of an equation. Letters and abbreviations were also used in expansion of an equation. For instance, a^4 was represented as $aaaa$.

Following is quoted from his manuscripts[11]. For the expansion of the multiplication $(b-a)(c-a)(df+aa)$, we could find out that the symbol '⌟' was used which was similar to the symbol in these days.

$$\left.\begin{array}{l} b-a \\ c-a \\ df+aa \end{array}\right| \rceil\lceil$$

$$\begin{array}{l} bcdf - bdfa + dfaa - baaa \\ \quad - cdfa + bcaa - caaa + aaaa \\ \qquad\qquad\qquad\qquad \rceil\lceil\ 0000 \end{array}$$

A symbol  was an equal sign to represent the expansion of an equation and four 0s at the last line showed a homogeneous expression. This meant that he dealt a homogeneous expression emphasizing calculability.

The solutions of the equation $(b-a)(c-a)(df+aa) = 0$ are $a = b$, $a = c$ and $aa = -df$. Harriot drastically represented $a = \sqrt{-df}$ from the solution $aa = -df$.

$a \,\text{II}\, b$
$a \,\text{II}\, c$
$aa \,\text{II} - df$
$a \,\text{II} \sqrt{} - df$

This was a challenging idea to existing mathematicians who disregarded negative roots.

There are differences between his manuscripts and <Artis Analyticae Praxis> which his colleagues published in the way of dealing roots of an equation. As we could find out from his manuscript in advance, Harriot dealt with negative and imaginary roots as same as positive ones. He received negative roots without any comments and he added an explanation of complex roots as 'noetic by rationality'[13]. He presented four roots of biquadratic equation as follows.

$5 \,\text{II}\, a, \quad -7 \,\text{II}\, a$

$a \,\text{II} +1+\sqrt{-32}, \quad a \,\text{II} +1-\sqrt{-32}$

He received positive, negative roots and two complex roots without reluctancy. In his manuscripts, we could find many imaginary roots from equations.

The following biquadratic equation is quoted from his manuscript[14].

$$aaaa - 6aa + 136a = 1155$$
$$- - - - - - - -$$
$$aaaa - 2aa + 1 = 4aa - 136a + 1156$$

This showed the process of solution of a biquadratic equation $a^4 - 6a^2 + 136a = 1155$. In the left side of the equation, he changed to the form of perfect square of the second degree and, as a result, he had a form of perfect square of the first degree. Then, he extracted the two square roots with $\pm$. Surprisingly, he adopted complex roots at the last line of two equations without reluctances.

$$aa - 1 = 3a - 34$$
$$33 = 2a - aa$$
$$aa - 2a = -33$$
$$aa - 2a + 1 = 1 - 33$$
$$a - 1 = \sqrt{} - 32$$
$$a = 1 + \sqrt{} - 32$$
$$a = 1 - \sqrt{} - 32$$
$$- - - - - - - - -$$

However, editors of <Artis Analyticae Praxis> were reluctant to receive negative roots of the equations. They excluded square roots saying 'unexplainable and impossible'. We could infer that editors' lack of understanding of Harriot's mathematics brought about the wrong revision of manuscripts. This led to further literature to mention Harriot's neglect of negative and complex roots[9].

Further Mathematical Achievements

Among Harriot's manuscripts, the mathematical contents except algebraic signs and roots of equations has been recently studied. A conic section, its problems and an observation of celestial sphere and related problems drew from research on Archimedes. Though there have been controversies whether Harriot fully understood the concept of infinitesimal, many literature showed that he solved Alhazen problem by using infinitesimal concept ahead of Barrow[11]. We could also find out the Pythagorean number and some calculations similar to differential and integral in his manuscripts[7, 11].

4 Comparison with Viéta and Descartes

At that time in Europe, Viéta(1540 1603) was more recognized by public. He, who was acknowledged as an originator of algebra using letters, was an algebraist and also had interests in geometry. Contrary to former times to substitute numbers, he was the first mathematician using letters and generalizing a quadratic equation. The former algebra was called as 'Logistica numerosa' and the latter was as 'Logistica speciosa'. History of algebra, in fact, has divided before-and-after algebra on the basis of the advent of Viéta.

Viéta represented a cubic equation as C, a quadratic equation as Q, and the unknown as N which were picked out from initial sounds of Latin. His use of letters and abbreviations was an epoch-making event in history. However, Harriot used more developed representation of a repetition of the same letters for expressing the degree[9]. Decartes(1596 1650), who was born thirty six years after Harriot, showed more progressive form of mathematics. Mathematics after the seventeenth century has developed by logic itself. Descartes, one of the prominent mathematicians at that time, showed his mathematical sense through his three paper <geometry> as appendixes of <Discours de la methode> dealing with philosophical problems. He was interested in mathematics because of its certainty definitude of an inference and he tried to apply rational considerations to studies of natural science. The invention of analytic geometry was the greatest achievement of Descartes. Analytic geometry, which related algebra and geometry, made our knowledge of space and spatial relations transfer to the language of numbers, and this allowed us to grasp the logic of geometric idea[3]. Though Descartes invented analytic geometry with Fermat, he was not ready to receive negative roots.

The representation of equations has used the abbreviation from Diophantus era and it has developed to easier way to deal with in the sixteenth and seventeenth centuries.

Viéta represented a cubic equation $x^3 - 8x^2 + 16x = 40$ as

$$1C - 8Q + 16N \text{ aequ. } 40.$$

and Harriot represented it as

$$aaa - 8aa + 16a = 40.$$

On the other hand, Descartes represented it as

$$x^{3^*} - 8x^{2^*} + 16x \propto 40$$

and it seemed like more modernistic one. Yet he represented the third and forth degree as x^{3^*}, x^{4^*} or x^3, x^4 the second degree was represented as xx for a long time[4, 9].

Viéta partly discovered the relations of roots and coefficients; he said 'solutions of a cubic equation

$$x^3 - (u + v + w)x^2 + (uv + vw + wv)x - uvw = 0$$

were u, v, w'. However, it was not complete that he only took positive roots in the actual extraction.

Harriot found out that 'if a, b, c were the solutions of a cubic equation, it could be represented as $(x - a)(x - b)(x - c) = 0$' and he showed the logic of the generalization of an equation of higher degree. The generalization of degrees of an equation was the progressive idea ahead of the times.

One equation could be represented in various ways. A sign could be changed when it transposes, and similar terms could be confused in order. However, the right side of an equation would be 0 when similar terms are put together and listed in descending order at the left side of an equation. This would be a standard type of an equation and a solution would be determined. It is no wonder in these days but in the sixteenth or seventeenth centuries people had various ways of solution. They had difficulties to receive negative numbers so that they had to transpose all negative numbers to the other side for a representation of positive numbers. Frend presented four types of quadratic equations in his book <The Principles of Algebra> (1796) as follows.

$$\begin{aligned} &x^2 = b \\ &x^2 + ax = b \\ &x^2 - ax = b \\ &ax - x^2 = b \end{aligned}$$

These four different types showed his difficulties to adopt negative numbers and a trial to remove it. Since Harriot dealt negative numbers the same as positive numbers, he had only one type of an equation. A standard type of an equation is the one and he only needed to factorize one. Thereafter, Descartes praised his achievement and called it as 'Harriot principle'.

5 Conclusion

Since Harriot had never published his achievements in lifetime, his works has gone unnoticed. He was one of the prominent mathematicians at that time in the aspect of discoveries in methodology and a mathematical sign and he led Britain mathematics to developed European one. His works were quoted by Stevin, Bombelli, Stiefel and Viéta at that time and later by Wallis and Descartes.

This paper revealed Harriot's mathematical achievements, especially on algebraic symbols and negative and complex roots in an equation, and restored his status by comparing it to Viéta's and Descartes'.

In the unacceptable atmosphere on negative roots in an equation, Harriot received even complex roots and said it was 'the perceptible roots only by a rational sense'. He also simplified the relations between roots and coefficients and generalized it. The representation of algebraic symbol was more modernistic than Viete's one which was famous at that time. A lot of literature mention that the first use of inequality signs of ‘>’ and ‘<’ was by Harriot but it has been controversial between Harriot and Viete.

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A DISCUSSION ON THE SIGNIFICANCE OF THE DISCOVERY OF MATHEMATICS BAMBOO SLIPS FROM THE WARRING STATES PERIOD, QIN AND HAN DYNASTY

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ABSTRACT

The recent acquisitions of archaeological findings of bamboo slips in many institutions in China provide us a good significance for the study of development of mathematics in Ancient China. These includes the Tsinghua University (arithmetic table, dated from the Warring States period), the Yuelu Academy (bamboo slips of the book "mathematics", Qin Dynasty), the Beijing University (bamboo slips of mathematics, dated in the Qin Dynasty), and the Hubei Museum (Han bamboo slips of mathematics at the found on "land of sleeping tiger", Han Dynasty). Firstly, it gives us rare first-hand information on the knowledge of mathematics in Ancient China. Secondly, it helps to clarify that the nihilistic attitude holds by some scholars on the development of early Chinese mathematics was unfounded. Thirdly, it provided strong evidence in the study of Chinese mathematics, that the major methods and mathematics problems in Jiuzhang Suanshu (Nine Chapters) was completed in early Qin period. Most importantly, it provides reliable literature to support that the first peak of mathematics achievement in China was in the Spring and Autumn and Warring States period. In the past, the statement was only supported by the evidence of the work of Jiuzhang Suanshu and its explanations and remarks made by Liu Hui.

1 The recent discovery of mathematics bamboo slips from the Warring States Period, Qin and Han Dynasty

From 1983 to early 1984, about 200 mathematical bamboo slips "Suanshu shu" were discovered in Jingzhou Zhangjiashan Han Tomb No. 247 in Hubei. There was less significant harvest for 20 years in unearthed mathematical bamboo slips since then. Then there were much news on discovery of mathematical bamboo slips from the Warring States Period, Qin and Han Dynasty in the past 10 years. The news of discovery excited the circle of the scholars in history of Chinese mathematics. Some of the discovered bamboo slips have been treated and studied, some are under treatment. We introduce as follow these discoveries according to the date of history of these mathematical bamboo slips:

Calculation table (suanbiao), Warring States Period, collection of Tsinghua University The Centre of research and protection for relics of the Tsinghua University has a collection of calculation table (suanbiao) dating from the Warring States Period. The book has 21 bamboo slips and is dated 2300 years from now. The whole table consists of 441 unit cells, in the form of 19 rows time 19 columns.

The right column and the upper column start from bottom to above and from left to right. It could be reads in order, from $\frac{1}{2}$ to the cardinal number 1 to 9 and then all two-digit number. Their products were recorded at the intersection of the cells. Therefore, the table is a multiplication in the form of 9 times 9; the rest is the extension of the table.[1]

Wooden Book of 9 times 9 table, Qin Dynasty, unearthed in Liye Town, Hunan In the year 2002, a multiplication table made from wood[2] dating from the Qin Dynasty in the form of 9 times 9 was unearthed from an old well in Liye Town of Hunan (湘西). There are in total 113 characters. Most of the bamboo slips of 9 times 9 multiplication table unearthed in 20th century were incomplete, and this table from the Qin Dynasty is a complete 9 times 9 table.

Bamboo slip ≪shu≫, Qin Dynasty, collection of Yuelu Academy Hunan University In December 2007, The Yuelu Academy in Hunan University had purchased a batch of bamboo slips from the antiques market in Hong Kong. Those bamboo slips were examined by a group of experts and it was confirmed that these were dated from Qin Dynasty. We are most interested in that collection of slips under the name "Shu"(mathematics). At present, 236 bamboo slips in Shu has been arranged and numbered, with 18 pieces of them not in good conditions (without numbering). The better-preserved slips are about 27.5 cm in length and 0.5 to 0.6 cm in width. There are three braided rope at the upper, middle and lower part of each bamboo slip. "Shu" was printed at the back of bamboo slip numbered 0956, which make the name of the book.[3]

In the format of "Shu", a complete problem (suan ti) generally consisted of four parts, conditions, questions, answers and method. Only a few of the problems was given a title. There are 19 exemplars of methods (shu) in the book. For example, "he fen shu" (xx) and pithy for multiplication. There are 34 bamboo slips with the record of ratio of volume and weight of grains and their rate of exchange. Another 3 bamboo slips recorded measurement and units. The content of ≪shu≫ from Qin Dynasty included arithmetic operation of fractions, areas of fields, crop yield, ratios of grains and its rate of exchange, shuaifen, shaoguang, volume, ying bu zu, gougu, yingjun and zu wu quan etc.[4] There was only one Gougu problem, and is in consistence with the gougu geyuan in GouGu Chapter in Jiuzhang Suanshu, with simple and unsophisticated wordings.[5]

Mathematical bamboo slip, Qin Dynasty, collection of the Beijing University In early 2010, a friend from Hong Kong donated a batch of bamboo slips to the Beijing University. Among these slips, there are in total 400 slips written with mathematics, a big proportion in this batch of slips. According to preliminary study, these mathematical bamboo slips is close to those bamboo slips Shu from Qin Dynasty by Yuelu Academy, Suanshu shu from Han Dynasty and the Jiuzhang Suanshu.

[1]Tsinghua's new development of bamboo slip. "The recent development of the jian", from the website of the Fudan University's Research Centre of Unearthed Literature and Ancient Text:
http://www.gwz.fudan.edu.cn/ShowPost.asp?ThreadID=3522, 2010-8-10

[2]Hunan Province Archaeological Research Office: Brief report of Hunan Long Shan Li Ye Warring States Period – ancient city of Qin Dynasty, No.1 well excavation. "Wen Wu" vol. 1, 2003.

[3]Chen Songchang : Content summarization of the bamboo slips from Qin Dynasty in Yuelu Academy. "Wen Wu" vol. 3, 2009.

[4]Xiao Can, Zhu Hanmin: Major contents and historical value of the bamboo slips "shu" from Qin Dynasty in Yuelu Academy. "Research of Chinese History" vol. 3, 2009.

[5]Xiao Can, Zhu Hanmin: New evidence of gougu – related research of the bamboo slips "shu" from Qin Dynasty in Yuelu Academy. "Research of the history of natural science" Paper 29, vol 3, 2010.

The problems of the slips are classified into different groups, which mean similar problems are put under the same group. There are two formats used for describing the mathematical methods and the structure of problems. One started by giving a description for algorithm, with the wordings "it is stated that..." and then listed the exemplar of problems, with some variation in the numerical values in the problem. The second one is by listing the problems, and then provides with the general form of algorithm with the wording "A says...". As we know, these two formats is the main body of style Jiuzhang Suanshu. Another point worth mentioning is that there is a passage around 800 words, which starts with "Lu jib inquire mathematics from chin quid", discussing the beginnings off mathematics, its role and significance. [6,7]

Mathematical bamboo slips from Shui-Hu-Di (Land of the Sleeping Tiger), Han Dynasty, collection of Hubei Museum There were 216 bamboo slips unearthed in a Han Tomb in Shui-Hu-Di (Land of the Sleeping Tiger) in Hubei. The book is named "Suan shu". These bamboo slips were slightly damaged, but the characters and the texts are clear.

And from the calendar unearthed at the same finding dated the seventh year of Hou-Yuan, the year of the emperor Wen-Di of the Han Dynasty (157 BC). There are 10 pictures of these bamboo slips on the "Jianghan kaogu". These bamboo slips are being compiled, and the preliminary information revealed that some of the questions were similar with Shu (Qin Dynasty), Suanshu-Shu (Han Dynasty) and Jiuzhang-Suanshu, but there are also content which has not been appeared in any of the Chinese traditional mathematical works.[8] In addition, pieces of mathematical bamboo slips were also unearthed from Fuyang of Anhui and Linyi Yinqueshan (mountain of silver bird) from Shandong .

2 The significance of mathematics bamboo slips from the Warring States Period, Qin and Han Dynasty

These mathematical bamboo slips unearthed are rich in mathematical content; and have great significance in the course of study in the history of Chinese mathematics. These have been discussed in Zou Taihai's papers.[9] This paper would like to share the following points.

2.1 It provides the firsthand literature from pre-Qin dynasty and Qin Dynasty

Over the past century, scholars in the field of history of Chinese Mathematics were much regret with the fact that very few mathematical literature were passed down from pre-Qin and Qin Dynasty. From the past relic exploration, only a few fragments of Chousuan and 9 times 9 multiplication table were unearthed. We could only deduce the achievement and development of mathematics in pre-Qin Dynasty and Qin Dynasty according to the shape, characterization and a few second hand information unearthed.

[6] Brief report of the work of Beijing University Unearthed Literature Research Office, vol 3, October 2010

[7] Discussion on China's earliest mathematical theories found from Beijing University's compilation of bamboo slips from Qin Dynasty (Guang Ming Daily), China News Web: `http://www.chinanews.com.cn/cul/2010/10-25/2609393.shtml`, 2010-10-25.

[8] Cai Dan: A report at the explanation meeting of bamboo slips from Qin Dynasty, September 2010.

[9] Zou Dahai: unearthed bamboo slips and the early Chinese mathematics history, "Humanity and Society" Paper 2, vol 3, 2008.

The discovery of mathematical bamboo slips from the Warring States Period, Qin and Han Dynasty provides researchers with the first hand information about mathematics research from pre-Qin Dynasty and Qin Dynasty, so that we can truly understand some of the real development of mathematics at that particular time. From these bamboo slips, we found that the development of mathematics was well-progressed in pre-Qin Dynasty and Qin Dynasty. There was a complete multiplication table in the form of 9 times 9, also complete methods and solution of arithmetic operation of fraction, fang tian, sumi, shuai fen, shaoguang, shanggong, ying bu zu, gougu. It can be said that in addition to technique of solving equation(fangcheng shu), which means the general solution of linear equations and the standard problem of junshu, other seven methods and problems in "jiushu" were discovered, also some difficult problems in "Jiushu" session in ≪Jiuzhang Suanshu≫ were found, these greatly enriched the content and range for study in the history of Chinese mathematics.

2.2 It breaks the nihilism towards the early development of Chinese mathematics

Certain people in Chinese and overseas academic sectors doubt about the existence of China's achievements in mathematics before Song and Yuan Dynasty. This does not refer to those Eurocentric academics and their followers in China who do not understand and not even try to understand the mathematics of ancient China. For example, there is one claiming to be a senior researcher at Chinese Academy of Sciences Shanghai Branch who made such a big mistake as identifying the well-known scientist Rene du Perron Descartes and Gottfried Wilhelm Leibniz as ancient Greeks. He also overblew the dark European Middle Age Era of mathematics, but ridiculed that in ancient China people only knew about gougu and achieved nothing in mathematics. In addition he blamed that the reason why China could not get the Nobel Prize was mainly because of its poor performance in ancient China.

Unfortunately, this kind of belief was constantly found in the publications, even including "China Science News"(now "Science Review"), "Dialectics of Nature" and other prestigious academic publications. This does not refer to ignorant people who are not afraid to give his or her opinion("Wu Zhi Zhe Wu Wei"), but to scholars who have in-depth study of mathematics in ancient China. Apart from the fragmentary information of Dunhuang Suanshu - the contents of these materials were relatively not rich enough, and all of them were the 5^{th} to 10^{th} century works -, there is no mathematical texts before the Song Dynasty handed down, therefore they suspected that the authenticity of achievements in mathematics in two Han Dynasty as well as Wei Dynasty of Southern and Northern Dynasties. Moreover, they also doubted about the existence of a mathematics book at that time, and saying that "Jiuzhang Suanshu" was a book claimed to have formed in Han Dynasty, but in fact, its earliest text was appeared in Southern Song Dynasty only. This implied that the mathematics information in ancient China is only reliable after formation of Song Ben Suanjing by Bao Huanzhi in Southern Song Dynasty Jiading period, all the previous information are unreliable.

By the end of 1983 to early 1984, "Suanshu shu" was unearthed; this to some extent refuted the above erroneous view. However, the latest period shown in li ri, which was unearthed on the same date as "Suanshu shu", was l hou er nian (186 BC). Saying that the vast majority of problems arose from pre-Qin and Qin Dynasty and existed in pre-Qin Dynasty[10], and there existed more than one

[10]Peng Hao: Explanation of Zhang Jiashan bamboo slips "Suanshu shu" from Han Dynasty, Beijing: Science Publishing Company, 2001.

mathematics book,[11] in pre-Qin Dynasty was a conclusion come up with scholars after verification and it was not obvious. The mathematical bamboo slips unearthed from Warring States Period, Qin Dynasty and Han Dynasty provide the world with the mathematics in pre-Qin Dynasty and Qin Dynasty without any change of the original text made by descendants. This not only refuted the wrong saying of no existence of any mathematics work in two Han Dynasty, but also broke the nihilistic attitude adopted by some scholars towards the early development of Chinese mathematics.

2.3 It provides a strong evidence in solving problems about formation of Jiuzhang Suanshu

Over the past 1700 years, views on the development of Jiuzhang Suanshu varied a lot. As Jiuzhang Suanshu is the most important classic mathematics book in ancient China, ranking the top of all canons, and therefore the development of Jiuzhang Suanshu is an important issue academically since the 20th Century. There are three interrelated issues with different focuses. The first one is the date of completion of the major methods and mathematics problems in Jiuzhang Suanshu. The second issue is whether there was a certain kind of form that Jiuzhang Suanshu in existence before the Qin Dynasty. The third issue is the completion date of the book Jiuzhang Suanshu which Liu Hui had referred.

2.3.1 When the main mathematics methods and problems in≪Jiuzhang Suanshu≫ were completed

On this issue, one of the founders of the history of Chinese mathematics branch Qian Baocong (1892–1974), said, "Without a doubt that the techniques in solving problems in fang tian, sumi, shuai fen, shaoguang, shanggong are mainly formed before Qin Dynasty." I believe that: "apart from data of equation(fangcheng) which was not found in books from pre-Qin Dynasty—it is very difficult to find information on mathematical methods from literature and history books—the mathematical methods of the remaining eight chapters and even some of the questions, evidence can be found from the texts and historical relics from pre-Qin Dynasty."[12] This saying is made before looking at the translated texts of ≪Suanshu shu≫ from Han Dynasty and ≪Jiuzhang Suanshu≫ from Qin Dynasty.[13] I actually think, "the main body of ≪Jiuzhang Suanshu≫ which means using texts with technical methods(shu wen) to group examples together as well as most of the examples in books were completed in Warring States Period and Qin Dynasty."[14] This is essential for understanding the compilation of ≪Jiuzhang Suanshu≫.

Although Suanshu shu was not the predecessor of Jiuzhang Suanshu[15], Suanshu shu had a lot of commonalities in the methods and problems with the bamboo slips of Shu from Qin Dynasty, the

[11]Suanshu shu" is not a systematically compiled publication, but rather one that is selectively compiled with many publications. See Guo Shuchun: Preliminary analysis of "Suanshu shu", "Research of Chinese Literature" Page 307–349, vol 11, June 2003.

[12]Qian Baocong: Abstract of Jiuzhang Suanshu. See Qian Baocong-edited "suan jing shi shu" volume 1. Beijing: Zhong Hua Book Store, 2003. "Li Yan, Qian Baocong The whole collection of science history" vol 4, Shen Yang: Liao Ning Education Publishing Company, 1998.

[13]Guo Shuchun: "Liu Hui, the world-class mathematics leader in ancient time" Ji Nan: Shan Dong Science Technology Publishing Company, 1992. revised edition in original complex, Tai Bei: Ming Wen Book Store,1995.

[14]Guo Shuchun: "Preface, explanation of Jiuzhang Suanshu" Shanghai Ancient Text Publishing Company, October 2009, April 2010.

collection of bamboo slips from Beijing University and Jiuzhang Suanshu. These commonalities obviously are the consensus of the academic circle in the early-Qin Dynasty. And this reinforces the saying that the main methods and questions of Jiuzhang Suanshu were completed in early-Qin Dynasty.

2.3.2 The existence of a particular form of ≪Jiuzhang Suanshu≫ in early-Qin Dynasty

It is commonly recognised in the academic circle that ≪Jiuzhang Suanshu≫ was the result of long-term accumulation, developed from "Jiushu,"[16]and was completed in Han Dynasty. However, mathematicians before Ming Dynasty have different arguments. Starting from the mid-Qing Dynasty, many scholars expressed their own views. In the existing data, the first one who talked about the compilation of ≪Jiuzhang Suanshu≫ was Liu Hui. He said: "Zhou Gong introduced the social norms and therefore "Jiushu" existed. The details of "Jiushu" are actually "Jiuzhang". As the Qin Emperor burnt so many books and the remaining ones were broken and incomplete. After Shi Jue, both Zha Chang, bei ping hou and Geng Shou Chang, da si nong zhong cheng of Han Dynasty were skillful in mathematics. They prepared the new edition of the books as the old ones were incomplete. When compared the content lists, they might be different from the old ones, but the actual contents were more or less the same.[17]

That is, Liu Hui believes that Jiuzhang Suanshu was developed from Jiushu and certain format of the text was formed in early-Qin Dynasty. This format of Jiuzhang Suanshu was damaged in Qin-fire. (I believe that the damage was come from the commotion in late Qin Dynasty, especially the burning and plunder by Xiang Yu and his men.)

It should be pointed out that we cannot put the views and comments by Liu Hui and the views of people after him to have the same weight on a balance. In other words, only by successfully refuting Liu Hui's argument, could we consider whether the argument held by people after Liu Hui being reasonable. If we negate the argument of Liu Hui without evidence of any loophole, and make speculations, is far from correct. We will argue later on, the negation opinion held by Dai Zhen on the Liu's historical material was wrong. From the analysis on the format of the Jiuzhang Suanshu and analysis on the information of the price of material of that time, Liu's argument is perfectly correct.

≪Jiuzhang Suanshu≫ can be divided into two forms, one is using texts with technical methods (shu wen) to group problems together, another one is the applied question bank .The style of using texts with technical methods (shu wen) to group problems together can be divided into three styles. The differences in styles illustrates that ≪Jiuzhang Suanshu≫ cannot be compiled in a single epoch. It was the efforts of many mathematicians from different generations. The three styles in using texts with technical methods (shu wen) to group problems together have a total of 82 methods (shu) and 196 problems (wen), covers all part of the six chapters included fang tian, sumi, shaoguang, shanggong, ying bu zu, fangcheng, also shuai fen, shuai fen in jun shu zhang, problem of jun shu and gougu shu in gougu zhang, gougu rong fang, rong yuan, ce yi zhu shu. Applied question bank is used in

[15]Guo Shuchun: "About the relationship between Suanshu shu" and "Jiuzhang Suanshu", "Zi, Qu Fu University of Education Journal", paper 34, vol 3, 2008.

[16]Zheng Xuan of Eastern Han Dynasty (127–200) referred to Zheng Zhong's(?–83) "Zhouli zho" to explain "Jiushu" and said, "Jiushu: fang tian, sumi, cha fen, shao guang,shang gong, jiu shu, fang cheng, ing bu zu, pang yao. Now it has zhong cha, xi jie, gougu." Lu Deming reckoned xi jie stream detailed article. See
"Zhouli", "Explanation of 13 books". Beijing, Zhong Hua Book Store, 1982.

[17]Guo Shuchun edited: Edited "Jiuzhang Suanshu" supplementary edition. Shen Yang: Liao Ning Education Publishing Company, Tai Bei: Jiuzhang Publishing Company, 2004.

the remaining non-shuai fen problem in shuai fen zhang, non standard jun shu problem in jun shu zhang, questions in solving gougu shape in gougu zhang and three questions[18]of yin mu wang shan. If non-shuai fen problem and non-jun shu problems from shuai fen and jun shu zhang are deleted respectively, the content of ≪Jiuzhang Suanshu≫ using texts with technical methods (shu wen) to group problems together match with the titles of different chapters and is surprisingly consistent with ≪Jiushu≫ mentioned by two Chengs. This proves that what Liu Hui said "jiu shu zhi liu equals to ≪jiu zhang shi yi≫"and "compared its content list might differ from the past" are supported by evidence. ≪Jiushu≫ is truly a source of≪Jiuzhang Suanshu≫.

Hori in Japan studied ≪Jiuzhang Suanshu≫ and the price of goods reflected from ≪Shiji≫, ≪Hanshu≫ and ≪Juyan Hanjian≫ .He concludes that it is rather weak to say that the price of goods found in ≪Jiuzhang Suanshu≫ equals to the price of goods in Han Dynasty. Basically the price of goods found in ≪Jiuzhang Suanshu≫ reflects to price of goods in Warring States Period and Qin Dynasty.[19] This conclusion is consistent with that of Liu Hui. To analyse after combining the difference between eras reflected by the price of goods shown in ≪Jiuzhang Suanshu≫ as well as its variation of problems, Liu Hui's view will be strengthened. There are 31 problems involved when compared and analyzed ≪Jiuzhang Suanshu≫ and the price of goods in Han Dynasty, 20 of them showed a large price difference when compared with that of Han Dynasty but rather close to the price in Warring States Period and Qin Dynasty. 18 out of these 20 problems are in the form of using texts with technical methods (shu wen) to group problems together. And 11 problems showed very close to the price in Han Dynasty but having a larger price difference when compared with that of Warring States Period and Qin Dynasty. 7 of them belong to the applied question bank; and 4 of them are in the form of using texts with technical methods (shu wen) to group problems together.

All in all, the present historical information not only did not contradict with Liu Hui's discussion on the compilation of ≪Jiuzhang Suanshu≫, but also prove that Liu Hui's discussion on the compilation of ≪Jiuzhang Suanshu≫ was totally correct. In addition, Liu Hui has a realistic and rigorous learning attitude as well as a high level of morality. We should believe the words of Liu Hui. He designed the mouhe fanggai and pointed out the right way to solve the volume of a sphere. Although the result fell short of his prediction and he could not calculate the volume of mouhe fanggai, he did not hide his failure, but spoke frankly, "When making the judgment and conclusion, there is the confusion of square and circle, and also the mixture of thickness and thinness, it is unable to set the equality. If a rough idea is simply made, it is afraid that the truth might be missed. Therefore the problem is not solved, it should wait for the capable one to get it done." "li shou zuo shu" was the traditional view at that time. But he said, "I haven't heard of its details." After the description of the shape of qiandu, he said," I haven't heard why it is named qiandu". The entire explanation of Liu Hui showed his supreme spirit of expressing his opinions with evidence and never tells unproved idea. Therefore Liu Hui's words do have the full reliability. In short, regarding the compilation of ≪Jiuzhang Suanshu≫, we should believe the words of Liu Hui. Freely denied the words of Liu Hui and even invented another argument was not a scientific attitude.

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[18]Guo Shuchun: "About Chinese traditional mathematics 'Shu"'. Li Wenlin & others edited: "Mathematics and mechanization of mathematics". Ji Nan: Shan Dong Education Publishing Company, 2001. This article made certain amendments on the related discussion basis of the author's "Liu Hui, the world-class mathematics leader in ancient time".

[19](Japan)Hori: "Study on price of goods in Qin and Han Dynasty". "Study and discussion on the history of legal system in Qin and Han Dynasty". Beijing: Law Publishing Company, 1988.

2.3.3 When the ≪Jiuzhang Suanshu≫ which Liu Hui read was formed

Liu Hui believed that ≪Jiuzhang Suanshu≫ which he read was compiled by Zhang cang (?–152 BC) and Geng Shou Chang (1st century BC) in Western Han Dynasty. As noted above, the words of Liu Hui are worth to believe. As compilation of ≪Jiuzhang Suanshu≫ was such a serious issue, if he did not have reliable information and had not read the concrete piece of the Zhang Cang and Geng Shou Chang compiled ≪Jiuzhang Suanshu≫, it was absolutely impossible for him to talk about this. It was unreasonable to reject the words of Liu Hui only based on the reason that it was a single case and no other circumstantial evidence was available. Because of years of delay and natural disasters, only very limited data which Liu Hui had seen were kept till mid-Qing Dynasty to today. Among those limited data handed down to the era of Dai Zhen, what Dai Zhen and other people could read and memorize was a very small proportion only. Therefore one can imagine how prejudiced it was when Dai Zhen and other people based on their own knowledge to reject Liu Hui's discussion. In fact, Dai Zhen and other people rejected the Zhang Cang compiled ≪Jiuzhang Suanshu≫ was mainly due to two reasons. Firstly, the existence of the name of a place called "Shang Lin".[20] Secondly, they said that problems of junshu existed only from the era of Han Wu Di, thus Zhang Cang was unable to involve in the compilation of ≪Jiuzhang Suanshu≫ . In fact, as early as the era of Qin Emperor, there was a Shang Lin Parkland[21], and junshu l was appeared in bamboo slip which was unearthed at the same time with ≪Suanshu shu≫.[22] As a result, the two main reasons for rejecting Liu Hui's discussion were no longer existed.

Conditions in ≪Jiuzhang Suanshu≫ also proved that Liu Hui's words are correct. The examples and styles used in the part that adopting applied question bank were totally different from the part that using texts with technical methods (shu wen) to group problems together. Moreover, differences were also found between the nature of questions and the nature of titles of chapters which they were classified, a clear patch nature could obviously be seen. Also, a great variation existed in ideologies for compilation.[23] By comparing with bamboo slips ≪Shu≫ from Qin Dynasty, ≪Suanshu shu≫ from Han Dynasty and problem of shaoguang from ≪Jiuzhang Suanshu≫, one would find that the former two slips were written by simple ancient words, while the later was written by the languages of Han Dynasty. This proved that what Liu Hui said "the actual contents were more or less the same" was supported by evidence.

In addition, from the guiding ideology for compilation of ≪Jiuzhang Suanshu≫, Qian Baocong[24] thought that the characteristics of the calculation techniques (suanfa) in ≪Jiuzhang Suanshu≫ as to solve practical problems as its fundamental purpose showed its realistic style. This reflected its acceptance of Xunzi's materialist ideology (wei wu zhu yi). On the other hand, ≪Jiuzhang Suanshu≫ did not define any mathematical concepts, and also no mathematical formula and explanation were derived and proved. This also reflected Xunzi's ideologies of "conventionalism (yue ding su cheng)"

[20](Qing) Dai Zhen: Abstract of "Jiuzhang Suanshu". In "Jiuzhang Suanshu" of "Wu Ying Hall collection of selected books". See Guo Shuchun edited: "Mathematics Paper - Collection of Chinese Science Technology Books" vol 1. Zheng Zhou: He Nan Education Publishing Company, 1993.

[21](Han) Si Maqian: "Shi Ji—Qin Emperor original record". Beijing, Zhong Hua Book Store, 1959.

[22]Li Xueqin: Significant findings in Chinese mathematics history". "Wen Wu Tian Di", vol 1, 1985.

[23]Guo Shuchun edited: "Mathematics Paper—Chinese Science Technology History", Science Publishing Company, October 2010.

[24]Qian Baocong: "Relationship between Jiuzhang Suanshu" and its explanation by Liu Hui and philosophical ideas". "Li Yan, Qian Baocong The whole collection of science history" vol 9, Liao Ning Education Publishing Company,1998.

and "there was an end in learning (xue you suo zhi)".[25] That is, ≪Jiuzhang Suanshu≫ is compiled under the guidance of Confucianism of the Xun School .There were very few historical records about the thinking of Zhang Cang. However, Xunzi (313BC–238 BC) taught ≪chun qiu zuo shi zhuan≫ to Zhang Cang. Zhang Cang taught ≪Zuozhuan≫ to Jia Yi.[26] It could be concluded that Xunzi, Zhang Cang and Jia Yi did have the lineal teacher-student relationship. As Jia Yi was the main representative of Confucianism of Xun School in the early Western Han Dynasty, therefore Zhang Cang did believe in Confucianism of Xun School.[27] This was consistent with the ideology for compilation of ≪Jiuzhang Suanshu≫.

In short, the fact that ≪Jiuzhang Suanshu≫ was compiled by Zhang Cang, Geng Shou Chang. should not be rejected.

2.4 It provides reliable literature to support the first climax of traditional Chinese mathematics occurred in Spring & Autumn and Warring States Period

In 1990s, I concluded that the first climax of traditional Chinese mathematics was occurred in Spring & Autumn and Warring States Period and Western Han Dynasty compiled ≪Jiuzhang Suanshu≫ was only a conclusion[28]of this climax after studying ≪Jiuzhang Suanshu≫ and its Liu Hui's zhu (commentary). Although I firmly believed that this view was correct, lack of evidence was the problem at that time. In 2000, interpretation of ≪Suanshu shu≫ was announced, in which the rich mathematical contents and Mr. Peng Hao's conclusion about ≪Suanshu shu≫ saying that the vast majority of problems arose from Qin Dynasty and pre-Qin Dynasty made me settled. Now, several batches of bamboo slips from Warring States Period and Qin and Han Dynasty were found, which provide more reliable literature for the study of mathematics in Qin Dynasty and pre-Qin Dynasty. This ultimately ends the situation of mainly relied on ≪Jiuzhang Suanshu≫ and its Liu Hui's zhu (commentary) to derive that the first climax of traditional Chinese mathematics was occurred in Spring & Autumn and Warring States Period.

3 Expectations and Recommendations

Currently, study in bamboo slips ≪Shu≫ from Yuelu Academy in Qin Dynasty is still very popular. While the mathematical bamboo slips from Qin Dynasty in Beijing University and the mathematical bamboo slips from Han Dynasty in Hubei Museum are still being compiled, so we are unable to glimpse the whole picture. We propose two expectations:

First is to accelerate the compilation of the mathematical bamboo slips from Qin Dynasty in Beijing University and the mathematical bamboo slips from Han Dynasty in Hubei Museum, so that interpretation of the text will be available as soon as possible.

[25](Warring States Period): Xun Qing: "Xun Zi" "Simplified explanation of Xun Zi", Shanghai People Publishing Company, 1975.

[26](Western Han) Liu : "Preface of Spring and Autumn". "Explanation of Spring and Autumn Zuo Chuan "Kong Yingda's explanation referred to Liu Xiang 'Bie Lu', See "Explanation of 13 books". Beijing, Zhong Hua Book Store, 1980.

[27]Guo Shuchun: "Zhang Cang and 'Jiuzhang Suanshu"'. In "Ke Shi Xin Chuan", Shen Yang: Liao Ning Education Publishing Company, 1997.

[28]Zou Dahai: "The rising of Chinese mathematics and Early Qin mathematics – the late report". Shi Jia Zhuang: He Bei Education Publishing Company, 2001.

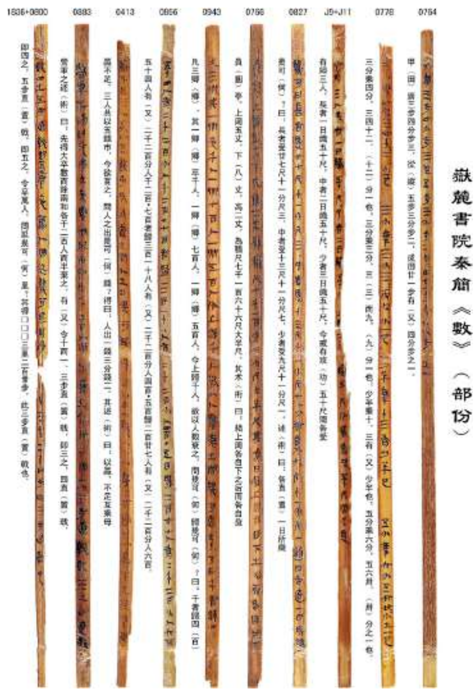
Second is hoping more colleagues in history of mathematics sector can participate in the study in mathematical bamboo slips from Warring States Period, Qin Dynasty and Han Dynasty. Mathematics in pre-Qin Dynasty is the origin of source and the fundamental stone of traditional Chinese mathematics. In the past, we were in the state of blurred understanding. With the findings and studies of mathematical bamboo slips from Warring States Period, Qin Dynasty and Han Dynasty, the mystery of mathematics in pre-Qin Dynasty was gradually unveiled, so that our knowledge on mathematics in pre-Qin Dynasty can truly get close to the history.

In the basis of in-depth study in mathematical bamboo slips from Warring States Period, Qin Dynasty and Han Dynasty, two tasks should be conducted:

Firstly, to co-ordinate experts of history of mathematics, archaeology and ancient text to conduct a comprehensive research and interpretation of mathematical bamboo slips from Warring States Period, Qin Dynasty and Han Dynasty.

Secondly, as proposed by Mr. Dao Benzhou, to launch an international academic symposium about mathematical bamboo slips from Warring States Period, Qin Dynasty and Han Dynasty at an appropriate time, in order to conclude and promote the study in mathematical bamboo slips from Warring States Period, Qin Dynasty & Han Dynasty, and the study in mathematics in pre-Qin, Qin and Han Dynasty and even the entire Chinese history.

湖南里耶出土的秦九九表
Multiplication table in the form of 9 times 9 in Hunan Liye Town



秦简≪数≫的部分简(岳麓书院供图)
Some Bamboo slips

RELATIONSHIP BETWEEN PRE-SERVICE MATHEMATICS TEACHERS' KNOWLEDGE OF HISTORY OF MATHEMATICS AND THEIR ATTITUDES AND BELIEFS TOWARDS THE USE OF HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION

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ABSTRACT

The purpose of this study was to investigate the relationship between pre-service mathematics teachers' knowledge of history of mathematics and their attitudes and beliefs towards the use of history of mathematics in mathematics education. Data were obtained from 1593 pre-service elementary mathematics teachers during the fall semester of 2010-2011 academic year by Attitudes and Beliefs towards the Use of History of Mathematics in Mathematics Education (ABHME) Questionnaire and Knowledge of History of Mathematics (KHM) Test. The main correlation between ABHME and KHM mean scores was found to be positive and statistically significant ($r=.18$, $p<.01$) which meant that the pre-service teachers who were more knowledgeable in history of mathematics topics had more positive attitudes and beliefs towards the use of history of mathematics in mathematics education. Furthermore, all of the seven correlations between the mean scores from each of the seven sub topics of ABHME Questionnaire and the original form of KHM Test were also positive and statistically significant at the .01 level, which were presented in details in the results. The findings were discussed with the relevant existing literature followed by implications for teacher education programs and policy makers, and suggestions for future research were addressed.

Keywords: History of mathematics, Attitudes and beliefs, Knowledge of history of mathematics, Mathematics education, Pre-service mathematics teachers

1 Introduction and Theoretical Framework

Teachers may enhance the standards and the quality of mathematics teaching by including various methods into their instruction through meeting varied learning goals (Hiebert & Grouws, 2007). Be-

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fore entering upon the professional career, they undergo the formal pre-service education process which gives opportunities to improve their knowledge and skills in their field. Having knowledge of history of mathematics (HM), being able to use HM and displaying a demand for this usage as a rooted special method for mathematics education may be pointed as important components of such knowledge and skills (McBride & Rollins, 1977) as HM illuminates the relationship among mathematical concepts, it introduces different perspectives to the learning of mathematics subjects and it clarifies the nature of mathematics and mathematical knowledge (Freudenthal, 1981; Furinghetti & Radford, 2002; Gulikers & Blom, 2002; Siu, 2000).

Mathematics teachers' knowledge and their attitudes and beliefs regarding the teaching and learning of mathematics have been foci of interest and frequently investigated on the grounds that both of them influence the quality of mathematics instruction (Alexander & Dochy, 1995). Apart from this common point, research in education indicates a natural interaction between individuals' knowledge of an issue (as a component of cognitive domain) and their related attitudes and beliefs (as a component of affective domain) (Maker, 1982; Thompson, 1992). Gilbert (1991) claimed that one must initially have some knowledge and experience on a topic before stating attitudes and beliefs related to that topic. In other words, attitudes and beliefs form the subjective form of one's objective knowledge of a topic (Pehkonen & Pietila, 2003). Thompson (1992) also stated that "disputability is associated with beliefs; truth and certainty is associated with knowledge" (p. 129).

The knowledge of HM is likely to present credible information about the mathematics topics at first hand (Freudenthal, 1981). The great part of the responsibility here may probably be on the pre-service education (Hill, Sleep, Lewis, & Ball, 2007). In order to practice upon HM in ME effectively in the future, pre-service mathematics teachers should firstly master the historical information of the mathematical concepts that they are going to teach (Fried, 2001) by taking curricula and textbooks into consideration. Yet, having enough HM background may not be adequate on its own for the pre-service teachers. If they do not hold positive attitudes and beliefs towards the use of HM in mathematics education, then they may not utilize it in the mathematics classrooms (Li, 1999).

Before illuminating the significance of these attitude and belief constructs in conjunction with the HM integration, it is necessary to primarily identify them. In respect of Philipp's (2007) metaanalysis on affective domain research, attitudes are defined as "manners of acting, feeling, or thinking that show one's disposition or opinion" where beliefs are the "lenses that affect one's view of some aspect of the world" (p.259). Correspondingly, attitude and belief are actually two different, but strongly engaged affective constructs (Ajzen, 2001; Goldin, Rösken, & Törner, 2009). More specifically, attitudes are related to senses such as "liking, disliking, being curious, being bored" (McLeod, 1992, p.581), whereas beliefs are more related to one's cognition as "psychologically held understandings, premises, or propositions about the world that are felt to be true" (Richardson, 1996, p. 103) which also form basis for the related attitudes (Koballa & Glynn, 2007). In comparison with beliefs, attitudes may change more easily and thus less permanent (McLeod, 1992). Indeed, the common point of these constructs, which is also the driving force for considering them together in this study, is that humans generally reflect their attitudes and beliefs towards an object, a situation, or another person into their related feelings and behaviors such as interest, demand for learning, and utilization (Koballa & Crawley, 1985). Considering these definitions, pre-service mathematics teachers' beliefs regarding the use of HM in mathematics education can be identified as their professed viewpoints for this usage, and their related attitudes are feelings and thoughts displaying these beliefs. Both of them are likely to influence the concern, excitement, trust, and knowledge related to this notable method, and thus also to affect the relevant choices and usages in future mathematics teaching.

As for the research pertinent to the relationship investigated in this study, Goodwin (2007) studied mathematics teachers' knowledge of HM together with their images of the mathematics discipline. In his study, knowledgeable teachers thought that learners of mathematics could discover mathematical ideas again, mathematics could be done even one was not a professional mathematician, and mathematics was a continuously developed subject by several cultures in the history (Goodwin, 2007). The parallelism of these thoughts with the essential arguments of history in mathematics education was notable. However, the relationship between pre- or in-service mathematics teachers' HM knowledge and their attitudes and beliefs about integrating HM in mathematics education has not been investigated in the accessible literature, indicating a gap in the mathematics teacher education. This limitation and the presented arguments in favour of the interplay between knowledge and attitude-belief leaded to the following research question examined in this study:

- Are there statistically significant correlations between pre-service elementary mathematics teachers' knowledge of HM and their attitudes and beliefs towards the use of HM in ME?

This study aimed to fill the addressed gap in the literature by finding a response to the above research question via data collected from Turkish pre-service elementary mathematics teachers.

2 Methodology

This study intended to raise a claim, which was the existence of a link between pre-service mathematics teachers' knowledge of HM and their attitudes and beliefs about using it in mathematics education, via generalizing from a large and representative sample to the population of interest considering the more general literature on the relationship between individuals' knowledge and their associated attitudes and beliefs. Therefore, it was based on quantitative methodology.

2.1 Context and Sample

The Elementary Mathematics Education (EME) programs in Turkey are four years teacher education programs which train future mathematics teachers for grades 4 to 8. These programs are similar in terms of courses as higher education is loosely centralized through a governing Council of Higher Education (CHE) in Turkey. CHE suggests that "History of Science", "History of Mathematics", and "Philosophy of Mathematics" should be considered as elective courses in EME programs and compulsory pedagogical courses such as "Methods of Teaching Mathematics" should include HM and its integration into mathematics education (CHE, 2007).

A total of 1593 pre-service elementary mathematics teachers (478 freshmen, 432 sophomores, 409 juniors, and 274 seniors; 1064 females and 529 males) from nine universities located in each of seven geographical regions of Turkey were the sample of the study. Clustered random sampling method was used in order to attain a representative sample of target population (Fraenkel & Wallen, 2006) who were all Turkish pre-service elementary mathematics teachers. Twenty per cent of the universities from all the regions were initially selected randomly, and the pre-service teachers enrolled in four years teacher education programs were reached as many as possible in the fall semester of 2010–2011 academic year.

2.2 Data Collection Tools

The data were gathered via Knowledge of History of Mathematics (KHM) Test (Alpaslan, Işıksal, & Haser, 2011b) and Attitudes and Beliefs towards the Use of History of Mathematics in Mathematics Education (ABHME) Questionnaire (Alpaslan, Işıksal, & Haser, 2011a) whose validity and reliablity procedures were completed. KHM Test was formed of 11 questions comprising 13 multiple choice, short answer, and true-false items which were determined with reference to mathematics teacher competencies suggested by Turkish Ministry of National Education (MoNE, 2011), Turkish elementary mathematics curricula (MoNE, 2009), and formal elementary mathematics textbooks (Durmuş, 2010a, 2010b, 2010c). ABHME Questionnaire was a Likert type scale containing 35 items in which the pre-service mathematics teachers could state their attitudes and beliefs concerning the use of HM in the context of its practicality, didactical and motivational contributions to mathematics education, importance for their future teaching (Alpaslan, Işıksal, & Haser, 2011a). In order to examine the correlation between knowledge of history of mathematics and the related attitudes and beliefs under discussion in details, the items of ABHME Questionnaire were grouped into seven sub topics. Considering the relevant literature, these subtopics were determined as *self-efficacy* (SE) beliefs towards the method (the use of history of mathematics in mathematics education) (items 11, 12, 17, and 29), attitudes and beliefs towards the *personal development* (PD) on the method (items 6, 7, 28, and 32) and *usability* (US) of the method (items 1, 15, 18, 21, 23, and 33), the method's contributions to *revealing the meta-issues* (RM) of mathematics (items 2, 5, 10, 13, 19, 24, and 30), to *motivations for learning* (ML) mathematics (items 4, 20, and 25), to *learning* (LE) mathematics directly (items 3, 8, 9, 14, 22, 26, 31, and 34). There were also items addressing *very general* (VG) attitudes and beliefs towards the method (items 16, 27, and 35). The items of KHM Test was not grouped as ABHME Questionnaire on the gorunds that it was designed for measuring the knowledge of the history of elementary mathematics topics in Turkish elementary mathematics curricula (MoNE, 2009) as a whole and thus it had a unity in itself. One sample question from KHM Test and seven sample items for each sub topic of ABHME Questionnaire were illustrated respectively in Table-1 on the following page.

3 Results

Pearson product-moment correlation analysis was run through PASW Statistics 18 software program with the intent of examining the possible relationship. Before conducting this parametric statistical method, its five assumptions which were *level of measurement, related pairs, independence of observations, normal distribution, linearity,* and *homoscedasticity* were checked in order to see whether the data is appropriate for this analysis (Pallant, 2007). Since all of the variables produced from pre-service elementary mathematics teachers' knowledge of HM and their attitudes and beliefs towards the use of HM in mathematics education were continuous at interval level, the *level of measurement* assumption was ensured. In the final form of the data, there was not any missing datum in the *related pairs* of scores on the two variables. The participants were presumed to have no interaction during the data collection addressed *independence of observations*. In addition, their mean scores on each of the variables were observed to be *normally distributed*. As for the *linearity* assumption, the scatterplots revealed a linear relationship for each of the examined correlations, which referred a trend of increase in one variable accompanied with the increase in the other variable, or the reverse. The scatterplots also clarified that the data pairs (e.g., ABHME mean scores-KHM mean scores, ABHME/SE mean

Table-1: Sample Items from KHM Test and ABHME Questionnaire

Instrument	Sub Topic	Item
KHM Test		**1.** – They have one of the known oldest number systems. - They developed a number system up to millions before approximately 5000 years ago. - Numerals in their mathematics are formed by juxtaposing some certain symbols. - 7 different symbols constituting their numeration system was given below:  Which antique civilization has the above mentioned characteristics? **A)** Mesopotamian Civilization **B)** Roman Civilization **C)** Egyptian Civilization **D)** Babylon Civilization
ABHME Questionnaire	SE	**11.** I do not have an idea about how to use history-based didactical materials (e.g., pantograph, tangram).
	PD	**7.** Prospective teachers must be given courses about how to use history of mathematics in mathematics education.
	US	**1.** It is difficult to integrate history of mathematics in mathematics education.
	RM	**2.** Having knowledge about history of mathematics gives an idea about why humans felt the need for mathematics.
	ML	**4.** Using history of mathematics in mathematics education causes students to lose their enthusiasm for learning mathematics.
	LE	**3.** The use of history of mathematics in mathematics education makes positive contribution to the learning of mathematics by providing a different standpoint and mode of presentation.
	VG	**16.** History of mathematics should be integrated into mathematics education.

scores-KHM mean scores) concentrated around the linear correlation line provided the last assumption called homoscedasticity (Pallant, 2007). These implied meeting all the required assumptions of Pearson product-moment correlation analysis.

The correlation analysis results revealed that all of the correlations tested for the pairs between ABHME mean scores and KHM mean scores were positive and statistically significant, whose coefficients (r values) were presented in Table-2 below:

Table-1: Sample Items from KHM Test and ABHME Questionnaire

	ABHME Mean Scores							
	ABHME Mean Scores	ABHME/SE Mean Scores	ABHME/PD Mean Scores	ABHME/US Mean Scores	ABHME/RM Mean Scores	ABHME/ML Mean Scores	ABHME/LE Mean Scores	ABHME/VG Mean Scores
KHM Mean Scores	.18*	.19*	.12*	.10*	.12*	.18*	.12*	.08*

*. Correlation is significant at the 0.01 level (2-tailed).

Considering the table above, the result that all of the positive correlations were also statistically significant could lead to the idea that pre-service teachers' higher scores on KHM Test accompanied with their higher scores on ABHME Questionnaire. Additionally, the relatively higher correlation coefficients between the three pairs of KHM mean scores-ABHME mean scores, KHM mean scores-ABHME/SE mean scores, and KHM mean scores-ABHME/ML mean scores were remarkable which were discussed in the next part. The coefficients of determination (r^2) were ranged between .01 and .04 referred that the different kinds of the attitudes and beliefs shared 1 through 4 per cent of their variance with the knowledge, or vice versa.

4 Discussion and Implications

The results of the study suggested a relationship between pre-service elementary mathematics teachers' knowledge of HM and their attitudes and beliefs towards the use of HM in mathematics education which supported the claims in the literature for the interplay between knowledge, attitudes, and beliefs (Gilbert, 1991; Maker, 1982; Thompson, 1992). It was also coherent with the relevant specific studies on the relationship between knowledge of HM and HM related attitudes and beliefs (Goodwin, 2007). It might be the case that better knowledge on the HM would be a key factor in the preferences for employing it in the classroom for mathematics teachers of the future. On the contrary, the reversibility of the correlation also pointed out that positive attitudes and beliefs about using HM seemed to lead the pre-service teachers to enrich the knowledge of HM. The positive attitudes and beliefs might have encouraged them to learn HM and this situation maybe resulted in an increase in the achievement got from the KHM Test.

In private, the relatively higher positive relationship between *self-efficacy* beliefs towards the use of history of mathematics in mathematics education and knowledge of history of mathematics pointed out the importance of mastering the history of mathematics before employing it for teaching mathematics. The pre-service teachers' decidedness for the future use of the method may be broken down due to poor knowledge of history of mathematics, and hence lower self-efficacy beliefs towards using it. The other relatively higher positive relationship between knowledge of history of mathematics and attitudes and beliefs towards the method's contributions to *motivations for learning* mathematics may be a result of that the Turkish pre-service teachers' attitudes and beliefs were a product of their self-perceiving the method as only a tool servicing the motivational purposes of learning mathematics (Alpaslan & Haser, 2012). As a result of this, the pre-service teachers who were more knowledgeable on the history of mathematics might have much displayed that they could use this knowledge for motivational aims. As for the relatively lower positive relationships, they might be an outcome of that the pre-service teachers were not raised awareness of the addressed sub topics (*personal development* on the method, *usability* of the method, the method's contributions to revealing the meta-issues of mathematics, *learning* mathematics directly, and the method in *general*) (Alpaslan & Haser, 2012). If they were adequately informed about the use of history of mathematics in mathematics education and were gained more positive attitudes and beliefs on this method, this might have directed them to enrich their knowledge of history of mathematics.

The dual relationship found here could guide teacher educators for presenting HM knowledge in order to train mathematics teachers who feel themselves familiar with and even competent in using this alternative method. Teacher education policy makers may also design undergraduate courses on HM addressing both the HM knowledge and the relevant attitudes and beliefs. Further studies should explore this relationship through other research designs in different contexts such as those directly seeking for a cause and effect relationship between knowledge of HM and different components of affective domain. In these studies, qualitative methodology also can be included to see the existing reality more vividly.

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THE INFLUENCE OF SOLVING HISTORICAL PROBLEMS ON MATHEMATICAL KNOWLEDGE FOR TEACHING

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ABSTRACT

This research investigated the role of solving historical problems on prospective mathematics teachers' mathematical knowledge for teaching. The primary research question was: *In what ways does a prospective secondary mathematics teacher's work on historical problems contribute to their developing mathematical knowledge for teaching?* In an effort to capture ways in which prospective secondary mathematics teachers (PSMTs; those who will teach pupils aged 10 – 18) engaged in solving problems found in historical sources during a history of mathematics course, I implemented an historical problems and analysis assignment. For the assignment each PSMT selected ten problems that they previously solved in the course. They then presented their solution and provided a reflection on how work on each problem informed their understanding of the underlying mathematical concepts addressed in the problem. The presentation will include a summary of the most often selected historical problems and will highlight the common themes identified as a response to the study's primary research question.

1 Mathematical knowledge for teaching

The concept of mathematical knowledge for teaching (MKT) is the most recent focus of trying to understand the special knowledge teachers must possess to teach mathematics – and to teach the subject matter well. This focus of mathematics education research began with large-scale efforts primarily focused on assessing teachers. Furthermore, the field has endeavored to define exactly what the nature of the special knowledge for teaching mathematics is, in much the same way that mathematics education (among other disciplines) sought to establish its unique definition of pedagogical content knowledge (PCK) in the wake of Schulman coining the term in 1986.

Ball, Thames, and Phelps (2008) described knowledge for teaching beyond the "obvious" knowledge of "topics and procedures that [teachers] teach" (p. 395). To do this, they concentrated on "how teachers need to know that content" (p. 395) and they sought to "determine what else teachers need to know about mathematics and how and where teachers might use such mathematical knowledge in practice" (p. 395). Several scholars with an interest in what history of mathematics contributes to teaching and learning mathematics (Clark, in press; Jankvist et al., forthcoming) have begun to focus on the potentiality of the history of mathematics to be one dimension of the "what else" described by Ball et al. and how this specialized knowledge contributes to prospective secondary mathematics teachers' (PSMTs') future practice.

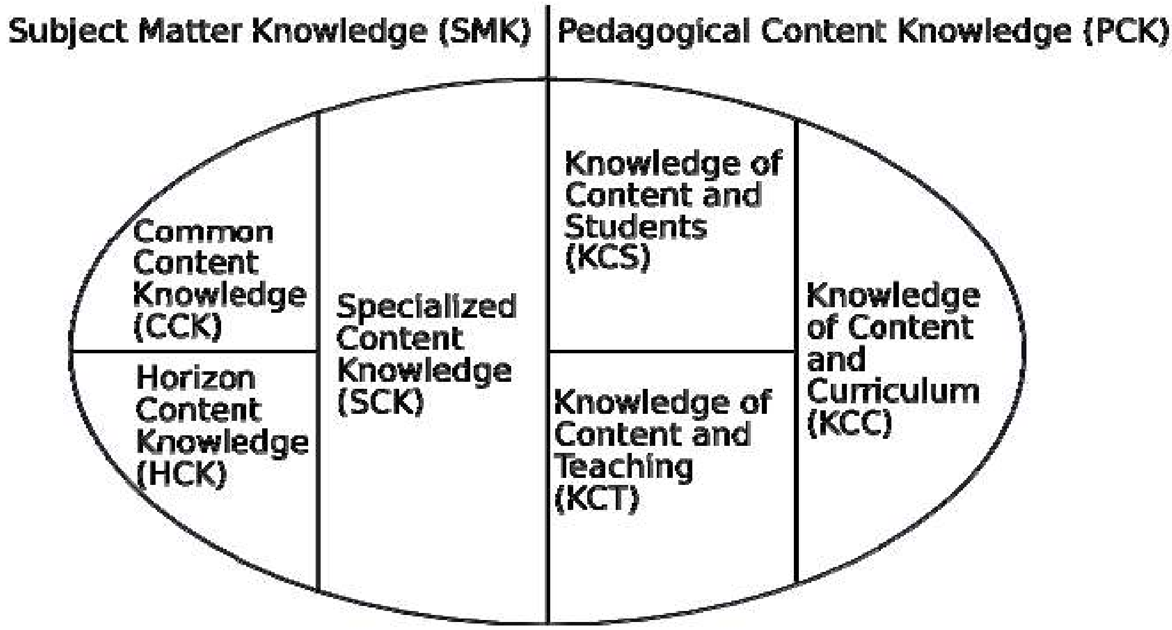

Figure 1. Division of Subject Matter Knowledge and Pedagogical Content Knowledge into further subdomains (Ball, Thames, and Phelps, 2008, p. 403).

Ball, et al. (2008) deconstructed Shulman's concepts of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) into further subdomains (or subcategories), as shown in Figure 1. Although the "egg model" for mathematical knowledge for teaching may continue to evolve, the six subdomains provide a framework from which to gain insight into the ways that history of mathematics may contribute to PSMTs' MKT. SCK and KCT may be the obvious types of knowledge for which PSMTs' historical problem solving has the most influence. HCK, however, may also have a role to play, especially since many scholars have remarked on the tentative nature (e.g., Ruthven, 2011) of this subcategory – and that it certainly has interplay with the other five subcategories. One interpretation of HCK is that it relates to knowledge influenced by awareness, dispositions, and orientations towards particular instructional practice. Thus, history of mathematics may have strong ties to influencing PSMTs' MKT along this domain; however, stronger definitions of the HCK subcategory and research aimed at testing such definitions is needed. Rowland and his colleagues (2010) introduced the analytical framework of The Knowledge Quartet, or the different forms of knowledge trainee teachers possess. Of the four types of knowledge, Rowland's notion of foundation knowledge "concerns trainees' knowledge, understanding and ready recourse to their learning in the academy, in preparation (intentionally or otherwise) for their role in the classroom. It differs from the other three units [of the Knowledge Quartet] in the sense that it is about knowledge possessed, irrespective of whether it is being put to purposeful use" (p. 1842). This framework is also promising for the ways in which teacher educators can examine and characterize such foundational knowledge of mathematics teacher candidates. In particular, if, when, and how history of mathematics contributes to the development of foundation knowledge, serious implications exist regarding the policies and practices governing the role of history of mathematics (e.g., course content or courses in general) of mathematics teacher preparation programs.

2 Participants and study details

Twenty-four PSMTs, enrolled in a *Using History in Teaching Mathematics* course, participated in this study. The course was required for all PSMTs within two certification tracks, either middle grades mathematics certification (for teaching pupils aged 10-14) or secondary mathematics certification (for teaching pupils aged 11 – 18). Student work from one of the course assignments, "Historical Problems and Analysis", was used as the primary data source in the study. The assignment was described to students as follows:

> During most class sessions we will work with historical problems – either in groups during class or individually (or with a partner) outside of class. You will not be handing in each assignment. Instead, you will keep track of the work that you do and for the final task you will select ten problems, tasks, or activities that you feel represent your best effort toward achieving the course objectives. For each problem selection you will: (1) state each problem, task, or activity; (2) present your solution, work, or explanation (as appropriate); (3) describe which of the objectives you feel you addressed when completing and reflecting upon the problem, task, or activity and why; and (4) provide a reflection of how your work on the historical problem contributed to your understanding of the underlying mathematical concepts within the problem, task, or activity.

Students in the course – PSMTs – could elect any format to present their work on this assignment. In most cases, PSMTs used a format that included the stated problem, presentation of accompanying work, and a written narrative in response to the third and fourth items outlined in the assignment. Of the 24 students enrolled in the course, 23 completed the course with a passing grade, and of the 23 completed "Historical Problems and Analysis" assignments 13[1] were analyzed for the purpose of this paper.

The primary research question for this investigation was: *In what ways does a prospective secondary mathematics teacher's work on historical problems contribute to their developing mathematical knowledge for teaching?* Analysis of the collection of PSMTs' responses to the problems they selected included three tasks: (1) reviewing the accuracy and completeness of the solutions, (2) coding each reflection for revelations of how working on the problems improved PSMTs' understanding of the underlying mathematics, and (3) coding each reflection for expressions of beliefs about mathematics prompted by PSMTs' study of history of mathematics.

2.1 Data analysis

Although the problem solution (presentation, accuracy, completeness) was important to assess each PSMT's work for a course grade, for the purpose of this investigation, I was more interested in what each PSMT articulated in their analysis with regard to how the particular problems influenced either their mathematical knowledge or they way in which they now thought about particular mathematics concepts. I note that this investigation represents a preliminary effort to assess the potential for future research and as a result caution is offered regarding the use of such self-reported data. That is, without triangulation (e.g., using additional data sources) of each PSMT's claim, my interpretation of how

[1]Twelve of the 13 PSMTs were pursuing secondary mathematics certification and one was pursuing middle grades mathematics certification.

solving historical problems might influence their developing knowledge for teaching is just that: an interpretation. In future research, pre- and post-assessments of content and micro-teaching tasks and subsequent reflections will provide substantiation of claims and enable possible framework building to use for identifying influences on PSMTs' mathematical knowledge for teaching.

Table 1 presents a brief description of the problems selected by the PSMTs and the frequency of each. For this discussion I highlight two problems (or, types of problems in the case where multiple examples were available) that were selected and discussed by a majority of the PSMTs: problems on the method of false position (solving linear equations) and problems on the method of completing the square (for solving quadratic equations).[2]

Table 1. Historical problem selections.

Historical problems selected	Number of students selecting
1. Babylonian area calculation (Ancient Babylonian problems)	9
2. Egyptian unit fractions	8
3.The power of zero: 0^0	5
4. Method of false position	**10**
5. Acceptance of negative numbers	2
6. Use of different values of π	7
7. Euclid's Elements (various problems)	6
8. "A square and things" (method of completing the square)	**13**
9. Trigonometric identities	6
10. Imaginary numbers	6
11. Stifel's symbols	11
12. Proofs of the theorem of Pythagoras	9
13. Italian abacists	1
14. Figurate number tasks	1
15. Solving cubic equations	4
16. Decimal fractions	1

3 Problem reflections

3.1 Method of false position

The first type of problems, those based on the method of false position (found in surviving papyri from Ancient Egypt), were selected, solved, and discussed by ten students. Two problem sets were assigned to students, one set from Sketch 9 of *Math Through the Ages* (Berlinghoff & Gouvêa, 2004) and one from a task I created that required students to solve problems such as: A quantity three times; the quantity's three-fifths and one-fifth is added to it. It becomes 19. What is the quantity? The goal of including this topic in the *Using History in Teaching of Mathematics* course was to encourage PSMTs to consider the origins of solving linear equations and to connect the method of false position

[2]The problems based upon the algebraic symbols introduced and used by Michael Stifel were not considered for this investigation (even though 11 students discussed these problems in their historical problems analysis task) because these problems were not considered to contain sufficient rigor.

to important underlying ideas when solving linear equations, including, but not limited to rate of change and "undoing" arithmetic operations.

However, many PSMTs did not discuss the potential mathematical content of the false position problems, nor did they discuss how their knowledge developed as a result of their problem solving. Instead, many revealed their beliefs about solving historical problems using the method of false position and these were often superficial in that the PSMTs focused on issues of ease or familiarity. For example, many PSMTs shared some aspect about how the method was difficult to understand, how a modern method of solving linear equeation was "easier" to perform, or they were simply unable to analyze their mathematical understanding:

> "…it is frustrating to use 'guess and check' for solving linear equations when we know a way that comes much easier to us" (Clara[3]).
>
> "I understood it but did not understand why that method would be used…
> The false position is about guessing and proving [the answer] is right" (Julie).
>
> "It was quite difficult to solve their way, but solving it in the modern way was simple….The false and double false position is not the most efficient way to solve for "x" (Katrina).
>
> "I found it more difficult to solve because it is a more involved method" (David).
>
> "The tricky part came when you had to decide how to go about solving the remaining problem based on whether your estimation gave an error above or under the desired number" (Chantal).

One PSMT, however, described her understanding of the method of false position by relating it to modern solution methods. Janine stated that, "I honestly feel more comfortable with the historical method…. The main difference with the two problems is the usage of a variable for the unknown [which is not used in the historical method]. For the most part, the method of solving for the "x" in the modern method is very similar to finding the solution in the historical version." Thus, instead of stating a key difference in the methods of solution, Janine tried to articulate that the actual solution process was the same – and that only symbolic representation set them apart.

In light of this sample of PSMTs' reflections, it is difficult to identify a subcategory of MKT (Ball et al., 2008) or Rowland's notion of foundation knowledge that was influenced by engaging in and reflecting about the false position problems. As a flawed application, aspects of each of the excerpts hint at a basic form of what Rowland described as the knowledge possessed – though in almost every case this may be interpreted as flawed knowledge possessed, or incomplete knowledge possessed.

3.2 Method of completing the square

As with the historical problems on the method of false position, PSMTs were assigned several problems to choose from for inclusion in their "Historical Problems and Analysis" assignment that focused on the method of completing the square. Problems from three different time periods were part of the course – from Ancient Babylonian mathematical texts, Euclid's *Elements*, and from the famous text of al-Khwarizmi. However, all 13 PSMTs who selected problems on the method of completing the square to solve quadratic equations selected those from al-Khwarizmi's text.

[3]All names are pseudonyms.

PSMTs' reflections in response to the prompt, "provide a reflection of how your work on the historical problem contributed to your understanding of the underlying mathematical concepts within the problem, task, or activity", about their work on the method of completing the square also lacked strong evidence of how their understanding was impacted. Furthermore, the strongest declaration expressed by these future teachers was that they had no previous experience with the historical roots of the method of completing the square, and applying that method to solve quadratic equations:

> "I had no clue that the quadratic formula and completing the square had actually anything to do with areas of square or rectangles" (Katrina).
>
> "Before these problems I never completely understood completing the square" (Julie).
>
> "Even though I was aware of completing the square method, I was completely unaware of where it originated" (Steven).

Other PSMTs discussed the connections between algebraic and geometric representations emphasized by the method of completing the square:

> "[Al-Khwarizmi's] use of the geometric representation, however, really helps you to see the completing of the square as you work through the problem" (Janine).
>
> "I have trouble considering a geometric representation to solve algebraic problems and [this method] really gave me ideas about having geometric representations for this and other sorts of problems" (Megan).
>
> "This task deepened my understanding through the use of geometrical methods and now I can finally say I understand completing the square" (Kevin).

Again, I anticipated that PSMTs would discuss what they now understood about solving quadratic equations after examination and successful application of the geometrical and numerical demonstrations provided by al-Khwarizmi. However, this proved difficult for most of the PSMTs. In addition to the two broad types of declarations (examples above), two students revealed incomplete understanding of the method of completing the square. Chantal, for example, began her reflection with "this was definitely something new for me". She added,

> I misunderstood at first, adding the missing portion to what the equation was equal to, just because I thought everything should equal that particular number (subtraction instead of addition of the two numbers given). I understand enough that I would feel confident if having to present this to students.

Chantal was able to identify how she incorrectly approached the problem; however, she still claimed that she only understood "enough" to teach the concept to students. Unfortunately, Chantal could not further articulate what her understanding was (or, was not).

Carrie's reflection on the method of completing the square revealed her incomplete mathematical understanding. She stated that

> ⋯the topic can show students the purpose of a topic that is constantly being drilled into them that they often do not understand. It uses the history of math to show what squares of quantities

were used for so that students can begin to make connections and understanding of exponential ideas and terminology.

Not only did Carrie's reflection shift to attention on students' understanding of completing the square (in fact, many PSMTs did this), she also incorrectly summarized the purpose of the method. Her conclusion that squares of quantities were used for understanding exponential ideas was mathematically incorrect compared to what other PSMTs discussed (e.g., geometrical representations of equations involving squares). Furthermore, this particular reflection raises the question of how history of mathematics – in particular, working on non-trivial historical problems – can actually highlight what mathematical knowledge prospective teachers do not possess (as opposed to the knowledge possessed that Rowland described) and that efforts to implement more historical problems in teacher training programs may prove beneficial.

4 Implications

This investigation intended to identify the ways in which history of mathematics influenced PSMTs' mathematical knowledge for teaching. I anticipated that PSMTs would choose problems from throughout the 15-week course and be able to discuss their understanding of the underlying mathematical concepts when viewed through the lens of their own work on the various historical problems. I was also hopeful that the prospective teachers would have sufficient practice at such written reflections given the numerous writing tasks assigned in the *Using History in Teaching Mathematics* course. Unfortunately, most of the 13 PSMTs' reflections did not contain explicit descriptions of what they really understood nor did they discuss how their work on historical problems assisted in that understanding.

Still, important lessons for future practice in mathematics teacher education can be learned from the initial analysis of the data. For example, prospective mathematics teachers must be provided ample opportunity to reflect on their own mathematical thinking and to articulate that thinking in order to prepare them for doing the same with their future pupils' mathematical thinking. Assigning such tasks to prospective mathematics teachers is not alone sufficient. Instead, as mathematics teacher educators interested in the role of history of mathematics in the preparation of future teachers, we must do a better job at modeling such reflective practices. Furthermore, we must create opportunities for which PSMTs can participate in public reflective practice. In this way PSMTs are called upon to listen and respond to their peers when undertaking these essential reflective tasks.

Finally, establishing the purpose of particular historical problems may yield important information regarding the question of whether historical problem solving contributes to mathematical knowledge for teaching. For example, explicitly presenting the method of false position as a way to analyze rate of change may have prevented many PSMTs' reflections focused on the perceived difficulty or inefficiency of the method. Instead, PSMTs could be prompted to focus on what mathematical ideas are present, why the mathematics "works", and the ways in which different mathematical ideas are related or represented. Such information may help reveal with more certainty the knowledge PSMTs possess (Rowland, 2010) and has the potential to tease out the influence of historical problem solving on the subdomains Ball, et al. (2008) described.

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AN ESSAY ON AN EXPERIMENT IN MATHEMATICS CLASSROOM

—the golden ratio related in the form of the Nautilus shell—

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ABSTRACT

What do children learn in mathematics classroom? Of course, it might be mathematical concepts as well as mathematical theories that they should learn. But usually the concepts and theories are quite abstract, and they cannot understand them so easily. In addition, they should also understand that they learn mathematics for the activities in their everyday life. For the purpose, they should also see concrete features of mathematics. Then, how can teachers give them the concrete subjects of mathematics? In this paper, we discuss an experiment in mathematics classroom, as an example of concrete subject.

Here we show the experiment about Nautilus shell. It is often said that the Nautilus shell has a logarithmic spiral whose growing rate is related to the golden ratio. From a viewpoint of biological investigation, Theodore Cook and D'Arcy Thompson published each of their works on the morphology of the nature at the beginning of the 20th century. In their works, they discussed the spiral of Nautilus shell. Recently, the subject of this kind is treated in many books on mathematics for the specialists and even for the general public.

From this experiment, we can find that the growth curve of the Nautilus shell is almost exactly a logarithmic spiral. Through this discussion, we also show that Nautilus shell is one of most applicable examples, because it can be examines according to children's level (both their school curriculum and their ability). Moreover, we can get the same results repeating the procedure with another type of Nautilus shell called *Nautilus macrompharus* (native to New Caledonia), etc.

Finally, we discuss the benefit of the experiment in mathematics classroom, as follows:

(1) Children can understand the feature (the form) of living things mathematically,

(2) Children can understand concrete features of mathematical subjects,

(3) Children can understand mathematics through various concrete features of mathematics

(4) Teachers can encourage students to interest themselves to mathematics

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In addition, the experiment presented here had been already conducted in the high school mathematics classroom, as well as even in the teacher-training course, and this trial also included the analysis applied with Excel software.

1 Introduction

1.1 Background

Nowadays, mathematics is considered to be universal. The universality of mathematics seems to be argued from the fact that mathematics has developed, especially after Descartes in the 17th century, with the aim of forming a conceptual system in spite of various aspects of its development process. We are now sharing almost the same mathematics all over the world and its globalization is very important to develop our scientific and technological civilizations at present. Thinking about the historical development, universality is not always true; on the contrary, it is sometimes a prejudiced perspective under eurocentrism[1]. After the Scientific Revolution of the 17th century, the framework of human thinking has kept a certain kind of universality. It has been based on the "new scientific thinking" of the 17th century, and we can find the features of our modern civilization in that extension realized through the 18th century. Mathematics is considered to have the same evolutionary process.

With this in mind, we can see the reason why mathematics is considered to be a universal discipline at present. Mathematics today has, in a sense, developed under eurocentrism. Although various kinds of mathematics were developed in each civilization, we perceive mathematics to be a conceptual discipline which has been formed by cutting off many concrete human-cultural parts and by rearranging the remaining conceptual parts into a logical and concise system. This kind of mathematics is surely convenient for mathematicians, but not so comprehensible to the public as well as mathematics education; for, universal mathematics formed under eurocentrism loses the features that represent the original character related to human life and culture.

All that human beings have built up should be considered to be a part of civilization, and therefore, mathematics, which is a product of human wisdom, must also be a kind of civilization. When we consider mathematics as a key element to understanding our civilization, we should see a society or a community including mathematics as an integrated system of multi-cultured dimensions just as human cultures, human life-styles, science and technology, etc. Here, we could find some elements for mathematics classroom; we could find concrete subjects which are related closely to children's direct understanding, for example in everyday life, in the nature, etc.

Then, how can teachers give such concrete subjects to children? When they choose a material, they can show its mathematical feature theoretically; this might be an explanation from a deductive point of view. But they can also show it in another way. Especially concerning materials seen in the nature, we can analyze them through an experiment from a viewpoint of mathematical theory.

[1]For example, let us discuss the historical process of the abstraction of mathematics. Generally speaking, there exist three typical periods of abstraction: ancient Greece, the 17th century and the 19th century. This gives us a composition that is most easy to understand when we look at mathematics from a macroscopic perspective. It is also from a viewpoint of eurocentrism that mathematics has been developed around Europe.

1.2 Problématique

In this paper, we discuss the possibility of "mathematical experiments". Here, by taking the case of Nautilus shell, we try to show how to investigate it from a mathematical viewpoint.

It is often said that the Nautilus shell has a logarithmic spiral whose growing rate is related to the golden ratio. The logarithmic curve is thought to have been developed by Rene Descartes, French mathematician. At that time, Jan Swammerdam (1637-1680), Dutch biologist, and Christopher Wren (1632-1728), English architect, studied the spirals of snail shells. From a biological viewpoint, Theodore Cook and D'Arcy Thompson published each of their works on this subject at the beginning of the 20th century[2]. In their works, they discussed the spiral of Nautilus shell. Recently, the subject of this kind is treated in many books on mathematics for the specialists and even for the general public. For example, Ian Stewart as well as other authors discussed the formation of the spiral of Nautilus shell[3].

We can investigate the features of the spiral that the Nautilus shell creates during its growth. By tracing the spiral curve that appears along the cross section of the cut Nautilus shell, and by drawing a series of the tangents of the curve which intersect with each other perpendicularly, we can see a series of the length of the tangents and even the radii. Then by calculating the rate of increase between each successive length, we can find that this sequence is a geometrical progression and that the lengths and the numbers of order (the value of the angles) are related to each other as an exponential function. And on further investigation we can also find that the coefficient of the growth is slightly larger than the golden ratio.

This is an example of the practicable experiments in the mathematics classroom. This experiment had been already conducted in the high school mathematics classroom, as well as even in the teacher-training course. This trial also included the analysis applied with Excel software.

In this paper, we also discuss how we can realize the experiment and how to examine the results according to children's level (children's school curriculum or even mathematical ability). As mentioned in the section 3, the result of this experiment can discuss in various manner. Therefore, we could handle the result depending on the subjects that children learn at school (i.e. the problems related to a geometrical progression, an exponential function, etc.). In addition, since the example is quite concrete, children could be convinced of the result even depending on each of their own abilities.

In conclusion, we discuss the benefit of the experiment in mathematics classroom.

2 Method of the Experiment on Nautilus shell

2.1 Preparation

(a) Material

- Nautilus pompilius shell (a species native to the Philippines), which is cut right down the middle into the two similar parts.

[2]Cook, Theodore Andrea, *The corves of Life*, Dover Publication, 1979 (originally published in 1914). Thompson, D'Arcy, On the Gowth and Form, Dover Publication, 1992 (originally publishd 1917).

[3]Stewart, Ian, *Nature's Numbers*, Basic Books, 1995

(b) Tool
- a flat vat, absorbent cotton, a clear glass plate (a clear acrylic plate),
a tracing paper (squared graph paper), a ruler (a caliper)

2.2 Experiment

The experiment shoud be carried out in the following procedures.
[1] Trace the spiral curve of the Nautilus shell on the tracing paper:

(1-1) cover the bottom of the vat with plenty of absorbent cotton
(1-2) put the cut Nautilus shell onto the absorbent cotton (the cross section upward) and cover it with a clear glass plate to fix the shell firmly on the absorbent cotton
(1-3) put a tracing paper on the glass plate and trace by hand the spiral curve of the shell on the paper (handwriting)

[2] Investigate the properties of the spiral curve:

(2-1) draw a series of the tangents of the curve which intersect with each other perpendicularly (see Fig. 1)
(2-2) measure the length of each tangent in ascending order from the shortest one (P_1P_2, P_2P_3, P_3P_4, $\cdots$, in the Fig. 1)
(2-3) draw a segment joining the points of contact on the horizontal tangents (T_9T_{11}) and another segment joining the points of contact on the vertical tangents (T_8T_{10})
(2-4) find the center of the spiral as the intersection of these segments (O)
(2-5) measure the length of each radius in ascending order from the shortest radius at the right angle (here radius means the length between the center and each point of contact, OT_1, OT_2, OT_3, $\cdots$)
(2-6) calculate the ratio between the lengths of two consecutive tangents (between horizontal tangents, between vertical tangents, etc.)

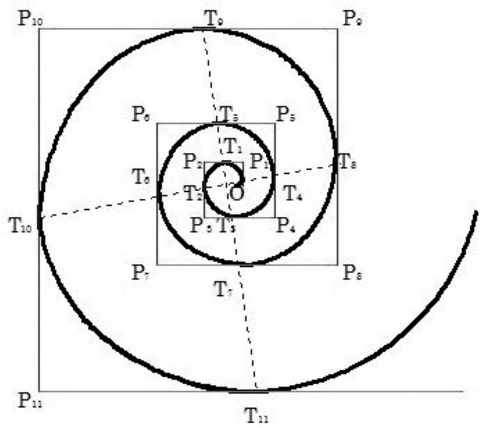

Fig.1 The tangents drawn of the spiral

We show the picture just about a specific sample (see Fig. 2).

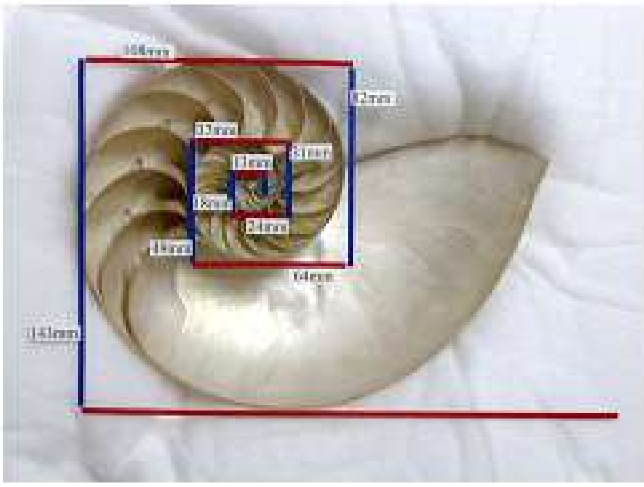

Fig.2 Nautilus shell and its tangents

3 Results and Discussion of the the Experiment

3.1 Result

According to the experiment mentioned above, we can get the result as follows (just about the specific sample shown in the Fig. 2):

Table 1 The length of tangents

Tangents	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4	a_5	b_5
Length(horizontal)	13		24		37		64		108	
Length(vertical)		18		31		49		82		143

(mm)

Table 2 The length of radii

Radii	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}
Length	6.9	9.6	12.7	16.4	19.6	26.0	34.3	43.5	57.3	75.9

(mm)

where a_i, b_j and r_k are defined as follows:

- a series of the length of the tangents
  horizontally: $a_1 = P_1P_2$, $a_2 = P_3P_4$, $a_3 = P_5P_6$, $\cdots$
  vertically : $b_1 = P_2P_3$, $b_2 = P_4P_5$, $a_3 = P_6P_7$, $\cdots$
- a series of the length of the radii
  $r_1 = OT_1$, $r_2 = OT_2$, $r_3 = OP_3$, $\cdots$

Then, the ratio of two consecutive horizontal tangents is as follows:

$$a_2/a_1 = 1.84\ldots,\ a_3/a_2 = 1.54\ldots,\ a_4/a_3 = 1.72\ldots,\ a_5/a_4 = 1.68\ldots$$

and concerning the vertical tangents,

$$b_2/b_1 = 1.72\ldots,\ b_3/b_2 = 1,58\ldots,\ b_4/b_3 = 1.67\ldots,\ b_5/b_4 = 1.74\ldots.$$

Moreover, the ratio of each two consecutive tangents is as follows:

$$b_1/a_1 = 1.38\ldots,\ a_2/b_1 = 1.33\ldots,\ b_2/a_2 = 1.29\ldots,\ a_3/b_2 = 1.19\ldots,\ b_3/a_3 = 1.32\ldots,$$
$$a_4/b_3 = 1.30\ldots,\ b_4/a_4 = 1.28\ldots,\ a_5/b_4 = 1.31\ldots,\ b_5/a_5 = 1.32\ldots.$$

In addition, concerning each two consecutive radii, we get the ratio between them as follows:

$$r_2/r_1 = 1.39\ldots,\ r_3/r_2 = 1.32\ldots,\ r_4/r_3 = 1.29\ldots,\ r_5/r_4 = 1.19\ldots,\ r_6/r_5 = 1.32\ldots,$$
$$r_7/r_6 = 1.31\ldots,\ r_8/r_7 = 1.26\ldots,\ r_9/r_8 = 1.31\ldots,\ r_{10}/r_9 = 1.32\ldots.$$

From these values, it is resulted that

(A) the ratio between two consecutive horizontal tangents seems to be nearly equal to the value 1.7,
(B) the ratio between two consecutive vertical taqgents seems to be nearly equal to the value 1.7,
(C) the ratio between two consecutive tangents seems to be nearly equal to the value 1.3.
(D) the ratio between two consecutive radii seems to be nearly equal to the value 1.3.

3.2 Discussion (1) – from a viewpoint of mathematics classroom

From the results of the experiment about the Nautilus shell, we can investigate various properties depending on the situation of children's understanding. For example, in the first year (or second year) of a high school, students study the theory of progression. Therefore, they can understand that the sequence of the length of the consecutive tangents (or the consecutive radii) appeared in the Nautilus shell is considered as a geometrical progression. Perhaps, as to the ratio between two consecutive tangents (or the consecutive radii), even junior high school students can understand the fact. But it is just high school students in a science and engineering course who can understand that the spiral possessed in Nautilus shell could be considered as a logarithmic spiral.

Thinking so, we can say that Nautilus shell is one of most applicable materials in mathematics classroom; for it includes various aspects that we show below.

(1) From a viewpoint of a geometrical progression
The sequence of the length of consecutive tangents is considered as a geometrical progression, with the value about 1.3 as the common ratio. In addition, the sequence of the length of consecutive horizontal (vertical) tangents is also considered to be a geometrical progression with the value about 1.7 as the common ratio. Here, it should be noted that the value 1.3 is nearly equal to the square root of 1.7. And the value 1.7 is considered to be nearly equal to the golden number $1.618\cdots$. Then, each tangent seems to increase at the rate of the square root of the golden ratio at each right angle, and therefore at the rate of the golden ratio at each 180 degrees.

(2) From a geometrical view point
In the experiment, the length of each radius is also measured. Then we can see the ratio between two consecutive radii is equal to the ratio between two consecutive tangents. This can be argued from a geometrical construction shown in the Fig. 3. Here, the triangles OPQ, OQR, ORS are similar to each other, then we can easily obtain the result as

$$PQ : QR : RS = OU : OV : OW.$$

According to this fact, it is also true that the ratio of two consecutive tangents is considered as the rate of the growth of Nautilus shell.

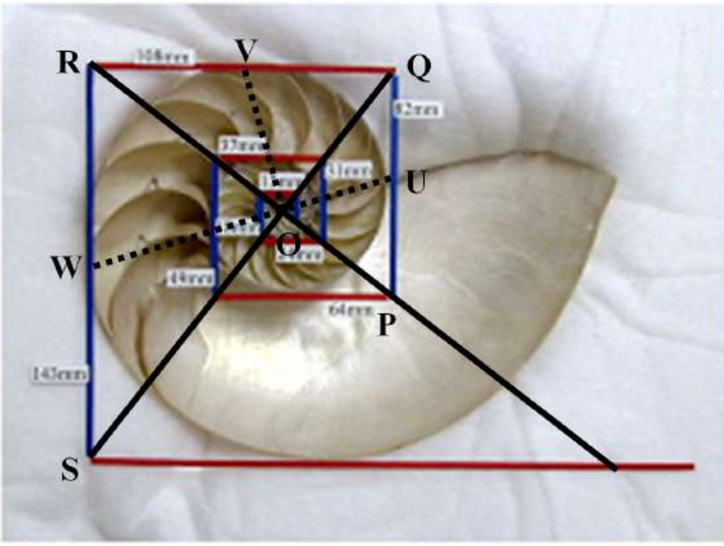

Fig. 3 Nautilus shell and its center

In the experiment, the procedures (2-3) and (2-4) can be attained on the hypothesis that the sipiral is a logarithmic. It is because all the points of contact could be arranged on the same segments only when the angle between the radius and each tangent are always equal to a constant value (an equiangular spiral). And only in this condition, the center of the spiral curve could be fixed (see the Fig. 4).

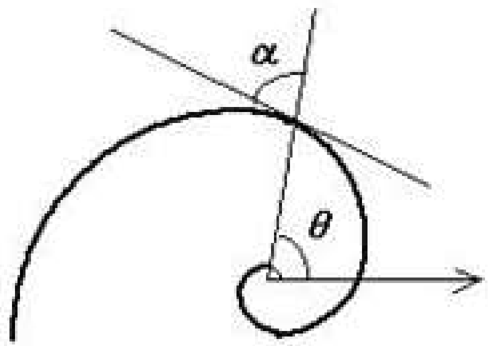

Fig. 4 Equiangular Spiral

In the Table 2, the length of each radius is related to the growth of the shell in every right angle. Thinking of the geometrical discussion mentioned above, the length of each tangent coincides with that of each radius. Then, we can investigate the growth rate of Nautilus shell by using the length of the tangents (Table 1).

Rewriting the Table 1 as follows:

Table 3 The length of tangents (rewritten)

No.	1	2	3	4	5	6	7	8	9	10
Length	13	18	24	31	37	49	64	82	108	143

(mm)

where each number of order signifies the value of the angle increasing in every right angle (here, we adopt the number of order for simplifying; i.e. the number k signifies the value $k * (\pi)/2$, for $k = 1, 2, \ldots, 10$).

When we approximate the relation between the number and the length to an exponential function, we can get the following graph (by means of Excel software).

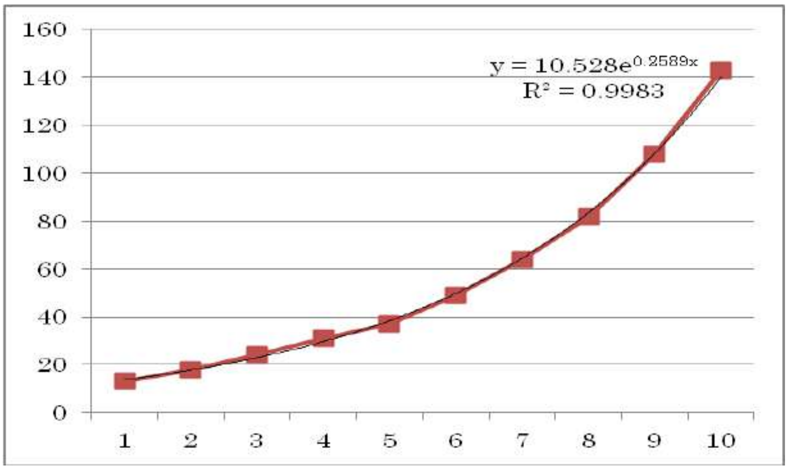

Fig. 5 Relation between the number and the length

This graph shows that the approximation is quite suitable, and then, the fact that Nautilus shell has a curve close to a logarithmic spiral can be clarified.

(4) From a viewpoint of the golden ratio

Considering the discussion of the fact that the spiral of *Nautilus* shell is logarithmic, we can continue to investigate if Nautilus shall has the golden ratio.

From the argument in (3), we can put the following expression as the formula of the spiral of Nautilus shell,

$r = Ae^{Kx}$ (where r=radius, x=angle; A, K : constants)

Here, suppose that the rate of the radius is dr $(= r'/r)$ depending on the difference of the angle $dx = x' – x$, and then,

$$dr = e^{Kdx}$$

therefore

$$\log dr = Kdx.$$

If the growing rate of the spiral of Nautilus shell is equal to the golden ratio, from the discussion in 3.1., the ratio between two tangents in every 180 degrees might be equal to the golden ratio; in consequence,

$dr = 1.618\ldots,\ dx = 3.1415\ldots,$ then $K = 0.1532$

In the case of the expression in the graph shown above, the value of exponent is equal to 0.2589. Since this value is calculated with the numbers as variable, by converting this variable into the value of the angle, we get, in this case

$$K = 0.2589 * 2/3.1415\ldots = 0.1648\ldots$$

In consequence, we find that the rate of the growth is slightly larger than the golden ratio.

4 Observation

4.1 Historical Comment

The experiment conducted here in itself is quite biological, because it clarifies the feature of Nautilus shell: we can find that Nautilus shell possesses a logarithmic spiral in its structure. However, this fact is simultaneously mathematical, because it should be one of the conspicuous examples that the form (or the figure) of living things can be analyzed and explained by means of mathematical theory.

Historically, the form of living things has not always been explained by means of mathematics. Of course, we can found some specific cases; for example, it is known that Pappus (B.C. 3C.) tried to explain the hexagonal section of a honeycomb geometrically. But, especially as to a logarithmic spiral, it seems to have been discussed as a mathematical problem. Maybe Leonardo da Vinch (1452-1519) tried to discuss a kind of spiral, but it might be concerned with the flow of water and the women's hair. At that time, Albrecht Dürer (1471-1528) is known to discuss a logarithmic spiral implicitly. After Descartes' mathematical discussion on this spiral, in the 17th and 18th centuries, there appeared some trials concerning the form of shell.

However, it is just in the second half of the 19th century or even early in the 20th century that the form of living things had become a subject of mathematical research. We could say that some kind of mathematical morphology had just established at that time. Concerning the curves of living things, Theodore Cook and D'Arcy Thompson are evaluated as pioneer of this field.

Nowadays, it is well said that Nautilus shell has the golden ratio in its form. It is true that Cook and Thompson suggested this matter in their works. But their methods are not so clear. From a viewpoint of research, the form and the growth of living things are so complicated that we cannot discuss easily. On the contrary, just about Nautilus shell, the spiral appeared in it is so harmonic that we can understand its feature even mathematically. It is considered as a suitable material for children. In this paper, we tried to show how we can handle its spiral. Naturally, with the present advanced technology, we can make use of various kinds of devices (a photocopy, a scanner, a digital camera, etc.) to get the image of the spiral. But here, we adopted a way to trace the spiral by hand, because we think that a hands-on activity should be important for children.

4.2 Experiment in Mathematics Classroom

From the experiment, we can conclude that the growth curve of the Nautilus shell is almost exactly a logarithmic spiral. Moreover, we can get the same results by repeating the procedure with another type of Nautilus shell called *Nautilus macrompharus* native to New Caledonia, as well as other type of the shells (for example, *Argonauta argo* called a Paper Nautilus).

This is an example of the experiment in the mathematics classroom. Well, what is the benefit of the experiment in mathematics classroom? In conclusion, we show four points as follows:

(1) To understand the feature (the form) of living things mathematically:
It is a biological experiment, in itself. Each student handles each sample and then get each result. But when students accumulate each of their results, they can find that nautilus shell (as well as other kinds of shells) would have a proper feature that its spiral is close to a logarithmic curve. This is merely a kind of a scientific experiment, to be sure, but students can understand the validity and the advantage

of mathematics which can integrate what they acquired and after all even their thinking.

(2) To understand concrete features of mathematical subjects:
Normally, in the classroom, students try to understand mathematical concepts and theories. Since mathematics is often abstract, students are apt to feel that mathematics is far from their activities. On the contrary, through an experiment, students handle concrete objects, which include mathematical features. Well, what should students learn in mathematics classroom? It is not only how to understand mathematics but also how to think their activities and how to live by means of mathematical thinking. When we face some mathematical objects, we can understand various aspects hiding these objects through the experiment. It is because the experiment promotes various kinds of perspective.

(3) To understand mathematics through various concrete features of mathematics
How can we understand objects that we are facing? When student try to understand a abstract concept, usually some concrete examples might be quite helpful. Because, it is easier to understand at first some concrete matters and then to induce general matter. Therefore, experiments could give to students the first starting points and the direction of thinking.

(4) To encourage students to interest themselves to mathematics
From a viewpoint of mathematical education, teachers should discuss the methods to encourage children to interest themselves in mathematics. At present, we can find some mathematical museums, where many kinds of visual devices and hands-on exhibits are set up. Perhaps, many students are interested in such museums, and even some adult are still eager to visit. The experiment in mathematics classroom could play the same role as the devices and the exhibits in mathematical museums.

Therefore, it is necessary and important for teachers to consider that mathematics is related to many things around us, our culture, our everyday life, the nature, etc. The trial of some experiments could become one of key elements for mathematics education in the future.

ON ALEXANDER WYLIE'S JOTTINGS ON THE SCIENCE OF THE CHINESE ARITHMETIC

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ABSTRACT

Starting from August of 1852 the British Protestant missionary and sinologist, Alexander Wylie (1815–1887), published in nine instalments an account *Jottings on the Science of the Chinese Arithmetic* in the newspaper *North China Herald*. He explained clearly the purpose of his account at the beginning:

> "The object of the following desultory notes, made from time to time, in the course of some researches entered upon, with another purpose in view, is to draw attention to the state of the arithmetical science in China, a subject which has not been so fully explored as it might with advantage, and on which some erroneous statements have been current in modern publications."

Alexander Wylie is a well-known figure in the last quarter of the Qing Dynasty for his contribution in transmitting Western science into China during the latter half of the 19th century. In mathematics he was known for translating three treatises in collaboration with the Qing mathematician Li Shanlan (1811–1882) — *Supplementary Elements of Geometry* in 1856 but published in 1865 (believed to be based on the English translation of Book VII to XV of *Elements* by Henry Billingsley in 1570), *Treatise of Algebra* in 1859 (based on *Elements of Algebra* by Agustus De Morgan in 1835) and *Analytical Geometry and Differential and Integral Calculus Step by Step* in 1859 (based on *Elements of Analytical Geometry and of the Differential and Integral Calculus* of Elias Loomis in 1850). He was also the author of *Compendium of Arithmetic* published in 1853.

This presentation will discuss the knowledge of Chinese science and mathematics which most European sinologists of the 18th and 19th centuries possessed and the low regard they held it in, but the viewpoint of which was critically examined by Wylie in his account.

Keywords: Alexander Wylie, Chinese mathematics, arithmetic, algebra

1 Introduction

Alexander Wylie (1815–1887) was a Protestant missionary of the London Missionary Society and later an agent of the British and Foreign Bible Society in China. He was sent to China by the London Missionary Society in 1847. His contribution was not only on spreading Christian faith to China, but

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perhaps more importantly on the intellectual exchange of scientific and mathematical knowledge between China and Western countries. He was well-known in transmitting Western science and mathematics into China by publishing and translating scientific books (in collaboration with Li Shan-lan 李善蘭) such as *Compendium of Arithmetic* (Wylie, 1853), *Supplementary Elements of Geometry* (Wylie and Li, 1865), *Treatise of Algebra* (Wylie and Li, 1859a), *Analytical Geometry and Differential and Integral Calculus Step by Step* (Wylie and Li, 1859b). On the other hand, he was also a sinologist who brought Chinese literary, philosophy, science and mathematics to the Western world especially Britain. During his 30 years of stay in China, Wylie collected many Chinese books in different disciplines. He published *Notes on Chinese Literature* (Wylie, 1867) which provides a bibliography with detailed explanatory notes on Chinese books. He had also written a number of articles related to China which were published in newspapers and periodicals. His colleague James Thomas selected some of these articles and edited as *Chinese Researches* (Wylie, 1897). These two books provided valuable sources for Westerners in the 19th century to know about China (in a new light).

In an accompanying workshop by the same authors, titled " *Chinese Arithmetic in the Eyes of a British Missionary and Calculus in the Eyes of a Chinese Mathematician*", we will focus on Wylie's introduction of (Western) algebra and calculus into China. The present paper supplements and complements this workshop. We will focus on how Wylie introduced Chinese mathematics to his own country. In particular, we will discuss a series of newspaper articles titled *Jottings on the Science of the Chinese Arithmetic*[1], which were first published in nine instalments from August to November of 1852 in *North China Herald* and later reprinted in *Chinese Researches* (Wylie, 1897, pp. 159–194). This series of articles played a pioneering role in the study of the history of Chinese mathematics in the Western world. It may be the only reliable (Western) source on the history of Chinese mathematics before the publication of Yoshio Mikami's *The Development of Mathematics in China and Japan* in 1913 (Wang, 1999). Dauben (2000) gives the following comment on *Jottings:*

> "This article is the first in English to give a reliable account, for the most part, of Chinese mathematics ⋯. Given the pioneering nature of this work, it is not surprising that it contains various errors and inaccuracies ⋯. Nevertheless, the "Jottings" is an important work for the history of Chinese mathematics, and was to have a significant influence upon such prominent historians of mathematics as Moritz Cantor, Florian Cajori, and David E. Smith." (Dauben, 2000, p. 781–782)

According to Wylie, the objective of this series of articles is to clarify some erroneous statements about the status of mathematics in China that were found in (Western) publications at his time. He explained clearly this purpose at the beginning:

> "The object of the following desultory notes, made from time to time, in the course of some researches entered upon, with another purpose in view, is to draw attention to the state of the arithmetical science in China, a subject which has not been so fully explored as it might with advantage, and on which some erroneous statements have been current in modern publications."

In this presentation, we will first outline Westerners' common views at Wylie's time on the status of Chinese mathematics. Then, we will discuss how Wylie responded to these views in his *Jottings* and

[1] In subsequent discussion, we will use "*Jottings*" as the abbreviation for the article *Jottings on the Science of the Chinese Arithmetic*.

evaluate his viewpoints in the light of contemporary literature on history of Chinese mathematics. Finally, we will discuss the implication of this historical document to current mathematics education. Since this paper is to be submitted five months before the actual presentation so that the authors lack the opportunity of benefitting from comments and views of colleagues in the audience, the present record will focus mainly on Wylie's response to the erroneous statements, while its pedagogical implication will be discussed in more detailed during the actual presentation.

The Chinese terms in the text are written in a system adopted by Wylie in his writings, which is not exactly the (older) Wade-Giles system nor the (more modern) Pinyin system.

2 Common views of Westerners in the 19th Century about Chinese mathematics

In this section, we will give a brief account (with support from excerpts of source materials) on the common views of Westerners in the 19th Century about Chinese mathematics. It will provide background information to the subsequent discussions on Wylie's *Jottings*. We are indebted to Wang (2004) in directing us to some of these source materials.

Generally speaking, Westerners in the 19th Century thought that Chinese possessed only very limited mathematical knowledge which was far behind them (the Europeans). They also thought that mathematics was a neglected discipline to the Chinese. These views are evidenced from the following quotations:

- "The knowledge of mathematics even among learned men is very small, and the common people study it only as far as their business requires." (William, 1848, p. 147)
- "For their [Chinese] acquaintance with the exact sciences cannot for a moment bear comparison with that of Europeans." (Murray et al., 1836, p. 224)
- "It happens that men of genius neglect that kind of knowledge [knowledge of mathematics and astronomy], and pursue the more popular branches which lead to honour and emolument." (Murray et al., 1836, p. 225)

2.1 Contribution of Jesuit missionaries to Chinese mathematics

The prevalent Western view at the time was that Chinese books on mathematics were based on contribution from the Jesuit missionaries. For instance, William (1848) pointed out that the *Swan-fah Tung Tsung* (*General Comprehensive Arithmetic*) and the *Tsuimí-shan Fang Sho Hioh* (*Mathematics of the Lagerstraemia Hill Institution*) contained a lot of material from the mathematical writings of the Jesuit missionaries. Similarly, Davis claimed that:

> "In the science of numbers, and in geometry, the Chinese have, as usual, nothing to teach us; being, on the contrary, indebted for a good deal to Europe, as may be seen from the logarithmic tables and other works prepared for the Emperor Kâng-hy by the Jesuits." (Davis, 1851, p.282)

However, the impact of Western mathematics transmitted by the Jesuits was small. For instance, Murray et al claimed that:

> "the progress which it had made in that country [China], when compared to the time it had been cultivated before the Jesuit missionaries obtained a footing among them, was extremely small.

⋯ it may be inferred, that there existed in that country no mathematics by which it could be improved." (Murray et al, 1836, p.231)

2.2 Chinese numeric notation, arithmetic and algebra

According to the understanding of Westerners, "the [numeric] notation of the Chinese is based on the decimal principle, but their figures are not changed in value by position, and it is difficult therefore to write out clearly the solution of a question." (William, 1848, p. 146). William continued to explain that this was overcome, in arithmetical calculations, by the assistance of an abacus. However, he pointed out that its disadvantage is that "if an error be made, the whole must be performed again, since the result only appears when the sum is finished" (William, 1848, p. 146). Therefore, he concluded that: "This mode of notation ⋯ falls far behind the Arabic system now in general use in the west" (William, 1848, p. 146).

Other literature of Westerners in this period shared similar opinion. For instance, Murray et al. (1836) gave the following comments on the abacus and Chinese numeric notation and arithmetic:

> "It must, however, be admitted, that although this machine [the abacus] be well adapted for explaining the principles of arithmetic, it would be a very inadequate substitute for our Arabic numerals, more especially in those laborious calculations which the progress of European science has rendered indispensable. Sir George Staunton says, that the Chinese have no characters, except those in their common language, to express sums in an abbreviated form, after the manner of the Arabic figures used by Europeans. When, however, they have occasion to introduce numbers in their writings, they have recourse to their ordinary terms, each of which denotes a numerical value, independently of its relative position, - a method less tedious indeed than the expression of the same numbers by the method of alphabetical writing, but which by no means equals the conciseness of the same process in the Arabic notation. The universal multiplication and subdivision of all quantities by decimal proportions, facilitates their calculations, and prevents the necessity of methods to abridge them." (Murray et al., 1836, p. 228–229)

Davis (1851) not only repeated the above opinions, he even claimed that: "No algebraic knowledge is to be found in China" (Davis, 1851, p. 282). Unfortunately, this erroneous statement was rather popular among Westerners in that period. Indeed, in his *Jottings*, Wylie put quite a lot of effort to correct this misunderstanding.

2.3 Summary

Based on the source materials above, common views of Westerners at Wylie's time can be listed below:

1. Chinese mathematics was far behind Western mathematics.
2. Nothing about Chinese Mathematics was worth learning by Westerners. On the contrary, Chinese mathematics benefited wholly from Western mathematics (for example, logarithm) which was transmitted by the (Jesuit) missionaries.
3. Chinese numeric notation was cumbersome. It fell far behind the Arabic numeric system used by Westerners. Although the notation was based on decimal principle, it did not have local value (that is, the numeric figures were not changed in value by positions).

4. Abacus was an apparatus which assisted the Chinese to do arithmetic calculation, but it was not very useful – at least, it was inadequate as a substitute for the Arabic numeric system.
5. There was no algebra in Chinese mathematics.

In the next section, we will describe how Wylie responded to these views in his *Jottings*.

3 Wylie's response in his *Jottings*

Wang (1998) gave a detailed analysis on the structure and content of Wylie's *Jottings*, with selected passages translated into Chinese. In this presentation, we analyse *Jottings* from another perspective, namely, how Wylie responded to Westerners' common (erroneous) views about Chinese mathematics. In the following discussion, the page numbers of *Jottings* refer to those in the *Chinese Researches* reprint edition.

3.1 The history of abacus

Wylie wrote: "It has been erroneously stated by some authors that the Chinese have used the 算盤 *Swan-pan* or abacus from time immemorial." (p.168). It seems that this erroneous statement was rather common among Westerners at that time (see for instances, Murray, 1836, pp. 227–228; Davis, 1851, p.283–284). Wylie pointed out that the abacus was indeed introduced in "comparatively recent date". He continued to introduce the *Show* or tallies which is a predecessor of abacus. "In ancient time calculations were carried on by means of 籌 *Show* or tallies made of bamboo" (p.168). We remark that the history of abacus and tallies mentioned by Wylie is basically correct. Martzloff (1997, Chapter 13) pointed out that the counting rods (tallies) can be traced as early as Former Han Dynasty (1st century, B.C.E.) and kept on playing an important role in Chinese mathematics until the Yuan Dynasty (13–14th century). It was also pointed out that "the abacus only entered into common use in China from the second half of the 16th Century [Ming Dynasty]" (p. 215).

The most interesting thing about the tallies which Wylie correctly pointed out is that "the written character is evidently a rude representation of these [the tallies] " (p. 168). He made an analogy of this kind of written representation with the Roman numerals and pointed out that both systems have a new symbol for the increment of 5. It provided evidence suggesting that the Chinese numeric notation depended on the theory of local values at a time much earlier than the European understood this theory.

3.2 Local values in Chinese written numbers

On p. 169 of his *Jottings*, Wylie quoted several books of his time, including *Penny Cyclopaedia of the Society for the Diffusion of Useful Knowledge* (edited by Charles Knight, 1833) and also Sir John Davis's works, which claimed that the Chinese written numeration does not have local value. Wylie disagreed and pointed out that "an example from any native work will be a sufficient reply to the above statements" (p.169). Then, he quoted a question from Chapter 8 of *Soo-shoo-kew-chang (Nine Sections of the Art of Numbers)*[2] 數書九章by Tsin kew-chaon (Song Dynasty, 13th century) as a "random" example. Wylie used this example as an illustration that the arithmetical work in (ancient) China was essentially

[2]Nowadays, this book is known as *Mathematical Treatise in Nine Sections*.

the same as what English did, except perhaps with different meanings in some terms. In particular, Wylie argued that "the author [of *Nine Sections of the Art of Numbers*] had the same view with regard to local value,⋯, as that universally adopted by modern civilized nations" (p.169).

It is interesting to note that in later part of *Jottings* when Wylie introduced the method of *Tien-yuen-yih* 天元一 (Chinese algebra of polynomials) found in the Yuan Dynasty, he has the following comments:

> "In the *Tien-yuen-yih*, unity is employed as the representative of an unknown number; this being combined with an extension of the theory of local value, in order to represent the successive powers of the Monad or unknown number" (p.182).
>
> "It is not a little remarkable, that while it has been gravely asserted by most respectable authorities in Europe, that the Chinese are ignorant of the meaning of local value, we find here on the contrary, that they have pushed the principle to a degree of refinement unpracticed in the west" (p.182).

In other words, Wylie pointed out that the polynomial representation in the method of *Tien-yuen-yih* is indeed a generalization of the theory of local value.

3.3 Algebra in ancient China

In order to respond to the claim that there is no algebra in China, Wylie provided some concrete algebraic methods found in ancient China. It is interesting to note Wylie's comments on the dates for the origin of these methods:

> "In examining the productions of the Chinese one finds considerable difficulty in assigning the precise date for the origin of any mathematical process; for on almost every point, where we consult a native author, we find references to some still earlier work on the subject" (p.175).

Nevertheless, this quotation suggests that Wylie believed that algebraic knowledge did indeed exist in China long time ago.

Ta-yen ("Great Extension") 大衍 (known as "Chinese Remainder Theorem" nowadays) may be the most well-known algebraic method introduced by Wylie in his *Jottings*. As a result of a German translation of *Jottings* (translated by K.L. Biernatzki), this method had drawn the attention of Western historians. Unfortunately, because of some misinterpretation in this German translation, some of these historians thought that this method was mathematically incorrect. After a long process of investigation (thanks to the work of L. Matthiessen in 1881) Westerners realized that *Ta-yen* method was indeed equivalent to the method devised by Gauss. Finally, this method was recognized as the Chinese Remainder Theorem. Readers who are interested in the details of this story may refer to Wang (2004). We now come back to the discussion on how Wylie introduced the method of *Ta-yen* in his *Jottings*. First, Wylie quoted the well-known problem of *Wuh-puh-chi-soo* ("Unknown Numerical Quantities") 物不知數appeared in *Sun-tsze Swan-king* (*Sun Tsze's Arithmetical Classic*)[3](Chin Dynasty, 1st Century):

> "Given an unknown number, which when divided by 3, leaves a remainder of 2; when divided by 5, it leaves 3; and when divided by 7, leaves 2; what is the number?" (p.175)

[3]Nowadays, this book is known as *Master Sun's Arithmetical Manual.*

After giving a brief discussion of the method of solution, Wylie proceeded to describe the general method given in Chapter 1 (*Ta-yen*) in *Nine Sections of the Art of Numbers*. It is interesting (but may not be so appropriate[4]) that he selected Problem 1 in the Chapter of *Ta-yen* as an illustration of this method. Despite the fact that this principle is not very clearly explained (for instance, the precise procedure of "finding unity"), Wylie's work played a pioneering role in introducing this method to Westerners.

Another method introduced by Wylie is the *Tien-yuen-yih*(unity)[5] as "the representative of an unknown number" (p.182). This was an ancient Chinese method of representing a polynomial of one variable. More precisely, ancient Chinese used different terms (such as *Yuen*元, *Tai*太, *Tai-kieh*太極) to represent the coefficients of different powers of an unknown quantity, that is, variable x (in today's terminology). As mentioned in a previous section, Wylie regarded it as "an extension of the theory of local value". Furthermore, he also pointed out that "the method invented by Hariot, of placing all the significant terms on one side, is precisely that used by the Chinese [as demonstrated by *Tien-yuen-yih*] some five centuries earlier; and although in itself but a variation in algebraic language, yet it is said by De Morgan to have been the foundation of most important branches of the science" (p.182).

Next, Wylie pointed out that Horner's method of "solving equations of all orders" which was first published in 1819 (some 30 years before the publication of *Jottings*) could be found in *Nine Sections of the Art of Numbers* (Song Dynasty, 13th century which was 6 century earlier). Again, this gives another example that many algebraic methods known by Westerners were already known by Chinese many centuries earlier. This serves as a refutation of the Westerners' usual claim that "no algebraic knowledge is to be found in China".

3.4 Chinese mathematics versus Western mathematics

The overall purpose of Wylie's *Jottings* is to respond to the common Westerners' view (at his time) that Chinese mathematics was far behind their Western mathematics and nothing in Chinese Mathematics was worth learning. As discussed above, Wylie provided some examples to support an opposite view, namely, quite an amount of mathematical knowledge known to Westerners at his time was actually discovered by ancient Chinese much earlier (some even several centuries earlier). The theory of local values in numeric representations, the concept of negative numbers, *Ta-yen* (Chinese Remainder Theorem), *Tien-yuen-yih* (the method of representing a polynomial), and solving polynomials of any degrees are some examples. Furthermore, detailed introduction of some classical Chinese mathematics books such as *Kew-chang-swan-shun* (*Arithmetical Rules of the Nine Sections*)[6] 九章算術 and *Soo-shoo-kew-chang* (*Nine Sections of the Art of Numbers*)數書九章 are included in *Jottings*. Despite the fact that it contains some erroneous descriptions on these books (see for instance, Wang 1998), it has opened up a new window for Westerners to know about Chinese mathematics.

Wylie held a balanced view on Chinese mathematics and Western mathematics. On the one hand, he did not underestimate Chinese mathematics; on the other hand, he recognized the contribution of Western mathematics transmitted by the missionaries to the progress of mathematics in China. In the last part of *Jottings* (p.188 and onwards), he gave a brief account on mathematics in Qing Dynasty

[4]See Wang (2004).

[5]Authors such as Wang (2004) pointed out that Wylie has (mistakenly) mixed up *Ta-yen and Tien-yuen-yih*. Indeed, they are actually not related.

[6]Nowadays, this book is known as *Nine Chapters on the Mathematical Art*.

and pointed out how Western mathematics influenced the development of mathematical ideas in that period. For instance, the work of Li Shan-lan, who became one of Wylie's close co-workers, on logarithm (a mathematical idea transmitted by the Jesuit missionaries) was introduced. As revealed in the following comment, Wylie paid rather high regard to Li's work:

> "This small indication of self-satisfaction may be very well overlooked, as quite pardonable in one who has had no better aid than that afforded by the *Leuh-lih-yuen-yuen*, and who has here given us, as the result of four years' thought, a theorem, which in the days of Briggs and Napier, would have been sufficient to raise him to distinction." (p.194)

The following closing remark in *Jottings* suffices to describe Wylie's view on Chinese mathematics and Western mathematics, which was indeed rather innovative in his time!

> "It is true the Celestials are disposed to look with a feeling akin to contempt on the mushroom antiquity of our Western lore; yet it is equally true that a spirit of inquiry still germinates among them, which if fostered by a greater freedom of intercourse, will doubtless tend much to smooth the asperities which now exist, and this prove mutually advantageous." (p.194)

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INDIAN PEDAGOGY AND PROBLEM SOLVING IN ANCIENT THAMIZHAKAM

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ABSTRACT

Indian pedagogies are not well known to research community outside India. The researcher refers to Indian literature to examine the pedagogies that have been in use in India since ancient Vedic times. She also identifies the use of these pedagogies even in contemporary classroom. She concludes that some of the current pedagogies such as memorization are culture-influenced pedagogies, culture being defined as a human learned behaviour.

Though pedagogies such as memorization and oral repetition which are considered in modern times as not very beneficial in enhancing the learners' intellect were used in ancient India, the researcher cites that how problem solving was part of the mathematics curriculum in Thamizhakam (Southern India), the province in which the mathematician Ramanujan was born. The researcher also cites how mathematics was taught and learnt in Thamizhakam. She quotes some of the ancient problem solving questions, which were transmitted orally for generations. She concludes that though traditional methods such memorization and oral repetition were in use in Indian mathematics classrooms, problem solving was also part of mathematics curriculum in Thamizhakam, which could have galvanized the mathematical reasoning of the learners.

Keywords: Indian pedagogy, problem solving, Indian culture in education

1 Introduction

Due to the phenomenal growth in the number of schools, the literacy rate in India has risen drastically in the past 50 years. According to the census of 2011, 821 out of every 1000 men and 655 out of every 1000 women could read and write (Department of School education). Guo (2005) praises India on witnessing "phenomenal educational development both in quantitative and qualitative terms, since independence" (p. 190). However, there are not many classroom studies on Indian classrooms focusing on teaching and learning.

Though India has produced many mathematicians, the pedagogy used in India is a mystery to many researchers as India did not take part in any international studies and there are not many studies on Indian pedagogies known to research community outside India. The researcher refers to literature to examine the pedagogies used in India and identifies certain key characteristic pedagogies such as oral transmission of knowledge, teacher questioning, memorization and oral repetition that have been in use since ancient Vedic times. The researcher also cites how mathematics was taught and some of the problem solving questions used in ancient Thamizhakam, the Tamil speaking part of South India, which could explain how the two extremes of the spectrum of pedagogy complemented each other in the development of students' mathematical abilities.

2 Hallmarks of Indian Education

Each country has its own philosophy of education. Unlike the educational philosophies of Greece and China, which separated education and religion, in Indian educational philosophy, they were intertwined to the extreme that "music and poetry became even handmaids to religion" (Venketeswara, 1980, p. 31). Mathematics known as Ganita was mainly used in ancient India to calculate auspicious time to perform religious rituals and prayers based on the movement of planets and to build temples and altars. On analyzing the pedagogy of Indian primary schools, Alexander (2000) deduces that developing character is more important in India than enhancing intellect. According to Asthana (2001), the main function of education in India "to develop virtues, socially accepted thoughts and habits" (p. 2). In the ancient Indian education system, "Culture not literacy, was the highest aim of education in India" (Venketeswara, 1980, p. 24).

2.1 Borrowing others' ideas and indianizing

Venketeswara (1980) claims that the main feature of Indian education was its comprehensiveness as the curricula never excluded anything unfamiliar or new. According to him, "the expansiveness of Indian culture was the adaptability of the old to altering conditions and new circumstances"(p. 26). Since ancient times, the Indians readily accept new things "without completely ringing out the old" (ibid: p. 27). In their quest to search the truth, the Indians readily accept foreign doctrines. Venketeswara (1980) suggests that the immensity of Indian culture is due to "a readiness to borrow" (p. 25) and an adaptation of the borrowed ideas to the conditions and climate of their own country. "Astronomical terms were borrowed, they were skillfully Sanskritised and incorporated in Indian Astronomy that they became flesh of its flesh and bone of its bone"(ibid: p. 26). Venketeswara (1980) claims "the expansiveness of Indian culture is illustrated by a readiness to borrow, and an adaptation of the borrowed details to the conditions of this clime and country" (pp. 25–26) and "the adaptability of the old to altering conditions and new circumstances" (p. 26). The Indian educators have borrowed the doctrine of constructivism and have introduced activities in mathematics teaching in recent years. However, to maximize the success rate of students benefiting from the activities, the activities are transformed into teacher-directed activities. The main aim of the activities is to make every one including the average and below average students understand the mathematical concepts. In other words the activities supplement the whole class teaching but are not substitute to teaching. Though we can notice the introduction of collaborative learning, the use of ICT for learning and the use of activities to teach mathematics concepts in Indian classrooms, we can also observe the ancient pedagogies such as the stress on memorization, mental computation, questioning and even oral repetition in late 20th and 21st century Indian classrooms. This shows how the Indian society values ancient pedagogy though welcomes some changes in teaching methods for the benefit of the students.

3 Prominent Pedagogies of Teaching and Learning in India

3.1 Learning by listening

In ancient India, knowledge was transmitted orally. Though the art of writing was developed, the teaching was mostly depended on verbal learning (Chatterjee, 1951, p. 189). Learning by listening in India has been mentioned by Venketeswara (1980), "India stands alone in the emphasis of Sruti, learning by the ear, even long after writing came into common vogue"(p. 25). Even *Kautilya*, the chief advisor to the first Maurya Emperor in his book on politics *Arthasastra* (written around 300 B.C.) sums up the object of study as follows:

"**From hearing ensues knowledge**, thence *Yoga* (steady application), thence *Atmavatta* (self possession) (as quoted in Venketswara, 1980. p 164, emphasis added).

As oral pedagogy was predominant, listening to learn became indispensable. Venketswara (1980) claims that " both in Hindu and Buddhistic schools, instruction was oral; text-books were seldom used" as Fa-Hien, a Chinese monk who visited India in 4th century "could not find a single copy of the precepts in North India, where teachers trusted entirely to oral tradition" (p. 213). It is proposed that most of the learned books might have been written by the 17th century only. In South India known as *Thamizhakam* "the children began their lessons in mathematics with the learning of Tamil numerals, **by hearing** the number names" (Senthil Babu, 2007, p. 25). Only some teachers had text books called 'Ponnilakkam' (ibid.).

The significance attached to learning by listening in Tamil Nadu, the southern part of India can be understood as *Thirukural* ackowledges the knowledge attained by listening as the prime knowledge.

செல்வத்துள் செல்வம் செவிச்செல்வம் அச்செல்வம்
செல்வத்துள் எல்லாம் தலை.

(Kural. 411)
(Translation: Of all wealths, listening is the best available wealth on earth.)

Subramanian (2007) confirms that people followed this practice by stating "the pattern of education in ancient Tamilakam (Thamizhakam) was not merely reading and understanding of texts, but also listening to learned persons" (p. 342).

3.2 Questioning as one of the main pedagogies

Questioning has been employed as the main teaching methodology even in ancient India. There were frequent mentions of terms like "*prasnin*" (questioner), "*abhi—-prasnin*" (cross—questioner), "*prasna—vivaka*" (answerer) (Gupta, 2007, p. 76). Many past studies have identified the Indian teachers' use of questioning in mathematics classroom (Alexander, 2000; Clarke, 2001; Clarke & Fuller, 1997; Rao & Cheng, 2001). Clarke and Fuller (1997) have reported observing the teachers in Chennai transmitting "what the students should know" by lecturing and "how they should know" (p. 54) by asking questions. Clarke (2001) has also identified "the teacher asking questions and the student answering" (p. 77) as the predominant model of interaction between teacher and student in mathematics teaching. Alexander's (2000) research throws light on Indian teachers' efforts to develop dialogue with students and scaffold understanding despite the crowded classroom (p. 558) with the help of questioning. Rao and Cheng (2001) have found the Indian teachers use of questioning “to involve children actively in the learning process” (p. 11). Sensarma (2007) has identified ‘question-answer dominated interaction pattern’ as one of the seven commonly used distinctive interaction pattern in mathematics classrooms at secondary level. Subramanian’s (2010) research analyzes extensive use of questioning in Indian mathematics classroom both to assess and assist students’ learning. Since ancient times, discourse, questioning and debate have been commonly practiced pedagogies in India, which could explain the natural prevalence of this type of verbal interaction in the classrooms even today.

3.3 Memorization as a pedagogy

The Vedas, the oldest texts of Hinduism, have been transmitted orally for three thousand years as “the Vedas as recited from memory by Brahmans” (Fuller, 2001, p. 1) were considered as authoritative. The priests needed to learn texts from the Vedas even before mastering the Sanskrit language. Hence memorization became the dominant pedagogy used in the Vedic period (1500 B.C.–500 B.C.) Repetition (*parayana*) and memorization (Venketeswara, 1980) were renowned pedagogies as Vedas were repeated every day so that students could learn and remember the texts.

Students listened to the guru and repeated his utterances without the aid of books and memorized the Vedic texts. Venketeswara (1980) even claims that there were prayers for memory in ancient India.

> As regards methods of education, the first noteworthy principle is that of memorizing and even learning by rote. There are prayers for memory (*medtha*). ‘May the Lord endow me with *medha*; may we learn much and learn by the ear, and may we retain what we have learnt’ (p. 88)

Ancient Indian mathematical works, mostly composed in Sanskrit, usually in *the form of Sutras* in which a set of rules or theorems were stated in verse in order to aid memorization. “Profuse use of the verse style as an aid to memory and to make the students learn verbatim” (Bara, 1998, p. 161) clearly exhibits the society’s belief in this pedagogy of learning. There was a strong stress on memorization of multiplication and conversion tables involving fractions and measurements in *Thamizhakam* where merchants memorized “all kinds of tables relating the various kinds of measures was the first task to be accomplished” (Samuel, 2005a, p. 59) to master arithmetic.

Kanita Nul (an ancient mathematics book in Tamil, recorded in palm-leaf manuscript) proves the significance attached to memorization to learn arithmetic in Thamizhakam (Samuel, 2005a). Senthil Babu (2007) claims that memory the mode of learning was central to education in the indigenous schools in Thamizhakam. He further asserts that memory was used not because of the lack of books but mainly good memory was considered as intelligence. Studies also show the presence of memorization as the common pedagogy in Indian classrooms even during the British period and the British could not as do much as the pedagogy has been deeply rooted in the Indian pedagogy (Clarke, 2001). The Vedic mathematics, which has been drawing a lot of attention in recent years, expounds the memorization of certain computation procedures to perform complex computation mentally. Memorization was a foundational pedagogy of learning in education systems in most cultures in ancient times; it is venerated in certain cultures, especially in India even today.

3.4 The use of oral repetition since the Vedic period

The dominant Pedagogy used in Vedic period (1500 B.C.–-500 B.C.) was repetition (*Parayana*) and memorization (Venketeswara, 1980). Vedas were chanted every day so that students could learn and remember the texts. Oral transmissions of texts were promoted, as the pronunciation of the texts with accuracy and correct intonation could not be achieved by learning from the texts. The pedagogy of oral repetition was not changed even under the Muslim rule, as "the oral transmission of the Quran" was "the backbone of Muslim education" (Robinson, 1996, p. 65). The numerical tables were recited twice a day in order to memorize them (Acharya, 1996). When there was a scarcity of textbooks and when many could not afford to buy textbooks, oral transmission of texts became inevitable and oral repetition became a strategy to memorize the texts and to pass on to the next generation. However, it is worth mentioning that even written texts are readily available at affordable prices, people prefer the oral transmission of knowledge. Oral repetition became an effective strategy when oral tradition was widely used. Oral repetition, which was one of the common pedagogies in ancient times, is still in use in Indian classrooms (Alexander, 2000; Clarke, 2001; J. Subramanian, 2010).

The researcher summarizes that stress on oral tradition, teacher questioning, oral repetition and memorization are pedagogies being in practice in India since ancient times and hence they are influenced by culture as these practices are 'learned behaviours', being transmitted socially.

4 History of Education in Ancient and Medieval Thamizhakam

The Tamil (a language) speaking community was formed nearly thirty five centuries ago (N. Subramanian, 1996) and lived in the southern part of Indian peninsula which is currently known as Tamil Nadu. Originally this part of India was known as *Thamizhakam*. As we find great quality of poetic literature, we understand that creativity was celebrated by the Tamils. At the same time, "the merchants and the royal servants were to learn accounting and arithmetic" (N. Subramanian, 2007, p. 343). We do not know much on curriculum or pedagogy. Subramanian (ibid) claims, "Much of the teaching was oral" (p. 345). Many literary pieces were saved due to the remarkable memory of the students of those days, as they did not have any other aids to preserve the literary texts. The students memorized the texts not only because there was a scarcity of textbooks but mainly because "it was believed that a strong memory was a chief virtue of a scholar" (ibid. p. 345).

4.1 Significance attached to Mathematics in Thamizhakam

The ancient Tamil literature, Thirukural (also known as Kural) proclaims numeracy and literacy as the two eyes of human being.

எண்ணென்ப ஏனை எழுத்தென்ப இவ்விரண்டும்
கண்ணென்ப வாழும் உயிர்க்கு.

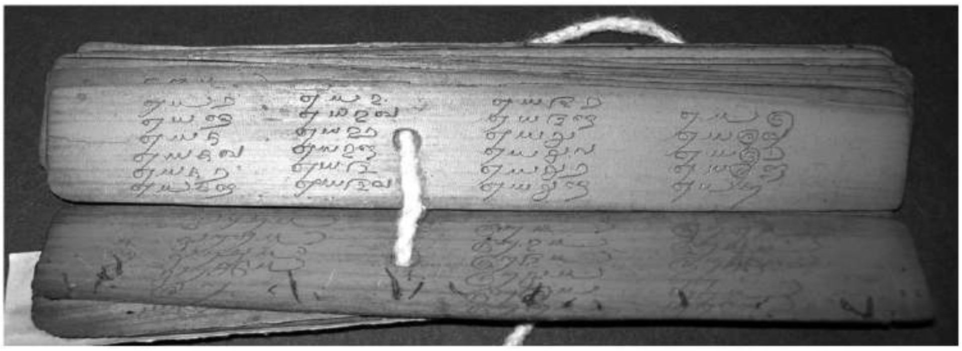
(Kural.392)
(Meaning: Numeracy and literacy are considered as eyes for human beings.)

In Western literature, we come across literacy and numeracy. However, in Tamil Nadu since ancient times, numeracy (*en*) is mentioned before literacy (*eluttu*), which the researcher argues shows the significance attached to numeracy. It is mentioned in *Pinkala Nikandu* (*an ancient book in Tamil*) that "*en*" and "*eluttu*" refer to "*Kanakku*"(*math*) (N. Subramanian, 1996, p. 157) as in Tamil, letters were used to denote numbers. It is argued that '*en*' refers to arithmetic and '*eluthu*' refers to algebra in which case, mathematics consisting of arithmetic and algebra is considered as the two eyes of a human being. In other words, the supremacy of mathematics has been acknowledged by the Tamils since ancient times.

4.2 Tinnai palli (Veranda schools)

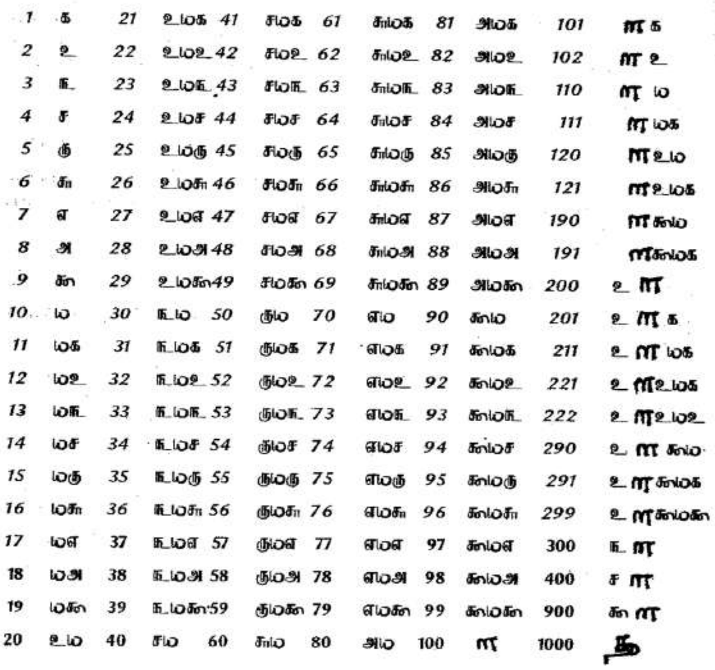
In Thamizhakam, 'Tinnai' schools became popular in the 18th to 19th century. Every village or a cluster of nearby villages had a *Tinnai* school. Students paid the fees in cash and kind to the teacher; students also worked on the land of the teacher to pay the fee (Senthil Babu, 2007). The children were not divided into classes based on their ages, but according to their abilities to "learn language and mathematics" (ibid: p. 20). There was no standardized curriculum and the aim of this education was to prepare the children to be competent in language and numbers practised by the society.

Letters were used to represent 1 to 10, 100 and 1000. Using these 12 letters, they represented all numbers. One followed by 14 zeroes was named as "*Maha koti*". Only some teachers had a book called 'Ponnilakkam' for elementary mathematics.



Ponnilakkam—elementary number primer for *Tinnai* schools (Courtesy French Institute of Pondicherry,Pondicherry). (Senthil Babu, 2007, p. 34)

Like Tamil letters, Tamil numerals were also memorized.



(Samuel, 2005a, p. 292).

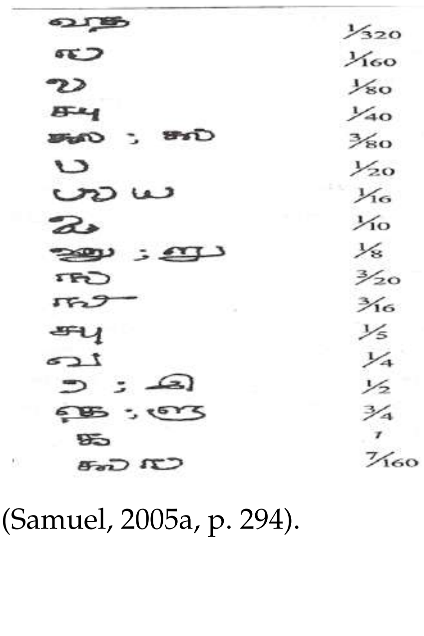
They had special terms for each fraction. For example, $\frac{1}{320}$ is termed as *muntri*. The lowest fraction with a term is called *immi* (little) $\frac{1}{1075200}$ (ibid; p. 49). Students memorized the fraction table [eg., 4 *mahani* $(\dfrac{1}{16}) = 1$ *kal* $(\dfrac{1}{4})$]. However there was meaning in naming the terms such as $\dfrac{1}{8}$ has been termed as "*ari kal*" (half of quarter).

Similarly, $\dfrac{1}{32}$ is known as 'erandu kaaniye ari kaani'. Kaani= $\dfrac{1}{80}$.

Erandu(twice) kaani= $2 \times \dfrac{1}{80} = \dfrac{1}{40}$. Ari (half) kaani= $\dfrac{1}{2} \times \dfrac{1}{80} = \dfrac{1}{160}$.

Erandu kaaniye ari kaani= $\dfrac{1}{40} + \dfrac{1}{160} = \dfrac{5}{160} = \dfrac{1}{32}$ (ibid; p. 81)

Tamil letters, which were used to represent fractions, are listed below.



(Samuel, 2005a, p. 294).

According to a manuscript dated 1693, there were six types of measurement: *En alavu*—numerals, *Kol alavu*—linear measurements, *Kal alavu*—measurements involving volume, *Tula alavu*—measurements in weighing, *Nal alavu*- measurement of time and *Teyva Atikaram*—dealing with celestials (astronomy) (Samuel, 2005b). Each of them had a conversion table which the students needed to master (for example, 60 *Nodi* (seconds) = 1 *Nimisham* (minute), 24 *Nimishams* (minutes) =1 *nazhi*, 60 *nazhis* = 1 day…). All tables and conversions were memorized and recited. Memorization was the predominant pedagogy of learning. The *Tinnai* schools in *Thamizhakam* trained many pupils to memorize procedures and shortcuts from *Kanita Nul* to solve problems, which were in verse format to become accountants honing their mathematical skills and logical reasoning. As they needed to do long computations quickly and accurately in either agriculture, commerce or astrology, they were taught certain procedures, which they memorized. *Paun* has been is use for measuring the weight of gold since ancient time (which is eight grams of gold). They had separate measurement conversions for gold.

31. மாவாகில் மஞ்சாடி மாகாணி பத்துமா
ஆமாகி முக்காணிக் காறுமா -- நேரே
குன்றிக் கரைமா பிறவுக்குக் காணியாய்
அரைக்காணிக் கோர்மாவாம் பொன்.

பொன் எண்ணின் தானம் அறிதல்.

ஒரு மாவுக்கு மஞ்சாடி என்றும், 1/16 க்கு 10 மா என்றும், 3/80 க்கு 6 மா என்றும், 1/40 க்கு குன்றி என்றும், 1/80 க்குப் பிளவு என்றும், 1/160 க்கு நெல் என்றும்; 1/320 க்கு அரை நெல் என்றும் சொல்லப்படும் என்றவாறு.

(Samuel, 2007, p. 65)

Translation:
1 Ma = 1/20 = I Manchadi Pon
1 Ma kani = 1/16 = 10 Ma Pon
1 Mukkani = 3/80 = 6 Ma Pon
1 Arama = 1/40 = 1 kunri Pon
1 Kani = 1/80 = 1 Pilavu Pon
1 Araikkani 1/160 = 1 Nel Pon
1 Muntiri 1/320 = 1/2 Nel Pon

Kannakkatikaram also has problems and solutions in verse format for farmers to find the area of land in the shape of an arrow (sector) and for government servants to calculate taxes, for goldsmiths and customers to find the purity of gold in ornaments. They also had formula to find the area of triangle and trapezium. The following verse informs us how to find the length of the yard-stick if the area is given.

குழியதி னாலே கோலறி குணர்ந்து
குழியின் சமனில் கோலடி பெருக்கி
இற்குழி சமனுக் கீய்ந்து பெற்ற
சொற்குழி கோலி னடியெனச் சொல்லுமே.

(Samuel, 2005a, p. 136)

Translation:
When measured by a 12 yard stick, the area is 100 *kuli* (unit for area). When the same area is measured by another stick of unknown length, the area becomes 25 *kuli*. Find the length of the yard stick.
The solution is given as follows:
$12\sqrt{100} = 12 \times 10 = 120$
Therefore the length of the second yard—stick= $120/5 = 24$.

Since ancient times, we had different names for multiples of ten, some of which are still in use in India such as 'Lakh' which is 100000 and 'Crore'(kodi in Tamil) which is 100 lakhs or 10000000 (10 millions). We also find verses explaining how to find the sum of numerals from 1 to 10 and sum of square numbers. The method to find the sum of squares from *Kanitha Nul* (an anicent mathematics textbook in Tamil) is given below. This verse given below explains how $1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2$ can be computed.

கூறிய மட்டுங் குழிமா றினதொகை கூறவென்றால்
கூறிய தின்குழி யாற்கூறல் மாறியே கூறல்தள்ளி
மீறிய மூன்றொன்றி லேதான் தொகைதனை மேல்வைத்து
ஏறிய இலக்க மிதாமென வேசொல்லும் ஏந்திழையே.

(Samuel, 2005a, p. 101)

Translation:
Multipling 10^2 (which is 100) by 10 to get 1000.
By subtracting 10 from 1000, we get 990.
Dividing 990 by 3, we get 330.
Adding the sum of numerals from 1 to 10 (which is 55) to 330, we get 385, which will be the answer for the problem.

Samuel (2005a) translates another question from the manuscripts of *Kanita Nul*..

கால்முதல் கொண்டு பின்னெட் டுந்தொகை காணவென்றால்
சாலவு மெட்டுக்கும் எட்டுக்கா லென்றுஞ் சாற்றினதைச்
சீலமதா கக்குழி மாறின தன்னினஞ்சேர் திப்பிளவைக்
காலினி லீய்ந்திடக் கண்டிடுங் காணிக் கணக்கினையே.

(p. 91)

The above verse tells how to compute the sum of the series $\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + \cdots + 8$.

Translation:
Divide 8 by quarter, which gives 32 (the number of terms).
Multiply 32 by 32 and add 32 and halve the result (which is $1024 + 32 = 1056 \div 2 = 528$.
Multiplying 528 by the quarter, the result is 132.

To conclude, memory has been used not as a mere strategy but as pedagogy to learn arithmetic in *Thamizhakam*. As agriculture and business were the main occupations, many short cuts for easier computation were taught. And the arithmetic they mastered was applicable to real life.

5 Problem-solving in ancient Thamizhakam

In mathematics teaching, students needed to memorize basic conversion (in measurement of length, weight etc.,), and tables involving numbers and fractions. Though arithmetic was the core of mathematics curriculum, problem solving was also a part of mathematics curriculum in ancient Thamizhakam (Senthil Babu, 2007). Several treatises on mathematics in Tamil are available in palm- leaf manuscript form. Palm leaves manuscripts that had been extensively used to learn arithmetical practice have been discovered and translated into formal Tamil and recently into English. There are 58 sums in verse format, 25 in prose style and another 25 sums known as prose style practice sums. The researcher cites from Samuel (2005b) ancient problem solving questions both in verse and prose style, which was part of curriculum in ancient Thamizhakam.

Earlier, the problem was stated in verse format and passed on from generation to generation. Students needed to be good in language to understand the meaning of the problem. The chameleon question stated below was a popular one even after centuries. Even those who could not memorize the verse passed on the questions in prose format.

The problem stated below involves the conversion of measurements in lengths; however, it also tests the problem solving skill.

3. ஓணான் கணக்கு

(வெண்பா)

முப்பத்தி ரண்டு முழம்உள முட்பனையைத்
தப்பாமல் ஓந்தி தவழ்ந்தேறிச் – செப்புமுடன்
சாணேறி நான்கு விரற்கிழியும் என்பரே
நாணா தொருநாள் நகர்ந்து. (3)

கருக்கு நிறைந்த ஒரு பனைமரம். அது 32 முழம் உயரமுடையது. ஒரு நாளைக்கு சாண் அளவு ஏறி நான்கு விரலளவு இறங்கும் ஒரு ஓணான் எத்தனை நாளில் அம்மரத்தின் உச்சியை அடையும்?

(ibid: p. 54).

The translation of the popular 'Chameleon' problem is as follows:
A chameleon climbs up a palm tree of height 32 feet. It climbs up the tree a span length speed every day but

slides down by a length of four fingers. How many days will it take to reach the top of the tree?

The problem solving questions in verse form involved many topics in Mathematics including rate, ratio and solving equations.

There were twenty—five questions written in prose style, which might have been written at a later period easing the burden on language. The twenty—five 'prose style practice style sums' are similar to 'prose style sums' involving "inductive reasoning" (ibid: p. 51). Some of these questions are more challenging than the verse style problems. These questions were passed on from generation to generation. The problem stated below involves solving equations with five variables, which could be reduced to two variables.

1. மாணிக்கத்தின் விலை காணல் கணக்கு

ஓர் ஊரில் ஓர் அரசன் இருந்தான். ஒரு நாள் ஒரு வியாபாரி ஒரு மாணிக்கத்தை அரசனுக்குப் பரிசாக அளித்தான். அந்த அரசனுக்கு 4 மந்திரிகள் இருந்தனர். அரசன் முதல் மந்திரியை அழைத்து மாணிக்கத்தின் விலை என்னவென்று கேட்டான். அதற்கு அந்த மந்திரி, தன் சம்பளத்தில் மூன்றில் ஒரு பங்கும், மற்ற மூன்று மந்திரிகளின் சம்பளமும் சேர்ந்தால் எவ்வளவு பணமோ அவ்வளவு என்றான். இரண்டாவது மந்திரியை அழைத்துக் கேட்டபோது, தன் சம்பளத்தில் நான்கில் ஒரு பங்கும், மற்ற மூன்று மந்திரிகளின் சம்பளமும் சேர்ந்தால் எவ்வளவு பணமோ அவ்வளவு என்றான். மூன்றாவது மந்திரியை அழைத்துக் கேட்டபோது, தன் சம்பளத்தில் ஐந்தில் ஒரு பங்கும் மற்ற மூன்று மந்திரிகளின் சம்பளமும் சேர்ந்தால் எவ்வளவு பணமோ அவ்வளவு என்றான். நான்காவது மந்திரியை அழைத்துக் கேட்ட போது, தன் சம்பளத்தில் ஆறில் ஒரு பங்கும் மற்ற மூன்று மந்திரிகளின் சம்பளமும் சேர்ந்தால் எவ்வளவு பணமோ அவ்வளவு என்றான். இந்த நான்கு மந்திரிகள் சொன்ன விலையும் ஒரே விலையாக இருந்தது என்றால் மாணிக்கத்தின் விலை என்ன? மந்திரிகள் ஒவ்வொருவரின் சம்பளம் எவ்வளவு?

(ibid: p. 93).

Translation:
Once there was a king in a town. One day a merchant presented a diamond to the king. The king wanted to know the price of the diamond. He had four ministers. He called his first minister and asked him the price of the diamond. He said the price of the diamond was equal to one third of his salary and the salaries of other three ministers. The king asked his second minister the cost of the diamond. He said that the cost of the diamond was equivalent to one fourth of his salary and the salaries of other ministers. Then the king called his third minister who said that the cost of the diamond was one fifth of his salary and the salaries of other ministers. The fourth minister told the king the cost of the diamond was equal to one sixth of his salary and the salaries of other three ministers. If the price quoted by all of them is the same, find the cost of the price of the diamond. What was the salary of each one minister?

Though there was a strong emphasis on memorization, the curriculum also developed problem-solving skills and included practical application questions. The problem given below gives a typical example testing intuitive reasoning.

௫. சேவகன் சம்பளக் கணக்கு

ஓர் அரசனிடத்தில் ஒருவன் சேவகனாக வேலைக்குச் சேர்ந்தான். அவனுக்கு ஒரு நாளைக்கு ஒரு வராகன் சம்பளம். அவன் 30 நாட்களுக்கு அரசன் இடத்தில் வேலை செய்வதாக ஒப்பந்தம் செய்து கொண்டான். சேவகன் எப்பொழுது வேலையில் இருந்து நின்றாலும் அத்தனை நாட்களுக்குச் சம்பளம் கொடுப்பதற்கு ஏற்றவாறு 30 வராகன் எடையில் அரசன் 5 மோதிரங்களைச் செய்து விரல்களில் அணிந்து கொண்டான். அந்த ௫ மோதிரங்கள் ஒவ்வொன்றும் எத்தனை வராகன் எடை கொண்டவை?

(ibid: p. 96)

Translation:
A king had made five rings of different weights in such a way that whenever he dismissed his servant, he could pay his salary exactly as ring(s). If the salary of the servant each day is one '*poun*' (8 g) gold, what are the weights of the rings (in terms of *pouns*) of the king?

According to Samuel (2005a), though memorization of conversion tables and short cuts from *Kanita Nul* helped merchants to perform computation effectively, to make 'school mathematics useful to real life' the problem—solving questions in real life situations might have been written. These kind of questions kindled students' interest in mathematics. They also complemented the 'memory driven mathematics curriculum', enhancing students' logical thinking.

6 Snapshot of an Indian mathematics classroom in 21st century

The researcher observed and video recorded thirteen consecutive mathematics lessons in Chennai, the capital of Tamil Nadu in 2006. While the teaching practices of a teacher in 8th grade mathematics classroom are analyzed, the teacher under study was found practicing the ancient pedagogies such as the use of memorization, oral repetition, and questioning. She asked every student questions following a rigid pattern and helped the students to verbalize mathematical arguments. In her interview, the teacher acknowledged the use of memorization, oral repetition and questioning as she claimed they have pedagogical values. In every lesson, she asked her students to recall different formulae that they have learnt. According to the teacher, memorization of basic mathematical facts and formulae are essential for mastering fundamental mathematics. She asked the same questions to many students, as she believed in use of oral repetition to enhance learning. She even made a student to repeat five times the formula, which he could not recall. While classrooms in some other cultures downplay the use of oral repetition in learning and some modern learning theories attach negative connotation to ancient pedagogies as old fashioned, in the classroom under study, oral repetition played a positive role in aiding memorization, developing students' understanding and helping learners master mathematical language and to communicate mathematics. According to an Indian educationalist, "oral repetition is another approach to involve the slow learner to the main stream and it would be a misinterpretation to consider this oral tradition as one lacking individual outcome" (Personal communication, D. Subramaniam, August 2010).

7 Conclusion

On referring to literature, the researcher concludes that the pedagogies that have been in use in India since ancient times are learning by listening, teacher questioning, memorization and oral repetition. The researcher identifies these pedagogies even in a 21st century Indian mathematics classroom as these pedagogies have been transmitted for generations. Both teacher and students in the observed classroom reported valuing these pedagogies. The researcher classifies them as culture influenced pedagogies, as they are learned behaviours, being transmitted for generations. The researcher also refers to mathematics curriculum to illustrate how mathematics was taught and learnt in ancient southern part of India and the significance attached to memorization by citing the ancient arithmetic book called *Kanita Nul*. She also cites some of the problem solving questions which were transmitted for generations orally and which were also part of the curriculum even in 18th to 19th centuries in Thamizhakam, the ancient South India.

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HPM AND PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS

Analysis of Mathematics Teaching in the View of the History of Mathematics

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ABSTRACT

History of mathematics is very important for mathematics education, but mathematics teachers make rarely use of mathematical history in mathematics class. For history of mathematics coming into mathematics classes，it is crucial that researching and advancing mathematics teachers' quality in mathematical history. The quality includes three aspects: the recognition on mathematical history, the knowledge about mathematical history and the ability to use mathematical history in teaching. According to the Evaluation Theory of SOLO Classification, we can classify teachers' quality in mathematical history into 5 levels. Through adopting appropriate measures, we can advance teachers' quality in mathematical history, and further promote mathematics teachers' professional development.

Keywords: mathematics teacher; quality in mathematical history; level of quality; advance; professional development of teacher.

Remark: The name marked an asterisk (*) is presenter.

Full Text in Chinese:

与数学教师专业发展

—谈数学教师的数学史素养及其提升

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摘要： 数学史在数学教育中陷入“高评价、低运用”误区。研究和提升数学教师的数学史素养，对推进数学史走进数学课堂至关重要。数学教师的数学史素养包含教师对数学史的认识、数学史知识、运用数学史教学的能力三个要素。根据 SOLO 理论，数学教师的数学史素养可划分为 5 个水平。教师在自身努力下，通过课题带动、专项培训、创建 HPM 资料库、开展反思性教学、发挥群体优势等措施，可提升数学史素养水平，促进专业发展。

关键词： 数学教师；数学史素养；素养水平；提升；教师专业发展

“数学史与数学教育（HPM）”是国际数学教育研究的热点论题。数学史的教育价值日益凸显。我国数学课程标准关注数学史，旨在把数学史引入数学课堂。然而，数学史在教学中“高评价、低运用”的现象普遍存在。调查显示，当前我国数学教师的数学史素养普遍偏低是其关键因素。[1] 然而，何谓数学教师的数学史素养？教师的数学史素养应如何提升？却是少有人问津。为此，本文首先探讨数学教师数学史素养的内涵，并在此基础上提出数学教师数学史素养的提升策略。

一、数学教师数学史素养的内涵

数学史素养是数学教师专业素养的重要组成部分，是数学教师全面了解数学、优化数学知识结构、提高数学教学能力的重要基础，也是搞好数学教学的重要保证。数学教师的数学史素养不同于数学史学家的数学史素养，它是为数学教学服务，并非纯粹研究数学史。该素养涉及三个要素：对数学史的认识、数学史知识、运用数学史教学的能力。本文主要论述中学数学教师的数学史素养。

数学教师对数学史的认识，是数学教师数学史素养的重要组成部分，它对教师教学中数学史的运用状况起到指挥和调控作用。例如，有教师认为，学生学习数学的目的是掌握数学知识，提高数学思维能力和解决实际问题的能力，而不是学习数学史。教学中掺入数学史只能侵占宝贵的课堂时间，增加学生学习负担。这样，他们在教学中将尽量少用或不用数学史；但是，如果教师认识到数学史对数学教育的重要作用：激发学生学习兴趣、深刻领会数学教育价值、指导教学实践（历史上数学家曾经遇到过的困难，课堂上，学生同样会遇到）等，他们设计教学方案时，就会有意识地思考数学史，教学中也主动运用数学史。香港大学的萧文强教授在总结教师教学中不愿用数学史的一些常见理由时，就指出数学教师对数学史的认识程度影响着数学教师学习和运用数学史的积极性 [2]。

数学史知识的广博程度，是衡量数学教师数学史素养的重要指标。若一位数学教师不知道祖冲之、刘徽为何许人，对阿基米德、牛顿、高斯闻所未闻，《几何原本》、《九章算术》不知为何物，很难说他具有较高的数学史素养。中学数学教师需要了解的数学史知识，主要指与教学内容相关的和“数学史选讲”中涉及的数学史知识。如数学概念及重要成果的产生背景、重要数学思想的诞生历程及著名数学家的感人故事及趣闻轶事等。

运用数学史教学的能力，是数学教师借助数学史提高教学效果所必需的技能。Fulvia Furinghetti 指出“不同作者对数学史作用得出的不同结论，并不是数学史自身作用的问题，而缘于不同数学教师对数学史的不同运用方式”。[3] 这显示了数学教师运用数学史教学能力的重要性。近几年，很多学者赞成“数学史融入数学教学”，其主要形式是教师结合数学史对教学内容重新设计和加工，制作适用于教学的“历史套装”，让数学史料“随风潜入夜，润物细无声”，学生于潜移默化之中便领悟到数学史上的数学思想、思维方式等。洪万生教授领导下的台湾 HPM 团队中，有很多教师开发了一些供课堂教学使用的“学习单”，并应用于教学实践 [4][5]。这些具体例证对提高教师运用数学史教学的能力

上值得借鉴。

二、数学教师数学史素养的水平划分

不难发现，不同教师的数学史素养存在较大差异。例如，对于勾股定理，有的教师仅把它看作一个抽象枯燥的数学结论；而有的教师更了解历史上勾股定理的来龙去脉和生动有趣的奇闻轶事；还有的教师不仅能把勾股定理置于古今东西方文化之中，还能把相关史料有机地融入数学教学，既提高了学习效果，又让学生感受到多元文化的魅力和学习数学的价值。对于不同水平的数学史素养，应该如何划分呢？根据 SOLO 理论 [6]，结合数学教师数学史素养的组成要素，我认为数学教师的数学史素养也可划分为与 SOLO 对应的五个水平。综合 Hans Niels Jahnke 诠释学理论与台湾苏意雯的相关研究，[7] 对这五个水平分别介绍如下：

第 0 水平 (前结构层次): 教师缺乏数学史意识，对数学史知识及数学史在数学教学中的作用毫无知晓，教学中从不利用数学史。此水平的教师可以说全无数学史素养。教师的教学主要借助数学知识逻辑关系（这里不排除做游戏、做实验等与数学史无关的手段）促进学生的数学认知，根本想不到运用数学史。（如图 1）

第 1 水平（单一结构层次）：教师已具备少量数学史知识 (图中用虚线表示)，但认为数学史与数学教学无关，在教学中不使用数学史。如图，教师虽对数学史有所了解，但仍是借助数学知识逻辑关系促进学生的数学认知，不触及数学史范畴。数学史对于数学教学来说，仍是“天外之物”。（如图 2）

第 2 水平（多点结构层次）：教师已具备局部的数学史知识。对数学史在教学中作用的认识还比较肤浅，仅停留于提高学生的学习兴趣的低层次上（图中用虚线表示。在某些章节教学时偶尔运用数学史，其方法或是直接提供史料文本，或是讲数学史故事、作数学史报告等，数学逻辑与数学史料未能有机整合，数学史的讲授与数学知识的教学可谓泾渭分明。由于不能较好地两者兼顾，经常是顾此失彼。（如图 3）

第 3 水平（关联结构层次）：教师拥有局部的数学史知识，对教学中数学史的作用已有较深刻认识，认识到数学史可以促进学生对数学知识的理解、有利于形成良好的人格和精神等（图中虚线加粗）。对于特定单元（如勾股定理等），了解较丰富的数学史知识，教学设计时能从数学知识逻辑与数学史料两个面向促进学生数学认知，能较好地把数学史融入数学教学（图中虚线加粗）。但由于教师数学史视野所限和经验不足，在教学中难免有所欠缺。（如图 4）

第 4 水平（抽象拓展层次）：教师拥有较广博的数学史知识，对数学史的作用已有深刻认识。HPM 教学不再局限于特定章节，整个教学都从数学知识逻辑、数学史料两个层面考虑促进学生的数学认知，对于课程内容与 HPM 的适切性已有全面了解，教学中能够高效地发挥数学史的作用。HPM 教学已成为一种教学观念，一种教学意识。如下图所示，此图与图 4 的不同之处在于左右两圈的扩大，表明教师实施 HPM 教学的章节不断拓展，对 HPM 教学的理解与实践能力也有很大提高。这一水平应是数学教师的数学史素养的理想层次，也是数学教师努力的目标。（如图 5）

若从动态的角度来看，上述五个水平可看做 HPM 视角下数学教师专业发展过程中的五个阶段。数学教师从数学史素养的 0 水平提升到第 4 水平，实际上是数学教师 HPM 概念形成的过程。在此过程中，教师的数学史知识不断增多，对数学史的认识逐步加深，教学中由想不到数学史，发展到尝试运用数学史，进而在任何教学单元都能进行有效的运用数学史，形成 HPM 教学观念。

三、数学教师数学史素养的提升策略

根据前文所述，数学教师的数学史素养是决定数学史能否有效融入数学教学的关键。通过对中学数学教师数学史素养的调查发现，当前中学数学教师的数学史素养大多停留在第 1 水平或第 2 水平。为探寻数学教师数学史素养的提升策略，我们开展了“基于教师专业发展的数学史与数学教学整合的研究”实验课题。结合课题实践，参照有关资料，笔者认为，要提高数学教师的数学史素养可从以下几个方面努力：

（一）利用课题带动或开展切实有效的专项培训

我们实施为期两年的“基于教师专业发展的数学史与数学教学整合的研究”实验课题后发现，参与教师的数学史素养都得到不同程度的提升。我们认为，开展课题研究是实现 HPM 概念形成的有效社会互动方式，是提升教师专业素养的重要途径。

继续教育研究显示，专项培训是提高教师专业水平的有效策略。要提升数学教师数学史素养，培训中应要切实好以下几项工作：第一，为教师推荐部分实用的、经典的数学史学习材料，如汪晓勤与韩祥临合写的《中学数学中的数学史》、克莱因撰写的《古今数学思想》等。同时通过听专家报告，撰写论文等形式提高教师对数学史的认识，增加教师的数学史知识；第二，培训工作应与教学实践相结合。培训结束前，每人至少准备一节“数学史融入数学教学”的观摩课，或课堂录像资料，以此促使教师提高运用数学史教学的能力；第三，制定科学、严格的考核制度，把数学史素养的三个要素均作为考核内容，走出只注重数学史知识的误区。把考核结果与教师利益挂钩，真正起到培训的效果。

（二）教师自身应提高认识，加强学习，自觉提升数学史素养

虽然外在环境是影响教师成长和发展的重要因素, 但就教师个体而言, 发展和成长的关键还在于自身的努力。

由前面分析可知，教师数学史素养的提升从心理学上讲，一般要经历堆积、复合思维、抽象化三个阶段。而每一阶段的跨越都需要教师艰辛的智力操作。数学史素养的三要素紧密相连，相互影响，教师必须统筹兼顾，才有可能实现数学史素养的整体提升。因此，教师平时要多读一些反映数学史价值、介绍数学史知识的书籍、期刊，经常浏览数学史网站，积极参加有关的报告会、讨论会，撰写学习心得。同时，在设计教学方案时，既考虑数学知识逻辑、学生认知水平，又要查阅相关数学史料，挖掘与教学有关的历史素材，提炼有价值的数学思想方法，制作出关照三维度的“历史套装”，实践于数学课堂。所有这些，没有教师的高度关切和全身心的投入，提升教师的数学史素养只能是天方夜谭

（三）创建 **HPM** 教学资源库，为教师运用数学史教学提供参考

让数学教师完整地经历数学史融入数学教学的设计、实施全过程，对数学教师数学史素养的提升很有益处。但是，由于广大数学教师毕竟不可能像数学史学家那样对数学史知识了如指掌，再加上平时工作繁忙，教学压力较重，有时试图运用数学史，却由于数学史料搜寻困难，设计教学需要花费很多时间，便望而却步；部分教师对 HPM 概念了解甚少，不知道在教学中如何运用数学史，对数学史只能敬而远之；也有教师根据自己的设想在教学中运用数学史后，发现数学史对数学教学没有作用，于是决定不再涉猎数学史。在课题实施中，与中学教师座谈时，很多教师反映“如果能提供一些数学史融入数学教学的范例，让我们有个参照就好啦”。可以看出，从事研究 HPM 的专家学者当务之急就

是设计出一线教师在教学中切实可行的数学史融入数学教学的范例，或者对善于运用数学史的教师的教学设计进行加工提炼，创建对中学教师真正有用的 HPM 教学资源库，既可吸引更多的教师关注 HPM，又能为尝试运用数学史的教师提供参考，这必将扩大 HPM 的影响，推动数学史融入数学教学的开展。

（四）提倡反思性教学，发挥群体优势，提高教师运用数学史的能力

研究发现，教学反思是推动教师成长的核心因素，也是促进教师专业成长的必由之路。例如，教师运用数学史教学后，发现效果并不理想，可反思教学设计理念、教学过程，探寻改进方案；教师对教学效果满意时，可反思成功的原因，提炼成功的经验。经过不断的反思，教师可做到扬长避短，逐步提高运用数学史教学的能力。实验课题也证实了反思性教学的重要作用。

因为每位教师都有自己长期形成的教学信念、教学风格，仅凭借自身的努力可能很难超越自我，教师数学史素养的提升更需要群体的智慧。学校要充分利用教研组这一平台，开展“我对 HPM 的理解”、“数学史在数学教学中的定位与作用”等专题讨论，集思广益、寻求共识；开展“数学史融入数学教学”讲课比赛，让每位教师经历教学过程；请优秀执教者上示范课，提供教学样板。它山之石，可以攻玉。相信通过采取有效措施，教师的数学史素养一定会得到提升。

（五）改革高师数学史教育模式

当前，多数高师院校把数学史作为选修课，普遍重视不够、要求不严。有研究者对即将走向工作岗位的高师毕业生做过调查 [8]，，结果显示，有相当一部分学生对数学史知识缺少基本的了解。因此，要提高未来数学教师的数学史素养，就必须改革目前的数学史教育模式。

首先，明确课程定位。高师院校数学教育专业的培养目标是为中学输送合格的数学教师。数学史对数学教学必不可少。因此，数学史应定位于必修课，作为学生必须具备的一种素养，在课程考评上要和专业基础课放到同等重要的位置。

其次，改革教学方式。教学中不应把数学史当成真正的“历史”来讲授，要把它作为一种研究活动过程、方法、技术和能力，让学生在学“史”中促进自己专业素养的提高。为此，建议大三、大四都应开设数学史课程。教学过程可分为两个阶段：第一阶段，安排在大学三年级。让学生获得中学数学教学必备的数学史知识，了解历史上数学思想方法的演变，体会数学与数学文化的联系；第二阶段，安排在大学四年级，请 HPM 教学经验丰富的教师做指导，通过开辟“数学教育实验室”开展教学实验，重点培养学生运用数学史教学的能力。

最后，加强专业师资队伍建设。数学史是一门博大精深的学科。任课教师需要有较深厚的数学文化底蕴，进行过系统的数学史学习，对数学的概貌有正确的理解和认识。而目前绝大部分高校缺少高水平的数学史教师，致使开设的课程没有发挥应有的作用。笔者认为，要提高数学史教育效果必须加强专业师资队伍建设，引进数学史方面的专业人才，为培养学生的数学史素养提供保障。

总之，数学史素养，是数学教师专业素养的重要组成部分，也是数学教师专业化的重要体现，更是能否实现数学史教育价值的关键因素。面对当前中学数学教师数学史素养普遍不高的现实，重视数学教师数学史素养理论及提升策略的研究，对适应课程改革，提高教学质量，更好地落实 HPM 的研究成果，推进数学教师专业发展都有着现实和深远的意义。

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文中的图表

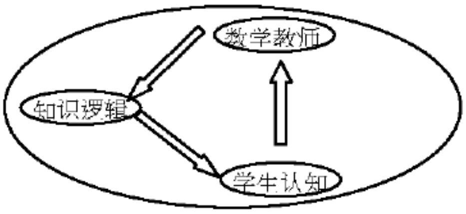

图 1 数学教师数学史素养第 0 水平

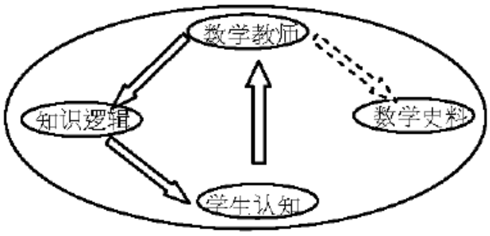

图 2 数学教师数学史素养第 1 水平

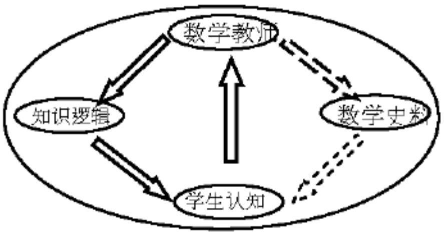

图 3 数学教师数学史素养第 2 水平

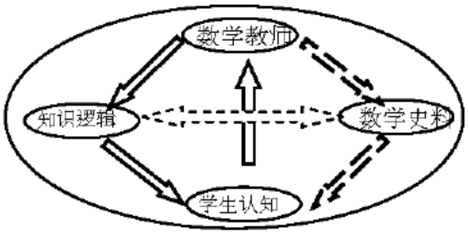

图 4 数学教师数学史素养第 3 水平

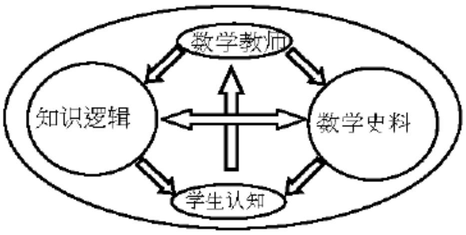

图 5 数学教师数学史素养第 4 水平

MATHEMATICS EDUCATION AND TEACHING PRACTICE TO BRING UP HISTORY OF MATHEMATICS CULTURE RICHLY

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ABSTRACT

本稿は，数学文化史における教育内容の一つとして，「日時計」について論究した．日時計は，赤道型日時計，地面水平型日時計，鉛直型日時計に分類されるが，赤道型日時計の製作に取り組ませてから地面水平型日時計の作製へと移行する授業を実施した．日時計を教材とした学習活動は，立体幾何，三角関数の実用，三角形の合同の応用，作図のまとめとなり，文化史を指導する上で有効であることを確認することができた．

1　はじめに

横地清によって，天文学とは別に，立体幾何の視点から，太陽の運行と時刻を指導する教材として，日時計の種類や原理が整理され数学化された．日時計は，算数や数学だけではなく，総合学習の教材として定着してきた．

筆者は，昨年の夏に中学校の数学科の教師の研修会で，実際に日時計を自作させる指導を，図1の様に中学校の教師たちに対して実施した．そこでは，論証幾何の導入としての定義，公理，定理によって証明の体系を整理し，日時計の製作原理に生かす指導のありかたについても指導した．また，文化史の観点から，今回の大震災のあった日本国宮城県塩竈市にある塩竈神社博物館に常設されている林子平考案とされている図2の地面水平型日時計と，日本国長崎県出島のオランダ商館にある日時計との関係について，小学校，中学校，高等学校の数学科の先生方に研究会の場で講演もした．これらのことから，学校教員が，数学文化史を理解する教材として，日時計に関する話題が有効であることが示唆されたので，その報告をする．

2　日時計と文化史

ここで，文化史として，日時計を選んだのかについて述べる．それには，数学文化史をどの様にとらえるのかということについての説明が必要である．そこで，本稿においては，数学文化を数学教育としての側面から捉え，教育的意義として，図形教育，モデル化，文化史，他教科との関連の4つの視点から説明する．

図１　中学校の数学の教師の研修会の様子

図２　林子平考案とされる地面水平型日時計

はじめに，日本においては，図形教育に対する体系が，必ずしも整っているとは言い難い状況にあることは，だれもが認めるところである．特に，立体幾何といった空間図形に関する指導内容は，外国と比較しても整っているとは言いにくい．これを日時計は，学習者にとって，自然な形で解決してくれる教材であると捉えている．具体的には，日時計の教材においては，図形に関する定理を多く利用する．例えば，平行線の同位角，円の半径と接線の垂直，三垂線の定理，球面上の位置関係，球と接平面，平面同士の平行，二面角等がある．これは，従来の立体幾何とは異なった視点から空間図形を扱えるような教材である．高校生を対象とした場合は，太陽の運行を空間の解析幾何として証明可能でもあり，一定レベルの数学の内容を扱うことも可能である．また，実際に時計を製作することによって，数学の実用性を学習者に理解させることが可能である．上で述べた様な視点からは，図形の論証と数学の実用性を体験させることが可能である．

次に，モデル化と言った視点から述べる．最も基本となる赤道型日時計の製作を終了すると，それを基に水平型日時計の製作へと進む．この時，時刻盤の時刻線の引き方が異なるといった課題が生じる．これは，水平型日時計から鉛直型日時計を製作する時にも生じる．この様にして，

数学モデルを発展させることができる．このことは，次の節で述べる林子平考案とされる塩竈神社博物館にある日時計と平戸の日時計との比較解析において，重要な役割を果たしている．

また，林子平考案とされる日時計を解析することにより，それが存在する塩竈ではなく，平戸の緯度に相当する位置に存在するのが妥当であるとの結論を導く．日本の歴史に触れながら，歴史上の出来事について数学を利用して，科学的に解明することの重要性について，実感をもって理解させることが可能である．また，例えば，赤道型日時計についても，緯度の異なる国においては，時刻盤と地面との二面角の大きさが違う．この様なことからも，日時計は，歴史や文化財へと結びつくと同時に，国際的視野を育成する数学文化史としての側面を持つ重要な教材になり得る．

さらに，従来日本において，日時計は，理科の教材として扱われてきた．しかし，日時計によって，数学と理科と言った異なる教科が，十分に連携可能である．最も，その前提としては，指導する教師の十分な教材研究が必要である．

3 世界の日時計と日本の日時計

日時計の基本的な原理は，日影をつくるノーモンと呼ばれる影とり棒とその影が落ちる時刻線で生成される時刻盤で構成される．時刻盤の地面に対する設置の仕方で，赤道型日時計，地面水平型日時計，鉛直型日時計の 3 つに大きく分類される．この日時計も地域や国によって，設置される日時計の種類に特徴がある．中国や韓国においては，赤道型日時計がみられる．日本においては，各地に図の様な地面水平型日時計がある．ヨーロッパにおいては，比較的緯度がたかいために，必然的に図 3 や図 4 の様な鉛直型日時計が設置されている．

今回の大震災のあった日本国宮城県塩竈市にある塩竈神社博物館に江戸時代の地元伊達藩の英雄で日本国内でも非常に有名な学者である林子平考案とされる日時計が常設されて，展示されている．その日時計は，ノーモンと呼ばれる影とり棒は，オリジナルかどうかの確実性は，はっきりとしてはいないが，時刻盤がジナルデアルであることは，間違いの無い事実としてはっきりとしている．実は，この文字盤に奇妙な事実がある．地面水平型日時計のノーモンの時刻盤に対する設置角度は，その日時計が置かれる緯度で決定される．つまり，ノーモンと文字盤で決定される角度を測定すれば，その日時計が緯度何度の地点に設置されるべき日時計であるかが分かるのである．

したがって，林子平考案とされる日時計が，緯度何度の地点に置かれる日時計であるかは，そのノーモンと時刻盤で決定される角度を測定すればわかるのであるが，上で述べた通り，ノーモンがオリジナルであるかどうかがはっきりしないために，それを測定しても確実性がない．しかし，幸いなことに，時刻盤は，オリジナルであることがはっきりとしているので，時刻盤から，その日時計が設置される緯度を計算によって求めることは可能である．　実際に，時刻盤の拓本を基に，7 時，8 時，9 時，10 時，11 時の各時刻における緯度の値を計算するとそれぞれ次の通りになる．33.7°，33.1°，32.4°，34.3°，33.8°．この算術平均は，33.4° となるが，塩竈神社博物館のある地域の緯度は，38.3° となっていて，明らかに誤差の範囲を超えている．具体的には，長崎県平戸付近の緯度に相当する．

図３　ニュールンベルグの日時計のある教会

図４　東側に設置されている鉛直型日時計

4 日時計の原理

4.1 立体幾何の視点からの時刻盤の作製原理

地面水平型日時計の数学的構造は，図 5 の通りである．図において，☆は，ノーモンを向けるべき北極星を表している．文字盤上における任意の時間におけるノーモンの影は，線分OR 上にある．したがって，同じ時刻を示す地面水平型日時計の時刻線は，線分PR と一致する．
そこで，PQ = 1 としても一般性を失わない．
いま，PQ = 1，緯度を
A，∠QOR = T，∠QPR = X とおく．
直角三角形において，POQ において，OQ = sin A．
直角三角形において，QOR において，QR = OQ · tan T = sin A · tan T，

$$\tan X = QR \text{ である．}$$

したがって，$\tan X = \sin A \cdot \tan T$ となる．
よって，$A = \sin^{-1}\left(\dfrac{\tan X}{\tan A}\right)$
これが，前章において緯度を求める時に利用した式である．

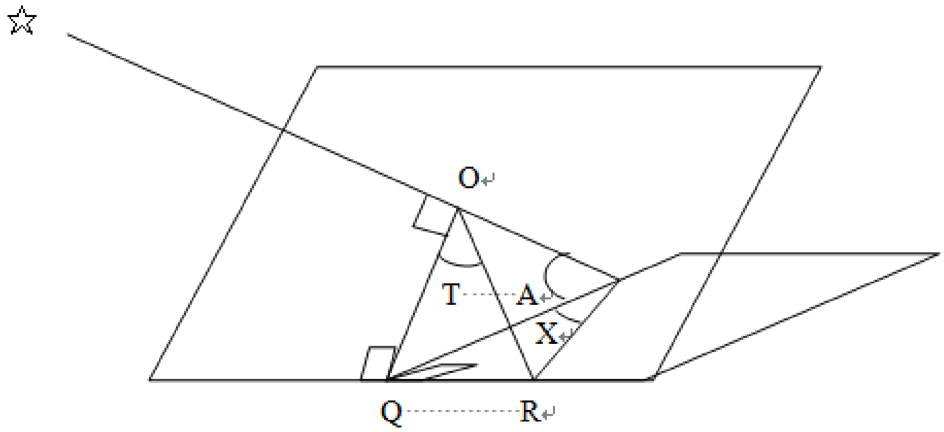

図 5　緯度と時刻線の関係

4.2 作図の視点からの時刻盤の作製原理

図 6 が，赤道型日時計と水平型日時計の時刻線の関係である．
図 7 を参考に，作図によって点O の位置を求める．次に，斜辺QP と一辺OQ の長さが分かっている直角三角形を作図したうえで，∠OPQ を計測して大きさを求めることになる．この計測値が，その日時計が置かれるべき緯度となる．
∠OQP =(90° −緯度)，∠OPQ = 緯度

尾道市の緯度は，北緯約 34° なので，ここで利用可能な日時計を製作する．生徒には，完成してある赤道型日時計を基にして，赤道型では 15° おきになっていた時刻線が，地面水平型日時計においては，どの様な線になるか考えさせると良いでしょう．ただし，その際に，日時計のノーモン線は水平面まで伸びるものとして考えさせる．
図 9 において，ノーモンAP 影は，赤道型日時計の上部では，△ARO を含む面となる．

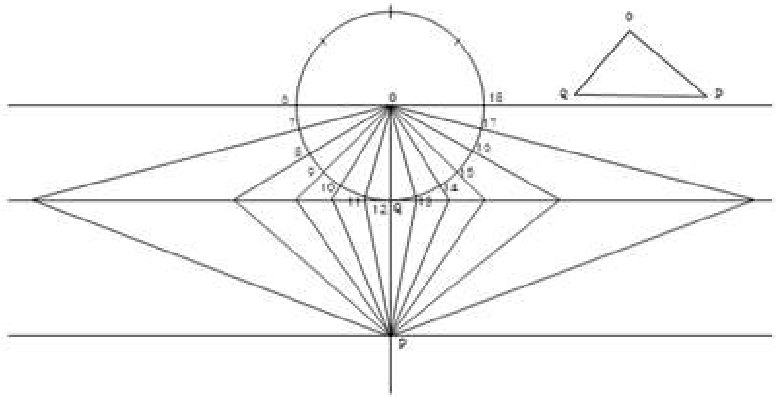

図 6　赤道型日時計と地面水平型日時計の時刻線の関係

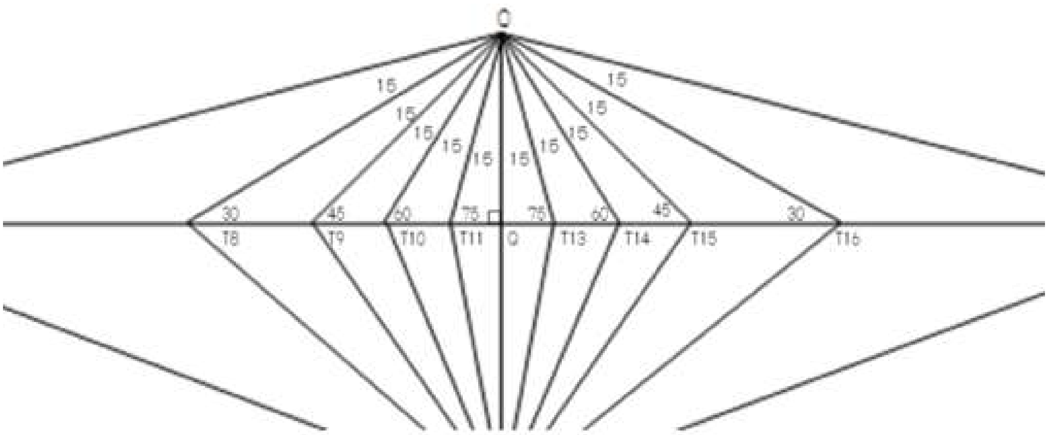

図 7　時刻線と点 O の位置関係

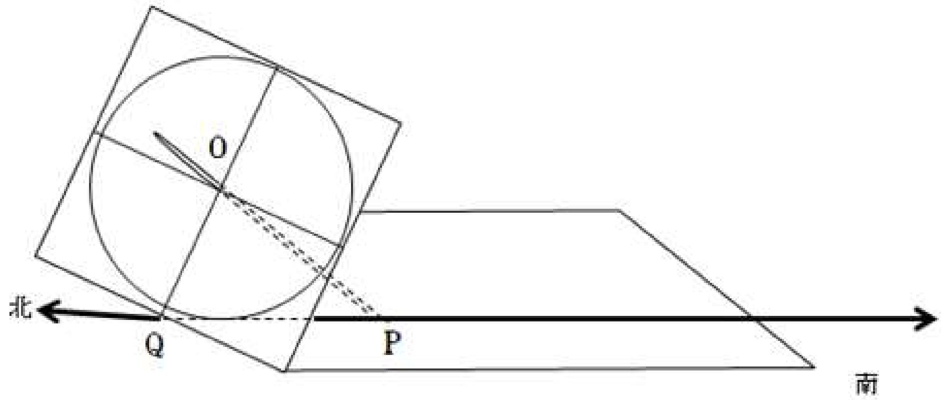

図 8

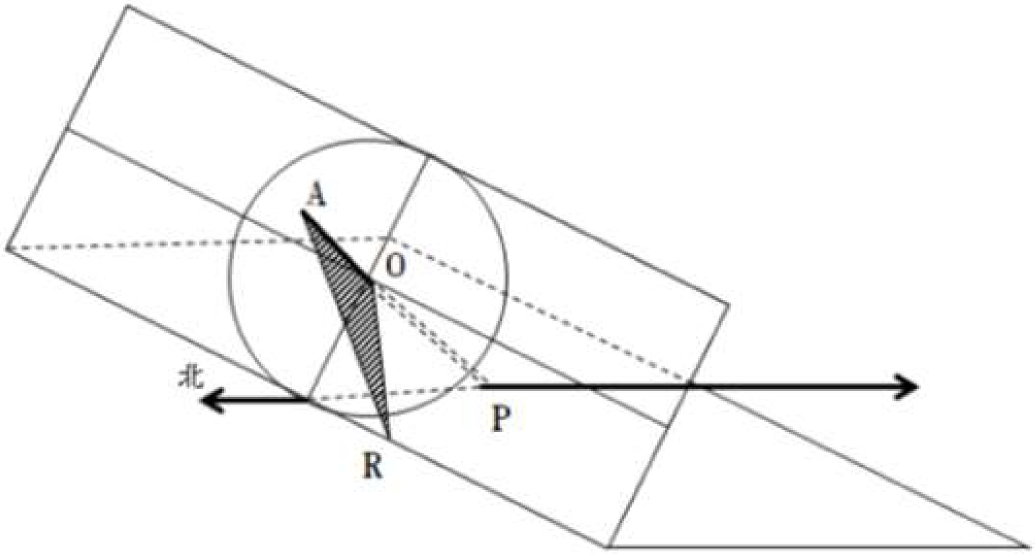

図 9

水平面に対して，ノーモンAP の影は，△ARP を含む面と水平面の交線となるので，

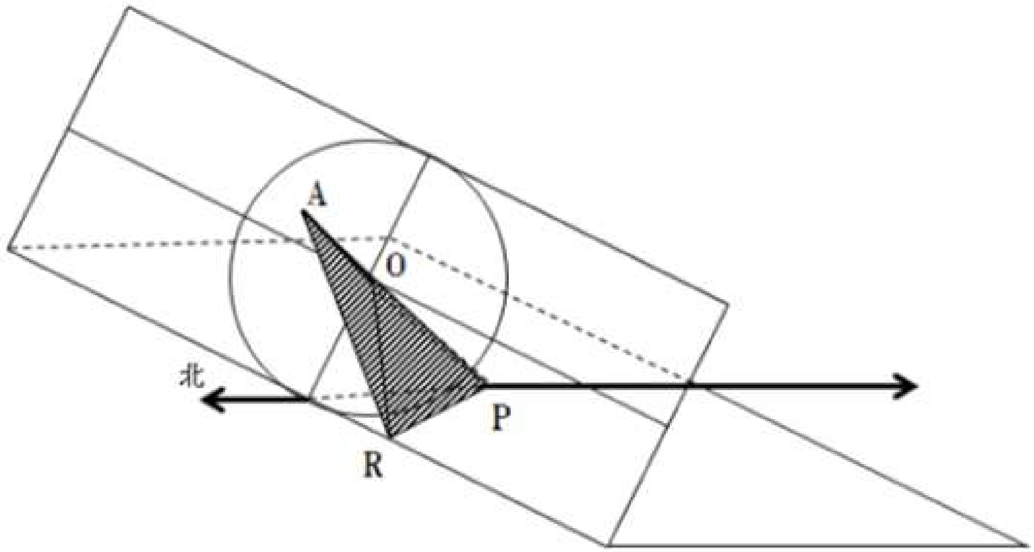

図 10

そこで，円を 15° ずつ 24 等分した赤道型日時計の文字盤上の時刻線を，赤道面と水平面の交線まで延長して交点を作った．図 6 の点P の位置を作図から決定し，先程の交点と点P とを結ぶ．点P の位置は，まず，OQ の長さを決めて，∠Q =90° －緯度，∠O =90° から△OQP を作図する．この図のPQ の長さを測るとて点P の位置が決定できる．

下の図 12 の様に，水平面を長方形で囲んで，時刻を記入すれば，図 13 のように地面水平型日時計の時刻盤ができあがる．

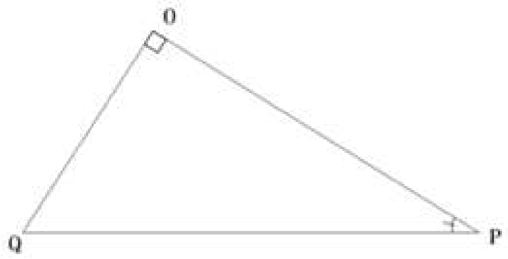
O
Q
P

図 11

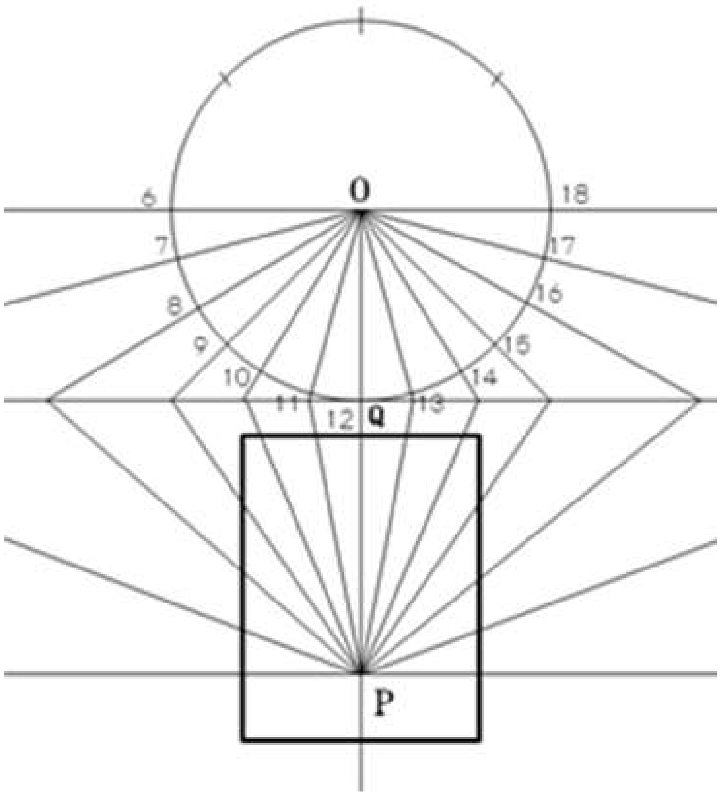
O
6
18
7
17
8
16
9
15
10
14
11
12
Q
13
P

図 12

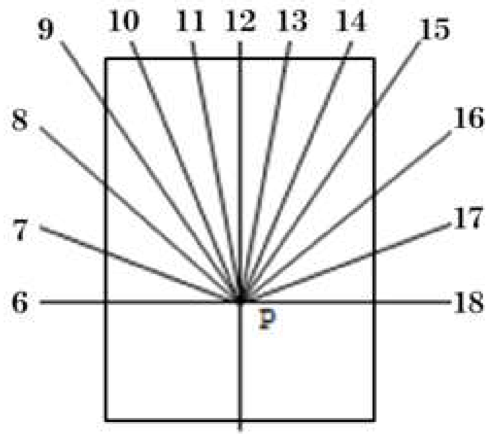
9
10
11
12
13
14
15
8
16
7
17
6
18
P

図 13

5　まとめ

前章までに述べてきたように，実際に中学校の数学科の教師を対象とした研修会を実施した．研修会後に，この教材に関して評価してもらった．その結果は，次の通りである．

教職経験 2 年目の男性の先生は，「赤道型日時計については，選択授業で活用したいと思います．地面水平型日時計については，ノーモンをいかにして水平に立てるか角度をつけて立てるかに注目するとさらにおもしろいと思います．」教職経験 2 年目の女性の先生は，「実際に授業で作ってみたいと思った．生徒も楽しみながら作れそうだと思った．地面水平型日時計は，なぜ，文字盤についてもう少し深く学習したかった．」教職経験 29 年目の男性の先生は，「実際に授業で作成させ，生徒に外に出て確認させたいと思った．」教職経験 26 年目の男性の先生は，「生徒は，興味を持つと思います．平行線をきちんと引く，接戦を引くなど作図の練習になると思いました．」教職経験 24 年目の男性の先生は，「中学生には，地面水平型日時計の方が，赤道型日時計よりも作りやすそうだと思った．実際に，地面水平型日時計を測らせてやっていけると思う」．研修会に参加された他の先生方もほぼ同様な意見であった．以上のことから，日時計は中学校において十分に有効な授業の教材になりうると考える．

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SUNDIAL AND MATHEMATICS
Analysis of oldest horizontal sundials in Japan by mathematics

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ABSTRACT

本稿では，人類最古の科学装置ともいわれる日時計，その中でも日本において作製された最古の屋外型日時計の数学的正確性を検証し，さらにその結果から当時の日本の数学の発展状況を探 た。検証の結果，作製された日時計は，的確な数学の原理を理解せずに作製していたことが示された。このことから，当時の日本では，いわゆるその背景となるユ クリ ド幾何学などの西洋数学を体系的に受容できていなか たことも示唆された。

1 はじめに

日時計は数学理論をバ クにして作製された人類最古の科学的装置であることはよく知られている。その日時計は，いろいろな種類が存在している。例えば，韓国で有名な仰釜型日時計 (Fig.1)，また中国で有名な赤道型日時計 (Fig.2)，イタリアやフランスの西洋で有名な垂直型日時計 (Fig.3)，日本で有名な地面水平型日時計 (Fig.4) などがある。

この日時計に関して，日本では現在，その内容を学校の算数教科書では特に扱 てはいないが，オリジナルに教育実践がなされ，教育効果があることが示唆されている。

さて，この日時計であるが，日本での開発は意外に遅く，日本最古の屋外型日時計といわれるのが出島和蘭商館跡にある日時計（以下，出島の日時計），日本人独自に開発した最古の屋外型日時計は塩竈神社にある日時計（以下，塩竈の日時計）とされるが，いずれも地面水平型日時計で，18 世紀後半の作製である。開発が遅れた原因は，日本における数学の発展とも大きく関係があると考えられる。

そこで本稿では，日本最古といわれる日時計に着目し，その数学的正確性の分析を行うこととする。さらに，その分析結果から，日時計作製時の日本における数学の発展状況を探ろうと考える。

2 日時計の数学的原理

以下で，原理が簡単な赤道型日時計と分析対象とする地面水平型日時計の数学的原理を関連づけてその数学原理を考察する。なお対象とするのは，「ノ モンのなす角」と「時刻目盛」である。

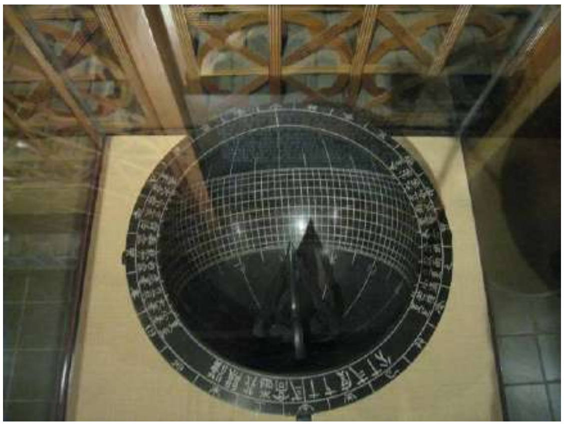

Fig.1 : 新羅歴史科学館 (韓国　慶州)

Fig.2 : 国子監 (中国　北京)

Fig.3 : サンタ　マリア　ノヴ　ラ教会（イタリア　フレンツ　）

Fig.4 : 四條畷東小学校 (日本　大阪)

2.1　赤道型日時計の原理

赤道型日時計は，地軸と平行な位置関係にあるノ　モン (影針) に対する太陽の影を，赤道面と平行に設置している文字盤に投影して時刻を表す時計である。なお，時刻目盛は文字盤の両面にかかれ，ノ　モンは文字盤の両面に垂直に設置される。すなわち文字盤を貫く独特な形となる（Fig.2）。

（1) ノ　モンのなす角について

以下，緯度，日時計（文字盤，ノ　モン）の関係を述べる (詳しくは渡邉 (2000) を参照されたい)。緯度$= \angle L$，日時計のなす角 (文字盤と地面のなす二面角)$= \angle T$，ノ　モンのなす角$= \angle\theta$ とすると，$\angle L = \angle\theta$ である。さらに，$\angle T = \angle R - \angle\theta$ となる。

（2）文字盤の時刻目盛について

赤道型日時計の場合，太陽は文字盤と平行に 1 日に 1 周する。すなわち，1 時間で中心角が$15°(360°\div 24 = 15°)$ で円周を動くこととなる。したが　て，時刻目盛は，円を$15°$ ずつ分割したもので，時刻は右回りに刻まれる。ただし，設置する場所の緯度により，正午 (12 時) 線（基準線) を調整する必要がある。日本においては，経度が135 度であれば日本標準時の経度のため，南中が正午とみなせる。したが　て，それ以外の場合は，基準線がずれることになる。

2.2 地面水平型日時計の原理

一方，地面水平型日時計は文字盤を接平面上に置き（すなわち，文字盤が地面上にある），ノ　モンは地軸と平行になるようにする (Fig.4)。

（1）ノ　モンについて

以下，緯度と日時計（文字盤，ノ　モン）の関係を述べる。ノ　モンのなす角を$\angle\theta$とする。緯度$= \angle$ Ｌとすれば，$\angle\theta = \angle$ Ｌとなる。

（2）文字盤の時刻目盛について次に地面水平型日時計の文字盤の時刻目盛 (Fig.5) について述べる。赤道型日時計とは，Fig.6 のような関係が成り立つ。赤道型日時計 (平面 α) における 12 時の時刻線から各時刻線の角度を ω，地面水平型日時計 (平面 β) における 12 時の時刻線から各時刻線の角度をϕとすれば，ノ　モンのなす角 (緯度)θについて，次の関係式が成り立つ。

$$\Theta = sin^{-1}(\tan\phi/tan\omega)\cdots \text{関係式 (1)}$$

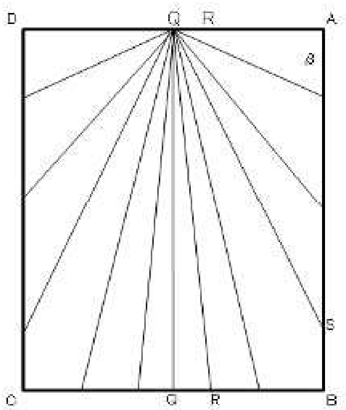

Fig. 5 : 地面水平型日時計の文字盤 (平面β)

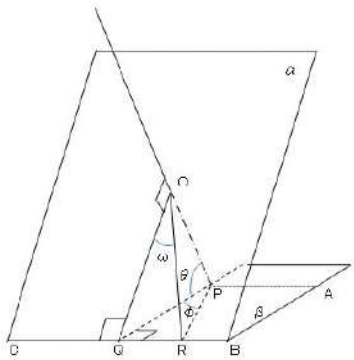

Fig. 6 : 赤道型日時計 (平面α) と地面水平型日時計 (平面 ββ の関係

3 日時計の教育実践

上述のような数学原理をもつ日時計であるが，先述したように日時計に関しては，教科書では扱われていないが，オリジナルにたびたび実践が行われ，その数学教育的な価値が認められている。以下，筆者の行　た主な教育実践を簡単に紹介する。

3.1 小学校における実践

（**1**）赤道型日時計作製の教育実践の実際

赤道型日時計の作製の実践を 1999 年に，大阪府四條畷市立四條畷東小学校 5 年生 (27 名) を対象に，25 授業時間 (1 授業時間は 45 分) 行　た。

主な教育内容は以下のようである。

- リンゴ地球儀による緯度　経度の理解 (Fig.7)
- 球面上の 2 地点間の距離の理解

- 日の出，日の入り，時差の理解
- 日時計の原理の理解
- 日時計の作製　設置 (Fig.8)

Fig. 7 : リンゴ地球儀で緯度　経度を表す

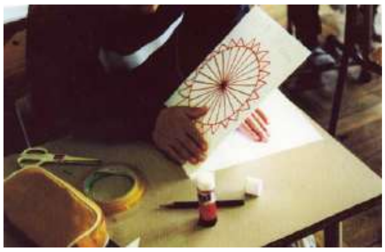

Fig. 8 : 赤道型日時計の作製

これらの実践から，小学校 5 年生で十分に赤道型日時計の原理を理解し，作製できることが示唆された。

3.2　中学校における実践

（**1**）地面水平型日時計作製の教育実践の実際 地面水平型日時計の作製の実践を 2000 年に，大阪府四條畷市立四條畷南中学校 1 年生 (5 名) を対象に，8 授業時間 (1 授業時間は 60 分) 行　た。主な教育内容は以下のようである。

- 赤道型日時計の原理の学習
- 地面水平型日時計の原理の学習
- 地面水平型日時計の作製 (Fig.9,10)
- 地面水平型日時計の設置

Fig. 9 : 文字盤の作製

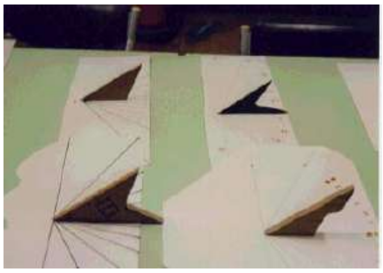

Fig. 10 : 作製した地面水平型日時計

(2）赤道型日時計作製　設置教育実践の実際 赤道型日時計の作製　設置の教育実践を 2006 年に，京都府私立立命館宇治中学校 2 年生 (33 名) を対象に 4 授業時間 (1 授業時間は 50 分) 行　た。

主な教育内容は以下のようである。

- 赤道型日時計の原理の理解
- 赤道型日時計の作製

- 赤道型日時計の設置
- 赤道型日時計の移動 (Fig.11,12)

なお，この実践は，日時計の任意の設置ができることを目標とし，そのために必要な南北線の平行移動の方法を，平行四辺形の性質の学習から理解するものである。

これらの実践から，中学校 2 年生で，南北線の移動による日時計の任意の設置が理解できることが示唆された。

Fig. 11 : 移動の方法の説明

Fig. 12 : 南北線の移動

4 日本最古の日時計の分析

以下，日本最古の屋外型日時計といわれる，出島と塩竈の日時計の正確性を分析することとする。

4.1 出島の日時計について

屋外に据え付けられた日時計として，日本で最古のものは，出島和蘭商館跡にあるものだといわれている。この日時計を作らせたのは 1766.11.1(〜1767.10.20) に出島に赴任したオランダ商館長ヘルマン　クリスチ　ン　カステンス (Kastens， Herman Christian) である。したが　て，この日時計は日本最古とは言われるが，日本人が独自に作製したものとは考えにくい。出島の日時計は，現在はそのレプリカが「出島和蘭商館跡」の庭園に展示され (Fig.13)，実物は倉庫に保管されている。今回はその倉庫に保管されている実物の日時計を，長崎市文化観光部出島復元整備室室長の山下氏をはじめ，山口氏，杉本氏他皆様のご協力により，実地調査することができた (Fig.14)。なお，日時計はおよそ縦 83cm× 横 68cm× 厚さ 4.5cm の大きさであり，大人 3〜4 人でないと動かせない重さがある。また，以前は個人所有で屋外に展示されていたため，すでに文字盤の表面の目盛りや時刻線はかなり劣化している。

4.2 出島 (長崎県長崎市) の日時計，塩竈（宮城県塩竈市）の日時計

（1）出島の日時計について

屋外に据え付けられた日時計として，日本で最古のものは，出島和蘭商館跡にあるものだといわれている。この日時計を作らせたのは 1766.11.1(1767.10.20) に出島に赴任したオランダ商館長ヘルマン　クリスチ　ン　カステンス (Kastens， Herman Christian) である。したが　て，この日時計は日本最古とは言われるが，日本人が独自に作製したものとは考えにくい。出島の日時計は，現在はそ

のレプリカが「出島和蘭商館跡」の庭園に展示され (Fig.13)，実物は倉庫に保管されている。今回はその倉庫に保管されている実物の日時計を，長崎市文化観光部出島復元整備室室長の山下氏をはじめ，山口氏，杉本氏他皆様のご協力により，実地調査することができた (Fig.14)。

なお，日時計はおよそ縦 83cm× 横 68cm× 厚さ 4.5cm の大きさであり，大人 3〜4 人でないと動かせない重さがある。また，以前は個人所有で屋外に展示されていたため，すでに文字盤の表面の目盛りや時刻線はかなり劣化している。

（2）塩竈の日時計について

一方，日本人が独自に作製した最古のものは，林子平 (1738–1793) によるものとされることが多い（出島のものを複写したものといわれている）。なお現在，林子平製作の塩竈の日時計は，塩竈神社博物館 (宮城県塩竈市) に保存されている (Fig.15)。

Fig.13 : 出島にある日時計のレプリカ

Fig.14 : 日時計の実物

Fig.15 : 塩竈神社博物館の日時計の目盛

4.3 日時計の正確性

（1）文字盤の時刻目盛について

出島の日時計について，オリジナルの日時計の文字盤の目盛 (Fig.16) を計測した数値から緯度 θ を逆算し，その正確性の検証を行う。

一方，塩竈の日時計については，オリジナルの日時計の文字盤の拓本の目盛 (Fig.17) からその正確性の検証を行う。

θ を算出する方法としては，次の 2 つの方法を採用する (表中の記号は Fig.5，6 に対応)。

Fig. 16 : 出島の日時計の文字盤

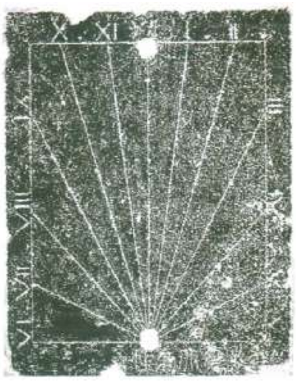

Fig. 17 : 塩竈神社博物館の日時計の文字盤の拓本

［方法 1］12 時の時刻線からの偏角 φ の大きさを実際に分度器で測定し前述の関係式 (1) を利用

［方法 2］文字盤にある時刻線等の長さの関係から逆正接関数を利用して φ の大きさを算出した上で前述の関係式 (1) を利用

分析の結果，次のことが明らかとな　た。

出島の日時計について，各時刻線から算出した緯度 θ の平均値をとると，方法 1 では，32.9 度，方法 2 では 32.3 度となる (Table1，2)。出島の緯度が 32.4 度であることを考えると，かなり正確に（正確な理論に基づいて）作製されていることがわかる。

一方，同様に塩竈の日時計を，方法 1，2 で θ を算出すると，平均値はそれぞれ，34.7 度，34.4 度となる (Table3，4)。塩竈市の緯度が 38.2 度であることを考えると，出島の日時計を多少補正した可能性はあるものの，正確ではない（正確な理論で作製されたものではない）ことは確かである。

Table 1 : ϕ を分度器で実測 (出島)

時刻線	7時	8時	9時	10時	11時	12時	13時	14時	15時	16時	17時
ω	75	60	45	30	15	0	15	30	45	60	75
ϕ（実測）	63.5	43.8	28.5	17	8	0	8.1	17.1	29	44.5	64
θ：緯度	32.51	33.6	32.9	32	31.6		32.1	32.2	33.66	34.6	33.3
θ（平均値）	32.85										

Table 2 : ϕ を逆正接関数の利用により算出 (出島)

時刻線	7時	8時	9時	10時	11時	12時	13時	14時	15時	16時	17時
ω	75	60	45	30	15	0	15	30	45	60	75
縦(AB,CD)	72.5	72.5	72.5	72.5	72.5		72.5	72.5	72.5	72.5	72.5
横(AP,PD)	29.6	29.6	29.6	29.6	29.6		27.5	27.5	27.5	27.5	27.5
SD	14.8	31.5	54.5						51.8	29.8	13.8
QR				22.2	10.3		10.4	22.3			
ϕ	63.43	43.2	28.5	17	8.09		8.16	17.1	27.96	42.7	63.4
θ :緯度	32.4	32.9	32.9	32	32		32.4	32.2	32.07	32.2	32.3
θ (平均値)	32.33										

Table 3 : ϕ を分度器で実測 (塩竈)

時刻線	7時	8時	9時	10時	11時	12時	13時	14時	15時	16時	17時
ω	75	60	45	30	15	0	15	30	45	60	75
ϕ（実測）	65	45	30	18	8.5	0	8.5	18	29.6	44.9	65
θ：緯度	35.07	35.3	35.3	34.2	33.9		33.9	34.2	34.62	35.1	35.1
θ（平均値）	34.67										

Table 4 : ϕ を逆正接関数の利用により算出 (塩竈)

時刻線	7時	8時	9時	10時	11時	12時	13時	14時	15時	16時	17時
ω	75	60	45	30	15	0	15	30	45	60	75
縦(AB,CD)	14	14	14	14	14		14	14	14	14	14
横(AP,PD)	6.1	6.1	6.1	6.1	6.1		6	6	6	6	6
SD	2.7	6.1	10.8						10.9	6.2	2.8
QR				4.6	2.1		2	4.5			
ϕ	66.12	45	29.5	18.2	8.53		8.13	17.8	28.83	44.1	65
θ :緯度	37.26	35.3	34.4	34.7	34		32.2	33.8	33.4	34	35
θ (平均値)	34.41										

5 なぜ日時計が正確に作製できなか たのか

ここで注目すべきは，これら 2 つの日時計が同時期に作製されたにもかかわらず，出島の日時計は正確で，塩竈の日時計は正確ではなか た点である。このことは，作製者の数学の力量に大きく関係していると考えられる。出島の日時計は，オランダ商館長（オランダ人）が作製させていることから，作製したのはその関係者であり，すでにそれを作製できる数学に明るか たのは間違いない。したが て，正確な日時計が作製できたと考える。一方，日本人では作製できなか たことを考えると，この時期，作製できるだけの数学が十分に広ま ていなか た状況にあ たと考えられる。

実際，この当時，日本は鎖国 (1639–1854) をしており，重要な学問などは，出島を通して，オランダや中国から輸入していたものの，十分に広まるまでの土壌はなか たと考えられる。例えば，日時計を作る際には幾何学（とくにユ クリ ド幾何学）が必要となるのはいうまでもないが，中国では 1600 年頃にイエズス会の宣教師マテオ リ チによ てもたらされ，1607 年頃には徐光啓によ て (前半部分が) 翻訳 出版されたが，日本では同時期に来日したイエズス会の宣教師カルロ スピノラ（マテオ リ チとカルロ スピノラはいずれも，グレゴリオ暦作成の中心人物であり，ユクリ ド原論の注解書を書いたドイツ出身の数学者クリストフ クラヴ ウスに学んでいる）が持 てきたとは考えられるが，翻訳 出版もされていないことをみると，十分にその必要性を理解できるまでには至 ていなか たとも考えられる（日本では，鎖国が解かれた以降にようやく，西洋数学が十分に取り入れられた）。

この現状が同じようにわかるのが，同じ幾何学を必要とする「遠近法」の日本での広まりである。ユ クリ ド幾何学を必要とする遠近法は，西洋ではすでに，15 世紀にレオン バ テ スタ アルベルテ によ て理論づけられ，広ま ていた。一方，日本では，遠近法はようやく 18 世紀にな て，平賀源内，佐竹曙山，司馬江漢らによ て確立され始めるが，それでも日本では遠近法が広まらなか た。これは，それに必要な数学の理解が十分に広ま ていないことによると考えられる。このことは，塩竈の日時計の作製時期とほぼ同時期の話である。

こうしたことから，日本では，数学が十分に発展していないことが推察され，日時計が正確に作製できなか たことも，このことが原因であると考えられる。

6 おわりに

本稿では，日本における最古の屋外型日時計（出島の日時計と塩竈の日時計）の正確性を数学的に考察した。その結果，オランダ商館長が作 た（作らせた）出島の日時計は正確であり，日本人（林子平）が作 た（作らせた）塩竈の日時計は十分に正確とはいえるものではないことが明らかとな た。また，このことから，その背景にある数学がこの時期には日本で十分には広ま ていなか たことが示唆された。

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PROJECTS FOR STUDENTS OF DISCRETE MATHEMATICS VIA PRIMARY HISTORICAL SOURCES: Euclid on his algorithm

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ABSTRACT

We analyze our student project *Euclid's Algorithm for the Greatest Common Divisor*. The project was written for students to decipher Euclid's verbal description of his famous algorithm for calculating the greatest common divisor of two numbers, convert it to a modern mathematical formulation, consider various issues that arise, and prove its correctness. We will discuss how the project design achieves specific pedagogical goals for teaching directly from primary historical sources.

Keywords: Euclidean algorithm, greatest common divisor, primary sources, original sources, pedagogy

1 Introduction

We analyze pedagogically the project *Euclid's Algorithm for the Greatest Common Divisor*, written for student study at beginning undergraduate or pre-college level. In the core of the project students are guided to decipher Euclid's verbal description of his famous algorithm for calculating the greatest common divisor of two numbers, convert it to a modern mathematical formulation, consider various questions and issues that arise, and prove its correctness. Along the way the Euclid source naturally raises questions about the nature of numbers, divisibility, algorithms, efficiency of computation, correctness, and proof. We will discuss how the project design achieves specific pedagogical goals.

This student project is part of a larger endeavor. Over the past nine years, with support from the US National Science Foundation, our interdisciplinary team of seven mathematicians and computer scientists has been developing and testing student projects based directly on primary historical sources for studying discrete mathematics and related subjects. Our 34 projects for students are based on primary sources encompassing discrete mathematics, combinatorics, abstract algebra, logic, and computer science, and have been extensively tested with students at varied institutions. The goal is to study mathematics directly from the minds of the pioneers, such as Euclid, Archimedes, Fermat, Pascal, Bernoulli, Lagrange, Cauchy, Cayley, Boole, Venn, Dedekind, Frege, Russell, Whitehead, and others. All our projects are available, along with guidance for instructors and our philosophy of teaching with them, at [**?**, **?**]. Our overall program for developing, testing, and evaluating the use of these projects is also discussed in [**?**].

Designed to capture the spark of discovery and motivate subsequent lines of inquiry, each project is built around primary source material close to or representing the discovery of a key concept. Through guided reading and directed questions and activities, students explore the mathematics of the original discovery and develop their own understanding of the subject. To place the source in context, a project also provides biographical information about its author, and historical background about the problems with which the author was concerned. Advantages include providing context and direction for the subject matter, honing students' verbal and deductive skills through reading the original work of some of the greatest minds in history, and the rediscovery of conceptual roots. Additionally, students practice the skill of moving from verbal descriptions to precise mathematical formulations, and must often recognize an organizing concept for a detailed procedure.

2 Mathematical aims of the Euclid project

Each of our projects provides a summary of the project along with suggestions about class activities for instructors. For our project *Euclid's Algorithm for the Greatest Common Divisor*, the notes to the instructor are:

> "The project is meant for use in an introductory computer science or discrete mathematics class. The project can be used to introduce students to the notion of "computation method" or "algorithm" and to explore concepts like iteration in a basic setting. It allows them to practice their skills in doing proofs but more importantly to observe the evolution of what is accepted as a valid proof or a well-described algorithm. The students will easily notice that the method presented by Euclid to compute the GCD and the proof of its correctness that he provided would not be formally accepted as correct today. They will also notice, however, that Euclid is somehow able to convey his ideas behind his method and proof in a way that they can be easily translated into a modern algorithm and proof of its correctness. In this way, it will provide them a sense of connection to the past.
>
> A basic knowledge of programming is essential to successfully complete some of the components of the project."

3 Pedagogical design goals

In [**?**] we distilled a set of pedagogical goals informing our selection of primary source material and the design of projects. We list these here in preparation for analyzing the Euclid project.

Fifteen Pedagogical Goals Guiding the Development of Primary Source Based Projects

1. Hone students' verbal and deductive skills through reading.
2. Provide practice moving from verbal descriptions of problems to precise mathematical formulations.
3. Promote recognition of the organizing concept behind a procedure.

4. Promote understanding of the present-day paradigm of the subject through the reading of an historical source which requires no knowledge of that paradigm.
5. Promote reflection on present-day standards and paradigm of subject.
6. Draw attention to subtleties, which modern texts may take for granted, through the reading of an historical source.
7. Promote students' ability to equally participate, regardless of their background or capability.
8. Offer diverse approaches to material which can serve to benefit students with different learning styles through exposure to multiple approaches.
9. Provide a point of departure for students' work, and a direction for their efforts.
10. Encourage more authentic (versus routine) student proof efforts through exposure to original problems in which the concepts arose.
11. Promote a human vision of science and of mathematics.
12. Provide a framework for the subject in which all elements appear in their right place.
13. Promote a dynamical vision of the evolution of mathematics.
14. Promote enriched understanding of subject through greater understanding of its roots, for students and instructors.
15. Engender cognitive dissonance (dépaysement) when comparing a historical source with a modern textbook approach, which to resolve requires an understanding of both the underlying concepts and use of present-day notation.

4 The Euclid project and its pedagogy

We will present the complete student project based on Euclid's text, and intersperse commentary discussing how it addresses our various pedagogical goals. We will find that every one of the goals in our list is addressed.

Euclid's Algorithm for the Greatest Common Divisor

Numbers, Division and Euclid

People have been using numbers, and operations on them like division, for a very long time for practical purposes like dividing up the money left by parents for children, or distributing ears of corn equally to groups of people, and more generally to conduct all sorts of business dealings. It may be a bit of a surprise that things like calculating divisors of numbers also form the core of today's methods ensuring security of computer systems and internet communications. The RSA cryptosystem that is used extensively for secure communications is based on the assumed difficulty of calculating divisors of large numbers, so calculating divisors is important even today.

A related and even more basic notion is that of multiples of quantities. A natural way to compare quantities is to "measure" how many times we need to aggregate the smaller quantity to obtain the larger quantity. For example, we may be able to compare two unknown lengths by observing that the larger length can be obtained by "aggregating" the smaller length three times. This provides a sense of how the two lengths compare without actually knowing the two lengths.

The larger quantity may not always be obtainable from the smaller quantity by aggregating it an integral number of times. In this scenario, one way to think would be to imagine each of the two quantities to be made up of smaller (identical) parts such that both the quantities can be obtained by aggregating these smaller parts an integral number of times. Obviously, we will need a greater number of these parts for the larger quantity than for the smaller one. For example, when comparing two weights, one might observe that the larger one can be obtained by aggregating some weight 7 times whereas the smaller weight can be obtained by aggregating the same weight 5 times. This provides a basis for comparing the two weights. Of course, in the above scenario, one can also observe that if we chose even smaller parts to "split" the weights (say a quarter of the first one), the first weight would be obtained by aggregating this even smaller weight 28 times and the smaller of the two original weights would be obtained by aggregating this smaller part 20 times, which also provides us a sense of the relative magnitudes of the two weights. However, using smaller numbers like 7 and 5 to describe relative magnitudes seems intuitively and practically more appealing than using larger numbers, like 28 and 20. This leads us to think about what would be the greatest magnitude such that two given magnitudes will both be multiples of that common magnitude.

This question was considered by Greek mathematicians more than 2000 years ago. One of those Greeks was Euclid, who compiled a collection of mathematical works called *Elements* that has a chapter, interestingly called a "Book", about numbers. During the course of this project you will read a translation of part of this chapter to discover Euclid's method (algorithm) to compute the greatest common divisor of two numbers. It is not clear if Euclid was the first person to discover this algorithm, but his is the earliest known written record of it.

Commentary. *This brief introductory mathematical discussion and thought experiment sets the stage both for modern practical applications and historical context. It provides a motivational point of departure for students, and highlights the issue of choice of unit for measurement, something that is often glossed over today, but whose importance is brought out naturally through reading ancient texts.*

Euclid of Alexandria

Euclid lived around 300 B.C.E. Very little is known about his life. It is generally believed that he was educated under students of Plato's Academy in Athens. According to Proclus (410–485 C.E.), Euclid came after the first pupils of Plato and lived during the reign of Ptolemy I (306–283 B.C.E.). It is said that Euclid established a mathematical school in Alexandria. Euclid is best known for his mathematical compilation *Elements* in which among other things he laid down the foundations of geometry and number theory. The geometry that we learn in school today traces its roots to this book, and Euclid is sometimes called the father of geometry.

Euclid did not study mathematics for its potential practical applications or financial gains. He studied mathematics for a sense of order, structure and the ideal form of reason. To him geometrical objects and numbers were abstract entities, and he was interested in studying and discovering their properties. In that sense, he studied mathematics for its own sake. One story that reveals his disdain for learning for the purpose of material gains concerns a pupil who had just finished his first geometry lesson. The pupil asked what he would gain from learning geometry. As the story goes, Euclid asked his subordinate to give the pupil a coin so that he would be gaining from his studies. Another story that reveals something about his character concerns King Ptolemy. Ptolemy asked the mathematician if there was an easier way to learn geometry. Euclid replied, "There is no royal road to geometry", and sent the king to study.

Euclid wrote several books such as *Data, On Divisions of Figures, Phaenomena, Optics*, and the lost books *Conics* and *Porisms*, but *Elements* remains his best known compilation. The first "book" [chapter] in this compilation is perhaps the most well-known. It lays down the foundations of what we today call "Euclidean" geometry (which was the only plane geometry people studied until the Renaissance). This book has definitions of basic geometric objects like points and lines along with basic postulates or axioms. These axioms are then used by Euclid to establish many other truths (*Theorems*) of geometry. Euclid's *Elements* is considered one of the greatest works of mathematics, partly because it is the earliest we have that embodies an axiomatic approach. It was translated into Latin and Arabic and influenced mathematics throughout Europe and the Middle East. It was probably the standard "textbook" for geometry for more than 1500 years in western Europe and continues to influence the way geometry is taught to this day.

Book 7 of *Elements* provides foundations for number theory. Euclid's Algorithm for calculating the greatest common divisor of two numbers was presented in this book. As one will notice later, Euclid uses lines to represent numbers and often relies on visual figures to aid the explanation of his method of computing the greatest common divisor (GCD) of two numbers. As such, he seems to be relating numbers to geometry, which is quite different from the present day treatment of number theory.

Today, erroneously, many different methods are called Euclid's algorithm. By reading the original writings of Euclid you will discover the real Euclidean algorithm and appreciate its subtlety. In any case, "Euclid's Algorithm" is one of the most cited and well-known examples of an (early) algorithm. To quote Knuth [**?**] :

By 1950, the word algorithm was mostly associated with "*Euclid's Algorithm*".

Commentary. *This biographical and historical background gives students a sense of the human aspect of the creation of mathematics, including the interplay between studying mathematics for its own sake and for applications, as well as a sense for the evolution of mathematics over a very long period. By pointing out the relationship between number and geometry for Euclid, it also fosters a framework in the mind of the student in which the different parts of mathematics are interrelated, unlike the way they are often taught today.*

Prelude

We say that a number[1] x divides another number y if y is a multiple of x. For example, $1, 2$, and 3 all divide 6 but 5 does not divide 6. The only divisors of 17 are 1 and 17. The notation $x|y$ is a shorthand

[1]The word number in this section means a positive integer. That is what it meant to Euclid.

for "x divides y". We denote by $divisors(x)$ the set of all the numbers y such that $y|x$. So, for example, $divisors(6) = \{1, 2, 3, 6\}$ and $divisors(60) = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$.

A number z is called a common divisor of two numbers x and y if $z|x$ and $z|y$. We denote by $cd(x, y)$ the set of all common divisors of x and y. For example, $cd(6, 8) = \{1, 2\}$ and $cd(40, 180) = \{1, 2, 4, 5, 10, 20\}$.

Exercise 4.1. *What is the set of divisors of the number* 315*?*

Exercise 4.2. *Calculate the set* $cd(288, 216)$*.*

While it is relatively easy to calculate the divisors of a number and common divisors of two numbers when the numbers are small, the task becomes harder as the numbers becomes larger.

Exercise 4.3. *Calculate* $divisors(3456)$*.*

Exercise 4.4. *Calculate* $cd(3456, 4563)$*.*

Exercise 4.5. *A rather naive method for computing the divisors of a number x is to test whether each number from 1 to x inclusive is a divisor of x. For integers n = 1, 2, 3, . . ., x, simply test whether n divides x. Using this naive algorithm, write a computer program in the language of your choice that accepts as input a positive integer x and outputs all divisors of x. Run this program for:*

(a) $x = 3456$,

(b) $x = 1009$,

(c) $x = 1080$.

Exercise 4.6. *The naive method for computing the common divisors of two numbers x and y is to test whether each number from 1 to the least of $\{x,\ y\}$ divides x and y. In modern notation, let m denote the minimum (least of) $\{x,\ y\}$. For n = 1, 2, 3, . . ., m, first test whether n divides x, and, if so, then test whether n divides y. If n divides both x and y, record n as a common divisor. Using this naive algorithm, write a computer program in the language of your choice that accepts as input two positive integers x, y, and outputs their common divisors. Run this program for:*

(a) $x = 3456$, $y = 4563$,

(b) $x = 625$, $y = 288$,

(c) $x = 216$, $y = 288$,

(d) $x = 147$, $y = 27$.

As you might have noticed the number 1 divides every number. Since there is no number smaller than 1, 1 is the **smallest** common divisor for any two numbers x and y. What about the **greatest** common divisor? The greatest common divisor of two numbers x and y, denoted by $gcd(x, y)$, is the largest number z such that $z|x$ and $z|y$. Finding the greatest common divisor is not nearly as easy as finding the smallest common divisor.

Commentary. *This prelude prepares students for reading the primary source directly, since Euclid does not provide definition, motivation or context for questions about the greatest common divisor of two numbers. This short section explores the issue through a number of concrete examples and exercises, and encourages students to program naive algorithms. We intentionally make the prelude no longer than necessary for students to dive into Euclid, since our goal is to have students engage the primary source as quickly and as deeply as possible.*

*We intersperse exercises throughout the project, encouraging students and instructors who wish to engage the project in small stages, as regular day-by-day classroom work and homework. The logistics of ways to use a project in class are discussed further at [**?**].*

Euclid's Algorithm

Here we present the translations of (relevant) Definitions, Proposition 1 and Proposition 2 from Book VII of Euclid's *Elements* as translated by Sir Thomas L. Heath [**?**]. Euclid's method of computing the GCD is based on these propositions.

BOOK VII of *Elements* by Euclid
DEFINITIONS.

1. A **unit** is that by virtue of which each of the things that exist is called one.
2. A **number** is a multitude composed of units.
3. A number is **a part** of a number, the less of the greater, when it measures the greater.
4. but **parts** when it does not measure it.[2]
5. The greater number is a **multiple** of the less when it is measured by the less.
6. An **even number** is that which is divisible into two equal parts.
7. An **odd number** is that which is not divisible into two equal parts, or that differs by a unit from an even number.
8. An **even-times even number** is that which is measured by an even number according to an even number.
9. An **even-times odd number** is that which is measured by an even number according to an odd number.
10. An **odd-times odd number** is that which is measured by an odd number according to an odd number.
11. A **prime number** is that which is measured by a unit alone.[3]
12. Numbers **prime to one another** are those which are measured by a unit alone as a common measure.

[2]While this definition is not relevant here, what is meant by this definition is quite subtle and the subject of scholarly mathematical work.

[3]Reading further work of Euclid, e.g. Proposition 2, it is clear that Euclid meant that a prime number is that which is measured only by the unit and the number itself.

13. A **composite number** is that which is measured by some number.
14. Numbers **composite to one another** are those which are measured by some number as a common measure.

Exercise 4.7. *Discuss how Euclid's "unit" relates to the number 1. Does Euclid think that 1 is a number?*

Exercise 4.8. *What is likely meant when Euclid states that a number "measures" another number? Express Euclid's notion of "measures" in modern mathematical notation.*

Exercise 4.9. *Does the number 4 measure number 72? Does 5 measure 72? Briefly justify your answer.*

Exercise 4.10. *Euclid never defines what is a "common measure," but uses that in definition 12 and 14. What is your interpretation of Euclid's "common measure"?*

Exercise 4.11. *Find a number (other than the unit) that is a common measure of the numbers 102 and 187. According to Euclid's definitions, are the numbers 102 and 187 composite to one another? Why or why not?*

Exercise 4.12. *According to Euclid's definitions, are the numbers 21 and 55 composite to one another? Justify your answer.*

Commentary. *Euclid's definitions provide considerable grist for mathematical questions, intentionally left unexplained by us, for students to grapple with in exercises. Such basic questions as whether "one" is a number, and what "measures" means and why Euclid leaves it undefined, reflect the rich intellectual stimulation a primary source can provide, even before anything particularly technical is encountered.*

Already here deductive skills through reading the primary source are naturally emphasized, as well as the challenge of moving between verbal and symbolic formulations, e.g., in divining the meaning of "measures" and comparing it to the concept of a "multiple".

We now present Proposition 1 from Euclid's book VII. The proposition concerns numbers that are prime to one another.

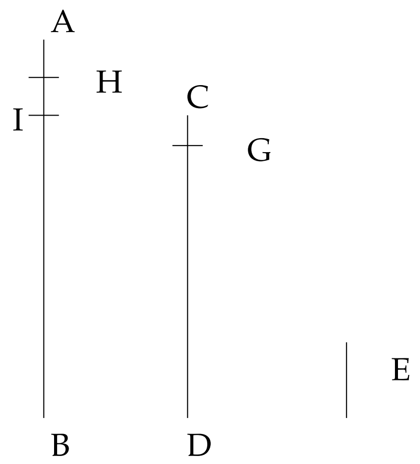
PROPOSITION 1.

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another.

For, the less of two unequal numbers AB, CD being continually subtracted from the greater, let the number which is left never measure the one before it until a unit is left;

I say that AB, CD are prime to one another, that is, that a unit alone measures AB, CD.

For, if AB, CD are not prime to one another, some number will measure them.



Let a number measure them, and let it be E; let CD, measuring BI, leave IA less than itself,

let, AI measuring DG, leave GC less than itself,

and let GC, measuring IH, leave a unit HA.

Since, then E measures CD, and CD measure BI, therefore E also measures BI.

But it also measures the whole BA;

therefore it will also measure the remainder AI.

But AI measures DG;

therefore E also measures DG.

But it also measures the whole DC;

therefore it will also measure the remainder CG.

But CG measures IH;

therefore E also measures IH.

But it also measures the whole IA;

therefore it will also measure the remainder, the unit AH, though it is a number: which is impossible.

Therefore no number will measure the numbers AB, CD; therefore AB, CD are prime to one another. [VII. Def 12]

Q. E. D.

Exercise 4.13. *Euclid begins with two unequal numbers AB, CD, and continually subtracts the smaller in turn from the greater. Let's examine how this method proceeds "in turn" when subtraction yields a new number that is smaller than the one subtracted. Begin with $AB = 162$ and $CD = 31$.*

(a) How many times must CD be subtracted from AB until a remainder is left that is less than CD? Let this remainder be denoted as IA.

(b) Write $AB = BI + IA$ numerically using the given value for AB and the computed value for IA.

(c) How many times must IA be subtracted from CD until a remainder is left that is less than IA? Let this remainder be denoted as GC.

(d) Write $CD = DG + GC$ numerically using the given value for CD and the computed value for GC.

(e) How many times must GC be subtracted from IA until a remainder is left that is less than GC? Let this remainder be denoted as HA.

(f) Is HA a unit?

(g) Write $IA = IH + HA$ numerically using the computed values of IA and HA.

Exercise 4.14. *Apply the procedure outlined in Proposition 1 to the numbers $AB = 625$ and $CD = 288$. Begin by answering questions (a)–(f) above except with the new values for AB and CD.*

(g) In this example, how should the algorithm proceed until a remainder is reached that is a unit?

Exercise 4.15. *Euclid claims that if the repeated subtraction algorithm of Proposition 1 eventually produces a unit as a remainder, then the original numbers AB, CD are prime to one another. He does so by using a "proof by contradiction." Suppose the result, namely that AB and CD are prime to one another, is false. In this exercise we examine the consequences of this.*

(a) If AB and CD are not prime to one another, must these numbers have a common measure E that is greater than 1? Justify your answer by using Euclid's definitions.

(b) From $AB = BI + IA$, why must E also measure IA ? Be sure to carefully justify your answer for general numbers AB and CD (not tied to one particular example).

(c) From $CD = DG + GC$, why must E also measure GC? Be sure to carefully justify your answer.

(d) From $IA = IH + HA$, why must E also measure HA? Carefully justify your answer.

(e) If according to Euclid, HA is a unit, what contradiction has been reached in part (d) ?

Commentary. *Euclid's Proposition 1 and the exercises achieve a number of our pedagogical goals. To decipher and understand Euclid students will use considerable verbal and deductive skills through reading, practice moving from a verbal description to a precise mathematical formulation, puzzle out the organizing concept behind a procedure, and gain perspective on present-day paradigms through translating to modern formulas. Moreover, Euclid's approach through imagining geometric measurement is quite different from a modern one,* *providing students with a diversity of viewpoints and enabling students with different backgrounds and learning styles to benefit. The exercises also explore the phenomenon of iterative procedures and when they terminate, raising questions about Euclid's description, which does not directly address this issue.*

We now present proposition 2 from Book VII of Euclid's elements. This proposition presents a method to compute the GCD of two numbers which are not prime to each other and provides a proof of the correctness of the method. Euclid's presentation intermixes the proof and the method to some extent. Despite this the elegance of his method and the proof is striking.

PROPOSITION 2.

Given two numbers not prime to one another, to find their greatest common measure.

Let AB, CD be the two given numbers not prime to one another.

Thus it is required to find the greatest common measure of AB, CD.

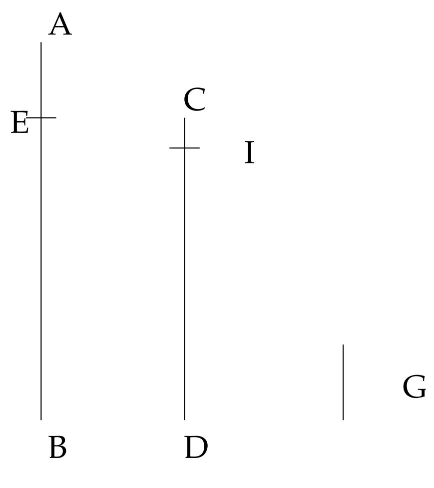
If now CD measures AB - and it also measures itself - CD is a common measure of CD, AB.

And it is manifest that it is also the greatest; for no greater number than CD will measure CD.

But, if CD does not measure AB, then, the less of the numbers AB, CD being continually subtracted from the greater, some number will be left which will measure the one before it.[4]

For a unit will not be left; otherwise AB, CD will be prime to one another [VII, I], which is contrary to the hypothesis.

Therefore some number will be left which will measure the one before it.



[4]This is the heart of Euclid's description of his algorithm. The statement is somewhat ambiguous and subject to at least two different interpretations.

Now let CD, measuring BE, leave EA less than itself, let EA, measuring DI, leave IC less than itself, and let CI measure AE.

Since then, CI measures AE, and AE measures DI,

therefore CI will also measure DI.

But it also measures itself;

therefore it will also measure the whole CD.

But CD measures BE;

therefore CI also measures BE.

But it also measures EA;

therefore, it will also measure the whole BA.

But it also measures CD;

therefore CI measures AB, CD.

Therefore CI is a common measure of AB, CD.

I say next that it is also the greatest.

For, if CI is not the greatest common measure of AB, CD, some number which is greater than CI will measure the numbers AB, CD.

Let such a number measure them, and let it be G.

Now, since G measures CD, while CD measures BE, G also measures BE.

But it also measures the whole BA;

therefore it will also measure the remainder AE.

But AE measures DI;

therefore G will also measure DI.

But it will also measure the whole DC;

therefore it will also measure the remainder CI, that is, the greater will measure the less: which is impossible.

Therefore no number which is greater than CI will measure the numbers AB, CD;

therefore CI is the greatest common measure of AB, CD.

PORISM. From this it is manifest that, if a number measure two numbers, it will also measure their greatest common measure.

Exercise 4.16. *In Proposition 2 Euclid describes a procedure to compute the greatest common measure of two numbers AB, CD, not prime to one another. The method again proceeds by repeatedly subtracting the smaller in turn from the greater until some number is left, which in this case divides the number before it. Let's examine this process for $AB = 147$ and $CD = 27$.*

(a) *Does CD measure AB? If so, the process stops. If not, how many times must CD be subtracted from AB until a positive remainder is left that is less than CD. Let EA denote this remainder.*

(b) *Write $AB = BE + EA$ numerically using the given value for AB and the computed value for EA. Also find a positive integer q_1 so that $BE = q_1 \cdot CD$.*

(c) *Does EA measure CD? If so, the process stops. If not, how many times must EA be subtracted from CD until a positive remainder is left that is less than EA. Let IC denote this remainder.*

(d) *Write $CD = DI + IC$ numerically using the given value for CD and the computed value for IC. Also, find a positive integer q_2 so that $DI = q_2 \cdot EA$.*

(e) *Does IC measure EA? If so, the process stops. If not, how many times must IC be subtracted from EA until a positive remainder is left that is less than IC ?*

(f) *Find a positive integer q_3 so that $EA = q_3 \cdot IC$.*

Exercise 4.17. *Apply Euclid's procedure in Proposition 2 to compute the greatest common measure of $AB = 600$ and $CD = 276$ outlined in the steps below.*

(a) *To streamline the process, let $a_1 = AB = 600$, $a_2 = CD = 276$, and $a_3 = EA$. Compute a_3 numerically for this example. Write the equation $AB = BE + EA$ entirely in terms of a_1, a_2 and a_3.*

(b) *Let $a_4 = IC$. Compute a_4 for this example. Write the equation $CD = DI + IC$ entirely in terms of a_2, a_3 and a_4.*

(c) *Does IC measure EA in this example? If so, the process stops. If not, how many times must IC be subtracted from EA until a positive remainder is left that is less than IC? Denote this remainder by a_5.*

(d) *Write an equation using* a_3, a_4 *and* a_5 *that reflects the number of times* IC *must be subtracted from* EA *so that the remainder is* a_5.

(e) *Does* a_5 *measure* a_4*? If so, the process stops. If not, how many times must* a_5 *be subtracted from* a_4 *until a positive remainder is left that is less than* a_5*?*

Exercise 4.18. *In modern notation, the Euclidean algorithm to compute the greatest common measure of two positive integers* a_1 *and* a_2 *(prime to each other or not) can be written as follows. Find a sequence of positive integer remainders* $a_3, a_4, a_5, \ldots, a_{n+1}$ *and a sequence of (positive) integer multipliers* $q_1, q_2, q_3, \ldots, q_n$ *so that*

$$
\begin{aligned}
&a_1 = q_1 a_2 + a_3, \quad 0 < a_3 < a_2\\
&a_2 = q_2 a_3 + a_4, \quad 0 < a_4 < a_3\\
&a_3 = q_3 a_4 + a_5, \quad 0 < a_5 < a_4\\
&\vdots\\
&a_{i-1} = q_{i-1} a_i + a_{i+1}, \quad 0 < a_{i+1} < a_i\\
&a_i = q_i a_{i+1} + a_{i+2}, \quad 0 < a_{i+2} < a_{i+1}\\
&\vdots\\
&a_{n-1} = q_{n-1} a_n + a_{n+1}, \quad 0 < a_{n+1} < a_n\\
&a_n = q_n a_{n+1}
\end{aligned}
$$

(a) *Why is* a_{n+1} *a divisor of* a_n *? Briefly justify your answer.*

(b) *Why is* a_{n+1} *a divisor of* a_{n-1}*? Carefully justify your answer.*

(c) *In a step-by-step argument, use (backwards) mathematical induction to verify that* a_{n+1} *is a divisor of* a_i, $i = n, n-1, n-2, \ldots, 3, 2, 1$.

(d) *Why is* a_{n+1} *a common divisor of* a_1 *and* a_2*?*

(e) *In a step-by-step argument, use (forwards) mathematical induction to verify that if G is a divisor of* a_1 *and* a_2*, then G is also a divisor of* a_i, $i = 3, 4, 5, \ldots, n+1$. *First, carefully explain why G is a divisor of* a_3*. Then examine the inductive step.*

(f) *From part (d) we know that* a_{n+1} *is a common divisor of* a_1 *and* a_2*. Carefully explain how part (e) can be used to conclude that* a_{n+1} *is in fact the* greatest *common divisor of* a_1 *and* a_2*. A proof by contradiction might be appropriate here, following Euclid's example.*

Exercise 4.19. *In Proposition 1 Euclid describes an algorithm whereby, given two unequal numbers, the less is continually subtracted in turn from the greater until a unit is left. While in Proposition 2, Euclid describes an algorithm, whereby, given two unequal numbers, the less is continually subtracted from the greater until some number is left which measures the one before it.*

(a) *To what extent are these algorithms identical?*

(b) *How are the algorithms in Proposition 1 and Proposition 2 designed to differ in application?*

(c) Does Euclid consider a unit as a number? Justify your answer citing relevant passages from the work of Euclid. Does Euclid consider a common measure as a number? Again, justify your answer from the work of Euclid.

(d) Why, in your opinion, does Euclid describe this algorithm using two separate propositions, when a single description could suffice?

Exercise 4.20. *In the modern description of the Euclidean algorithm in Exercise* (4.18)*, the last equation written is*

$$a_n = q_n a_{n+1}\,,$$

meaning that after n-steps, the algorithm halts and a_{n+1} divides (measures) a_n. Given any two positive integers a_1 and a_2, why must the Euclidean algorithm halt in a finite number of steps? Carefully justify your answer using the modern version of the algorithm.

Exercise 4.21. *Write a computer program in the language of your choice that implements Euclid's algorithm for finding the greatest common divisor of two positive integers. The program should accept as input two positive integers a_1, a_2, and as output print their greatest common divisor. Run the program for:*

(a) $a_1 = 3456$, $a_2 = 4563$,

(b) $a_1 = 625$, $a_2 = 288$,

(c) $a_1 = 216$, $a_2 = 288$.

Commentary. *The core of the project addresses many of our pedagogical goals, guided in exercises. Students must engage considerable subtleties, since there is more than one way to interpret what Euclid is saying. Moreover, it is intellectually useful to ask why for Euclid the algorithm is separated into two procedures, when today we make no distinction whether the GCD is one or greater. Justifying the algorithm from a modern viewpoint requires double mathematical induction, connecting ancient with modern methods and promoting understanding of modern standards and paradigms. And justifying the correctness of Euclid's claims provides an authentically important and worthwhile challenge for students, along with a clear sense for the roots of a critical piece of modern mathematics. Finally, the difference between Euclid's verbal presentation and our modern terminology and methods engenders considerable healthy cognitive dissonance for students to resolve.*

Acknowledgement

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THE JOURNEY TO A PROOF: IF F' IS POSITIVE, THEN F IS AN INCREASING FUNCTION

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ABSTRACT

When moving on from secondary to tertiary education, students are - in most countries - faced with new challenges in terms of proof: all theorems are, from then on, proved by the lecturer (which calls for proof-understanding skills); student are now expected to devise proofs of a more or less formal nature. As a consequence, the issues of proof-understanding and proof-writing have long been focal points in the research on AMT (Advanced Mathematical Thinking). Numerous strategies have been put forward - and sometimes tried out with students -, among which: (1) to distinguish between proof-ideas (or proof-germs) and formal proofs, and have students write formal proofs from informal ones [Downs & Mamona-Downs 2010], (2) to study historical proofs [Robert & Schwartzenberger 1991].

We will present a series of historical texts which lead to the now standard proof of the fact that, for a differentiable function of one real variable, the sign of the derivative determines the variations of the function (on an interval). Several features of this historical file are relevant from a maths-education perspective : (1) it illustrates the role of "*local*" counter-examples (to use Lakatosian terminology), a role which may not be familiar to students (although some students may be used to dealing with "global" counter-examples); (2) the various proofs (or proof-attempts) are based on at least *two* pretty different proof-ideas; (3) even a *proper* (meaning, both intuitive and formal) understanding of the concepts involved in the statement of the theorem may lead to a faulty proof scheme; (4) it helps understand the necessity of such intricate concepts as "uniform upper bound" or the completeness of **R**.

For this workshop, we will provide translations of French and German original sources, with excerpts from Lagrange, Cauchy, Serret, Peano, Darboux and Weierstrass.

Keywords: mathematical analysis, calculus, AMT, proof design, proof analysis.

1 Rationale

In the twentieth century, most tertiary-level textbooks of mathematical analysis prove the following theorem: let f be a differentiable real-valued function defined on an interval, if its derivative f'

*Large parts of this paper derive from joint work with Anne Michel-Pajus, and Philippe Brin. A fundamental starting point was [Dugac 1979].

is positive, then f increases over this interva[1]. Its standard proof is a rather straightforward application of the "mean value theorem"[2] ("*égalité des accroissements finis*" in French, "*Mittelwertsatz*" in German); the proof of which is a rather straightforward application of the "Rolle theorem"Let f be a differentiable real-valued function, defined over some interval [a,b], such that f(a) = f(b). There is a value c between a and b for which de derivative vanishes.[3], which, in turn, depends on the fact that a continuous real-valued function defined on a closed and bounded (i.e. compact) interval has a maximum or a minimum. The latter fact, although quite intuitive, depends on not-so-trivial properties of the set of real numbers (completeness of the metric space, local compactness). Historically speaking, this proof-chain can be found in the textbooks of Jordan [Jordan 1893, 65-67], Stolz [Stolz 1893, 51-], Osgood [Osgood 1912, 26-28]; in his chapter on differential and integral calculus for the famous *Enzyklopädie der mathematischen Wissenschaften* [Voss 1899, 65-66], Voss states in the clearest of ways that the mean value theorem in calculus depends on Weierstrass' theorem on the existence of extrema for continuous functions.

With this example, we can see that the proof of a rather intuitive qualitative fact (namely: if all the tangents point upward, the curve has to move up) requires several layers of sophisticated concepts (differentiability, continuity, properties of the numerical continuum), and a few silly tricks (affine changes of variable). In this workshop, we will present some of the proofs given, over the 19^{th} century, either of this mathematical fact, or of some key points in its proof.

We must stress the fact that this paper was designed for a *reading workshop*: it is by no means a research report on the "history" of that theorem - whatever that may mean. Among other things, we do not aim for a comprehensive overview; the historical connections between the various authors are hardly mentioned; nor are the institutional and intellectual contexts of the various research or teaching programs. Our main goal is to make a few important texts available to an audience of teachers and researchers who are not familiar with the French or the German languages. We hope this selection of texts, and the points we will highlight as we read them, will provide food for thought, and trigger further work, be it classroom work, or more theoretical work in the teaching and learning of (advanced) mathematics.

2 Lagrange's proof (1806)

As we saw earlier, any function $f(x+i)$ can be expanded into the series

$$f(x) + if'(x) + \frac{i^2}{2!}f''(x) + \frac{i^3}{3!}f'''(x) + \cdots ,$$

which naturally goes on to infinity, unless the derived functions vanish, which is the case when f(x) is an entire rational function of x.

As long as this expansion is used for the sole generation of derived functions, it is indifferent whether the series goes to infinity or not; it is also the case, when the expansion is seen as a mere analytical transformation of the function; but if one wants to use it to get the value of the function in particular

[1]For the analysis of a teaching experience, see [Praslon 1994].

[2]Let f be a differentiable real-valued function, defined over some interval $[a,b]$, there exists a value c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Geometrically speaking: on the arc of curve joining the points $(a, f(a))$ and $(b, f(b))$, there is a point where the tangent is parallel to the chord joining the two endpoints.

[3]Let f be a differentiable real-valued function, defined over some interval $[a,b]$, such that $f(a) = f(b)$. There is a value c between a and b for which de derivative vanishes.

cases, in which case it displays an expression of a simpler form - quantity i having been released from the function - then, since only a given number of terms can be taken into account, it is important to have means to assess the remainder of the series which we neglect, or, at least, to find bounds to the error that we make by neglecting this remainder.
The determination of these bounds is above all important in the application of the Theory of functions to the Analysis of curves, and to Mechanics, so as to impart on this application the rigour of ancient geometry, as can be seen in part two of the *Theory of analytic functions*.
In the solution which I gave in the above mentioned work, I first found the exact expression of the remainder of the series, then determined bounds for that expression. But these bounds can also be found in a more elementary way, which is just as rigorous. For this purpose, we shall establish this general principle, which can be of use in several occasions:
A function which vanishes when the variable vanishes, will, as the variable increases positively, have finite values of the same sign as that of its derived function; or of the opposite sign if the variable increases negatively, as long as the values of the derived function keep the same sign and do not become infinite.
This principle is very important in the theory of functions, since it establishes a general relationship between the state of primitive functions and that of derived functions, and also helps determine bounds[4] for functions for which only the derivatives are known.
We shall prove it rigorously.
Let us consider the function $f(x+i)$, whose general development is

$$f(x) + if'(x) + \frac{i^2}{2!}f''(x) + \cdots .$$

As we saw in the former lesson, the form of the development may be different for some specific values of x; but we saw that, as long as $f'(x)$ is not infinite, the first two terms of the expansion are exact; and that the other terms will, consequently, contain powers of i greater than the first, so that we shall have

$$f(x+i) = f(x) + i[f'(x) + V],$$

V being a function of x and i, which vanishes when $i = 0$.
So, since V vanishes when i vanishes, it is clear that, should i be made to increase from zero through insensible degrees, the value of V would also increase from zero by insensible degrees, either positively or negatively, up to a certain point, after which it may decrease; consequently, one will always be able to assign to i a value such that the corresponding value of V - regardless of the sign - is less than any given quantity, and that for lesser values of i, the values of V are also lesser.
Let D be a given quantity, which may be chosen as small as one pleases; one can always assign to i a value so small that the values of V are bounded by the limits D and $-D$; so, since we have

$$f(x+i) = f(x) + i[f'(x) + V],$$

It follows that the quantity $f(x+i) - f(x)$ will be bound by these two

$$i[f'(x) \pm D].$$

[4]We chose the work "bound" to translate Lagrange's use of the word "*limites*" in this context.

Since this conclusion holds for any value of x, as long as $f'(x)$ is not infinite, it will hold when

$$x+i, x+2i, x+3i, ..., x+(n-1)i$$

are substituted for x; so that one can always choose i positive and small enough for all the quantities

$$\begin{aligned}&f(x+i)-f(x),\\&f(x+2i)-f(x+i),\\&f(x+3i)-f(x+2i),\\&...,\\&f(x+ni)-f(x+(n-1)i),\end{aligned}$$

to be respectively bound between the limits

$$\begin{aligned}&i[f'(x)\pm D],\\&i[f'(x+i)\pm D],\\&i[f'(x+2i)\pm D],\\&...\\&i[f'[x+(n-1)i]\pm D],\end{aligned}$$

taking the same quantity D in all these limits, which is allowable so long as none of the quantities

$$f'(x), f'(x+i), f'(x+2i), ..., f'(x+(n-1)i)$$

is infinite. So, if all these quantities are of the same sign, that is, all positive of negative, it is easy to conclude that their sum, which amounts to

$$f(x+ni)-f(x)$$

is bounded by the sum of the bounds, that is by the quantities

$$if'(x)+if'(x+i)+if'(x+2i)+\cdots+f'[x+(n-1)i]\pm niD.$$

So, if the arbitrary quantity D is chosen less than the sum

$$f'(x)+f'(x+i)+f'(x+2i)+\cdots+f'[x+(n-1)i]$$

divided by n, then, if we do not take into account the sign of this sum, the quantity $f(x+ni)-f(x)$ will necessarily be bound between zero and the sum

$$2i[f'(x)+f'(x+i)+f'(x+2i)+\cdots+f'[x+(n-1)i]].$$

So, if P is the largest positive or negative value of the quantities

$$f'(x), f'(x+i), f'(x+2i), ..., f'(x+(n-1)i),$$

the quantity $f(x+ni)-f(x)$ will be bound between zero and $2niP$.
And yet, since when taking i as small as we wish, n can - at the same time - be taken as large as we

wish, we can assume that in is equal to any given quantity z, positive or negative, since quantity i can be taken positive or negative.
The quantity $f(x + ni) - f(x)$ will thus become $f(x + z) - f(x)$, and can be used to represent any function of z which vanishes for $z = 0$, quantity x being now seen as an arbitrary constant. Similarly, the quantity $f'(x + ni)$ will become $f'(x + z)$, and will represent the derived function of the same function of z, since $f'(x + z)$ is also the derived function of $f(x + z)$, either with respect to x or to z. Hence, one may conclude generally that, if $f'(x+z)$ constantly takes on finite values of the same sign, and if P denotes the largest of these values - regardless of the sign - the primitive function will be bound between 0 and $2zP$; consequently, it will also remain finite, and of the same sign as the derives function if z is positive, or of the opposite sign if z is negative. [Lagrange 1884, 86-89]
This passage clearly demonstrates that Lagrange was not a proponent of a *purely* formal analysis. Of course, in the preface of the *Théorie des fonctions analytiques* (and even in the subtitle), he rejected the notion of limit as a proper foundation and starting point for a systematic development of mathematical analysis. Indeed, he defined the derivative f' of a function f as the coefficient of i in the power series expansion

$$f(x + i) = f(x) + if'(x) + \frac{i^2}{2!}f''(x) + \frac{i^3}{3!}f'''(x) + \cdots$$

However, he was also concerned with numerical aspects in which issues of convergence, and lower and upper bounds are of the essence. In particular, he determined upper bounds for the integral remainder in the Taylor-Lagrange expansion, in order to assess the degree of approximation given by a partial series expansion, and to establish convergence in some important cases[5].
In this passage, we can see that Lagrange also had a proper numerical understanding what the value of the derivative at a given point represents, and that he did interpret limits as relationships of dependence between inequalities. For instance, he rephrased "V being a function of x and i, which vanishes when $i = 0$" as "one will always be able to assign to i a value such that the corresponding value of V - regardless of the sign - is less than any given quantity (...)".
However, in spite of the fact that the theorem Lagrange set out to prove is correct, and that the proof relied on a correct numerical understanding of the derivative construed as a limit, something does not sound right in the proof. The modern, 21st-century reader probably feels a bit uneasy when reading the summing argument, and the conclusions derived by passing to the limit. Even if we do not know exactly where things go wrong, we feel too many variables depend on one another in more than one way for the final limiting arguments to be safe and sound.
However, this "gut feeling" of disbelief, this red signal flashing before your eyes as we read the proof, is ascribable to our maths education: we were taught to distrust this kind of reasoning. In §5 of this paper, we will endeavour to shed some light on the historical roots of this distrust.

3 Cauchy's proof (1823)

> Problem. Assuming that the function $y = f(x)$ is continuous relative to x in the neighborhood of specific value $x = x_0$, one asks whether the function increases or decreases as from this value, as the variable itself is made to increase or decrease.

[5] For a recent historical analysis of Lagrange's work, see [Ferraro & Panza 2012].

> Solution. Let Δx, Δy denote the infinitely small and simultaneous increments of variables x and y. The $\Delta y/\Delta x$ ratio has limit $dy/dx = y'$. It has to be inferred that, for very small numerical values of Δx and for a specific value x_0 of variable x, ratio $\Delta y/\Delta x$ is positive if the corresponding value of y' is positive and finite. $(\cdots)$
> This being settled, let's assume function $y = f(x)$ remains continuous between two given limits $x = x_0$ and $x = X$. If variable x is made to increase by imperceptible degrees from the first limit to the second one, function y shall increase every time its derivative, while being finite, has a positive value. [Cauchy 1823, 37]

Unlike Lagrange, Cauchy defined the derivative as a limit; just like Lagrange, he was able to derive proper numerical conclusions from this numerical conception of the derivative. So what makes his argument so different from Lagrange's ? Actually, they do not have the exact same understanding of what the *conclusion* to be reached is; both have implicit definitions[6] of what it is for a function to be increasing, but their definitions do not match exactly. Lagrange's definition is closer to the one we find in today's textbook: a real valued function defined over some interval I is an increasing function if, a and b being any elements of I, $a < b$ implies $f(a) < f(b)$. Lagrange's (implicit) definition reads slightly differently, since he compared the values of f at 0 and at any other *given* value[7].

Cauchy's implicit definition of an increasing function can be rephrased as follows: a real-valued function f defined over some interval I is an increasing function if, a being any element of I, there is a neighbourhood N_a of a such that, for any x in N_a, the order between $f(a)$ and $f(x)$ is the same as that between a and x. Lagrange's definition is global, point-wise, and refers to two (arbitrarily, independently) given points; Cauchy's definition is one in which some local property holds in the neighbourhood of every (arbitrarily) given point. It can be shown - but it takes a little work - that both definitions are actually equivalent from a mathematical viewpoint. However, they differ significantly, both from an epistemological viewpoint (in which, for instance, the difference between local and global properties are put to the fore), and from a cognitive viewpoint [Chorlay 2007, 2011].

The fact that both definitions coincide from a mathematical viewpoint does not imply that proving that the first holds involves the same kind (and amount) of work than proving that the second holds. The information we start with (sign of the derivative) being of the everywhere-local-type[8], a mere rephrasing of the hypotheses leads to Cauchy's definition of increasing functions, hence to the conclusion. Reaching Lagrange's conclusion involves patching up local pieces of information to reach global conclusions, an endeavour which the modern reader knows to be usually tricky.

To conclude this paragraph, we must add that, on other occasions, Cauchy himself used the same kind of reasoning that Lagrange used in the above quoted proof. For instance, in the third volume of his *Cours d'analyse à l'école Polytechnique*, he set out to prove the existence of the solution to an ordinary, first degree differential equation $y' = f(x, y)$ in the neighbourhood of a regular point [Cauchy 1981[9]]. His proof relied to some extent on the same idea as Lagrange's: the derivative provides local affine

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[6]We could use the notion of "in-action definitions" [Ouvrier-Buffet 2011]. For an analysis of the notion of functional variation, see [Chorlay 2010]. For a more detailed analysis of Cauchy's proof, see [Chorlay 2007, 2011].

[7]It is however completely equivalent to the $a - b$ definition, since the specific value 0 plays no part in the proof.

[8]We do not need to distinguish here between ≪local≫ and ≪infinitesimal≫ [Chorlay 2007].

[9]These lectures were probably delivered in the 1820s, but were not published by Cauchy nor included in the *Oeuvres complètes*.

approximations of the required function; these affine approximations are to be patched up to form a piece-wise linear function; these are then taken to the limit as the subdivision step tends to zero. For the 21st-century reader, this proof has basically the same flaws as Lagrange's proof: continuity is assumed to be uniform; same for the convergence of the sequence of functions.

4 Bonnet's proof (in J.-A. Serret's textbook, 1868)[10]

Theorem I.- *Let $f(x)$ be a function of x which remains continuous for values of x between two given limits, and which, for these values, has a well-determined derivative $f'(x)$. If x_0 and X denote two values of x between these same limits, the following*

$$\frac{f(X)-f(x_0)}{X-x_0}=f'(x_1),$$

will hold, with x_1 a value between x_0 and X.
Indeed, the ratio

$$\frac{f(X)-f(x_0)}{X-x_0}$$

has, by hypothesis, a finite value; and, if A denotes this value, we will have

$$[f(X)-AX]-[f(x_0)-Ax_0]=0. \tag{1}$$

Let $\varphi(x)$ denote the function of x defined by the formula

$$\varphi(x)=[f(x)-Ax]-[f(x_0)-Ax_0], \tag{2}$$

then, from equality (1),

$$\varphi(x_0)=0,\varphi(X)=0,$$

so that $\varphi(x)$ vanishes for $x=x_0$ and for $x=X$. Let us assume, for instance, that $X>x_0$, and let x increase from x_0 to X; at first, the value of $\varphi(x)$ is zero. If we assume that this function is not everywhere zero, for values of x between x_0 and X, it will have to either begin to increase, thus taking on positive values, or begin to decrease, thus taking on negative values; be it from $x=x_0$, or from some other value of x between x_0 and X. If these values are positive, since $\varphi(x)$ is continuous and vanishes for $x=X$, it is obvious that there will be a value x_1 between x_0 and X such that $\varphi(x_1)$ is greater than or equal to the neighbouring values

$$\varphi(x_1-h),\quad \varphi(x_1+h),$$

h being an arbitrarily small quantity. (...)
This, in either cases, the value x1 will be such that the differences

[10]Unfortunately we did not use the 1868 edition, but a later edition. We know the editions do not differ, as far as the quoted passages are concerned (see [Dugac 1979]).

$$\varphi(x_1 - h) - \varphi(x_1), \quad \varphi(x_1 + h) - \varphi(x_1)$$

Will be of the same sign; consequently, the ratios

$$\frac{\varphi(x_1 - h) - \varphi(x_1)}{-h}, \quad \frac{\varphi(x_1 + h) - \varphi(x_1)}{h} \tag{3}$$

will be of opposite signs. (...)
Both ratios (3) tend to the same limit when h tends to zero, since we assumed $f(x)$ has a well-determined derivative; hence, so does $\varphi(x)$; besides, these two ratios are of opposite signs: hence their limit is zero. So, one has,

$$\lim \left[\frac{\varphi(x_1 + h) - \varphi(x_1)}{h}\right] = 0,$$

or, taking equation (2) into account,

$$\lim \left[\frac{f(x_1 + h) - f(x_1)}{h} - A\right] = 0,$$

i.e.

$$A = \lim \frac{f(x_1 + h) - f(x_1)}{h} = f'(x_1).$$

Therefore

$$\frac{f(X) - f(x_0)}{X - x_0} = f'(x_1),$$

as we claimed. (...)
Comment.- The above proof is due to Mr. Ossian Bonnet. It should be noticed that no assumptions are made as to the continuity of the derivative $f'(x)$; one merely assumes that it exists and has a well-determined value.
Theorem II.- *If function $f(x)$ is constant for all the values of x between two given limits, the derivative $f'(x)$ vanishes for these values of x. Conversely, if the derivative $f'(x)$ vanishes for all values of x between two limits, the function $f(x)$ has a constant value for the values of x between these limits. (...)*
Theorem III.- *If the derivative $f'(x)$ of function $f(x)$ remains finite for all the values of x between the limits x_0, if $X > x_0$, and if x is made to increase from x_0 to X, the function $f(x)$ will increase as long as the derivative $f'(x)$ will not be negative, and it will decrease as long as $f'(x)$ will not be positive.*
Indeed, since x lies between x_0 and X, the ratio

$$\frac{f(x \pm h) - f(x)}{\pm h}$$

has limit $f'(x)$, which is a finite quantity; so it will of the same sign as that of the limit, for values of h between zero and some sufficiently small positive quantity ε. Consequently, for these values of h, the following will hold

$$f(x-h) < f(x) < f(x+h)$$

if $f'(x)$ is > 0, and

$$f(x-h) > f(x) > f(x+h)$$

if $f'(x)$ is < 0.
Thus, the function $f(x)$ will increase, as from any value of x for which $f'(x)$ is > 0; and decrease, as from any value of x for which $f'(x)$ is < 0. [Serret 1900, 17-22]

In this passage, Serret introduced Ossian Bonnet's proof of the mean value theorem, a proof idea which relied on an affine change of variable and the vanishing of the derivative at a local extremum. The existence of the extremum is not proved (at least when one compares with later rewritings of this proof), but made obvious in the *narrative* style which is so typical of the first half of the 19^{th} century. Strikingly, Serret did not use the mean value theorem to establish the relationship between the sign of f' and the variations of f; he relied on Cauchy's argument, hence on Cauchy's notion of functional variation. However, the mean value theorem was used to establish theorem II. Actually, quite a few textbook writers made the same choice in the second half of the 19^{th} century. For instance, in the very Weierstrassian textbook by Genocchi and Peano, Cauchy's proof is given first; then comes the proof of the mean-value theorem, from which Serret's theorem II is derived [Genocchi & Peano 1889, 43].

5 Proof-analysis and regressive analysis

5.1 Proof-analysis: the role of uniform convergence

We identified in Lagrange's proof a flaw which can be described in several ways: implicit assumption of uniform differentiability; failure to notice that some variable is dependent on some other, while trying to consider the limit of second while leaving the first fixed; exchange, without due caution, of two limiting processes. The same flaws were common to most proofs in analysis which dealt with the numerical aspect of functions (as opposed to formal aspects); to name a few: Ampère's proof of the inequality form of the mean value theorem[11] [Ampère 1806], Cauchy's proof of the same [Cauchy 1823, 44-45[12]], Cauchy's proof that the limit of a sequence of continuous functions is continuous [Cauchy 1821, 120], Cauchy's proof of the existence of a local solution to a first degree differential equation in the neighbourhood of a regular point [Cauchy 1981, chapter 7] etc.
It is well known that the difference between point-wise and uniform[13] continuity (for a function), and point-wise and uniform convergence (for a sequence of functions) was stated in the clearest of

[11]Namely, that $|f(b) - f(a)|$ over $|b - a|$ is less than or equal to the maximal value of $f'(x)$ on $[a, b]$.
[12]The page numbers refer to the *Œuvres*.
[13]"*gleichmässig*" in German.

ways in Weierstrass' Berlin lectures on the foundations of analysis; a distinction which he attributed to his master Gudermann. The awareness of the importance of this distinction spread in the 1870s and 1880s among students, followers or readers of Weierstrass (Hurwitz, Cantor, Schwartz, Dini, Peano, Pincherle, du Bois Reymond, Heine, Thomae etc.). Of course, this awareness was displayed in a new generation of textbooks and research papers; it also spread through criticism of faulty proofs found in papers or textbooks written by the most distinguished mathematicians.
An instance of this is given by Peano's criticism of the proof of the mean value theorem, which he read in the first edition of Jordan's *Cours d'analyse de l'école Polytechnique*. The exchange of letters was published in 1887, in the *Nouvelles annales de mathématiques*:

> Mr. Jordan gave a not quite rigorous proof of the following theorem:
> "Let $y = f(x)$ be a function of x whose derivative remains finite and well-determined when x varies in some interval. Let a and $a+h$ be two values of x in this interval. We will have
>
> $$f(a+h) - f(a) = \mu h,$$
>
> where μ denotes a quantity between the largest and the smallest values of $f'(x)$ in the interval between a and $a+h$."
> Indeed, Jordan writes, let x take on a series of values a_1, a_2, ..., a_{n-1} between a and $a+h$; let us set
>
> $$f(a_r) - f(a_{r-1}) = (a_r - a_{r-1})[f'(a_{r-1}) + \varepsilon_r].$$
>
> Let us now assume that the intermediate values a_1, ..., a_{n-1} are indefinitely multiplied (and brought closer together). The quantities ε_1, ε_2,... will all tend to zero, since ε_r is the difference between $\frac{f(a_r)-f(a_{r-1})}{a_r-a_{r-1}}$ and its limit $f'(a_{r-1})$.
> The latter assumption is not correct; for
>
> $$f'(a_{r-1}) = \lim \frac{f(a_r) - f(a_{r-1})}{a_r - a_{r-1}}$$
>
> when a_{r-1} is assumed to be fixed, and a_r variable and approaching a_{r-1} infinitely closely; but one cannot claim the same when both a_r and a_{r-1} vary, unless the derivative is assumed to be continuous.
> Indeed, for instance, let us set
>
> $$y = f(x) = x^2 \sin \frac{1}{x},$$
>
> with
>
> $$f(0) = 0;$$
>
> its derivative

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

for $x \gtrless 0$, and $f'(0) = 0$, remains finite and well-determined, but it is discontinuous.
Let

$$a = 0, \quad h > 0;$$

let us set

$$a_1 = \frac{1}{2n\pi}, \quad a_2 = \frac{1}{(2n+1)\pi},$$

a_3, a_4, ... any numbers.
One then has

$$\varepsilon_2 = \frac{f(a_2) - f(a_1)}{a_2 - a_1} - f'(a_1);$$

but

$$f(a_1) = 0, \quad f(a_2) = 0, \quad f'(a_1) = -1;$$

hence

$$\varepsilon_2 = 1,$$

which does not tend to zero.
Nearly the same mistake was made by Mr. Hoüel (*Cours de Calcul infinitesimal*, t.I, p.145).
Eventually, I shall add that the formula

$$f(x_0 + h) - f(x_0) = hf'(x_0 + \theta h),$$

Can be established very easily, and without assuming the continuity of f'. [Peano 1884, 45]

In fact, very similar objections had been made a few years earlier to Jules Hoüel himself ! In the 1870s, Hoüel and Darboux co-edited the *Bulletin des sciences mathématiques*, and Hoüel asked Darboux for comments on the drafts of his lectures on analysis, which he would eventually publish in 1878. Darboux was well aware of the recent developments in Berlin, and criticized many of the "classical" proofs Hoüel planned to rely on. Unlike the Peano-Jordan exchange of letters, this correspondence remained private, and was partly published in the 1970s and 1980s [Gispert 1983]. Here is Darboux's view on Hoüel's Lagrange-style proof of the mean value theorem, in a letter dated February 4, 1874:

As to §52, which plays a fundamental part in your argument, I also find fault with it, namely: the ε quantities are functions of two variables. For instance, setting $x_3 - x_2 = h$,

$$f(x_2 + h) - f(x_2) = hf'(x_2) + h\varepsilon_2.$$

Clearly, ε_2 is infinitely small, and is a function of x and h about which *only* the following is known: it tends to zero with h, when x remains fixed; but then, I claim that you do not know what becomes of it when, as h tends to zero, x_2 varies with h, which is the case in your decomposition. For instance, consider

$$\frac{h_2}{x_1 - a + h}.$$

For any x_1, *so long as it remains fixed*, this expression vanishes with h. But if x varies, for instance if we have

$$x_1 = a - h + h^4$$

Then the expression simplifies into $\frac{1}{h^2}$, which becomes infinitely large as h tends to zero.
I am telling you this for I am deeply convinced that if you stuck more closely to rigour, you would come up with a treatise of infinitesimal calculus of exceptional interest.
If I were you, I would give up on the theorem on the limit of sums, which is worthless, just as many other things. With the mean value theorem, such as established by Serret, you could build a strong structure. This, along with the definition of the integral, is all one needs. This is how Weierstrass does it, I believe. [Gispert 1983, 89-90]

Needless to say Hoüel failed to be convinced, even though Darboux repeatedly voiced dissent and disbelief. He sent out several other counter-examples, and rephrased the main argument in various ways. Even though he did not use the terms "uniform convergence", the notion was perfectly clear to him, as we can see in this letter, dated January 18, 1875:

Here is where I find fault with your reasoning, which no one deems rigorous any more[14].
When setting

$$\frac{f(x+h) - f(x)}{h} - f'(x) = \varepsilon,$$

ε is a function of the two variables x and h which tends to zero when, leaving x fixed, h vanishes. But if x and h vary, as in your proof; even more, if every new subdivision $x_1 - x_0$ generates new ε quantities, I cannot see anything clearly any more, and your proof becomes only seemingly rigorous. (...) You could get out of this predicament in one of two ways, 1. By changing proofs altogether, which I advise you to do[15]. 2. By proving that if a function always admits a derivative between x_0 and x_i, one can find a quantity h such that for all values of x between x_0 and x_i, and all values x_0 and h_1 of h less than some limit value, one has

$$\frac{f(x+h) - f(x)}{h} - f'(x) < \varepsilon,$$

[14]This is an optimistic overstatement from Darboux.
[15]Darboux strongly recommended the Bonnet proof.

> where ε has a value which is fixed but chosen as small as one wishes; which is difficult[16]. [Gispert 1983, 99-100]

5.2 Regressive analysis: the role of the existence theorem for extrema

A critical mind might object to Bonnet's proof of the mean value theorem that it depends on the existence of a maximum or a minimum, an existence which is implicitly taken for granted. It seems clear that if the function is piece-wise monotonous (as seems to be assumed in the text), it will indeed admit either a local maximum or a local minimum; but a differentiable function needs not be piece-wise monotonous, as the ever useful example $f(x) = x^2 \sin \frac{1}{x}$ shows.
In fact, the existence of a maximum can be grounded without piece-wise monotony, or continuous differentiability, as Weierstrass established, for instance in his 1878 lectures on the theory of functions. The following passage has nothing to do a *priori* with calculus. It comes after the construction of the set of real numbers **R** (or, more precisely, the affinely extended real number line $\overline{\mathbf{R}} = [-\infty, +\infty]$) starting from rational numbers. The theorem about nested intervals was established on this basis, as well as the existence theorem of an upper bound (*obere Grenze*) for any non-empty subset of the extended real number system.

> Let a value y correspond to every point $(x_1, \ldots, x_n)$ of some domain; then y is also a variable quantity, hence has a lower and an upper bound; let g denote it. Then, there exists at least one point in the x-domain (that point needs not belong to the defined domain)[17], with the following property: if we consider however small a neighbourhood of that point, and consider the values of y corresponding to that x-domain, then these values of y also have an upper bound, this upper bound being exactly g. Similarly for the lower bound. [Weierstrass 1988, 91]

What we have here is a purely set-theoretic theorem; the function does not have to be continuous; no hypotheses are made on the domain (if it is not closed, the point with the remarkable property may lie on its boundary). We skip the proof, which relies on the definition of the upper bound, and the (now familiar) technique of nested intervals. Weierstrass then turned to an application of this very abstract result in "everyday" analysis:

> One is commonly faced with the question: among the values taken on by some magnitude, is there a maximum or a minimum (maximum or minimum in the absolute sense[18]). Let y be a continuous function of x, $y = f(x)$. Here, x must remain between two given limits a and b. In which circumstances is there a maximum and a minimum for y? There is an upper bound for y. According to our proposition, there must be some point x_0 in the x-domain such that the upper-bound of the values of y for x between $x_0 - \delta$ and $x_0 + \delta$ is

[16] The fact that, if the derivative is continuous, then $\frac{f(x+h)-f(x)}{h}$ *does* tend to $f'(x)$ *uniformly* on every closed and bounded interval was proved, for instance, in the second edition of Jordan's textbook [Jordan 1893, 68].

[17] Indeed, it may lie on the boundary of the first domain.

[18] i.e. not a local maximum or minimum.

> also g. Point x_0 either lies inside $a...b$, or on its border ($x_0 = a$, or $x_0 = b$).
> In the first case, $f(x_0)$ is a maximum. Indeed, $f(x_0)$ must be equal to g: for $f(x) - f(x_0)$ can be made as small as we wish, by choosing an adequately small $|x - x_0|$; on the other hand, since x lies between $x_0 - \varepsilon$ and $x_0 + \varepsilon$, $f(x)$ can be chosen arbitrarily close to g; hence $f(x_0) = g$. (If we had $f(x_0) = g + h$, we would have $f(x) - f(x_0) = f(x) - g - h$, and $f(x)$ could not come arbitrarily close to g if h was not 0).
> If x_0 coincided with either a or b, then we could only claim that $f(a)$ (resp. $f(b)$ is a maximum if $f(x)$ was continuously at a (resp. b) as well. [Weierstrass 1988, 91-92]

A similar proof can be found in other texts, sometimes published before 1878, but all deriving directly or indirectly from Weierstrass' Berlin lectures: [Cantor 1871], [Heine 1872 186], [Darboux 1872]. Fifty years later, the full conceptual clarification of the notion of maximum would still be considered one of the achievements of Weierstrass' work on the foundation of analysis, as is shown by the first lines of Hilbert's famous 1925 paper *On the Infinite*:

> Weierstrass, through the critique elaborated with the sagacity of a master, created a firm foundation for mathematical analysis. By clarifying, among other notions, those of minimum, function, and derivative, he removed the remaining flaws from the calculus, cleansed it of all vague ideas concerning the infinitesimal, and conclusively overcame the difficulties that until then had their roots in the notion of infinitesimal. [Hilbert 1967, 369]

6 Conclusion

Let us attempt to summarize the pretty intricate network of definitions, proof-ideas (or proof-germs), proof-techniques, and proof-analyses displayed in this sample of texts.
At least two *definitions* of what it means for a real-valued function to "increase" can be found in the 19^{th}-century: a point-wise and global definition which can be found in Lagrange; a definition that relies on an everywhere-valid local property, which can be found in Cauchy. If we stick to Cauchy's definition, then the proof of the theorem about the relationship between the sign of f' and the variations of f is pretty trivial. If we want to reach the Lagrange-style conclusion, then much more work is needed, since one has to start from an everywhere-valid local property (sign of f') and reach a global conclusion.
To reach that conclusion, we saw two very different *proof-ideas*, namely Lagrange's and Bonnet's. In the proof we studied, and in quite a few other parts of his work, Lagrange distanced himself from the formal manipulation of formulae (finite or infinite), and engaged in numerical proof: he relied on the correct numerical understanding of the notion of limit; on this basis, he cautiously built networks of inequalities; he finally endeavoured to ground his reasoning on the determination of upper bounds for the errors in a process of affine approximation. In the first half of the 19^{th} century, many proofs of the most important theorems in function theory were written along this line. Distrust of this proof-scheme spread as mathematicians grew aware of the distinction between point-wise and uniform (continuity, convergence). They spread all the more slowly since the theorems were correct,

the building blocks of the proofs showed a proper understanding of the notions at stake, and local counterexamples[19] were hard to find. As Darboux insightfully (but to no avail, as far as Hoüel was concerned) stressed, there were only two ways out of this predicament: either to change proof-germs, or to establish uniformity[20].

For the theorem on which we chose to focus, an alternative proof became available in the 1860s, which relied on a completely different proof-idea; unlike Lagrange's proof, it did not rely on what the derivative of a function at a point *is* (a limit, which provides some local affine approximation), but on a *property* of the derivative (stated in the mean value theorem). Some elements of Bonnet's proof were later seen as insufficiently grounded, in particular the existence of a minimum or a maximum; in the 1890s, mathematicians such as Jordan or Stolz used Weierstrass' analysis of the set-theoretic properties of the real line to back up that weaker step in Bonnet's proof.

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USING HISTORY FOR *MATHS IN EFL* TEACHING
Original texts and cultural approach in the classroom

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ABSTRACT

A new kind of mathematics course in France is a part of "European studies", including one or two hours a week of math in English as a Foreign Language (EFL). The major aims are to provide the students with a solid basic knowledge of words, methods and practices of mathematics that are studied in other European countries and/or Anglophone countries, and to allow a cultural approach to mathematics.

The use of original English texts is perfectly adapted to this purpose, as parts of the general culture and history of England; it's a good opportunity of working together with teachers of English civilization and History teachers.

In this paper we show several examples of original texts that we have used in the classroom with students from 10th to 12th grade (aged 15 to 18), as well as the use of this kind of texts in final exams.

1 English as a foreign language (EFL) and mathematics teaching

As it may be the case in all other countries, teaching foreign languages in France is a specialty of foreign language teachers. However, a recent change gave high school students the opportunity of taking a part of their mathematics course in foreign language (namely English, which is the most important, but also German or, more rarely, Spanish); this part is not necessarily linked with the ordinary curriculum, because it was mostly intended to improve the students' abilities in speaking and writing English.

The "European sections" are not widely spread, as only several high schools in every school district are allowed to create one, despite the constant increase of the students' and families' interest. Even if the original intentions of the Government were generous (and then rather costly), the high schools were not free to open European sections, and consequently, their number is desperately small.

Indeed, the abilities in foreign language speaking have traditionally also been rather inadequate in France (the fact is well-known) and several attempts to correct this have been made, including the creation of European sections; but the families tend to sign their children up for these classes to allow them to be part of the elite classes in the school···

1.1 Mathematics in the European section

The main problem (and the most interesting challenge!) for math teachers in "Euro" sections is their freedom to choose the contents they will teach. The exam is part of the Baccalaureate pupils take at the end of their third year at the *lycée* ("Terminale") when they are 18, and the special euro exam is created by the group of teachers themselves, which is not ordinary in France.

So, the *euromaths* teachers do not have to follow any curriculum, and they tend to favour "good old mathematics", like geometry, as well as useful (discrete maths, graph theory ⋯) or "foreign" ones (exotic calculations, matrices ⋯). The Baccalaureate is also a good opportunity to let the students face everyday problems using probability or investigating some kind of recreational mathematics, as flight charges or shoe lacing.

In my practice, I take a cultural approach to foreign civilizations trough historical texts, books or papers on the history of mathematics. Inspired by previous works by Leo Rogers and Peter Ransom[1], I use different texts from the 16th century England in the classroom, especially practical texts allowing students to apply ancient techniques in computations and measurements as well.

1.2 The choice of original texts

There are many ways to link mathematical contents to the English-speaking cultures, when you choose your documents amongst the huge quantity of books you can find on *Google Books* for instance, or in National Libraries websites[2]

One of the tricks I use to get the students involved is to choose the books according to the destination of our school trips: John Napier's *Rabdologia* when we sailed to Edinburgh, Leonard Digges' *Tectonicon* because we need to know the height of the main tower in Doune Castle (in order to attack it as a tribute to Monty Python's film about King Arthur's quest[3]), Voster's *Arithmetic* published in Ireland, and so on.

But why do we need original texts when present day textbooks would be enough to introduce all the notions we use? Well, the problem with our textbooks is that they do not focus on language and thus it is not easy to gain knowledge of vocabulary and idiomatic expressions through a mere reading. Moreover, you hardly learn anything about Anglo-Saxon cultures through contemporary literature on maths!

2 Getting to know the vocabulary: John Kersey's *Algebra*

The first year students know but few words about mathematics: the first hundred numbers, the classical shapes; but in geometry as well as in arithmetic or algebra, we have to know the right words, and use them properly. A simple French-English dictionary would be inappropriate, because we want our students to progressively become able to think in English naturally, and not to think in French first and then translate their ideas into English.

This objective makes it necessary to practise the language in situation, reading and analysing texts without exactly knowing what the vocabulary is. A cover/discover/uncover strategy is used with success with extracts of the *Algebra* of John Kersey; at the beginning of this book, Kersey stresses the relationship between formulae and expressions in ordinary language. For the classroom, a slideshow

[1] See bibliography: Rogers (2006) and Ransom (2006).

[2] example, the website of the Royal Society of London offers a large part its Philosophical transactions online. You can find a variety of references on the website of the (virtual) European Library too.

[3] *Monty Python and the Holy Grail* (1974) is also about mathematics: a famous scene deals with logic and burning witches⋯

is made using pages of the original book, covering the answers; every student of the class has to fill in one of the blanks by telling the answer aloud. Everyone can try and prepare their answer.

In the example below, we use chapter 10 (*A collection of easie questions*, that is a set of exercises), in which the author the author begins with the definition of two quantities, *whereof the greater is a (or 3), the lesser is e (or 2)*, and follows with several questions, namely English expressions that must be translated into formulae (*by Letters*) or numerical results (*by Numbers*). The columns on the right are covered first and uncovered gradually to check the correctness of the answers (or give them when no answer comes).

	Anſwers by Letters,	*by Numbers.*
1. The Sum of the two Quantities propoſed is . . .	$a+e$	5
2. Their Difference, or the exceſs of the greater above the leſs, is	$a-e$	1
3. The Product of their Multiplication is	ae	6
4. The Quotient of the greater divided by the leſs is . .	$\frac{a}{e}$	$\frac{3}{2}$

As the chapter contains a variety of exercises, from very basic algebraic expression to more complex ones, we can browse the complete vocabulary for beginners, and sometimes students can discuss the answers.

3 Old English texts and practical mathematics

One of the first activities I propose to my second year students consists in analysing an old text about geometry, focusing on the language and trying to find the present equivalents of some weird words.

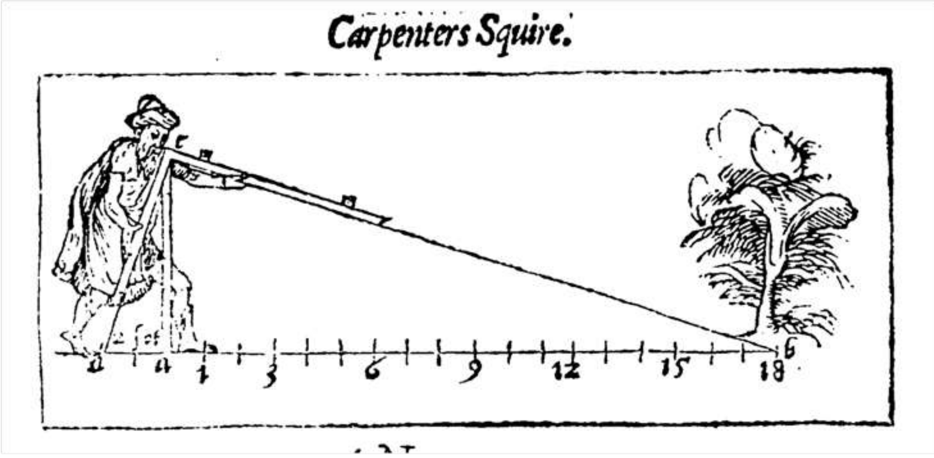
The first contact is always a sort of amazement, both about the typographic issues and the expressions that are used. Indeed, it is not that easy to decipher Digges on measurements of the towers height! This is the object of the first activity sheet: can you correct the vocabulary and make sense of the maths?

3.1 Leonard Digges' *Tectonicon* and *Stratioticos*

In his book *Tectonicon*, Leonard Digges gives general definitions of the geometric shapes and concepts, along with several methods to measure lengths, heights, widths (because these different lines are not considered the same) and then areas and volumes. The text is to be read aloud in order to understand the words⋯

How lengths in plaine Ground are searched by the Carpenters or Masons Squire. The staffe .a.c. in this fygure is imagined 6 fote, & *the space .a.d. 2. Foote. Consideryng nowe that .6. the lengthe of the staffe conteyneth .2. thrise, therfore the longitude desiered .a.b. of force muste conteyne thre tymes the staffe (whiche staffe is .6. foote) that maketh .18. foote. As this is proved true by a small grounde in the fygure folowynge: so the Arte fayleth not in a greater space, whiche the good speculator and diligent practiser by anye waye canne not denye.*

Does the Art really prevent failure? The practice of geometry on the field is highly recommended as a sequel to the study of texts in the classroom; and it is a good way to test the accuracy of actual measures of distant lengths or heights. I organised several out-of-classroom workshops for 20 years, always pretending we had an urgent need of measures in order to attacks castles or fortified cities. In



February 2012, during our school trip to Scotland already mentioned, the challenge was to evaluate the height of the main tower of the Castle of Doune: the students had studied a part of *Tectonicon* before, as well as other books about practical geometry, and they knew the principal techniques they could use: proportionality of triangles or use of the mirror. Actually, we discovered they used a lot of different tricks to meet with the challenge: as they had no stick, no astrolabe nor any mathematical instrument, they used their cameras (the height of a pupil they know, compared to the height of the tower), and their own bodies (someone standing on a bench, someone lying on the ground and searching alignment between the top of the tower and the top of the other one). Of course they found different values (from 20 to 30 meters), and we will have to draw conclusions about this; fortunately, we were able to get entrance to the Museum of the University of Saint Andrews and have a look at showcases presenting astronomical instruments; the students then could see by themselves that more and more preciseness was a constantly growing concern for instrument makers.

For the same purpose of interpreting old words, the other book by Digges that we use in the classroom is *Stratioticos* (1579), which deals with military matters, including problem solving from an arithmetical point of view. Students find it more difficult to read because of the lack of pictures (!), and the old units of measure they have to translate into modern ones:

Certaine Questions touching the Office of the High Marshall and Campe Maister

The firste question: Admit I finde by experience that 3000 footemen may commodiously be encamped in a plat of grounde 300 pace Square. I demaund how many pace the grounde shold be square that shall receyve 1[0]000. Footemen. [Answer: between 549 and 550]

In both cases, the teacher has to help students find their way through the old problems, old units and old fashion in general; who cares about the depth of a well nowadays? Who needs to know the space occupied by the infantry on the battlefield?

3.2 Robert Record's and John Napier's calculations

It is interesting to have the students read original texts in their original form, especially when they are printed in Old English font: just allowing the teacher to keep the contents at a certain distance and let the students cross the language door to reach the hidden mathematics: turn familiar into uncanny.

For instance, hereafter is the detailed multiplication of 2036 by 23 *by a checker table*, as it is explained by Robert Record in *The Ground of Arts*, which would be currently published in the same period as Shakespeare's *Romeo and Juliet*, but with greater success!

First I consider that my greatest nomber hath .iiii.fygures or places, and therefore I make so many roumes betwene lynes, thus. Then I see that of my multyplyers there are .ii. Wherefore I drawe so many lynes a crosse the other, that there may be 2 roumes betwene them. Then draw a crosse barre thorough every close square, so that it may reche down to the lowest overth warte lyne, as in this fourme.

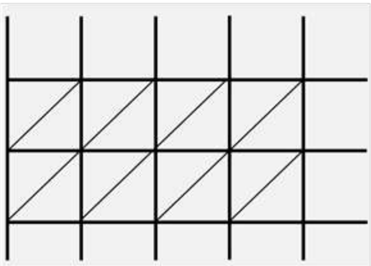


Figure 1: *

The empty checker table

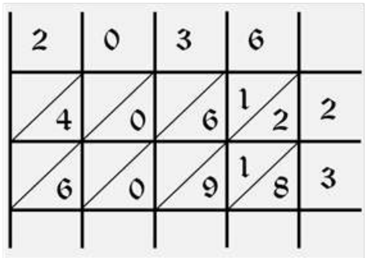


Figure 2: *

The complete one

As you can see, this is the well-known multiplication *per gelosia*; you can verify the partial products and the final result that is given by adding the numbers in the oblique sets of cells [NB: Record gives result 46,828]

This activity leads us naturally to the study of Napier's bones; logarithms should be a logical sequel, but it has to be left for the third year (and we haven't had time to undertake this activity before writing this paper). Actually, it was a part of the project of the same school trip to Edinburgh: the students saw two copies of Napier's bones sets in The National Museum of Scotland (see pictures below), and they could note that the artefact they had learned the use of was really in use.

4 Strange mathematics

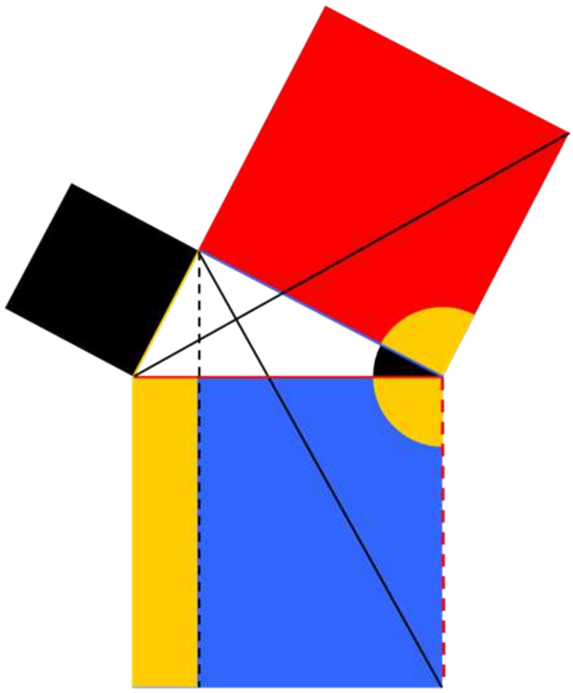
Some of the matters we study in the euroclass are different from the mathematics in the main course in French. We pick subjects in newspapers, on YouTube or in recreational or serious books, but none of them has the special taste of the 19th century England inventions or strange and forgotten ways of thinking the usefulness of mathematics.

Oliver Byrne's edition of Euclid's *Elements* was printed in colours, which was quite rare at the time; moreover the book was written in favour of the girls, who were illiterate then and consequently not able to read mathematics; that is the reason why Byrne wrote but a few words, letting the figures speak for themselves. In the same period, Thomas Fowler invented a system to perform calculations rapidly for Poor law Unions (Parish charity), but this ternary system has been completely forgotten, after the success of binary system in computers. In both cases, the context of 19th century England is worth the study, in order to link maths to social studies in a historical point of view.

4.1 Oliver Byrne's Euclid

Studying this book with can show students how they should use colours in their drawings or think differently the relation between the text and the illustrations. An interesting follow-up is the redesigning of usual geometry theorems, using only strictly necessary words, without naming the points but showing the shapes in the text.

The picture on the right is used to sketch the proof of Pythagoras's theorem, as done in the 47th proposition of Euclid's *Elements*, book I. The points are not labelled and the shapes are quoted as they are: for instance, the theorem would be expressed in terms of red square, black square and blue/yellow square. The text following the picture is full of shapes: the segment lines according to their colours; the squares, with respect to their colours and positions (but not sizes); the yellow angles, each one paired with the black one; the similar triangles that are used in the proof.



The main idea in the proof is expressed very simply: the (area of) the yellow rectangle is equal to the (area of) the black square, and similarly the (area of) the blue rectangle is equal to the (area of) the red square.

Finally, as the (area of) square on the hypotenuse is the sum of (the area of) the two rectangles, the theorem follows. Isn't that an interesting way of simplifying the text?

4.2 Fowler's *Balanced Ternary*

The main part of the lesson was about bases in general, with examples in binary and hexadecimal systems; it didn't use original texts except the famous paper by Leibniz about binary (Leibniz, 1703). In a search of topics for Baccalaureate subjects, I read a paper in Scientific American, it was about "the most efficient integer base", which appeared to be base 3; a remark was made about Fowler's machine and allowed me to close my slideshow on bases with a historical presentation.

Thomas Fowler (1777-1843) was a self-taught mathematician, who lived in Great Torrington, Devon, UK. As he explains in the preface of his *Tables for Facilitating Arithmetical Calculations* (Fowler, 1838), he published them *chiefly for the purpose of facilitating the very troublesome Calculations, which occur every Quarter in making up the Accounts of Poor Law Unions. Having [him]self been employed in the Torrington Union, to make up the Accounts, at the commencements, [he] found the most troublesome part of the business [⋯].* The Balanced Ternary system doesn't use the digits 0, 1 and 2 as does the usual ternary system, but 0 (which is denoted by $\odot$), $+$ and $-$, provided that every integer can be written in a unique manner as the sum or difference of powers of 3.

For instance, if you want to express the number twenty-three in ordinary ternary, you will write it as $212_{(3)}$, because $23_{(10)} = 2{\times}9 + 3 + 2 = 2{\times}3^2 + 1{\times}3^1 + 2{\times}3^0$, whereas in balanced ternary, as you don't use the digit 2, you write twenty-three as $+\odot --$, because $23_{(10)} = 27 - 3 - 1 = 1{\times}3^3 + 0{\times}3^2 + (-1){\times}3^1 + (-1){\times}3^0$. When the students are familiar with the system, I can show them an excerpt from Fowler's *Tables*:

I may now conclude, with an Example of Multiplication, by the Ternary Scale, which scarcely requires any mental exertion whatever; no Multiplication, nor even Addition, is required, as ordinarily practised.

Multiply	$+\odot--+-+$	=	628
By	$+-+\odot\odot-\odot$	=	564
	$-\odot++-+-\odot$		2512
	$+\odot--+-+\odot\odot$		3768
	$-\odot++-+-$		3140
	$+\odot--+-+$		
	$+-\odot\odot\odot\odot\odot\odot--+-\odot$	=	354192

Did the reader try to verify the result in Balanced Ternary? That is what the students have to do, and then construct the addition and multiplication tables in Balanced Ternary; eventually, they have to perform several computations and conversions in Balanced Ternary: great fun with strange objects! Who would have time and motivation to create such a system nowadays?

5 The final touch: History of mathematics in the exams

The euromaths exam takes place every year in June, a couple of weeks before the other session of the Baccalaureate; it is an oral examination and, more precisely, an oral examination on foreign language. The students are given (short) texts on a variety of subject in everyday mathematics, history of mathematics, etc., and they have to comment the text they've got to study, answering several questions too; the total time is 20 minutes for each student; the jury is composed of a teacher of mathematics and a teacher of English

The original texts are usually adapted because of their difficulty, and they deal with rather easy matters, because the English teacher might be alone (official communication!) In this purpose, simple historical texts are highly valued, because usually make it easy for the English teacher to engage discussion on non-mathematical matters⋯

5.1 Record's multiplication

It is a subject from the 2009 Baccalaureate session: *In 1542, the Welshman Robert Record published The Ground of Arts, in which he showed how to multiply two numbers between 5 and 10. Here is the multiplication* *of 8 by 7:*

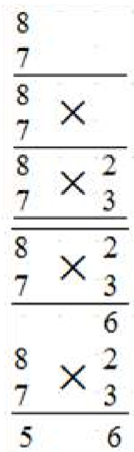
First set your digits one over the other:

Then from the uppermost downwards, and from the nethermost upwards, draw straight lines, so that they make a St. Andrew's cross:

Then look how many each of them lacks of 10, and write that against each of them at the end of the line, and that is called the difference.

Multiply the two differences, saying, "two times three make six", that you must ever set down under the differences.

Subtract one difference from the other digit (not from his own),as the lines of the cross warn you, and write it under the digits. You can take one or another, for all is like: if you subtract 3 from 8 or 2 from 7, it remains 5. So 7 multiplied by 8 is 56.



The text was adapted from Richard J. Gillings' *Mathematics in the times of the Pharaohs*; the follow-up questions were these:

1. Using Record's method, multiply 6 by 9.

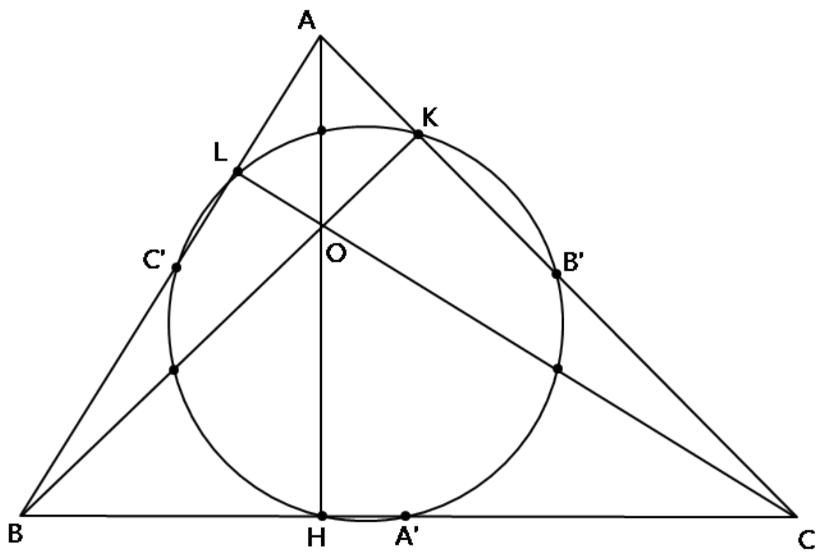
2. In Record's last explanation, we can read: You can take one or another, for all is like. Why is it true?

3. Prove that Record's method is true, by choosing two digits a and b and showing that the final line contains the product ab.

You can see that the examination is based on the comprehension of the text.

5.2 The Feuerbach circle

Another example of subject, from the 2010 Baccalaureate session: *In every triangle, the three midpoints of the sides, the three base points of the altitudes, and the midpoints of the three altitude sections touching the vertices lie on a circle.*



The proof consists of two steps: in the first we demonstrate that the circle circumscribing the triangle of the three midpoints of the sides passes through the base points of the altitudes; and in the second we show that the circle circumscribing the triangle of the altitude base points passes through the midpoints of the altitude sections.

<u>Step 1</u>*: Let A', B' and C' represent the midpoints, respectively, of sides BC, AC and AB. Let H be the base point of the altitude AH. Then the trapezoid $HA'B'C'$ is isosceles and it is therefore a quadrilateral inscribed in*

a circle, that we will name C. In the same manner we would demonstrate the other altitudes base points, namely K and L, lie on circle C, circumscribing triangle $A'B'C'$.

Step 2: Let the altitudes of the triangle ABC be AH, BK, CL, and O their point of intersection. We will now show that the centre of each altitude section touching a vertex, let us say section OC, also lies on circle C. [End of this proof to be completed in q3]

The text was adapted from Heinrich Dörrie's books on the great problems of Mathematics, published in 1965. That is already history, isn't it? (Mind your answer, I was already born in 1965⋯) The questions were these ones:

1. What are the "midpoints of the three altitude sections touching the vertices"?

2. In step 1, explain why $HA'B'C'$ is a trapezoid and why it is isosceles [Hint: HC' is the radius of the circle which has AB as its diameter.]

3. Show that an isosceles trapezoid is always inscribed in a circle (you can sketch the situation; it may be useful to consider some perpendicular bisectors.)

4. Completing step 2: a) Consider triangle OBC: what are its altitude bases? What is their circumscribed circle? b) Apply the result of step I to triangle OBC: what is the circumcircle of these 3 points?

But this subject wouldn't be given anymore, because the present day students are not trained in geometry as much as they used to be some years ago. In fact, the euromaths programme is a space of freedom for the teachers and the students that are eager to learn deeper mathematics. Our current reform of the curriculum tends to skip the difficulties for all the students to succeed in their high school exams, but the counterpart of this is that the pleasure of meaningfulness little by little disappears. Teaching maths in English as a foreign language allows teachers to be creative and lead our students to "old fashioned" mathematics, including historical texts. Old fashioned maybe, but so tasty!

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FORTIFYING FRANCE: LES VILLES DE VAUBAN

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ABSTRACT

This workshop will focus on work done with many students aged between 12 and 14 over the past 4 years. It will be presented by the Marquis de Vauban (aka Peter Ransom), a 17^{th} century Marshal of France in period costume.

The workshop deals with various topics: the geometry of fortification, based on plates from an 18^{th} century French mathematics book by Du Chatelard, P. (1749) and associated works; 3D transformation geometry; enlargement (dilation); ratio and quadratic formulae leading to the study of projectiles. It stresses the fact that mathematics is an integrated subject and one in which students should appreciate the interconnectedness of the various topics.

Accompanied by images (stills and videos) of French towns and cities fortified by Vauban, participants will have many opportunities for creative work. They will make a pair of proportional dividers and experience how these have been used in the mathematics classroom. It will be explained how the work has expanded to include the use of siege engines.

Handheld wireless technology will be used and the impact this has on students' learning will be evidenced. Participants will be given a CD-ROM of all materials used over the past 4 years.

KEYWORDS

3D transformation geometry, cross-curricular, enlargement (dilation), fortification, geometry, graph transformation, projectiles, quadratics, ratio, Vauban.

1 The impetus of this work

I incorporate events from history into mathematics lessons because I find it very interesting to see the practical applications of mathematics set into the period when it was used.

Using Jankvist's and Grattan-Guiness' previous works (Jankvist 2009, Grattan-Guiness 2004), Tzanakis & Thomaidis (2011) classify the arguments and methodological schemes for integrating history in mathematics education and this episode fits into the two-way table mainly as *History-as-a-tool* and *Heritage* though there are overlaps into the *History-as-a-tool* and *History* cell (*op.cit.* section 4, Table 1). The over-riding concept in my work is *History-as-a-tool*.

The reliability of my work in the sense of reproducibility by someone else is impossible to quantify, since teachers use such episodes in different ways with different students and probably not in costume! Every session I do with students is different according to local conditions and the knowledge students bring to the sessions, so any classroom outcomes will, of course, vary.

2 Vauban and his fortifications

It was Frédéric Metin who first introduced me to French fortifications. His paper (Metin 2002) deals with how fortification was integrated into the curriculum at the Jesuit colleges. Then and now this is an incredibly rich area for geometry. Marshal Sébastian le Prestre de Vauban, or Vauban (1633-1707) as he is better known, spent years under Louis XIV, the Sun King, fortifying towns and cities in France. He rose to prominence as an engineer during campaigns of the 1650s then set to work to reconnoitre the defences of France. After the War of Devolution (1667-68) he took the lead in planning the sieges and fortress building on the Belgian frontier. I was fortunate to acquire an old mathematics text (Du Chatelard, 1749) that contained a treatise on fortification (as well as gnomoniques (sundials) - another interest of mine) and the plates intrigued me so much I developed a series of lessons based on them.

The number of fortresses that Vauban designed is unknown. Estimates vary between 60 and 330 - it all depends on how much work Vauban put into the fortresses before one counts them. There are considerable differences between the number of complete new fortresses built from scratch such as Neuf Brisach (normally quoted as 8 or 9), the number of improvements to existing fortresses, and the number of ideas for future work that he laid out for others to build.

3 The school-based work

Four of the five plates on which the initial work was based are shown in the appendix together with the questions asked about each one. The other one with which I start the work is shown here.

I use this plate with students asking them about the symmetries of the shapes and how they would draw them accurately using dynamic geometry software or otherwise. For this we use the TI-nspire

Fig 1: The author (left) with Frédéric Metin below the Vauban statue in his natal town of Saint Léger Vauban

CX hand-held technology since it allows them to use appropriate software within their own classroom without having to move to an IT room. By using the wireless Navigator software I can capture their screens and see how students are progressing.

I give the following information to students about the above plate. It refers to campaign forts which are used when armies are on the move. 1 toise was exactly 6 pieds (feet), i.e. about 1.949 metres in France before 10 December 1799, so I ask them to take 1 toise to be 2 metres. I ask them to describe the symmetries of shapes f.23 to f.30. There is a short discussion about f.25 and we realise that the printer has probably deliberately omitted part of the fort at corner b to get the whole plate onto one page. The construction marks on one side of f.28 allow students to see how each side is divided to

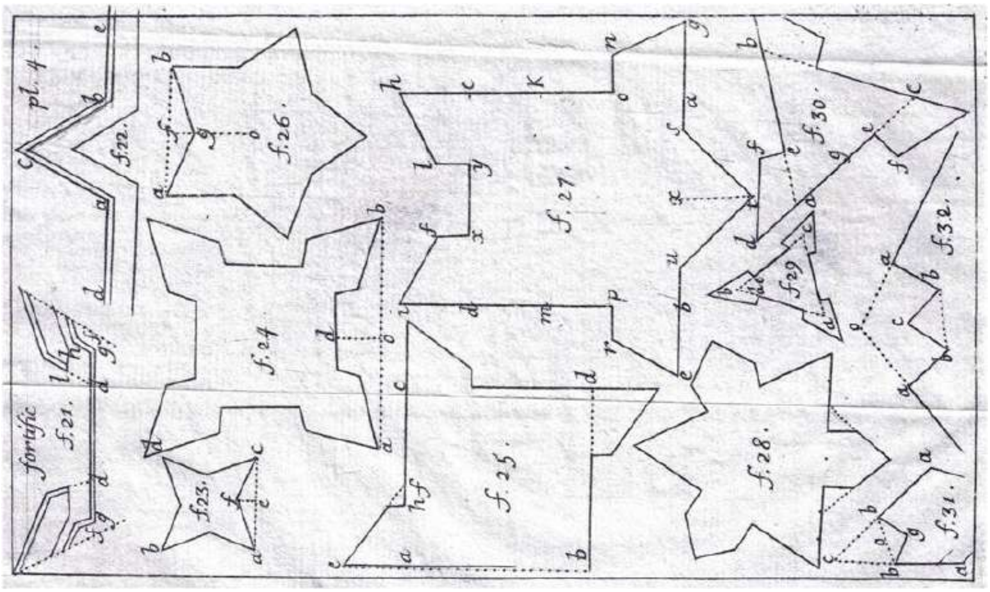

Fig 2: Plate of campaign forts, rotated for convenience. Note construction marks on f28

obtain the bastions. Students are asked to describe how this fort is constructed since mathematical communication is important. I ask students to draw the 'four star' fort of f.23, given that ac is 12 toises, and *ef* is 2 toises (Vauban's work states that the indent should be 1/6 of the side length), then to calculate the area of the fort in m^2.

3.1 Developing mental geometry

This is a neglected part of school mathematics. To encourage students to develop their mental geometrical skills I show them the following simple figure based on a 5 by 5 square array of dots, drawn large enough for the whole class to see.

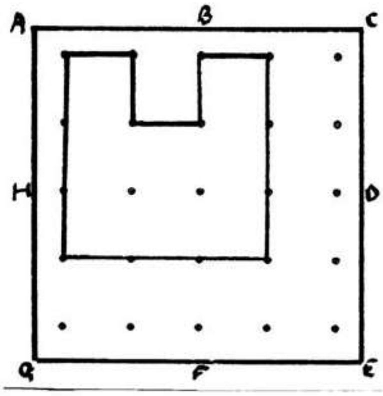

Fig 3: A simple figure ready to be rotated about an axis of symmetry

I hold the board at B and F and give it a half turn so that the class now sees the back of the board. They then have to draw exactly what I can see. Students can use IT and wireless technology or draw the result on a piece of dotty paper - both systems work well. After they have drawn their attempt, I turn round so that they can see the result. This is repeated with hands at HD, then at CG. For a challenge rotate the board 180° with hands at AE then follow this immediately by a 90° rotation with the picture still facing you!

Initial success with this task is rare - however repeating this every month or so soon sees increasing success with mental geometry and students need to appreciate that since we live in a 3D world things can appear differently depending on our viewpoint.

3.2 Proportional dividers, enlargement (dilation) and ratio

Few mathematics teachers have heard of proportional dividers (and even fewer used them), yet they date back many years. Heron of Alexandria (1st century AD) has been credited with a device that was probably an early version of a fixed proportional compass. In 1565 Jacques Besson in Lyons describes and illustrates a slotted variety with legs engraved with linear scales.

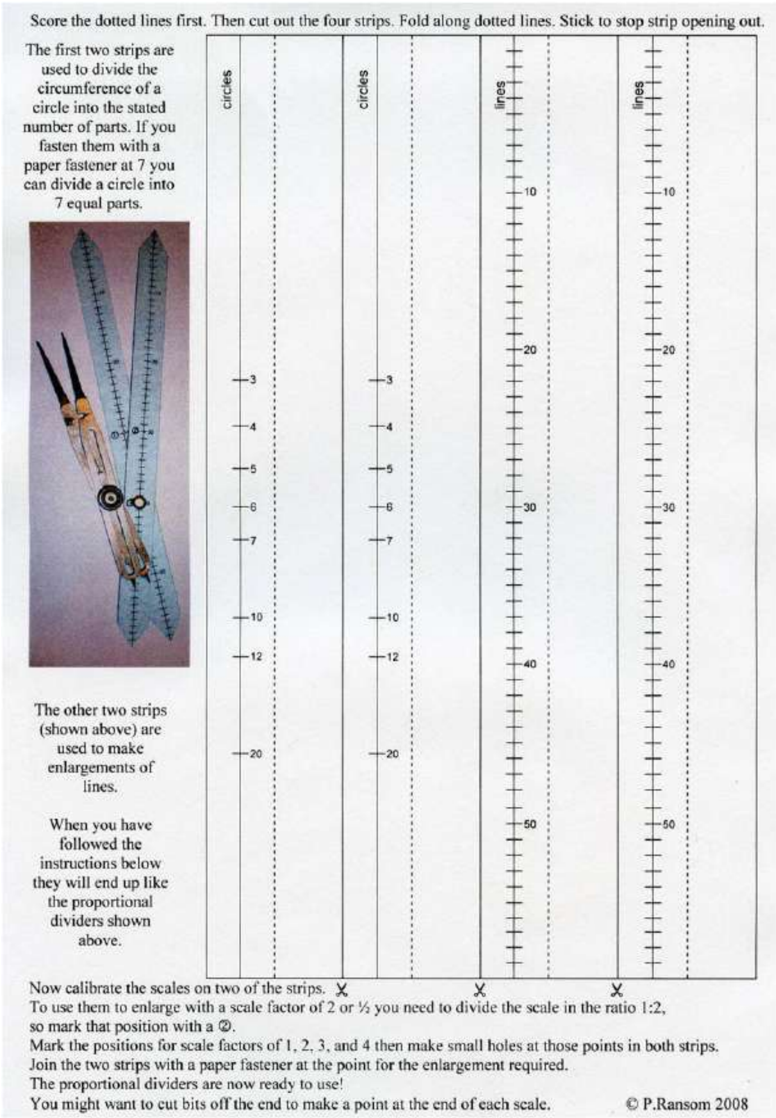

Fig 4: 9 inch brass proportional dividers and student made version from card with instructions for making one from card

I have a set of brass proportional dividers that students use to divide a circle into seven equal parts. The proportional dividers have four scales: *lines* that enlarge lines with a given factor; *circles* that divide circles into a given number of equal parts; *planes* that enlarge areas with a given factor and *solids* that enlarge volumes with a given factor. Once students have divided their circle into seven parts they then construct a seven pointed star fort based on Vauban's indent of one sixth of the side and the use of the brass proportional dividers. Here are the instructions the students follow.

Today you will draw a plan of a fort in the style of Vauban, the French engineer who fortified many French towns in the 17*th* *century.*

You will use an old mathematical instrument called 'proportional dividers' to construct a regular heptagon (a 7-sided shape with equal sides and angles).

BE CAREFUL!

These instruments are sharp and expensive so take care not to damage them or yourselves!

1 Loosen the screw

2 Move the slider so that the line on moveable part lines up with the 7 on the CIRCLES scale.

3 Tighten the screw - not too tight!

4 Open the dividers so that one of the long points touches the centre of the circle and the other touches the circumference of the circle.

5 Place one of the small points on the circumference so it makes a small mark.

6 Keeping that point fixed rotate the dividers and make a small mark where the other point meets the circle.

7 Keeping that new point fixed rotate the dividers again and continue until you are back where you started.

8 Using a ruler and pencil join up the marks you made.

9 Find the perimeter of the heptagon (that's the distance all the way around)

10 Now close the dividers.

11 Loosen the screw and set the slider so that it lines up with the 2 on the LINES scale.

12 Tighten the screw - not too tight!

13 Open the dividers so that the long points touch the ends of one side of the heptagon.

14 Now place one of the small points on a vertex and make a mark with the other point on a side.

15 Repeat with all the sides.

16 Join these mid-points to the centre of the circle with a feint line.

17 Now close the dividers.

18 Loosen the screw and set the slider so that it lines up with the 6 on the LINES scale.

19 Tighten the screw - not too tight!

20 Open the dividers so that the long points touch the end of one side of the heptagon.

21 Now place one of the small points on the mid-point of a side and make a mark with the other point on the line that goes to the centre.

22 Repeat with all the sides.

23 Join these new points to the vertices of the heptagon.

24 Find the perimeter of this 14 sided shape.

They then make their own pair of proportional dividers for enlarging lines from a sheet of A4 card. This introduces the concepts of dividing a scale in a given ratio and similar triangles. To get the proportional dividers to enlarge with a factor of 2, the scale needs to be divided in the ratio 1:2. Since the scale divides the length into 60 parts it is equivalent to dividing 60 in the ratio of 1:2 and so students add the ratio parts and divide 60 by that sum and so can pin the two legs at either 20 or 40 to give the desired enlargement. They then work out where to put the pin to enlarge with a factor of 3, 4, 5 etc. and as an extension with a factor of $1\frac{1}{2}$. There is the 'wow!' factor when they check that it actually does work with the scale factor of 2 - the number of times one hears students say 'It works!' never ceases to please me. Part of the fact that is does work is that it does not matter about how accurately the student cuts out the pieces because the central scale is not affected - the other reason it does work is because it is mathematics!

3.3 Projectiles, quadratic graphs and their transformation

The first plate in the appendix gives a scaled cross section of a fortification and the final lesson revolves around using IT to get a projectile from a given point outside the defences to a specific point inside using a quadratic graph. The document used is described in a series of screen shots shown below and students work through these dynamic pages answering questions about the graphs and their transformations. It finishes with a worksheet for them to identify the graphs of various quadratics.

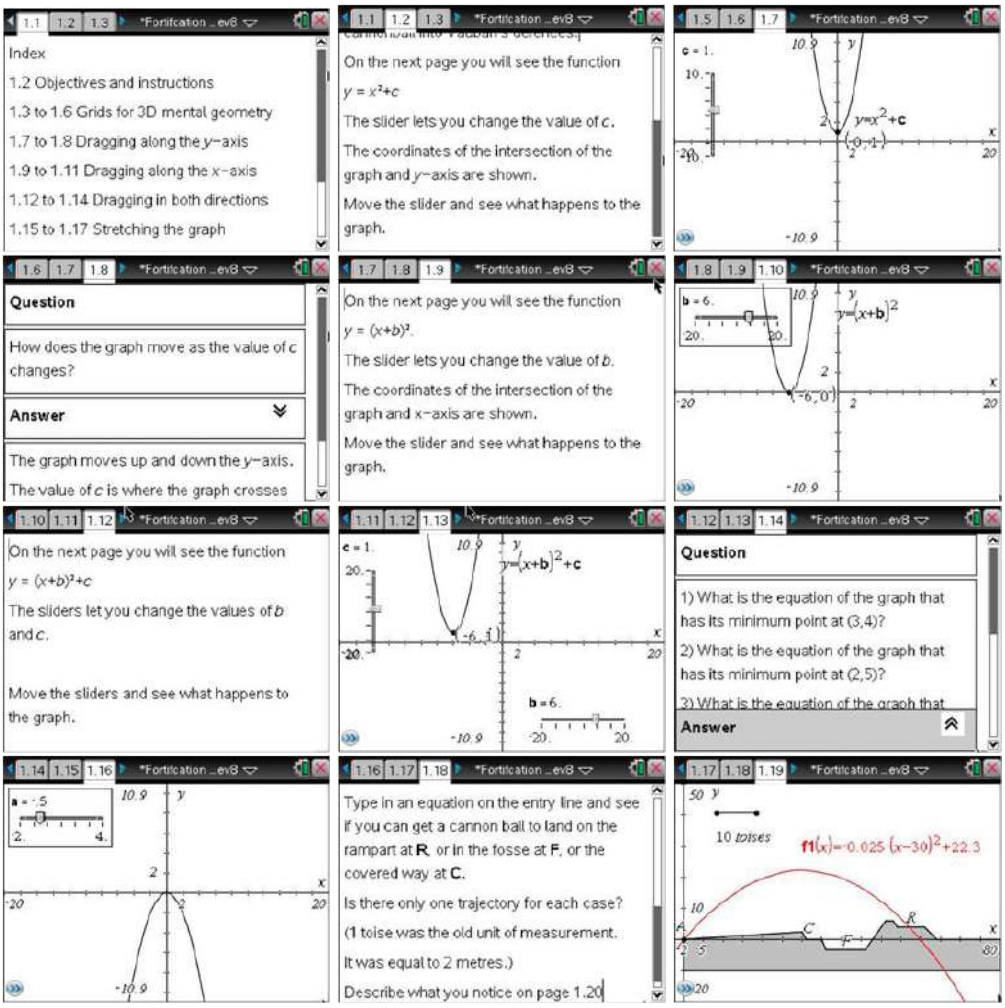

Fig 5: A series of screenshots from a TI-nspire CX

4 Conclusion

When used in the classroom with 12/13 year old students there was a surprise result. Students worked in groups of three on these activities and I had asked them to produce a plan for defending the mathematics area in school from invaders. Three groups went one better than just a plan - they also built a scale model over a week's half term. I was amazed at how much time and effort went into their work since one group had met up and worked for a total of 20 hours together. Imagine the uproar if I had asked them to spend that amount of time on mathematics homework.

In fact unknown to them Louis XIV ordered similar things, with Louvois (his minister of war) producing maquettes for each town that Vauban fortified. There is a collection of 15 of these in the basement of Le Palais des Beaux-Arts in Lille.

Following on from this my 14/15 year old students explored the mathematics of siege engines such as the *ballista* (a large Roman crossbow), *onager* (large Roman catapult) and *trebuchet* (a mediaeval counterweight catapult). They experienced small scale models in the classroom to see how they worked, then explored the effect of how far they could throw a ball when extending their arm length. This was achieved by throwing a ball outside with and without using a dog throw (a piece of plastic with a scoop at one end which in effect provides a longer arm). The data was then analysed using IT and students wrote a report on their findings.

The workshop at HPM will focus on the mental geometry, use of IT and proportional dividers exercises with participants receiving a CD-ROM with all the materials used.

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Appendix - the plates and questions

All the questions about the plates start with the following information: echelle is French for 'scale'

1 toise was exactly 6 pieds (feet) i.e. about 1.949 meters in France before 10 December 1799.

For the purpose of this sheet, take 1 toise to be 2 metres.

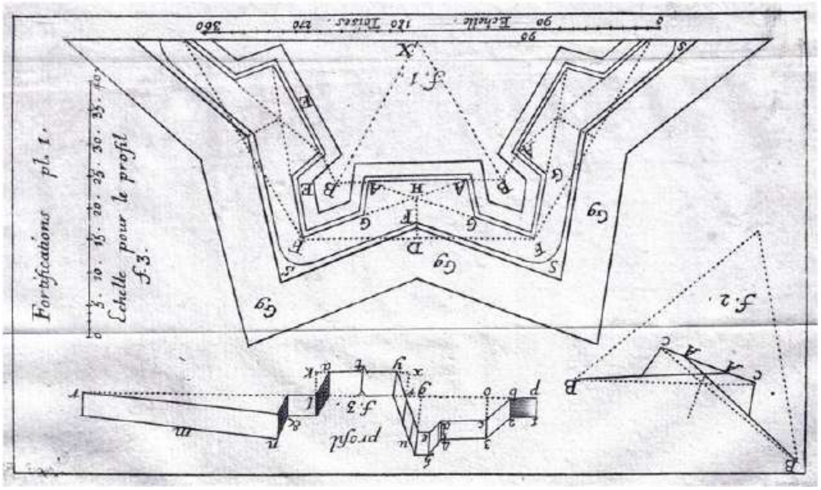

Plate 1 1. On what basic shape is this fort based? 2. What kind of triangle is triangle BXB?

3. What kind of triangle is triangle FHF?

4. Draw an enlargement of triangle FHF with scale factor 2. Measure $\angle FFH$.

5. Use the information you have gathered to calculate $\angle GFG$.

6. Now draw the full fort (you can omit the glacis Gg)

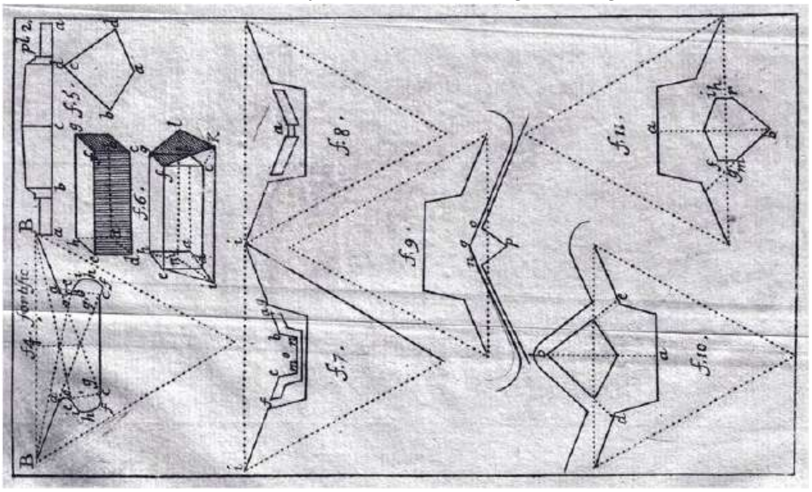

Plate 2 1. What is the name of the solid above f.6? 2. On what basic shape is f.10 based?

Concentrate on f.10 - all the following questions refer to this figure.

3. On what type of triangle is this figure based? 4. What is the (mathematical) name of shape b?

(Its defensive name is a *ravelin*.)

5. What are the properties of the diagonals of the shape that touches b?

6. Construct f.10, taking the side of the triangle to be 12cm.

You need to know that the distance ab is half the side of the triangle and the short diagonal of the shape at b is one-third the side of the triangle.

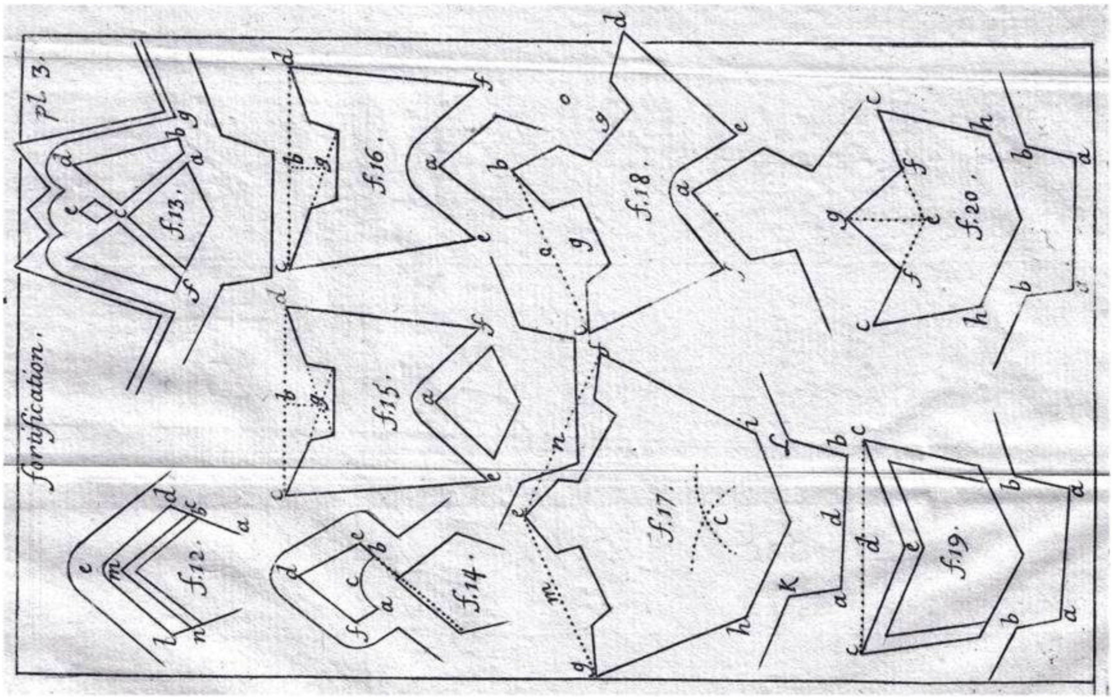

Plate 3 This plate refers to the construction of *ravelins* (f.17-20) and *hornwork* (f.15 & 16)

Here are the instructions on how to draw the horn in f.15.

Draw diagonal of the ravelin from a then from a on the ravelin continue the line for 85 toises to b.

At b draw a perpendicular line cd such that cb and bd are each 60 toises.

From b measure 20 toises to get g.

Join cg and dg to get the angles of the contrascarp. The faces are 38 toises long.

Use a scale of 1cm to 5 toises and draw the five edges of the front face of the horn.

You have two cannons to put on this horn.

Where do you put them to defend as much of this face as possible?

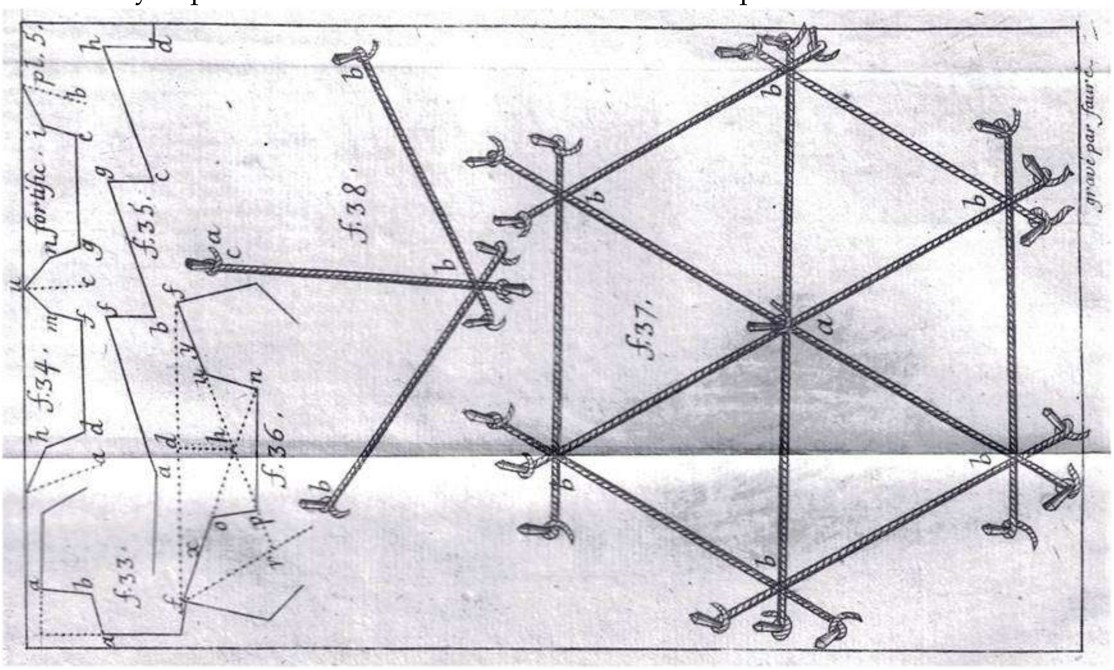

Plate 5 This plate gives more details on bastions (f.33, f.34, f.36), redans (f.35) and laying out equilateral triangles. You need to work with f.36 since you are trying to calculate all the angles in this figure, which

is based on a square fort with bastions. You are given the following.
fr passes through the centre of the square. $rphuf$ is a straight line perpendicular to fr.
fhn is a straight line.
$\angle opn$ is 100°.
Sketch f.36 and work out the size of all the angles that you can find. Give reasons for your answers.

BOTTLED AT THE SOURCE:
The Design and Implementation of Classroom Projects for Learning Mathematics via Primary Historical Sources

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ABSTRACT

As mathematics instructors, it is not unnatural for us to be tempted to provide students with clear and precise presentations, both in our teaching and in the written materials which we provide to them. But just as a water filtration process intended to remove impurities can also remove healthy minerals and their interesting tastes, efforts to remove potential impediments to student learning can inadvertently strip a subject down to a set of facts and formulas lacking in context, motivation and direction. Beyond this, teaching something very distilled is unlikely to help students see how they can develop and reason with ideas on their own.

Going back to the source from which a mathematical subject originally sprang is one means of restoring these vital ingredients to student learning. In this paper, I describe my own experiences with a particular approach to using primary historical sources in mathematics to promote student learning, and share some of the challenges and rewards that I have personally found the most exciting in using classroom projects based on original sources with my students.

Keywords: Primary Historical Sources, Original Sources, Classroom Project Design, Pedagogy, Elementary Set Theory, Boolean Algebra, Group Theory

1 Introduction

The audience of this HPM conference and readers of its *Proceedings* are likely to be familiar with various ideas for bringing a historical dimension to bear on students' learning of mathematics; many of you may also have used primary source materials in your teaching in some way. This paper describes my own experiences with a particular approach to using primary source material in mathematics to promote student learning. The specific classroom projects I use to illustrate the challenges and opportunities afforded by this approach are part of a larger compendium of such projects that have been developed and tested since 2008, with support from the US National Science Foundation (NSF), by an interdisciplinary team of mathematicians and computer scientists at New Mexico State University, Old Dominion University and Colorado State University-Pueblo. The full collection of these projects is located on our web resource [6]; additional projects developed with prior NSF support can be found both in print [7] and on a separate web site [5].

As the title of this paper suggests, the pedagogical approach our team has adopted is not simply to bring primary source material to the students and ask them to "drink" that material in. Instead, original source material is "bottled" for student consumption as part of a classroom project which includes a discussion of the historical context and mathematical significance of each primary source selection, as well as a series of tasks designed to illuminate the source material and prompt students to develop their own understanding of the underlying concepts and theory. By guiding students through the reading of an original source in this way, our classroom projects fall short of Fried's call to adopt a 'radical accommodation' strategy of (directly) studying (original) mathematical texts as a means of avoiding the danger of trivializing history when using it as a teaching tool [12].[1] We never-the-less embrace the commitment to humanizing mathematics which Fried contends requires one to take history seriously by "look[ing] at [mathematics] through the eyes and works of its practitioners, with all their idiosyncrasies" and "as far as possible, [to] read *their* texts as *they* wrote them" [12, p. 401].

Returning to the water processing metaphor of the title, one of the major challenges of our approach is thus to design projects which avoid (or at least minimize) the amount of filtering which occurs as a result of the "bottling" process. Of course, as mathematics instructors, it is not unnatural for us to be tempted to provide students with clear and precise presentations. But just as a water filtration process intended to remove impurities can also remove healthy minerals and their interesting tastes, efforts to remove potential impediments to student learning can inadvertently strip a subject down to a set of facts and formulas lacking in context, motivation and direction. Beyond this, teaching something very distilled is unlikely to help students see how they can develop and reason with ideas on their own, or to help them develop mathematical competencies as well as mathematical techniques. As argued by Kjeldsen & Blomhøj [15], Jahnke [13] and others, going back to the source from which a mathematical subject originally sprang is one means of restoring these vital ingredients to student learning.

At the same time, our ultimate pedagogical goal is to develop projects for learning core material from the curriculum of contemporary courses in discrete mathematics and computer science.[2] Adopting Jankvist's terminology in [14], our primary focus is thus on 'history-as-a-tool,' rather than on 'history-as-a-goal.' Within the 'history-as-a-tool' category, my own emphasis has been on the role of history as a cognitive tool to support the learning of mathematics (versus its motivational potential). While pursuing that goal, I am repeatedly amazed by the ways in which an original source reading selected to address some specific content objective can both stimulate and quench a student's thirst for deeper understanding that reaches beyond that objective.

In the rest of this paper, I share some of the challenges and rewards that I have personally found the most exciting in my work with original source projects, focusing primarily on the project "Origins of Boolean Algebra in the Logic of Classes: George Boole, John Venn and C. S. Peirce." In the next section, I use excerpts from that project to illustrate how primary source selections and student tasks can be combined to address a specific curricular topic in ways that go beyond simple mastery of that topic. In the final section of this paper, I more briefly describe design issues surrounding the project "Abstract awakenings in algebra: Early group theory in the works of Lagrange, Cauchy, and Cayley," relating those issues back to the water processing metaphor of my title.

[1]'Radical accommodation' is one of two alternatives that Fried identifies as a means to avoid this danger; the second alternative is that of 'radical separation' in which the study of the history of mathematics is placed on an entirely different track from the regular course of study.

[2]More detailed discussion of our pedagogical design goals is found in [8]; an overview of how our projects can be used as replacement units for specific topics within a discrete mathematics course or to teach such a course in its entirely is found in [9].

2 Unexpectedly refreshing! "Origins of Boolean Algebra in the Logic of Classes: George Boole, John Venn and C. S. Peirce"

The project "Origins of Boolean Algebra in the Logic of Classes" [1] was initially designed for use in a first course in discrete mathematics which includes both elementary set theory and boolean algebra as part of its curriculum; in its current incarnation, it can be used simply as a unit on elementary set theory. Arranged in five sections, the project begins with a historical introduction describing the general context of Boole's work. The second section of the project employs extensive excerpts from Boole's 1854 *An Investigation of the Laws of Thought* [10] as a means to introduce students to the operations of logical addition (i.e., set union), logical multiplication (i.e., set intersection) and logical difference (i.e., set difference). Current terminology and notation for set operations, however, is intentionally *not* employed. Instead, students use Boole's deliberately algebraic notation — an integral part of his effort to develop a symbolic algebra for logic — to complete project tasks which explore the basic laws governing this algebra (e.g., commutativity, idempotency) and Boole's justification for those laws. Certain restrictions imposed by Boole on the use of the operation symbols (now long since lifted) are also explored in ways that raise important mathematical themes that I have found difficult to pursue in conjunction with a standard textbook treatment of elementary set theory.

Boole's justifications for these restrictions rely in part on his definitions of the operations, and in part on the analogy of his symbols with those of 'standard algebra.' The following excerpt from the project illustrates how Boole's own writing is woven together with project tasks that prompt students to examine and evaluate Boole's arguments:[3]

Proposition I.

All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz:

1st. Literal symbols, as x, y &c., representing things as subjects of our conceptions.

2nd. Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the elements.

3rd. The sign of identity, $=$.

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra.[4]

...

6. Now, as it has been defined that a sign is an arbitrary mark, it is permissible to replace all signs of the species above described by letters. Let us then agree to represent the class of individuals to which a particular name or description is applicable, by a single letter, as x.

[3] To set them apart in this paper, project excerpts are indented. Within project excerpts, primary source selection are set in sans serif font and bracketed at their beginning and end by the following symbol :

[4] Emphasis added.

⋯By a class is usually meant a collection of individuals, to each of which a particular name or description may be applied; but in this work the meaning of the term will be extended so as to include the case in which but a single individual exists, answering to the required name or description, as well as the cases denoted by the terms "nothing" and "universe," which as "classes" should be understood to comprise respectively "no beings," and "all beings." ⋯Let it further be agreed, that by the combination xy shall be represented that class of things to which the names or descriptions represented by x and y are simultaneously applicable. Thus, if x alone stands for "white things," and y for "sheep," let xy stand for "white sheep;" and in like manner, if z stand for "horned things," and x and y retain their previous interpretations, let zxy represent "horned white sheep," i. e. that collection of things to which the name "sheep," and the descriptions "white" and "horned" are together applicable. Let us now consider the laws to which the symbols x , y , &c., used in the above sense, are subject.

⋯

10. We pass now to the consideration of another class of the signs of speech, and of the laws connected with their use.

11. *Signs of those mental operations whereby we collect parts into a whole, or separate a whole into its parts.*

We are not only capable of entertaining the conceptions of objects, as characterized by names, qualities, or circumstances, applicable to each individual of the group under consideration, but also of forming the aggregate conception of a group of objects consisting of partial groups, each of which is separately named or described. For this purpose we use the conjunctions "and," "or," &c. "Trees and minerals," "barren mountains, or fertile vales," are examples of this kind. **In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words "and" "or" are analogous with the sign + in algebra, and their laws identical.**[5] Thus the expression "men and women" is, conventional meanings set aside, equivalent with the expression "women and men." Let x represent "men," y, "women;" and let $+$ stand for "and" and "or," then we have

$$x + y = y + x, \tag{3}$$

an equation which would equally hold true if x and y represented numbers, and $+$ were the sign of arithmetical addition.

Task 4

Note that Boole imposed a restriction on the use of the addition symbol "+" in his system by asserting that "the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another." In this task, we consider the question of whether this restriction reflects standard language usage, as Boole claimed.

[5]Emphasis added.

Consider, for instance, the following expressions:

(I) infants and teenagers	(III) lying or confused
(II) dancers and singers	(IV) conqueror of Gaul or first emperor of Rome

For which of these is the conjunction (*or* , *and*) used in the exclusive sense specified by Boole? That is, is there an implication in standard language usage that a particular individual under discussion will belong to at most one of the named classes, but not both classes simultaneously?

Task 5
In his section 11 above, Boole referred to the analogy of symbolic logic with arithmetical algebra as further justification for restricting the use of "+" to disjoint classes:

> In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words "and" "or" are analogous with the sign + in algebra, and their laws identical.

Recall that the standard definition of addition for whole numbers m, n defines $m + n$ to be the total number of elements in the union (or aggregate) of a set containing m objects with a set containing n objects. Provide one or more specific examples to illustrate why it is important to use disjoint sets in this definition.

In my own classroom implementation of this project, student responses to the two tasks above have been as intriguing as their responses to Boole's claims themselves! One exciting features of this project which is exemplified by their responses is the way in which Boole's attention to issues related to language use and set operations (e.g., inclusive versus exclusive 'or') ensures that the inherent subtleties of these issues are made explicit. In contrast, contemporary textbook authors often downplay (or ignore) these subtleties in a way that can exacerbate their difficulty for students. Beyond the reassurance which an acknowledgement that the matter is not altogether unproblematic provides for some, all students are explicitly required to wrestle with these subtleties in completing Task 4 and other related project tasks. Because Boole's own writing both recognizes that there are decisions to be made about how symbols are used *and* offers criteria for making these decisions, the question of what constitutes sufficient evidence for a claim also naturally arises during class discussion of these and later tasks. In this way, students' introduction to elementary set theory is elevated above the level of simply applying basic set operations and their laws.

Of course, the context of Boole's own motivation for studying the laws of thought was an influential factor in the criteria he applied for making decisions about symbol usage.[6] In particular, his work in logic was strongly influenced by the general state of nineteenth century British mathematics, and especially by the concept of a 'symbolical algebra.' Although today's students acquire their mathematical ideas in a quite different context, Boole's development of the *algebraic* aspects of of symbolic

[6]The importance of considering context when reading original source material is discussed at come length in Jahnke [13].

logic is not completely foreign to their experience. Their own experience with algebra and their work on Tasks 7 and 8 below, for example, make them especially sympathetic to the restriction which Boole imposes on the usage of '$-$' in the following passage:

11. ⋯The above are the laws which govern the use of the sign $+$, here used to denote the positive operation of aggregating parts into a whole. But the very idea of an operation effecting some positive change seems to suggest to us the idea of an opposite or negative operation, having the effect of undoing what the former one has done. Thus we cannot conceive it possible to collect parts into a whole, and not conceive it also possible to separate a part from the whole. This operation we express in common language by the sign *except*, as, "All men except Asiatics," "All states except those which are monarchical." **Here it is implied that the things excepted form a part of the things from which they are excepted.**[7] As we have expressed the operation of aggregation by the sign $+$, so we may express the negative operation above described by $-$ minus. Thus if x be taken to represent men, and y, Asiatics, i. e. Asiatic men, then the conception of "All men except Asiatics" will be expressed by $x - y$. And if we represent by x, "states," and by y the descriptive property "having a monarchical form," then the conception of "All states except those which are monarchical" will be expressed by $x - xy$.

⋯

13. ⋯Let us take the Proposition, "The stars are the suns and the planets," and let us represent stars by x, suns by y, and planets by z; we have then

$$x = y + z. \tag{7}$$

Now if it be true that the stars are the suns and the planets, it will follow that the stars, except the planets, are suns. This would give the equation

$$x - z = y, \tag{8}$$

which must therefore be a deduction from (7). Thus a term z has been removed from one side of an equation to the other by changing its sign. This is in accordance with the algebraic rule of transposition.

Task 7

Note Boole's comment (in his section 11) that the operation *minus* will require "that the things excepted form a part of the things from which they are excepted." For example, if x represents men and y represents women, then the expression $x - y$ is meaningless in Boole's system since the class of women does not form part of the class of men.

(a) Suppose we wanted to drop Boole's restriction in the particular example just described. What class would the expression $x - y$ denote? Based on your reading of Boole up to this point, could this class be represented symbolically in some other way within his system? Explain.

[7] Emphasis added.

(b) Now consider the expression $d-p$ in the case where d represents drummers and p represents pianists. Explain why Boole would consider the expression $d-p$ to be meaningless, then describe the class which the expression $d-p$ would denote if Boole's restriction were removed. Based on your reading of Boole so far, how could this class be represented symbolically within his system? Explain.

(c) Based on these examples, do you agree or disagree with Boole's restriction, and why? What argument, if any, does he give for adopting this restriction in his section 11?

Task 8
Now consider Boole's equations (7) and (8) in his section 13 above, and recall his earlier restriction that "+" can only be applied to classes which share no members. Suppose we were to drop this restriction, and let y represent men, z represent doctors, and $x = y + z$ as in Boole's equation (7). Can we still deduce Boole's equation (8) in this case? That is, does $x - z = y$? Explain.

As with Boole's restriction on '$+$' to classes which are disjoint, his restriction on the use of '$-$' is not part of today's set theory. One concern about using an original source in which the author's conception differs in this way from that currently in use is that students may be unduly confused when they are eventually required to adopt current usage. But in truth, many beginning students of set theory unconsciously adopt Boole's 'common language' argument for restricting the use of '$x - y$' to cases where y is a part of x; after all, how can one remove something that is not already there?? Thus, a potential difficulty to understanding our current definition of set difference which exists independently of instructional materials is again made explicit through reading of the primary source. Boole's use of the notation '$x - xy$' to represent 'All states except those which are monarchical' in the preceding excerpt also anticipates a way of sidestepping the restriction on subtraction which my students have picked up on in responding to Task 7(b). In their responses to Task 8, many students also identify "$x = y + z \Rightarrow y = x - z + yz$" or "$x = y + z \Rightarrow y = x - z(1 - y)$" as legitimate 'subtraction rules' in the case where y and z are not disjoint.[8] Later in the project, students read excerpts from Boole's *Laws of Thought* in which he justifies the use of the symbol '1' to represent the 'Universe' and begins to use expression '$x(1 - y)$' as an (algebraic) equivalent to '$x - xy$' which allows the aggregate of non-disjoint sets x, y to be 'legally' expressed as '$y + x(1 - y)$.' Long before that point, however, many of my students have anticipated this algebraic strategy!

In short, Boole's explicitly algebraic notation meshes with students' experiences in algebra in ways that allow them to quite naturally anticipate the current definition of set difference as $A - B = A \cap \overline{B}$. In the third section of the project, which examines refinements to Boole's system made by John Venn in his 1894 edition of *Symbolic Logic* [17], students are thus unsurprised by the following pronouncement by Venn:

[8]Task 8 also begins an exploration of a second compelling reason for adopting Boole's restricted interpretation of '$-$'; namely, an unrestricted use of'$+$' would not allow for a well-defined inverse operation. Later project tasks based on original source excerpts from Boole, Venn and Peirce further explore the concept of inverse operations for logical operations.

It will now be seen how it is that the process of subtraction, with the corresponding symbolic sign, can be dispensed with. If y be excepted from x it must be a part of x, and may therefore be written $xy \cdots$. But this, as we have just explained, is the same as to 'multiply' x by $1-y$, or by not-y. In other words, the legitimate exception of y from x is the same thing, in respect of the result, as taking the common part of x and not-y. 'The clergy, except the teetotalers', means, when the exception is duly interpreted, the class common to those of 'the clergy' and 'the non-teetotalers'.

Nor are students particularly surprised by Venn's removal of the restriction on '$+$' as applicable to disjoint sets only, a modification to Boole's system which Venn justifies as follows:

Boole, as is well known, adopted the ⋯plan [of] making all his alternatives mutually exclusive, and in the first edition of this work I followed his plan. **I shall now adopt the other, or non-exclusive notation:– partly, I must admit, because the voting has gone this way, and in a matter of procedure there are reasons for not standing out against such a verdict**;[9] but more from a fuller recognition of the practical advantages of such a notation. ⋯as a rule and without intimation of the contrary, I shall express 'X or Y,' in its ordinary sense, by $X+Y$. I regard it as a somewhat looser mode of statement, but as possessing, amongst other advantages, that of very great economy. [10]

By this point in the project, students have completed various project tasks calling for reflection on Boole's arguments in favor of an exclusive interpretation for '$+$', and analyzed specific examples which illustrate the 'very great economy' to be gained by a non-exclusive interpretation of '$+$'. What is surprising to many students, however, is the 'non-mathematical' reason that Venn gives for having changed his mind about this issue between the first and second edition of his *Symbolic Logic*; namely, by popular demand! This example of the ways in which non-technical factors can (and do) play a decisive role in the development of mathematics affords students a superb view of mathematics as a human endeavor.

One last design comment before leaving discussion of this particular project. Because I personally use this project to lay the ground work for a more abstract treatment of boolean algebra as a discrete axiomatized structure later in the course, the project concludes with primary source material from a C. S. Peirce, who took a more formal approach to the algebra of logic.[11] In fact, Peirce was one of those mathematicians who voted for a non-exclusive interpretation of '$+$' prior to Venn's 1894 adoption of

[9]Emphasis added.

[10]Interestingly, Venn does not remove Boole's restriction on '$-$', nor does he dispense with the process of subtraction altogether.

[11]This section of the project could be omitted by an instructor seeking only to introduce elementary set theory.

that interpretation. Although chronological earlier than Venn's work, Peirce's notation and general approach in his 1867 *On an Improvement in Boole's Calculus of Logic* [16] is more abstract than that of Boole or Venn, as illustrated by the following excerpt:

Logical addition and logical multiplication are doubly distributive, so that

(8) $$(a +_{,} b),c =_{,} a,c +_{,} b,c$$

and

(9) $$(a,b) +_{,} c =_{,} (a +_{,} c),(b +_{,} c).$$

Proof. Let

$$\begin{aligned} a &=_{,} a' + x + y + o \\ b &=_{,} b' + x + z + o \\ c &=_{,} c' + y + z + o \end{aligned}$$

where any of these letters may vanish. These forumulæ comprehend every possible relation of a , b and c; and it follows from them that

$$a +_{,} b =_{,} a' + b' + x + y + z + o \qquad (a +_{,} b),c =_{,} y + z + o$$

But

$$a,c =_{,} y + o \qquad b,c =_{,} z + o \qquad a,c +_{,} b,c =_{,} y + z + o \qquad \therefore (8)$$

In this excerpt, '$+$' indicates the two sets being joined are disjoint, while '$+_{,}$' is used for the union of non-disjoint sets. In contrast to the expository writing styles of Boole and Venn, Peirce offers no rationale for the use of two separate symbols and simply states the basic dual algebraic laws (i.e., commutativity, associativity, idempotency), asserting that they are "evident." His proof (above) of the first of the dual distributivity law is also more formal than anything in either Boole or Venn. Yet students' grounding in the more concrete arguments of Boole and Venn allows them to make sense of Peirce's formalism, while Peirce's use of somewhat different notation in turn prepares students to shift to the standard set theoretical notation. The decision to place excerpts from Peirce in non-chronological order relative to the work of Venn was thus a deliberate design choice based on pedagogical, rather than historical, considerations. In the subsequent closing section of the project, the standard terminology and notation for set operations in use today is then (finally!) introduced and the algebraic properties of those operations are compared with the standard axioms for an abstract boolean algebra.[12]

[12] A companion project entitled "Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization" [2] goes on to explore the early axiomatization of boolean algebra as a fully abstract structure and introduces students to the use of a model to establish the independence and consistency of an axiomatic system, while a second companion project entitled "Applications of Boolean Algebra: Claude Shannon and Circuit Design" [3] explores the application of boolean algebras to the problem of circuit design. All three projects can be completed independently of each other.

3 Preservatives Added? "Abstract awakenings in algebra: Early group theory in the works of Lagrange, Cauchy, and Cayley"

The goal of the Boolean Algebra project discussed in the preceding section is to develop an understanding of the modern paradigm of elementary set theory as a specific example of a boolean algebra. The design of that project and its two companion projects (footnote 12) also provides an opportunity for students to witness how the process of developing and refining a mathematical system plays out, the ways in which mathematicians make and explain their choices along the way, and how standards of rigor in these regards have changed over time. To achieve these goals, all three projects rely on student reading of primary sources and completion of associated project tasks, with little commentary provided on that source material apart from historical information.[13] The central design issue was thus to ensure a selection (filtration) of original source excerpts sufficient to build a modern conception of set theoretical operations and the construction of project tasks to frame (bottle) those sources while remaining faithful to their original meanings.

In this concluding section, the challenges of achieving a proper balance between "filtration" and "bottling" are considered in the context of the project "Abstract awakenings in algebra" [4]. Designed for use in a first course in abstract algebra, my original intention was to base this project on just one original source, Cayley's 1854 paper *On the theory of groups, as depending on the symbolic equation* $\theta^n = 1$ [11]. In that paper, Cayley explicitly recognizes the common features of various (apparently) disparate mathematical developments of the early nineteenth century and defines a 'group' to be any (finite) system of symbols subject to certain algebraic laws. Asserting (without proof) that the concept of a group corresponds to the "system of roots of [the] symbolic equation [$\theta^n = 1$]," Cayley states (without proof) several important group theorems and proceeds to classify all groups up to order seven. While focusing on the classification of arbitrary (finite) groups and their properties, Cayley does not neglect to motivate this abstraction through references to specific nineteenth century appearances of the group concept, including "the system of roots of the ordinary equation $x^n - 1 = 0$" and the theory of elliptic functions. He also remarks that "The idea of a group as applied to permutations or substitutions is due to Galois, and the introduction of it may be considered as marking an epoch in the progress of the theory of algebraical equations."

The fact that Cayley's paper provides such a powerful lens on the process and power of mathematical abstraction makes it simultaneously attractive and difficult to use in a classroom project aimed at developing an understanding of elementary group theory. Unlike Boole's introduction of symbolic algebra as a tool for studying logic, Cayley's paper does not make a radical departure from previous work in an existing field of study that requires no particular knowledge of previous treatments of that field. Instead, Cayley's insight into the common features of a variety of existing mathematical objects was dependent on his familiarity with those objects. Even at the time, his insight was premature and his paper attracted little attention from other mathematicians until late in the nineteenth century. Thus, simply reading Cayley's paper as one's first introduction to group theory seems unlikely to lead to a robust understanding of that theory.

One obvious alternative to using Cayley's paper as students' introduction to group theory is for students to study basic group theory using a contemporary textbook prior to reading Cayley's paper

[13] The interested reader should consult the projects at [6] to ascertain the extent to which commentary on the source material is employed.

in an effort to glean additional insights into that theory from his treatment of it. My own experiences with trying this approach many years ago, however, suggested that the few insights which students ultimately gained from reading the paper justified neither the time spent doing so nor the frustration experienced along the way. As beautiful as the paper is, students were coming to it without a proper historical context for Cayley's ideas, and the modern treatment to which they had already been exposed seemed to interfere more than it assisted with understanding those ideas.

With respect to the pedagogical goals of our NSF grant, another disadvantage of my first attempts to use Cayley's paper in teaching abstract algebra was the way in which it became simply an add-on to the course, rather than the primary vehicle by which students acquired an understanding of group theory. My experiences with the design and implementation of the Boolean Algebra and other primary source projects further suggested that something far more powerful could result from reading Cayley within a properly framed historical context. This led me to the question "What did Cayley himself read?" as a means of identifying prior source material that could lead up to a successful reading of Cayley's own paper. Among the mathematical works on which Cayley was building, that of two authors emerged as especially important precursors to Cayley's definition of abstract group: Lagrange's writing on algebraic solvability and Cauchy's writing on permutation theory.

The 90-page project which resulted from this research develops a significant portion of the core elementary group theory topics from the standard curriculum of a first course in abstract algebra, and has been successfully tested at three institutions as a textbook replacement for this part of the curriculum.[14] Structured in four major sections containing a total of 84 project tasks, the project also includes an introduction that provides a broad overview of the historical roots of group theory in the theory of equations, a conclusion which states the theorem now known as 'Cayley's Theorem' in modern terminology and sketches its proof in a project task, and an appendix that provides current definitions of 'coset' and 'normal subgroups.' Although not explicitly discussed by Cayley, each of these "modern" concepts is implicit in Cayley's paper; Cayley's Theorem, for instance, is an almost trivial observation after working with the numerous Cayley tables in the paper.

In fact, one of the most exciting aspects of implementing this project with students has been the way in which so many theorems of elementary group theory stated by Cayley become obvious as a result of having read Lagrange and Cauchy in the first half of the project. Even the proof of "Lagrange's Theorem" — typically established in contemporary textbooks using cosets in a fashion that causes many students to struggle — becomes straightforward after working through Cauchy's proof of this same result for permutation groups.[15] This phenomenon was not wholly unexpected; after all, the rationale for including works from Lagrange and Cauchy in the project was to provide students with concrete examplars (roots of unity and permutations respectively) of the abstract group concept identified by Cayley. What was unexpected (but welcome!) was the extent to which those examplars prepared students for that shift in abstraction.

The use of this project as a replacement for such a substantial portion of a required course did, however, raise new tensions for me in terms of design decisions.[16] Ultimately, my students need to

[14]See http://www.cs.nmsu.edu/historical-projects/Blog/cayley-notestoinstructor.pdf on [6] for a detailed overview of the project contents.

[15]See [4, pp. 43 - 46] for Cauchy's proof of this theorem and related projects tasks.

[16]A few words about implementation seem appropriate at this point. Depending on the project(s) selected, instructors should generally allow one to several weeks to implement a project in class. (The Cayley project is an exception in that full implementation takes approximately 10 weeks.) While a project is being implemented, several strategies are possible.

know that they are leaving the course with a firm understanding of the current paradigm of group theory and an ability to read and write proofs that meet today's standard of rigor and formality. This concern led me, for example, to include more commentary on the original sources than is the case with the Boolean Algebra projects. This additional commentary includes "modern" proofs of certain propositions, as well as certain examples (e.g., infinite groups) and terminology (e.g., isomoporhism) that are not found in the original sources.

Besides fleshing out the historical context of Cayley's work by including original source material from his predecessors, I have also tried to exploit the ways in which my students' mathematical context is different from that of Cayley to promote their understanding of group theory in its current form. For example, although Cayley's pioneering work in matrix theory does not seem to have influenced his thinking about groups, my students have completed a pre-requisite course in linear algebra which makes invertible matrices a natural example of a non-commutative group for them.

In short, I have not only filtered the sources in this particular project in a variety of way, but have also mixed a few additives and preservatives into the bottle before offering it to students. Yet student learning and assessment results (as measured by homework and exam performance) in my classes have been so positive as a result of completing this project, that I now plan to extend it into a full-length abstract algebra textbook. As I continue to wrestle with the tension between remaining true to my sources while fulfilling the demands of the curriculum in this work, I remain convinced that doing so is merited not only by the pedagogical value which primary sources provide with respect to providing context, motivation, and direction for students' mathematical endeavors, but also by the deep and robust mathematical understanding which can flow from those sources as a result of those endeavors.

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[2] Barnett J., 2008, *Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization*, 19 page curricular module based on primary historical sources, suitable for use in undergraduate courses in Discrete Mathematics, Model Theory, and Abstract Algebra. Available at [6].

Students could work on the project entirely in class, either individually or in small groups, as the instructor monitors and assists their progress. Whole class discussions or brief lectures may also be appropriate at certain junctures. In preparation for class activities, many instructors assign select project tasks for students to complete based on their own reading. Some type of student writing or presentation is also recommended; again, instructors have considerable flexibility in how this is done. Our experience further suggests that each assigned project should count for a significant portion of the course grade.

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TEACHING HISTORY OF MATHEMATICS TO TEACHER STUDENTS

Examples from a short intervention

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ABSTRACT

Two years in a row, I have developed six-hour teaching sequences on history of mathematics for prospective teachers for pupils age 11-16. The development was based on the research literature of the HPM group as well as on a questionnaire given to the students before the teaching. But at the same time, the opportunity to teach these students was used to do further research on the teacher students' conceptions of history of mathematics, by means of pre- and post-teaching questionnaires.

Based on the research literature and on the questionnaire results, I set up the following goals for the teaching sequence: It should give the students examples of different ways of teaching with history of mathematics (more than just "telling stories"), it should be connected to the students' curriculum, it should give ideas that are suitable for different age levels in the 11-16 bracket, it should show both how mathematics has been developed and how it has been used and it should give pointers to further studies for interested students.

In this workshop, I will show how I tried to meet these goals and give examples of activities included in the teaching.

More detail and background to the development of the teaching sequence is included in my talk at the TSG20 at ICME the week before this conference.

1 Background

The 2010 teacher education reform in Norway created a new course: A one-year (60 ECTS) course in mathematics education for students who want to teach mathematics in grades 5-10 (ages 11-16). The course as it is taught in my institution does not have much of a historical perspective, but I was asked to give a short (six hour) course on history of mathematics for these students. I decided to try to combine the task of developing a meaningful six-hour "taster" of history of mathematics with trying to develop an understanding of teacher students' conceptions of history of mathematics. This workshop will give examples of what I decided to include. A paper at the TSG20 at ICME12 details both the development part and the research part of the project. (Smestad, 2012b)

The six hour course has so far been held twice, for the first year students in academic years 2010/11 and 2011/12. This workshop will be based on both iterations. I will first state the goals and give a short description of the intervention. The main part of the paper will be concrete examples of what was done in the classroom.

2 Information on the interventions

2.1 Goals for the first iteration

The first year, my goal was to create a teaching sequence of six hours which

1. fits into the context in the particular course, in particular the course curriculum and the target group: pupils in the 11-16 age bracket.
2. gives the students examples of different ways of teaching with history of mathematics (more than just "telling stories").
3. stresses issues that are important in the literature on HPM
4. takes into account students' conceptions of history of mathematics before the course, and thereby
5. gives students an introduction to history of mathematics that motivates them to search for more knowledge

2.2 Outline of the first iteration

I decided to organize the teaching around the second goal above, and to give the students examples of history of mathematics based on different methods of working with pupils. I still kept a little bit of chronology but leapt from topic to topic. The plan turned out like this (the references are to papers where I have written about this before):

- Using old techniques: Russian peasant multiplication, Gelosia method of multiplication, casting out nines (Smestad & Nikolantonakis, 2010)
- Using old concrete materials: Navigation
- A play in the classroom: Plato's Meno
- Exercises based on history: Pascal's triangle, the Pascal-Fermat correspondence, unit fractions
- Etymology: Etymological crossword
- Working on original sources: Babylonian clay tablets (Smestad, 2012a)
- Multi-curricular: Perspective drawings (Smestad, 2011)
- Biography: Niels Henrik Abel and Florence Nightingale

Of course, this is far too much for a six-hour lecture, thus giving opportunities for choice as the teaching evolved.

2.3 Outcome of the first iteration

There was no test at the end of the teaching, only an informal evaluation. In this, the students made several points. I had made the error of having most of the „talking" in the first three hours with much more group work in the last three. A better balance is easy to achieve. More importantly, the students found it difficult to take in so many different topics, and would have preferred to concentrate on a smaller number of topics.

Surprisingly, many of the students insisted that the etymological crossword had been the most entertaining part. Navigation, on the other hand, became more of a lecture than a discussion or a hands-on-activity, partly due to lack of time.

2.4 Goals for the second iteration

In the next academic year, circumstances dictated that I did my teaching not long before the exam, at a time when the students were working on combinatorics and probability. Based on the feedback from the students the year before, I felt it would be a good idea to concentrate on one topic, and this had to be the history of probability. But I didn't want to let go of the other goals, for instance I still wanted to introduce the students to „different ways of teaching with history of mathematics". Thus, I had to find mathematically interesting materials on probability with a variation of methods.

2.5 Outline of the second iteration

I included parts where I lectured and discussed with students (l), group work on exercises based on history (e), a play (p), work on original sources (os) and a game (g). These were the contents:

- Introductory remarks on the history of probability (l)
- Remarks on the history of statistics (l)
- A story of St. Olaf, King of Norway (Smestad, 2011) (l)
- The two problems of de Méré (The Pascal-Fermat correspondence) (e/os)
- Expected value, insurance, the Archbishop of Bremen (e)
- The St. Petersburg Problem (Smestad, 2011) (e)
- Buffon (e)
- The Monty Hall problem (p)
- Combinatorics: Varahamihira, Mahavira's formula, Bhaskara (l)
- Pascal's triangle (os)
- Pepys and Newton (e/os)
- Bertrand's chord paradox (Smestad, 2011) (l/e)
- D'Alembert's misconception (e)
- Leibniz and Galileo (e)
- Montmort's paradox (e)
- Etymological crossword (g)

Again, this is far more than could fit into six hours, leaving me with choices to make in the classroom.

2.6 Outcome of the second iteration

This time, the mathematical learning was more obvious to the students. Actually, at times, the students got so involved in the mathematics that they may have forgotten that they were working on history of mathematics. The meta-issues of different ways history of mathematics can be included in teaching was probably lost on most of the students.

3 Examples from the work

Here, I will give some examples of what I did with the students. The examples are chosen to give an overall idea of the teaching I did. At the same time, I try to avoid repetition of examples that I've published elsewhere.

3.1 Navigation

The history of navigation is a rich source for teachers, as navigation was essential knowledge for a long time and involved many different kinds of mathematics. In this short introduction, I chose to start out by asking students how they would find their way from A to B at sea, given a map but no GPS. There are at least four answers which can come out of such a question:

1. Keep to the shore until you see your destination.
   (But if there is no shore connecting A and B, you could)
2. find the right direction from A and try to stick to that.
   (Sadly, you tend to drift away from the set direction, so you should also)
3. try to keep track of where you have sailed.
   (Which happens to be quite difficult, so it would be better to)
4. try to figure out where you are at any given moment.

There's not much mathematics in the first point, but it is still worth discussing that keeping close to the shore has its problems, as there may well be people there that you don't want to see you. The second point gives opportunities to discuss when the compass was invented (and the difference between magnetic North and geographical North), as well as other ways of telling directions at sea.

To keep track of where you have sailed, you need to know both your direction and your speed. In the 1500s, the "chip log" was developed. This consisted of a line with knots on it which was attached to a piece of wood and dropped in the water. As the line went out, the number of knots would tell the speed. This is also the origin of the measuring unit *knot*. (Jeans, 2004) Obviously, in this way you could only measure the speed relative to the water, which could be quite a serious problem.

Seamen also used their local knowledge to tell if they were at the right place. A "lead line" was a nice tool - basically just something heavy that you sunk to the bottom, covered in wax so you could find out what was on the bottom. At the same time, you also got to know the depth. Eventually, whole books were published describing the conditions of the seabed, including descriptions such as this: "Betwixt latitude $3° 40'$ S. and $1° 40'$ S. the soundings are from 22 to 18 fathoms, sand, with some few cafts, muddy soundings, extremely regular; but when you come to pass this latitude, namely, $1° 40'$ S. you will meet with soundings 18 fathoms, fine red sand." (Wright & Herbert, 1804, p. 452)

But the mathematically most interesting part appears when we start discussing how to get to know where you are at a certain moment. You need only two pieces of information; the longitude and the latitude. Moreover, if you travel at the right latitude, you can just stay on it and you will eventually find your destination. The geometry of longitude and latitude is not all that well-known to students, and bears repetition at this point. I will not go into that in this article.

The simplest way to measure the latitude is to measure the angle between Polaris and the horizon. At day, you could measure the solar altitude - but this would only give the latitude if you had tables of solar altitudes for different latitudes. Astronomical almanacs including solar altitudes were standard equipment for any navigator for centuries.

Explaining why the angle between Polaris and the horizon equals the latitude includes a bit of geometry and is all very well. But it is something else to stand on a ship at night and actually do the measurements. Over time, instruments were devised to make these measurements as accurate as possible.

The khashabat, developed in the 9^{th} century and later developed into the kamāl, were such instruments. The kamāl consisted of a wooden rectangle with a hole in the middle, as well as a string. With one part of the string in his mouth, the seaman would place the rectangle so that the top just covered Polaris while the bottom was even with the horizon. Measuring the length of the string from the seaman's teeth to the board gave a measurement of the latitude. The kamāl was not really useful at higher latitudes, though, as it would end up very close to the face, which would lead to inaccurate measurements. (McGrail, 2004) Thus, other instruments were devised here, for instance the quadrant, the cross-staff, the Davis' quadrant (a backstaff), the reflecting octant, the sextant and so on. For our purpose, it may be enough to see pictures of the different inventions to understand how important these measurements were. But it would also be possible to do experiments with pupils trying to find the latitude by means of the kamāl or other instruments. These experiments could also be done, or at least discussed, with the students.

Finding the longitude is more difficult. The theory is simple: when the sun is at its highest, you know that the local time is 12, and if you know the local time of some other place, you can calculate how far east or west you are from that place. The problem, though, was to know the local time of the other place. Until the beginning of the 1700s, even the best clocks had a margin of error of about 10 minutes per day, which would translate into an error of 278 km a day. In 1764 John Harrison invented a clock that was usable at sea, which was used by James Cook as he sailed around the world in 1779. (Library, 2004)

In connection with the mathematics discussed while working on navigation, it makes sense to include Eratosthenes' calculation of the circumference of the Earth. That is a simple calculation giving impressive information about the globe we live on. It is too well-known to be included in this article.

After all this talk about different ways to navigate, we discussed why these issues are relevant for teaching. Mathematics is obviously included, and the topic also includes knowledge of the world we live in that every child should be aware of. The topic gives clear examples of how mathematics is (and has been) important, and how people have been using mathematics to expand human knowledge in other areas.

When doing this kind of classroom discussion, the teacher should preferably have quite broad knowledge of the field. In my case, I had to answer some of the questions by promising to check things and give the answer later. Probably, most other ways of working on history of mathematics with students demand less of the teacher on the spot than whole-class discussions like this.

3.2 Dramatization: Two plays

In both the iterations, I included some dramatization. The first time, the students and I „performed" (without any rehearsals) part of Plato's dialogue Meno, wherein there is a discussion of what happens to the area of a square when the side is doubled. The second time, I wrote my own „play", based closely on the story of the Monty Hall problem and the letters sent to Marilyn von Savant.

Plays are beneficial in that they may make history of mathematics „come alive“, as the reasonings from history are given by people with flesh, blood and voices. See for instance Hitchcock (1992) for more on theatre in mathematics teaching.

The Monty Hall play had the added benefit of including 18 roles, making more students active in the performance. Moreover, many of the students had recently debated the problem heatedly, which contributed to their interest. The way the attitude towards Marilyn von Savant turned around as more and more people experimented on their own, was a useful reminder of the role of experiments and simulation in the teaching of probability.

3.3 St. Olaf, King of Norway

(Artist: Erik Werenskiold)

In *Heimskringla*, there is the following story about St Olaf, King of Norway:

> Thorstein Frode relates of this meeting, that there was an inhabited district in Hising which had sometimes belonged to Norway, and sometimes to Gautland. The kings came to the agreement between themselves that they would cast lots by the dice to determine who should have this property, and that he who threw the highest should have the district. The Swedish king threw two sixes, and said King Olaf need scarcely throw. He replied, while shaking the dice in his hand, "Although there be two sixes on the dice, it would be easy, sire, for God Almighty to let them turn up in my favour." Then he threw, and had sixes also. Now the Swedish king threw again, and had again two sixes. Olaf king of Norway then threw, and had six upon one dice, and the other split in two, so as to make seven eyes in all upon it; and the district was adjudged to the king of Norway. We have heard nothing else of any interest that took place at this meeting; and the kings separated the dearest of friends with each other. (Sturlason)

I take this story as a vehicle to discuss the concept of probability. Just as the kings did not consider the dice to be decided by chance, but rather by God, the students may also encounter pupils who consider events to be decided by fate, by gods or by who "deserves" to win. This is a complication in teaching probability that the students need to be aware of and ready for.

3.4 Pacioli's Problem of Points

The Pascal-Fermat correspondence, which many regard as the beginning of probability theory, provides interesting questions for pupils. I have developed a few exercises based on the history, which leads students through some of the problems. The example here is on what is known as the problem of points.

Exercise 1
De Méré's second problem was as follows:
Two persons are playing a game consisting of a series of rounds, and in each round each player has the same chance of winning. The winner of the game is the one to first win six rounds. But suddenly the game is (for some reason) stopped. At that time, A has won four rounds and B has won three.

a) How should the prize money be divided? Try to find a solution to the problem. (Maybe you have to look at a simpler problem first to get going - for instance the numbers can be changed to make it easier.)
b) Check if your way of solving the problem gives reasonable solutions when you use different numbers in the exercise (for instance if the number of rounds they have to win is a hundred or only two, or if A has a big lead or a small one⋯)

Luca Pacioli also looked at this problem, in his book Summa, and the problem is therefore sometimes called "Pacioli's problem of division". Pacioli solved the problem like this: as A has won four rounds of the seven they have played, he should have 4/7 of the prize money.

c) This solution was criticized 60 years later by Niccolo Tartaglia, who referred to what would happen if the game was stopped after only one round. Can you explain what Tartaglia meant?

Exercise 2 Pacioli's solution was criticized by Tartaglia, who felt it was unfair that A would get the whole prize if he had a 1-0 lead as the game was stopped. This would also make A very eager to stop the game if he was leading 1-0, and this would leave B feeling unhappy, as he would otherwise have a fair chance of overtaking A's lead and get the whole prize.
Tartaglia solved the problem like this: A leads by one point, which is one sixth of the number needed to win. Therefore, he should have one sixth of B's stake, which would give A 7/12 and B 5/12 of the prize.

a) Do you find this reasonable? Do you find it fair that a 1-0 and a 5-4 lead should be treated the same way?

Tartaglia himself commented this: "The resolution of such a question must be judicial rather than mathematical, so that in whatever way the division is made there will be cause for litigation". (Hacking, 1975, p. 51)

Later, it was agreed that it was necessary to look at what the chance each would have of winning the game if it had continued.

b) What is A's probability of winning the game (at the point when the score is 4-3)? (Or: what is A's expected prize at the score 4-3)? Remember that each player is supposed to have the same chance of winning each round.

Here are some of the proposed solutions:

- Pascal argued like this:[1] Suppose the score was 5-4 and the prize 64 dollars. If A wins the next round, he has won, so he gets all the 64 dollars. If B wins the round, they are even, so they can split the prize with 32 dollars each. So A can say to B: "I am sure to get the 32 dollars even if I lose this round, and as far as the other 32 dollars go, maybe I will get them and maybe you, the chance is the same for both. So let us share the 32 dollars equally and give me also the 32 dollars which I'm sure of winning". Thus, A has a right to 48 dollars and B 16 dollars.
  Suppose now that the score was 5-3. Either A wins the next round and will get all the 64 dollars, or he loses, and is entitled to 48 dollars as we just established. Thus, he should have 56 dollars.
  Now suppose that the score was 4-3. Either A wins the next round and should have the 56 dollars by the argument above (as the score is then 5-3), or he loses, and the score is then 4-4 and they have a right to 32 dollars each. Thus A should have 44 dollars.
- Pascal sent this solution to another mathematician, Pierre de Fermat. We don't have Fermat's answer, but based on Pascal's answer back, we can reconstruct Fermat's solution: Fermat saw that the game would necessarily have to stop by the time four more rounds had been played. Therefore, we could easily list all possible outcomes of those four rounds:

  aaaa aaab aaba aabb
  abaa abab abba abbb
  baaa baab baba babb
  bbaa bbab bbba bbbb

  Of these 16 possibilities, there are 11 where A wins. He thus has 11/16 chance of winning, and should have 11/16 of the prize. (This is the same answer as Pascal's solution.)
  This method of solution was criticized by several mathematicians, Roberval and d'Alembert among others.[2] d'Alembert pointed out that the cases (aaaa, aaab, aaba, aabb, abaa, abab, baaa, baab in this example) would never happen, as the game is finished when A has won another two rounds. The only real possibilities are therefore aa, aba, abba, abbb, baa, baba, babb, bbaa, bbab, bbb. Of these, A wins in 6 cases, which means that his chance is 6/10. Roberval's argument was similar: "It is wrong to base the division method on the supposition that there will be four more games; when the one needs two points and the other three, they will not necessarily play four rounds since they may happen to play only two or

[1] I have tried to keep the solution method (after Todhunter, I. (1865). *A history of the mathematical theory of probability from the time of Pascal to that of Laplace*.: Chelsea Publ. Co.), but both the numbers and the wording is changed.

[2] I know only d'Alembert's solution of similar problems, I don't have access to what he did on this problem in particular.

three." (Based on Pascal's description of Roberval's reasoning in his answer to Fermat.)

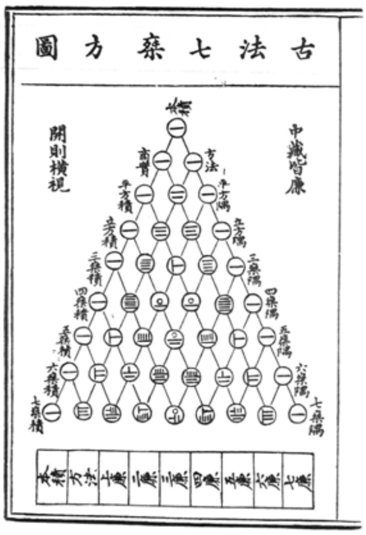
c) Which solution do you agree with?

Pascal's answer (August 24th, 1654) was as follows: "It is not clear that the same gamblers, not being constrained to play the four throws, but wishing to quit the game before one of them has attained his score, can without loss or gain be obliged to play the whole four plays, and that this agreement in no way changes their condition? For if the first gains the two first points of four. will he who has won refuse to play two throws more, seeing that if he wins he will not win more and if he loses he will not win less? For the two points which the other wins are not sufficient for him since he lacks three, and there are not enough [points] in four throws for each to make the number which he lacks." (Smith, 1959, pp. 556-557)

In this example, the exercises are based on history, but the students do not work on original sources directly. Working on a condensed summary like this may be more effective and also simpler for the students, but of course also risks introducing inaccuaracies and does not give the students the „direct" connection to history that original sources do. See for instance Glaubitz (2007) for more on using original sources.

3.5 Pascal's Triangle

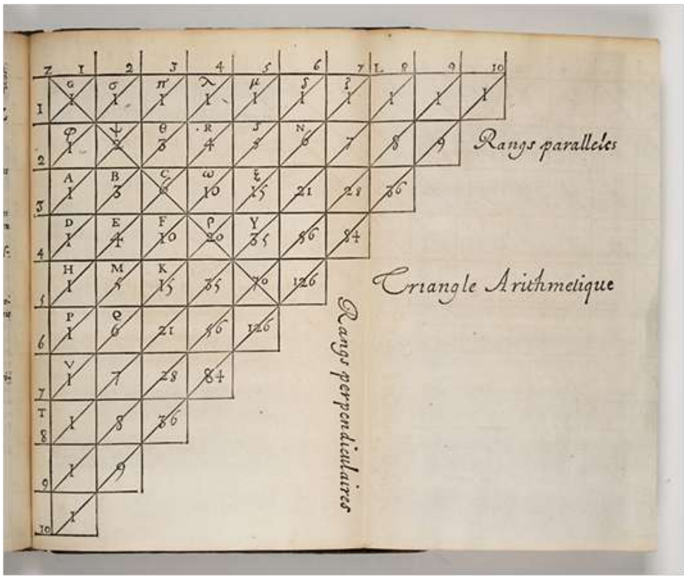
The work on Pascal's triangle includes work on a Chinese table as well as Pascal's treatise. We start by looking at a Chinese original source:



The numbers in the circles are written in Chinese numerals. Try to find out what they can mean. Also try to find out what the pattern in the table is. There is at least one error in the table. Find it.

The table in this exercise was writted in 1303 in China. It is known from many places and times. Today it is known as "Pascal's triangle". It is a well-known phenomenon that a little too much is named after famous (and Western) mathematicians instead of unknown mathematicians, but in this case there are reasons for this. Pascal wrote a whole book on the triangle - *Traité sur le triangle arithmétique* - which went further than anyone before him in the study of the triangle.

Here is the triangle as given in Pascal's treatise. Before we continue our work on the treatise, please work out the connection between this table and the Chinese table above.



This rest of my work on Pascal's triangle part is heavily inspired by David Pengelley and Janet Heine Barnett's workshop at the HPM2008 (Barnett, Lodder, Pengelley, Pivkina, & Ranjan, 2012), see also Pengelley (2009). In this work, students work on the original text from Pascal (although translated into English). By working on the original sources in this way, there may be a risk that the students ' overwhelming idea is that today's notation is much better than the one Pascal used. However, I will discuss with them whether the modern notation, which is very compact, is always a better one, and whether we could also be inspired by these original texts to use more words in our teaching with children.

3.6 Pepys and Newton

Another exercise is based on the correspondence between diarist Samuel Pepys and Sir Isaac Newton, which is entertainingly summarized in McBride (2007).

In 1693, Isaac Newton was a living legend in England. His main work *Principia* had been published six years earlier, and had revolutionized how the laws of physics were viewed. At the same time, he used mathematics in new ways. Samuel Pepys was a far less important person, even though he belonged to the upper classes and was a naval administrator. For posterity, he is mainly known for his diary, which described his life in a wealth of detail through nine years (1660-1669). His diaries were not published until after his death.

In 1693, Pepys sent a letter to Newton about a problem he had: What is most probable: to get at least one six when you throw six dice, to get at least two sixes when you throw twelve dice, or to get at least three sixes when you throw eighteen dice. He asked Sir Isaac Newton about this, and after several letters to and fro, Newton managed to convince Pepys of the answer. What do you think it was?
(A hint: To calculate the probability of at least one six when you throw six dice, it is enough to calculate the probability of not getting any sixes. To calculate the probability of getting at least two sixes when throwing twelve dice, it is enough to calculate the probability of not getting any sixes and of getting exactly one six. And so on.)
Newton answered thus:

> What is ye expectation or hope of A to throw every time one six at least wth six dyes?
> What is ye expectation or hope of B to throw every time two sixes at least wth 12 dyes?
> What is ye expectation or hope of C to throw every time three sixes at least wth 18 dyes?
> [. . .]
> If the Question be thus stated, it appears by an easy computation that the expectation of A is greater then that of B or C, that is, the task of A is the easiest. And the reason is because A has all the chances of sixes on his dyes for his expectation but B & C have not all the chances on theirs. For when B throws a single six or C but one or two sixes they miss of their expectations.

Pepys was not convinced, and asked to see the calculations. Newton wrote a long letter where he calculated that the probability for A was 31031/46656, the probability of B was 1346704211/2176782336, and he was satisfied to say that the probability for C was even less. When Pepys again asked for further explanations, Newton returned to the start:

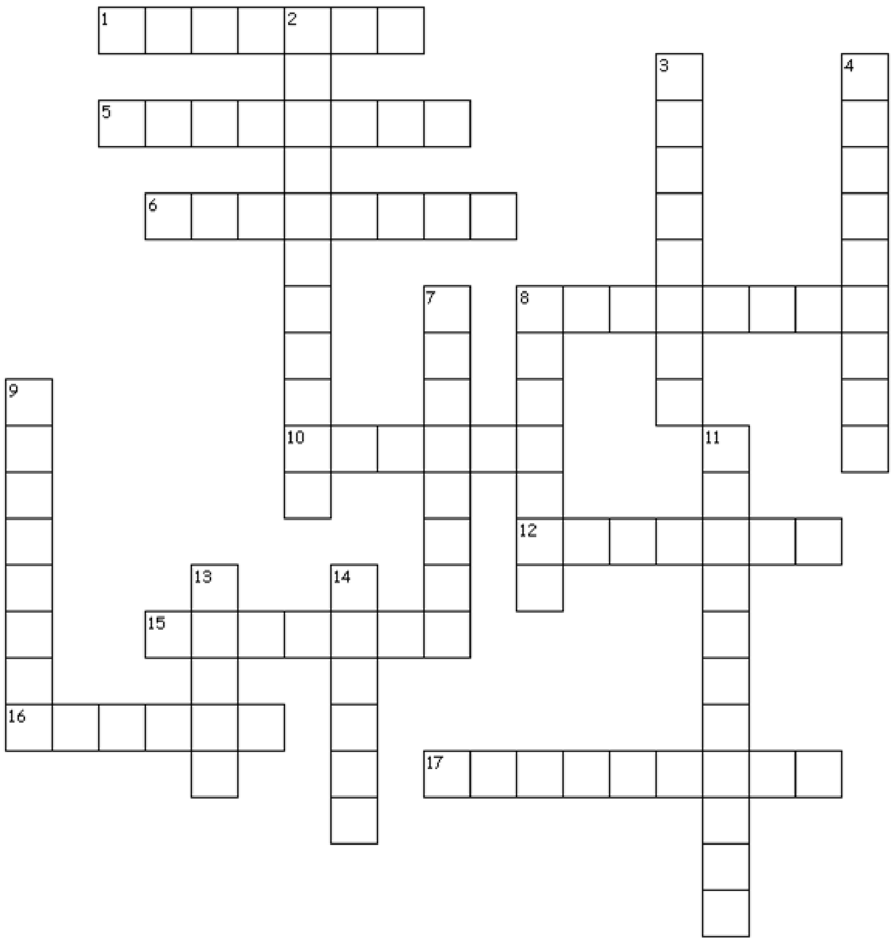
> As the wager is stated Peter [A] must win as often as he throws a six but James [B] may often throw a six & yet win nothing because he can never win upon one six alone. If Peter flings a six (for instance) four times in eight throws he must certainly win four times, but James upon equal luck may throw a six eight times in sixteen throws & yet win nothing.

Newton's reasoning has a logic flaw - can you find it? (A hint, possibly: Newton's reasoning works just as well (or badly) in this situation: A is to have at least one six in 12 dice, while B is to have at least two sixes in 24 dice. And so on.)

The subtle error in Newton's reasoning is detailed in Stigler (2007). This correspondence is a (perhaps unnecessary) warning to the students that basing their decisions on what seems reasonable instead of on calculations may at times be a bad idea. On the other hand, finding Newton's error may give the students confidence in their own abilities.

3.7 Etymological crossword

As mentioned above, a favorite part of the course has been the etymological crossword. Here is an English version made especially for this article:



Across

1. from Latin “tenth”
5. from Latin “standing firm, stable, steadfast, faithful”
6. from French “space between palisades or ramparts”
8. from French “increase, augmentation”
10. from Latin “small ring”
12. from Greek “spinning top”
15. from French “two”+“million”
16. from Latin “staff, spoke of a wheel, beam of light”
17. from Latin form of Al-Khwārizmī

Down

2. from Greek, “knowledge, study, learning”
3. from Latin “broken”
4. from French “right”+“angle”

7. from Latin “perform, execute, discharge”
8. from Arabic “reunion of broken parts”
9. from Greek “across”+“measure”
11. from Latin “from under”+“to pull, draw”
13. from Latin “fold in a garment, bend, curve.” (from Sanskrit “bowstring”)
14. from Latin “to separate”

While etymological crosswords should certainly not be at the core of mathematics teaching, I do believe that teachers should reflect on the fact that most words used in mathematics have interesting etymologies. The etymologies sometimes are connected to the concept itself, at other times they point to the history of the concept’s development. In both cases, the etymology may be used to demystify the mathematical words.

4 Concluding remarks

The point of this workshop was to share some ideas and to give opportunities for discussion on how (very) short courses on history of mathematics for teachers could be structured. As mentioned before, I still hope that even six hours may be enough to broaden students’ view of what history of mathematics is and how it could be included in teaching. Learning enough to develop their own materials, however, is left to their own enthusiasm.

Looking back at the goals set up for the first iteration, I think I succeeded in combining both traditional and less traditional ways of teaching with history of mathematics, while at the same time keeping close to important parts of the curriculum for the teacher students. However, it would be interesting to look more into the tendency I thought I noticed: that a closer focus on the mathematics may make the students less aware of the history, while organizing the teaching around different ways of using history of mathematics may make the students more aware of the pedagogical questions, but less aware of the mathematics.

Some readers will have noticed that there are ways of working on history of mathematics that are missing in this article. For instance, music is not found here, neither is project work. And even working with concrete materials, which I do mention, is not really included in the examples. It is a continuous challenge for the community to keep publishing a variety of examples and to discuss them, to bring the art of teaching with history of mathematics forwards.

5 Acknowledgements

Teaching of and with history of mathematics, like any other teaching, benefits from inspiration from other educators. Most of the ideas in the teaching mentioned here, leans on ideas from the HPM literature. I have included many references to articles and books where I have borrowed ideas, but there may well be other influencers that are not mentioned, because I’ve heard the ideas in passing many years ago. My thanks to them all.

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ON ALEXANDER WYLIE'S JOTTINGS ON THE SCIENCE OF THE CHINESE ARITHMETIC

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ABSTRACT

Starting from August of 1852 the British Protestant missionary and sinologist, Alexander Wylie (1815–1887), published in nine instalments an account *Jottings on the Science of the Chinese Arithmetic* in the newspaper *North China Herald*. He explained clearly the purpose of his account at the beginning:

> "The object of the following desultory notes, made from time to time, in the course of some researches entered upon, with another purpose in view, is to draw attention to the state of the arithmetical science in China, a subject which has not been so fully explored as it might with advantage, and on which some erroneous statements have been current in modern publications."

Alexander Wylie is a well-known figure in the last quarter of the Qing Dynasty for his contribution in transmitting Western science into China during the latter half of the 19th century. In mathematics he was known for translating three treatises in collaboration with the Qing mathematician Li Shan-lan (1811–1882) — *Supplementary Elements of Geometry* in 1856 but published in 1865 (believed to be based on the English translation of Book VII to XV of Elements by Henry Billingsley in 1570), *Treatise of Algebra* in 1859 (based on *Elements of Algebra* by Agustus De Morgan in 1835) and *Analytical Geometry and Differential and Integral Calculus Step by Step* in 1859 (based on *Elements of Analytical Geometry and of the Differential and Integral Calculus* of Elias Loomis in 1850). He was also the author of *Compendium of Arithmetic* published in 1853.

This presentation will discuss the knowledge of Chinese science and mathematics which most European sinologists of the 18th and 19th centuries possessed and the low regard they held it in, but the viewpoint of which was critically examined by Wylie in his account.

Keywords: Alexander Wylie, Chinese mathematics, arithmetic, algebra

1 Introduction

Alexander Wylie (1815–1887) was a Protestant missionary of the London Missionary Society and later an agent of the British and Foreign Bible Society in China. He was sent to China by the London Missionary Society in 1847. His contribution was not only on spreading Christian faith to China, but

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perhaps more importantly on the intellectual exchange of scientific and mathematical knowledge between China and Western countries. He was well-known in transmitting Western science and mathematics into China by publishing and translating scientific books (in collaboration with Li Shan-lan 李善蘭) such as *Compendium of Arithmetic* (Wylie,1853), *Supplementary Elements of Geometry* (Wylie and Li, 1865), *Treatise of Algebra* (Wylie and Li, 1859a), *Analytical Geometry and Differential and Integral Calculus Step by Step* (Wylie and Li, 1859b). On the other hand, he was also a sinologist who brought Chinese literature, philosophy, science and mathematics to the Western world especially Britain. During his 30 years of stay in China, Wylie collected many Chinese books in different disciplines. He published *Notes on Chinese Literature* (Wylie, 1867) which provides a bibliography with detailed explanatory notes on Chinese books. He had also written a number of articles related to China which were published in newspapers and periodicals. His colleague James Thomas selected some of these articles and edited as *Chinese Researches* (Wylie, 1897). These two books provided valuable sources for Westerners in the 19th century to know about China (in a new light).

In an accompanying workshop by the same authors, titled " *Chinese Arithmetic in the Eyes of a British Missionary and Calculus in the Eyes of a Chinese Mathematician*", we will focus on Wylie's introduction of (Western) algebra and calculus into China. The present paper supplements and complements this workshop. We will focus on how Wylie introduced Chinese mathematics to his own country. In particular, we will discuss a series of newspaper articles titled *Jottings on the Science of the Chinese Arithmetic*[1], which were first published in nine instalments from August to November of 1852 in *North China Herald* and later reprinted in *Chinese Researches* (Wylie, 1897, pp. 159–194). This series of articles played a pioneering role in the study of the history of Chinese mathematics in the Western world. It may be the only reliable (Western) source on the history of Chinese mathematics before the publication of Yoshio Mikami's *The Development of Mathematics in China and Japan* in 1913 (Wang, 1999). Dauben (2000) gives the following comment on *Jottings:*

> "This article is the first in English to give a reliable account, for the most part, of Chinese mathematics⋯.. Given the pioneering nature of this work, it is not surprising that it contains various errors and inaccuracies⋯.. Nevertheless, the "Jottings" is an important work for the history of Chinese mathematics, and was to have a significant influence upon such prominent historians of mathematics as Moritz Cantor, Florian Cajori, and David E. Smith." (Dauben, 2000, p.781–782)

According to Wylie, the objective of this series of articles is to clarify some erroneous statements about the status of mathematics in China that were found in (Western) publications at his time. He explained clearly this purpose at the beginning:

> "The object of the following desultory notes, made from time to time, in the course of some researches entered upon, with another purpose in view, is to draw attention to the state of the arithmetical science in China, a subject which has not been so fully explored as it might with advantage, and on which some erroneous statements have been current in modern publications."

In this presentation, we will first outline Westerners' common views at Wylie's time on the status of Chinese mathematics. Then, we will discuss how Wylie responded to these views in his *Jottings* and

[1] In subsequent discussion, we will use "*Jottings*" as the abbreviation for the article *Jottings on the Science of the Chinese Arithmetic*.

evaluate his viewpoints in the light of contemporary literature on history of Chinese mathematics. Finally, we will discuss the implication of this historical document to current mathematics education. Since this paper is to be submitted five months before the actual presentation so that the authors lack the opportunity of benefitting from comments and views of colleagues in the audience, the present record will focus mainly on Wylie's response to the erroneous statements, while its pedagogical implication will be discussed in more detailed during the actual presentation.

The Chinese terms in the text are written in a system adopted by Wylie in his writings, which is not exactly the (older) Wade-Giles system nor the (more modern) Pinyin system.

2 Common views of Westerners in the 19th Century about Chinese mathematics

In this section, we will give a brief account (with support from excerpts of source materials) on the common views of Westerners in the 19th Century about Chinese mathematics. It will provide background information to the subsequent discussions on Wylie's *Jottings*. We are indebted to Wang (2004) in directing us to some of these source materials.

Generally speaking, Westerners in the 19th Century thought that Chinese possessed only very limited mathematical knowledge which was far behind them (the Europeans). They also thought that mathematics was a neglected discipline to the Chinese. These views are evidenced from the following quotations:

- "The knowledge of mathematics even among learned men is very small, and the common people study it only as far as their business requires." (William, 1848, p.147)
- "For their [Chinese] acquaintance with the exact sciences cannot for a moment bear comparison with that of Europeans." (Murray et al, 1836, p.224)
- "It happens that men of genius neglect that kind of knowledge [knowledge of mathematics and astronomy], and pursue the more popular branches which lead to honour and emolument." (Murray et al, 1836, p.225)

2.1 Contribution of Jesuit missionaries to Chinese mathematics

The prevalent Western view at the time was that Chinese books on mathematics were based on contribution from the Jesuit missionaries. For instance, William (1848) pointed out that the *Swan-fah Tung Tsung* (*General Comprehensive Arithmetic*) and the *Tsuimí-shan Fang Sho Hioh* (*Mathematics of the Lagerstraemia Hill Institution*) contained a lot of material from the mathematical writings of the Jesuit missionaries. Similarly, Davis claimed that:

> "In the science of numbers, and in geometry, the Chinese have, as usual, nothing to teach us; being, on the contrary, indebted for a good deal to Europe, as may be seen from the logarithmic tables and other works prepared for the Emperor Kâng—hy by the Jesuits." (Davis, 1851, p.282)

However, the impact of Western mathematics transmitted by the Jesuits was small. For instance, Murray et al claimed that:

> "the progress which it had made in that country [China], when compared to the time it had been cultivated before the Jesuit missionaries obtained a footing among them, was extremely small.

…it may be inferred, that there existed in that country no mathematics by which it could be improved." (Murray et al, 1836, p.231)

2.2 Chinese numeric notation, arithmetic and algebra

According to the understanding of Westerners, "the [numeric] notation of the Chinese is based on the decimal principle, but their figures are not changed in value by position, and it is difficult therefore to write out clearly the solution of a question." (William, 1848, p.146). William continued to explain that this was overcome, in arithmetical calculations, by the assistance of an abacus. However, he pointed out that its disadvantage is that "if an error be made, the whole must be performed again, since the result only appears when the sum is finished" (William, 1848, p.146). Therefore, he concluded that: "This mode of notation…falls far behind the Arabic system now in general use in the west" (William, 1848, p.146).

Other literature of Westerners in this period shared similar opinion. For instance, Murray et al (1836) gave the following comments on the abacus and Chinese numeric notation and arithmetic:

> "It must, however, be admitted, that although this machine [the abacus] be well adapted for explaining the principles of arithmetic, it would be a very inadequate substitute for our Arabic numerals, more especially in those laborious calculations which the progress of European science has rendered indispensable. Sir George Staunton says, that the Chinese have no characters, except those in their common language, to express sums in an abbreviated form, after the manner of the Arabic figures used by Europeans. When, however, they have occasion to introduce numbers in their writings, they have recourse to their ordinary terms, each of which denotes a numerical value, independently of its relative position,—a method less tedious indeed than the expression of the same numbers by the method of alphabetical writing, but which by no means equals the conciseness of the same process in the Arabic notation. The universal multiplication and subdivision of all quantities by decimal proportions, facilitates their calculations, and prevents the necessity of methods to abridge them. " (Murray et al, 1836, p.228–229)

Davis (1851) not only repeated the above opinions, he even claimed that: "No algebraic knowledge is to be found in China" (Davis, 1851, p. 282). Unfortunately, this erroneous statement was rather popular among Westerners in that period. Indeed, in his Jottings, Wylie put quite a lot of effort to correct this misunderstanding.

2.3 Summary

Based on the source materials above, common views of Westerners at Wylie's time can be listed below:

> 1. Chinese mathematics was far behind Western mathematics. 2. Nothing about Chinese Mathematics was worth learning by Westerners. On the contrary, Chinese mathematics benefited wholly from Western mathematics (for example, logarithm) which was transmitted by the (Jesuit) missionaries. 3. Chinese numeric notation was cumbersome. It fell far behind the Arabic numeric system used by Westerners. Although the notation was based on decimal principle, it did not have local value (that is, the numeric figures were not changed in value by positions). 4. Abacus was an apparatus which assisted the Chinese to do arithmetic calculation, but it was not

very useful-—at least, it was inadequate as a substitute for the Arabic numeric system. 5. There was no algebra in Chinese mathematics.

In the next section, we will describe how Wylie responded to these views in his *Jottings*.

3 Wylie's response in his *Jottings*

Wang (1998) gave a detailed analysis on the structure and content of Wylie's *Jottings*, with selected passages translated into Chinese. In this presentation, we analyse Jottings from another perspective, namely, how Wylie responded to Westerners' common (erroneous) views about Chinese mathematics. In the following discussion, the page numbers of *Jottings* refer to those in the Chinese Researches reprint edition.

3.1 The history of abacus

Wylie wrote: "It has been erroneously stated by some authors that the Chinese have used the 算盤 Swan-pan or abacus from time immemorial." (p.168). It seems that this erroneous statement was rather common among Westerners at that time (see for instances, Murray, 1836, pp. 227–228; Davis, 1851, p.283–284). Wylie pointed out that the abacus was indeed introduced in "comparatively recent date". He continued to introduce the *Show* or tallies which is a predecessor of abacus. "In ancient time calculations were carried on by means of 籌 *Show* or tallies made of bamboo" (p.168). We remark that the history of abacus and tallies mentioned by Wylie is basically correct. Martzloff (1997, Chapter 13) pointed out that the counting rods (tallies) can be traced as early as Former Han Dynasty (1st century, B.C.E.) and kept on playing an important role in Chinese mathematics until the Yuan Dynasty (13–14th century). It was also pointed out that "the abacus only entered into common use in China from the second half of the 16th Century [Ming Dynasty]" (p.215).

The most interesting thing about the tallies which Wylie correctly pointed out is that "the written character is evidently a rude representation of these [the tallies] " (p.168). He made an analogy of this kind of written representation with the Roman numerals and pointed out that both systems have a new symbol for the increment of 5. It provided evidence suggesting that the Chinese numeric notation depended on the theory of local values at a time much earlier than the European understood this theory.

3.2 Local values in Chinese written numbers

On p. 169 of his *Jottings*, Wylie quoted several books of his time, including *Penny Cyclopaedia of the Society for the Diffusion of Useful Knowledge* (edited by Charles Knight, 1833) and also Sir John Davis's works, which claimed that the Chinese written numeration does not have local value. Wylie disagreed and pointed out that "an example from any native work will be a sufficient reply to the above statements" (p.169). Then, he quoted a question from Chapter 8 of *Soo-shoo-kew-chang (Nine Sections of the Art of Numbers)*[2] 數書九章 by Tsin kew-chaon (Song Dynasty, 13th century) as a "random" example. Wylie used this example as an illustration that the arithmetical work in (ancient) China was essentially the same as what the English did, except perhaps with different meanings in some terms. In

[2]Nowadays, this book is known as *Mathematical Treatise in Nine Sections*.

particular, Wylie argued that "the author [of *Nine Sections of the Art of Numbers*] had the same view with regard to local value,⋯, as that universally adopted by modern civilized nations" (p.169).

It is interesting to note that in later part of Jottings when Wylie introduced the method of *Tien-yuen-yih* 天元一 (Chinese algebra of polynomials) found in the Yuan Dynasty, he has the following comments:

> "In the Tien-yuen-yih, unity is employed as the representative of an unknown number; this being combined with an extension of the theory of local value, in order to represent the successive powers of the Monad or unknown number" (p.182).
>
> "It is not a little remarkable, that while it has been gravely asserted by most respectable authorities in Europe, that the Chinese are ignorant of the meaning of local value, we find here on the contrary, that they have pushed the principle to a degree of refinement unpracticed in the west" (p.182).

In other words, Wylie pointed out that the polynomial representation in the method of *Tien-yuen-yih* is indeed a generalization of the theory of local value.

3.3 Algebra in ancient China

In order to respond to the claim that there was no algebra in China, Wylie provided some concrete algebraic methods found in ancient China. It is interesting to note Wylie's comments on the dates for the origin of these methods:

> "In examining the productions of the Chinese one finds considerable difficulty in assigning the precise date for the origin of any mathematical process; for on almost every point, where we consult a native author, we find references to some still earlier work on the subject" (p.175).

Nevertheless, this quotation suggests that Wylie believed that algebraic knowledge did indeed exist in China long time ago.

Ta-yen ("Great Extension") 大衍 (known as "Chinese Reminder Theorem" nowadays) may be the most well-known algebraic method introduced by Wylie in his *Jottings*. As a result of a German translation of *Jottings* (translated by K.L. Biernatzki), this method had drawn the attention of Western historians. Unfortunately, because of some misinterpretation in this German translation, some of these historians thought that this method was mathematically incorrect. After a long process of investigation (thanks to the work of L. Matthiessen in 1881) Westerners realized that *Ta-yen* method was indeed equivalent to the method devised by Gauss. Finally, this method was recognized as the Chinese Remainder Theorem. Readers who are interested in the details of this story may refer to Wang (2004). We now come back to the discussion on how Wylie introduced the method of *Ta-yen* in his *Jottings*. First, Wylie quoted the well-known problem of *Wuh-puh-chi-soo* ("Unknown Numerical Quantities") 物不知數 appeared in *Sun-tsze Swan-king (Sun Tsze's Arithmetical Classic)*[3](Chin Dynasty, 1st Century):

> "Given an unknown number, which when divided by 3, leaves a remainder of 2; when divided by 5, it leaves 3; and when divided by 7, leaves 2; what is the number?" (p.175)

[3]Nowadays, this book is known as *Master Sun's Arithmetical Manual.*

After giving a brief discussion of the method of solution, Wylie proceeded to describe the general method given in Chapter 1 (*Ta-yen*) in *Nine Sections of the Art of Numbers*. It is interesting (but may not be so appropriate[4]) that he selected Problem 1 in the Chapter of *Ta-yen* as an illustration of this method. Despite the fact that this principle is not very clearly explained (for instance, the precise procedure of "finding unity"), Wylie's work played a pioneering role in introducing this method to Westerners.

Another method introduced by Wylie is the *Tien-yuen-yih*(unity)[5] as "the representative of an unknown number" (p.182). This was an ancient Chinese method of representing a polynomial of one variable. More precisely, ancient Chinese used different terms (such as *Yuen* 元, *Tai* 太, *Tai-kieh* 太極) to represent the coefficients of different powers of an unknown quantity, that is, variable x (in today's terminology). As mentioned in a previous section, Wylie regarded it as "an extension of the theory of local value". Furthermore, he also pointed out that "the method invented by Hariot, of placing all the significant terms on one side, is precisely that used by the Chinese [as demonstrated by *Tien-yuen-yih*] some five centuries earlier; and although in itself but a variation in algebraic language, yet it is said by De Morgan to have been the foundation of most important branches of the science" (p.182).

Next, Wylie pointed out that Horner's method of "solving equations of all orders" which was first published in 1819 (some 30 years before the publication of *Jottings*) could be found in *Nine Sections of the Art of Numbers* (Song Dynasty, 13th century which was 6 century earlier). Again, this gives another example that many algebraic methods known by Westerners were already known by Chinese many centuries earlier. This serves as a refutation of the Westerners' usual claim that "no algebraic knowledge is to be found in China".

3.4 Chinese mathematics versus Western mathematics

The overall purpose of Wylie's *Jottings* is to respond to the common Westerners' view (at his time) that Chinese mathematics was far behind their Western mathematics and nothing in Chinese Mathematics was worth learning. As discussed above, Wylie provided some examples to support an opposite view, namely, quite an amount of mathematical knowledge known to Westerners at his time was actually discovered by ancient Chinese much earlier (some even several centuries earlier). The theory of local values in numeric representations, the concept of negative numbers, *Ta-yen* (Chinese Remainder Theorem), *Tien-yuen-yih* (the method of representing a polynomial), and solving polynomials of any degrees are some examples. Furthermore, detailed introduction of some classical Chinese mathematics books such as *Kew-chang-swan-shun (Arithmetical Rules of the Nine Sections)*[6] 九章算術 and *Soo-shoo-kew-chang (Nine Sections of the Art of Numbers)* 數書九章 are included in *Jottings*. Despite the fact that it contains some erroneous descriptions on these books (see for instance, Wang 1998), it has opened up a new window for Westerners to know about Chinese mathematics.

Wylie held a balanced view on Chinese mathematics and Western mathematics. On the one hand, he did not underestimate Chinese mathematics; on the other hand, he recognized the contribution of Western mathematics transmitted by the missionaries to the progress of mathematics in China. In the last part of *Jottings* (p.188 and onwards), he gave a brief account on mathematics in Qing Dynasty

[4]See Wang (2004).

[5]Authors such as Wang (2004) pointed out that Wylie has (mistakenly) mixed up *Ta-yen and Tien-yuen-yih*. Indeed, they are actually not related.

[6]Nowadays, this book is known as *Nine Chapters on the Mathematical Art*.

and pointed out how Western mathematics influenced the development of mathematical ideas in that period. For instance, the work of Li Shan-lan, who became one of Wylie's close co-workers, on logarithm (a mathematical idea transmitted by the Jesuit missionaries) was introduced. As revealed in the following comment, Wylie paid rather high regard to Li's work:

> "This small indication of self-satisfaction may be very well overlooked, as quite pardonable in one who has had no better aid than that afforded by the Leuh-lih-yuen-yuen, and who has here given us, as the result of four years' thought, a theorem, which in the days of Briggs and Napier, would have been sufficient to raise him to distinction." (p.194)

The following closing remark in *Jottings* suffices to describe Wylie's view on Chinese mathematics and Western mathematics, which was indeed rather innovative in his time!

> "It is true the Celestials are disposed to look with a feeling akin to contempt on the mushroom antiquity of our Western lore; yet it is equally true that a spirit of inquiry still germinates among them, which if fostered by a greater freedom of intercourse, will doubtless tend much to smooth the asperities which now exist, and this prove mutually advantageous." (p.194)

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HISTORICAL ALGORITHMS IN THE CLASSROOM AND IN TEACHER-TRAINING

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ABSTRACT

A new national curriculum in French high-schools highlights the importance of algorithms in mathematics, and explicitly requires that elementary but varied work on algorithms be carried out in the classroom. To help teachers face this new demand, we decided to give our teacher-training sessions - set up on the use of historical mathematical sources- a more algorithmic flavor, in order to show them that *understanding*, *writing*, and *justifying* algorithms had been tasks of prime importance throughout the history of mathematics; tasks which could be carried out in the classroom on the basis of historical sources, and not only with a computer. We give in this paper some examples and commentaries.

The *history of mathematics* groups of the IREM (Institute for Research on Mathematics Education) have been involved in teaching (at the secondary level) and teacher-training since the 1980s, with an emphasis on the use of original sources. In the 1990s, the national IREM network published a large selection of historical texts presenting algorithms ([3], [4]), from the most elementary (reckoning) to the pretty sophisticated (approximation of solution of ODEs, inversion of matrices).

A new national curriculum in French high-schools, which is currently being implemented, highlights the importance of algorithms in mathematics, and explicitly requires that elementary but varied work on algorithms be carried out in the classroom. To help teachers face this new demand, we decided to give our teacher-training sessions a more algorithmic flavor, in order to show them that *understanding*, *writing*, and *justifying* algorithms had been tasks of prime importance throughout the history of mathematics; tasks which could be carried out in the classroom on the basis of historical sources, and not only with a computer.

Our involvement in these teacher-training sessions, along with work from professional historians, helped us deepen our understanding of what is at stake when working with algorithms in the math class.

1 The demands of a new curriculum

The new curriculum (here for the 5th Year of Secondary School) emphasizes the role of algorithms in all of the mathematics: “The algorithmic process is, since the beginning of time, an essential part of mathematical activity. In the first years of secondary education, pupils met algorithms (Algorithms of

Elementary Arithmetic Operations, Euclid's Algorithm, Algorithms in Geometrical Constructions). What is proposed in the curriculum is formalization in natural language.[1]"

Although some *notions* have to become familiar to students (such as loops, conditional branching, etc.), the emphasis is on *tasks*: understanding, writing, modifying algorithms. Both computer languages and "natural" language are considered to be relevant semiotic environments. Hence, the general spirit of this new curriculum is not to introduce computer science in the math class, but to promote an *algorithmic form of mathematical thinking*, in addition to (and not in place of) more traditional forms of mathematical activity (algebraic problem-solving, Euclidean-style geometry, calculus etc.).

This take on algorithms is highly consistent with the work on historical sources, since most of them up until the 17th century are of a more or less algorithmic nature, and these ones, and many others, will be improved and formalized in the following centuries.

2 Historical origin of the word and concept of "algorithm"

The first outbreak of the word I know comes from the *Carmen de algorismo , by* Alexandre de Villedieu (circa 1220) : "haec algorismus ars praesens dicitur, in qua talibus Indorum fruimur bis quinque figures : 0. 9. 8.7. 6. 5. 4. 3. 2. 1."[2]

Here "algorismus" refers to the "art" of Algus (or Argus, or Aldus), latinized name of Al Kwharizmi, whose "*The Book of Addition and Subtraction According to the Hindu Calculation*" survived only in its Latin translation . The first words of the manuscript (untitled) are Dixit *aldorizmi* :"So said al-Khwārizmī".

During the course of time, the meaning extended from routine arithmetic procedures "to mean, in general, the method and notation of all types of calculation. In this sense we say the algorithm of integral calculus, the algorithm of exponential calculus, the algorithm of sinus, etc." as wrote D'Alembert, in the article "Algorithme" of his *Encyclopédie*.

Now, here is the definition in our curriculum[3] : "An algorithm is defined as an operational method allowing one to solve, with a number of clearly specified steps, all the instances of a given problem. This method can be carried out by a machine or a person."

Today, the idea of finiteness has entered into the meaning of algorithm as an essential element. This concern arose in a more general context in Hilbert's 10th Problem (1900): given a polynomial equation with arbitrary rational coefficients, "a method is sought by which it can be determined, in a finite number of equations, whether the equation is solvable in rational numbers"[4]. This defines an "effective procedure" (which effectively achieves a result in a finite time).

But this does not give a formal definition of the notion of an algorithm.
In the 17th century Leibniz dreamed of a universal language that would allow reducing mathematical proofs to simple computations. Then, during the 19th, logicians such as Babbage, Boole, Frege, and Peano tried an "algebrization" of logic, but only the definition of a recursive function (Gödel and Church, between 1931 and 1936) gave a satisfactory formal definition to an algorithm.([2] [3] [4]).

[1] *Ressources pour la classe de seconde* : Algorithmique online http://media.eduscol.education.fr/file/Programmes/17/8/Doc_ress_algo_v25_109178.pdf

[2] This new art is called the algorismus, in which we derive such benefit out of these twice five figures of the Indians : 0 9 8 7 6 5 4 3 2 1,

[3] 7th year, students majoring in Informatique et Sciences du Numérique

[4] The answer is no (Matijasevic, 1970)

3 A brief theoretical analysis

Teachers and students are used to formulas for solving problems. A formula is, from the Latin etymology, a "small form" condensing the relations between different objects. According to the Online Etymology Dictionary, modern sense is colored by Carlyle's use (1837) of the word for "rule slavishly followed without understanding"⋯but of course an algorithm too can be slavishly followed without understanding!

Our work with historical texts presenting algorithms prompted us to *describe* the texts, and *characterize* the various mathematical tasks involved in dealing with them. So, a rough sketch would go as follows: at the core of each text lies a mathematical procedure, which calls for:

I. Expression/transmission(in a given semiotic and instrumental context.)
I.a "Algebraic type" : a mathematical relation between different elements (formula) a-i) given in natural language (rhetorical) a-ii) given in an algebraic form with symbols (symbolic)
I.b "Algorithmic type" : a list of instructions (algorithm) b-i) in natural language, given on an example (generic example) b-ii) in natural language, given on undetermined data b-iii) in a programming language

II. Justification (in a given epistemic and social context)
II.a) Simply checking the algorithm with a few examples
II.b) Justifying the mathematical procedure
II.c) Establishing properties of the algorithm itself (seen as a syntactic object): proving that it terminates, proving that it does what it is supposed to do, estimating its size (for comparison purposes), its rapidity of convergence (in case of approximation), etc. (But only finiteness is asked in our curriculum.)

Although this descriptive framework may seem sketchy – and calls for refinement – it turned out to be useful in teacher training. It is useful when it fits, and very useful when it doesn't fit the actual historical sources, since it helps raise interesting questions, of both historical and pedagogical natures.

4 Some Examples of Algorithms (to be discussed during the workshop).

4.1 Chinese and Indian iterative algorithms for the extraction of square roots

As told in the curriculum, we can look at "Algorithms of Elementary Arithmetic Operations". These operations have been performed in a variety of ways (tables for Babylonian and Egyptian, pebbles, knots on strings, bead on an abacus frame, tokens on counting board, marks in the dust⋯). Our pupils have learnt them by using paper and pencil, and the decimal system - like the method named "algorism" in Medieval Books. The procedures are not far from Indian and Chinese ones, but they were only taught on examples. That is not the case for Indian and Chinese ones ([5]).

In both cases, the algorithms indicate the flow of operations to be performed on the computing surface (dirt for instance).The results of operations, receive specific names (dividend, divisor, quotient, given by parallelism with the algorithm of division) and are put in assigned places. In each position, the numbers may vary, but are subjected to the same transformation. This allows an iterative algorithm. And that is what we do in a modern algorithm by assignations of the variables. Of course, we sometimes have to interpret the denominations by using the analogy with the division.

4.2 Heron of Alexandria gives a Method of Successive Approximations for square root on a generic example (circa 50 AD)

Heron gives first an algorithm, on a generic example, to find the area of a triangle, knowing the length of its 3 sides:

"For instance, let the sides of the triangle be of 7, 8, 9 units.

Compose the 7 and the 8 and the 9: the result is 24;

from this take the half: the result is 12;

subtract the 7: 5 remaining.

Again from the 12, sub tract the 8: 4 remaining;

and again the 9: 3 remaining.

Make the 12 by the 5: the result is 60;

these by the 4 : the result is 240;

these by the 3 : the result is 720; from these take a side and it will be the area of the triangle."

It is easy to recognize the corresponding formula, precisely known as "Heron's formula. Then, always on the same example, Heron gives an algorithm in order to find the "side" of 720:

"Since 720 has not a rational square root, we shall make a close approximation to the root in this manner. Since the square root nearest to 720 is 729, having a root of 27, divide 27 into 720; the result is $26\frac{2}{3}$; add 27; the result is $53\frac{2}{3}$; Take half of this; the result is $26\frac{2}{3} + \frac{1}{3} = 26\frac{5}{6}$; therefore the square root of 720 will be very nearly $26\frac{5}{6}$.
For $26\frac{5}{6}$ multiplied by itself gives $720\frac{1}{36}$; so that the difference is $\frac{1}{36}$. Il we wish to make the difference less than $\frac{1}{36}$, instead of 729, we shall take the number now found, $720\frac{1}{36}$, and by the same method, we shall find an approximation differing by much less than $\frac{1}{36}$.[5]

There is no justification for the algorithm. Note that Heron names it a "method", and tells it is a "synthesis".

We find another way of presenting this algorithm by Theon of Alexandria (circa 370 BC).The procedure is described following a geometrical figure in the style of Euclid's Elements (II.4)

This algorithm (along with the following one, from Al Khwarizmi) is a favorite of teachers. It is easy to program with a recurrent sequence, and offers interesting ways of justification, and of error evaluation, in the frame of the curriculum.

[5]Heron , *Metrica*, in Ivor Thomas, *Greek mathematical Works*, vol.II, p.471, and [2], p. 202

4.3 Al-Kwharizimi : *al-Kitab al-mukhtasar fi hisab al-jabr w'al-muqabala or The Compendious Book on Calculation by Completion [or Restoring] and Balancing.*

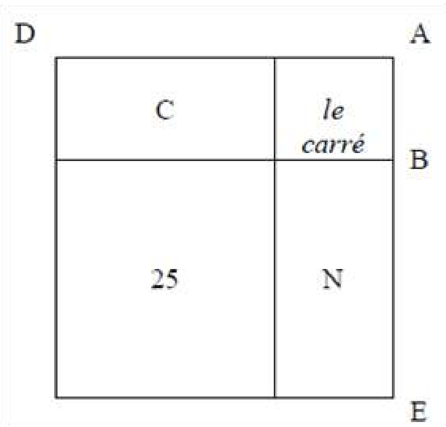
"*Squares and numbers are equal to roots*: for instance, 'one square and ten roots of the same, amount to thirty-nine dirham'; that is to say, what must be the square which, when increased by ten of its own roots, amount to thirty-nine.

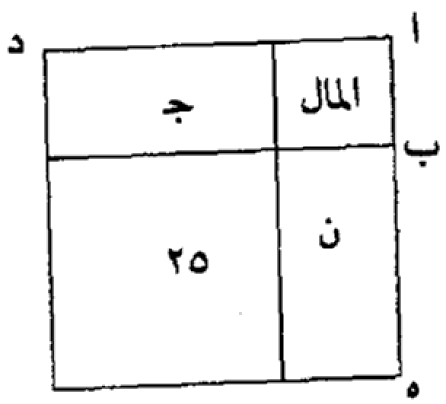
The solution is this: you halve the number of the roots, which in the present instance yields five. This you multiply by himself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for; the square itself is nine. [⋯]"

We have said enough so far as numbers are concerned [⋯] Now, however, it is necessary that we should demonstrate geometrically the truth of the same problems which we have explained in numbers "

Here is a first geometrical proof; Al Kwharizmi gives a second one:

There is another figure which leads to this. Let the area AB, which is the *square*. We are seeking to add to it ten of its roots. We take half of ten: we'll get five.,of which we make two area applied to each side of AB ; let it be the two areas C et N. The length of each of these areas will be five, which is the half of the ten roots., and its breadth is equal to the side of AB. It remains a square from an angle of AB, Which is five times five, and five is the half of the ten roots that we added to each side of the first area. We know then that the first area is the *square*, that the two areas added on each side are ten roots, and the whole is thirty-nine, and that it remains to complement the grater area the square of five by five – which is twenty-five, that we add to thirty-nine to complement the grater area, which is DE. We get from this sixty-four ; we take its root, which is eight, and which is one side of the greater area ; if we subtract from this an equal quantity to the one we have added, which is five, it remains three, which is the side of the area AB – which is the *square* – and which is its root ; the *square* is nine. Here is the figure:





4.4 : Diophantus (III^{rd} century AD) : "formula" vs" program"

We chose this text, because Diophantus gives three presentations for the same mathematical property: one with a "rhetorical formula", and one with an algorithm, then the inverse algorithm.[1]

In the short Treatise *About polygonal numbers (De polygonis numeris*), with the first four propositions and a subtle additional argument, Diophantus proves the validity of the following defining relation of each type of polygonal numbers:

> "Every polygonal, multiplied by the octuple of the number less by a dyad than the multiplicity of the angles, and taking in addition the square on the number less by a tetrad than the multiplicity of the angles, makes a square "

In "symbolic formula" language, a polygonal number P with v angles satisfies the relation $8P(v-2)+(v-4)^2$ =*square*. Prop. 4 makes it explicit what is the side of the *square*; its expression contains the side l of the polygonal number P: the full-fledged relation is: $8P(v-2)+(v-4)^2 = [2+(v-2)(2l-1])^2$ amounting to a definition of a *specific* polygonal number, that is, to an identification of any number greater than 2 as a specific polygonal one.

Diophante explains ensuite how to find, for a polygone number from a given type (ie that the multiplicity of angles v is fixed), a polygone number P whose side l is given and the reverse. The description is formulated by a mix of grammatical participle and future.

> 'Thus, taking the side of the polygone [number], always doubling it, we'll subtract one unity, and multiplying the remainder by the number lesser by a dyad than the multiplicity of angles, we'll add a dyad to the product; and taking the square on the result, we'll subtract from it the square of the number lesser by a tetrad than the multiplicity of angles, and dividing the remainder by eight times the number lesser by a dyad than the multiplicity of angles, we'll find the required number.
>
> Again, given the polygone itself, we'll find the side like this: Multiplying it by eight times the number lesser by a dyad than the multiplicity of angles , and adding to the product the square on the number lesser by a tetrad than the multiplicity of angles, we'll thus find a square (assuming that the given number is a polygone), and subtracting always a dyad to the side of this square ,we'll divide the remainder by the number lesser by a dyad than the multiplicity of angles, and adding to the result one unit, and halving the result, we'll get the side required.

Here, the "rhetorical formula" of the beginning justifies the algorithms.

Moreover, we can analyze how Diophantus writes the "inverse" algorithm, by inverting exactly the order and the operations (add/subtract, etc.).

We are used to the formal manipulation of formulae, but, we can analyze here a formal manipulation of algorithms. There are other documents: in the *Nine chapters*, for instance, we can find that two algorithms are proved to be equivalent because some steps cancel in pairs.

The text of Al-Khwarizmi is quite often used in the classroom to express an algorithmic solution to one that is summarized in a formula. Teachers are used to going from the algorithm to the formula, as formula is more familiar to them (and to current students). We see here how Diophantus extracts an algorithm from a formula. This is a commonplace task in elementary mathematics.

4.5 Jordanus de Nemore (circa 1220)

"(*a*) *If a given number is separate in two parts whose difference is known, then , each of the parts can be found.*

Since the lesser part and the difference equal the larger, the lesser with another equal to itself together with the difference make the given number.
Subtracting therefore the difference from the total, remains is twice the lesser. Halving this yields the smaller and, consequently, the greater part.
For example, separate 10 in two parts whose difference is 2. If that I subtracts from 10, 8 remains, whose half is 4."

This is not as easy as algebra to follow, but it gives both method and justification. But, for the Problem (b) it is more difficult! So Jordanus introduces letters for intermediary results.

"(*b*) *If a given number is separated in two parts such that the product of the parts is known, then each of the parts can be found.*
Let the given number a be separated into x and y so that the product of x and y is given as b. Moreover, let the square of $x+y$ be e, and the quadruple of b be f. Subtract this from e to get g, which will then be the square of the difference of x and y. Take the square root of g, and call it h. h is also the difference of x and y. Since h is known, then x and y can be found.
The mechanics of this is easily done thus; For example, separate 10 into two numbers whose product is 21. The quadruple of this is 84, which subtracted from the square of 10, namely from 100, yields 16. 4 is the root of this and also the difference of the two parts. Subtracting this from 10 to get 6, which halved yields 3, the lesser part; and the greater is 7."[6]

Here, there is no more justification. We can notice that this algorithm comes back to the previous, using it as a procedure .

4.6 Euler, (1774) an algorithm to find the square root; on a generic example and with letters for undetermined data.

Elements of algebra, Chap XVI, On the Resolution of Equations by Approximation

784. When the roots of an equation are not rational, whether they may be expressed by radical quantities, or even have not that resource, as in the case with equations which exceed the fourth degree, we must be satisfied with determining their values by approximation; that is to say, by methods which are continually bringing us nearer to the true value, till at least the error being very small, it may be neglected. Different methods of this kind have been proposed, the chief of which we shall explain.

785. The first method which we shall mention, supposes that we have already determined, with tolerable exactness, the value of one root; that we know, for example, that such as 4, and that it is less than 5. In this case, if we suppose this value $= 4 + p$, we are certain that p expresses a fraction. Now, as p is a fraction, and consequently less than unity, the square of p, its cube, and in general, all the higher powers of p, will be much less with regard to the unity ; and, for this reason since we require only an approximation, they may be neglected in the calculation; When we have, therefore, nearly determined the fraction p, we shall know more exactly the root $4 + p$; from that we proceed to determine a new value still more exact, and continue the same process till we come as near the truth as we desire.

786. We shall illustrate this method first by an easy example, requiring by approximation the root of the equation $x^2 = 20$.

Here we perceive, that x is greater than 4 and less than 5 ; making, therefore, $x = 4 + p$, we shall have $x^2 = 16 + 8p + p^2$; but as p^2 must be very small, we shall neglect it, in order to that we may have only the equation $16 + 8p = 20$, or $8p = 4$. This gives $p = \frac{1}{2}$, and $x = 4\frac{1}{2}$, which already approaches nearer the true root. If, therefore, we now suppose $x = 4\frac{1}{2} + p$; we are sure that p expresses a fraction much smaller than before, and that we may neglect p^2 with greater propriety. We have, therefore, $x^2 = 20\frac{1}{4} + 9p = 20$, or $9p = -\frac{1}{4}$; and consequently $p = .-\frac{1}{36}$; therefore $x = 4\frac{1}{2} - \frac{1}{36} = 4\frac{17}{36}$.
And if we wished to approximate still nearer to the true value, we must make $x = 4\frac{17}{36} + p$, and should thus have $x^2 = 20\frac{1}{1296} + 8\frac{34}{36}p = 20$; so that $8\frac{34}{36}p = -\frac{1}{1296}$, $322p = -\frac{36}{1296} = -\frac{1}{36}$ and $p = -\frac{1}{36\cdot 322} = -\frac{1}{11592}$, therefore, $x = 4\frac{17}{36} - \frac{1}{11592} = 4\frac{4473}{11592}$, a value which is so near the truth, that we may consider the error as of no importance.

787. Now, in order to generalize what we have here laid down, let us suppose the given equation $x^2 = a$, and that we previously know x to be greater than n, but less than $n + 1$. If we now make $x = n + p$, p must be a fraction, and p^2 may be neglected as a very small quantity, so what we shall have $x^2 = n^2 + 2np = a$; or $2np = a – n^2$; and $p = \frac{a-n^2}{2n}$ and consequently $x = n + \frac{a-n^2}{2n} = \frac{n^2+a}{2n}$. Now , if n approximated towards the true value, this new value will approximate much nearer ; and , by substituting it for n, we shall obtain a new value, which may again be substituted, in order to approach still nearer ; and the same operation may be continues as long as we please."

5 Questions and Perspectives

. We shall mention two here, which we plan to study in some detail in the future:

A. Algorithms in geometry: to what extent can construction procedures be read as algorithms? For instance, in Euclid's elements, can the *ekthesis* be seen as an initializing phase? Can *diorisms*, (related to the numbers of solutions of a problem in function of the value of the "givens" of the problem) and *case distinctions* be seen as instances of conditional branching?

B. Formulae vs algorithms : Beyond issues of cognitive flexibility ("translation" tasks), one can investigate the following issues:

- Compare the two semiotic environments (formulae / algorithms) in terms of manipulation potential
- Finiteness issues: many iterative algorithms (for square root approximation for instance, or iterated Euclidean division to find a continued fraction expansion) lead to infinite formulae. Could we see the later as the counterpart of algorithms? This lead to a wealth of questions, beyond that of convergence.

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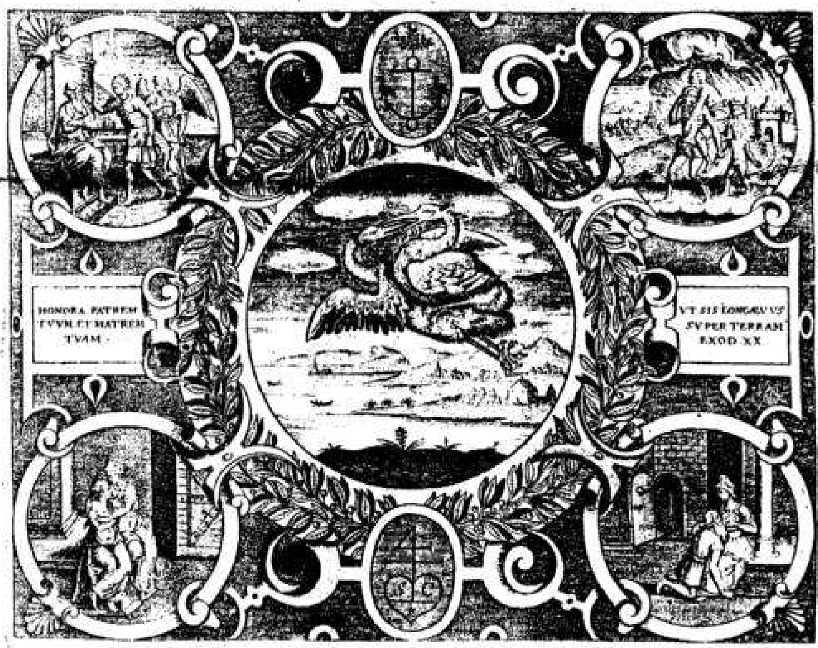

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Mathematics of the 19th century engineers: methods and instruments

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ABSTRACT

Traditionally, history of mathematics was more interested in the production of abstract concepts and major theories than in the one of methods and tools of calculation. As a result, it has often neglected to consider the mathematical practices of engineers. I want to sketch here an overview of these practices, at least as they can be identified in the 19th century, at the time when the professional community of engineers structures itself and deepens its mathematical culture[1].

1 Different mathematical practices

The mathematical needs of engineers seem very different from those of mathematicians. To illustrate this with a significant example, consider the problem of the numerical solution of equations, a pervasive problem in all areas of mathematics intervention.

In the early 19th century, the Joseph-Louis Lagrange's *Traité de la résolution des équations numériques de tous les degrés* is authoritative. This treatise, which went through three editions in 1798, 1808 and 1826, can be seen as a gigantic algorithm, fully established from the theoretical point of view, to detect all the roots of a polynomial equation and to calculate each of them numerically by using a always converging process. Given a polynomial equation, Lagrange's method consists in calculating a lower bound $\triangle$ of absolute values of differences between the distinct real roots. For this, one constructs an auxiliary equation whose roots are the differences between all ordered pairs of distinct roots. Then, using $\triangle$ and transforming possibly the equation by a change of scale, one determines a set of integers p such that each interval $[p, p+1]$ contains a single real root and each real root is contained in one of these intervals. Finally, on each interval $[p, p+1]$, one uses a continued fraction expansion which converges always to the root and provides an estimate of the error, contrary to what happens in some cases with Newton's method.

[1]For this text, I borrowed heavily in the work of three researchers with whom I had the pleasure of collaborating for several years: Konstantinos Chatzis, Marie-José Durand-Richard, and Joachim Fischer. I thank them for all they have brought me.

This algorithm, justified with the utmost rigor, is undoubtedly perfect for the algebraist who does not solve equations actually, but proves simply that it is possible to solve them numerically with arbitrary precision. That is why Lagrange's method quickly raised criticisms by some mathematicians like Joseph Fourier and Charles Sturm. These names are not trivial: indeed, they are scholars for whom mathematics is not an end in itself but a tool for natural philosophy, that is to say, the understanding of the physical world. It is precisely to study the propagation of heat that Fourier imagined how to model the evolution of temperature by using the trigonometric series that now bear his name. Similarly, Sturm focused on various problems of mechanics and physics, such as the compressibility of liquids or, following Fourier's tradition, the heat, which led him to his famous results on the "Sturm-Liouville problem". Sturm and Fourier are interested primarily in mathematical problems they encounter in the context of natural philosophy, and do not hesitate to use all the resources of calculus to deal with problems in the theory of equations which some other geometers of their time should have addressed by purely algebraic methods. On this occasion, they do not refrain from criticizing Lagrange.

In the *Analyse des équations déterminées*, published posthumously in 1831, Fourier writes: "Lagrange and Waring have offered to find the smallest difference of the roots of the equation, or a lesser amount than the smallest difference. Considered theoretically, the solution is correct [...]. But it is easy to judge that we cannot accept such a solution method. 1° Indeed the calculation that would furnish the value of the limit $\triangle$ is impracticable for equations of somewhat high degree [...]"[2] A few years later, in his "Mémoire sur la résolution des équations numériques" published in 1835, Sturm is no less severe: "This method, considered from a purely theoretical point of view, leaves nothing to be desired on the side of rigor. But, in the application, the length of the calculations necessary to form the equation for the squared differences, and the multitude of substitutions that one may have to perform, make it almost impracticable, and although Lagrange made there some simplifications, it still requires very painful calculations, so we have tried other solutions"[3].

These mathematicians-physicists, knowing well by nature what is actually the practice of mathematics within applications, are sensitive to the effectiveness of the calculation methods they study. But Sturm and Fourier are nevertheless academicians and professors at the École polytechnique; they are not practitioners. Looking at the side of engineers, the situation is radically different. Léon-Louis Lalanne, a French civil engineer who, throughout his career, sought to develop practical methods for solving equations, wrote what follows as a summary when he became director of the École des ponts et chaussées: "The applications have been, until now, the stumbling block of all the methods devised for solving numerical equations, not that, nor the rigor of these processes, nor the beauty of the considerations on which they are based, could have been challenged, but finally it must be recognized that, while continuing to earn the admiration of geometers, the discoveries of Lagrange, Cauchy, Fourier, Sturm, Hermite, etc., did not always provide easily practicable means for the deter-

[2]Joseph Fourier, *Analyse des équations indéterminées, première partie*, Paris: Didot, 1831, p. 116-117.

[3]Charles Sturm, "Mémoire sur la résolution des équations numériques", *Mémoires présentés par divers savans à l'Académie royale des sciences de l'Institut de France, section sciences mathématiques et physiques*, 6 (1835), p. 274.

mination of the roots"[4].

Lalanne says that as politely as possible, but his conclusion is clear: the methods advocated by mathematicians, that they be "pure" or "applied" (provided this distinction be meaningful in the 19th century), are not satisfactory. These methods are complicated to understand, long to implement and sometimes totally impracticable for ground engineers, foremen and technicians, who, moreover, did not always receive a high-level mathematical training.
Given such a situation, 19th century engineers were often forced to imagine by themselves the operational methods and the calculation tools that mathematicians could not provide them. The objectives of the engineer are not the same as those of the mathematician, the physicist or the astronomer: the engineer rarely needs high accuracy in his calculations, he is rather sensible to the speed and simplicity of their implementation, especially since he has often to perform numerous and repetitive operations. He needs also methods adapted for use on the ground, and not just for use at the office. Finally, priority is given to methods that avoid performing calculations by oneself, methods that provide directly the desired result through a simple reading of a number on a numerical or graphical table, on a diagram, on a curve or on the dial of a mechanical instrument.

2 A growing need for calculation

The 19th century is the moment of the first industrial revolution, which spreads throughout the Western world at different rates in different countries (Great Britain: 1780-1850, France: 1815-1850, Germany: 1835-1870, United States: 1860-1885, Japan: 1870-1900, Russia: 1880-1905). Industrialization causes profound transformations of society. In this process, the engineering world acquires a new identity, marked by its implications in the economic development of industrial states and the structuration of new professional relationships that transcend national boundaries. The engineer then changes its status: it was formerly a practitioner of the arts in service of princely courts; it is now becoming a professional working for the civil society that develops simultaneously with the appearance of the nation-states. The constitution of a specific milieu of engineers, resulting among other things in the creation of numerous professional associations and many specialized journals, is closely related to major public works and ambitious industrial projects that accompany the political changes implemented in Europe, the United States and in the emerging colonial empires.

In this context, the engineer is faced with ever more numerous calculations, longer and more complex, requiring an increasing mathematization as well as a collective work organization more effective. This leads him to create by himself mathematical tools adapted to the new problems and the new implementation constraints that face him. Let us mention here briefly some of these problems and tools.

During the years 1830-1860, the sector of public works experiences a boom in France and more generally in Europe. The territories of the different countries are covered progressively by vast networks of roads, canals, and, after 1842, of railways. These achievements require many tedious calculations of

[4]Léon-Louis Lalanne, "Exposé d'une nouvelle méthode pour la résolution des équations numériques de tous les degrés (troisième partie)", *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 82 (1876), p. 1487.

surfaces of "cut and fill"[5] on cross-sections of the ground. Civil engineers try then different methods of calculation more or less expeditious. Some, like Gaspard-Gustave Coriolis, calculate numerical tables giving the surfaces directly based on a number of features of the road and its environment, such as roadway width or slope. Other engineers, especially in Germany and Switzerland, design and build mechanical devices, planimeters and integraphs, used to quickly calculate all kinds of surfaces on a plane. These instruments are the perfect mechanical translation of the calculus principles (continuous summation of infinitesimal surfaces for the planimeters, continuous drawing of a curve whose slope is equal at every moment to the ordinate of a given curve for the integraphs), and for this reason they will have significant applications in many other scientific fields. Still others, like Lalanne, imagine replacing numerical tables by graphical tables, cheaper and easier to use. It is within this framework that a new mathematical discipline, called "nomography", develops itself and will be deepened throughout the second half of the 19th century and beyond.

Another large field of problems is that of construction. The study of conditions of stability and resistance of structures (beams, bridges, roofs, arches, vaults, retaining walls, etc.) also gives rise to a new discipline, called "graphic statics", which uses systematically the concepts of polygon of forces and funicular polygon, and whose goal is to replace analytical methods by graphical methods exploiting the achievements of projective geometry, a branch of mathematics that was booming at this time. Graphic statics takes immediately on an international character, since it is developed mainly by engineers from France (Jean-Victor Poncelet, Gabriel Lamé, Paul-Émile Clapeyron), Germany (Carl Culmann, Otto Mohr, Wilhelm Ritter), Great Britain (William Rankine James Clerk Maxwell), Italy (Luigi Cremona, Antonio Favaro) and United States (Henry Turner Eddy). Graphic statics arrives timely, when metal structures of all types are multiplying: suspension bridges, railway stations, exhibition halls, etc. These metal structures are composed of a multitude of individual elements. Going through the algebraic calculation to estimate the internal forces working inside this type of structure is a tedious process, which may simultaneously give rise to errors because of the large number of pieces that form these structures. This type of construction culminated with the spectacular achievements of Gustave Eiffel, who, in collaboration with the engineer Maurice Koechlin, a student of Culmann, realized notably the Garabit Viaduct (1878-1923) and the Eiffel Tower (1887-1889). The famous Eiffel Tower has been calculated completely by using the techniques of graphic statics: "To give an idea of the importance of the labor of studies, it suffices to say that the principal drawing office, directed by Maurice Koechlin, made for the backbone of the Tour alone, not including elevators and ancillary works such as basements, stairways, tanks, restaurants, etc., more than 1,700 general drawings; and that the auxiliary office headed by Mr. Pluot established 3,629 drawings for execution. The total surface of these 5,300 drawings exceeds 4,000 m2. The number of different pieces that are detailed is 18,038. This considerable work has required hard work of thirty draughtsmen for eighteen months"[6].

If we look now on the military side, the engineers from the artillery schools face difficulties in constructing firing tables from the fundamental equation of ballistics. The increasing ranges and speeds of projectiles make it impossible to keep the old assumption that air resistance is proportional to

[5]"Cut and fill" is the process of earthmoving needed to construct a road, a canal or a railway: to minimize the construction labor, the amount of material from cuts should roughly match the amount of fills.

[6]Gustave Eiffel, *La Tour de trois cent mètres*, Paris: Société des imprimeries Lemercier, 1900, p. 101.

the square of the velocity. The determination of ballistic trajectories is a place of tension between mathematicians and engineers. Here as elsewhere, the theoretical solutions are not satisfactory for the artilleryman on the battlefield, who must determine very quickly the firing angle and the initial speed to give to his projectile to reach a given target, and needs for this firing tables accurate and easy to use. In 1892, Francesco Siacci, a major figure in Italian ballistics, writes: "Our intention is not to present a treatise of pure science, but a book of immediate usefulness. Few years ago ballistics was still considered by the artillerymen and not without reason as a luxury science, reserved for the theoreticians. We tried to make it practical, adapted to solve fast the firing questions, as exactly as possible, with economy of time and money"[7].

To calculate their firing tables, military engineers conceive numerous numerical and graphical methods for integrating by approximation the ballistic differential equation. In 1834, Jacob Christian Friedrich Otto uses integration by successive arcs to compute firing tables that will experience a great success and will be in use until the early 20th century. In France, an original approach is due to Alexander-Magnus d'Obenheim in 1818. His idea was to replace the numerical tables by a set of curves carefully constructed by points calculated with great precision. These curves are drawn on a portable instrument called the "gunner board". The quadrature method used to construct these curves is highly developed. Obenheim employs a method of Newton-Cotes type with a division of each interval into 24 parts. In 1848, Isidore Didion, following Poncelet's ideas, constructs ballistic curves that are not a simple graphic representation of numerical tables, but are obtained directly from the differential equation by a true graphical calculation: he obtains the curve by successive arcs of circles, using at each step a geometric construction of the center of curvature. During the last part of the 19th century, there is a parallelism between the increasing speeds of bullets and cannonballs, and the appearance of new instruments to measure these speeds. Ballisticians are then conducted to propose new air resistance laws for certain intervals of speeds, and they used almost 40 different empirical laws to calculate tables.

It would be easy to multiply examples, speaking also, among other subjects, of hydraulics and, more generally, of fluid mechanics. This domain forced engineers to imagine many new approaches to problems that are modeled by partial differential equations. But the key is to have highlighted the enormous computational requirements which appeared during the 19th century in all areas of engineering sciences and which caused an increasing mathematization of these sciences. This leads naturally to the question of engineering education: how were engineers prepared to use high-level mathematics in their daily work and, if necessary, to create by themselves new mathematical tools?

3 Mathematical education of the engineers in the 19th century

The French model of engineering education in the early 19th century is that of the École polytechnique, founded in 1794 through the main impetus given by Gaspard Monge. Although it had initially the ambition to be comprehensive and practice-oriented, this school promoted quickly a high-level teaching dominated by mathematical analysis. This teaching only theoretical was then completed,

[7]Francesco Siacci, *Balistique extérieure*, trad. fr. par P. Laurent, Paris & Nancy: Berger-Levrault, 1892, p. x.

from the professional point of view, by two years in application schools with civil and military purposes: École des ponts et chaussées, École des mines, École de l'artillerie et du génie de Metz, etc. This training model, which subordinates practice to theory, has produced a corporation of "ingénieurs savants" capable of using the theoretical resources acquired during their studies to achieve an unprecedented mathematization of the engineering art. In particular, the engineers of the Ponts et Chaussées played a leading role in the creation of new instruments and methods of calculation: indeed, in the years 1830-1860, they had to face, as we saw it before, the many technical and organizational problems posed by the massive achievement of modern infrastructure (roads, canals, railways) throughout the French territory.

This model is considered to have influenced the creation of many polytechnical institutes throughout Europe and to the United States. However, unlike France, the teaching of the theory and practice are held together in these new schools, and the training they offer is far from monolithic. In particular the teaching of mathematics, while occupying an important place, is less centered on mathematical analysis than in France. In the first half of the century, polytechnical institutes are created on the model of elite military school. They train mostly State military engineers, who may also be in charge of certain public works. Later, polytechnical institutes better meet the growing needs of industry and train mainly civil engineers in the strict sense. They are more similar to the École Centrale des Arts et Manufactures in Paris and are integrated, sometimes into the higher technical schools, sometimes into the universities. These institutes offer courses directly adapted to the future professional practice of engineering students, such as courses of descriptive geometry, graphic statics, or graphical computation.

Unlike continental Europe, there are no polytechnical institutes in England for the training of civil engineers; the latter is organized by employers inside an apprenticeship system. This is done so first and foremost in the business, before being integrated later, towards the end of the century, into the new universities of industrial cities: the "red-brick universities" or "civic universities". The result is a much less intense standardization of the engineering profession than on the Continent, and a less formal training. Thus, in reports made in 1889, 1892 and 1893 for the British Association for the Advancement of Science, Henry Selby Hele Shaw noted the delay of the British engineers in the operation of new calculation methods, including graphic statics, and calls for reform of engineering education, including the introduction of specialized courses in high schools incorporated into new universities. He notes in particular that the higher geometry is taught widely on the Continent: "This kind of geometry, which is called, under the various names of 'modern geometry', 'higher geometry', 'projective geometry', or, best of all, 'geometry of position', has advanced enormously in importance in recent years, and has become a subject of instruction chiefly on the Continent, in the polytechnical schools"[8], and then he regrets that this is not the case in Great Britain: "The fact is, that very few engineers in this country understand even the nature of projective, modern, higher geometry, or geometry of position, by all of which names it is variously called"[9]. The lack of specialized engineering schools and

[8]Henry Selby Hele Shaw, "Second report on the development of graphic methods in mechanical science", *Report of the Sixty-Second Meeting of the Bristish Association for the Advancement of Science Held at Edinburgh in August* 1892, London: Murray, 1893, p. 385.

[9]*Ibid*., p. 423.

the lack of high-level mathematical courses offered to engineering students in the universities then explain the fact that the English engineers, at least for a certain period, have contributed less than their continental counterparts to develop new mathematical methods.

4 An example of a mathematical discipline created by engineers: nomography

Unable to deal with everything in the limited framework of this conference, I will devote myself to nomography. Indeed, this is the paradigmatic example of a corpus of mathematical tools, constituting an autonomous discipline, which was created from scratch by engineers themselves to meet their needs. Moreover, the theoretical foundations of this discipline are almost entirely due to French engineers who came out the École polytechnique and then worked in civil engineering: Léon-Louis Lalanne, Charles Lallemand, Maurice d'Ocagne, Rudolph Soreau, etc. We must add to this list the Belgian engineer Junius Massau, an ancient student and then professor at the school of civil engineering of the University of Ghent, where training was comparable to that of the École polytechnique, with high-level courses of mathematics and mechanics.

The main purpose of nomography is to construct graphical tables to represent any relationship between three variables, and, more generally, relationships between any number of variables. Isolated examples of graphical translation of double-entry tables are found already in the first half of the 19th century, mainly in the scope of artillery, but this is especially Lalanne, engineer of the Ponts et Chaussées, who gave a decisive impetus to the theory of graphical tables. In 1843 was published in Paris, in a French translation, the *Cours complet de meteorologie* by Ludwig Friedrich Kämtz, professor of physics at the University of Halle. In an Appendix to this course, Lalanne provides consistent evidence that any law linking three variables can be graphed in the same manner as a topographic surface using its marked level lines. In this same year 1843, Lalanne presents at the Académie des sciences de Paris a memoir in which he outlines the idea of using non-regular scales on the x-axis and the y-axis: by replacing the primitive variables by auxiliary functions of those ones, suitably chosen, it is possible in some cases, to reduce to straight lines the marked level lines. For example, for multiplication, after remarking that the relationship $\gamma = \alpha\beta$ can also be written $\log\gamma = \log\alpha + \log\beta$, just graduate the axis with the new variables $x = \log\alpha$ and $y = \log\beta$, and then the bundle of hyperbolas $\gamma = \alpha\beta$ becomes a bundle of straight lines with equations $x + y = log\gamma$. By analogy with an optical phenomenon, Lalanne names "anamorphosis" this transformation. He also introduces the word "abacus" in this context to designate the new graphical tables by inscribing them into the historical continuity of the art of computation (previously an abacus was a calculating table on which jetons could be displayed and moved inside columns)[10].

Lalanne obtained a massive spread of graphical tables in the sector of public works, where his ideas came to a favorable moment. Indeed, the Act of June 11, 1842 had decided to establish a network of major railway lines arranged in a star from Paris. To run the decision quickly, one felt the need for

[10]This new sense of the word $\ll$ $abacus$ $\gg$ was not adopted in England and the United States, where this kind of graphical table was called $\ll$ $contourlinechart$ $\gg$.

new ways of evaluating the considerable earthworks to be carried out. In 1843, the French government sent to all engineers involved in this task a set of graphical tables for calculating the areas of cut and fill on the profile of railways and roads.

After Lalanne, the graphical tables resting on the principle of concurrent lines spread rapidly, until becoming, in the third quarter of the 19th century, very common tools in the world of French engineers. The Belgian engineer Junius Massau succeeded Lalanne to enrich the method and its scope of application. Professor at the school of civil engineering of the University of Ghent, Massau distinguished himself in the field of engineering sciences by contributing in a creative manner to rational mechanics, to nomography and especially to graphical integration, a discipline of which he is considered as the creator. Essentially, his contributions to the theory of graphical tables are contained in two large memoirs published in 1884 and 1887.

After recalling the work of Lalanne that inspired him, Massau introduces a notion of generalized anamorphosis, seeking what are the functions that can be represented using three pencils of lines, without requiring the first two pencils to be parallel to the coordinate axis. In this case, the lines of α, β and γ shall have respective equations of the form

$$
\begin{aligned}
f_1(\alpha)x + g_1\alpha)y + h_1(\alpha) = 0,\\
f_2(\beta)x + g_2\beta)y + h_2(\beta) = 0,\\
f_3(\gamma)x + g_3\gamma)y + h_3(\gamma) = 0,
\end{aligned}
$$

and the concurrency condition of these lines, coming from the elimination of x and y, is

$$
\begin{vmatrix}
f_1(\alpha)\, g_1(\alpha)\, h_1(\alpha) \\
f_2(\beta)\, g_2(\beta)\, h_2(\beta) \\
f_3(\gamma)\, g_3(\gamma)\, h_3(\gamma)
\end{vmatrix} = 0.
$$

When we can put in this form a relationship $F(\alpha, \beta, \gamma) = 0$, this relationship is precisely representable by an abacus with concurrent straight lines (called in English a “straight line chart”). Determinants of the above type, called “Massau determinants”, played an important role in the subsequent history of nomography; they are encountered in research until today.

A variant of straight line charts was designed by Charles Lallemand, an engineer of Mines. The type of graphical table he invented, called “abaque hexagonal” in French and “hexagonal chart” in English, prolonged undoubtedly the interest in the charts with concurrent lines. We are at a time when a broad program of public works is elaborated in France. The execution of this program required more precise knowledge of the relief of the ground, hence it was decided to undertake what geodesists call the leveling of the whole country: complementary to the triangulation which fixes the position of points on the ground in horizontal projection, the geodetic leveling consists in determining their elevations above mean sea-level. From 1880, Lallemand was responsible for creating a Service du nivellement general de la France, which officially began in 1884. It is within this context that he invented the hexagonal charts, designed as a graphical method to automate the long and tedious calculations necessary for the operation of numerous measurements on the ground. Lallemand’s efforts allowed tripling the accuracy of previous results, while significantly reducing costs. Hexagonal charts operate a method of graphical addition based on the fact that the sum of the projections of a straight line segment on two axis forming between them an angle of 120°, is equal to the projection of the same segment on the internal bisector of these axis. By graduating the three axis with non-regular scales $x = f(\alpha)$, $y = g(\beta)$ and $z = h(\gamma)$, one can represent by such a chart any equation with three variables of the form $f(\alpha) + g(\beta) = h(\gamma)$.

With Massau's and Lallemand's publications, the theory of contour lines charts was entering into a mature phase, but in the same time a new character intervened to orient this theory towards a new direction. Philibert Maurice d'Ocagne entered the École Polytechnique in 1880, and then made his entire career in the corps of Ponts et Chaussées. In particular, he was called from 1891 to 1901 at the Service du nivellement, to help Lallemand. Meanwhile, he taught tirelessly for 45 years at the École polytechnique, at the École des ponts et chaussées and at the Sorbonne (University of Paris). Closely linked to this dual activity as an engineer and a teacher, Ocagne continued during all his life an important research that resulted in over 400 publications. Within this burgeoning work that touches on many themes, it is essentially the part about graphical charts that won him fame.

In 1884, when he was only 22 years old, Ocagne observes that most of the equations encountered in practice can be represented by an abacus with three systems of straight lines and that three of these lines, each taken in one system, correspond when they meet into a point. His basic idea is then to construct by duality, by substituting the use of tangential coordinates to that of punctual coordinates, a figure in correlation with the previous one: each line of the initial chart is thus transformed into a point, and three concurrent lines are transformed into three aligned points. The three systems of marked straight lines become three marked curves. To clarify this, consider three arbitrary curves defined by parametric equations

$$x = \frac{f_i(t)}{h_i(t)}, \quad y = \frac{g_i(t)}{h_i(t)} \quad (i = 1,\ 2,\ 3).$$

Three points marked with $t = \alpha$, $t = \beta$ and $t = \gamma$, taken on these three curves respectively, are aligned when

$$\begin{vmatrix} f_1(\alpha)\, g_1(\alpha)\, h_1(\alpha) \\ f_2(\beta)\, g_2(\beta)\, h_2(\beta) \\ f_3(\gamma)\, g_3(\gamma)\, h_3(\gamma) \end{vmatrix} = 0.$$

A given relationship between three variables is then representable by an "alignment chart" if and only if, it can be put into the form of a determinant of the above type. One recognizes unsurprisingly a Massau determinant, because it is clear that the problem of the concurrency of three straight lines and the problem of the alignment of three points, dual to each other, are mathematically equivalent. Using an alignment chart is particularly simple: in practice, to avoid damaging the chart, one does not draw actually the auxiliary straight line on the paper: one uses either a transparency marked with a straight thin line or a thin string tightened between the points to join.

After this first achievement in 1891, Ocagne deepened the theory and applications of the alignment charts until the publication of a large treatise in 1899, the famous *Traité de nomographie. Théorie des abaques. Applications pratiques*, which became for a long time the reference book of the new discipline.

In this treatise, Ocagne still employs the accepted term of "abacus" to refer to any graphical table; however, a little later, he introduced the generic term "nomogram" to replace "abacus", and the science of graphical tables became "nomography". From there, alignment charts were quickly adopted by many engineers for the benefit of the most diverse applications. At the turn of the 20th century, nomography is already an autonomous discipline well established in the landscape of applied sciences.

5 Engineering mathematics as a source of new theoretical developments

The mathematical practices of engineers are often identified only as "applications", which is equivalent to consider them as independent from the development of mathematical knowledge in itself. In this perspective, the engineer is not supposed to develop a truly mathematical activity. We want to show, through some examples, that this representation is somewhat erroneous: it is easy to realize that the engineer is sometimes a creator of new mathematics, and, in addition, that some of the problems which he arises can in turn irrigate the theoretical research of mathematicians.

To return to nomography, the problem of general anamorphosis, that is to say, of characterizing the relationships between three variables admitting a representation by straight lines charts (or, in an equivalent formulation, relationships that may be put into the form of a Massau determinant), has inspired many theoretical research to mathematicians and engineers: Augustin-Louis Cauchy, Paul de Saint-Robert, Junius Massau, Leon Lecornu, and Ernest Duporcq have brought partial responses to this problem before that in 1912 the Swedish mathematician Thomas Hakon Gronwall gives a complete solution resulting in the existence of a common integral to two very complicated partial differential equations.

Beyond the central problem of nomographic representation of relationships between three variables, which define implicit functions of two variables, there is the more general problem of the representation of functions of three or more variables. Engineers have explored various ways in this direction, the first consisting in decomposing the functions of any number of variables into a finite sequence of functions of two variables, which results in the combined use of several charts with three variables, each connected to the next by means of a common variable. Such a practical concern was echoed unexpectedly in the formulation of the Hilbert's 13th problem, one of the famous 23 problems that were presented at the International Congress of Mathematicians in 1900. The issue, entitled "Impossibility of the solution of the general equation of the 7th degree by means of functions of only two arguments" is based on the initial observation that up to the sixth degree, algebraic equations are nomographiable. Indeed, up to the fourth degree, the solutions are expressed by a finite combination of additions, subtractions, multiplications, divisions, square roots extractions and cube roots extractions, *i. e.* by functions of one or two variables. For the degrees 5 and 6, the classical Tschirnhaus transformations lead to the reduced equations $f^5 + xf + 1 = 0$ and $f^6 + xf^2 + yf + 1 = 0$, whose solutions depend again on one or two parameters only. The seventh degree is then the first actual problem, as Hilbert remarks: "Now it is probable that the root of the equation of the seventh degree is a function of its coefficients which does not belong to this class of functions capable of nomographic construction, i. e., that it cannot be constructed by a finite number of insertions of functions of two arguments. In order to prove this, the proof would be necessary *that the equation of the seventh degree* $f^7+xf^3+yf^2+zf+1=0$ *is not solvable with the help of any continuous functions of only two arguments*"[11].

In 1901, Ocagne had found a way to represent the equation of the seventh degree by a nomogram involving an alignment of three points, two being carried by simple scales and the third by a double scale. Hilbert rejected this solution because it involved a mobile element. Without going into details, we will retain that there has been an interesting dialogue between an engineer and a mathematician reasoning in two different perspectives. In the terms formulated by Hilbert, it was only in 1957 that the 13th problem is solved negatively by Vladimir Arnold, who proved to everyone's surprise that every continuous function of three variables could be decomposed into continuous functions of two variables only.

I will take another example in ballistics. As we saw it before, in the second half of the 19th century, ballisticians were conducted to propose new air resistance laws for certain intervals of speeds. The fact that some functions determined by artillerymen from experimental measurements fell within the scope of integrable forms has reinforced the idea that it might be useful to continue the search for such forms. It is within this context that Francesco Siacci resumed the theoretical search for integrable forms of the law of resistance. In two papers published in 1901, he multiplies the differential equation by various multipliers and seeks conditions for these multipliers are integrant factors. He discovers several integrable equations, including one new integrable Riccati equation. This study leads to eight families of air resistance laws, some of which depend on four parameters. In his second article, he adds two more families to his list. The question of integrability by quadratures of the ballistic equation is finally resolved in 1920 by Jules Drach, a brilliant mathematician who has contributed much in Galois theory of differential equations in the tradition of Picard, Lie, and Vessiot. Drach puts the ballistic equation into a new form that allows him to apply a theory developed in 1914 for a certain class of differential equations, which he found all cases of reduction. Drach exhausts therefore the problem of theoretical point of view, by finding again all integrability cases previously identified.

From a more general point of view, experimental, numerical and theoretical research on the ballistic equation has nevertheless played the role of a laboratory where the modern numerical analysis was able to develop. Mathematicians have indeed been able to test on this recalcitrant equation all possible approaches to calculate the solution of a differential equation. There is no doubt that these tests have helped to organize the domain into a separate discipline at the beginning the 20th century.

I could also highlight fruitful interactions between the development of graphic statics and the one of projective geometry. I could cite Karl Pearson who used some ideas from graphic statics and nomography to give a new impetus to mathematical statistics. I could also mention Massau's research on graphical integration that was the source of important theoretical developments in the field of partial differential equations, but it is unnecessary to multiply further examples to be convinced that knowledge and practices of 19th century engineers were in constant interaction, immediate or delayed, with

[11]David Hilbert, "Mathematical problems", translated by Mary Winston Newson, *Bulletin of the American Mathematical Society* 8 (1902), p. 462.

those of pure and applied mathematicians.

Conclusion

Recent research by historians shows more clearly that mathematical knowledge and mathematical representations are part of various social groups in interaction, in which they find various legitimacies. Within this new framework, history of mathematics should enrich itself by taking greater account of the engineering community, within which specific mathematical practices, original and fruitful, did exist. Moreover, as these old practices are often based on numerical, graphical and instrumental methods translating in a simple and concrete manner the key concepts of mathematics, these practices should constitute a fruitful source of inspiration for creating relevant activities to be exploited nowadays in mathematics education[12].

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[12] I began to conceive and experiment such activities in classrooms. In the bibliography, see my papers dated from 2005, 2007, 2010, and 2012.

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THE DIVISION OF THE TONE AND THE INTRODUCTION OF GEOMETRY IN THEORETICAL MUSIC IN THE RENAISSANCE: AN HISTORIC-DIDACTICAL APPROACH

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ABSTRACT

This presentation intends to show the role of the division of the tone and of mathematical conceptions underlying such a procedure in a substantial change undergone by the conception of western music throughout its history from Antiquity to the Renascence. It aims at showing therefore that such a change comprises also a significant extension in the spectrum of techniques used in theoretical music, which begin to include explicitly geometry, subsequently widely used, among the mathematical tools of solving problems. In a wider sense, during such a period, western music came from a cosmological-mathematical-speculative understanding, in which the main attention was placed on a rational activity of speculation and the purpose of the musical sound was to imitate a supramusical order and regularity to assume a mathematical-empirical concept, in which the main emphasis was set on the quality of the sound itself and music was examined by means of its laws and effects in people. The possibility of division of the tone, proscribed since the Pythagoreans not only into halves, but also into any number of parts, triggered new possibilities hindered by the authority and legitimacy of the Platonic-Pythagorean conception of music, according to which only whole numbers and ratios of integers should participate of the discourse concerning theoretical music, thereby promoting greater interaction between arithmetic and geometry in musical contexts. Such a possibility also unchained significant changes in the conceptions of theory of ratios underlying theoretical music, promoting the emergence of arithmetical conceptions of ratio in such contexts. This presentation also intends to raise the didactical potential of such changes approached in musical context inasmuch as such a context is fertile to differentiate ratios and proportion from structural analogous, but semantically different, ones, concepts sometimes approached indifferently in the dynamics of learning/teaching.

1 Division of the tone

The equal division of the tone played an important part in the historical process that led to the emergence of equal temperament. Mathematically, the equal division of the tone $8 : 9$ provides incommensurable ratios[1] underlying musical intervals. Attempts to divide the tone were already made

[1] The equal division of the tone $(8 : 9)$ means mathematically to find x so that $8 : x = x : 9$; that result, anachronistically speaking, in irrational numbers, inconceivable in the Pythagorean musical system.

in Antiquity, for instance by Aristoxenus (fourth century B.C.). In contrast with the Pythagoreans, who defended the position that musical intervals could properly be measured and expressed only as mathematical ratios, Aristoxenus rejected this position, asserting instead that the ear was the sole criterion of musical phenomena (Winnington-Ingram, 1995, 592). In preferring geometry to arithmetic in solving problems involving relations between musical pitches, Aristoxenus sustained, also against the Pythagoreans, the possibility of dividing the tone into two equal parts, conceiving musical intervals—and indirectly ratios—as one-dimensional and continuous magnitudes, making possible in this way their division. This idea provoked a large number of reactions expressed for instance in the *Sectio Canonis*[2] (Barbera 1991, 125), which was in Antiquity attributed to Euclid[3] and much later in the *De institutione musica*[4] (Bower and Palisca 1989, 88) of Boethius in the early Middle Ages, which gave birth to a strong Pythagorean tradition in theoretical music throughout the Middle Ages. Following the Platonic-Pythagorean tradition, a great part of medieval musical theorists sustained the impossibility of the equal division of the tone, which would mathematically lead to incommensurable ratios underlying musical intervals. Gradually, the need to carry out the temperament gave birth to different attempts to divide the tone.

The division of the tone is directly linked to the arithmetization of theories of ratio in the context of mathematics and music, a process which developed throughout Middle Ages up to the Renaissance. During Latin, Byzantine and Arabic Middle Ages, such a process had already received meaningful contributions, culminating in the Renaissance in a strong confluence of such traditions, which brought about during this time an unprecedented acceleration of the arithmetization of theories of proportions.

Up to the Renaissance, the utilization of ratios and proportions did not occur in a well-defined structure. Sometimes, such a use had arithmetic features, other times, geometric-musical features or even occurred as a combination of such tendencies. To such different structures, which kept up with the concepts of ratio and proportion since Antiquity, corresponded theories on these concepts underlying music and mathematic treatises up to the Renaissance.

Throughout history of the controversy involving arithmetization of ratios, diverse theoreticians contributed to shaping the theories mentioned above. In the 4^{th} century b.C., Aristoxenus used musical intervals—and indirectly ratios—as uni-dimensional continuous magnitudes, making possible with this the division of musical intervals in any number of parts. In the 4^{th} century a.C., Theon of Alexandria inserted interpolations in the book VI of "The Elements" of Euclides, modifying with this the original sense of compounding ratios. In the 11^{th} century, Psellus proposed the geometric division of the tone. Starting from theoretic-musical contexts, such a conception implied in the interpretation of ratios as continuous magnitudes. In the translation of book V of "The Elements" of Euclid in the 13^{th} century, Campanus of Novara conferred to the definition 5 an arithmetic interpretation, in which he introduced for instance the terminology "denominatio", although such a concept was not present in the original text. In the end of the 15^{th} century, Erasmus Horicius published for the first time in musical context a treatise, which made use of ratios as a continuous quantity. In the 16^{th} century, the

[2]Neither one nor many mean numbers will fall proportionately between a superparticular interval. Superparticular intervals are those produced by an epimoric ratio, i.e. a ratio of the type $m + 1 : m$.

[3]For discussions supporting the attribution of the *Sectio Canonis* to Euclid, see Menge, 1916, pp. 39–40.

[4]Demonstration against Aristoxenus that a superparticular ratio cannot be divided into equal parts, and for that reason neither can the tone.

process of arithmetization accelerated so that in the 17^{th} century the arithmetic theory of ratio and proportion became predominant.

Goldman suggests that Nicholas Cusanus (1401–1464) was the first to assert in *Idiota de Mente* that the musical half-tone is derived by *geometric division* of the whole-tone, and hence would be defined by an irrational number (Goldman, 1989, 308). As a consequence, Cusanus would be the first to formulate a concept that set the foundation for the equal temperament proposed in the work of the High Renaissance music theorists Faber Stapulensis (1455–1537) and Franchino Gafurius (1451–1524), published half a century later (Goldman, 1989, 308). Nevertheless, one can find in the Byzantine tradition Michael Psellus (1018–1078), who suggested in his *Liber de quatuor mathematicis scientijs, arithmetica, musica, geometria, [et] astronomia* (Psellus, 1556) a geometrical division of the tone, whose underlying conception implies an understanding of ratio as a continuous magnitude. Also concerning the division of the tone before Cusanus, Marchetus of Padua (1274 ? –?) proposed, in his *L*ucidarium in Arte Musice Planae written in 1317/1318, the division of the tone into five equal parts (Herlinger, 1981, 193), an innovation of extraordinary interest which made Marchettus the first in the Latin tradition to propose such a division, but without any mathematical approach. At the end of the fifteenth century and the beginning of the sixteenth century, Erasmus Horicius, one of the German humanists gifted in musical matters, wrote his *Musica* (Erasmus Horicius, 1500?, fo. 66v), where he suggested a geometric division of the whole tone (Palisca, 1994, 160). Erasmus stated that any part of any superparticular ratio can be obtained, in particular the half of $8 : 9$, which corresponds to divide equally the whole tone (Palisca, 1994, 159). Theoretically based on many geometrical propositions and, unusually, modeled on Euclidean style, his *Musica* dealt with ratio as a continuous quantity, announcing perhaps what would emerge as a truly geometric tradition in the treatment of ratios in theoretical music contexts during the sixteenth century. Such a change from an arithmetical to a geometrical basis in the theory of music represents a meaningful structural transformation in the basis of theoretical music, strongly tied to the change in the conception of western music mentioned at the beginning of this article.

2 The introduction of geometry in theoretical music in the Renaissance

The period from the end of the fifteenth century to the end of sixteenth century witnessed more intense structural changes in the conceptions underlying ratios and proportions in the contexts of theoretical music. With the need of equal temperament which brings together the need of the division of the whole tone and consequently structural changes in the conceptions of ratios, treatments with such concepts in theoretical music ceased to be a subject exclusively of arithmetic and became a subject of geometry.

In this context, Erasmus Horicius contributed immensely to the introduction of geometry as an instrument for solving structural problems in theoretical music. Notwithstanding the announcements of the need for geometry in theoretical music by previous authors, Erasmus could be considered the first in the Renaissance to apply Euclidean geometry extensively in his *Musica* (Erasmus Horicius, 1500?) for the resolution of structural problems in theoretical music. Relying mainly on books V and VI of Euclid, Horicius used geometry in different ways to solve musical problems, applying it to intervals, in contradiction to the Boethian arithmetical tradition. He used in his *Musica* the *denominatio* terminology taken from Campanus's Latin translation of the Elements, a procedure which contributed to the emergence of an arithmetical theory of ratio in the context of theoretical music. Making use of geometrical resources hitherto unusual in musical contexts, Erasmus showed that the intervals of the fifth ($3:2$) and the whole tone could be divided through a proportional mean, namely by finding a magnitude b between a and c so that $a:b$ is proportional to $b:c$ considering the whole tone mathematically expressed by $a:b$, although such resources involved potentially irrational numbers. Procedures like those in musical contexts intensified the conflicts associated with the Pythagorean tradition concerning theoretical music, according to which only whole numbers and ratios of whole numbers could serve as the basis for theoretical music, whether through a stiff distinction between consonance and dissonance defined by the first four numbers or through the search for a perfect system of intonation based on commensurable ratios.

Erasmus represents an intensification in the conceptual change undergone by theoretical music at this time, and his contribution is relevant to the research on mathematics and music at the end of the fifteenth century and beginning of the sixteenth century at the University of Paris, inasmuch as one can find the use of geometry in the solution of musical problems, for instance in the geometric division of superparticular intervals[5] presented in Faber Stapulensis's *Elementalia musica*, first published in 1494. This work had influence in the Spanish tradition of theoretical music in the sixteenth century, with authors like Pedro Ciruelo (1470–1548) and Juan Bermudo (1510–1565), who also presented respectively in the works *Cursus quatuor mathematicarum artium liberalium*, published in Alcalá de Henares in 1516 (Ciruelo, 1526) and *Declaración de Instrumentos*, published in 1555 in Osuna (Santiago Kastner, 1957) the same division of the tone with the geometrical mean presented by Faber Stapulensis. In the Iberian Peninsula, the tendency to use geometry occurred also in Salinas's *De Musica* published in Salamanca in 1577, which contains a geometrical systematization for the equal Temperament that makes extensive use of Euclid's *Elements*.

Such a tendency spread also to the German and Italian production in theoretical music. For instance, the German mathematician Heinrich Schreiber (1492-1525) published in the appendix *Arithmetica applicirt oder gezogen auff die edel kunst Musica* of his "Ayn new kunstlich Buech⋯" of 1521 (Bywater, 1980) a geometric division of the tone into two equal parts making, use of the Euclidean method for finding the geometric mean. He also operated with ratios with a very arithmetical structure, for instance, compounding them as one, anachronistically, multiplies fractions.

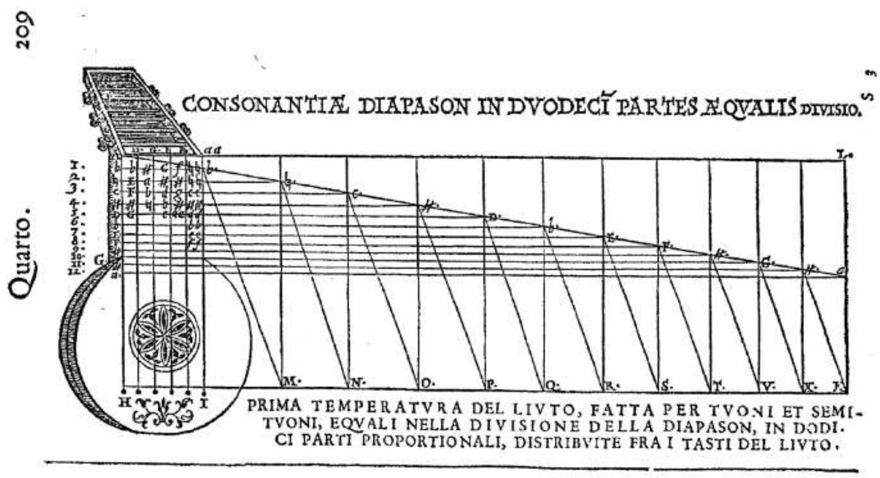
In the Italian tradition the tendency to use geometry was also strong. A representative example of such a tendency is Gioseffo Zarlino, a leading Italian theorist and composer in the sixteenth century. One of the most important works in the history of music theory, Zarlino's *Le istitutioni harmoniche* (1558), represents an important attempt to unite speculative theory with the practice of composition on the grounds that "music considered in its ultimate perfection contains these two parts so closely joined that one cannot be separated from the other (Palisca, 1995, 646). The tendencies for reconciling theory and practice also manifested themselves in this period in the context of structural problems underlying theoretical music. Such a reconciliation seemed to be incompatible with a Pythagorean perspective on theoretical music, in which there was no place for geometry, an essential tool for modeling a new language claimed by practical music.

In this context, it is worthwhile to mention Zarlino's *Sopplimenti musicali* (1588), in which the Italian theorist demonstrated much greater penetration into the ancient authors, particularly Aristoxenus and Ptolemy, than in *Le istitutioni harmoniche* (Palisca, 1995, 648). In spite of the still existing authority

[5]Superparticular intervals are those produced by an epimoric ratio, i.e. a ratio of the type $m+1:m$.

of Pythagoreanism in the context of theoretical music in the sixteenth century, Zarlino's *Sopplimenti musicali* already gave evidence of the tension between speculative theory and practice in the contexts of structural problems in theoretical music, inasmuch as it presented geometrical solutions for the equal temperament but was also based on Pythagorean foundations.

The figure below shows Zarlino's first proposal for a theoretical accomplishment of the equal temperament displayed on the lute, which is presented in chapter 30 of book 4 of the third volume of *Zarlino's Sopplimenti musicali* (Zarlino, 1588, 208). Entitled *Come si possa dirittamente diuidere la Diapason in Dodici parti ò Semituoni equali & proportionali*, this chapter presented the first theoretical possibility for the equal temperament as the *temperament of the lute, made by tones and semitones, equally made in the division of the diapason*[6]*, in twelve proportional parts, distributed between the keys of the lute*.



First theoretical proposal for the accomplishment of the equal temperament, from Zarlino, Gioseffo, Sopplimenti musicali del rev. M. Gioseffo Zarlino da Chioggia. Venetia: Francesco de Franceschi, 1588, fo. 209.

3 Concluding remarks

The 16^{th} century saw a major revolution in the production of treatises on theoretical music. In contrast with the Pythagorean tradition, it witnessed the introduction of geometry as a tool to solve the problem of division of the tone and consequently to solve theoretical problems related to the systematization of the temperament, representing a considerable change in the base of theoretical music. Such facts are representative, in a wider sense, of a greater change undergone by the foundations of theoretical music in this period, which gradually ceased to be based on an arithmetical dogmatism to assume a geometric-physical approach, resulting in a weaken of the Platonic-Pythagorean tradition in musical contexts.

Nevertheless, it is worthwhile to mention that in spite of the new conceptions of theoretical music, Pythagorean ideas are still present in music contexts in another sense even in the 17th century, when such ideas seemed to be less present and less prominent. A representative example of such a presence can be found in book V of Kepler's *H*armonices mundi (1619), where the Platonic-Pythagorean cos-

[6]The diapason means the musical interval of the octave.

mos received a magnificent restatement, before being withdrawn (Werner, 1966). On the one hand, this procedure represents possibly a last evidence of the Platonic-Pythagorean speculative tradition in music rescuing the old doctrine of the music of the spheres; on the other hand, it also equips the old doctrine with a mathematical-empirical conception under which, in a wider sense, western music had been approached since the late Middle Ages in detriment to the persistent mathematical-cosmological-speculative conception of music predominant since Antiquity.

The resistance to the use of geometry for solving the problem of the division of the tone and consequently other structural problems in music can be put down in greater extent to the authority and legitimacy of the Platonic-Pythagorean tradition in musical contexts, since carrying out such a division would be impossible making use only of whole numbers and would demand to handle ratios as continuous quantities, and/or as a number. Such a practical need eventually demanded an arithmetization of ratios in music contexts, despite its incompatibility with the Platonic-Pythagorean tradition in music. It is worthwhile thus to mention the didactical potential of music contexts in dealing with such changes involving the concepts of ratio, proportion, number, equality as well as of continuous and discrete quantities, inasmuch as at the light of the historical changes mentioned above and also considering the corresponding musical meanings of such concepts, the semantic difference between such concepts stands out, despite the structural similarity, which brings about occasionally the overlooking by the didactic approach some inherent difference they have.

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THE FORMATION OF MATHEMATICS CURRICULUM CHARACTERISTICS BY AUGUSTUS DE MORGAN IN UNIVERSITY COLLEGE, LONDON

On the Boundary between Mathematics and Natural philosophy

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ABSTRACT

In the early nineteenth century, mathematics professors did not teach the same areas of mathematics in various institutions. The boundary problem between the adjacent fields and the institutional matters could be important in the constitution of the different educational curriculums. This article examines the important factors which had influence on the constitution of mathematical curriculums in UCL. So this research will then suggest that for the better understanding of a mathematician, it is necessary to investigate the systematic nature of the institution in which the mathematician worked at, the nature of the students who that mathematician taught, the mutual relationships between the other professors of the adjacent areas in the same institution, and the distinctive educational features of the adjacent areas in that institution.

Keywords: Augustus De Morgan; Dionysius Lardner; pure mathematics; mixed mathematics natural philosophy; experimental philosophy; boundary between adjacent areas; curriculum; University College, London

1 Introduction

Until the mid-seventeenth century, the status of mathematics was considered rather low comparing with that of natural philosophy, and the boundary between them was relatively clear. Natural philosophy was the field which studies about the natural reality and explores the cause of the natural phenomena while mathematics was the field which attends to ideal objects and seeks to save the phenomena.[1] The proper developments of mathematical tools and experimental instruments were necessary for the boundary change besides the problems of epistemological status of mathematical representation and truthfulness of the results by experiments.

With the success of Issac Newton, the attempt to describe and analyze the natural phenomena and the motion of natural bodies mathematically came to gain credibility, and the trial was expanded

[1]Peter Dear, *Discipline & Experience: The Mathematical Way in the Scientific Revolution* (Chicago: University of Chicago Press, 1995), pp. 35–46, 161–168.

into the other areas of natural and experimental philosophy.[2] Especially, the French scholars' contribution was successful in those attempts[3] and so the range of mathematics became very wide from elementary mathematics to advanced mathematical physics parts in the late 18^{th} century. And the boundary between mathematics and natural philosophy was very vague since then. This disposition continued until early 19^{th} century. An example of which was in 1814, English mathematician Charles Butler classified mathematics as pure and mixed mathematics in his book, *An Easy Introduction to the Mathematics*. The former included arithmetic, geometry, differential and integral calculus, and analysis. And the latter encompassed the mathematical physics parts like astronomy, optics, mechanics, hydrodynamics, and pneumatics, and the practical parts such as acoustics, surveying, architecture, navigation, pyrotechnia, chemistry, and electricity.[4] Then, what was the mathematics curriculum of various universities in the early 19^{th} century England?

This article will focus on the first mathematics professor, Augustus De Morgan(1806–1871), in the newly established university in 1920s, University College, London (UCL)[5] to examine the academic boundary surrounding mathematics in the early 19^{th} century. Boundary problem was not acute in the traditional institutions like Oxford or Cambridge compared to the new university because teaching students was not an important duty of professors in the traditional university and so the already employed professors studied on the areas of their interest regardless of the boundaries of adjacent disciplines. On the contrary, it had to be newly done to subdivide academic disciplines, to name each professorship, and to decide teaching areas and curriculums in the new college. Thus, UCL can be helpful in understanding the change of contemporary academic situation surrounding mathematics and the process of drawing boundaries between adjacent areas.

2 Augustus De Morgan's Pure Mathematics Curriculum

From the beginning, UCL adopted a professor-teaching system in contrast with Oxford and Cambridge University. Thus teaching students was the most important duty of professors in UCL, and so the class courses and curriculums had to be properly planned. Directly after the appointment on 23^{th} February, 1828, De Morgan drew up a mathematics curriculum and the curriculum was inserted in the *Second Statement by the Council of the University of London, Explanatory of the Plan of Instruction* which was printed in June 1828. Considering the facts that he was only 21 years old, just graduated from Cambridge University, and the *Second Statement* was printed right after his belated appointment, we can assume that his curriculum would be very similar with that of Cambridge University and that the curriculum would include from the pure mathematics to the mixed mathematics parts.

However, his curriculum was organized with mainly pure mathematics parts. His mathematics classes were composed of 2 years courses, a lower and a higher division. According to the *Second Statement* of 1828, during his first year, he mainly taught arithmetic, algebra, the plane, solid and descriptive geometry, and the plane and spherical trigonometry. The second year's course embraced

[2] Thomas Kuhn, "Mathematical versus Experimental Traditions in the Development of Physical Science", *Journal of Interdisciplinary History* 7 (1976), pp. 1–31.

[3] E. Garber, *The Language of Physics: The Calculus and the Development of Theoretical Physics in Europe, 1750–1914* (Boston: Birkhäuser, 1999), pp. 31–34, 78–86.

[4] Charles Butler, *An Easy Introduction to the Mathematics* (Oxford: Bartlett and Newman, 1814), vol. 1., pp. xxxi–xxii.

[5] UCL was established as 'London University' in 1826. However the name was changed into 'University College, London' when 'University of London' was newly established in 1836 as the administrative institution for the colleges in London.

the field like conic sections, transcendental algebra, trigonometric analysis, algebraic geometry, calculus, the theory of projection, and probability.[6] Over time, his teaching areas were confined to pure mathematics. For example, in the London University Calendar of 1831, the theory of projection and probability were excluded in the curriculum.[7] Later on, his students also remembered him as a pure mathematics professor.[8] In addition, his professorship was regarded as the pure mathematics position. When natural philosophy professor, Richard Potter, retired from the office in 1865, his professorship was divided into two professorships: 'Mathematical Physics' and 'Experimental Physics'. For Mathematical Physics, Thomas A. Hirst was employed. However, the name of Hirst's professorship was changed into 'Pure and Applied Mathematics' when De Morgan resigned from his position in 1867.[9] Considering that both mathematical physics and applied mathematics were used to signify the same academic areas, the change of the name of Hirst's professorship means that the mathematical realms of De Morgan were thought of as pure mathematics.

De Morgan's pure mathematics curriculum was a very peculiar one in comparison with the curriculum of the other universities or colleges. The traditional Cambridge and Oxford Universities had taught the wide range of mathematics parts from pure to mixed mathematics.[10] This way was applied to the Military academy or colleges like Royal Military Academy, Woolwich and Royal Military College too.[11] Then was the pure mathematics curriculum a distinguishing feature in the new universities of London? For this, it is necessary to investigate the curriculum of King's College, London (KCL) which was a very similar institution with UCL. The first mathematics professor of KCL was Thomas G. Hall(1803–1845). He graduated from Cambridge University as 5^{th} wrangler in 1824 and also applied to the mathematics professorships of UCL before De Morgan. Hall could not be the first mathematics professor in UCL because he withdrew his application for a religious cause, but he was an excellent mathematician.[12] However, when Hall was appointed in KCL, his curriculum was very different with De Morgan's.

Hall included mixed mathematics parts including pure mathematics in his curriculum. His classes were composed of 3 years courses. For the first year, he taught arithmetic, geometry, algebra, plane trigonometry, logarithm, conic sections, and the chief propositions in mechanics. The next year, he introduced the first three sections of Newton's *Principia* with the higher parts of algebra, the theory of equations, the application of algebra to geometry, and differential and integral calculus which would enable students to comprehend those theoretical parts of mechanics. For the final year, his teaching

[6] London University, *Second Statement by the Council of the University of London, Explanatory of the Plan of Instruction*, (London: John Taylor, 1828), pp. 42–45

[7] London University, *The London University Calendar of 1831* (London: John Taylor, 1831), pp. 53–57.

[8] Mr. Taylor recollected his old teacher as a pure mathematics professor in the *Cambridge University Reporter*. "As Professor of Pure Mathematics at University College, London, De Morgan regularly delivered four courses of lectures,⋯and His course embraced a systematic view of the whole field of Pure Mathematics, from the book of Euclid and Elementary Arithmetic up to the Calculus of Variations.", Sophia De Morgan, *Memoir of Augustus De Morgan* (London: Longman, 1882), pp. 98–99.

[9] H. Bellot, "Chart 4. Growth of University of London, University College from 1826 to 1926, Faculty of Science" in *University College London 1826–1926* (London: University of London Press, 1929)

[10] For the curriculum of Cambridge University, Rouse Ball, *A History of the Study of Mathematics at Cambridge* (Martino Publishing, 2004), pp. 190–192.; John Wright, *Alma Mater, or, Seven Years at the University of Cambridge* (London: Black Young, and Young, 1827), Vol. 1, p. 9, 206, 207, 225–226, Vol. 2, pp. 25–29, 15–58.; For Oxford University, the following document can be helpful, Anonymous, "On University of Education-Oxford", *Quarterly Journal of Education* 2 (1831), pp. 23–29.

[11] Niccolo Guicciardini, *The Development of Newtonian Calculus in Britain, 1700–1800* (Cambridge: Cambridge University Press, 1989), pp. 108–123.; Charles Hutton, *A Course of Mathematics* (London: F.C.&J. Rivington, 1811), 3 vols.

[12] Adrian Rice, "Inspiration or Desperation? Augustus De Morgan's Appointment to the Chair of Mathematics at London University in 1828", *The British Journal for the History of Science* 30 (1997), pp. 261, 264.

areas were composed with spherical and solid trigonometry, the higher parts of the differential calculus, physical astronomy, the theory of the earth, and the analytic parts of hydrostatics, optics, and astronomy.[13] The areas which Hall taught in his class were very wide, and similar with the Cambridge University. So the numbers of the students who studied with Hall, and then went to the Cambridge University for the advanced study were many, and their scores in the Mathematical Tripos of the Cambridge University were very high.[14]

3 Why Did De Morgan Teach Only Pure Mathematics?: The Boundary Problem Between Mathematics and Natural Philosophy

Why did De Morgan teach only pure mathematics parts in his classes? It is safe to say that studying in the Cambridge University didn't seem to affect his teaching because his curriculum was different from the professors who came from the same University. Then did he have an ardent interest on pure mathematics parts? The memoirs of De Morgan by his wife shows that he has more interest on mixed mathematics or mathematical physics fields than pure mathematics parts.[15] Then did he not possess the sufficient mathematical ability to teach the advanced mixed mathematics parts? Considering the testimony of his teachers and the contract for Statics text with SDUK (Society for Diffusion of Useful Knowledge), he had excellent knowledge about mixed mathematics.[16]

And so, did UCL committee not intend to provide mixed mathematics? For this, we have to examine in detail the process of the employment of De Morgan. In the earlier time, UCL had planned two mathematics professorships, 'Elementary Mathematics' and 'Higher Mathematics and Mathematical Physics'. For the latter position, UCL committee tried to employ Charles Babbage.[17] When Babbage did not accept the proposal, Dionysius Lardner(1793–1859) applied for the two mathematics professorships. However UCL suggested natural philosophy professorship to Lardner. After Lardner was appointed as 'Natural Philosophy and Astronomy' professor, the name of Higher Mathematics and Mathematical Physics professorships were changed into 'Higher Mathematics'.[18] And then UCL committee secretly contacted John Herschel for the Higher Mathematics professorship.[19] De Morgan was selected by UCL after Herschel politely refused this position and Higher Mathematics professorship was united with Elementary Mathematics professorship into just one 'Mathematics' professorship. For what UCL intended for Mathematics professor, we have to think what 'higher mathematics' meant back then. If higher mathematics meant higher pure mathematics areas, it is difficult to explain how UCL committee could propose pure mathematics position to the noted astronomer, John Herschel. Thus if we regard higher mathematics as mixed mathematics, then De Morgan's curriculum had nothing to do with his intent and ability, and the UCL's plan. In the end, we have to find the different factors of the pure mathematics curriculum of De Morgan in UCL. Then social, institutional, economic, and interdisciplinary point of view can help to understand another side surrounding the

[13] King's College, London, *Calendar of King's College, London, for 1833–34* (London: John W. Parke, 1834), p. 14.
[14] Rice, "Mathematics in the Metropolis: A Survey of Victorian London", *Historia Mathematica* 23 (1996), p. 390.
[15] S. De Morgan, *Memoir*, pp. 18, 24.
[16] S. De Morgan, *Memoir*, pp. 19, 28, 41–69.; Rice, "Inspiration or Despertion?", pp. 268–271.
[17] Rice, "Inspiration or Despertion?", p. 266.
[18] London University, *Statement by the Council of the University of London, Explanatory of the Nature and Objects of the Institution* (London: Longman, 1827), p. 10.
[19] Rice, "Inspiration or Despertion?", pp. 66–68.

constitution of the curriculums.

The Boundary between mathematics and natural philosophy was not clear in the 1820, 30s. The way of defining each discipline was different and the terms were not consolidated. On one hand mathematical physics parts were called mathematics, and the other natural philosophy. So it was difficult to divide the boundaries between mathematics and natural philosophy. However ordinary scholars did not mind dividing the academic boundary and just continued their study depending on their own interests. There was no classification between academic journals or societies which were used for publication or presentation of the research about mathematics and natural philosophy.

However, the boundary problem could be very acute in UCL. UCL had to start the whole things afresh but could not get any support from the government because of its secular nature. Fund was not enough. Given these circumstances, the wages of professors were determined in proportion to the fees of students in classes. Then it could be an important matter for professors to secure the wider areas for their curriculum and sufficient attendees. Thus drawing boundary could be a more sensitive problem between the professors of adjacent fields in UCL.

Under this circumstance, Lardner was employed as the first natural philosophy professor in UCL. Lardner was a very attractive scholar considering the current academic state of UCL. He had frequently contributed to the very popular journals like *Edinburgh Review* and *Metropolitan Cyclopaedias*, published a lot of mathematics textbooks including *The Differential and Integral Calculus*, and had outstanding skill in popular lecture to take the gold medal from Dublin Royal Society for the lecture about steam engine.[20] So UCL committee persuaded Lardner into getting natural philosophy professorship, although he had applied for the professorships of Elementary Mathematics, and Higher Mathematics and Mathematical Physics first of all.

But Lardner did not immediately accept the natural philosophy professorship. Natural philosophy was considered to be more noble discipline than mathematics, but teaching natural philosophy in UCL was a different problem with the academic status. The regular natural philosophy class was permitted for the students who had sufficient knowledge of mathematics. However the educational level of the UCL students was not good because the situation of mathematics education at elementary and middle levels was very poor in the early 19^{th} century London. That meant the number of students who would attend the natural philosophy classes might be small, and it could be connected with the poor wage.

So Lardner requested an exact explanation about his salary and the terms of his employment when he got the proposal for natural philosophy professor through Henry Brougham who was the principal member of the UCL committee. With his request, Brougham sent the following letter in May 24^{th}, 1827.

> "The class you will teach cannot be of less value than 1200*l.* a year. Our plan prevents us from securing a salary larger than 300*l.*; but there will be pupils to pay five or six guineas each, say six for two courses of six or three months; and I look to three hundred pupils as the very last number which may be expected.[21]
>
> It is probable that five hundred will attend the Experimental Philosophy and higher mathemat-

[20] James McMullen Rigg, "Lardner, Dionysius", *Dictionary of National Biography, 1885–1900*, vol. 32, pp. 145–147.

[21] John Conolly et al., *Statements Respecting the University of London, Prepared, at the Desire of the Council, by Nine of the Professors* (London, 1830), p. 20.

> ical physics, and that a junction could be in effect of yourself with some popular experimental lecturer, securing to you two thirds of the profits, which leave 2000*l*. for you, and 1200*l*. or 1500*l*. for him."[22]

This letter meant that experimental philosophy and higher mathematical physics would be Lardner's share. After this letter, Lardner included both mixed mathematics and experimental philosophy in his natural philosophy curriculum. He showed his viewpoint about the boundary of natural philosophy in the first lecture of 28^{th} October 1828. He divided natural philosophy into mechanical philosophy and chemical philosophy, and put the mechanical philosophy between pure mathematics and chemistry. And then he defended natural philosophy from the criticism against it through confronting natural philosophy with not just mathematics but pure mathematics. For him, even experimental philosophy parts like electricity were one of mixed mathematics, and mixed mathematics was the branch of natural philosophy.[23] After then, his regular classes were planned with the high standard, and were permitted only for students who "attended the lectures of the Mathematics professor during the first session" or already possessed "a sufficient knowledge of the elements of mathematics science to enable them to join" his class.[24]

In this situation, the young De Morgan could not teach the same mixed mathematics areas in his class. Lardner was his senior and a revered scholar in academic world. By comparison, De Morgan was very young and had no academic career besides the 4^{th} wrangler of the Cambridge University. And Lardner was employed earlier than De Morgan by nearly six months. This meant that Lardner had been planning his classes when De Morgan was just employed. In the fee-related salary system, it was not easy for the young professor to have the lecture area which senior professor already held, and to overlap the curriculum of the adjacent fields.

In addition, Lardner had difficulty in maintaining a sufficient salary although he had preoccupied the whole mixed mathematics parts. At first, Brougham made Lardner feel at ease about his salary promising a considerable sum of money, but it was not official. So Lardner asked for the mediation about his salary problem to Leonard Horner who was employed as the Warden of UCL. However LCL did not pay the wage, 300 pounds, to Lardner in the first session and the student fees were also decreased to 4.10 pounds per class, when UCL actually opened the courses.[25] Lardner began to convey the contents of the engagement with Brougham to the UCL committee through Horner. However, Horner did not report the matter to the committee well. On the 20^{th} June, 1829, the answer which Lardner received through Horner was that UCL committee determined to allow him "a salary of 300 pounds for the first two years that is, until the 1^{st} of November, 1830, but not longer, and refused to allow the stipulated fee."[26] Lardner objected to this decision at once and demanded for reconsideration. But the Council declined his request. And then he was notified that no money would be paid to him, unless he would sign a legal defeasance of his claims under the original agreement. He could not but subscribe to the document prepared by the UCL Council. His salary was, however, not regularly paid even after then. He continuously sent the letters for the delivery of his discontent and

[22] Conolly et al., *Statements Respecting the University of London*, p. 23.

[23] Lardner, *A Discourse on the Advantages of Natural Philosophy and Astronomy, As Part of a General and Professional Education, Being and Introductory Lecture Delivered in the University of London, On the 28^{th} October, 1828*, (London: John Taylor, 1829), pp. 8–16.

[24] London University, *Second Statement*, p. 46, 52–63.

[25] Conolly et al., *Statements Respecting the University of London*, p. 25.

[26] Conolly et al., *Statements Respecting the University of London*, pp. 33–34.

the amicable settlement of wage problem to the UCL Council. After that, the UCL committee planned the arrangements for remunerating to, at least, a certain extent nearly all the professors for the next session, but made an exception of Lardner. And the problem of expulsion was also mentioned in the process of discussion about salary.[27]

Although ULC concerned natural philosophy in the early stages, the condition surrounding natural philosophy was poor. Natural philosophy lectures were located on the higher stages in the whole educational course. There was little inducement for studying natural philosophy in UCL. For the most part, UCL students had much interest on the professional discipline like law or medicine. And many students in UCL were non Anglican, and had difficulty in going on to Cambridge University which gave the first rate mathematics education.[28]

While Lardner had trouble with his salary, De Morgan felt tired because of the overflowing of students. For the first session, the students present at his lecture class were above one hundred. And the number of the students increased on the following sessions.[29] In these situations, he had no choice but to adjust his curriculums not to overlap with Lardner's.

4 The Educational Impact of the Teaching style of Certain Area on the Adjacent Areas

In the early nineteenth century, the teaching condition surrounding mathematical sciences was very poor. Considering the religious leanings or vocational aptitudes, the students who would have interest on mathematical natural philosophy were not many. In that time, the evaluation on mathematical sciences was negative and the popular experimental lectures made public to regard mathematical approach more difficult and unnecessary through experimental demonstration with interesting instruments or mechanical models.[30]

So, independently of the regular courses of mathematical natural philosophy, Lardner was concerned about the preparation for popular and experimental lectures for the student who did not have mathematics knowledge or interest on mathematical physics parts. He requested the provision of experimental instruments and laboratory to UCL, and the UCL committee accommodated his request and approved 200 pounds for the first budget. The expenditure gradually increased and the room for instruments and experimental demonstration was opened on Percy Street in the late 1827.[31] The

[27]Conolly et al., Statements Respecting the University of London, pp. 25–30.

[28]Bellot, *University College, London*, pp. 47–59.; A. Craik, *Mr Hopkins' Men: Cambridge Reform and British Mathematics in the 19th Century* (London: Spring-Verlag, 2007), pp. 27–33.; Christopher Phillips, "Augustus De Morgan and the Propagation of Moral Mathematics", *Studies in History and Philosophy of Science* 36 (2005), pp. 105–133.;

[29]S. De Morgan, *Memoir*, p. 30, 34.

[30]For the popular experimental lectures, Larry Stewart, *The Rise of Public Science: Rhetoric, Technology, and Natural Philosophy in Newtonian Britain, 1660–1750* (Cambridge: Cambridge University Press, 1992); Stewart, "Other Centers of Caclulation, or where the Royal Society didn't Count, Commerce, Coffee-houses and Natural Philosophy in Early Modern London", British Journal for the History of Science 32 (1999), pp. 133–53.; Laurence Brockliss, "Science, the Universities, and other Public Spaces: Teaching Science in Europe and the Americas", R. Porter ed., *The Cambridge History of Science*, pp. 65–66.; Mary Fissell and Roger Cooter, "Exploring Natural Knowledge: Science and the Popular", Porter ed., The Cambridge History of Science, pp. 134-139.; Turner, "Eighteenth-Century Scientific Instruments and Their Makers", Porter ed., *The Cambridge History of Science*, pp. 521–525.; For the critical attitudes on the mathematical science, Richard R. Yeo, *Defining Science: William Whewell, Natural Knowledge and Public Debate in Early Victorian Britain* (Cambridge University Press, 2003), pp. 75–78.; S. De Morgan, *Memoir*, p. 41.

[31]J. W. Fox, "From Lardner to Massey: A History of Physics, Space Science and Astronomy at University College, London, 1826–1975" http://www.phys.ucl.ac.uk/department/history/BFox1.html

budget for natural philosophy lectures was a significant part of total expenditure in UCL.[32]

With the opening of UCL classes, Lardner began to give experimental and popular lectures. As a reminder, UCL seemed to plan to appoint a popular experimental lecturer, remembering the letter from Brougham to Lardner on 24, May 1827. But UCL did not employ any experimental lecturer and instead Lardner undertook the job directly. In the *Second Statement*, Lardner explained that he would "deliver short courses of lectures, in a popular style, on particular subjects, more particularly on those departments of the science which have derived interest from recent discovery and improvement or from their useful application in the arts, manufactures, and commerce."[33] The popular and experimental nature of his lecture increased over time. In the *London University Calendar* of 1831, He explained his popular lectures would be "adapted for medical students, and various persons already engaged in professions or businesses, and in general to all who do not desire to pursue the science into minute detail, or mathematical investigation," and "be copiously illustrated by experimental apparatus, models, drawings, &c."[34] And then he just introduced only popular lectures on astronomy, mechanics, hydrostatics, pneumatics, optics, and heat in detail.[35]

While Lardner's popular lectures were well-received, UCL students began to lose interest in the regular lectures about mathematical natural philosophy by Lardner. Many students already had gotten some knowledge about natural philosophy through popular experimental lectures by Lardner before they could attend his regular mathematical lectures. In this situation, they did not have a necessity taking difficult lectures. After all, the number of students who attended Lardner's regular natural philosophy classes in 1830 was only 8.[36]

As the circumstances of Lardner's regular mathematical lectures began to worsen, explaining and persuading the usefulness or value of studying mathematics became more complex and urgent problem in UCL. How De Morgan worried about it can be shown in the lecture delivered at the opening of the classes of mathematics, natural philosophy, and chemistry in late 1830. At first, De Morgan pointed out that the foundation of science is mathematics, and mathematics has the similar experimental features with other science parts. For example, he explained the efficiency of geometry can be promoted when students make a reasonable inference based on the facts by observation.[37] He deplored the UCL situation that "the costliness and unusual nature of the apparatus employed, the time and skill required for many of the processes, and above all, the interesting and popular nature of the results lead many who are unacquainted with the real state of the case to suppose that these branches of knowledge are wholly dependent upon experiment and not at all upon reasoning and demonstration." And then, he stressed that mathematical thinking or theoretical reasoning is necessary for the study of physical sciences as well.[38] He persuaded that the difficulties of mathematics are much exaggerated.[39] He thought mathematics as the foundation of accurate knowledge, and regretted the study

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[32] "Statement of Receipt and Expenditure from 1^{st} January to 31^{st} December, 1827", *Hume Tract 215*, UCL Special Collection, p. 11.

[33] London University, *Second Statement*, p. 51.

[34] London University, *London University Calendar of 1831*, pp. 59–60.

[35] London University, *London University Calendar of 1831*, pp. 60–64.

[36] Fox, "From Lardner to Massey"

[37] De Morgan, *Remarks on Elementary Education in Science, An Introductory Lecture, Delivered at the Opening of the Classes of Mathematics, Physics, and Chemistry in the University of London, November 2, 1830* (London: John Taylor, 1830), p. 3.

[38] De Morgan, *Remarks on Elementary Education in Science*, P. 5.

[39] De Morgan, *Remarks on Elementary Education in Science*, P. 12.

of mathematics as to being delayed so late.[40]

In this situation, De Morgan could not justify the value of mathematics by the application of mathematical method to natural philosophy because teaching mathematics as the preparation for mathematical natural philosophy was futile in UCL. An urgent need was not to bring mixed mathematics parts from natural philosophy professor and to teach the advanced mathematics but to make young students to get the basic knowledge of mathematics, and to understand the exact meaning of mathematical terminology and the basic principles of mathematical demonstration. Since then, De Morgan's intellectual activities were concentrated on teaching pure mathematics, publishing texts on pure mathematics parts, contributing articles to educational journal about elementary mathematics education, and pursuing proper methods for training students to reason logically through mathematical logic.[41]

5 Conclusion

While De Morgan served for more than thirty years in UCL, the education of mathematics in UCL was consolidated and developed, and some students who received the teaching from De Morgan established the pure mathematics-centered academic society, London Mathematics Society in 1865. De Morgan's case shows that the curriculum can be constituted not by the intention of the professor or the institution, but by the accidental, systematic, or adjacent area related factors. And then, this study suggests that for the better understanding of the intellectual activities by the certain mathematician, it is necessary to examine the institutional and interdisciplinary contexts surrounding the mathematician besides mathematical matters.

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[40] De Morgan, *Remarks on Elementary Education in Science*, P. 11.

[41] I examined De Morgan's intellectual activities minutely in the 4th , 5th, and 6th chapters of my Ph.D. dissertation. Su Nam Cho, *The Understanding of Pure Mathematics Developments in England through the Mathematics Education by Augustus De Morgan: On the Boundary between Adjacent Areas* (Seoul National University, 2012)

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THE FLATTENING OF THE EARTH: Its Effect on Eighteenth Century Mathematics

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ABSTRACT

In the sixteenth and seventeenth centuries, as long sea voyages became common, travelers began to report that pendulum clocks consistently ran slow near the equator. Isaac Newton mentioned this phenomenon in the Principia and suggested (Book III, Proposition 18) that the earth was not quite spherical; it bulged slightly near the equator and thus the force of gravity was less there. The theory was confirmed in the 1730's by measurements taken by Maupertius in Lapland and by La Condamine in what is now Ecuador.

We discuss how Euler and Lagrange incorporated the knowledge that the earth was not a perfect sphere in the development of the theory of conformal mapping, and suggest ways in which this case study of the interplay of technology (navigation and geodesy) with mathematical theory (conformal mapping) might be used with students.

Keywords: geodesy, cartography, history of mathematics

1 Flattening the Earth

The sixteenth-century round-the-world voyages of Fernão de Magalhães (Ferdinand Magellan), Francis Drake, Ignacio de Loyola, and others put an end to any lingering doubts about the correctness of Aristote's theory that the earth was round. However, two observations in the late seventeenth century led to a suspicion that the round shape was not a perfect sphere.

The first observation was by Giovanni Cassini in 1665, taking advantage of improvements in telescope making to make very precise measurements of the disc of Jupiter, discovering that the planet was in fact an oblate spheroid, with the equatorial radius about $1/15$ larger that the polar radius. Ironically, Cassini and his descendants were to later lead a faction that believed the earth, unlike Jupiter, was a prolate spheroid with a larger polar than equatorial radius.

The second observation did not seem at first to have any relationship to the shape of the earth. In 1672-3, the Académie Royale sent Jean Richer on an expedition to Cayenne, "pour la perfection & et l'avancement de l'Astronomie;" while there, Richer conducted a number of astronomical and physical experiments. Curiously, he found that a pendulum clock which had been precisely calibrated in Paris, lost more than two minutes per day in the tropics. Perhaps it was Christian Huygens, then resident at the Paris Observatory, who suggested that the force of gravity was in fact less at the equator.

But why would gravity be diminished at the equator? Huygens thought it might be centrifugal force, caused by the earth's rotation. Isaac Newton published the *Principia* in 1687. In Book III, Proposition XVIII of that work, he proposed a different explanation: that the earth was, like Jupiter, an oblate spheroid. Objects at the equator were further from the center of the earth than those at the poles and, therefore, by the inverse-square law, the effect of gravity was less. Newton said: "If our earth was not higher about the equator than at the poles, the seas would subside about the poles, and rising towards the equator, would lay all things there under water."[13, p.426] In Proposition XIX, Newton, assuming that "the matter of the earth is uniform, and without movement" uses geometrical results on the inverse- square law from Book I to compute the ratio of the equatorial to the polar axes, and arrives at a value of 230:229. In Proposition XX, the theory is extended to planets other than earth.

Newton's work did not convince everyone. Especially in France, adherents of the Cartesian camp distrusted the notion of action at a distance. Jacques Cassini (son of Giovanni) measured the distance of one degree of latitude in the south of France, and found (erroneously) this was longer than the degree measured near Paris in an earlier survey. Thus, they argued for a prolate, rather than an oblate, spheroid. To resolve the dispute, the Académie obtained royal permission, and funding, to send expeditions to Lapland and to Spanish Peru (in a region which is now part of Ecuador).

The Lapland expedition, including Maupertuis, Clairaut, and Anders Celsius from Sweden, returned in 1638. Their results showed that a degree of latitude was significantly longer than in France. Maupertuis was acclaimed by Voltaire as the "flattener of the earth" [19]. Voltaire wrote the following epithet:

> This poorly known world which he knew how to measure
> Becomes a monument from which he derives his glory,
> His destiny is to describe the world,
> To please and enlighten it.[1]

The story of the two expeditions is told elsewhere ([9],[8], [1]); and Todhunter ([18]) still appears to be the authority on more refined measurements, and on further improvements to Newton's theory, especially by Stirling and Clairaut. The rest of this paper will treat the specific problem of how the model of earth as an oblate spheroid was adopted for use in cartography. But one last incident needs to be be mentioned.

A survey to measure the length of a degree of latitude in the the Paris-Amiens region had been undertaken by Jean Picard in 1669.[2] Then a 1720 survey in the south of France reported that the length of a degree of latitude was slighty longer than Picard's figure. This was one reason the Cassinis held out so long for the prolate earth theory. But, in 1740, Cassini de Thury (grandson of Giovanni) admitted to the Academy that a resurvey had shown significant errors with Picard's original measurements. This was interpreted as a concession by the Cartesian party that the earth might indeed be flattened at the poles.

Voltaire now addressed Maupertuis as "flattener of the earth, and of the Cassinis" [17].

2 Euler Looks at the Data

The return of Maupertuis and the Lapland expedition, and their findings confirming Newton, were a major event in eighteenth century science. Leonard Euler, at the Academy in St. Petersburg, would

[1] "Le Globe mal connu qu' il a sçu mesurer, Devient un Monument où sa gloire se fonde; Son sort est de fixer la figure du Monde, De lui plaire, et de l' éclairer.

[2] Newton mentions this survey in the *Principia*.

certainly have heard of it. Moreover, Euler would have had a interest in the cartographic applications of the expedition's findings, since a large part of his duties then involved work at the Academy's Department of Geography, working to produce a map of the Russian empire. In 1740, Euler produced a short paper[2], which used Maupertuis' figures to compute the length of one degree of latitude and one degree of longitude at several different places on the globe. Euler's method was apparently used by C.N. de Winsham[3] in the construction of a book of tables[18, p. 132].

In 1754, while at the Berlin Academy, Euler presented another paper[3] on the calculation of distances on the surface of the globe. This paper extends the methods of a prior paper on spherical trigonometry to an ellispoidal earth. The paper begins with an analysis of the problem, where general formulae are derived, followed by an estimate of the size and shape of the earth, using data from four surveys at different latitudes, and then a solution of a some specific arithmetical problems.[4]

The first section, interesting for its general treatment of elliptical geometry, uses only simple calculus to derive an expression for the radius of curvature, which is needed to compute the surface distance along a meridian. An approximation to surface distance as a function of the equatorial radius and eccentricity is found, again by simple calculus.

He shows that, at latitude ϕ, the distance on the surface of the ellipsoid corresponding to a small displacement in latitude, $d\phi$, is equal to

$$\frac{2\sqrt{2}\,a^2b^2\,d\phi}{(a^2+b^2)^{3/2}}\left(1-\frac{3}{2}\,\delta\cos(2\phi)\right), \quad \text{or} \quad A\cdot\left(1-\frac{3}{2}\,\delta\cos(2\phi)\right),$$

where a and b are the major and minor semi-axes, and δ is a constant related to the eccentricity.[5] In this case, we are measuring the length of one degree, so $d\phi = \pi/180$.

In the second section, Euler estimates the equatorial radius and eccentricity, He uses data from four different surveys from four different latitdudes:

- the La Condamine/Bougeur expedition to South America;
- the Maupertuis expedition to Lapland;
- a new measurement by Nicolas-Louis de La Caille, at the Cape of Good Hope in South Africa;
- A survey of France from near Paris to near Amiens (this appears to be the 1740 survey mentioned previously).

Here are the data Euler uses:

[3] or Winsheim

[4] Example: Knowing the meridian at a point L, determine the latitude by observing the stars, then walk in a straight line keeping a fixed angle with the meridian, to another point M, and determine the distance LM after a new observation of the stars.

[5] Euler used e for the equatorial radius(semi-major axis) and a for the polar radius (semi-minor axis). To make it easier for the modern reader, and to conform to current usage, we changed these to a for the semi-major and b for the semi-minor axis.

Location	Latitude	Length[6] of One Degree
South America	$-0°30'$	56753
Cape of Good Hope	$-33°18'$	57037
France	$49°23'$	57074
Lapland	$66°20'$	57438

Thus, the four approximate equations to resolve are

$$\begin{aligned} A\cdot\left(1-\frac{3}{2}\,\delta\cos\,1°\right) &= A\,(1-1.4997715\,\delta) = 56753+p\\ A\cdot\left(1-\frac{3}{2}\,\delta\cos\,66°36'\right) &= A\,(1-0.5957219\,\delta) = 57037+q\\ A\cdot\left(1-\frac{3}{2}\,\delta\cos\,98°46'\right) &= A\,(1-0.2286183\,\delta) = 57074+r\\ A\cdot\left(1-\frac{3}{2}\,\delta\cos\,132°40'\right) &= A\,(1-1.0165980\,\delta) = 57438+s \end{aligned}$$

where the quantities p, q, r, s are unknown errors in the measurements. Euler subtracts the first equation from the other three, than divides the last two equations by the last two, to obtain

$$\frac{321+r-p}{284+q-r}=\frac{65}{34}, \quad \text{and} \quad \frac{685+s-p}{284+q-p}=\frac{437}{157},$$

or

$$31p-65q+34r=7546 \quad \text{and} \quad 280p-437q+157s=16563$$

or, finally, eliminating p,

$$-150q+307r-157s=51594.$$

So far, so good. But now a subjective judgment enters. Euler says "if one wanted to suppose these three errors equal, each would be 84 toises, which would be too much, given the great exactness with which the second and fourth of the measurements were made." Somewhat arbitrarily, it seems, he takes r equal to 125 toises, and assumes that q and s are equal, so that $q=s=-43$, and so $p=15$.

Euler plugs his estimates into the above equations, and concludes that $\delta=0.00436055$. Since

$$\delta=\frac{a^2-b^2}{a^2+b^2},$$

this means that the ratio of major axis to minor axis is

$$\frac{a}{b}=\sqrt{\frac{1+\delta}{1-\delta}}=1.00437,$$

or about 230 to 229, which is precisely what Newton predicted. One gets the impression that Euler has set out to prove Newton right and the Cassinis wrong.

It is tempting look at these errors with the benefit of current knowledge. The current figures[7] for the major and minor axes of the earth are 6378136.6 and 6356751.6 meters, or 3272470 and 3261498

[7]The current standard for the figure of the earth, known as WGS84, is essentially a compromise ellipsoid based on many thousands of satellite measurements. The data used here is taken from [8], p. 264, who in turn cites [7].

toises. These figures agree well with Euler's estimates, although the earth appears somewhat less flattened than was assumed in Euler's time. However, it is ineteresting to look at errors in the four surveys in the light of current knowldege.

	Latitude	Euler	WGS84
South America	$-0°30'$	15	-20
Cape of Good Hope	$-33°18'$	-43	-132
France	$49°23'$	125	-11
Lapland	$66°20'$	-43	-224

Euler thought that the most accurate surveys were the second and the fourth; these were in fact the least accurate. The French survey, which Euler assumed was the least accurate, turns out to have the smallest error.

Of course, it is unfair to criticize Euler for lack of access to data which did not yet exist. The French survey had some notoriety, as suggested above, and it is likely that its accuracy was widely questioned. On the other hand, the very large errors in the Lapland survey were not known until the 1920s.

Euler's data analysis appears somewhat ad-hoc. But in his defense, the method of least squares had not yet been invented. Yielding to temptation again, let us see what a least-squares analysis might tell us. We start with the four equations above, in linear form as

$$A - A\delta x_i = y_i + \epsilon_i,$$

with A and δ as above, the x_i are equal to $-3/2$ times twice the cosines of each latitude, and the y_i the estimated survey measurements.

A standard least-squares calculation[8] shows

$$A = 57019.50896, \qquad A\delta = 191.222296, \quad \text{so that} \quad \delta = 0.00335363.$$

Imitating Euler's calcualtions, the result are (units in toises)

	Euler's estimates	Least Squares estimates	WGS84
Major axis	3281168	3280417	3272470
Minor axis	3266892	3266134	3261498
Flattening Ratio	230	229.66	298.25

Compared to Euler's predictions, the method of least squares shows the earth slightly smaller, but with the same shape (flattening ratio). But again, what is interesting is the least-square residuals at each data point, compared with Euler's estimates:

	Latitude	Euler	Least Squares
South America	$-0°30'$	15	2
Cape of Good Hope	$-33°18'$	-43	-57
France	$49°23'$	125	111
Lapland	$66°20'$	-43	-56

[8]Computation was performed using the `lm()` module in the open-source statistical package *R*. For information about *R*, see[14].

The least-squares residuals show the Africa and Lapland surveys about equal in goodness of fit, while the French survey is much worse than the others. This is precisely the same pattern that Euler achieved in his manipulations. Perhaps what may have seemed subjective judgment was really an instinctive feel for how the data fit together.

In summary, later data, unavailable at the time, would eventially prove Euler wrong in his assessments of errors. But he did impressively well, based on the data he had to work with.

3 Practical Cartography

Euler published, in 1775, three treatises[4, 5, 6] which constitute his principal contribution to mathematical cartography. However, all of these papers assume a spherical, rather than an ellipsoidal earth. Euler's interest was in the abstract properties of a mapping from one surface to another, and not so much in the tools of day-to-day cartography. He closes the first treatise[4] with these word: "⋯it is not easy to derive methods of practical use from our formulae. Nor, indeed was the intention of the present work to dwell on practical uses, especially since, for the usual projections, these matters have been explained at length by others." [9]

Both Lambert[11] and Lagrange[10] published papers in the 1770s which did attempt to incorporate knowledge of an ellipsoidal earth into their work on angle-preserving (conformal) cartographic maps. Their treatments are similar. We shall describe Lagrange's work here since his treatment is more systematic.

Lagrange considers the question of a mapping from points on the globe with latitude s and longitude t to points on the plane (x, y). Specifically, "let us consider two points infinitely close, which are determined on the surface of the globe by the variables s, t, $s + ds$, $t + dt$, and on the map by the variables corresponding variables x,y, $x + dx$, $y + dy$, and look for the distances of these two points on the gobe and on the map. Evidently, the distance on the globe is expressed by $\sqrt{ds^2 + q^2 dt}$, where $q\,dt$ is the arc on the parallel [of latitude] between two meridians [of longitude], and the distance on the map is expressed by the usual formula $\sqrt{x^2 + y^2}$." Lagrange sets as a fundamental condition, that such a mapping should satisfy

$$\sqrt{ds^2 + q^2\,dt^2} : \sqrt{dx^2 + dy^2} = 1 : m,$$

or

$$dx^2 + dy^2 = m^2 ds^2 + q^2\,dt^2$$

where m is some arbitrary constant. Now Lagrange prepares for the possibility of mapping from an ellipsoidal shape. Observing that "the ordinate q of the curve of the meridian is given by the nature of the curve, so that $\dfrac{ds}{q}$ is integrable", he replaces the latitude s by the unspecified function $u(s)$. Setting $n = mq$, the previous condition becomes

$$dx^2 + dy^2 = n^2 du^2 + dt^2$$

The next fifteen pages are devoted to finding mappings from (u, t) to (x, y) which satisfy this condition under suitable restrictions. Finally Lagrange returns to consider the shape of the globe. If the earth is

[9] "⋯minus facile est methodos usu receptas ex formulis nostris generalibus elicere. Neque vero institutum praesens permittit, ut huic negotio immoremur, praecipue cum consuetae proiectiones ab aliis iam abunde sint explicatae"

in fact a sphere, then

$$q = \sin s, \quad \text{so that} \quad du = \frac{ds}{\sin s}, \quad \text{and} \quad u = \ln\left(k \tan\frac{s}{2}\right)$$

(for an arbitrary constant k), a value which can then be inserted in previously found expressions for $x(u,t)$ and $y(u,t)$. If, on the other hand, the meridians are elliptical, then taking z (rather than s) for the distance to the pole, and ε for the (linear) eccentricity, the expression for u becomes

$$u = \log\left(k \tan\frac{z}{2}\left(\frac{1+\varepsilon\cos z}{1-\varepsilon\cos z}\right)^{\frac{\varepsilon}{2}}\right).$$

Now simply let ζ be an angle which satisfies

$$\tan\frac{\zeta}{2} = \tan\frac{z}{2}\left(\frac{1+\varepsilon\cos z}{1-\varepsilon\cos z}\right)^{\frac{\varepsilon}{2}};$$

if the ellipse is near a circle, ζ can be approximated by a rapidly convergent series. Now the formulae previously found for the sphere can be used, with the slightly different angle ζ in place of z.

Lambert, writing seven years earlier, had come up with almost identical results to those of Lagrange, but his treatment is much briefer. (One speculates that Lagrange simply took Lambert's results and packaged them into a more readable format.) In an analysis of angle-preserving maps, Lambert comes up with an with an expression like Lagrange's(above) for the value of u. He defines a (small) auxiliary angle as a function of latitude and shows how to use this to make corrections to the latitude via a radidly convergent series.

Modern cartographers, when designing maps where the ellipsoidal shape of the earth is relevant, use the same technique pioneered by Lambert and Lagrange. For each given latitude, a slightly corrected "auxiliary latitude" is computed, and then the auxiliary is used in the projection computations. Different auxiliary latitudes are used, depending on the desired propoerites of the map. Detailed information on auxiliary latitudes can be found in [12] and [15].

4 Uses in Pedagogy

The Euler and Lagrange works discussed above show how to model the earth as an ellipsoid. Each author develops the geometry of the ellipse; in particular, the relationship between a the coordinates of a point on the curve and the angle of the tangent line at that point. The development requires only the fundamental equation and some simple calculus, but offers material not often seen in an undergraduate calculus class.

The Euler paper in the second section offers other intriguing opportunities for discussion, including:

- How do we incorporate subjectively known differences of error in an analysis?
- Alternatively, how do we read and assess the merits of a subjective data analysis?
- What are the advantages and potential drawbacks of the least-squares method?
- Are there alternatives to the least-squares method?

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MATHEMATICAL FOUNDATIONS OF COGNITIVE SCIENCE

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ABSTRACT

A would-be cognitive scientist needs to get familiar with three mathematical landmarks: Turing machines, Neural networks, and Gödel's incompleteness theorems. This paper explores the mathematical foundations of cognitive science, focusing on these historical works. We begin by considering cognitive science as a metamathematics. The following parts address two mathematical models for cognitive systems; Turing machine as the computer system and Neural networks as the brain system. The last part investigates the retrospective and prospective implications of Gödel's works for cognitive science.

Keywords: mathematical models of cognitive system, Turing machines, neural networks, Gödel's incompleteness theorems, quantum cognitive science

1 수학에서 인지과학으로

현대 과학의 가장 특징적인 면은 창의적 융합(creative convergence)을 강조하는 노력에서 찾아 볼 수 있다. 융합은 일정한 분야와 그와는 다른 분야의 경계를 넘나들면서 일어나는 사건이며, 창의성은 다른 사람이 보지 못한 새로운 연결과 생성을 함의한다. 융합이 수평적인 차원의 문제라면, 창의성은 수직적인 차원의 문제이다. 이러한 창의적 융합을 위하여 수학 분야에서는 일정한 하부시스템의 부족한 성질들 자체를 새로운 개념으로 추가하여 활용함으로써 보다 풍부한 능력을 갖춘 새로운 상위시스템의 확장(extensions)을 이룩해 왔다 ([32][33][34]).

인지과학(cognitive science)은 20세기 중반에서 후반 사이에 성립되었고, 창의적 융합을 추구하며 인지시스템을 연구하는 학문이다([25]). 인지과학의 역사를 살펴볼 때, 어떤 인지과학자도 수학자와 수학의 공헌으로부터 자유로울 수 없다는 사실을 발견할 수 있다([16][18][19][23][26][27][28][29]). 본 연구에서는 수학사의 전망에서 먼저 인지과학의 생성과 발전 과정을 분석한다. 그 다음으로는 인지과학의 기본을 이루는 두 가지 수학적 구조를 밝히고, 인지과학의 과제를 전망하고자 한다.

2 메타수학으로서의 인지과학

현재의 과학이 현실세계(real world)에서 다룰 수 있는 인지시스템은 (1) 계산하는 기계(Computers), (2) 뇌(Brain), 그리고 (3) 인간의 마음(Mind) 세 가지이다. 인지과학은 컴퓨터와 뇌와 마음을 융합적으로 다루는 학문이며, 그 연구의 결과는 전체 과학에서 수렴가능하다는 비전을 전제하고 있다. 이러한 비전이 실현될 수 있기 위해서는 무엇보다도 먼저 학문적 의사소통을 위한 언어가 필요했다. 다양한 분야 상호 간에 내용을 정확하게 전달할 수 있고, 특정 분야에 편중되지 않으며 보편적으로 충분히 사용할 수 있는 도구로서 추상적 언어(formal language)가 필요했는데, 그러한 조건을 만족시킬 수 있는 것은 바로 수학이었다. 인지과학 연구결과의 소통을 위해서는 내용이 수학적으로 표현가능하고 수학적으로 이해될 수 있어야 했다.

하나의 인지시스템으로 마음과 뇌와 컴퓨터를 이해하고 그러한 시스템을 구현하기 위해서는 융합이 가능한 수학적 모델을 찾거나 구성하는 일에 성공해야 한다. 곧 인지시스템에 대한 타당하고 포괄적인 수학적 모델이 존재할 때, 인지시스템의 구조와 기능에 관한 충분한 이해를 추구하는 인지과학의 목표는 달성될 수 있다. 여기에서 수학적 모델은 현실세계(real world)가 아니라, 형식세계(formal world) 또는 추상적 세계(abstract world)에 속하는 대상이다. 그러므로 현실세계의 뇌를 형식세계의 수학적 모델과 대응시켜 다루는 것, 또는 현실세계의 컴퓨터를 형식세계의 계산기계와 대응시켜 다루는 것이 인지과학에서의 수학적 해석(interpretation)이자 검증작업이다.

인지과학은 마음과 뇌와 컴퓨터라는 인지시스템(cognitive system)을 연구하는 과학이다. 그러나 현재까지 인지(cognition)에 대한 단일한 정의는 확립되지 못했다([19][23]). 역설적으로 인지에 대한 진정한 과학적 정의가 내려지는 순간이 인지과학이 그 임무를 다하는 시점이 될 것이다. 심리학자들은 인지를 정신적 세계 내의 정신적 기능(mental function)으로 정의한다. 신경과학자들은 생물학적 세계 내의 물리적 기능(physical function)으로 정의한다. 인공지능학자들은 기계적 세계 내의 계산적 기능(computational function)으로 인지를 정의한다. 그러나 인지과학자들은 더 나아가서 인지를 마음, 뇌, 컴퓨터와 관련하여 하나의 틀 내에서 추상적으로 정의하고자 시도한다([23]). 이것은 인지과학자들이 마음에서의 인지라는 기능을 '뇌'와 '컴퓨터'의 어떤 수준(level)에서의 정보처리(information processing)로 이해하고 있음을 의미한다. 인지는 의향이나 목표지향적 행위(goal-oriented behaviors)를 내포해야 한다. 예를 들면, 조건반사에 따른 감각이나 행동들은 인지에 포함되지 않는다. 일반적으로 지능, 추론, 학습, 직관, 언어, 기술, 이해, 문제해결, 의사결정, 기억, 지각, 동기화, 감성, 정서가 인지에 포함된다([18][19]). 논의의 집중을 위해서 이 글에서는 인지의 범위를 수학적 사고(mathematical thinking)에 한정하여 다룰 것이다([21]).

인지과학의 역사에서 인지에 대한 가장 중요한 정의는 계산(computation)의 개념으로부터 시작되었다([18][19][23]). 물론 이러한 이해는 명백하게 수학의 개념으로부터 도출된 것이다. 인지를 일종의 계산으로 이해하면서 인지과학의 성립이 가능했고, 계산은 인지과학의 키워드이자 연구방법을 결정하는 도구였다. 예를 들어, '표상'과 '처리'라는 인지과학의 핵심 개념에 가장 영향력이 있고 표준적인 정의를 소개하면 다음과 같다([15]).

표상(representation). 표상은 어떤 정보의 타입과 실체를 명시화해주는 형식체계(a formal system)로 정의된다.

처리(process). 처리는 한 형식체계로부터 다른 형식체계로의 함수(a mapping)로 정의된다.

여기에서 인지란 하나의 정보처리과정(information processing)으로서 이해되고 있고, 이것은 형식체계에서 형식체계로의 함수와도 같이 이해되고 있다. 여기에서 함수는 일정한 구조들 사이의 구조함수에 해당된다고 할 수 있다.

신경생리학과 컴퓨터과학을 전공했던 인지과학자 마(D. Marr)는 인지과학을 위하여 (1) 계산이론, (2) 표상과 알고리즘, (3) 하드웨어 구현이라는 세 가지 수준의 연구가 꼭 필요하다고 강조하였다([15]). (1)과 (2)의 관계는 타입 동일(type identity)의 관계로 볼 수 있고, (2)와 (3)의 관계는 실현(realization)의 관계로 볼 수 있다. 수학과 뇌과학을 전공했던 인지과학자 아비브(M. Arbib)는 인지적 행동(높은 수준을 의미)과 물리적 뉴론(낮은 수준을 의미) 사이의 연결 관계를 설명해 줄 수 있는 계산의 스키마(schema)를 통해서 기능적 수준과 구조적 수준 사이의 연결 관계까지 설명할 수 있어야 한다고 강조했다([1]).

과학에서 행동(action)은 시간($Time$)에서 공간으로의 함수(mapping)로 이해된다. 그렇다면 인지($cognition$)는 시간에서 마음이라는 특정 공간($cognition$)으로의 함수로 이해될 수 있다.

$$Time \xrightarrow{cognition} Mind$$

그리고 현재 인지과학에서 전제하는 마음이란 시스템은 뇌와 컴퓨터의 곱(product)으로 설정되어 있다.

$$Mind = Brain \times Computer$$

여기에서 뇌에 대한 연구를 통하여 마음을 이해하려는 접근법이 가능하며, 또한 컴퓨터를 통하여 마음을 이해하려는 접근법이 가능하다. 인지과학에서는 전자를 연결주의적 접근 c(connectionist approach)라 하고, 후자를 기호주의적 접근 s(symbolic approach)라고 한다([20][23]). 두 접근은 적절한 사영함수(projection)

$$Brain \xleftarrow{c} Mind \xrightarrow{s} Computer$$

를 찾는 과정으로 볼 수 있다. 1980년대와 1990년대 중반까지는 기호주의적 접근과 연결주의적 접근이 서로 대결의 구도를 이루고 있었다. 그러나 그 후로부터 현재까지는 서로의 장점을 인정하면서 부분적인 연합 또는 새로운 인지시스템을 구축하고자 노력하는 양상이었다. 향후 인지과학의 궁극적인 목표는 두 접근 사이의 동일구조관계(isomorphism)를 밝히는 것이라고 할 수 있다.

3 컴퓨터의 수학적 모델

역사적으로는 먼저 기호주의적 접근이 인지과학을 주도하였다. 기호주의적 접근은 순차적 처리(sequential process)를 전제한다. 기호주의적 접근에 의하면, 인지시스템은 순차적으로 정보를 처리하는 시스템으로 설명된다. 이러한 시스템은 기호를 조작하여 인지기능을 수행한다([18]).

기호주의적 접근의 계산모델은 바로 '튜링기계'(Turing Machines)로 불리는 형식시스템(formal system)으로부터 직접적으로 유래되었다([3]). 그런데 튜링기계는 수리논리학자 튜링(Alan Turing)에 의해서 고안된 수학적 시스템이다(On Computable Numbers with an Application to the Entscheidungsproblem, 1936). 튜링은 계산하는 인간에 대한 수학적 모델링을 시도하였고 그때 '계산하는 인간'을 '컴퓨터(computer)'라고 명명했다. 그러므로 튜링이 사용했던 용어 '컴퓨터'는 오늘날 '계산하는 기계'를 지칭하는 것과는 전혀 다른 의미로 사용된 것이었다.

튜링에 의하면 생각하는 인간을 수학적으로 분석하면 세 가지의 요소에 의해서 설명될 수 있다((1) a list of states, (2) a finite alphabet of symbols, (3) a finite list of instructions). 이 요소에 의해서 인간의 생각을 수학적으로 설명할 수 있으며, 그 설명을 모은 것이 튜링기계에 해당된다. 튜링기계의 수학적 정의는 다음과 같다.

정의(Turing Machine). $\sum$를 기호들의 유한집합이라 하고, B(Blank를 표시)와 |(Stroke을 표시)는 $\sum$의 원소라 하자. $q_1, q_2, \ldots$를 상태(states)의 기호라고 하자. (단 $q_1, q_2, \ldots$는 $\sum$의 원소가 아니다). 그때에 튜링기계 TM은 오원 (q_i, s, t, Φ, q_i)의 유한집합이다. 여기에서 s와 t는 $\sum$의 원소이고, Φ는 기호 L(왼쪽으로 하나 이동) 또는 기호 R(오른쪽으로 하나 이동)의 하나이다. 기호 q_i는 i상태를 나타낸다.

그러므로 튜링기계 TM은 자연수 $0, 1, 2, \ldots, n$에 대하여 다음과 같은 함수이다.

$$TM : \{0, 1, 2, \ldots, n\} \times \{B, |\} \to \{B, |\} \times \{L, R\} \times \{0, 1, 2, \ldots, n\}$$

예를 들어, $x + u$를 계산하는 튜링기계 TM_{x+u}는 다음과 같이 정의될 수 있다.

$$TM_{x+u} = \left\{ \begin{array}{ccc} (q_1, |, |, R, q_1), & (q_1, B, |, R, q_2), & (q_2, |, |, R, q_2), \\ (q_2, B, |, L, q_3), & (q_3, |, B, L, q_4), & (q_4, |, B, L, q_5), \\ (q_5, |, B, L, q_6), & (q_6, |, |, L, q_6), & (q_6, B, B, R, q_0) \end{array} \right\}.$$

정의(Universal Turing Machine). 리커시브 함수 F에 대하여, 다음을 만족하는 유니버설 튜링기계 UTM이 존재한다. 임의의 TM_n과 임의의 자연수 n과 x에 대하여,

$$F_{TM_n}(x) = F_{UTM}(n, x)$$

유니버설 튜링기계 UTM은 임의의 튜링기계 TM을 구현할 수 있는 튜링기계를 의미한다. (이하, 튜링기계는 유니버설 튜링기계를 의미한다.)

튜링은 튜링기계가 계산하는 인간의 수학적 모델이므로, 튜링기계라는 형식시스템을 생각하는 기계라고 보았다. 1948년 수리논리학자 폰노이만(von Neumann)은 튜링기계에 내장된 프로그램을 구상함으로써 자기 증식(self-reproduction)이 가능한 최초의 컴퓨터를 제작할 수 있었다([2]). 기호주의적 접근을 하였던 다수의 인지과학자들은 튜링의 수학적 모델을 물리적으로 구현한 폰 노이만의 컴퓨터를 '생각하는 기계'로 여겼다. 그들에게는 인류 역사상 처음으로 생각하는 기계, 즉 생각하는 도구를 가지게 된 것을 의미했다. 이 도구를 통하여 인간이 생각하는 방법과 구조를 이해할 수 있을 것으로 기대했다. 이러한 기대는 '물리적 기호 시스템 가설(physical symbol system hypothesis)'에 잘 반영되어 있다. 이 가설은 기호주의적 접근의 기본적인 전제에 해당하며, 그 내용은 물리적 기호 시스템에는 일반적인 지능 행위를 위한 필요충분조건을 만족하는 수단이 있다는 것이다([18]). 즉 이 가설에 따르면 인간의 인지행위는 튜링기계의 처리과정을 물리적으로 구현한 것과 동등하다. 결국 수리논리학의 연구과정에서 나온 튜링기계는 오늘날 컴퓨터의 모델이 되었을 뿐만 아니라 인지시스템의 주요 모델이 되어, 생각하는 기계의 모델과 연구의 방향을 제공하였다.

인지과학의 역사상 중요한 사건으로 평가되는 'Logic Theorist'(1955)는 뉴웰(A. Newell), 쇼(J. Shaw)와 사이몬(H. Simon)이 만든 최초의 인공지능 프로그램이었는데(Empirical Explorations with the Logic Theory Machine: A Case Study in Heuristics), 러셀(B. Russell)과 화이트헤드(A. Whitehead)의 Principia Mathematica의 제2장을 증명할 수 있는 프로그램이었다. 그 후에 뉴웰과 사이몬이 일반적인 문제까지 확장시켜 일반문제를 해결할 수 있는 프로그램을 제시하고자 한 것이 프로그램 'General Problem Solver'(1961)였다. 생성시스템(Production system), 전문가시스템(Expert system), ACT, SOAR 등은 기호주의적 접근에서 이론적으로 주요한 모델들이다([18]). Oracle 튜링기계, 계산복잡성(complexity) 등 튜링기계와 관련된 다양한 수학적 계산능력에 관하여는 [13]에 잘 소개되어 있다.

4 뇌의 수학적 모델

인지과학 내에서 연결주의 접근은 뇌를 통하여 마음을 이해하려는 과학적 연구들을 묶어서 표현한 것이다. 이 접근에서는 뇌의 구조와 계산방식에 의해 인지시스템을 규명하고자 한다. 연결주의에 의하면 인지시스템은 뉴론과 같은 단위들의 연결망과 병렬분산적 동시처리 방식에 의해 설명될 수 있다([17]). 그러므로 연결주의적 계산 시스템은 다음의 표와 같이 튜링기계의 구조와 순차적 계산방식과는 여러 가지 면에서 대조된다.

	기호주의적 접근 (Symbolic Approach)	연결주의적 접근 (Connectionist Approach)
수학적 모델	튜링기계	신경망
모델의 기초	논리 시스템	뇌 시스템
정보처리방식	순차적 처리	병렬적 처리
추론 방식	연역적 추론	귀납적 추론
기반형	규칙 기반형	사례 기반형

프로그래밍 분량	많은 프로그래밍	적은 프로그래밍
학습	역동적	고정적
지식 표상	명시적	암묵적
알고리즘 성격	정확한 일치	근사적 일치
수학적 근거	수리논리	통계, 확률
필요 사항	전문가 필요	데이터 필요
고장방지 능력	약함	강함
사용자 인터페이스	화이트 박스형	블랙 박스형

연결주의 접근이 인정을 받으며 인지과학의 주요한 축으로 자리잡은 것은 1980년대이지만, 역사적으로는 1943년 신경생리학자 머컬로크(Warren McCulloch)와 수학자 피츠(Walter Pitts)에 의해서 뇌의 수학적 모델이 구성되었다(A Logical Calculus of the Ideas Immanent in Nervous Activity, 1943). 이것은 뇌에 대한 연구

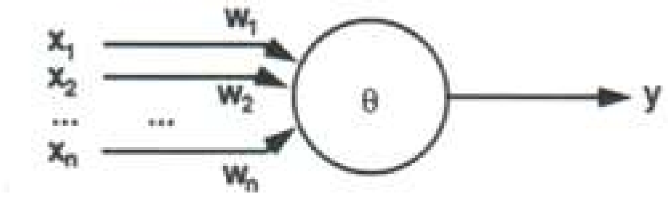
의 역사에서도 획기적인 사건이었다. 이 논문에서 뉴론(neurons)의 수학적 모델로 임계논리단위(threshold logic unit)가 제공되었다. 그들의 연구는 튜링의 연구결과에 자극을 받아 이루어진 것이었으며, 화이트헤드(A. N. Whitehead)와 러셀(B. Russell)의 수리논리학에서 사용된 용어를 채택한 것이었다. 머컬로크와 피츠는 튜링기계의 계산을 구현할 수 있는 신경망(neural networks)의 수학적 모델을 제공해 주었다([9]).

머컬로크-피츠 신경망(McCulloch-Pitts Neural network)은 머컬로크와 피츠가 정의한 뉴론의 연결망이다.

정의(McCulloch-Pitts Neuron). 뉴론은 m개의 입력 $x_1, x_2, \ldots, x_m$과 하나의 출력 u를 가진 임계논리단위이다. 뉴론은 시간 스케일 $t = 1, 2, 3, \ldots$에 따라 동작하며, 뉴론의 입력값이 임계값보다 클 경우에 발화(fire)한다. 이러한 규칙은 다음과 같다.

$$y(t+1) = 1 \leftrightarrow \sum_i x_i(t) \geq \theta$$

이러한 구상은 보다 발전되어 다음과 같이 뉴론의 수학적 모델로 일반화되었다.



여기에서 x_i는 입력(the inputs), w_i는 가중치(the weights, 생물학적 뉴론의 시냅스 연결강도(strengths)를 의미함), θ는 임계(the threshold), u는 출력(the output)이다.

$$y(t+1) = \left\{ \begin{array}{ll} 1 & \text{if } \sum_i w_i x_i(t) \geq \theta \\ 0 & \text{otherwise} \end{array} \right\}.$$

이러한 뉴론의 네트워크에 대한 수학적 계산모델이 인지과학의 신경망(neural networks)모델이 되었다. 특히 1980년대에는 활성화 함수(activation function)로서의 다양한 임계함수의 종류와 연결패턴을 나타내는 아키텍처(architecture)에 따라 여러 가지의 신경망이 개발되었다. 여기에서 신경망의 아키텍처는 방향을 가진 그래프(a directed graph)와 같다. 기본적인 신경망은 다음과 같이 정의될 수 있다.

정의(Neural Networks). 신경망은 오원구조 (N, X, Y, C, q)이다. 여기에서 N은 뉴런들의 유한집합, N는 입력의 집합, Y는 출력의 집합, $C = N \times N$는 시냅스의 연결(connections), $q : N \to F$는 뉴런할당함수이다(F는 뉴런에서의 임계함수들의 집합).

그러므로 신경망에서 $(N \cup X, C)$는 방향을 가진 (혹은 방향을 가지지 않은) 가중치 그래프(weighted graph)이다. 그래서 여러 가지 신경망의 모델은 가중치 그래프의 종류와 활성화 함수의 종류에 따라 구별될 수 있다. 예를 들면, 유명한 최적화 문제였던 여행판매원 문제(traveling salesman problem)를 풀어서 기호주의적 접근을 능가하는 모델로 많은 관심을 모았던 홉필드 신경망(Hopfield Network)은 방향을 가지지 않은 그래프(undirected graph), 다층(multi-layered) 아키텍처, $f : R^n \to \{1, -1\}$ 임계함수 (R은 실수의 집합, n은 자연수) 등의 특징을 가지고 있었다.

튜링기계와 비교해 볼 때, 신경망의 출력은 뉴런의 활성화(activation)를 설명해주고 주어진 시간단위에서 활성화의 패턴은 신경망의 구성(configuration)에 해당된다. 신경망 내에서 구성이 단기기억(short-term memory)에 해당되면, 가중치의 패턴은 장기기억(long-term memory)에 해당된다. 이러한 구조에서, 가중치의 변화 즉 시냅스 강도의 조정을 통해서 발생되는 것이 바로 신경망에서 '학습'(learning)의 정의이다. 여기에서 가중치를 조정하는 방법이 신경망의 알고리즘에 해당된다. 그러므로 기호주의 접근의 튜링기계와는 달리 연결주의의 신경망 모델은 역동적인 학습을 설명할 수 있다는 새로운 면을 보여주었다.

뇌를 수학적으로 모델링한 신경망은 튜링기계와는 달리 중앙처리장치(CPU)가 전혀 필요하지 않으며, 수많은 뉴론들의 연결로 이루어져 있으므로 고장방지능력(fault tolerance)을 지니고 있다고 평가된다. 신경망의 종류는 매우 다양하다. 예를 들면, 이론적으로 중요한 모델로는 홉필드(Hopfield) 신경망, 해밍(Hamming) 신경망, ADALINE 신경망, 자기조직화(Self-Organizing) 신경망, 다층퍼셉트론(Multi-Layer Perceptron)이 대표적이며, 신경망의 다양한 수학적 계산능력에 관하여는 [24]에 잘 소개되어 있다.

5 괴델의 불완전성 정리와 인지과학

인지시스템의 본질과 기능을 해명하고 검증하려는 역사적 흔적은 유클리드(Euclid)의 '공리화'(axiomatization), 아리스토텔레스(Aristotle)의 '논리학'에서 그 기원을 찾아 볼 수 있다. 라이프니츠(G. Leibniz)는 보편적 개념을 표현할 수 있는 언어, 즉 과학의 보편언어(characteristica universalis)를 기호(symbol)라고 보고, 기호의 시스템 내에서 모든 계산을 처리할 수 있을 것으로 기대했다.

부울(G. Boole, *An Investigation of the Laws of Thought*, 1854)은 논리학의 대수화를 체계화했다. 그는 사고의 연산법칙이 존재한다면, 사고에 대한 대수적 계산이 가능하다고 생각했다. 부울의 대수는 현재의 컴퓨터과학을 포함하여 모든 이항시스템과 이항연산의 중요한 기틀이 되었다.

부울과 달리, 프레게(G. Frege, *Begriffsshrift, a formal language, modeled upon that of arithemtic, for pure thought*, 1879)는 논리의 수학화보다는 수학의 논리화를 지향했다. 개념을 명료하게 표기하고자 기호를 사용하여 새로운 논리시스템을 제안했다. 그의 공헌으로 논리학은 (1) 명제함수(propositional function)와 (2) 양화(quantification)를 갖춘 술어논리학(predicate logic)으로 발돋움하여 명제논리 시대를 마감하고 술어논리 시대를 열게 되었다. 프레게는 명제함수 즉 술어와 칸토르의 집합개념을 통하여 모든 수학적 내용을 논리학으로 환원할 수 있다고 생각하였다. 이러한 구상은 러셀과 화이트헤드(Principia Mathematica, 1910-1913)에 의해 계승되었다.

이러한 움직임에 대하여, 힐베르트(D. Hilbert)는 논리적 공리로부터 자유로운 공리를 지닌 형식시스템(axiomatic formal system)을 수학의 기초로 삼아야 한다고 생각했다. 그는 이 형식시스템이 세 가지 조건 (1) 무모순성(consistency), (2) 결정가능성(decidability), 그리고 (3) 완전성(completeness)을 만족하면, 수학의 유토피아를 건설할 수 있을 것이라고 확신했다.

괴델(K. Gödel, 'On Formally Undecidable Propositions of Principia Mathematica and Related Systems I,' 1931)은 제1불완전성정리를 통하여 러셀과 화이트헤드의 논리시스템(PM) 내에서 결정불가능한 명제가 존재함을 증명하였다. 이는 곧 힐베르트의 프로그램이 달성될 수 없음을 의미했다. 한편, 괴델은 불완전성정리를 증명하면서, 정수 위에서의 인코딩(encoding)과 디코딩(decoding)기법을 소수를 이용하여 완벽하게 구사함으로써 정보이론과 컴퓨터 과학에 중요한 공헌을 남기기도 하였다.

괴델의 제1불완전성정리(1931). 형식시스템 PM이 무모순(consistent)일 때, PM 내에서 증명될 수도 없고 반증될 수도 없는 결정불가능한 명제(undecidable propositions)가 존재한다.

괴델의 제2불완전성정리(1931). 형식시스템 PM이 무모순(consistent)일 때, PM 자체의 무모순성은 PM 내에서 증명될 수 없다.

인지과학을 위하여 계산하는 기계의 수학적 모델을 제공하게 되었던 튜링의 1936년 논문은 사실 괴델이 증명했던 PM의 불완전성이 튜링기계 형식시스템 내에서 정지불가능 문제(unhalting problem)와 수학적으로 동등함을 보이기 위한 것이었다. 괴델이 직접 정의했던 형식시스템(1934)과 재귀함수(general recursive function)는 인지과학을 구성하는 일에 결정적인 공헌을 하게 되었다. 뒤를 이어 나왔던 처치(A. Church)의 람다시스템, 튜링(A. Turing)의 튜링기계, 포스트(E. Post)의 생성시스템(production system)과 같은 형식시스템들과 그 시스템들이 정의하는 계산가능 함수들은 모두 수학적으로 동치인 것으로 증명되었다.([6][7]). 여기에서 함수들의 계산가능성과 함수를 정의하는 형식시스템은 인지과학의 기호주의적 접근에서 기본적 모델이 되었던 것이다. 가령, 포스트의 생성시스템은 인지과학의 성립에 중요한 역할을 하였던 언어학자 촘스키(N. Chomsky, *Logical Structure of Linguistic Theory*, 1955)의 생성문법에 직접적인 영향을 주었다.

괴델의 불완전성정리는 인지과학의 연구에 가장 핵심적인 문제와 직결되어 있었다([1][14][21][29]). 괴델의 정리에 따르면, 인지시스템이 모순이 없는 형식시스템이라면, 인지시스템은 결코 완전할 수 없다는 것을 의미한다. 그렇다면 인공지능의 프로그램과 그 물리적 구현이 제한적이거나 불가능함을 함의하게 된다. 그래서 인지과학의 기호주의적 접근이나 연결주의적 접근이 피해갈 수 없는 과제가 되었던 것이다. 괴델의 정리와 관련하여 이른바 '인공지능 논쟁'이 벌어졌다. 이 논쟁은 다음과 같이 두 가지로 범주화될 수 있다.

(1) 괴델의 정리에 의해서 수학자와 동등한 인공지능 시스템은 불가능하다는 주장 : 루카스(J. Lucas), 펜로즈(R. Penrose) 등이 대표적 학자이다.

(2) 괴델의 정리에 의해서 (1)의 결론을 얻을 수 없다는 주장 : 호프슈태터(D. Hofstadter), 아비브(M. Arbib), 생커(N. Shankar) 등이 대표적 학자이다.

괴델 자신은 다음의 두 명제가 모두 필요한 진리명제라고 생각했다([10]). (A) "수학자의 마음은 어떠한 유한기계도 무한히 능가한다." (B) "절대 해결될 수 없는 수학적 문제가 존재한다." 그러므로 괴델에게는 다음의 논리합이 성립한다. (C) "수학자의 마음은 어떠한 유한기계도 무한히 능가하거나(or), 절대로 해결될 수 없는 수학적 문제가 존재한다." 명제(A)와 명제(B)는 서로 배타적이지 않으며, 논리합(C)에서 양립가능하다. 논리합(C)의 양립가능성은 괴델의 불완전성 정리 이후 수학자와 튜링기계의 동등성에 관한 괴델 자신의 결론이었다. 그러므로 괴델은 어느 편의 손도 들어주지 않은 셈이다([31]). 괴델의 명제에도 불구하고 괴델의 정리를 둘러싼 인공지능의 논쟁은 아직도 계속되고 있다.

6 전망

역사적으로 인류에게 수학은 인간의 인지적 활동이 내부에서 개념화되고 형식화되는 과정이었다. 그리고 오늘날 인지과학은 다양한 인지적 과정들이 외부에서 개념화되고 형식화되는 메타과정이다. 지금까지 인지과학의 수학적 기초를 조명하면서, 인지과학의 형성과 발전과정에 나타난 수학적 공헌을 살펴 보았다. 수학은 형식시스템이란 이론적 기반을 제공했고, 다양한 학문의 소통을 가능하게 해주는 언어의 기능을 담당함으로써 인지과학의 형성과 발전을 가능하게 하였다. 특히, 수학자들은 기호주의적 접근에서 핵심적인 수학적 모델이 되었던 튜링기계를 제공하였고 연결주의적 접근에서 핵심적인 수학적 모델이 되었던 신경망을 제공하여, 인지과학의 이론의 양대 축을 이루는 데 결정적인 공헌을 하였다. 튜링과 피츠 외에도 인지과학의 역사에서는 다양한 수학자들이 등장하고 그들의 다양한 공헌들이 있지만, 이 글에서는 인지과학의 가장 근본적 과제와 관련하여 괴델과 그의 공헌이 인지과학 내에서 갖는 의미를 중심으로 논의하였다.

괴델 정리에 의해 제기된 튜링기계와 동등한 시스템의 한계 문제를 둘러싼 논쟁이 계속되자, 인지과학자들은 새로운 차원의 수학적 가능성을 모색하게 되었다. 괴델의 불완전성 정리의 정보이론적 의미는 정지확률의 수(halting probability number) $\Omega = \sum_{u\ halts} 2^{-|p|} (0 < \Omega < 1)$에 의해서 표현가능하다([5]). 여기에서 p는 프로그램을 나타내고, $|p|$는 프로그램 p의 비트(bits)의 크기를 나타낸다. 정보이론에서의 괴델 정리는 정지확률 Ω가 정수에 대하여 우리에게 계속 복잡성의 새로운 성질, 즉 무작위성(randomness)을 보여준다는 것을 함의한다. 이것은 수학자가 단지 튜링기계와 동등하다는 튜링의 기계주의를 괴델이 비판한 바와 같다([11][30]). 비트에 근거한 기계적 방식에 의해서 수학을 정복할 수 있는 인지시스템은 존재할 수 없다.

0과 1로 구성된 이진(binary) 기반의 시스템에서는 괴델의 불완전성 정리를 피해갈 수 없으므로, 고전논리의 시스템을 넘어서는 새로운 시스템이 필요할 수밖에 없었다. 이에 가장 근접한 조건을 보여주는 대표적인 시스템은 양자논리(Quantum Logic)와 양자계산(Quantum Computing)이다. 괴델의 불완전성 정리는 바로 결정론적 계산보다는 비결정론적 양자계산의 필요성을 함의한다고 할 수 있다([4]). 버코프(G. Birkhoff)와 폰 노이만(J. von Neumann)의 기념비적인 논문("The Logic of Quantum Mechanics," 1936)에서 시작된 양자논리는 그동안 다양하게 정의되어 왔다. 일반적으로 양자논리는 힐베르트 공간의 닫힌 부분공간의 격자(lattice)로 해석되었고, 고정된 힐베르트 공간에서 유한한 변수들을 허용하고 $\wedge$(meet), $\vee$(join), $'$(negation)을 사용하는 다양한 방정식들을 의미했다([8]). 양자논리의 중요한 특징은 두 개의 값만을 허용하는 비트가 아니라 세 개의 값을 허용하는 큐비트(qubits)를 바탕으로 하는 상태공간(state space) C^n을 전제한다는 점이다.

C^1의 양자논리는 고전적인 부울 논리(Boolean logic)과 동등하다. 그러나 C^2의 양자논리부터는 분배법칙

이 성립되지 않는다. 그러므로 C^2의 양자논리는 표준적인 일차논리시스템(First order logic)과는 전혀 다른 계산을 하는 시스템임을 알 수 있다. 가령, 뇌와 같은 인지시스템을 양자논리시스템으로 본다면, 인지는 수학적으로 유한 차원의 힐베르트 공간을 가진 양자시스템의 양자계산으로 새로이 해석될 수 있다. 인지시스템을 양자적 관점에서 이해하는 접근(quantum cognitive science)은, 기존의 기호주의적 접근과 연결주의적 접근이 보였던 한계를 보완하고, 더 나아가 새로운 차원에서의 이론적 융합가능성을 제공할 수 있을 것으로 기대되고 있다([12][22]).

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ANALYSIS ON LINES AND CIRCLES IN SECONDARY SCHOOL MATHEMATICS TEXTBOOKS ACCORDING TO THE TYPES OF CONCEPT

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ABSTRACT

This paper treats how to teach mathematical concepts. We focus to a balanced, unified achievement for two patterns of concepts, intuitive concepts and logical concepts. Mathematical thinking is simply a combination of intuition and logic. In this paper we analyze certain textbook in secondary school mathematics from the viewpoint of patterns of concepts. We provide a concrete and practical investigation with basic geometrical concepts, that is line and circle and their position relation.

수학적 사고는 단적으로 직관과 논리의 결합이다. 본 논문에서는 효과적인 개념 학습을 위하여 직관적 개념과 논리적 개념으로 구분하여 중고등학교 교과서를 분석하였다. 구체적인 분석은 직관적 개념이 잘 반영되는 기하학 개념인 직선, 원, 직선과 원의 위치 관계를 대상으로 적용하였다.

1 서론

인류가 도구를 사용하면서부터 셈은 시작되었고 문명의 발전과 더불어 셈은 수의 개념으로 발전했으며 더 나아가서 수의 학문으로 자리했다. 이처럼 수학은 인류 문명의 발전과 함께 해왔다. 복잡하고 다양한 문명으로 발전한 오늘날의 수학은 무엇이며 수학을 어떻게 배우고 가르칠까는 단지 수학자나 수학교육자만의 고민은 아니다. 우리나라 중등 수학에서는 특히 대학입시라는 현실적인 상황에서 볼 때 더욱더 큰 관심사가 된다.

효과적인 개념학습을 위하여 박홍경 등은 수학적 개념을 직관적 개념과 논리적 개념, 형식적 개념으로 나누었다 [P-K-L]. 본 논문에서는 중등수학을 대상으로 하기 때문에 형식적 개념은 거의 나타나지 않는다. 이러한 의미에서 여기서는 형식적 개념을 논리적 개념으로 포함시킴으로써 수학적 개념을 직관적 개념과 논리적 개념으로 나누어 다루어도 충분하다.

*First Author

입시중심의 학교현장에서는 개념학습보다는 문제해결학습이 치중되고 있음은 부인할 수 없다. 그러한 문제해결도 계산위주의 시험문제해결이 강조됨으로써 개념학습의 측면에서 볼 때 논리적 개념에 편중되어 있다고 할 수 있다. 그로 인해 직관적 개념이 소홀하게 되어 균형적이고 통합적인 개념학습이 이루어지지 않게 되고 이것은 효과적인 개념형성을 어렵게 만든다.

한편 수학의 지도 순서에 대하여 김용운은 3가지 유형, 즉 역사적 순서, 이론적 체계 및 그 양자의 결합으로서 강의적 체계 순서를 제안하였다 [K 1986]. 이러한 강의적 체계순서를 수립하는 구체적인 사례연구로서 박홍경 등은 문제해결학습 측면에서는 각의 강의적 체계순서 [P-K-J, 2005] 를 다루었고 개념학습 측면에서는 벡터나 이를 포함한 선형대수학의 주된 개념 또는 연속성을 대상으로 강의적 체계순서를 연구하였다 [P-K-N, 2006], [P-K-L, 2007], [P-K, 2007].

수학적 사고는 단적으로 직관과 논리로 대별된다. 이에 대응하여 수학적 개념을 직관적 개념과 논리적 개념으로 구분하는 것은 자연스러운 발상이다. 이러한 개념유형의 구분은 양자의 통합적이고 균형적인 개념학습을 통하여 효과적인 개념형성을 도와준다. 따라서 효과적인 개념학습의 일환으로 수학적 개념을 직관적 개념과 논리적 개념으로 분석하는 것이 필요하다. 본 논문에서는 중등학교 현장에서 사용되고 있는 수학교과서를 대상으로 분석을 시도한다. 특히 구체적이고 실제적인 분석을 위하여 직관적 개념이 잘 반영되는 기하학적 개념을 대상으로 적용한다. 이러한 시도는 앞서 언급한 입시수학으로 인한 개념학습의 문제점을 해결하려는 하나의 출발점으로 볼 수 있다.

분석의 대상이 되는 수학 교과서는 중학교 약 16종, 고등학교 약 42종이 있다. 그 중 본 논문의 분석대상인 직선과 원의 개념은 중고등학교에 걸쳐 있기 때문에 내용 분석의 일관성을 고려하여 저자가 동일한 교과서중 한 종을 임의로 선정하였다. 선정한 교과서를 중심으로 직선의 개념, 원의 개념, 직선과 원의 위치관계를 직관적 개념과 논리적 개념으로 분석하고자 한다.

본 연구와 관련하여 동일한 분석을 다른 교과서에 적용하여 비교하는 것은 흥미롭다. 그것은 직관적 개념과 논리적 개념 사이의 균형성을 고찰하는 자료로 활용할 수 있기 때문이다. 또한 기하학적 개념만이 아니라 다른 수학적 개념으로 확대하는 것도 흥미로울 것으로 보인다. 이를 통해 수학 전반에 있어서 통합적이고 균형적인 개념학습의 방안이 될 수 있을 것이다.

2 수학적 개념의 유형

이절에서는 [P-K-L] 에 의거하여 수학적 개념의 유형에 대해 살펴본다. 박홍경 등은 효과적인 개념학습을 달성하기 위해서 수학적 개념을 직관적 개념, 논리적 개념, 형식적 개념의 3가지 유형으로 분류하고 이들은 각각 3가지 수리철학인 직관주의, 논리주의, 형식주의에 의거한다고 주장하였다 ([P-K-L]). 본 논문은 중등수학을 대상으로 삼기 때문에 중고등학교에서 나타나는 형식적 개념은 논리적 개념에 포함시킴으로써 직관적 개념과 논리적 개념으로 나누어 분석하고자 한다.

직관적 개념은 감각적, 정서적, 신체적 활동이나 사고와 분명한 연계성을 가진다. 그 연계성이 기원에서든 응용에서든. 공간의 입장에서는 볼 때 감각적 공간과 수학적 공간 사이에 유클리드 공간이 있다. 따라서 이러한 연계성은 특히 2차원, 3차원 유클리드공간에서 고려되어질 때 자연스럽게 형성된다. 표현기법에 있어서는 감각을 통해 개념을 이해하는 방식으로서 다이어그램이나 함수의 그래프 등 다양한 그림으로 표현된다.

논리적 개념은 직관적 개념이 갖는 불확실성이나 오류가능성을 제거하기 위하여 다루고자 하는 대상과 그들 사이의 관계를 명백하게 논리적으로 기술하는 것이다. 그러기 위해서는 대상이 명확히 기호화되어야 한다. 이러한 논리적 개념은 통상 고차원 공간에서 구체적인 성분표시로 주어진다. 실제 현대수학의

기초인 집합론은 공리적 체계에 입각하여 수리논리에 의해 전개된다. 표현기법에 있어서 논리적 개념은 대상을 기호화함으로써 수식, 논리식, 도형의 방정식 등으로 표현된다. 이를 정리하면 <표 1> 와 같다.

<표 1> 2가지 유형의 개념 비교

유형	직관적 개념	논리적 개념
기본 철학	직관주의	논리주의
활동과 사고 측면	감각적 활동, 사고 →수학적 활동, 사고 연결	수학적 활동, 사고에서 직관의 불확실성, 오류가능성 제거
공간	저차원 공간	고차원 공간
표현기법	다양한 그림 함수(방정식)의 그래프	수식, 논리식 도형의 방정식(함수)

이러한 개념 유형은 개념이해, 계산기능, 응용의 개념학습의 3가지 요소와 다음과 같이 관계한다. 직관적 개념은 개념이해에 있어서 동기를 부여한다. 동기부여는 역사적으로나 현재시점이거나, 보편적이거나 개인적이거나, 학술적이거나 생활상이거나, 다양한 방식이 가능하다. 계산 기능에는 의미를 부여한다. 그래서 계산이 단지 기계적으로 행해지지 않고 의미를 통하여 개념이해를 돕는다. 또한 응용에 대한 다양한 아이디어를 제공한다. 이것은 동기부여로 환원하거나 실제적 필요성에 따라 새롭게 제기되기도 한다.

논리적 개념은 직관적 개념과 더불어 개념이해를 더욱 명확히 해준다. 직관적 이해만으로는 개념이 불완전하며 오류를 범할 위험성이 항상 내재한다. 이를 상보적으로 보완하는 것이 논리적 개념이다. 계산기능에 실제적인 힘을 부여한다. 훈련과 연습은 이러한 기능을 더욱 강화시킨다. 응용에 있어서는 이론적 객관적 근거를 제공한다. 이를 정리하면 <표 2> 와 같다.

<표 2> 개념의 유형과 요소 사이의 관계

요소	직관적 개념	논리적 개념
개념이해	동기 부여	명확화
계산기능	의미 부여	실제적 힘 부여
응용	새로운 아이디어 제공	이론적 근거 제공

3 **Euclid** 기하학과 해석 기하학

본 논문의 분석대상은 기본적인 기하학적 개념인 직선과 원이다. 또한 분석방법은 직관적 개념과 논리적 개념의 구분이다. 기하학의 입장에서 볼 때 직관적 개념은 유클리드기하학에 논리적 개념은 해석기하학에 밀접하다. 이러한 이유로 다루고자 하는 내용과 관련하여 유클리드기하학과 해석기하학의 역사적 순서를 고찰하는 것은 자연스럽다. 주된 내용은 [P-K-J]를 참조한다.

유클리드기하학은 그리스 고전적 논리주의에 기인한다. 인간과 자연을 대립적으로 간주하고 객관적 존재인 자연의 통일성, 공통성에 주목하여 로고스로부터 출발하는 논리적 연역을 중시하는 입장이다. 주된 연구대상은 점, 직선, 평면을 구성요소로 한 직선도형과 원, 구를 추가하여 구성할 수 있는 특수한 곡선도형이다. 가령 두 점을 지나는 직선은 단하나 존재한다는 결정조건에 의해 직선을 규정하거나, 원은 한 점으로부터 거리가 일정한 점들의 모임으로 규정한다. 이러한 대상에 대해 직관에 의해 직접적인 방법으로 길이(거리), 각, 면적, 체적 등의 기하학적 계량을 주로 고찰한다.

반면 해석기하학은 유클리드기하학의 초등기하학적 방법의 개선에서 나온 것으로 대수적 고찰을 통하여 기하학을 연구하는 것이라 할 수 있다. 이러한 고찰방법의 전환은 좌표에 의한 수와 점의 동일시에 의거한다. 가령, 직선 위에 좌표를 도입하여 점과 실수를 동일시함으로써 직선은 1차방정식과 동일시된다. 또한 유클리드평면은 좌표계를 도입함으로써 순서쌍 전체의 집합 과 동일시할 수 있다. 이로 인해 기하학은 대수적 방법에 의해 직관에 의한 도형의 연구에서 범하기 쉬운 방법적 오류를 피할 수 있고 대수학은 기하학적 해석을 통해 대수 연산에 의미를 부여할 수 있게 되었다.

유클리드기하학에서 해석기하학으로의 전환은 기하학의 연구 대상인 도형과 대수학의 연구 대상인 방정식이 동일한 대상의 다른 표현임을 깨닫게 해 주는 전기를 마련해 주었다. 즉 방정식은 도형의 대수적 표현으로 볼 수 있으며 도형은 방정식의 그래프로 본다.

4 직선과 원, 직선과 원의 관계에 관한 개념

이제 여기서는 II절에서 언급한 개념유형에 따라 직선의 개념, 원의 개념, 직선과 원의 위치관계를 직관적 개념과 논리적 개념으로 분석하고자 한다. 1절에서 밝힌 바와 같이 연구대상의 교과서는 동일한 저자를 가진 교과서 중의 하나를 임의로 선정하였다. 그 교과서를 [Le]로 표기한다.

4.1 직선의 직관적 개념과 논리적 개념

[Le]를 대상으로 <표 1>에서 정의한 직관적 개념과 논리적 개념에 입각하여 중학교 과정과 고등학교 과정에서 다루어지는 직선 개념을 살펴본다. 이로부터 직선의 개념은 직관적 개념에서 논리적 개념으로 이행하고 있음을 관찰할 수 있다.

1) 감각적 활동에서 수학적 활동으로의 연결

직선의 직관적 개념을 도입하기 위하여 [Le]에서는 먼저 감각적 활동에서 수학적 활동으로 연결로서 다음과 같은 내용을 소개하고 있다.

> 종이 위에 연필을 세워 놓고 누르면 점이 찍히고, 연필을 누른 상태에서 움직이면 선이 된다. 이와 같이 점이 움직인 자리는 선이 된다. 선이 움직인 자리는 면이 되고 면은 곡면과 평면으로 나눌 수 있으며 평면과 평면이 만나면 직선이 된다."
>
> (중학수학 1학년, p179)

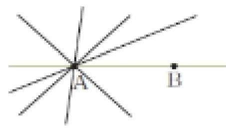
활동과 사고의 측면에서 볼 때 인용문에서 종이, 연필, 세워놓고 누른다, 움직인다 등은 감각적 활동이며, 점, 선, 만난다 등은 수학적 활동이다. 이러한 점에서 이 문단은 감각적 활동에서 수학적 활동으로의 연결로 볼 수 있다.

이러한 연결은 수학적 경험의 입장에서 볼 때 Davis와 Hersh가 말한 사고실험에 해당한다 [D]. 사실 수학적 경험은 감각적 활동에서 수학적 활동으로의 연결을 통해 시작되며 사고실험에 의해 경험을 쌓아가기 때문이다.

2) 유클리드기하학적 직관적 정의

수학적 활동이나 사고의 대상이 된 직선의 개념은 3절에서 다룬 유클리드기하학의 입장에서 평면에서 직선의 정의를 도입하고 있다. 이는 다음 문장에서 찾을 수 있다.

> "오른쪽 그림과 같이 한 점 A를 지나는 직선은 무수히 많지만, 두 점 A, B를 지나는 직선은 오직 하나뿐이다."
> (중학수학 1학년, p. 181)



이는 2차원 공간인 평면에서 그림을 통하여 직선의 결정조건을 설명하고 있어서 직관적 정의라 할 수 있다. 다만 [Le]에서는 이를 적선의 정의로 명확히 밝히지는 않고 있다.

3) 해석기하학적 직관적 정의

이와 같은 직선의 직관적 정의는 좌표계를 도입하여 다시 고려한다. 즉 해석기하학의 입장에서도형의 대수적 표현으로서 직선의 개념을 설명한다. 먼저 일차함수를 배운 뒤 일차함수의 그래프가 직선이 됨을 소개한다.

> "서로 다른 두 점을 지나는 직선은 오직 하나뿐이다. 이와 같이 좌표평면 위에 두 점의 좌표가 주어지면 이들 두 점을 지나는 직선을 그래프로 하는 일차함수의 식을 구할 수 있다."
> (중학수학 2학년, p. 130)

이 인용문에서 먼저 2)에서 언급한 유클리드기하학적 직관적 정의를 토대로 일차함수의 그래프가 직선임을 말한다. 다만 이것만으로는 수직선과 같은 직선을 표현할 수 없다는 점에서 직선의 정의에 해당하지는 않는다. 이에 대해 직선의 정의는 일차방정식의 그래프임을 아래와 같이 소개하고 있다. 일차함수에서 일차방정식을 통해 직선의 (직관적) 개념을 소개하는 것은 함수의 개념이 방정식보다 늦다는 점에서 역사적으로는 반대이다 하지만 간단한 것에서 복잡한 것으로, 쉬운 것에서 어려운 것으로 나아가는 이론적 체계에서 볼 때에는 부합하는 순서라 할 수 있다.

> "일반적으로 x, y가 수 전체의 집합의 원소일때, 일차방정식의 해는 무수히 많고, 이들 해를 좌표평면 위에 나타내면 직선이 된다. 또 이 직선 위의 모든 점의 좌표는 일차방정식의 해이다.
> (중학수학 2학년, p133)

4) 논리적 개념

3)에서 살펴본 해석기하학적 직관적 정의에 의해 이제 직선의 논리적 개념이 명확히 주어진다. 사실 일차함수의 일반형이나 일차방정식의 일반형이 바로 직선의 논리적 개념이 된다. [Le]에서는 3)에서 인용한 문장들에 각각 아래 문장들이 이어서 등장하고 있다.

> "일차함수 $y = ax + b(a \neq 0)$의 그래프는 직선이다"
> (중학수학 2학년, p. 114)

"일차 방정식 $ax+by+c=0$(a, b, c는 상수, $a \neq 0$ 또는 $b \neq 0$)을 직선의 방정식이라고 한다" (중학수학 2학년, p133)

3) 에서 언급한 바와 같이 전자는 일차함수의 그래프가 직선이 된다는 것을 말한다. 하지만 모든 직선을 표현하지는 못한다. 후자는 해집합으로서 일차방정식의 그래프는 정확히 직선을 정의하는 것을 말한다.

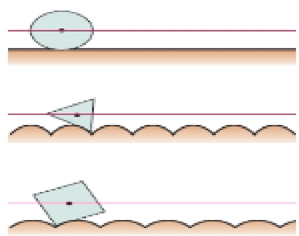
이처럼 [Le] 에서는 중학교 2학년 과정에서 직선의 직관적 개념과 논리적 개념이 모두 소개되어진다. 한편 고등학교 과정에서는 직선에 대한 논의가 논리적 개념으로 다루어진다. 이는 중학교에서 이미 직관적 개념과 논리적 개념 모두를 학습한 것으로 보기 때문이다. 따라서 고등학교 과정에서 직선의 직관적 개념을 상기시키는 일은 통합적이고 균형적인 개념학습을 위해 중요하다.

4.2 원의 직관적 개념과 논리적 개념

기본적인 직선도형의 하나인 직선과 마찬가지로 기본적인 곡선도형의 하나인 원에 대하여 개념 유형을 분석하자. 그러면 [Le]에서는 직선과 마찬가지로 직관적 개념에서 논리적 개념으로 이행하고 있음을 관찰할 수 있다.

1) 감각적 활동에서 수학적 활동으로의 연결

> "우리는 자동차 바퀴가 삼각형이나 사각형이 아닌 원 모양인 것을 당연하게 생각한다. 그런데 바퀴는 왜 원 모양일까? 그것은 효율성과 안정성 때문이다. 바퀴는 일정한 힘을 가해 굴릴 수 있어야 하고 덜컹거리지 않게 물건을 운반할 수 있어야 하는데,
> 이것은 원 모양일 때만 가능하다. 즉, 바퀴의 축과 가장자리 사이의 거리가 항상 일정하기 때문에 길만 평평하게 잘 닦여 있다면 중심이 직선으로 움직여 덜컹거릴 일이 없다." (중학수학 1학년, p. 229)



자동차 바퀴, 굴린다, 평평하다 등은 감각적 활동이며 삼각형, 사각형, 원 등은 수학적 활동이다. 이러한 표현에서 감각적 활동에서 수학적 활동으로의 연결을 주는 것으로 볼 수 있다.

2) 직관적 개념

직선의 경우와 마찬가지로 원의 정의도 먼저 유클리드기하학의 입장에서 주어진다. 하지만 직선과 달리 정의로서 설명하고 있다.

> **"평면 위의 한 점 O로부터 일정한 거리에 있는 모든 점의 집합을 원이라 하고, 이것을 원 O로 나타낸다. 이때, 원의 중심 O와 원 위의 한 점 A를 이은 선분 OA를 원의 반지름이라고 한다."**
> **(중학수학 1학년, p. 232)**

3) 논리적 개념

직관적 개념에 이어 좌표를 도입함으로써 해석기하학의 입장에서 도형의 방정식으로서 원의 개념을 소개한다. 이것은 원의 논리적 정의에 해당한다.

> "좌표평면에서 중심이 $C(a,b)$이고 반지름의 길이가 r인 원 위의 임의의 점을 $P(x,y)$라고 하면
> $|CP|=r$이므로
> $$\sqrt{(x-a)^2+(y-b)^2}=r$$
> 양변을 제곱하면
> $$(x-a)^2+(y-b)^2=r^2$$
> 과 같은 원을 나타내는 방정식을 원의 방정식이라고 한다."
> (고등수학, p. 215)

4) 2차 곡선으로서 원의 논리적 개념

원은 원추곡선의 하나이다. 이러한 측면에서 원의 논리적 개념은 아래와 같이 2차 곡선의 특수한 형태로서 표현된다.

> "$(x-a)^2+(y-b)^2=r^2$을 원의 방정식의 표준형이라 하고, 방정식
>
> $$x^2+y^2+Ax+By+C=0$$
>
> 을 원의 방정식의 일반형이라고 한다"
>
> (고등수학, p. 215)

4.3 원과 직선의 위치관계에 대한 직관적 개념과 논리적 개념

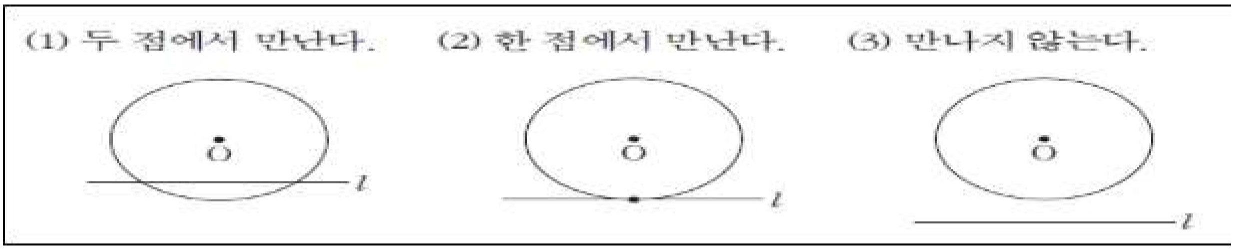
앞에서 직선과 원의 직관적 개념과 논리적 개념을 살펴보았다. 이것의 활용으로서 이들의 위치관계를 살펴보자. [Le]에서는 직선과 원의 경우와 마찬가지로 직관적 개념에서 논리적 개념으로 이행하고 있다.

1) 감각적 활동에서 수학적 활동으로의 연결

(고등수학, p.214)

수평선을 직선으로 태양을 원으로 보는 것을 통하여 이들 사이의 위치관계에 대한 개념이 감각적 활동에서 수학적 활동으로 자연스럽게 연결된다. 또한 산과 달을 통해서도 마찬가지 연결을 꾀하고 있다.

2) 직관적 개념



(중학수학 1학년, p. 239)

유클리드기하학의 입장에서 볼 때 먼저 직선과 원의 직관적 개념을 사용하여야 한다. 이 때 직선과 원의 위치관계는 위에서와 같이 이들의 교차성을 통해 쉽게 설명할 수 있다. 만나거나 만나지 않거나. 만나는 경우를 한 점이거나 두 점이거나에 따라 다시 나누면 모두 3가지 경우를 고려할 수 있게 된다.

3) 논리적 개념

2)에서 살펴본 위치관계에 대한 직관적 개념은 좌표를 도입하여 해석기하학의 입장에서 고려할 수 있다. 그러면 직선과 원의 논리적 개념을 사용하면 이들의 교차성은 그들 도형의 방정식의 교차성에 해당한다. [Le]에서는 아래와 같이 일차함수와 원의 방정식의 교차성을 소개하고 있다.

원 $x^2+y^2=r^2$과 직선 $y=mx+n$의 교점의 개수는 두 도형의 방정식에서 y항을 소거하여 만든 x에 대한 이차방정식 $x^2+(mx+n)^2=r^2$의 실근의 개수와 같다.

(고등수학, p.219)

우리의 논의를 따르면 일차방정식과 원의 방정식의 교차성을 다루는 것이 보다 일반적이다. [Le]에서는 소개하고 있지는 않지만.

나아가 이러한 위치관계는 원만이 아니라 원추곡선의 경우로 확장할 수 있음을 쉽게 이해할 수 있다. 이 경우 논리적 개념으로는 원 대신에 2차 곡선이 사용된다.

5 제언

지금까지 논리적 개념 치중의 문제해결학습의 문제점을 해결하고 효과적인 개념학습을 위하여 중고등학교 교과서를 대상으로 개념유형의 측면에서 직관적 개념과 논리적 개념으로 나누어 분석하였다. 구체적이고 실제적인 분석이 될 수 있도록 적용대상은 직관적 개념이 잘 반영되는 기하학적 개념 중에서 가장 기본이 되는 직선과 원의 개념을 선정하였다.

우리의 분석으로부터 몇가지 결과를 도출할 수 있다. 우선 [Le]에서는 직선의 원의 개념 모두 직관적 개념에서 논리적 개념으로 이행함을 관찰할 수 있었다. 또한 직관적 개념은 감각적 활동에서 수학적 활동으로의 연결을 통하여 자연스럽게 도입하고 있었다. 이는 기하학의 역사적 순서에 부합할 뿐만 아니라 이론적 체계에 있어서도 감각적이고 직관적인 내용에서 추상적이고 논리적인 내용으로 나아간다는 점에서 동기부여나 흥미유발을 일으킨다고 할 수 있다.

다음으로 개념형성에 있어서 정의는 명확해야 한다. 이러한 점에서 직관적 개념과 논리적 개념이 명확히 구별되기 위해서는 무엇보다 직관적 정의가 명확해야 한다. [Le]에서는 원과는 달리 직선의 경우는 정의로서 명확히 언급하고 있지 못한 것은 아쉬운 부분이라 할 수 있다.

서론에서 제안한 바와 같이 본 연구와 관련하여 동일한 분석을 다른 교과서에 적용하여 비교하는 것은 흥미롭다. 그것은 직관적 개념과 논리적 개념 사이의 균형성을 고찰하는 자료로 활용할 수 있기 때문이다. 또한 기하학적 개념만이 아니라 다른 수학적 개념으로 확대하는 것도 흥미로울 것으로 보인다. 이를 통해 수학 전반에 있어서 통합적이고 균형적인 개념학습의 방안이 될 수 있을 것이다. 게다가 이러한 개념학습은 계산기능만이 아니라 개념이해와 응용력을 향상시킴으로써 문제해결학습에도 더욱 효과적임은 자명하다.

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ON THE EIGENVALUES OF THREE BODY IN EARLY 20th CENTURY
20세기초의 삼체문제에서 고유치에 관해서

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ABSTRACT

In the past 200 years, three body problem in classical approaches have been applied to Lagrangian and Hamiltonian. In the resolution of the three body problem more general representative of the way there is the statistical methods and the perturbation methods, here assume the foundation of resolution by the combination of analytical calculations of orbit and semi-analytic method.

For a closed solution three mass points in three dimension require 18 independent integrals. But it is possible to resolve 12 integrals: the six integrals of motion of center of mass, the three integrals of angular momentum, one energy integral, one elimination of the time and one elimination of the ascending node. So the integral system reduces to sixth order. Thus the n body problem constituting a system of order 6n can be reduced to a system of order $(6n - 12)$. In 1887 Bruns proved the ten classical integrals are the only independent algebraic integrals of the problem by the rectangular coordinates and all others can be formed by a combination of them. By extending this idea Poincaré establish no existence of new single-valued transcendental integral providing the masses of two of the bodies are very small compared the third. Concerning the stability of the solar system, the problem become the convergence or divergence of series then Poincaré firstly prove the general divergence of series selecting the achievements of Delaunay, Lindstedt, Gyldén. Poincaré open the new possibility of using a statistical approach in three body problem concluding an example of chaos in nature.

The key word of Poincare in celestial mechanics is periodic solutions, invariant integrals, asymptotic solutions, characteristic exponents and the non existence of new single-valued integrals.

Poincaré define an invariant integral of the system as the form which maintains a constant value at all time t, where the integration is taken over the arc of a curve and Y_i are some functions of x, and extend 2 dimension and 3 dimension.

Eigenvalues are classified as the form of trajectories, as corresponding to nodes, foci, saddle points and centre. In periodic solutions, the stability of periodic solutions is dependent the properties of their characteristic exponents.

Poincaré called bifurification that is the possibility of existence of chaotic orbit in planetary motion. Existence of near exceptional trajectories as Hadamard's accounts, say that there are probabilistic orbit. In this context we study the eigenvalue problem in early 20 century in three body problem by analyzing the works of Darwin, Bruns, Gyldén, Sundman, Hill, Liapunov, Birkhoff, painlevé and Hadamard.

지난 200년 전부터 삼체문제에 고전적 접근방식인 라그랑즈안과 하밀토니안이 적용되어왔다. 일반삼체문제는 통계적인 방식과 섭동적인 방식이 대표적인 해의 방법이고, 여기에는 해의 기초가 되는 궤도의 계산과 반 해석적 (semi-analytic) 방법의 결합을 가정하는 것이다. 삼체문제의 거장 프엥카레 (1854-1912) 는 1890년 오스카 2세의 60회 생일 기념수상논문과 그의 저서 천체역학에서 중요한 진보가 20세기 초에 이루어졌다.

결정론적인 행성의 궤도와는 다른 카오스적인 궤도를 증명하였다. 프엥카레의 천체역학의 주요 키워드는 주기해, 적분불변, 점근해, 적분불가능성으로 볼 수 있다. 프엥카레는 적분불변을 정의하기를 $\int \sum Y_i \, dx_i$ 가 모든 적분시간상에서 상수값을 가지며, 적분은 커브의 호상에서 이루어지고, Y_i 는 x의 함수라고 하였다. 이를 2차원과 3차원으로 확장하였다.

주기해에서는 고유값에 해당하는 특성지수에 따라서 주기해를 갖는다고 하였다. 이는 해당 급수의 수렴 여부에 따라서 안정성과 불안정성을 판단하였다. 즉, 주기해의 안정성은 특성지수의 성질을 조사하는 것과 동일한 것이다. 미분방정식 $dx_i/dt = X_i$ 의 주기해에서 $\phi_i(t) + \zeta_i(t)$ 에서 선형항 $\zeta_i(t)$ 만을 제외하고 무시하면, 여기서 나오는 특성지수는 $\zeta_n = e^{\alpha_k t} S_{nk}$ 로 쓰는 바 α_i 를 고유값 또는 특성지수라고 부른다. S_{ik} 는 주기함수로서 동일한 주기의 $\Psi_i(t)$ 시간 t 에 해당한다. 해밀톤계가 자율적이면, 특성지수의 크기는 반대부호를 갖는 동일한 크기의 짝을 이루어 두 개의 특성지수가 영 (zero) 이 된다. 물체의 고유한 성질을 나타내는 고유값은 영년방정식을 풀어서 구할 수 있고, n차 다항식의 근을 찾는 것과 같다. 대각행렬일 경우 고유값은 대각선의 행렬요소와 같다. 적분불변과 주기해에서 특성지수가 실수이며 양의 값을 가지면 불안정하다고 판정하였다. 불안정성은 바로 카오스라 불리는 것으로 리아프노프 지수로 불리는 정의된 수이고, 주기해의 근방에 돌아오지 못하는 것을 말한다. 원래 주어진 주기해에서 근소한 차이를 보이는 해를 고려한 바, 급수는 파라메터인 질량 μ 나 $\sqrt{\mu}, \mu$ 의 멱제곱에 의존하여전개된 다고 하였다. 삼차원 공간상에서 세 개의 질점은 18개의 독립적인 적분의 닫힌 해를 필요로 한다. 그러나 10개의 적분만이 가능하다. 즉, 고전적 적분 개념에 의하면 질량중심의 운동에 대한 6개의 적분, 각운동량에 대한 3개의 적분, 그리고 1개의 에너지적분이다. 여기서 시간과 승교점의 2개의 적분이 소거되어 18개의 적분식을 갖는 삼체문제는 6개의 적분으로 축약이 된다. 따라서 n체 문제는 $(6n-12)$ 계의 시스템을 갖는 적분식으로 축약이 된다. 1887년 브린스는 변수를 직각좌표계로 잡으면, 10개의 고전적 적분이 서로 독립적임을 증명하였다. 나머지 적분들은 상호조합에 의해서 이루어 질 수 있다고 하였다. 이 점에 착안해서 프엥카레는 세 번째 물체에 비해서 두 물체의 질량이 매우 작은계에서는 유일의 초월적분값이 존재하지 않는 다고 하였다. 수학적으로 삼체문제의 해답은 무한급수의 전개문제이지만 급수가 천천히 수렴함으로 사용하기 힘들다. 뉴턴이래 200년간 지속된 삼체문제의 해법문제는 프엥카레가 자연의 카오스 문제로 결론을 내림으로서 통계적 접근의 길을 열어놓았다. 분지라고 불리는 천체궤도의 카오스적 존재 가능성을 프엥카레는 예외적 궤도의 존재로 주장하였고, 이는 아다마르의 견해대로 우연에 의한 확률적 궤도의 존재를 말하는 것이다. 호모크리닉점의 존재는 삼체문제의 이중 점근해를 말하고, 이것은 궤적이 카오적임을 말해주는 것이다. 주어진 조건에 따라서 엑스포넨셜함수의 고유값인 특성지수가 계속 변함으로, 매우 작은 간격에서도 분지들은 얻게 되고, 원래의 주기와는 다소 멀어지는 것이다. 주기해의 안정성문제는 특성지수를 연구하는 것과 같다. 점변환이 규칙적임을 가정하고, 선형항만을 고려해서 근사치를 구할 수 있다. 프엥카레는 궤적의 거동이 선형변환의 고유값 성질에 의존하고 이 고유 값들과 서로 다른 특이점들 사이에 매우 밀접한 관련이 있음을 발견하였다. 프엥카레는 처음으로 적분불가능성을 증명하였다. 이를 위해 일정시간동안 삼각급수와 멱급수를 이용한 근사해를 시도하였고, 뉴턴의 법칙이 제한된 시간뿐 만이 아니라 영원히 적용이 될 수 있을지에 대한 이론적 근거를 제공하였다. 여기서는 주로 프엥카레가 제기한 뉴턴적 세계관에 입각한 삼체문제를 살펴보는 바 그 이유는 프엥카레가 삼체문제의 연구에서 당대뿐이 아니라 그 이후에도 큰 영향을 미쳤기 때문이다. 뷔른스, 질덴, 순드만, 힐, 다윈, 벌코프, 하이테커, 아다마르등의 이론전개는 프엥카레의 이론과 불가분의 관계를 가는다.

1 시대적 맥락

세 질점 또는 세 천체가 중력하에서 어떤 운동을 그리는가가 삼체문제의 핵심이다. 프엥카레는 자연에서 삼체문제는 카오스의 한 예라고 결론내리면서 통계적 접근에 새로운 길을 열었다. 프엥카레의 천체역학은 1890년 오스카 2세의 60회생일 기념수상논문인 삼체문제의 확장으로 이루어졌다 .[1] 고전물리학의 한계가 들어난 20세기 초 프엥카레는 그의 천체역학에서 전통의 결정론적 우주관과는 다르게 행성운동이 카오스 적임을 증명하였다. 주기해, 적분불변, 점근해, 적분불가능성은 프엥카레 천체역학의 주요내용이다. 삼체문제의 고전적인 접근방식인 라그랑즈안과 하밀토니안 방식은 200년 전부터 계속되어져왔다. 일반삼체문제의 해는 통계적인 방법과 섭동적인 방식이 대표적인 해의 방법이고, 이것은 해의 기초가 궤도계산과 반 해석적 (semi-analytic)

[1] H. Poincare, Sur le probleme des trois corps et les equations de la dynamique, Acta Mathematica 13, 1890, pp. 1–270; OEuvres VII, pp. 262–479.

적 방법의 결합을 가정하는 것이다.

삼체문제의 9개 미분방정식중에서 속도상수와 위치상수를 고려하면 18번의 적분을 필요로 한다. 고전적으로 10개의 적분이 알려져 있고, 질량중심에 관한 6개의 적분, 각운동량에 관한 3개의 적분, 1개의 에너지 보존법칙에 의한 적분이다. 여기에 시간과 교점소거에 의해서 2개의 적분상수가 알려져 12개의 적분만이 가능하다. 3체 문제는 적분상수가 6개가 부족하므로 일반해를 얻을 수 없다. 계가 음의 에너지를 갖게 되면 모든 물체는 중력에 묶여있다. 계가 양의 에너지를 갖는 다면 물체를 구성하는 군은 불안정하며, 팽창한다. 뷔룬스, 펭르베, 프엥카레는 더 이상의 닫힌 해가 존재하지 않음을 보였다. 수학적으로 무한급수의 전개이나, 급수가 천천히 수렴해서 사용하기 힘들다. 실제로 수치적분이 정확하여 초기위치와, 초기속도, 그 순간의 힘의 세기를 앎으로서 다음순간의 위치가 계산된다.

태양 주의를 도는 지구와 행성들은 엄밀히 말해서 2체 문제는 아닌 것이다. 다른 행성에 의한 중력은 추가적인 힘이 작용해서 행성을 타원궤도에서 밀어내려고 한다. 그래서 18세기 과학자들이 추가적인 힘이 지구가 태양에 붙어버리든가, 아니면 태양계 밖으로 밀어내지 않을까 하고 걱정했던 것이 무리가 아니었다. 이러한 우려는 지구의 나이가 수 천년뿐이 안 된다고 여겨졌을 당시는 당연한 것이었다. 더구나 이때는 지구궤도에 미치는 다른 행성영향의 가능한 조합이 발생하지 않는 다고 여겨질 때였다.

2체 문제는 뉴턴의 만유법칙이래로 오일러에 의해서 완전한 해를 얻었다. 오일러는 제한삼체문제로서 동일직선상에서 태양-지구의 질량중심(두 고정점)을, 질량을 무시할 수 있는 달을 가정하여 5차방정식의 근이 달의 일정한 합충에 해당하는 해를 얻었다. 두 고정점 방식은 1903년 칼리에가 다시 다루었고, 최근에 인공위성의 기준궤도 축에 간접적으로 응용이 되고 있다. 그러나 뉴턴의 운동법칙에 따라서 임의의 질량과 임의의 초기조건에 따라서 푸는 3체문제의 일반해는 없다. 다만 1772년 라그랑즈는 삼체문제의 특수해로서 임의의 질량이 동일직선상의 평형점(L_1, L_2, L_3)에 놓일때와 정삼각형의 꼭지점(L_4, L_5)에 있을때 만 해당하는 특수해를 구했을 따름이다. 134년이 지난후 태양-목성을 밑변으로하는 정삼각형의 꼭지점, 즉 삼체문제의 해에 해당하는 지점에 소행성이 존재하고 있다는 것이 관측에 의해 확인되었다. 여기서 고띠에(A. Gautier)가 달의 이론에 대해서 프랑스 학사원으로부터 오일러와 라그랑즈가 공동수상한 주제인 삼체문제에 의한 달의 이론을 해석한 대목을 1817년의 그의 저서 "삼체문제의 역사적 시론" 9장 첫 부분에서 살피는 것이 맥락을 파악하는데 있어서 의미 있는 일이라고 생각한다.

> 오일러와 라그랑즈가 동시상황에서 삼체문제의 미분방정식에서 중요변수를 직각좌표계에서 극좌표로 바꾸려한 것은 특이한일이다. 1772년 상을 받은 라그랑즈 방법이란 물체 A, B, C의 궤도를 결정하는데 있어서, 이 물체들의 상호간 거리인 r, r', r''을 이용하는 것 이었다. 라그랑즈의 논문은 두 부분으로 구성되어있다; 첫 논문은 삼체문제를 일반적으로 다루는 2장으로 구성되고, 둘째 논문 특별히 달의 이론을 다루는 바 역시 2장으로 구성이 되어 있다. 첫 부분에서는 물체 A주위를 도는 물체 B와 C의 상대적인 궤도를 결정하는 것이고, 이어서 물체 B주위를 도는 C의 궤도를 결정하는 바, 라그랑즈는 4개의 원적분식을 정밀하게 만들었다; 이어서 6개의 처음원시방정식을 세 개의 대칭방정식으로 줄였다. 거리를 시간에 대한 2계미분방정식과, 질량과 거리함수의 이중적분기호를 생략하게 되었다. 이식은 원시적분식보다 더 복잡한데, 그 이유는 각각 2개의 적분기호가 필요하기 때문이다. 그러나 어떠한 근도 포함하질 않기 때문에 라그랑즈는 이것을 달 이론에 적용을 한 것이었다. 직각좌표계에서 나타나는 r, r', r''의 시간에 대한 거리를 대수적 연산으로는 푸는 것이 불가능 하지만, 적분을 통해서는 가능한 것이다. 즉, A, B, C의 물체가 통과하는 면을 고정시키면, 이 물체들이 지나는 A주위를 도는 B와 C의 궤도식을 세우게 된다; 라그랑즈가 증명한 바는 이 고정된 모든 면 들 중 처음의 두 물체가 그리는 가장 단순한 면을 찾아서, 물체 B와 C가 그리는 궤도상의 서로의 위치를 나타내는 공식을 결정할 수 있게 된 것이다.

클레로, 달랑베르, 오일러, 라그랑즈, 라플라스에 대한 천체운동론은 따통(R. Taton)과 윌슨(C. Wilson)이 중심이 되어 요약되어 상세한 내용은 여기서 생략한다.[2] 자코비는 1843년 '삼체문제의 교점소거'라는 논문에서

[2](1) R. Taton and C. Wilson, The General History of Astronomy, (Cambridge University press, 1989)Vol. II. -(2)C. Wilson, The dynamic of Solar System, Companion Encyclopedia of the Theory and Philosophy of the Mathematical Sciences(Routledge, Newyork, 1994)vol2, pp. 1044–1053. C Wilson, The three-body problem; Companion Encyclopedia of the Theory and Philosophy of the Mathematical Sciences(Routledge, Newyork, 1994)vol2, pp. 1059–1061.

일반삼체문제를 태양중심의 두개의 가상의 물체의 운동으로 환원하였다. 두 가상의 물체의 질량, 위치, 속도를 좌표중심과 원점이 일치 하도록 하여 힘 함수로부터 가속도를 얻도록 하였다. 운동에너지가 보존되고 면적이 보존되도록하여, 라그랑지안 형식을 얻었다.[3]

$$\mu_1 x_{1i} = \partial U/\partial x_{1i} \quad \mu_2 x_{2i} = \partial U/\partial x_{2i} \quad i = 1, 2, 3$$

여기서 가상질량 μ_1과 μ_2는 임의의 상수에 의존하고, 이 값이 행성의 질량과 좌표와 거의 동일하도록 하였다. 이를 바탕으로 하여 베트랑(J. Bertrand)은 1852년 첫 번째 가상질량을 m_1과 m_2의 질량중심에 위치하도록 하여, $m_1 m_2/(m_1+m_2)$와 같도록 하였다. 두 번째 질량은 m_3의 원래위치에 놓아서 $m_3(m_1+m_2)/(m_1+m_2+m_3)$와 같도록하였다. 자코비처럼 베트랑도 계를 6차로 줄였다. 이 뿌리는 라그랑지안이고, 1808년 라그랑즈는 섭동함수의 궤도매개변수에 대한 편미분이 이 매개변수에 대한 시간의 미분과 선형함수임을 발견하였다.

$$\partial H/\partial a_j = \sum_{j \neq i} [a_i, a_j](da_j/dt)$$

H는 섭동함수이고, 괄호안에 있는 항은 시간에 무관한 함수로 라그랑즈 괄호라고 하는 것이다. 라그랑즈는 궤도매개함수의 선택에 따라서 단일 시간 도함수가 됨을 보였다. 이에 대한 역관계를 1809년 포아송이 유도하였다.

$$da_j/dt = \sum_{j \neq i} [a_i, a_j](\partial H/\partial a_i)$$

괄호안에 있는 항은 시간과 무관한 함수로 포아송 괄호라고 하는 것이다. 하밀톤은 1834년과 1835년에 발표한 두 논문에서 이 형식을 확장하여, 이 전에는 힘 함수를 좌표만의 항으로 정의하였는데, 운동량 P_i와 일반화된 좌표 Q_i를 임의의 함수 H로부터 편미분하였다. 이 함수는 시간에 무관하고, 최소작용원리를 적용하여, n쌍의 1계의 편미분방정식에서 n개의 자유도를 갖는 운동방정식을 유도하였다.

$$\dot{P}_i = -\partial H/\partial Q_i \quad \dot{Q}_i = -\partial H/\partial P_i$$

자코비는 이것을 정준운동방정식으로 불렀고, 하밀톤은 이 함수의 적분이 주 함수 S라고 불리는 편미분으로 나타낼 수 있음을 보였다. 드로네이(Charles Delaunay)는 20년의 노력 끝에 하밀톤니안을 급수전개하여, 하밀토니안 함수에서 주기항을 소거하여 적분한 끝에 달의 경도, 위도, 시차를 무한급수로 표현한 바, 시간 t가 주기항의 편각으로 만 나타나게 하였다. 드로네이의 결과는 1초의 정밀도를 보였고, 급수의 느린 수렴으로 제한되었다. 그러나 영년항이 주기항의 편각밖에 t가 나타나는 것을 피 할 수 있었다. 1874년 뉴콤브(Simon Newcomb)는 삼체문제가 주기항으로만 이루어진 무한급수에 의해서 풀릴 수 있음을 증명하였다. 1883년 린드스테드(A. Lindstedt)는 라그랑즈 방정식으로부터 삼체문제에 대한 급수해를 구할 수 있음을 보였다. 그러나 이 급수들이 수렴하지 않는 비 주기적임을 보여 프엥카레의 카오스론에 영향을 주었다.[4] 앞서 언급한대로 프엥카레는 1882–1884년에 함수가 임의 큰 값을 갖게 되면 급수가 균일하게 수렴하지 않을 수 있음을 증명하였다. 1884년 뷔른스는 급수가 장주기항에서 수렴과 발산사이를 진동하면, 상수는 시간에 따르는 계수가 매우 작은 값에서도 변할 수 있음을 증명하였다. 이어서 1887년 뷔른스는 삼체문제문제의 대수함수가 보존되지 않는 양일 수 있음을 보였다.[5]

힐(George William Hill)은 처음으로 달 이론에서 무한 행렬식을 도입했다. 힐은 제한 삼체문제와 회합좌표계를 이용하여, 태양, 지구 질량을 무시한 달의 주기해를 구하였다. 프엥카레는 힐의 주기해에 대한 아이디어를 정리한바, 이는 크로네커정리를 삼체문제에 응용한 것이고, 무한수의 주기해의 존재를 밝힌 것이었다.

개략적으로 이러한 천체역학자들이 프엥카레의 '천체역학의 신이론'이 출간되기 이전의 삼체문제의 중요이론을 냈다고 볼 수 있다. 구체적으로 프엥카레의 천체역학이 출간된 맥락을 살펴보면 다음과 같다. 프엥카레의 천체역학의 신 이론이라는 저서가 나온 시기는 상대론과 양자론이 나오기 직전의 시기였다. 프엥카레는

[3]C Wilson, The three-body problem; Companion Encyclopedia of the Theory and Philosophy of the Mathematical Sciences(Routledge, Newyork, 1994)vol2, pp. 1059–1061.

[4]http://en.wikipedia.org/wiki/Anders_Lindstedt.

[5]http://www.gap-system.org/~history/Biographies/Bruns.html.

스웨덴의 왕 오스카 II세의 60세 생일기념으로 낸 삼체문제의 공개적인 현상응모에서 1887년 수상했다. 심사위원은 당대의 수학의 대가인 바이어스트라스, 에르미트 그리고 Acta Mathematica의 창간자인 미타그-레플러로 구성이 되었다. 삼체문제와 관련된 제반의 수학적 문제들을 J. B. Green 박사가 엄밀하게 분석하여 본 논문에서 상당부분 참고하였다.[6]

1902년 미국천문학자 물톤(F. R. Moulton)은 프엥카레의 1890년 Acta Mathematica에 발표한 삼체문제에 대한 이논문에 대해서 다음과 같이 평하고 있다: "프엥카레가 사용한 방법은 천체역학에서 이전에 나온 어떠한 논문보다도 강력하고 비교할 수 없을 정도로 심오하고, 과학발전에 신기원을 이루었다".

1925년 미국수학자인 벌코프(George Birkhoff)는 "프엥카레는 삼체문제에서 동역학에서 처음으로 적분불가능 문제를 다루었다" 고 하였다.

필립 호르메스(Philip Holmes)는 프엥카레의 논문이 동역학계의 최초의 질적인 이론이라고 하였다'.

프엥카레의 천체역학의 신이론(I권 1892년간, II권1893년간, III권 1899년간)은 1282쪽에 달하는 바, 여기서 동력학계의 카오스적 거동을 수학적으로 다루고 있다. 오스카 수상논문의 오류가 발견돼 큰 물의를 일으킨바 있지만, 1890년 수상논문의 오류를 수정해서 Acta에서 발표했고, 천체역학의 신 이론에서 카오스적 궤도라고 불리는 호모크리닉 점(이중점근해)에 관해서 다루었다. 호모크리닉점의 발견은 오늘날 카오스 이론의 창시자로서 프엥카레의 업적을 높이고 있다. 프엥카레는 19세기말 힐의 연구에 영향을 받았고, 점근해에서 급수의 수렴에 대한 오류에서 일련의 물의를 일으켰다. 핀란드의 질덴(Hugo Gylden)같은 천문학자는 이에 대해서 이의 주장했지만, 대부분의 수학자나 천문학자들은 프엥카레의 업적을 인정하고 찬사를 보내는 입장이었다. 당시 저명수학자 바이어스트라스나 에르미트까지도 이를 신중하게 여길 정도였다.

프엥카레의 천체역학 신 이론은 위의 관련 인물 이외에도, 리아프노프, 아다마르, 펭르베, 다윈, 레비 시비타, 순드만, 모르스, 멜니코프등에 영향을 주었고, 해밀톤계에 준주기해에도 영향을 끼친 KAM(Kolmogrorov, Arnold, Moser)이론이 그 이후에 나왔다.[7]

영국의 화이테커는 1899년 삼체문제의 해에 대한 발전의 보고서라는 제목으로 요약 정리하였고, 이후 1904년 교과서적 저서인 '입자와 강체의 동역학적 해석[8]'에서 약술하였다. 프엥카레의 삼체이론은 현대의 삼체문제의 시작이었다는 점에서 의의가 깊다.

2 프엥카레의 이론과 20세기 전후의 삼체문제

수학사가 벨이 마지막 만능의 수학자라고 평했던 앙리 프엥카레(Henri Poincare; 1854-1912)는 삼체문제의 주기해를 주장하면서, 궤적의 안정성을 엄밀히 논하였다. 앞서 소개한대로 주기해, 적분불변, 점근해는 프엥카레 천체역학의 주요형식이다. 예외적 궤도의 주장은 아다마르의 견해대로 우연에 의한 확률적 궤도의 존재를 말하는 것이다. 아인쉬타인은 "신은 주사위 놀이를 하지 않는다" 고 하였다. 프엥카레는 최고의 수학자에게 있어서 확률이란 있을 수 없다고 하였다.[9]우선 벨이 평하는 프엥카레의 강력한 무기는 다음과 같다.

> 뉴턴과 뉴턴의 후계자 이래로 천문학은 수학자들이 해결해 낼 수 있는 것보다 많은 문제를 제공해 주었다. 19세기 후반까지 천문학을 공격하기 위한 수학자의 무기는 뉴턴, 오일러, 라그랑즈 와 라플라스 같은 수학자들이 발견한 것의 변형에 지나지 않았다. 그러나 19세기를 지나면서, 특히 코시가 복소변수의 함수론과 무한급수의 수렴연구 등을 발전시키면서 아직까지 사용되지 않은 무기로 쌓인 거대한 병기고가 순수 수학자의 노력에 의해서 이루어 지게 되었다. 프엥카레와 같은 해석학이 자연적 사고로부터 발생하는 사람에게는, 이 거대한 사용되지 않은 수학의 이 거대한 파일(pile)이 천체역학이나 행성진화의 중대한 문제에 새로운 공격에 사용될 것처럼 보인다. 그는 산더미로부터 뽑아내 선택하고 개량하여, 100년동안

[6](1) June Barrow Green, Poincaré and the Three Body Problem(American Mathematical Society, Providence, 1997); (2) Barrow-Green, June (2010). The dramatic episode of Sundman. Historia Mathematica, 37(2), pp. 164–-203; (3) Barrow-Green, June (2005). Henri Poincare, memoir on the threebody problem (1890). In: Grattan-Guinness, I. ed. Landmark Writings in Western Mathematics 1640-1940. Amsterdam: Elsevier, pp. 627-–638.

[7]앞의 문헌, 6-(1), chap. 1.

[8]E. T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies(Reprinted, Cambridge University Press, 1970).

[9]일리야 프리고진, 이자벨 스텐저스(신국조 옮김), 혼돈으로부터의 질서(고려원 미디어, 1993) 360쪽.

> 공격에 사용되지 않은 자신의 새로운 무기를 발명하여 이론천문학을 공격하여 큰 인기를 끌게 될 것이다. 그의 현대식 공격은 전역(campaign)에서 참으로 천체역학의 대부분의 전문가들에게는 초현대식이므로, 프엥카레의 공격이후 40년이 지난 오늘날 까지도 프엥카레의 무기에 정통한 사람이 없다는 것이고, 어떤 수학자는 실질적 공격에 무력함을 암시하고 있다.[10]

오늘날 우리는 동력학계들이 하나의 매개화(parametrized)된 족이 구조적으로 불안정한 지역을 통과할 때 분지(bifurcation)가 발생한다고 한다. 직관적으로 말해서, 분지가 발생하는 곳은 서로 다른 거동을 보이는 동역학계의 집합이 분리되는 곳에서 발생이 되고 있다. 프엥카레는 1885년 분지라는 용어를 도입하였는데, 회전하는 유체의 각운동량이 변하는 모형의 천이를 기술하기 위함이었다. 아다마르는 주기해의 수의 변화나 선형적 안정성의 변화를 설명하기위해서 분지라는 용어를 사용하였다. 이것은 현재 국소적 분지론으로 불리고 있다. 프엥카레는 천체역학의 핵심주제는 삼체문제라고 하였고, 삼체문제의 연구는 뉴턴력으로만 천체운동의 설명이 가능한지를 확인하는데 있다고 하였다. 여기서 이 문제에 대한 프엥카레의 결론은 먼저 살펴보면 보면 다음과 같다.

> 연구자들이 가장 큰 관심을 가진 문제 중의 하나는 태양계의 안정성에 관한 것이다. 진솔하게 말한다면 이 문제는 물리학의 문제라기보다 수학적인 문제 인 것이다. 비록, 우리가 일반적이고 엄밀한 증명을 발견한다고 할 지 라도, 우리는 태양계가 영원하리라고는 결론을 내릴 수 없을 것이다. 사실 뉴턴력 이외의 다른 힘들에 의해서 지배를 받을는지 모르는 것이다.[11]

프엥카레(Henri Poincare; 1854-1912)의 중력장에서의 삼체문제의 일반해의 증명에서는 닫힌 수학적 해를 갖지 못하는 것으로, 단지 해의 일 단계에 해당된다고 볼 수 있다. 프엥카레에게 실제로 중요했던 것은 해석적인 해가 존재하질 않더라도 계의 운동을 어떻게 기술하느냐에 있었다. 위상공간을 사용함으로서 프엥카레는 서로 다른 질량비와 초기조건 하에서 삼체문제의 다루는 방법을 고안했다. 미분방정식으로 일정한 초기조건을 갖는 삼체문제는 주기적인 운동을 수행한다. 그렇지만 초기조건을 조금 변경하면 삼체의 궤도는 전혀 달라진다. 새로운 초기 조건하에서 삼체의 궤도가 미분방정식에 의해서 예상되기도 하지만, 때때로 원래의 방정식에서 예상 할 수 없는 혼돈(chaotic)이 일어난다. 그래서 프엥카레는 중력장내 에서의 삼체가 뉴턴식의 결정론적인 궤도와 예측할 수 없는 비뉴턴적 궤도로 나타남을 발견하였다. 이것은 사고실험으로 쉽게 설명할 수 있는데, 베아링에 강체봉을 달고 어떻게 미느냐에 다라서 강체봉은 회전 할 수도 있고 좌우로 진동 할 수 있는 것 과 같다.

프엥카레의 삼체문제 연구 이후로 많은 사람들이 주어진 문제의 파라메터 공간상을 설정하여 잘 정의된 닫힌 해가 존재하느냐의 여부를 연구하였다. 그래서 삼체문제의 혼돈해(chaotic solutions)가 존재함을 발견되었는데, 삼체중 일체가 충분한 속도를 얻으면 이체를 탈출한다는 것이다. 일체(一體)의 탈출은 삼체계(三體界)가 음의 에너지 값을 가질 때 일어난다. 프엥카레도 이 사실을 일고 있었고 혼돈해를 태양계의 안정성과 관련시키려 했다. 삼체계가 음의 에너지를 가짐으로서 일체가 탈출하는 것은 불안정한 상태를 의미하는 것이었다. 그래서 프엥카레는 삼체문제를 태양의 안정성과 관련시켜서 다시 검토 작업에 착수하였다.[12]

19세기 초 라플라스(P. S. Laplace; 1749–1827)가 태양계의 안정성 여부를 판정하기 위하여 섭동론을 이용하였다. 라플라스는 다체문제로서 태양계를 다루었고, 태양의 영향력뿐 만 이아니라 행성들의 상호간 영향력을 다루었다. 라플라스의 목적은 이러한 행성간의 섭동이 영년 발산(secular divergences=행성의 탈출)에 영향을 미치지 못한다는 것을 증명하는 일 이었다. 라플라스는 프리에 급수 전개를 이용하여, 행성 상호간의 섭동은 주기적으로 발생하므로, 태양계는 안정하다고 증명하였다.[13] 라플라스는 1차 근사에 의하여, 태양계의 안전성을 증명하였고, 프와송은 2차 근사에 의하여 태양계의 안정성을 증명하였다. [14] 여기서 필자는 2008년

[10] E. T. Bell, Men of Mathematics, (Victor Gollancz, London, 1939)pp. 599–600.

[11] Henri Poincaré, Henri Poincaré, Les méthodes nouvelles de la Mécanique Céleste; 영역본, New Methods of Celestial Mechanics, Edited and Introduced by Daniel L. Goroff, I 80.

[12] Victor G. Szebehely, Hans Mark; Adventure in Celestial Mechanics, John wiley & son, 1998, 1장과 13장.

[13] P. S. Laplace, Mécanique céleste, vol I-V.

[14] Jacques Hadamard, L'oeuvre mathématique de Poincaré; The Mathematical Heritage of Henri Poincaré, Proceedings of Symposia in Pure Mathematics(American Mathematical Society, usa, 1983)volume 39 part 2, p. 422.

8월의 수학사 학회지에서 이미 라플라스가 물분자력(입자력)의 세기가 중력에 비해서 대단히 큼에 놀란 바 있다고 기술하였다.

프엥카레는 한 점의 궤도가 안정하다함을 그 점이 원래의 임의의 위치근방에 무한하게 자주 되돌아오는 것이라 하였고 프와송으로 부터 기인한다고 하였다. 라플라스나 프와송의 이같은 추론은 영구적 추론이 아닌 일시적 추론인 바, 프엥카레는 라플라스의 연구를 다시 검토하였다. 그래서 태양계가 절대적으로 안정한 것이 아니라고 추론 하였다. 이런 맥락에서 천체역학의 신이론이 출간되기 직전인 1890년에 Acta 에 출간되고, 1887년에 오스카 II세의 60세 생일 기념으로 프엥카레가 수상한 삼체문제를 보자. 1885년 Nature지에는 다음의 광고기사가 났다. 오스카 II세 전하의 1889년1월21일 생일 기념으로 다음문제의 해를 상금으로 내걸었다. 수리과학의 진보를 위한 폐하의 지대한 관심으로 다음문제의 명쾌한 증명을 기대하면서... 심사위원회는 다음 문제를 공모한다.

> 뉴턴의 법칙을 따라서 상호간에 인력이 작용되는 임의의 다체계가 주어졌을 때, 두 개의 입자가 전혀 충돌하지 않는 다는 가정하에서 각각의 점을 좌표로 나타내고, 시간에 대한 기지의 함수에 일치해서 진행되는 급수를 전개하고, 시공간상에서 균등하게 수렴하는 것을 찾아라.[15]

바이어스트스의 이 수상논문의 코멘트에서와 같이 프엥카레의 해가 완전하게 끝을 보지는 못하였지만 천체역학의 새로운 시대를 열었다고 논평하였다. 1912년 순드만이 출제된 문제를 $n = 3$인 삼체상에서는 해결했다고 볼 수 있고, 수학적으로는 $n > 3$인 n체에 대해서는 1990년대에 왕기동에 의해서 해결(특수하게) 되었다고 볼 수 있다.

벡터장이 사라지는 점을 평형점 또는 특이점이라고 한다. 라그랑즈점, 말 안정점등과 같은 특이점의 존재는 초기조건에 따름을 프엥카레는 강조하였다. 동일미분방정식에 대한 서로 다른해에 대한 커브의 관계를 분석하면서, 특이점근방에서의 이들 커브에 대한 거동을 검토하고 국부적(local)분석을 시작하였다. 특이점에 대해서 네 개의 서로 다른 타입이 있을 수 있음을 보였다; 무한수의 해곡선이 통과하는 노드. 단지 두 개의 해곡선이 통과하는 말 안장점으로서 두 곡선은 이웃 해곡선에 대해서 점근선처럼 작용한다. 셋째로 로가리즘한 나선형태로 해곡선에 접근하는 초점으로서 이고, 넷째로 중심으로서 폐해곡선이 서로서로를 둘러싸고 있는 것이다. 이러한 특이점의 분포를 $N + F - C = 2 - 2p$로 주어진다고 하였다. N은 노드수이고, F는 초점수, C는 말안장수이고, p는 곡면 S상에서 제너스(genus)수 이다.[16] 라그랑즈와 라플라스의 결정론적 궤도론이 프엥카레가 카오스적임을 제기하여 확률적인 수리물리적 궤도론을 제기한 바는 의의가 깊다고 하겠다.[17]

2.1 특성지수(고유값)와 주기해

본 논문에서는 주기 해를 중심으로 프엥카레의 천체역학을 살펴보기로 한다. 1889년 프엥카레는 제한삼체문제에서는 쟈코비안 이외에는 적분이 존재하지 않는다고 하였다. 그 후 제한 삼체문제의 주기 해에 관심을 기울였다. 1890년 수상한 에세이에서 처음으로 회귀정리를 증명하였다. 즉, 상공간의 임의의 지점 r_o가 매우 작다고 할지라도, 이 지점을 통과하는 궤적이 무한수로 존재한다는 것이다. 더구나 r_o에서 이 궤적들이 나타나지 않는 점들은 r_o보다 무한히 작은 부피를 갈게 되는 것이다. 그래서 프엥카레는 주어진 주기해로부터 다소 다른 해를 고려하게 되었다. 프엥카레는 섭동질량 혹은 섭동질량에 비례하는 인자인 파라메터μ로 전개한 하밀토니안을 가정하였다.[18]

$$F = F_o + \mu F_1 + \mu^2 F_2 + \cdots$$

$\mu = 0$이면, 즉, 섭동을 무시하면 해 $x_i = \phi_i(t), y_i = \psi_i(t)$는 주기함수가 된다. 여기서 프엥카레는 오늘날 자주 쓰는 하밀토니안기호 H 대신에 F를 썼다. 이 해는 알려져 있는 바, 케플러의 법칙에 따라서 타원궤도를 돌게 되는 것이다. 이것은 서로 간에 통약 가능함을 말한다. 하다마르는 이런 무한해가 쟈코비안이 영이 되는 난점을

[15]앞의 문헌, 6-(1), p. 229.
[16]앞의 문헌, 6-(1), pp. 31-32.
[17]앞의 문헌, 3
[18]위와 같은 문헌.

유발한다고 해석하였다. 따라서 음함수에 대한 심화연구가 요구되는 것이다. 기하학적으로 말해서, 구한 해를 정의하는 초기좌표들에서 μ값에 이 초공간의 점을 연결하면, 해가 주기로 나타나는 방정식은, $\mu = 0$인 영역에 위치한 연속된 어떤 부분인 다양체를 정의하게 되는 것이다. 그래서 μ가 변하는 방정식의 해는 연속인 급수를 얻을 수 없게 된다. μ의 해석에 따라서 초공간의 커브를 문제에 해당되는 다양체로서 얻게 되는 것이고, $\mu = 0$ 인 정의역을 절단지점에서 이 다양체의 중복점이 결과로 나타나는 것이다.[19] 프엥카레는 μ가 작은 값을 갖게 되면, 주기 해에서 항이 다음과 같이 추가된다.

$$x_i = \Phi_i(t) + \zeta_i, \quad y_i = \Psi(t) + \eta_i$$

여기서 $\zeta_i = S_i exp(\alpha_k t), \quad \eta_i = T_i exp(\alpha_k t)$가 되고, S_i와 T_i는 t의 주기함수이다. $\alpha\alpha_k$를 특성지수라고 하는 것으로, 대수방정식의 근이며, 안정하기위한 해로는 허수이어야만 한다. μ가 작거나 정수의 제곱에 따라서 전개될 때 해결책으로 고안한 것이 바로 분지형식 또는 안정성의 계수인 것이다. 이는 무한의 다양체성에 대응하기 위한 것이다. 분지형식은 이들 다양체의 이중점에 해당하고, 결과적으로 구축하려는 핵심요소를 구성해 주는 것이다. 안정성의 요소들은 바로 특성지수들의 제곱 값인 것이다. 이는 임의의 주기 해에 대한 리아프노프 안정성에 달려있는 것이다. 행성의 모습에 대한 이론에서와 마찬가지로, 분지형식에 나타날 때마다 매번 일종의 안정성의 교환이 나타나는 것이다. 게다가 동일한 차원에서 발생하는 분지의 관점에서 반대가 되는 경우라도, 즉, μ가 변화하는 중인 경우라면 주기 해는 사라지게 되는 것이다. 주기해의 소멸은 대수방정식에서 실근들의 짝으로 쌍을 이루는 것과 같은 것이고, 같이 사라지는 해는 다른 안정성을 갖게 되는 것이다. 초공간상을 그리는 곡선의 호에서 동일한 방향으로 μ가 계속 변한다면, 안정성의 교환정리에서는 반대로 역 과정을 받아들이는 것이다. 즉, 분지에 의해 진행되는 안정성의 변화만 나타나게 되는 것이다. 이에 프엥카레는 특성지수 역할에 또 다른 예로서 제2 장르(second genre)의 주기 해를 들어 적용하고 있다. 이는 결정된 주기 T 근방에 k 회전 후에 생긴 주기 kT를 말한다. 이것은 첫 번째 주기에 새로운 주기를 접합하는 것을 말하고, 더 큰 이중점을 말한다. 이것은 최소작용과 관련된 것으로 극대극소의 연구에서 난점을 적분불변으로 해결하게 된 것이다. 이 경우의 특성지수는 $2\pi i/kT$의 배수가 되는 것이다. $k =$정수이므로 $2\pi i/kT$의 배수는 원하는 만큼 서로 서로 근접시킬 수 있다. μ에 따라서 특성지수가 계속 변함으로, 매우 작은 간격에서도 분지들은 얻게 되고, 원래의 주기 T와는 다소 멀어지는 것이다.[20]

프엥카레에 있어서 적분불변이란 동역학적인 의미에서 임의의 시간 T에서 차지하는 부피 V가 일정함을 말하는 것으로, 시간변화에 따른 부피가 일정함을 의미한다.

프엥카레의 점근 해란 기본형의 적분이 불가능함으로 해서 만든 것이다. 점근해란 무한으로 주기 해에 접근하거나 주기 해에 무한으로 멀어지는 것을 말한다. 이 중점근해란, $t = -\infty$에서 주기 해에 무한히 근접함을 말하고, 멀어졌다가 다시 근접하여 $t = +\infty$서 주기 해에 근접하는 것으로 말한다. 프엥카레의 삼체문제는 결정론적인 행성궤도론에 확률을 도입하여, 분지라는 용어를 도입하여 카오스론의 수학적, 동역학적인 시초을 마련하였다는 점에서 의의를 찾을 수 있고, 이후 '천체역학의 신이론' 은 주기적 궤도에 대한 이후의 천체역학자들이 후속연구를 유발하였다.

1901년 레비 시비타(Tullio Levi-Civita)는 제한삼체문제에 있어서 질량없는 질점과 다른2체의 평균운동이 통약가능하면, 그 운동은 불안정하다고 하였다. 다윈(George Darwin), 물톤(Forest Ray Moulton), 브라운(E. W. Brown), 스트롱그렌(Elis Stromgren)등이 제한삼체문제에 대한 여러족을 연구하였다. 1912년 버코프(George David Birkhoff)는 제한삼체문제에 있어서 모든 안정한 궤도는 어느 정도의 회귀적 성질이 있다고 하였고, 이러한 성질에 따라서 점근적으로 접근했다가 멀어지는 것이라고 하였다. 버코프는 프엥카레의 연구를 이론 동역학적으로 깊이 연구하였다. 아다마르(Jacque Hadamard)는 1901년의 논문에서 측지적 궤적의 거동은 적분상수의 산술적 불연속성에 의존할지 모른다고 하였다. 아다마르는 태양계의 안정성을 논하는 것은 천체역학에서 좋은 질문이 아니라고 하였다. 태양계의 안정성에 대한 연구를 음의 곡률을 갖는 측지적 곡면과 관련된 유사한 질문으로 대체한다면, 안정한 궤도일지라도 초기조건의 미세한 변화에 의해서, 닫힌 측지선의 점근적 궤적과 같은 완전히 불안정한 궤도나 더 일반적인 임의 궤적으로 바뀌게 될 것이라고 하였다. 천문학에

[19]문헌 13번, p. 416.
[20]문헌 10, Chapitre XXX-XXXII

서 초기조건이란 물리적으로만 알려져 있는 것이고, 관측에 의한 오류는 줄일 수 있지만, 오류를 없앨 수는 없는 것이다. 관측상의 작은 오류일지라도 결과에서는 전부이자 절대적인 것이다. 1912년 순드만(Karl Sundmann)은 삼체문제는 풀릴 수 있는 것이지만, 급수가 천천히 수렴하여 실제로 사용하기엔 어렵다고 하였다. 순드만은 일반삼체문제에 있어서, 운동이 단일 평면에 구속되지 않는다면 삼체의 상호간 거리는 일정한 양(positive)값을 항상 초과할 것이라고 하였다. 순드만의 이론이 나온 후 40년 후, 차지(Z. Chazy)는 다음과 같이 분석하고 있다.

> 순드만의 해는 이미 삼체간의 충돌과 근접에 대한연구를 촉발하였다. 그리고 삼체간의 궤적에 대한 무한의 분지에 대한 연구를 유발하였다. 이미, 특이궤적의 결정은 삼체의 설명과 분포에서 주목할 만한 결과를 이끌어 내었다. 특이점의 연구로서 해석함수의 연구의 고려가 필요하게 되었다. 삼체문제에 의해 제기된 질적인 문제가 해결되지 않았더라도, 순드만의 해는 이문제의 발전에 있어서 긴요한 것이다. 프엥카레의 업적과 마찬가지로 순드만의 업적에 대한 질문은 열려있는 것이다.[21]

1960년경 수학자 콜모고로프(A. N. Kolmogorof), 아놀드(V. I. Arnold) 그리고 모세(J. Moser)는 프엥카레의 직관과 반대로 어떤 초기조건에서는 급수가 수렴할 수 있고, 따라서 준주기해를 제안하였다. KAM 이론은 전통적인 섭동론이 실패 했을 때 해당되며, 무한시간에 대한 계의 안정성을 증며해 준다. KAM 이론의 단점은 섭동매개변수가 극도로 작아야한다는 데 있다.

2.2 적분불변

앞서 소개한 대로 적분불변, 주기해, 고유값은 상호연관되어있다. 1890년 Acta에서 프엥카레가 전개한 정의–정리–증명(definition-theorem-proof)단계는 천체역학의 신이론에서 근간이 되어 응용되었기에 소개를 하면 다음과 같다. 프엥카레는 제한삼체문제의 해에서 안정성(태양의 안정성)과 관련해서 다음과 같이 정의를 내리고 있다. 프엥카레는 정성적인 증명을 시도하고 있다.

> 안정성의 정의 : 점 P의 운동은 그 점이 원래의 위치의 임의적 근방에 무한하게 되돌아 올 때 안정하다고 정의 한다.

이에 대한 정의를 회귀정리 라고 하고 주어진 계에서 자유도가 3일 때 부피는 보존이 되어 무한수의 해가 존재한다는 것이 안정성이라는 것이다. 그러나 9년 후인 1899년에 출간한 천체역학 제3권에서는 안정성의 정의를 3가지로 정의하여, 위에 기술한 것에 추가하고 있다:

> 1. 점간의 거리는 무한히 클 수가 없다. 2. 점간의 거리는 일정거리 이하로 작아 질 수 없다. 3. 계는 원래의 위치에 무한히 자주 임의적으로 가까이 돌아온다.[22]

> 정리 **I(**회귀정리**)** : 공간상 유한한 한 점 P의 좌표 x_1, x_2, x_3를 가정하게 되면 적분불변 $\iiint dx_1\, dx_2\, dx_3$가 존재 한다; 공간상의 임의의 영역 r_o가 아주 작다고 할 지라도, 여기를 통과하게 되는 궤적이 무한히 자주 존재하게 된다. 즉, 미래의 시간에는 원래의 위치에 임의로 가깝게 그 계는 되돌아 올 것이고, 무한히 자주 되돌아오게 될 것이다.

이에 대한 프엥카레의 증명은 단순하다.

> 증명 : 점 P가 있는 곳의 부피 V와 영역 R을 고려하자. 그러면 t시간에 무한수로 움직이는 점들로 구성된 부피 를 가진 영역 r_o를 R에서 고려할 수 있다. 이들 점들은 영역 r_1을 τ시간 동안 채울 것이고, 영역 r_2를 2τ 시간 동안 채우는 식으로 해서, r_n 영역을 $n\tau$시간동안 채우게 될 것이다. 여기서 r_1과 r_2는 공통되는 점이 없고, r_n는 r_o가 n번째 까지 반복되는 것이다. 부피가 보존됨으로 영역 $r_o \cdots r_n$는 같은 부피 V를

[21] 문헌 6-(1), p. 191.
[22] 문헌 6-(1), p. 160; Henri Poincaré, Les méthodes nouvelles de la Mécanique Céleste, Vol III, 1899, p343.

갖게 될 것이다. 만일 $n > V/v$ 이라면, 적어도 두 영역은 공통부분을 갖게 되는 것이다. 그래서 공통영역의 연속적인 반복을 고려한다면, 다른 영역을 확장 할 수 있는 r_o 에 동시에 속하는 점들을 모을 수 있음을 보일 수 있고, 이 점들의 모음(집합) 자체는 하나의 지역(region) σ 를 형성하게 해준다. 이 지역 σ 의 정의로부터, 한 점으로부터 시작하는 모든 궤적은 영역 r_o 를 무한히 자주 통과 하게 되는 것이다.[23]

프엥카레가 의미하는 예외적 궤도는 확률을 말하는 것이고, 영역을 통과하는 궤적이 k번보다 작으면 영(zero)를 의미하는 것이다. 비록 k값이 매우 크고, 영역가 매우 작아도 적용이되는 것이고, 보조정리와 보조정리의 증명은 1890년 acta에서 추가된 것이다. 이에 대한 보조 정리(corollary)는 다음과 같다.

보조정리 : 위로부터 따른다면, 무한수의 궤적이 영역 r_o 를 무한히 통과할 것이고; 유한한 시간동안에만 통과하는 다른궤적이 있을 수 있다. 이 후자의 궤적들은 예외적 궤도로 간주 될 수 있는 것이다.

이 예외적 궤도는 근사예외적 궤도로 불리는 것으로 앞서 언급한 바와 같이 안정한 궤도가 압도적으로 불안정한 궤도 보다 수적인 우세를 보이는 바 그 확률이 극도로 작을지라도 이는 무시할 수 없다는 것이다.

3 결론

수학적으로 삼체문제의 해답은 무한급수의 전개문제이지만 급수가 천천히 수렴함으로 사용하기 힘들다. 현실적으로 수치적분이 비교적 정확하여, 초기위치, 초기속도, 그 주어진 순간의 힘의 세기를 앎으로서 다음순간의 위치가 계산된다.[24] 그러나 본 논문은 프엥카레와 그 주변인물의 삼체문제에 대한 해의 상세하게 요약하지 못하였고, 그 이유는 미분방정식이 초기조건과 경계조건에 의해서 무한히 많은 해가 존재하기 때문이고 많은 지면을 요하기 때문이다. 최근 발렌토넨과 칼투넨은 뉴턴 이래의 전통적인 방식과 통계적인 방식으로 삼체문제를 다루었다.[25] 프엥카레의 삼체문제가 자연에 존재하는 카오스의 대표적인 한 예라고 하면서, 삼체문제는 통계적 접근의 이용에 대한 새로운 가능성을 열어놓았다고 하였다. 한국에서는 최규홍이 삼체문제을 중심으로 미국과 구 소련의 연구자들의 내역을 요약한 바 있다.[26] 1960대말과 1970년대 초 이래로, 영국의 Aarseth, 독일의 Wielen, 미국의 Szebehely등이 250체 문제이상을 컴퓨터에 의해서 수치적으로 풀기에 이르렀다. 이 이론은 항성역학, 성단의 역학적 진화에 응용되고 있다.

오늘날 탄도학이나 행성간 탐사선, 위성궤도론에서 삼체문제의 실용성은 극히 중요하고 핵심적인 역할을 한다. 천체역학에서 이처럼 핵심적인 역할을 하는 문제임에도 불구하고, 한국의 초중고의 교육에서는 삼체문제가 언급이 되지 않고 있다. 따라서 우주로의 여행뿐만 이아니라 근본적인 자연의 이해라는 관점에서 대학이나 고등학교에서 삼체문제에 대한 소개는 필요하다고 판단된다. 대학수준의 공업수학이나 수학물리에서는 고유치나 특이점문제에 대해서 프엥카레와 같은 20세기초의 대수리물리학자들보다 더 간단명료하게 미분방정식이나 선형대수학에서 설명해놓고 있다. 그러나 역사적인 맥락에서의 삼체문제연구는 실증적 차원이 아니더라도 자연현상의 수학적 이해 뿐 만이 아니라, 고전역학과 현대의 물리학의 근간을 이루는 양자역학의 수학적 배경을 어느 정도 지니는 흥미있는 주제로 판단된다: 수학교육학적인 측면에서 삼체문제는 고전역학과 양자역학이 막 분리되려고 하던 시점에서 나온 문제이기 때문에 현재에도 전문 수학자와 수리물리학자에게 중요한 관심대상이 되고 있는 것이다.

[23] 문헌 6-(1), p. 86.
[24] Hans-Heinrich Voigt, Outline of Astronomy I, II; 유경로외 5인 공역, 천문학강요(일신사, 1992) 50–51쪽.
[25] Mauri Valtonen and Hannu Karttunen, The Three-Body Problem(Cambridge University Press, 2006)
[26] 최규홍, 천체역학(민음사, 1997).

THE TOPOLOGICAL INTUITION OF LEONARDO DA VINCI

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ABSTRACT

In the early 16th Century was not strange at all that artists were interested in mathematics, which most notable area was Euclid's geometry. The purpose of the painters to simulate the depth of the space on flat surfaces—be it table, wall or canvas—required to explore resources quite unusual in this craft. It was not possible that scenery, characters, buildings and other painted figures compose an harmonious and credible work, if each one of them had not the appropriate proportion, as they were close or distant from the main level, and according to the angular variation under they were represented related to the front view. The technique required to get this was patiently developed in multiples empirical workshops over three Centuries. Each finding was highly significant and artists were proud of their hability to compete for contracts. Leon Battista Alberti, Piero della Francesca, Albrecht Dürer and Jean Pélerin—only pointing out some of the most significant contributors—wrote treatises of this emerging science, then called *Prospettiva* or perspective, in which a new type of geometry, not yet systematized in its principles, began to appear.

By that time, Leonardo da Vinci finished two decades of a successful work at the Duchy of Milan. In the last three years, he had the company and friendship of the mathematician Luca Pacioli. Leonardo collaborated with Pacioli, drawing the carefully elaborated illustrations the mathematician used in his treatise about the golden ratio, entitled *De Divina proportione*. In the years of working out the treatise, Leonardo studied not only the foundations of both the two-dimensional and the spatial geometry, as reflected in his manuscripts, but studied also the *Physics* of Aristotle, at least the first book, which explores the notions of *continuity* and *divisibility to infinity*. In fact, in the *Elements of geometry* of Euclid, *continuity* was only named once. Leonardo believed that, as well as the sculptor deals with discrete quantities, since each piece carved is separated from the rest, the work of the painter is related to the continuity of space, in which representation should appear all the things made by the Creator, with no space for vacuum.

So, the *Prospettiva* studied by Leonardo is not only the one that two centuries later would be known as Projective Geometry, but also he incorporated the alterations on the texture of the object observed, suffered by the density of the air which is interposed, as well as the changes of tonality suffered by the color in the distance. That is because he could say, in analogy to Dedekind, when he state his principle of continuity, that the surface, despite constituting "the limit of the body", "is not part of the same body", since it is also "the beginning of another body" (Br. M. 132a). In addition to analyze this sensitive issue, Leonardo made it a hallmark of his painting, known as *sfumato*.

At some moment of clairvoyance, between the years 1487 and 1490, Leonardo da Vinci wrote: "A point is not part of a line."[1] It must have been a singular observation among all those that illuminated his mind, for he made the annotation in a small pad, identified today as the *Codice Trivulziano*, whose 60 sheets, tight of profuse lists of terms extracted from literary works, give the impression of configuring a sort of private dictionary, to be judged by the way in which many of the 10,000 words that appear there are accompanied by an attempt of personal definition.

Of what source would the idea come to Leonardo that "a point is not part of a line"? By that time, the artist and inventor already stood out in the Duchy of Milan, and his personal library came close to the important figure of 40 volumes, among which was the *Physics* by Aristotle. Nevertheless, it is not there where Aristotle displays his interest for affairs as this one and others related (whether the surface forms part of the body it limits, for example), but properly in the *Metaphysics*.[2] And, true, we do not know for sure that Leonardo ever got to study this work,[3] but, even if he had, by other notes from his hand, that we will examine hereafter, we can state that, in what relates to this topic, Leonardo would take a direction different from that of the Greek sage, whom he so admired. To the pointed out library, soon enough would be added the *Elements* by Euclid, in whose study Leonardo would occupy a considerable time in the following years, especially under the impulse of the friar Luca Pacioli, with whom Leonardo would closely work on the creation and publication of the book *De divina proportione (About the divine proportions)*. But, as is known, already from the third definition of his First Book, Euclid leaves established a principle contrary to Leonardo's observation: "The extremes of a line arepoints."

It would not be in excess to point out that Leonardo knew to distinguish between the "mathematical" point and that other denominated "natural" in that time, equivalent to the minimum lump of dye the finest pen could leave over a board. After stopping on an anonymous manuscript, located today at the Laurentian Library, Leonardo makes the following reasoning: "The smallest natural point is larger than all mathematical points, and this is proved because the natural point has continuity, and any thing that is continuous is infinitely divisible; but the mathematical point is indivisible because it has no size."[4] As is fit to expect, Leonardo's attention was not circumscribed to the end points of a line but, in the same sense, it extended to the lines by which a figure is demarcated. Well now, Leonardo was not interested only in the conceptual understanding of these topics, for above all he saw himself as artist. And, besides an exquisite draftsman, Leonardo was a fast draftsman, as is appreciated in his tens of drawings about the birds in full flight, or in the ones of the turbulences formed by the fall of a water jet, or in the ones of the faces of protuberant features, caught in the brief passing of a tavern. Therefore, it must have resulted astonishing, to the fine and exact strokes draftsman, the discovery that lines do not exist in the world. "The line has in itself neither matter nor substance," he writes, with his specular calligraphy, on a sheet that isin the Royal Library of Windsor, in London, "and may rather be called an imaginary idea than a real object; and this being its nature it occupies no space."[5]

[1] *Codice Trivulziano*, 35r.

[2] Aristotle, *Metaphysics*, book III, 1001b–1002b.

[3] In fact, the discussion about the authorship of Aristotle initiated on that time, and the doubts would persist for several centuries. To cite a significant case, the young and brilliant contemporary of Leonardo, Giovanni Pico della Mirandola, considered the work apocryphal as he published it in his writing *Examinatio vanitatis doctrinae gentis* (IV, 5); and Pico was assiduous to the Neoplatonic Academy of Florence, institution that had exerted, for better and for worse, an important influence on Leonardo's formation.

[4] Laurentian Library, 27b.

[5] Royal Library, Windsor, 19151v.

In fact, what the careful investigator had observed is that lines, as it happens with numbers, belong to the exclusive field of thought and for that cannot be perceived with the senses. They can, indeed, be applied by our mind to the representation we make ourselves of that which we judge to be outside it. That we unthinkingly believe that a body on which we concentrate our gaze (a cloud, a table, the needles of a clock) is limited by the lines of its contour, is just fruit of a lightness of our observation, the same that would take us to conclude that the sun is smaller than the palm of the hand. What we call "contour" is no other thing that our discerning capacity applied to distinguishing an object from what is not it. Two volumes that, from our point of sight, are in the same line of vision differ from one another firstly by the characteristic of their color, not by their shape. In fact, at a certain distance, if they were partially superimposed and they both had the same color tonality on their whole surface, we would not accomplish distinguishing them. But every colored surface induces a shape, that of what the color embraces. And to the cerebral memory it results more eloquent the assimilation of that shape with the shapes it already knows; or, said in other words, it is more economical to identify than to discover. Thereof we believe to detect first the shapes than the colors.

The order of priority in the catching of those two factors, first the shape or first the color, is something that no person feels invited to determine, not just because the brevity of the interval in which they succeed each other makes them appear in an almost simultaneous manner, but because in any case the brain integrates them in the whole of a represented image, that of the perceived object. Just the painters, and in particular those that have seen themselves animated by the intention of representing the scenes of the world "such as the eye sees them," have hit upon to the need to take a position in this respect. The impressionists, for example, fascinated by the momentary effect of light, that on incidence over objects produced a myriad of minute shades, executed their works with uncountable touches of the brush over the fabric, to the way of infinitesimal stainlets of color that, in their abundance, would give the sensation of continuity. On their part, artists of the 15^{th} century used to first delineate on the board or on the wall all the significant figures, both those of the characters as well as those of the furniture, the architecture and the landscape, all that after having solved the problem of proportions according to the different planes of perspective, and then indeed dedicate themselves to filling them with color, task that many times they delegated on their young assistants.

To Leonardo, the inexistence of the contours did not obey a technical or stylistic criterion, but a discovery of the manner the spectacle of reality becomes manifest before human eyes. "Do not make the contours of your figures of a color different from that of the field in which they stand out,"[6] prescribes Leonardo to the apprentices of his way of painting. And following, explains: "this is: do not demarcate you figure off its field by means of a prominent stroke." Leonardo considered fallacious the representation of an ordinary object, if to represent it the object was isolated from its surroundings. In fact, his certainty that painting outdid—as art—sculpture came from the capacity of painting to recreate the totality of a scene the eye sees, including in it the variations of color suffered by the surfaces of the bodies, not only with respect of the angle from which light reaches them, but derived from the diverse layers of air interposed between the observer and the contemplated objects, depending on them being more or less distant from his eye. That is to say that, in Leonardo da Vinci's intuition, the world the eyes see was populated of those entities thick within but lacking borders, which in the beginning of the 20^{th} century would be known as open sets.

[6]Codice Urbinas, 46r–v.

Austrian physicist Fritjof Capra is without doubt the scientist that with more lucidity has centered his attention on Leonardo's manuscripts related with topology. In the appendix of one of his more recent books, *The Science of Leonardo*,[7] Capra studies with detail Leonardo's drawings that appear on a big folio of the Codex Atlanticus under the title *De ludo geometrico*[8] *(About the game of geometry)*, and that in the research of the noteworthy artist make part of what more broadly Leonardo denominated "geometry that is demonstrated by movement"[9]. It is about graphic transformations of geometric figures in part rectilinear and in part curved, built with rectangles, triangles and circles, by means of which a figure is transformed into another keeping invariable the area of its surface. They are processes similar to the quadrature of the lunules that centuries back Hippocrates of Chios realized, applied here to an exuberant variation of floral shapes, and taking advantage of resources so ingenious that Capra does not hesitate in qualifying this sort of geometric metamorphoses as "primitive forms of topological transformations"[10], and Veltman, with his formation more oriented towards the graphic computation, qualifies Leonardo for it as "mathematico-morphoses", and his work on surfaces and volumes as "a vision of 2-D and 3-D morphing *avant la lettre*"[11]. This same game of transformations Leonardo practiced in his studies of the human face, realizing variations of the more notorious features, like the diverse classes of nose, with the interest of characterizing facial typology according to the dominant of the mood[12].

It may not be fortuitous that, in his book *The Heritage of Apelles*, Gombrich included, in addition to the already cited article about the "grotesque heads", another essay about Leonardo, this one dedicated to the study of the waters. As has been registered since the most remote antiquity, water is, of the four elements of tradition, the only one that adapts to the shape of the recipient where it lodges. It was to be expected, then, that Leonardo saw in water the perfect means to examine the way in which the shape of a body changes without altering its volume. In incompressible liquids (in general conditions water is so), the preservation of volume entails with it the preservation of mass, so that the phenomenon of continuous transformation of a body in its external appearance, leaving invariant its mass, supposed for the "geometry that is demonstrated by movement", as Leonardo denominated that live science of the changes of nature, a field very rich in exploration. Leonardo saw implicit the continuity of the transformation in the economy of nature while realizing its movements. In folio 85v of the *Codice Arundel* it is read: "Every natural action is carried out by the shortest way."[13] It is all that takes physicist Fritjof Capra to consider this kind of Leonardo's investigations as "primitives of this important field of mathematics" that preceded Poincaré's formalization "by five hundred years".[14]

But not only in his painting was Leonardo consequent with his discovery that, as they are dis-

[7]Doubleday, 2007. References of this book that appear ing the present work have been taken from the text in standard Spanish *La ciencia de Leonardo*, Ed. Anagrama, 2008. Translation is mine.

[8]It is worthwhile to contextualize the sense of "game" as it was employed between the 12^{th} and 16^{th} centuries. While Creator, God was assimilated to the Great Geometer. Thereof Alberti denominated one of his works *Ludi matematici* (1452) and Nicolás de Cusa denominated *De ludo globi* one of his (1463). This position was adopted by artists and philosophers that in their own work did not see themselves as God's rivals but as his emulous. Cf. Veltman, Kim H., *Leonardo da Vinci and Perspective*, 2007, Maastricht McLuhan Institute.

[9]*Geometria che si prova col moto*, Codice Madrid II, folio 107r.

[10]Capra, p. 345.

[11]Veltman, op. cit., Transformational Geometry.

[12]The best reference is the essay *The Grotesque Heads* (1954), by prestigious art historian E. Gombrich, which made part later of his classic book *The Heritage of Apelles* (1976).

[13]"Ogni azione naturale è fatta per la via brevissima".

[14]Capra, op. cit., p. 271.

played in the world, things come devoid of contours.[15] As well as in one of his works, the border zone up to where the delicate veil covering with modesty the head of hair of a woman seems to extend,belongs both to her figure as well as to the remote little path winding between the rocky formations made out from her balcony, or that it may be about an indefinite tenuous zone that does not belong to either of them but to the luminous air of that unreal afternoon, so as well Leonardo abolished the artificial borders that separated the diverse disciplines of investigation. In the conception of those times, that celebrated with joy the rediscovery of the Greek's knowledge, nothing more distant than two contiguous elements, as were air and water. Precisely because their periodic table just consisted of four elements that, arisen since the night of the myth delivered their little splendor in the dawn of science, every natural philosopher knew that merely in the mystery of alchemy could a secret way to pass from one to the other be found. And Leonardo found it. The key was the same secret that life breathes in any of its forms: movement. And in spite of being aware that water and airopposed in a fundamental attribute, since the former is expansible and compressible while, in constant conditions, the latter does not expand nor compress, Leonardo was capable of observing that "the movement of water in the bosom of the water" was similar to that of "air in the bosom of the air".

Facing statements of such amplitude and certainty, even those having shown being more reluctant to grade as scientific Leonardo's investigative work, as is the case of the prestigious mathematician, philosopher and science historian Clifford Truesdell, find themselves compelled to recognize that the tireless and silent artist, that living never got to publish the results of his studies and that received no remuneration whatsoever for doing them, was the founder of the important field known as fluid mechanics.[16] Of course, to Leonardo it turned out to be impossible to make controlled experiments on wind strength. Instead, after scattering colored seeds on a water course, he could observe with much precision the effects produced upon collision with an obstacle in the route, or upon narrowing of the channel banks; and he knew the conclusions he was obtaining kept being valid when applied to the invisible movement of air, for example when it raised its impetus upon being compelled to go through the narrow pass between two mountains. But also the observation of air power led him to make discoveries on water movement, as in his observation that, in the surf of the sea, it is not water that advances. Leonardo writes: "Impetus is much faster than water, but very often runs the wave from where it was created, without water moving from its place, in fashion similar to how happens in May with the waves wind creates over the wheat fields; we see the waves run by the fields without the stems moving from their place."[17]

What had been the finding of *sfumato* in the field of painting, so was the analogy in the field of thought. It is this integrating way of vision, so characteristic of the freedom Leonardo's spirit never renounced, in spite of being in service of one and other and other of the more powerful monarchs, that which allowed the investigator to approach a phenomenon from a plurality of viewpoints. For that

[15]Writes Leonardo: "1, The superficies is a limitation of the body. 2, and the limitation of a body is no part of that body. 3, and the limitation of one body is that which begins another. 4, that which is not part of any body is nothing. Nothing is that which fills no space." British Museum, 131v.

[16]Truesdell, C., Essays in the History in Mechanics, Springer-Verlag, 1968, p. 71. The page here alluded corresponds to the version in standard Spanish, *Ensayos de Historia de la Mecánica*, Ed. Tecnos, Madrid, 1975. In spite of the insurmountable faults he finds with the investigation method followed by Leonardo, Truesdell also recognizes that Leonardo was the first to formulate the principle of communicating vessels, as well as the first to give a statement for the law of free fall. Cf. op. cit., p. 45.

[17]*Codice M*, Institut de France, 87v.

Capra is right in making us see that that not compartmented thought is a real foretaste of what today is known as Complex Thought Theory, promoted by philosopher and sociologist Edgar Morin.Who makes the exercise of going into Leonardo's manuscripts feels entering a surprising–but at the same time very close–universe. On each page will be found some result surprising to the most diverting mind. The closeness is not due so much to the familiarity with the topics–which for their diversity usually exceed the habitual contact today's man might have with them–, but rather to the simple manner, almost innocent, under which Leonardo discovers them at the very moment of treating them. It is that, facing all the treatises that have been published in all times and that teach something about any subject, Leonardo's manuscripts present an essential difference that characterizes them: they have been written in the rough and, in general, have no addressee.

In fact, Leonardo's manuscripts are not redacted to produce an impression in someone but to gather an impression of the mechanism that makes the universe act. Because, daily, as the nomad gatherer previous to our societies, Leonardo's mind went out to face the fruits offered it by the prodigality of this world. The folds on a linen shroud, the shadow of a cloud over the roofs of a church, the length of the forearm, the distribution of the branches on the stem, a mollusk shell at the top of mount Albano, the resistance of a beam, the digestion of food, the stillness of the sun, the membrane of the bat, the neck of the lute, the intrauterine life, the erosion of a barrack tower, the number of stamens of a flower, the relief of a gravel, the shape of the acorn, the sediment of a basin were affairs on which his mind stopped with the same happiness as the dessert dweller when watering through. Each thing was a door that led to the entire orb, not only because each one was associated with the others in the mental map of his personal inquiry but because, to Leonardo, knowing an object did not consist in providing it a sense but in waiting for the object to reveal it to him. Knowledge was not achieved with haste but by inserting oneself in the rhythms of life, because all things of the world came from life. Because of that, he contemplated with the same respect the flight of a butterfly and the womb of a cadaver, and with the same patience wrote down on his notebooks what one and other showed him.

Almost always, a page by Leonardo is an opening of annotations and drawings traced at the live instant of the observation. So that on occasion he writes on the margins, with poor orthography and dubiously, in vertical way, and so that he goes back on his word as he observes better, that he makes amends, cross outs and corrections, so that many of his texts lack the impeccable coherence of the printed treatise after the magnifying glass of revisions. It is not difficult, around one same topic examined by Leonardo in different years, to find discrepancies in his thought and even contradictions in his conception. It was not exactly logical limpidity what concerned Leonardo in the exercise of studying, and little did he care that his notebooks ended seeming a raiment of tatters, so long as under the patches was preserved in pure form the face of truth. In his case, well apply Morin's words: "Employing logic is necessary for intelligibility; surpassing it is necessary for intelligence. Reference to logic is necessary to verification; overcoming logic is necessary to truth."[18] Leonardo's pages are a cross cut to the very act of thinking.

In the thousands of pages of his notebooks almost no registry is found about that molasses of emotions and feelings that is usually denominated "the personal life". And nevertheless, how much life of the author pulsates in each one of them. Observation, even if it is about a moth around a candle, is a finding the contemplator of life thanks since it is the epiphany of his walk, the fashion in which

[18] E. Morin, *La Méthode*, tome 4, Points, p. 207.Translation is mine.

life manifests itself in any whatsoever of its folds. And to that, it just has to be brushed with looking, without imposing on it a meaning, because in pretending to light in artificial form that which in itself is light, its truth is inhibited and retracts, and we will just achieve seeing the spectrum of our own illusion. “What the light of the eye sees is seen by that light,” says Leonardo, “and what the light sees is seen by its pupil.”[19]

The everyday variegation of Leonardo’s notes only on occasion alternated with moments of reflection. It was the diastole of his thought, which found rest sketching the index of treatises he would never publish. And even in those passages appear imbricated the dimensions that interested his chore. If he conceived a treatise on light it would have to contain the thorough studies he had done on the diverse classes of shades—primary, projected, derivative—and the way in which luminous rays propagated by the atmosphere. Moreover, his investigations on optics should appear, since there was explainedthat related to the organ of sight, addressee of light. But then also the science of perspective, that though not ruler of the harmony of the world gives order to vision. And not just linear perspective, but aerial and of color, and that which he called declining, thatdiluted solidity of the bodies made out at great distance. And by account of perspective, astronomy, that is the manner in which man contemplates creation... For that his notes resemble his thought, and one and other resemble that discovery he gave form to in his pictures, and of which only centuries later its topological richness could be seen: that in the world there are no individual things, but that it is inhabited by entities without contours, each one open towards the others. Hence knowing well the nature of any whatsoever of them, being equivalent to knowing the very nature of existing.

[19]Royal Library, Windsor, 19152r.

Illustration1. Codice Trivulziano

Illustration2. La Gioconda, detail

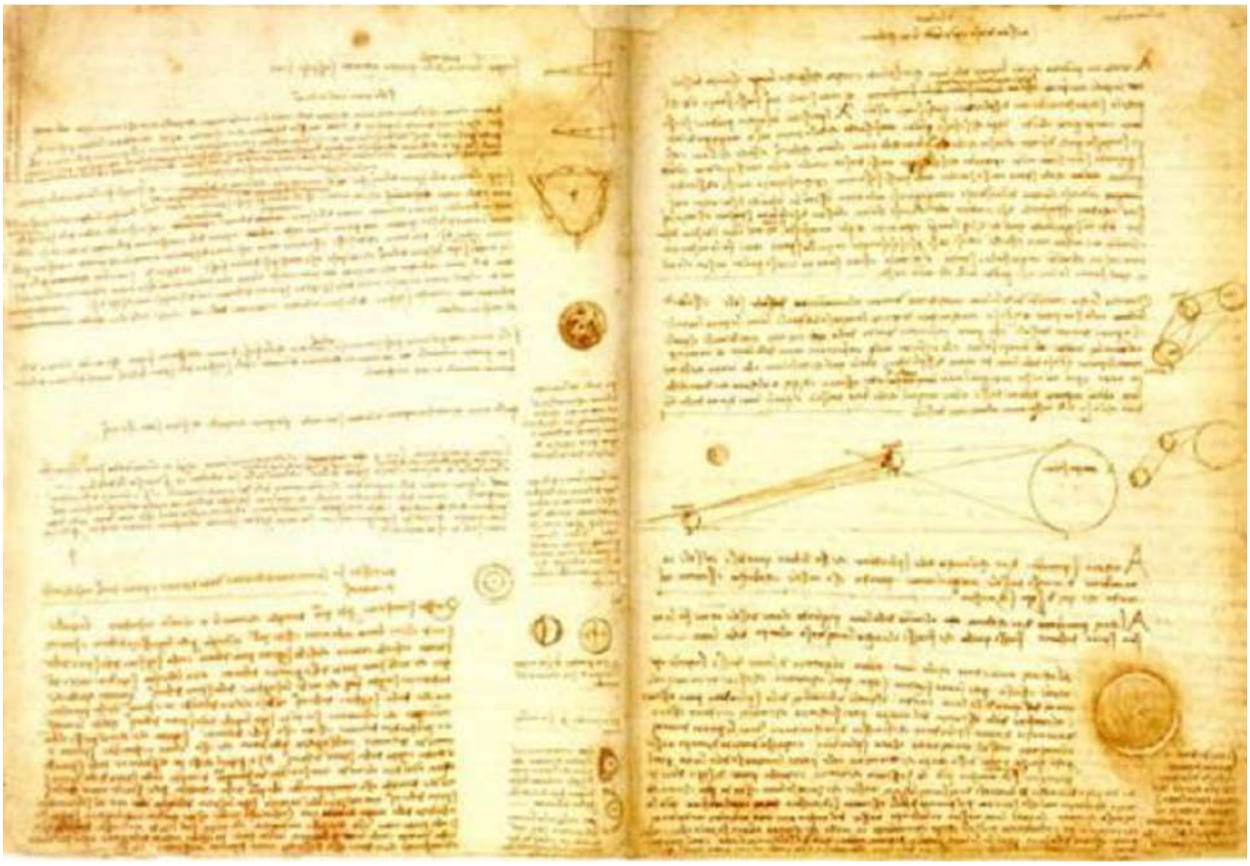

Illustration3. Codice Leicester

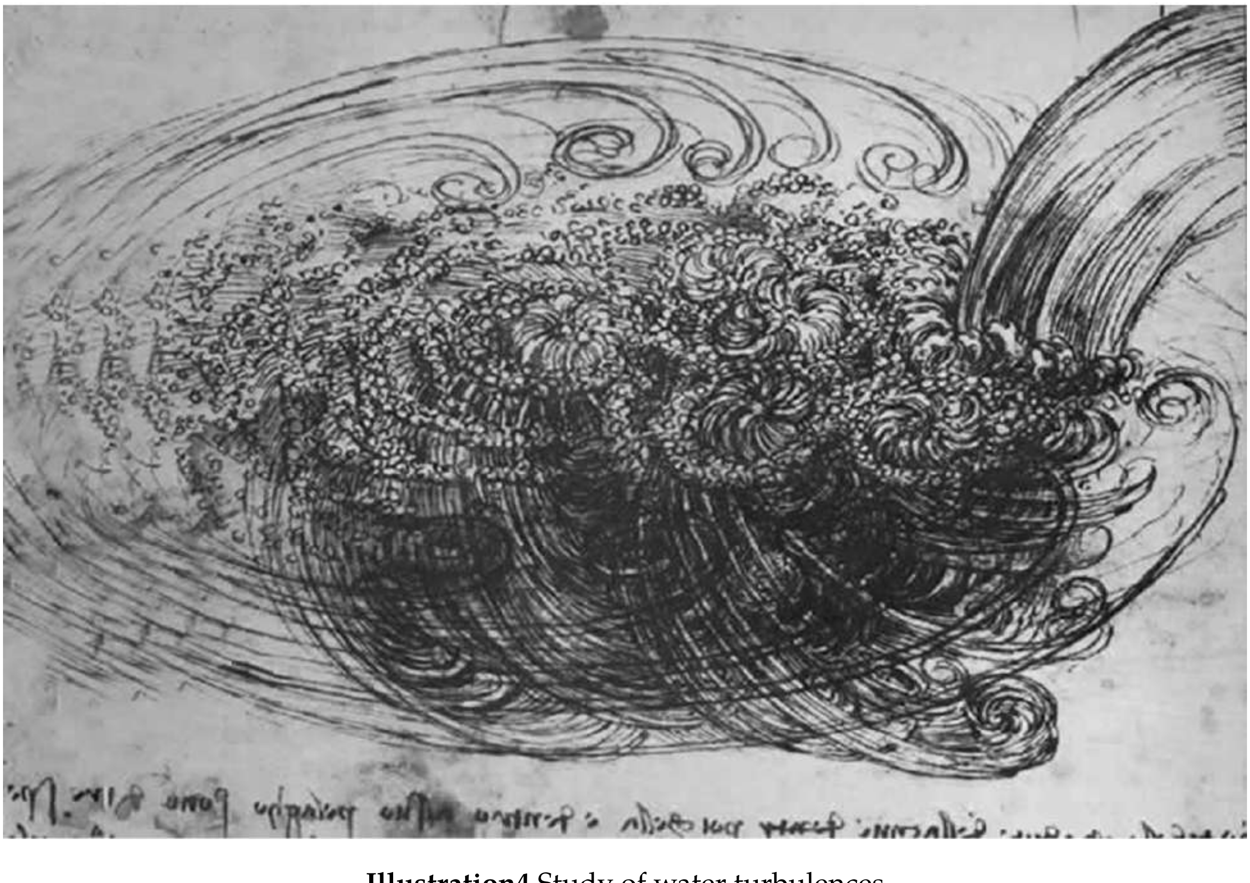

Illustration4. Study of water turbulences

VISUALIZATION OF THE MECHANICAL DEMONSTRATION TO FIND THE VOLUME OF THE SPHERE USING DYNAMIC GEOMETRY

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ABSTRACT

This research examines the mechanical demonstration of the mathematical process developed by Archimedes to determine the volume and surface area of the sphere. The process was divided into two parts: a mechanical demonstration and a mathematical proof (Archimedes, 2005). From an epistemological perspective we can see how mathematical knowledge arose from other sciences, specifically physics. For Archimedes the most important thing was to first understand, and then demonstrate. Motivated by the desire of enjoying the delight of the logical process, it was discovered that dynamic geometry can be used, Cabri 3D, to visualize the construction of the mechanical demonstration. These actions are guided by a didactic perspective developed in this historical study. Research does show that it is possible to combine the history of mathematics and computer sciences for the study and better understanding of the concepts that emerged throughout history.

Keywords: Archimedes, Volume of the sphere, Mechanical Method, dynamic geometry.

Theoretic framework

Philosophy of Mathematics

In the last three decades new types of evidence and argument have been created in the development of the mathematical practice, changing the rules in the area (Hanna, and Pulte Jahnke, 2006). Changes have been produced by the use of computers (as a heuristic device or as a means of verification), due to a new relationship of mathematics with empirical sciences and technology, and a strong unconsciousness on the social nature of the processes that guide the acceptance of a test. These changes have been reflected in new trends in the philosophy of mathematics. For years, philosophers have tried to define the nature of mathematics taking into account its logical foundations and its formal structure. In the past 40 years the search has drifted. The first one to highlight these changes was Imre Lakatos in the late sixties of last century, his work remains highly relevant to the philosophy of mathematics and mathematics education.

Imre Lakatos and the heuristic style

The philosophical approach that guides this work is that proposed by Imre Lakatos (1978), who emphasized the historic nature of mathematics, human activity that is not discovered but constructed.

Lakatos takes a quasi-empirical philosophical conception of mathematics, which gives it a more practical nature. He performed a comparison of two styles in the development of a mathematical proof. The first one, the deductivist, is developed by the Euclidean method. At first it mentions often artificial and mystically complex axioms, lemmas, and/or definitions whose origin is not explained, and must be accepted. Following is stated the theorem, full of harsh conditions, and finally the test. The expectation is that the student has some mathematical maturity, i.e. that he/she is endowed with the ability to accept the Euclidean arguments with no interest in the underlying problem and the heuristic aspect of the argument. Here mathematics is seen as a set of eternal and immutable truths, where counterexample, refutation and criticism are not accepted, amid an authoritarian environment. The deductivist style hides the struggle and covers the adventure in an environment where history fades, as well as the successive tentative formulations of the theorem throughout the test, the outcome being sacred.

In contrast, the heuristic style highlights the definitions generated by the test, the definitions generated by the previous tests and also the counterexamples that have led their discovery. This style emphasizes the problematic situation in the logic that has given birth to the new concept. The heuristic order would be given by stating conjecture, show proof and counterexamples to the theorem, and finally the description generated by the test.

Explanation vs. Justification

Work product of Lakatos and others led to conceptions of mathematics in general and the particular test, based on studies of mathematical practice, often combined with epistemological views and cognitive approaches. In this context, there has been a reconnaissance of central importance to mathematical comprehension, the way it is expressed, and what is considered as mathematical explanation. With these two changes in the focus attention has shifted from the supporting role to the explanatory role of proof.

> "The focus is not only why and how a test validates a proposition, but how it contributes to a proper understanding of the proposition in question, and what role it plays in this process by factors beyond logic" (Hanna et al., 2006).

Based on analysis of different studies on the problem of teaching of the proof, Hanna (1989) states that there is a growing trend away from formal proofs in the curriculum, and give more emphasis to the social criteria for acceptance of a proof. In this context she draws a distinction between mathematical proofs that prove and mathematical proofs that explain, both being legitimate, because they meet the requirements of mathematical proofs, although there are differences of opinion on the degree of rigor. The mathematics proofs that explain highlight in the convincing argument, in the role of the proof as a means of communicating mathematical ideas, generating new ways of teaching of the proofs. The proof that proves shows only that a theorem is true, a proof that explain shows that it is true and also shows why it is true, then highlights the characteristics mathematical properties that imply the theorem to be proved, ie the result depends on properties. It is therefore necessary to replace the non explanatory proofs by another equally legitimate as the first, but with an explanatory power provided by the mathematical properties on which they are based and by the mathematical message in the theorem. Mathematics educators have the task to make students understand mathematics and, therefore it is important to give a more prominent place in the mathematics curriculum to proofs that explain.

Visualization and proof from the Philosophy of Mathematics

For over two decades researches on the use of visual representation and their potential contribution to mathematical proofs are more numerous and important, mainly because computers have increased the possibilities of visualization (Hanna & Sidoli, 2007). Besides the traditional role of visual representations as evidence or source of inspiration for a mathematical statement, it analyze to what extent the visual representations can also be used in its justification, as substitutes for the traditional proof.

Researchers in mathematics have studied this issue and the different positions in this respect range from the traditional view that regards as useful complement to the proof, which can show the way for a rigorous proof but is unable to replace the rigor in the verification of knowledge gained in this way. In an intermediate term are located researchers that believe visual representations can play an essential role, though limited in the proofs, noting that in some cases visual proofs may constitute proofs if they meet certain requirements. At the other extreme, some researchers say that the visual representations may constitute evidence itself. The information can be presented in a linguistic and non linguistic way. Propositional logic is only linguistic representation. The task proposed is to extract the information implicit in a visual representation so as to obtain a valid proof. Some researchers do not believe that visual and propositional reasoning are mutually exclusive, and have developed the concept of "heterogeneous proof", where it facilitates reasoning with visual objects. It draws attention to the content of proof, it teaches logical reasoning and the construction of the proof by visual manipulation and propositional information in an integrated manner.

In summary we can see that there is no consensus on the role of visualization in mathematics, specifically in their roles of explanation and justification of the proof, although its role as an aid to mathematical understanding if accepted by all. In mathematics education there are research showing that the dynamic geometry is successful in improving students' ability to notice the details, conjecture, reflect and interpret relationships and provide provisional explanations and proofs, research that support the mathematics literature on the visualization and its help in mathematics understanding.

Prove and visualization from the Mathematics Education

In the classroom the key role of the test is to promote mathematical understanding (Hanna, 2000). One of the most effective ways to reach it is the use of dynamic geometry, helps students develop mathematical thinking, produces valid evidence and improves understanding of mathematics. In relation to the proof, even for experienced mathematicians, the rigorous nature of it, despite being defined, is secondary to understanding. It is compelling and legitimate if leads to understanding. Among the various functions of the proof, verification, explanation, systematization, discovery, communication, construction, exploration and incorporation, the first two are fundamental, but in education which comes first is the explanation, so is valued more proof that best explains.

There is a consensus among mathematicians on intuition, speculation and heuristics are useful in the preliminary stages of obtaining mathematical results, and that intuitive reasoning without proof is not a speculative branch, separate from mathematics. The dynamic geometry software has given a boost to intuition, speculation and heuristics, thanks to the exploration of mathematical objects by dragging. It encourages exploration and testing as it is easy to pose and prove conjectures. However, these potentials should not base an entirely experimental approach of mathematical justification, nor despise the focus of the proof in theory and mathematical practice. What you should do is use both exploration and prove because they are complementary and mutually reinforcing. Exploration leads to discovery and proof is to confirm the discovery. The use of dynamic geometry software on heuristics, exploration and visualization promote understanding of the test.

In both philosophy of mathematics and in mathematics education there is a mutual awareness of new types of proofs and explanation, mainly due to visualization and dynamic geometry, which is necessary to strengthen, and also it is imperative to develop a converging theoretic framework based on recent developments in both fields, in the light of the strong and realistic empiricists trends now shared and worked on by philosophers mathematicians and mathematics educators in different institutions and different research programs.

Miguel de Guzman and the History of Mathematics

Guzman (1993) considers that the History of Mathematics provides a truly humane vision of science and mathematics, not deified, sometimes crawling and sometimes painfully fallible, but also able to correct their mistakes. This view is very different from that perceived by the student when the theorems are presented as truths emerging from the darkness and lead to nowhere. The theorems acquire perfect sense in theory, after having studied it further, including its biographical and historical context. The History of Mathematics provides a temporal and spatial framework to great ideas, along with its justification and precedents; points out open problems of every age, their evolution, the situation in which they are now; and serves to point out the historical connections of mathematics with other sciences, in whose interactions traditionally have emerged many important ideas. The study of history of mathematics allows us to appreciate how the logical order of the theory mismatches the historical order, as seen in this study, and the didactic order mismatches the other two.

Knowing the history of mathematics will allow the professor to better understand the difficulties of generic man, of humanity, in the development of mathematical ideas, and through it those of their own students. History can be used to understand and to make a difficult idea be understood in a better way. Mathematical thinking has gone a long way before giving the rigorously formalized notion of the concepts. If those who gave birth to the concepts did wonderful things with them even though they did not reach the rigor, how can we intend to introduce these concepts to students with unnatural and hard to swallow structures that only after several centuries of work could eventually reached formalization.

Greek mathematics

The Greeks were interested in drawing tangents to curves, in defining and calculating lengths, areas, volumes and centers of gravity (Thiele, 2003). Instead of using algebra, operations with letters, they used the ratios and magnitudes instead of variables, and the role of the field of real numbers was played by the Eudoxus' s Theory of Proportions.

The dilemma analysis vs. synthesis is present in the Greek thought, understanding analysis as the splitting up of a given problem using accurate and logic steps until ending up in something that is already known to be true or a contradiction. With the synthesis completes the proof inverting the process of analysis, deducing the thesis that has been found true in the analysis. They have opposite meanings. Both are the two essential parts of the scheme of a proof in Greek mathematics developed in the field of geometric constructions. Thiele asserts that the Greeks make a distinction between the discovery or invention of mathematical concepts, *inventio* in Latin, and the proof that a given fact is true, verificatio in Latin. One could argue that there is a correspondence between analysis / synthesis, and inventio / verificatio. In relation to the interpretation of Greek mathematical texts, a basic problem is given by its geometric-verbal character.

With regards to concepts of number and magnitude, for the Greeks the number is not only the meaning of thought but the object of thought itself. They conceived three different kinds of numbers: the natural numbers and two ratios, the ratio of natural numbers (positive fractions), and the ratio of magnitudes (positive real numbers). The magnitude is characterized by the property of being able to increase and decrease. The magnitude has a dual interpretation: as a mathematical object and, as an object that can be measured, and the result of such a measurement is a magnitude.

The Greeks distinguish between numerical magnitudes or natural numbers and geometric magnitudes or continuous quantities. The first ones cannot be split, but the second ones (lines, surfaces, solids, angles) can be divided indefinitely. The Eudoxus' s Theory of Proportions, the problem of quadrature, the method of exhaustion, the method of compression, and others are contributions of the Greek mathematics, all of them of extreme importance to the subsequent development of mathematics.

Archimedes

Considered the greatest mathematician of antiquity, and one of the greatest in the history of mathematics, is well known not only for his work as a geometrist but for his inventiveness in the field of engineering, thus in the field of Physics and Technology. He combined theory with practice.

Extant works of Archimedes are all theoretical, both on geometry and mathematical physics (Archimedes, 2005). They have two characteristics: depth and originality, due to the topics treated as well as the methods employed. His works are not compilations of previous mathematical discoveries but actual scientific essays, whose purpose was to communicate his discoveries to the scientific community. His findings derived from his work in research lines proposed by the Greek mathematics, as in the case of finding the solution to the measurement of the circle, through triangulation and the compression method; the Quadrature of the parable, seeking equivalence between plane figures and areas of curvilinear figures, problem known as application areas; and, in the field of solid the seeking of equivalences between them (*On the sphere and cylinder, On conical and spheroids*).

He also ventured into new fields, studying different curves besides the circumference in *On Spiral Lines*; he applied the rigor of geometrical methods to experiments in the field of statics and hydrostatics (Mathematical Physics) as read *On the Balance of the Figures* and *On Floating Bodies*; in *The Sandreckoner* he addresses the problem of expression and large numbers notation; and *the Problem of the Oxen*, he presents a solution that in terms of current mathematics uses a diophantine equation of the Pell-Fermat type.

The Method

Archimedes stated in Method (1986) that the order of the demonstration is not the order of discovery. In the analysis of his works a methodological dualism in mathematical research is clearly differentiated, represented by the opposition between ars inveniendi, the path of discovery, and ars disserendi, the path of the demonstration, as a supported presentation of what is already known for sure, being both complementary. In the specific case of the volume of the sphere, Archimedes, in the path of discovery, performs a mechanical demonstration, using a heuristic logic in explaining the methodology used in the treatment of geometric issues with the help of mechanical notions, such as lever, center of gravity. Archimedes said that this mechanical method is not less useful with regards to the proof of the theorems themselves.

The Method is the only case in the Greek mathematics literature in which the heuristic system is exposed by an author. In the Letter to Erastosthenes, Archimedes states that he wanted to publish The Method because he believes that it could make a not-little-benefit contribution to mathematical research, while others using the method described will be able to come up with other theorems he never thought about (Archimedes, 1986). This assertion implies an educational perspective, communicating his work so it can be learned and used by others.

Fifteen examples are developed in The Method, being the most significant for Archimedes the demonstration that the cylinder circumscribed to the sphere has a volume one and a half times the volume of the sphere. These demonstrations are of special interest due to the fact that for the first time it was possible to find equivalence between figures made up of flat surfaces and figures made up of curved and flat surfaces.

According to Thiele (2003), the basic idea of the mechanical method is based on atomistic concepts evaluated mechanically. In the process, Archimedes sectioned the solids of revolution into "slices" infinitesimal. Each of these circular regions is made of physical matter, with a weight proportional to its volume, which allows the application of the law of the lever.

The Mechanical Demonstration

Previous assumptions

1. A1. If a magnitude is removed from another magnitude and the center of gravity of both the full magnitude and the magnitude removed is the same point, this same point is the center of gravity of the remaining magnitude.
2. A2. If a magnitude is removed from another magnitude with the center of gravity of the full magnitude and the removed magnitude not being the same point, the center of gravity of the remaining magnitude is in the prolongation of the line joining the centers of gravity of the full magnitude and the removed magnitude, placed at a distance whose ratio with the straight line between the centers of gravity is the same ratio between the weight of the magnitude that is removed and the weight of the remaining magnitude (*On the Equilibrium of the planes*, I, 8).
3. A3. If the centers of gravity of whatever number of magnitudes are on the same line (line segment, in this context), the center of gravity of the magnitude made up of all these magnitudes will be also found on the same line (Ibid., I, 4 and 5, II, 2 and 5).

4. A4. The center of gravity of any line is the point dividing the line into two equal parts (Ibid., I, 4).
5. A5. The center of gravity of any triangle is the point where straight lines drawn from the angles of the triangle to the midpoints of the sides intersect (Ibid., I, 14).
6. A6. The center of gravity of any parallelogram is the point where the diagonals converge (Ibid., I, 10).
7. A7. The center of gravity of the circle is the very center of the circle.
8. A8. The center of gravity of any cylinder is the point which divides the axis into two equal parts.
9. A9. The center of gravity of any prism is the point which divides the axis into two equal parts.
10. A10. The center of gravity of any cone is on the axis at a point which divides it so that the portion situated towards the vertex triples the remaining part.

Archimedes uses this theorem as well [established in the previous post *On Conical*]:

T1. If whatever number of magnitudes and other magnitudes in equal number relate to one another, taking in pairs the ones ordered in a similar way, in the same rate; if, in addition, all or some of the first magnitudes have any rates with other magnitudes, and the second ones have the same rates with other magnitudes taken in the same order, the set of first magnitudes is to the set of magnitudes placed in connection with them as the set of second magnitudes is to the set of the related to them.

The Volume of the sphere

Archimedes takes as a basis the construction of a sphere in which he plots the great circle ABCD and two perpendicular diameters AC and BD (Gonzalez, 2006). From the diameter BD he constructs a circle perpendicular to the circle ABCD and, based on the circle BD, he constructs a cone with its vertex on point A (Fig. 1). The ABD cone surface is extended to where it cuts a plane parallel to the base of the cone passing through point C. The intersection of the cone with the plane is a circumference perpendicular to AC with diameter EZ (Fig. 2).

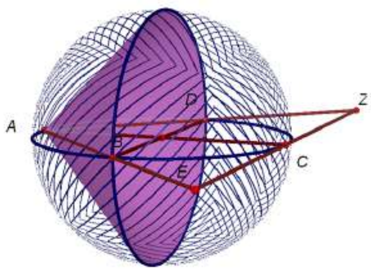

Fig. 1

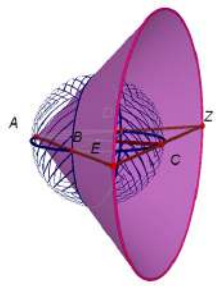

Fig. 2

A cylinder is constructed from EZ diameter having as axis AC and as generatrixes the segments EL and ZH (Figs. 3 and 4).

CA diameter is extended to construct AT so that AT=AC, considering CT as a lever, with A as midpoint. A line MN is drawn parallel to the diameter BD, so that it intersects the circumference ABCD in Q and O, which cuts the diameter AC in S, the segment AE in P and AZ in R (Fig. 5). A plane is drawn through MN, perpendicular to AC, which intersects the cylinder at the circumference with diameter MN, that intersects the sphere ABCD at the circumference with diameter QO and that intersects the cone AEZ at the circumference with diameter PR (Fig. 6).

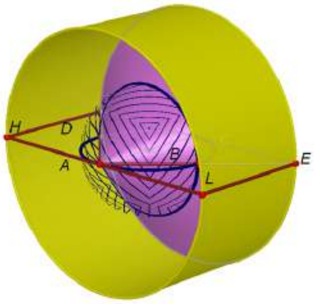

Fig. 3

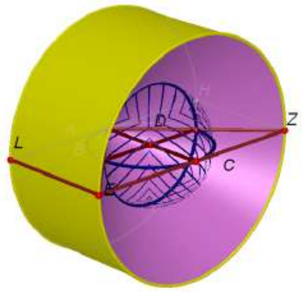

Fig. 4

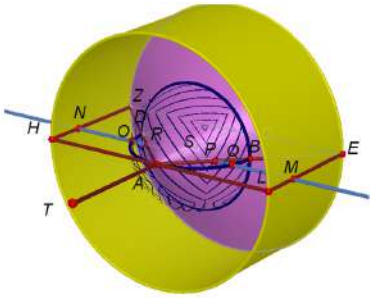

Fig. 5

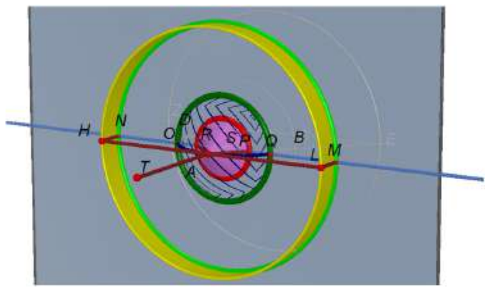

Fig. 6

The intersection of the plane and the cylinder becomes a circumference that delimits a circular region. The same applies to the intersections of the plane with the cone and the sphere. Taking point A as the midpoint of the lever, the circular regions formed by the intersection of the plane with the cone and the sphere are moved to the point T. Multiplying the segment AT by the sum of the areas of the circular regions which are in T, the result will be the same as the multiplication of segment AS by area of the circular region formed by the intersection of the plane with the cylinder (Fig. 7).

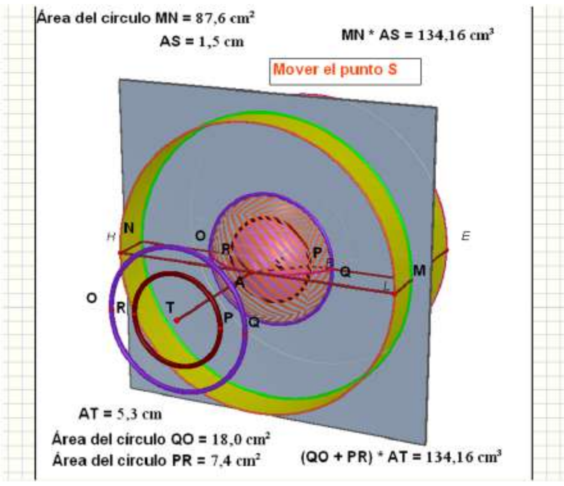

Fig. 7

The sphere and the cone are filled as the circles are moved, both of which, keeping the cylinder in the same place, are balanced about point A, being the center of gravity of the cone and the sphere at point T, and the center of gravity of the cylinder at point K (Fig. 8). From this process we can say that,

(Volume of sphere + volume of cone) $\times$ AT = Volume of cylinder $\times$ AK

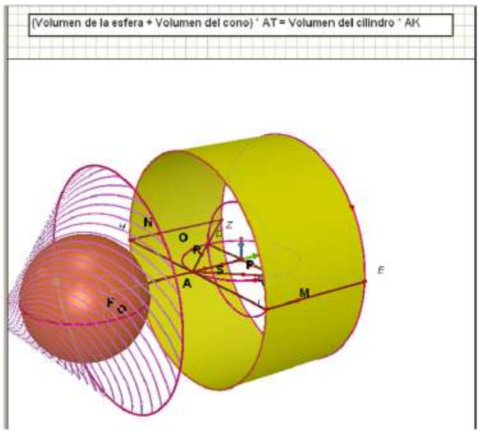

Fig. 8

Resulting that the volume of the cylinder is to the volume of the sphere and the volume of the cone together, as TA is to AK. But TA is twice AK, therefore the volume of the cylinder is double the volume of the cone and the sphere, and in turn the volume of the cylinder is three times the volume of the cone (Euc., XII, 10). Then the volume of the cone AEZ equals twice the volume of the sphere. But the volume of the cone AEZ equals to 8 times the volume of the cone ABD because EZ is twice BD (Euc., XII, 12). Then, 8 times the volume of the cone ABD equals twice the volume of the sphere; therefore, the volume of the sphere ABCD is equal to the volume of 4 cones ABD (Fig. 9)

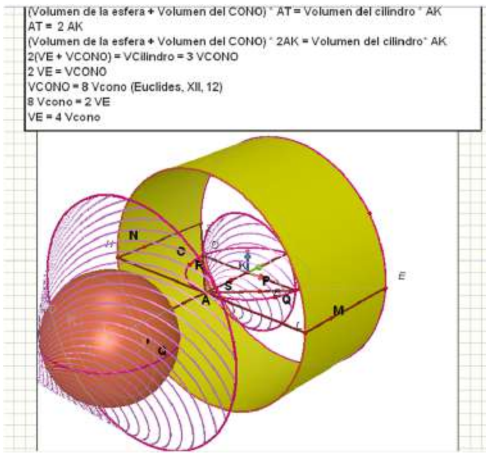

Fig. 9

Segments are drawn through B and D parallel to AC generating a cylinder that circumscribes the sphere. The volume of this cylinder is double the volume of the cylinder circumscribing the cone ABD. In turn, the latter cylinder volume is three times the volume of the cone ABD. Therefore, the volume of the cylinder circumscribing the sphere is six times the volume of the cone ABD. But as was shown earlier, the volume of the sphere is equal to four times the volume of the cone ABD. Therefore, the cylinder volume is one and a half times the volume of the sphere, or, the volume of the sphere is two thirds of the volume of the cylinder (Fig. 10).

$$V_c = \Pi r^2 * h$$
$$h = 2r$$
$$V_e = \frac{2}{3} * \Pi r^2 * 2r$$
$$V_e = \frac{4\Pi r^3}{3}$$

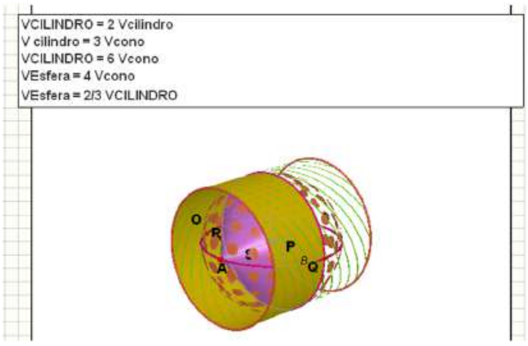

Fig. 10

Surface area of the sphere

To calculate the surface area of the sphere Archimedes established an analogy. As the area of every circle is equal to the area of a triangle whose base is the circle's circumference and a height equal to the radius of the circle (Fig. 11), he assumed that the volume of any sphere is equal to the volume of the cone whose base is the surface of the sphere and whose height is the radius of the sphere (Fig. 12). As the volume of the sphere and the length of the radius are known, it can be deduced therefrom that the area of the surface of the sphere is 4 times the area of its great circle.

$$A_{\text{O}} = A_{\triangle} = \frac{\text{circumferece} - \text{length} * \text{height}}{2}$$
$$A_{\triangle} = \frac{2\Pi r * r^2}{2}$$
$$A_{\triangle} = \Pi r^2$$

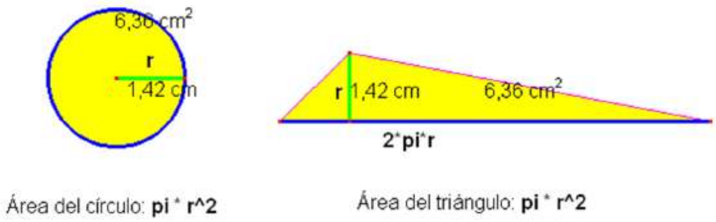

Fig. 11

$$V_e = V_c = \frac{\text{Area} - \text{surface} - \text{sphere} * \text{radio}}{3}$$
$$\frac{4\Pi r^3}{3} = \frac{\text{ASE} * r}{3}$$
$$\text{ASE} = 4\Pi r^2$$

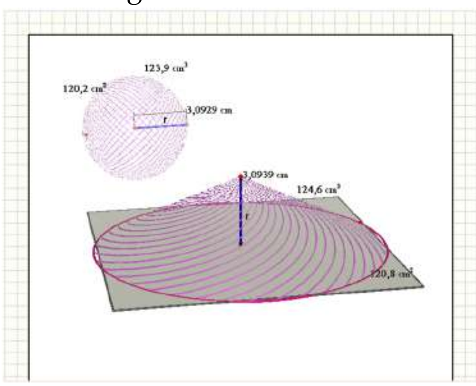

Fig. 12

Conclusions

- From the perspective of the philosophy of mathematics, regarding the dilemma deductivist style versus heuristic style, the mathematical practice proposed in this research work is strongly tinged with the heuristic style, as Archimedes explore, conjecture, experiment, deduced and build knowledge when using the mechanical method to determine the volume of the sphere.

It is possible to visualize this process through the manipulation and dragging of mathematical objects, actions that are facilitated by the dynamic geometry, thus allowing optimize the comprehension of the process that led to the discovery of these mathematical notions. Imre Lakatos actually picks up, put back in place, the heuristic style typical of the construction of mathematical knowledge, which is appreciated in its fullness in The Method of Archimedes, and which was devalued by the formalist trends of Modern Mathematics.

- In relation to the dilemma explanation versus justification, history of mathematics demonstrates the increased importance of understanding over justification, assertion proven in the work of Archimedes. To determine the volume of the sphere he develops the mechanical demonstration to understand the problem to be solved, and once the solution is found out he develops a mathematical proof and the axiomatic justification. Regarding the distinction between proof that prove and proof that explain, the mechanical demonstration is a proof that explains, because based on logical reasoning, exploration, experimentation and heuristics Archimedes was able to established mathematical properties and of other sciences, described in the process, which allowed to find the volume of the sphere. The mathematical demonstration that subsequently made is an example of proof that proves.
- Regarding the dilemma that analyzes how visual representations can also be used in justification, as substitutes for the traditional proof, mechanics demonstration of the volume of the sphere that has developed in this work using dynamic geometry software, is located at an intermediate point, as propositional and visual reasoning are used in an integrated manner. This demonstration is to constitute a *heterogeneous proof* proposed by Barwise and Etchemendy (Hanna, 2007).
- From an epistemological perspective the mechanical method shows that the emergence of many mathematical concepts has been given from an interdisciplinary field, in a context in which it has had to resort to other sciences, in this case, Physics. On the other hand, the ideas of atomism, one of the principles of chemistry as a science that arise in Greek thought, involve in the heuristic that seeks to solve the problem. Perhaps one could argue that three sciences converge in the mechanical method: mathematics, physics and chemistry. Only a man of genius as Archimedes could, over two thousand years ago, design and develop such a complex process. In particular, the infinitesimal employed by Archimedes show him as a precursor of analytical thinking.
- With regards to didactics, taking into account how Archimedes managed to make knowledge emerge, one must ask whether it would be more effective that students from different educational levels, work in an interdisciplinary context to achieve the construction of mathematical notions.
- In relation with the computer sciences, this study shows the usefulness of dynamic geometry, especially the Cabri 3D for visualization and a better understanding of complex processes that have led to mathematical notions, such as the case of volume and surface area of the sphere.

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HOW TO INTEGRATE HISTORY OF MATHEMATICS INTO MATHEMATICS TEXTBOOKS: Case Study of Junior High School Textbooks in China and France

数学史怎样融入数学教材：以中、法初中数学教材为例*

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ABSTRACT

构建分析框架，比较中、法两国初中数学教材对数学史的使用情况，发现数学史已进入我国教材正文的各个环节，数学历史名题的“复制式”与“顺应式”使用做到了联系学生实际。而对于符合学生认知需要的“重构式”融入，相比于法国教材却有明显不足。建议数学史专家应成为教材编写专家团队成员；史料呈现多种方式并举；数学史应成为衡量教材质量的一个维度。

Keywords: 关键词：数学教材；数学史；融入方式；重构式；比较研究

一、研究背景与问题

数学史具有重要的教学价值，已得到理论与实践两个层面的普遍认同。然而在实践教学中，却出现了史料及意识的“无米之炊”以及对数学史“高评价，低利用”的现象。教材中运用数学史可直接为教学提供史料素材，改变“无米之炊”的现状；而以何种方式呈现将决定数学史的使用水平，对数学教育目标的达成具有重要影响。数学史进入数学课程有显性和隐性两种形式 [1]，而尤以隐形融入为瓶颈。一些学者认为我国教材对数学史的处理方式，因存在简单化倾向，即对数学史料理解单一、内容选择单一、史料编排形式单一等不足，使得数学史内容未能真正“融入”教材，数学史料和教学主题与内容之间在形式及本质上仍处于分离状态。[2–3] 另外，因教师认识水平等因素，数学史在教学中常处于低水平使用、甚至被忽略的状态。数学史激发学习兴趣、帮助学生深入理解数学本质等多重资源价值与教学功能未能得到充分发挥。新课程的深入实施，使得数学史融入数学教材成为一个倍受关注、颇有争议并富于挑战意义的课题。

数学史融入数学教材的“正文”[4–5] 的“各个环节”[6] 已成为理论研究与实践需要的共同呼声。如今，新课程实施已逾 10 年，我国教材亦几经改进，教材中的数学史使用情况如何？怎样的方式融入教材才能更好地发挥数学史的多重价值与功能？扎根本土，深入分析我国教材已有做法的成功与不足之处，学习借鉴他国长处不失为一条客观、有效的途径。本研究将在比较、分析中国与法国初中数学教材运用数学史的内容、方式和水平的基础上，探讨为什么要对数学史内容进行重构、怎样重构等问题。期望能对数学史素材在教材中的融入提供思路借鉴和内容参照。

二、中、法初中数学教材运用数学史情况及分析

本研究中，我们选取了中国人教版教材 6 年级上册–9 年级下册 [7]（鉴于我国多把 6 年级作为初中预备班的事实）以及法国初中数学教材 Maths (Nouveau programme)3–6 [8]，根据我们对两国教材内容的统计、

*基金项目：国家社会科学基金“十一五”规划 2010 年度教育学重点课题“主要国家高中数学教材比较研究”（ADA100009）子课题九之部分研究成果．作者简介：蒲淑萍（1971–），华东师范大学数学系博士研究生，主要从事数学史与数学教育研究。汪晓勤（1966–），华东师范大学数学系教授，博士生导师，主要从事数学史与数学教育研究。

比对，涉及知识点吻合度近 90%，具有可比性。在对中、法两国初中数学教材中的数学史成分进行梳理的基础上，重点关注两个方面：1) 中、法两国初中数学教材的数学史内容及各自特色；2) 两国教材中数学史的呈现方式与使用水平。

（一）我们的发现

通过对两国初中数学教材进行汇总整理、分类分析，我们发现中、法两国初中数学教材中的数学史料具有如下特征：

1) 两国都重视数学史在教材中的作用，使用数量较多：中国为 93 处，法国为 108 处；

2) 两国教材共有的特征是对某些数学史内容的集中使用以及在某些知识点上数学史相对集中。如“几何代数方法”在两国教材中都得到充分利用；使用数学史内容相对集中的章节有“勾股定理”、“方程”、“不等式”等；

3) 鲜明的民族特色。两国教材虽都注意吸收了世界范围内的数学史素材，然而，史料内容多以本国为主，如，我国教材对中国古代问题的大量使用，法国教材对法国本土古代数学史料的使用；

4) 从使用的史料所涉范围来讲，法国的史料内容视角更宽，史料来源也更广泛，对相应知识点融合了世界多个国家和地区的数学史料；而中国教材的史料内容及来源略显单薄。

（二）运用方式的分类与分析

为获得对中、法两国初中数学教材数学史的呈现方式和使用水平的认识，本文选用作者之一汪晓勤教授建立的分析框架。该框架的分类方式按数学史与数学知识的关联程度，将数学教材运用数学史的方式分成五类：点缀式、附加式、复制式、顺应式、重构式。显然，这五种类型在数学史的使用水平上成逐步递升的趋势，笔者将 5 种类型对应水平分别记作：A、B、C、D、E 五个层次。可以看到，前面两类数学史独立于教材正文内容以外，而后三类则是将数学史融入教材正文。具体分类、水平及判定标准如表 3 所示：

类别	水平	描述
点缀式	A	孤立的图片，如数学家画像、数学图案、反映数学主题的绘画或摄影作品等
附加式	B	文字阅读材料，包括数学家生平、数学概念、符号、思想的源泉、历史上的数学问题、思想方法等
复制式	C	正文各栏目中直接采用历史上的数学问题、问题解法、定理证法等
顺应式	D	正文各栏目中对历史上数学问题进行改编，使之具有适合于今日课堂教学的情境或属性
重构式	E	正文各栏目中借鉴或重构知识的发生、发展历史，以发生法来呈现知识

数学教材运用数学史的五种方式及水平

依据该表所列分类、水平及标准，我们对两国教材中的数学史进行定量统计与定性分析。

（**1.**）中法两国数学教材中的数学史各类所占比重

按照表 1、表 2 所列结果，两国初中数学教材中的史料内容所占比重如下图 1，2：从统计图中可以看到：

1) 五种分类方式涵盖了两国教材数学史的使用情况 (所占比例如上图所示)；

2) 数学史都已经进入了两国教材的“正文”（复制式、顺应式和重构式），中国人教版教材占到了 41%；法国教材占到了 49%；

3) 在使用形式上，“附加式”是两国教材中史料的主要形式，都超过了使用量的半数；

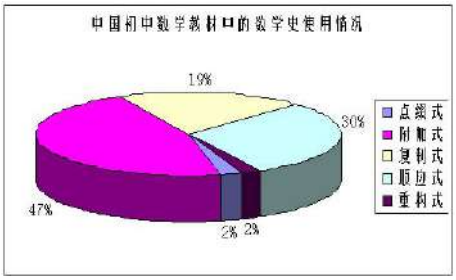


图 1

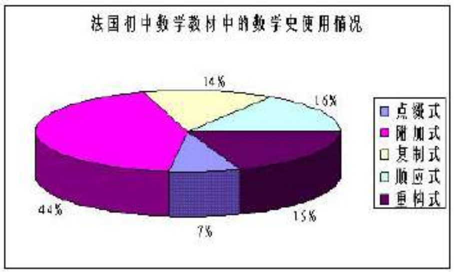


图 2

4) 两国差别较大的是顺应式与重构式两种方式，尤其后者差别明显；

5) 法国教材对数学史料在各个层次上的使用相对更均衡。

（**2.**）中国教材在数学史融入方面已取得的进步

1) 数学史已进入我国人教版教材正文的各个环节
我们将两个国家教材使用数学史的情况放在一起进行对照，如图 3 中表格。从图表中可以看到，我国教材中进入正文的史料内容多以直接搬用古代数学名题的“复制式”，或者新瓶装老酒，以历史名题为模板，将情境或属性换成学生熟悉的现代场景的“顺应式”为主。对数学史教材进入正文的环节多集中在“题目”—— 例题和习题上。

2) 对史料的使用体现了新课程理念，符合学生认知规律
“顺应式”实现了数学史内容的“古为今用”。结合学生经验等创设情境，对古代问题进行改编，或使内容和形式稍有变更，能使学生更好理解与接受。这体现了“内容的选择要贴近学生的实际, 有利于学生体验和理解、思考与探索”的新课程理念 [9]。

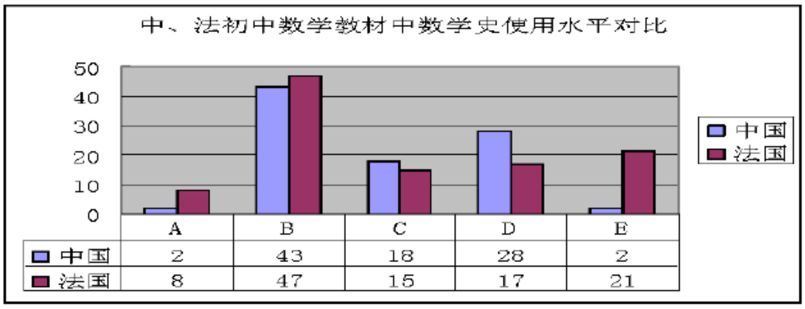


图 3

可以说，中国教材在数学史融入方面已经取得了长足的进步，然而存在的问题也是十分明显、突出的：史料使用水平较高形式的“重构式”内容所占比例相对不足，没有充分实现让学生从史学角度深刻理解数学本质的目的。

（**3.**）中国教材在数学史融入方面已取得的进步

何为“重构式”? 一般来讲，“重构式”数学史材料或出现于新概念的引入中，或隐含于某个知识点的整个脉络中，体现数学知识的历史形成过程。其优点是因遵循发生法，从而更符合学生的认识，更易于学生理解。如，代数学的发展经历了“修辞代数 —— 缩略代数 —— 符号代数”的演变过程，中、法两国教材都注意做到了间接地运用代数历史，进一步将学生从修辞代数引向符号代数，符合学生的认知。

两国教材对数学史料的使用方式，差别最大的就是“重构式”。中国教材“重构式”数学史料仅有 2 处，只占教材中使用数学史内容的 2%，而法国教材却在 103 处史料中的 21 处使用了“重构式”，占总数

的近 20%，是我国“重构式”史料使用量的 10 倍。“重构式”史料使用的严重不足，不能不说是我国教材一个很大的缺憾。相比之下，法国教材“重构式”史料不仅在量上要明显多于我国，而且在使用水平上也高于我国教材。

综合以上，我们认为，对史料进行隐性的“重构”融入教材是当今数学史融入数学教材应该重点关注的方面及努力的方向，而这也是数学史角度的教材编写面临的最大困难与瓶颈。

三、为什么要在教材中对数学史进行“重构”?

英国数学史家 John Fauvel 总结了应用数学史于数学教学的大致 15 条理由，其中包括：增加学生的学习动机；改变学生的数学观；历史发展有助于安排课程内容顺序；告诉学生概念如何发展，有助于他们对概念的理解；提供探究的机会，等等。[10] 如何实现以上目标？弗赖登塔尔（H. Freudenthal, 1905–1990）在《数学教育的主要问题》的报告中指出：“数学史是一个图式化不断演进的系统化的学习过程，儿童无需重蹈人类的历史，但他们也不可能从前人止步的地方开始。从某种意义上说，儿童应该重蹈历史，尽管不是实际发生的历史，而是倘若我们的祖先已经知道我们今天有幸知道的东西，将会发生的历史。” [11] 这提醒我们，教学中需要使用数学史，但不是原原本本地照搬数学史。我们须依据学生的数学现实，“重构”历史融于教材，这既是理论演绎的必然，也是实践需要的使然。为何需要对融入数学教材的史料进行“重构”，则涉及学生认知需要以及与之直接相关的教学需要的理论与实践两个层面。

（一）理论层面

（**1.**）历史发生原理

历史发生原理 (Historical-genetic-principle) 是指导对数学史进行“重构”的主要理论依据。该原理可以上溯到 18 世纪。法国实证主义哲学家、西方社会学创始人孔德 (A. Comte, 1798–1857) 认为：个体教育必然在其次第连续的重大阶段，仿效群体的教育。19 世纪，人们将德国生物学家海克尔 (E . Haeckel, 1843–1919) 所提出的生物发生学定律 ——“个体发育史重蹈种族发展史”运用于教育中，得出“个体知识的发生遵循人类知识发生的过程”，历史发生原理因此而形成。[12] 基于历史发生原理的相似性研究表明：学生对概念的理解与历史上该概念的历史发展具有相似性。因此，“知识的历史发展顺序符合学习心理的序进原则，符合个体的认知顺序，微观认知论是宏观认识论的具体而微. 按照历史发展顺序编写教材能实现科学知识的序和学习心理的序的有机统一”。 [13]

（**2.**）“再创造”理论

弗赖登塔尔认为认为存在两种数学，一种是现成的或已完成的数学，另一种是活动的或者创新的数学。完成的数学在人们面前以形式演绎的面目出现，它完全颠倒了数学的思维过程和实际创造过程，给予人们的是思维的结果；活动的数学则是数学家发现和创造数学的过程的真实体现，它表明了数学是一种艰难曲折又生动有趣的活动过程。弗赖登塔尔认为有效的学习要求每个学习者回溯所学学科历史演进的主要步骤。他反复强调：数学学习的唯一正确方法是实行“再创造”。“再创造”是“重构”数学主题的重要理论基础之一。[14]

（**3.**）发生教学法

发生教学法是一种借鉴历史、呈现知识自然发生过程、介于严格历史方法和严格演绎方法之间的一种方法，它关注主题的必要性和可接受性，要求在学生具备足够的学习动机、在学生心理发展的恰当时机教授该主题。这里，知识的自然发生过程不是历史过程的还原，而是历史知识的重构。

另外，发生教学法也得到了众多名家的重视与认同。英国哲学家、生物学家家斯宾塞（H. Spenser, 1820–1903）认为：个体知识的发生必须遵循人类知识的发生过程。历史上的教育方法，有助于为我们今天的教育提供指南。德国教育家、思想家第斯多惠（F. A. W. Diesterweg, 1790–1866）认为：之所以需要

采用发生方法，乃是因为这是所教学科兴起或进入人类意识的方式。德国数学家、数学教育家 F・克莱因（F. Klein, 1849–1925）认为：“发生教学法”是自然的真正科学的教学方法；美国数学家、数学教育家波利亚（G. Polya, 1887–1985）也曾说过：在教一门科学分支（理论、概念）时，我们应该让儿童重演人类心理演进的重大步骤。当然，我们不应该让他重复过去一千零一个错误，而只是重复重大步骤。波兰数学家托普利茨（O. Toeplitz，1881–1940）：发生法的本质是追溯一种思想的历史起源，以寻求激发学习动机的最佳方式，研究这种思想创始人所做工作的背景，以寻求他试图回答的关键问题。纽约大学著名教授爱德华（H. M. Edwards，1936–）认为：发生法倾向于从一个“虚构的”视角来呈现历史。以上名家名言对于我们理解“发生教学法”提供了不同的视角，对于如何实施发生教学法提供了可供参照的方法和路径。

（二）实践层面

（1.）为教师教学提供史学视角直接的素材与方案

很多的研究表明，教师在教学中较少使用数学史，主要原因在于缺乏“直接能用的数学史”。表现在两个方面：一是缺乏相应史料；二是有史料，却不知怎样使用。融入教材正文的数学史，展现了融入教学的具体形式，为教师设计教学提供了“直接可用的数学史”，解决了“无米之炊”的问题，为教师使用史料素材进行教学设计提供了素材与方案。

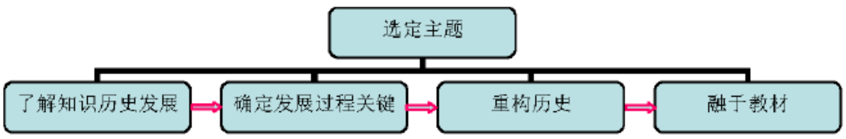
（2.）改变以往低水平使用、甚至被忽略的状况，发挥应有价值

在教材中“重构式”地使用数学史，突破了仅仅满足于“为历史而历史”的附加式等对数学史料的浅表使用，而是在一个较高的水平上无声地融入历史，是对发生法的恰当运用。这样编写教材，使得即使没有多少数学史素养的教师也能遵循知识形成过程的科学路径进行教学，充分发挥了“数学史是教学的指南”（Morris Kline，1908Ī992）的作用。

四、“重构”史料融入数学教材

（一）如何重构

以发生教学法为参照，对于选定主题，我们可沿如下步骤“重构”历史融于教材：首先需要了解主题的历史发展过程，寻找并确定知识历史发展过程中的关键步骤与环节，对知识的历史进行重构，使其适合学生认知和课堂教学，并设计一系列由易至难、环环相扣的问题。 [15] 其具体步骤如下图所示：



在教材的编写过程中，应组成一支以专职教材编写人员、数学史专家、数学教育专家、一线专家教师组成的研发团队，选定教材的核心主题，围绕相应内容，如概念、公式、定理等，开发研制数学史在教材中的“重构式”融入。为增强在实践中的可操作性，我们还可以在融入教材前，进行实践检验：将设计方案拿到课堂进行试验，根据教学反馈进一步修改、完善，再入编教材”。需要重点指出的是，“重构式”数学史料的研发，数学史专家的力量是不可忽略的重要因素。[16]

（二）哪些内容、哪些环节需要“重构”历史

我们知道无理数、负数、指数、对数、复数等许多重要概念都是经历了漫长的历史发展逐过程逐渐形成。在诸如此类的概念引入环节，应以“重构”历史史料的方式为主，展示概念在演进过程中的重要步骤，把握知识发展脉络，感受不断修正、完善的演进过程中古人的丰富智慧及所付出的艰辛与努力，使学生在理解概念、公式及定理本质的同时，培养他们勤奋、坚韧的学习品质。

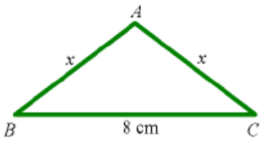


图 4 周长问题之一

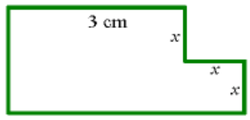


图 5 周长问题之二

对待公式和定理，“证明一个定理还是探索一个定理”[1]的态度反应了是将数学看作一个已经完成了、固定的数学知识体系，还是将数学看作一个动态的不断完善的演进过程两种截然不同的数学观。显然。后者才是我们的课程理念所倡导的。为展现数学的动态演进过程，我们可重构历史展示公式、定理的探索发现过程，如此可对学生掌握正确的学习方法以启迪，且能将逻辑推理还原为合情推理，将逻辑演绎追溯到归纳演绎；多种历史方法的展示能够体现不同时空数学家对同一课题的探求过程、多种不同文化的启发、育化作用，可以激励学生探索规律, 发现定理，体验数学发现的成就与快乐。[6] 典型的例子如勾股定理、乘法公式、一元二次方程的公式求解等。

（三）学习借鉴：法国教材“重构式”的两个典型案例

他山之石可以攻玉，学习借鉴可为我们提供最为直接做法，在此介绍法国初中教材“重构式”数学史料的两个案例：

《数学 4》“不等式、序与运算”一章对不等式的处理遵循了发生法。在“方法”栏目中，一道应用题相当于已知$145 \leq x \leq 155$，求$1000 - 5x$ 的范围。“习题”栏目中有很多类似问题，如：

- 已知$-3 < x < 5$，求$2x + 4$，$\frac{x-2}{3}$, $\frac{7-3x}{8}$ 的范围；
- 已知$4.5 < x < 4.6$，求图4 中等腰三角形ABC 周长的范围；
- 已知$0.8 < x < 0.9$，求图5 中多边形周长的范围。

在这些问题之后，设置了两道相反的问题，即已知 x 的一次多项式的范围，求 x 的范围。于是，一元一次不等式悄然出现，尽管其正式内容安排在下一册。这种由一次多项式所属区间的问题过渡到相反的问题（不等式)，正如由多项式求值问题过渡到相反的问题（方程) 一样，是对数学史的借鉴和重构，是对发生法的恰当运用。

另外法国教材在《数学 3》“乘法公式与零积方程”一章的活动栏目引入了“零积方程”[17] 概念 ——一边为两个一次因式乘积、一边为零的方程。据相关数学史料，零积方程先于因式分解法产生，正是零积方程导致了因式分解法的诞生。因此先于因式分解呈现“零积方程”既符合知识的历史发展顺序，也能符合学生的认知过程，使得“因式分解法”对于学生来说不再是“天上掉下的馅饼”，而找到了“生产馅饼”的作坊。

史料的“重构式”直接将概念等的历史发展融入数学教材，没有显性的历史素材，却在悄无声息中以最完美的方式体现数学史料所提供的思想、方法等，这应是数学教材使用数学史追求的最好境界。对此，郑毓信先生认为：“历史的理性重建”为彻底改变数学史向数学教学渗透方面所存在的“高评价、低应用”现象指明了可能的前进方向。[18]

五、思考与建议

以恰当的方式将数学史融入数学教材，还有许多需要有待进一步思考、完善的地方。对此我们的思考与建议是：

[1]引自 2011 年 5 月“第四届数学史与数学教育国际研讨会”，国际数学教育委员会前秘书长、加拿大 Laval 大学 B. R. Hodgson 教授所做“作为教师教育组成部分的数学史与数学文化 ——F · 克莱因《高观点下的初等数学》启发下的一些观点”报告内容.

（一）教材编写与数学史融入数学教材的 5 种形式

融数学史于教材的 5 种方式：点缀式、附加式、复制式、顺应式、重构式，彼此只有使用水平上的差异，却无严格的孰优孰劣的区分。5 种方式各有所长，配合使用更能从不同层次、角度展示数学内容的历史发展及其教育价值。我们应根据所要达到的教育教学目标选择使用。另外，五种方式在使用的量和比例上应有所均衡，力戒只以点缀式、附加式、复制式等对数学史“为历史而历史”的浅层使用，增加促进学生认知发展的复制式、顺应式和重构式的“为教育而历史”的融入使用。

（二）数学史专家应成为教材编写的团队成员

“重构式”需要对史料进行艰苦的再创造，非一人之力所能为，故而建议教材编写的专家团队，除专职编写人员、数学家、数学教育家外，还应吸纳数学史专家加入进来。这样的编写团队构成更加合理，力量更加雄厚。他们能对应融入教材的数学史内容在总体上进行统筹规划，注意做到前后同类知识使用史料的连续性，并就具体史料内容及其深度和范围、融入方式等进行充分论证，使得教材中数学史料的呈现更为科学合理，更加符合学生的认知、方便教师的教学。

（三）数学史对教材编写的评价作用探讨

数学史视角的教材编写能够按照历史演进顺序呈现知识，数学史理应成为教材编写质量好坏的一个衡量指标[2]。因为“对孩子的教育在方式和顺序上都必须符合历史上人类的教育，换言之，个体知识的发生必遵循人类知识的发生过程。”[19] 参照了知识的历史发展顺序的教材编写更加符合学生的认知过程与认知规律。

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