# History and Epistemology in Mathematics Education

**Proceedings of the 9th EUROPEAN SUMMER UNIVERSITY**

**Edited by**
Evelyne Barbin, Roberto Capone, Michael N. Fried, Marta Menghini, Helder Pinto, F. Saverio Tortoriello

Edizioni Nuova Cultura

# History and Epistemology in Mathematics Education

Proceeding of the
9th EUROPEAN SUMMER UNIVERSITY

18-22 July 2022

*Edited by*
Evelyne Barbin • Roberto Capone • Michael N. Fried
Marta Menghini • Helder Pinto • F. Saverio Tortoriello

**ESU-9**
**18-22 July 2022**
University of Salerno – Department of Mathematics, Fisciano (SA), Italy

**Edited by**
Evelyne Barbin, *University of Nantes, France*
Roberto Capone, *University of Bari Aldo Moro, Italy*
Michael N. Fried, *Ben Gurion University of the Negev, Israel*
Marta Menghini, *Sapienza Università di Roma, Italy*
Helder Pinto, *University of Aveiro & Piaget Institute, Portugal*
F. Saverio Tortoriello, *University of Salerno, Italy*


ISBN: 9788833656014

Copertina: a cura di Sofia Montani
Composizione grafica: a cura dell'Autore
Revisione a cura dell'Autore

MISTO
Carta da fonti gestite in maniera responsabile
FSC® C007287

Questo libro è stampato su carta FSC amica delle foreste. Il logo FSC identifica prodotti che contengono carta proveniente da foreste gestite secondo i rigorosi standard ambientali, economici e sociali definiti dal Forest Stewardship Council

# History and Epistemology in Mathematics Education

Proceeding of the
9th EUROPEAN SUMMER UNIVERSITY

18-22 July 2022

*Edited by*
Evelyne Barbin • Roberto Capone • Michael N. Fried
Marta Menghini • Helder Pinto • F. Saverio Tortoriello

Edizioni Nuova Cultura

# ESU 9: TABLE OF CONTENTS

**Workshops**



**Oral Presentations**



**Short Oral Communications**

## THEME 4: MATHEMATICS AND ITS RELATION TO SCIENCE, TECHNOLOGY, AND THE ARTS: HISTORICAL ISSUES AND SOCIO-CULTURAL ASPECTS IN RELATION TO INTERDISCIPLINARY TEACHING AND LEARNING

### Workshops



### Oral Presentations



### Short Oral Communications



## THEME 5: TOPICS IN THE HISTORY OF MATHEMATICS EDUCATION

### Workshops



### Oral Presentations

**Oral Presentations**

# ESU 9 (2022)
# Proceedings Presentation

E. Barbin, R. Capone, M. Fried, M. Menghini, H. Pinto, F.S Tortoriello

The 9th European Summer University on the History and Epistemology in Mathematics Education (ESU-9) took place at Department of Mathematics of the University of Salerno, Campus of Fisciano (SA), from Monday, July 18 to Friday, July 22, 2022 (https://esu9.unisa.it).
The University of Salerno has an important history. Its origins go back to the 8th century when the Salerno medical school was created. After the second half of the 9th century, with the constitution of the Lombard principality of Salerno, the institutional status of the medical school was fully established and was given the authority to produce and train doctors. The Salerno medical school remained a celebrated institution throughout the Middle Ages, and it was an institution where not only medicine was taught but also philosophy, theology, and law.
The city of Salerno lies between Pompei (Roman Empire) and Paestum (Magna Graecia), suggesting, therefore, an intellectual link across the centuries between the traditional Salerno medicine of the Lombard Age and older medical practice of the Greco – Roman period. In general, Salerno and its surroundings are marked by much historical richness and natural beauty. Indeed, it includes several UNESCO World Heritage Sites, including the *Amalfi coast* and *Paestum*, both of which were destinations for excursions during the Summer University.
A Summer University (SU) on the History and Epistemology in Mathematics Education began as an initiative of the French Mathematics Education community in the early 1980's. From those meetings the SU developed into an organization on a European scale and became the European Summer University (ESU) on the History and Epistemology in Mathematics Education. The first ESU was organized in Montpellier (France), 1993. Since then, ESU has been organized in different places in Europe: Braga (Portugal), 1996; Louvain-la-Neuve and Leuven (Belgium), 1999; Uppsala (Sweden), 2004; Prague (Czech

Republic), 2007; Vienna (Austria), 2010; Copenhagen (Denmark), 2014, Oslo (Norway), 2018. Its activities have now been integrated into those of the international HPM Group, which, since 2010, holds a conference every four years.

The principal aims of the ESU are:

- To provide a forum for presenting research in mathematics education and innovative teaching methods based on a historical, epistemological and cultural approach to mathematics and their teaching, with emphasis on actual implementation.
- To offer an opportunity for mathematics teachers, educators and researchers to share their historical knowledge, their teaching ideas and classroom experience related to this perspective.
- To motivate further collaboration along these lines, among members of the mathematics education community in Europe and beyond.

ESU sees mathematics as a human intellectual enterprise with a long history and a vivid present. Besides its "polished" products, those that can be communicated, criticized and incorporated into the body of mathematical knowledge, the process of "doing mathematics" is equally important, especially from a didactical point of view. From this perspective, the meaning of mathematical knowledge is determined not only by its deductively structured theory, but also by the procedures that led, or may lead to it and which are indispensable for its understanding. With that in mind, learning mathematics should include grasping implicit motivations, sense-making and reflection aiming towards the construction of meaning, while teaching mathematics should encourage these processes by providing learners with opportunities to "do mathematics." Perceiving mathematics both as logically structured collections of intellectual products and as processes of knowledge production, should, accordingly, form the core of the teaching of mathematics as well as the image of mathematics spread to the outside world.

It is here that history and epistemology have a role. Emphasizing the integration of historical and epistemological issues in mathematics teaching and learning constitutes a natural way for exposing mathematics in the making. This, in turn may lead to a better understanding of specific parts of mathematics and a deeper awareness of what mathematics as a discipline is. This history and epistemology of mathematics are thus important for mathematics education in that they can help students understand that mathematics:

- Is the result of contributions from many different cultures.
- Has been in constant dialogue with other scientific disciplines, philosophy, the arts and technology.
- Has undergone changes over time according to shifting views of what it is and how it should be taught and learnt.
- Has constituted a constant force for stimulating and supporting scientific, philosophical, technical, artistic, and social development.

The ESU is a framework for teachers and researchers to meet and work together. It is also a one in which beginners, more experienced researchers and teachers may present their teaching experience for the benefit of the participants and receive constructive feedback from one another. It is, moreover, directed towards all levels of education, primary and secondary schools, universities, and in-service teachers' training.

The program and activities of ESU-9 in Salerno were structured around the following main themes:
Theme 1: Theoretical and/or conceptual frameworks for integrating history and epistemology of mathematics in mathematics education.
Theme 2: History and epistemology in students and teachers' mathematics education: Curricula, courses, textbooks, and didactical material of all kinds - their design, implementation and evaluation.
Theme 3: Original historical sources in teaching and learning of and about mathematics.
Theme 4: Mathematics and its relation to science, technology, and the arts: Historical issues and socio-cultural aspects in relation to interdisciplinary teaching and learning.
Theme 5: Topics in the history of mathematics education.
Theme 6: History of mathematics in Italy.

Producing the Proceedings of the ESUs has always been a major task and a great responsibility since they have become standard references in its domain. Indeed, the Proceedings of ESU form a collection that constitutes a source used by teachers and researchers. The papers of all ESU proceedings are available in the website *Publimath* (https://publimath.univ-irem.fr/publimath.php?r=ESU&b=biblio&db=0).

The present volume collects papers or abstracts stemming from all types of activities that were accepted and included in the scientific program of ESU-9.

For each main theme, one plenary lecture was delivered. One plenary session was dedicated to the role of history of mathematics in forming an image of mathematics among students and general public, and a special plenary was given in the memory of Ubiratan d'Ambrosio.

There are also papers based on workshops that focused on specific historical, epistemological, or didactic topics. The role of the workshop organizer at the conference itself was to prepare and present the historical/epistemological (2-hour workshops) or pedagogical/didactical material that integrates historical elements (1.5-hour workshops), which motivates and orients the exchange of ideas and the discussion among the participants. The papers reflect the material presented by the organizers and the subsequent discussions with the participants. Texts and abstracts based on 30-minute oral presentations and abstracts of 15-minute short oral communications are also given below. Limitations of length prevented us from presenting more of these valuable and often fascinating talks. For each paper the theme which it faces is indicated in brackets, after the title.

Unfortunately, we cannot publish the entire text of the theater play by Gavin Hitchcock "Cracking the cubic", about the Italian history of the solution of the third-degree equations, nor, of course, the magnificent performance of the play by Italian ESU-9 participants (Riccardo Bellè, Davide Crippa, Maria Rosaria del Sorbo, Marta Menghini, Elena Scalambro, Luba Softova, Salvatore Tramontano, Antonio Veredice). Proceedings can never fully substitute for the living experience of conference!

We warmly thank all members of the Local Organizing Committee: Maria Giuseppina Adesso, Roberto Capone, Maria Rosaria Del Sorbo, Oriana Fiore, F. Saverio Tortoriello, who made ESU-9 a pleasant scientific and cultural experience. We also thank the University of Salerno for hosting and supporting ESU-9.

The Editors

# ROLES OF THE HISTORY OF MATHEMATICS IN THE MATHEMATICAL KNOWLEDGE FOR TEACHING

**Abraham ARCAVI**
Weizmann Institute of Science, Rehovot, Israel
abraham.arcavi@weizmann.ac.il

## ABSTRACT

"Mathematical Knowledge for Teaching" (MKT) is the knowledge required to practice and accomplish the work of teaching mathematics. In this contribution, I describe its essential components and I propose that history of mathematics can be a bountiful source to enhancing of teachers' MKT through examples and the morals thereof.

## *1 General Introduction*

I am very grateful to Marta Menghini and to all the members of the organizing committee for honoring me with the invitation to deliver this opening lecture at the $9^{th}$ European University on the History and Epistemology in Mathematics Education. I am humbled by this honor.

I begin with a proper disclosure: I am not a historian of mathematics and it has been many years since I worked on issues of history of mathematics in mathematics education. However, history of mathematics has been always close to my mind and to my heart and I salute the European Summer University as well as the HPM for the ongoing relevant and excellent work. A large part of my work in the last two decades is in the area of mathematics teacher knowledge, teaching practice and teacher professional development, and it is from that perspective that I would like to talk today. Thus, I am addressing the roles of history on the knowledge required for teaching mathematics, in the spirit of the work I did in my PhD many years ago (Arcavi, 1985). For that purpose, I am relying on the construct "Mathematical Knowledge for Teaching" (MKT) which is receiving much attention in the last two decades, as a framework for both research and the practice of mathematics education.

## *2 Mathematical Knowledge for Teaching (MKT)*

Mathematical Knowledge for Teaching (MKT) is defined as the <u>specialized</u> mathematical knowledge required to practice and accomplish the work of teaching mathematics (Ball et al, 2008). Its two main components are Subject Matter Knowledge and Pedagogical Content Knowledge.

Subject Matter Knowledge consists of

- *Common Content Knowledge*: Competence with the mathematical topics (concepts, procedures and their underlying ideas); meta-mathematical ideas and nature of mathematical activity and problem solving.
- *Specialized Content Knowledge*: Ways of presenting mathematical ideas, answers to "why" questions; resourcefulness to find appropriate examples and counter-examples; acquaintance with nature and limitations of different representations and knowing how to link among them; knowledge of applications.
- *Horizon Content Knowledge*: "Horizontal depth" that covers extensions, which may go beyond the topic required by the curriculum and acquaintance with contents students will meet in their future academic studies (in order to unpack and stress the required relevant "predecessor" knowledge).

Pedagogical Content Knowledge consists of

- *Knowledge of Content and Students*: Anticipation of (and sensitivity towards) student idiosyncratic ways of knowing, thinking and doing; competence with attentively listening and interpreting student questions and their often unpredicted answers; discernment of difficulty levels and a repertoire of pedagogical resources to deal with them.
- *Knowledge of Content and Teaching*: Design and sequencing of instruction; judicious choice for appropriate tasks and problems, opportune assignment of different modes of student activity (group-worthy activities, digital labs, individual enquiry) according to the nature of the contents and their affordability.
- *Knowledge of Content and Curriculum*: Familiarity with diverse curricular approaches' and proficiency in comparing and contrasting them in order to make thoughtful selections of materials for their classes and having the versatility for implementing them flexibly.

Figure 1 is a graphical representation of these components.

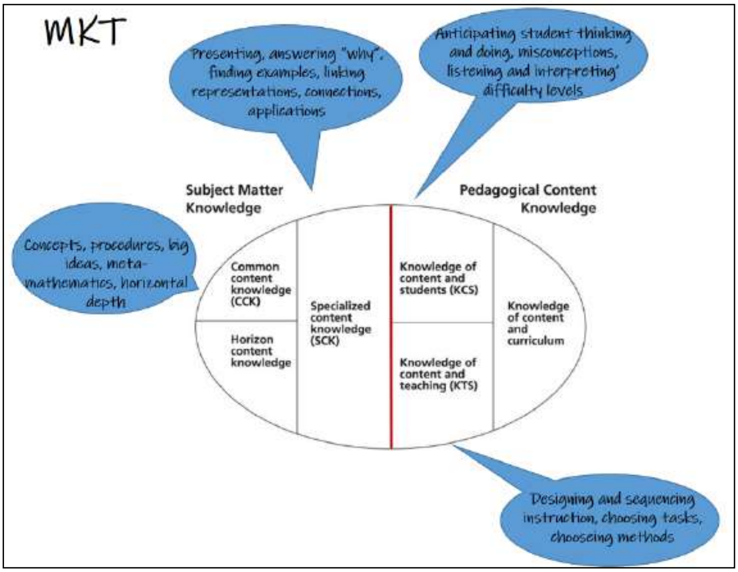



**Figure 1.** Graphical representation of the components of Mathematical Knowledge for Teaching.

### *3 Roles of History of Mathematics*

History of mathematics can play several roles in supporting and enhancing the mathematical knowledge for teaching. In the following, I describe and exemplify some of these roles:

- Source of problems
- Learning to listen
- Revisiting what is taken for granted
- Original texts as interlocutors

### 3.1 Roles of problems

"*Where can I find some good problems to use in my classroom?" is a question I am often asked by mathematics teachers. My answer is simple: "the history of mathematics*." (Swetz, 2000, p. 59)

*The history of mathematics contains a wealth of material that can be used to inform and instruct in today's classroom. Among these materials are historical problems and problem solving situations"* (Swetz, 2000, p. 65)

The following are just a few vivid examples of the above quotes.

The Rhind Papyrus is one of the oldest extant mathematical documents, it is dated around 1500-1600 B.C. and it became widely known less than 200 years ago. It contains arithmetic and geometric problems, some of which are intriguing even today, for example the arithmetic of unit fractions (fractions whose numerator is 1). Consider the following problem, written in modern decimal notation, and borrowed from the Papyrus: "decompose 2/9 into the sum of two unit fractions". An obvious answer is 1/9 + 1/9. However, if we add the restriction that the two unit fractions be different, the problem becomes more interesting. One may resort to trial and error, and then go on to search for more systematic methods, for example looking for a decomposition of 2/9 into different unit fractions, such that in our initial obvious solution of 1/9 + 1/9 we can replace one of the addends by the sum of different unit fractions. We may remind of the following general property:

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} \longrightarrow \frac{1}{9} = \frac{1}{10} + \frac{1}{90}$$

in order to find a first decomposition as follows: 2/9 = 1/9 + 1/10 + 1/90.

Another result can be helpful, by remembering that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$$

which serves as the basis for

$$\frac{1}{n} = \frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n} \longrightarrow \frac{1}{9} = \frac{1}{18} + \frac{1}{27} + \frac{1}{54}$$

Thus 2/9 = 1/9 + 1/18 + 1/27 + 1/54.

History provides us with yet another way called the Fibonacci – Sylvester method, in honor of these famous mathematicians Leonardo of Pisa, a.k.a Fibonacci (c.1170 – c.1250) and James Joseph Sylvester (1814 – 1897). According to this method, in each step we add the largest unit fraction, which is smaller than what remains. In our case, the largest unit fraction smaller than 2/9 is 1/5. Then, in this case, what remains is a unit fraction – problem solved since 2/9 = 1/5 + 1/45, which is already a third decomposition.

The Egyptian table for the decomposition of 2/n (n ≤101), shows yet another possible decomposition (2/6 + 2/18), for which no method is provided (see Figure 2).

| n | a | b | c | d |
|---|---|---|---|---|
| **5** | 3 | 15 | | |
| **7** | 4 | 28 | | |
| **9** | 6 | 18 | | |
| **11** | 6 | 66 | | |
| **13** | 8 | 52 | 104 | |
| **15** | 10 | 30 | | |
| **17** | 12 | 51 | 68 | |
| **19** | 12 | 76 | 114 | |
| **21** | 14 | 42 | | |
| **23** | 12 | 276 | | |
| **25** | 15 | 75 | | |
| **27** | 18 | 54 | | |
| **29** | 24 | 58 | 174 | 232 |
| **31** | 20 | 124 | 155 | |
| **33** | 22 | 66 | | |
| **35** | 30 | 42 | | |
| **37** | 24 | 111 | 296 | |
| **39** | 26 | 78 | | |
| **41** | 24 | 246 | 328 | |
| **43** | 42 | 86 | 129 | 301 |
| **45** | 30 | 90 | | |

| n | a | b | c | d |
|---|---|---|---|---|
| **47** | 30 | 141 | 470 | |
| **49** | 28 | 196 | | |
| **51** | 34 | 102 | | |
| **53** | 30 | 318 | 795 | |
| **55** | 30 | 330 | | |
| **57** | 38 | 114 | | |
| **59** | 36 | 236 | 531 | |
| **61** | 40 | 244 | 488 | 610 |
| **63** | 42 | 126 | | |
| **65** | 39 | 195 | | |
| **67** | 40 | 335 | 536 | |
| **69** | 46 | 138 | | |
| **71** | 40 | 568 | 710 | |
| **73** | 60 | 219 | 292 | 365 |
| **75** | 50 | 150 | | |

| n | a | b | c | d |
|---|---|---|---|---|
| **77** | 44 | 308 | | |
| **79** | 60 | 237 | 316 | 790 |
| **81** | 54 | 162 | | |
| **83** | 60 | 332 | 415 | 498 |
| **85** | 51 | 255 | | |
| **87** | 58 | 174 | | |
| **89** | 60 | 356 | 534 | 890 |
| **91** | 70 | 130 | | |
| **93** | 62 | 186 | | |
| **95** | 60 | 380 | 570 | |
| **97** | 56 | 679 | 776 | |
| **99** | 66 | 198 | | |
| **101** | 101 | 202 | 303 | 606 |

**Figure 2.** The modern notation of the Egyptian table for decomposition of 2/n into unit fractions

These problems are borrowed from history. We note and stress the "borrowing from" in order to illustrate how history can be a bountiful source of nice problems, and nice solutions, using our knowledge and symbolism. Furthermore, we can also exemplify a question of historic-mathematical interest: how did the Egyptians arrived at their result (not obtained by any of the methods proposed above) and why did they prefer it to others.

These examples illustrate how history may support and enhance the mathematical knowledge for teaching by providing a repertoire of interesting problems (Specialized Content Knowledge). Moreover, this example enriches the knowledge of a seemingly simple topic like fractions, and enacts aspects of mathematical activity such as evoking connections to previous knowledge, bringing it to bear in order to develop systematic methods of expanding fractions into unit fractions and to ponder about their generality (Common Content Knowledge).

History shows how ancient this mathematical topic is and yet in spite of being elementary, it includes unsolved problems as of today, such as the Erdös-Strauss conjecture proposed in 1948 and still unproven: for every $n \in N$, $n>1$, $4/n$ can be expressed as the sum of three unit fractions (e.g. Graham, 2013).

Another very interesting problem from historical sources and its use in a classroom is the following: "In 1355 the Italian professor of law Bartolus of

Saxoferrato (1313-1357) wrote a treatise on the division of alluvial deposit. The problem he discussed is the following… Some landowners … Gaius, Lucius and Ticius, have neighbouring properties besides the bank of a river. The river deposits silt so that the new land is formed at the riverside. How is the new fertile soil to be divided up?" (Van Maanen, 1992, p.37).

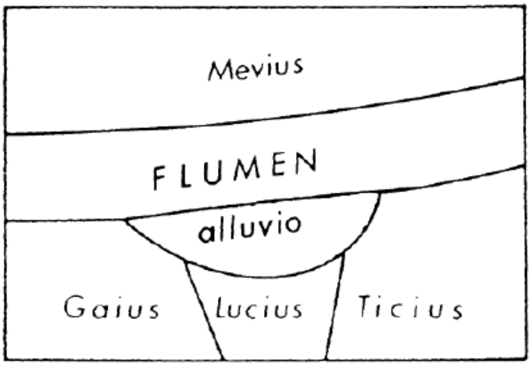



**Figure 3:** Map of the division of the alluvial deposit problem

The following are some of the goals of using this problem in the classroom: "to demonstrate the importance of mathematics in society; …to integrate disciplines; to let pupils to discover a number of constructions with ruler and compasses… to let pupils solve some juridical problems using the constructions that they had discovered earlier" (van Maanen, 1992, p. 42).

Laurence Sherzer, a mathematics teacher in a Florida school reported on an eight-grade mathematics lesson he had taught about the betweenness property of rational numbers. The class worked on methods of finding a rational number between two given rational numbers. They focused on the average and on how to calculate it (adding the two given numbers and dividing the sum by 2). The students not only practiced the procedure of adding fractions and dividing by 2, but they also had a procedural way to be convinced that since it is always possible to find an average between two given rational numbers, there is always one number between two given ones. Then "a student who had not been paying much attention but had been scribbling furiously suddenly interrupted. "Sir, you don't have to go to all that trouble to find a fraction between two fractions, all you have to do is add the tops and the bottoms." (Sherzer, 1973, p. 229). Sherzer admitted that he was going to reject outright the idea, possibly having in mind the typical erroneous procedure many students have for adding two fractions (i.e. numerator plus numerator over denominator plus denominator). On second thoughts, he decided to go along with the student suggestion. He suggested to go back and to apply the student's procedure to the examples they had already worked out by finding averages, and the new method indeed yielded a number in between. The class became excited and they tried many examples, until the teacher suggested to try to find a general proof to show that the procedure proposed indeed always yields a number in between two givens. The algebraic proof is rather simple (for a visually very convincing geometrical proof, see Arcavi, 2003). The teacher acknowledged that he did not know such property and the class was over, he "thought of that one moment when I was about to tell Mac Kay [the student's name], 'No, that's not the way it's done'" (Sherzer, 1973, p. 230).

This property, which the teacher named as "MacKay's Theorem", was firstly documented in the book *Le Triparty en la Science de Nombres* by the French mathematician Nicolas Chuquet (1445?-1488?). The manuscript of this book remained in private hands for about four centuries and was published in 1880 in Italy. "La rigle des nombres moyens" in its English version (Flegg et al., 1985, p. 91) reads as follows:

whoever would want to find an intermediate between $3\frac{1}{2}$ and $3\frac{1}{3}$ should add 1 to 1, which are the two numerators, and they come to 2 for numerator, and then 2 to 3 which are the two denominators, making 5 for denominator. Thus I have $3\frac{2}{5}$ as an intermediate between $3\frac{1}{2}$ and $3\frac{1}{3}$. For $\frac{2}{5}$ is more than $\frac{1}{3}$ and less than $\frac{1}{2}$.

**Figure 4:** English translation of Chuquet's Rule of Intermediate Numbers

Had the teacher known about episodes from the history of rational numbers, and in particular Chuquet's rule (*Common Content Knowledge*), he could have saved to himself the indecision about pursuing the student suggestion, and the risk of rejecting it, which he was at the verge of doing. Fortunately, the teacher did pursue the student proposal and, thanks to his opportune decision to listen to the student, he uncovered a piece of mathematics new to him and to his students (except for MacKay…).

### 3.2 Learning to listen

When students genuinely engage in learning and doing mathematics, they frequently proceed in idiosyncratic yet reasonable and productive ways, which are not always aligned with what teachers expect, and such was the case of

MacKay. Thus, the important component of MKT "*Knowledge of Content and Students*" includes (a) the anticipation of (and sensitivity towards) student distinctive ways of knowing, thinking and doing and (b) the competence of listening attentively and interpreting student questions, their often unpredicted ways of reasoning, their answers and their unexpected suggestions and conjectures.

By "listening to students", we mean giving careful attention to students, trying to understand what they say and do and the possible sources and entailments thereof. Such listening should include:

- detecting, taking up and creating opportunities for students to engage in expressing freely their mathematical ideas;
- questioning students in order to uncover the essence and the sources of their ideas;
- analyzing what one hears (sometimes in consultation with peers), making the enormous intellectual effort to adopt the 'other's perspective' in order to understand it on its own merits; and
- deciding in which ways to integrate productively students' ideas into the development of the lesson.

The importance of listening as a teaching skill cannot be overstated. It may be a strong manifestation of "a caring, receptive and empathic form" (Smith, 2003, p. 498) of teacher-student interactions. If often modeled by teachers, students as well as their mathematical productions would feel respected and valued. Moreover, the habit of listening as modeled by the teacher may be internalized by the students and become a habit in their repertoire of learning techniques and interpersonal skills. Above all, listening enables teachers to understand better student thinking for the benefit of good teaching and robust learning, and may benefit the teachers themselves. "Thinking ourselves into other persons leads us to reflect about our own relationship to mathematics" (Jahnke, 1994, p. 155). In other words, effective listening may influence 'listeners,' by making them re-inspect their own knowledge, against the background of what was heard from others. Such re-inspection of the listener's own understandings may promote the re-learning of some mathematics or meta-mathematics. There are several candid self-reports of this phenomenon even by mathematicians (e.g. Aharoni, 2005; Henderson, 1996).

Listening to students poses several challenges. For example, once we understand a complex idea, we may tend to forget (or even dismiss) the process we underwent while learning that idea. Listening requires unpacking that process. Listening also requires "decentering", namely the capability to adopt another person's perspective discarding as much as possible our own.

Given the importance of 'listening' towards understanding the students' point of view and in spite of the challenges it poses, it is a learnable ability. History of mathematics can provide rich scenarios for such learning, for example by approaching certain primary sources. Primary sources often offer ways of doing mathematics different from what is common nowadays and may conceal the thinking behind them. When facing a historical source with an approach foreign to us, we cannot dismiss it as 'incorrect', in the same way that we as teachers may dismiss an unexpected student approach. When facing an initially cryptic source, an effort may be required to make sense of it, and this activity of "deciphering" requires exercising a similar kind of decentering and unpacking needed for listening to students. Thus, working with teachers on activities of reading and understanding idiosyncratic ways of doing mathematics is a way of learning to listen. Such an activity taken from an extract of the Rhind Papyrus in which, what we call today, a linear equation with one unknown is solved was tried with prospective teachers. A detailed description of the activity and the findings of the experience can be found in Arcavi & Isoda (2007). The activity of deciphering the primary source was shown as promising in supporting the effort and nurturing the capability of understanding the other's perspective.

### 3.3 Revisiting was taken for granted

"I have observed, not only with other people but also with myself […] that sources of insight can be clogged by automatism. One finally masters an activity so perfectly that the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question." (Freudenthal, 1983, p. 469)

The development of ideas in history provides a repertoire of intricacies that may illuminate aspects of mathematics around questions that "are not asked anymore". Consider, for example, the following text taken from a letter by the mathematician Antoine Arnauld (1612-1694) to a colleague Jean Prestet (1648-1691) as it appears in Schrecker (1935).

*"…1 is to 3, as 4 is to 12. And 1 is 1/3 as 1/4 is 1/12. But I cannot adjust this to multiplications of two minus. For will we say that +1 is to -4, as -5*

*is to +20? I do not see it. Because +1 is more than -4. And conversely -5 is less than +20. Whereas in all other proportions, if the first term is greater than the second, the third must be greater than the fourth.*"

This text was presented to teachers in several workshops, and they were asked to formulate an answer to this contradiction between the idea of proportionality and the rule for multiplying two negative numbers, as if a student (Arcavi, 1985) posed it. This task was aimed at enhancing teachers' Specialized Content Knowledge.

### 3.4 Original texts as interlocutors

*The Principles of Algebra* by William Frend (1757-1841) was published in London in 1796. In this book, there is a virulent attack on the use of negative numbers. The following are extracts reflecting the arguments.

The ſirſt error in teaching the principles of algebra is obvious on peruſing a few pages only in the firſt part of Maclaurin's Algebra. Numbers are there divided into two ſorts, poſitive and negative; and an attempt is made to explain the nature of negative numbers, by alluſions to book-debts and other arts. Now, when a perſon cannot explain the principles of a ſcience without reference to metaphor, the probability is, that he has never thought accurately upon the ſubject. A number may be greater or leſs than another number; it may be added to, taken from, multiplied into, and divided by another number; but in other reſpects it is very untractable: though the whole world ſhould be deſtroyed, one will be one, and three will be three; and no art whatever can change their nature. You may put a mark before one, which it will obey: it ſubmits to be taken away from another number greater than itſelf, but to attempt to take it away from a number leſs than itſelf is ridiculous. Yet this is attempted by algebraiſts, who talk of a number leſs than nothing, of multiplying a negative number into a negative number and thus producing a poſitive number, of a number being imaginary.

This is all jargon, at which common ſenſe recoils; but, from its having been once adopted, like many other figments, it finds the moſt ſtrenuous ſupporters among thoſe who love to take things upon truſt,

**Figure 5:** Extracts from Frend's Algebra

The text's arguments are the following:

- Rejection of "reference to metaphors" ("debts and other arts")
- Numbers as magnitudes ("one will be one")

- Rejection of an extended version of operations (taking away only the small number from the greater, otherwise ridiculous)
- Rejection of an extended version of number ("a number being imaginary")

These claims may open up productive discussions about the very nature of mathematics, the place of generalizing beyond the concrete, and the role of didactical resources in presenting and concretizing abstract ideas.

Frend's book has many mathematical developments in which he juggles in order to avoid the use of negative numbers. Of special interest is his treatment of the solution of quadratic equations which purposefully avoid negative numbers. Not only are his mathematical arguments worth following but also the vivid experience of how the separation of cases (to avoid negative numbers) losses the efficiency and elegance of generality.

## 4 *Final remarks*

In this presentation, I attempted to illustrate the crucial roles that history of mathematics can play in supporting and enhancing the development of Mathematical Knowledge for Teaching. Further examples and further roles can be found, explored and tried out in the many environments for teacher education and teacher professional development forums.

**REFERENCES**


Aharoni, R. (2005). What I learned in elementary school. *American Educator, 29*(3). Available at https://www.aft.org/periodical/american-educator/fall-2005/what-i-learned-elementary-school Accessed 2 December 2022.

Arcavi, A. (1985). *History of Mathematics as a Componenet of Mathematics Teachers Background*. Unpublished PhD Dissertation, Weizmann Institute of Science, Rehovot.

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*(3), 215-241. DOI: 10.1023/A:1024312321077

Arcavi, A. & Isoda, M. (2007). Learning to listen: from historical sources to classroom practice. *Educational Studies in Mathematics*, *66*(2), 111-129. DOI: 10.1007/s10649-006-9075-8

Ball, D. L., Thames, M. H. & Phelps, G. (2008). Content knowledge for teaching: What makes its special? *Journal of Teacher Education, 59*(5), 389-407. DOI:10.1177/0022487108324554

Flegg, G., Hay, C. & Moss, B. (1985). *Nicolas Chuquet, Renaissance Mathematician*.

Reidel.
Freudenthal, H. (1983) *Didactical Phenomenology of Mathematical Structures*. Kluwer.
Graham, R. L. (2013). Paul Erdös and Egyptian Fractions. In L. Lovász, I. Z. Ruzsa & V. T. Sós, (Eds.) *Erdös Centennial*. Springer, 289-309. Available at https://link.springer.com/chapter/10.1007/978-3-642-39286-3_9 Accessed 2 December 2022.
Henderson, D. W. (1996). I learn mathematics from my students – Multiculturalism in action. *For the Learning of Mathematics, 16*(2), 46–52.
Jahnke, H. N. (1994). The historical dimension of mathematical understanding – Objectifying the subjective. In J. P. da Ponte, & J. F. Matos (Eds.), *Proceedings of the 18th International Conference for the Psychology of Mathematics Education – PME18*, 1. Lisbon, Portugal, 139–156. Available at https://www.igpme.org/wp-content/uploads/2019/05/PME18-1994-Lisbon.pdf Accessed 2 December 2022.
Sherzer, L. (1973). McKay's Theorem. *Mathematics Teacher*, *66*(3), 229-230.
Schrecker, P. (1935). Arnauld, Malebranche, Prestet et la théorie de nombres négatifs (d'après une correspondance retrouvée). *Thalès, 2*, 82-90. Available at https://www.jstor.org/stable/pdf/43861530.pdf?refreqid=excelsior%3A2b769455b923259d464a3df72f859ee4&ab_segments=&origin=&acceptTC=1 Accessed 2 December 2022.
Smith, T. J. (2003). Pedagogy as conversation: A metaphor for learning together. Mathematics Association of Victoria Annual Conference, Monash University. Melbourne, 491-505. Available at http://w.mav.vic.edu.au/files/conferences/2003/Smith_paper.pdf Accessed 2 December 2022.
Swetz, F. J. (2000). Problem solving from the history of mathematics. In Katz, V. (Ed.) *Using History to Teach Mathematics. An International Perspective*. The Mathematical Association of America, 59-68.
Van Maanen, J. (1992). Teaching Geometry to 11 Year Old "Medieval Lawyers". *The Mathematical Gazette*, *76*(475), 37-45.

# WHAT HISTORY TRAINING FOR FUTURE MATHEMATICS TEACHERS?

## PERSONAL EXPERIENCES AND REFLECTIONS

**Dominique TOURNÈS**
University of Reunion Island, Laboratory of Computer Science and Mathematics
dominique.tournes@univ-reunion.fr

## ABSTRACT

In this paper, I present my way of training future teachers in the use of history to improve students' mathematical learning. The strategy that I recommend includes three components in constant interaction. First of all, there is the manipulation of artefacts that the teacher can use in a process of semiotic mediation to favour the appropriation of the mathematical knowledge embedded in these objects. This importance given to gestures, procedures and instruments is reinforced by an opening towards ethnomathematics. Secondly, there is the study of short original texts taken mainly from six major works covering almost completely the contents of secondary education. This study is done in close connection with specific curriculum items and the conception of scenarios for the classroom. Finally, the future teachers are responsible for designing pedagogical sequences inspired by history and experimenting with them in their classes during the internships. This devolution phase, which is the subject of didactic analyses, seems to me essential for a sustainable integration of the training's achievements. The main objective of this triptych of activities is not to train future teachers in history, but to train them in the teaching of mathematics by deepening their knowledge of the discipline on the cultural, epistemological and didactic levels.

### *1 Context and guidelines of my training practice*

As a historian of mathematics, my research focuses on the history of numerical analysis in the broadest sense of the term. I am interested in the methods and instruments of calculation in the period before the computer, between the middle of the 18th century and the middle of the 20th century. In particular, I have worked on differential equations, graphical calculus, nomography, numerical tables and engineering mathematics (Tournès, 2022).

For many years I have been involved in the training of secondary school teachers in the history of mathematics. In this training, I do not see history as an end in itself, but as an entry point to change teachers' and students' representations of mathematics, and to foster students' learning. The aim of this paper is to give a testimony of the personal way in which I put this teacher training into practice.

### 1.1 My environment: Two French islands in the Indian Ocean

Before getting into the heart of the matter, I would like to complete these elements of context with a few words about the places where I teach, which are located in the Indian Ocean, the third largest ocean on the planet, bordered by many states with very different cultures and languages. My immediate environment is in the southwest, where there are about one hundred and fifty islands grouped into five states: Madagascar, the Seychelles, Mauritius, the Comoros and France.

All these islands are multi-ethnic, multi-cultural and multi-lingual societies, with populations from Asia, Africa and Europe. Two of these islands are French departments: Reunion, which is part of the Mascarene archipelago, and Mayotte, which is part of the Comoros archipelago. In these two islands, the educational system, the curricula, the university structures and the teacher training system are the same as in France, so what I am going to talk about is representative of what is currently being done in France to integrate the history of mathematics into teaching.

### 1.2 My institutions and sources of inspiration

In Reunion Island, teacher training is carried out within INSPE (National Higher Institute of Teaching and Education), an institute belonging to the University of Reunion Island. As in other parts of France, this university also has an IREMI (Research Institute for Mathematics and Computer Science Education), where action research is carried out to support training. In Mayotte, there is a University Centre and an IREMIS (IREMI + S as Science), but teacher training depends on the University of Reunion Island. That's why I also teach there. In both islands, I am in charge of a history of mathematics unit in the master's degree course for secondary school teachers. In each of the islands, there are about twenty students in each year of the master.

The training that I have designed and that I am implementing is the result of a long experience. It has been constantly nourished by my participation in the work of the inter-IREM commission in France, the HPM meetings and the

European Summer Universities at international level. Numerous studies presented at HPM and ESU meetings have examined the epistemological, methodological, practical and didactic objections that could slow down the integration of history into mathematics teaching. In return, these studies have highlighted the cognitive, epistemological, didactic, affective and cultural contributions of the integration of history, and have addressed the crucial problem of rigorously evaluating the effects of this integration on teachers' practices and students' mathematical learning (for a synthetic overview of these studies, see Clark et al., 2019; Chorlay et al., 2022).

### 1.3 An initial assessment

My aim here is not to go back over these high-quality theoretical works, to which I would have little new and relevant to add, but to explain how I have appropriated them in order to put them into practice effectively in my activity as a secondary school teacher trainer. As a starting point, I would simply like to take up the data and analysis presented by Marc Moyon at the 2021 HPM meeting (Moyon, 2021, 2022).

Based on a survey of 646 in-service secondary school teachers, he showed, first of all, that the vast majority of teachers do not introduce history of mathematics in their lessons or only introduce it very occasionally, fewer than three times a year. Only 9% of teachers use it regularly, in almost every lesson. If we take a closer look at what they do, it is principally to introduce a teaching sequence, in particular by means of anecdotes, short biographies or elements of context, in other words, historical snippets with little didactic impact on mathematical learning. On the other hand, when teachers were asked if they were interested in the history of mathematics and if they would like to use it, a large majority (71%) answered yes. Teachers are therefore willing, even enthusiastic, to use elements of history in their lessons.

We are thus faced with a complex challenge: how to get teachers to move from intention to action. To do this, as Marc Moyon has shown, we must meet their expectations by providing them with historical knowledge, of course, but also with didactic reflections, exchanges of practices, and an evaluation of the effects of history on the mathematical representations and learning of pupils. In this task, I think that we must remain modest. The right solution could be to start from the ordinary practices of teachers, their teaching programmes, their school textbooks and the usual documents at their disposal. Without upsetting their familiar pedagogical context, we can help them to develop a critical view of the historical data present in their primary resources and provide them with ways to transform and enrich these resources.

## *2 My overall strategy*

The strategy I advocate has three constantly interacting components that complement my course. First of all, there is the manipulation of artefacts that the teacher can exploit in a process of semiotic mediation to favour the appropriation of the mathematical knowledge embedded in these objects. This importance given to gestures, procedures and instruments is reinforced by an opening towards ethnomathematics.

Secondly, there is the study of short original texts. This work is done in close connection with specific curriculum items and the design of scenarios for the classroom. Here, it seems important to me not to limit ourselves to textual sources. In the history of mathematics, the sources are, on the one hand, texts and textual inscriptions, but also artefacts and instruments whose manufacture and use testify to mathematical activity.

Finally, the future teachers are asked to design pedagogical sequences inspired by history and to experiment with them in their classes during their practical training. This devolution phase, which is the subject of a priori and a posteriori didactic analysis, seems to me to be essential for a lasting integration of the training's achievements.

The evaluation of my teaching unit includes two tests:

- A three-hour written examination consisting of three to five exercises on original artefacts or texts, each with historical, mathematical and pedagogical questions.
- The writing and defence of a memoir on the design, classroom experiment, and didactic analysis of an educational sequence inspired by history.

### 2.1 The outline of my course

We will go through these three components of my training in turn. Before, here is the general outline of the course (see figure 1). As I only have a limited amount of time, I have decided to organise my course around six mathematicians and six books which mark crucial stages in the evolution of mathematics and which allow me to cover almost all the contents which appear in the secondary school curricula. These six mathematicians are:

- Euclid, in link with the axiomatic and deductive method, and the essential results of elementary geometry and number theory;

- al-Kwhārizmī for the constitution of algebra as an autonomous discipline;

- Descartes who, in a way, synthesised Greek geometry and Arabic algebra to give birth to coordinate or analytical geometry;

- Newton for the development of infinitesimal calculus;

- Cauchy and the constitution of the classical analysis based on the notion of limit;

- and finally, Jakob Bernoulli for his founding work of probability theory.

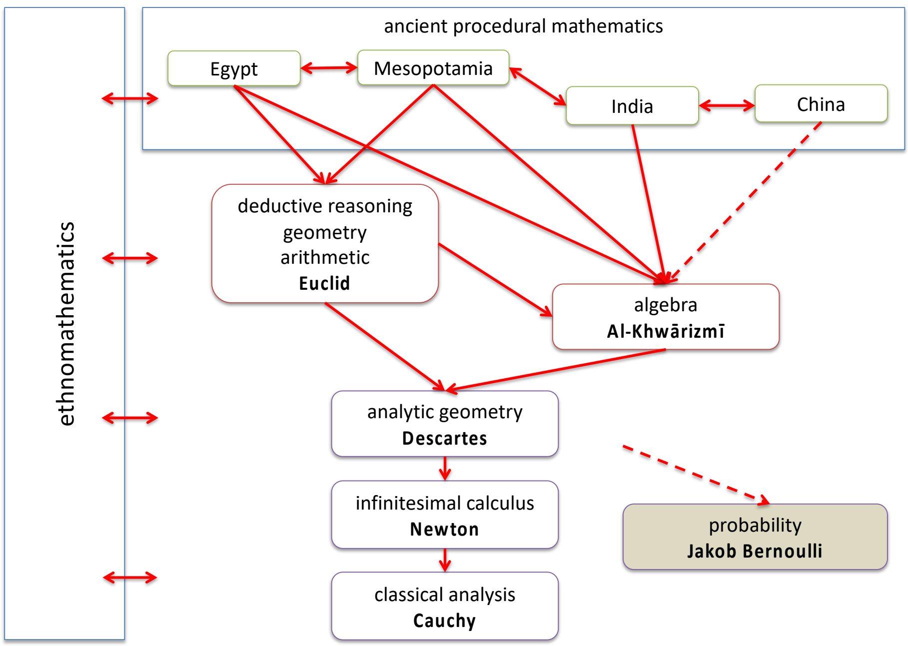



**Figure 1.** The outline of my course

If I focus my course on these six mathematicians so as not to disperse myself, this does not mean that I am talking only about them. I am, of course, placing them in a wider context. For example, in my lecture on Descartes, I also talk about Fermat, who at the same time formalised coordinate geometry. Or, for another example, I don't give my lecture on Newton without talking about Leibniz, the other inventor of the infinitesimal calculus.

Alongside this content, I have included in the top margin the ancient mathematical traditions of Egypt, Babylon, India and China. For lack of time, I have chosen not to devote specific lectures to them. However, of course, when I talk about Greek mathematics, I mention its Egyptian and Babylonian origins, and when I talk about Arabic mathematics, I mention in addition the influence of Indian mathematics. Students also have the opportunity to explore these ancient traditions on their own and to talk about them to the other students when defending their memoirs.

The margin on the left concerns ethnomathematics. Originally, ethnomathematics was seen as the study of mathematical ideas and practices in societies without writing. Now the definition has been broadened. It is, in general, the study of culturally specific uses of mathematical concepts and knowledge, especially those developed outside the scholarly and institutional field. In this sense, we can say that, insofar as they are inscribed in a specific culture, society and language, Egyptian, Babylonian, Indian or Chinese mathematics are ethnomathematics. In another direction, the mathematical practices specific to certain professional circles that have not studied academic mathematical content, such as sculptors, stone cutters or carpenters, are part of ethnomathematics. Finally, ethnomathematics is concerned with the primary mathematical knowledge acquired by children in their family or their cultural environment, a knowledge that may conflict with school learning.

Ubiratan d'Ambrosio (2006), one of the founders of ethnomathematics, has highlighted that in the history of mathematics, three types of knowledge should be taken into account: firstly, the scholar mathematics developed by mathematicians, secondly, the mathematics taught at school, and thirdly, the mathematics used in the street and in the workshop. This triptych seems to me very important: when we conceive a school activity, we must take into account, on the one hand, the scholarly knowledge and its didactic transposition, and on the other hand, the cultural roots, representations and primary knowledge of the children.

## 2.2 Correspondence between texts, curriculum and artefacts

As I said, the historical sources that I use, whether texts or instruments, are closely linked to items in the secondary school curriculum. I hope that the following table makes it possible to see more clearly the interest of the six books that I exploit in my training.

- The *Elements* of Euclid allow us to work on elementary geometry, including the Pythagorean theorem and the intercept theorem, which are at the heart of Middle School geometry. They also provide an opportunity to review the main concepts of arithmetic such as divisors, greatest common divisor, Euclid's algorithm and prime numbers. The associated artefacts are, of course, the ruler and the compass, which correspond directly to Euclid's geometry, puzzles to demonstrate the Pythagorean theorem, and cultural patterns, such as rosettes, friezes and pavings, which constitute a good entry into the elementary objects of geometry and usual geometric transformations.

| **Texts** | **Curriculum of Middle School and High School** | **Artefacts** |
|---|---|---|
| Euclid<br>*The Elements* | Elementary geometry<br>Pythagorean theorem<br>Intercept theorem<br>Divisors, gcd, prime numbers | Ruler and compass<br>Puzzles<br>Cultural patterns |
| Al-Khwārizmī<br>*al-Kitāb al-Mukhtaṣar fī Ḥisāb al-Jabr wal-Muqābalah* | Indian calculation<br>Elementary algebra<br>Linear and quadratic equations | Token abacuses |
| Descartes<br>*La Géométrie* | Coordinate geometry<br>Equations of straight lines, circles, and other curves | Ruler, compass and conics<br>Linkages<br>Nomograms |
| Newton<br>*The Method of Fluxions and Infinite Series* | Numerical solution of equations<br>Tangent and quadrature problems<br>Extremum problems | Planimeters<br>Integraphs |
| Cauchy<br>*Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal* | Limits, continuity<br>Derivatives, integrals<br>Common functions | Graphic construction of differential equations |
| Jakob Bernoulli<br>*Ars Conjectandi* | Combinatorics<br>Probability, expected value<br>Law of large numbers | Dices, cards<br>Spreadsheets |

**Table 1.** Correspondence between texts, curriculum and artefacts

- Concerning al-Khwārizmī, I talk about his book on algebra, which allows us to work on the introduction to algebra and on linear and quadratic equations, but I also talk about his lost book on Indian calculation, which allows us to go back to numeracy and operating techniques. Manipulations with token abacuses, which make it possible to understand that Indian calculation is an abstract transcription on paper of the manipulations that one used to do on the abacus, prove to be fruitful, as much with future teachers as with the pupils (Daval & Tournès, 2018).

- Descartes' *Geometry* offers the opportunity to study coordinate geometry, equations of straight lines, circles and other curves. Conics are no longer directly on the French High School curriculum, but there is still the parabola as the representative curve of the square function and the hyperbola as the representative curve of the inverse function. These curves are sufficient for geometric construction activities with ruler, compass and conics, instruments that allow solving equations up to the fourth degree. Parabolic and hyperbolic nomograms are another type of instruments that can be used to solve equations up to the fourth degree (Tournès, 2018b). Finally, there is also the possibility of manipulating linkages that give access to other algebraic curves.

- In Newton's *Method of Fluxions and Infinite Series*, there are interesting extracts for introducing the numerical solution of equations, the questions raised by the handling of infinity in mathematics, tangent problems, quadrature problems and extremum problems. In parallel, one can propose manipulations with planimeters and integraphs, those mechanical instruments that perform the operations of the integral calculus in an exact manner.

- Cauchy's *Course* is particularly relevant because it is very close, in its presentation, to what is currently done in High School, where the notions of limit, derivative and integral are defined in natural language, without using the formalism that was introduced later by Weierstrass. In connection with the Euler-Cauchy method used by Cauchy for the definition of integrals and for the proof of the fundamental existence theorem for differential equations, one can propose graphical constructions of integral curves of differential equations, which allows one to acquire a kinaesthetic knowledge of what is an integral curve of a differential equation (Tournès, 2018a).

- Finally, Jakob Bernoulli's *Ars Conjectandi* provides a good reflection on the beginnings of probability theory. Classical historical problems, such as the

Duke of Tuscany problem or the Chevalier de Méré problem can be experimented with dices and then simulated with a spreadsheet.

## *3 Manipulation of artefacts to construct mathematical knowledge*

After this general presentation of my course, I will go into a little more detail about each of the types of activities proposed to future teachers. The first is the manipulation of artefacts. In this regard, I would like to mention the work of Maria Bartolini Bussi and Michela Maschietto (2007), who have written extensively on the use of artefacts to construct mathematical knowledge using semiotic mediation theory. I am particularly grateful to Michela Maschietto who trained us in this theory when she came as a visiting professor to University of Reunion Island a few years ago. It was under her influence that I introduced more and more artefact manipulations into my training. I would also like to mention the work of Pierre Rabardel (1995), which has had a great impact on mathematical instrumentation. In particular, he introduced the distinction between artefact and instrument, as well as the notions of instrumentation and instrumentalisation, and the process of instrumental genesis. In France, in a book on mathematical constructions directed by Évelyne Barbin (2014), the inter-IREM commission also worked on the consideration of gestures and instruments. On the same theme, I would like to mention a very rich work by John Monaghan, Luc Trouche and Jonathan Borwein (2018) which contains a lot of information on the relationship between material tools and mathematical concepts. Finally, recent research in ethnomathematics can also shed light on the role of gestures, procedures and instruments in the emergence and transmission of mathematical ideas.

### 3.1 Ruler and compass

With these references in mind, let’s take a closer look at some of the artefacts I introduce in my training. The photos I will use to illustrate this paper sometimes show my own students, who are future teachers, and sometimes show my students’ students, who are middle school or high school students. It doesn’t matter, because I do with the future teachers the same manipulations that they later do with their students.

In connection with Euclid’s *Elements*, a reflection on the ruler and compass is necessary. Understanding the material sources of the definition of the circle and the third postulate in the *Elements*, and conversely the practical implementation of these abstract notions, deserves reflection. On figure 2, you can see students constructing the centre of a circle with a compass alone. This is the famous Napoleon's problem. Geometric activities in macro-space and meso-space seem to be necessary, because the micro-space of the sheet of paper is already an abstraction. The examination of a compass used by a Malagasy sculptor (see figure 3) is also fruitful. We can see that the usual compass used on paper can be problematic: it is not a plane linkage, and the radius of the circle is not materialised, it remains virtual. On the other hand, with a taut rope or a rigid rod, we have a plane linkage, as simple as possible, which materialises the radius of the circle.

**Figure 2.** Napoleon's problem

**Figure 3.** The compass of a Malagasy sculptor

### 3.2 Puzzles for Pythagorean theorem

In parallel with the Euclid's proof of the Pythagorean theorem, it can be fruitful to examine other demonstrations. I studied all the puzzles I could find for the demonstration of this theorem, and I came to the conclusion that there were 11 fundamentally different ones, based on distinct ideas, the others being just variations of those (Tournès, 2017). I had these 11 wooden puzzles made with a laser machine (see figure 4).

This is how I use these puzzles with my future teachers. Each group is tasked with studying a particular puzzle by answering the following questions:

- Complete the puzzle.

- Prove that the pieces of the two small squares fill the large square exactly, without loss or overlap.

- Write a construction algorithm for the cutting of the two small squares and execute it, either with geometry instruments or with dynamic geometry software.

- Search on the internet for the historical origin of the puzzle: Author? Period? Context?

- Compare with Euclid's proof of the Pythagorean theorem.

- What pedagogical use could you make of this puzzle in the French 4th grade (13-14 years old)?

At the end of the workshop, each group presents its findings to the others.

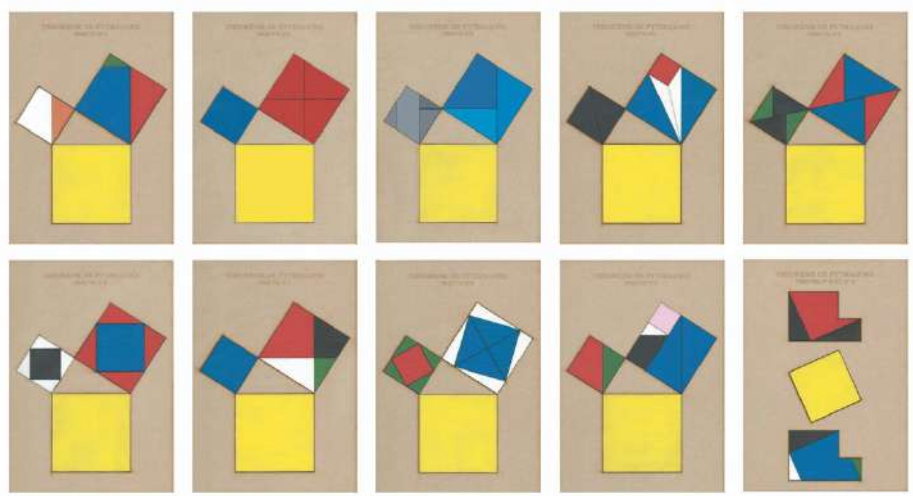


**Figure 4.** Eleven puzzles for the Pythagorean theorem

## 3.3 Cultural patterns from Reunion, Mayotte and Madagascar

In the local decorative arts, make-up, embroidery, basketry, woodcarving, etc., in Reunion, Mayotte and Madagascar, one finds patterns that allow one to work on the elementary figures of geometry and on the isometries of the plane (see figure 5). In particular, rosettes, friezes and pavings can be found. It is known that there are two infinite groups of rosettes, seven groups of friezes and seventeen groups of pavings. Working on these local artefacts as part of their culture is a great motivation for teachers and students.

In passing, I would like to read a quotation from Descartes, in the *Rules for the Direction of the Mind*, which recommends that we start from the observation of craft objects with regularities to extract abstract ideas:

> "Still, since not all minds have such a natural disposition to puzzle things out by their own exertions, the message of this Rule is that we must not take up the more difficult and arduous issues immediately, but must first tackle the simplest and least exalted arts, and especially those in which order prevails – such as weaving and carpet-making, or the more feminine arts of embroidery, in which threads are interwoven in an infinitely varied pattern." (Descartes, 1985, p. 35)

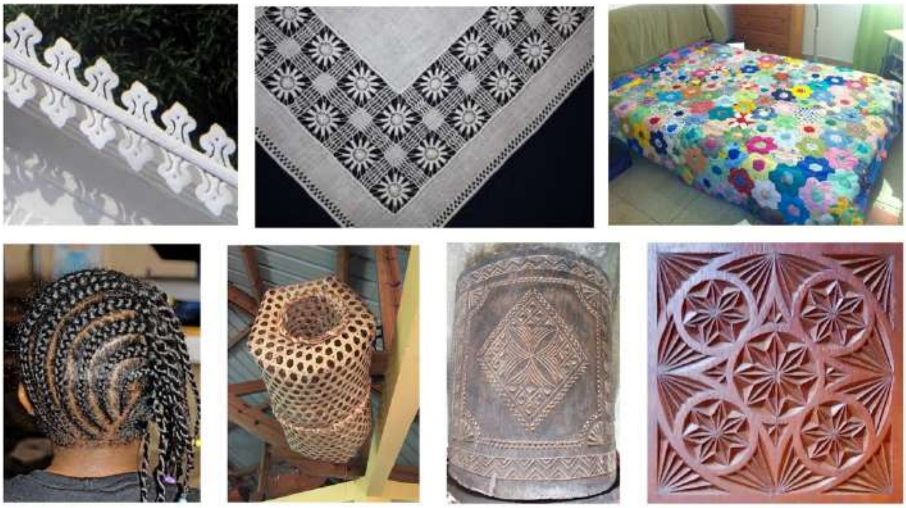


**Figure 5. Cultural patterns from Reunion, Mayotte and Madagascar**

### 3.4 Token abacuses

As I said, I use the token abacuses a lot, to make the future teachers work on numeration, operating techniques and the history of calculation, which goes from token or ball abacuses to electronic calculators through Indian written calculation. We look successively at addition, subtraction, multiplication and division. From a didactic point of view, the token abacus is more interesting than the Chinese abacus, because one can put as many tokens as one wants in a column, which makes it possible to completely separate the inscription of the numbers on the abacus from the later manipulations on these numbers. As tokens, I use Cape peas. They are quite large seeds that children can handle easily (see figure 6).

### 3.5 Nomograms

Another medium that I like very much is the graphic tables, also called nomograms. These tables played a big role in the history of numerical calculation in the 19th century and the first half of the 20th century (Tournès, 2022, Chap. 4). They were the preferred calculation tool of engineers and other professions. They are still used today in medicine. In my training, I use graphical tables formed from graduated hyperbolas or parabolas (see figure 7). A relationship between three numbers is expressed by the intersection of three lines or the alignment of three points. With this type of table, multiplication, division, extraction of square roots, and solving equations up to the fourth degree can be carried out. The manipulation of these tables involves most of the concepts of coordinate geometry present in high school, as well as the representative curves of the square function and the inverse function.

**Figure 6.** Token abacuses

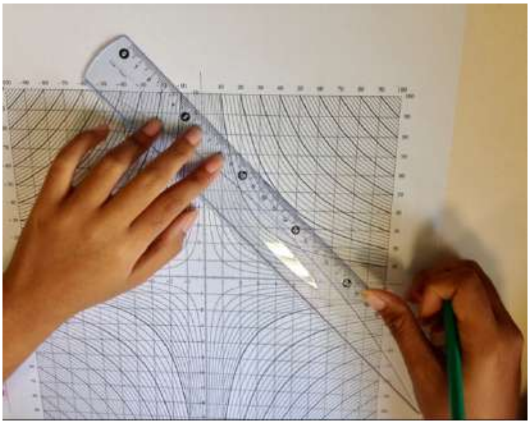


**Figure 7.** Use of a nomogram

### 3.6 Linkages

In addition to the ruler and compass, linkages allow all algebraic curves to be drawn, as suggested in Descartes' *Geometry* and demonstrated in the 19th century by Alfred Bray Kempe. Linkages can also perform geometric transformations: translation, rotation, axial symmetry, homothety, affinity, inversion, etc. We have in Reunion a collection of mathematical machines from the Laboratory of Mathematical Machines in Modena (see figure 8). Michela Maschietto taught us how to use them pedagogically.

### 3.7 Planimeters and integraphs

In the field of infinitesimal calculus, graphomechanical instruments are also available. Planimeters allow the exact calculation of the area of a surface. Integraphs make it possible to draw exactly the primitive curves of a given curve and, more generally, the integral curves of differential equations (Tournès, 2022, Chap. 6-7). These instruments played an important role in the first half of the 18th century in legitimising transcendental curves by tracing them with a single continuous motion, just as linkages had previously legitimised algebraic curves. I think it's important to talk to future teachers about this and have them manipulate it as much as possible. The difficulty is that it is not easy to have a large number of these instruments in cheap versions for concrete use in the classroom with pupils. In Reunion, we have started to make Prytz planimeters with 3D printers and to experiment with them in the classroom (see figure 9). Similar research is underway in France at the IREM des Pays de Loire (Guillet, Moureau & Voillequin, 2019; Tournès & Voillequin, 2022) and in Italy, where Pietro Milici is designing and building integraphs, which he is studying from a semiotic and didactic point of view with Michela Maschietto (Maschietto, Milici & Tournès, 2019).

**Figure 8.** Linkage

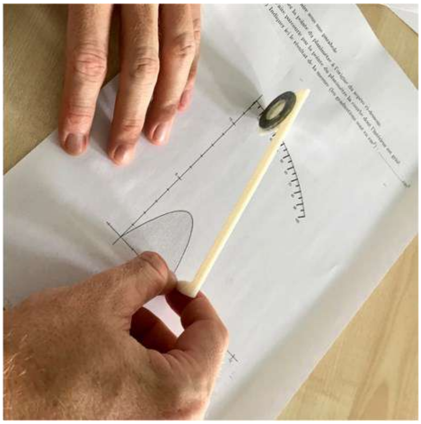


**Figure 9.** Prytz planimeter

### *4 Study of short original texts*

Let's move on to the second type of activity I offer my students. These are workshops in reading original texts. These workshops are organised in groups of three to five students. Each group is given a short excerpt from an original text directly related to curriculum items, with some questions to guide their research. Students are asked to study the text from a historical and mathematical perspective and then use it to develop a teaching scenario for a given level.

For example, I give them an extract from Descartes' *Geometry* on the geometric construction of the roots of a second-degree equation. Students should understand Descartes' text, which is rather elliptical, and justify in detail all the results it contains. Then they have to put Descartes' construction into practice on a particular case that I provide them with, the equation $z^2 = 9.1\,z - 7.2$. For that, I give them a sheet of paper on which are drawn three-line segments, one which is the unit of length and two others which measure 9.1 and 7.2 in relation to the chosen unit. They have to construct the roots of the equation with a ruler and a compass. At the end, they can measure the segments obtained, deduce approximate values of the roots and compare them with the values provided by their calculator. On figure 10 you can see some of the constructions made. In general, there are almost as many different constructions as there are students, which offers the opportunity for an interesting debate.

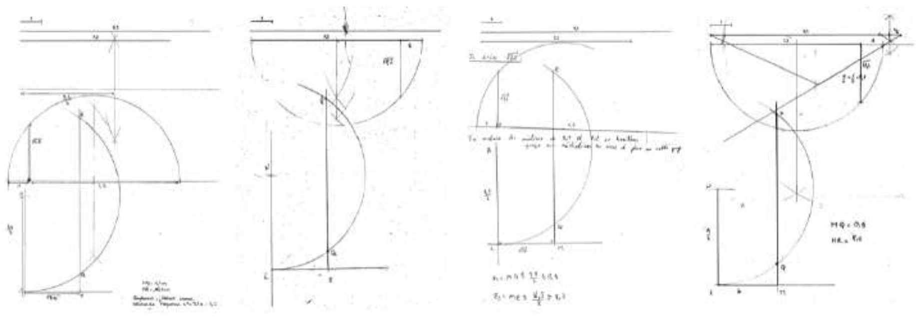


**Figure 10.** Four constructions of a second-degree equation

In the third and final stage, the students are asked to link this activity to the high school curriculum and to write the pedagogical scenario of a class session. In this case, they see that the curriculum talks about al-Khwārizmī and his solution of second-degree equations, and Descartes in relation to coordinate geometry and the equation of a circle.

The workshop ends with each group presenting its pedagogical scenario and a collective discussion on how to use Descartes' text in class. Should the text be read to the students? At the beginning or at the end of the session? In original or modernised spelling? Should they just use it as inspiration to design a meaningful problem? As an introduction to quadratic equations or as an application after studying them? Etc.

Such a workshop can last between two and three hours. This is the time needed for the students to really take ownership of the text and produce substantial output. Students are usually enthusiastic about this way of working, as they feel that they come away with simple ideas that are really applicable in the classroom.

I won't say more about my reading text workshops, because I think this example is enough for everyone to understand how they work. So now I'm going to talk about the third type of activity I offer to students to complement my course.

## *5 Examples of memoir topics chosen by students*

In this third type of activity, each student has to write a memoir on the integration of a historical topic in the classroom. For this memoir, the student has to contextualise the topic, design a teaching sequence inspired by history, experiment with this sequence in the classroom during his or her practical training, and write a didactic report on the experimentation, including an analysis of student work.

The topics can be very diverse. The memoirs are posted on the digital platform of my teaching unit following the documents of my course, and serve as complements, especially on questions that we did not have time to deal with during the course. They are the subject of a defence during which each student presents his or her memoir to the others for a collective discussion. As there are about twenty students, each student will leave at the end of the year with about twenty ideas for historical activities experienced in the classroom.

Concerning the didactic analysis of the sessions, I ask the students to use the concepts and tools they have assimilated in their didactics course. Indeed, they follow a didactics of mathematics course with another teacher in parallel with my history of mathematics course. It is therefore natural that they analyse the lessons integrating history in the same way as any other mathematics learning situation. Thus, they can bring in the notions of didactic transposition, a priori and a posteriori analysis, framework changes, tool-object dialectic, the anthropological theory of didactics, etc.

## 5.1 From false position to algebra

I will now present two examples of the content of these memoirs. The first example is a memoir about two one-hour sessions in a French fifth grade class (students between 12 and 13 years old) whose objective is the transition from the false position method to algebra.

In the first session, the students work on two Egyptian problems taken from the Rhind papyrus. The first problem, problem 26, is formulated as follows: “A quantity, its quarter is added, that makes 15”. The second problem, Problem 24, is of the same type: “A quantity, its seventh is added, that makes 19”. Students work in groups for one hour to understand and reformulate the scribe’s solutions in their own way. To extend this activity, they are given two problems in everyday life to solve at home using the false position method to prepare for the next session. On figure 11, you can see some of the students’ work. The essays vary from student to student, which allows for fruitful discussions.

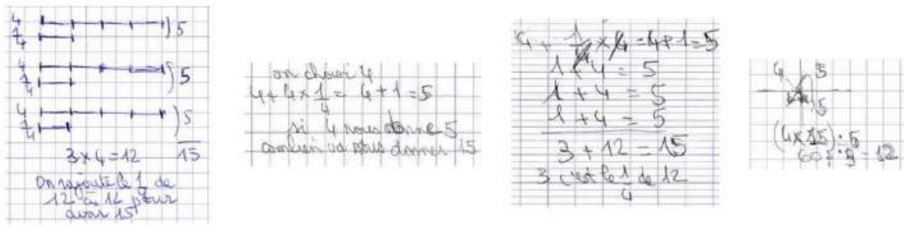


**Figure 11.** Reformulations by students of problem 26 from the Rhind papyrus

In the second session, the teacher introduced them to algebra by suggesting that they replace the numerical value used as a test with a letter and conduct the calculation similarly. In this way, they solved again the two problems of the Rhind papyrus and the two problems in everyday life. In all of this, more than the false position method or the algebra, the main obstacle came from the calculation of fractions, which was very difficult for some students.

At the end of each session, students are asked to write a few lines in their notebooks to summarise what they have done and what they have learned. Here are some examples of responses.

After the first session, a student has written:

*“During this session I learned how to do a problem by the false position method like the scribe Ahmes long ago in Egypt. I take a false starting value. I take care*

*that this value simplifies the calculations. And then by applying proportionality, I find what I am looking for."*

You can also read two personal summaries of the second session:

*"Today, I took the problem 26 that I had done with the false position, and I did it by taking a letter that represents the sought quantity. This is called the algebraic method."*

*"In this session I learned how to solve a problem using the algebraic method. We replace the quantity we are looking for with a letter, whereas in the other session, we took a false starting value. We do the calculations as if the letter was a number and by doing the calculations well, we find the quantity we are looking for."*

And here are some assessments of the whole sequence:

*"I understood better with the history, and I liked better the class and the way of working."*

*"I like history classes and I enjoyed doing history in maths. I followed it well and it helped me to understand better what to do."*

*"It was good to go back in time and see how problems were made. I liked this class."*

*"I liked to work in groups, to exchange with my classmates. We can correct errors and explain to each other or ask the teacher. I liked the history with the maths, the problem written in ancient Egyptian."*

I was very pleased with this memoir because, a month before her experimentation in the classroom, this teacher had never heard of false position methods and also had no idea about the origins of algebra. After discovering all this in my class, she herself made considerable progress in her understanding of several elementary concepts of the Middle School program and proved to be able to teach them in a relevant and effective way.

## 5.2 A cultural artefact for teaching the regular hexagon

The second example of memoir is ethnomathematical in nature. In a village in Reunion called Cilaos, there is a tradition of making carpets, bedspreads and other fabric handicrafts known as "beggar's carpets" ("tapis mendiants" in French). These objects are made from the recovery of various pieces of fabric and are presented in the form of pavings, usually square or hexagonal. The basic element is a regular hexagon, of which the seamstress has a cardboard model. She sews fabric hexagons and then assembles them seven by seven to form a kind of flower, and then these flowers are assembled into paving patterns to make, for example, a bed-spread (see figure 5, photo on the top right corner).

A future teacher chose to focus her memoir on these beggar's carpets. She first showed them to the students by asking the questions: What are they? Do you know these objects? Have you seen them before? What are they used for? Where do they come from? Who makes them? What geometric shapes do you recognise? And so forth.

Once the students have identified a regular hexagon, the teacher asked them to draw it first with their hands and then with their geometry instruments. Some students used the property of equal sides, others thank of putting the hexagon in a circle, but this was not enough. Some used the ruler instead, others the compass, by trial and error. It was only after a long time, after several collective discussions, that the expert construction emerged (see figure 12). The teacher then asked students to write the program for the construction that had just been discovered.

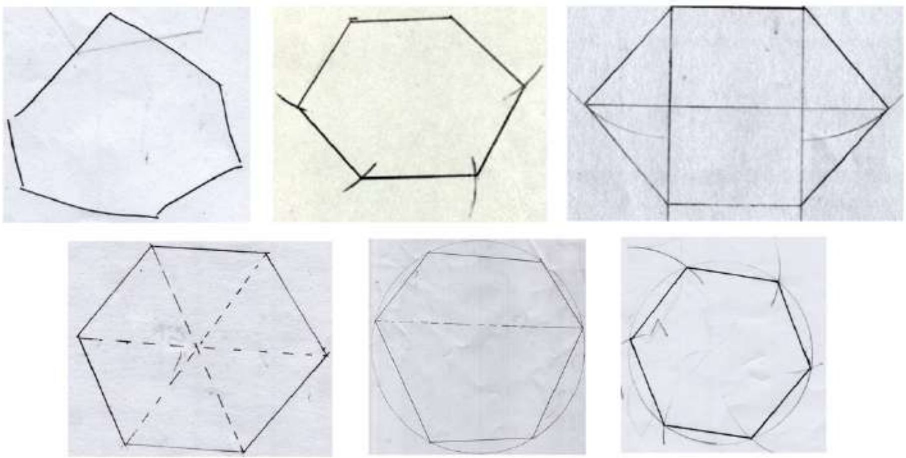


**Figure 12.** In search for a construction of the regular hexagon

The activity was also an opportunity to talk about the hexagon with equal sides and equal angles as presented in Book IV of Euclid's *Elements*, and to talk about the history of regular pavings in connection with crystallography and art history.

The sequence ended with a sewing workshop (see figure 13). To evaluate this geometry lesson based on local cultural content, the teacher asked one of her colleagues to propose a sequence on the regular hexagon in another class, but in a purely geometrical way, without any cultural context. In the class using ethnomathematics, all the pupils mastered the expert construction of the regular hexagon, whereas in the other class, only a third of the pupils demonstrated this mastery. There is no doubt here that the ethnomathematical entry created a powerful motivation and facilitated learning, both for the teacher and for her students.

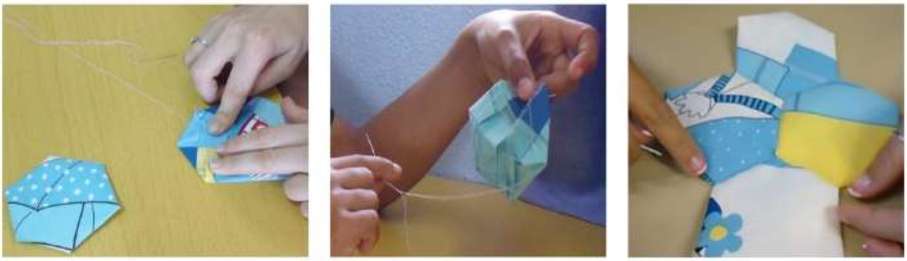


**Figure 13.** A sewing workshop

### *6 Concluding remarks*

To conclude this testimony, I could enumerate, from personal experience, some ingredients that seem to be relevant and effective in encouraging teachers to use the history of mathematics:

- consider ethnomathematics as part of the history of mathematics;
- manipulate artefacts to physically experience mathematical ideas;
- study short original texts directly related to curriculum items;
- practice with future teachers the same activities that they might offer in their classrooms;
- use the tools of the didactics of mathematics to analyse teaching and learning situations that incorporate history;
- foster a positive attitude and critical thinking towards textbooks and other historical resources;
- and… only offer very simple things in training... otherwise nothing will get into the classroom!

Finally, to find many ideas for classroom activities in the same spirit of what I have presented in this paper, I would like to point out the latest books from the French inter-IREM commission (Barbin 2010, 2012, 2018; Moyon & Tournès, 2018; Chevalarias et al., 2019).

## REFERENCES


Barbin, É. (Ed.) (2010). *De grands défis mathématiques d'Euclide à Condorcet*. Paris: Vuibert.

Barbin, É. (Ed.) (2012). *Les mathématiques éclairées par l'histoire. Des arpenteurs aux ingénieurs*. Paris: Vuibert.

Barbin, É. (Ed.) (2014). *Les constructions mathématiques avec des instruments et des gestes*. Paris: Ellipses.

Barbin, É. et al. (2018). *Let History into the Mathematics Classroom*. New York: Springer.

Bartolini Bussi, M. G., & Maschietto, M. (2007). *Macchine matematiche: dalla storia alla scuola*. New York: Springer.

Chevalarias, N., Gandit, M., Morales, M., & Tournès, D. (Eds) (2019). *Mathématiques récréatives. Éclairages historiques et épistémologiques*. Les Ulis: EDP Sciences & Grenoble: UGA Éditions.

Chorlay, R., Clark, K. M., & Tzanakis, C. (2022). History of mathematics in mathematics education: Recent developments in the field. *ZDM Mathematics Education*, *54*(7). https://doi.org/10.1007/s11858-022-01442-7.

Clark, K. M., Kjeldsen, T. H., Schorcht, S., & Tzanakis, C. (2019). History of mathematics in mathematics education – An overview. *Mathematica Didactica*, *42*(1), 1–26.

D'Ambrosio, U. (2006). *Ethnomathematics. Link between Traditions and Modernity*. Rotterdam: Sense Publishers.

Daval, N. & Tournès, D. (2018). De l'abaque à jetons au calcul posé. In M. Moyon & D. Tournès (Eds), *Passerelles. Enseigner les mathématiques par leur histoire au cycle 3* (pp. 39-64). Bouc-Bel-Air: ARPEME.

Descartes, R. (1985). *The Philosophical Writings of Descartes, Vol. 1*. Translated by J. Cottingham, R. Stoothoff & D. Murdoch. Cambridge: Cambridge University Press.

Guillet, C., Moureau, M.-L. & Voillequin, I. (2019). Mathematics and experiment: How to calculate areas without formulas. In É. Barbin et al. (Eds.), *Proceedings of the Eighth European Summer University on History and Epistemology in Mathematics Education* (pp. 227-243). Oslo: Oslo Metropolitan University.

Maschietto, M., Milici, P. & Tournès, D. (2019). Semiotic potential of a tractional machine: A first analysis. In U. T. Jankvist, M. van den Heuvel-Panhuizen & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (Utrecht, 5-10 February 2019)* (pp. 2133-2140). Utrecht: Utrecht University and ERME

Monaghan, J., Trouche, L., & Borwein, J. M. (2018). *Tools and Mathematics*. New York: Springer.

Moyon, M. & Tournès, D. (Eds). (2018). *Passerelles. Enseigner les mathématiques par leur histoire au cycle 3*. Bouc-Bel-Air: ARPEME.

Moyon, M. (2021). I would like to introduce history in my mathematics lesson but I do not know how to do it! A challenge for in-service teachers training. *History and Pedagogy of Mathematics ICME14 Satellite Meeting (19 July 2021, session online)*. https://www.youtube.com/watch?v=NTIi44eCMLM&list=PLtabqCKbZVB1sFeEpzcUpWPAQpRKz5SIQ&index=5. Accessed 21 November 2022.

Moyon, M. (2022). Desire of teachers and realities in textbooks: dealing with history of mathematics in the new French curriculum and its impact on teacher training. *ZDM Mathematics Education, 54*(7). https://doi.org/10.1007/s11858-022-01427-6.

Rabardel, P. (1995). *Les hommes et les technologies. Approche cognitive des instruments contemporains*. Paris: Armand Colin.

Tournès, D. (2017). *Onze puzzles pour le théorème de Pythagore*. IREM de la Réunion. https://irem.univ-reunion.fr/spip.php?article954#forum1317. Accessed 28 November 2022.

Tournès, D. (2018a). A graphic approach to Euler's method. In É. Barbin et al., *Let History into the Mathematics Classroom* (pp. 87-100). New York: Springer.

Tournès, D. (2018b). Calculating with hyperbolas and parabolas. In É. Barbin et al., *Let History into the Mathematics Classroom* (pp. 101-114). New York: Springer.

Tournès, D. (Ed.) (2022). *Histoire du calcul graphique*. Paris: Cassini.

Tournès, D., Voillequin, C. & Voillequin, I. (2022). Planimètres, intégraphes, tractoriographes: les instruments de la géométrie transcendante. In N. Chevalarias (Ed.), *Géométries d'hier à demain. Pratiques, méthodes, enseignement* (pp. 185-202). Poitiers: IREM&S.

# THE USE OF ORIGINAL SOURCES IN THE CLASSROOM FOR LEARNING MATHEMATICS

**Maria Rosa MASSA-ESTEVE**
Universitat Politècnica de Catalunya, Av. Diagonal, nº 647, Barcelona, Spain
m.rosa.massa@upc.edu

## ABSTRACT

The teaching and learning of the history of mathematics contributes to the overall education of students, whether they are prospective mathematicians, engineers or teachers. The use of the history of mathematics as an implicit and explicit resource makes it possible to improve the teaching of mathematics and the comprehensive training of students. The subjects that deal with the history of mathematics convey a perception of mathematics as a useful, dynamic, human, interdisciplinary and heuristic science, while complementing the thematic study of the different parts of mathematics. It is important to think of mathematics as a discipline linked to society and culture, as shown by the many people who have advanced the discipline by solving the problems of the society at each time and place. History shows that mathematics can be considered as a scientific and cultural activity that helps to solve problems in every period. This particular contribution will be focused on the use of original sources in the classroom; that is, on practical activities based on the history of mathematics for learning mathematics. Such practical activities using original sources drawn from the history of mathematics can provide students with a broader comprehension of the foundations and nature of the discipline, as well as a deeper approach to the understanding of the mathematical techniques and concepts used every day in the classroom. The aim of this paper is therefore to reflect on the use of original historical sources for learning mathematics by means of practical activities, in order to provide new resources and ideas for teachers of mathematics.

## *1 The history of mathematics for scientific education in the classroom*[1]

The teaching and learning of the history of mathematics contributes in two ways to the comprehensive training of students, whether they are prospective mathematicians, engineers or teachers of mathematics. On the one hand, it enhances the understanding and learning of some mathematical concepts and

[1] This research is included in the project "Mathematics, Engineering and Heritage: New Challenges and Practices (XVI-XIX)" (PID2020-113702RB-I00) of the Ministerio de Ciencia e Innovación. I am grateful to Fàtima Romero-Vallhonesta, Iolanda Guevara Casanova and Carles Puig-Pla. All us are members of the history of mathematics group at the ABEAM (Barcelona Association for the Teaching and Learning of Mathematics). Some of the practical activities presented have been worked together.

methods, and on the other provides a more authentic and accurate perspective of mathematics (Massa-Esteve, 2003).

The history of mathematics can be used as an explicit resource for introducing and achieving a greater understanding of certain mathematical concepts, methods and processes through the analysis in the classroom of selected original historical sources (Jahnke et al., 2000; Demattè, 2006 and Barbin, 2022). In addition, this analysis of historical sources enables students to acquire a vision of mathematics, not as a final and finished product but as a useful, dynamic, humane, interdisciplinary and heuristic science.

- A useful science. It is important to explain to students that mathematics has been an essential tool in the development of different civilizations. It has been used since antiquity for solving problems of counting, for understanding the movements of the stars and for establishing a calendar. There are many examples right down to the present day in which mathematics has proved to be vital in spheres as diverse as computer science, economics, biology, and in the building of models for explaining physical phenomena in the field of applied science, to mention just a few of the applications.

- A dynamic science. It is also necessary whenever appropriate to teach students about problems that remained open in a particular period, how they have evolved and the situation they are in now, as well as showing that research is still being carried out and that changes are constantly taking place.

- A human science. Teachers should reveal to students that behind the theorems and results there are remarkable people. It is not merely a question of recounting anecdotes but rather that students should learn something about the mathematical community; human beings whose work consisted in providing us with the theorems we use so frequently. Mathematics is a science that arises from human activity, and if students are able to see it in this way, they will probably perceive it as something more accessible and closer to themselves.

- An interdisciplinary science. Wherever possible, teachers of history of mathematics should show the historical connections of mathematics with other sciences (physics, biology, engineering, medicine, architecture, etc.) and other human activities (trade, politics, art, religion, etc.). It is also necessary to remember that a great number of important ideas in the development of science and mathematics itself have grown out of this interactive process.

-A heuristic science. We should analyze with students the historical problems that have been solved by different methods, and thereby show them that the effort involved in solving problems has always been an exciting and enriching activity at a personal level. These methods can be used in teaching to encourage students to take an interest in research and to become budding researchers themselves.

It is necessary to think of mathematics as a discipline rooted in society and culture, as shown by the many mathematicians who have made advances in the field by solving problems in society at each time and place. Indeed, the history of mathematics shows that the subject can be understood as a cultural activity, since societies develop as a result of the scientific activity undertaken by successive generations and that mathematics is being a fundamental part of this process. At the same time, the cultural and social influences involved in this historical development provide students with a view of mathematics as a subject closely linked to time and place, thereby contributing an additional value to the discipline itself (Radford, 2006).

Thus, when teaching the history of mathematics in the classroom, it is essential to analyze scientific discoveries in the context of both the time and place in which they occurred. To this end, the past, its contemporaries and its social and economic context must be taken into account. Some chronological accounts of the history of mathematics may make such discoveries appear as a mere linear correlation and give an impression of continuity that is not real. The objective of the historian of mathematics is not simply to compile lists of events and an enumeration of authors, but rather to shed light on influences and interactions, thereby helping to understand the origin of concepts and the different transformations of mathematics (Calinger, 1996).

The aim of this paper is to reflect on the use of original historical sources for learning mathematics through practical activities using new resources and ideas. In what follows, I would like to address this way of introducing the history of mathematics, which will lead to the analysis of some new resources and ideas, and also examine many of the practical activities, especially some drawn from the transformation of mathematics in the 17th century. The crucial aspects of this period will serve to attain improvements in the mathematical education of students by learning new mathematical ideas, procedures and proofs.

## *2 Using original historical sources for learning mathematics: practical activities.*

I have been teaching the history of mathematics in many university courses for twenty years. The practices analysed herein have been implemented in the Interuniversity Master of Formation of Prospective Teachers of Mathematics (UPC). I teach the compulsory subject “Mathematics from a Historical Perspective”, in two groups, each consisting of approximately 30 students. This subject is included in the Module of Complements of Formation of the abovementioned Master. In addition, in the University Degree in Superior Engineering (ETSEIB, UPC), I also teach the subject "The History of Applied Mathematics to Engineering" for the prospective engineers, in one group of approximately 25 students. In this case, the subject is not compulsory and the students show a keen interest in both the history of mathematics and the history of science. Finally, in the University Degree in Mathematics (FME), I teach the elective subject "History of Mathematics", in one group of approximately 15 students. I share the teaching of this course with Mónica Blanco (UPC), I teach the three first periods.

The basis of my courses on the history of mathematics consists of the use of practical activities for learning mathematics and for reflecting on its development. The practical activities using original historical sources provide students with a deeper approach to the understanding of the mathematical procedures and concepts used every day by the teachers in mathematics or engineering classes. History may serve as an explicit resource to introduce or understand better certain mathematical concepts through the analysis in the classroom of selected historical texts.

Therefore, it is necessary to explain how the practical activities are conducted and why, by the analysis of some examples, we regard this way of introducing the history of mathematics as very fruitful. In my courses on the history of mathematics I undertake a practical activity every week using original sources in accordance with a script of questions on the subject of the source, with the intention of clarifying any doubts or problems that may arise and together discuss the development of mathematical thought in each historical period. One aspect is the use of images or videos for introducing the source. It is not necessary to insist too much that the image in the source is not a mere complement to the explanation, but that, on many occasions, it has a leading role in the discourse or in the proof.

A further key point consists in the selection of these original sources. This is a complex matter, especially if we wish to ensure that it is reliable and really conveys the context or the idea that is to be communicated. Therefore, we want to remark that the choice of the sources for teaching the history of mathematics has to be considered carefully. We may pose some questions such as: Is the source related to the historical content of the curriculum? Does it contribute significantly to the improvement of the learning of mathematics? Is it essential to understand the origin of the mathematical concept being taught? Does it stimulate mathematical reflection? Has it represented the solution to any of the real problems of society? Does it arouse curiosity? Does it teach new methods? Does it enhance mathematical reasoning? (Jankvist, 2009; Mosvold, Jakobsen & Jankvist, 2014; Romero-Vallhonesta & Massa-Esteve, 2016).

The relevance of using historical sources in our courses is clear. From the results of our experience in the practical activities, we have observed that when students are faced with the historical mathematical text, they make it their own and create their own knowledge, which is the best way to learn mathematics. At the same time, thanks to these practical activities, students are able to learn, remember or review theorems, formulas or mathematical rules from another perspective. Finally, this way of introducing the history of mathematics transforms the class into a kind of laboratory in which ideas and concepts flow and are debated.

In the following, based on the sources, I briefly describe some practical activities with the aim of illustrating affirmative answers to some of the questions posed above. In my courses, the first three specific periods in the history of mathematics are regarded and will be addressed chronologically: “Mathematics in Antiquity”, as example of this period, the three first practical activities; “From Arab science to Renaissance algebra”, with the activities fourth and fifth as example and “The Birth of Modern Mathematics”, with the last five activities, as example. Therefore, I have selected 10 practical activities from my courses.

1. Pythagoras’ theorem in Euclid’s *Elements* (300 BC)
2. The measurement of the circle in Archimedes’ work (287 BC)
3. The distances to the Sun and the Moon using plane geometry in Aristarchus’ work (280 BC)
4. The geometrical justifications of the solution of equations in Al-khwarizmi’s work (813)

5. The measure of inaccessible distances in Tartaglia’s *Nova Scientia* (1537)
6. The negative roots of an equation in Girard’s work (1629)
7. The “specious” algebra in François Viète’s *In Artem Analyticen Isagoge* (1591)
8.The construction of quadratic equations in Descartes’ *Géométrie* (1637)
9. The Arithmetical Triangle in Pascal’s work (1654)
10.The quadrature of figures by using triangular tables in Mengoli’s work (1659 and 1672).

### 2.1 Analysis of these practical activities

1.“Pythagoras’ theorem in Euclid’s *Elements* (300 BC)”

The source is Euclid's *Elements* (300 BC), consisting of 13 books, which brings together the mathematical knowledge of different Greek schools and shows some geometric propositions that can be interpreted in terms of a second-degree equation or the Pythagorean theorem. This work, which is believed to be a collective endeavour, is second only to the Bible in the number of editions published (more than a thousand), being one of the most culturally influential works in the entire history of science.

There are many practical examples using propositions taken from Euclid's text. The Pythagorean theorem has been chosen because it is well known and it is demonstrated with equal geometric figures (triangles) that are compared with quadrilaterals, in an original and rigorous way (see Fig. 1).

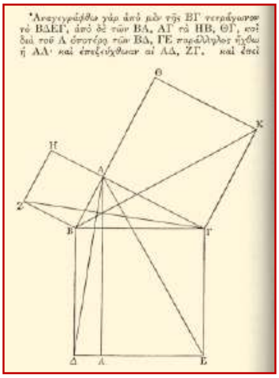


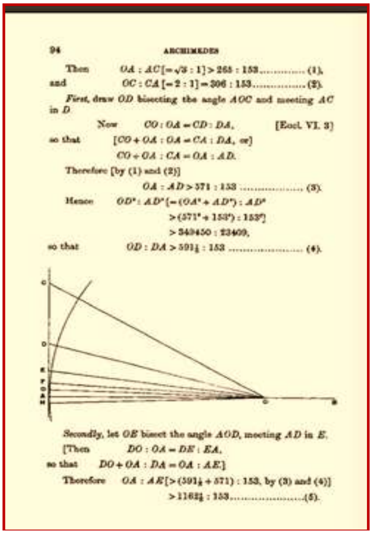
**Figure 1.** Proposition 47 of Book I in Euclid’s *Elements*

2. “The measurement of the circle in Archimedes’ work (287 BC)”

The source is *The Measure of the Circle* (approx. 287 BC) by Archimedes, where an approximation to the number π is calculated thus helping to understand its origin.

Proposition III in this book shows that the relationship between the length of the circumference and its diameter is between 3 10/71 and 3 1/7, which represents an approximation of the number π between 3.1408 and 3.1428.

Archimedes began by inscribing and circumscribing triangles in a circle, and by doubling the number of sides he arrived at polygons of 96 sides (see Fig. 2). To find this approximation he uses the bisector and the Pythagorean theorems and the relationship between the inscribed angles and the central angle, among other properties of the circumference.



94 ARCHIMEDES

Then $OA : AC [= \sqrt{3} : 1] > 265 : 153$ .............. (1),

and $OC : CA [= 2 : 1] = 306 : 153$ .............. (2).

*First*, draw *OD* bisecting the angle *AOC* and meeting *AC* in *D*.

Now $CO : OA = CD : DA$, [Eucl. VI. 3]

so that $[CO + OA : OA = CA : DA$, or$]$

$CO + OA : CA = OA : AD$.

Therefore [by (1) and (2)]

$OA : AD > 571 : 153$ .............. (3).

Hence $OD^2 : AD^2 [= (OA^2 + AD^2) : AD^2$

$> (571^2 + 153^2) : 153^2]$

$> 349450 : 23409$,

so that $OD : DA > 591\frac{1}{8} : 153$ .............. (4).

*Secondly*, let *OE* bisect the angle *AOD*, meeting *AD* in *E*.

[Then $DO : OA = DE : EA$,

so that $DO + OA : DA = OA : AE$]

Therefore $OA : AE [> (591\frac{1}{8} + 571) : 153$, by (3) and (4)]

$> 1162\frac{1}{8} : 153$ .............. (5).

**Figure 2**. The approximation of the number π (Archimedes, 1921)

3. “The distances to the Sun and the Moon using plane geometry in Aristarchus’ work (280 BC)”

The work “On the Sizes and Distances of the Sun and the Moon” by Aristarchus of Samos is an attempt to calculate the distances Sun-Earth and Earth-Moon, with an original, rigorous and correct method, using similar triangles, the bisector theorem and the Pythagorean theorem (see Massa-Esteve, 2005b for more). The geometric propositions that Aristarchus used are found mostly in Euclid's *Elements*. Eudoxus's theory of proportions from Book V of the *El-*

*ements* is used consistently and its properties of inverting, alternating, composing and multiplying are applied for both equal and unequal proportions. Aristarchus' work is also implicitly based on other relations, which we now see and identify as trigonometric, as if he knew them or considered them trivial.

An example can be seen in the proof of Proposition VII, where Aristarchus states: "The distance from the Earth to the Sun is greater than eighteen times, but less than twenty times the distance from (the Earth) to the Moon." Aristarchus builds a right triangle with vertices at the centres of the Earth (B), the Moon (C) and the Sun (A) with given angles or known by observation (see Fig. 3). As the Moon is shown to us split in two, the BCA angle is a right angle, the ABC angle is 87º (by observation) and the CAB is 3º. In fact, he shows that: 18CB < AB < 20CB, which we would say with trigonometry: 1/18> sin 3º = CB: AB > 1/20, where CB is the Moon-Earth distance, AB the Sun-Earth distance, and here the ratio of the distances is interpreted as the sine of the angle complementary to that between them. These ratios allowed Aristarchus to determine the upper and lower bounds of the value we are looking for.

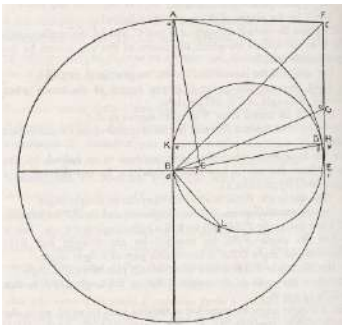


**Figure 3**. Image of Proposition VII (Aristarco de Samos, 2007: 50)

With this approach, it is necessary to highlight the four mathematical strategies required for the development of the proof of the first inequality (18CB < AB): the passage from the analysis of the problem of the triangle Sun-Earth-Moon to a similar triangle; the use of the relationship, as if it was trivial, between the tangents (current expression) and the angles (tg $\alpha$: tg $\beta$> $\alpha$: $\beta$, with the angles $\alpha$, $\beta$ of the first quadrant, and $\alpha > \beta$); the establishment of a proportion between the segments that determines the bisector of an angle and the sides of the triangle (applying the proposition VI.3 of the *Elements*) and, the last one, the approximation of square root of 2, by 7: 5. At the end, Aristarchus transfers the result obtained in the similar triangle to the initial triangle ABC or the Sun-Earth-Moon triangle and concludes that AB > 18 CB.

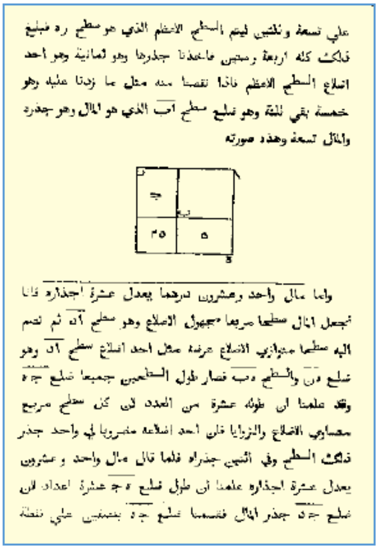
A comprehensive assessment of this work on astronomy must consider the close relationship that the beginnings of astronomy had with the origins of trigonometry, an aspect that contributes to a better understanding of the text and the evolution and utility of trigonometry.

4.“The geometrical justifications of the solution of equations in Al-khwarizmi’s work (813)”

Arabs have played a fundamental role in the development of many branches of science. Arabs collected the abstraction of Greek knowledge and the pragmatism and calculation of Hindu knowledge, for growing and transforming this assimilated knowledge, creating new ideas based on the resources of their own civilization. Baghdad emerged as the great scientific centre that enabled the translation of the great Greek works such as *The Elements* of Euclid and the *Almagest* of Ptolemy, thanks to which it was also possible to draw up new astronomical tables. After Baghdad, other focal points of culture were: Cairo, Cordova, Samarkand, Isfahan, and others. The Arabs made important contributions to physics, observational astronomy, alchemy, medicine, geometry and especially in algebra (Romero *et al*., 2015).

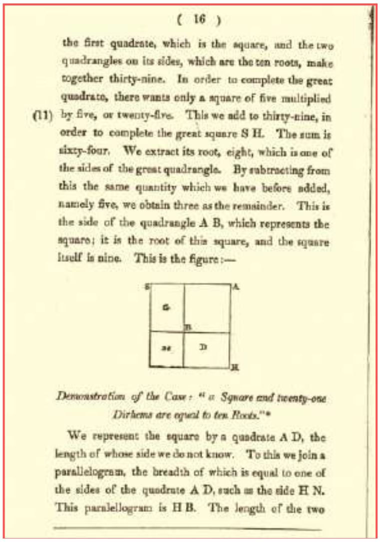
Abu Ja'Far Mohamed Ben-Musa al-Khwārizmī, mathematician, astronomer and member of the House of the Wise of Baghdad, died in 850 (AD) and is regarded as the creator of the rules of algebra. His work *Kitāb al-Mukhtasar fī hisāb al-jabr wa'l-muqābala* (ca. 813) was translated into Latin by Robert of Chester with the title *Liber algebrae et almucabola* (Segovia, 1145), where the current name of algebra comes from. The work of al-Khwarizmi consisted of a theoretical part with the method for solving equations with positive coefficients (classified into six types, up to the second degree) and a practical part that contained problems concerning numbers, trade, dowries and inheritance.

The language was rhetorical, without the use of symbols and with some geometric justification of found solutions. The geometric justification used by the Arabs for the solutions of the equation of second-degree is based on the construction of a square of side “*x*”, completing it with two rectangles of measures “*x*” and “b/2” and one square of side “b/2”, in order to obtain a square of side “$x$ + b/2”, as may be checked in the example of following figures (Figs. 4 and 5).


على تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ
ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته



واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا
نجعل المال سطحا مربعا مجهول الاضلاع وهو سطح اد ثم نضم
اليه سطحا متوازي الاضلاع عرضه مثل احد اضلاع سطح اد وهو
ضلع دن والسطح هب فصار طول السطحين جميعا ضلع جه
وقد علمنا ان طوله عشرة من العدد لان كل سطح مربع
متساوي الاضلاع والزوايا فان احد اضلاعه مضروبا في واحد جذر
ذلك السطح وفي اثنين جذراه فلما قال مال واحد وعشرون
يعدل عشرة اجذاره علمنا ان طول ضلع هج عشرة اعداد لان
ضلع جد جذر المال فقسمنا ضلع جه بنصفين على نقطة


( 16 )

the first quadrate, which is the square, and the two quadrangles on its sides, which are the ten roots, make together thirty-nine. In order to complete the great quadrate, there wants only a square of five multiplied
(11) by five, or twenty-five. This we add to thirty-nine, in order to complete the great square S H. The sum is sixty-four. We extract its root, eight, which is one of the sides of the great quadrangle. By subtracting from this the same quantity which we have before added, namely five, we obtain three as the remainder. This is the side of the quadrangle A B, which represents the square; it is the root of this square, and the square itself is nine. This is the figure:—



*Demonstration of the Case: “a Square and twenty-one Dirhems are equal to ten Roots.”*

We represent the square by a quadrate A D, the length of whose side we do not know. To this we join a parallelogram, the breadth of which is equal to one of the sides of the quadrate A D, such as the side H N. This parallelogram is H B. The length of the two

**Figure 4.** Geometrical Justification in the translation by Rosen (1831)

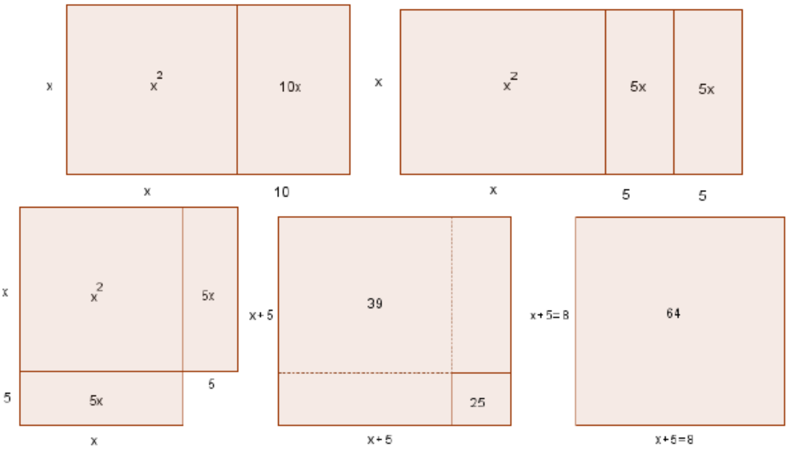



**Figure 5.** Solution of $x^2+10x=39$ with visual reasoning

In this activity, students attempt to represent and solve equations of the second-degree using geometry in the Arab way, while reflecting on the relationships between two parts of mathematics that often complement each other in order to advance mathematical knowledge: algebra and geometry.

5.“The measure of inaccessible distances in Tartaglia’s *Nova Scientia* (1537)”

The following historical activity deals with the work *Nova Scientia* (1537) by Nicolò Fontana (Tartaglia) (1499/1500-1557). In order to implement the activity in the classroom, it is recommendable: to begin with a brief presentation of the epoch, the Italian Renaissance, and the character of Tartaglia himself; the aims of the author as well as the features of the work would then be analyzed; finally, students are encouraged to construct an instrument for measuring degrees and follow the reasoning of a significant proof in order to acquire new mathematical ideas and perspectives. This classroom activity would be implemented also in the last cycle of compulsory education (14–16-year-old) with the aim of introducing and motivating the study of trigonometry (see Massa-Esteve, 2014, for more).

Tartaglia constructs two gunner’s quadrants, one with a graduate arc to measure the inclination of the cannonball and the other an instrument for solving the problem of measuring the distances and height of an inaccessible object. In the third book, from the first proposition to the fourth proposition, he describes the material required for constructing the second gunner’s quadrant: the rule and the set square, and checks its angles in the following propositions (see Fig. 6).

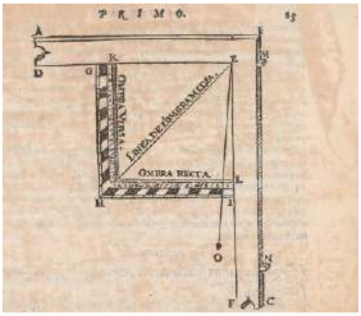
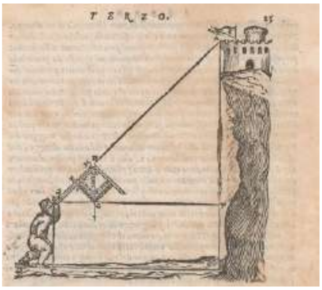


**Figure 6.** The gunner’s quadrant

This gunner’s quadrant is used by Tartaglia for measuring the height of inaccessible objects in the propositions of the third book. Tartaglia uses this gunner’s quadrant, while at the same time employing geometry in similar triangles in the proof for measuring the distances and height of an inaccessible object. In the implementation, students could be prompted to reproduce the reasoning of this proof with the geometry of triangles before introducing the trigonometry.

In Proposition VIII of third book, Tartaglia proves how to obtain the height of a visible but inaccessible object. The image of this proposition clarifies the geometric reasoning (see Fig. 7a):



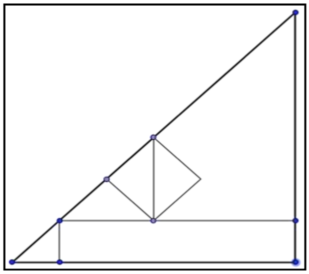


**Figure 7a.** Image of Proposition VIII. Tartaglia, 1537, 25r and **Figure 7b**. Reproduction of the mathematical problem

After explaining the construction of gunner's quadrant, accurately, together with the students the teacher could follow the reasoning of the proof using the similarity of triangles. For example, they could draw a figure with triangles that reproduces the geometric problem (see Fig. 7b). Together with the students, the teacher can reproduce the geometrical proof using similar triangles, Pythagoras' theorem and Thales' theorem. The teacher can also demonstrate the use of this figure to solve other problems in the classroom; for instance, the height of a house, using this procedure. In fact, these kinds of problems are solved today by trigonometry, and furthermore this historical activity also justifies the introduction of the teaching of trigonometry.

6.“The negative roots of an equation in Girard's work (1629)”

Through our experience, we have realized that many students have difficulty understanding and handling negative numbers (see Romero-Vallhonesta *et al.*, 2021, for more). Accordingly, first we present an analysis of some relevant historical texts that deal with the product of negative signs like in Pacioli's work (1494) or with the negative numbers like in Cardano's work (1545). On the basis of these texts, classroom activities can be designed with contents related to numbering, which forms part of the numbering and calculation block of the curriculum. However, a deep understanding of negative numbers by students will come past the conceptual barrier of symbolic reasoning and once solving algebraic equations is introduced. For example, Girard wished to solve the quadratic equation: $5x^2 = 18x + 72$, using some rhetorical instructions that reminding the modern formula. In this activity they realize a clearer acceptance of the negative roots of a quadratic equation in this work by Albert Girard (1595-1632), *Invention Nouvelle en l'algèbre* (1629), situated after publication by Viète's work and before by Descartes' work.

> *"When x2 (2) equals to x (1) and number (0).*
> *For example, if 5 x2 is equal to 18 x + 72.*
> *The half of the number of the x's is +9.*
> *Its square +81.*
> *To which we add the product of 5 times +72, which is +360.*
> *The sum + 441.*
> *The root of the sum is +21,*
> *which added, and subtracted from the first in this order will give 30 and -12.*
> *Each of these divided by 5 will give 6 and also -12/5 values of x".*

7."The "specious" algebra in François Viète: *In Artem Analyticen Isagoge* (1591)"

The implementation of this historical practical activity in a mathematics history course is appropriated for my courses and also for the bachelor's degree in mathematics or in the last cycle of compulsory education (14–16 years old) (see Massa-Esteve, 2005a and 2020, for more). This activity contains singular geometric constructions solving quadratic equations by François Viète (1540-1603), in the process of algebraization of mathematics, which was mainly the result of the introduction of algebraic procedures for solving geometrical problems; in turn, this process led to two fundamental transformations in mathematics: the creation of what is now known as analytic geometry, and the emergence of infinitesimal calculus (Mahoney, 1980; Mancosu, 1996). These disciplines became exceptionally powerful when connections between algebraic expressions and curves and between algebraic operations and geometric constructions were established.

In his *In Artem analyticen Isagoge* (1591), Viète used symbols to represent both known and unknown quantities, and was thus able to investigate equations in a completely general form. Viète introduced the specious logistic, a method of calculation with "species", kinds or classes of elements. The symbols of this analytic art (or algebra) could therefore be used to represent not just numbers but also values of any abstract magnitude, line, plane, solid or angle. In my courses, I analyze Viète's analysis as a method of solving all problems.

In fact, it is important to explain to students that Viète solved equations geometrically using the Euclidean idea of proportion: proportions can be converted into equations by setting the product of the medians equal to the product of the extremes (Viète, 1591: 2). This Viète's principle was taken directly from Euclid's *Elements* VII.19. (Euclid, 1956: 318-320). In Chapter 2 of *Isagoge,* Viète states: "And so, a proportion can be called the composition (*constitutio*) of an equation, an equation the resolution (*resolutio*) of a proportion".

In the classroom, Viète's claims concerning the quadratic equation ($x^2+bx=d^2$) and how he solved a geometrical problem with a singular construction are analyzed (see Fig. 8). In this construction, Viète set up the quadratic equation *A quadratum plus B in A, aequari D quadrato* by means of a proportion (A + B): D = D: A, using Viète's principle.

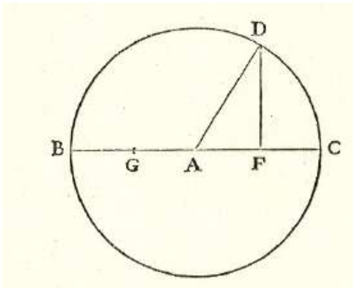



**Figure 8.** Viète's three proportional construction (Viète, 1646: 234)

Then, in the classroom, it is emphasized that Viète's geometrical construction procedures are based on the identification of terms of an equation, both known and unknown quantities, as terms of a proportion, or proportional lines through the height theorem.

8. "The construction of a quadratic equation in Descartes' *Géométrie* (1637)"

The other singular example that I analyze concerning the algebrization of mathematics is the geometrical construction in a quadratic equation found in *La Géométrie* (1637) by René Descartes (1596-1650). In the classroom, first, I explain the significance of Descartes' work and describe the contents of the three books in *La Géométrie*. I begin with the book I by describing the creation of an *algebra* of segments by Descartes and showing how Descartes adds, multiplies, divides and calculates the square root of segments with geometrical constructions (see Bos, 2001; Allaire &Bradley, 2021 and Massa-Esteve, 2020, for more). In the classroom, it is emphasized the use of Tales theorem for the product of segments, the introduction of the segment unity for the operations between segments and the height theorem for the extraction of the square root.

Next, I show how a quadratic equation ($x^2$ = a$x$ +bb) may be solved geometrically by Descartes, reproducing the singular geometric construction (see Fig. 9):

"For example, if I have $z^2=az+bb$, I construct a right triangle NLM with one side LM, equal to $b$, the square root of the known quantity $b^2$, and the other side, LN, equal to ½ $a$; that is, to half the other known quantity which was multiplied by $z$, which I suppose to be the unknown line. Then prolonging MN, the hypotenuse of this triangle, to O, so that NO is equal to NL, the whole line OM is the required line $z$. This is expressed in the following way: $z= \frac{1}{2} a + (\frac{1}{4} aa + bb)^{1/2}$."

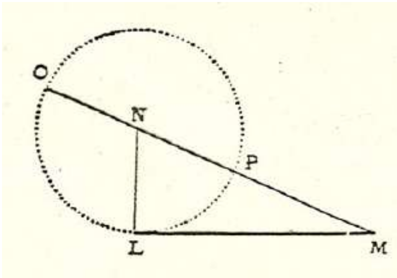



**Figure 9.** Descartes' geometrical construction (Descartes, 1637: 302)

In the classroom, after analyzing Descartes' geometrical construction, I could hold a discussion with the students. Note that the symbolic formula appears explicitly in Descartes' work. His geometrical construction corresponds to the construction of an unknown line in terms of some given lines without numerical coefficients. Therefore, the solution of the equation is given by the sum of a line and a square root, which has been obtained using the Pythagorean theorem. However, Descartes ignores the second root, which is negative, and did not mention that this geometrical construction could be justified by Euclid's *Elements* III. 36, in which the power of a point regarding a circumference is shown (Euclid, 1956: 75-77).

I ask questions for comparing the two geometrical constructions and reflect on the relationship between algebra and geometry. I reproduce Viète's and Descartes' geometrical construction and explain the procedure; questioning whether this geometrical construction could be used for any quadratic equation. Students should give suitable reasons to the following questions. What about negative solutions? How are the Pythagorean and the height theorem used? Explain their relationship with the solution of the equation. What is the main difference between Viète's geometrical construction and that by Euclid? What is the difference between Viète's and Descartes' geometrical constructions? Can we say that geometric reasoning reaches its full potential by relating algebra with geometry? One student of the course for prospective mathematicians (FME) answers this last question with these remarks:

"Thus, the tool that emerges from the fusion of algebra and geometry makes it possible to select the best properties of both sciences; from the first (algebra), the optimization of the treatment of mathematical concepts, obviating the need to represent the respective procedures and results of a demonstration, and at the same time providing more information intrinsic to the symbolism itself. From the second (geometry), the possibility of visualizing in a particular case the object studied algebraically, and at the same time having a large number of properties that could be used as an axis or complement to a proof. But this is not all; this combination not only allows for the construction of the mentioned method, but also catalyzes a much more effervescent development of both sciences, and consequently the creation (to be constructed later) of new fields of study within mathematics, as would be the case of analytical geometry or the convulsion that trigonometry triggered in the seventeenth century".

Students through this activity can learn that at the end of the process of algebrization, algebra and geometry became complementarians and that was from the coordination and conjunction of both branches that new fields of mathematics developed in the path of modern mathematics.

9. "The Arithmetic Triangle in Pascal's work (1654)"

The arithmetical triangle is the most famous set of numbers in mathematics arranged in a triangular table. It was useful in many fields and had been studied since ancient times and by many civilizations. Despite being used since the eleventh century, I may see that it is not until the seventeenth century when I find the first definitions of the arithmetic triangle, and where its properties are explained by Blaise Pascal (1623-1662). Indeed, the source of this activity, written in 1654 and published in 1665, is Pascal's work: *Traité du*

*Triangle arithmétique, avec quelques autres petits traités sur la même matière. Usage du Triangle Arithmétique pour les ordres numériques, pour les combinaisons, pour trouver les puissances des binômes et des apotomes….* After defining the arithmetical triangle, Pascal wrote and subsequently published three further treatises in which he put forward and explained, in a very clear style, these three interpretations, their properties and uses (see fig. 10).

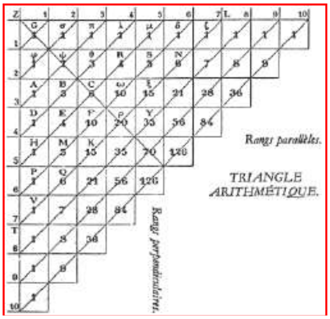



**Figure 10.** Pascal's Triangle (1654)

The rule for forming the arithmetical triangle is simple: every row begins and ends with 1, and the other numbers are obtained by the addition of two numbers closest to the row immediately above (see fig. 11).

1<br>
1 1<br>
1 2 1<br>
1 3 3 1<br>
1 4 6 4 1

**Figure 11.** The arithmetical triangle

The numbers that form the arithmetical triangle, arranged diagonally, are well known and date back at least as far the ancient Greeks, if not earlier. They are known as the figurate numbers (triangulars, tetrahedrals or pentagonals). The numbers in the rows of the triangle were subsequently recognized as the terms of a binomial development (now called binomial coefficients), and later on, as may already be seen in Pascal's arithmetical triangle, the numbers apply to solving combinatorial problems (see Edwards, 2002). In the classroom, the triangle is a source of ideas and enables us to calculate with combinatorial numbers (see fig. 12).

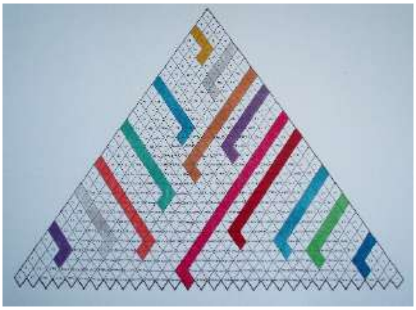


**Figure 12.** Pascal's triangle in the classroom (see Massa & Romero, 2009, for more)

$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}=\binom{n+3}{2}$$

As far as their applications can be appreciated, it is not only used to make combinatory or to find the coefficients of the Newton binomial, but can also be used to generalize, to calculate summations of powers or later summations of series, and even to calculate areas, as I explain in Mengoli's works and Leibniz's excerpts (see Massa-Esteve, 2017 and 2018, for more).

10. The quadrature of figures by using triangular tables in Mengoli's work (1659 and 1672)

The source is Mengoli's *Geometriae Speciosae Elementa* (Bologna, 1659), a 472-page text in pure mathematics with six *Elementa* whose title: "Elements of Specious Geometry" already indicates the singular use of symbolic language in this work and particularly in Geometry (see Massa-Esteve, 2006 for more). He unintentionally created a new field, a "specious geometry" modelled on Viète's "specious algebra" since he worked with "specious" language, that is to say, symbols used to represent not just numbers but also values of any abstract magnitudes.

Indeed, throughout the book he introduced triangular tables as useful algebraic tools for calculations. In the *Elementum primum,* the terms of the triangular tables are numbers and they are used to obtain the development of any binomial power (see fig. 13).

u

*a* *r*

*a*2 2*ar* *r*2

*a*3 3*a*2*r* 3*ar*2 *r*3

*a*4 4*a*3*r* 6*a*2*r*2 4*ar*3 *r*4

**Figure 13.** Table of binomial power

In the *Elementum secundum,* the terms are summations and are used to obtain their values (see fig. 14).

*O.u*

*O.a* *O.r*

*O.a*2 *O.ar* *O.r*2

*O.a*3 *O.a2r* *O.ar*2 *O.r*3

*O.a*4 *O.a*3*r* *O.a*2*r*2 *O.ar*3 *O.r*4

**Figure 14.** Table of summations

Finally, in the *Elementum sextum* of *Geometria* and in the *Circolo,* the terms are geometric figures or forms and triangular tables are used to obtain the quadratures of these geometric figures (see fig. 15).

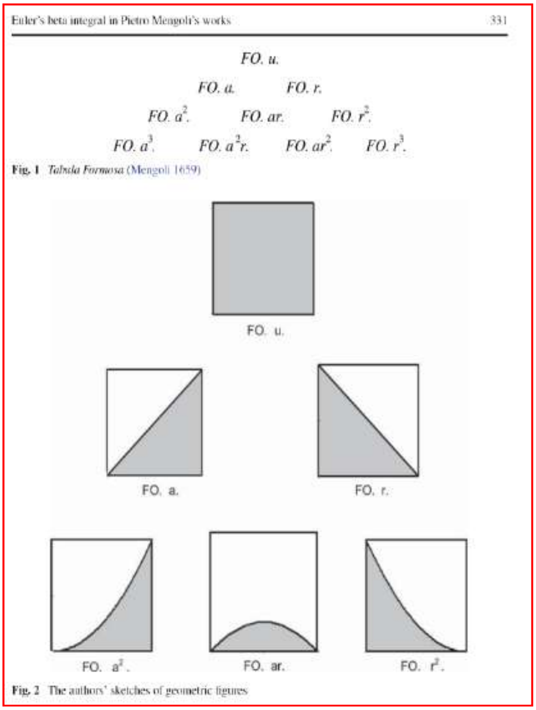



**Figure 15.** Table of geometric figures (Massa-Esteve & Delshams, 2009: 331)

I analyze with the students (prospective mathematicians) that Mengoli's originality did not stem from the presentation of these tables but rather from his treatment of them. On the one hand, he used the combinatorial triangle and symbolic language to create other tables with algebraic expressions, clearly stating their laws of formation; on the other hand, he employed the relations between these expressions and the binomial coefficients to prove results like for instance the sum of the pth-powers of the first *t-1* integers. Mengoli found a rule in which the value of the sum of the *p*th powers is obtained. However, in addition to stating the rule, Mengoli also proved it and used it to perform these values expressing all calculations in symbolic language.

Mengoli's idea was that letters could represent not only given numbers or unknown quantities, but variables as well: that is, determinable [but] indeterminate quantities. The summations are indeterminate numbers, but they are determinate when we know the value of *t*. By assigning different values to *t*, Mengoli explicitly introduced the concept of "variable", a notion that was quite new at the time. He applied his idea of variable to calculate the "quasi ratios" of these summations. The ratio between summations is also indeterminate, but is determinable by increasing the value of $t$. From this idea of quasi ratio, he constructed the theory of “quasi proportions” taking the Euclidean theory of proportions as a model, which enabled him to calculate the value of the limits of these summations. This theory constitutes an essential episode in the use of the infinite and would prove to be a very successful tool in the study of Mengoli’s quadratures and logarithms.

Nevertheless, Mengoli’s principal aim was the computation of the quadrature of the circle. Instead of just computing it, Mengoli created a new and fruitful algebraic method which involved the computation of countless quadratures. He explicitly identified these geometric figures with the values of their areas, which were also displayed in another triangular table (now called the harmonic triangle) (see fig. 16 and 17). It is noteworthy that in the *Geometria,* there are only three drawings of the geometrical figures whereas in the *Circolo,* he did not include any drawing.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1/2 & & 1/2 & & & \\
 & & 1/3 & & 1/6 & & 1/3 & & \\
 & 1/4 & & 1/12 & & 1/12 & & 1/4 & \\
1/5 & & 1/20 & & 1/30 & & 1/20 & & 1/5
\end{array}
$$

**Figure 16.** Values of the quadratures. Harmonic Triangle

$$
\begin{array}{c}
\int_0^1 1\,dx = 1 \\
\int_0^1 x\,dx = 1/2 \qquad \int_0^1 (1-x)\,dx = 1/2 \\
\int_0^1 x^2 dx = 1/3 \quad \int_0^1 x(1-x)\,dx = 1/6 \quad \int_0^1 (1-x)^2 dx = 1/6 \\
\int_0^1 x^3 dx = 1/4 \quad \int_0^1 x^2(1-x)\,dx = 1/12 \quad \int_0^1 x(1-x)^2 dx = 1/12 \quad \int_0^1 (1-x)^3 dx = 1/4
\end{array}
$$

**Figure 17.** Identification of quadratures and his values in modern notation

In other Mengoli’s work, *Circolo* (1672), basing in the harmonic triangle, by interpolation, he computed quadratures between 0 and 1 of mixed-line geometric figures determined by $y = x^{n/2}(1\text{-}x)^{(m\text{-}n)/2}$, for natural numbers $m$ and $n$. Note that in the special case $m$=2 and $n$=1, the geometric figure is the semicircle of diameter 1.

However, I argue in my courses that the most innovative aspect of Mengoli's algebraic procedure was his use of letters to work directly with the algebraic expression of the geometric figure. On the one hand, he expressed a figure by an algebraic expression, in which the ordinate of the curve that determines the figure is related to the abscissa by means of a proportion, thus establishing the Euclidean theory of proportions as a link between algebra and geometry. On the other hand, he showed how algebraic expressions could be used to construct geometrically the ordinate at any given point. This allowed him to study geometric figures via their algebraic expressions and calculated its areas.

### *3 Some reflections*

These kinds of practical activities are very rich in terms of competency-based learning, since they allow students to apply their knowledge in different situations and from different points of view, rather than to reproduce exactly what they have learned.

The practical activities based on the analysis of historical texts using original sources contribute to improving the students' overall education, providing them with additional knowledge of the social and scientific context of the periods involved. Students acquire a vision of mathematics, not as a final product, but as a science that has been developed on the basis of seeking answers to questions that mankind has been asking throughout history about the world around us.

These practical activities oblige students to tackle some significant historical demonstrations with different procedures, while at the same time encouraging debate and reflection, thereby transforming the classroom into a laboratory of ideas. Showing the difficulties that have been encountered throughout history in answering certain questions can help to motivate students who sometimes believe that mathematics consists of a series of formulas and rules that understanding is preserved for privileged minds.

Geometry has a great visual and aesthetic value and offers a beautiful way of understanding the world. The elegance of its constructions and proofs makes it an area of mathematics that is highly appropriate for developing the student reasoning process and providing proofs, as well as for incorporating geometrical constructions as a part of the heuristic in solving problems. Geometric proofs have a great potential for linking geometrical and numerical reasoning in some of the activities proposed, and geometrical and algebraic reasoning in others. In this way, students are able to establish connections among numbers, figures and formulas; that is to say, calculations, geometric constructions and algebraic expressions.

In addition, with these practical activities, students can work with problem-solving, reasoning and proof processes, thus addressing connections, communication and representation. By analysing historical texts, students are introduced to different ways of working from different perspectives (transversal competences), which enables them to tackle mathematical problems by developing their mathematical thinking.

Finally, I conclude that this "way of introducing" the history of mathematics will enable prospective engineers, mathematicians and teachers of mathematics to more readily recognize the most significant changes taking place in the mathematical discipline, and above all to reflect more deeply on the formation of their scientific thought.

**REFERENCES**

Allaire, Patricia R. & Bradley, Robert E. (2021). Geometric approaches to Quadratic Equation from Other Times and Places. *Mathematics Teacher,* 94 (4), 308-319.

Al-Khwarizmi (1986). *The Algebra of Mohammed ben Musa*. (Rosen, Frederic (ed. and Trans.) (1th ed. London, 1831). Hildesheim/Zürich/New York: George Olms Verlag.

Archimèdes (1921). *Les œuvres complètes d'Archimède*. Traduites du grec en français avec une introduction et des notes, par Paul Ver Eecke. Desclée.

Aristarco de Samos (2007). *Sobre los tamaños y las distancias del Sol y la Luna.* (2da. Edición 2017). Introducción, traducción y notas de Mª Rosa Massa Esteve. Servicio de Publicaciones de la Universidad de Cádiz.

Barbin, É. (2022). On the role and scope of historical knowledge in using the history of mathematics in education. *ZDM Mathematics Education*. DOI: 10.1007/s11858-022-01410-1.

Bos, H. J. M. (2001). *Redefining geometrical exactness*. Sources and Studies in the History of Mathematics and Physical Sciences. New York: Springer-Verlag.

Calinger, R., (ed.) (1996). *Vita Mathematica. Historical research and Integration with teaching.* Washington: The Mathematical Association of America.

Demattè, A. (2006). *Fare matematica con i documenti storici. Una raccolta per la scuola scondaria de primo e secondo grado*. Trento: Editore Provincia Autonoma di Trento – IPRASE del Trentino.

Descartes, René. (1637). *La géométrie.* «The geometry of René Descartes» by Smith, D.E.; Latham, M. L. (Trad. & Edit). Dover, 1954.

Edwards, A. W. F. (2002). *Pascal's Arithmetical Triangle. The story of a Mathematical Idea.* The Johns Hopkins University Press.

Euclid (1956). *The Elements*. T. L. Heath (trad.), 3 vols. Dover.

Girard, A. (1629). *Invention nouvelle en l'algèbre*. Guillaume Iansson Blaeuw.

Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., El Idrissi, A., et al. (2000). The use of original sources in the classroom. In: J. Fauvel & J. Van Maanen (Eds.) *History in Mathematics Education. The ICMI Study* (291-328). Kluwer Academic Publishers.

Jankvist, U. T. (2009). A categorization of the 'whys' and 'hows' of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.

Mahoney, M. S. (1980). The beginnings of algebraic thought in the seventeenth century. In: Gaukroger, S., (ed.). *Descartes' philosophy, mathematics and physics* (141–156). Barnes and Noble/ Harvester.

Mancosu, Paolo (1996). *Philosophy of Mathematics & Mathematical Practice in the Seventeenth Century.* Oxford University Press.

Massa-Esteve, Maria Rosa (2006). Algebra and Geometry in Pietro Mengoli (1625-1686). *Historia Mathematica,* 33 (1), 82–112.

Massa-Esteve, Maria Rosa (2003). Aportacions de la història de la matemàtica a l'ensenyament de la matemàtica. *Biaix,* 21, 4–9.

Massa-Esteve, Maria Rosa (2005a). Les equacions de segon grau al llarg de la historia. *Biaix,* 24, 4–15.

Massa-Esteve, Maria Rosa (2005b). L'Ensenyament de la trigonometria. Aristarc de Samos (310-230 BC). In: *Actes de la I Jornada sobre Història de la Ciència i Ensenyament "Antoni Quintana Marí"* (95–101). Institut d'Estudis Catalans.

Massa-Esteve, Maria Rosa (2014). Historical activities in the mathematics classroom: Tartaglia's Nova Scientia (1537). *Teaching innovations,* 27 (3), 114–126.

Massa-Esteve, Maria Rosa (2017). Mengoli's mathematical ideas in Leibniz's excerpts. *British Journal for the History of Mathematics*, 32 (1), 40–60.

Massa-Esteve, Maria Rosa (2018). The harmonic triangle in Mengoli's and Leibniz's works. *Quaderns d'història de l'enginyeria*, vol. XVI, 233–258.

Massa-Esteve, Maria Rosa (2020). The Algebraization of Mathematics: Using Original Sources for Learning Mathematics. *Teaching Innovations,* 33 (1), 21–35.

Massa-Esteve, Maria Rosa & Delshams, Amadeu (2009). Euler's Beta integral in Pietro Mengoli's works. *Archive for History of Exact Sciences,* Vol. 63, 325–356.

Massa-Esteve, Maria Rosa & Romero-Vallhonesta, Fàtima (2009). El triangle aritmètic de

Blaise Pascal (1623-1662). *Biaix,* 28-29*,* 6–17.

Mengoli, Pietro (1659). *Geometriae Speciosae Elementa.* Bologna.

Mengoli, Pietro (1672). *Circolo*. Bologna.

Mosvold, R., Jakobsen, A. & Jankvist, U.T. (2014). How Mathematical Knowledge for Teaching May Profit from the Study of History of Matematics. *Science & Education* **23**, 47–60. DOI: 10.1007/s11191-013-9612-7.

Radford, L. (2006). The cultural-epistemological condition of the emergence of algebraic symbolism. In: Furinguetti, F., Kaijser, S. and Tzanakis, C. (Eds.): *Proceedings of the 2004 History and Pedagogy of Mathematics Conference & ESU4* (509–524). Uppsala, Sweden.

Romero, F., Guevara Casanovas, I., Massa-Esteve, M. R., Puig-Pla, C. & Roca-Rosell, A. (2015). Teacher training in the history of mathematics. In: Barbin, E., Jankvist, U. T. and Kjeldsen, T. H. (eds.): *History and Epistemology in Mathematics Education. Proceedings of the Seventh European Summer University* (113–128). Danish School of Education.

Romero-Vallhonesta, F. & Massa-Esteve, M. R. (2016). Història de les matemàtiques per a l'ensenyament de les matemàtiques. Analitzant les fonts. In: Grapí, P. & Massa-Esteve, M. R. (eds). *Actes de la XIII Jornada sobre la Història de la Ciència i l'Ensenyament* (59–70). SCHCT- Institut d'Estudis Catalans. DOI: 10.2436/10.2006.02.8

Romero-Vallhonesta, F., Massa-Esteve, M. R., Guevara-Casanova, I. & Puig-Pla, C. (2021) Textos històrics per a l'aprenentatge de les matemàtiques: el cas dels nombres negatius. In: Ferragud, C. & Massa-Esteve, M. R. (ed.) *Actes de la XVII Jornada sobre la Història de la Ciència i l'Ensenyament* (17–28). SCHCT-Institut d'Estudis Catalans. DOI: 10.2436/10.2006.04.3.

Tartaglia, N. 1558 (First Edition 1537). *Nova Scientia*. Venice.

Viète, F. (1646). *Opera Mathematica.* Schooten, F. A. (ed.). Georg Olms Verlag.

# ALGORITHMS BEFORE COMPUTERS

**Michel ROELENS**
Faculty of Teacher Education, University Colleges Leuven-Limburg (University of Applied Sciences), Diepenbeek, Belgium
Maria-Boodschaplyceum (secondary school), Brussels, Belgium
Uitwiskeling (magazine for mathematics teachers), Belgium
Michel.Roelens@ucll.be

## ABSTRACT

In the new curricula that are now gradually being implemented in secondary schools in Flanders, the Dutch-speaking part of Belgium, there is a shift towards mathematics as a part of STEM (Science, Technology, Engineering, Mathematics) and towards computational thinking in mathematics: programming, logic and electronic gates, graph theory, and for some pupils linear programming. Meanwhile, the new curricula contain less geometry and fewer references to history and art. As a first reaction, I welcome the introduction of graph theory, but I regret the loss of some beautiful parts of geometry. Instead of complaining, I try to relate the new subjects with geometry and the history of mathematics. There have always been algorithms in mathematics, long before there were electronic computers that could be programmed. Moreover, geometrical and visual thinking can be used to discover many algorithms.
In this article, I would like to illustrate historical and geometrical aspects of algorithms with some examples: the calculation by hand of the digits of a square root and the algorithm for constructing a Eulerian circuit in a multigraph in which the degree of each vertex is even.

## *1 Introduction*

### 1.1 My personal situation

I teach mathematics - and especially geometry - on a half-time basis to future secondary school teachers. In our teacher training in Flanders, prospective teachers for pupils aged 12 to 16 years take two subjects: my students will become teachers of for example mathematics and biology, or mathematics and French. I also teach mathematics to pupils aged 16 to 18 years, in a secondary school in the centre of Brussels, three days a week. At night and weekends, since 1984, I work on the magazine *Uitwiskeling* for mathematics teachers.

### 1.2 Recent tendencies in Flemish curricula

New curricula are introduced in secondary education in Flanders. Due to a decision by the Constitutional Court, these curricula are likely to be reduced, but I will not go into these political issues here.

The new curricula present the following trends. First, more attention is paid to the role of mathematics as a part of STEM (Science, Technology, Engineering and Mathematics). Second, a new term has been introduced in the mathematics curricula: 'computational thinking'. In the literature on mathematics and technology education, this idea has been around for 16 years (Wing, 2006), but it is now making its appearance in our mathematics curricula. This leads to new chapters in mathematics text books: logic and electronic gates, graph theory, algorithms and programming, and for pupils in some economics classes also linear programming.

I find many of these topics interesting, especially graph theory. But new topics always come at the expense of other topics. What is being covered less? Alas, geometry (e.g. inscribed angles in circles). And the history of mathematics is not even mentioned in the new curricula, nor the link between mathematics and art.

These trends also affect my personal work situation. In my secondary school in Brussels, I teach according to the new curricula. In teacher training, I still teach strong geometry courses to future teachers, but they will have to teach less geometry. My history of mathematics course, in which students developed workshops for colleagues, has regrettably been dropped. The curriculum makers' motivation is that the new topics are important in this computer age: "Algorithms are becoming important because there are computers everywhere, even in pupils' school bags."

This brings me to the question: is it true that algorithms and computational thinking are typically associated with computers? Algorithms *for* computers? Or are algorithms of all times? Algorithms *before* computers? Students and young colleagues mostly see algorithms as something to feed computers with. Let's look to some examples of algorithms, using material developed by my students.

## *2 Algorithms for (square) roots*

How to calculate a root? If you ask a pupil, the predictable answer is: with a calculator. But there have not always been calculators. Another way is with your naked hands. In Dutch, ‘wortel’ (root) is the same word as ‘carrot’. The calculation of roots is ‘the extraction of carrots’. Two students of mine made a workshop about this. As a joke, participants who calculated the roots correctly were allowed to pull real carrots from a tub of potting soil and take them home.

We will look in detail at two historical algorithms for calculating square roots: the Babylonian and the Chinese. There were also Indian ones.

## 2.1 Babylonian algorithm

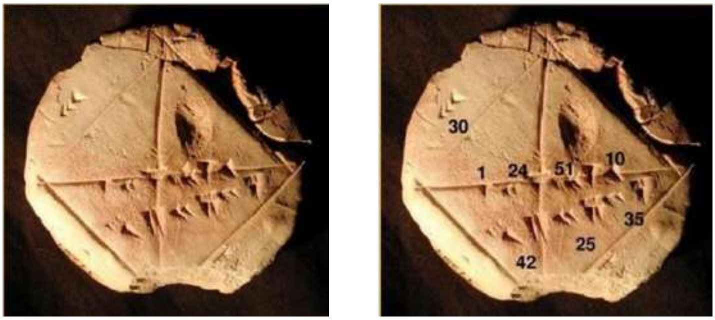



**Figure 1**. Tablet YBC 7289

Figure 1 shows a tablet from about 4,000 years ago (Yale Babylonian Collection 7289, USA). It testifies that the Babylonians of the time had an algorithm for calculating square roots. According to Fowler and Robson (1998), this was a hand-held tablet on which a student solved school problems. It could be erased and reused. Let us call it the smartphone of the Old Babylonian age.

Babylonians used a numeral system with base sixty, or ‘sexagesimal’ numbers. This is a very convenient numeral system because $60$ has a large number of divisors.

On one of the diagonals, we can read the sexagesimal number $1;24;51;10$, which means $1+\frac{24}{60}+\frac{51}{60^2}+\frac{10}{60^3}\approx 1.41421296$, a good approximation for $\sqrt{2}\approx 1.41421356$. Above, we read the number $30$ and under the previous number, the sexagesimal number $42;25;35$, which equals $42+\frac{25}{60}+\frac{35}{60^2}\approx 42.42638889$, an approximation for $30\sqrt{2}\approx$

$42.42640687$. So, the pupil multiplied the side 30 of the square with $\sqrt{2}$ to determine its diagonal.

Another clay tablet (Vorderasiatische Abteilung Tontafeln 6598, Berlin, Germany) presents the following problem: "Find the length of the diagonal of a door 10 units wide and 40 units high." According to the (later) Pythagorean theorem, this amounts to: find $\sqrt{1700}$.

According to Swerts (2012) and others, the method used by the Babylonians for calculating $\sqrt{1700}$ is essentially the same as that of the later Heron of Alexandria (1st century AD). We make a guess, e.g. $40$. This guess serves as a first approximation. The second approximation is then the average of 40 and $\frac{1700}{40}$. If we stop here we find 41.25, written decimally, or 41;15 in sexagesimal notation. We can continue, taking as a third approximation the average of $41.25$ and $\frac{1700}{41.25}$ and so on.

If we interpret this algorithm geometrically, it becomes much clearer, and I would not be surprised if the Babylonians invented it by thinking geometrically. The problem is: "Given a square with area 1700, find the side." The guess $40$ is too small. If we make a rectangle with area 1700 and that side 40, the other side $\frac{1700}{40}$ is too large because the area is still $1700$ (Figure 2). The side of the square must be somewhere in between. And the simplest number between two numbers is the average.

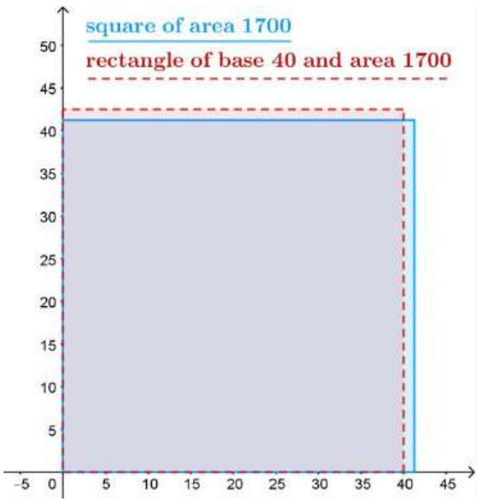



**Figure 2.** Geometric interpretation of the Babylonian algorithm

Note that this ancient method is equivalent to Newton's method. The approximation of $\sqrt{1+x}$ by replacing the graph of the square root function by its tangent in $\mathrm{x}=1$ gives $\sqrt{1+x} \approx 1+\frac{1}{2}\mathrm{x}$ and this is the average of the first guess 1 and the second guess $1+x$ (Figure 3). Applied to our example of the door, this gives:

$$\sqrt{1700} = \sqrt{1600+100} = 40\sqrt{1+\frac{1}{16}} \approx 40\left(1+\frac{1}{32}\right) = 41.25.$$

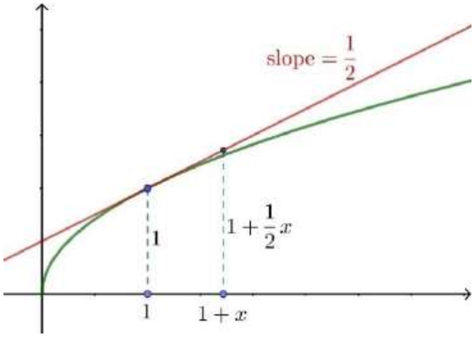



**Figure 3.** Newton's method

## 2.2 Calculating the decimal digits of a square root

When I was in primary school in the 1960s and 1970s, we studied an algorithm to calculate the decimal digits of a square root one by one. This algorithm originates from the Ancient Chinese. You can find an example in chapter 4 of the Nine Chapters of Mathematics (Jiuzhang Swanshu, 10th to 2nd century BC). The method is called *Kai Fang*, opening the square (Burgos & Beltrán-Pellicer, 2018). In the 3rd century AD, Liu Hui gives a detailed description of the method. Again, I am convinced that the algorithm is geometrically inspired.

Let's look at an example: $\sqrt{7356\ 796}$. This is not the original example of the Ancient Chinese; it is the example of my students Vankriekelsvenne and Vanmarsenille (2003). We invite the reader to follow the steps in Figure 4, starting with a square determined by the first digit and adding *gnomons* in order to find the next digits of the square root. A gnomon is a figure to be added to a square to make a larger square. A gnomon can be transformed into a long rectangle.

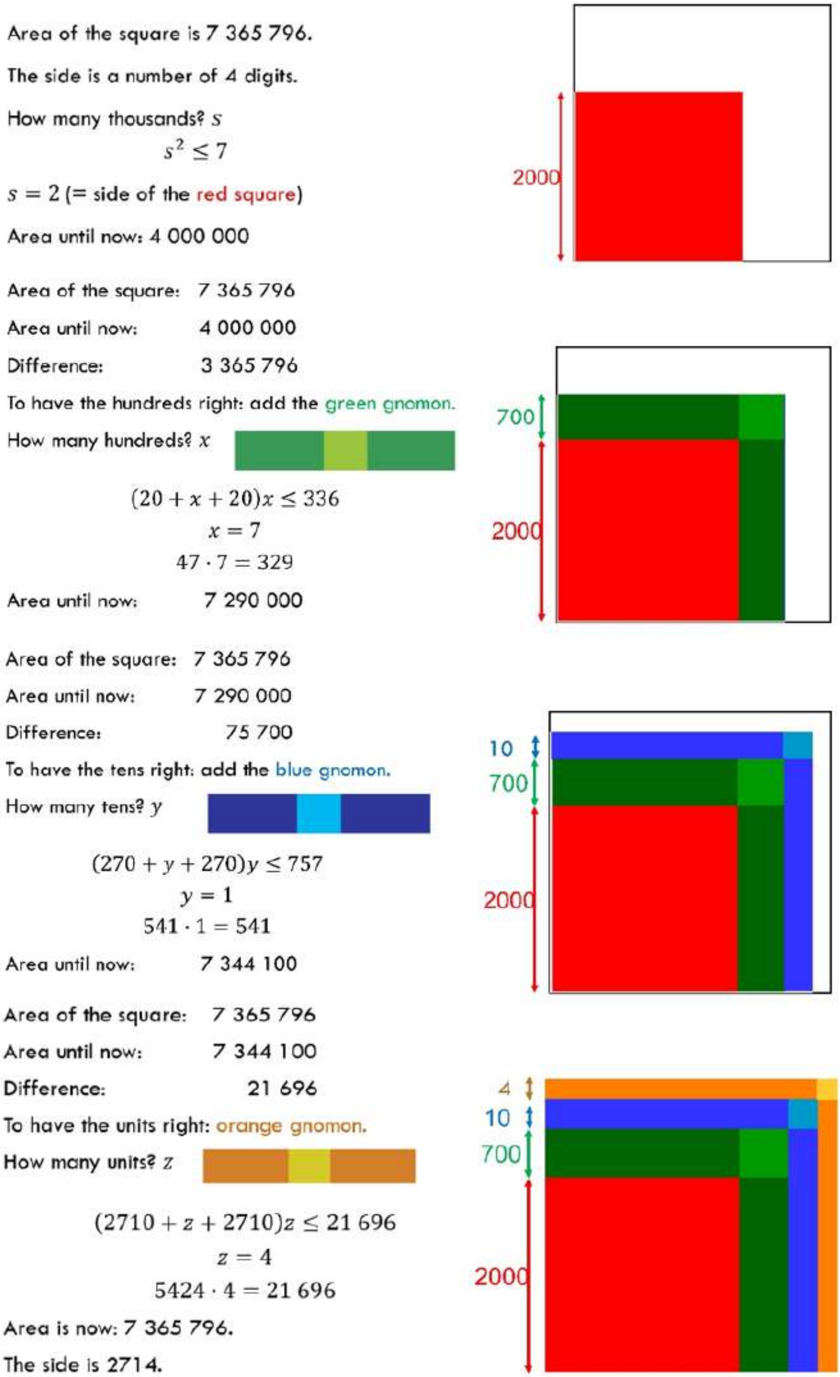



**Figure 4.** Geometrical search for the digits of the square root

At school, I learned the algorithm without the geometric explanation. Figure 5 shows the calculation for the same example as in Figure 4, with the same steps. Next to it, the figure shows the calculation, with the same algorithm of the first digits of $\sqrt{2}$.

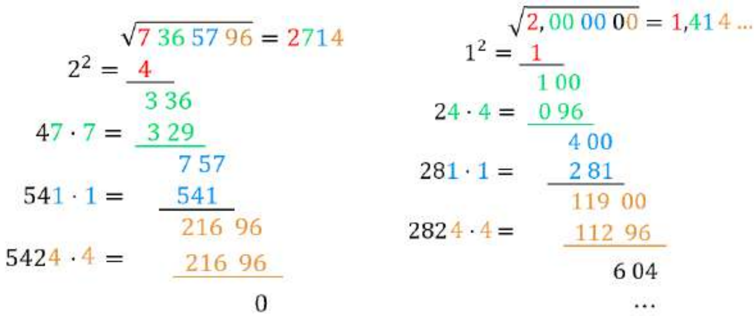



**Figure 5.** Same algorithm, as we wrote it down at school

There is a similar algorithm to compute the digits of the cubic root of a number, also inspired by geometry, starting with a cube and adding solid gnomons. For more details, see TwoPi (2008).

### 2.3 Roots of quadratic equations

In the 9th century, the Persian scholar Al Khwarizmi gave a systematic algorithm for the solution of the six types of quadratic equations. In fact, the word 'algorithm' is derived from his name. He had to distinguish different types because the numbers had to be positive. A negative term of a modern equation appeared on the other side of the equal sign in Al Khwarizmi's equation. For example, one of the types was $ax^2 = bx + c$, with $a, b$ and $c$ positive.

The solution recipe was explained geometrically. In figure 6, this is illustrated with the equation $x^2 = 3x + 4$ (Roelens & Van den Broeck, 2015). Al Khwarizmi argues that geometric visualization helps to understand his algebraic algorithms (quoted by Siu, 2002): "*We have now explained these things concisely by geometry in order that what is necessary for an understanding of this branch of study might be made easier. The things which with some difficulty are conceived by the eye of the mind are made clear by geometric figures*."

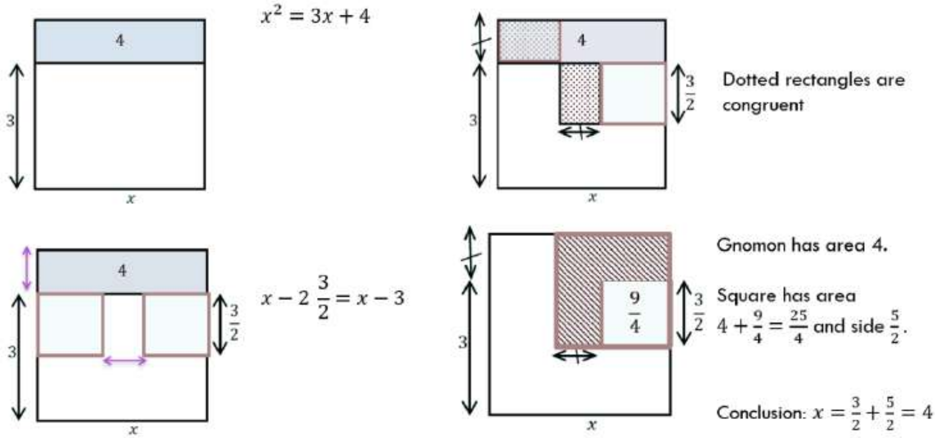



**Figure 6.** Al Khwarizmi's geometric solution of the equation $x^2 = 3x + 4$

## 2.4 Conclusion so far

The algorithms to calculate square roots and solutions of quadratic equations are much older than electronic computers. They are largely inspired by geometry and it is through geometry that they can be understood. The practical *use* of the algorithm to calculate the digits of a square root, has been rendered obsolete by the advent of computers and calculators. Still, we think that the geometric discovery, guided reinvention and explanation of the algorithm can be an interesting activity for pupils.

## *3 Euler graphs*

Let's move on to another branch of mathematics. Graph theory has now been added to the mathematics curriculum in Flemish secondary schools because the internet, Facebook, ... are all big graphs, as is the road network in which the GPS (Global Positioning System) has to find the shortest or fastest routes. Graph theory is really something of the computer age...

Again: this theory may not date back to the Babylonians, but it certainly predates computers.

## 3.1 Euler's theorem

The origins of graphs are usually attributed to Leonhard Euler (18th century), the incredibly versatile Swiss mathematician. Euler wrote so much that, according to the Dutch Wikipedia, it would take an estimated 50 years to transcribe all his works by hand at a rate of eight hours of writing a day.

In Euler's time, there were seven bridges across the river Pregel connecting the various parts of the Prussian city Königsberg (figure 7). After World War II, Königsberg was added to Russia as an exclave between Poland and Lithuania, and has since then been called Kaliningrad.

Euler posed the question whether it was possible to make a walk in Königsberg that passes precisely once over each bridge and returns to the starting point.

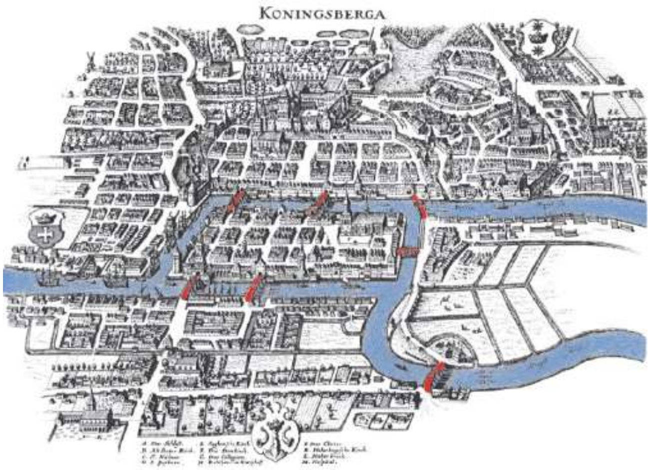



**Figure 7.** The seven bridges of Königsberg

The solution with graph theory has become a classic. The parts of the city separated by the river are represented by vertices and the bridges by edges connecting these vertices (Figure 8). It is not a single graph but a multigraph, because a pair of vertices can be connected by more than one edge. A walk that goes precisely once over each edge and returns to the starting vertex is called a Eulerian circuit. The degree of a vertex is the number of edges adjacent to that vertex. To have a Eulerian circuit, the degrees of all vertices must be even. In the multigraph represented by

Königsberg, the degrees of vertices $a, b, c$ and $d$ are 5, 3, 3 and 3, respectively. So the requested walk is not possible.

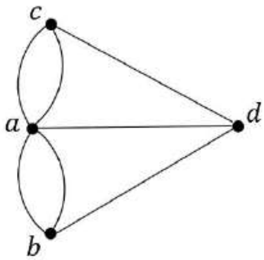
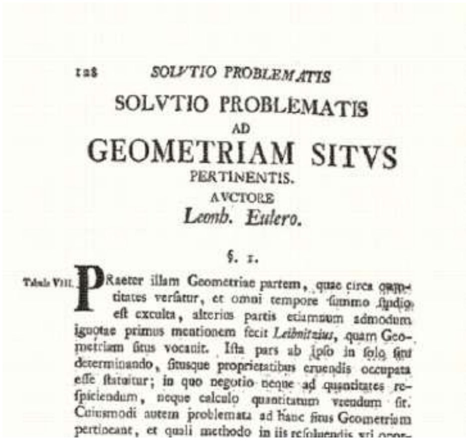



**Figure 8.** Graph representation of Königsberg

I was convinced that this solution was Euler's and that Euler effectively used a graph with vertices and edges. When I read Euler's text (Euler, 1741, Figure 9), I was surprised that Euler did not actually draw a graph, but continued to reason with city regions and bridges. He worked on his simplified drawing of the city, figure 10.


128 *SOLVTIO PROBLEMATIS*

SOLVTIO PROBLEMATIS
AD
GEOMETRIAM SITVS
PERTINENTIS.
AVCTORE
*Leonb. Eulero.*

§. 1.

Tabula VIII. PRaeter illam Geometriae partem, quae circa quantitates verſatur, et omni tempore ſummo ſtudio eſt exculta, alterius partis etiamnum admodum ignotae primus mentionem fecit *Leibnitzius*, quam Geometriam ſitus vocauit. Iſta pars ab ipſo in ſolo ſitu determinando, ſitusque proprietatibus eruendis occupata eſſe ſtatuitur; in quo negotio neque ad quantitates reſpiciendum, neque calculo quantitatum vtendum ſit. Cuiusmodi autem problemata ad hanc ſitus Geometriam pertineant, et quali methodo in iis reſolvendis

**Figure 9.** Euler's text about the bridges of Köningsberg

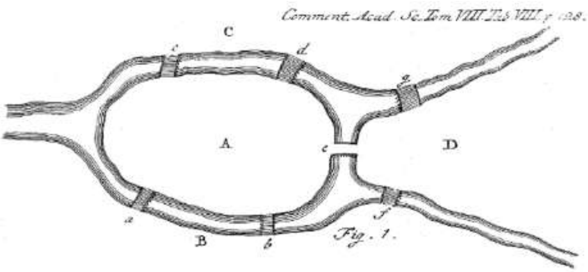


**Figure 10.** Euler's drawing of the city of Königsberg

"[The problem] could be solved by a complete enumeration of all the walks" he writes, but he prefers a much simpler method. He codes each walk by a sequence of letters, e.g. $ABA$ means start in $A$, take a bridge to go to $B$, return to $A$ with another bridge. He proves the impossibility of the requested walk by reasoning on the basis of the number of occurring letters in these codes. For more details, see Barnett (2009) or the original article Euler (1741). Ultimately, he concludes that the number of bridges for each area must be even to make a requested walk possible. He ends by remarking: "The question remains how the walk is to be carried out."

Euler's theorem, in modern terms, is the implication: "*If in a connected multigraph there is a Eulerian circuit, then all vertices have even degree.*" His other theorem is about a Eulerian *trail*, a walk going precisely once over each edge but not returning to the starting vertex: "*If in a connected multigraph there is a Eulerian trail, then exactly two vertices have odd degree.*"

His last comment can be understood as admitting that he does not actually prove the converse theorems. This will be done more than a century later by Hierholzer.

### 3.2 Hierholzer's algorithm

Carl Hierholzer (1873), in a short article, proves the converse theorems:
"*In a connected multigraph: if all vertices have even degree, then there is a Eulerian circuit; if exactly two vertices have odd degree, then there is a Eulerian trail.*"

He repeats the theorems and proofs of Euler, but in a version with vertices and edges, much easier to read than Euler's text. Then he proves his theorems, the converse theorems of Euler's, by describing an algorithm for constructing a Eulerian circuit or trail.

Let's illustrate Hierholzer's algorithm with an example. Take the multigraph with vertices $A, B \dots J$ of Figure 11. The degrees of the vertices are even, so we should be able to make a Eulerian circuit. Start somewhere and walk on the edges, without repeating the same edge, until you get stuck. For example, start in $A$ and make the walk $ABCDAIJJA$. Colour the edges you walk on in a first colour, e.g. red. Because all degrees are even, you cannot get stuck elsewhere than in the starting vertex $A$. The walk had to be a sub*circuit*. If all edges were coloured, you would have finished a Eulerian circuit, but in the example this is not the case. Now, start at a vertex where some but not all edges are coloured, e.g. $B$, and make a new subcircuit on non-coloured edges until you get stuck again, for example $BDEGIDB$ (green). Repeat this procedure, e.g. make subcircuit $EFGHIE$ (blue). Now in the example all edges are coloured. To make a Eulerian circuit from our subcircuits, you replace $B$ in the first subcircuit by the second subcircuit: $A\ BDEGIDB\ CDAIJJA$ and then you replace E in the second subcircuit by the third subcircuit: $A\ BD\ EFGHIE\ GIDB\ CDAIJJA$. This is a Eulerian circuit in the given multigraph.

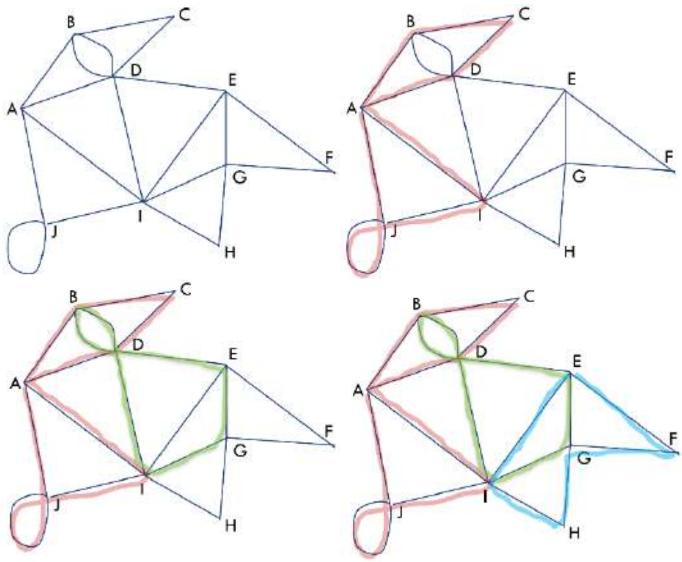



**Figure 11.** Example of Hierholzer's algorithm

## 4 *Algorithms before computers*

### 4.1 No mathematics without algorithms

Long before there were electronic computers, algorithms formed an important aspect of mathematics.

In section 2, we discussed geometrically underpinned Babylonian and Old Chinese algorithms to calculate square roots and a still-used medieval Persian algorithm to solve a second degree equation.

There are many other examples. Euclid (300 BC) devised an algorithm for the greatest common divisor of two natural numbers. Any Greek construction with ruler and compasses can be viewed as an algorithm in which, starting from a finite number of given points, circles and lines are added and intersected step by step to obtain the desired result. The Chinese mathematician Liu Hui (3rd century) systematically solved systems of first-degree equations using what later came to be called Gauss's pivot method (19th century).

Many proofs also contain an algorithm. In section 3 we explained Hierholzer's proof of the existence of a Eulerian circuit in a multigraph in which the degrees of all vertices are even. Hierholzer proves this by giving an algorithm to construct such a Eulerian circuit. When Euclid proves that there are infinitely many prime numbers, he does so by describing an algorithm to produce, for any list of prime numbers, an additional prime number missing from the list.

Man-Keung Siu (Siu, 2002) discusses the distinction between 'algorithmic' and 'dialectic' mathematics. Algorithmic mathematics is about finding solutions, using fixed methods and algorithms. Dialectic mathematics is about explaining and proving. When I read Siu's article, I was thinking that the recent trend in our curricula is towards more algorithmic mathematics and less dialectical mathematics. On the other hand, the examples I elaborated or cited above show that dialectic and algorithmic mathematics are inseparable and intertwined. This is also Man-Keung Siu's conclusion: they are two aspects of a same reality, like Yin and Yang in the Chinese tradition: "*In the teaching of mathematics we should not just emphasize one at the expense of the other. When we learn something new we need first to get acquainted with the new thing and to acquire sufficient feeling for it. A procedural approach helps us to prepare more solid ground to build up subsequent conceptual understanding. In turn, when we understand the concept better we will be able to handle the algorithm with more facility*." (Siu, 2002)

## 4.2 Inventing and explaining algorithms geometrically

So, no mathematics without algorithms. But the essence of mathematics is not the execution of algorithms. In some first-grade mathematics textbooks, many solution procedures are laid down in 'step-by-step plans' that students have to perform literally. In part, of course, it is necessary for pupils to automate certain calculations or solution methods so that they do not have to think about them from scratch every time they need them. But let's not overdo it: for executing algorithms, there are now computers, and pupils are no computers. Running an algorithm is less interesting than searching for solutions and explanations, coming up with algorithms...

Let us not turn mathematics lessons into merely 'applying algorithms'. Let's teach about algorithms, their history, their geometric inspiration. Let's give pupils the opportunity and the time to invent algorithms.

### 4.3 More or less algorithms in our computer era?

Obviously, computers have made algorithms *more relevant*. Algorithms and computer programmes play a big role in the background of the internet, social media, search engines, navigation systems... They determine our lives and sometimes threaten our privacy.

On the other hand, computers have made algorithms *less relevant*. In my own job as a mathematics teacher, I had to program much more often in the previous century than in this one. This is because many things are now pre-programmed, in GeoGebra, in graphing calculators, in all kinds of apps. Having a graph drawn by a computer, investigating the effect of parameters: I remember that this required programming. In the last 25 years, I hardly ever had to program. And now it will be necessary again, because the curriculum makers consider it part of the computer age.

I want to end this article with some quotes from Donald Knuth. He is 84 now. He is considered the father of programming and the inventor of TEX. He also wrote about the history of Babylonian algorithms (Knuth, 1972).

"*Programming is the art of telling another human being what one wants the computer to do.*"

"*An algorithm must be seen to be believed*."

"*Programs are meant to be read by humans and only incidentally for computers to execute.*" (Knuth, 1969)

## REFERENCES


Barnett, J.H. (2009). Early Writings on Graph Theory: Euler Circuits and The Königsberg Bridge Problem. *Resources for Teaching Discrete Mathematics. MAA Notes #74* (pp. 197-208). The Mathematics Association of America. https://www.maa.org/sites/default/files/pdf/pubs/books/members/NTE74.pdf

Burgos, M., Beltrán-Pellicer, P. (2018). On Squares, Rectangles and Square Roots. *Convergence 15.*

Chabert, J.-L, Barbin, E., Guillemot, M. (1995). *Histoire d'algorithmes: du caillou à la puce.* Paris: Belin.

Euler, L. (1741). Solutio problematis ad geometriam situs pertinentis. *Euler Archive - All Works. 53* https://scholarlycommons.pacific.edu/euler-works/53 (Latin text). https://www.cantab.net/users/michael.behrend/repubs/maze_maths/pages/euler_en.html (English translation).

Fowler, D., Robson, E. (1998). Square Root Approximations in Old Babylonian Mathematics: YBC 7289 in Context. *Historia Mathematica 25*.

Goldberg, D. (2000). Ode to the Square Root: A Historical Journey. *Humanistic Mathematics Network Journal 23*.

Hierholzer, C. Wiener, C. (1871). Ueber die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechung zu umfahren. *Mathematische Annalen VI*, 30–32. https://eudml.org/doc/156599

Kindt, M. (2007). *√ met een staart*. Non-published hand-out Nationale WiskundeDagen.

Kindt, M. (2022). Algoritmen met een staart. *Euclides* 98(2) *and 98*(3).

Knuth, D.E. (1968). *The Art of Computer Programming* (3rd ed.). Addison Wesley.

Knuth, D.E. (1972). Ancient Babylonian Algorithms. *Communications of the ACM* 15(7), pp. 671-677.

'Leonhard Euler.' (2022, December 22) In *Wikipedia*. https://nl.wikipedia.org/wiki/Leonhard_Euler

Moore, C., Mertens, S. (2012). *The Nature of Computation.* Oxford University Press.

Moyon, M. (2007). La tradition algébrique arabe d'al Kwarizmi au Moyen Age latin et la place de la géométrie. Barbin, E., Bénard, D. (Ed.) *Histoire et enseignement des mathématiques: rigueurs, erreurs, raisonnements* (pp. 289-318).Clermont-Ferrand: Institut National de Recherche Pédagogique.

Perucca, A. (2020). De zeven bruggen van Koningsbergen. *Uitwiskeling 36*(1).

Roelens, M., Van den Broeck, L. (2015). Veeltermvergelijkingen vroeger en nu. *Uitwiskeling 31*(4).

Roelens, M., Vanlommel, E. (2020). Redeneren en puzzelen met grafen. *Uitwiskeling 36*(2).

Siu, M-K (2002). *'Algorithmic mathematics' and 'dialectic mathematics': the 'yin' and the 'yang' in Mathematics Education.* Text of plenary lecture at the 2nd International Conference on the Teaching of Undergraduate Mathematics, Crete, July 2002. https://hkumath.hku.hk/~mks/AlgorithmicDialectic.pdf

Swetz, F. (2012). *Mathematical Expeditions. Exploring Word Problems across the Ages.* Baltimore: The John Hopkins University Press.

Vankriekelsvenne, K., Vanmarsenille, J. (2003). *Handmatig worteltrekken*. Diepenbeek: non-published workshop material.

Tycho's Wiskunde: [Handmatig worteltrekken] Deel 2 - Decimaal voor decimaal. https://www.youtube.com/watch?v=JjUJua1Gptg

Tycho's Wiskunde: [Handmatig worteltrekken] Deel 6 – Derdemachtswortels. https://www.youtube.com/watch?v=gpma8dhD69Q

TwoPi (2008). *Root extraction, part II: cube roots*. https:/threesixty360.wordpress.com/2008/02/11/root-extraction-part-ii-cube-roots/

Wing, J.M. (2006). Computational Thinking. *Communication of the ACM*.

# A HIDDEN THREAD: IDEAS AND PROPOSALS ON CHILDRENS' MATHEMATICS EDUCATION THROUGHOUT HISTORY

**Ana MILLÁN GASCA**
Roma Tre University, via del Castro Pretorio 20, Rome, Italy
anamaria.millangasca@uniroma3.it

## ABSTRACT

> Dipoi imprendano l'abaco e quanto sia utile geometria; le quali due sono scienze atte e piacevoli agli ingegni fanciulleschi, e in ogni uso ed età non poco utili saperle. Poi ritornino a' poeti, agli oratori, a' filosofi ...
> Leon Battista Alberti, *I libri della famiglia* (1433-1441)[2]

> Children are virtually mute in the sources of history.
> Willem Frijhoff (2012)

### *1 Beyond the history of teaching arithmetic: mathematics for children 1800-1950*

The first steps in mathematics of the child in primary school and preschool are today an issue of considerable concern of parents, teachers, and education researchers all over the world. Educational overall methods and specific recipes are on the market in many countries, with national and cultural specificities.

This concern emerged distinctly in Europe around 1800 and for two hundred years it pushed the conception of many educational ideas regarding mathematics for young children as well the marketing of an array of proposals

[2] Then they should take up the abacus [written arithmetic] and what is useful in geometry; which two are suitable and pleasant sciences for childish minds, and in every use and age it's not a little useful to know them. Then they return to poets, orators, philosophers ... [Eugenio Garin quotes this passage in his *Education in Europe 1400-1600*, Garin 1976, p. 143]

– textbooks addressed to adults (schoolteachers or parents) or to pupils, picture books, toys and educational aids. Contributions were put forward in several countries and languages, which expressed civil commitment, showed trust the child's early “elective affinity” with number and form, and displayed the authors’ creativity in devising methods and designing materials. Several among the innovators – starting from the late 18th century – deployed their proposals in close alliance with the publishers and manufacturers of books and other gadgets for children, who aimed at potential relevant sales in the growing childhood-oriented market. At the turn of the 20th century there was a golden age of reflection on mathematics and children, with the contribution of several women.

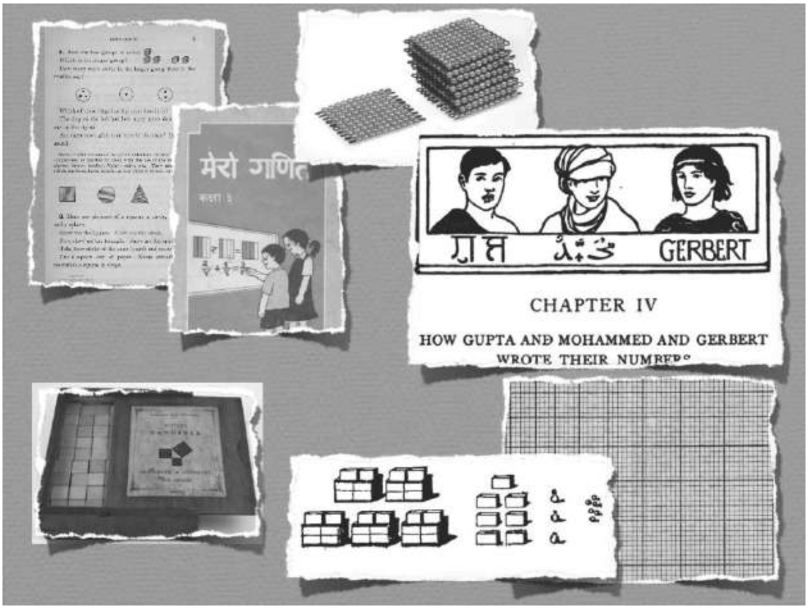



**Figure 1.** Objects and books display the increasing attention to children as mathematical learners, with a playful dimension of enjoyment. Some examples from the 19th-20th centuries: educational aids and toys, children’s books with illustrations, books for educators, primary school textbooks, conveying new views on elementary mathematics for early education and approaches.

The issue of children's first steps in mathematics throughout history belongs to a certain extent to the history of teaching arithmetic,[3] as the teaching of oral numeration, measures, and written symbols for quantities (numerals) has always been (until nowadays) the core of the early contacts of children with mathematical knowledge. The tradition with which the innovators in the Late Modern age were confronted focused on numeracy.

On the other hand, innovators did not restrict themselves to written arithmetic, but paid attention to oral counting and calculation and (most importantly) considered also geometry as suitable for a smooth and enjoyable – at the same time, more far-reaching – contact with mathematics. Pure reckoning was insufficient to meet the ambition, in certain sectors of European intellectuality, to offer a generalist education to all, particularly in primary school, a school level that emerged as a school of the people.

The growing sensitivity and understanding of the world of children, especially in early and middle childhood (up to age 10), led several men and women to seek new educational solutions. To *improve* (methods of teaching), to *innovate* (contents and methods), to *adapt* (to children's minds and sentiments) have been keywords all along this evolution. Innovators were confronted with a *tradition* inherited from the past that conflicted with *the new*: with children themselves, "new to the world" as Hannah Arendt writes,[4] as well as with the very conditions of an ever-changing world.

Since the late 18th century, the first contact of children with number and form happened in a complex educational universe including both schooling in institutions of various kinds (including infant schools) and domestic learning with tutors or parents; from a social point of view, it included both males from affluent families before attending secondary school (where they would further study Euclid), and the children of low classes in charity or popular schools. An audience continuously growing to include the poor, the girls, and even disabled children, which was to merge in unified state systems of compulsory and universal primary education.

Let's consider briefly some aspects of the *status quo* that innovators focusing on young children had to face. Next, I introduce some examples of ideas and proposals in Great Britain around 1800, before considering some general

[3] See Bjarnadóttir 2014 for the state of the art on the history of teaching arithmetic.

[4] Arendt 1961.

trends emerging from available studies. We shall see that ideas and proposals often aimed at developing a genuine “mathematics for children”.

## *2 Children learn to reckon: contexts and contents*

Starting from the pioneering contribution by Augustus De Morgan (1806-1871), the concern with and attention to the teaching of arithmetic to the young has been a powerful impulse to explore the historical roots of current practices and contents of arithmetic in elementary education (that involving minors). Around the turn of the 20th century historical and bibliographical research made it possible to identify the persistence, in the study programs of both secondary and primary schools, of the contents transmitted by what was identified as an European successful printed book genre in its own right during the XVI-XVII centuries, the commercial elementary arithmetics.[5]

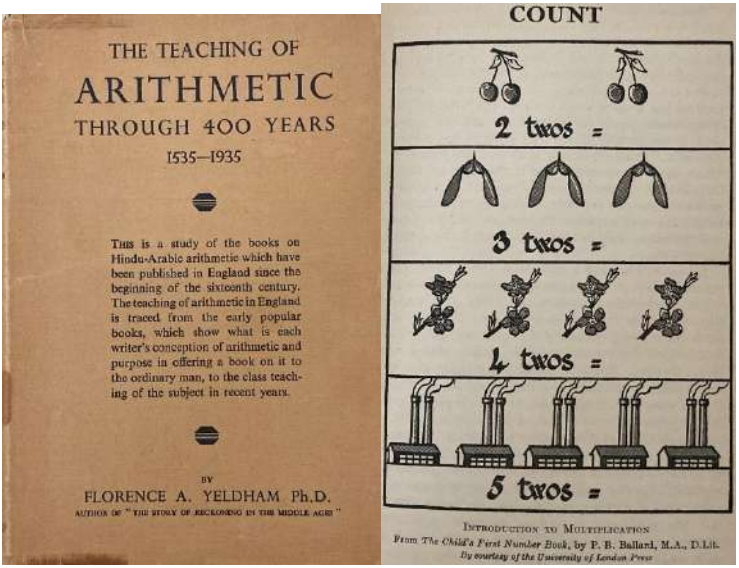



**Figure 2.** (left) Cover Florence A. Yeldham's (1877-1945) essay on English arithmetic books, a companion to her book on reckoning in the Middle Ages (London/Bombay/Sidney: George G. Harrap &Co, 1936). She made wide use in it of De Morgan's *Arithmetical books* (1847). (right) The reproduction of an illustrated page on multiplication using numerals, number words and pictures from *The child's first*

[5] See Schubring 2014. Many contributions were published in Germany and the USA, countries where the extension of literacy was ahead, as underlined by Cipolla (1969).

*number book* (2nd ed. 1933) by Philip B. Ballard (1865-1950), included in the last chapter of Yeldham's essay, devoted to "Arithmetic in recent years": "Two other new movements, namely, the custom of sending children to school at a much earlier age than before and the better equipment for school buildings, are in great measure responsible for the important advance of modern years, the approach to the subject through practical work." (Yeldham 1936, p. 131)

Printed books were written and published with a view to adult buyer-readers, it was in fact "mathematics for sale"; let's read the synthetic reconstruction by Frank Swetz (1992):[6]

While commercial arithmetics emerged as a mathematical genre in the late fifteenth century and throughout the sixteenth century, their influence prevailed in the teaching of arithmetic up until the beginning of the twentieth century. Their content, format and instructional style, with a great reliance on the use of problems, set the standard for arithmetic teaching for centuries. The impact of the commercial texts had raised the popular understanding of arithmetic to a new level; [...] it became the basis for accessible, lucrative careers. Quite simply, arithmetic became noticeably useful. Unfortunately, in a sense, this usefulness focused it as a kind of vocational training. [..] Throughout the seventeenth and into the eighteenth century, arithmetic books grew larger in content and became more comprehensive but still closely resembled their fifteenth and sixteenth century ancestors.

Thus, a long-standing tradition was at hand for any family or teacher (both tutoring at home or in schools) facing the first steps with numbers of a child in the 18th-19th centuries. This European tradition was conserved and transmitted by means of arithmetic books; its vitality can be linked to some trends in European cultural, social and economic evolution: the diffusion of literacy, the development of national written cultures in vernacular languages; commercial and manufacturing development in conditions of relative freedom, including the book market pushed by the increasing number of consumers.

As for contents, basic arithmetic started from the reading and writing of natural numbers using the decimal numeration system (with digits 1, 2,.... , 8, 9 and the use of 0), measures, and then the operations and its use in the solution of problems consisting of short quantitative scenarios of practical daylife

[6] These contributions, encouraged by the aim to improve teaching, are among the first in the history of mathematics education.

and work life, also with “broken” numbers, culminating with proportion for coin change and measure conversion, as well as distribution of profits, payments and so on (see Bjarnadóttir 2014, p. 434).

Content and methods can be been explored mainly through arithmetic books, and through other sources (manuscript notebooks, autobiographical writing), as well as through the criticisms and alternative proposals of the innovators. As for methods, they appear to be based more on practice and remembering than on understanding. For example, John Denniss (2009), considering both arithmetic books and pupils’ notebooks in the case of Great Britain, writes:

*Both Recorde and Newton had long ago pleaded for more imaginative teaching of arithmetic, based on understanding rather than rote learning. At the end of the 19th century such ideals had still barely begun to be put in practice.*

The lack of a “more imaginative teaching” could potentially have a greater negative impact as the age of the learners/pupils decreased. Now, to what degree and in which modalities were young children granted numeracy? Institutionalization of schooling needs to be put in the picture, considering both the early stages of secondary education and the schools for the people (popular schools and vocational schools). It is one of those aspects of daily life of the past which risks going unnoticed by historical studies, hidden from attention by issues of a wider nature, within such fields as childhood history, the history of literacy or the history of elementary education. And yet, even scattered examples reveal a world of practices, of feelings, of meanings and expectations linked to the first steps of children in an adult world of number and measure.[7]

Medieval reckoning schools were a turning point, as their (vocational) training was addressed to children. In the pioneer essay by Carlo Cipolla, *Lit-*

[7] There is an ancient history also regarding children and mathematics. For example, Werner Jaeger has pointed out that Plato, in his last work, *Laws*, writes about the learning of number and measure in childhood by play and with pleasure (in his 1944 essay *Paideia*). Henri Marrou depicted lively scenes of the introduction to numeration of children in his *History of education in antiquity* (1948), and has brought up that Augustin, in his *Confessions* (I, 13), remembers “the odious song: one and one, two; two and two, four”. Thus, Marrou stressed the struggle of young learners.

*eracy and development in the West* (1967), numeracy and specifically children's education as one aspect of literacy spreading is not a central issue. Yet, he quotes an episode dating back to the 14th century regarding the Italian reckoning teaching to the youngest: a document from the Italian town Lucca dated 1382 proclaimed that prosperity of citizens depended upon trade and it was thus indispensable that they learnt to read and reckon, and after four years city authorities put in practice their advice hiring a master so that "children would be taught arithmetic, so as to became wiser and shrewd in business."[8] Yet the fact that the schools devoted to the teaching of the new Hindu-Arabic reckoning technique have as attendants children (not adults) has a relevance in itself in the history of children and mathematics in Europe. Raffaella Franci has paid attention to this circumstance:

*The topics of the curriculum were finely divided into teaching units called [in Italian] muta (shift). Thus, for example, reading and writing numbers formed one pack, learning the multiplication tables formed another. The multiplication between integers was divided into several shifts according to the number of digits of the multiplier, and similarly the division relative to the digits of the divisor. The pupils, upon entering the school, were placed in a muta according to the level of their knowledge, they passed to the next wetsuit only when they demonstrated full mastery of the techniques taught in the previous one. The presence in the school lasted the whole day, the teaching was based on the repetition of numerous exercises, both written and oral. In addition to the numerous exercises done at school, homework was also provided. The teaching, aimed at learning the techniques rather than understanding the methods, was very repetitive, many practical rules were taught, above all to deal with the complicated system of coins, weights and measures. The techniques relating to commercial operations (companies, barters, merits, etc.) were taught by proposing problems of gradually increasing difficulty, in which an attempt was made to envisage all the cases that could have arisen in the effective exercise of the trade.* [Franci 2000, p. 130-131]

[8] He quotes from a 1905 essay by Paolo Barsanti on public teaching at Lucca in the 14th-18th centuries.

**Figure 3.** A page from Filippo Calandri's (1468-1517/18) *Arithmetic*, manuscript conserved at the Florence library Biblioteca Riccardiana. Source: Ulivi 2002

Recent historical research has ascertained that the first European arithmetic printed books in turn stemmed from the content and teaching practices of medieval reckoning schools attended by children.[9] A working hypothesis is that method or "learning path" developed by Italian teachers (*maestri d'abaco*) would last for centuries, being extended as recipients both to self-taught

[9] See the overall presentation in Bjarnadóttir 2014. Historical research in past decades has thoroughly analyzed the origin of reckoning schools for trading professions in Italy in the late Middle ages: their relevance for the history of mathematics, for the history of accounting techniques, for European economic history, and even for the history of vocational education had been considered. Contents and teaching methods were developed by reckoning masters) in an extraordinarily fruitful cultural season, thanks to the confluence of social factors (the flourishing of urban centres); to the activation of economic life (manufacturing, commerce, finance); to the recovery of classical mathematical knowledge. Reckoning included learning the oral numeration of course, but the main focus was the written numeration of Indian origin that – after its diffusion in the Islam area – had reached Europe.

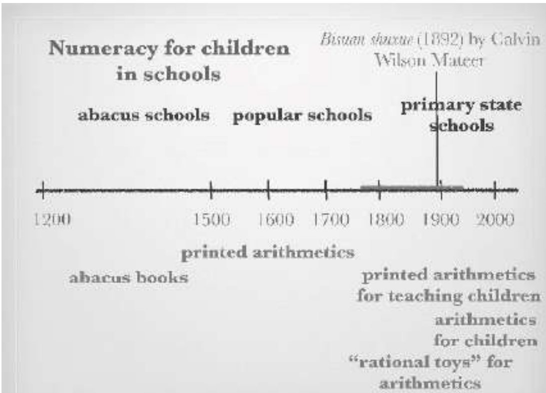
adults and (indirectly, through teachers) to children. Let's read the synthetic reconstruction by Swetz, who underlines the contribution of Pestalozzi:

*The applied arithmetic taught in secondary schools and private academies of the nineteenth century was, for all practical purposes, commercial arithmetic. When the educational reformer Johann Pestalozzi (1746-1827) established a curriculum for the education of young children, he drew on the subject content of the higher schools. Thus, the primary arithmetic studies he designed were heavily influenced by commercial arithmetic and monetary transactions. This overriding influence remained until reforms early in the twentieth century refocused education on the child as an individual with æsthetic and intellectual need beyond mere vocational training. Although arithmetic today is taught as a subject from which a child may draw many meanings and uses, its teaching had been irrevocably shaped by the commercial needs of the early Renaissance.* [Swetz 1992, p. 377]

Now, Pestalozzi's curriculum for the educational establishments founded and managed by him was not only a direct adaptation of the subject content of gymnasia or collèges and the same. Early arithmetical training for children of higher classes could be part of the initial curriculum in secondary schools; besides, in modern Europe, arithmetic was combined with reading and writing in popular schools, created and maintained by charitable privates or towns or religious groups. And Pestalozzi's initiatives were originally intended for orphans and children of the working class (Horlacher 2011).

For example, the Catholic area, in the Piarist schools of José de Calasanz (1557-1648) the children were taught free of charge reading, writing, arithmetic and Catholic doctrine. Moreover, the idea of "readiness" of very young children, as put forward in the Protestant area by Comenius (Jan Amos Komensky, 1592-1670), included mathematics: he envisaged in his *Didactica magna* a mother school (the first six years of a child's life) where the seeds of arithmetic will be planted if the child understand what is meant by "much" and "little," can count up to ten, can see that three are more than two, and that one added to three makes four" [...] He will possess the elements of geometry if he know what we mean by "large" and "small," "long" and "short," "broad" and “narrow,” “thick” and “thin”; what we signify by a line, a cross, or a circle, and how we measure objects (in different length measures).[10]

Swetz pointed out that possibly both the persistence of the educational practical goals (commercial/arts and crafts) needs and the fossilizing of an emphasis on procedures, receipts, and formulae hindered the exploration of a first approach to number – to mathematics as a whole – from a “æsthetic and intellectual” encounter, emphasizing conceptual clues (such as decomposition, equality, comparison, ratio), problems as challenges, beauty enclosed by regularities in mathematical properties and patterns, and the pleasure that could be obtained from number and form. Exploring precisely this new dimension in the encounter of young children with mathematics was a fascinating task in front of several scholars who shared their commitment to education and the nearness to the child's world. Their work was developed in the decades in which the goal of a state universal, free of charges primary school spread in many countries in the world. While the Modern age brought both the *sentiment de l'enfant* (Ariès 1960) and popular schools, mass education and the schooling of young children during the 19th century prompted coping with the need to adapt the arithmetic curriculum and training methods to the specificities of children's minds.




[10] Comenius 1896 (1657), p. 426. Besides, the elements of statics will have been learned if the children see objects weighed in scales or acquire the power of telling the approximate weight of objects by weighing them in their hands.

**Figure 4.** A tentative time line regarding educational contexts and the literary tradition handing down content. The timeline does not suggest continuity in aims, but different contexts which should be taken into account. An example is the textbook *Bisuan shuxue* (1892) on pen-calculation arithmetic by the U. S. A. missionary Calvin Wilson Mateer, the "dawn of modern Chinese mathematics education" (Ma 2010, pp. xiv-xv; see Fisher 1911, Dauben 2003).

### *3 "Making minor mathematicians": a glimpse at some early contributions*

The following dialogue between a 5 year old child and his father appears in the first pages of the lengthy treatise *Practical education* (1798) by Maria Edgeworth and her father Richard Lovell Edgeworth. This essay is an early example of the aim of developing a genuine science of education in Europe. It offers us a shot of a young child approaching numbers at home in late 18th century Great Britain:[11]

*We will give you an instance: arithmetic is one of the first things that we attempt to teach children. In the following dialogue, which passed between a boy of five years old and his father, we may observe that till the child followed his father's train of ideas he could not be taught.*

*Father. S —, how many can you take from one?*

*S—. None.*

*Father. None! Think; what can you take nothing from one?*

*S—. None, except that one.*

*Father. Except! Then you can take one from one?*

*S—. Yes, that one.*

*Father. How many then can you take from one?*

*S—. One.*

*Father. Very true; but now, can you take two from one?*

*S—. Yes, if they were figures I could, with a rubber-out. (This child had frequently sums written for him with a black lead pencil, and he used to rub out his figures when they were wrong with Indian rubber, which he had heard called rubber-out)*

*Father. Yes, you could; but now we will not talk of figures, we will talk of things. There may be one horse or two horses, or one man or two men.*

*S—. Yes, or one coat or two coats.*

[11] Edgeworth and Edgeworth 1798, pp. 58-59. On Maria Edgeworth, see Fantaccini, L.,& Leproni, R. (Eds.) (2019).

*Father. Yes, or one thing or two things, no matter what they are. Now, could you take two things from one thing?*

*S—. Yes, if there were three things I could take away two things, and leave one.*

This narrative is presented by the Edgeworths as one of their first examples of the need to enter in the cognitive and sentimental sphere of the child, and it regards precisely arithmetic. The scene happens in a full oral framework, using (counting) number words and expressions such as "take from", "how many", together with objects and the hand gesture of rubbing out. Yet a reference is made to written arithmetic: figures and sums. The Edgeworths, as any other educated European, were heirs to a consolidated tradition of training in reckoning. The time was ripe, however, for taking account of a child's encounter with numbers in its specific characteristics. For example, through the awareness on the part of the teachers of the fact that it was a question of acquiring a technique carried out in writing, but nevertheless learning passed through a relationship with the pupil where the spoken words and the body (the hand, the eye, motion, and rhythm) played a crucial role.

In the Edgeworths' essay, moreover, the chapter on arithmetic is followed by one on geometry, and both contained references to the "rational games" mentioned in the chapter *Toys*, that opens the 2 volume-work with a critique of expensive and fashionable play objects for children:

*The first toys for infants should be merely such things as may be grasped without danger, and which might, by the difference of their sizes, invite comparison: round ivory or wooden sticks should be put into their little hands; by degrees they will learn to lift them to their mouths, and they will distinguish their sizes: square and circular bits of wood, balls, cubes, and triangles, with holes of different sizes made in them, to admit the sticks, should be their playthings. No greater apparatus is necessary for the amusement of the first months of an infant's life. [...] To gratify the eye with glittering objects, if this be necessary, may be done with more safety by toys of tin and polished iron, a common steel button is a more desirable plaything to a young child then many expensive toys; a few such buttons tied together, so as to prevent any danger of their being swallowed, would continue for some time a source of amusement.*

A contemporary of the Edgeworths, Ellenor Fenn (1744-1813), an author of readings for children who published her works – under pseudonyms – with John Marshall and Newbery, promoted what she called "teaching in sport", and for this purpose she designed a «Set of toys», produced in the 1790s by Marshall, consisting of three wooden boxes or trays included in a large box, each of them divided into compartments containing a variety of materials in cardboard or wood, dedicated respectively to grammar, spelling and figures. In the Figure Box, compartments presented a purse of counters, a sack with beans for merchandise, and a set of cards for multiplication (pairs of cards with cuts of animals which when matched together produce the entries multiplication tables) published by J. Aldis in London.[12]

Fenn's boxes are an early example of educational aids which sharply marks the distinction between self-taught adults and children, together with the books to be directly used by children, such as *Marmaduke multiply's merry method of making minor mathematicians*, published by John Harris, around 1816, using of pleasant story-images as *aide-mémoire* for numerical facts (Fig. 1).

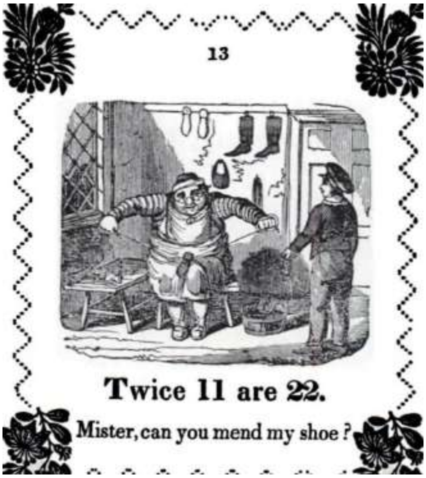



**Figure 5.** A page of the booklet *Marmaduke multiply's merry method of making minor mathematicians* (see Denniss 2009).

The possibilities of making money on children's goods in Britain at the turn of the 1800s gave wings to the creativity of authors in alliance with publishers. Consider the box «Tangible arithmetic & geometry» designed and sold by Henry Butter, author of primers for spelling and reading (Fig. 6).

[12] Fossa 2021; on Fenn see Stocker 2007.

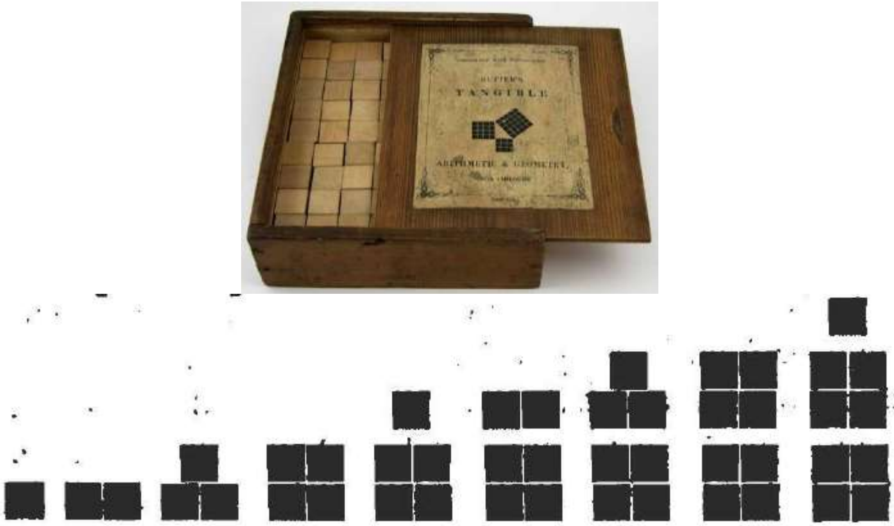



**Figure 6**. The box of cubes Tangible arithmetic and geometry by Henry Butter.

Explanations were presented in a booklet (35 pp, with some end pages of advertising of other works by Butter) *Tangible arithmetic & geometry: The most effectual method of teaching addition, multiplication, subtraction, and division, and the analysis and composition of numbers; also the formation of squares, parallelograms, triangles, rhomboids, cubes, parallelopipeds, prisms, pyramids, &c. It will likewise convey, with ease and certainty, a far more accurate knowledge of fractions than has been attainable by any means hitherto devised. Illustrated by figures, and by one hundred and forty-four cubes in a box. Forming a Permanent fund of amusement and instruction for all ages*, London (printed by J. S. Hodson). The booklet was illustrated with geometric drawings (squares representing cubes). The introduction opens with the following recommendation:

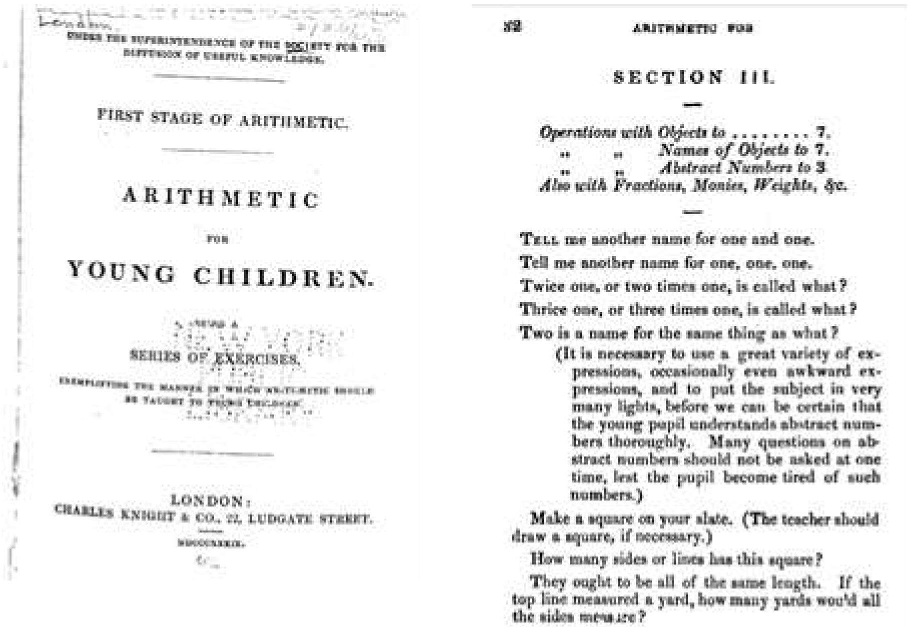
*“The Author strongly recommends that young children should not be required to learn the numerical characters till they are very familiar with numbers, for which those characters are mere arbitrary signs. Instead of which he begs to suggest that for Addition, Subtraction &c., they should be accustomed to place the cubes in regular groups, for the various numbers up to nine, so that they may at a glance see what number of cubes are in a group, without having to count them.”*

Imagination, narration, and play were introduced in mathematics thanks to drawings, stories, counters and dice, sticks and cubes to compare sizes, for composition and decomposition to be linked to numerical operations. This was part of the general development of toys and children's books, which has received increasing attention in recent years. Educational aids for learning mathematics would become a kind of rational toys, and a flourishing commercial production until our days.

The books published by Horace Grant (1801-1859) represent both the focus on young children and the fact that the extension of early mathematics instruction to working-class children also fostered innovations (Calabrese 2015). In fact, he belonged to the radical milieu which in Great Britain sought to obtain a law establishing public primary education, and which founded (in 1826) the Society for the Diffusion of Useful Knowledge. While De Morgan wrote treatises on mathematics for the Society, Grant was entrusted with a «Library for the young children». In a biographical note published in 1861, Edwin Chadwick wrote: "[he] will be found worthy to be classified with Comenius, Pestalozzi, the Abbé Gaultier, and the Abbé Girard, men of great humanity and eminent ability, who devoted themselves zealously to the special study of the minds of children, and to the best means of cultivating them" [Chadwick 1861, p. v].

The child's first steps in oral counting were for Grant a moment to be seized with care, through a shelling of deliveries and small problems, between observation and imagination, between sounds, movements and small problems. The experience with children of various social backgrounds led him to an overall cultural project composed of three works: *Arithmetic for young children* (1835), *Exercises for the improvement of senses* (1835), which contains many exercises in geometry with objects such as wooden cubes or through measurements, and an unpublished work, *Course of exercises in the first elements of form, adapted for the use of mothers and early teachers (Geometry for children)*.[13]

[13] He also published *Drawing for young children* (1838), *Second stage of arithmetic for schools and families* (1841) and *The elements of practical geometry for schools and workmen* (1852) (Calabrese 2018)


UNDER THE SUPERINTENDENCE OF THE SOCIETY FOR THE DIFFUSION OF USEFUL KNOWLEDGE.

FIRST STAGE OF ARITHMETIC.

ARITHMETIC

FOR

YOUNG CHILDREN.

SERIES OF EXERCISES,

EXEMPLIFYING THE MANNER IN WHICH ARITHMETIC SHOULD BE TAUGHT TO YOUNG CHILDREN.

LONDON:
CHARLES KNIGHT & CO., 22, LUDGATE STREET.

22 ARITHMETIC FOR

SECTION III.

*Operations with Objects to . . . . . . . . 7.*
*" " Names of Objects to 7.*
*" " Abstract Numbers to 3.*
*Also with Fractions, Monies, Weights, &c.*

Tell me another name for one and one.

Tell me another name for one, one, one.

Twice one, or two times one, is called what?

Thrice one, or three times one, is called what?

Two is a name for the same thing as what?

(It is necessary to use a great variety of expressions, occasionally even awkward expressions, and to put the subject in very many lights, before we can be certain that the young pupil understands abstract numbers thoroughly. Many questions on abstract numbers should not be asked at one time, lest the pupil become tired of such numbers.)

Make a square on your slate. (The teacher should draw a square, if necessary.)

How many sides or lines has this square?

They ought to be all of the same length. If the top line measured a yard, how many yards would all the sides measure?

**Figure 7.** Cover page and sample page of *Arithmetic for young children* (1835), published under the superintendence of the Society for the Diffusion of Useful Knowledge by the publisher Charles Knight. The dialogue form was present in arithmetic books, but in this case the purpose is to exemplify how to introduce number to the young child through graduated questions using number words, involving a variety of examples such as the body, real life objects, geometrical figures, coins and measures. Figures are introduced in the final pages. Educational comments are included in between questions. Thus, before the production of books for children on number, educational books were published with a focus not only on contents, but also on what we would call teaching methodology.

### *4 The traces of the past in our present: researching ideas and proposals regarding mathematics and children.*

A research program has been developed on the subject of this paper in the Department of Education of Roma Tre University, leading to several Master theses in history of mathematics education regarding single authors. The research was tightly connected with the development of courses and workshops in mathematics and mathematics education for the preschool/primary school education degree (ca. 170 teaching hours in the first and third years, following the Italian 2011 state regulations). Historical research work was coupled with the design of mathematical activities for children inspired by ideas and proposals, which were tested in practice during the traineeship in preschools and primary schools.

The issue of the relationship between methods and conceptions presented in university courses and "traditional" practices in school is deeply felt among students, some of them actually already working in schools with temporary jobs. Trying to avoid dogmatic blame on school habits, historical perspective appeared potential useful. Some examples analyzed in single master thesis were presented to students in experiential workshops, where the historical case was coupled with the discussion of school activities inspired by the historical material.[14] Subsequently, a panorama of the evolution in history of contacts between children and mathematics, was introduced in the syllabus of the Roma Tre courses and practical lessons.[15] On the one hand, tracing the historical origin of present practices (such as the limit of 20 for natural numbers in 1st degree (6 years olds), the emphasis put on written algorithms and

[14] Those experiential workshops constantly arose strong engagement, self questioning and active participation of many students with comments and questions. One should take into account, besides and independently of the historical contents, the fact that listening to a course mate as well as storytelling, photographs and videos regarding actual school and children were important factors of success. Experiential workshops have been developed further in the context of the Erasmus + project ANFoMAM (www.unavarra.es/anfomam)

[15] Millán Gasca 2016, chapters 2 and 3, presents an overview of some trends among innovators, considering their interaction with practices and contents in the tradition of introduction to elementary arithmetic. The course and workshops include also an overview of the evolution of mathematics in history with some detail on the history of classic geometry, of numerations systems and of measure.

on conversion of measures in the decimal metric system, the lack of geometrical contents) was helpful to give meaning to a student's puzzlement or to their personal records. On the other hand, considering and comparing the work of innovators could help in analyzing contemporary proposals (including marketed ones); and in enhancing students' skills in designing activities.

The research program on ideas and proposals on mathematics for children in 18th-20th centuries started from this remark on Pestalozzi and Friedrich Fröbel (1782-1852) in a reflection by the Italian mathematician Federigo Enriques (1871-1946) in a chapter devoted to teaching in his 1938 essay *Mathematics in history and culture* induced to investigate early childhood and popular elementary schools:

*The educational value of mathematics is revealed [...] also in the first grades of childhood and working class education; because mathematical intelligence is very precocious. Two pedagogues above all worked to bring mathematical knowledge into the education of the child, as an element of his intellectual development: Pestalozzi and Fröbel. The first teaches "as Geltrude teaches her children", showing them early on the awareness of the relationships of number and measure, which they must learn quickly and clearly. The second, already in his first gifts, in the games and exercises in his gardens, offers children a vision of geometric figures and their symmetries, and interests them in increasingly difficult observations, with a methodical progression that responds to a precise educational design. For infant schools as for popular schools, what was observed above is above all true, that the educational direction is not separated from the utilitarian, which creates the acceptance of the things taught with interest.*

Moreover, the essays included in the 1988 edition of Condorcet's arithmetic book by Gert Schubring and Charles Coutel, and Renaud D'Enfert's discussion of Francoeur's linear drawing for mutual schools (D'Enfert 2014), showed that Pestalozzi's contribution should be considered in a wider context, instead of considering him essentially an isolated figure and focusing on his followers.[16] Moreover, was there a Pestalozzi's revival (Denniss 2009) in the late

[16] Such as José Mariano Vallejo (1770-1846) *Aritmética de niños para uso en las escuelas del reino* (1804) in Spain, Warren Colbourn (1793-1833) *An arithmetic on the plan of Pestalozzi* (1821) in the United States (Kirkpatrick), Hippolyte Léon Denizard Rivail (1804-1869) *Cours d'arithmétique pratique et théorique d'après la méthode de Pestalozzi* (1824) and Hip-

19th century? The attention to two books, Jean Macé's (1815-1894) children's book *Grand papa's arithmetic* (Colella 2012) and Charles Laisant's *Initiation mathématique* (Lamandé 2011, Schiopetti 2015), was suggested by a short essay published in 1907 by the Spanish mathematician Zoel García de Galdeano (1846-1924).[17] A PhD thesis at the University of Zaragoza on mathematics with children with Trisomy 21 included a historical analysis of Édouard Séguin's (1812-1880) geometrical educational aids and principles, with applications to the design of activities. The analysis has a starting point the goal of researching an author that had inspired some among Maria Montessori's (1870-1952) educational aids for mathematics with young children, designed around 1900.[18]

Subsequently, a group of works have been developed in the last five years on selected topics regarding several periods and countries, considering single books or single authors.[19] A landscape has begun to be depicted, starting from

---

polyte Vernier's *Petite arithmétique raisonnée* (1832) in France.

[17] See Millán Gasca 2015. I am working on a paper containing an analysis of the correspondence between Macé and his publisher Hetzel on the children's book *Arithmétique du grand papa*.

[18] Gil Clemente 2016; Cogolludo-Augustin & Gil Clemente 2019. Some aspects of the role of geometry in the awakening of conciousness in intellectually disabled children (between plane and solid geometry are considered in Gil Clemente, Millán Gasca 2021).

[19] John M. Colaw and John K. Ellwood's textbook *School arithmetic* (1900) (Di Clemente 2018) in the USA; on Jules Dalsème's (1845-1904) conception of "natural geometry" (Zannoni 2018); on the Adriana Enriques' 1928 schoolbooks (Peccarino 2018); on mathematics in the primary school state unique book in fascist Italy (Cammillucci 2020); on Montessori's treatment of ratio and proportions in *Psychoarithmetic* (1934) (in Parenti 2017); on Margalida Comas' (1892-1972) contribution to primary school mathematics (Mazza 2018); on Horace Grant's (1801-1859) books for young children for the Society for the Diffusion of Useful Knowledge (Calabrese 2018) and on Mary Boole's Lectures on the logic of arithmetic (Tomasello 2019); on the origins of George Cuisenaire's proposal of colored rods for reckoning (Di Tella 2021); on the toybox «Initiateur Mathématique» designed by Jacques Camescasse (Panichelli 2020); on mathematics elementary instruction in the pedagogical reflection of Emilia Santamaria Formiggini (1877-1971) in the early 20th century (Stramaccioni 2021); mathematics in Steiner's thought and pedagogy (Di Marco 2022): on mathematics in Richard L. and Maria Edgeworth's *Practical education* (1798) (Fossa 2022). See also in this volume the workshops Gil Clemente, Migliucci & Panichelli, Magrone.

contributions in the late 18th century and arriving to the 1920s and 1930s. In the 1930 the first contributions by Alina Szeminska (1907-1986), working at Geneva with Jean Piaget, were published, leading to the essay *La genèse du nombre dans l'enfant* (1941). The working hypothesis is that of a discontinuity – the age of new math – that made many among these contributions fall into oblivion.

A cultural thread can be distinctly identified linking ideas and proposals, as different authors (mathematicians and educators, women and men) shared a common tradition of mathematics teaching that they try to cope with, taking into account the needs of the child and the changing times. In some cases, the aim was to prepare and easing the struggle with written numbers and operations; in other cases, an overall proposal of radical, ambitious transformation was put forward.[20] This thread was linked to the evolution of views and social practices regarding childhood as well as to numeracy as a social goal, including the self education of adults. Entrepreneurial initiatives were prompted by these ideas and proposals, and at the same time their supporters were also encouraged by political visions of a modern, equal society.

Some general, interwoven trends are the following:

- the attention to the specificities of young children, thus of initiation or preparation to number (or to mathematics proper) rather than the teaching or "study" of mathematics;
- the changing emphasis from literacy (written notations and procedures) to orality in the educational setting;
- the extension of aims from raw numeracy to a general introduction to mathematics, with a pivotal role of geometry,[21] also supporting the understanding of number and measure;
- the design of mathematical gadgets intended as "rational games"

[20] The consideration of the impact and effective diffusion of these ideas and the commercial success of the proposals, as well as the impact on school practices – all certainly aspects of great interest – remain outside the scope of this contribution, but many contributions can be easily placed in the context of intellectual- political networks supporting the diffusion of ideas, or can be linked to the fortune of specific publishers/manufacturers. One should notice that, as innovators often proposed counter-measures to soften the rigidity of established practices, they offer indirect information of the persistence and varieties of such practices.

[21] Millán Gasca 2015.

openness to the multiple dimensions of playfulness and beauty (also thanks to the connection with the entertainment and pleasure market)

And yet it’s a hidden thread, because of several circumstances. On one side, the issue of children, education and mathematics in history is placed at the crossroads of several historiographical perspectives. Historians of education and of the pedagogical thought in the contemporary age have neglected arithmetic (compared to grammar, history and other subjects) in the evolution of education and schooling (but see for example Roggero 1994 for popular numeracy, Terrón Bañuelos, & Alonso Velázquez 1999 for arithmetic as a school subject); and geometry may be hidden behind the teaching of drawing (which has recently received considerable attention). Besides, the role of mathematics (not only as a basic school subject, but from a more general point of view) in educational thought has been sometimes overlooked: for example, even for the iconic Pestalozzi and for the father of the Kindergarten, Friedrich Fröbel (but see Bullynck 2008, Spranger 1939, Friedman 2021), or in Rudolf Steiner, as well as in pioneers such as Édouard Séguin for special needs education or Wilhelm Lay for experimental pedagogy).[22]

Historians of mathematics and of mathematics education may have disregarded this issue as little relevant for the reconstruction of the mathematical universe throughout the ages, concentrating rather on secondary and higher education. Increasing attention to the sphere of childhood can be considered as part of a more general shift towards taking into account the history of non-elite mathematics. Yet, as teaching arithmetic (Bjarnadóttir 2014) and in particular numeracy in history regarded both adults and children, the young addressees risk to be concealed, as in the case of the arithmetic books, that did not reach the hands of children but reached them rather through adults.

On the other side, sources that could be helpful in unveiling the issue of children, education and mathematics – such as toys and educational aids, notebooks, school textbooks or picture books (Denniss 2009, 2012, Moyon 2016) – are often difficult to find, as they have not been properly archived and preserved in libraries and museums because they do not belong to “high” culture. Besides, single's contributions appear mainly to arise from a cultural and

[22] On mathematics in Steiner's pedagogy, see De Marco 2021; on Séguin, see Gil Clemente, & Millán Gasca 2021.

political background even if mainly as isolated initiatives: a single figure appears to have had a very influential role, Pestalozzi, who inspired innovators until the early 20th century. In that period, the French mathematician Charles Laisant explicitly acknowledged authors inspiring his work, which started from mathematics but lead him to promote a much more ambitious program regarding science for children.[23] Of course, further research could throw light on cultural connections.

The evolution of conceptions in the 18$^{th}$-20$^{th}$ century was accompanied by myths of success (from Pestalozzi to Montessori, Rudolf Steiner or Georges Cuisenaire), by believers and by fashions driven by advertising and storytelling: sometimes these myths are still at work[24] in present days nourished by the above mentioned deep concerns of the general public regarding children's mathematics education, and implied the risk of a biased historical analysis. Conversely, the challenge coping with an early introduction to maths for the social goal of an inclusive, equal society, as well as for our digital future, is encouraging a recognition of the evolution in different countries or cultural areas – national and linguistic specificities need to be taken in to account – which in turn may represent the basis for comparative analysis so as to find the traces of the past in the present.

## REFERENCES

Arendt, H. (2006/1961). "The crisis in education". In *Between past and future* (pp. 170-193). London: Penguin.

Ariès, Ph. (1960). *L'enfant et la vie familiale sous l'Ancien Régime*. Paris: Plon.

Auvinet, J. (2013). Charles-Ange Laisant. Paris: Herman.

Becchi, E. & Julia, D. (Eds.) (1998). *Histoire de l'enfance*. Paris: Seuil.

Bjarnadóttir, K. (2014). History of Teaching Arithmetic. In A. Karp &, G. Schubring (Eds.), *Handbook of the history of mathematics education* (pp. 431-457). New York: Springer.

Brueckler, F. M., Stilinović, V. (2013). Teaching arithmetic in the Habsburg Empire at the end of the 18th century–A textbook example. *Historia Mathematica*, *40*(3), 309-323.

[23] Lamandé 2011, Schiopetti 2015; on Laisant see Auvinet 2013.

[24] See Dhont et al 2015 for Rudolf Steiner.

Bullynck, M. (2008). The transmission of numeracy: integrating reckoning in Protestant North-German elementary education (1770-1810). *Pedagogica Historica 44*(5), 563-585.

Calabrese, F. (2019). *Horace Grant (1801-1859) e l'istruzione matematica nella prima infanzia, nel contesto delle iniziative della Society for the Diffusion of Useful Knowledge (1826 - 1848)* [Horace Grant (1801-1859) and early mathematics education, in the context of the initiatives of the Society for the Diffusion of Useful Knowledge (1826-1848)] (Master of Education Dissertation, Roma Tre University).

Chadwick, E. (1861). A biographical note of the author. In Grant, H., *Second stage of arithmetic. Arithmetic for schools* (New edition). Bell and Daldy, pp. v-xxxv.

Cipolla, C. (1969) *Literacy and development in the West*. London: Penguin.

Comenius, J. A. (1896) *The Great Didactics of John Amos Comenius*, translation by M. W. Keatinge. London: Adanmand Charles Black.

Condorcet, J.-A.-N. de Caritat (1988/1798). *Moyens d'apprendre à compter sûrement et avec facilité*. Paris: Art, Culture, Lecture Éditions.

Dauben, J. (2003). Internazionalizing mathematics east and west. In K. H. Parshall, &A. C. Rice, *Mathematics unbound: The evolution of an international mathematical research community, 1800-1945* (pp. 253-280). Providence: American Mathematical Society.

Decayeux-Cuvillier, M. (2019). *Des mathématiques pour les filles? L'exemple de l'enseignement primaire dans la Somme de 1881 à 1923*. Villeneuve d'Ascq: Presses Universitaires du Septentrion.

D'Enfert, R. (2003). Inventer une géométrie pour l'école primaire au XIXè siècle. *Tréma*, 22, pp. 41-49.

D'Enfert, R. (2014). Jomard, Francœur et les autres… Des polytechniciens engagés dans le développement de l'instruction élémentaire (1815-1850). *Bulletin de la Sabix*, 54, 81-94.

Denniss, J. (2009). Learning arithmetic: textbooks and their users in England 1500-1900. In E. Robson, & J. Stedall (eds.), *The Oxford Handbook of the History of Mathematics* (pp. 448-467). Oxford: Oxford University Press.

Denniss, J. (2012). *Figuring it out. Children's arithmetical manuscripts 1680-1880*. Oxford: Huxley Scientific Press.

Dhondt, P., Van de Vijver, N., & Verstraete, P. (2015). The possibility of an unbiased history of Steiner/Waldorf education? *Contemporary European History*, *24*, 639-649.

Di Clemente, C (2018). *Analisi storico-didattica di "School Arithmetic" (1900) di*

*John M. Colaw e John K. Ellwood nel contesto dell'evoluzione delle idee sull'istruzione matematica infantile in età contemporanea* [Historical-didactic analysis of "School Arithmetic" (1900) by John M. Colaw and John K. Ellwood in the context of the evolution of ideas on children's mathematics education in the contemporary age] (Master of Education Dissertation, Roma Tre University).

Di Marco, M. (2022) *La matematica nella pedagogia steineriana, dagli scritti di Rudolf Steiner (1861-1925) alla lettura attuale di Ernst Schuberth* [Mathematics in Steiner pedagogy, from the writings of Rudolf Steiner (1861-1925) to the current interpretation by Ernst Schuberth] (Master of Education Dissertation, Roma Tre University).

Di Tella, E. (2020). *Indagine sulle "reglettes" di Georges Cuisenaire (1891-1976)* [Investigation on Georges Cuisenaire's (1891-1976) rods] (Master of Education Dissertation, Roma Tre University).

Edgeworth, M., & Edgeworth, R. L. (1798). *Practical Education*. London: J. Johnson.

Fantaccini, L.,& Leproni, R. (Eds.) (2019). S*till blundering into sense. Maria Edgeworth, her context, her legacy*. Firenze: Firenze University Press.

Fisher, D. W. (1911). *Calvin Wilson Mateer, Forty-five years a missionary in Shantung China: A biography*. Philadelphia: Westminster Press.

Fossa, V. (2022). *Giocattoli, matematica e scienza: l'istruzione matematica infantile nell'opera* Practical Education *(1798) di Richard Lovell Edgeworth (1744-1817) e Maria Edgeworth (1767-1849)* [Toys, mathematics, and science: mathematics education in childhood in *Practical Education* (1798) di Richard Lovell Edgeworth (1744-1817) and Maria Edgeworth (1767-1849)] (Master of Education Dissertation, Roma Tre University).

Franci, R. (2000). L'insegnamento dell'aritmetica nel Medioevo. In P. Freguglia, L. Pellegrini, & R. Paciocco (Eds.) *Scienze Matematiche e insegnamento in epoca medioevale. Atti del Convegno internazionale di studi Chieti 2-4 maggio 1996*. Napoli-Roma: Edizioni Scientifiche Italiane, pp. 111-132.

Friedman, M., & Muñoz Alvis, J., (2021). Haüy, Weiß, Fröbel: the influence of nineteenth-century crystallography on the mathematics of Friedrich Fröbel's Kindergarten. (in two parts), *Paedagogica Historica* on line

Frijhoff, W. (2012). Historian's discovery of childhood. *Paedagogica Historica*, *48*(1), 11-29.

Garin, E. (1976). *L'educazione in Europa 1400-1600. Problemi e programmi*. Roma-Bari: Laterza, 3rd edition (1st ed. 1957, 2nd ed. 1966).

Gil Clemente, E., & Millán Gasca A. (2016). Integrating history of mathematics with foundational contents in the education of prospective elementary teachers. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.) *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics* (pp. 427-440). IREM de Montpellier.

Gil Clemente, E., & Millán Gasca A. (2021). Geometry as “forceps of intelligence”: lines, figures and the plane in Édouard Séguin's pedagogical thought. *Bollettino di storia delle scienze matematiche*, *41*(2), 315-339.

Horlacher, R. (2011). Schooling as a means of popular education: Pestalozzi's method as a popular education experiment. *Pedagogica Historica*, 47 (1-2), 65-75.

Jackson, L. L. (1906). *The educational significance of sixteenth century arithmetic from the point of view of the present time*. New York (printed for the author)

Karp, A. (2014). The history of mathematics education: Developing a research methodology. In A. Karp &, G. Schubring (Eds.), *Handbook of the history of mathematics education* (pp. 9-24). New York: Springer.

Lamandé, Pierre (2011). Une personnalité du monde de l’Éducation nouvelle: Charles Ange Laisant (1841-1920) et son combat politique pour une éducation rationnelle fondée sur la science. *Pedagogica Historica*, 47 (3), 283-301.

Ma, L. (2010). *Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States*. New York: Routledge (original edition 1999)

Magrone, P., & Millán Gasca, A. (2023). Children shall know what lies at the heart of genuine Mathematical science: The *Lectures on the logic of arithmetic* (1903) by Mary Everest Boole (1832-1916), to appear on Bollettino di storia delle scienze matematiche.

Massenzi, S. (2016). *Il contributo di Mary Everest Boole (1832-1916) all’istruzione matematica infantile* [Mary Everest Boole's (1832-1916) contribution to mathematical education in childhood] (Master of Education Dissertation, Roma Tre University).

Millán Gasca, A. (2015). Mathematics and children's minds: The role of geometry in the European tradition from Pestalozzi to Laisant. *Archives Internationales d'Histoire des Sciences, 65*(2)-175, 261-277

Millán Gasca, A. (2017). Zoel García de Galdeano y las matemáticas para niños hacia 1900, *Gaceta de la Real Sociedad Matemática Española*, 20 (3), 611-634.

Millán Gasca, A. (2021). Educational writing by women in the early 20th century: mathematics for children in Mary Everest Boole, Maria Montessori, Grace Chisholm Young and Margalida Comas Camps. In R. Leproni (Ed.) *Women on*

*women. De-gendering perspectives* (pp. 141-153). Franco Angeli.
Moyon, M. (2016). Initiation au calcul et éducation nouvelle: la "méthode Havránek" au catalogue du Père Castor. *HISTEMAT. Revista de História da Educação Matemática*, *2* (3).
Panichelli, V. (2021). *L'innovazione didattica a scuola come questione culturale: il caso di Jacques Camescasse e della scatola-gioco* Initiateur Mathématique (1910) [Educational innovation at school as a cultural issue: the case of Jacques Camescasse and the toy box *Initiateur Mathématique* (1910)] (Master of Education Dissertation, Roma Tre University).
Pasquali, C. (2020). *Augustus De Morgan (1806-1871) nella storia della istruzione matematica* (Master in Mathematics Dissertation, Roma Tre University).
Peltzman, B. R. (1998). *Pioneers of early childhood education. A biobibliographical guide*, Westport, Greenwood Press, 1998.
Roggero, M. (1994). Conti sulle dita, calcoli a penna. L'aritmetica elementare a fine Settecento. *Studi Storici*, 35(4), 1039–1060.
Sanford, V. (1930). *A short history of mathematics*. New York: Houghton Mifflin Company.
Schiopetti, V. (2015). *Charles-Age Laisant nella storia dell'istruzione matematica infantile* [Charles-Age Laisant in the history of mathematical education in childhood] (Master of Education Dissertation, Roma Tre University).
Schubring, G. (2014). On historiography of teaching and learning mathematics. In A. Karp &, G. Schubring (Eds.), *Handbook of the history of mathematics education* (pp. 3-8). New York: Springer.
Spranger, E. (1939). *Aus Friedrich Fröbels edankenwelt.* Berlin: de Gruyter, Berlin.
Stedall, J. (2012). *The history of mathematics. A very short introduction*. Oxford University Press.
Stoker, D. (2007). Ellenor Fenn as "Mrs. Teachwell" and "Mrs. Lovechild": A Pioneer Late Eighteenth-Century Children's Writer, Educator, and Philanthropist. *The Princeton University Library Chronicle*, 68 (3), pp. 817- 850.
Stramaccioni S. (2021). *L'istruzione matematica infantile nell'opera di Emilia Santamaria Formìggini (1877-1971)* [Children's mathematical education in Emilia Santamaria Formìggini's (1877-1971) work] (Master of Education Dissertation, Roma Tre University).
Swetz, F. (1992). Fifteenth and sixteenth century arithemtic texts. What can we learn from them?, *Science & Education*, *1*, pp. 365-378.
Tomassoni, L. (2019) *Little Mathematicians: approccio narrativo e approccio con-*

*cettuale nelle* Lectures on the Logic of Arithmetic *(1903) di Mary Everest Boole* [Little Mathematicians: narrative and conceptual approach in Mary Everest Boole's *Lectures on the Logic of Arithmetic* (1903)] (Master of Education Dissertation, Roma Tre University).

Ulivi, E. (2002). "Scuole e maestri d'abaco in Italia tra Medioevo e Rinascimento", in Giusti (ed.), *Un ponte sul Mediterraneo. Leonardo Pisano, la scienza araba e la rinascita della matematica in Occidente*, Firenze, Edizioni Polistampa, pp. 121-159.

Ulivi, E. (2014). Masters, questions, and challenges in the abacus schools. *Archive for the History of Exact Sciences*, *69* (6), pp. 651-670.

Zannoni, I. (2017). *La geometria naturale di Jules Dalsème: sforzo divulgativo e contributo all'istruzione matematica primaria nella Francia della Terza Repubblica* (Master of Education Dissertation, Roma Tre University).

Yeldham, F. A. (1936).*The teaching of arithmetic through 400 years 1535-1935*. London/Bombay/Sidney: George G. Harrap &Co.

# THE HISTORY OF MATHEMATICS IN ITALY THROUGH THE AGES:

## Sources, correspondences and editions

**Maria Teresa BORGATO**
Department of Mathematics and Computer Science, University of Ferrara, Italy
bor@unife.it

## ABSTRACT

The study of primary sources is the starting point for the research in the history of mathematics, and increasing those resources is one of the most significant and enduring contributions that scholars can make to the discipline. In past decades, Italian historians of mathematics have contributed to it with critical editions of works, and the publication of numerous inedited documents, manuscripts and correspondences. The enhancement of primary sources for the history of mathematics in Italy was a determining element for the foundation of our scientific journal *Bollettino di Storia delle Scienze Matematiche*. In the teaching of mathematics, quoting meaningful passages from these sources can be a useful strategy to introduce arguments and concepts, or to raise awareness of historical development. In my talk, I will present a certain number of examples and suggestions, which go from the late Middle Ages and the Renaissance, to the first half of the twentieth century. Starting from original and recent research, it is possible to propose less usual themes connected to different mathematics topics and directed to different age groups, including: abacus mathematics and Archimedean tradition, the resolution of algebraic equations of third and fourth degrees, the pre-Newtonian systems of the world, the foundation of infinitesimal calculus, the science of waters and the origin of hydrostatic and hydrodynamic laws, new approaches in elementary geometry (fusionism, paper folding).

## *1 Introduction*

I should like to thank the organizers of this Summer School for their invitation to hold this lecture. This is a particularly important occasion for those like me, who have made the history of mathematics their main field of research and have always sustained the importance of the history of mathematics in the teaching of mathematics. I feel that my presence here today is also due to my position as President of the Italian Society of the History of Mathematics and consequently it is my duty to provide as wide a picture as possible of the research works that have so far been carried out.

As the community of historians of mathematics has been quite active since the last quarter of the past century (1980s) it would not be possible here today to give an adequate account of its output, so I shall limit myself to the contents of research on primary sources. The enhancement of primary sources is one of the main objectives of the *Bollettino di Storia delle Scienze Matematiche*, which includes many of these contributions. The use of historical sources in the introduction of topics in the history of mathematics in the classroom is, moreover, a persistent theme in the international literature in the teaching of mathematics, which runs through other interventions in this conference. As the title of section 6 indicates ("History of Mathematics in Italy"), I will focus on manuscripts and works of Italian mathematicians, investigated by Italian historians of mathematics, which may provide students with accessible material and which are therefore of use in the context of teaching. As a result, I will not be covering research works, which do not concern the history of mathematics in Italy, or mathematical works of advanced content, or historical essays of synthesis or interpretation.

## 2 Long tradition of historical mathematical studies in Italy

First of all, let me remind you that research in the history of mathematics in Italy is based on a secular cultural heritage, which preceded the unity of the nation itself. The historiography of mathematics in Italy has a long tradition dating back to the Renaissance (Bernardino Baldi, 201 biographies in his *Vite de' matematici*). Moreover, Italian scholars contributed to the transmission and history of science with translations, commentaries and editions of ancient classics, and chronologies were included in encyclopedic treatises in the seventeenth century (Giuseppe Biancani, Giovanni Battista Riccioli, and Claude François Milliet Deschales). In a modern sense, however, the historiography of mathematics begins in the eighteenth century, when the history of mathematics was considered an area of the history of human thinking. In Italy it was initially developed as a part of Italian literature and inserted into general works (Giovanni Andres, Girolamo Tiraboschi).

Critical analysis of mathematical theories and their historical foundations can be found in works of mathematicians like Joseph-Louis Lagrange, Gregorio Fontana, Pietro Cossali, and Giambattista Guglielmini. An important work, which combined general overview, technical and archival investigation came out later: the famous *Histoire des sciences mathématiques en Italie* by

Guglielmo Libri (4 volumes 1838-41, see Del Centina & Fiocca, 2010). For this first period you can see (Borgato, 1992) and the articles collected in the volume of the conference proceedings (Barbieri & Cattelani, 1989).

The historiography of mathematics developed in the second half of the nineteenth century, after the political unification of Italy, with Baldassarre Boncompagni's foundation of the *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche* (20 volumes), which supplied both an international diffusion and primary sources. Antonio Favaro was the editor of Galileo's collected works, and Pietro Riccardi published a bibliographic work still of great importance, the *Biblioteca matematica italiana*. In the first half of the twentieth century, two mathematicians became the main historians of mathematics: Gino Loria and Ettore Bortolotti.

The Second World War constituted a sort of breakdown also for this type of studies so Italian historians of mathematics had to re-launch their activity in the light of a new relation with the international environment. In the last quarter of the century, the community of Italian historians of mathematics developed so that Italy became known as one of the main centres for the historiography of mathematics. In particular, their research was supported by the publication, from 1981 on, of the *Bollettino di storia delle scienze matematiche* (40 volumes so far) and, in 2000, the foundation of the national society: Società Italiana di Storia delle Matematiche (SISM web site: http://www.sism.unito.it/), which, among the objectives of its statute, has the training of teachers and the promotion of mathematical culture in the country. Besides numerous contributions in journals and volumes (see Barbieri & Pepe, 1992), in recent times Italian historians of mathematics have collaborated with historians and philosophers of science to important editorial projects:

- Archivio della Corrispondenza degli Scienziati Italiani: https://www.olschki.it/catalogo/collana/acsi/p2
- Edizione Nazionale Mathematica Italiana: http://mathematica.sns.it/
- Edizione Nazionale dell'Opera Matematica di Francesco Maurolico: www.maurolico.it/Maurolico/index.htm
- Edizione Nazionale Ruggero Giuseppe Boscovich: http://www.brera.inaf.it/~boscovich/pagine/

### *3 Importance of primary sources for the history and teaching of mathematics*

The study of primary sources is the starting point for the research in the history of mathematics, and increasing those resources is one of the most significant and enduring contributions that scholars can make to the discipline. In past decades, Italian historians of mathematics have contributed to it with critical editions of works, and the publication of numerous inedited documents, manuscripts and correspondences. Correspondences, in particular, allow us to investigate the circulation of scientific thought and the origin of mathematical ideas. Primary sources are also a useful tool for introducing topics of history of mathematics into mathematics teaching, with some precautions, however.

The benefits of a historical approach in the teaching and learning of mathematics are widely recognized, as:

- The lack of interest in mathematics can be overcome through the emotional involvement of the story
- In conjunction with epistemological obstacles, it promotes learning and self-esteem
- By placing the mathematical discipline within the more general history of culture promotes holistic learning
- Through the awareness of intrinsic difficulties in the discipline, it gives reasons for formal definitions and choices that may appear obscure and excessively artificial or abstract
- With the awareness that the development of mathematics is not the merit of a single country and a single civilization, it favors integration
- It places mathematics alongside the other sciences, in which theories, hypotheses, languages, and representations, are linked to the time and place (to the society) that produced them.

The risks in the historical approach in the teaching and learning of mathematics are sometimes underestimated by teachers, they include:

- Failure to adapt to the level of pupils in proposing the themes
- Trivialization in an attempt to simplify a complex topic
- Introduction of misconceptions
- Superficiality due to little knowledge of the history of mathematics as a discipline.

In conclusion, not all mathematics topics lend themselves to a historical approach, which is not limited to narrative, and biographical or bibliographic references, and requires attention on the part of teachers in choosing the topics that must be commensurate with the students' age and level of understanding, and also the sources to refer to, which must be scientifically valid.

The reference to original works emphasizes advantages and difficulties, since it requires the interpretation of a language that is not immediately translatable and must be contextualized. On the other hand, it emotionally involves the student even more, in that it allows the most direct contact with the author and the social and cultural environment of the time. Finally, it enriches the mathematical topic with other elements of historical investigation and lends itself to an interdisciplinary development.

### *4 An overview of the workshop project*

To provide a broad picture of topics to be developed in the classroom, connected to recent research on primary sources, conducted by Italian historians of mathematics, I proposed a workshop divided into two sessions (of two hours each) connected to my lecture.

However, as I had to make a selection for this, I thought of a temporal distribution of the themes aimed at covering the fourteenth to seventeenth centuries in workshop n.1, and in workshop n. 2, the eighteenth to the twentieth centuries. The first session is entitled:

*Starting from the history of mathematics in Early Modern Italy: From primary sources to mathematical concepts*

It includes the following topics:

- *Abacus mathematics and Archimedean tradition*
- *The resolution of algebraic equations of third and fourth degrees*
- *The law of free fall*
- *The pre-Newtonian systems of the World.*

The organization of this workshop is shared with Alessandra Fiocca (University of Ferrara) and Veronica Gavagna (University of Florence) and makes use of the collaboration of Elena Lazzari (University of Ferrara).

The second session is entitled:

*Starting from the history of mathematics in Late Modern Italy: From primary sources to mathematical concepts*

And deals with:

- *The dissemination of infinitesimal calculus in Italy*
- *The science of waters as the main field of applied mathematics*
- *Re-launching Italian education and research after political unification*
- *The geometry of paper folding and the resolution of problems of third degree*

The organization of workshop 2 is shared with Maria Giulia Lugaresi (University of Ferrara) and Paola Magrone (University of Rome) with the collaboration of Elena Lazzari (University of Ferrara) and Elena Scalambro (University of Turin).

### 4.1 Abacus mathematics and Archimedean tradition

Abacus mathematics developed in Tuscany and then spread over a period of time that traditionally goes from 1202 (*Liber abbaci* by Leonardo Fibonacci) to 1494 (*Summa* by Luca Pacioli.)

The meticulous research of manuscripts and documents carried out in Tuscan archives for forty years by Raffaella Franci, Elisabetta Ulivi and Laura Toti-Rigatelli has allowed both the publication of various texts used in these schools and the extensive reconstruction of this network of schools, devoted to the formation of professional figures in the fields of commerce, crafts and finance. These schools contributed to the growth of middle-class professions, which were to become the foundation of the new Renaissance society.

**Figure 1.** Gentile da Fabriano, *Arithmetic*, fresco. Foligno, Palazzo Trinci 1411-1412

The *Liber Abbaci* by Leonardo Pisano, which marks the start of the dissemination of abacus mathematics in the West, is a very rich and complex text, the first critical edition of which was recently published by Enrico Giusti in collaboration with Paolo D'Alessandro (2020). From the *Liber Abbaci* by Leonardo Pisano various rules may be extracted in order to carry out operations with numbers, as well as problems of mercantile mathematics (purchases and sales, barters, corporations, and coins), or linked to the Fibonacci series. From the *Pratica Geometrie* of the same author, we find other problems of practical geometry. Many teaching sequences and materials were designed on these themes, the first by *Il Giardino di Archimede*, mostly experimented in Italy. Franco Ghione and Laura Catastini have recently set up a website devoted to the *Liber Abba*ci with a translation in Italian and numerous teaching proposals: https://www.progettofibonacci.it/

In parallel, a "cultured" mathematics was developing in medieval universities and courts of enlightened princes, which originated from the rediscovery and retrieval, and at times from the reconstruction of works from classical antiquity. Later on, the invention of printing favored the process of enlargement of the scientific community. Among the leading protagonists of this retrieval is the mathematician Francesco Maurolico (works of Archimedes, Theodosius, Menelaus, Euclid and Apollonius). Maurolico's legacy included printed works and manuscripts: after several years of partial editions and essays, an editorial enterprise under the direction of Pier Daniele Napolitani with the collaboration of Veronica Gavagna, was undertaken to produce the National Edition of the Mathematical works of Francesco Maurolico: *Edizione Nazionale dell'Opera Matematica di Francesco Maurolico*.

In spite of the philological competence that the reading of the documents requires, some topics suitable for use in secondary schools can be drawn from themes of medieval mathematics, as well as from passages of the ancient classical works rediscovered. Veronica Gavagna develops this topic, showing, with direct examples from the texts, the influence of the Archimedean tradition in the works of Piero della Francesca and Luca Pacioli.

### 4.2 The resolution of algebraic equations of third and fourth degrees

Bringing together different traditions, of commercial mathematics and the mathematical culture of universities and academies, the sixteenth century saw the conquest of the Italian algebraists Del Ferro, Tartaglia, Ferrari, Cardano; i.e. the general resolution of third and fourth algebraic equations. The contrasting events of this discovery and the mathematical challenges that have accompanied it lend themselves to an engaging narrative for students, and have been the subject of presentations, including theatrical ones. See in particular the performance "La formula segreta" (The secret formula) staged by Daniele Squassina and Maurizio Lovisetti, which, in addition to the pleasant show, presents a careful reconstruction of some details of the life of Tartaglia and figures connected to him. The theme also lends itself to mathematical insights, even if not trivial. Tartaglia's famous poem containing the resolving rule can be an example of a text to be interpreted and the methods of resolution can be re-proposed.

*A mathematical formula in poetry*

The resolution formula of the cubic equations of the form $x^3 + bx + c = 0$ is expressed in the first nine verses of the poem that Tartaglia communicated to Cardano on the night of March 25, 1538, shown below with the paraphrase alongside the modern algebraic symbolism:

| | |
|---|---|
| *Quando chel cubo con le cose appresso* | When $x^3 + bx$ |
| *Se agguaglia à qualche numero discreto* | $= c$ |
| *Trovan dui altri differenti in esso.* | find $u$ and $v$ such that $u - v = c$ |
| *Che'l lor produtto sempre sia eguale* | and $u \times v =$ |
| *Al terzo cubo delle cose neto,* | $(b/3)^3$ |
| *El residuo poi suo generale* | Then follows |
| *Delli lor lati cubi ben sottratti* | $\sqrt[3]{u} - \sqrt[3]{v}$ |
| *Varrà la tua cosa principale.* | $= x.$ |

In his 1545 treatise *Ars Magna*, Cardano showed that the cubic equations also containing the second degree term (of the general form $ax^3 + bx^2 + cx + d = 0$) could be transformed with suitable artifices into equations of the reduced form without this term : $x^3 + bx + c = 0$.

We can then go on with the *imaginary numbers*, introducing the work by Rafael Bombelli who in *Algebra* (1572, 1579) examines the solutions of the various cases of the third degree equations, including the so-called *irreducible case*, which in Cardano's formula presents the square root of a negative number. The imaginary roots, called "sylvan quantities" and complex numbers (referred as to "plus of minus" and "minus of minus" for + *i* and -*i*) are then examined, establishing their calculation rules (addition and multiplication). Later Descartes was to coin the term *imaginary.*

Bombelli's *Algebra* represents a more mature result of sixteenth century algebra, remaining for over a century the most authoritative text on advanced algebra. Unlike several of his contemporary mathematical writers, in his printed editions and manuscripts Bombelli used a sophisticated form of mathematical notation, introducing, in particular, the exponents to indicate the powers of the unknown.[25]

To use this work as a primary source in classroom, we can cite some passages from the first book of Bombelli's *Algebra* in which syncopated algebraic language is used (so p. stands for "plus", m. for "minus", R. q. means "square root" etc. for example see p. 95 of the first edition). We can read the rules to operate with imaginary numbers (addition, subtraction, multiplication) for example on p. 169, 189, 192, of Book I. The irreducible case can be studied using resolving algebraic formulae and the graph of the corresponding polynomial.

Alessandra Fiocca, who has published some inedited works by Ludovico Ferrari and a new edition of the third book of Bombelli's *Algebra* (Fiocca & Leone, 2017), deals with this part of workshop n. 1.

### 4.3 Galileo's law of free fall

The seventeenth century is Galileo's century, and we can certainly take cues from his works and the mathematicians of his school, such as Torricelli, Viviani, Cavalieri and Borelli, introducing for example, Torricelli's hyperbolic solid as a famous example of an unlimited solid with finite volume and an application of Cavalieri's principle. We prefer to focus on the law of free fall. The law of uniformly accelerated motion, the space proportional to the square of time (*the time-squared law*) as well as the *law of odd numbers*, which follows from the former, are to be found in a letter from Galileo to Paolo Sarpi dated 16th October 1604:

[25] Bombelli's *Algebra*, like many other reproductions of fundamental and rare printed works of Italian mathematicians can be downloaded from the site of the project carried out by the Scuola Normale Superiore of Pisa: *Mathematica Italiana*: http://mathematica.sns.it/opere/9/, where you can also find a description of the main features of the work. The project was carried out by a scientific board that included Luigi Pepe, Mariano Giaquinta, and Paolo Freguglia.

*"gli spazi passati dal moto naturale esser in proportione doppia de' i tempi et per conseguenza gli spazi passati in tempi uguali esser come i numeri impari ab unitate".*

Galileo later developed a mathematical theory of accelerated motion, which he derived from a fundamental principle, firstly identified in the proportionality of speed to space, and only later in the proportionality of speed to time (Giusti, 1990).

We are used to representing the Galilean law of fall as a single result, summarized in a formula that describes a vertical motion, in which the traveled space grows proportionally to the square of time, and the constant of proportionality is the same for all bodies, corresponding to half the acceleration of gravity. Each of these assumptions, however, has registered different opinions and positions and has been the subject of debate and experimentation (Borgato, 2014).

In fact, the questions raised by Galileo's law in relation to scholastic philosophy are many and of diverse nature which are to be considered separately:

- non uniformity (*difformity*) of the fall and the distinction between levity and gravity
- the law of uniformly accelerated motion ('uniformiter diformiter') of heavy bodies, that is the Times Square Law, and the equivalent Odd-Number Law
- the constant of proportionality, that is the ratio between the distance traversed and the square of time–intervals
- velocity independent of weight and the simultaneous fall (in a vacuum or in the air)
- the trajectory of a freely falling body in 'absolute' space

An in-depth study of these issues will require the contribution of the philosophy teacher. However, a useful exercise in classroom, starting from Galileo's direct quotation, could also be to compare the two forms of Galileo's law, that is the *Times Square Law* and *Odd-Number Law* and try to demonstrate their equivalence, even without using the infinitesimal calculus:

$$s = \frac{1}{2}gt^2 \qquad\qquad s(t+1) - s(t) = \frac{1}{2}g(2t+1)$$

$\Rightarrow$ Trivial

$\Leftarrow$

$$s(0) = 0,\ s(1) = \frac{1}{2}g\,,$$

$$s(2) = \frac{1}{2}g(3+1) = \frac{1}{2}4\,g\,,$$

$$s(3) = \frac{1}{2}g(5+4) = \frac{1}{2}9\,g\,, \ldots$$

$$s(t) = \frac{1}{2}g(2t-1) + s(t-1) = \frac{1}{2}g(1+3+5+\cdots+2t-1) = \frac{1}{2}gt^2$$

## 4.4 The pre-Newtonian systems of the World

I would like to point out another source, which refers to the school in opposition to the Galilean school, that Jesuit school which also has merits in the physics-mathematical research of the seventeenth century, especially in those disciplines that did not interfere with the dictates of Catholic orthodoxy. This theme brings us to celestial mechanics and cosmological systems before the definitive affirmation of the Copernican system according to Kepler's hypothesis. Giovanni Battista Riccioli was a famous astronomer of his day, who is currently living a sort of renewed celebrity thanks to the many volumes and articles devoted to him.[26] One of his merits was the first direct experimental proof of Galileo's law of free fall, carried out by Riccioli with the help of many members of his brotherhood in the 1650s, as they performed various launches of heavy spheres from churches and buildings in Bologna and in particular from the Asinelli Tower.

But now let's turn to the proposal of a teaching activity. In his major work, the *Almagestum Novum*, two great folio volumes, Riccioli compares the various astronomical systems of the world proposed from antiquity to his day and discusses all the evidence in favor or against the motions of the Earth.

[26] The first of these books, was: (Borgato, 2002).

Riccioli’s aim is represented in the extremely famous and often reproduced frontispiece of the *Almagestum Novum* (1651).

**Figure 2.** In it, two characters confront each other: Astraea representing theoretical astronomy, and Argus Panoptes, the many-eyed giant in Greek mythology, representing observational astronomy. In the center a balance in which the heliocentric Copernican system and the Tychonic system are opposed. The balance hangs in favor of the latter, with the variants introduced by Riccioli: the Sun, Jupiter and Saturn revolve around the Earth while Mercury, Venus and Mars orbit around the Sun. The Ptolemaic system lies on the ground neglected.

The representations of the various cosmological systems in comparison can be the starting point for introducing the students to the various hypotheses that followed one another: the Ptolemaic system, the Copernican system, the Tychonic system, the Ricciolian system and others.

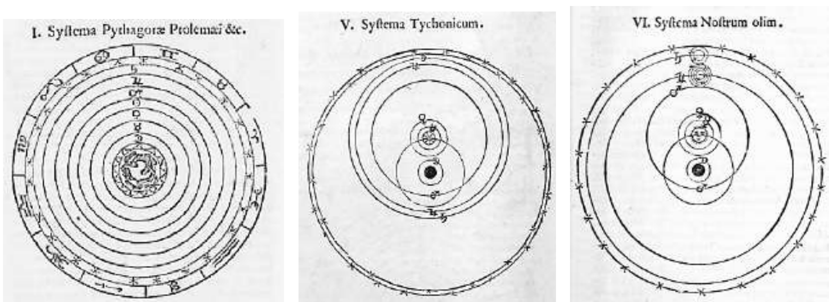



**Figure 3.** The Ptolemaic Tychonic and Ricciolian systems.

In the pre-Newtonian systems, the planetary orbits that explained the apparent motions of the celestial bodies with respect to the Earth, were obtained by composing different circular motions according to a complex system of epicycles, deferents, eccentrics and equants, which can be explained to the students through figures and then reconstructed in the simplest cases using the GeoGebra software (roulettes, epicycloids). Elena Lazzari presents a laboratory on this type of curves.

T the center of the Earth
AOPF the eccentric
LK the epicycle
C the equant
HRV the ecliptic
HV the line of apsides

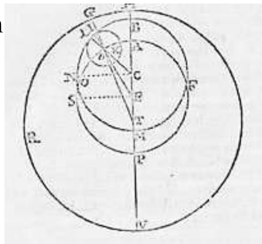


**Figure 4.** The system of eccentric epicycles in Riccioli's Almagestum Novum

The epicycloid is the trajectory of a point on a circumference that rolls outside of another circumference. The rolling circumference could represent the epicycle, while the deferent would be the circular trajectory of the center of the epicycle. If the ratio *k* between the radii of the major and minor circles is rational the curve is closed (the trajectory is *periodic*). Otherwise, it is open (*aperiodic*).

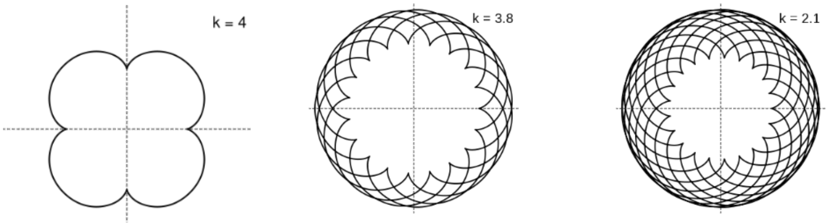



**Figure 5.** Epicycloids corresponding to different values of k


**Figure 6.** The aperiodic trajectory of Mars

In this figure, the complicated and aperiodic trajectory of Mars, drawn by Kepler using Tycho Brahe's data, for the model in which the Earth is still and at the center of the universe.

## 4.5 The dissemination of infinitesimal calculus in Italy

This proposal of classroom activity is related to the first half of the eighteenth century, and inspired by the research carried out by Luigi Pepe and Silvia Clara Roero, on the dissemination of infinitesimal calculus in Italy, as well as a treatise by Lagrange, which remained unknown until the 1980s and edited by myself. As is well known, in the October of 1684, Leibniz published, in the *Acta eruditorum*, his *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus*. This is traditionally considered as the official birth of infinitesimal calculus. In this short memoir with a long title, Leibniz introduced directly the rules of differentiation.

Almost twenty years before the publication of Leibniz's *Nova Methodus*, Newton had already formulated the fundamental elements of his calculus, based on the systematic use of series expansions. In works written in the 1670s but published only later, he presented the problem in terms of finding the relation between "fluxions" (that is, the velocity of variations) of given "fluent" quantities (i.e the variables). The first publication of Newton's results did not take place until 1687, under the title *Philosophiae naturalis Principia mathematica*, so shortly after the appearance of Leibniz's *Nova Methodus*. The bases of the calculus are contained in the first book, where some lemmas introduce the method in the form of "first and last ratios of evanescent quantities", and in the second book with the algorithms of differentiation. In the first edition of the *Principia*, Newton recognized in these the foundation of his own method as well as that of Leibniz, but in the third edition, this reference disappeared. In the meantime, the well-known controversy on the priority of discovery had developed, which involved and divided the mathematicians for decades.[27]

The dissemination of the infinitesimal calculus in Italy is characterized in a first period by an adherence to the Leibnizian method, favored by Leibniz's trip to Italy. This journey, started by Leibniz to reconstruct the origins of the House of Hanover of the Dukes of Brunswick-Lüneburg, for whom he served as librarian (and related to the Este princely family), was documented in his meetings with Italian mathematicians by André Robinet (2007).

The initial spread of infinitesimal calculus in Italy owes much to the many works written by Guido Grandi (1671-1742), one of Leibniz's correspondents, as well as to Gabriele Manfredi, the author of the first work on infinitesimal calculus ever published in Italy (*De constructione aequationum differentialium primi gradus*, Bologna, 1707). Other figures who also greatly contributed to this dissemination were Jacopo Riccati (1676-1754), for his studies on differential equations, and Maria Gaetana Agnesi (1718-1799), author of the first treatise devoted to a wider public of students and scholars: *Instituzioni Analitiche*, published in 1748.[28]

However, when it was necessary to establish "rigorous" theoretical bases, mathematics scholars turned to the Newtonian conceptualization of the "ultimate ratios of vanishing quantities". By the expression *ultima ratio* (*ultimate ratio* in the English translation) in the final scholium of Section I, Book I of *Principia*, Newton attempted to give a meaning to the ratio 0/0 which two variable quantities assume when they become equal to zero. A strong criticism of the foundations of both the Leibnizian infinitesimal calculus and the Newtonian theory of fluxions arose in Great Britain following the publication of a pamphlet by George Berkeley: *The Analyst: or, a Discourse addressed to an Infidel Mathematician...,* London 1734.

In Italy, Joseph-Louis Lagrange composed a treatise for the Artillery Schools of Turin in which he outlines his personal conception of the founda-

[27] More details in (Giusti 2007), and useful teaching materials in the related exhibition: web.math.unifi.it/archimede/archimede_NEW_inglese/mostra_calcolo/pannelli/3.html

[28] The reconstruction and documentation of this phase of the Leibnizian calculus in Italy, through correspondences and unedited documents, in (Pepe 1981) and (Mazzone-Roero 1997).

tions of infinitesimal analysis deriving from the Newtonian theory of fluxions and series: *Principi di Analisi Sublime* (1754~).[29]

An attempt to found the calculus on purely algebraic basis, with no assumption of motion or consideration of compound quantities of infinitely small parts, was developed by John Landen, who also influenced Lagrange's later research on this matter: J. Landen, *A Discourse Concerning the Residual Analysis: A New Branch of the Algebraic Art*, London, Nourse, 1758.

Turning to primary sources of the Italian scientific panorama, to compare the two different approaches, students can start from the treatise by Agnesi (*Instituzioni Analitiche*, 1748), read some passages on the study of curves, and others with the rules of calculus, and draw some curves with GeoGebra (in particular the "Versiera" attributed to her). Then examine some passages from the *Principi di Analisi Sublime* (1754~), by Lagrange in which differential and integral calculus is introduced starting from the calculus of differences and finite sums, according to a definition that goes back to Newton.

The rules of differential calculus are preceded by the algebraic calculus of finite differences:

$$\Delta(x+y) = \Delta x + \Delta y$$

$$\Delta(x \cdot y) = x\Delta y + y\Delta x + \Delta x \cdot \Delta y$$

$$\Delta\left(\frac{x}{y}\right) = \frac{y\Delta x - x\Delta y}{y(y+\Delta y)}$$

$$\Delta x^m = \Delta x[(x+\Delta x)^{m-1} + x(x+\Delta x)^{m-2} + x^2(x+\Delta x)^{m-3} + \cdots + x^{m-1}]$$

$$\Delta\sqrt[m]{x} = \sqrt[m]{x} - \sqrt[m]{x+\Delta x} =$$
$$= \frac{\Delta x}{\left(\sqrt[m]{x+\Delta x}\right)^{m-1} + \sqrt[m]{x}\left(\sqrt[m]{x+\Delta x}\right)^{m-2} + \left(\sqrt[m]{x}\right)^2\left(\sqrt[m]{x+\Delta x}\right)^{m-3} + \cdots + \left(\sqrt[m]{x}\right)^{m-1}}$$

The differential calculus is then introduced in accordance with Newtonian conceptualization as: "the calculus of the ultimate ratios of the differences, i.e. the ultimate terms, to which the general ratios of the differences continually approach, while these continually decrease".

[29] This treatise, which remained in private hands until the 1950s, was published for the first time only some years ago (Borgato 1987).

The resort to infinitesimals, just as in the Leibnitzian tradition, is instead accurately avoided, and is introduced only at the end of the demonstration as a useful tool to simplify the calculus.

The integral calculus, instead of being introduced as the inverse of differentiation (as in Agnesi) is derived from the rules of summation:

$$\int x\Delta x = \frac{x^2 - x\Delta x}{2}$$

$$\int x^2\Delta x = \frac{x^3}{3} - \frac{x^2\Delta x}{2} + \frac{x(\Delta x)^2}{6}$$

Then the integral is presented in geometrical terms clearly inspired by Newton (cfr. *Principia*, Book I, Section I, Lemma II).

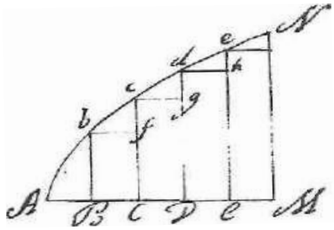



**Figure 7.** The integral presented in geometrical terms

## 4.6 The science of waters as the main field of applied mathematics

In Italy there is a solid tradition of hydraulic studies in all eras, and mathematicians had an extremely important role as theoretical and practical hydraulic experts in territorial planning. Leonardo da Vinci designed some of the most important hydraulic works of the 16th century in Lombardy and France, Bombelli was an expert hydraulic engineer who worked in Bologna and Rome. Galileo, Castelli, Torricelli, Cassini, Guglielmini, Grandi, Manfredi, Poleni, Zendrini all wrote about hydraulics and were used as technicians in the different states into which Italy was divided.

The territory of our peninsula underwent many hydraulic works over the centuries, which greatly modified its structure. Exploitation of the waterways, important not only for irrigation, but also as channels of communication and commerce, caused controversy among the states and involved works, at times damaging, on the course of rivers, which protected the interests of some to the detriment of others.

Some examples of century-old hydraulic problems and interventions include: the Lagoon of Venice and the deviation of rivers that flowed into it, floods of Po River, Adige River and the entire Po Valley hydraulic system, the convergence of the River Reno into the River Po, vast extensions of marshlands (Valli Grandi Veronesi, Paludi Pontine, …), conflict for irrigation (for the use of Tartaro River, between Mantua and Verona…) and so on.

**Figure 8.** Benedetto Castelli. *Della misura delle acque correnti*, In Roma, nella Stamperia Camerale, 1628, frontispiece

Some historians of mathematics such as Luigi Pepe, Alessandra Fiocca and myself have devoted part of their research works to the science of hydraulics in Italy. Maria Giulia Lugaresi has also made in-depth studies on the theoretical aspects of the science of water, or rather the laws which have been formulated for the study of fluid mechanics in ideal conditions, on the one hand, and, on the other, the empirical laws and instruments of measurement which have been devised for practical hydraulic works (Lugaresi 2015). In this field, too, many inedited works taken from the abundance of documentation in the Archives of Venice, Rome, Milan, Paris and Vienna have been published.

**Figure 9.** A manuscript map with details of the Po River and the Panaro River, 1808 (State Archive of Ferrara).

In her workshop, Maria Giulia Lugaresi has designed a teaching sequence starting from a historical perspective of practical problems and direct citations to introduce students to basic hydrodynamic laws.

### 4.7 Re-launching Italian education and research after political unification

The unification of Italy was preceded by the simplification of the political framework operated in the Napoleonic period and by the impulse given to the sciences and to scientific teachings (Patergnani & Pepe, 2011). After political unity (1861), there was a strong resumption of mathematical research and education. On the one hand, a connection with the most advanced sectors of European mathematical research, on the other, a colossal commitment to the creation of a national education system.

As for the teaching of mathematics at secondary school level, there was, as in the rest of Europe, a return to synthetic geometry, without the admixture of algebra. The original Euclidean text was initially revived but later on new texts were produced and a new didactic proposal emerged to blend plane and solid geometry (*fusionism*) and introduce new results in elementary geometry.

The 1867 ministerial programs written by Betti and Brioschi made Euclid's *Elements* compulsory in the teaching of geometry in gymnasiums and lyceums. It was also Betti and Brioschi who edited the famous publication of the *Elements* the following year, which was based on the seventeenth century translation by Vincenzo Viviani with notes by Luigi Cremona, and an appendix containing Archimedean results on cylinders, cones and spheres. Following the attempt to re-propose Euclid's text directly in secondary schools, there appeared new presentations of elementary geometry, always however within a purist approach, edited by some of the leading mathematicians of the end of the nineteenth century. These texts, besides their teaching aims, were guided by a concern to maintain the utmost rigor, since studies on the foundations of geometry had revealed the incompleteness of Euclid's system of axioms (Giacardi, 2006). The phenomenon of *fusionism* arrived in Italy later than other European countries, and dates back to the publication of the treatise by Riccardo De Paolis in 1884: *Elementi di Geometria* (Loescher, Turin). More than a scholastic text, it is a treatise on fusionism and the foundations of geometry. The text by De Paolis was addressed to first grade secondary schools, but without detracting from its rigor and completeness, it was too difficult for the students and too innovative for some teachers. The *Elementi di Geometria* by Giulio Lazzeri and Anselmo Bassani, published firstly in 1891, and then in 1898, was more successful. They wrote a book which was better suited to students as it was the result of direct experimentation, and was used as a textbook in various secondary schools (Borgato 2016). The structure of the book is the same as that of De Paolis, with a few variations. The postulates are divided into twelve groups, with an additional group regarding points, lines and surfaces. It is divided into five books. What is original was the introduction of the theory of radical axes and planes, and the theory of the homothetic figures, free from the theories of equivalence and proportions. This text formed the basis of some experimentations in various schools and also had an influence on other texts destined for use in secondary schools. A heated debate arose within Mathesis, the Italian association of mathematics and physics teachers founded in 1895, on the didactic value or otherwise of fusionism.

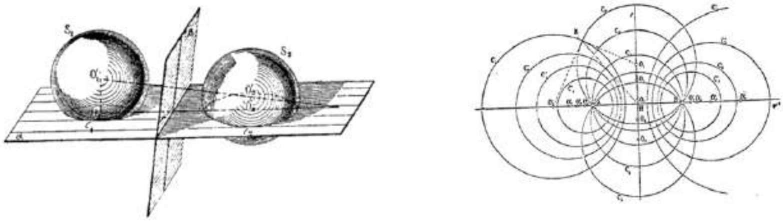


**Figure 10.** Lazzeri & Bassani, Elementi di Geometria, Livorno, 1891, pp. 188, 192

I do not want to go further on this theme, studied by Livia Giacardi and Elisabetta Ulivi as well as by myself. More recently, Marta Menghini has also made some contributions. In the laboratory proposed by Elena Lazzari, the starting point is one of the most modern parts of the treatise by Lazzeri and Bassani, who wanted to include, from a fusionist point of view, new results in elementary geometry resulting from the research of Poncelet, Moebius, Gergonne, Klein… of that century. Elena focuses on the theory of circles and spheres, that is, of the infinite systems of these entities, in which the fusionist method allows us to deduce plane theorems from the three-dimensional theorems in a more evident and direct way.

## 4.8 The geometry of paper folding and the resolution of problems of third degree

The last laboratory proposal concerns elementary geometry of the twentieth century. This research originates from an exhibition I organized on Women and Mathematics in Italy, which highlighted some aspects of Margherita Beloch's teaching activity in Ferrara. At the same time in Rome, other research was being carried out by Paola Magrone on this professor of descriptive and projective geometry, who also linked her name to photogrammetry and roentgen photogrammetry, the forerunner of CT (computed tomography). Beloch also made an interesting contribution to the geometry of paper folding.

This geometry was introduced by the Indian mathematician Tandalam Sundara Row in 1893 and was favourably received in the West particularly following its appraisal by Felix Klein. In this geometry, the use of the ruler and the compass is replaced by the folding of paper, and Sundara Row makes use of five fundamental folds to obtain all the results of Euclidean geometry and some results of the geometry of algebraic and transcendent curves.

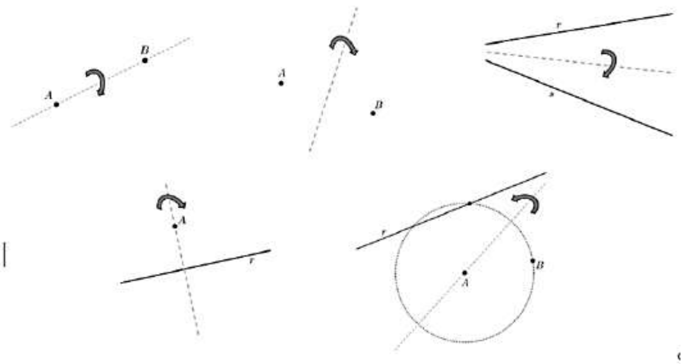


**Figure 11.** Sundara Row's basic folds

Beloch introduced a new folding that made it possible to solve classic problems impossible with ruler and compass, such as the trisection of an angle or the duplication of a cube. It corresponds to the possibility to find, by folding, the common tangent to two parabolas (Magrone &Talamanca 2018; Borgato & Salmi 2018).

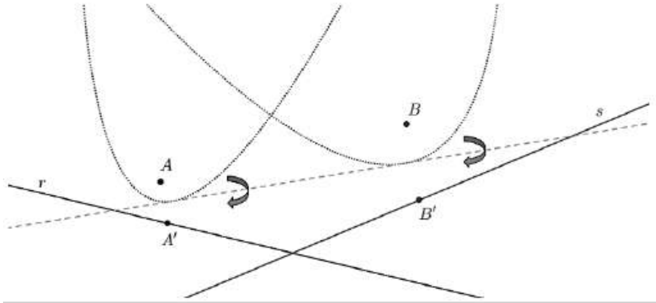



**Figure 12.** Margherita Beloch's new fold

In the laboratory proposed by Paola Magrone we find some applications of what is now called *origami geometry* in the construction of curves, and also the general Beloch method to solve all the geometric problems of third degree.

## 5 *Conclusions*

Even in the limited space of this conference, I have tried to give a fairly broad picture of the resources offered by the history of mathematics in Italy to build educational paths starting from the original sources. I based myself on topics accessible to secondary school students, and mainly on materials of my own direct investigation. I have also tried to include different historical periods and different mathematical disciplines, favoring interdisciplinarity. I hope that my contribution will serve to foster collaboration between teachers, educators and historians of mathematics in relation to the teaching and learning of this discipline, designing new proposals which are inspired by the most current research in an attempt to offer original ideas. This collaboration also guarantees the necessary competence to build quality teaching materials and to avoid superficiality and misconceptions.

**REFERENCES**

Barbieri F., Cattelani F. (1989). *Pietro Riccardi (1828-1898) e la storiografia delle matematiche in Italia. Atti del Convegno*. Modena, Università degli Studi.

Barbieri F., Pepe L. (Eds.) (1992), Bibliografia italiana di storia delle matematiche: 1961-1990. *Bollettino di Storia delle Scienze Matematiche*, 1*2* (1).

Borgato, M.T., Salmi, R. (2018). La Matematica Origami e la risoluzione delle equazioni algebriche, *Periodico di Matematiche*, *10*(3): 57-71.

Borgato, M.T. (2016). Il fusionismo: moda didattica o riflessione sui fondamenti della geometria? *Periodico di Matematiche*, *8*(2): 45-65.

Borgato, M.T. (2014). Gli esperimenti di Giambattista Riccioli sulla caduta libera e il pendolo. *Giornale di Fisica 55*/4: 267-295.

Borgato. M.T. (Ed.) (2002). *Giambattista Riccioli e il merito scientifico dei gesuiti nell'età barocca*. Firenze, Olschki, 2002.

Borgato, M.T. (1992). On the History of Mathematics in Italy before Political Unification. *Archives Internationales d'Histoire des Sciences*, *42*: 121-136.

Borgato M.T. (Ed.) (1987). Lagrange: Principj di Analisi sublime. *Bollettino di Storia delle Scienze Matematiche* 7(2): 45-200.

Del Centina A., Fiocca A. (2010). *Guglielmo Libri matematico e storico della matematica*. Firenze, Olschki.

Fiocca, A., Leone E. (Eds.) (2017). *L' inedito terzo libro de l'Algebra di Rafael Bombelli*. Pisa. Scuola Normale Superiore.

Gavagna, V. (2020). The Edizione Nazionale of Francesco Maurolico's Mathematical Work: history and perspectives. *Physis*, 54: 479-490.

Giacardi, L. (Ed.) (2006). *Da Casati a Gentile. Momenti di storia dell'insegnamento secondario della matematica in Italia*. Lugano, Lumières Internationales.

Giusti, E. (1990). Galilei e le leggi del moto. I: G. Galilei, *Discorsi e dimostrazioni matematiche: intorno a due nuove scienze*. Torino, Einaudi: ix-lviii.

Giusti, E. (2007). *Piccola storia del calcolo infinitesimale dall'antichità al Novecento*. Pisa, Ist. Editoriali e Poligrafici.

Lugaresi, M.G. (2015). Le raccolte italiane sul moto delle acque. *Bollettino di Storia delle Scienze Matematiche* 35(2): 201-304.

Magrone, P., Talamanca V. (2018). Folding cubic roots: Margherita Piazzolla Beloch's contribution to elementary geometric constructions. *Proceedings of 16th Conference on Applied Mathematics*. Bratislava, Spektrum STU: 971-984.

Mazzone, S., Roero, S.C. 1997. *Jacob Hermann and the diffusion of the Leibnizian calculus in Italy*. Firenze, Olschki.

Patergnani E., Pepe L. (2011). *Insegnamenti matematici e istruzione tecnica dalla legislazione del Granducato di Toscana alla legge Casati*. Pisa, Fabrizio Serra editore.

Pepe, L. (1981). Il calcolo infinitesimale in Italia agli inizi del secolo XVIII, *Bollettino di Storia delle Scienze Matematiche*, *1*(2): 43-101.

Robinet, A. (1988). *G.W. Leibniz: Iter italicum (Mars 1689-Mars 1690). La dynamique de la République des Lettres. Nombreux textes inédits*. Firenze, Olschki.

# ETHNO + MATHEMA + TICS: THE LEGACY OF UBIRATAN D'AMBROSIO


**Pedro PALHARES**

CIEC, Institute of Education, University of Minho, Campus de Gualtar, Braga, Portugal

palhares@ie.uminho.pt


## Abstract


Ubiratan D'Ambrosio has been considered as the father of ethnomathematics. Above all, in his conception of the term Ethnomathematics. In his own words: 'the adventure of the human species is identified with the acquisition of modes, styles, arts and techniques (tics) of explaining, learning, knowing and coping with (mathema) the natural, social, cultural and imaginary environment (Ethno)'. Having worked in the fields of Mathematics Education (for which received the Felix Klein medal from ICMI) and the History of Mathematics (he also received the Kenneth O. May medal from ICHM), he considered ethnomathematics as a subfield of both the History of Mathematics and Mathematics Education, enriched by its connections with cultural studies and political domains. In this lecture, I will try to highlight Ubiratan's most important ideas regarding ethnomathematics and discuss some of their implications for both Mathematics Education and the field of History of Mathematics.


## *1 Introduction*

Ubiratan D'Ambrosio has been considered as the father of ethnomathematics: above all, in his conception of the term *Ethnomathematics*. The word, *father*, however creates an image that he had the ideas and then the ethnomathematics movement was created. It was not really that way. Ubiratan's ideas were encapsulating what was already in motion, reshaping it and giving it a new purpose. In the follow-up, I will highlight Ubiratan's most important ideas regarding ethnomathematics and discuss some of their implications for both Mathematics Education and the field of History of Mathematics.

## *2 His career*

I will start with just a few important themes from his lifelong career.

### 2.1 His PhD

First, I think it is important to mention his PhD:

In 1963, D'Ambrosio finished his doctorate in pure mathematics and defended his thesis entitled Generalized Surfaces and Finite Perimeter Sets, from EESC, at USP. In January 1964, he was invited to develop his postdoctoral work, from 1964 to 1965, as a Researcher Associate in the Department of Mathematics at Brown University, in Providence, Rhode Island, USA. (Rosa & Orey, 2021, p. 438)

Considering the direction his work took, it is a bit surprising that his doctorate and postdoctoral work were both in pure mathematics.

At the same time, this initial degree meant that he could speak of mathematics from solid ground and with the respect of other mathematicians.

## 2.2 His efforts to the establishment of ethnomathematics

Ubiratan has made considerable effort to the establishment of ethnomathematics, some of these efforts are considered of vital importance:

It was in 1977 that the term ethnomathematics was first used, in a lecture Ubi gave at the Annual Meeting of the American Association for the Advancement of Science, in Denver, USA. In 1984, the term ethnomathematics was consolidated in an opening lecture entitled "Sociocultural Bases of Mathematics Education" given by D'Ambrosio at ICME-5, in Adelaide, Australia. This is important, as this is when he officially instituted the Program Ethnomathematics as a field of research. ( Rosa & Orey, 2021, p. 441).

This period between 1977 and 1984 was decisive to the denomination of the surging movement and to its establishment as a field of research. In this period many different names were proposed but it was Ubiratan's proposal that was widely accepted.

And the program Ethnomathematics, established as a field of research, definitely grew up.

Perhaps the next phase in Ubi's development of Ethnomathematics was his creation of the International Study Group on Ethnomathematics (ISGEm). The ISGEm Newsletter, which often had contributions from Ubi, morphed into the Journal of Ethnomathematics and then the peer-reviewed Journal of Mathematics and Culture (Scott, 2012, p. 242).

We have to say that ISGEm was not really created by Ubiratan, but it was his idea, and together with a few others, it was finally created. ISGEm is the international organization that oversees the organization of the International Conferences on Ethnomathematics (ICEM) every four years.

## 2.3 His awards

Ubiratan has received three major international awards:

In 1983, D'Ambrosio was honored with the title of Fellow of the American Association for the Advancement of Science (AAAS) for his imaginative and effective leadership in the evolution of Mathematics Education in Latin America and also for his efforts aimed at the development of international cooperation. In 2001, D'Ambrosio was awarded, by the International Committee of History of Mathematics (ICHM), the Kenneth O. May Award for his important contributions to the History of Mathematics. The award announcement stated that the ICHM awarded this medal to him for his never-ending efforts through writing and lectures that promote ethnomathematics and thereby contributing intensely to the establishment of this research field. In 2005, D'Ambrosio was honored by the International Committee of Mathematics Instruction (ICMI) with the second Felix Klein Medal for the recognition of his contributions to the field of Mathematics Education. In 2016, D'Ambrosio was awarded with the title of Emeritus Member of the Brazilian Society of Mathematics Education (SBEM) for his contributions to the development of mathematics education in Brazil. (Rosa & Orey, 2021, p. 443)

Two things to make us wonder:

1) How come a person who had a doctorate in pure mathematics ended up with awards of his work in the history of mathematics and mathematics education?

2) Why it was only in 2016 that he finally received an award in Brazil?

Both answers have something to do with the military coup that established a dictatorial regime in Brazil. It happened while he was on his post doctorate in the United States which then made him stay there with the family.

## 3. His ideas

In this section I will bring forth his ideas on some important themes. I will do it by taking excerpts from his most relevant publications in English.

## 3.1 What is Mathematics?

As to the main ideas of Ubi, let us first look at what he really thought of Mathematics. Here is a first impression, elaborated on a dialogue with another well-known ethnomathematician, Marcia Ascher (she has studied, with her husband, an anthropologist, the Quipu, from the Incas civilization, elaborating on the mathematics it contains):

When I look at the etymology of "mathematics" I recognize, in *mathema*, or *mathemata*, what is usually recognized as explaining, understanding, a broader sense of coping with the many aspects and challenges that reality presents (Ascher & D'Ambrosio, 1994, p. 37). First there is this identification with the Greek civilization. Mathematics is what was constructed based on the Greeks ways, tied with philosophy and a certain kind of game. The problems that were philosophical impasses to the Greeks are indicators that Greek mathematics was very close to the idea of a game. These intellectual games were related to divine activities, activities which were typical of the gods. I am very much influenced by the scenario given by Herman Hesse in his "Magister Ludi". The most important intellectual activity in Castalia was the glass-beads game, which has much of the same characteristics of the mathematical games the Greeks were playing. (Ascher & D'Ambrosio, 1994, p. 38). And finally, in an enlargement, Mathematics is identified with the region where the roots of Greek mathematics are. So, Mathematics started with coping with the challenges that reality presented, but a reality that belonged to a region around the Mediterranean Sea.

Other regions of the world started differently and could have evolved to a different 'Mathematics':

I understand Mathematics as broad category which is an abstract construct originated in the cultures of the Mediterranean Basin and the Mesopotamic (Ancient Iraq) and Nile valley civilizations. We might say that Academic (School) Mathematics is the Ethnomathematics of that region. This category of knowledge is sometimes referred as the Euclidean style and it is supported by tertium-non-datur, is insufficient and even inadequate as a strategy to deal with facts and phenomena of other different natural and socio-cultural environments. As we learn from eminent historian of mathematics Wu-Wen Tsun, ancient mathematics in China had a different method of thinking and style of presentation of Greek mathematics. (D'Ambrosio, 2020, p. 573)

### 3.2 On the universality of mathematical ideas

All people in every culture have the need for transcending. That is the basis for the intellectual exercises, or games. So all people end up developing some sort of 'Mathematics':

Yes, we can use the word play in this broader sense of transcending, which is pure existence. It leads to the concepts of religion, which are clearly of the same nature as those of the arts, sciences, and everything else. In every culture you find some form of god. Every culture tries to reach above the mere earthly needs of survival. We have found in every culture these kinds of intellectual exercises, these plays and these games, as practices that allow people to approach the considerations that go beyond pure survival. This I see as the need to transcend one's existence. This is always associated with the search for explanations, for understanding and meeting the challenges, which I call the *mathema*. (Ascher & D'Ambrosio, 1994, p. 39)

### 3.3 On the word ethnomathematics

This is one grand achievement by Ubiratan: the name, but not only the name, the construction that led to the name, that gives a different sense than other similar names, such as ethnomusicology. It is appealing and has a construction that enables a clear definition of its range:

Browsing a well-known classical Greek etymological dictionary, I found three interesting words: techné (for ways, arts, and techniques), mathemá (for understanding, explaining, and learning), and ethno (for a group within the same natural and sociocultural environment that has compatible behavior). These roots combined would make for techné of mathemá in an ethno. A little modification gives tics of mathema in different ethnos and a different ordering gives ethno-mathematics. (D'Ambrosio, 2016, p. 8)

Different formulation, basically the same construction:

Throughout history and throughout their existence, individuals and peoples have created and developed instruments for reflection and observation, material and intellectual instruments [which I call tics] to explain, understand, come to know, and learn to know and do [which I call mathema] in response to the needs for survival and transcendence in different natural, social, and cultural environments [which I call ethno. Thus, from this derives the name Ethno-mathematics. (D'Ambrosio, 2016, p. 46)

### 3.4 On ethnomathematics

These next citations are devoted to ethnomathematics, what is, together with some clarification for its connection to culture.

Ethnomathematics is a research program about the history and philosophy of mathematics, with obvious implications for teaching. (D’Ambrosio, 2006, p. 17)

Ubi elaborated on the reason why it is a research program:

Putting all this in the context of ethnomathematics: this is the reason I call ethnomathematics a program in the history and philosophy of mathematics. It's a program with a holistic approach, much broader than current historiography and epistemology which have clearly selected only a few variables for analysis. (Ascher & D’Ambrosio, 1994, p. 40)

And now with a different focus on the connections, again what is ethnomathematics:

Making a bridge between anthropologists and historians of culture and mathematicians is an important step towards recognizing that different modes of thoughts may lead to different forms of mathematics; this is the field which we may call ethnomathematics. (D’Ambrosio, 1985, p. 44)

Different modes of thought may lead to different forms of mathematics: Some ethnomathematicians have taken this further. Two examples follow.

The first is from Paulus Gerdes who has invented a new type of matrices from the study of sand drawings from the north of Angola. And another example, the Tahitian and Maori languages, when locating an object, use two points of origin, the speaker and the listener, and two angles, one at each of the origins. This can be developed into a mathematically valid system. Barton’s global conclusion on this point is that each language contains its own mathematical world (Shirley & Palhares, 2016).

This next citation is very important because of the clarification of the manifestations of culture. I don’t see it however as restrictive, only as opening possibilities:

As we have mentioned above, culture manifests itself through jargons, codes, myths, symbols, utopias, and ways of reasoning and inferring. (D’Ambrosio, 1985, p. 46)

But what is a culture? Here we find a clarification:

Upon recognizing that the individuals of a nation, community, or group share their knowledge, such as language, systems of explanation, myths and spiritual gatherings, customs and culinary habits, and that their behaviors are made compatible with and subordinated to value systems agreed to by the group, we say that these individuals pertain to a culture. In sharing knowledge and making behavior compatible, the characteristics of a culture are synthesized. Thus we speak of the culture of the family, the tribe, the community, the association, the profession, the nation. (D'Ambrosio, 2006, p. 10)

Some of the groups are less obvious than others. Of course, nation, tribe, community are perhaps more obvious. Not quite so for the association or the profession. Even these have been used with an expanded meaning: Terezinha Nunes has studied the arithmetic capabilities of a group of children selling items in the streets of Brazil (It is problematic to argue they had a profession, because being children they were not legally allowed to work - yet they were working in practice) [Shirley & Palhares (2016)]

### 3.5 The Program Ethnomathematics

This citation is also about ethnomathematics. However, I respect Ubi's specification of Program ethnomathematics. It has been defined before that ethnomathematics is a research program. But Ubi thinks it is important to place 'Program' together with 'ethnomathematics' to stress even more the fact that it is a research program.

The great motivator for the research program known as Ethnomathematics is to seek to understand mathematical knowing/doing throughout the history of humanity, in the contexts of different communities, interest groups, communities, peoples and nations. (…). Why do I talk about Ethnomathematics as a research program and, at the same time, often use the term Program Ethnomathematics? The principal reason results from my concern regarding attempts to propose an epistemology, and as such, an explanation for Ethnomathematics. Upon insisting on the name Program Ethnomathematics, I seek to make evident that the intention is not to propose another epistemology, but rather to understand the adventure of the human species in the search for knowledge and the adoption of behaviors. (D'Ambrosio, 2006, pp. 8-9)

### 3.6 On mathematical activities

What do we look for when researching in a culture? What is related to mathematics and what is of interest for the ethnographer but not for the researcher in ethnomathematics?

Ubi was giving some ideas, but at the same time probably didn't want to restrict the possibilities and what results is that different sets were advanced, sometimes in the same article or book.

Let us start with these three citations below, where I underline the activities he mentions:

Of course this concept asks for a broader interpretation of what mathematics is. Now we include as mathematics, apart from the Platonic ciphering and arithmetic, mensuration and relations of planetary orbits, the capabilities of classifying, ordering, inferring and modelling. This is a very broad range of human activities which, throughout history, have been expropriated by the scholarly establishment, formalized and codified and incorporated into what we call academic mathematics. But which remain alive in culturally identified groups and constitute routines in their practices. (D'Ambrosio, 1985, p. 45)
...recent research, mainly carried on by anthropologists, shows evidence of practices which are typically mathematical, such as counting, ordering, sorting, measuring and weighing, done in radically different ways than those which are commonly taught in the school system. (D'Ambrosio, 1985, p. 44)
(…) we have practises such as ciphering and counting, measuring, classifying, ordering, inferring, modelling, and so on, which constitute ethnomathematics. (D'Ambrosio, 1985, p. 46)

The three citations above were all from the same article, and we could even see differences between them. Let us see some more:

Ethnomathematics are these corpora of knowledge derived from quantitative and qualitative practices, such as counting, weighing and measuring, comparing, sorting and classifying. (D'Ambrosio, 1999, p. 35)

Among the different ways of doing and knowing, some privilege comparing, classifying, quantifying, measuring, explaining, generalizing, inferring, and, in some way, evaluating. We are then talking of a knowing/doing mathematics that seeks explanations and ways of dealing with the immediate and remote environment. Obviously, this knowing/doing mathematics is contextualized and responds to natural and social factors. Everyday life is impregnated in the knowledge and practices of a culture. At all times, individuals are comparing, classifying, quantifying, measuring, explaining, generalizing, inferring, and, in some way, evaluating, using material and intelectual instruments that belong to their culture. (D'Ambrosio, 2006, p. 13)

The first article shown above, with the three different lists, was from 1985, the next from 1999, and this last one is from 2006, it seems that Ubi finally came to terms with this list.

It is comparable with the list of universal mathematical activities from Bishop (1988):
counting, measuring, localizing, designing, explaining, and playing.

I have seen Bishop's list more used in research than Ubi's list. I personally prefer it too, due to the inclusion of playing, and locating. And yet no one can ignore Ubi’s list, that contains some important activities that are not in Bishop’s list (generalizing!). Possibly in the future someone can come up with a synthesis of the two that is more satisfying.

### 3.7 On knowledge

How is knowledge produced, considering the list of mathematical activities above?

Basically, artifacts and mentifacts are produced, which become part of the reality of the group:

My reflections on multicultural education have led me to see the generation of knowledge as primordial in this whole process. The truth is, this generation occurs in the present, in the moment of transition between the past and the future. That is, the acquisition and elaboration of knowledge occur in the present, as a result of an entire past, individual and cultural, projected into the future. The future is understood as immediate and, at the same time, very remote. As a result, reality is modified, incorporating new facts into it, i.e. "artifacts" and "mentifacts". This behavior is intrinsic to the human being, and results from the drives for survival and transcendence. (D’Ambrosio, 2006, pp. 37-38)

In the next citation, an explanation of what mentifacts are:

We simply assume reality in a broad sense, natural, material, social and psycho-emotional. Now, we observe that links are possible through the mechanism of information (which includes sensorial and memory, genetic and acquired systems) which produces stimuli in the individual. Through a mechanism of reification these stimuli give rise to strategies (based on codes and models) which allow for action. Action impacts upon reality by introducing facti into this reality, both artifacts and "mentifacts". (We have introduced this neologism to mean all the results of intellectual action which do not materialize, such as ideas, concepts, theories, reflections and thoughts.) (D’Ambrosio, 1985, pp. 45-46)

## 3.8 On Education

Ubiratan has written extensively about education, and has some bold proposals. I selected here three citations from different sources, the first is about his big proposal of a new trivium.

My proposal has been to reorganize school curricula in three strands: Literacy, Matheracy, and Technoracy.
Literacy. Clearly, reading has a new meaning today (...) Nowadays, "reading" includes also the competency on numeracy, interpretation of graphs, tables, and other ways of informing the individual. But, if dealing with numbers is part of modern literacy, where has mathematics gone?

Matheracy is the capability of drawing conclusions from data: inferring, proposing hypotheses, and drawing conclusions. It is a first step towards an intellectual posture, which is almost completely absent in our school systems. Regrettably, even conceding that problem solving, modeling and projects can be seen in some mathematics classrooms, the main importance is given to numeracy, or the manipulation of numbers and operations. Matheracy is closer to the way Mathematics was present both in classical Greece and in indigenous cultures. The concern was not with counting and measuring, but with divination and philosophy. Matheracy, this deeper reflection about man and society, should not be restricted to the elite, as it has been in the past.

Technoracy is critical familiarity with technology. Of course, the operative aspects of it are, in most of the cases, inaccessible to the lay individual. But the basic ideas behind the technological devices, their possibilities and dangers, the morality supporting the use of technology, are essential questions to be raised among children in a very early age. History shows us that ethics and values are intimately related to technological progress. (D'Ambrosio, 1999, p. 36)

One reflexion about multiculturalism so present in schools nowadays:
Multiculturalism is becoming the most notable characteristic of education today. With the great mobility of people and families, intercultural relations will become more intense. Intercultural encounters will generate conflicts that can only be resolved based on ethics that result from the individual knowing him/herself and knowing his/her culture, and respecting the culture of the other. (D'Ambrosio, 2006, p. 32)

Finally, an apology for creativity:

The adoption of a new educational posture, in truth the search for a new paradigm of education that substitutes for the worn-out teaching. Learning, based on an obsolete cause-effect relation, is essential for developing creativity that is uninhibited and leads to new forms of intercultural relations, providing the appropriate space for preserving diversity and eliminating inequality in a new organization of society. (D'Ambrosio, 2006, p. 64)

### 3.9 On Mathematics Education

Mathematics Education is a subset of Education, and deserves some specific attention from Ubiratan. Let us see three reflections about mathematics education, the first about the dichotomy ethnomathematics/mathematics from the utilitarian point of view:

From a utilitarian point of view, which cannot be ignored as a very important goal of school, it is a big mistake to think that ethnomathematics can substitute good academic mathematics, which is essential for an individual to be an active being in the modern world. In modern society, ethnomathematics will have limited utility, but at the same time, much of academic mathematics is absolutely useless in this society, as well. (D'Ambrosio, 2006, p. 31)

Second is about the pursuit of equality:

Mathematics education is deeply affected by priorities of this period of transition to a planetary civilization. The pursuit of equity in the society of the future, where cultural diversity will be the norm, demands an attitude without arrogance and prepotency in education, particularly in mathematics education. (D'Ambrosio, 2006, p. 55)

Third is about contextualizing:

Contextualizing mathematics is essential for everyone (D'Ambrosio, 2006, p.59). Probably this one is the most useful for research intending to apply ethnomathematics findings in schools; the justification is often to help contextualize mathematics that would otherwise be presented abstractly. And let us not forget that some believe that ethnomathematics, the mathematics of identifiable cultural groups (D'Ambrosio 2006), can help in this process of contextualization and, furthermore, the humanization of mathematics (Shirley & Palhares, 2016).

### 3.10 On the history of mathematics

We know that Ubiratan was a historian of mathematics but he was quite critical, as we can understand in this conversation:

Regrettably, the history of mathematics, and history in general, has put so much emphasis on the need of man to survive, as if survival and transcendence were separate states of human behavior. The originality of man among the other species is the association of drives towards survival and towards transcendence; man's behavior reveals both components. (Ascher & D'Ambrosio, 1994, p. 39)

And he has a proposal for the transformation of History of Mathematics:

About History of Mathematics, there is need of a broader historiography. History of Mathematics can hardly be distinguished from the broad history of human behavior in definite regional contexts, recognizing the dynamics of population exchanges. This is a way of identifying the origin of exclusion of populations and entire civilizations through denial of knowledge, which allows for the proposal of corrective measures. By looking into the bodies of knowledge which have been integrated in the syncretic evolution of Mathematics, Ethnomathematics allows for a better understanding of the cultural dynamics under which knowledge is generated. The proposed historiography can be seen as a transdisciplinary and transcultural approach to the History of Mathematics. (D'Ambrosio, 1999, pp. 35-36)

And now more specifically, connecting with the universal mathematical activities:

A broad view of the history of mathematics, focusing on anthropological, social, political, religious, and other issues as well the cultural dynamics of encounters, is a very clear illustration of the full cycle of knowledge. It looks into how the processes of observing, comparing, classifying, evaluating, quantifying, measuring, counting, representing, and inferring originated in different cultures. It also examines how cultural dynamics played an important role in the development of these forms of knowledge, leading, as a result, to local institutionalization and local ways of thinking and doing. (D'Ambrosio, 2016, p. 10)

### 3.11 Educating for peace

Ubiratan was very fond of peace and thought peace should be the absolute priority for educators. I found this theme broader than formal education and so I separated it. The absolute priority of our mission as educators is to obtain

PEACE in future generations. We cannot forget that these generations will live in a multicultural environment, that their relations will be intercultural, and their day-to-day lives will be impregnated with technology. (D'Ambrosio, 2006, p. 33)

And what does it have to do with mathematics education:

In the current state of civilization, it is fundamental to focus on our actions, as individuals, as a society, in the realization of an ideal of Education for Peace and a for a happy humanity. When I speak of Education for Peace, many come with the question, "But what does this have to do with mathematics education? and I respond, "It has everything to do with it. (D'Ambrosio, 2006, p. 65)

Most important is achieving the state of inner peace:

Achieving a state of inner peace is difficult, above all due to all the problems we face in our daily lives, particularly in our relationship with the other. Could it be that the other also finds it difficult to achieve a state of inner peace? Without a doubt, the state of inner peace can be affected by material difficulties, such as the lack of security, lack of employment, lack of a salary, and often even the lack of housing or food. Social peace is a state in which these difficulties do not present themselves. Solidarity with our fellow man to overcome these difficulties is a first manifestation in order for us to feel part of society and move toward social peace. (D'Ambrosio, 2006, p. 65)

And what should the effort for scientific and technological advancement be for:

The multiple dimensions of peace [inner peace, social peace, environmental peace, and military peace] are the first objects of any educational system. The greatest justification for efforts for scientific and technological advancement is to achieve total peace, and, as such, it should be the substrate of every planning discourse. This should be the dream of the human being. (D'Ambrosio, 2006, p. 66)

## 4. On the person

Final theme, how did Ubi saw himself as a mathematics educator?

It is beautiful and it is the best way to finish this lecture:

How do I see myself as a mathematics educator? I see myself as an educator whose field of ability and competence is mathematics, and who uses it, but not as a mathematician who uses his position as an educator to impart and transmit his mathematical abilities and competencies. My science and my knowledge are subordinated to my humanism. As a mathematics educator, I seek to utilize what I have learned as a mathematician to realize my mission as an educator. In very clear and direct terms: the student is more important than programs and content. Spreading this message is my aim as a teacher of teachers. (D'Ambrosio, 2006, pp. 67-68)

**Acknowledgements**

I acknowledge valuable comments on a previous version by Gail Fitzsimons and Sara Ribeiro.

This work was financially supported by Portuguese national funds through the FCT (Foundation for Science and Technology) within the framework of the CIEC (Research Center for Child Studies of the University of Minho) projects under the references UIDB/00317/2020 and UIDP/00317/2020.

## REFERENCES

Ascher, M., D'Ambrosio, U. (1994), Ethnomathematics: a dialogue. *For the Learning of Mathematics,* 14, 2 (36-43). Available at: *https://flm-journal.org/Articles/6D5AE98E5864B7EE4924BFBFB3A8A.pdf*

Bishop, A. J. (1988). Mathematical enculturation. A cultural perspective on mathematics education. Dordrecht: Kluwer Academic Publishers.

D'Ambrosio, U. (1985), Ethnomathematics and its place in the history and pedagogy of Mathematics, *For the Learning of* Mathematics, 5, 1 (44-48). Available at: https://flm-journal.org/Articles/72AAA4C74C1AA8F2ADBC208D7E391C.pdf

D'Ambrosio, U. (1999), Mathematics, History, Ethnomathematics and Education: A comprehensive program, *The Mathematics Educator*, Vol. 9 No. 2, Spring 1999, pp. 34-36. Available at: https://openjournals.libs.uga.edu/tme/article/view/1806/1714

D'Ambrosio, U. (2006). Ethnomathematics: Link between Traditions and Modernity, Rotterdam. Sense Publishers.

D'Ambrosio, U. (2016). An Overview of the History of Ethnomathematics. In Milton Rosa, Ubiratan D'Ambrosio, Daniel Clark Orey, Lawrence Shirley, Wilfredo V. Alangui, Pedro Palhares, Maria Elena Gavarrete, (Org.). Current and Future Perspectives of Ethnomathematics as a Program. Hamburg: Springer Open, p. 5-10.

D’Ambrosio, U. (2020). An essay on Philosophy, mathematics and culture, *Qualitative Research Journal*. São Paulo (SP), v.8, n.18, p.571-594. Special Edition: Philosophy of Mathematics.

Rosa, M. and Orey, D. C. (2021). “Ubiratan D’Ambrosio: Celebrating his life and legacy”, *Journal of Humanistic Mathematics*, Volume 11 Issue 2 (July 2021), pages 430-450. DOI: 10.5642/jhummath.202102.26. Available at: https://scholarship.claremont.edu/jhm/vol11/iss2/26/

Scott, P. (2012). “The Intellectual Contributions of Ubiratan D’Ambrosio to Ethnomathematics”, *Cuadernos de Investigación y Formación en Educación Matemática*. Año 7 Número 10. pages 241-246. Costa Rica. Available at https://revistas.ucr.ac.cr/index.php/cifem/article/view/10608/10010

Shirley, L. & Palhares, P. (2016). Ethnomathematics and its diverse pedagogical approaches. In: Milton Rosa, Lawrence Shirley, Ubiratan D'Ambrosio, Daniel Clark Orey, Wilfredo V. Alangui, Pedro Palhares, Maria Elena Gavarrete. (Org.). Current and Future Perspectives of Ethnomathematics as a Program. 1ed. (13-17). Hamburg: Springer Open.

# THE ROLE OF HISTORY OF MATHEMATICS IN FORMING AN IMAGE OF MATHEMATICS AMONG STUDENTS AND GENERAL PUBLIC

**Discussion panel held during ESU-9, Salerno**

**Snezana LAWRENCE[1], Helena DURNOVÁ[2], Jean-Michel DELIRE[3]**

[1]Middlesex University, London, s.lawrence@mdx.ac.uk

[2]Masaryk University, Brno, Czech Republic, hdurnova@ped.muni.cz

[3]University of Brussels (ULB), jeanmicheldelire@gmail.com

## ABSTRACT

The panel which this contribution summarises, discussed the role of the history of mathematics in education focusing on the image of mathematics and mathematicians that students in education and more generally some societies, have. We looked at several such societies: those we come from, and those from our respective historical societies of interest, and compared some notes and made some conclusions.

## 1 *Preamble and Contributors*

The organisers of the ESU asked the authors of this report to form a panel for the ESU9 dedicated to the images of mathematics and mathematicians in education, and their historical and contemporary portrayal. We interpreted this question in similar ways but different to the point that each one of us teaches different groups of students and is interested in different historical societies. We also all come from different European cultures, but also bring interests in other cultures, most notably cultures from the ancient Indian subcontinent.

We therefore begin this report by listing our plans and contributions during the discussion, and further the details of conversations that emerged followed with some major questions posed, and finally make a conclusion about the role history of mathematics can play in forming images of mathematics and mathematicians in education in general.

### 1.1 Jean-Michel Delire

Jean-Michel is a mathematician and a philologist, specialized (PhD in 2002) in Sanskrit mathematics. Delire is still doing research in India, especially about Jai Singh II's (1689-1743) achievements in astronomy and mathematics. After teaching mathematics in secondary schools, Delire is, since 2007, training students-future maths teachers in geometry and history of mathematics. While trying to improve their general knowledge of the historical contexts in which mathematics were produced, Delire's aim is help such students to prepare lessons that will lead their pupils to a better understanding of mathematics.

Delire's credo is that history of mathematics gives both mathematics and mathematics education a more human attire, so that pupils who declare themselves to have skills and interests more in the "literary" than "scientific" disciplines could also begin to like and have a greater interest in mathematics. For that purpose, Delire's method is to take advantage of ancient texts and instruments, in order to immerse pupils in the context and background of the problems they consider mathematically.

In so doing, Delire's hope is that pupils will imagine the situation of the ancient researcher, project themselves into his problem, and give more sense to the mathematical problem than by making the use of the usual (and more abstract) method of learning mathematics. The pupils will so understand that mathematics are not a punishment imagined by a cruel god to torment them, but have been built, little by little, by women and men like themselves.

### 1.2 Helena Durnová

Helena Durnová teaches mathematics and history of mathematics at Masaryk University in Brno, Czechia. She has written on history of computing in Czechoslovakia in the 1950s and 1960s, including early programming practices there. She is interested in the intimate connections at the intersection of computing, mathematics, and language, and she seeks to incorporate these in her teachings of both mathematics and history of mathematics.

In Czechoslovakia, mathematics as a school subject only appeared in the 1950s. Before that time, textbooks and classes were devoted to arithmetic's (including algebra) and geometry. In 1950, a new journal was founded, named *Matematika a fysika ve škole* (*Mathematics and physics at school*). Probably due to the involvement of top mathematicians (including the topologist Eduard Čech, who was involved in the design of the new curriculum as well as the textbooks of the period), mathematics education in Czechoslovakia in the early post-war years comprised of lots of purely mathematical topics; especially geometry was represented more than before.

By 1969, a new trend in mathematics teaching arrived also to Czechoslovakia. Since then, children were exposed to traces of mathematics as a science, not just to arithmetics, algebra, and geometry, applied to real-world problems. With the increased level of sophistication involved, it became necessary to justify the topics historically (cf. Znám et al., 1986) and later also from the socio-cultural point of view (Kuřina, 2012,).

History of mathematics, especially as practised by mathematicians, played an instrumental role in positioning mathematics as a universal science through pointing out mathematics (calculations and geometry) in Ancient Egypt and Babylonia. Many of those now counted among mathematicians may have described themselves as philosophers, from Pythagoras to Leibniz and beyond. In the late 19$^{th}$ century, mathematics was seen as the example of correct reasoning, and thus as an example to be followed. This view has penetrated also into the way mathematics education is still thought about: mathematics is primarily a subject that teaches logical reasoning. Pointing to its beauty, on the other hand, is thought of as a good way to make mathematics more attractive for pupils and students who are not mathematically gifted. History of mathematics serves as provider of such connections.

### 1.3 Snezana Lawrence

Snezana is Senior Lecturer at the Department of Mathematics and Engineering Design, Middlesex University, London. She is a mathematical historian and is interested in the creativity, identity, and engagement in the learning of mathematics.

Snezana has been, for the past two decades, involved in various national initiatives in the UK, and international programmes and institutions (mostly via the HPM) to promote the use of the history of mathematics in mathematics education. She is now the Chair of the History and Pedagogy of Mathematics Study Group for 2020-2024. In the UK, Snezana has been an active member of The Institute of Mathematics and Its Applications and she is their Diversity Champion. She is also the Associate Editor of the BJHM (British Journal for the History of Mathematics). In all these roles, engagement with mathematics is an incredibly important aspect of the interraction with the educational as well as wider public. The images of mathematics and mathematicians are therefore at the centre of Snezana's work most recently, and the critical reflection on the role such images of mathematics play in society in general have been a major driver for her publishing output.

It is this focus that also drove her to write the book *A New Year's Present from a Mathematician* (Lawrence, 2019), which is more broadly dedicated to two questions: «What is it that mathematicians do?» and «Who gets to be called a 'mathematician' and why?». These questions, in turn, arise from her work with mathematics teachers, and more recently mathematics undergraduate students.

Preconceptions about mathematics and mathematicians can of course be positive, negative, and anything in between. Their influence on students' perceptions of mathematics are an important motivating force in their learning processes. These influences can directly be linked to the engagement strategies that involve developing students' critical analysis of mathematical techniques, contexts, and tools at the disposal of historical mathematicians so that students become more able to identify what they have at their own disposal, as well as what it is that is of interest, and therefore most appropriate to their own quest of finding mathematical 'truths'.

## Contributions from panellists

In this section we note our own reflections on how mathematics can help students form a positive and productive image of both mathematics and mathematicians. The reflections are given as they were presented and are therefore reported in the 'first person' manner.

### 1.4 Jean-Michel Delire

Mathematics is generally considered as a very abstract subject that only supermen and super-women can understand, people who has something in the brain called 'bosse des maths' in French. Of course, this conception is related to the ancient and erroneous theory (developped by G. Lavater, *L'art de connaître les hommes par la physionomie*, Paris, 1775) that the head of a human being is covered by humps that could reveal his/her defects and qualities. This theory has been abandoned today, but the conception remains that if you are attracted by mathematics, you must be bizarre somehow.

Another very popular – and erroneous – conception is that you are either a 'scientific' or a 'literary' person (see C. P. Snow, *The Two Cultures*, 1963, Cambridge University Press - French translation *Les deux cultures*, Paris 1968). Many journalists in Belgium, for instance, declare themselves as 'literary' and, sometimes, admit that they never understood anything in mathematics. But, as good 'literary' as they could be, they still don't see the difference between a number and a digit. When they have to quote a numerical information, they always say 'le chiffre (of the unemployment, of the Covid epidemy, etc.) est 15%', by instance. This is probably due to their difficulty with abstraction, for a digit is a concrete sign, while a number cannot be shown.

In any case, these examples show that mathematics are misunderstood, and that the journalists – and not only them – give a very bad image of our discipline. Some weeks ago, on a Belgian program RTBF1, one of them who had to talk about mathematics began his presentation by apologizing for addressing this subject.

The teachers themselves have their responsibility in this bad image. I have been teaching mathematics and I am still teaching history of mathematics to future teachers and I try to guide them so that they could tell their pupils when, why and how a certain mathematical problem occurred. But, very often, they choose the 'easy way' of teaching the rules, the steps their pupils should follow to solve a certain type of problems. And, of course, when the problem appears in another guise, the pupils are no more able to find their steps, for they don't understand the reason of these steps.

In many of the cases exposed so far, history of mathematics could be of some help: the 'literary' could find something else than terse and abstract rules when discovering the context and even the mathematical texts themselves; the future teacher could understand some of the pupils' difficulties when he/she is placed before a text for the reading of which he doesn't know the rules yet – not even a text in an ancient language (Latin or Greek for example), but a text written three or four centuries ago, by instance.

Of course, I could go on and enumerate all the qualities history of mathematics has – we are all convinced that it is a very useful approach which complement our understanding of mathematics, a complement that for many of us has taken the main place in our research. I developed some of these qualities in the introduction of a bulky book I wrote some years ago. It is a mathematical book with a lot of history (J. M. Delire, *Mathématiques multiculturelles I* –

*Arithmétique, Algèbre, Géométrie élémentaire*, volume 3 de la Collection *Sciences, Arts et Cultures*, Les Editions HE2B, Bruxelles, 2018, 431 p.).

The real problem is not the usefulness of history of mathematics, but its implementation in the cursus of our students. History of mathematics is still not compulsory – for future maths teachers – in too many European countries. I will begin by exposing the situation in my own country, Belgium. As far as I know, it is not compulsory for future maths teachers to take a history of maths course during their studies, at least in the French speaking Belgian universities. In the Hautes Ecoles where maths teachers for the lower secondary school (Collèges in France) are educated, the situation depends of the school. By instance, I succeeded, several years ago, to create a little course (24 h/year) of history of maths at the Haute Ecole de Bruxelles. I could even double the number of hours for this course last year, but I really don't know if this good situation would last after my retirement. It is very often like that: by instance, the history of maths course given by Jean Doyen at the University of Brussels is in latency since his retirement seven years ago…

The situation of the history of maths courses is too often dependent on one person, who devoted him/herself to the subject but has rarely a successor.

I can also say a word about some other countries, with information from contacts I have there; these reflections follow below.

**Germany.** Prof. Krömer and Klaus Volkert at Wuppertal University: the students-future maths teachers at the University follow a 6 hours/45 min (3h Vorlesung – 3h Seminar = oral presentation by students) in M1 or M2. And Prof. Krömer considers it as «probably the heaviest package in a German university (…) it depends certainly heavily on the competences and interest of the persons in post». He esteems that there are 30 to 40 researchers-teachers in the history of maths in Germany and Wuppertal has the greatest team: 6 persons, of which 2 are retired + doctorants and postdocs. In Siegen, he says, there is a history of maths course (4 hours) for the future primary school teachers, but the future Lycée teachers have rather course on the philosophy of maths. The education program of the Nordrhein-Westfalen Bundesland mentioned only once history of mathematics «Schülerinnen und Schüler erfahren, dass Mathematik eine historisch gewachsene Kulturleistung darstellt.» i.e. pupils experienced that mathematics has a historically grown cultural achievement; Prof. Krömer adds that every Bundesland has its own program.

**France.** It is comparable in a way, since the history of maths/science courses and research are mostly the prerogative of philosophy Faculties. But the INSPE, which prepare the future maths teachers, have their own courses on history of maths, depending again on personal interest (because the teachers are generally recruited on mathematics only oriented list = 25, but there is a certain liberty to create courses – generally 15 hours). The IREM are also to be taken into consideration as organisms of education through research, but they are open only to already diplomed students. They do a tremendous work and I must say that it is through their publications that I personally learned a lot after my first steps in Aarhus (Denmark) with K. Andersen and K. P. Moesgaard.

**Netherlands.** I have recently been informed that it is compulsory there for a future maths teacher to take a course on history of maths… Which is a good thing, comparable with the situation in Denmark.

**Denmark.** As far as I know (from my personal experience there 30 years ago), it is also compulsory, at least at the University of Aarhus. But, there again, there was a particular person, a historian of astronomy and mathematics, Olaf Pedersen, who heavily influenced the evolution, back in the sixties.

I would like now to listen to the reactions of the auditors about the situation in their countries, but I would also like afterwards to ask the question of what we could do, as a European organized body, in order to reinforce the teaching and use of the history of maths in our universities, Hautes Ecoles, INSPE (France), etc.

## 1.5 Helena Durnová

Should we teach history to mathematicians and mathematics teachers? If the answer is yes, what history should we teach and why do we teach it? Before unfolding the plethora of considerations, I would like to express my thankfulness to all my colleagues and also students with whom I have discussed the topic over the years.

Historians often organise their findings chronologically, so let me start with my own 1990s experience of being taught history of mathematics systematically for the first time. Then, each future upper-secondary school mathematics teachers had to go through a two-semester course in history of mathematics. It was presented to be a novel activity, starting around 1980 upon the initiative of the mathematician and didactician of mathematics Jaroslav

Šedivý (1934-1988). The people (usually mathematicians) interested in history of mathematics gathered at summer schools since 1980. The summer schools were called World-view Education in Mathematics, which the participants remembered as a clever cover name for history of mathematics, a field allegedly not supported by the ruling party of the 1980s. The participants of the 1980s summer schools thus understood history of mathematics as a dissident field which could only enter mathematics teachers' education under a cover-up title, namely world view problems of mathematics. Thus, history of mathematics, the story has it, could only enter the programme for upper-secondary school mathematics teachers through a back door.

There are a few question marks connected with this, which I will try to sketch below.

The first puzzling question is, how is it possible that history of mathematics and history of science would not be promoted by a regime that was based on science and the scientific world view? Would it not be only natural for a society based on the scientific worldview to promote history of science as well as history of mathematics?

Another puzzling question concerns the resources already available two decades before 1980. In 1963, *A Concise History of Mathematics* by Dirk Jan Struik was translated into Czech by Jaroslav Folta (1933-2011) and Luboš Nový (1929-2017), who were both originally trained as mathematicians and mathematics teachers. Those two also co-operated on the comprehensive volume published in 1961, *Dějiny exaktních věd v českých zemích do konce 19. století* (*History of the Exact Sciences in the Czech Lands until the end of the 19th century*) and on the overview published in 1979, *Dějiny exaktních věd v datech* (*History of the Exact Sciences to Date*). In other words, contrary to what the participants of the World-View Problems in Mathematics summer schools believed, history of mathematics was cultivated in Czechoslovakia between 1948 and 1989.

Yet another puzzling question might be why history of mathematics was (and is) taught to the future teachers, but not to the future mathematicians, but that is a quite different story.

As for the topics discussed in history of mathematics lectures, they often depend on the lecturer. Some prefer ancient mathematics, while others are fond of discussing late nineteenth and early twentieth century mathematics. In Czech history of mathematics and physics, there are several canonical figures, some of whom are internationally renowned, while others are also mentioned because of their contact with Czechia. They include René Descartes for his introduction of Cartesian coordinate system, but also for his alleged participation in the Battle of White Mountain (1620), a key event in Czech history; Johannes Kepler, who spent some time in Prague; Bernard Bolzano as a precursor of set theory and a native of Prague, who belongs to Czech history of mathematics, even though he wrote in German; and Albert Einstein, whose three-semester stay in Prague has been repeatedly described. Rather recently, Kurt Gödel, who was born in Brno, is also remembered, although he studied in Vienna and worked mainly in Princeton. Faculty of Informatics of Masaryk University in Brno chose his face to be on the medal worn at festive occasions by the dean of the faculty.

In general, these topics reveal the purpose of teaching history of mathematics to future teachers, which is creating a shared body of knowledge about history of mathematics among mathematics teachers. In creating it through the courses, it becomes apparent that such a shared body of knowledge is necessarily selective and often culturally biased (for better or worse).

Let us now take it for granted that all future mathematics teachers should know something about history of mathematics, as they themselves will be the point of contact in this matter for the pupils and students. They approach the topic differently, ranging from classroom decoration to using historical problems in their teaching. Different history of mathematics courses may look very different, depending on the personality of the teacher. While some teachers emphasize the dates and names, others emphasize the mathematical content. Also, different teachers may wish to emphasize different periods and different geographical regions. In what follows, I would like to present the vision I follow in the course I have been teaching, with the hope that it will stimulate some discussion.

For my students in the secondary mathematics teacher training programme at the Faculty of Education of Masaryk University, the one-semester compulsory course in history of mathematics (worth 5 ECTS) is often their first conscious contact with the field. In the one-hour lecture, they receive an overview of history of mathematics, whereas in the seminar, we read extracts of almost original texts, i.e. texts in translations closely following the original. These include Simon Stevin's *De Thiende*, Christianus of Prachatitz's, *Algorismus Prosaycus*, and Jacob Bernoulli's *Ars Conjectandi*. The aim of the course is rather to make students more open-minded than to teach them how to do history of mathematics themselves.

Mathematics teacher-training students will soon become mathematics teachers, and as such, they will also serve as a point of first contact in history of mathematics for their pupils and their parents. They will become a part of one of the three substantial groups of audience of historians of mathematics namely: mathematicians, mathematics teachers, and the general public. The interest of the last section of audience in history of mathematics lies, in particular, in biographies of mathematicians, but they also show interest in the mathematics hidden behind computers and artificial intelligence.

Let me take future history teachers as another example of the general public: not having had much mathematics education, the mathematical content presented to them has to be either elementary, or greatly simplified. As anyone who has ever tried to explain Gödel's results to the layman will probably agree, such a simplification is not at all a trivial task. However, let me use this example to elaborate on a different point, namely that of popularisation mathematics and especially history of mathematics. The books are in the shops and in the libraries, so rather than avoid them and claim they contain mistakes, I have found it more useful to make future teachers aware of their existence and encourage them to use these books with their students, as they have a value in speaking to young people and children in their language. At the same time, popularizations of history of mathematics can be used as material for emphasising what a good history of mathematics should look like. Historians should namely give us a chance to travel into the past with them, let us see with their interpretation how differently mathematicians thought in the past from the way we think now. Equipped with this, students may find it easier to phrase their own questions about history of mathematics and try and answer them. My role as a teacher therein is warning them that for particular topic, they may not be able to find the sources to finish their essay within the semester and redirecting them to a different topic. In marking their essays, the clarity of their expression has an indispensable value.

Having said all this, let me finish with a provocative question: How does knowing about Pythagoras help pupils master Pythagoras' theorem?

### 1.6 Snezana Lawrence

I found the most intriguing aspects of mathematics always to be those related to the reality and context of mathematicians who invented or discovered some new mathematics, and the reasons they were impelled to do so. For me, this is the most interesting part of mathematics, and I think we should do much more about talking with students about how new mathematics gets created or discovered, and then communicated, and how it sometimes travels huge geographical and temporal distances to connect people from very different backgrounds.

In the book about which I wanted to talk today (Lawrence, 2019), I chose to try to answer two questions:

- What is it that mathematicians actually do?
- Who gets to be called a mathematician and why?

I tried to answer these questions by adopting a structure based on a calendar, that is familiar to many amateur historians of mathematics, or rather many of the people who engage in popular history of mathematics on social media, like those on twitter for example (or following my twitter handles @snezanalawrence and @mathshistory). So, my book answers the above two mentioned questions through a series of stories from ancient to modern mathematics, but not in a chronological order. It rather uses calendar months to anchor stories about famous mathematicians and their contributions to mathematics in a particular fashion. I've presented therefore twelve stories – I tried to report on mathematical inventions as if I was an observer of a scene from another time and place. I imagined opening this metaphorical door and looking at the mathematicians or their inventions or discoveries, and reporting on what I saw just as a journalist would report from a field.

It is possibly not obvious to mathematics educationalists, or rather to us in the HPM network, that lots of mathematicians, in particular the mathematicians in industry and government, obviously like mathematics, but do not have enough of the material to talk about its importance to others without going into the detail of their, often highly technical or skilled work. So, this book was dedicated to this audience also. I came across this large group of mathematicians whilst engaged on the various committees and being the Diversity Champion of the Institute of Mathematics and Its Applications (UK). So partially, this book was dedicated to this group of people too, other than general public interested in history of mathematics. Most of all, it aimed to enchant and engage a reader who knows a little bit of mathematics, and also wishes to learn more about how various developments happened in the history of our discipline.

There is, in fact, no prior knowledge needed to read this book. I wanted to write as if writing for a wide-raging audience, with concepts explained in an easy-to-read-and-digest way without trivialising their nature. The book is Euro-centric but another method I used in writing was to describe the whole journey through time as also a journey through a geographical area. I hope that I would, in time, write another book of another area – Middle East is something I have started to look in the recent years.

## Discussion

After our presentations, we had a discussion which we summarise here from the notes taken by Delire.

A question from an Italian delegate came about the expansion of history of science after WWII. The delegate asked why no similar expansion of courses in the history of mathematics occurred in universities of Europe. Snezana answered that there are not many modules in UK (England, Scotland, Northern Ireland, and Wales) – some years ago, the British Society of Mathematics undertook a survey and found that across the whole of the country, and in around 130 universities in total, had only less than twenty such modules/courses in the history of mathematics. We discussed how students would choose history of mathematics if there is an offer. But a problem of having such a small number of courses, and hence positions in the history of mathematics, there is not enough qualified people to be employed as purely historians of mathematics, which is something we should work on as a community.

Another question was posed about the history and philosophy of mathematics and their joint role in education. Snezana argued here that there is a useful international group on philosophy of math practice called Association for the Philosophy of Mathematical Practice (Association, 2022) which has much to offer in terms of considering precisely such questions and with whom we should perhaps have more contact. Helena proposed that the teachers, in their professional practice, also constantly develop their priorities and purposes, and that this is not something that remains the same all the time for each individual teacher. For example, when teaching, maths teachers will constantly re-examine their priorities and think about what is most important for them: to help develop their pupils‘ minds, select what is important or not in mathematical thinking, look at ancient texts (this too comes with different aims, depending on the historical period). Jean-Michele added his comments here about Indian mathematics culture and the fact that Indian mathematics didn't develop in the same way as Greek (ancient) mathematics did, and was not of the same hypothetico-deductive systém. This contrast between two ancient systems of mathematics could bring interesting questions if presented in the classroom. He further added that our own, contemporary philosophical thinking, is also always evolving and, thus is also our way of teaching.

A question then arose about the social images of mathematics, by instance through films (which show mathematics is made by some extraordinary genius) and what we thought about that. We answered that we all thought it is necessary to present maths as a collective construction, and offered an example of Perelman's conjecture solved by several mathematicians. Snezana further added a comment about a fantastic resource for the classroom on this particular aspect of popular images of mathematics in the movies, made by a Harvard academic, *Mathematics in Movies* (Knill, 2022), which has numerous clips from many films from the last two-three decades in which mathematicians are portrayed in different ways. This resource she often uses in teaching mathematics as it can offer good starting points for conversations in a classroom. Helena alluded to theatre plays about Gödel, for example one by William Hugel, and a more recent one by Marcus du Sautuy, *I is a strange loop*.

A fourth question was about the fact that secondary maths teachers and their pupils, even when interested in the history of mathematics, don't know about libraries, but use Wikipedia instead. Snezana's answer was about the methodology and nature of a course in the history of mathematics. Such course/module has to cover both history *and* mathematics. These two subjects sometimes can confuse students, so they need some training also in the way they conduct historical research, if they are in fact doing a mathematics degree (or post-graduate course in mathematics primarily rather than social sciences). Helena added that there are a lot of discussions among French colleagues about historical teaching and sources. In terms of mathematics and philosophy or mathematics and history courses, as in Hungary and Czechia, pluridisciplinarity is appreciated, she believes.

## Conclusions

In some of the other discussions and questions we came to the conclusion that we should, perhaps, as a community develop a greater and more accessible platform to enable teachers from around the world to access resources and communicate with each other online. Although our network is large, there are many more people who may want to be able to access resources in the history of mathematics and use them in their every-day work in the classroom.

## REFERENCES


Association for the Philosophy of Mathematical Practice (2022). http://www.philmathpractice.org (accessed 25th Dec 2022).

Delire, J. M. (2015). Mathematical Activities in the Classroom on the Occasion of the Exhibition "Art et Savoir de l'Inde" During the Festival Europalia-India (Brussels, October 2013 – January 2014), in Barbin, É., Jankvist, U., & Kjeldsen, T. H. (Eds), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* pp.627-636.

Folta, J. and Nový, L. (1961). *Dějiny exaktních věd v českých zemích do konce 19. století* (*History of the Exact Sciences in the Czech Lands until the end of the 19th century*). Prague: Nakl. Československé akademie věd.

Folta, J. and Nový, L. (1979). *Dějiny exaktních věd v datech* (*History of the Exact Sciences to Dates*). Prague: Nakl. Československé akademie věd.

Knill, O. (2022). *Mathematics in Movies*. https://people.math.harvard.edu/~knill/mathmovies/ (accessed 25th Dec 2022).

Kuřina, F. (2012). *Elementary matematika a kultura. (Elementary mathematics and the arts).* Hradec Králové: Gaudeamus.

Lavater, G. (1775). *L'art de connaître les hommes par la physionomie*. Paris: Prudhomme.

Lawrence, S. (2019). *A New Year's Present from a Mathematician.* CRC Press, Routledge, New York.

Snow, C. P. (1963). *The Two Cultures*, Cambridge: Cambridge University Press

Snow, C. P. (1968). *Les deux cultures*, Paris: Pauvert.

Struik, D. J. (1954). *A Concise History of Mathematics*. London: G Bell and Sons.

Struik, D. J. (1963). *Dějiny matematiky*. (*A Concise History of Mathematics*). Translated by Jaroslav Folta and Luboš Nový . Prague: Orbis.

Znám, Š. et al. (1986). *Pohľad do dejín matematiky* (*A Glance at the History of Mathematics*). Bratislava: Alfa.

# THEME 1

# THEORETICAL AND/OR CONCEPTUAL FRAMEWORKS FOR INTEGRATING HISTORY AND EPISTEMOLOGY OF MATHEMATICS IN MATHEMATICS EDUCATION

# THROUGH A HISTORICAL DOCUMENT THE AUTHOR CAN GUIDE THE STUDENTS

**Adriano DEMATTÈ**
Centro Ricerche Didattiche "Ugo Morin"
Via S. Giacomo 4, Pieve del Grappa (TV), Italy
adrdematte@gmail.com

## ABSTRACT

In the workshop, I described to the participants an episode of a classroom experiment in which historical documents was used with upper secondary school students. I asked the participants to write their remarks about both the work made by two students and some theoretical reflections. In the present article, I report and comment on these remarks. The purpose of the experiment and of the activity proposed in the workshop was to examine the relation that the students established with the author of the document who, although passive, can guide the students towards cultural considerations about mathematics and its evolution.

## *1 Introduction*

A text is, for students, phenomenality of the author. If it is an author lived in the past, the historical distance can help students to highlight elements that connote the author, such as discussing in an unusual way or referring to the everyday life of the past. I consider these elements as traces of the author, that enrich phenomenality and "reveal" him/her as a human being. This contributes to creating the Student–Author relation. The students do not search for these elements which represent a possibility for their "unpredictable" encounter with the author (Demattè, 2019).

Based on these considerations, my research question is: Through the document, can the Other/Author summon students, so that they pose questions about him/her? In the present article, I discuss this question and I try to answer, also through the outputs of the workshop.

## *2 The experiment in the classroom*

The document considered in this article is a problem taken from Filippo Calandri's *Aritmetica*, an Italian treatise of the 15th century, originally used to teach mathematics in the Abbacus schools whose main purpose was to prepare for mercantile, commercial, and artistic activities; see (Demattè & Furinghetti, 2022).

The problem was proposed as part of a competition between classes of ninth grade (14–15-year-old students), inspired by *Rallye Mathématique Transalpin* (http://www.rmt-sr.ch/). Each class had to tackle a number of problems, difficult to be solved in the allotted 50 minutes. So, the class had to distribute the problems among its members. The problem taken from Filippo Calandri was chosen considering that the students were facing algebra (polynomials and linear equations) and it was aimed at verifying whether students had internalized the use of letters for formalizing a general procedure. The competition took place in February, that is, shortly after the middle of the school year.

In Fig. 1, the English translation of the problem with a premise (common to each problem of the competition) is reported. For the students, the archaic Italian language has been modified to confine the difficulties of interpreting the text to mathematics, avoiding linguistic complications, but without changing spirit and meaning of the original text.

*In addition to the text of the problem, the authors also report the resolution which, however, does not include all the steps, or in any case does not always explain the reason for the operations that are carried out and the results obtained. Write explanations to complete their reasonings.*
Filippo Calandri (ca. 1467 –?) belonged to a family of Florentine abacists (the accountants and mathematics teachers of that time, too); his brother Pier Maria, his father and a great-grandfather also practiced the same profession. He is the author of one of the first printed mathematics texts: the *Aritmetica* (1491).

From Calandri's *Aritmetica*, Problem **LXXX**: **How to find the natural number your friend thought up**. If he would think 10, tell him to double that, and that's 20, then tell him to add 5, and that's 25, then multiply by 5, and that's 125, tell him to add 10, and that's 135, multiply by 10, and that's 1350. From this, subtract 350 and that's 1000 which is 10 hundreds for each of which you take 1. So, you say that the friend thought 10. And the same way you could do if he thought another number: every hundred is worth 1.

**Figure 1.** The problem delivered to students (material for participants, too).

In class, Calandri’s document was assigned to the students Matilda and Benedetta. In the workshop, we focused on their work. Here it is a very short excerpt, sufficient to understand how they repeated the author's reasoning, using the number 9 instead of 10. "Ask your friend to pick a number, ex. 9. Tell him to multiply x 2, ex. $9 \cdot 2 = 18$ […] Now each hundred is worth 1 so the result will be 9".

About 20 days later, I interviewed Matilda and Benedetta. I asked them questions regarding Calandri’s document. One of the questions was: "What questions (to ask an expert/teacher) did this problem raise you?". This question asked to the students was aimed precisely at investigating whether they had wondered *why*, whatever number the friend thinks, at the end of the game a value one hundred times greater than that number is obtained. I supposed their lack in mathematical tools and their consequent difficulties to answer in written form that *why*. They said: "Did I understand what the author is doing? Does the game change if I use other numbers and rules? Why did Filippo want to do that research on the number that his friend thought?"

The first one of these three questions can be interpreted as their way to get information about correctness of what they wrote as an explanation. The second can be an indication of their (partial) awareness of possibility to analyse the game in terms of variables and operations, that is, in the terms of the algebra they were facing at that time in class. I will discuss their third - unpredictable - question in the following pages.

## *3 The workshop*

The audience consisted in teachers (in service or retired) from various school grades, researchers in education or history, and doctoral students.

After a 15–minute oral presentation in which the experiment in the classroom was described, I proposed to the participants to reflect individually, to discuss in small groups, and finally to write personal remarks about the materials reported in Fig. 1 and Fig. 2. The following premise introduced all the material delivered to them: "What would you say to Matilda and Benedetta? What is your opinion about the Theoretical reflections [see Fig. 2]? Suggestions and critical remarks are welcome!".

**Theoretical reflections**

In reading a mathematical text, students are called:

- to accept the *author*'s proposal, in its alterity (particularly evident in a historical document),
- to share the mathematical *content* (i.e., to get an agreement with the author through interpretation – hermeneutical process),
- to overflow *themselves* (through acceptance of a teaching and propensity to change their own foreknowledge).

Obedience to the request of the teacher who proposed reading the document is not enough to explain why that can happen. There must be a relation with the text and with the author, that excludes the teacher who has only created the conditions for this relation to take place. We may say that a main goal of mathematics education is to find ways so that students pass from their institutional tasks to their work with the Other; see (Demattè, 2021).

Even considering any mathematical document (not just historical) I think that not because it is manifestation of culture the students can be involved in analysing it, but because it belongs to the Other (I use the capital O considering that students have established a relation with that Other). Culture is a sort of surface.

According to Levinas (1991, p.191, endnote 10): "It is as possessed by a neighbor, as relics, and not as clothed with cultural attributes, that things first obsess. Beyond the "mineral" surface of things, contact is an obsession by the trace of a skin, the trace of an invisible face, which the things bear and which only reproduction fixes as an idol".

In conclusion, despite his passivity, the author can guide the students: in adherence to the mathematical reasoning he expounds, and in the questions he can raise about him and his work. Reminding Matilda and Benedetta, the fact that the author is a mathematician of the past gives them the indication that his work is inside the flux of the history of mathematics, so they consider it reasonable to pose questions about his "research" (of something new).

**Figure 2.** Theoretical reflections: material for participants.

## 4 *Participants' remarks and questions*

Each of the participants chose to focus on some aspects of the material delivered during the workshop, that is: the document, its educational role, the students' explanation, their interview, the theoretical reflections. No one has treated them all. Here, I use different letters to identify each individual participant or group who have written their remarks or communicated them orally.

Participants of group A discussed the choices made by the teacher. They considered the student–teacher–author triangle. They then asked a series of questions: With what criteria did the teacher choose the document? At what stage of class work did you introduce it to the students? For what purposes? Perhaps to introduce or to reinforce the use of algebra to generalize?

Participant B chose to work individually and proposed an observation regarding the student's difficulties in front of the document, establishing in some sense a continuity with the discourse introduced by group A: "There is a problem to translate the question in a formal language (algebraic) through the input of a variable. This is not trivial at that age, considering that the historical text doesn't use it".

Participant C worked in a group but produced his own contribution in which he formalizes Calandri's procedure with an equation and mentions historical and educational aspects: "$((2x+5)\cdot 5+5+10)\cdot 10=1350$. Find $x$. Linear equation. $1350-350=1000$ $1000:100=\boxed{10}$. Shifting from "rhetoric" to symbolic algebra? What do students need to know? The use of $x$ (the "thing")?".

With D, I indicate two people who worked together. In his writing, one of them formalizes the problem with an identity: $(((((x\cdot 2)+5)\cdot 5)+10)\cdot 10)-350= 100\cdot x$. Then he focuses on the three questions posed by Matilda and Benedetta. He points out that the first one concerns "the process", the second the "logic behind the game", the third "the why" and "it poses some interesting insights on the motivational problem behind mathematics, and the things students usually do in schools". "Regarding the bond with the author", he considers "more interesting" to "read the explanations that the author gives himself" instead "just solving an ancient problem". This is related with what his mate writes. She says about the "relation with the text and with the author" that "excludes the teacher"; "the main goal of mathematics education is to find ways so that students pass from their institutional tasks to their work with the Other". She continues by observing that the activity with students suggests an expansion through interdisciplinary work to introduce them to the "historical context of fifteenth-century Florence".

Participant E worked with another person, who, however, did not produced a contribution. He insists on "Theoretical reflections" and highlights his point of view oriented to a "Philosophical practice", also with reference to (Radford & Sabena, 2015, p.159). He expresses his disagreement with the following statement contained in the "Reflections": "relation with the text and with the author [...] *excludes* [italics added] the teacher". He feels that by saying this, one forgets that the teacher is a mediator. He suggests clarifying the meaning of the term "passivity" in the context in which it is used here. He then underlines that the text belongs to the Other, it contains the trace of the Other – mostly a missing person – and it is the only link between student and author.

Participant F also focuses on the “Theoretical reflections” thinking about links with the Bakhtinian perspective (Guillemette & Radford, 2022). He tries to think about a dialogue between Matilda and Benedetta, the teacher, and Calandri. This raised questions about autonomy of the students, role of the teachers who can accompany them, what to "give" them, and what they have to discover/encounter by themselves.

## 5 *Responses to participants' remarks*

Some questions posed by the participants are answered hereinafter, and others have already been answered elsewhere in this article.

Participant C wrote some questions that suggest part of the possible answers. His question "Shifting from rhetorical to symbolic algebra?" recalls Nesselman’s three historical phases of algebra: rhetorical, syncopated and symbolic. The questions: "What do students need to know?" and "The use of $x$ (the "thing")?" – posed differently by A participants – could open a long discourse regarding epistemological and educational aspects on the use of the variable: for example, see (Bråting, 2019). Note that the students did not use variables in their explanation.

When in the "Theoretical reflections" I affirm that the "relation with the text and with the author [...] excludes the teacher" I want to refer to the relation that I define "ethics" in (Demattè, 2021) and that I consider a necessary condition for the student’s learning in using a written mathematical text. The teacher has the role of creating the conditions for this relation to be established. He is only momentarily excluded. The student will return to the teacher for other phases of classroom work in which the teacher will possibly assume the role of mediator.

I used the term "passivity" recalling Levinas when he focuses the ethical relation between two human beings; see (Levinas, 1989) and (Guillemette, 2018). That relation is established without either of the two performs acts aimed at conditioning the Other. Referring this relation to the case of the au-

thor of a document and the student, I consider that the author offers a content without having the power to force the reader. Thus, he exposes himself to the violence of a distorted use of the document: let's think about when the student uses it for a mnemonic study, and therefore without a substantial adherence to the author's reasoning. The author is a defenceless Other – passive – but, when the ethical relation with the student is established, it happens that the author involves the student in his own reasoning (even if the temporal distance and the different personal experience do not allow to speak of coincidence of reasoning); see (Guillemette & Demattè, 2022).
Matilda and Benedetta asked a question regarding author's choices, namely the motivation that led him to the game. The two students moved the discussion beyond the strictly cognitive aspects. This is a case of the ethical events that connotate mathematics classroom; more situations in (Radford, 2021).

## 6 Further considerations regarding the "Theoretical reflections"

By the participants, there were no objections to the statement "Even considering any mathematical document (not just historical) I think that not because it is manifestation of culture the students can be involved in analysing it, but because it belongs to the Other [...]. Culture is a sort of surface". I expected that there could be a disagreement on the part of most of the participants. In fact, one of the reasons why the use of history in the teaching of mathematics is supported in the HPM group lies precisely in the fact that mathematics is included in the culture: mathematics "is the result of contributions from many different cultures" (Clark et alii, 2016). The approach I would like to propose, instead, is aimed at putting the relation between people in the foreground, in order to identify the origin of the possible motivations of a student facing a historical document. In this approach, the basic educational problem becomes helping students to see the author as an Other to be questioned and guided by. In the case of Matilda and Benedetta, the use of the term "research" ("ricerca", in the original report of the interview, in Italian) strikes me, and suggests some reflections and questions regarding the reasons for which they used it, even about a recreational mathematics problem. The term "research" indicates a process that has in itself something unfinished, not yet defined. It seems then that Matilda and Benedetta see exactly this in what Calandri illustrates - the first two questions formulated by the students testify that they do not fully master the situation of the game, which therefore they perceive as not defined. Research is a process that takes place over time and therefore it seems that the two students have chosen the term also considering Calandri and his work within the flow of the history of mathematics. The fact that the students got a biography of Calandri and the fact that they found an unusual and unsettling situation in the document gave them elements for their question about the reasons for the author's research. They thus expressed the desire to know something more about him. They made themselves open to be guided, first by retracing the author's reasoning – using another number – and then developing curiosities that go beyond what the document exposes.

Involvement of the students cannot derive from culture if there are no other previous conditions. Referring to mathematics education what Levinas affirms in the passage reported in paragraph 3, we can say that it is since it belongs to an Other that an element of culture – a mathematical text – involves ("obsessions"), not because it has a cultural garment ("clothed with cultural attributes"). Texts bear author's trace ("of an invisible face") and it is in this that the students find motivation to access mathematics in the document. History shows that mathematics is a product of culture; the complex of institutions, activities and spiritual manifestations that constitutes culture is produced by people. Thanks to the "obsessive" relations between people, culture grows. Mathematics is not an exception. What I have said about the historical document and the author's trace can also be referred to the use of any mathematical text, even if not taken from history. For a document of the history of mathematics, the attribution to an author is a salient fact. On the contrary, in the case of the school texts, it is a fact usually considered irrelevant; this deprives the students of one of the elements to establish the relation with the author.

## *7 Concluding remarks and didactical implications*

We might wonder if the same question asked by Matilda and Benedetta about the research of Calandri would have been formulated by the students if the game had appeared in their textbook, having as author a person they do not know at all (from informal interviews with students of various classes, I have realised that they almost never know the name of the author of their textbook).

During the workshop, no one directly answered the question written as a first introduction of the material given to the participants, that is: "What would you say to Matilda and Benedetta?". One of the participants of the couple D highlighted the importance of motivation in learning mathematics. This seems a suggestion for teachers to give to students directions and information in sight of their possible "whys", similar to "Why did Filippo want to do that research on the number that his friend thought?" by Matilda and Benedetta.

A balance of the workshop allows to highlight the complexity of the activity proposed to the participants: from aspects regarding the specific historical document to theoretical reflections concerning the philosophy of mathematics education. According to their sensitivity and their interests, the participants oriented themselves towards some of those aspects. Also in this paper, only some aspects of the material delivered to the participants have been deepened. For example, the criteria on the basis of which the document was transcribed – modifying certain specific parts of the original, considered too difficult for the students – have not been examined. We reflected on the role of an author of a historical document and of an author of a modern text. We considered the student's relation with the author through the text in both cases. From this, we can remark that working with history can suggest aspects transferable to didactic situations that do not involve its use: educators can derive from history reflections on the pedagogy of mathematics in a broader sense, to improve mathematics education at all levels. Coming back to the students' questions in Section 2, we can find suggestions for projects of activities with the class. For example: a) use of letters for formalizing and generalizing (in different situations, for a long-term intervention), b) problems of recreational mathematics in the history, about which see (van Maanen, 2003) and (Moyon, 2019), c) historical context in which Calandri lived and worked (as a participant in the workshop suggested: see Section 4 of the present article).

## REFERENCES

Bråting, K. (2019). Development of school algebra – A comparison between the 1980 and 2011 Swedish mathematics curricula. In E. Barbin, U.T. Jankvist, T.H. Kjeldsen, B. Smestad, & C. Tzanakis (Eds.). *Proceedings of the 8th European Summer University on History and Epistemology in Mathematics Education*, 711-726. Oslo: Metropolitan University.

Clark, K. M., Kjeldsen, T. H., Schorcht, S., Tzanakis, C., & Wang, X. (2016). History of mathematics in mathematics education: Recent developments. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting – HPM 2016*, 135–179. IREM de Montpellier.

Demattè, A. (2019). Searching for the Other in a Historical Document. *Philosophy of Mathematics Education Journal*, No. 35.

Demattè, A. (2021). On student's ethical relation with a mathematical written text: meaning and empirical evidence. *Philosophy of Mathematicd Education Journal*, No. 38.

Demattè, A & Furinghetti, F. (2022). Today's students engaging with Abbacus problems. *ZDM – Mathematics Education*, 54(7), 1521–1536.

Guillemette, D. (2018). The ethical subject of Levinas. In E. Bergqvist, M. Österholm, C. Granberg & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology in Mathematics Education (Otherness in Mathematics Education*), Vol. 1, 95–123. PME.

Guillemette, D. & Demattè, A. (2022). A Discussion on Levinas's Ethics of Otherness, Originals from the History and Teaching-Learning Mathematics. *For the Learning of Mathematics*, vol. 42, num. 2, 22–28.

Guillemette, D. & Radford, L. (2022). History of mathematics in the context of mathematics teachers' education: a dialogical/ethical perspective. *ZDM-Mathematics Education*, 54(7), 1493–1505.

Levinas, E. (1989). Ethics as First Philosophy. In S. Hand (ed.), *The Levinas Reader*, 75–87. Oxford: Basil Blackwell.

Levinas, E. (1991). Otherwise than Being or beyond Essence. Dordrecht: Kluwer Academic Publishers.

Moyon, M. (2019). Teaching mathematics and algorithmics with recreational problems. The Liber Abaci of Fibonacci. In E. Barbin, U.T. Jankvist, T.H. Kjeldsen, B. Smestad, & C. Tzanakis (Eds.). *Proceedings of the 8th European Summer University on History and Epistemology in Mathematics Education*, 417–436. Oslo: Metropolitan University.

Radford, L. (2021). Mathematics teaching and learning as an ethical event. *La matematica e la sua didattica*, 29(2), 185–198.

Radford, L. & Sabena, C. (2015). The Question of Method in a Vygotskian Semiotic Approach. In Bikner-Ahsbahs et al. (eds.), Approaches to Qualitative Research in Mathe*matics Education, Advances in Mathematics Education*, 157–184. Dordrecht: Springer Science+Business Media

Van Maanen, J. (2003). Research on history in mathematics education in the Netherlands: The 'Reinvention Studies'. In F. L. Lin (Ed.), *Common sense in mathematics education*, 191–201. Taipei: National Taiwan Normal University.

# TRANSCENDENTAL CURVES BY THE INVERSE TANGENT PROBLEM:

## HISTORICAL AND DIDACTICAL INSIGHTS FOR CALCULUS

**Davide CRIPPA*, Pietro MILICI ****
*Ca'Foscari University, Malcanton Marcorà
Calle Contarini, Dorsoduro 3484/D, 30123 Venezia, and Institute of Philosophy, Czech Academy of Sciences, Jilská 1, 110 00, Prague.
**University of Insubria, Via O. Rossi 9, Varese, Italy
davide.crippa@unive.it
pietro.milici@uninsubria.it

## ABSTRACT

In this paper, based on a workshop held at ESU 2022 (Salerno), aimed at secondary school and university teachers, as well as historians and didacticians, we suggest a new approach to calculus from a constructive geometric perspective. Specifically, we will provide a historical presentation of certain geometric instruments related to calculus, and we will introduce a new device for hands-on experiences. After that, we will describe the activities of the workshop presented at the latest ESU 9 in Salerno, in which we alternated the introduction of historical sources and laboratory activities.

## 1 *Introduction*

Mathematical instruments are an important focus in recent research in the didactics of mathematics (cf. Monaghan, Trouche & Borwein, 2016). In this regard, the history of instruments to trace curves (or "Zeicheninstrumente" in German) has also been recently studied by Bartolini Bussi & Maschietto (2007), Tournès (2009), and van Randenborgh (2015).

Historically, these instruments have played a key role in the constitution of analytic geometry and calculus. This role gradually waned: during the 18th century geometric instruments were marginalized, separated from "pure" mathematics except when they were used as a resource for teaching, and confined in some cases to treatises on instruments in general.[30] This process went

[30] Among these treatise, we can mention: Nicolas Bion (1652-1723), *Traité de la construction et des principaux usages des instrumens de mathématique*, Paris 1709 (cf. also Turner, 2014); Jakob Leupold (1664–1727), *Theatrum arithmetico-geometricum, Das ist:Schau-Platz der Re-*

alongside with the separation of finite and infinitesimal analysis from geometry to form an independent discipline, the so-called "algebraization of analysis" culminated in the 19th century with Bolzano, Cauchy and Weiestrass.

In this paper, we shall focus, on the basis of historical examples, on the epistemological implications of geometric instruments with respect to the understanding of fundamental concepts of analytic geometry and calculus.

Moreover, we shall explore the fruitfulness of a concrete approach to calculus for today's teaching by proposing learning experiences involving machines that implement historical ideas.

## 2 History

One of the explicit goals of the pioneers of calculus, such as Leibniz, Huygens, and the brothers Bernoulli was the construction and study of curves, particularly those that could not be associated with algebraic equations (Blåsjö, 2017). According to Leibniz and his friends, discovering new curves through already known instruments, or finding new artifacts to construct curves that had been previously observed in natural phenomena were still paramount.

The episode, recounted by Leibniz and represented in Fig. 1, of Perrault's posing of the problem of finding the curve traced by a watch attached to a chain and dragged along a tabletop is just an example of how the manipulation of an artifact, initially not devised for geometrical purposes, could lead to the discovery of new geometric objects and new insights into the solution of open problems (Bos, 1988; Tournès, 2009).

In addition to providing new materials for the geometer, the use of instruments in the early modern period also had more theoretical goals. As in Descartes (1637), instruments could frame the relations between symbolic computations in finite and infinitesimal analyses and the study of curves, as well as answer foundational questions, such as those about the admissibility of curves in geometry (Bos, 2001; Panza, 2011).

---

*chen- und Meßkunst*, Leipzig 1727; Gianbattista Suardi (1711–1767), *Nuovi istrumenti per la descrizione die diverse curve antiche e moderne*, Brescia 1752; George Adams (1750–1795), *Geometrical and geographical essays containing a general description of the mathematical instruments used in geometry, civil and military surveying, leveling, and perspective*, London 1791. See Randenborgh (2015, p. 46).

Broadly speaking, a geometric instrument can be defined as an “artifact”, that is a material object that, among other things, makes a mathematical idea concrete (Randenborgh, 2015, pp. 6ff., and Vollrath, 2003). For example, the Euclidean compass is an object employable in various ways and with different goals; however, when used by students to perform the constructions prescribed in Euclid’s Elements, it encapsulates Euclid’s third postulate, and embodies the real definition of a circle.

**Figure 1.** Perrault’s problem as it was imagined by Giovanni Poleni around 50 years later (Poleni, 1729, table).

However, one may doubt whether a watch dragged upon a table should be counted as a geometric instrument at the same level as the compass. What mathematical idea does this object realize? What is the nature of the curve described by the trajectory of the watch case? Perrault posed this question as an explicit challenge. During a controversy that saw the involvement of Huygens, Johann and Jakob Bernoulli, Leibniz declared himself to be the first to have answered correctly. Leibniz identified the curve traced by the moving watch with a “tractrix” or “tractoria” and described it as the curve having constant tangent-length (Leibniz, 1693, also Bos, 1988).

This was not a small discovery, since it shows that Perrault’s instrument realizes a solution to a fundamental problem for the development of analysis: to find a curve whose tangents have constant length. More generally, problems that involve the determination of a curve starting from given properties of their tangents, or relations among tangents and other segments, are called “inverse-tangent problems”.

Christian Huygens, who had also studied Perrault's curve, was less prone to accept the system formed simply by a watch moved on a table as a geometric instrument (Bos, 1988, p. 29ff.; Tournès 2009, pp. 15-16). According to him, a "dragged watch" should fulfill certain criteria to function as a proper curve-tracing instrument, which are not trivially met. First, Huygens demanded that the plane of the dragging be horizontal to avoid the effects of gravity. Second, the dragging of the weight from one position to the next should be reversible so that the purported instrument can change from position A to B and vice versa. Third, the motion should be sufficiently slow to prevent inertia. Due to the existing friction, the direction of the motion, made "real" by the dragged chain, would always be tangent to the curve. Only in this way, and because the length of the chain is invariant during the motion, the traced curve would have tangents of constant length.

Without suitable modifications to the physical configuration, manipulating the watch could easily get out of hand and fail to produce the desired curve. For this reason, Huygens proposed models of alternative machines that ensured the fulfillment of the three conditions described above, and thus would be fully geometrical (cf. Bos, 1988, p. 30; Tournès, 2009, p. 16; Blåsjö 2017). Unfortunately, Huygens never published his results on this topic. Leibniz, who nevertheless addressed the construction of transcendental curves in his published works and, at greater length, in the unpublished ones, eschewed the problem of constructing concrete machines. For him, the tractrix was a legitimate curve once the possibility of its construction via its tangent properties was conceded. It did not matter much whether a device that ensured such motion could be constructed. The issue was still open during the first half of the 18th century when the British mathematician John Perks and the Italians Giovanni Poleni and Gianbattista Suardi designed, and in some cases produced, artefacts which could trace the tractrix and the logarithmic. Thanks to them, during the first half of the century, new machines to trace transcendental curves entered mathematics.[31]

These efforts show that, for these authors, ideal machines were just not enough; mathematical ideas ought to be embodied into physical objects.

[31] For a short introduction and the construction of a new machine cf. (Crippa & Milici, 2019)

Poleni’s machines are described in all their technical details in a letter to Hermann published in 1729 in Padua and reprinted in the *Fasciculus epistolarum mathematicarum* (1729), together with other letters for eminent Italian mathematicians. The machines described in the letter are also mentioned in a printed catalogue of mathematical and physical instruments that belonged to Poleni's “Cabinet of physics” (Talas, 2013, p. 58), demonstrating that they were also constructed and used. The explicit goal behind Poleni’s project was to improve known methods to construct the tractrix and the logarithmic, such that these curves could be generated in a way not more complex than those employed for the conic sections, and thus fulfill the demands for constructability laid by previous authors, *in primis* Huygens himself.

The construction of the tractrix depicted in Fig. 2 (left) exemplifies this task (cf. Tournès 2009, pp. 72ff). In the instrument, a solid bar replaces the original chain in Perrault’s experience, and a toothed wheel (“rotula signatoria”) orthogonal to the plane of the curve replaces the weight. It is the wheel, and not the weight, that traces the curve on the plane with the correct property of tangents (“cujus rotatione curva signatur”, Poleni, 1729, § 26), acting through a single continuous motion.

The fundamental difference with respect to previous models and descriptions is the introduction of a wheel to guide the tangent to the curve. The wheel replaced Perrault’s original watch and served as a fundamental component to obtain a precise mechanical device to guide the tangent. With respect to the dragged point of Perrault’s construction, a wheel (working in a similar way as when we turn the front wheel of a bike) also greatly enhances the precision. In this way, the wheel maintains the bar always tangent to the curve, so that a tractrix can be traced.[32]

[32] This technical innovation is not merely related to the accuracy of the output. As highlighted in (Dawson, Milici & Plantevin, 2021), the principle underlying the wheel and the dragged point are mathematically different, even though they coincide in the simplest cases.

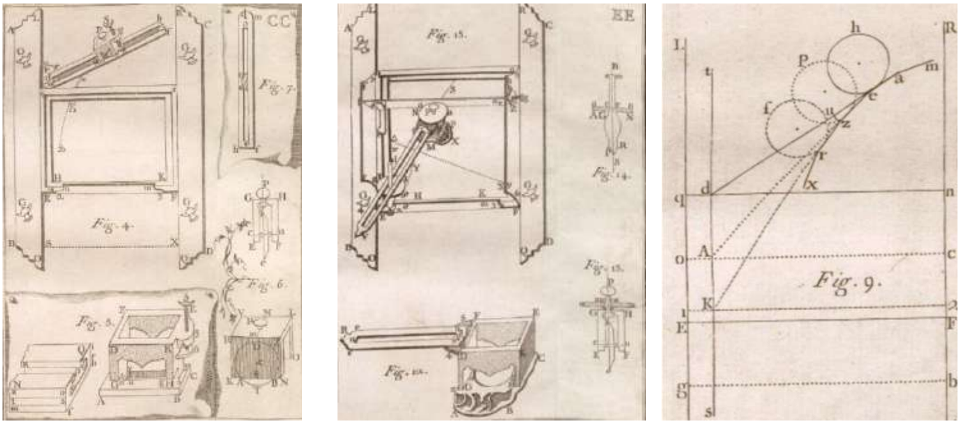



**Figure 2.** Poleni's drawings from his "Fasciculus". Left: a machine to trace the tractrix (the original chain in Perrault's watch has been replaced by a bar and the watch-case by a wheel). Center: a machine to trace the logarithmic curve. Right: the wheel marks infinitesimal segments corresponding to the direction of the tangents at each point.

Poleni employed a similar procedure to construct the logarithmic curve. As shown in Leibniz (1684), the logarithmic (or exponential) is a curve whose subtangents have constant length. The machine for the logarithmic, depicted in the center of Fig. 2, is then a variation of the instrument for the tractrix. The bar representing the tangent to the curve has a variable length, whereas a horizontal bar, namely the side of a moving rectangle in the figure, ensures that the subtangent to the curve is constant for any of its points. After a period of oblivion, geometric methods to solve the inverse tangent problem were independently rediscovered in the second half of the 19th century (Tournès, 2009, pp. 271ff.). At that time, the mechanical resolution of the inverse tangent problem was adopted not to legitimate or introduce specific curves, but to perform transformations related to the resolution of differential equations. Such machines are called "integraphs". They are devices that integrate functions introduced as geometrical curves. The main aim of these devices was to solve practical problems (e.g. finding solutions to differential equations that do not allow symbolic resolution). In at least two periods, during the 18th, and 20th centuries, machines for the construction of curves became part of cabinets with a pedagogical function. We find these in the 18th century "Cabinet of experimental philosophy" of Giovanni Poleni in Padua (Talas, 2013) and in the 20th century "Cabinet of Differential Calculus" of Ernesto Pascal in Naples (Tournès, 2009, p. 271ff.). It is significant that both scholars designed and constructed new devices. Even though there is no evidence that Poleni used them in his lectures, it is attested that he employed instruments in his classes of experimental philosophy (Talas, 2013, pp. 52ff.). Furthermore, machines in brass to trace the logarithmic and the tractix are mentioned in a printed list of Poleni's machines, and Poleni himself had samples built in order to circulate them among his colleagues.[33] Therefore, we cannot even exclude the possibility that Poleni might have used his machines to enhance students' understanding of the fundamental concepts of calculus. Manipulating curves with a machine would not justify the principles of calculus, but it may have helped pupils accept operations involving counter-intuitive objects such as infinitesimal segments and familiarize themselves with abstract relations and definitions, such as the new definition of tangent as the line connecting two infinitely close points. Thanks to the action of the wheel, in particular, these machines may have made concrete and visible a fundamental principle of the Leibnizian calculus: "a curve line can be considered composed by infinitely many small lines, or elements, infinitely small … which include angles, from which the curvature is generated" (Poleni, 1729, §44).[34] As Poleni himself noticed, the wheel marks the infinitesimal segments $ae$, $ez$ … (Fig. 2, on the right), corresponding to the directions of the tangents at each point of the curve, so that these "infinitely small lines are described by the motion of our instrument" (Poleni, 1729, §49). This example shows that, by making abstract

[33] Poleni sent exemplars of his machines to the mathematicians Gabriele Manfredi and Jacopo Riccati, and to his friend Antonio Conti (Tournès, 2009, p. 79). Although an exemplar of an alleged machine to draw transcendental curves is preserved in the Museum of History of Physics in Padua, its function and design are presently unknown. We thus have to conclude that none of Poleni's geometrical machines has survived until today in its original form. Following Poleni's tables and descriptions, models of the machines for the tractrix and the exponential have been recently reconstructed (Milici & Plantevin, 2022). A video of such a reconstruction is available online at the link www.youtube.com/watch?v=LIsQkML2Tis

[34] Poleni's definition ("lineam curvam … mex confiderari posse ceu compositam ex infinitis lineolis (sive elementis) rectis, infinite parvis, ma, ae, ez, zr, rx comprehendentibus inter se angulos, ex quibus lineæ curvature progignitur") is actually a postulate in L'Hopital treatise: "We suppose that a curved line may be considered as an assemblage of infinitely many straight lines, each one being infinitely small, or (what amounts to the same thing) as a polygon with an infinite number of sides, each being infinitely small, which determine the curvature of the line by the angles formed amongst themselves." (Bradley, Petrilli & Sandifer, 2015, p. 3).

notions of differential calculus concrete and more manipulable than the original devices using weights and strings, Poleni's machines can represent a suitable basis for didactical experiences.

## 3 The material artifact

With the idea that the historical machines discussed above can be possible vectors of "didactical ideas" (van Radenborgh, 2015, p. 6), in the workshop we merged a presentation of selected excerpts from the historical sources surveyed above with related laboratory activities involving material artifacts. Specifically, we proposed a "kit for calculus" (patented, invented and constructed by the second author) integrating the 18th and 20th centuries resolutions of inverse tangent problems, with a specific focus on the simplicity of the design. The machine, that can be called a "T-sliding integraph" (because there is a T-shaped rod with two perpendicular guides), is introduced in www.machines4math.com (also with some related videos); it is built by digital construction tools, and the source files (together with assembling instructions) are freely available at www.thingiverse.com/thing:5532958. A previous version of the kit is described in more details in the same volume (Maschietto and Milici, 2023).

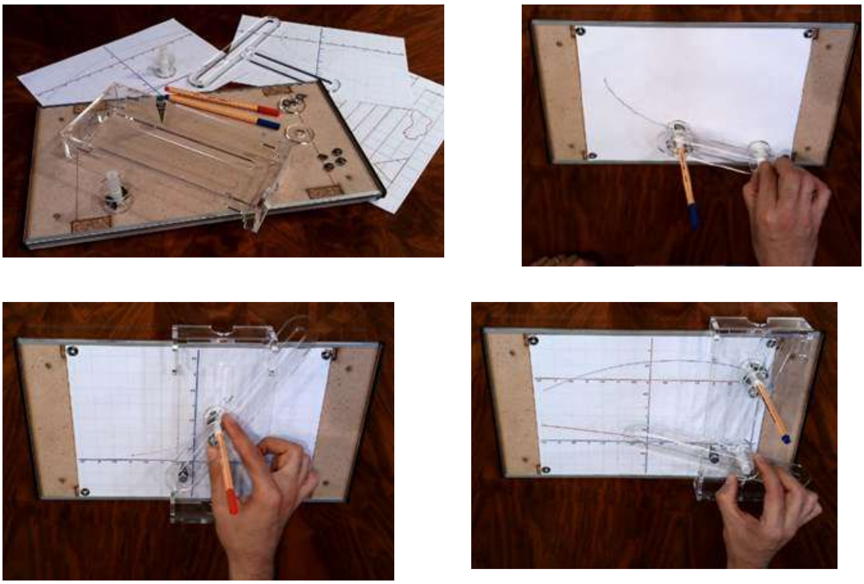


**Figure 3.** The proposed kit for calculus: its components and various uses.

The components of the proposed device are shown at the top left of Fig. 3. There are two pointers (one with a couple of wheels) that can be used to follow an already traced curve or trace a curve-to-be using a marker. There is also a wooden base (where paper sheets can be attached) and a transparent plastic case that can slide on it. Finally, there are two rods that can be joined to make a “T": these rods can be used as guides for the pointers. These components can be assembled in several ways as required for various possible activities.

### *4 The workshop*

In the workshop, we alternated the introduction of historical sources (with guided discussions) and laboratory activities (participants were divided into small groups with one kit per each group).

**4.1** First activity: ***the tangent***. After a historical introduction to organic geometry, we considered the historical problem of justifying the existence of transcendental curves in an organic way, that is, through an instrument. To demonstrate how the limits set by Descartes can be overcome, we proposed the following laboratory activity. After tracing an arbitrary curve on a sheet of paper, the audience was invited to move the pointers on the traced curve. The aim of this activity is to focus on the difference of use between the “smooth pointer” (the one whose bottom is smooth) and the “wheeled pointer” (the one whose bottom has two parallel wheels). The introduction of the wheels imposes the restriction that, to move the wheeled pointer on a curve, the piece must be rotated so that its direction is tangent to the traced curve. The audience then experienced the passage from direct to inverse tangent problems, by linking the direction of the wheels to a rod and moving the other extremity of the rod along a line (cf. top right of Fig. 3). After a brief description of the experience of Perrault’s watch and some historical references to Huygens and Leibniz, the audience was guided to reenact the construction of the tractrix in the manner of early modern geometers and link the mechanical components to the possibility of tracing a curve given its tangent properties.

**4.2** Second activity: ***the exponential*** (cf. bottom left of Fig. 3). In this activity, the audience was invited to assemble the kit for calculus in a way that imposes the constant subtangent. After presenting Poleni’s letter to Hermann and the tables reproduced in fig. 2, the audience realized that they were handling a modern version of Poleni’s machine for the logarithmic curve. The guided analysis of the machine resulted in the construction of an exponential curve without introducing notions of limits or summations (from an analytic perspective, the machine geometrically solves a differential equation). As a variation, if one imposes the direction of the wheel not as the one passing through the peg fixed on the sliding case but perpendicular to it, the device can trace parabolas (cf. Maschietto, Milici & Tournès, 2019). In this case, with a suitable choice of reference frame, the differential equation is easily converted into an integral.

Third activity: ***derivatives and antiderivatives*** (cf. bottom right of Fig. 3). In this case, a sheet with two reference frames, one for each pointer, was adopted (cf. Maschietto & Milici, 2023, §2). This activity ideally follows the first one, in which one can show that, when the wheeled pointer follows a curve, the direction of the wheels must be the tangent to the curve. Furthermore, the T-shaped rod imposes the restriction that the direction of the tangent must be parallel to the line passing through the peg fixed on the case and the other pointer. With a suitable reference frame, the ordinate of the smooth pointer can be shown to represent the slope of the tangent to the curve followed by the wheeled pointer. This geometric configuration corresponds to the definition of the derivative. In contrast to the previous case, if the smooth pointer is moved, the wheeled pointer traces an antiderivative. This activity is in the direction suggested by Blum (1982), according to which integraphs may be used to make students discover the fundamental theorem of calculus by themselves (e.g., by approximating piecewise the function to be integrated by many constant functions). The third activity was integrated with historical notes on integraphs.

## *5 Conclusions*

Geometric instruments for curve tracing can play an important role in didactics, because they are artifacts that embody multiple ideas: historical, mathematical, technical, and pedagogical. In this article, we presented a possible convergence of historical considerations and hands-on activities. We began with a case study to introduce historical machines for the construction of certain transcendental curves. These machines embody a mathematical idea, namely the concept of a curve known through the property of its tangents (for a modern mathematical setting of these machines see Milici, 2020). Then, we proposed a series of activities presented during a workshop at the latest ESU-9 in Salerno,  based on the suitable use of a machine to present some key concepts of differential and integral calculus. Even though these experiences do not directly reproduce historical episodes, they are devised in the same spirit: to use concrete artifacts to visualize and anchor abstract concepts, such as the notions of tangent and slope, or epistemologically difficult ones, such as that of an infinitesimal segment. The workshop highlighted the fact that most participants, who are teachers or researchers in history and didactics, were sympathetic with the rationale behind this workshop, and with the importance of developing more concrete, engaging and intuitive approaches to calculus than the standard ones. They also agreed that the activities presented at the workshop  could be fruitfully applied in secondary education. However, this step is far from obvious, as several questions during the discussion following our presentation have pointed out. A major critical aspect concerns the use of the calculus kit as an exploratory device and can be summarized as follows. To enhance students' understanding of calculus, didactical activities should be carefully designed so that students are guided in the process of discovery and each step of their activity is duly motivated in terms of what the questions they should pose and the conclusions they should achieve through the manipulation of the device. Therefore, we also plan to find new collaborations to improve these underdeveloped aspects of our proposal and to bring this historical and tangible approach to calculus in high schools.

Acknowledgements: This article was made possible thanks to the support of the European Union’s Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement
No 101024431-LEGITIMATH.

**REFERENCES**

Bartolini Bussi, M. G., & Maschietto M. (2007). *Macchine matematiche: Dalla storia alla scuola*. Springer Science & Business Media.

Bos, Henk J. M. (2000). *Redefining Geometrical Exactness*. Springer.

Bos, H.J.M. (1988). Tractional Motion and the Legitimation of Transcendental Curves. *Centaurus*, 31, 1,9–62. DOI:10.1111/j.1600-0498.1988.tb00714.x

Blåsjö, V. (2017). *Transcendental Curves in the Leibnizian Calculus*. Elsevier Academic Press.

Blum, W. (1982). Der Integraph im Analysisunterricht - Ein altes Gerät in neuer Verwendung. *ZDM Zentralblatt für Didaktik der Mathematik*, 14, 25–30.

Bradley, R. E, Petrilli S. J., & Sandifer C. E. (2015). *L'Hôpital's Analyse Des Infiniments Petits*. Vol. 50. Science Networks. Historical Studies. Cham: Springer International Publishing. DOI:10.1007/978-3-319-17115-9.

Crippa, D., & Milici, P. (2019). A relation between tractrix and logarithmic curve with mechanical applications. *The Mathematical Intelligencer*, 41 (4), 29–34. DOI: 10.1007/s00283-019-09895-7.

Dawson, R., Milici P., & Plantevin F. (2021). Gardener's hyperbolas and the dragged-point principle. *American Mathematical Monthly*, 128:10, 911–921. DOI:10.1080/00029890.2021.1982634.

Descartes, R. (1637). *Discours de la méthode pour bien conduire sa raison, & chercher la verité dans les sciences. Plus La dioptrique, les météores et la géométrie, qui sont des essais de cete méthode*. Leyden: De l'imprimerie de Jan Maire.

Leibniz, G. W. (1684). Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, & singulare pro illis calculi genus. *Acta Eruditorum Anno MDCLXXXIV [-Anno MDCLXXXV*]: 467–473.

Leibniz, G. W. (1693). Supplementum Geometriae Dimensoriae, Seu Generalissima Omnium Tetragonismorum Effectio per Motum : Similiterque Multiplex Constructio Lineae Ex Data Tangentium Conditione. *Acta Eruditorum Anno MDCXCIII,*: 385–92.

Maschietto, M., & Milici, P. (2023) Get in touch with calculus: A new material device collecting a historical legacy, *ESU9 proceedings* (this volume).

Maschietto, M., Milici, P. & Tournès, D. (2019) Semiotic potential of a tractional machine: a first analysis, In: U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (CERME11, February 6 – 10, 2019), Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME, pp. 2133–2140.

Milici, P. (2020). A differential extension of Descartes' foundational approach: A new balance between symbolic and analog computation. *Computability*, 9(1), 51–83. DOI:10.3233/COM-180208.

Milici, P., & Plantevin, F. (2022). Reconstructing Poleni's machines for transcendental curves. *Bulletin of the Scientific Instrument Society*.

Monaghan, J., Trouche L., & Borwein, J. M. (2016). *Tools and Mathematics*. Vol. 110. Mathematics Education Library. Cham: Springer International Publishing. DOI:10.1007/978-3-319-02396-0.

Panza, M. (2011). Rethinking Geometrical Exactness. *Historia Mathematica*, 38, 1: 42–95. DOI:10.1016/j.hm.2010.09.001.

Poleni, G. (1729). *Epistolarum Mathematicarum Fasciculus*, Padova.

Randenborgh, C. van. (2015). *Instrumente der Wissensvermittlung im Mathematikunterricht*. Wiesbaden: Springer Fachmedien. DOI:10.1007/978-3-658-07291-9.

Talas, S. (2013). New Light on the Cabinet of Physics of Padua, In Bennet, J., Talas, S. (Eds.), *Cabinets of Experimental Philosophy in Eighteenth-Century Europe*, Brill: 49–67. DOI:10.1163/9789004252974_005.

Tournès, D. (2009). *La construction tractionnelle des équations différentielles*. Blanchard, Paris.

Turner, A. (2014). Nicolas Bion, Globe-Maker, Instrument-Maker, Author and Business-Man. *Globe Studies*, 59/60: 198–218.

Vollrath, J. (2003). Zur Erforschung mathematischer Instrumente im Mathematikunterricht, in Hefendehl-Hebeker, L., Hußmann. S (Eds.), *Mathematikdidaktik zwischen Fachoreientierung und Empirie, Festschrift für Norbert Knoche, Hildesheim (Franzbecker):* 256–265.

# GEOMETRICAL ACTIVITIES FOR CHILDREN WITH INTELLECTUAL DISABILITIES INSPIRED IN EDOUARD SEGUIN´S (1812-1880) APPROACH.

**Elena GIL-CLEMENTE, Almudena AGUDO CARNICER**
Universidad de Zaragoza, Pedro Cerbuna 12, Zaragoza, Spain
elenagil@unizar.es, almuagcar@gmail.com

## ABSTRACT

This workshop takes inspiration from the work of Édouard Séguin (1812-1880), the French pedagogue and philosopher of education generally considered as the outstanding pioneer of special needs education and disability studies and is addressed to Primary and Lower Secondary teachers (both future and in-service).
In the workshop the participants will experience his research on the human mind and geometrical concepts. Trying to overcome the isolation of the so-called "idiot" children (in fact, the Greek stem of the word makes reference to their difficulties understanding and communicating with the outside world by means of speech), he devised an educational path focused on the awakening of awareness. His physiological method included smart, carefully designed activities intended to awake the five senses and develop "notions". The passage from notions to "ideas" asked for connections between perceptive notions: for this, Séguin turned to geometry: equal, greater than, composition and decomposition were the starting actions and comparisons that he proposed to children, using wooden bricks, rods, and frames (a context of 3D geometry). These materials were crucial to establishing a joint frame of intentions, to prompt the child's activity, intelligence, and finally his will. The workshop is organized in three steps and encapsulates the reading of some excerpts from his work and the presentation of a leisure time mathematical workshop held in Spain with children with Trisomy 21 following Seguin´s approach

## 1 *Who was Edouard Séguin? A pioneer in the use of geometrical means to break the isolation of idiot children in nineteenth century.*

It is convenient to start the workshop with some information on the figure of Edouard Séguin, his personality, his broad culture, and his humanity being moved by the situation of cognitively impaired children and adults in France in the nineteenth century.

Edouard Séguin had a polyhedric personality (see Gil-Clemente & Millán Gasca, 2022 for more information). He was a doctor and pedagogue, a man of action but also a deep thinker. His convictions that children with disabilities can be educated and deserve, like the rest of children, an integral education that makes them free people with their own thoughts permeates all of his work.

His books are tightly linked to life. He used to say that they are a unique book rewritten in several moments, taking into account his progressive experiences working with "idiot" children.

Personal circumstances lead him to work with the renowned physician Jean Itard, in the education of Adrien, a so-called "idiot". This changed his life devoting it to the upbringing of children that were institutionalized and mixed with people who had different mental illnesses. He wrote his two first short book in 1839 after the experience with Adrien and the third in 1842 after three months of work at Hospice des Incurables in Paris with a group of nine children (see references for the tittles). He ended his first block of books with the one he wrote in 1843, two years after leaving Bicetre Hospital, because of disagreements with the director.

In 1846, he wrote his last book in French where he reaffirmed and further developed the ideas stated in the previous books. In 1850, he emigrated to the USA due to a mixture of political reasons and lack of recognition. There, in 1866, he systematized his method and in 1873, he also wrote a final book, a bit different from the others, after Vienna World's Fair.

His ideas about education are born out of this personal experience of encounter with "idiots" and from considering them as human beings that deserve an integral education. He put all his previous background (he was a cultivated person, with studies of medicine, law and art) at the service of their education.

He opposed method to chance, and he designed a clear path to follow in the training of idiots: beginning from the education of the muscular system, following by the education of the nervous system, by which they acquire notions for arriving to ideas first and finally to morality. The bases of his method are: 1) the triad *action, intelligence, will*, the goals of his paideia. Acquisition of will is a guide for him. 2) The crucial opposition *myself /not-myself*, being the transition between both, a key in the process of growing. This opposition is a guide for the design of educational exercises based upon imitation: personal imitation for the developing of notions on *myself*, and impersonal or objective for the developing of notions on *not-myself.* In this educational path, Séguin attributes a central role to Geometry. Geometry act as a bridge to construct the necessary relationship, between myself and my non-self His work is better understood in the wider context of the early initiatives to stablish compulsory primary education for citizens in modern states in XIX century. The introduction of geometrical contents to make the instruction evolve from rote training to methods that help understanding is one of the most relevant aspects in these initiatives. In this way the formative role of mathematics is extended to all and not only to those who will continue higher-education. In this “all”, the originality of Séguin was to include “idiots”.

## *2 The workshop proposed*

The workshop is organized in three steps: 1) A starting activity to enable participants to get to grips with the meaning of disability; 2) A journey through the learning path proposed by Séguin with special emphasis on the role he attributes to Geometry. This phase is conducted by the leader of the workshop and it combines reading some excerpts from Séguin’s work with commenting the most significant issues with participants; 3) An audiovisual presentation, of a collection of geometrical activities for intellectually disabled children, designed following Séguin’s approach and held in Spain.

The workshop has been designed following the experiential *ANFoMAM* workshops, for training Pre-school and Primary School teachers (Lizasoain Iriso, Magrone, Millan Gasca, et al. 2022) and it is recommended for 20 participants. It is to be held in a classroom with an open plane space, chairs and tables equipped with a projector. During the workshop, participants will be provided with a dossier with a wide selection of texts to help them understand the learning path proposed by Séguin which includes a timeline of his life and work

For the presentation of the workshop in these proceedings we have opted to: 1) explain the starting activity; 2) describe the main stages of the path proposed by Séguin introduced by one of the selected excerpts and including some keys about the activities developed in Spain for children with intellectual disabilities.

### 2.1. Starting activity: a blind comparison of lengths.

We ask participants to form groups of four people, look for a place in the room and sit on the floor. One of them is going to be the observer and the others are going to close their eyes using a blindfold.

We give the observer a set of ten wooden bars of increasing length from 10 cm to 1 m. The members of the group have the mission of ordering them. While they face the challenge, the observer can help the group only with practical issues. We ask the observer to watch carefully what the group does and which is the process they follow. We give them time to order the bars (more or less 10-15 minutes) and afterwards we invite the participants to share their experience. The leader of the activity says: "You have experienced firsthand what it means to face the challenge of being deprived of one sense: the sight. We encourage you to share: 1) the process followed for ordering the bars, 2) what other senses or capacities did you have to put into play; 3) any preliminary reflection of what disabilities means".

We expect certain ideas to appear such as the use of tact or the need to match the ends in order to compare their length.

We finish the activity explaining that this is an exercise that Séguin describes as infallible to help "idiotic children" to connect their inner and outer worlds, and finally reading together the following excerpt from his 1843 book written after his work with nine children in the Hospice des Incurables in Paris

*On fait scier vingt règles: la première a 5 centimètres de longueur, la seconde en a 10, la troisième, 15, etc., jusqu'à la vingtième, qui en a 100. Chaque intervalle de 5 centimètres est indiqué sur les quatre faces de chaque règle par un trait [à la scie et au crayon] noir. On commence par poser l'une à côté de l'autre la plus grande, le mètre, et la plus petite, celle de 5 centimètres. On demande alternativement à l'enfant la plus grande et la plus petite. On en ajoute une troisième, la moyenne; puis on rapproche les extrêmes jusqu'à ce que ces règles [offertes à l'appréciation visuelle de l'enfant] ne différent plus [entre elles] que par leur différence progressive de 5 centimètres, toujours en demandant la petite, la moyenne, la grande. Enfin on jette confusément à terre toutes ces règles, et on demande à l´enfant la plus petite, la suivante, la suivante, etc., jusqu'à la dernière, la plus petite. Enfin on jette confusément à terre toutes ces règles, et on demande à l'enfant la plus petite jusqu'à la plus grande, ou la plus grande jusqu'à la plus petite, et quand il les prend et les range ainsi progressivement de la première à la dernière, on peut compter que le regard, habitué à ce genre de comparaison, saura l'appliquer ensuite à tous les objets, ce dont on s'assure d'ailleurs par l'expérience (Séguin, 1843)*

The last sentence is especially interesting because points out his broad goals: he uses an educational material but aims to getting children used to comparing objects in general.

### 2.2. Educational path proposed by Séguin. Selected excerpts and recent reformulation for children with Trisomy 21

In order to describe the method Séguin proposed, we have chosen his first four works collected by Bourneville in 1897: *Premiers memoires de Séguin sur l'idiotie (1838-1843).* In them, we can see the young Séguin, the curious Séguin who learns after observing, the courageous Séguin who goes against the tide, moved by what he sees at the Hospice des Incurables and at Bicétre, the one that uses all his background to go out to the encounter of the young people living there and to try to bring them out of their isolation.

In this paragraph we describe the main steps of the educational path proposed by Séguin trying to bring to light their geometrical basis. In each step we combine the more relevant selected texts that can be discussed by the participants and the results of a research developed in Spain about the mathematical education of children with Trisomy 21 aged between 3 and 15 years old (Cogolludo & Gil-Clemente, 2019; Agudo, Cogolludo & Gil-Clemente,2021; Gil-Clemente, 2020; Gil-Clemente, 2022), that appear in this audiovisual presentation (https://www.youtube.com/watch?v=mEXPc-rr3Oc)

#### 2.2.1 Standing still, alignment and march.

> *Trouvant un corps agité de mouvements convulsifs et incessants, je l'ai condamné à une immobilité d'un mois; et l'immobilité était le seul point du levier sur lequel on pût s'appuyer pour obtenir une action régulière. La marche du soldat, l'imitation de divers mouvements de la tête et des bras ont commencé à donner à l'enfant les notions du moi. (Séguin, 1838)*

In his first encounter with Adrian and the other nine children in Hospice des Incurables, Séguin speaks about their convulsive or chaotic movement. He designed exercises, meant to help children in the passage from this convulsive movements to regular movements in time and space. They have clear geometrical basis and are the core of the so-called *personal imitation,* for the formation of notions of *myself.* He designed also some stopgaps for facilitating the immobility needed to learn (see figure 1a)). Séguin links these exercises with the arising of awareness (linked with the modern concept of "proprioception", that means to be conscious of the position of our limbs and our movements).

*Activities with children with Trisomy 21*

The work with children with trisomy 21 begins with motor activities of geometrical inspiration. The possibility of learning starts when children are able to distinguish stillness from motion, when they are able to control their convulsive movement and line up in a straight line with other children and march in a coordinated manner "like soldiers". In Geometry, the primitive concepts are point and line, as Euclides in s.II b.c and Hilbert in XIX century pointed out, in a clear parallelism intuited by Séguin. The practice with these exercises help them to interiorize the primitive concepts: point, straight line, and plane and prepare them to abstraction. (see figure 1b)


**Figure 1.** a) Stopgaps for the progression to standing still (our design), b) Children aligned prepared for marching

### 2.2.2 Generation of lines

> *Liant la base d'une verticale à une horizontale, je les réunis aux extrémités opposées par une oblique, et l'élève exécute un triangle rectangle; quatre triangles réunis à leur sommet nous donnent un carré parfait, du centre duquel on efface ensuite les lignes obliques qui lui donnent la figure d'un sablier: puis on le trace avec des parallèles seules, après quoi, déployant les courbes autour d'une ligne droite quelconque, l'enfant produit le cercle complet; et enfin, compliquant toutes ces notions (si simples en apparence, mais si précieuses dans l'espèce), l'élève produit des figures infiniment agglomérées sans omettre les moindres détails, sans confondre les directions, les points de conjonction, les rapports de grandeur et de disposition des parties entre elles et du tout, et cela se conçoit: toutes ces figures sont exécutées méthodiquement, en partant d'une ligne*

*qui sert de base à une seconde sur laquelle s'appuie une troisième, etc.* (Séguin, 1843, p.133*)*

Séguin devoted much attention to the problem of *generation of lines* with a double intention: firstly as a way to prepare them for the graphical and motor aspects of reading and the painful activity of writing; secondly as a way to stablish communication in absence of speech.

He describes his initial work with Jean Itard trying to teach a single boy to draw a square and their failure (see figure 2a)). Although this boy knows how to draw lines it was impossible for Séguin and Itard to communicate with him and explain how to close the square. This leads Séguin to the discovery of the problem underlying this failure: the lack of awareness of the concept of plane and the need to design exercises to make children reach a reasoned stroke: to draw a line with a determined direction implies choice and will. A geometrical concept contributes again in his vision to the development of a free individuality.

*Activities with children with Trisomy 21*

Séguin identified the problem as lack of communication with the child: how to explain without words the need to cut two straight lines in order to close the square? Research with children with intellectual disabilities (see 2.2. for references) shows that the use of mimesis is a key to awake understanding. In a story of superheroes where the straight lines are lightning rays that have to touch each other to defeat a villain, children understand the idea (see figure 2b)) and afterwards are able to draw it on paper.

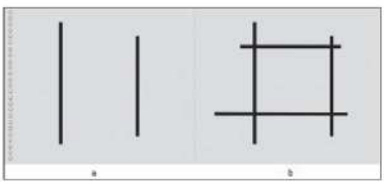


**Figure 2a.** A reconstruction of Séguin´s exercise *drawing a square*, **Figure 2b.** Children play to touch their lightning rays

### 2.2.3 Dimension, configuration and layout

Progressively Séguin evolves from a utilitarian use of form (reading and writing) to a more formative one that conceives the work with geometry as a key for the awakening of awareness. Reading and commenting the following sentence can be very valuable.

Il a fallu user des moyens que je crois pouvoir appeler les *forceps de l'intelligence (Séguin, 1842,p.45)*

Séguin proposes the transition to exercises of *objective or impersonal imitation* as a turning-point in the educational process. These exercises involve 3D geometrical operations and relationships that contribute to the awakening and development of intelligence. He notes that these exercises that *connect the myself with what is non-myself*, contribute to build the individuality, and in this way they open the door to building ideas, that are particular of each person

a) The first exercises refer to *dimension.*

> *Les enfants semblent posséder cette notion plus que les autres; mais elle n'implique pas en eux comme chez l'homme, l'idée d'une échelle métrique…*
>
> *…. Aussi n'ont-ils pas de degré de comparaison, et ne connaissent-ils que les extrêmes. Quant aux idiots, ils mesurent les distances en raison de leur paresse, et les trouvent toutes trop longues: la quantité de leurs aliments en raison de leur gourmandise, aussi la trouvent- ils toujours insuffisante...(Séguin, 1843, p.119)*

He says that for idiotic children it is easier to perceive the dimension of objects, not with a metric scale with numbers, but by comparison, one of the central intellectual operations in his view. He is speaking about additive comparison: which one is biggest, which ones are equal. Finding the longer one between a pair of rods is the first step to ordering them by length (see figure 3a)). From a mathematical point of view it corresponds to a "total order relation".

*Activities with children with Trisomy 21*

To compare is an abstract process that implies to find a similarity between two objects. Geometrical comparison is an intellectual operation that is linked to physical actions (putting the ends of two strips side by side to compare the unmatching ends or superimposing two similar surfaces to see which one is bigger; or using a scale to decide which object is heavier…) that helps the encounter between myself and not-myself. The research (see 2.2. for references) shows that it is easier for children with intellectual disabilities to compare for example the length of rods similar to those proposed by Séguin than to compare their own height. (see figure 3b)). This regular material leads also naturally to the idea of geometrical ratio (being a half, a quarter, a third, the double, and so on) which we have also explored. Discovery of ratio relationships encourage children to talk, which also helps them get out of the isolation Séguin spoke about. Geometry actually links myself and not-myself.

**Figure 3a.** Rods set for additive comparison proposed by Séguin, **Figure 3b.** Playing to order rods

b) The second group of exercises refer to *configuration.*

> *Tous, en effet, connaissaient par leur usage un certain nombre d'objets usuels comme une table, un couteau, un habit, etc.; mais ils ne les connaissaient pas par l'analyse raisonnée de leurs diverses parties; il n'y avait, par conséquent, nulle différence pour eux entre une veste et un habit, une table ronde ou carrée. Cette confusion semble au premier coup d'œil peu importante, [...]; il fallait l'ajouter aux notions, précédemment acquises à l'aide des figures simples* (Séguin, 1842, p.50)

They are designed to perceive the form of the objects starting from contrast to analogies (see figure 5a)). Exercises for analyzing the parts of the objects are very useful to distinguish shapes by the form and not only by their use. Awareness of form is a key in the transition to ideas; and regular forms are a mean for understanding the physical world. Thus the formative value Seguin attributes to geometry appears clearly.

*Activities with children with Trisomy 21*

The recognition of regular shapes in everyday objects is a rich source of activities with them. These activities help them put order in a world that they perceive as chaotic due to their sensorial difficulties. Being able to discover what different objects have in common independently of their use helps them become more in charge of their own lives (see figure 5b)).

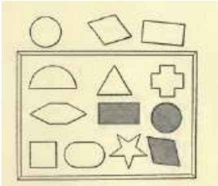


**Figure 5a.** Séguin form -bord (Hull 1913, p.1) **Figure 5b.** Discovering the form in everyday objects

c) The final block of exercises corresponds to *layout* or *arrangement,* that is to put objects in any position you can imagine.

> *The child being in front of the teacher, a table between them, a few blocks piled near their right hands, the teacher takes one, puts it flat before him on the table, and makes the child do the same. The T. puts his block in various positions relatively to the table and to himself, and shows, not directs, the C. to do the same. The T. puts two blocks in particular relative positions, and the C. does the same each time. What was done with two blocks is done with three, with four, with more, in succession, till the exercise of simple imitation becomes quite intellectual, requiring at least a good deal of attention and power of combination. Later, the T. creates combinations of two or more blocks at once, and the C. must imitate all of it at once; and finally the T. creates a combination of a few blocks, destroys it, and asks the C. to make the same construction, the model of which he can now find only in his mind.* (Séguin, 1866, p.166)

The key idea is that of *constructing* (which geometrically means *adding*, or involves an idea on position in space linked to dihedral angle) and *destroying* (geometrically, *decomposition*). Séguin resorts again to a 3D object, the planchettes or dominoes (see figure 6a)).

*Activities with children with Trisomy 21*

The aspect of playing is very important. By playing, children are working, without knowing, with Eulidean concepts linked to physical actions: by pulling the bricks together, they can see they are congruent, for example. In this way children learn to combine parts to obtain a whole (see figure 6b)). The combinations of objects they make with their hands will later be reproduced in the intellectual order, combining ideas.

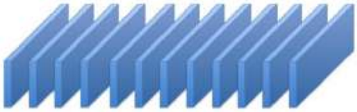


**Figure 6a.** The bricks set, ca. 27 x 13,5 x 2,7 cm: 12 dominos for Adrien, 60 planchettes for the Hospice des Incurables, **Figure 6b.** A boy with Trisomy 21 exploring symmetry

## Final remarks

This workshop was attended by about 12 participants, among them some university colleagues of different nationalities who deal with the history and didactics of mathematics and some schoolteachers. They valuated the possibility of empathize with people with disabilities in the first activity. They also stated that the activities proposed by Séguin and updated in the workshop in Spain were of great help for all students, not only those with intellectual disabilities, to develop some skills. The ongoing research based on the historical analysis of Séguin´s work presented in this workshop is pioneer in the field of mathematical education for people with intellectual disabilities. Séguin ´s views about the formative value of Goemetry and of the virtues of mimesis as a way of learning are inspiring now a research involving the development of expressive skills in children with Trisomy 21.

## REFERENCES


Agudo Carnicer, A, Cogolludo Agustín, J.I.. & Gil -Clemente, E.(2021). Awareness of symmetry, order, and stability in children with Down syndrome. *Journal Of Applied Research In Intellectual Disabilities*, *34 (5)* 1223-1223.

Cogolludo Agustín, J.I. & Gil-Clemente, E (2019). The Effectiveness of Teaching Geometry to Enhance Mathematical Understanding in Children with Down Syndrome. *International Journal of Disability, Development and Education, 66(2*) 1–20.

Gil-Clemente, E. (2021). *Matemáticas que suman: didáctica para la iniciación matemática de los niños con discapacidad intelectual*. Editorial Horsori.

Gil Clemente, Elena (2022) *Talleres temáticos para la educación matemática de niños con discapacidad intelectual: guía multimedia*. Servicio de Publicaciones. Universidad de Zaragoza. (French and Italian translation in https://www.unavarra.es/anfomam/recursos/aula-inclusiva)

Gil-Clemente, E. & Millán Gasca, A. (2022). Geometry as "forceps of intelligence": lines, figures and the plane in Édouard Séguin's pedagogical thought. *Bollettino di storia delle scienze matematiche, 41(2)*.315-339.

Hull Sylvester, R. (1913) *The form board test*. Thesis presented to the Faculty of the Graduate School of the University of Pensylvania for the degree of Doctor of Philosophy

Lizasoain Iriso, M.I., Magrone, P., Millan Gasca, A., et al. (2022). *Growing with mathematics: experiential math workshops for education of pre-school and primary school teachers a teaching guide for professional development and instruction of future teachers*, Universidad Pública de Navarra, Pamplona. https://academica-e.unavarra.es/xmlui/handle/2454/42415

Séguin, É. (1838). *À Monsieur H***. Résumé de ce que nous avons fait depuis 14 mois, du 15 février 1838 au 15 avril 1839*. Porthman.

Séguin, É (1839). *Conseils à M.O. sur l´education de son fils.* Porthman.

Séguin, É (1842). *Théorie et pratique de l'éducation des enfants arrierés et idiots (leçons aux jeunes idiots de l'hospice des incurables).* Germer Baillière.

Séguin, É (1843). *Hygiène et éducation des idiots.* Jean Baptiste Baillière

Séguin, É (1846). *Traitement moral, hygiene et éducation des idiots et des autres enfants arriérés ou retardés dans leur développement*. Jean Baptiste Baillière

Séguin, É (1866). *Idiocy, and its treatment by the physiological method*. William Wood &Co.

Séguin, É (1875). *Report on education*. (Facsimile reproduction, 1976 con una introducción de ROSEN, M.)

# DIRECT COMPARISON BETWEEN OBJECTS

## Discrepancies between the ancient and the modern world

**Ernesto ROTTOLI, Gianstefano RIVA**

Gruppo di Ricerca sull'insegnamento della matematica per la scuola primaria, Università Milano Bicocca, Milan, Italy

ernerott@tin.it

**ABSTRACT**

Philosophical practice on "direct comparison between objects" indicates an unusual approach to the familiarization of primary school children with fractions.

## *1 Introduction*

The workshop is divided into three parts, each of which is based on activities of direct comparison between objects: 1) Direct comparison as addressed by Davydov. 2) Direct comparison as performed by the Pythagoreans. 3) Activities of direct comparison as the source of the concept of logos/ratio. The aim is to propose moments of philosophical practice related to these activities. As an introduction to the workshop, after some brief information about our research group, we have presented: 1) our idea of philosophical practice and 2) the basic activity of direct comparison between objects.

### 1.7 Philosophical practice

Accompanying classroom practice with philosophical practice characterizes our Group, which has met for more than twenty years at Milano Bicocca University. Our effort aims to keep philosophical practice at the heart of teaching practice.

We have introduced this topic, proposing to discuss the prominent meanings we attribute to the term "philosophical": a) bringing out questions, b) pursuing new answers, c) transmitting new meanings.

### 1.8 Direct comparison between objects

We met direct comparison activities while addressing the question of the long persistence of unsatisfactory results in teaching and learning fractions. This question was both directly affecting our didactic practice and persistently emerging from the scientific literature. At the same time, we found in the scientific literature Davydov's proposal about direct comparison activities: he indicated these activities as the source of the concept of fractions. He proposed a series of situations that led primary school children from direct comparison to familiarization with the concept of fractions. (Davydov et al., 1991) We have explored Davydov's situations, starting to apply them in some primary school classes and accompanying this classroom practice with philosophical practice.

## *2 First part of the workshop: Direct comparison between objects as addressed by Davydov.*

### 2.1 Direct comparison between two mugs of different shapes

We started this part of the workshop by proposing the activity of direct comparison between two mugs of different shapes. The answer was of complete "dépaysement". This answer is not surprising because it is like the answer given by our students in laboratories of "Scienza della Formazione Primaria" at the University of Milano Bicocca. Having to compare real objects instead of mathematical objects, brings out meanings of the adjective "direct", to which the mathematics teacher is often not accustomed. We will see that this "dépaysement" is closely linked to the history of direct comparison in the Western world.

### 2.2 From direct comparison between objects to comparison of their measurements.

To understand how Davydov proceeds to answer the problem of direct comparison, we have proposed to the workshop some texts by Sierpinska.

*a)* *"Davydov spent a lot of time thinking about the meaning and the sense of fundamental mathematical concepts taught in grades 1-3, such as number, multiplication, and fractions, coming up with activities on which these concepts have their source."* (Sierpinska, 2019)

b) *"Children were first introduced to numbers in the context of direct comparison (by juxtaposition, superposition) of objects relative to qualities such as length, width, height, weight, etc."* (Bobos & Sierpinska, 2017)

*c)* *"Children were taught to record the results of their comparisons using symbols such as +, <, >."* (Bobos & Sierpinska, 2017).

d) *"Next, the situations were changed to make the direct comparison of objects difficult or impossible; this forced the children to measure each object using sticks, pieces of string, and other devices and compare the measurements, not objects."* (Bobos & Sierpinska, 2017)

So Davydov solves the problem by moving from the direct comparison of objects to the comparison of their measurements.

This solution proposed by Davydov is identical to that generally proposed in the modern Western world. It is the same solution we have adopted in our classroom practice: grade 3 children (8-year-old) carry out direct comparison activities by comparing the measurements of the objects. We showed it by photocopies of the record of activities carried out in the classroom.

Bringing the direct comparison between objects back to the comparison of their measurements, makes the aforementioned "dépaysement" disappears.

## 2.3 Two key features of Davydov's proposal

Before moving on to the second part of the workshop, we have presented some texts that highlight two very significant features of Davydov's proposal: 1) his idea of abstraction/generalization and 2) his use of symbolic language in primary school.

As for abstraction/generalization, we have proposed the following texts:

*"... The generalization reduces the diversity in the specific examples. Davydov argues that we ought to conceive of learning differently. The specific examples should be seen as carrying the generalizations within them; the generalization process ought to be one of enrichment rather than impoverishment."* (Kilpatrick, 1990)

*"Davydov's views on abstraction, his ascent to the concrete, refers to the development of an idea via a dialectical "to and fro" between the concrete and the abstract."* (Monaghan, 2016)

In our didactic practice of fractions in primary school, we proposed abstraction/generalization as an ascent to "the objective content" of fractions.

As for use of symbolic language in primary school, we followed Davydov's suggestion, but not as systematically as he does. Our aim is to develop "friendliness with symbols" in children. At this point, we presented to the workshop our idea of "friendliness with symbols", which results from the synthesis of two utterances: a) "*Friendliness with numbers precedes number sense*" by Howden, and b) "Making students friends with symbols" by Arcavi.[35] In the second part of the workshop, we will see the role of friendliness with symbols in coming to a writing that favors the ascent to the objective content of fractions.

## *3 Second part of the workshop: Direct comparison between objects as addressed by Pythagoreans.*

As was customary throughout the ancient world, the Pythagoreans performed the comparison between objects directly, without bringing it back to the comparison of their measurements. That comparison, which has the goal to seek the highest common unit between the two compared quantities, was named "Anthyphairesis" by the Greeks.

### 3.1 Activities on anthyphairesis

First, we presented to the workshop the following text by Fowler: "*Anthyphairesis = anti-hypo-hairesis, 'reciprocal sub-traction'. It is described in Euclid Elements X,2): "… when the less of two unequal magnitudes is continually subtracted in turn from the greater, …". The phrase 'is continually subtracted in turn from', describes the anthyphairesis*".

Then a PowerPoint has been projected, in which the mutual subtraction of two quantities is shown step by step.

Subsequently, the participants performed the direct comparison of two natural numbers, being careful to write the development in two columns. This development will constitute a central text in the third part of the workshop.

### 3.2 Philosophical practice on anthyphairesis

We drew the attention of the participants to the fact that while the objective practice of anthyphairesis indicates this activity as elementary and apparently scarcely significant, the philosophical practice brings out its profound meaning. To this, we showed to the workshop two Leont'ev's statements [Sierpinska, 2002].

[35] The path towards friendliness with symbols develops through activities on change: a) to register the change; b) to discuss and describe the change; c) to recognize the change; d) to get the quantity that produces the change. [Bonetto et al., 2008]. We have shown the workshop some of these activities.

The first statement is: "*Meaning belongs first of all to the world of objective-historical phenomena*". As for the "objective" aspect of phenomena, Davydov points to the direct comparison as the source from which the search for the "objective content" of the concept of fractions must begin.

As for the "historical" aspect of phenomena, Leont'ev's second statement clarifies the direction in which Davydov has developed his interpretation: "*Individual mind is a result of an assimilation of the experience of the previous generations of people*". But the discovery of incommensurability, moving the direct comparison of objects to the comparison of their measurement, causes the forgetfulness of the Pythagorean comparison [Fowler, 1979]. Consequently, the assimilation of the experience of direct comparison by the generations of people before the Pythagoreans, is cut off. That forgotten knowledge stays "*deposited in the culture*" [Asenova et al., 2020]. It demands an interpretation of history that goes beyond Leont'ev's assimilation.

In search of this interpretation, we resort to Toth.

### 3.3 Toth: "There is something else"

We met with Imre Toth in 1999. His suggestion to listen to *hidden meanings that could still be kept in Pythagorean mathematics* and the reading of his book "Lo Schiavo di Menone" have introduced us to the anthyphairesis, and have initiated us into philosophical practice.

Just at that time, we were starting to practice in some classes Davydov's situations about the familiarization of primary school children with fractions. The intertwining of Davydov-inspired classroom practice and Toth-inspired philosophical practice has led us, over the years, to make changes both to the practice and to the interpretation of Davydov's situations.

### 3.4 A split in the development of Greek mathematics.

To show the changes, we proposed to the workshop some considerations on incommensurability, taking up the previous text by Fowler. *The definition of incommensurability is found in Euclid Elements X, 2: "If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable."*

The discovery of incommensurability by the Pythagoreans introduced a split in the development of Greek mathematics. This split makes it possible to distinguish between “mathematic before the discovery of incommensurability” and “mathematics after the discovery of incommensurability”.

The activity carried out in the workshop on anthyphairesis helps to highlight how, before the discovery of incommensurability, Greek mathematics was characterized by two founding and independent acts: measurement and comparison. A measurement is a number that requires the concept of unit of measurement. A comparison is a “pair of numbers in relation” - a logos and requires the concept of the highest common unit. The discovery of incommensurability has led to abandoning the anthyphairesis, to neglecting the direct comparison, and to downgrading the search for the highest common unit.

After the discovery of incommensurability, measurement becomes the founding concept, while comparison, as it is reduced to two measurements, becomes a derived concept; the comparison of two objects is moved to the comparison of their measurements; the “highest common unit” is subordinated to the choice of a “unit of measurement”. The crisis of incommensurability channels in this direction the development of Greek mathematics. This direction is preserved in the development of Western “academic” mathematics.

We kept these considerations in mind as we addressed the question of “the long presence of unsatisfactory results in teaching and learning fractions”. This resulted in our didactic proposal regarding the familiarization of primary school children with fractions.

### 3.5 Hypothesis

Philosophical practice led us to link the “long persistence” with the split created by the discovery of incommensurability and with the forgetfulness that ensued. Our hypothesis, presented at the workshop, is the following: In the forgotten procedure of anthyphairesis there are indications that allow treating the question of “long persistence” in a different way from the usual ones. But be careful: In our classroom activities, we have never made any direct recourse to the procedure of Pythagorean comparison. Rather, through Davydov's situations, we have rediscovered the activities of direct comparison between objects that characterize the Pythagorean comparison. Thanks to that, we returned to the centrality of the search for the highest common unit, making use of two tools: 1) A "new" language. During the actions of each activity carried out in the classrooms, the language is built on three keywords, two referring to the quantities to be compared, the third referring to the common unit. 2) Symbolic writing. In familiarizing primary school children with fractions, we have introduced symbolic writing to record the comparison activities.[36]

Our classroom practice develops in a movement between the objective phenomenon and the symbolic writing, mediated by the new language. During the workshop, we traced the structure of our classroom activity by means of photocopies of the children's notebooks. There are three steps. 1)The first consists of direct comparison activities. These are developed in the language of the three quantities: the two compared quantities and the common unit. Symbolic writing is M;R = 9;3. Children read this form in this way: “The comparison between the two mugs "M" and "R" is the pair of numbers 9;3. That is, mug M contains 9 times the common unit, and mug R contains 3 times the common unit.” 2) The next step represents the transition to measurement. It is obtained by introducing an order between the compared quantities, with the choice of the reference quantity W. Symbolic writing is C/W = 16/4 = 4. Children read: “The measurement gives me a pair of numbers that determine how many egg cartoons I can pack.” This is the measurement by comparison. 3) Finally, as activity evolves toward division, symbolic writing evolves toward Euclidean division: Z/W = 17/5 = 3+2/5. “The fraction 17/5 equals 3 integers plus 2 common units.”

Symbolic writing is not formal. It evolves with classroom practice by crossing the “sub-constructs” ratio, measurement, and division of the “construct” fraction.[37] So, children become accustomed to thinking of “ratio, measurement, and division” as related to each other in the concept of fractions.

## *4 Third part of the workshop: Logos / Ratio*

The third part, the shortest, has focused on the concept of logos/ratio. The main tools of this part were some texts by Toth and Fowler.

[36] For this purpose, we took advantage of children's friendliness with symbols, which had already been developed previously.

[37] Kieren [1980] introduced five sub-construct of the construct rational number: ratio, measure, division, part-whole, and operator.

### 4.1 Al tempo dei Pitagorici il logos ha una natura puramente aritmetica.[38]

This text by Toth emphasizes the purely arithmetic nature of logos, before the discovery of incommensurability.

What happened after the discovery of incommensurability is described by Toth as follows: "*La penetrazione dell'àlagon nell'universo puramente aritmetico del logos ... produce un salto nella concezione del logos… Il linguaggio aritmetico scompare e il discorso risulta trascinato da un flusso verbale di sostanza differente*"[39].

In our teaching practice with primary school children, thanks to the direct comparison between objects, we tried to restore the purely arithmetic nature to the concept of logos/ratio: this concept is associated with a pair of numbers.

### 4.2 A ratio is a relationship between two numbers, not just 'two numbers'

This text by Sierpinska introduced the workshop to the theme of the relationship between the two numbers of logos. The following two texts indicated how, after the discovery of incommensurability, the search for this relation has resorted to a language that has lost its arithmetic character:

*(In the Elements) no alternative definition of ratio apart from V, Definition 3 is proposed: "A ratio (logos) is a sort of relation in respect of size between two magnitudes of the same kind"*. (Fowler, 1979)

"*L'espressione ποια σχέσις (una certa relazione) rinvia a una relazione il cui carattere concreto è forse fluido e resta ancora nel vago.*"[40] (Toth, 1988)

The philosophical practice led us to attempt to extend the purely arithmetic nature to the relationship between the two numbers of logos.

### 4.3 Purely arithmetic nature of the relationship

To eliminate the vagueness of the meaning of the relationship between the two numbers of the logos, we have represented the anthyphairesis by "a

[38] "*At the time of the Pythagoreans, logos has a purely arithmetic nature*".

[39] "The penetration of the àlagon into the purely arithmetic universe of logos ... produces a leap in the conception of logos ... The arithmetic language disappears, and the speech is dragged by a verbal flow of different substance".

[40] "The expression ποια σχέσις (a sort of relationship) refers to a relationship whose concrete character is perhaps fluid and still remains vague."

modern symbolic writing". We first showed to the workshop the following text by Fowler "*We now consider the suggestion that the ratio of two numbers or magnitudes was defined by their anthyphairesis*". We then proposed the following activity: 1) Take up again the sheet on which, in the second part of the workshop, you had written the development of the anthyphairesis in two columns. 2) Add two outer columns. 3) Write in these columns the letter S, if the subtraction has been performed in that column, or the letter C if the number has remained constant. In this way, they got a writing that reproduces the Pythagorean subtractive procedure step by step. The double column that survives in this writing is the key tool for our conclusions. We showed the workshop the following text by Fowler which contains another writing of the anthyphairesis: "*We call this procedure 'anthyphairesis', and refer to the sequence* $n_0, n_1, \ldots$ *as 'the anthyphairesis of A and B\ sometimes writing Anth(A, B) =* $[n_0, n_1, n_2, \ldots]$." We are faced with two different writings: 1) The two-column writing reproduces step by step the subtractive nature of the Pythagorean comparison. 2) The one-line writing translates the Euclidean algorithm and its multiplicative character. The first writing needs some considerations. It reproduces the feature of the Pythagorean comparison of moving between two columns. It is the "story" of this "movement" and generates the unusual property of the unique additive partition of the pair of numbers. As an example, we consider the pair (11; 3). This pair has the unique additive partition (1 + 1 + 3 + 3 + 3; 1 + 2). When the number 11 belongs to the pair (11; 7), it has another partition: (1 + 3 + 7; 1 + 1 + 1+ 4). The unique partition reproduces step by step the subtractive procedure of the anthyphairesis. Academic mathematics and its extraordinary effectiveness are founded on the unique factoring of natural numbers. They are generated by the choice of the Euclidean algorithm and by the abandonment of the Pythagorean comparison. But does the attention to Pythagorean comparison suggest rethinking foundations? The Pythagorean comparison suggests new concepts of measurement and quantity: the number 11 considered in the abovementioned example takes on meaning by the number it is paired with. Could this observation open a dimension in which to investigate dual phenomena? (Rottoli & Riva, 2021)

## REFERENCES

Arcavi, A. (1994). Symbol Sense: Informal Sense-making in Formal Mathematics. *For the Learning of Mathematics* 14(3) 24-34

Bobos, G., Sierpinska A. (2017). Measurement Approach to Teaching Fractions: A Design Experiment in a Pre-service Course for Elementary Teachers. *International Journal for Mathematics Teaching and Learning* 18(2) 203-239

Bonetto, M., Bonissoni, P., Soffientini, D., Longoni, P., Riva, G., & Rottoli E. (2008). From concept of quantity to early algebra: an example of requalification and recontextualization of teaching in primary mathematics. In *Proceedings of 3rd EME elementary mathematics education*. Edited by Alexandra Gomes (pp. 94-104). FCT, Braga.

D'Ambrosio, U. (1994). Ethnomathematics, the Nature of Mathematics and Mathematics Education. In *Mathematics, Education and Philosophy: An International Perspective*. Edited by Paul Ernest. (pp. 230-242). The Falmer Press London Washington, D.C.

Davydov, V.V. (1990). *Types of generalization in instruction: logical and psychological problems in the structuring of school curricula*. J. Kilpatrick, Editor & J. Teller, Translator. Reston, VA: National Council of Teachers of Mathematics

Davydov, V.V, & TSvetkovich, ZH. (1991) On the objective origin of the concept of fractions. *Focus on Learning Problems in Mathematics* 13(1) 13-64.

Fowler, D.H. (1979) Ratio in early Greek mathematics. *Bulletin of the American Mathematical Society* 1(6) 807-846.

Howden, H. (1989) Teaching Number Sense. *Arithmetic Teacher*, 36(6) 6-11.

Kieren T.E. (1980) The Rational Number Construct. Its Elements and Mechanisms. In *Recent Research on Number Learning* edited by Kieren T.E. (pp. 125-149). University of Alberta

Kilpatrick, J. (1990). Introduction to Davydov's *Types of generalization in instruction*,

Monaghan, J. (2016). Activity Theoretic Approaches. In *Tools and Mathematics. Instruments for learning*. Edited by Monaghan J., Trouche L., & Borwein J.M. Chapter 9. Springer International Publishing. Switzerland

Rottoli, E., & Riva, G. (2021). On an "Old" Concept of Quantity. *Academia Letters, September.*

Sierpinska, A. (2002) Reaction to *Abstraction: Theories about the Emergence of Knowledge Structures*. By Boero, P., Dreyfus, T., & Gravemeijer, K. Research Forum PME 26.

Sierpinska, A. (2019). Materials for teaching a course on research in mathematics education about students' difficulties in mathematics. © Sierpinska A.

Toth, I. (1998). *Lo schiavo di Menone. Il lato del quadrato doppio, la sua misura non-misurabile, la sua ragione irrazionale*. Editore: Vita e Pensiero.

# WRITING A BOOK FOR TEACHERS:

## an introduction to history of mathematics and its connection to education

**Bjørn SMESTAD**
Oslo Metropolitan University, Oslo, Norway
Volda University College, Volda, Norway
bjorsme@oslomet.no
smestadb@hivolda.no

## ABSTRACT

I am in the process of writing a book that will to be an introduction to history of mathematics for teachers in primary and lower secondary school. In this workshop, I outlined a series of objectives in writing the book, and then invited participants to discuss (in groups) concrete examples of sections of the book.

## *1 Introduction to the book project*

On a rainy day in Salzburg, after an inspiring ESU6 in Vienna, I opened my laptop and started writing an outline of a book on history of mathematics (HM) for teachers. Nine years later, in 2019, I visited Crete and Costas Tzanakis, where I talked about my ideas and got valuable feedback and additional references. In 2022, as another attempt to jumpstart the project, I presented ideas from the book at the ESU9 conference in Salerno, Italy. At the current speed, the book will be key task for me when I retire in some 20 years' time.

The book is supposed to be a textbook for teachers in primary and secondary school in Norway and be in Norwegian. I have several objectives in writing the book:

- I want to give teachers rich examples of how HM can be included in mathematics teaching (see for instance Smestad (2011) for a number of such "hows").
- I want examples that are a fertile ground for discussing important reasons for including HM in mathematics teaching (which many publications in the HPM group, for instance Clark et al. (2019); Grattan-Guinness (2004); Jankvist (2009), have discussed).
- I want to show how the development of mathematics as a science, the use of mathematics in different cultures and ages, and the develop-

ment of mathematics as a school subject (or more general, as an object of learning), are all fertile areas for inclusion in mathematics teaching.
- I assume that the reader has little previous knowledge of HM, and that the book will provide a first introduction.
- Of course, one major objective is to be able to write in a "popular" way, without simplifying too much.

It is important to note that it is not a goal to be completely original. If one idea has worked wonderfully in the hands of Adriano Dematte, Peter Ransom or Frederic Metin – or other inventive educators who have shared their ideas – that makes the ideas more, not less, relevant for Norwegian teachers. Of course, it is essential to refer to where the ideas came from when possible.

The chapters will be arranged according to different "hows" of including HM. The outline of chapters is like this:

1. Introduction (incl. "hows", reasons, three perspectives, an overview of HM)
2. Using old notations and solution strategies
3. Working with manipulatives
4. Drama/theatre as a method
5. Original sources: Pictures
6. Tasks based on history of mathematics
7. Etymologies
8. Original sources: Mathematical texts
9. Cross curricular work
10. Original sources: Textbooks
11. Biographical information
12. Project work
13. Texts on history of mathematics
14. HM games
15. The road ahead

The design of the book is a bit ambitious, as in addition to this arrangement according to "hows", I want to discuss the reasons for including HM in teaching throughout the book, I want to illustrate three aspects of HM throughout (mathematics as science, mathematics in use, and mathematics as a school subject). Finally, to reduce confusion, I want the main examples used to be in chronological order. These ambitions perhaps partly explain why the book has been in the making for 12 years so far. All of the chapters that discuss a way of including HM in teaching, have a similar structure: 1) Why use this way of including HM?; 2) Historical introduction to a period; 3) Main HM example (including suggested teaching unit and comments from a mathematics education perspective); 4) Perhaps another main HM example; 5) Other (short) examples (not necessarily from the same period).

In the workshop, the participants formed groups and selected chapters were distributed to the groups. The goal of the groupwork was to elicit insightful discussions of importance for the HPM group. To newcomers, it might serve as a good introduction to established theory in the field, discussed in concrete cases. Moreover, the discussions might provide valuable input to me. The concrete tasks for the groups were:

For each of the four examples (in early drafts, of course), discuss:

- In what way does the example help in reaching the goals I set?
- Suggestions for improvements to reach the goals in a better way?
- Suggestions for other parts of history to include?

And finally, please comment on the overall ideas of the book.

## 2 *Example*

The four examples were:

- Chapter 6 Tasks based on history of mathematics. Main example: The Pascal-Fermat correspondence, in the form of tasks based on their correspondence.
- Chapter 8 Original sources 2: Mathematical texts. Main example: The Pascal-Fermat correspondence, in the form of longer extracts of original text from the correspondence.
- Chapter 10 Original sources 3: Textbooks. Main example: Regula de tri
- Chapter 11 Biographical information. Main example: Niels Henrik Abel (and the quintic equation)

The examples handed out at the workshop totalled 18 pages. For issues of space, I will include just two of the examples here – the text on using textbooks as original sources and the text on including biographical information. I need to point out that the chapters are written in Norwegian, and that the translations into English have been done for the purpose of this workshop. This translation may be inaccurate in some details.

# Chapter 10 Original sources 3: Textbooks

## 10.1 Why work on old textbooks?

A text on the "whys" – based partly on what Jan van Maanen has written. Also a reminder about depaysement.

## 10.2 Historical introduction: Mathematics education through the times

A historical introduction will go here.

## 10.3 Main example 1: Regula de tri

### 10.3.1 Introduction

Regula de tri has been a common part of mathematics education for centuries, in many different cultures. For instance, it has been treated in Bhaskara's *Lilivati* from India and in Fibonacci's *Liber abaci* from Italy. It was also considered part of standard education in Norway. For example: from 1817, to be accepted for *Sjøkadettinstituttet* (Sea cadet institute) you needed to be a boy no older than 15 years of age, be able to read and write and to calculate with the "4 species" (addition, subtraction, multiplication, and division) and regula de tri (Botten, 2009).

The text we will be looking at, is from *Arithmetica Danica* by Tyge Hanssøn (not to be confused with other books of the same name). This was the first textbook in mathematics written in Norway, published in 1645, written for students of Trondheim's cathedral school. It includes topics such as addition, subtraction, division, multiplication, regula de tri, square roots and cube roots. It also includes motivational poems and some parts written especially for girls.

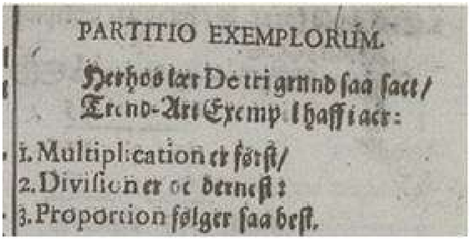
Using such a source in teaching may very well challenge the teacher's usual role in the classroom. While mathematics teachers are often prepared for many of the questions students will come up with, in working on an original source, there is a multitude of questions that may come up that just experts who are well versed in such sources can answer. While the teacher needs to prepare for the obvious questions, the task probably needs to be framed as an exploration of the past where the teacher takes part in the exploration.

### 10.3.2 Teaching unit

| Calculation | Grade 7 | 1-2h | 1600s | School subject | Mathematics<br>Nature<br>Depaysement |
|---|---|---|---|---|---|

Below is a text with excerpts from *Arithmetica Danica* by Tyge Hanssøn from 1645. When students see this text, much will seem difficult or incomprehensible. Students are asked to work in groups. First, they are asked to make a list of what they are curious of about this excerpt. Thereafter, they are asked to find out as much as they can about these excerpts, both concerning the content and concerning the context. At the end of the lesson, students present their findings and remaining questions.

In the first Norwegian textbook in mathematics, *Arithmetica Danica* by Tyge Hanssøn from 1645, *regula de tri* was treated over more than fifty pages. The start was like this:


PARTITIO EXEMPLORUM.
Herhos lær De tri grund saa sact/
Trend-Art Exemp l haff i act:
1. Multiplication er først/
2. Division er oc dernest:
3. Proportion følger saa best.

PARTITIO EXEMPLORUM

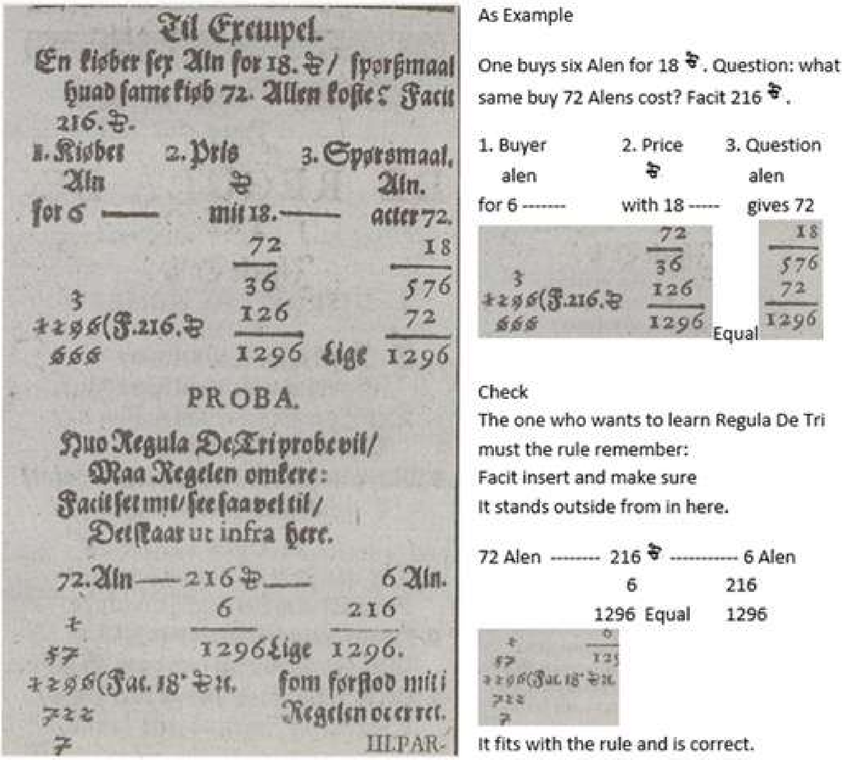
Learn «de tri» for reasons given
Three arts example have in mind:

1. Multiplication is first
2. Division is also thereafter:
3. Proportion then follows best.

PARTITIO EXEMPLORUM

Herhos lær De tri grund saa sact
Trend-Art Exempel haff i act:

1. Multiplication er først
2. Division er oc dernest:
3. Proportion følger saa best




Students may become curious about many issues, such as:

- Why are parts of the text written in verse? Was that normal for mathematics textbooks?
- What kind of school was this book meant for? Who were the students?
- What does Regula de tri mean and why haven't we heard of it before if it used to be in textbooks?
- What is the strange symbol?
- What is "Aln" (or "Alen")?
- What goes on in the calculation? Where can we find the multiplication and the division that is mentioned in the rule? And where does the number 216 come from?
- Why was the answer (fact) given immediately next to the question – and was that normal?
- How do the algorithms for multiplication and division work? (See more about this division algorithm in chapter 2.)

- Why is 126 placed in the way it is below 36? (My personal idea: maybe the printer moved 126 a bit to make space for the division to the left.)

The excerpts are from Geir Botten: Min lidle norske regnebog: noen dypdykk i ei lærebok i matematikk fra 1645, Universitetsforlaget 2009, p. 65. https://urn.nb.no/URN:NBN:no-nb_digibok_2019120207035

#### 10.3.3 Comments from a mathematics education perspective on *regula de tri* tasks

Nowadays, students would perhaps regard *regula de tri* tasks as equations that they would solve by setting up an equation involving fractions and then solve for x. Learning about other solution methods, such as *regula de tri*, might provoke some discussion on why this works and what happens when solving the equations.

From quite another perspective, it is interesting to look at *regula de tri* tasks from the point of view of authenticity. Many mathematics tasks lack authenticity, and in regula de tri tasks there is often an unmentioned assumption that the rate (price per unit, speed per hour etc) is constant. To help discuss this, some humorous examples of non-authentic regula de tri tasks can be used:

- A plumber takes 40 minutes to repair the plumbing under the kitchen sink. How long time would 10 plumbers spend on the same job?
- A woman spends 9 months to carry a child. How much time would three women spend?
- Four weightlifters take five minutes to carry a one-ton-car 200 metres. How long would one weightlifter need to carry the car 50 metres?

### 10.4 Other examples

Here, there will be other examples of pieces of textbooks of interest – each example taking no more than half a page. Possible examples include: old algorithms used in Norway's first mathematics textbook, old Norwegian measuring units, New Maths, …

# Chapter 11 Biographical information

## 11.1 Why work on biographical information?

History of mathematics is, of course, so much more than just mathematicians' biographies. But still students should get to know one or more mathematicians during their school years to help "humanize" the subject. Including information about the people who developed or used mathematics throughout history, can have the effect on some students of turning mathematics from a stale, technical issue to a human endeavour. Moreover, getting to know the people, we also find out how these people have at different points faced and solved problems. If we are careful in our selection of examples, we may also be able to show that mathematicians are a diverse crowd, counteracting some unhelpful stereotypes about who can succeed in mathematics.

## 11.2 Historical introduction: Mathematics in the 19$^{th}$ century

| |
|---|
| A historical introduction will go here. |

## 11.3 Main example 1: Florence Nightingale

| |
|---|
| Here, there will be an example on Florence Nightingale, with introduction, a teaching unit and some comments from a math. ed. perspective – on statistics, mathematics' role in war etc. |

## 11.4 Main example 2: Niels Henrik Abel (and the quintic equation)

### 11.4.1 Introduction

One of the greatest mathematicians of all time was the Norwegian Niels Henrik Abel (1802-1829). I will give a sketch of his life, stressing some episodes that can give an insight into the person Abel.

### 11.4.2 Teaching unit

| | | | | | |
|---|---|---|---|---|---|
| Algebra | Grade 10 | <1h | 1800s | Development | Attitudes, view |

The teaching "idea" in this teaching unit is simply to tell the story of Niels Henrik Abel.

He was born in Finnøy on August 5$^{th}$, 1802, grew up on Gjerstad and in 1815 he became a student at Oslo Cathedral School. Niels Henrik's father was an important person, both a priest and parliamentarian. However, both he and his wife developed a drinking habit. The family's economy deteriorated, and it got even worse when the father died in 1820. Then Niels Henrik was left to take care of his little brother.

In November, 1817, one of Abel's fellow students, Henrik Stoltenberg, died after having been beaten up by one of the teachers, Hans Peter Bader. Bader lost his job, and Abel got a new mathematics teacher, Bernt Michael Holmboe. Holmboe was a mathematics teacher who understood mathematics and could see Abel's talent. Today, a prize for good mathematics teachers is named after Holmboe.

In 1820, Abel believed that he had found a general solution for the quintic equation. He showed it to his teacher and to the professors at the university, and thereafter to professor Degen in Copenhagen. Not even Degen could find a mistake, but he asked Abel to give the proof in a more detailed form and to try some concrete numbers. Abel eventually realized that he had made a mistake. Some years later, in 1824, he was able to publish a proof that there is no such general solution.

At that time, he had already been a student at the university in Christiania for a while, and he got a travel scholarship to go to Copenhagen, Göttingen and Paris to learn from the major mathematicians at the time. However, he had a tendency to travel where his friends were rather than traveling to the great mathematicians – he described it like this: "I happen to be like this – I can absolutely not, or at least with great difficulty, be alone. I tend to become melancholy and then I am not in the best mood to get anything done".

In the summer of 1823, he had met a girl, Christine Kemp. They got engaged at Christmas time in 1823, and thereafter the only thing that was needed was for him to get a permanent position before they could get married. After the trip abroad mentioned above, he got a temporary job, but a permanent position did not come his way. He spent the summer of 1828 on Froland, where Christine had gotten a position as a nanny. Here Niels Henrik Abel died of tuberculosis on April 6$^{th}$, 1829. Two days later, the university in Berlin offered him a permanent position, but it was too late.

### 11.4.3 Comments from a mathematics education perspective on identity, on proving impossibilities etc.

The story of Abel is of course especially relevant for Norwegian students, since he is considered our foremost mathematician, in a close tie with Sophus Lie. In the story of his life, one can see that a child from modest conditions can become a great mathematician, but also that even mathematicians can make major mistakes. And that even mathematicians can fall in love… Research shows that many students are positioned as less able and that many students believe that making mistakes is a sign of low abilities – history of mathematics may help.

The biography of Abel may also be a start of a discussion on the non-solution of the quintic equation, and on how mathematics may at times be about what is not possible.

### 11.5 Other examples

Here, there will be other examples of biographical information that could be useful in school, no more than half a page per example. Relevant examples: al-Khwarizmi, Ramanujan, Ahmes (the scribe who created the Ahmes papyrus), Archimedes, Pythagoras [where the idea would be to point out how little we know], Lewis Carroll, Andrew Wiles and the proof of Fermat's theorem, Turing and code theory, Galois, Benjamin Banneker… [I need some biographies of users and learners of mathematics as well…]

## *3 Postscript*

As expected, the group discussions went in many different directions, and no two groups discussed exactly the same issues. Groups discussed:

- Issues in history of mathematics
- How history of mathematics can enrich mathematics education – and how suited the examples are (and whether other examples would be better)
- Issues in mathematics education that can be discussed in light of the examples
- What teachers need in order to teach with history of mathematics
- Ideas on how to develop such a book

I will give examples from each of these directions. However, let me repeat that part of the goal of the workshop was to elicit insightful discussions of importance for the HPM group, and to be welcoming to newcomers. As this workshop was on the first day of the conference, I was happy to see that people got to know each other by sharing their own experiences, knowledge, and views on including HM in teaching mathematics. I heard several references to this workshop later in the conference, which is a sign that the groups had interesting discussions.

Obviously, some of the discussion concerned the history of mathematics in itself – people were talking about other examples of *regula de tri* and provided pointers to other articles to read, details to check and so on.

People also discussed the goals of including history of mathematics in mathematics education, including their own experiences. The Abel biography includes little mathematics, and some participants would prefer to include mathematicians where the usefulness of mathematics – to sciences or practical work – was more obvious. One example could be Isaac Newton.

Issues in mathematics education were discussed in connection with the question about giving the answer directly after the question. Mathematics education researchers advocate for less focus on the answer and more on the process. Giving the answer in advance could be a way of ensuring that students did not believe that the point of mathematics is to find the answer. However, most likely the students in the 1600s did not have a book each – it is more likely that the questions were read to the students and that the answer in the book was to the benefit of the teacher.

Another such issue concerns my use of modern, "humorous" examples of "non-authentic regula de tri tasks". Discussions at the conference – and feedback from a reviewer – have made me uncertain about whether the use of these are wise, as they are clearly created to be funny, and can be easily dismissed. It would perhaps be better to find authentic regula de tri-tasks with problematic modelling assumptions (for instance, that 1 kg and 1000 kg of a product would have the same cost per kg, with no room for bargaining, or that it is possible to travel at a constant speed).

Yet another question is what a teacher needs to know in order to start teaching (meaningfully) with history of mathematics. To avoid serious mistakes, a teacher needs to know history quite well, but few teachers will include history of mathematics if they have to study history of mathematics for years first. Participants discussed whether the examples gave enough information to be used safely.

There was also a discussion in the workshop and after the workshop on how such a book should be developed. The idea of a website instead of a book was proposed. A website can be easily accessed, which is both good and bad. More teachers will see it, but more teachers will also see just one part of it without bothering to look at the whole. A website makes it possible to include "work in progress", but this may lead teachers to believe that nothing on the website is "finished". Still, I consider creating a website with preliminary versions of chapters, partly to get more input from teachers and classrooms, and partly to get things "out there" while waiting for the book to be finished. One major issue at the moment, is that each example has been tested little or not at all, and a website could help with that.

My conclusion is that the workshop helped to raise many questions, some of which were very helpful to me, but some were probably also interesting for the participants.

## REFERENCES


Botten, G. (2009). *Min lidle norske regnebog: noen dypdykk i ei lærebok i matematikk fra 1645*. Universitetsforl.

Clark, K., Kjeldsen, T., Schorcht, S., & Tzanakis, C. (2019). History of Mathematics in Mathematics Education—An Overview. *Mathematica Didactica*, *42*, 1-26.

Grattan-Guinness, I. (2004). History or Heritage? An Important Distinction in Mathematics and for Mathematics Education. *The American Mathematical Monthly*, *111*(1), 1-12. https://doi.org/10.2307/4145010

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics*, *71*(3), 235-261. https://doi.org/10.1007/s10649-008-9174-9

Smestad, B. (2011). History of mathematics for primary school teacher education. Or: Can you do something even if you can't do much? In V. J. Katz & C. Tzanakis (Eds.), *Recent Developments on Introducing a Historical Dimension in Mathematics Education* (pp. 201–210). Mathematical Association of America.

# POTENTIAL OF COLLABORATION BETWEEN HISTORY AND MATHEMATICS TEACHERS: EMPERICAL INVESTIGATION

**Abdelrahman AFFAN, Michael. N. FRIED**

The Program for Science and Technology Education, School of Education, Ben-Gurion University of the Negev, Israel.
affanabd@gmail.com,
mfried@bgu.ac.il

ABSTRACT

At the ESU-8 conference (Affan & Fried, 2019), we emphasized the rationale of our research concerning the potential and importance of collaboration between history teachers and mathematics teachers in the project of incorporating history of mathematics (HoM) in classroom. In this talk, we present the exact structure of the workshop and a partial empirical result from analyzing the data collected from a questionnaire distributed to 557mathematics and history teachers. Initial categories constructed according to clinical semi-structured interviews, observations, video records and log analysis with 10 teachers who participated in a course based on history of Islamic mathematics are also set out. The goal of the analysis is to identify characteristics of the collaboration between the two populations and see how these characteristics are reflected in the construction and implementation of a joint learning unit in HoM.

## *1. Introduction*

The basic assumption (not always acknowledged) guiding most studies on using HoM in mathematics teaching is that it is indeed feasible to integrate HoM into mathematics teaching and learning so that the main consideration is how to do it (Jankvist, 2007, 2009, Gulikers & Blom, 2001), however, it is indeed feasible to integrate HoM into mathematics teaching and learning. But this cannot be taken for granted. Fried (2001), for example, claims that along with difficulties such as lack of time and material supply constraints in the dense curriculum, the integration of the HoM contains within it a theoretical difficulty expressed in the tension between anachronism and relevance a dilemma in which the teacher is forced to choose between adopting an approach which sees the HoM as merely a “tool” (Jankvist, 2009) to be used in mathematics classes to teach mathematics relevant to the modern world and typical mathematical curricula and a historically sensitive approach that focuses on historical context, original sources, etc.

Since this dilemma is founded on the different demands made by the disciplines of mathematics and history and subsequently in the teaching of these disciplines, we need to examine the relationship between mathematics teachers and history teachers, the assumption being that teachers embrace the norms of their respective disciplines. In other words, this research sees the problem of incorporating HoM into mathematics education as a problem of interdisciplinary education or even transdisciplinary education (Clark et al. 2018; Thomaidis & Tzanakis, 2022; Barbin, 2022). Such studies take the nature of "collaboration" as something that is not self-evident.

Keeping in mind the framework of interdisciplinarity/transdisciplinarity, our study aims to bring history teachers and mathematics teachers together in order to see, first of all, what kind of considerations and presuppositions they have when they come to HoM, and second, whether they are able to work together to produce a chapter on the HoM for mathematics classes and perhaps for history classes. Our hypothesis is that integrating history of mathematics into mathematics education will benefit from the interaction between these communities of teachers, mathematics and history teachers. This has been done before to some extent (Moyon, 2013), but we have taken it up with the awareness that the attempt may involve a problem similar to that it is meant to solve, namely, a tension between the commitments and basic assumptions of history and mathematics teachers (Fried, 2001). That said, the awareness itself together with the particular choice of historical material may allow for the creation of genuine cooperative community, so that the solution to Fried's dilemma may be to think of HoM as fundamentally a trans-disciplinary subject that requires enlarging the community that deals with it.

Our overall aim is to investigate characteristics of the collaboration between mathematics and history teachers and how they are reflected in the construction and implementation of a joint learning unit in history of mathematics.

## 2. *Research questions*

Our overall research project is guided by three research questions: The first refers to the significant differences on considerations, presuppositions, and beliefs between teachers for mathematics and history regarding the integration of HoM into the classroom. The second refers to the impact of a course in the HoM on considerations and beliefs of the two teacher populations regarding mathematics, history, and the HoM and what characterizes the learning experience of the two populations during their joint participation. The third refers to feasibility of an interdisciplinary collaboration between mathematics and history teachers, what characterizes this collaboration in transferring of a joint learning unit in HoM. In this paper, we focus mostly on the second and third research question.

### 3. *Participants and Research Tools*

In the present study, we focus on a group of mathematics and history teachers from Arab schools in Israel. In the first stage, a questionnaire distributed to 419 mathematics teachers and 138 history teachers to find out what are the considerations, presuppositions and beliefs of teachers for mathematics and history regarding the integration of HoM into the classroom. For the analysis of quantitative data, a descriptive analysis, paired samples t-test, correlation and regression analysis were used.

In the second stage, a group of 10 mathematics and history teachers (5 mathematics teachers and 5 history teachers) participated in a course-based on history of Islamic mathematics. The goal of the course, which included a joint activity, was to find out the teachers` views of collaboration, what are the characteristics of the collaboration between the two communities and how they are reflected in the construction and implementation of a joint learning unit in HoM.

The text used was "*On the Geometric Constructions Necessary for the Artisan*" by the Muslim mathematician Abu'l-Wafa Buzj'ani (940-998). This particular work was chosen because it contained genuine mathematical content, but the mathematics also had a clear social and cultural context. The treatise concerns the work of ordinary people, the artisans, and, therefore, reflects a distinctly Islamic emphasis on activities benefitting society. On the mathematical side, many of the examples in the book *were original*. The range of problems is very wide, from the simplest planar constructions (the division of a segment into equal parts) to polyhedrons inscribed in a given sphere (Affan et. al., 2019). Thus, the text should appeal to the needs of both groups of teachers. The course consist of five workshops of about 1.5-2 hours each session. In the first meeting, the researchers explained the rationale of the course and listened to the teachers' expectations, considerations and beliefs connected to it. In the second meeting, we presented Abu'l-Wafa's book including activities and original texts from the book; later, the participants were divided to pairs (each pair consisting of a history teacher and a mathematics teacher) focusing on an activity from the book. In the third meeting, we gave them three original historical texts from the Abbasid era (10th century) about the connection between the translation movement and the development of mathematics and sciences. In the fourth meeting, the teachers participated in further activities with original texts from Abu al-Wafa's book such as constructing polygons inscribed in a circle. In the last meeting, each pair was presented with examples of ornaments from Islamic art and were asked to construct the ornament on the basis of what they learned in the previous meetings and present the activity in the class.

In the final stage of the research project, the teachers were divided into groups of two (each pair included a math teacher and a history teacher) and were asked to construct and implement a study unit on the history of Islamic mathematics inspired by the texts from Abu'l-Wafa 's book studied in the course given in the first stage. Three groups implemented their unit in the 10th grade and one is still in the construction process. The fifth group could not agree on the exact topic for the teaching unit and ended their collaboration.

The data set of our analysis included questionnaires, observations and video recordings of the activities in the course and the joint study unit, and interview transcripts. In our analysis of the data, we were especially interested in teachers' views regarding collaboration, how collaboration developed during the in the course and in design of the study unit afterward.

### 4. Initial Findings

As mentioned previously, the questionnaire, which was designed principally to gain information regarding the first research question and to provide background for the course, contained 48 questions falling under four categories: considerations and attitudes concern mathematics; history and history thinking; history of mathematics; collaboration between history teachers and mathematics teacher. Since our main focus is the course itself, we will only summarize briefly the main conclusions from the questionnaire. Similar to previous surveys (Tzanakis & Arcavi, 2000), and others, mathematics teachers and history teachers were found to have positive attitudes about the integration of HoM. However, unlike mathematics teachers, history teachers think that integrating HoM in classroom can contribute more to the understanding of the culture/ social/ human needs and less to mathematical thinking or to the understanding of mathematics concepts, which is the opposite to Mathematics teachers' views. Namely, the combination of the two subjects develops an awareness to human culture, and exposing the students to the contributions of mathematics to culture and humanity develops a historical thinking and critical arguments rather than a direct contribution to mathematical thinking or to mathematics itself. To answer the second and third research questions we analyzed the data using qualitative methods to determine categories describing the tensions, emotions and gestures among the teachers. The exact categories are still being revised. That said, from our observations and the videotapes recordings, we were able to discern four initial categories characterizing the degrees of cooperation or tension between the history and mathematics teachers.

### 4.1 Discomfort and Ease

At the beginning of the session, also during the first meetings of the course, most of the history teachers felt uncomfortable. Several facial expressions and gestures (such as combining hands, hands on the cheek or forehead, tense facial expressions…) indicated lack of ease or self-confidence with respect to the meeting. One of the teachers whispered to one of the researchers:" How did you convince me to come, I am a history teacher, what I suppose to do with mathematics". Another teacher wrote in a reflection after class, "As a student I was weak in mathematics and didn't like it, the thought that, after many years, I am going back to math again, gives me goosebumps….."

**Figure 1**: Discomfort and Ease expressed by facial expressions and body movements

### 4.2 Arrogance and Respect

In the activities and dialogues, we noticed a tendency for the mathematics teachers to be overbearing during the math activities and the history teachers during history lessons. As mentioned earlier, there was tensions between the teachers of the two subjects and we noticed tension within some of the pairs (the mathematics teacher during the mathematical activity and the history teacher during the historical activity) who tried impress the other side, to show mastery on the material and sometimes trying to imprint there opinion. In contrast, in other pairs, we noticed a mutual respect among the participants and tried to help each other without hurting the other's feelings.

### 4.3 Compassion and Embarrassment

Some teachers expressed understanding for the other side and tried not to make them feel embarrassed. In one case, the participants were asked to draw a circle with a given diameter, a history teacher asked, "*what is a diameter ?*". Some of the history teachers laughed, and a part of mathematics teachers smiled, and he teacher looked very embarrassed (figure 2). On the other side, his "partner" (a mathematics teacher) explained to him patiently what a diameter is and how to draw a circle with a given diameter.

**Figure 2**. History teacher look embarrassed after he asked what a diameter is.

### 4.4 Dominance and Passivity

In the mathematical activities, each pair was asked to read a [geometric] text from Abu al-Wafa's book and apply the activity in the class. The mathematics teachers were more dominant when performing the activity while the history teachers were passive, watched them and followed their instructions. On the other hand, in the history activity of analyzing texts in history we noticed that history teachers took over the discussion most of the time, while mathematics teachers barely participated in the discussion.

**Figure 3**. Mathematics teachers more dominant in HoM activities

The aim of the joint work was to examine what characterizes the collaboration between the two communities during constructing and transferring a learning unit based on Abu al-Wafa's book. From our observations and the videotapes recordings, we were able to discern four initial categories characterizing the degrees of collaboration between the history and mathematics teachers. These types can be divided into four categories:

### 4.5 Complementing Pairs

In this type of collaboration, both teachers complemented each other. The history teacher explained to the mathematics teacher the interpretation of the historical text and discussed the significance of the historical events in it, in the other side, the mathematics teacher guided the history teacher in carrying out the mathematical activity.

### 4.6 Domineering Teacher

Mathematics teachers are more domineering in HoM activities (especially in mathematical activities), they were the first to hold and use the ruler and compass, draw the figures and explain to the history teacher what to do. On the other side, they swap roles in the historical activities.

### 4.7 Together But Each Separately

Both teacher sitting around the table, but everyone works alone and only at the end of the activity, they discuss what they do separately and compare the results. In the case of this pair, they did not succeed and failed in performing the activity.

### 4.8 Fully cooperating pair

Both teachers worked in harmony with full collaboration, motivating and encouraging one each other in order to produce a perfect product.
The following diagram (figure 4) shows the categories mentioned above about kinds of collaborating between mathematics and history teachers while participating a history of mathematics course.

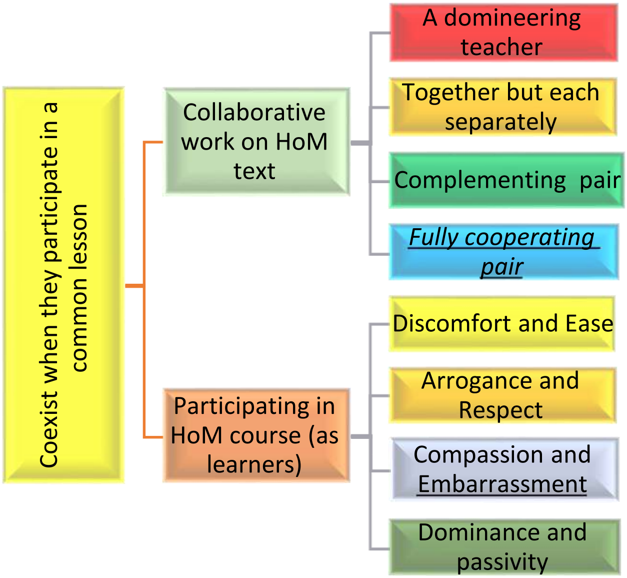



**Figure 4.** Kinds of collaborating Categories between mathematics and history teachers.

## 5. *Conclusions*

In this study, we observed that mathematics teachers and history teachers are two different communities according to their attitudes and considerations with respect to HoM in the classroom. Thus, a course in the history of Islamic mathematics to examine the feasibility of interdisciplinary collaboration between them was set up. Initial findings indicate that difficulties arose during the course and the joint unit study afterward. However, there was at least one fully collaborating pair. This shows that despite the difficutlies, it might, nevertheless, be feasible to enlarge the HoM community in order to promote collaboration between both communities. Further research is necessary to determine the exact mechanism for such successful collaboration.

## REFERENCES

Abū'l-Wafā` Buzj'ani. *Book on what is necessary from Geometric Construction for the Artisans*, Istanbul: Suleymaniye Kutuphanesi, Ayasofya 2753, copied in the first half of the 15th century. [Arabic manuscript], available online at https://www.alukah.net/library/0/88512/

Affan. A & Fried. M. (2019). Potential of collaboration between history and mathematics teachers: investigation and framework based on a text by Abu'l-Wafa Buzj'ani . In É. Barbin, U. T. Jankvist, T. H. Kjeldsen, B. Smestad & C. Tzanakis (Eds.), *Proceedings of the Eighth European Summer University on History and Epistemology in mathematics Education (ESU-8)* Oslo Metropolitan University.

Barbin, É. (2022). On role and scope of historical knowledge in using the history of mathematics in education. *ZDM – Mathematics Education*, *54*(7), this issue. https://doi.org/10.1007/ s11858-022-01410-1

Clark, K. M., Kjeldsen, T. H., Schorcht, S., & Tzanakis, C. (2018). Introduction: Integrating history and epistemology of mathematics in mathematics education. In K. M. Clark, T. H. Kjeldsen, S. Schorcht, & C. Tzanakis (Eds.), *Mathematics, education and history: Towards a harmonious partnership* (pp. 1–23). Springer.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist?. *Science & Education, 10*, 391-408.

Jankvist, U.T. (2007). Empirical research in the field of using history in mathematics education: Review of empirical studies in HPM2004 & ESU4. NOMAD 12(3), 83–105.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics*, *71*(3), 235–261.

Gulik-Gulikers, I. & Blom, K. (2001). 'A historical angle', a survey of recent literature on the use and value of history in geometrical education. *Educational Studies in Mathematics*, 47(2), 223–258.

Moyon, M., (2013). Diviser en multipliant les approches. Quand les mathématiques remontent aux sources, *Repères-IREM*, 93, 47-77.

Tzanakis, C., & Arcavi, A. (2000). Integrating history of mathematics in the classroom: an analytic survey. In Favuel, J., & Van Manen, J. (Eds.), *History in mathematics education* (pp. 201-240), Netherlands: Kluwer Academic Publishers.

Thomaidis, Y., & Tzanakis, C. (2022). Historical knowledge and mathematics education: A recent debate and a case study on the different readings of history and its didactical transposition. *ZDM – Mathematics Education*, *54*(7), this issue. https://doi.org/10.1007/s11858-022-01370-6 (correction https://doi. org/10.1007/s11858-022-01387-x)

# MATHEMATICAL OBJECTS WITHIN A TRANSITORY EPISTEMOLOGY

**Miglena ASENOVA**
Free University of Bolzano, Italy
miglena.asenova@unibz.it

## ABSTRACT

In this paper we present a category-theoretical characterization of mathematical objects as synthetic objects within a transitory epistemology. This allows us to take into account pragmatic and dynamic issues as in the definitions of mathematical object in mathematics education research and in the historical evolution of mathematical practice. We discuss the relations of such objects to mathematical objects considered from an objective, set-theoretical perspective, as well as the implications of the category-theoretical characterization for teaching and learning mathematics.

## *1 Introduction*

In Mathematics Education (ME) research there are a lot of definitions of mathematical object (MO) (e.g. Chevallard, 1991; D'Amore, 2001; Duval, 2009; Font et al., 2013; Lavie et al., 2019; Radford, 2008). These definitions highlight pragmatic aspects (e.g., reference to human activity), epistemic constraints (e.g., reference to individuals that display routines), recourse to semiotic resources (e.g., objects seen as invariants behind semiotic transformations), and dynamicity (e.g., evolution of MOs over time). These definitions are far away from the ones used in mathematics that shape the definitions used in textbooks and taught in math classroom (Asenova, 2021).

How these two different kinds of definitions of MOs can be linked to each other is an important ontological issue in ME research (Asenova, submitted; Asenova et al., submitted), but it has also interesting implications for the teaching-learning process in the classroom. We suggest that starting from a suitable epistemology of mathematical practice that fits the historical emergence of new MOs, it is possible to frame MOs theoretically in a way that is close to the 'pragmatical' needs of the classroom but that also consider the relation to their 'objective' mathematical definition.

## *2 Theoretical framework*

Charaterizing the main issues of the philosophy of mathematical practice, Giardino (2017) sums up four aspects on which this current of thought focuses: (1) the dynamicity of MOs in mathematical practice; (2) the importance of semiotics; (3) the emphasis on epistemological aspects that go beyond construction of formal systems; (4) the emphasis on pragmatic issues, in reference to the use of objects and tools in context. The phylosophy of mathematical practice expresses the more recent developments in the history of phylosophy of mathematics and could be useful for better frame the way MOs should be introduced and could be conceptualized in mathematics classroom.

Indeed, the four aspects mentioned by Giardino are very close to the concerns that characterize the definitions of MOs in MER presented in the introduction. Indeed, teaching-learning phenomena: (1) Are strictly related to cognitive processes and require a dynamic approach to mathematics; (2) Are rooted in the web of semiotic transformations that underly mathematical thinking; (3) Are necessarily involved in epistemic constrains that deal more with knowing why and how rather than with systhematysing; (4) Are related to the use one is able to make of MOs, rather than to their abstract meaning the discipline as a whole. In this sense, the phylosophy of mathematical practice allows to connect the viewpoint on mathematics as a discipline to the viewpoint on ME as prexeology in the classroom.

In order to operationalise the very general ideas rooted in the phylosophy of mathematocal practice, we need to choose a specific philosophy of mathematical practice that is able to provide a suitable epistemology for characterizing MOs. In this sense, in the following we refer to the Synthetic Philosophy of Contemporary Mathematics (SPhoCM) (Zalamea, 2012).

## *3 MOs within a transitory epistemology*

From the viewpoint of the SPhoCM, mathematical practice claims for a transitory epistemology that considers knowledge pragmatically, in evolution over time, and MOs as synthetic „(quasi-)objects“ (Zalamea, 2012, p. 323), that means dynamic objects defined by their relations with other objects, rather then as analytic objects, considered in themselves. The idea of synthetic vs. analytic object is expressed in Figure 1.

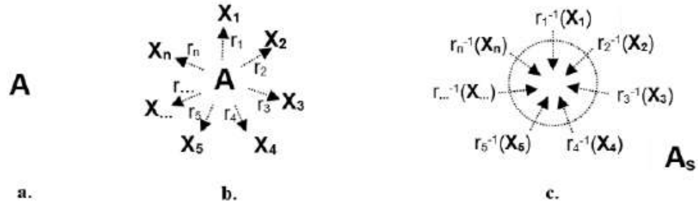



**Figure 1a**. The object $A$ from analytic viewpoint (as object in itself); **1b**. The context in which $A$ is immersed (composed by the objects $X_1$, ..., $X_n$); **1c**. $A$ as synthetic object ($A_s$).

In Figure 1a the MO $A$ is represented from analytic viewpoint: It is an object in itself, without relations to other objects. In Figure 1b, the object $A$ is condidered as immersed in a context, where $X_1$, ..., $X_n$ represent the objects related to it. In this sense, $A$ is not considered ‚in itself‘, but by its relations to the objects belonging to its context ($r_1$, ..., $r_n$). By considering the counter-images ($r_i^{-1}(X_i)$) and the relations between them (dotted circle), the object $A$ ‘disappears‘ and is seen from synthetic viewpoint ($A_s$), as mirrored by its context (Figure 1c).

According to Zalamea (2012), the transitory epistemology of contemporary mathematics can be framed by a conceptual use of mathematical tools. This use does not refer to the formal aspects of the mathematical tools but to their characteristics as means of universal thought: For example, a category is interpreted as a suitabel context, focusing on relations between objects that can be composed in an associative manner. Asenova (2021) transposes this way to use category-theoretical tools in ME research and characterizes it as *metaphorical*, according to the idea of structural metaphor, based on an analogy (Pimm, 1981). In this sense, an anlogy between the category-theoretical tools and the concepts in the discursive language of ME research is established (Asenova, 2021): A category is a coceptual context, in the sense of a web of objects or concepts; an arrow is a relation; a functor is a way of meaning-making; a natural transfomation between functors is a translation between two diferent ways of meaning-giving to the same context.

In this way, the category-theoretical model used to frame MOs from synthetic viewpoint can be immersed in the category of the ME research-practice by an immersion-functor. This functor transferes the relations of the category-theoretical model into the discursive language of ME research and justifies the metaphorical use of category-theoretcal terms (e.g., we cen stalk about a functor that ‚gives meaning to *A*').

According to this metaphorical use of the category-theoretical tools, to know an object belonging to a context, means to give meaning to its relations to all the other objects belonging to the context. This brings us back to the idea of MO from synthetic viewpoint. To better explain this point, I use the idea of representable functor. Let us consider the context of the object *A* as a category (the category ***C***), in the way it is represented in Figure 1b. Let us immerse ***C*** in the category ***Set***[41] in the following way: each object of ***C*** is represented by the set of relations (arrows) that have *A* as domain-object and the object itself as codomain-object; each relation (arrow) in ***C*** is represented by the functions that map between those sets. The functor that creates this translation of ***C*** 'from the viewpoint of *A*' is a functor representable by the object *A* and it creates a copy of ***C*** in ***Set***. A special kind of representable functor, the hom-fuctor, is a contravariant functor that inverts the directions of the relations (arrows) with respect to the ones present in ***C***. To know the object *A* means to give meaning to the representations of ***C*** from the different viewpoints of the objects belonging to the context expressed by ***C*** and to the ways they can be translated to each other. Since a functor is a way of meaning-making and different functors express different ways to give meaning to a context, a functor also expresses the way meaning is given to a single object belonging to that context. Coming back to Figure 1, we can state that if we consider the MO *A* as an object in itself, we can return to its characterization as analytical object (1a). From the other hand, if we consider the feedbacks turned back by the context, the object can be interpreted as evolving in a temporal sequence, according to the indices of the relations $r_i$, and we are able to recover also the dynamicity of evolution over time. As in this paper the focus is on ME as praxeology of the classroom and not on the epistemology of ME as research domain, we focus only on the idea of MO from synthetic viewpoint because it is the most fruitful for the purpose to explain how the transitory epistemology of mathematical practice can support teaching and learning

[41] ***Set*** is the category whose objects are sets and whose arrows are functions.

mathematics in the classroom. For the complete definition of MO specific to MER, the reader can refer to Asenova et al. (submitted).

## *4 Discussion*

The characterization of a MO from synthetic viewpoint, based on the transitory epistemology of the SPhoCM, mirrors the way MOs emerge in the history of mathematics: They first arise as a tool to solve problems within mathematical practice in a context composed by other objects, and than gradually acquire meaning from the "feedbacks" send back from the context they arise from: "The new discourse started emerging when people realized that a number of routines displayed the same pattern" (Lavie et al., 2019, p. 164). The transitory epistemology of the SPhoCM fits well this idea of emergence of new MOs from historical and epistemic perspective and the idea of MO from synthetic viewpoint represents a model of this way to conceive MOs. Furthermore, the knowledge of a MO from synthetic viewpoint is a potentially complete knowledge (Asenova, 2021) and this supports the idea that a new MO should be introduced in the classroom as a solution of suitable mathematical problems that defines it in a synthetic way by mirroring its characteristics by all the other objects involved in the problem and the relations between them. In this way, the objects emerge from the practice, from the routines and by the semiotic transformations carried out by the students. This, conversely, fits the way new MOs emerge from the mathematical practice seen as a historical process. The usual (Bourbaki-style) definition acquires meaning only as the end-point of such a gradual emergence from social problem-solving-practices that require to carry out semiotic transformations.

## *5 Conclusions*

The characterization of MOs from synthetic viewpoint seems to fit well the features of MOs emerging from mathematical practice from a historical perspective but it also fits the way which is usually considered 'sustainable' in ME while introducing new mathematical objects in the classroom. It is particularly interesting to see that what is usually stressed by scholars in MER and by philosophers of the mathematical practice can be suitably modelled by mathematical tools, provided we consider them as universal means of thought, rather than as formal objects. This seems to be a promising way to find technical tools to model an analyse aspects related to ME research (Asenova et al., submitted), but it also can be seen as a backing of what is well known from research practice: From an epistemic viewpoint, meaning making is something very pragmatic and is, at least at the beginning, far away from the idea of set of elements that satisfies a certain property.

## REFERENCES


Asenova, M. (2021). *Definizione categoriale di Oggetto matematico in Didattica della matematica*. Pitagora.

Asenova, M. (submitted). Is Theoretical Topic-Specific Research "old fashioned"? An epistemological inquiry about the ontological creativity of Mathematics Education Research.

Asenova, M., Bagossi, S., & Arzarello, F. (submitted). Using category-theory in Mathematics Education research: Theoretical definitions of mathematical objects.

Chevallard, Y. (1991). Dimension instrumentale, dimension sémiotique de l'activité mathématique. *Séminaire de Didactique des Mathématiques et de l'Informatique de Grenoble*. LSD2-IMAG, Université Joseph Fourier.

D'Amore, B. (2001). Une contribution au débat sur les concepts et les objets mathématiques. *Scientia Paedagogica Experimentalis, 38*(1), 17–46.

Duval, R. (2009). «Objet»: Un mot pour quatre ordres de réalité irréductibles? In J. Baillé (Ed.), *Du mot au concept: Objet* (pp. 79–108). PUG.

Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics, 82*(1), 97–124. https://doi.org/10.1007/s10649-012-9411-0

Giardino, V. (2017). The Practical Turn in Philosophy of Mathematics: A Portrait of a Young Discipline. *Phenomenology and Mind*, 12, 18–28.

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: from ritual to exploration. *Educational Studies of Mathematica, 101*(2), 153–176. https://doi.org/10.1007/s10649-018-9817-4

Pimm, D. (1981). Metaphor and analogy in mathematics. *For the Learning of Mathematics, 1*(3), 47–50.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in Mathematics Education: Epistemology, History, Classroom, and Culture* (pp. 215–234). Sense Publishers. https://doi.org/10.1163/9789087905972_013

Zalamea, F. (2012). *Synthetic Philosophy of Contemporary Mathematics*. Sequence Press.

# HISTORICAL JOURNEY OF THE CONCEPT OF PERIODICITY AND ITS DIDACTIC IMPLICATIONS IN TRIGONOMETRY

**Maria Rosaria DEL SORBO**[(1)], **Roberto CAPONE**[(2)], **Maria Rosaria ENEA**[(3)]

(1) IS Leonardo Da Vinci, via Turati, Poggiomarino (NA), Italy
mariarosaria.delsorbo@isisleonardodavincipoggiomarino.it

(2) Department of Mathematics, University of Bari Aldo Moro, Bari, Italy
roberto.capone@uniba.it

(3) Dept. of Eng., Computer Sc. and Mathematics, University of Aquila, Italy
mariarosaria.enea@univaq.it

## ABSTRACT

The teaching innovation proposal about the study of one of the most important but sometimes undervalued topics in High School Mathematics consists of a multidisciplinary approach, specifically based on the historical evolution of the concept of periodicity and in multiple applications. This article summarizes the results of a study that evaluated the effectiveness of the historical and epistemological approach in teaching the concept of periodicity in Mathematics, which students often perceive as a hard topic. The study was carried out with 62 high school students aged 16-18 years in a public scientific high school in Poggiomarino, Naples (Italy). After observing the results collected from different running kinds of tests on the students, we concluded that the approach based on these advanced perspectives and methodologies was significantly helpful in making trigonometry easier and digging deeper into the concept of periodicity.

## 1 *Introduction*

The concept of periodicity is rather familiar to young people from an early age. Periodicity is profoundly inherent in the cycles of many phenomena of nature, inside our body, such as heartbeat and breathing, or outside us, such as seasons and day cycles. But repetition is also typical of patterns and scheduling conceived by the human mind and widely spread everywhere, not only in the works of all arts of any historical age but also in many organizing solutions of everyday life. There are many examples of the human trend to conceive and think in terms of periodicity and to exploit periodic schemes.

Calendars have been the common tool to rationalize the flowing of time for ancient and recent civilizations worldwide, even though different kinds of arrangements have been adopted (Zerubavel, 1985). Observing so many periodicity examples in nature and arts is also very interesting. “Many different artworks mimic the statistical properties of natural fractal patterns – in particular, self-similarity at different scales” (Balmages A. et al., 2021). Repetition of colors, shapes, and forms is the fundamental principle of painters like William Morris, Piet Mondrian, and Andy Warhol. In (Neeman S. & Maharshak, A., 2006), there is a specific reference to the beneficial psychological effect of repetition and variation in arts: “When we are introduced with an irregularity, a piece of information that is out of place, we try to make sense of it, and when we succeed we have a feeling of tension reduction, which gives us pleasure”. According to this assumption, our perception of repeated similar shapes in space or recurrent events in time transfers to the brain signals, processed to seek organized and regular patterns or a generalization: as in the Gestalt theory (Wertheimer & Riezler, 1944), our mind finds a sort of fulfillment in finding regularities. It is not by chance that Islamic patterns, inspired by vegetal patterns, in Granada’s Alhambra elicit a sense of beauty, perfection, and wonder and have inspired the artistic compositions of artists like Escher. Nevertheless, regarding the music, “Large- and small-scale patterns are the essential building blocks for a musical composition, be it improvised or slowly conceived and notated […]. Perhaps the sense of order suggested by patterns in a given composition brings a certain satisfaction to the listener, while a temporary disruption of those patterns promotes a sense of drama” (Wilson, D., 1989). As in many historiographers' conjectures, periodicity is also recurrent in historical interpretations of the events. From prehistoric times onwards, many ancient cultures had some idea that the natural and human worlds moved in cycles of origin → growth → prosperity → decline → fall, with various repeating patterns. In Greece, Thucydides, who lived in the 2nd half of the V century b.C., most typically embodies the concept of cyclic human history. After this vision, he believes that the collection of past speeches and actions will have a profitable use in the future. He says: “If my work is judged useful by any who shall wish to have a clear view both of the events which have happened and of those which will someday, according to the human condition, happen again in such and such-like ways, it will suffice for me [...] But those who want to look into the truth of what was done in the past-which, given the human condition, will recur in the future, either in the same fashion or nearly, so those readers will find this History valuable enough, as this was composed to be a lasting possession and not to be heard for a prize at the moment of a contest" (Thucydides). Also, even though it is not a science, astrology is based on eternal repetition. "Socrates recommended above all to learn astrology, in order to know the time of the night, of the month or the year, in case of travel, navigation or service, or for everything that is done at night, in the month or the year; it is about having benchmarks to distinguish the moments of these different times, but it is easy to learn them from night hunters, sailors and many other people who have an interest in knowing them". (Thucydides)[42] Due to the previous considerations, periodicity can be considered a common factor in many disciplines. Thus, it is suitable to be the topic for a transversal learning unit.

Therefore, it makes sense to propose a deeper analysis and reflection on periodicity to students of a scientific high school. To put it another way, practices related to the construction of the concept of periodicity and the introduction of analytic instruments to deal with it along the history of mathematics and human thought can make students aware of a model and a category of the human spirit. This is why, we proposed to a group of students aged 16 to 18, supervised by teachers and experts while studying trigonometric functions, research the origin and development of the theory of periodic functions in the past centuries.

## 2 *The mathematical prototypes of periodic functions are trigonometric functions.*

The origin of trigonometry goes back to astronomy studies at the geometric school of Alexandria (Barbin et al., 2015). These studies were motivated by the requirements to build a *quantitative* astronomy that could be used to predict the motions and positions of celestial bodies and to aid in the determination of time, the compilation of calendars, navigation, and geography (Heath, 1921; Rogers, 2010). This explains why spherical trigonometry historically precedes plane trigonometry against the natural scale of difficulties. The founder of trigonometry was probably Hipparchus of Nicaea (II century BC). Fundamental contributions to spherical trigonometry are also due to Theodosius of Tripoli (1st century BC) and Menelaus of Alexandria (1st-2nd century AD). But most of the information on the Alexandrian trigonometric methods

[42] Freely translated by the authors.

comes from the *Almagest* by Ptolemy (II century AD), who laid the foundations of the astronomical theory, which dominated the scientific scene until the seventeenth century. The key difference between Greek and modern trigonometry is that Alexandrian trigonometry used the chords of a circle instead of sines.

From the $9^{th}$ century, the natural successors of the Greek geometers were the Arab mathematicians. They quickly assimilated most of the studies known at that time, unifying them in an original method, which was transmitted a few centuries later to European scholars. The first innovation with respect to Alexandrian trigonometry was the use of the sine instead of the chord and a systematic study of circular functions, so defined because radian measures of angles are determined by the lengths of arcs of circles. Trigonometric functions, a special type of circular functions, are defined using the unit circle. For the Arabs, spherical trigonometry was particularly important also for religious reasons: the direction of Mecca, marked on all public sundials, was determined by solving the spherical triangle, which has the position of the beholder, Mecca, and the north pole as vertices. As we said, trigonometry reached the West mainly by translating Arabic sources into Latin. Medieval development was slow, and European scholars made no interesting contributions before the fifteenth century. A further boost to the development of trigonometry comes from topography, which, unlike astronomy, is based on rectilinear trigonometry. The first formalization of the plane and spherical trigonometry is contained in *De triangulis omnimodis* by Regiomantanus, written around 1464 but printed only in 1533. Numerous treatrises followed it; among these, we mention that of Nicolò Copernicus contained in his famous work *De revolutionibus orbium caelestium* published by G. J. Rhaeticus (1542).

The introduction of logarithms by Napier gave a strong boost to the development of trigonometric techniques: trigonometric calculations could be greatly simplified by combining the tables of circular and logarithmic functions. Rhaeticus himself prepared a series of tables of the six circular functions. Until the middle of the seventeenth century, sines, cosines, tangents, etc., were numbers given by tables, which provided for each value of the angle the value of the sine, or later its logarithm.

Around 1650, a different point of view began to spread: the functional one, or rather, since the concept of function was not yet well defined, the geometric one. Thus, the curves of sines, cosines, tangents, and others were studied.

With the creation of infinitesimal calculus and the formulation of the concept of function, the mathematics of trigonometric functions was also systematized. Studying the irregularities of the motions of Jupiter and Saturn, Euler gave a systematic and complete treatment of the trigonometric functions: their periodicity is clearly expressed in his *Introductio in analysin infinitorum* (1748), in which the measure of angles in radians is also introduced. To keep up with achievements in navigation, astronomy, and geography, greater precision was needed in the interpolated values of trigonometric, logarithmic, and nautical tables: this problem led to the series expansion of functions (Kline, 1972).

## 3 *Periodic functions by a didactic point of view*

The first and most suitable question about a periodic function is what is actually "periodic"? The most common answer could be that it consists of or contains a series of repeated stages, processes, or digits at regular intervals, as the ones depicted in figure 1a) below. In order to avoid confusion in the students' learning cognitive operation, it is important to specify how it is possible to find the "regular interval", i.e., the "period" of the function: it is the least repetition pattern's length that can be detected as can be seen in figure 1b). So the prevailing challenges and criticality in learning periodicity, on the one hand, can be to detect confidently which is the period of the function and how long it takes for the pattern to be repeated (Inan, 2013; Kamber & Takaci, 2018). But on the other hand, periodicity hides the extraordinary potential of future forecasting: once detected the period, it is possible to predict the trend of the function at any upcoming point.

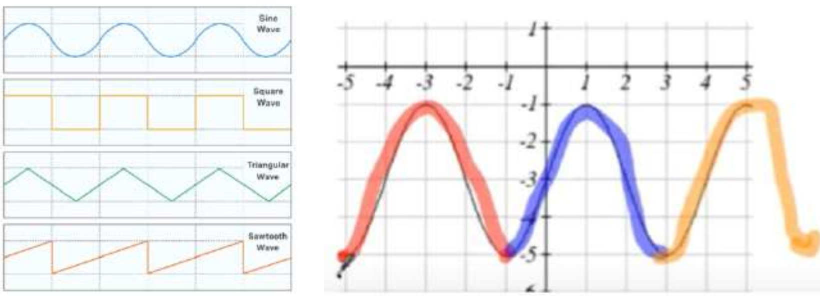



**Figure 1.** a) Examples of periodic signals b) The periods of the function are highlighted using different colors

To better delimit the set of periodic functions, two limit cases are shown: the constant function (periodic of period $t$, with $t$ any real number) and a non-periodic function, such, for example, an exponential function of the type $e^x$, which turns out to be periodic of the infinite period. This consideration recalls students' attention to the real significance of periodicity, besides eliciting their curiosity about other special cases of periodic functions, for instance, quasi-periodic functions, depicted in figure 2, and almost periodic functions, which understanding needs a deeper knowledge, most likely out of high school student's outlook. For the sake of congruence with the concepts and definitions known to the students to whom the experimentation was directed, the concept of quasi-periodic functions was taught in a intuitive and graphical way.

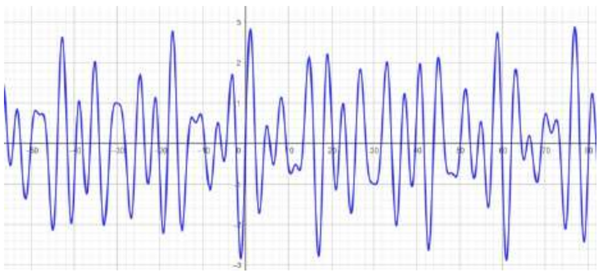


**Figure 2.** Graphic of the function $\boldsymbol{f(x) = \sin x + \sin(\sqrt{2}x) + \sin(\sqrt{3}x)}$

For personal experiences of the authors and as reported in Maor (2013), a concept that students in the last years of high school or at university usually accept with high interest is that almost any periodic function can be represented by a sine and cosine convergent series, such as:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right)$$

Which is known as the Fourier series. It means that as more and more components of the series or "harmonics" from the series are summed, each successive partial Fourier series sum will better approximate the function and will equal the function with a potentially infinite number of harmonics.

## *4 Experiment with students*

Our experiment was carried out with a total of 62 students of three similar class groups of 21, 21, and 20 members in their fourth year of a Scientific High School in a small town in Poggiomarino, near Naples (Italy). The first group of students was taught with a traditional approach, frontal lessons, guided exercises in the classroom, and individual homework followed by a collective correction. The second group of students was taught using a flipped classroom methodology, with large historical enhancements and many lab applications. The third group received lectures based on an experimental historical background, with frontal lessons and study material given by the teacher. The experimentation started in October 2020 and has been continued since April 2021. In order to establish their starting conditions and to assess their prerequisites, an entrance test was given to all the students. The groups were formed based on the test results, with maximum homogeneity and similarity criteria.

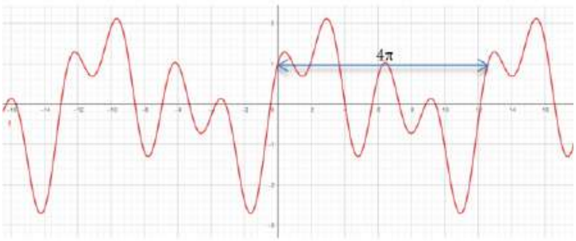



**Figure 3.** Example of a periodic function

For the second and third groups, the first question asked to students was about their idea of periodicity: what does the adjective 'periodic' mean? They could have an introductory immersion in the topic with a brainstorming session.
The first group, according to a traditional teaching, followed a different approach.
Students of the second and third groups were then urged to draw what they figured out to be a periodic function by hand or use a dynamic geometry tool, such as *Geogebra*. Then teachers showed them many kinds of function graphs (see figure 3 above for an example) and asked them to classify the functions, possibly detect the period, and link the periodicity to their previous experiences. Then, with the established methodologies, each group followed its specific path, as previously specified.

### 4.1 Students' misconceptions

As specified above, the preliminary activities highlighted the students' ideas about periodicity. In particular, it seems interesting to focus on their naive, most common conceptual mistakes about what periodic means.

- Misconception 1: Periodic functions are exclusively trigonometric functions.
- Misconception 2: If the expression of a function is a combination of sine and cosine functions, the function is necessarily periodic.
- Misconception 3: association with a periodic table. The periodic table is called periodic because of the regular variation of one or more parameters, such as atomic radius and electronegativity.

### 4.2 Activities and Results

Among others, the most important activities performed with students in our experiment on an action-research modality were the application of the periodicity concept to Physics, specifically to electromagnetic radiation of oscillation in space and in time and to wave packet; the creation with a *3D* printer and the finding of 3-dimensional periodic artifact, as in figure 4 a) below, and informatics labs, where CORDIC algorithm to compute in an elementary way the trigonometric functions were experimented by students and some machine learning algorithms to detect the periodicity of a function was tested as shown in figure 4 b).


**Figure 4.** a) Examples of periodicity in 3D artifacts; b) AI application

The results, collected using written and oral tests, simulations, and a challenge between groups, are resumed in the table below, wherein the third column EMR stands for Electromagnetic Radiation, whose representation is a set of periodic functions.

| | Periodic functions | Goniometric equations | EMR |
|---|---|---|---|
| Group 1 | 68 % sufficient or more | 51 % sufficient or more | 62 % sufficient or more |
| Group 2 | 85% sufficient or more | 54 % sufficient or more | 72% sufficient or more |
| Group 3 | 76% suffucient or more | 53% suffucient or more | 70% suffucient or more |

**Table 1.** Outline of the results of the experiment

Observing this table, there is evidence that the most part of students of any of the groups showed good results by the testing phase after the activities about periodic functions and the applications on electromagnetic radiation. Less positive was the impact of experimental teaching had on the solving strategies for goniometric equations.

## *5 Conclusions and future developments*

In our study, we tested the effectiveness of the historical and epistemological approach in teaching the concept of periodicity in Mathematics and trigonometry, which students often perceive as a hard topic. We organized the activities to explore the evolution of the concept of periodicity, starting from the history of trigonometry. The study was carried out on 62 students of a scientific high and was quantitative. From the observations of the results of the tests, we concluded that the approach based on history and epistemology, with the observation of cognitive values expressed by mathematics, went out to help make learning trigonometry easier. As shown earlier, we collected proofs that after the experiment, students' awareness about periodic functions and periodicity in general improved by a significant amount.

In the future, we plan to widen the experiment to a larger number of students, to focus our attention on the periodic but not trigonometric functions, and to better characterize students' difficulty with periodicity in Mathematics.

## REFERENCES

Balmages, A., Schiffman, L., Lyle, A., Lustig, E., Narendra-Babu, K., & Elul, T. (2021). Quantifying patterns in art and nature. *Journal of Mathematics and the Arts*, 15(2), 188-197.

Barbin, É., Jankvist, U., & Kjeldsen, T. H. (Eds.) (2015). *History and Epistemology in Mathematics Education: Proceedings of the 7$^{th}$ ESU*. Copenhagen.

Heath, T. (1921). *A History of Greek Mathematics*, Oxford at the Clarendon Press.

Kamber, D. N., & Takaci, D. (2018). On problematic aspects in learning trigonometry. International Journal of Mathematical Education in Science and Technology, 49, 161 - 175.

Kline M., (1972). *Storia del Pensiero Matematico, dall'antichità al Settecento*, Edizione italiana a cura di Alberto Conte, Einaudi.

Inan, C. (2013). Influence of the Constructivist Learning Approach on Students' Levels of Learning Trigonometry and on Their Attitudes towards Mathematics.

Maor, E. (2013) Trigonometric Delights, Princeton: Princeton University Press.

Neeman, S. and Maharshak, A., (2006). Order and disorder, entropy in math, science, nature and the arts. *Proceedings of the 3rd WSEAS/IASME International Conference on Engineering Education*, Vouliagmeni, Greece, pp. 198–202.

Rogers, L. (2010). *The History of Trigonometry Part 2*. https://nrich.maths.org/6853&part=

Thucydides, The Peloponnesian War I, 22.

Wertheimer, M., & Riezler, K. (1944). Gestalt theory. *Social Research*, 78-99.

Wilson, D., (1989). The Role of Patterning in Music. *Leonardo*, 22(1), 101–106.

Zerubavel, E. (1985). *Hidden rhythms: Schedules and calendars in social life.* Univ of California Press

# HISTORICAL PARALLELISM AND THE DIDACTICAL TRANSPOSITION OF HISTORICAL KNOWLEDGE

**Yannis THOMAIDIS[a], Constantinos TZANAKIS[b]**
[a]Central Administration of Education, Thessaloniki, Greece
[b]Department of Education, University of Greece, Rethymnon, Greece
gthom54@gmail.com[a], k.tzanakis@uoc.gr[b]

## ABSTRACT

The idea of *parallelism* between the way mathematics has evolved through the mathematicians' creative work and the way students learn mathematics (called the "*parallelism*" issue) is well-known and its naïve formulations have been rightly criticized as untenable oversimplifications. By means of a case study based on an example from Euler's *Algebra*, we provide more nuanced views, thus pointing to the complementary character of similarities and dissimilarities between past and present, its significance for appreciating the complementarity of the historians and mathematics educators' aims and commitments, and the need for and basic features of an effective didactical transposition of historical knowledge.

## 1 *Introduction: The parallelism issue*

The idea that teaching and learning mathematics should follow or/and correspond to the historical development, is a whole spectrum of views and methodological prompts or recipes appearing under various names since the late 19th century (Furinghetti & Radford, 2008). For brevity, we call it the "parallelism issue". In the last decades its naïve formulations have been rightly criticized as untenable oversimplifications, but appreciation of the inherent subtleties is implicit in early publications, albeit not discussed in detail (Freudenthal, 1973, pp. 101, 103; Memorandum, 1962, pp. 190–191; Vergnaud, 1990, p. 16). Among several relevant interesting ideas, Sierpinska (1990) pointed to a negative and a positive aspect of parallelism, corresponding respectively, to overcoming epistemological obstacles and understanding, as two complementary perspectives in Bohr's sense introduced via quantum physics as an epistemological principle to understand reality (Bohr, 1934, p. 10). This notion of complementarity can be useful for understanding deeper what so far has been considered as incompatibilities or clash of commitments between the history of mathematics (HM) and mathematics education (ME); i.e., manifestations of complementary perspectives not to be used simultaneously, but equally legitimate and necessary in order to teach and learn mathematics both as a structured corpus of human intellectual products and as a cultural endeavor leading to them (Thomaidis & Tzanakis, 2022, section 2). Here, this notion is considered only in relation to the similarities and dissimilarities between past and present.

## 2 *Complementarity between similarities and dissimilarities of the past and the present, and the didactical transposition of historical knowledge*

In the 1980s-1990s the HPM domain expanded, leading to an influential ICMI study (Fauvel & van Maanen, 2000). Theoretical issues became essential and the need to theorize from actual implementations went hand in hand with criticism as to the role of the HM in ME, including the parallelism issue. A characteristic example of such a criticism is Fried's (2007) pointing to the subtleties inherent in any use of the HM in ME that - for practical reasons - adopts an anachronistic perspective of the historical development in an educational context. Fried points to a "clash of commitments" between HM and ME, by stressing the distortions of historical knowledge caused by considering similarities between past and present without due attention to their dissimilarities.

This debate stresses the subtleties inherent in the attempt to serve "properly" the aims and commitments of both ME and the HM, hence the subtle balance that has to be achieved so that none of them gets distorted or/and ineffective. According to Fried the historical and the mathematical ways of knowing should be conceived as complementary epistemologies (Fried, 2007, p.204).

In a series of papers advocating a multiple-perspective approach to history, Kjeldsen went further, calling for a more nuanced view of history in ME, that (just as for school mathematics) requires a didactical transposition of scholarly historical knowledge in order to capture the variety of ways in which HM can be beneficial for learning of and about mathematics (Kjeldsen, 2012, pp. 333-334).

In the light of this critical discourse, further discussion of the parallelism issue raises several more subtle points: Are there similarities between past mathematicians' creative work and students' ways of learning mathematics? If so, how could they be beneficial both for ME and for understanding further the historical development? What are the limitations imposed by the differences between these two worlds? These questions complement the preceding critical account: History gets distorted by considering similarities between past and present without attention to their dissimilarities, and ME gets mis-focused by considering dissimilarities between past and present without attention to their similarities. Here, similarities refer to drawing possible parallels between, either obstacles, misconceptions, errors, difficulties, premature formulations met by students and also by past mathematicians (a kind of *negative parallelism*), or/and innovative, idiosyncratic ways to cope with questions, problems, etc., that cannot be adequately treated with the (available at the time) knowledge of the mathematicians' community, or in the students' classroom (a kind of *positive parallelism*; Thomaidis & Tzanakis, 2007, 2022). Dissimilarities concern taking account of the differences between the social, cognitive, cultural, and scientific conditions of the past mathematicians' world and that of students today. Thus, in the context of ME, *similarities* and *dissimilarities* between past and present are *complementary*. This is what could be understood as an appropriate *didactical transposition* of historical knowledge inspired and guided by similarities between past mathematicians' creative work and students' ways of learning, however, on the condition of both being properly contextualized, i.e., dissimilarities between past and present are carefully accounted for.

In addressing these issues systematically, historical analysis has to be compared with empirical data on how students conceive and use specific pieces of mathematical knowledge. Below we report on a case study, illustrating these ideas about complementarity and didactical transposition, and serving as an example of how historical research motivated by didactical problems can contribute to still open or debated historical issues.

## 3 *A case study: On Euler's "mistake" and its didactical significance*

Euler (1770), in his essentially didactic treatise on algebra, develops about one third of volume 1 without introducing the equality symbol, though it includes all basic operations, exponentiation and roots' extraction. In order to apply algebraic operations to the transformation of proportionality relations and the solution of equations, = is introduced in chapter 20 (ibid, Vol. 1, §206). To our surprise, this remarkable and peculiar – from a modern perspective – "delayed" introduction of = seems to have passed unnoticed by historians of mathematics.

Without =, standard algebraic rules, including the square roots of negative numbers, are *verbally* formulated in this part of the book; for instance, for the product of two square roots of positive numbers, or the square roots of negative numbers Euler writes:

> …if it is required to multiply$\sqrt{a}$ by $\sqrt{b}$, the product is $\sqrt{ab}$…the square root of the product *ab*…is found if the square root of *a*…is multiplied by the square root of *b*… (Euler, 1770, Vol. 1, §132 p. 56)

> Since $-a$ is as much as $+a$ multiplied by $-1$ and the square root of a product is found by multiplying together the roots of its factors, so the root of *a* multiplied by $-1$, that is $\sqrt{-a}$ is as much as $\sqrt{a}$ multiplied by $\sqrt{-1}$. But $\sqrt{a}$ is a possible number, therefore the impossibility appearing therein, always can be led to $\sqrt{-1}$. Consequently, because of this, $\sqrt{-4}$ is as much as $\sqrt{4}$ multiplied by $\sqrt{-1}$: But $\sqrt{4}$ is 2, hence $\sqrt{-4}$ is as much as $2\sqrt{-1}$ and $\sqrt{-9}$ [is] as much as $\sqrt{9}\sqrt{-1}$, that is $3\sqrt{-1}$... (Euler, 1770, Vol. 1, §147 p. 61)

Euler uses the expression "is as much as", ("ist so viel als"), not "is equal to" ("ist gleich"), indicating an operational meaning of the two factors giving the product as a result via multiplication. Moreover, the above two citations lead to *opposite* results for the product of the square roots of negatives (Thomaidis & Tzanakis, 2022, section 4.2). This is the reason for attributing to Euler grave elementary mistakes (Cajori, 1993, p. 607; Grattan-Guinness, 1997, §6.15, p. 334; Katz, 2009, §19.1.3, p. 670; Kline, 1980, p. 121) [43]. But Euler's formulation is not "relational", but "procedural", conveyed by standard expressions not containing the word "equal" or the equality symbol. Thus, if one wants to symbolize Euler's verbal expressions "is as much as", "is found", "can be led to", in principle a *different* symbol should be used, denoting the reduction of the left-hand side to the right-hand side (we call this the *reduction-conception* of equality); e.g., for the two citations above one can write respectively $\sqrt{(-2)}\times\sqrt{(-3)}\rightarrow\sqrt{6}$, $\sqrt{(-2)}\times\sqrt{(-3)}\rightarrow\sqrt{2}\times\sqrt{(-1)}\times\sqrt{3}\times\sqrt{(-1)} \rightarrow -\sqrt{6}$.

After introducing =, Euler deals with transforming equations in equivalent forms, and not just with algebraic or arithmetic operations that produce a result. Therefore, = concerns operations between equations: Equalities now become symbolic objects, hence = is necessary for their representation, with equality acquiring here its standard meaning of an *equivalence relation*.

Thus, there are two distinct equality conceptions: a *procedural* "*reduction*

[43] Martinez (2007) is the first to question this interpretation, however, following a different rationale.

*conception*” and a *relational* “*equivalence conception*”, clearly separated by the introduction of = in Euler’s book. This is crucial because it questions the established historical interpretation about Euler’s elementary mistakes.

In fact, there are clear indications that Euler was not constrained by any single-valuedness of the square root either before, or after the introduction of the square root symbol (Euler 1770, Vol.1 §§122, 150). Therefore, since in evaluating the square root of the product of two numbers, Euler uses only the reduction-conception of equality, $\sqrt{(-2)} \times \sqrt{(-3)} \to \sqrt{6}$ and $\sqrt{(-2)} \times \sqrt{(-3)} \to -\sqrt{6}$ are equally valid *reductions* (not *equivalences*)! In Euler’s *Algebra*, the two equality conceptions, though coexistent, are carefully separated with distinct operative roles, and used in a contradiction-free manner before and after the introduction of =. However, from didactical research it is well-known that though both conceptions are used by school students, they are often muddled. For instance, Kieran notes that

In elementary school the equal sign is used more to announce a result than to express a symmetric and transitive relation. In attempting to solve the problem

> *Daniel went to visit his grandmother, who gave him \$1.50. Then he bought a book costing \$3.20. If he has \$2.30 left, how much money did he have before visiting his grandmother?*
>
> $6^{th}$ graders will often write 2.30 + 3.20 = 5.50 – 1.50 = 4.00
>
> …the equal sign … is read as “it gives”, that is, as a left-to-right directional signal. (Kieran, 1990, p.98)

Freudenthal notes that = is “…primordially read as a task, or a question… as unilaterally directed towards a ‘reduction’.” (Freudenthal, 1983, pp. 477, 481).

Although the *reduction-conception* of equality is evident in the above chain of “equalities”, the pupils understood the problem and how to solve it. Their solution can be formulated without =: “adding 2.30 to 3.20 gives 5.50”, then “subtracting 1.50 from 5.50 gives 4.00”.

Bearing in mind the systematic use of this conception in Euler’s *Algebra*, this is an example of “*positive parallelism*”. That is, the reduction-conception of equality is used successfully – albeit in a symbolically idiosyncratic way – to tackle a problem by students who developed this conception apparently without having been taught it. But the inadequacy of this conception is also a case of “*negative parallelism*”:

> [Many college students] …continue to view the equal sign as a separator symbol rather than as a sign for equivalence [as] seen in their shortcutting of steps in equation solving, and in their staggering of "adding the same thing to both sides": Solve for $x$: $2x + 3 = 5 + x$ , $2x + 3 - 3 = 5 + x$, $2x = 5 + x - x - 3$, $2x - x = 5 - 3$, $x = 2$. (Kieran, 1990, pp. 100–101)

Students solving $2x + 3 = 5 + x$ in this way use the reduction-conception of equality in the successive transformation of each individual member, thus violating the very notion of an equation and the logical consistency of the solution's method. Hence, they meet insurmountable difficulties to proceed effectively in situations involving more elaborate algebraic manipulations.

Being aware of the primitive character of the *reduction-conception* of equality and its inadequacy for issues more elaborate than simple algebraic calculations, Euler went beyond it after introducing =, that he henceforth used to denote the deeper and effective *equivalence relation conception* of equality.

## 4 *Concluding remarks and comments on the didactical transposition of historical knowledge*

Since students hold and mix up the two equality conceptions, what could be done? In teaching mathematics today, it is impossible to ignore the established meaning of equality and to avoid setting as a principal aim its understanding by the students as an equivalence relation. But it is here that historical and educational research in cooperation could help from several perspectives that outline main points of the didactical transposition of historical knowledge in this case, namely: to appreciate the coexistence of the *reduction* and *equivalence conceptions* of equality; to point to the pitfalls and misinterpretations resulting when the two notions are muddled; to help teachers get aware of their existence in students' mathematical understanding. In view of this didactical problem, there are many alternatives for realizing this didactical transposition, adapted to the target population's characteristics and limitations (Thomaidis & Tzanakis, 2022, section 5).

With the example from Euler's *Algebra*, and the critical discussion that preceded, we emphasized some issues we consider important for understanding better the connections between HM and ME: the nuances related to the "parallelism issue" (in particular, the "positive-negative" aspects and their limitations); the complementary nature of similarities and dissimilarities between past and present and their educational significance; the fact that this complementarity can be a key idea for appreciating that the historians and mathematics educators' aims and commitments are complementary rather than in conflict; and the need of a constructive collaboration of these communities for the appropriate didactical transposition of historical knowledge.

## REFERENCES


Bohr, N. (1934). *Atomic theory and the description of nature*. Cambridge University Press.

Cajori, F. (1993). *A history of mathematical notations*. Dover.

Euler, L. (1770). *Vollständige Anleitung zur Algebra*. Euler Archive-All Works. 387 (Erster Theil); 388 (Zweyter Theil).

Fauvel, J., & van Maanen, J. (2000). *History in mathematics education. The ICMI Study*. Kluwer.

Freudenthal, H. (1973). *Mathematics as an educational task*. Reidel.

Freudentahl, H. (1983). *Didactical phenomenology of mathematical structures*. Reidel.

Fried, M. N. (2007). Didactics and history of mathematics: Knowledge and self-knowledge. *Educational Studies in Mathematics*, *66*(2), 203–223.

Furinghetti, F., & Radford, L. (2008). Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics. In L. English (Ed.) & M. Bartolini Bussi, G. A. Jones, R. A. Lesh, B. Sriraman, & D. Tirosh (Assoc. Eds.), *Handbook of international research in mathematics education* (pp. 626–655). Routledge.

Grattan-Guinness, I. (1997). *The Fontana history of the mathematical sciences*. Fontana Press.

Katz, V. (2009). *A history of mathematics. An Introduction*. Addison-Wesley.

Kieran, C. (1990). Cognitive processes involved in learning school algebra. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 96–112). Cambridge University Press.

Kjeldsen, T.H. (2012). Reflections and benefits of uses of history in mathematics education exemplified by two types of student work in upper secondary school. In B. Sriraman (Ed.) *Crossroads in the History of Mathematics and Mathematics Education*, 333–356. Information Age Publishing.

Kline, M. (1980). *Mathematics. The loss of certainty*. Oxford University Press.

Martinez, A. (2007). Euler's "mistake". The radical product rule in historical perspective. *The American Mathematical Monthly*, *114*(4), 273–285.

Memorandum (1962): On the mathematics curriculum of the high school. *The American Mathematical Monthly*, *69*(3), 189–193.

Sierpinska, A. (1990). Some remarks on understanding in mathematics. *For the Learning of Mathematics*, *10*(3), 24–36.

Thomaidis, Y., & Tzanakis, C. (2007). The notion of historical "parallelism" revisited: Historical evolution and students' conception of the order relation on the number line. *Educational Studies in Mathematics*, *66*(2), 165–183.

Thomaidis, Y., & Tzanakis, C. (2022). Historical knowledge and mathematics education: A recent debate and a case study on the different readings of history and its didactical transposition. *ZDM – Mathematics Education*, *54*(7). https://doi.org/10.1007/s11858-022-01370-6 (correction https://doi.org/10.1007/s11858-022-01387-x)

Vergnaud, G. (1990). Epistemology and psychology of mathematics education. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education* (pp. 14–30). Cambridge University Press.

# WHITEHEAD'S PHILOSOPHY OF MATHEMATICS AND EDUCATION AS A FOUNDATION OF DIALOGICAL TEACHING

**Arto MUTANEN**
Finnish Naval Academy & Finnish National Defence University, Suomenlinna, Helsinki, FINLAND
arto.mutanen@gmail.com

In the education of mathematics, and also in education more generally, a dialogical approach has been emphasized for several years (Hofmann & Ruthven 2018; Mercer, Dawes & Kleine Staarman 2009; Radford 2011; Koterwas, Dwojak-Matras, & Kalinowska 2021, Prottas 2018). However, dialogical teaching has long historical roots, as the Socratic Questioning Method shows. The examples of Bacon and Kant demonstrate that a rational questioning method must be based on some presuppositions, showing the methodical foundation of dialogical teaching. (Hintikka 2007.)

The dialogical methods of teaching emphasize the role of language in general (Radford 2011), the role of symbolism (Kitcher 1986), and reasoning (di Toffoli 2021), which is deeply connected to the philosophy of mathematics (Korhonen 2013; Hintikka 1973). Hence, it is illustrative to more closely consider A. N. Whitehead's ideas of the education of mathematics. Whitehead, in *Principia Mathematica* (written together with Russell), explicated the logicist philosophy of mathematics. The book was intended as a sequel to Russell's *Principles of Mathematics* (1903), but it "became increasingly evident that the subject is a very much larger one" than the authors had supposed.

A main intention in the logicist program was to reduce mathematics to logic (Steiner 2006; Benacerraf & Putnam 1964). Whitehead and Russell followed the logicist project, which originates from Frege's and to Peano's ideas. Besides logicism (Frege, Russell, Whitehead), two other main approaches in the philosophy of mathematics, namely intuitionism (Brower) and formalism (Hilbert), can be identified. The huge task of the *Principia Mathematica* was to make the reduction explicit (Steiner 2006; Benacerraf & Putnam 1964). So, it is not a surprise that the Principia became such an extensive work of three large volumes.

The *Principia* is a detailed analysis of mathematical concept formation and mathematical reasoning, in which logico-mathematical rigor was emphasized. The foundational ideas of it can be rooted into the Kantian idea of intuition (Anschauung), which is related to perception. Hence, intuition in mathematics is closely connected to the Kantian intuition-based epistemic approach (Korhonen 2013), which entails an important parallelism between the learning of mathematics and the learning of science (Hintikka 2007). The whole picture changed in the 1930s because mathematics was proved to be both incomplete (Gödel 1931) and undecidable (Turing 1936). The epistemological questions, such as questions about mathematical practice, became especially central in the 1960s (Murawski 2010; Korhonen 2013; Kitcher 1986).

In the study of the education of mathematics, the epistemological questions have been central ones which implies that the leading philosophy has been a gathering of different philosophies of mathematics (Steiner 2006). Of course, the logicist philosophy of mathematics has been of extreme importance for educators of mathematics, especially, for instructionists who emphasize, not toward learning but toward "compartmentalized and decontextualized facts", which are intended to be accomplished by clear and effective instructional techniques. There is no reason to believe that such education produces deep conceptual understanding (Sawyer 2014, 2-3), which was emphasized by Whitehead (1911; 1929). According to Whitehead without understanding, mathematics is only an aggregate of unintelligible theorems, which is "fatal in education". The conclusion is that "mathematics, if it is to be used in general education, must be subjected to a rigorous process of selection and adaptation", which entails that, in the education of mathematics, we have to "deal directly and simply with a few general ideas of far-reaching importance" (Whitehead 1929). Whitehead (1911) emphasizes that formal character allows us to extend "the number of important operations which we can perform without thinking about them", which is an essential step in cultivating "the habit of thinking of what we are doing". The formalism makes reasoning visible (de Toffoli 2021). Whitehead emphasizes the role of historical and philosophical knowledge in understanding the true character of mathematical knowledge, as in the case of geometry or trigonometry (Whitehead 1911). A central idea of Whitehead's philosophy of mathematics is the possibility to explicate semantical ideas behind formalisms, which are used as presuppositions of questions in dialogical education. The formal structure of the textbooks of mathematics

(Steiner 2006) entails the important distinction between procedural and conceptual knowledge (Hiebert & Lefevre 1986), which are closely connected: there is no procedural knowledge without some conceptual knowledge and vice versa. In education sciences, it is usual to study dialogical methods empirically (Hofmann & Ruthven 2018). There is also need for philosophico-conceptual study of strategic dialogical teaching of mathematics. Whitehead (1929; 1-2) emphasizes that the "ideas that are merely received into the mind without being utilized, or tested, or thrown into fresh combination" cannot be useful; Whitehead refers to such ideas as *inert ideas*. Socratic dialogues enliven mathematical knowledge. Whitehead does not refer to Socrates nor to Plato. But he has, in his mind, the method of analysis and synthesis, which has the same roots (Niiniluoto 2018). The method of analysis and synthesis can be seen as the foundation of the dialogical method, which can also be seen from the oral tradition, when oral dialogue was enriched by drawings in the sand (Radford 2011), which takes place in Plato's *Meno*. Plato argues that written language ends proper dialogue: the text answers questions always in similar words, which stops the dialogue. (Radford 2011.) Whitehead's idea of the foundational role of historical and philosophical knowledge for learning mathematics allows us to overcome the tension between the formal and substantial understandings of mathematics (Whitehead 1929), which allow us to develop dialogical methods of teaching and learning.

**REFERENCES**

Benacerraf, P. & Putnam, H. (eds.) (1964). Philosophy of mathematics: Selected Readings, Cambridge: Cambridge University Press

Davis, M. (ed.) (1993). The Undecidable. New York: Dover Publications.

De Tofoli, S. (2021). Reconciling Rigor and Intuition, Erkenntnis (2021) 86:1783–1802 https://doi.org/10.1007/s10670-020-00280-x

Gödel, K, (1931/1993). On Formally Undecidable Propositions of the Principia Mathematica and Related Systems. I. In Davis (1993), (p. 5-38).

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural Knowledge in Mathematics. In Hiebert, J. (1986). Conceptual and procedural Knowledge: The Case of Mathematics (p. 1-27). Lawrence Erlbaum Associates.

Hintikka, J. (1973). *Logic, Language Games, and Information: Kantian Themes in the Philosophy of Logic*. Clarendon Press.

Hintikka, J. (2007). *Socratic Epistemology.* New York: Cambridge University Press

Hofmann, R. & Ruthven, K. (2018). Operational, interpersonal, discussional and ideational dimensions of classroom norms for dialogic practice in school mathematics. British Educational Research Journal Vol. 44, No. 3, June 2018, pp. 496–514 DOI: 10.1002/berj.3444.

Kitcher, P. (1986). Mathematical Change and Scientific Change. I T. Tymoczko, (ed.), *New Directions in the Philosophy of Mathematics* (p. 215-242). Birkhäuser

Korhonen, A. (2013). Logic as Universal Science: Russell's Early Logicism and its Philosophical Context, Palgrave Macmillan; London; https://doi-org.libproxy.helsinki.fi/10.1057/9781137304858

Koterwas A, Dwojak-Matras A, & Kalinowska K. (2021). Dialogical teaching of research integrity: an overview of selected methods. FACETS 6: 2138–2154. doi:10.1139/facets-2021-0045.

Mercer, N., Dawes, L. & Kleine Staarman, J. (2009). Dialogic teaching in the primary science classroom, Language and Education, 23:4, 353-369, DOI: 10.1080/09500780902954273

Murawski, R. (2010). Essays in the Philosophy and History of Logic and Mathematics, Brill

Niiniluoto, I. (2018). Truth-Seeking by Abduction. Springer

Prottas, N. (2018). Between Practice and Theory Dialogical Teaching and Art as Performative, Museum Worlds: Advances in Research 6 (2018): 60–73; doi:10.3167/armw.2018.060106

Radford, L., (2011). Dialogism in Absentia or the language of Mathematics, in Silvia Sbaragli (ed.), *La Mathematica e la sua Didattica Quarant'anni di Impegno; Mathematics and its Didactics: Forty Years of Commitment. In Occasion of the 65 Years of Bruno D'Amore*. Pitagora Editirice Bologna

Russell, B. (1903). *The Principles of Mathematics*. Merchant Books.

Sawyer, K. (Ed.) (2014). The Cambridge Handbook of the Learning Sciences.2nd ed. DOI: https://doi.org/10.1017/CBO9781139519526.

Steiner, Mark, 2006, Teaching Elementary Arithmetic through Applications, in Curren, Randall (ed.), A Companion to the Philosophy of Education, Blackwell

Turing, A. (1936/1993). On Computable Numbers, with an Application to the Entscheidungsproblem. In Davis (1993), pp. 116-154.

Whitehead, A. N. (1911/2005). Introduction to Mathematics, New York: Barnes & Noble Books

Whitehead, A. N. (1929). The Aims of Educations and Other Essays. New York: MacMillan Company

Whitehead, A. N. & Russell, B. (2010). Principia Mathematics, Merchant Books.

# THEME 2

# HISTORY AND EPISTEMOLOGY IN STUDENTS AND TEACHERS MATHEMATICS EDUCATION: CURRICULA, COURSES, TEXTBOOKS, AND DIDACTICAL MATERIAL OF ALL KINDS - THEIR DESIGN, IMPLEMENTATION AND EVALUATION

# A PIONEER EDUCATIONAL AID FOR THE LEARNING OF THE FIRST NOTIONS OF GEOMETRY: JULES DALSÈME'S *MATÉRIEL-ATLAS* (1882)

**Paola MAGRONE**
Roma Tre University, Via della Madonna dei Monti 40, Rome, Italy

Paola.magrone@uniroma3.it

## ABSTRACT

This workshop is inspired by the work of the French normal school teacher Jules Dalséme (1845-1904) and is addressed to primary and lower secondary school teachers (future and in-service ones) and it also stems from Ilaria Zannoni's master's degree thesis and internship work, (Zannoni 2018), addressed to 4th to 7th grade pupils, entitled "Geometry as the beginning of scientific thinking: observing, moving, comparing, drawing, representing". Dalséme's *Éléments de Takymétrie (géométrie naturelle): à l'usage des instituteurs primaires, des écoles professionnelles, des agents des travaux public, etc* (1880) presents an original way to teaching geometry to children, going beyond the simple numeracy and proximate to synthetic Euclidean geometry. His goal was to encourage and make science more accessible for everybody, in the spirit of the intentions of the French Third Republic state education. Participants are engaged in hands-on activities taken from the educational aid *Matériel-Atlas* (1882), to calculate the volume of an object commonly found in construction sites.

## *1 Who was Jules Dalséme? An intermediate figure in the XIX century French national mathematical community*

It is appropriate to start the workshop with some information on the figure of Dalséme, in order to understand his proposals to facilitate the general access of mathematics to pupils and to citizens (primary and vocational schools). Jules Dalsème (1845-1904)[44], a French engineer trained at École Polytechnique, devoted most of his life to children's education in mathematics and to the training of future teachers. In 1869 he left his just started military career to devote himself to that of a teacher, starting as a *répétiteur de mathématique* at

[44] A more accurate insight on Dalséme and his cultural profile will appear in the article (Magrone, Millán Gasca, & Zannoni, 2023)

Collège Chaptal in Paris; in 1872 he became a mathematics professor in *École normale d'instituteurs de la Seine* and in 1874 he published his first work *Premières notions de géométrie*, (Hèment & Dalsème 1874), in collaboration with Felix Hément, his former teacher. This first work was adopted as a text in the schools of Paris and marked the beginning of Dalsème's activity as a writer of textbooks for primary, secondary, normal and vocational schools. Dalséme lived in a period of great changes that affected public education: the minister Jules Ferry's (1832-1893) then new laws were the foundation of the conception of the state school system, no longer conditioned by religion and were the basis for a more democratic and progressive society.

Dalséme's *Éléments de Takymétrie* (Dalséme 1880) is inspired by the geometry of the arts and crafts in order to propose an introductory, intuitive geometry, possibly propaedeutic to demonstrative Euclidean geometry; his work is part of a broader current of thought, which saw in science education a way to help society to improve, and individuals to become "reasoning citizens" (Millán Gasca 2015; Magrone, Millán Gasca, & Zannoni 2022). Several editorial proposals in this framework chose a *fusionist* approach (Barbin & Menghini 2014, Karp & Schubring 2014, Menghini 2015), putting together the 2d and 3d geometrical shapes which follow similar rules for the calculation of area, perimeter, and volume. He was directly inspired by the *pont et chaussée* engineer Édouard Lagout (1820-1885), who in 1857 invented the takymetric[45] method which aimed to quickly train uneducated personnel at the time when he was chief engineer in the building of the Adriatic railway in Italy (Fig 1 left). Dalsème intended to bring the takymétric method (ready, immediate, exact) as teaching for primary school teachers, maintaining the main mechanism features: the terminology of the construction yard, which relates to the ancient tradition of the mathematics of arts and crafts, and, in this specific case, to the geometry involved in the construction works; the use of "educational aids" (addressing both intuition and understanding) and the combination of those flat and solid shapes whose measure is calculated according to similar rules (fusionist approach).

Dalsème's proposal is that of an everyday geometry, which is intuitive, and make it accessible to primary education as well as to vocational schools.

---

[45] From the Greek *tachys*, meaning fast, quick, and *métron*, measure: "Fast measure". See also (Leme da Silvia and Moyon) for the diffusioni of Lagout's takymétrie.

*... la méthode intuitive [...] se propose d'agir sur les sens pour pénétrer jusqu'à l'esprit et s'adresse aux facultés intellectuelles par l'entremise d'objets matériels, d'images visibles, de faits, méthode dont l'application et les avantages peuvent s'étendre bien au-delà du cadre modeste de l'enseignement primaire et qui peut se résumer d'un mot la mise en éveil incessante des facultés d'observation. [...] La répétition des choses qui se comptent fit naître les premiers linéaments du calcul. Les idées de forme et de grandeur, issues de tant d'objets environnants, engendrèrent les premières conceptions géométriques. (Dalsème 1889, 5).*


**Figure 1.** From left to right the front covers of (Lagout 1874) and (Dalséme 1880)

## *2 A hands-on workshop from a XIX century cut-out atlas*

The workshop is based on the educational aid contained in the album *Matériel-Atlas de takymétrie* (Fig. 2) which was probably inspired by the richness of the educational materials associated with the takymetry books devised by Lagout.


**Figure 2.** Dalseme's Atlas front cover, source : "*Matériel-Atlas de takymétrie à l'usage des écoles primarires*. Collection de 32 figures tirées en Chromo-typographie à decouvrir et à assembler pour l'enseignemente de la géométrie usuelle". Paris : Librairie classique Belin. Educational aid, designer Jules Dalsème, producer Librairie classique Belin, 1882.

The collection of bi-colored cardboard cards (12 cards, not bound, 14x25cm.), which had to be cut out, folded, and assembled, is a forerunner of the albums we find today in stationery shops. The workshop was designed for 20 participants, to be held in a classroom with tables and chairs; participants had scissors, tape and we suggested that they work in pairs. This workshop was set up according to the experiential *Anfomam* workshops, for the training of pre-school and primary school teachers, see (Lizasoain Iriso, Magrone, Millan Gasca, et al., 2022). In particular this workshop belongs to the category TMb: "*Playing with* primary school math activities. The participants place themselves simultaneously in two worlds, as if they were children, while at the same time they have the perspective of a teacher" (TM from the Spanish: *taller de matemáticas*). We use a comparison of teaching methodologies, educational aids, and show how they evolved throughout history in order to achieve as a main purpose the diffusion of mathematical culture to the *multitude*[46]. It combines historical elements, manual activity, and comments on what was previously done in the classroom. We showed and discussed some images of Ilaria Zannoni's experience with primary school children, taken

[46] The word *multitude* is intended as in (Price 1994)

during her internship, addressed to 4th up to 7th grade pupils; the activities were inspired by (Dalséme, 1880). In Fig. 3 a picture of the decomposition of a cube into three pyramids is shown, which she did together with her pupils.

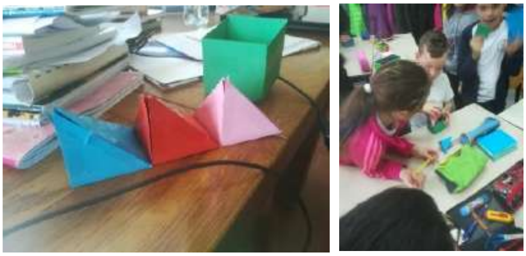


**Figure 3.** Activities in the classroom. Pictures by Ilaria Zannoni

Participants were provided with copies of the Atlas pages relating to the "tas de cailloux" (pile of pebbles), a geometric figure which, according to both Dalséme and Lagout (both mention this object) encompasses an entire geometry course (Fig. 4).

Dalséme introduces the truncated figures in the sixth lesson of (Dalséme 1880); namely, trapezoids and truncated pyramids. In a fully fusionist spirit, he associates, 2d and 3d figures whose measurement rules are similar. The geometric object that encloses the essence of all truncated shapes is the "tas de cailloux":

*Parmi ces volumes, le principal, ou du moins celui que l'on rencontre le plus souvent, nous est offert par les tas de cailloux s'élevant de distance en distance sur les routes et servant à leur entretien [...] C'est la forme que l'on retrouve dans l'auge du maçon, le tombereau du terrassier, le pétrin du boulanger, etc. C'est aussi la forme des gros poids en fonte.* (Dalséme 1880, p.38).

Lagout shares the same enthusiasm for the didactic power of this object: "Le tas de cailloux résume à lui seul toute la géométrie des figures terminées par des lignes droites et des plans, de sorte que sa règle contient l'ensemble des formules déjà trouvées directement." (Lagout 1874, p. 24).


**Figure 4.** Dalséme argues that the tas de Cailloux is found very often, along roadsides and in many other objects that belong to everyday life. From left to right, top to bottom: “tas de Cailloux”, “l’auge di maçon”, “le tomberau du terrassier”, “le pétrin du boulanger”, a big “poid en fonte”

Figures 5-6 must be enlarged and printed in color[47], and distributed to every couple of participants, in particular: figure 5 left represents the entire shape and one of the corner pyramids, whereas 5 right displays the remaining three corner pyramids; figure 6 left and center shows the 2 small and the 2 big sloped sides, which Dalséme calls “talus de longeur” and “talus de largeur”; finally, the central cuboid is in figure 6 right.

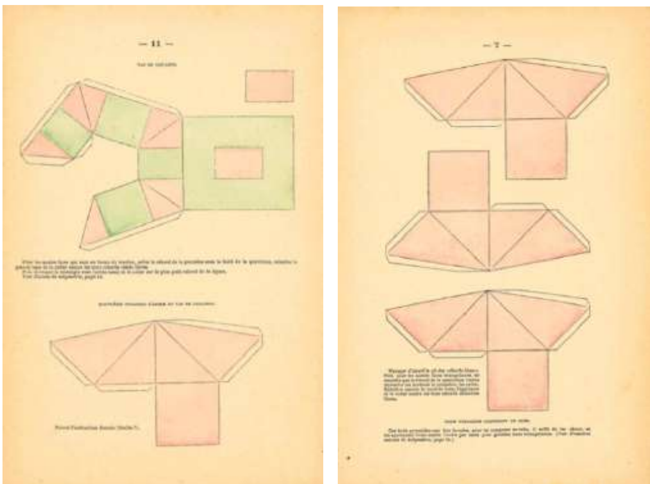


**Figure 5**

[47] The ideal is a 100-120 gr paper (the usual photocopy paper is 80 gr. and it’s too thin)

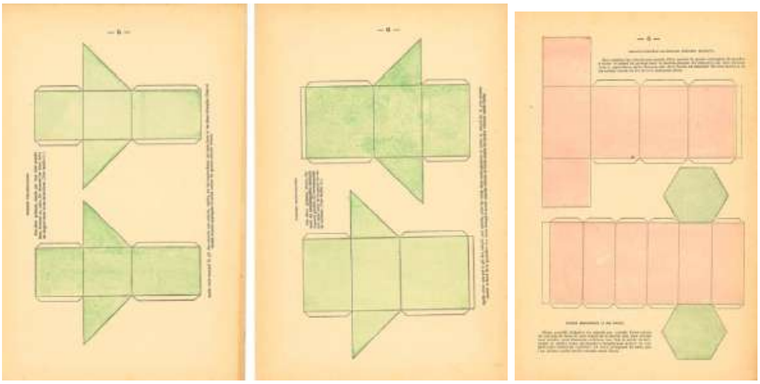


**Figure 6**

First step: cut and assemble the tas de cailloux, as a single object, taken from figure 5 (upper picture); this is necessary in order to have the entire shape in one's hands, as a reference model for the second phase (Fig. 7, left). Second step: cut out from figures 5-6 all the nine parts which will form the entire shape; fold them and assemble them by fixing the single pieces with clear tape (Fig. 7 right).

**Figure 7.** The tas de cailloux as a single object; the solid assembled with the nine parts

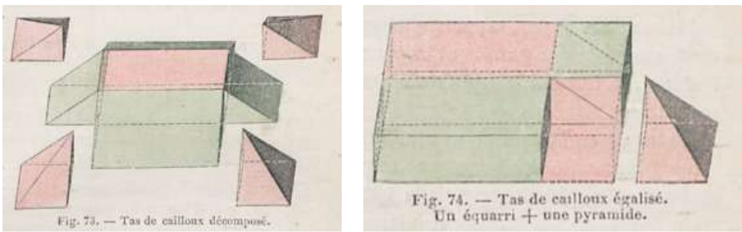



**Figure 8.** The re-assembling of the tas de cailloux, recomposing it in a different way, in order to calculate the volume: on the right it is easy to visualize that the volume

corresponds to that of an equivalent cuboid, plus one of the corner pyramids (Dalséme 1880, 38).

We are following now Dalséme's takymetric reasoning, which he himself describes in pages 38-40 of (Dalséme 1880): in order to measure the volume of a figure, we equalize it to another shape, for which the rule for the volume is known. This process is described with a lot of details, specifying the physical movement to be done: "Je transporte maintenant à droite le talus solide de gauche, mis sens dessus-dessous. Les deux plans de talus, égaux, s'appliquent l'un sur l'autre (comme deux équerres pour former un rectangle). Le tas se trouve ainsi équarri dans le sens de sa longueur" (Dalséme 1880, p 39 and fig. 8). At the end of this procedure, we will obtain two figures, whose total volume is equivalent to that of the *tas*: a cuboid and a pyramid.

Dalséme uses a kind of "spoken algebra": he describes the formulas by words ("multiply", "add", "height" instead of using letters and symbols) and only at the very end of the lesson he writes down a formula with only symbols.

The *tas* is a trapezoid in both length and width, in other words its lateral walls are slanted, and only its ceiling and floor are parallel to each other. So Dalséme proposes to compute its volume by re-assembling the nine pieces in a more regular shape, an "equarri" (a cuboid, Fig 8, right), plus a pyramid. The issue is to compute the right dimensions of the resulting cuboid. We call the two horizontal dimensions of the *tas* "length" and "width" and for each of them, there will be a small one and a big one, since the solid is trapezoidal. By reading carefully pages 38-40 of (Dalséme 1880), or simply observing the geometrical shape, one can deduce that the resulting cuboid has these dimensions:

- The height is the same as the *tas*
- The length is: ½ (big length+ small length)
- The width: ½ (Big width + small width)

The dimensions of the corner pyramid (whose volume must be added) are:

- The height is the same as the *tas*
- The length is: ½ (big length - small length)
- The width is: ½ (big width - small width)

Dalséme's words for the volume of the *tas*: "La 1 /2 somme des longueurs multipliée par la 1/2 somme des largeurs et par la hauteur, plus la 1/2 différence des longueurs, multipliée par la 1 /2 différence des largeurs et par le tiers de la hauteur" (Dalséme 1880, p. 40); then, in formulas:

$$\text{Formule}: \left(\frac{L+L'}{2}\right)\times\left(\frac{l+l'}{2}\right)\times h+\left(\frac{L-L'}{2}\right)\times\left(\frac{l-l'}{2}\right)\times\frac{h}{3}$$

**Figure 9.** The formula for the volume of the tas de cailloux. L= largeurs, l= longeurs; the capital letters mean to distinguish between the minor or major base of the trapezoid; h= height

### *3 Final remarks*

This workshop was attended by about 15 participants, among them university colleagues of different nationalities who deal with the history and didactics of mathematics, and some schoolteachers. Although the subject is elementary geometry, such as the calculation of the volume of a solid with flat faces, the participants showed interest and enthusiasm in carrying out the cutting and assembling activities; hands-on activities are themselves challenging, since they break the usual pattern of the frontal lecture, fostering group work and discussions among participants; the fascination of working with paper and scissors using an educational aid designed more than 100 years ago should not be underestimated. The fusionist approach is an educational aspect that should be retrieved: the geometry books dedicated to schools of every level are nowadays filled with repetitive formulas, which are reproposed, as if they were new each time and therefore to be learned, while instead they deal with already discussed cases. The approach of Lagout and Dalséme of grouping geometric figures that respond to the same calculation rules, such as rectangles and parallelepipeds, cylinders and prisms, trapezoids and truncated pyramids, etc., could be taken as examples to eliminate excess formulas.

**REFERENCES**


Barbin, É., Menghini, M. (2014). History of teaching Geometry. In Karp, Schubring (eds.) 2014, (pp. 473-492).

Dalsème, J. (1880). *Éléments de Takymétrie (géométrie naturelle): a l'usage des instituteurs primaires, des écoles professionnelles, des agents des travaux public, etc*. Paris: Libraire classique d'Eugène Belin.

Dalsème, J. (1889). *Enseignement de l'arithmétique et de la géométrie*. Paris: Imprimerie Nationale.

Hément, F. and Dalsème, J. (1874). *Premierès notions de géométrie*. 2nd ed. Paris: Librairie Ch. Delagrave

Karp, A., Schubring, G. (Eds.) (2014). *Handbook on the History of Mathematics Education*. New York: Springer.

Lagout, É. (1874). *Takymétrie. Géométrie concrète en trois leçon. Accessible-Inaccessible-Incalculable. Cahier d'un soldat du génie. Rédaction des conférences faites par ordres du ministre de la guerre à l'école régimentaire du génie de Versailles*. Paul Dupont, Parigi.

Leme Da Silva, M.C. and Moyon, M. (2020). Tachymetry Production and Circulation Background, *Acta Scientiae*, 22(6), 185-205.

Lizasoain Iriso, M.I., Magrone, P. & Millan Gasca, A., et al. (2022). *Growing with mathematics: experiential math workshops for education of pre-school and primary school teachers a teaching guide for professional development and instruction of future teachers*, Universidad Pública de Navarra, Pamplona. https://academica-e.unavarra.es/xmlui/handle/2454/42415

Magrone, P., Millán Gasca, A. & Zannoni, I., (2023). Science for the multitude: Jules Dalsème's (1845-1904) "natural geometry" for the education of all, to appear on Almagest

Menghini, M. (2015). From Practical Geometry to the Laboratory Method: The Search for an Alternative to Euclid in the History of Teaching Geometry, in S. J. Cho. *Selected Regular Lectures from the 12th International Congress on Mathematical Education,* 561-587. Cham: Springer.

Millán Gasca, A. (2015). Mathematics and children's minds: The role of geometry in the European tradition from Pestalozzi to Laisant, *Archives Internationales d'Histoire des Sciences* 65(2)-175, 261-277.

Price, M. (1994). *Mathematics for the multitude? A History of the Mathematical Association*. Leicester (UK): The [Mathematical] Association.

Zannoni, I. (2018). *La geometria naturale di Jules Dalsème: sforzo divulgativo e contributo all'istruzione matematica primaria nella francia della terza repubblica,* unpublished master degree dissertation, Roma Tre University

# HANDS ON EUCLIDEAN GEOMETRY

**Daniele PASQUAZI**
L.S.S. B. Touschek
Viale Kennedy snc, Grottaferrata, Italy
Departments of Mathematics, Tor Vergata University,
Via della Ricerca Scientifica 1, Rome, Italy
pasquazi@mat.uniroma2.it

## ABSTRACT

The formation of rational thinking is one of the most important general objectives to be pursued in learning. The study of Euclidean geometry makes a valuable contribution in this regard, especially if students are able to grasp the meaning of propositions independently. For these reasons, we consider it essential to make students use teaching tools that simulate mathematical entities and can be manipulated directly by them. Various teaching activities are described in the paper with the overall aim of solving concrete problems using Euclidean propositions.

## *1 Introduction*

The Italian National Guidelines emphasise that it is absolutely necessary to propose problems in school practice, whose search for a solution strategy stimulates productive rather than reproductive thinking. Furthermore, it is considered appropriate for students to develop a positive attitude towards mathematics through meaningful experiences. In fact, it is very important that they understand how mathematics is necessary to develop those processes that are useful for problem solving.

We believe that drawing inspiration from Euclid's Elements for the design of teaching activities enables us to achieve these objectives. The nature of the Elements and the contributions of neuroscience suggest that it is absolutely necessary to make activities truly concrete in order to develop and amplify geometric perceptual skills in students. The manipulation of concrete tools that simulate mathematical objects has many advantages including driving the autonomous discovery of geometric properties of figures. Furthermore, teaching tools built according to mathematical rules allow for self-correction during the discovery process. Hands, therefore, assume a fundamental role in learning mathematics. Such learning, which significantly contributes to the construction of rational thought, absolutely must precede a formalization process if the latter is to become a useful language for understanding mathematics and not an obstacle.

This article describes the main steps of a teaching activity during which students had to estimate how many people could actually occupy a town square. After an initial common part, the activity will be divided into two possible paths that differ according to the age of the target students, 12 - 13 years old (seventh grade) or 14 – 15 years old (tenth grade).

## *2 Theoretical background*

The study of Euclidean geometry helps develop productive thinking (Di Martino, 2017), which is useful in the search for problem-solving strategies. The 'power' of Euclid's proofs lies in the explicit link between visual-spatial skills and verbal reasoning (Enriques et al., 2006). The continuous interaction between the two skills is realized (Russo et al., 2017) by constructing, through the practice of drawing, various geometric entities in order to ensure their existence (propositions called “problems”) and establish the validity of their properties by reasoning on them (propositions called “theorems”).

However, geometric design produces static figures, which do not contribute to imagining their movements as a whole, or of their parts. Such movements are, under certain circumstances, useful in determining a strategy for solving a problem. This can indeed be facilitated by the concrete construction of “artificial manipulatives” (Bartolini Bussi & Martignone, 2020) that simulate geometric figures. Our cognition would thus be amplified, as it is now known that manipulation and perception influence each other (Craighero et al., 1999, Rizzolatti & Luppino, 2001). Specially designed artificial manipulatives are used in the activities described in this document. An important effect of manipulatives is their ability to facilitate the discovery of invariant elements, which we call 'stable structures', that persist despite the movements performed. It is these stable structures that constitute the properties of geometric figures (Pasquazi, 2020).

Manipulatives are primarily designed for preadolescents. For complex topics more suitable for adolescents, it is usually more appropriate to use interactive geometry software. In this case, you lose the advantages of direct manipulation. However, one is facilitated to make more complex deductions due to the possibility of exploring many more cases than with manipulatives. The positive effects of using interactive geometry software were explored by Miragliotta & Baccaglini-Frank (2021) in the field of geometric prediction. The theoretical control provided by the software (as well as manipulatives) facilitates students' research and discovery of geometric properties.

It is hypothesised that the improvement in geometric perception skills due to the manipulatives and software also affects the ability to argue the discovered properties. Property deductions using manipulatives or software are not true Euclidean proofs. But they are preparatory to these. Both allow a continuous interaction between the object (concrete or virtual) and the deduction, between intuition and logic, preparing the groundwork for subsequent formalization.

### *3 Aims of the teaching activity*

The history of mathematics offers many educational insights. An effective way to capture students' attention and interest is to start with concrete problems from which the first mathematical ideas developed. Among these is the problem of estimating very large surfaces. This is a very topical problem whose solution allows, for example, an estimate of how many people are contained in a town square. A concrete problem, therefore, whose resolution is by no means trivial. To solve it, we will work on a map of a town square that has an irregular shape and to calculate the area of these surfaces, we will use some Euclidean propositions. These propositions will be learned following axiomatic deductive logic by means of "figurative proofs" performed directly on manipulatives or on figures simulated by interactive geometry software. After having calculate the area of the town square represented on a map, using a scale factor, the area of the real square will be estimated. We speak of estimation because the accuracy of the calculation will naturally depend on how much the regular polygon chosen to approximate the square is. You will find that increasing the sides of the irregular polygon will give a better approximation of the town square.

### *4 Description of the main steps of the activity*

Based on these assumptions, the mathematical prerequisites are the concept of area as the extension of a plane surface, the calculation of the area of triangles, common notions 2 and 3 of Euclid's Elements, the characteristics of parallelograms and proportions.

All activities were carried out in laboratory mode, proposing open problems to the students divided into groups. The first step in the activity was to pose a realistic problem: Piazza San Giovanni in Rome (where the schools in which the activity took place are located) has always been a venue for public events. By examining various sites that reported the number of participants at such events, the students found that the same square was judged to be full from a minimum of 150.000 people to a maximum of 1.000.000. The question posed was: what is the true capacity of the square? How can mathematics help us answer these questions?

To effectively define the problem, it is necessary:

1) a plan of the town square and its relative scale;

2) a count of how many people can be contained in one square metre;

3) a method for estimating areas that cannot be measured directly.

Looking for a plan of the town square and its scale is now a very easy problem to solve thanks to geographical applications. The students took a screenshot of the town square, including the scale, then turned it into an image to work with.

Knowing how many people can be contained in one square metre is also a very easy problem to solve. After creating a square on the floor with a one-metre side using paper tape, they counted how many people could fit into it. Generally, a minimum of four and a maximum of six people could fit.

Finally, the students had to learn a method to estimate the area of a very large surface. This is by no means an easy problem to solve.

## 4.1 Euclid's propositions.

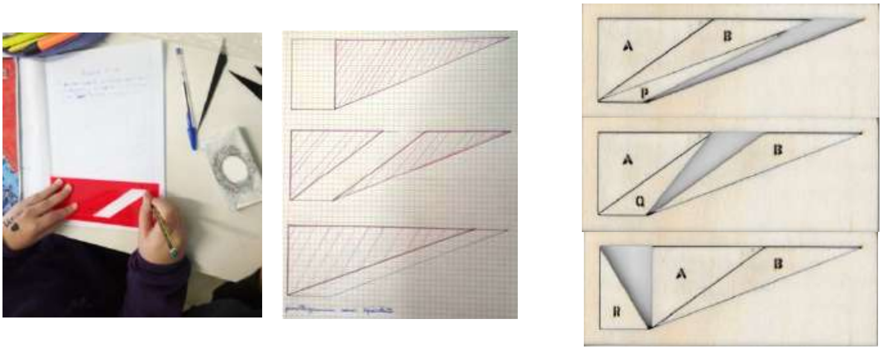
The shape of Piazza San Giovanni is irregular. Therefore, there are no formulas to determine such an area. This forces students to work on geometric rather than arithmetic aspects. It is therefore necessary to transform the town square into an equivalent polygon for which it is easy to calculate the area. To solve this problem we used Proposition I.35 of Euclid's Elements which states: *parallelograms which are on the same base and in the same parallels are equal to one another (read equivalent).*

Students have to discover the previous property for themselves. To this end, they were provided with mathematical manipulatives (Figure 1) consisting of a trapezoid-shaped structure we call the *base*, which always remains fixed, and two triangles (indicated with “A” and “B” in figure), called *mobile figures*. These triangles must be inserted into the base and through their movement it is possible to discover the formation of parallelograms obtained from the difference between the trapezoid and all the mobile figures in it.


**Figure 1.** The equivalent parallelograms within the base

The students had to discover that the resulting parallelograms have three stable structures (Figure 2a and b). They have the same area (consequence of Common Notion 3). They also have base sides and are contained in the same parallels, i.e. they all have the same height. Learning is facilitated by the possibility of rotating the base. What was not observed according to a certain orientation of the base could be discovered with respect to another orientation.

Through appropriate observations, students discovered that the equivalence of parallelograms is a direct consequence of the equality of the base side and height of parallelograms (and not vice versa).




(a) (b) (c)

**Figure 2.** (a), (b) Students at work. (c) The equivalent triangles within the base

The following proposition I.37 of Euclid's Elements was the next goal to be achieved. This states: *triangles which are on the same base and in the same parallels are equal to one another* (read equivalent). The activity will be carried out in exactly the same way as the previous one. The difference will be that three triangles, will be added to the previously available manipulatives. It can be verified that each triangle inserted into the corresponding parallelogram will be its half (Figure 2c). Therefore, the remaining triangular empty spaces will always have the same area.

### 4.2 From a polygon to an equivalent triangle: a method for estimating an area that cannot be measured directly.

In the following activity, the map of Piazza San Giovanni to scale (1:50) and a tracing paper booklet were used. Initially, the last sheet of the booklet was placed on the map tracing the perimeter of the town square. We assume that the town square has been approximated by a non-regular pentagon ABCDE (Figure 3a and b) to be transformed into a quadrilateral with the same area.

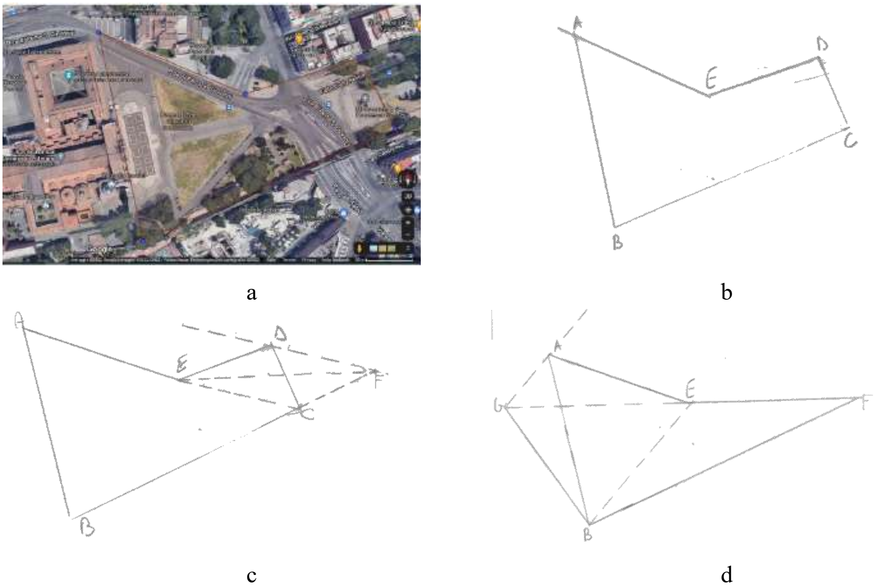



**Figure 3.** (a), (b) Piazza San Giovanni drawn on the last sheet of glossy paper of the booklet; (c) the construction to transform the pentagon ABCDE into the equivalent quadrilateral ABFE; (d) the construction to transform the quadrilateral ABFE to equivalent triangle BFG.

To do this, draw the diagonal CE of the pentagon (Figure 3c); then, draw the parallel to the diagonal CE from point D. This parallel intersects the extension of BC at point F. It then joins point F with point E. It is recognised that triangle CEF has the same area as triangle CED, according to Proposition I.37. Thus, according to Common Notion 2, the initial pentagon ABCDE has the same area as the quadrilateral ABFE.

After redrawing the ABFE quadrilateral on the next sheet of tracing paper in the booklet, the same procedure must be repeated. Having drawn the diagonal BE, the parallel to the diagonal BE is drawn from A (Figure 3.d). The latter will intersect the extension of side FE at point G. Since triangle ABE and triangle GBE are equivalent according to Proposition I.37, triangle BFG is equivalent to quadrilateral ABFE. In conclusion, the final triangle BFG is equivalent to the initial pentagon ABCDE.

At this point, to determine the area of the triangle BFG the length of its side base and height were measured. The side base of the triangle measures $b = 18\,cm$ cm while the relative height measures $h = 6\,cm$. According to the scale used, 50 metres of the town square is equivalent to 2 cm of the map. Therefore, the effective dimensions correspond to $b_R = 450\,m$ and $h_R = 150\,m$. In conclusion, according to the approximation considered, the area of Piazza San Giovanni is $A = \frac{b_R \cdot h_R}{2} = 33.750\,m^2$. Taking into account this low approximation of the square's surface area and the fact that one square metre can hold a minimum of four and a maximum of six people, the students concluded that a minimum of 135,000 and a maximum of 202,500 people can be contained in the entire square.

A further step was to ask the students if and how the approximation of the calculated area could be improved. They had to note, in fact, that by choosing a pentagon to approximate the town square, many spaces in the latter were left outside the pentagon itself. They realized, therefore, that it was necessary to increase the number of sides of the polygon approximating the area of the town square in order to obtain a better estimate of its area.

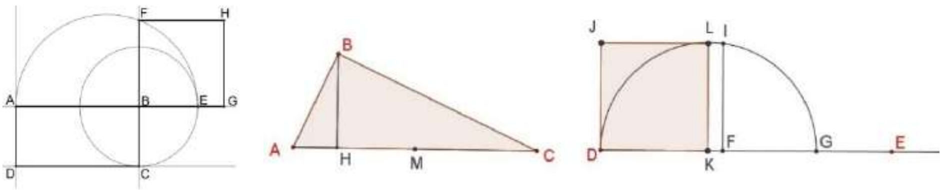
### *5 Summary description of the activity in a 10th grade classroom.*

We believe that the activities carried out up to section 4.1 are also suitable for 10th grade students. The subsequent activity for them is described in section 5.1. The objectives of this activity are: to learn Euclidean theory as a scientific theory by acquiring the general principles of the deductive axiomatic model. To this end, the students first worked on Geogebra and then proceeded to formalize the discovered propositions. As prerequisites, it was necessary to know and be able to apply the Pythagorean theorem (Proposition I.47).

### 5.1 Description of the main steps of the activity.

The students, following instructions on a worksheet, performed the required construction with Geogebra (Figure 4a). The first activity consisted of squaring a rectangle, i.e. constructing a square equivalent to a given rectangle. It is about proposition II.14 of the Elements.

The second activity consisted of constructing a square equivalent to a given triangle. Given triangle ABC (Fig. 4b) with base side AC and relative height BH, the segment DF = AM is constructed on the line DE (M is the midpoint of AC) and the segment FG = BH. Draw the semicircle of diameter DG, from point F draw the perpendicular FI and from vertex D draw the square of side FI, i.e. DKLJ. This square is equivalent to triangle ABC.




a b

**Figure 4.** (a) The students' construction with Geogebra relating to proposition II.14; (b) The construction to transform triangle ABC into an equivalent square DKLJ.

The third activity consisted of transforming the irregular polygon approximating Piazza San Giovanni into an equivalent square. Students search the web for a map of the square with its scale, which is then uploaded to a Geogebra file (Figure 5). The polygon approximating the square, let us assume a pentagon, is outlined. Drawing its diagonals is divided into triangles. Knowing that each triangle is equivalent to the corresponding square, using the Pythagorean theorem the students found the area of the square FGHI as the sum of the areas of the different squares. Finally, the area of Piazza San Giovanni is determined based on the scale factor provided.

Using the software, the approximation soon becomes satisfactory. In fact, by approximating Piazza San Giovanni with a pentagon, it is established that it has an area of approximately 40,000 square metres. Again, by increasing the sides of the polygon approximating the town square, the approximation was greatly improved.

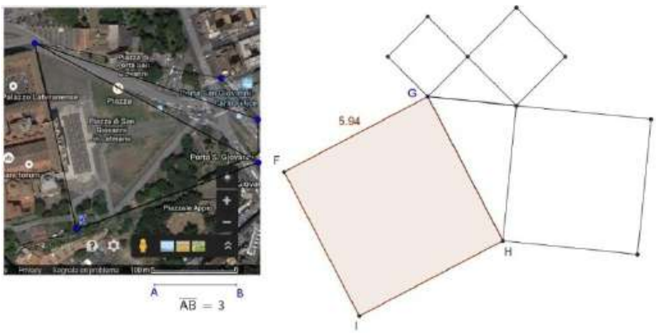



**Figure 5.** The algorithm to approximate the Piazza San Giovanni.

### *6 Conclusions and discussions*

At the end of these activities, we have always noticed great satisfaction among the students. The reasons, in our opinion, are several. Firstly, students discover that the Euclidean geometry they study in school helps build their reasoning skills to the point where they are able to verify the validity of the information they learn from the media. They realise, therefore, that their critical thinking is greatly enhanced. Furthermore, they were actively involved in the activities, as protagonists. The discoveries they made were the result of their efforts. The students constructed their learning through own conjectures, subsequent verification and self-correction. Our belief in the effectiveness of manipulatives comes from studies of primates (Rizzolatti et al., 1998) which allow us to make inferences about humans by analogy. We know that manipulation of concrete objects allows for a 'pragmatic' mental representation of the object that facilitates the retrieval of associations established between movements performed on the object and related concepts (Jeannerod et al., 1995). Thus, the development of perceptual skills makes an important contribution to our cognition, because what is learnt during the use of mathematical manipulatives can generate the ability to imagine the same movements when observing drawn figures. Moreover, these skills appear to be long-lasting (Pasquazi, 2020). Therefore, knowledge and skills acquired during interaction with manipulatives can also be transferred to similar contexts. This transfer is all the more effective and persistent over time the more stable structures can be identified during the interaction with mathematical manipulatives. In conclusion, we believe that the learning of Euclidean geometry, which is fundamental for the development of rational thinking and is considered in some respects a scientific theory based on the concreteness of design, is certainly facilitated by the use of manipulatives. Therefore, we consider it necessary to further investigate the effectiveness of these methodologies.

## REFERENCES


Bartolini Bussi, M.G., Martignone, F. (2020). *Manipulatives in Mathematics Education*. In: Lerman S. (eds) Encyclopedia of Mathematics Education. Springer, Dordrecht. https://doi.org/10.1007/978-3-030-15789-0_93.

Craighero, L., Fadiga, L., Rizzolatti, G., Umiltà, C. (1999). Action for perception: a motor-visual attentional effect. *Journal of experimental psychology: Human perception and performance*. 25.6, 1673. https://doi.org/10.1037/0096-1523.25.6.1673.

Di Martino, P. (2017). Problem solving e argomentazione matematica [Problem solving and argumentation]. *Didattica della matematica. Dalle ricerche alle pratiche d'aula*, (1), 23–37. https://doi.org/10.33683/ddm.17.1.2.

Enriques, F., Ghione, F., Moretti, M. (2003). Insegnamento dinamico [Dynamic teaching]. *Agorà*. 93–103.

Jeannerod M, Arbib M.A, Rizzolatti G, Sakata H. (1995). Grasping objects: the cortical mechanisms of visuomotor transformation. *Trends in Neuroscience*. 18(7), 314-20. https://doi.org/10.1016/0166-2236(95)93921-J.

Miragliotta, E., Baccaglini-Frank, A. E. (2021). Enhancing the Skill of Geometric Prediction Using Dynamic Geometry. *Mathematics*, 9(8), 821. https://doi.org/10.3390/math9080821.

Pasquazi, D. (2020). Capacità sensoriali e approccio intuitivo-geometrico nella preadolescenza: un'indagine nelle scuole [Sensory skills and intuitive-geometric approach in pre-adolescence: a survey in schools]. *Cadmo* (1), 79-96. https://doi.org/10.3280/CAD2020-001006.

Rizzolatti G, Luppino G. (2001). The cortical motor system. *Neuron*. 27, 31(6), 889-901. https://doi.org/10.1016/S0896-6273(01)00423-8.

Rizzolatti, G., Luppino, G., Matelli, M. (1998). The organization of the cortical motor system: new concepts. *Electroencephalography and clinical neurophysiology*, 106(4), 283-296. https://doi.org/10.1016/S0013-4694(98)00022-4.

Russo L., Pirro G., Salsiccia E., (2017). Euclide, Il I libro degli Elementi. Una nuova lettura [Euclid, The First Book of the Elements. A new reading]. *Carocci Edit*

# TEACHING SIMILAR TRIANGLES IN HISTORICAL PERSPECTIVE

**Polimnia ANDREADOU, Konstantinos NIKOLANTONAKIS**
University of Western Macedonia, Florina, Greece
pandreadou@sch.gr, knikolantonakis@uowm.gr

**ABSTRACT**

This study presents a research designed in order to investigate the way that the implementation of history of mathematics in the teaching process affects the understanding of the concept of similar triangles by the students. It was carried out during the 2020-21, and twenty-one 14-year-old students participated in it. This didactical intervention helped students to acknowledge not only the importance of the concept of similar triangles, but its historical evolution as well.

## *1. The integration of the History of mathematics in the teaching of the similarity of triangles.*

Many researchers have referred to the benefits of using the History of Mathematics in teaching. The reasons why incorporating history is useful have been categorized from time to time and presented in various articles. Fried (2001) summarized these arguments into three categories. The first has to do with the fact that the history of mathematics makes mathematics more human by connecting the study of mathematics with human motives. The second is that it stimulates students' interest by giving variety to the approach of different concepts and by showing the role of mathematics in society. The third category refers to the fact that history provides knowledge about concepts, problems and problem solving.

### 1.1 Ways of using the history of mathematics in teaching similar triangles.

In this particular teaching intervention, we made an approach to teaching the similarity of triangles, inspired by the historical evolution of the concept. The highlights of this development were identified and given to students through a sequence of problems adapted to modern teaching contexts. Thus, in this way as reported by Tzanakis and Thomaidis (2000), history offers a deeper and more comprehensive understanding of the specific subject.

Also, an authentic historical source was used, namely an excerpt of J. Errard's book *La géométrie et pratique générale d'icelle*. In this excerpt, Errard describes the construction of an instrument for measuring inaccessible distances as well as how measurements are made on a flat surface.

According to Jahnke et al. (2000), three concepts best describe the teaching effect of using authentic sources: The concept of replacement, the concept of reorientation and the concept of cultural understanding.

In the first case, the integration of history offers something different from the usual, because mathematics is seen as an intellectual activity rather than a set of knowledge and techniques. In the sense of reorientation, in understanding a historical text, we remember that concepts were invented and did not just emerge. According to the cultural understanding, mathematics is placed in a specific scientific and technological context of the history of ideas and societies. By studying this passage, the students had the opportunity to understand the social need that led to the construction of Errard's tool. They could also understand that the concept of similarity of triangles is not only limited to the techniques of calculating the sides of a triangle but has important applications in problematic situations of everyday life.

### 1.2. The Errard's historical instrument and its role in teaching similar triangles.

An instrument in mathematics is the result of an invention and its use can certainly give more knowledge about it than a simple description of it in words (Barbin, 2016). The introduction of cultural artefacts into the classroom also strengthens the connection between school mathematics and the knowledge of everyday life (Bussi, 2000). In addition, it is important that the construction of artefacts helps the interaction and communication between students who work in team to achieve the goal (Bussi et al., 2014).

For the above reasons, therefore, in addition to studying the passage in which Errard's instrument is described, the students had the opportunity to construct and use it.

## *2. The research*

### 2.1. Purpose and research question.

Our research intervention study the similar triangles. It was implemented in the 2020-2021 academic year to twenty-one students in ninth grade. The purpose of the research was to study how the integration of the History of Mathematics in the teaching of similar triangles, affects the degree of understanding of the concept by the students.

The research question was: In what way does students' dealing with historical problems and constructing/using a historical instrument affect their understanding of the concept of similar triangles. The research tools were: seven work sheets, a cognitive test and researcher's notes during the intervention. We gave one teaching hour to each worksheet.

## 2.2. The intervention.

In the first worksheet, as an introduction to the concept of similar triangles, we prepared activities based on the Thales' conclusion that "multiple gnomons have equally multiple shadows". Students worked in a GeoGebra file (Fig.1) and found the equal ratio of the homologous sides of two right triangles. After that, they compared the angles of the two triangles, and they concluded that they are equal. In that way, they came up with the definition of similar triangles.

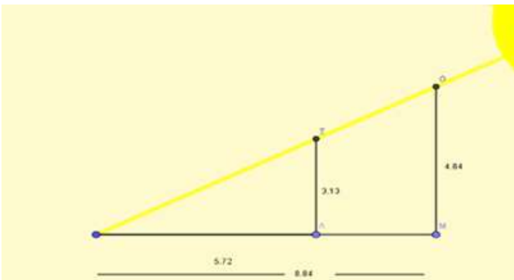



**Figure 1.** An image from student's work on GeoGebra

After that, the students had to write down the differences between similar and equal triangles. They also had to draw a similar and an equal triangle to one triangle designed in their worksheet, using geometrical tools.

We noticed that the students pointed out the differences and easily constructed the shapes. They seemed to have understood the stability of the ratio of the homologous sides of similar triangles. However, they met difficulties in describing the process they followed to design the two shapes. Only four students made an appropriate description. Some students could not find the appropriate words for their description, while others only mentioned the definitions of equal and similar triangles and then sketched the triangles.

In the second worksheet students had to deal with two tasks. In the first task students had to draw, without using geometrical tools, two similar triangles in the plane and two similar triangles in space. Students draw similar triangles in the plane successfully, but it was difficult enough for them to imagine and make such shapes in space. Unfortunately, students of that age are not familiar with the shapes in space or with the concept of perspective in general.

In the second task students worked on a 3-D GeoGebra file (Fig.2). At first, they had to find that the two triangles are similar, using the definition of similar triangles. Then they could move M and N point (Fig.2) and they noticed that the corresponding angles are still equal. In that way they concluded that two triangles with equal corresponding angles are similar.

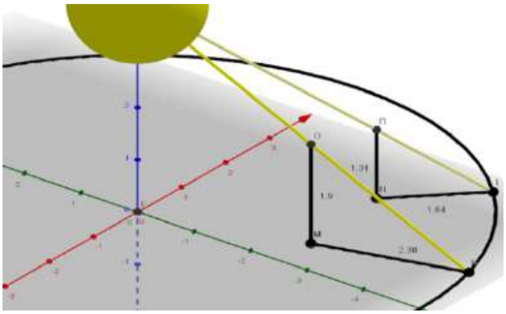


**Figure 2.** Another image from student's work on GeoGebra

In the third worksheet, students studied an excerpt of an original historical source from Plutarch's work "Morals". They tried to understand the way that Thales measured the height of the Great Pyramid of Giza, thus understanding the need for finding a way of measuring inaccessible heights. After that, they were given the appropriate distances and they had to find the height of the pyramid and describe the procedure (Fig.3).

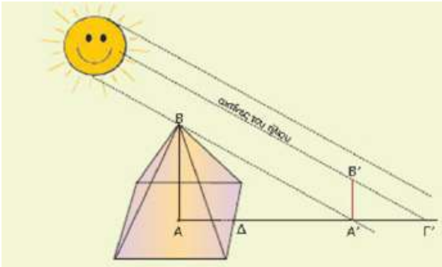



Figure 3.

The discussion between the students about the method followed by Thales in measuring the height of the pyramid, stimulated their interest and most of them managed to calculate it. They used the criterion of similarity to prove that the two right triangles were similar and then they found the height using the equal ratio of the homologous sides. However, only 11 students managed to describe verbally that procedure.

Two examples of the descriptions of the students are given below:

"He used the similar triangles which have proportional sides and equal angles. Thales found the ratio of the shadows and thus managed to find the ratio of the rods, i.e. the height of the pyramid."

"Thales calculated the pyramid's height using the triangles ABA′ and A′B′Γ′. Both triangles are formed by the rays of the sun. As we already know, the rays are not far apart, so the angles B and B′ are equal. The angles A and A′ are right angles therefore according to the similarity criterion the two triangles are similar." For the fourth worksheet, we used an original historical source translated in Greek, some excerpts from the book, *La géométrie et pratique générale d'icelle*, by the 16th century French engineer Jean Errard. In this book, Errard describes the design and construction of an instrument for measuring inaccessible lengths, based on the theory of similar triangles (Fig.4). The aim was for students to learn about another way of measuring inaccessible distances and to think about aspects of everyday life that we can use this instrument. The students liked the study of this original source but found it quite difficult to understand the contents of the passage.

Nevertheless, all the students after processing the text information about the Errard's instrument managed to make a drawing of it on their worksheet.

32 LIVRE I. DE GEOMETRIE

OR. ie deſirerois que la compoſition fuſt telle: Que deux reigles de leton bien droictes fuſſent conioinctes en forme de compas commune AB, AC, & de lõgueur chacune d'vn pied & demy ou enuiron, & de largeur d'vn poulce, ſe mouuans & tournans au centre A, apres que ſur la reigle AB, (que nous appellerons theſe) ſoit vne graueure pour couler vn demy anneau marqué icy D, lequel enclorra vne autre regle EF (appellée baſe) laquelle ſera de ſemblable longueur, mais de largeur egale aux autres, ou vn peu plus eſtroicte, & qui ſe coulera auec le demy anneau au long de AB, & en ſorte que ſur ſon centre E, (qui ſera iuſtement ſur la ligne droicte AB,) elle pourra s'incliner & faire tel angle qu'on voudra auec la reigle AB, au long & ioignant les ſuperficies des reigles de la theſe & de la regle mobile AC, & laquelle neantmoins pourra eſtre arreſtée par le moyen de quelque vis appliquée audit demy anneau.

Cela ainſi compoſé, & les pinnulles eſtant miſes aux poincts ABC (comme on a accouſtumé faire à tous inſtrumens) Que chacune deſdites trois reigles ſoit diuiſée en 300 parties egales, ou en autre nombre tel qu'il plaira. Finalement ſoit attaché à la reigle de la baſe vn pied pour ſouſtenir, & ſur lequel ſe puiſſe mouuoir tout l'inſtrument de coſté & d'autre en telle inclination qu'on voudra, attaché, di-je, à la baſe, en tel endroit qu'icelle eſtant en peſanteur egale aux deux autres, puiſſe ſeruir de contrepois, afin que le mouuement de l'inſtrument en ſoit plus doux & aiſé. Ie laiſſe à diſcretion la fabricature du pied auec les ioinctures qui y doiuent eſtre: Seulement i'ay icy depeint audit pied vn globe enchaſſé dans vne concauité, parce qu'il

Figure 4.

Based on the previous worksheet, as well as on the instructions of the teacher (fifth and sixth worksheets), the students were divided in groups and each group made its own instrument which they used to measure an inaccessible height in the school building (Fig.5). To construct the instrument, they used: three numbered pieces of modelling paper 15 cm long each, a bolt, a plastic sight, a clamp and a pin. Students could move two of three pieces and they had to keep the other one horizontal by using a level thread.

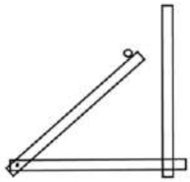


Figure 5.

In the seventh worksheet students dealt with three historical mathematical problems which were all solved by using the similar triangles concept.

The first problem was solved by Thales himself and involved measuring of the distance of a ship from the harbour (Fig.6). In this problem, about half of the students described verbally the process followed by Thales, but found it difficult to prove that the triangles are similar. They could not find the pairs of equal angles, because no one mentioned in his description the parallelism of the two sides of the triangles.

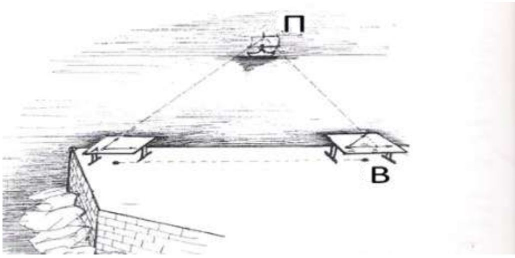



Figure 6.

The second problem was the measuring of the depth of a well, which is included in the Chinese book, *Jiuzhang suanshu*, written between 100 BCE and 100 CE. In this problem, the students very easily recognized the similar triangles and using the appropriate equality of ratios calculated the required depth (Fig.7).

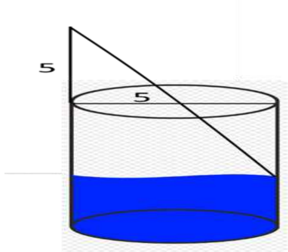



Figure 7.

The third problem was about the "Efpalinos tunnel", a tunnel constructed in the Greek island of Samos in order to irrigate the city of Samos from a spring high in the mountains (Fig.8). First, the students watched a relevant video and then, based on this, they had to describe how Efpalinos measured the length of the tunnel. They had a lot of difficulty, so it was necessary to watch the video twice and have a discussion with the teacher so that they could be able to describe the process Efpalinos followed using the similarity of triangles.

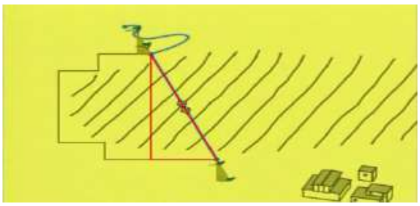


**Figure 8.**

In the cognitive test, there were four problems. The first two problems (Fig.9) were about two different ways of calculating an inaccessible height. The students had to calculate the height of these two buildings and to describe how they did it. Almost everyone calculated the height using the equality of the ratio of the homologous sides, in both problems. However, in the first shape was difficult for them to find the pairs of the equal angles, in order to prove the similarity of the triangles. Only six students finally proved it. Furthermore, seven (other?) students described verbally the process that someone could follow to calculate the height of the two buildings.

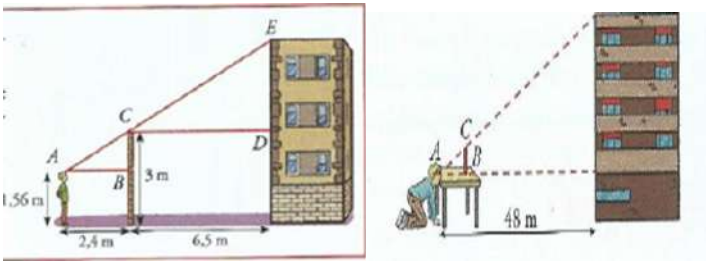



Figure 9.

In the third problem, students were asked to calculate the length of the diving board (Fig.10). 19 students successfully calculated this length. 12 of them used the similarity of the right triangles, that one is contained into the other. They found it easier to find the equal angles.

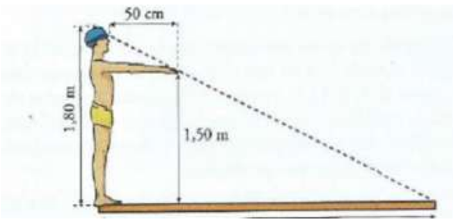



Figure 10.

In the fourth problem, students had to calculate the unknown lengths x and y of the sides of the second triangle (Fig.11). 19 students proved the similarity of the triangles and found x and y, using the equal ratios of the homologous sides.

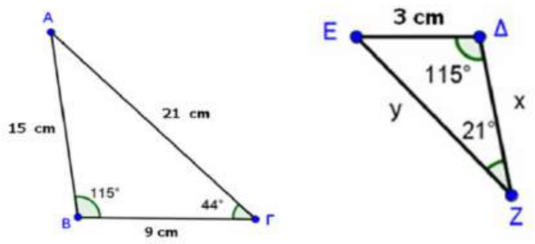



Fig.11

### *3. Conclusions*

The purpose of our intervention was to investigate the extent to which the implementation of the History of Mathematics in the teaching process will help the students to understand better a specific subject. The subject we chose was the similarity of triangles. It was obvious that the enrichment of the lesson with the historical references stimulated the interest and the curiosity of the students and helped students to cooperate with each other and also to create a positive climate about dealing with the subject of similar triangles. The fact that we used a hands-on approach, in particular the students' construction of Errard's tool, contributed to a better understanding of the concept of the ratio of the homologous sides of the triangles, as they had to do the measurement and design process themselves. With this construction and also by solving historical problems, the students understood why the concept of similarity of triangles is important and where it is used, which contributes positively to its understanding. Another thing that we noted during the intervention was that the students struggled in the questions where they needed to develop a verbal proof. They also faced difficulties when they had to transfer the real situation to paper, during the measurement of an inaccessible distance. This research took place during pandemic. The schools in Greece were closed for almost three months. This fact affected the planning of the teaching intervention and limited our available time as it had to be carried out when the schools were open. It would be interesting if this research happened again under normal conditions.

**REFERENCES**

Barbin E. (2016). L'instrument mathematique comme invention et connaissance en action. *Menon. Online Journal of Educatonal Research*, UOWM.

Bartolini Bussi, M. G. (2000). Ancient instruments in the mathematics classroom. *History in mathematics education: The ICMI study*, 343-351.

Bussi B., Chiappini G., Paola D., Reggiani M., Robbuti O. (2014). Learning Mathematics with tools. *History in mathematics education: The ICMI study.*

Deligiannidis, T., & Nikolantonakis, K. (2018, July). *Developing geometric proportional thinking to 6th Grade students with the use of a historical instrument of Errard de Bar le Duc*. Paper presented at the 8th European Summer University on Epistemology and History in Mathematics Education, Oslo Metropolitan University, Oslo, Norway.

Errard, J. (1602). *La géométrie et practique generalle d'icelle* [Geometry and its general practice] (2nd ed.). Paris: Auvray.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science &Education*, 10, 391–408.

Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., El Idrissi, A., Weeks, C. (2000). The use of original sources in the mathematics classroom. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education. The ICMI Study* (pp. 291-328). Dordrecht: Kluwer Academic Publishers.

Tzanakis, C., & Thomaidis, Y. (2000). Integrating the close historical development of mathematics and physics in mathematics education: Some methodological and epistemological remarks. For the Learning of Mathematics, 20(1), 44–55.

# COLLEGE GEOMETRY FROM AN ADVANCED HISTORICAL STANDPOINT FOR MATHEMATICS EDUCATION

**Celil EKICI**
Texas A&M University-Corpus Christi, 6300 Ocean Dr. #5825, Corpus Christi, Texas
celil.ekici@tamucc.edu

## ABSTRACT

Following Felix Klein, an advanced historical standpoint is here presented for teaching college geometry for teachers. Three main ways of developing an advanced historical standpoint are discussed with classroom experiments. One is building connections among geometries by developing an inquiry into definitions of geometric objects such as rhombus, their extensibility with their family relationships across Euclidean and non-Euclidean geometries. Second is on the multiplicity and extensibility of transformations as represented by two historical approaches advocated by Klein and Usiskin. The third way to develop an advanced standpoint is by developing a critical look into a geometry practice tracing its change with the reforms in school geometry. The practice of constructions to connect geometry and algebra is impacted by two historical efforts. One is a supportive effort by Hilbert on the practice of constructions by Hilbert's Algebra of Segments dating back to 1902 to connect geometry and algebra. The other historical reform effort is by School Mathematics Study Groups (SMSG) during 1960s, which led to weakening the axiomatic foundations of the practice of constructability and exactness. The case of SMSG's angle construction axiom is criticized in their revision of axiomatic foundations of school geometry. Three approaches to develop an advanced standpoint informing research and practice of geometry teacher education towards a more historically connected stance.

## *1 Advanced Historical Standpoint on Geometry for Teacher Education*

It is important for mathematics teachers to know some of the history of mathematics, but also the history of mathematics education. Felix Klein's *Elementary Mathematics from an Advanced or Higher Standpoint* (1908/2016) made a historical impact towards advancing mathematical preparation of teachers by analyzing elements of mathematics with its fundamental concepts informing school mathematics. Kilpatrick (2019) brought to the attention of history mathematics education researchers that Felix Klein's double discontinuity between university-to-school mathematics and triple approach to address it, by a unified approach to show how problems in branches of mathematics are connected (e.g., geometry, algebra), and how they are related to the problems of school mathematics. This approach focuses on improving teacher education and mathematical knowledge for teaching with a scholarship on practice by revising the mathematics content courses for teachers to help them gain a higher standpoint. Following Klein's epistemological approach, a higher stance for school geometry is targeted here with future teachers by focusing on the connections within sub-disciplines of mathematics, rather than treating them separately, and providing more unified view through common elementary constructions. Also integrating the practice of developing advance perspective as exemplified by Usiskin, Peressini, Marchisotto and Stanley (2003), the advanced perspective on geometry is pursued here by focusing on alternative definitions of familiar geometric objects, their extensions, and connections. This approach aligns with the recommendations made by Conference Board on the mathematical preparation of teachers suggesting future teachers to complete three courses with a focus on school mathematics from an advanced viewpoint (CBMS, 2012).

This paper with its three parts provides a contribution on the mathematical education of teachers by building a scholarship on teaching geometry with an advanced historical standpoint through classroom experiments. First, an inquiry-based approach is offered by revisiting familiar geometric objects in alternative geometries, with an attention to rhombus by exploring its definitions and connections within and across geometries. Instructional artifacts are given to describe this novel perspective in teacher education to develop knowledge of connections between different geometries. Second part focuses on the transformational geometry as advanced by Klein (1906) and Usiskin et al. (2003) for teaching geometry. The extensibility of transformation perspectives such as isometries and dilations across geometries are discussed. Third part is about developing a critical stance on the practice of construction as impacted by Hilbert's Algebra of Segments and the changes on axiomatic foundations of geometry.

## *2 Classroom Experiments in Building Advanced Perspectives*

This work builds on author's scholarly mathematics teaching and learning research on geometry courses mainly for teachers integrating historical perspectives. Instructional materials incorporating higher standpoint on school geometry are experimented in undergraduate and graduate courses for teachers.

### 2.1 Building connections across geometries with familiar geometric objects towards developing an advanced standpoint

College Geometry course for teachers integrates historical perspectives by emphasizing alternative axiomatic foundations for geometry including Euclid’s, Hilbert’s, SMSG, and transformational geometry. The extensibility of the geometric objects such as rhombus or parabolas across Euclidean and non-Euclidean geometries are investigated to gain higher standpoint.

Preservice teachers explored the extensible definitions of the geometric objects exploring their alternative definitions in alternative geometries. Rhombus is a shared parent object for equilateral quadrilaterals subsuming squares in Euclidean and quasi-squares in spherical and hyperbolic geometries, providing a contrast to Saccheri squares building on perpendicular adjacent sides. The observed characteristics common to all three constructions of rhombus was that the diagonals are perpendicular and bisect each other in all three geometries. Students used this property as a defining characteristic of squares/quasi-squares in Euclidean and Non-Euclidean geometries. A square is redefined extensibly as a geometric object across alternative geometries as a rhombus with congruent diagonals (See Fig 1).

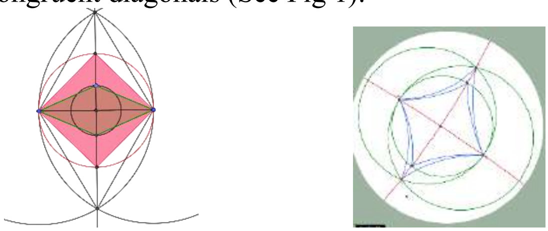


**Figure 1.** Quasi-Squares from rhombus as equilateral quadrilaterals with congruent diagonals extensible to Hyperbolic and Spherical Geometries

Higher perspective is gained by revising a familiar geometric object in alternative geometries and gaining a new sense by extending the geometric object into other geometries redefining it through its viable manifestations.

### 2.2 Advancing higher stance by historical perspectives on transformations for teaching school geometry

Elementarization of transformation perspective and groups were two main drivers of historical shifting efforts during the reform efforts in school geometry in USA (Schubring, 2019). Klein defines Euclidean geometry as a science that studies those properties of geometric figures that are not changed by similarity transformations. Historical perspectives on definitions of isometries were compared by students analyzing textbooks comparing the alternative definitions of isometries as defined by experts such as F. Klein (2004), and Z. Usiskin et al. (2003). Students generated their own definitions of reflection, rotation, and translation. Students worked on alternative definitions of reflection avoiding common circular definitions. Students examined the definition by the textbook defining reflection about line *l* as a transformation of the plane which, for every point P on the plane: P = P' (if P is on *l*) and, *l* is the perpendicular bisector of PP' (if P is not on l). Students criticized this definition since it used the reflected point P' as a part of reflection, which is clearly not an operative definition to construct a reflection of P but it helps us validate/refute a point P' if it is a reflection. To advance their perspective, students developed a non-circular definition of reflection for a given P and *l* based by constructing congruent triangles APB and AP'B for any two points A and B along *l*, forming a kite APBP' with perpendicular diagonals, which helped to build students' inquiry into relevant propositions to justify.

### 2.3 Advanced Stance by Studying Historical Changes in the Practice of Constructibility and Algebra of Segments in Geometry Education

D. Hilbert (1906) and School Mathematics Study Group (SMSG) during 1960s advocated two opposing historical perspectives related to the practice of constructability. SMSG developed a revised axiomatic system on school geometry during the New Math reforms in 1950s. Hilbert's *Foundations of Geometry* (1906) presented a revised Euclidean axiomatic system containing the Algebra of Segments that can be traced back to Euclid's Elements and Descartes' geometric method of constructing segments to solve polynomial equations (Bos, 2001). Building on Hilbert's Algebra of Segments, students were here given segments a and b to produce segment c corresponding to the addition, subtraction, division and multiplication. Students constructed geometric multiplication (Fig. 2). Given lengths were placed along axes. Parallel lines were constructed. Depicted by the circle, the congruency of x*y and y*x indicated the commutativity of geometric multiplication.

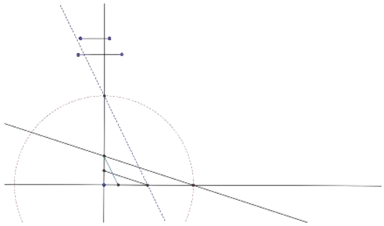


**Figure 2.** Geometric multiplication of given two segments

Students realized that constructing the division is a multiplication with an inverse. Division was built on finding inverse of a point with respect to a circle with a given radius. This path of development helped students understand how one can build segments corresponding to a polynomial equation as introduced by Descartes (Bos, 2001). Both Euclid's or Descartes' geometry produces the exact measure of length or area through geometric constructions (Bos, 2001). Students analyzed Descartes' method on using geometry to solve quadratic equations. A higher stance was gained by highlighting connections between geometry and algebra that was lost in school geometry. Next, students approached the angle related axioms with a critical stance comparing axiomatic systems. Hilbert's Congruence Axiom (Postulate III.4) states that if ∠ ABC is an angle and if B'C' is a ray, then there is exactly one ray B'A' on each "side" of line B'C' such that ∠ A'B'C'≅∠ABC. Half-century later, SMSG's Angle Construction Axiom suggests that for every r between 0 and 180 there is exactly one ray AP with P in H such that m∠PAB=r. This axiom is essentially about existence and uniqueness of a terminating side of an angle for a given measure. While it is not about its constructability, it was named as "Angle Construction Axiom" with consequences to trivialize the construction of angles for any measure. While the exactness had been an essential feature of geometry throughout history (Bos, 2001), the exact constructions of angles lost its sense with the SMSG axioms. The students' focus was on geometric construction of segments with irrational lengths with regular polygons and their connections to solving polynomials, connecting golden triangles, pentagons, and golden ratio and solving $x^2$ -$x$+1=0 with geometric approach.

## 3 *Discussion and Conclusion*

This presentation provides a contribution on the mathematical education of teachers by developing an advanced stance for geometry teachers by building connections across geometries, axioms, integrating algebra and geometry with historical perspectives. Algebra of Segments is a neglected historical component during the school mathematics reforms in 20th century. Elementarization and Klein's "historical shifting' are complementary processes in the transformations of school mathematics responding to advances in mathematics and mathematics education (Schubring, 2019). It is exemplified here that historical shifting process does not always yield the desired consequences.

**REFERENCES**


Bos, H.J.M. (2001). Redefining geometrical exactness: Descartes' transformation of the early modern concept of construction. New York: Springer.

Conference Board of the Mathematical Sciences (CBMS) (2012). The Mathematical Education of Teachers II. Providence: American Mathematical Society.

Euclid, Heath, T.L., & Densmore, D. (2002). Euclid's Elements. Santa Fe, NM: Green Lion.

Hilbert, D. (1902). Foundations of geometry (EJ Townsend, Trans.). Chicago. Open Court.

Kilpatrick, J. (2019). A Double Discontinuity and a Triple Approach: Felix Klein's Perspective on Mathematics Teacher Education. In H.-G. Weigand et al. (eds.), The Legacy of Felix Klein, ICME-13 Monographs, (pp.215-226). Springer. https://doi.org/10.1007/978-3-319-99386-7_15

Klein, F. (2016). Elementary mathematics from a higher standpoint. Volume II: Geometry (G. Schubring, Trans.). Berlin, Heidelberg: Springer.

Klein, F. (2004). Elementary mathematics from an advanced standpoint: Geometry. Mineola, NY: Dover Publications.

Usiskin, Z, Peressini, A., Marchisotto, E., & Stanley, D. (2003). Mathematics for High School Teachers: An Advanced Perspective. New York: Prentice Hall.

Schubring, G. (2019). Klein's conception of 'elementary mathematics from a higher standpoint.' In H.-G. Weigand, W. McCallum, M. Menghini, M. Neubrand, & G. Schubring (Eds.), The legacy of Felix Klein (pp. 169–180). Springer.

# A PARTICIPATIVE RESEARCH WITH IN-SERVICE SECONDARY SCHOOL TEACHERS ON THE INTRODUCTION OF THE HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION: AN OVERVIEW AND SOME PRELIMINARY RESULTS

**David GUILLEMETTE**
Université du Québec à Montréal, C. P. 8888, succursale Centre-Ville, H3C 3P8, Montréal (Québec), Canada
guillemette.david@uqam.ca

## ABSTRACT

In this paper, we present an overview of a study that seeks to better understand the contribution of the historical and cultural dimension of mathematics in the context of secondary school mathematics teaching. Seeking to give mathematics teachers a voice, the study takes the form of participatory research in which teachers and researchers come together to reflect on the educational potential of the history of mathematics and how to implement it within the classroom. The purpose is both to document the didactic and pedagogical reflections of secondary school teachers and to support their practice. A group of four teachers engaged in this collaborative process was formed, and three collaborative meetings were organized. We present here an overview of the study and preliminary results on how teachers see the role and the potential of the history of mathematics in their classroom, their different ways of developing tools, and some of the challenges they face.

## 1 *Problematizing elements*

In the proceeding of the 2016 HPM meeting, Clark et al. (2016) published an important paper concerning recent developments in the field of research on the history of mathematics in mathematics education. In their concluding remarks (p. 175), the authors mention the needs and issues that are currently central to researchers: 1) emphasizing pre- and in-service teacher education, (2) designing, making available, and disseminating a variety of didactic source material, 3) systematically and carefully performing applied empirical research to examine in detail and convincingly evaluate the effectiveness of the HPM perspective, and 4) acquiring a deeper understanding of theoretical ideas put forward in the HPM domain to carefully develop them into coherent theoretical frameworks and methodological schemes. The study presented here addresses the first three issues more precisely.

### 5.2 Empirical studies in the HPM literature

In the HPM literature, Guliker and Blom (2001) observe, on the one hand, that theoretical research provides important conceptualizations and, on the other hand, that empirical research tries "to put to the test" the development of certain tools without considering theoretical developments. Theoretical and empirical research seem to walk side-by-side but have a hard time informing each other.

With respect to empirical studies, Jankvist (2009) noted the low number of empirical studies matched with theoretical studies and called for empirical discussions in the field and more rigorous, meticulous, and well-founded approaches. As for teacher education, we have observed that studies on practicing teachers are quite rare. We nevertheless acknowledge the contributions of Moyon (2021) and Vincentini et al. (2019) who show, using large-scale statistical analyses in various European contexts, how in-service secondary teachers are generally interested in introducing mathematical history into their classrooms but encounter difficulties (e.g., lack of time, resources, and mastery).

One way to 1) galvanize the dialectic between theoretical and empirical research and 2) more finely examine the ambiguous relationship that high school teachers seem to have with history may be to develop participatory field research (see Guillemette, 2021). Participatory research focuses on research "with" rather than "on" participants and would not attempt to "test" or "provide" history-based teaching tools to teachers, but to develop these in conjunction with research input and teachers' knowledge and skills. This would improve the relevance, effectiveness, and feasibility of the teaching tools produced, but also the credibility of the research findings. That said, a recent review of the literature (Bellefleur & Guillemette, to be published) shows that of the 37 empirical studies published since 2010 in HPM literature, none takes the form of a proper participatory research.

### 5.3 History of mathematics in curricula in Quebec, Canada

In terms of teaching practice, the history of mathematics is now an integral part of the Quebec (Canada) mathematics secondary school curriculum, which states that "students should be able to place mathematical concepts in a historical and social context, to understand their evolution and identify the issues that led to their development and the concepts served by this process, and to recognize the contribution of mathematics to science, technology, and the culture of societies and individuals" (Ministère de l'Éducation du Loisir et du Sport, 2003, p. 248, our translation). The purpose of a historical perspective is to place mathematics in a broad socio-historical context, while humanizing mathematics.

### 5.4 Research questions

Given these problematizing elements, we pose the following research questions: How do secondary school mathematics teachers intend to meet these expectations? How can research knowledge be mobilized to support their practice while considering the constraints and difficulties experienced and expected?

## *2 Theoretical positioning*

Our study is theoretically based on the cultural-historical approach to mathematics education (Radford, 2021). Inspired by Vigotskian psychology, this approach in mathematics education places importance on cultural artifacts (objects, instruments, literary and scientific productions, etc.) and social interaction in teaching-learning. The mathematics classroom is not perceived as a neutral and closed space because the modes of *activity* (objectives, actions, operations) that take place there are mediated by objects, history, and culture (Radford, 2021).

More specifically, our study is theoretically based in a dialogical-ethical perspective on the history of mathematics in mathematics education (Guillemette, 2019; Guillemette & Radford, 2022). In this perspective, the main idea is to make dialogical interaction between voices (participants, researchers, voices from the past, curricula, etc.) accessible, assess engagement critically, and investigate the experience and actions of the participants, focusing primarily on the experience of "otherness."

## *3 Research objectives*

Within this theoretical framework and having in mind the problematizing elements discussed earlier, we formulated the following research objectives: 1) document the epistemological, didactic, and pedagogical reflections of secondary school teachers around social and political issues in mathematics education, 2) describe dialogically how didactic and pedagogical tools are developed for their classroom through jointly produced activities, and the challenges they face in this regard, and 3) describe their teaching constraints and difficulties experienced and apprehended in the classroom.

## *4 Methodological framework: participative research*

Our objectives call for implementing a qualitative study anchored in a comprehensive paradigm. Moreover, our objectives address both the production of research knowledge and the professional development of practicing secondary school teachers in a unified manner. This prompted us to adopt a participatory approach. As presented above, this approach aims to better understand the relationship between research and professional practice through a reciprocal lens. (Desgagné & Bednarz, 2005).

### 4.1 Context of the study

Four secondary school mathematics teachers were recruited (they had taken part in a professional workshop on reading historical texts led by us in 2019). Participants were mid-career teachers (10–25 years of experience; two women, two men). A meeting was held to set goals and working arrangements. Three other meetings (150 minutes) took place in the next month. We discussed "historical situations" to explore together the role and potential of mathematical history, possibilities for classroom implementation, and potential challenges.

These "historical situations" (Cavalieri (volume of the ball), Arbalestrille (Jacob's staff), al-Khawarizmi ($4^{th}$ model), Ptolemy's trigonometry, Mesopotamian numeral system) consisted of short excerpts of historical texts (sometimes only one image) derived from HPM literature. There were no historical or pedagogical explanations but only brief introductions by us. The excerpts were essentially a basis for discussion, a catalyst to create dialogue. We were also open to proposals from the group, but no proposals were made and the group focused essentialy on the above "historical situation".

### 4.2 Data collection and analysis

Videotapes of the collaborative meetings, transcription of the audio, and a research diary constituted the data. At this stage, we planned to conduct a dialogical analysis of the transcription focusing on the interaction between emerging "voices," including the researchers' (see Guillemette, 2019).

## 5 *Preliminary results*

In this section, we present, as introduced above, some preliminary results based on the notes that we took in the research diary.

In terms of the role and potential of the history of mathematics for teaching and learning mathematics, participants mentioned the need for teachers to demonstrate a certain attitude toward the students, i.e., as teachers interested in various topics and able to think about mathematics more broadly. They were also interested in learning about "how hard it used to be" and emphasizing the effectiveness of modern mathematics. They also mentioned the desire for a feeling of wonder, the need for renewing the mathematics classroom, and striving for interdisciplinarity.

In terms of ways to develop tools, participating teachers looked for ways to go beyond the mere anecdotal. They mentioned the need for a type of "story line" that could help make connections between various historical developments. They also mentioned the need to sequentially and incrementally explore the history of mathematics by creating different transition situations (such as addition and the subdivision of arguments). In terms of difficulties and reticence, they mentioned that the history of mathematics is not suitable for all students, that testing was not feasible, that they did not have sufficient knowledge, and that their resources were limited. In addition, they pointed out the conflicts between the existing curriculum and the inclusion of the history of mathematics.

## 6 *Conclusion*

In this paper, we introduced an ongoing study that seeks to document the epistemological, didactic, and pedagogical reflections of secondary school teachers related to social and political issues in mathematics education, the emerging ways to implement teaching activities along these lines, and the difficulties encountered. We highlighted the need for descriptive and participative approaches to galvanize the relationship between theoretical and empirical research in the field and to examine more finely the ambiguous relationship that teachers seem to have with the history of mathematics. Our study is an attempt in this sense. The preliminary results presented here are limited, but further analysis and research results are forthcoming.

## REFERENCES


Bellefleur, J.-P. & Guillemette, D. (to be published). Les récentes études empiriques sur l'histoire dans l'enseignement des mathématiques : un regard sur la méthodologie. *Petit x.*

Clark, K., Kjeldsen, T. H., Schorcht, S., Tzanakis, C., & Wang., X. (2016). History of mathematics in mathematics education: Recent developments. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.)*, Proceedings of HPM 2016* (pp. 135–179), IREM de Montpellier.

Desgagné, S., & Bednarz, N. (2005). Médiation entre recherche et pratique : faire de la recherche « avec » plutôt que « sur » les praticiens. *Revue des sciences de l'éducation*, *31*(2), 245–258.

Guillemette, D. (2019). Being in research and doing research on history and mathematics education in a Bakhtinian dialogical perspective. In E. Barbin, U. T. Jankvist & T. H. Kjeldsen (Eds.), *Proceedings of ESU 8* (pp. 91–106), Metropolitan University.

Guillemette, D. (2021, July). *Towards qualitative and participative research on the history of mathematics in mathematics education: some arguments and possible paths*. Oral communication at ICME-14 [TSG 27], Shanghai, China.

Guillemette, D., & Radford, L. (2022). History of mathematics in the context of mathematics teachers' education: A dialogical/ethical perspective. *ZDM - Mathematics education* [Special issue], *7,* Online first.

Gulikers, I., & Blom, K. (2001). "A historical angle": Survey of recent literature on the use and value of history in geometrical education. *Educational Studies in Mathematics*, *47*(2), 223–258.

Jankvist, U. T. (2009). On empirical research in the field of using history in mathematics education. *Relime*, *12*(1), 67–101.

Ministère de l'Éducation du Loisir et du Sport. (2003). *Programme de formation de l'école québécoise : enseignement secondaire*. Les publications du Québec.

Moyon. M. (2021, July). "*I would like to introduce history in my mathematics lessons, but I don't know how to do it"*. Oral communication at HPM 2021 [held online].

Radford, L. (2021). *The Theory of Objectification: A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Brill.

Vicentini et *al*. (2019). History, epistemology, and teaching mathematics: A challenging partnership. In E. Barbin, U. T. Jankvist & T. H. Kjeldsen (Eds.), *Proceedings of ESU 8* (pp. 205–220), Metropolitan University.

# ENGAGING PRE-SERVICE MATHEMATICS TEACHERS WITH HISTORICAL SOURCES: THE "MODERATOR IN DIALOGUE" ATTITUDE AND SOME ANTINOMIC ASPECTS OF EDUCATIONAL PRACTICES

**David GUILLEMETTE**
Université du Québec à Montréal, C. P. 8888, succursale Centre-Ville, H3C 3P8, Montréal (Québec), Canada
guillemette.david@uqam.ca

## ABSTRACT

In this paper, I discuss the "moderator in dialogue" attitude introduced by Michael Fried in his plenary talk at the 2021 HPM meetings. The talk was about Edmond Halley, particularly about his posture towards the work of Apollonius. Fried qualified Halley's view of the past as "moderator in dialogue," since Halley recognized both the potential of modern mathematics and the thoughtful study of the past when done deferentially, without imposing modern ideas. Fried draws a parallel with mathematics educators who may also adopt this attitude towards the past. Based on research findings in mathematics education and my own experience as a teacher educator, I examine more closely the tensions surrounding Fried's proposal and highlight some of the antinomic aspects of these educational practices, particularly regarding the use of original historical sources in this context.

## *1 Introduction*

The 2021 HPM meeting was held online and, due to the pandemic, was reduced to a series of five plenary lectures. The first was given by Michael Fried (2021) and entitled: *Edmond Halley's posture towards Apollonius's works and its relevance for teaching historical material in modern mathematics classroom*s. The talk addressed the HPM theme *Theoretical and/or conceptual frameworks for integrating history in mathematics education*. I will summarize the presentation and focus on its main object, the "moderator in dialogue" attitude that Fried described in the work of Edmond Halley. The talk was very inspiring and raise many questions on the very relation between the study of the history of mathematics and mathematics education, but also important questions related more precisely to, as I will try to show, the role of the teachers and the way to accompany learners in the encounter with the history of mathematics.

## 2 *Halley as a moderator between past and present*

Edmond Halley (1656–1742) was a Savilian Professor in Geometry at Oxford University. In his talk, Fried focused on Halley's reconstruction of Book VIII of Apollonius's *Conics* and on Halley's way of relating to the past. The main idea of Fried was that Halley "brings out questions to a historical approach to mathematics education and ways of pursuing it." He observed that Halley, on the one hand, regards the work of mathematicians from the past "with great interest and faithfulness" and, on the other hand, understands the great advantage of modern tools. For Fried, it is necessary to resist the temptation to impose modern ideas on the past or to devalue the present in relation to the past, but rather to maintain a "healthy position" between the past and the present. He found in Halley a good example of this position.

### 2.1 Halley's way of relating to the past

To describe Halley's way of relating to the past, Fried closely examined Halley's additions and comments to Apollonius's work. Fried was attentive to the tone, to how these additions and commentaries were stated by Halley, and to the types of expressions Halley used. Fried found in Halley a kind of middle ground between historical sensitivity and mathematical interest. As introduced above, Halley avoids treating the ancients as inferior while, at the same time, appreciates modern mathematical conceptions and tools. Halley found in the past, particularly in Apollonius's work "a font of intelligent treatments of problems and ideas by thoughtful people." For Fried, Halley was, in a way, in search of broadening his own horizon in mathematics.

### 2.2 Knowledge, self-knowledge, and the humanist perspective

In his talk, Fried's aim was to explore how educators who are interested in the history of mathematics and want it to be included in their classrooms could or should deal with the past. Elsewhere, Fried has elaborated on the need to develop such thinking, particularly in the context of building appropriate theoretical and conceptual frameworks in the field (see Fried, Guillemette, & Jahnke, 2016).

More precisely, the attention given by Fried to the "healthy position" between the past and the present is also echoed in his earlier works on the notion of *self-knowledge* (see Fried, 2007). Indeed, he has suggested that the back and forth movement and dialogue between modern mathematical understandings and ancient understandings can bring learners to a deeper understanding of themselves: "… a movement towards self-knowledge, a knowledge of ourselves as a kind of creature who does mathematics, a kind of mathematical being" (p. 218). He proposes that this self-knowledge, the knowledge of oneself as a "mathematical being," should be the primary objective of all forms of mathematics education based on the history of the discipline. Fried does not hesitate to emphasize the background of his thinking around these considerations by stating that "[Education], in general, is directed towards the whole human being, and, accordingly, mathematics education, as opposed to, say, professional mathematical training, ought to contribute to students' growing into whole human beings" (p. 219).

### 2.3 The "moderator in dialogue" attitude towards the past

Fried noted that Halley "gives a fair chance to the old and learns from it." To do so, there is the need to establish a kind of dialogue between the old and the new and to "represent faithfully, respectfully and fairly the works of the mathematicians from the past." Such an attitude, as discussed above, would allow seeing all human practitioners of mathematics as "a genuine human community with all of its wealth and all of its diversity." This is what the "moderator in a dialogue" attitude towards the past seems to imply. In this sense, it is not a method or an approach, but simply an attitude, or a way of being, that Fried invites educators to adopt.

## *3 In search of a moderator attitude for mathematics teachers*

At this point, I would like to acknowledge my agreement with Fried's proposal and the need to question and problematize how we relate to the past as educators. As teacher educators and researchers, we try to position ourselves as moderators in the dialogue and generally share this view of the role and potential of the history of mathematics. The following section is not intended to be critical, but to share my own experience of trying to achieve these goals, and to examine more closely the tensions surrounding this perspective, partly from research, but also from a kind of introspection and analysis of my own practice.

### 3.1 Some empirical and theoretical research findings

My research with pre-service teachers has shown that engagement with mathematicians of the past does not happen automatically. Maintaining an "empathetic relationship" with past mathematicians is challenging, and students have a strong tendency to read historical texts synchronically (from a modern synchronic plan) and have much difficulty engaging with these mathematicians on their own terms. (Guillemette, 2017). There is a serious need to find ways to support and guide learners to avoid what we have called the "violence of modern synchronization." Other studies have made similar observations (e.g., Arcavi & Isoda, 2007; Fried, 2000).

Furthermore, in teacher education, we have found that there is a certain nuance and complexity to be brought to bear when seeking a kind of middle ground between historical sensitivity and mathematical interest. Indeed, we have argued that educators seem to read texts *differently*, displaying a different form of engagement and answerability, notably by focusing on the potential estrangement from historical texts and on the vicarious aspects around different ways of *being* in mathematics and *doing* mathematics. (see Guillemette & Radford, 2022). It seems that there is another possible position, neither that of historians nor that of mathematicians, but that of educators, and that there is a need to investigate more closely how mathematics teachers, for example, engage with the past.

### 3.2 Making the moderator in dialogue attitude more effective

The above remarks are important if we want pre-service teachers or learners to adopt this moderator in dialogue attitude themselves, and if we consider, as teacher educators, that it is not enough to simply show this attitude in the presence of learners. One could say that as educators we should situate ourselves as moderators between the past and present in preparation for the encounter with the past in our classroom. But again, this dialogue between the old and new must engage the class if we want learners' horizons to expand. We would like the whole class to dialogue with the past.

To do this, we must face the difficulties that lie in the experience of "otherness" inherent in encounters with the past. These difficulties are numerous and include types of language, notations, unusual argumentative or discursive forms, implicit theorems, new definitions, unusual arguments, unusual typography, and so on. These are direct barriers that must be "surmounted" to understand historical texts. Moreover, there is much to know about the historical, cultural, social, and mathematical context of the period of historical texts if we are to embrace past mathematicians on their own terms. Personally, in the context of teacher training, I feel more like a *facilitator* than a moderator.

## *4 Tensions and problems*

Indeed, it is as if the attitude of "mediator in a dialogue" towards the past seemed somehow ancillary to the attitude of "facilitator" towards the learners. These tensions could possibly be linked to antinomic aspects of pedagogical practices linked to the exploration of the history of mathematics in mathematics teaching, especially around the reading of historical texts.

First, there is a need to prepare the learner for this encounter. As reported extensively in research (see Clark et al. 2016), the reading of historical texts cannot be done without a minimum of introductory instructions or explorations around the historical, cultural, or mathematical context of past mathematicians. Second, there is a need to set the encounter pedagogically, to *feel* a distance, a fruitful *disorientation* in the classroom (*cf.* Barbin, 1997), in order to highlight the terms specific to the mathematicians.

On the one hand, if the encounter with the text is too "prepared," the experience of otherness may be diminished, since students would be presented with a predetermined entity, and the reading activity would be reduced to a matching or identifying game. On the other hand, if the encounter is not sufficiently prepared, there is a risk that the encounter itself will fail due to a far too great semantic distance between the students and the text, making it impossible to engage with distant voices.

## *5 Conclusion*

The "moderator in dialogue" attitude towards the past, as Fried describes it in Halley, is inspiring but raises important questions. Indeed, how do we prepare the encounter between students and texts? How do we guide students in the experience of otherness? In what contexts or types of courses should historical reading take place? Should it remain in the hands of the educator?

We hope to have highlighted the importance of these questions for the research community.

## REFERENCES


Arcavi, A., & Isoda, M. (2007). Learning to listen: From historical sources to classroom practice. *Educational Studies in Mathematics*, *66*(1), 111–129.

Barbin, É. (1997). Histoire et enseignement des mathématiques : Pourquoi ? Comment ? *Bulletin de l'Association mathématique du Québec*, *37*(1), 20–25.

Clark, K., Kjeldsen, T. H., Schorcht, S., Tzanakis, C., & Wang., X. (2016). History of mathematics in mathematics education: Recent developments. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of HPM 2016* (pp. 135–179), IREM de Montpellier.

Fried, M. N. (2000). Some difficulties in incorporating history of mathematics in teaching-training. In A. Ahmed, J. M. Kraemer, & H. Williams (Eds.), *Proceedings of the CIEAEM 51* (pp. 73–78). Horwood Publishing Limited.

Fried, M. N. (2007). Didactics and history of mathematics: Knowledge and self-knowledge. *Educational Studies in Mathematics*, *66*(2), 203–223.

Fried, M. N. (2021, July). *Edmond Halley's posture towards Apollonius's works and its relevance for teaching historical material in modern mathematics classrooms.* Plenary talks at the 2021 HPM meeting [held fully online, with the help of the British Society for the History of Mathematics].

Fried, M. N., Guillemette, D., & Jahnke, H. N. (2016). Theoretical and/or conceptual frameworks for integrating history in mathematics education. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of HPM 2016* (pp. 211–230), IREM de Montpellier.

Guillemette, D. (2017). History of mathematics in secondary school teachers' training: Towards a nonviolent mathematics education. *Educational Studies in Mathematics*, *96*(3), 349–365.

Guillemette, D., & Radford, L. (2022). History of mathematics in the context of mathematics teachers' education: A dialogical/ethical perspective. *ZDM - Mathematics education* [Special Issu : *Exploring the significance of the history of mathematics in mathematics education: recent developments in the field*], *54*(7), 1493–1505.

# PROVING WALLIS FORMULA FOR π IN A PROBABILITY CONTEXT WITH PROSPECTIVE PRIMARY SCHOOL TEACHERS

**Michael KOURKOULOS[a], Constantinos TZANAKIS[b]**
Department of Primary Education, University of Crete, Rethymnon, Greece
mkourk@uoc.gr[a], k.tzanakis@uoc.gr[b]

## ABSTRACT

Motivated by the historical connections between the Wallis' product formula for $\pi$ and the approximation of the binomial by the normal distribution in probability theory, we discussed with our students - prospective elementary school teachers - an elementary proof of this formula, which, though initially given in a geometric context, admits a probabilistic interpretation as we showed.

## 1. *Introduction*

In 1656, Wallis published his product formula for $\pi$ (W.P.) and the relevant investigation in his *Arithmetica infinitorum* (Stedall, 2004, pp. xvii-xx, proposition 191). W.P. is an important mathematical result that historically had equally important applications, particularly in relation to the early development of probability theory in the 18th century. Specifically, in 1733 De Moivre used it as a key tool in his pioneering work on the normal approximation of the symmetric binomial distri-bution, which in turn was historically the first normal approximation (Hald, 2003, pp. 468-484; Khrushchev, 2006; Stigler, 1986, ch. 2). Moreover, in this context both Stirling and De Moivre found in 1730 (slightly different) approximations of $n!$ (Hald, *ibid*). This use of the W.P. shows its crucial role in the development of prob-ability, so that even in the 18th century the connection between analysis and proba-bility was already coming to light. Therefore, given the importance of the W.P. in this context, we looked exhaustively at its historical development, for getting aided by history to find out how to teach its proof in our introductory course on probabil-ity and statistics addressed to prospective elementary school teachers (no publica-tion seems to exist on empirical didactical work concerning its proof). Our students have a very limited (and in many cases, weak) mathematical background, but never-theless, have to cope with elementary probability and statistics, including the bino-mial distribution, an understanding of the significance of the law of large numbers, the normal distribution and its significance in relation to the central limit theorem and in particular as an approximation to the binomial distribution. This led us to wonder whether there is an elementary probabilistic proof of this formula adequate for our purpose. Although in our historical investigation we identified 15 proofs of W.P. since Wallis' time (most after 1980), none was an elementary probabilistic one[48], and all but two, use advanced mathematics: Yaglom and Yaglom (1987, pp.24, 36-37) use complicated trigonometry; Wästlund (2007) uses elementary algebra and geometry (both proofs employ the squeezing theorem of limits). However, we realized that a feature implicit to the latter (not mentioned by Wästlund) is that basic quantities in the proof can be interpreted as probabilities, hence related algebraic relations as probabilistic properties.

So, with Wästlund's proof translated into an elementary probabilistic one and with a variant of his geometric argument, we taught the proof in the context of the normal approximation to the symmetric binomial. This allowed to comment on how W.P. was used in De Moivre's work even though we did not develop the connection in the way he did. Nevertheless, along these lines it became possible through history to provide the students with hints on the historical connections between probabilistic concepts and mathematical analysis.

## 2. *The teaching framework*

In the teaching activity 26 volunteers participated, 17 having followed no introductory course on calculus. All were taught in high school the algebra and geometry necessary for the proof, though in an initial test (including questions on polynomial expansion, solving a linear system of equations, properties of powers and fractions) 7 gave answers that indicated significant weaknesses in elementary algebra. Discussion on the proof and applications of W.P. started on the $5^{th}$ course week for 12 hours.

Until then students had been taught basic elements of combinatorics, the additive and multiplicative rules of probability and its extension to more events (the chain rule of probability) and the binomial and hypergeometric distributions, together with examples and applications.

## 3. *The implemented teaching approach*

Originally, the teacher told the students that they were going to consider the limit

---

[48] Though several recent proofs employ advanced tools and results of probability theory and mathematical analysis (Chin, 2020; Kovalyov, 2011; Miller, 2008; Wei et al., 2017).

of an important product, $\frac{1\times3}{2^2}\times\frac{3\times5}{4^2}\times\ldots\times\frac{(2n-1)\times(2n+1)}{(2n)^2}$, and that this limit was initially found by Wallis in 1656 and provided information about Wallis, his work, and the importance of W.P. (Stedall, 2004, pp. xi-xxxiii; Khrushchev, 2006). He also said that they were going to study a variant of Wästlund's proof, because it needs only elementary mathematics.

### 3.1 The first properties discussed with the students

(a) The probability at the center of the symmetric binomial with even number of trials is $B(n,2n,0{,}5) = \frac{(2n)!}{(n!)^2}\times\frac{1}{2^{2n}} = \frac{(2n-1)!!}{(2n)!!}$ with $b_n = B(n,2n,0{,}5)$ for $n \geq 1$, and $b_0 = 1$ by definition. (b) $2n\times b_n = b_{n-1}+\ldots+b_1+ b_0$.

### 3.2 The partial Wallis product and its relation to $b_n$

The teacher explained that, for $\varphi_0 = 0$, $\varphi_n = b_n^2 \times (2n+1)$, $n\in\mathbb{N}$, and rearranging the factors in the numerator of $b_n^2$, the inverse of the partial W.P. becomes

$$\varphi_n = \frac{1\times3}{2^2}\times\frac{3\times5}{4^2}\times\ldots\times\frac{(2n-1)\times(2n+1)}{(2n)^2}.$$

Moreover, setting $a_n = b_n^2 \times 2n$, $a_0 = 0$, $a_1 = \frac{1}{2}$, and for $n \geq 2$ rearranging the factors in the denominator of $b_n^2$, gives $a_n = \frac{1}{2}\times\frac{3^2}{4\times6}\times\frac{5^2}{6\times8}\times\ldots\times\frac{(2n-1)^2}{(2n-2)\times2n}$.

Then he discussed that (i) $\varphi_n > \varphi_{n+1}$; (ii) $\alpha_{n+1} > \alpha_n$; (iii) $\frac{a_n}{\varphi_n} = \frac{2n}{2n+1}$

Thus $\varphi_n > \alpha_n$ and $\alpha_n$ approaches $\varphi_n$ as $n$ increases; (iv) by (i)-(iii), $\alpha_n$ and $\varphi_n$ have the same limit $C$, still to be found, and $\varphi_n > C > \alpha_n$.

### 3.3 A simple Pólya-Eggenberger urn model

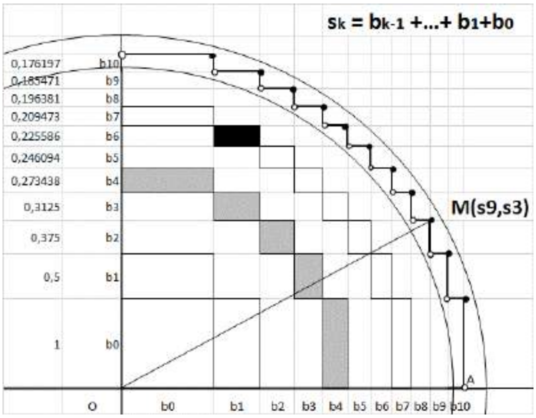
The following property is of central importance in Wästlund's proof

$$b_0b_n + b_1b_{n-1} + \cdot \quad \cdot \quad \cdot + b_nb_0 = 1 \qquad (1)$$

He proves it algebraically, but we proved it probabilistically. For this we conceived a simple Pólya urn model, running as follows: Initially, an urn contains a black and a red ball. At each trial, we draw randomly a ball from the urn and return it to the urn plus two new balls of the same color. The model was discussed with the students; in particular, finding the probability Pl($k$,$n$) to get $k$ black balls in $n$ successive random trials, with their order of occurrence being immaterial. We proved that Pl($k$,$n$) = $b_k\times b_{n-k}$. Then we got Pl(0,$n$) + Pl(1,$n$) +…+ Pl($n$,$n$)= 1, since it is the sum of probabilities of all possible events, all being mutually exclusive. Finally, substituting the probabilities in this sum and using Pl($k$,$n$) = $b_k\times b_{n-k}$ the desired property resulted.

The generalization of the urn model with three parameters was discussed next and information on Polya's work in mathematics, physics, and mathematics education was given (e.g., Alexanderson, 2000). Students' work on Polya's urn models, though limited, was a significant introduction to the subject.

### 3.4 A grid for representing the probability Pl($k$, $n$)=$b_k \times b_{n-k}$ and its properties




The first column of the grid has width $b_0$, the next has width $b_1$, etc. The rows are determined similarly. Columns and rows are enumerated starting from 0. So, column $k$ has width $b_k$ and row $m$ has height $b_m$. Their intersection is a rectangle, *Rec*($k$, $m$), with dimensions $b_k$, $b_m$ and area *AreaRec*($k$, $m$) = $b_k \times b_m$, e.g. in black color the *Rec*(1, 6). Let $A_k$ be the sequence of rectangles *Rec*(0,$k$), *Rec*(1,$k$-1), …, *Rec*($k$,0) (e.g., $A_4$ is the sequence in grey). Since *AreaRec*($i$, $j$)= $b_i \times b_j$, the union of the surfaces of the rectangles of $A_k$ has area: $AreaA_k = b_0 \times b_k + b_1 \times b_{k-1} + \ldots + b_k \times b_0 = 1$, by (1). Let $\Pi_n$ be the polygon whose surface is the union of the surfaces of the rectangles of the sequences $A_0$, $A_1$, $A_2$, ..., $A_{n-1}$ (e.g., in the figure the last bold perimeter is the perimeter of polygon $\Pi_{11}$.). Since $AreaA_k = 1$, the area of $\Pi_n$ is $n$. We set $s_k = b_{k-1} + ... + b_1 + b_0$ for $k \geq 1$ and $s_0 = 0$, $s_k = 2k \times b_k$ (recall §3.1). The outer corners of the polygon $\Pi_n$ have coordinates ($s_k$, $s_{n+1-k}$), with $k$ integer from 1 to $n$. The inner ones have coordinates ($s_k$, $s_{n-k}$), with $k$ integer from 0 to $n$. These results were discussed with the students step by step and using many examples.

### 3.5 Circular quadrants containing and contained in $\Pi_n$

The teacher explained that the quadrant of a circle (O, $R_n$), with delimiting radii on O$x$ and O$y$, contains $\Pi_n$ if and only if $\sqrt{s_k^2 + s_{n+1-k}^2} \leq R_n$, for $k \in \mathbb{N}$ $1 \leq k \leq n$. With $s_k = b_{k-1} + ... + b_1 + b_0 = 2k \times b_k$, $\alpha_k = b_k^2 \times 2k$ and $\alpha_k$ an increasing sequence, it was obtained that if $R_n = \sqrt{(2n+2) \times \alpha_n}$, then the above quadrant contains $\Pi_n$. Since the area of this quadrant is greater than that of $\Pi_n$, which is $n$, we have

$$n < \frac{1}{4}\pi \times \left(\sqrt{(2n+2) \times \alpha_n}\right)^2 = \frac{1}{4}\pi(2n+2) \times \alpha_n \Leftrightarrow \frac{2n}{\pi(n+1)} < \alpha_n \Rightarrow$$

$$\lim_{\nu \to +\infty} \frac{2n}{\pi(n+1)} \leq \lim_{\nu \to +\infty} \alpha_n = c \Rightarrow \frac{2}{\pi \times C} \leq 1 \quad (2)$$

Then, the teacher discussed that similarly there is a quadrant of the circle (O, $r_n$) with $r_n = \sqrt{(2n-2)\varphi_n}$, contained in $\Pi_n$, and from this it was obtained that:

$$1 \leq \frac{2}{\pi \times C} \quad (3)$$

(3) and (2) imply that $C = \frac{2}{\pi}$ which is the limit sought $\lim_{\nu \to +\infty} \varphi_n = \lim_{\nu \to +\infty} \alpha_n$

### 3.6 The approximation of $b_n$

Using this limit and the properties in §3.2 the teacher discussed with the students the derivation of the approximation of $b_n$;

that for large $n$ (i) $b_n \approx \frac{1}{\sqrt{\pi \times n}}$, (ii) $b_n \approx \sqrt{\frac{2}{\pi(2n+1)}}$ and $\frac{1}{\sqrt{\pi \times n}} > b_n > \sqrt{\frac{2}{\pi(2n+1)}}$.

He also discussed the historical importance of this approximation; that in 1729, after Stirling's suggestion, De Moivre used W.P. to obtain (i); that this was an important step in his work leading to the normal approximation of the symmetric binomial (the historically first normal approximation); and that in 1730 Stirling and De Moivre used W.P. for approximating $n$! - still another important result (Hald, 2003, pp. 468-484).

## 4. *On the evaluation of the implemented teaching approach*

In the final test, 17 students answered correctly or with minor errors the questions on the proof of W.P., and 9 made important errors and/or gaps. Among the 7 students, with significant weaknesses in elementary algebra in the initial test (group B) only one answered correctly these questions, whereas, among the 19 students with no such weaknesses in the initial test (group A), 16 answered correctly or with minor errors. This difference between groups A and B is significant at the level of 0.01 (Fisher exact test $p = 0.00223$).

## 5. *Final remarks*

Teaching a version of Wästlund's proof about W.P. in a probabilistic context, was a journey based on the interplay among elementary algebra, probability, and geometry. In its context history played a double role:

(1) At the meta-cognitive level by contributing to the development of our didactical background as teachers/researchers (Tzanakis et al, 2000, section 7.2(c), p. 206) in the sense (i) of enriching our didactical repertoire; (ii) in getting aware of how "advanced" may be the subject to be taught in relation to the students to whom its teaching is addressed; and (iii) in getting involved into the creative process of "doing mathematics".

(2) Part of (iii) above, and a lot of information on the role of W.P. in De Moivre's and Stirling's related work were presented and discussed in the classroom, both in the form of "historical snippets" (Tzanakis & Arcavi, 2000, section 7.4.1) and strictly as a mathematical subject. For instance, we examined the application of W.P. for approximating $b_n$ and discussed its importance in the context of De Moivre's work for the normal approximation to the symmetric binomial. Furthermore, we discussed some interesting properties of probabilities and their geometric representations, including an introduction to Pólya's urn models. This in turn, motivated further discussion on the origin and importance of these models and Polya's multi-faceted contributions, including mathematics education. By devoting enough teaching time and a significant amount of work, the proof of W.P. presented here was accessible to students with no significant weaknesses in elementary algebra, though this was not possible for the weaker students.

In summary, our approach could be understood as an example of an "illumination approach" in the sense of Jankvist (2009, section 6.1, pp. 245-246).

## REFERENCES

Alexanderson, G. L. (2000). *The random walks of George Pólya*. The Mathematical Association of America.

Chin, W. (2020). A probabilistic proof of a Wallis-type formula for the gamma function. *The American Mathematical Monthly*, *127*(1), 75–79.

Hald, A. (2003). *A history of probability and statistics before 1750*. Wiley.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics*, *71*(3), 235–261.

Khrushchev, S. (2006). A recovery of Brouncker's proof for the quadrature continued fraction. *Publicacions Matemàtiques*, *50*(1), 3-42.

Kovalyov, M. (2011). Removing magic from the normal distribution and the Stirling and Wallis formulas. *The Mathematical Intelligencer*, *33*(4), 32–36.

Miller, S. J. (2008). A probabilistic proof of Wallis's formula for π. *The American Math-*

*ematical Monthly*, *115*(8), 740–745.

Stedall, J. A. (2004). *The Arithmetic of Infinitesimals, John Wallis 1656*. Springer.

Stigler, S.M. (1986). *The history of statistics: the measurement of uncertainty before 1900*. Harvard University Press.

Tzanakis, C., Arcavi, A., de Sá, C. C., Isoda, M., Lit, C-K., Niss, M., … & Siu, M.-K. (2000). Integrating history of mathematics in the classroom: an analytic survey. In J. G. Fauvel, & J. van Maanen (Eds.), *History in Mathematics Education: The ICMI Study*, New ICMI Study Series, vol. 6 (pp. 201–240). Kluwer.

Wästlund, J. (2007). An elementary proof of the Wallis Product Formula for pi. *The American Mathematical Monthly*, *114*(10), 914–917.

Wei, Z., Li, J., Zheng, X. (2017). A probabilistic approach to Wallis' formula. *Communications in Statistics-Theory and Methods*, *46*(13), 6491–6496.

Yaglom, A. M., & Yaglom, I. M. (1987). *Challenging mathematical problems with elementary solutions* (vol. II, problem 147). Dover.

# DESIGNING A HISTORY AND PHILOSOPHY OF MATHEMATICS COURSE FOR PRESERVICE TEACHERS: AN EMERGING FRAMEWORK

**Magdalini LADA, Iveta KOHANOVÁ, Melih TURGUT**
Norwegian University of Science and Technology, Trondheim, Norway
magdalini.lada@ntnu.no, iveta.kohanova@ntnu.no, melih.turgut@ntnu.no

**ABSTRACT**

In this paper, we introduce an emerging framework for designing a History and Philosophy of Mathematics course at master's level for preservice mathematics teachers (PMTs). The components are i) knowledge of mathematics and its history, ii) belief about/attitudes towards mathematics and its history, iii) beliefs about/attitudes towards using history in teaching mathematics and iv) design capacity of history-inspired mathematics lessons. We provide two examples of topics we used in our course and discuss how those contributed to PMTs´ knowledge and beliefs related to the above components.

## *1 Introduction*

Epistemology and historical developments of ideas, notions and mathematical objects have received particular attention from educational researchers. One can underline three main directions: *a*) the use of historical developments/ideas as a resource in teaching school mathematics (e.g., Ernest, 1998; Johansen & Kjeldsen, 2018); *b*) the use of historical developments/ideas as a tool and mediator for preservice and in-service teachers' and teacher educators' professional development (e.g., Jankvist et al., 2020; Turgut & Kohanová, 2021), and *c*) the interrelation between students' beliefs and historical developments of mathematical notions (e.g., Charalambous et al., 2009; Lada & Kohanová, 2022). As a synthesis, the results underline the potential power of historical resources in teacher training and how this impacts their mathematical knowledge in teaching (Jankvist et al., 2020). In this paper, we report on ongoing research by focusing on b) and c) above and discuss an emerging framework (with its tentative four components) that can be used as a guide for designing a History and Philosophy of Mathematics (HPM) course for PMTs. Our aim is to explore dialectics through the use of history of mathematics, beliefs, and elaborating didactical designs as part of PMTs' professional development.

## 2 *Timeline and Background*

The framework introduced in this paper emerged from the research, which was initiated in 2018, under a pilot study entitled "Understanding student teachers' learning and development in the Historical and Philosophical Aspects of Mathematics course". The course is designed as a master-level course for PMTs with a two-fold aim. On the one hand, the aim is to provide information about historical, ontological and epistemological foundations for mathematical concepts and algorithms, as well as knowledge of what mathematics is, how its nature and methods developed, and what it constitutes today. On the other hand, it is aimed to develop PMTs' skills in transforming knowledge of the history of mathematics into didactical and pedagogical designs for teaching mathematics (grades 5-10). In the course we have covered topics from algebra and geometry (e.g. solution of polynomial equations in different cultures; the road from Euclidean geometry to non-Euclidean geometries). In the first years of the pilot study, we observed that PMTs had issues in designing lessons which incorporated the historical context meaningfully and that PMTs formed unexpected beliefs about mathematics and its development. This directed us toward systematic intentional inquiry to improve the practice and PMTs' outcomes, which is known as action research (Koshy et al., 2011). We have adopted an iterative approach embracing problem identification, action planning, implementation, evaluation, and reflection. The insights gained from the initial cycle fed into the planning of the second cycle, for which the action plan was modified, and the research process was repeated the next year


(Figure 2).

**Figure 2.** Iterative cycles applied in our action research

## 3 *The emerging framework*

As a first step for creating a framework for teaching history and philosophy of mathematics to PMTs, one needs to ask what the goals of such a course are. To answer this question, one needs to know what the goals of mathematics education for teachers are and how history of mathematics can contribute to achieving those. For the case of Norway, this has been discussed by Bjørn Smestad in the panel discussion of ESU6. Starting with two main goals of mathematics education for teachers: 1) to develop students' knowledge of mathematics and attitudes towards it, and 2) to develop their ability to teach mathematics, Smestad argued how history of mathematics can play a part in the whole spectrum of teacher knowledge (Barbin et al., 2011). The first of the above goals is a prerequisite for achieving the second one, which we view as the main goal of our teacher education program. Then, with 2) as a goal of the mathematics education for teachers, we set the main goal of the HPM course to be: *to develop the students' ability to teach mathematics using history of mathematics*. By "using history of mathematics" we mean that the students will become able to make choices informed by history of mathematics in planning, implementing and evaluating their lessons.

With the goal of the HPM set, the next question to ask is what a teacher needs to know to teach mathematics using history of mathematics. One needs *knowledge of mathematics and its history* which is the first component of our framework. By knowledge here we mean what Ball et al. (2008) call subject matter knowledge (SMK). SMK alone is, of course, far from enough. If the PMTs are to use history of mathematics in their future work as mathematics teachers, they need to develop positive attitudes towards this approach. They need to believe that history has an important role to play in mathematics teaching and learning and that it is worth overcoming all the obstacles (Tzanakis & Arcavi, 2000) they might meet. *Beliefs about and attitudes towards using history in teaching mathematics* are therefore the second component of our framework. The third component is PMTs' *beliefs and attitudes towards mathematics and its history*. It is often argued that history can be used to develop views on the nature of mathematics and mathematical activity (Tzanakis & Arcavi, 2000). But what type of views are cultivated within an HPM course? Special attention is needed, for example, to ensure that experimental and not formalist views on mathematics (Charalambous et al., 2009) will be strengthened. Knowledge and beliefs/attitudes about mathematics and its history and about using history in mathematics do not automatically turn into lesson plans. Even if PMTs through an HPM course get inspired to use history in their future teaching, the course should also aim in equipping them with concrete methods for doing so. Thus, the fourth component of our framework is the PMTs' *design capacity of history-inspired mathematics lessons*. To summarize, the emerging framework´s components are the domains of PMT´s knowledge and beliefs/attitudes we believe are important for teaching mathematics using history of mathematics and which can be affected by an HPM course (Figure 1).

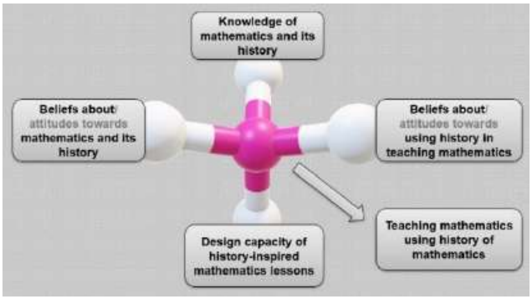



**Figure 1.** The emerging framework

The framework is theoretical in the sense that at its core lies a set of questions (Fried et al., 2016). What type of knowledge, beliefs/attitudes do teachers need to use history in teaching? How can an HPM course contribute to building those? As any theoretical framework though, it also has practical implications. The teacher educator can, by seeking to answer the questions above, use it as a guide for setting goals and choosing topics to include in an HPM course.

## 4 *Examples*

One of our intentions with including the three famous construction problems of Greek antiquity (quadrature of the circle, duplication of the cube, trisection of an angle), was that our PMTs discover the historical, social and cultural dimensions of mathematics (Hersh, 1994). In the session devoted to the topic, these three problems were presented to the PMTs and they worked hands-on with some of the efforts of Greek mathematicians of the time. The session ended with an outline of the mathematical advances that had to be made before an answer to the problem was given in the 19$^{th}$ century. In the written assignment that followed the PMTs were asked *what the construction problems teach us about mathematics and its development.* Analysis of their answers revealed that few PMTs grasped the sociocultural dimension of mathematics (Lada & Kohanová, 2022) making evident that the expectation that certain beliefs will be formed just by exposing PMTs to history was unrealistic. In redesigning the course, we included discussions that brought to the surface the connection between mathematics and the cultural context it is born in. Our second example is related to the lesson design component of our framework, and it is the product of a group of four mathematically strong PMTs' work within the workshop we organized in the spring semester of 2022.

The group designed a lesson about Archimedes' Arbelos which was discussed in class in connection with the lunes of Hippocrates and the intention of showing how old mathematical problems can become the source of inspiration for newer ones. The PMTs created a task where GeoGebra was used to investigate the properties of the Arbelos. The only historical information included though was that "Archimedes found out this more than 2000 years ago". They were then challenged to reflect on how the historical aspect of their lesson supports pupils' learning. In the revised version they submitted, the PMTs added an introduction to their activity: "we start by asking the pupils how they thought a shoemaker's knife looked 2000 ago [and] ask them if they manage to see the knife [in the figure]". They claimed they used history as spices to increase pupils' motivation and that through their lesson "pupils can get an idea that mathematics is not universal truths but human-made conclusions", which is again doubtful considering that the problem was removed from its historical context.

## 5 *Concluding remarks*

Our framework emerged within a particular context. All choices, results and conclusions are tightly connected to the mathematical and pedagogical background of our PMTs, and the goals were set to fit the existing description of the HPM course and the expectations of our PMTs. We believe though it can potentially be used in designing an HPM course in different contexts. Last, as a next step, it would be interesting to follow our PMTs in practice and observe how they would manage to implement their lessons.

**Acknowledgement:** The paper was financially supported by the H2020 project MaTeK, no. 951822.

### REFERENCES

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education, 59*(5), 389–407.

Barbin, E., Furinghetti, F., Lawrence, S., & Smestad, B. (2011). Teachers and history of mathematics in Italy. In E. Barbin, M. Kronfellner, & C. Tzanakis (Eds.), *History and Epistemology in Mathematics Education. Proceedings of the Sixth European Summer University ESU 6* (pp. 27–32). Verlag.

Charalambous, C.Y., Panaoura, A. & Philippou, G. (2009). Using the history of mathematics to induce changes in preservice teachers' beliefs and attitudes: insights from evaluating a teacher education program. *Educational Studies in Mathematics 71*, 161–180.

Hersh R. (1994). Fresh Breezes in the Philosophy of Mathematics. In P. A. Ernest (Ed.), *Mathematics, Education and Philosophy: An International Perspective* (pp. 11–20). Routledge.

Ernest, P. (1998). The history of mathematics in the classroom. *Mathematics in School*, *27*(4), 25–31.

Fried, M., Guillemette, D., & Jahnke, H. N. (2016). Theoretical and/or conceptual frameworks for integrating history in mathematics education. In L. Radford et al. (Eds.) *History and Pedagogy of Mathematics. Proceedings of the 2016 ICME Satellite Meeting* (pp. 211–230). Montpellier, France: IREM de Montpellier.

Jankvist, U.T., Clark, K.M., & Mosvold, R. (2020). Developing mathematical knowledge for teaching teachers: potentials of history of mathematics in teacher educator training. *Journal of Mathematics Teacher Education*, *23*, 311–332.

Johansen, M.W., & Kjeldsen, T.H. (2018). Inquiry-reflective learning environments and the use of the history of artifacts as a resource in mathematics education. In K.M. Clark et al. (Eds.), *Mathematics, Education and History* (pp. 27–42). Springer.

Koshy, E., Koshy V., & Waterman, H. (2011). *Action Research in Healthcare*. Sage Publications.

Lada, M., & Kohanová, I. (2022). Student teachers' beliefs shaped by History of Mathematics: the case of the Construction Problems of Antiquity and Euclidean Geometry. In G. Nortvedt et al. (Eds.) *Proceedings of NORMA20 - The ninth Nordic Conference on Mathematics Education* (pp. 137–144). Stockholm, Sweden: SMDF.

Turgut, M., & Kohanová, I. (2021). Students' perceptions and challenges regarding mathematics teaching cycle in practices of historical and philosophical aspects of mathematics course. In G. Nortvedt et al. (Eds.) *Preceedings of NORMA20 - The ninth Nordic Conference on Mathematics Education* (pp. 273–280). SMDF.

Tzanakis, C., & Arcavi, A. (2000). Integrating history of mathematics in the classroom: An analytic survey. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education: The ICMI Study* (pp. 201–240). Kluwer Academic Publishers.

# IMPACT OF THE USE OF HISTORY IN SECONDARY SCHOOL MATHEMATICS EDUCATION: AN EMPIRICAL STUDY

**Gaëlle LOUAKED**
Université de Lille, Cité Scientifique, 59650 Villeneuve-d'Ascq, Lille, France
gaelle.louaked@ac-lille.fr

## ABSTRACT

Within a recent French reform (2019), the history of mathematics (HM) is mentioned in the students' curriculum (from grade 10 to grade 12). Through an overview of the curriculum and textbooks, a gap emerges: a need to create activities that perceive history as a goal in itself addressing concepts prescribed by the curriculum.
My research questions at this stage are: 1) What are students' perceptions of mathematics before HM introduction? 2) How does this perception change as the activities progress? 3) Does working with activities specifically designed in collaboration with researchers and teachers to meet everyone's expectations lead to increased motivation, in the sense of perseverance to complete the task? Will the students choose to pursure further studies of mathematics at the end of the school year ("long-term" motivation)? At the end of this article, we will describe how our activities, which will be used for data collection in 2023-2024, convening of the history of mathematics are created.

## 1 *Introduction: a reform context*

French education has undergone a major reform, implemented at the start of the 2019 school year. As we can read in this extract from the official bulletin (OB)[49]: «*The problems proposed to the students can be internal to mathematics, [or] come from the history of mathematics [...]. In all cases, they must be well designed and motivating, in order to develop mathematical knowledge and skills of the programme».* The new syllabus suggests the use of HM in the class and seeks to motivate the students. France is not isolated, the introduction of HM in the students' curriculum concerns many other countries (Charbonneau, 2006; Jankvist, 2010; Halmaghi, 2013). Now in France, after grade 10 (15-16 years), students can choose whether or not to continue with mathematics. They have the common modules including science (2 hours per week) and three (out of 13 proposed) specialised subjects to choose from, including mathematics. In a report by the national statistical service in March 2022, a comparison is made between the system before and after. They considered that « *mathematics is not, or not sufficiently,*

[49] The implementation of ministerial measures of national education are the subject of regulatory texts called official bulletins.

*taught in the framework of science »*. It is mentioned that today 36% of the pupils did not do mathematics in the grade 11 (16-17 years old) against 13% in the old system.

Many research studies have suggested that the use of HM can improve students' learning outcomes in cognitive and affective domains (e.g. De Vittori, 2022; Fauvel, 1991; Lim and Chapman, 2015). Others have advocated the inclusion of this practice in national curricula (Fauvel and Van Maanen, 2000) and in teacher education courses (Charalambous, Panaoura and Phillippou, 2009; Clark, 2012; Guillemette, 2017).

The OB of National Education revised the integration of mathematics teaching into the science curriculum for grade 11 (16-17 years old) on 07/2022. It specifies its major intentions: the consolidation of the mathematical culture of the students, situating mathematics within its social context through the insertion of elements of HM and science.

Finally, the secondary school final examination (Baccalaureat) has been modified. From now on, students have to take a « grand oral » (big talk) on a topic related to the speciality teaching they are taught by. For this, each teacher has to work on the oral with their students. How to work on oral expression in mathematics? Lim and Chapman (2015), Perkins (1991) and Siu (1997) suggest that stories about mathematicians could be used to engage students in mathematics lessons. All these works, show a willingness of research and institutions to use HM in the classroom.

### *2. The integration of HM in the classroom.*

Despite the institutional demand and research showing its potential interest, teachers rarely provide students with the reading of a historical text, a historical problem, the history of a concept or a mathematical tool, limiting themselves to using HM in the form of anecdotes, 'story bites' (in the sense of Tzanakis, 2000) to introduce a teaching sequence (Moyon, 2022). Moyon explains that teachers rarely use HM to establish a proof, to show the usefulness of mathematics, or for training exercises. Very few of them use HM for presentations or biographies (2%) and during homework (1%). Moyon concludes: *« Although it is comforting that mathematics teachers are willing to introduce a historical perspective in their teaching, the fact remains that there is still a long way to go»*. So, according to Moyon teachers *« suffer from a lack of didactic expertise in this area » and « a lack of historical knowledge »* (p.4).

Moreover teachers are not convinced that HM can lead to better learning outcomes (Fauvel, 1991; Gulikers and Boom, 2001), or that reading historical texts can help students to better understand mathematics (Moyon, 2022)

Yet most studies testing the effectiveness of HM in improving achievement (Clark, 2012; Gulikers and Boom, 2001) suggest that the use of HM improves students' learning outcomes in the cognitive and affective domains.

Lim and Chapman (2015) showed students in the experimental group felt more motivated to learn mathematics than those in the control group, but,the post-test indicated that this effect was time-limited. Bütüner and Baki (2020) explained that students' perceptions changed after the implementation of activities related to the history of mathematics. There is a real need to understand this phenomenon better. Several researchers ask more empirical study (HPM[50], Clark & al (2018))

In my thesis, I will only focus on the affective part and in particular on the motivation (or not) of students after their teachers' use of historically supported activities. I will study several 10th grade classes (15-16 years old) when faced with tasks involving HM. For this purpose, several activities will be created and then tested. I have chosen this level in order to observe if there is an impact on the choice to continue mathematics[51] the following year or not.

At this stage of my explorations my research questions are:

1) What are the students' perception of mathematics before the introduction of HM?
2) How do these perceptions change as the activities progress?
3) Does working with activities specifically designed through collaboration between researchers and teachers lead to increased motivation, in the sense of perseverance to complete the task? What about motivation at the end of the year?

I am also interested in the teacher's perception of their class. Will their feelings about the motivation of their students be in line with the students own perceptions of their motivation?

For my research, activities will be designed in a collaborative mode in order to best fit with the prescriptions of this French curriculum. The researcher first creates a draft activity which is reviewed by historians of mathematics and didacticians of mathematics. It is then modified a first time. Then it is submitted to

[50] the International Study Group on the Relations between the History and Pedagogy of Mathematics
[51] In the context of the French reform described above

teachers and reworked a second time. This second version is tested on classes and a first feedback from teachers is made. This second activity and this feedback are discussed with historians of mathematics and didacticians in order to provide the final version of the activity which will allow me to collect data and carry out my study.

### *3. Conclusion*

Data collection is planned for the academic year 2023-2024. I hope that this research can help teacher practice and inform research on the impact of the introduction of HM on secondary school mathematics classes. Activities created may be reused later for further or initial training.

## REFERENCES


Barbin, E. (2006). Apports de l'histoire des mathématiques et de l'histoire des sciences dans l'enseignement. *Tréma*, (26), 20–28

Bütüner, S. Ö. & Baki, A. (2020). The use of history of mathematics in the mathematics classroom: An action study. *International Journal of Education in Mathematics, Science and Technology, 8*(2), 92–117

Charalambous, C. Y., Panaoura, A., & Phillippou, G. (2009). Using the history of mathematics to induce changes in preservice teachers 'beliefs and attitudes: Insights from evaluating a teacher education program. *Educational Studies in Mathematics, 71(*2), 161–180.

Charbonneau, L. (2006). Histoire des mathématiques et les nouveaux programmes au Québec: un défi de taille. In L'enseignement des mathématiques face aux défis de l'école et des communautés: Actes du colloque EMF. Sherbrooke: Faculté d'éducation, Université de Sherbrooke.

Clark, K. M. (2012). History of mathematics: Illuminating understanding of school mathematics concepts for prospective mathematics teachers. *Educational Studies in Mathematics, 81*, 67–84.

Clark, K. M., Kjeldsen, T. H., Schorcht, S., & Tzanakis, C. (Eds.) (2018). *Mathematics, education and history: towards a harmonious partnership*. Springer International Publishing

De Vittori, T. (2015). Les tâches des élèves dans une activité mathématique à dimension historique. Petit x, 97, 5-26. France: IREM de Grenoble.

De Vittori, T. (2022). Relevance of a history-based activity for mathematics learnings. *Discover Education*, 1(1)

Doz, D. (2021) Using the History of Mathematics as a Motivational Factor in Teaching Math in *Nauka, Nastava, Učenje u Izmenjenom Društvenom Kontekstu*, 471. University of Kragujevac, Faculty of Education in Uzice, 2021

Fauvel, J. (1991). Using History in Mathematics Education. *For the Learning of Mathematics*, *11*(2), 3–6. http://www.jstor.org/stable/40248010

Fauvel, J., & Van Maaren, J. (Eds.). (2000). History in mathematics education. *The ICMI Study.* Dordrecht, The Netherlands: Kluwer.

Gulikers, I., & Blom, K. (2001). 'A historical angle', a survey of recent literature on the use and value of history in geometrical education. *Educational Studies in Mathematics, 47*(2), 223–258.

Guillemette, D. (2017). History of mathematics in secondary school teachers 'training: towards a nonviolent mathematics education. *Educational Studies in Mathematics, 96*(3), 349-365.

Guillemette, D., & Radford, L. (2022). History of mathematics in the context of mathematics teachers 'education: a dialogical/ethical perspective. *ZDM: the international journal on mathematics education.* 54.

Halmaghi, E. (2013). Inequalities in the history of mathematics: from peculiarities to a hard discipline. In S. Oesterle, D. Allan & P. Liljedahl (Eds.), P*roceedings of the 2012 GCEDM/* Acte de la réunion annuelle 2012 du GCEDM (pp. 171-178), Burnaby, Canada : CMESG/GCEDM.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235-261.

Jankvist, U. T. (2009a). Using History as a 'Goal 'in Mathematics Education Ph.D. Thesis, University of Southern Denmark.

Jankvist, U. T. (2009b). On empirical research in the field of using history in mathematics education. ReLIME, 12(1), 67-101.

Jankvist, U. T. (2010). An empirical study of using history as a "goal". *Educational Studies in Mathematics, 74*(1), 53-74.

Jouve, G., Marmier, A.-M., Moyon, M., Recher, F., Tazzioli, R., & Tournès, D. (dir.). (2020). Mathématiques en perspectives - Hommage à Rudolf Bkouche. Presses Universitaires de Limoges.

Lim, Y. S. & Chapman, E. (2015). Effects of using history as a tool to teach mathematics on students 'attitudes, anxiety, motivation and achievement in grade 11 classrooms. *Educational Studies in Mathematics, 90*(2), 189–212.

Moyon, M. (2022). Desire of teachers and realities in textbooks: dealing with history of mathematics in the new French curriculum and its impact on teacher training. *ZDM –*

*Mathematics Education*
Perkins, P. (1991). Using history to enrich mathematics lessons in a girls 'school. *For the Learning of Mathematics,* 11(2), 9–10.
Siu, M. K. (1997). The ABCD of using history of mathematics in the (undergraduate) classroom. *Bulletin of the Hong Kong Mathematical Society, 1*(1), 143–154.
Tzanakis, C., Arcavi, A., Correia de Sa, C., Isoda, M., Lit, C. K., Niss, M., Pitombeira de Carvalho, J., Rodriguez, M., & Siu, M. K. (2000). Integrating history of mathematics in the classroom: An analytic survey. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study* (pp. 201–240). Springer

# GET IN TOUCH WITH CALCULUS

## A new material device collecting a historical legacy

**Michela MASCHIETTO, Pietro MILICI**

Department of Education and Humanities, University of Modena e Reggio Emilia, via Timavo 93, Reggio Emilia, Italy

Department of Theoretical and Applied Sciences, University of Insubria, Via O. Rossi 9, Varese, Italy

michela.maschietto@unimore.it

pietro.milici@uninsubria.it

## ABSTRACT

We introduce an analog device designed for laboratory activities related to calculus. Such a device recollects the legacy of historical instruments to find the area by solving inverse tangent problems (*integraphs*), that analytically corresponds to the resolution of differential equations. We present an analysis of its exploration by a high-school mathematical teacher with experience in mathematical machines.

## 1 *Introduction*

In this paper, we introduce a new analog device that recollects the legacy of historical instruments to find the area by solving inverse tangent problems (*integraphs*). We present an analysis of it by a high-school mathematical teacher with experience in mathematical machines and analyze the exploration within the theoretical framework of the semiotic mediation in mathematics education from the perspective of its use in the classroom.

The starting geometrical problem is to construct a curve given the properties of its tangent, the so-called "inverse tangent problem" (cf. Bos, 1988; Tournès, 2009; Milici, 2015; Milici, 2020; Crippa & Milici, 2019). To mechanically solve an inverse tangent problem, we must constrain a point so that it moves along a direction. Considering a wheel rolling on a curve, the direction of the wheel is tangent to the curve. By guiding that direction, in the first half of the 18th century scholars like Perks, Poleni and Suardi were able to trace transcendental curves. In the late 19th century, similar technical ideas were independently rediscovered; for example, while moving a pointer along a traced curve, a machine was able to trace the integral of a previously traced curve. Machines of this kind were named integraphs. More historical details are available in this ESU9 volume (cf. Crippa & Milici, 2023).

## 2 *Our integraph*

We introduce an integraph invented by the second author and built by typical FabLab tools (laser cutting, 3D printing, CNC milling). Following the numbering on the left-hand side of Fig. 1, it is made up by the following components.

1. *The frame*. This stands on a sheet of paper and allows the plate [2] to slide.
2. *The plate*. This is a rectangular piece of transparent plexiglass with three guides carved out (the two little ones perpendicular to the big one).
3. *Two rods*. These act as linear guides to be put on the plate. They can be joint to form a T (the short rod contains the piece [6], the long one pieces [4] and [5]).
4. *The peg*. After fixing the peg in a point of the two little guides of the plate [2], it slides inside the long rod [3].
5. *The positional pointer*. This is a pen holder that can slide inside the long rod [3] and the big guide of the plate [2] (see also Fig. 1, top right).
6. *The directional pointer*. This is a pointer on which the bottom two parallel wheels can rotate at different speeds when touching the paper sheet. It has a cap to constrain the direction of the rod through which it passes to be parallel or perpendicular to the direction of the wheels (see also Fig. 1, bottom right).

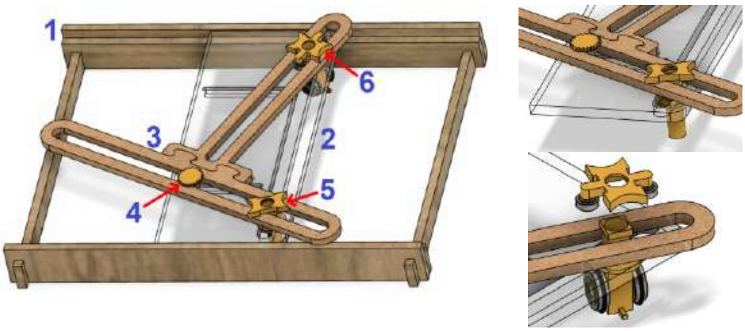



**Figure 1.** Left: our integraph and its components. Top right: the positional pointer and the peg (note that for the positional pointer the orientation of its top cap is irrelevant). Bottom right: the directional pointer (note that the square head allows the top cap to be right-angle rotated).

Pointers have a hole that can be used as a viewfinder (to move the pointer along a curve) but also to hold a marker (so that the pointer leaves a trace). To understand how this machine is related to calculus, let us introduce two reference frames such that abscissae correspond with the little guides on the plate [2 in Fig.1] and the two ordinate axes are superimposed. In Fig. 2, we represent in red the Cartesian axes related to the directional pointer D [6 in Fig.1], and in blue the ones for the positional pointer P=(x,y) [5 in Fig.1]. We also take as unit the distance between the peg [4 in Fig.1] and the big guide of the plane: thus the peg has coordinates (x-1,0) in the blue reference frame and the ordinate of P corresponds to the slope of its line through the peg. According to the configuration on the left of Fig. 1, the direction of D is set perpendicular to the short rod, that is perpendicular to the long rod: that implies that the line through P and the peg must be parallel to the direction of D. But if D follows the graph of a function *f*, its direction corresponds to the tangent to the curve, therefore the slope of the line through P and the peg is exactly the derivative $f'$. To sum up, if we move D along the graph of a function in the red reference frame (recall that, to move along a curve, we have to guide the direction of D), the point P can trace by a marker its derivative in the blue reference frame. Conversely, when P moves along the graph of a function in the blue reference frame, D traces one of its anti-derivatives (according to the initial position of D).

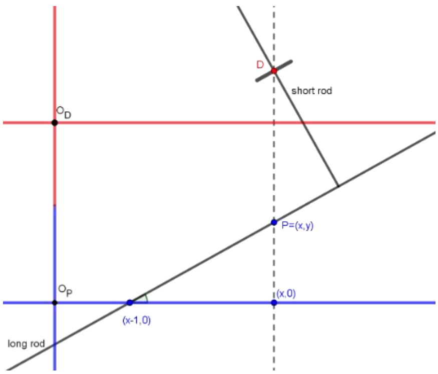



**Figure 2.** D is the directional pointer (the gray segment represents the direction of D) and its position is relative to the red axes; P is the positional pointer and its position is relative to the blue axes. Note that, in every configuration, the abscissae of the two pointers coincide.

## 3 Theoretical background and methodology

Our study of the use of the integraph in mathematics teaching and learning is based on the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008), in which an artifact is used for mediating mathematical meanings. The educational choice of an artifact is based on the analysis of its semiotic potential, i.e. how its use to accomplish a task is linked with emerging personal meanings and mathematical meanings embedded in the artifact. This analysis is essential for constructing tasks for students. It is mainly based on the exploration of the machine which is guided by four questions (Bartolini Bussi et al., 2011): How is the machine made? What does the machine make? Why does it make it? What could happen if ...?

Within these historical and educational references, we aim to answer to the following research questions:

1) What are the cognitive processes during the exploration of the machine?

2) Which, when and how do mathematical meanings emerge, in particular the idea of tangent line during the exploration of the machine?

For answering these questions, we proposed the exploration of the machine to a mathematics teacher (V) who is an expert in mathematical machines (she collaborates with the University of Modena e Reggio Emilia). This choice is based on three main reasons: the mathematical knowledge embedded in the machine is part of teacher's mathematical background; her expertise in mathematics laboratory with mathematical machines; sharing of the mathematics education theoretical framework. The teacher participated to the exploration of another inverse-tangent machine some years before (Maschietto, Milici & Tournès, 2019). Nevertheless, the two artefacts are very different.

The process of exploration was guided by one of the authors (P) of this paper. The analysis is based on the videotape of the session and V's drawings.

## 4 Analysis

In this paper, we analyze the first part of the exploration. For our analysis, we take into account four steps:

1. Start of the exploration and the emergence of the components;
2. Conjectures on movements of the machine without moving it;
3. Gestures performed during the manipulation;
4. The emergence of the idea of a tangent line.

1. The exploration starts with the description of the machine, corresponding to the answer to the question "How is the machine made?". At the beginning, V identified: «a plane [the frame, 1 in Fig.1], there are two rails on which I suppose this structure can shift…then there is a kind of set square … composed of two rods [3 in Fig.1] forming a right angle. Inside these rods, there are sliders [pointers 5 and 6, in Fig.1]. […]. Below, one of the two sliders [6 in Fig.1] has some small wheels, the other has not». The peg [4 in Fig.1] is mentioned by V only after P's question. After this, V paid attention to another characteristic of the machine: two pointers touch the plane, while the peg does not. About 5 minutes after the beginning of the exploration, V saw the holes into the pointers and highlighted the possibility to insert a pencil. After a certain time period, V asked about the role of the "transparent structure" [plate]. Even though V knew how the exploration of a machine should be carried out, the components emerged little by little through the interaction with P.

2. Concerning the movement of the machine, P asked how the machine could move and what were the constraints before moving the machine. First, V made conjectures about the relationships between the components. Then, V tested her conjectures directly by moving the plate. In particular, we have distinguished:

- Conjecture about movement: «I think that the structure on the two rails, in fact, moves vertically following the rails and ... we think ... if this structure moves […] this slider [the positional pointer] will move to the left and this one [the peg] will move upwards, I think».

- Conjectures about constraints: «The constraints are…, probably because there is another groove here, the machine can be turned, I think. You can turn it over by inserting this rod into this groove [the big guide with the directional pointer inside], I think. The constraints are given by ... these ... small grooves, I guess, the slider [pointer] can only shift in there and got to the bottom ... Then…».

3. During the exploration, we have identified three kinds of gestures, concerning how the different components are grasped and moved:

- Sliding the plate by moving fingers placed upon it. There are gestures of usage for discovering the possibility of movement of the machine.

- Following a drawn straight line. V was asked to follow a straight line drawn on the paper at the plane of the machine.

- Following a curve. After changing the structure of the machine by taking off the positional pointer and putting a single rod in the direction of the wheels of the directional pointer, P traced a curve on the paper sheet and asked V to follow it.

With respect to the previous gestures, V directly grasped the pointer and followed the curve.

4. Concerning the idea of tangent line, the task of following a drawn curve seems to be crucial. The controlled movement of following a curve seems to suggest to pay attention to the wheel and the relationship between the rod and the curve, but P's intervention is fundamental at this point.

P: «What does this rod represent? With respect to the drawn curve?».

V: «At each point, it is the tangent line… yes, at each point of the curve it is the tangent line at the curve in the point».

## *5 Conclusion*

This article presents a step of our study of the machine from the perspective of its use in mathematics education. It considers the analysis of the semiotic potential, which is based here on the exploration of the machine by a high-school mathematical teacher.

The first research question concerns the processes activated during the use of the machine. The analysis highlights that the exploration of the machine is quite complex, because of its different and several components and their manipulations. Indeed, the components emerge little by little and at different steps; gestures depend on the configurations and constraints of the machine. The guidance of the researcher results essential during all the process, both to make the components emerge and to support the exploration. For instance, when the task of following a curve with the directional pointer is proposed, the focus on a particular component is necessary. It thus emerges that the rod evokes the idea of a tangent line in a strong way.

Our analysis aims to provide insights for constructing tasks for students and teachers who are not experts in exploring this kind of machines. It also suggested the construction of a new version of the machine, available at www.machines4math.com/; it can be made by downloading files for free from www.thingiverse.com/thing:5532958

**REFERENCES**


Bartolini Bussi, M.G., Garuti, R., Martignone, F., & Maschietto, M. (2011). Tasks for teachers in the MMLAB-ER Project. In B. Ubuz (Ed.), *Proceedings of PME35* (Vol. 1, pp. 127–130). PME.

Bartolini Bussi, M.G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English et al. (Eds.), *Handbook of international research in mathematics education* (II ed., pp. 746–783). Routledge.

Bos, H.J. (1988). Tractional motion and the legitimation of transcendental curves. *Centaurus*, 31(1), 9–62. DOI:10.1111/j.1600-0498.1988.tb00714.x

Crippa, D., & Milici P. (2019). A relationship between the tractrix and logarithmic curves with mechanical applications. *The Mathematical Intelligencer*, 41(4), 29–34. DOI:10.1007/s00283-019-09895-7

Crippa, D. & Milici, P. (2023). Transcendental curves by the inverse tangent problem: historical and didactical insights for calculus. *ESU9 proceedings* (this volume).

Maschietto, M., Milici, P., and Tournès, D. (2019). Semiotic potential of a tractional machine: a first analysis. In Jankvist, U.T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.), *Proceedings of CERME 11*. Freudenthal Institute, Utrecht University and ERME.

Milici, P. (2015). A geometrical constructive approach to infinitesimal analysis: Epistemological potential and boundaries of tractional motion. In G. Lolli, M. Panza, & G. Venturi (Eds.), *From logic to practice* (pp. 3–21). Boston studies in the philosophy and history of science 308. Springer. DOI:10.1007/978-3-319-10434-8_1

Milici, P. (2020). A differential extension of Descartes' foundational approach: A new balance between symbolic and analog computation. *Computability*, 9(1), 51–83. DOI:10.3233/COM-180208

Tournès, D. (2009). *La construction tractionnelle des équations différentiell.* Blanchard

# HISTORY OF MATHEMATICS IN PORTUGUESE TEXTBOOKS OF VICENTE GONÇALVES (1896-1985) AND JOSÉ SEBASTIÃO E SILVA (1914-1972)

**Hélder PINTO[1], Cecília COSTA[2] and Helmuth R. Malonek[3]**
[1] Piaget Institute, RECI and CIDMA, Portugal, helder.pinto@ipiaget.pt
[2] University of Trás-os-Montes e Alto Douro and CIDTFF, Portugal, mcosta@utad.pt
[3] University of Aveiro and CIDMA, Portugal, hrmalon@ua.pt

## ABSTRACT

Vicente Gonçalves (1896-1985) and José Sebastião e Silva (1914-1972) were two Portuguese mathematicians' university professors who have written textbooks for undergraduate levels. Those books became famous by their mathematical accuracy but also by their notes about History of Mathematics. In this study we identify and analyse in detail the characteristics of those historical notes.

## *1 Introduction*

During the time of the Estado Novo («The New State», synonym for the dictatorship installed by A. Salazar in Portugal in 1933 and lasting until 1974), two Portuguese mathematicians and university professors – Vicente Gonçalves (1896-1985) and José Sebastião e Silva (1914-1972) – wrote mathematical textbooks for high school. This was unusual in Portugal at the time. Only few textbooks by J. Sebastião e Silva were co-authored by the high school teacher Silva Paulo.

These textbooks are from distinct periods: those by Vicente Gonçalves are from the beginning of the Estado Novo (published between 1935 and 1939), while those by J. Sebastião e Silva are from the end of the dictatorship (1957-1978) (Fig. 1).

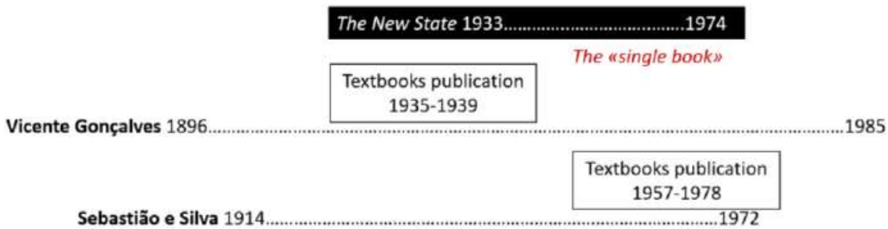



**Figure 1**. Time scheme

The textbooks written by Sebastião e Silva were, in general, the *livro único* [single book]. Being *livro único* means that they were the only books that were allowed to be used to teach mathematics in all of the country´s schools. The approval period for a single book was five years, during which the authors could propose, in new editions, changes that they considered important. On each copy offered for sale, it would appear on the back cover, the mention "OFFICIALLY APPROVED AS A SINGLE BOOK", with the indication of the Government Diary in which it was instituted. In addition, they were numbered and approved by the Ministry of National Education (Almeida, 2011).

Several of Vicente Gonçalves and Sebastião e Silva textbooks have the particularity of containing notes about the History of Mathematics, a fact that was uncommon in high school education in Portugal. The aim of this study is to identify and analyse in detail the characteristics of those historical notes.

Vicente Gonçalves and J. Sebastião e Silva were important members of the mathematical community of their time; it's possible to find more information about them, for instance, in (Costa, 2001) and (Guimarães, 1972). Their works were widely disseminated in Portugal and are still recognized today for their scientific accuracy. Given their importance in the Portuguese panorama, we consider pertinent the description, study and understanding of this option of inserting History of Mathematics' topics in the educational context through textbooks. These examples of using History of Mathematics are interesting because they are consistent with the goals of the HPM group although they are from an earlier period (mid-twentieth century).

## *2 Methodology*

The corpus of analysis was constituted by the textbooks for Mathematics Secondary Education, five writen by Vicente Gonçalves and three by Sebastião e Silva, two of them in co-authorship with Silva Paulo. The categories considered were the ones established in (Fauvel, 1991): (i) as brief historical notes; (ii) as introduction of new concepts; (iii) as a pedagogical tool; (iv) as a content referring to some historical facts; (v) as resource of exercises or examples.

## *3 Findings*

Three of the five textbooks by Vicente Gonçalves have historical remarks: (Gonçalves, 1937a, 1937b, 1939). All the three textbooks by Sebastião e Silva have historical remarks: (Sebastião e Silva & Silva Paulo, 1968a, 1968b) and (Sebastião e Silva, 1978), an after death compilation. Considering Fauvel's categorization (1991), the textbooks of both authors have:

i) **brief historical notes** (many as footnotes):

This notes focus mathematicians' biographies, introduction of notations, and the creation and evolution of mathematical symbols. Next we present some illustrations [52]:

*"Algebra comes from the Arab word al-jabr, reduction, decomposition (see further on no. 137). The al-jabr is one of the operations treated by Al-Kwarizmi in his work Al-jabr w' al moqabalah, translated in the XII century by Leonardo de Pisa and followed in Western Europe till the XVI century."* (Gonçalves, 1937a, p. 240)

*"The sign x appears for the first time in 1631, in Clavis Mathematicœ published by Oughtred; the dot appears, on the same date, in Artis Analyticœ Praxis, by Harriot. Descartes simply joined together the factors (as in the text)."* (Gonçalves, 1939, p. 62)

*"The word "cartesiano" is derived from "Cartesius", the Latin name of the great French mathematician and philosopher Descartes (1596-1650), one of the greatest thinkers of all time. It was Descartes who introduced this method of representation, founding analytic geometry, which will be studied in more detail in 7th grade."* (Sebastião e Silva & Silva Paulo, 1968a, p. 122)

Both Vicente Gonçalves and Sebastião e Silva gave prominence to ancient Portuguese mathematicians and explain, with certain detail, the inovation of their works. The notes concerning Pedro Nunes and Daniel Augusto da Silva, should be highlighted.

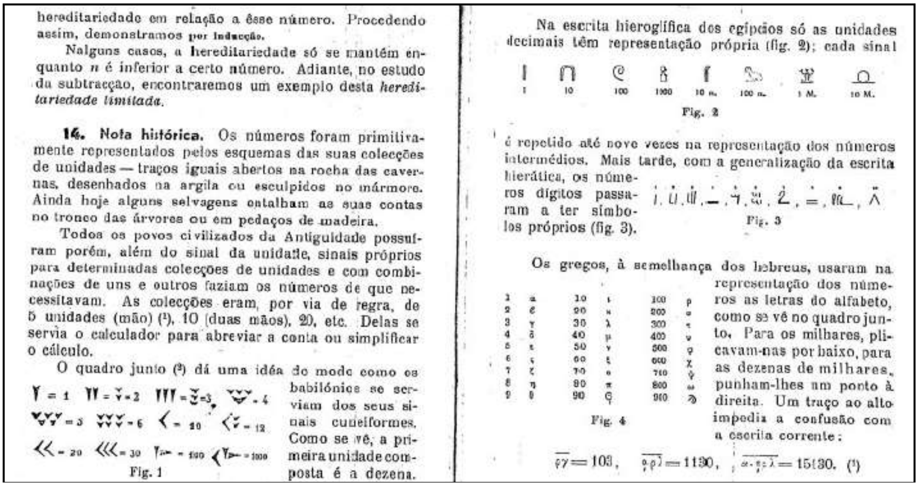
ii) **contents referring to some historical facts**:

These textbooks contain several sections entirely devoted to different topics of History of Mathematics. In the following are presented some of the contents treated.

- Chapter Metric System, in (Gonçalves, 1937a, p. 148);
- Numbers Representation, in (Gonçalves, 1939, pp. 26-31), see Fig. 2;

[52] All the quotes presented in this paper were translated by the authors from the Portuguese original text.

- Distribution of prime numbers, in (Gonçalves, 1939, pp. 169-170);
- Origin and evolution of the concept of fraction, in (Gonçalves, 1939, pp. 196-200);
- Mensurable and incommensurable quantities, in (Sebastião e Silva & Silva Paulo, 1968a, pp. 71-74);
- The infinity and infinitesimals, in (Sebastião e Silva & Silva Paulo, 1968a, pp. 180-184), see Fig. 3;
- History of Algebra, in (Sebastião e Silva & Silva Paulo, 1968b, pp. 211-220).
- History of Logarithms (and slide rule), in (Sebastião e Silva & Silva Paulo, 1968b, pp. 276-281).


hereditariedade em relação a êsse número. Procedendo assim, demonstramos por indução.

Nalguns casos, a hereditariedade só se mantém enquanto $n$ é inferior a certo número. Adiante, no estudo da subtracção, encontraremos um exemplo desta *hereditariedade limitada*.

**14. Nota histórica.** Os números foram primitivamente representados pelos esquemas das suas colecções de unidades — traços iguais abertos na rocha das cavernas, desenhados na argila ou esculpidos no mármore. Ainda hoje alguns selvagens entalham as suas contas no tronco das árvores ou em pedaços de madeira.

Todos os povos civilizados da Antiguidade possuíram porém, além do sinal da unidade, sinais próprios para determinadas colecções de unidades e com combinações de uns e outros faziam os números de que necessitavam. As colecções eram, por via de regra, de 5 unidades (mão) (¹), 10 (duas mãos), 20, etc. Delas se servia o calculador para abreviar a conta ou simplificar o cálculo.

O quadro junto (²) dá uma idéa do modo como os babilónios se serviam dos seus sinais cuneiformes. Como se vê, a primeira unidade composta é a dezena.



Na escrita hieroglífica dos egípcios só as unidades decimais têm representação própria (fig. 2); cada sinal



é repetido até nove vezes na representação dos números intermédios. Mais tarde, com a generalização da escrita hierática, os números digitos passaram a ter símbolos próprios (fig. 3).



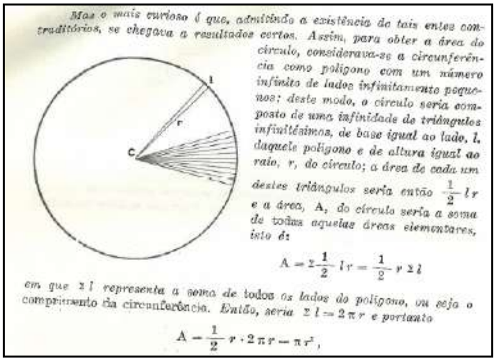
Os gregos, à semelhança dos hebreus, usaram na representação dos números as letras do alfabeto, como se vê no quadro junto. Para os milhares, plicavam-nas por baixo, para as dezenas de milhares, punham-lhes um ponto à direita. Um traço ao alto impedia a confusão com a escrita corrente:

| | | | | | |
|---|---|---|---|---|---|
| 1 | α | 10 | ι | 100 | ρ |
| 2 | β | 20 | κ | 200 | σ |
| 3 | γ | 30 | λ | 300 | τ |
| 4 | δ | 40 | μ | 400 | υ |
| 5 | ε | 50 | ν | 500 | φ |
| 6 | ς | 60 | ξ | 600 | χ |
| 7 | ζ | 70 | ο | 700 | ψ |
| 8 | η | 80 | π | 800 | ω |
| 9 | θ | 90 | ϙ | 900 | ϡ |

Fig. 4

$\overline{\rho\gamma} = 103$, $\overline{\alpha\rho\lambda} = 1130$, $\overline{\alpha\cdot\epsilon\rho\lambda} = 15130$. (¹)

**Figure 2.** Historical section about numbers in (Gonçalves, 1939, pp. 26-27)


*Mas o mais curioso é que, admitindo a existência de tais entes contraditórios, se chegava a resultados certos. Assim, para obter a área do círculo, considerava-se a circunferência como polígono com um número infinito de lados infinitamente pequenos; deste modo, o círculo seria composto de uma infinidade de triângulos infinitésimos, de base igual ao lado, l, daquele polígono e de altura igual ao raio, r, do círculo; a área de cada um destes triângulos seria então* $\frac{1}{2} l r$ *e a área, A, do círculo seria a soma de todas aquelas áreas elementares, isto é:*

$$A = \Sigma \frac{1}{2} l r = \frac{1}{2} r \Sigma l$$

*em que* $\Sigma l$ *representa a soma de todos os lados do polígono, ou seja o comprimento da circunferência. Então, seria* $\Sigma l = 2\pi r$ *e portanto*

$$A = \frac{1}{2} r \cdot 2\pi r = \pi r^2,$$



**Figure 3.** Part of the historical section in (Sebastião e Silva & Silva Paulo, 1968a, p. 182), where were used infinitesimals to find the area of the circle.

Some of the historical sections were very long and complete; for instance, in the section about the "*History of the Algebra*" were pointed out all the following names (the order of the list respect the order in the original text): Alkarismi, Euclides, Diofanto, Fibonaci, Dedekind, Cantor, Pedro Nunes, Viéte, Jordanus, Stiefel, Harriot, Descartes, Del Ferro, Tartaglia, Cardano, Ferrari, Bombelli, Abel, Ruffini, Galois and Liouville; and with the warning: "the History of Algebra does not end here: it continuous, following the history of man on Earth" (Sebastião e Silva & Silva Paulo, 1968b, p. 220).

The study and the analysis of some historical parts from these textbooks can provide current teachers some ideas for introducing new concepts as well for using it as source of exercises or problems.

## *4 Final remarks*

This study shows that the uses of History of Mathematics in the textbooks of Vicente Gonçalves and Sebastião e Silva, two very prominent 20th century Portuguese mathematicians, are mostly as brief notes and as content referring to historical facts.

The brief historical notes focus on mathematicians (their life and work), and the introduction of notations and their creators, as well as creation and evolution of mathematical symbols. The reference to some historical facts, in various cases is very extensive and complete. There are detailed explanations of various topics that are seldom currently found in a textbook. For example: mensurable and incommensurable quantities; the approximation of the number Pi; Zeno's paradoxes; and Zermelo's axiom. There are also very detailed notes on some of the most important figures of all time such as Fermat, Leibniz, Newton, Galois, as well as references to several of their works.

History of Mathematics was not mandatory and it was printed in small letters; in small letters are also printed the most difficult and theoretical exercises that only should be applied to the "good" students. Teachers were advised at the Preface of the book to present these topics only to the more interested students (Sebastião e Silva & Silva Paulo, 1968a and 1968b). Nevertheless, as it was *livro único* [single book], all students had access to the historical information contained in those textbooks.

Note that these historical notes were intended, mainly, to humanize the discipline of mathematics:

"The detailed clarification of certain issues, as well as the insertion of the subjects within the framework of a general culture, which temper and humanize the abstractionism inherent to mathematics, trying to explain it as a historical process – all of this is a considerable enterprise, which can only be attempted in a book." (Sebastião e Silva & Silva Paulo, 1968a, Preface)

Finally, notice that these historical notes were quite complete and thorough in comparison to those in current textbooks in Portugal. Notice also that there are examples in other countries of the use of History of Mathematics in footnotes (for one example, in 19th Century (Spain), see (Muñoz-Escolano & Oller-Marcé, 2021)).

## REFERENCES


Almeida, M. (2011). O livro único em Portugal. 1947-1974. In L. Saraiva (Ed.), *Livro de Atas: V Encontro Luso-Brasileiro de História da Matemática* (pp. 305-320). Castelo Branco: C. M. Castelo Branco.

Costa, C. (2001), *José Vicente Gonçalves: Matemático. . . porque Professor!*. Funchal: Centro de Estudos de História do Atlântico.

Fauvel, J. (1991). Using History in Mathematics Education. *For the learning of Mathematics*, *11*(2), 3-6.

Gonçalves, J. V. (1937a). *Aritmética prática e Álgebra para as 1.ª, 2.ª e 3.ª classes do curso dos liceus*. Braga: Livraria Cruz.

Gonçalves, J. V. (1937b). *Compêndio de Álgebra para o 3.º ciclo (7.ª classe) do curso dos liceus*. Braga: Livraria Cruz.

Gonçalves, J. V. (1939). *Compêndio de Aritmética para o 3º ciclo (7ª classe) do curso dos liceus*. Braga: Livraria Cruz.

Guimarães, A. A. (1972). *Vida e Obra do Professor José Sebastião e Silva*. Porto: FCUP.

Muñoz-Escolano, J. M., & Oller-Marcén, A. M. (2021). Notas al pie en la Aritmética (1884) de los Coroneles Salinas y Benítez. *Matemáticas, educación Y Sociedad, 4*(1), 17-33.

Sebastião e Silva, J. & Silva Paulo, J. (1968a). *Compêndio de Álgebra para o Ensino Liceal, 6.º ano* (Tomo 1). Braga: Livraria Cruz.

Sebastião e Silva, J. & Silva Paulo, J. (1968b). *Compêndio de Álgebra para o Ensino Liceal, 7.º ano* (Tomo 2). Braga: Livraria Cruz.

Sebastião e Silva, J. (1978). *Compêndio de Matemática para o Ano Propedêutico*. Lisboa: GEPMEIC.

# RANDOM WALKS IN THE HISTORY AND EPISTEMOLOGY OF MATHEMATICS

**Jorge SOTO-ANDRADE[1], Dandan SUN[2], Daniela DIAZ-ROJAS[3], Amaranta VALDES-ZORRILLA[1]**

[1]University of Chile, Santiago, Chile; [2]Shandong Normal University, China;

[3]University of Oxford, United Kingdom.

sotoandrade@uchile.cl, dandan-sun@qq.com, daniela.diazrojas@wolfson.ox.ac.uk, amvaldes@uchile.cl

## ABSTRACT

The random walks in the title go from eleventh-century Chinese mathematician Shao Yong's Xiantian diagrams, illustrating the genesis of the Yi Jing hexagrams, up to Darwin's tree of life and the contemporary "eternal symmetree" of Stanford physicists in cosmology. They also involve the old Chinese 5-element cycles via random whole numbers. We present illustrative examples of classroom activities inspired by these random walks, with humanistic as well as scientific students, including pre-service and in-service teachers, where we implement an experiential way of teaching. Our final discussion takes into account some caveats regarding Whig history.

## *1 The Yi Jing, Shao Yong, Bouvet and Leibnitz*

Our first random walk starts with the launching of the Jesuit Mission to China (1552-1715) by François Xavier. Among the Jesuits travelling to China ca. 1700 was mathematician and astronomer Joachim Bouvet (1656 – 1730). While cleverly trying to find bridges between Chinese culture and Christianity, a prerequisite to evangelisation, Bouvet stumbled upon Yi Ying, (Book of Changes), the ancient Chinese oracle. Recall that Yi Jing hexagrams (as answers to questions asked) were selected by a traditional random procedure (Wilhem/Baynes, 1950), which provided equally likely Yang (unbroken) and Yin (broken) lines for the hexagrams.

Shao Yong (1011-1077) was an extraordinary Chinese thinker, whose complex and metaphorical thinking blended Confucianism, Taoism and Buddhism, with a unique twist of mind. He has been described as an alien, of another place and time (Birdwhisttell, 1989, p. 52). His main concern was *change* in the Universe, rather than matter and substance, *processes* rather than *structures*. He developed an iconic and philosophical approach to Yi Jing, as a combinatorial encoding of the varieties of change in nature and in human society (Birdwhistell, 1989). Notice how his rectangular and circular Xiantian diagrams (Fig. 1) provide a *synthetic* view of the 64 hexagrams.

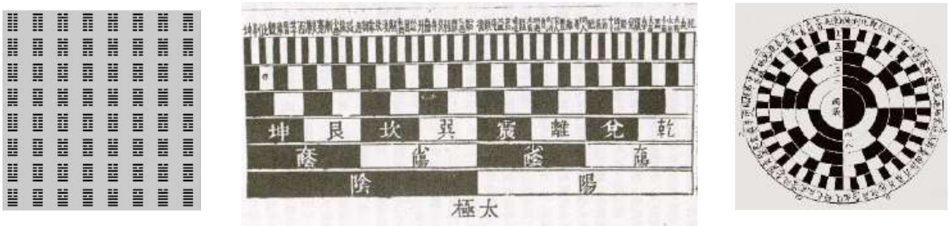



**Figure 1.** Shao Yong's 8x8 square and Xiantian diagrams (Marshall, 2010)

Bouvet discovered the isomorphism between Shao Yong's 64 hexagram sequence and the numbers 0 to 63 written by Leibnitz in the binary way, enabling Leibnitz to trace back his construction of the binary system 4 500 years ago embodied in the Yi Jing hexagrams. This resonated with his dream of linking Chinese "natural theology" and his Christian faith.

We have here a remarkable epistemological phenomenon: the convergence of an ancient Chinese insight and a more recent European one on the binary nature and dynamics of the Universe, which was a common concern of Shao Yong and Leibnitz. (Aiton & Shimao, 1981; Birdwhistell, 1989). Xiantian is very likely the first explicit avatar of the binary tree in human civilisation. It appears as a pictorial "explanation" of the generation, through binary branching, of the hexagrams, where black=Yin, white=Yang (cf. Needham, 1956, pp. 276-277). Shao Yong appears then as a forerunner of Darwin: both share the remarkable idea of fathoming the current state of a system as the outcome of a *branching process* unfolding in time. Shao Yong saw the 64 hexagrams as the outcome of his Xiantian and Darwin, the sundry living species on earth, as coming out of the Tree of Life (Gould, 1997).Interestingly Xiantian surfaces much later again, in the West, as the "eternal symmetree", a discrete combinatorial model for a multiverse in eternal inflation in cosmology (Harlow et al., 2012). See Fig. 2.

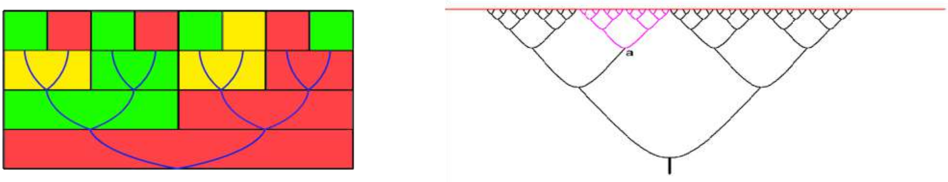



**Figure 2.** Three-step causal tree and causal future of *a*, for *p=2.*

## 2 *The Chinese five element cycles and random numbers*

After the Chinese Five Element Theory (Wu Xing), dating back to ca. 400 BC (Needham, 1956, p. 242), the world changes according to the five elements' *generating* or *overcoming* cyclic relationships (ibid. pp. 253 ff). See Fig. 3. Generating Interactions may be worded as: Wood fuels Fire; Fire forms Earth; Earth contains Metal; Metal carries Water; Water feeds Wood. Overcoming Interactions may be worded as: Fire melts Metal; Metal penetrates Wood, Wood separates Earth; Earth absorbs Water; Water quenches Fire.

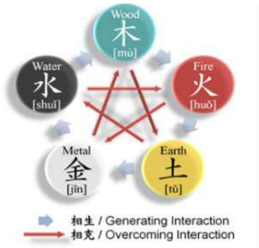



**Figure 3.** Generating and overcoming cycles in Wu Xing

The mathematical question arises: how could we model in an elementary way these "intransitive" cycles, using just whole numbers, so that the cyclic orderings emerge naturally? One idea is to use *random whole numbers*, i. e. (finitely supported) probability distributions on the whole numbers, which can be embodied in *dice*. A usual dice embodies the random number taking values 1, 2, 3, 4, 5, 6 equally likely. We have a natural order relation among random numbers: A random number *m dominates* (or *overcomes*) another random number *n* if *m* is bigger than *n* "most of the time". Now, to model the 5-element overcoming and generating cycles, we can concoct the following weird dice in Fig. 4 (Grime, 2019). Here yellow is dominated by red, red by green, green by pink, pink by blue and blue by yellow. The opposite cycle goes: yellow – pink – red – blue – green – yellow, where each dice dominates the next.

**Figure 4.** The Grime dice modelling Wu Xing.

### *3 Illustrative examples of related classroom activities*

Our activities involve the following learners at the University of Chile: (a) First year humanistic students, taking a one-semester mathematics course; (b) Pre-service secondary mathematics-physics teachers; (c) Prospective mathematicians and mathematics-physics teachers taking a history and epistemology of mathematics course; (d) In-service primary school teachers enrolled in a professional development program. Our didactic methodology is to have the students work autonomously on given or self-constructed problems in random small groups of 3 or 4 monitored by the teacher and assistants.

### 3.1 Discovery approach to Shao Yong's diagrams

How would a *synthetic* (big) picture of the whole binary 64 hexagram sequence look like?

Here learners (d), used to work with concrete material, spent half an hour working in groups, cutting out a paper printout of the hexagrams and trying to put them together side by side. Four of thirty teachers came up with Xiantian like diagrams, after coding Yang as white and Yin as black.

### 3.2 Archeological exploration of Shao Yong's square

Learners received a copy of Shao Yong's square and tried to make sense of it. They saw the square as an 8x8 matrix indexed by trigrams! Some recognized the binary expansion of numbers by interpreting Yang or Yin in the *n*-th line as the *presence* or *absence* of $2^n$, as Leibnitz did.

### 3.3 Grasping Xiantian in a glimpse

We showed a glimpse of Xiantian to learners (a), (c) and (d) and asked them to reconstruct it. See Fig. 3 (Soto-Andrade, 2008).

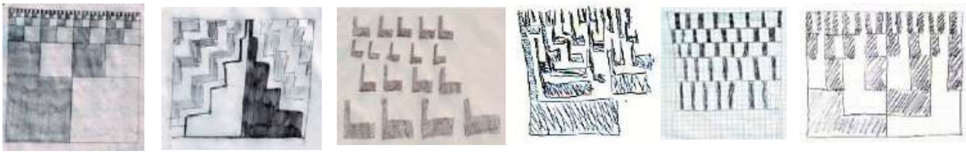


**Figure 5.** Students' reconstructions of Xiantian

### 3.4 Numbers as ascendings paths in a rooted binary tree

Learners (c), metaphorised numbers from 0 to 63 as ascending walks in the 6-generation binary tree and went further to the infinite binary tree (Needham, 1956, p. 276) and its *boundary,* rediscovering Hensel's notation of 2-adic numbers as series of powers of 2 (Dickson, 1910).

### 3.5 A baby avatar of the Chinese 5-element cycle

Our students concocted tetrahedral dice to model the well-known intransitive 3-cycle: scissors, stone, paper. See Fig. 6.

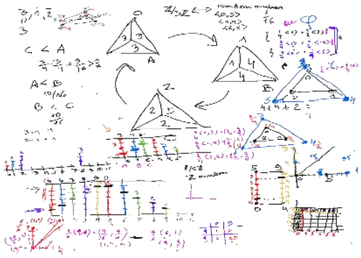


**Figure 6.** Student production: tetraedral dice to model the 3-cycle

## *4 Discussion and caveats*

In our classroom activities, particularly with students C on a separate historical track (Fried, 2001), we used Shao Yong's diagrammatic mathematical and philosophical insights as *triggers* of autonomous mathematical and epistemological reflexions. As spin-offs of our activities, we noticed that the work the students did in a separate history of mathematics track fed back, in a circular way, to their understanding of previous mathematical contents they were supposed to master. They often reported having understood for the first time contents they had learned in a formal, abstract and disconnected way. An enactivist caveat is required though: we do not see the binary tree as a well-defined mathematical object "standing out there", which is fathomed and *represented* by different cultures in different guises (avatars). To us, the binary tree is a typical Western mathematical construction, which we automatically project onto Xiantian. We intend nevertheless to avoid being "Whiggish" in this historical context (Butterfield, 1931), particularly avoiding the "Brownian mirage" of seeing a direction in our path backwards, which from the perspective of the past is indeed the path of a random walker (Fried, 2001). We try instead to listen to Shao Yong (Arcavi & Isoda, 2007), suspending our preconceptions (*epoché*) and accept that this may enhance our understanding of "something" that we have constructed as a binary tree nowadays in our culture. Indeed, trees appear often as hierarchical *structures,* contrary to *rhizomes* (Deleuze & Guattari, 1980), but Xiantian suggests a (dichotomous branching) *process*, pertaining to a changing and flowing cosmos, a Taoist view indeed (Birdwhistell, 1989). Epistemological disorientation (Clark et al., 2018) may help us to acknowledge that other cultures have had insights ours is blind to.

**Acknowledgements.** Funding from PIA-CONICYT Basal Funds for Centres of Excellence Project FB0003 and DAAD Project 573 35022 D (Universität Bielefeld - Universidad de Chile) is gratefully acknowledged.

## REFERENCES

Aiton, E. and Shimao, E. (1981). Gorai Kinzō's study of Leibniz and the I Ching hexagrams, *Annals of Science*, 38:1, 71-92.

Arcavi, A. and Isoda, M. (2007). "Learning to listen: from historical sources to classroom practice" *Educational Studies in Mathematics, 66*(2), 111-129.

Birdwhistell, A. D. (1989). *Transition to neo-Confucianism: Shao Yung on knowledge and symbols of reality.* Stanford U. Press.

Butterfield, H. (1931) *The whig interpretation of history,* G. Bell and Sons.

Clark, K. M. et al. (Eds) (2018). *Mathematics, Education and History.* ICME-13 Monographs. Springer.

Deleuze, G. and Guattari, F. (1980). *Mille Platteaux*, Éditions de Minuit.

Dickson, L. E. (1910). Hensel's Theory of Algebraic Numbers. *Bull. Amer. Math. Soc.*, *17*(1), 23-36.

Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, *10*(4), 391–408.

Gould, S.J. (1997). Redrafting the Tree of Life. *Proceedings of the American Philosophical Society 141*(1), 30-54.

Grime, J. (2019). *Intransitive Dice*. https://singingbanana.com/dice/article.htm

Harlow, D., Shenker, S. H., Stanford, D., and Susskind, L. (2012). Eternal symmetree, *Phys. Rev. D 85*, 063516

Leibniz, G. W., Cook. D.J. and Rosemont, H. (1994). *Writings on China*. Open Court.

Marshall, S. (2015). *Reading hexagrams off the Xiantian diagram*. www.biroco.com/yijing/xiantian.htm

Needham, J. (1956), *Science and Civilization in China*, vol. 2, Cambridge U. Press.

Ryan, J. (1966). Leibniz' Binary System and Shao Yong's Yijing. *Philosophy East and West*, 46(1), 59-90

Soto-Andrade, J. (2008). Mathematics as the art of seeing the invisible. *Proceedings ICME 11, Monterrey.*

Wilhem, R./Baynes, C. (1997). *The I Ching or Book of Changes*. Princeton U. Press

# TEACHING-LEARNING OF MATHEMATICAL CONCEPTS THROUGH PODCASTS: AN ATTEMPT TO LINK HISTORY OF MATHEMATICS (HM) AND DIGITAL INFORMATION AND COMMUNICATION TECHNOLOGY (DTIC*)*

**Adriana DE BORTOLI**
Lins Technology Collegee- FATEC, Estrada Mário Covas Jr, s/n. CEP: 16400-000. Lins (SP), Brazil. adrianadebor-toli1@hotmail.com

**Zionice GARBELINI MARTOS RODRIGUES**
Federal Institute of Education, Science and Technology of São Paulo - Birigui campus, Rua Pedro Cavalo, Nº 709 - Residencial Portal da Pérola II - CEP:16201-407Birigui-SP - Brazil
zionice@ifsp.edu.br

## ABSTRACT

In this short paper, we aimed to make considerations about the researches that were made in the Mathematic Education field, whose aim will be to achieve an articulation between two teaching trends: History of Mathematics (HM) and Digital Information and Communication Technology (DTIC), aiming to promote reflections regarding Mathematics teaching in order to enhance the learning process of the discipline. For that purpose, we will present a description of the construction of a platform named Mathpods*,* whose virtual learning environment (VLE) consists of a platform developed by (three) 3 students from the course of Analysis and Development of Systems and Systems for the Internet in the form of a undergraduate research. The developer students are from a public college in the state of São Paulo/Brazil. Such virtual learning environment enhances the teaching-learning process of mathematical concepts through *podcasts*. We will present the creation process and its uses and describe the potential of this technological educational product. Furthermore, the choice of ma- thematical concepts guided by historical procedures (such as the equations of second degree by the false position method) and considerations of extension actions in which we propose courses for Basic Education students will be addressed. We concluded that the platform has been helping teachers and students in the teaching and learning process at a local level in the state of São Paulo/Brazil, as podcasts make the teaching methodology more flexible and meet the interests of students. In addition, the extension actions by the authors have enabled the interaction between the college/institute and state public schools, which is believed to have added additional possibilities in times of remote education. Besides this, one of the authors seeks to investigate the contributions of the podcast elaboration with undergraduates in Mathematics from an institute of education, science and technology in the northwest of the state of São Pau-lo, in two curricular components, namely the Laboratory of Mathematics Education and Non-Euclidean geometries. As preliminary results, we verified the students' interest in knowing mathematical concepts produced by other peoples, in addition to the desire to learn about new cultures, including the enhancement of a methodological procedure that consists of making available historically produced mathematical subjects forwarded through podcasts that were made available on the platform. Additionally, it was pointed out by

the participants of the audio productions that Podcasts contribute to inclusion regarding education (mainly for needs related to low vision) and that it stimulates those who are producing in various areas sine qua non for teaching practice, in addition to if it consists of a medium that becomes more accessible to the general public, as currently the appropriation of virtual resources successfully captures the attention of young people.

## 1 Introduction

In this paper considerations are described about researches, performed the scope of Mathematics Education, whose intention will be to concretize an ar- ticulation between two tendencies of teaching: Math History (MH) and Digital Technologies of Information and Communication (DTIC), aiming to promote actions of learning and teaching Math, in order to enhance these processes.Our standpoint speech is about a reality which the technology is starting to set in, in educational terms. While some of Brazilian federal government's programs have been historically invested on technologies, it is still needed a lot to approximate the DTICs and the Brazilian schooling. Authors, such as Javaroni (2015), Borba; Scucuglia & Gadanidis (2014) describe the paths and the Brazilian reality face on new tech challenges in classroom. Mendes, Fossa, Valdés (2006), they can observe how important is to recover the historical information from a specific subject, because it allows the student to develop by him/herself math and investigation abilities, andalso, in special highlight, raise scientific awareness willing to intellectual autonomy. Moreover, this bibliographical review deliberates on a Brazilian author, Sousa (2020), who proposes a new alliance between Mathematics history, Digital technologies and Mathematics Investigation. We propose the usage of podcasts as a technological instrument in this background, in addition to those two fields of study, in order to improve the teaching/learning process. The podcast will be available on a digital platform called MathPods. The investigative point of this project consists of: is it possible to enhance how to learn and teach Math, using a method historically produced, andsupported by podcasts?

To indicate answers, we based some arguments on a legal Brazilian document, the National Common Curricular Base (CNCB), which recommends using various investigation fields, such as approach and didactical propose: "Besides the different didactical sources, like abacus, graph paper, games, calculators, Electronic worksheets, and software of dynamic geometry, it is important to include the Mathematic History as a resource that can raise awareness and represent meaningful context to learn and teach math." (BRASIL, 2018, p. 296). Moreover, the International Evaluation Program of Students (IEPS) 2022 shows the importance of using smartphones as a resource to learn math.

The program emphasizes the usage of cellphones in math classes. In face of all this scientific literature, we have elaborated and taught a course of university extension offered to students of 9th grade of Brazilian elementary school, to present and use the platform that accommodates math podcasts in order to elicit the possibility of autonomy in studying mathematic conceptions.

## *2 The University Extension Course*

We offered a course to 19 (nineteen) students from a public school, in the city of Lins, SP (Brazil), with the average age range around 14 years old. There were 3 meetings of 2 (two) hours long each one. Firstly, we administered a Pré-test, to get a notion of the engagement from those students with the technology, and if they already had familiarity with podcasts. The method of false position was used to teach and learn 1st degree equa- tions in this extension act. The students had not had any previous contact with calculation and solving first degree equations by the current algorithmic pro- cedure. In the end, we administered a post-test willing to verify how the activ-ity was taken from the perspective of students. The results gotten to the fol- lowing questions: After listening this podcast about equations resolution by “false position” method, do you think this method/podcast turned the resolution easier for you?And the second was: Would you use podcast as a resource for studying math? According to the answers, it is seen that 74% of students affirmed that the activity provided them more easiness to solve de algebraic equations, and we also realized that they were for using podcasts to study math, because 84% of students affirmed yes to that question.

## *3 Concluding Remarks*

It is concluded that the platform has been helping professors and students with the learning and teaching, in a local sphere in one inland region of the São Paulo state, Brazil, because the podcasts ease the teaching methodology, and appeal to the student interests. Therefore, the actions of extensions by the authors have allowed the interaction between the college and public schools, and it is believed that those schools have been well-completed with great new possibilities for this time of remote schooling. On the other hand, one of the authors investigates the contributions of pod-casts developing, cooperating with the under-graduate students, majoring in Mathematics. The developments and contributions are studied in two compo- nents: non-Euclidean geometries introduction, and in Education Lab of Math. It is verified in the preliminary results the interest from the students in com- prehending mathematic concepts produced by different peoples, aside from the appetite for meeting new cultures, and including the appreciation of a methodologic procedure which consists in listening to the podcasts about his- torical and technical math subjects, produced and uploaded in the platform application.

## REFERENCES


Brazil. (2018). National Common Curriculum Base. Source: http://basenacionalcomum.mec.gov.br/#/site/início Accessed: 22 jul. 2022

Borba, M. C., & Gadanidis, G. (2014). *Four phases of digital technologies inMathematics Education*. Belo Horizonte: Authentic .

Javaroni, S. (2015). The use of ICT in the practices of Mathematics teachers in the basic education network: the Mapping project and its consequences. *Bolema*, 998-1022.

OECD. (2022). PISA 2022 *Mathematics Framework* . Source: PISA 2022: https://pisa2022-maths.oecd.org/index.html Acessed : 20 out. 2022.

Mendes, I. A., Fossa, J., & Valdès, J. (2006). *History as an agent of cognition inMathematics Education*. Porto Alegre: Sulina.

Souza, G. (2020). Alliance between *History of Mathematics and Technologies ofMathematical Research*. São Paulo : Livraria da Físics.

# THEME 3

# ORIGINAL HISTORICAL SOURCES IN TEACHING AND LEARNING OF AND ABOUT MATHEMATICS

# THE TANGENT LINE TO THE PARABOLA ACCORDING TO GREEK MATHEMATICS AND GALILEO GALILEI

**Riccardo Bellé**
Dipartimento di matematica, Università di Pisa, Italy
riccardo.belle@gmail.com

## ABSTRACT

Our workshop aims to present the tangent line with the didactical use of historical sources from Greek mathematics and Galileo Galilei. The discussion starts with an analysis of the definition of the tangent line in Greek mathematics in the case of the circle, from Euclid's Elements. Then a method to draw the tangent line at a point on a parabola is presented. The description is taken from the Fourth Day of *Discorsi e dimostrazioni matematiche intorno a due nuove scienze* by Galileo Galilei (1638). Galilei's approach comes from Greek mathematics and is different from the algebraic one usually used at school. An application of this geometrical technique to high school exercises is carried out, with examples from textbooks.

## 1 *Using original sources in the history of mathematics*

In my approach the didactical use of original sources from the history of mathematics should possess the following characteristics: (a) the historical treatment should be neither too far nor too close to what is done in school today; (b) the sources, if possible, must be read in their original language (or at least in a "faithful" translation); (c) avoid the translation of mathematics into modern notation (some exceptions are possible but with some care, and always with warnings).

### 1.1 Our proposal

The first aim is to raise students' awareness that some geometrical objects could have different representations; parabola and tangent line in this case. Moreover, thanks to this approach, they should realize that mathematics is not a static science but evolves and changes over time. This will be achieved by reading a real mathematical book by G. Galilei *Discorsi e dimostrazioni matematiche intorno a due nuove scienze*, printed in Leiden in 1638. In this text, the author gave a geometrical construction for the tangent to the parabola that students will try to apply in some exercises taken from their textbooks.

The complete activity needs five or six hours and is addressed to an 11th-grade class of “Liceo scientifico” (however, it could be adapted to other schools). There are some prerequisites, especially for the geometrical proofs carried out by Galilei: properties of triangles and circles, and the intercept theorem. It is advisable (but not strictly needed) that students also know the parabola according to the usual algebraic approach used in high school. I proposed this activity in the school where I teach, and I thank my colleagues Sabrina Rossi, Maria Grazia Marzario, and Saverio Vignali for the opportunity they gave to me.

## 2 *The tangent line in Greek mathematics*

The activity starts recalling Euclid’s definition of line tangent to a circle: (Heath, 1956, p. 1): “A straight line is said to **touch a circle** which, meeting the circle and being produced, does not cut the circle.”

We need to pay attention to the reformulation that this definition could produce in the mind of students: “a line that intersect the circle only at one point is the tangent to the circle in that point”. This *oversimplification* is correct for the circle, but it is not generally true. In this sense, in the workshop we read the answers to a questionnaire on the tangent I gave to my 13th grade; the questionnaire is adapted from (Maracci & Marzorati, 2019), (Biza, 2006) and (Biza, 2007). For example, the first question was:

> *Prova a spiegare, in parole semplici, a cosa pensi quando senti il termine “retta tangente”. (Puoi scrivere liberamente quello che ti viene in mente, puoi fare un disegno, scrivere simboli o formule e usare qualsiasi mezzo più o meno formale per comunicare le tue idee).*
>
> *Try to explain, in simple words, what you think of when you hear the term “tangent line”. (You can write freely whatever comes to your mind, you can draw a picture, write symbols or formulas, and use formal or informal means to describe your ideas).*

Among all the students’ answers, we discussed the following one:

> *Una retta tangente è una retta che si interseca a una curva in un solo punto. In una circonferenza ad esempio la retta tangente in un punto è anche perpendicolare al raggio.*

*A tangent line is a line that intersects a curve at only one point. In a circumference, for example, the tangent line at a point is also perpendicular to the radius.*

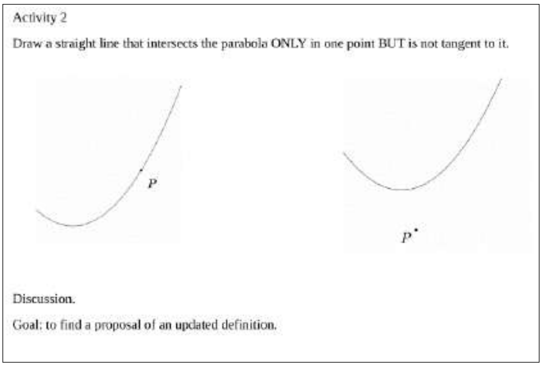
This statement is closely linked to the circumference. We shall see the case of the parabola gives an opportunity to broaden this point of view.

## 2.1 How to draw a tangent to a circle (Euclid III.16)

After the discussion on Euclid's definition, we proceed with the geometrical construction of the tangent to a circle, according again to Euclid (Heath, 1956, p. 37): "The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed." The proof is read thoroughly, because it is a good example of indirect proof (by contradiction) and it will be useful for the case of the parabola. In the workshop we discussed on the difficulties in proofs by contradiction (see for example, Antonini & Mariotti, 2008) and concluded that great care should be taken on this side.

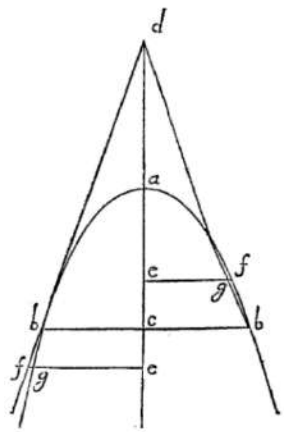
## 2.2 The tangent line to the parabola: definition

Having examined the case of the circumference, we introduce the tangent line to




the parabola with Activity 2 (see figure). The first step is to find a good definition of tangent line for the parabola. In the material used in classrooms two parabolas are given (and one point); students are asked to draw a straight line that intersects the parabola only at one point but it is not tangent to the parabola. The aim is to understand that in the case of the circle a line that intersects the circle only once is always a tangent line; for the parabola the same does not happen. In fact, the straight lines parallel to the axis intersect the parabola once, but they are not tangents. In the case of the parabola, a tangent line intersects the curve only once; but not all the lines that intersect the curve in only one point are tangents. Then, there are other curves (those in which the curvature changes sign) in which it is not even true that the tangent cuts the curve only once. This should make students aware that the definition of tangent line evolves along with the curves to which it is applied.

### 2.3 Galilei's construction

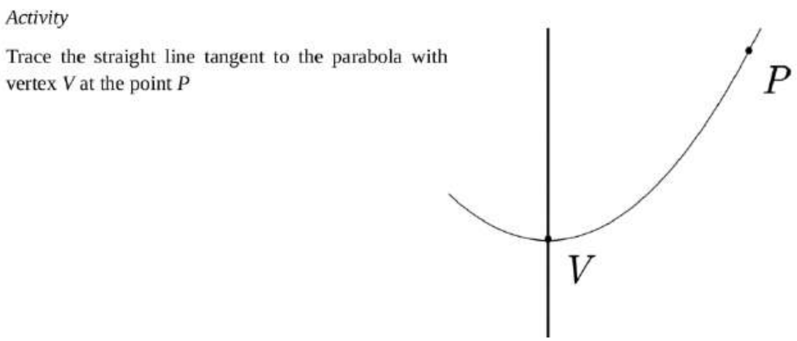



In this part, we show a geometrical construction to draw the tangent to a parabola. The text from *Discorsi e dimostrazioni matematiche intorno a due nuove scienze* (Galilei, 1638) is given, and the students are invited to read and discuss the following passage:

"Segniamo la Parabola, della quale sia prolungato fuori l'asse *ca* in *d*. E preso qualsivoglia punto *b*, per esso intendasi prodotta la linea *bc* parallela alla base di essa Parabola. E posta la *da* eguale alla parte dell'asse *ca*, dico, che la retta tirata per i punti *d, b*, non cade dentro la Parabola ma fuori, sì che solamente la tocca nell'istesso punto *b."* (Galilei, 1638 pp. 239-240).

“Draw a parabola with its axis CA extended to D, and take any point B [on the parabola], drawing through this the line BC parallel to the base of this parabola. Take DA equal to the part CA of the axis, and draw a straight line from D to B. I say that the line DB does not fall within the parabola, but touches it on the outside only, at point B.” (Drake, 1974, pp. 219-220). The students have to draw, with pencil and ruler, according to Galilei’s construction, the tangent line in the case presented in the following figure.

Figure



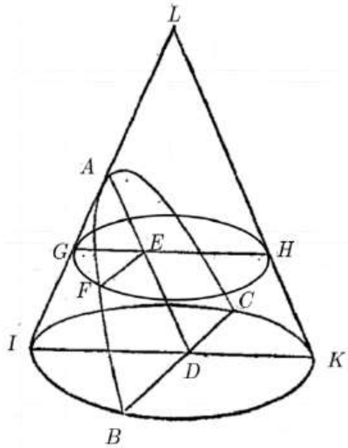

## 2.4 Application to textbook exercises

Then, I propose to solve two exercises with Galilei’s method: (a) find the tangent line to the parabola $y = 2x^2 - 4x$ at its point $Q$ with abscissa 3; (b) find the tangent to the parabola with axis parallel to the $x$-axis and the point $V(1,1)$ as its vertex, passing through the point $Q(4,4)$ on the parabola. At the beginning, it is not clear how to draw the parabola following Galilei, since these exercises are stated from a Cartesian perspective. It is not hard to find the focus and directrix, given the points $V$ and $Q$, and a horizontal axis; then it is possible to proceed in a Galilean manner, once the parabola is drawn. The construction can also be carried out with a dynamic software (GeoGebra, for example), following the steps of Galilei: (a) trace the perpendicular $PT$ from the point $P$ to the axis; (b) take $T'$, the point symmetric of $T$ with respect to $V$; (c) trace the line $T'P$: it is the tangent to the parabola at the point $P$. These examples give the opportunity to illustrate the differences between the Galilean and Cartesian approaches. For Galilei, following the Greek point of view, the parabola is a curve obtained by cutting the cone, while the Cartesian method identifies the parabola with its equation.

### 3 *Background to the proof of Galilei's construction*

A discussion concerning what the parabola was for Galilei is necessary before moving on to the proof of the tangent's construction. First, the teacher should make clear to the students why Galilei was interested in parabolas: a brief discussion on projectile motion (possibly already examined in physics) is carried out (Drake, 1974, p. 217): "When a projectile is carried in motion compounded from equable horizontal and from naturally accelerated downward [motions], it describes a semi-parabolic line in its movement."

But to understand Galilei's proof it should be clarified that the parabola at that time was not "our parabola": there was no equation, no analytic expression, like $y = 3x^2 - 2x + 1$ and not even a coordinate system. Following the




tradition of classical Greek mathematics, the parabola is the curve obtained by cutting a cone with a plane. So, there is no equation, but a characteristic property, expressed with a proportion between some geometric magnitudes $BD^2 : FE^2 = DA : AE$. In the material for the classes, I present a diagram of a cone cut by a plane in such a way as to obtain the parabola (see figure). Then, in the workshop we read and discuss the proof of this result (Galilei, 1638 pp. 238-239):

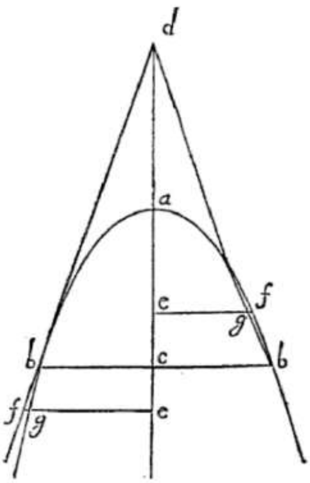
> Dico che il quadrato della *bd* al quadrato della *fe* ha la medesima proporzione che l'asse *da* alla parte *ae*.

Students also have at their disposal three-dimensional models of the cone and its sections (see figures below). These models were used in classes to follow and visualize every step of Galilei's proof. The models were provided by Monica Guadagni, a student writing her degree thesis about introducing conics at high school level with the use of models and mathematical machines. This representation is very important for students that can touch with their hands the cone and see the parabola as Galileo imagined it.

Figure

### 3.1 *Galilei's proof*

Galilei's proof is a proof by contradiction which assumes that the straight line constructed with the procedure described above (see section 2.3) cuts the parabola in two different point, and so it is not tangent to the parabola. This will lead to a statement that is known to be false (Galilei, 1638, pp. 239-240, Drake, 1974, p. 220):




> [la retta tangente] caschi dentro, segandola sopra, o, prolungata, segandola sotto, ed in

essa sia preso qualsivoglia punto *g*, per il quale passi la retta *fge*.

For if possible, let it [*DB*] fall within and cut [the parabola] above [*B*], or below when extended. Take in it [extended] some point *G*, through which draw the line *FGE*.

It is important to note that the figure contains both cases: on the left side the point *g* is below *b*, on the right side the point is above the point of tangency *b*.

Then Galilei, making use of the theory of proportion and of the property of the parabola previously proved (see section 3), shows that:

> Maggior proporzione ha la *ea* alla *ac*, che'l quadrato *ge* al quadrato *bc*, cioè che'l quadrato *ed* al quadrato *dc*.

The last step is to prove this impossible. Galilei proceeds: "la linea *ea* alla *ac* ha la medesima proporzione che 4 rettangoli *ead* a 4 quadrati di *ad* cioè al quadrato *cd*. […] adunque 4 rettangoli *ead* saranno maggiori del quadrato *ed*". But this is impossible because the point *a*, which bisect the line *dc*, could not bisect also the line *de*.

There are no symbols in this proof (with the exception of letters that indicate geometrical points) therefore, students are "forced" to carefully read the text. Some expressions used by Galilei raised questions and discussion with the teacher and among students. For example, "il rettangolo *ead*" means the rectangle constructed on *ea* and *ad*. This way of expressing geometrical quantities is very different from what students are used to. They would have expected the product of two lines and not the construction of a rectangle, as Galilei stated.

## 4 *Conclusion*

The parabola is a well-known curve, studied at school. However, usually, in school it is defined as a set of all points that satisfy a specific condition (locus of points) or as the graph of a quadratic function $f(x) = ax^2 + bx + c$. These representations lead to the development of an algebraic method to find the tangent line to the curve: setting the discriminant of some equation equal to zero. Instead, in Greek mathematics – and still for Galileo – the parabola is the curve obtained intersecting a cone with a plane. Consequently, it is possible to construct geometrically the line tangent to the parabola, as in the case of the circumference. In our proposal, this analogy is carried out to make students aware of the different representations of mathematical objects.

## REFERENCES


Antonini, S., & Mariotti M. A. (2008). Indirect proof: what is specific to this way of proving? *ZDM: the international journal on mathematics education* (40), 401-412.

Biza, I., et al. (2006). Students' Thinking about the Tangent Line, in Novotná, J. et al. (eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, vol. 2, pp. 177-184.

Biza, I., (2007). Is the Tangent Line, Tangible? Students' Intuitive Ideas about Tangent Lines, in D. Kuchemann (ed.), *Proceedings of the British Society for Research into Learning Mathematics*, pp. 6-11.

Drake, S. (1974). *Galileo Galilei. Two New Sciences. Translated, with a New Introduction and Notes.* Wall & Thomson.

Heath, T. H. (1956). *Euclid: The thirteen books of the Elements*. Vol. 2 (Books III-IX). Dover.

Galilei, G. (1638). *Discorsi e dimostrazioni matematiche intorno a due nuove scienze.* In Leida. Appresso gli Elzevirii.

Maracci, M., & Marzorati, A. (2019). Uno studio sulla concept-image di nozioni di base dell'analisi non standard, *Didattica Della Matematica. Dalla Ricerca Alle Pratiche d'aula*, (6), 9-34. https://doi.org/10.33683/ddm.19.6.1

# INTRODUCTION AU CALCUL DIFFERENTIEL PAR UNE ETUDE DES TANGENTES

**Carène GUILLET, Marie-Line MOUREAU**
IREM des Pays de la Loire, UFR des sciences et techniques, Nantes, France
carene.guillet@gmail.com, mlmoureau@aol.com

**ABSTRACT**

Dans cet atelier les participants ont pu découvrir, d'abord en se plaçant en situation d'élèves puis en les analysant en tant qu'enseignants, deux séquences autour de l'entrée dans le calcul différentiel. Ces séquences, mises en œuvre auprès d'élèves de 16 ans, sont pour l'une basée sur la méthode des tangentes de René Descartes et pour l'autre sur la notion de « différence » et le calcul des sous-tangentes du Marquis de L'Hospital.

## *Introduction*

Lors de l'atelier, nous avons présenté deux approches élaborées pour amoindrir les difficultés rencontrées par les élèves dans leur apprentissage du calcul différentiel. Construites après la lecture de l'article de Maggy Schneider-Gilot (1991), elles ont pour point commun l'appui sur des éléments géométriques tangibles, tangente ou sous-tangente, *a contrario* du triangle différentiel qui se réduit à un point en cours de calcul. Après avoir testé les dispositifs mis en place pour nos élèves, les participants ont pu les analyser, les comparer et discuter de leur pertinence par rapport à l'objectif pédagogique annoncé.

## *1 Une progression s'appuyant sur la tangente à un cercle*

Avant de présenter le travail mené en classe, nous avons rappelé le contexte français pour l'enseignement de la dérivation. Depuis 2019, la tangente à un cercle n'est plus un objet d'étude obligatoirement proposé aux élèves. Ils doivent donc en classe de première (16-17 ans) à la fois découvrir cette notion et, de manière intuitive selon les termes des programmes, comprendre que la tangente à une courbe en un point est la limite des sécantes et que sa pente est la limite des taux de variation. Cette limite s'obtient en posant égales à 0 des quantités préalablement définies comme non nulles. La tangente, objet nouveau, n'est pas elle-même une sécante, sa pente n'est pas non plus un taux de variation. Cela occasionne de nombreuses incompréhensions. Finalement pour les élèves, une tangente existe-t-elle vraiment, c'est-à-dire la limite est-elle atteinte ? Est-elle unique ?

Nous avons donc élaboré et mis en œuvre en classe une progression commençant par l'étude de la tangente à un cercle dans les *Éléments* d'Euclide, se prolongeant par la tangente à une courbe via les cercles tangents, en suivant la méthode exposée par René Descartes dans sa *Géométrie* (1637), pour terminer par la position limite des sécantes, mais définie comme une droite ayant un point d'intersection double avec la courbe. C'est cette progression qui a été présentée et soumise à discussion dans la première partie de notre atelier.

### 1.1 Première étape: tangente à un cercle

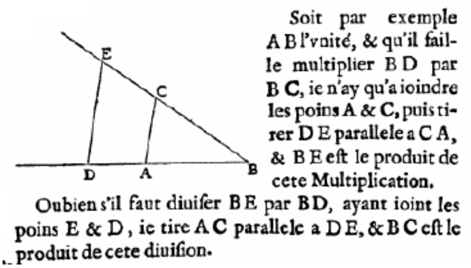
À travers la lecture des définitions 8, 10, 15, 16 et 17 du Livre I des *Éléments* d'Euclide (Euclide, 1804) il s'agit avec les participants, comme avec nos élèves, de prendre conscience de la pérennité des objets géométriques (cercle, droite), mais aussi des pièges d'un vocabulaire ne recouvrant pas les mêmes concepts : la droite euclidienne est infinie en puissance mais pas en acte, la notion de segment n'a donc pas de sens, et le cercle euclidien est un disque. Cette vigilance sur le vocabulaire est indispensable pour ne pas commettre de contresens dans les lectures qui suivent. Nous donnons la définition de la tangente à un cercle : « Une droite qui touchant le cercle et qui étant prolongée ne le coupe point, est appelée tangente du cercle. » (Euclide, 1804, livre III, p.107). Puis nous abordons le cœur de l'activité avec le théorème :

« Proposition 16. Une droite perpendiculaire sur le diamètre d'un cercle et menée par une de ses extrémités, tombe hors de ce cercle ; il est impossible qu'il y ait une droite dans l'espace qui est compris entre cette perpendiculaire et la circonférence » (Euclide, 1804, livre III, p. 141).

Lors de l'atelier, nous avons placé les participants dans les mêmes conditions que nos élèves : la démonstration d'Euclide leur a été distribuée et ils ont dû légitimer chacune des étapes en référence aux connaissances antérieures dans la scolarité. Cela correspond aux renvois dans le texte à des propositions précédentes de l'ouvrage. Les élèves ont compris à cette occasion la structure logique des *Éléments*, où chaque proposition s'appuie sur les définitions et propositions précédentes. Ce travail offre en classe un moment de réflexion sur le mode démonstratif utilisé, c'est-à-dire un raisonnement par l'absurde et la forme axiomatico-déductive de l'argumentation repérée à travers les connecteurs : puisque, donc, or, mais. À la fin de la séance, les élèves ont écrit le théorème : « Une droite est la tangente en un point à un cercle si et seulement si elle est perpendiculaire au diamètre dont ce point est une des extrémités. »

### 1.2 Étape 2: tangente à une courbe dans la *Géométrie* de Descartes

Lors de l'atelier nous avons pris le temps, avant d'aborder la méthode de Descartes pour déterminer les tangentes, de présenter son essai *La Géométrie* (Descartes, 1637) en soulignant les trois grands apports de Descartes : l'arithmétisation de la géométrie, son algébrisation et une nouvelle classification des courbes. Descartes opère une rupture totale avec la géométrie grecque en définissant des opérations sur les lignes (segments de droites) dont le résultat est encore une ligne. Si cela va de soi pour l'addition ou la soustraction, pour les Grecs le produit de deux longueurs est une surface. Descartes, lui, définit le produit de deux lignes comme une ligne : le produit des lignes (grandeurs) *BD* et *BC* est la ligne *BE* (figure 1) obtenue par construction géométrique.



Soit par exemple A B l'vnité, & qu'il faille multiplier B D par B C, ie n'ay qu'a ioindre les poins A & C, puis tirer D E parallele a C A, & B E eſt le produit de cete Multiplication.

Oubien s'il faut diuiſer B E par B D, ayant ioint les poins E & D, ie tire A C parallele a D E, & B C eſt le produit de cete diuiſion.

**Figure 1.** Multiplication et division de deux lignes (Descartes, 1637, p. 298)

Ainsi tout problème de géométrie se ramène au calcul de quelques lignes et il peut être traité comme un problème d'algèbre. Ce qui signifie qu'il faut le considérer comme déjà fait, nommer les lignes connues et inconnues et produire des équations entre ces lignes, équations qu'il suffit alors de résoudre pour pouvoir construire la solution du problème. Descartes dit pouvoir résoudre ainsi « tous les problèmes de géométrie » (Descartes, 1637, p. 297). Pour ceux relatifs à une courbe il faut que cette courbe soit « géométrique » au sens cartésien, c'est-à-dire pour laquelle tous les points « ont nécessairement quelque rapport à tous les points d'une ligne droite, qui peut être expliqué par quelque équation, en tous par une même. » (Descartes, 1637, p. 319). Par exemple considérons la figure 2. Les points de l'ellipse passant par *C* et *E*, sont appliqués sur la droite (*GA*), *A* étant choisi « pour commencer par lui les calculs » (Descartes, 1637, p. 320), selon une certaine direction, ici perpendiculaire. La droite (ici l'axe de l'ellipse) et le point sont choisis de sorte que l'équation de la courbe soit la plus simple possible. Pour chaque point de la courbe, Descartes produit deux grandeurs (*AB* égale à *CM,* notée *x* et *CB* égale à *AM* notée *y*) qui, puisque la courbe est géométrique, satisfont une équation polynomiale, indépendante du choix de *C* sur la courbe. Dans l'étude de cette méthode, les participants de l'atelier ont pu se convaincre qu'il n'y a pas de « repère cartésien » qui serait défini préalablement à la courbe étudiée. De plus, *x* et *y* ne désignent pas des nombres mais des segments et il n'y a ni axes ni orientation.

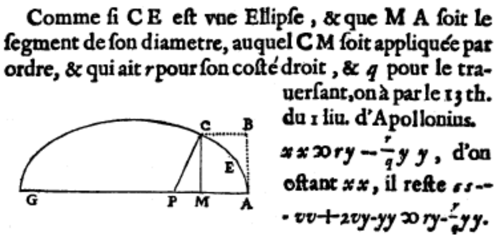

Comme ſi CE eſt vne Ellipſe, & que MA ſoit le ſegment de ſon diametre, auquel CM ſoit appliquée par ordre, & qui ait $r$ pour ſon coſté droit, & $q$ pour le trauerſant, on à par le 13 th. du 1 liu. d'Apollonius. $xx \propto ry - \frac{r}{q}yy$, d'on oſtant $xx$, il reſte $ss - vv + 2vy - yy \propto ry - \frac{r}{q}yy$.


**Figure 2.** Mise en équation pour une ellipse (Descartes, 1637, p. 343)

Ce long préambule nous a paru nécessaire lors de l'atelier s'adressant à des formateurs mais a été réduit pour nos élèves au fait que $x$ et $y$ étant des longueurs, ils ne peuvent désigner *in fine* que des nombres positifs. Nous avons bien conscience de procéder à un amalgame entre nombre et longueur mais cela nous a paru difficile à éviter à ce stade de la formation mathématique de nos élèves.

Le cadre étant posé, nous avons soumis le problème de Descartes aux participants. La lecture d'extraits significatifs des pages 342 à 347 de sa *Géométrie* leur a permis de comprendre que son but est de déterminer la normale à la courbe en un point *C,* sachant que cette normale est le diamètre du cercle tangent de centre *P*. Il explique que, un point *P* étant placé sur la droite sur laquelle les points de la courbe sont appliqués, le cercle de centre *P* et passant par *C*, la recoupe « nécessairement » en un autre point *E*. Ce point est confondu avec *E* lorsque le cercle est tangent. Descartes traduit cette configuration par le fait que l'équation, vérifiée par les points d'intersection de la courbe et du cercle, a alors une solution double. Il s'agit donc de déterminer la position de *P* assurant cette contrainte. Les participants ont donc tout d'abord été invités à suivre pas à pas les étapes du raisonnement et des calculs de Descartes sur le cas particulier d'un arc de parabole. Ensuite une étude de l'activité proposée à nos élèves leur a permis d'évaluer les adaptions réalisées et juger de leur pertinence.

> Exprimer en fonction de ses coordonnées $(x, y)$ le fait que le point $E$ est sur le cercle de centre P passant par C, puis le fait qu'il est sur la parabole.
> **Par le moyen d'une de ces deux équations, j'ôte de l'autre équation l'une des deux quantités indéterminées $x$ ou $y$ de façon qu'il reste après cela une équation en laquelle il n'y a plus qu'une seule quantité indéterminée $x$ ou $y$.**
> [Éliminer $y$ pour trouver une équation en $x$ ]
> **Or après qu'on a trouvé une telle équation, au lieu de s'en servir pour trouver les quantités $x$ ou $y$, on la doit employer à trouver $v$ ou $s$ qui déterminent le point P qui est demandé. Puis aussi il faut considérer que lorsque le cercle touche la courbe sans la couper, les deux solutions de l'équation sont égales.** [Autrement dit, il faut trouver $v$ tel que l'abscisse de $C$ soit une solution double.]
> Montrer que l'équation équivaut à $(x^2 - 9)(x^2 + 10 - 2v) = 0$
> Déterminer $v$.

**Figure 3.** Extrait de la fiche élève

En effet d'une part, même si, autant que faire se peut, les questions sont portées par les mots mêmes de Descartes (en gras, sur la fiche) le texte est allégé et réécrit avec notre orthographe. D'autre part les calculs sont fortement modernisés et le problème est placé dans le cadre de la géométrie repérée. Enfin, une fois obtenues les coordonnées de *P*, les calculs sont prolongés par la recherche de la pente de la normale, puis celle de la tangente. Cette activité aboutit ainsi sur la formule donnant la fonction dérivée de la fonction carré.

En 1638, dans une lettre au mathématicien Claude Hardy (Descartes, 1898, p. 170), Descartes précise que les calculs peuvent être simplifiés s'il ne s'agit que de déterminer la tangente, en remplaçant le cercle par une droite sécante en *C*. Reprenant cette idée, nous avons pu établir avec les élèves les formules de dérivation des fonctions cube, inverse et racine carrée. Les élèves ont fait d'eux-mêmes le lien entre les calculs et l'obtention d'une position limite de la sécante. Nous avons donc retrouvé les attendus du programme.

### 1.3 Bilan de cette approche par la tangente à un cercle

Lors de la discussion qui a suivi cette présentation, nous avons pu argumenter que, malgré la difficulté de l'activité basée sur le texte de Descartes, éprouvée en classe et soulignée par les participants, celle-ci a permis aux élèves d'accéder plus naturellement à l'idée de position limite. En cela notre objectif a été atteint.

## *2 Une approche de la tangente comme objet d'analyse*

Notre expérience d'enseignantes nous a montré à quel point les obstacles à la compréhension des liens entre dérivation et tangente sont nombreux, d'un point de vue épistémologique comme didactique. Nous avons donc exploré plusieurs pistes. Et, parallèlement à l'expérimentation relatée en première partie, notre groupe a travaillé sur une approche plus analytique, objet du second temps de l'atelier.

Nous avons ainsi partagé avec les participants une séquence expérimentée dans d'autres classes et qui cette fois prend appui sur l'ouvrage du Marquis de l'Hospital, *L'analyse des infiniment petits, pour l'intelligence des lignes courbes,* publié en 1696 (L'Hospital, 1696). Dans son ouvrage, celui-ci présente en français le nouveau calcul de Leibniz, auquel il a été initié par Jean Bernoulli au sein du groupe des malbranchistes, sous une forme axiomatico-déductive très pédagogique, adaptée aux "commençans", et donc également à nos élèves.

### 2.1 Les « différences » et leur calcul

Durant l'atelier, la lecture d'extraits de la préface de l'ouvrage, des premières définitions, « demandes » ou « suppositions », a permis de mettre en évidence plusieurs points-clés de la théorie du Marquis.

L'« analyse » mise en exergue dans le titre de son livre ne se limite pas à celle des grandeurs finies, comme le fait l'algèbre, mais aux grandeurs infinies, et même au-delà : « L'analyse ordinaire ne traite que des grandeurs finies, celle-ci pénètre jusque dans l'infini même. […] On peut même dire que cette analyse s'étend au-delà de l'infini. ». Il s'agit d'analyse au sens de « calcul infinitésimal » (L'Hospital, 1696, préface).

Sa conception de la ligne courbe est celle d'un « polygone d'une infinité de côtés […] infiniment petits », les positions qu'ont ces côtés entre eux formant alors la « courbure », qu'il associe aux tangentes de la courbe (L'Hospital, 1696, préface). Après avoir distingué les quantités variables des quantités constantes, il définit une « différence » comme « la portion infiniment petite dont une quantité variable augmente ou diminue continuellement » (L'Hospital, 1696, p. 2). Cette définition est ensuite mise en œuvre sur plusieurs grandeurs géométriques issues d'une même figure (figure 4) : segments de droite (*PM*), portions de lignes courbes (*AM*), aires curvilignes (*APM*).

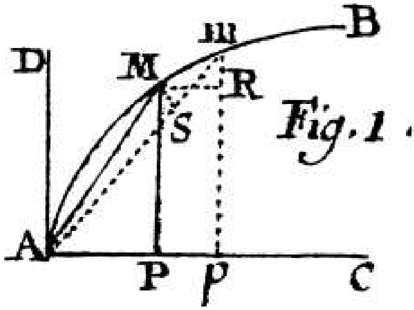



**Figure 4.** Définition des « différences » (L'Hospital, 1696, planche 1$^{\text{ère}}$, p. 24)

Les règles de calcul sur ces différences sont assez facilement établies. Elles dépendent des règles usuelles de l'algèbre et du principe qui indique que « *dxdy* est une quantité infiniment petite par rapport aux autres termes » (L'Hospital, 1696, p. 4). Durant l'atelier, les participants ont pu tester l'efficacité de cette méthode en déterminant la formule de la différence d'un quotient *x*/*y*. Ils ont découvert que ce texte offre une méthode simple et astucieuse pour démontrer les formules de dérivées des opérations courantes.

## 2.2 La méthode des tangentes

Un peu plus loin dans l'ouvrage, le Marquis expose sa méthode de détermination des tangentes. Au préalable, il définit une tangente de la façon suivante : « Si l'on prolonge un des petits cotés *Mm* du polygone qui compose une ligne courbe, ce petit côté ainsi prolongé sera appelé la *Tangente* de la courbe au point *M* ou *m*. » (l'Hospital, 1696, p. 11).

Dans le sillage du travail de Leibniz, la méthode exposée par le Marquis repose sur la similitude de deux triangles. En effet (figure 5), les triangles *MRm* et *MPT* sont semblables, d'où l'égalité des rapports : $mR/RM = MP/PT$, soit encore $dy/dx = y/PT$, en posant $MR = dx$, $mR = dy$, $MP = y$ et $AP = x$. La sous-tangente *PT* est alors calculée ainsi : $PT = y.dx/dy$. Puis la méthode est énoncée en ces termes (L'Hospital, 1696, p. 11) :

« Or par le moyen de la différence de l'équation donnée, on trouvera une valeur de *dx* en termes qui sont tous affectés par *dy*, laquelle étant multipliée par *y* et divisée par *dy*, donnera une valeur de la sous-tangente *PT* en termes entièrement connus et délivrés des différences, laquelle servira à mener la tangente cherchée *MT*. »

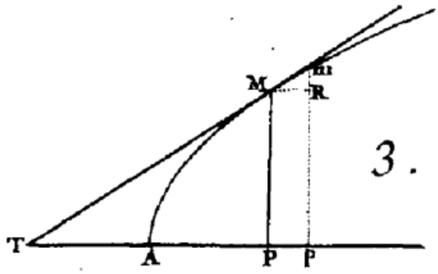



**Figure 5.** Méthode des tangentes (L'Hospital, 1696, planche 1ère, p. 24)

Sa méthode, certes fondée sur un calcul, est donc une méthode de construction, qui s'appuie sur la détermination de la sous-tangente *PT* pour mener la tangente cherchée. C'est cette particularité qui permettra de lever en classe une partie des difficultés liées à la compréhension du nombre dérivé.

## 2.3 Application de la méthode

Durant l'atelier, nous avons souhaité faire expérimenter cette méthode sur une courbe non triviale pour des participants aguerris en mathématiques. Nous avons donc choisi le folium de Descartes, que le Marquis traite dans son ouvrage (L'Hospital, 1696, p. 15). Cette courbe d'équation $y^3 - x^3 = axy$ (avec différentes valeurs de $a$ fixées) a été présentée dans un repère, avec un anachronisme assumé. L'objectif était alors de calculer la sous-tangente *PT* avec la méthode du Marquis, puis de construire la tangente demandée. Nous avons même été jusqu'à proposer cette situation pour des coordonnées de point négatives, ce qui n'est pas envisageable dans la vision purement géométrique du Marquis. La méthode fournit néanmoins encore des résultats corrects.

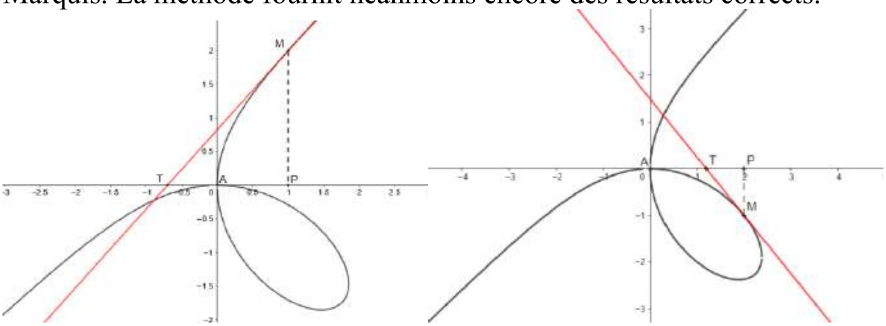
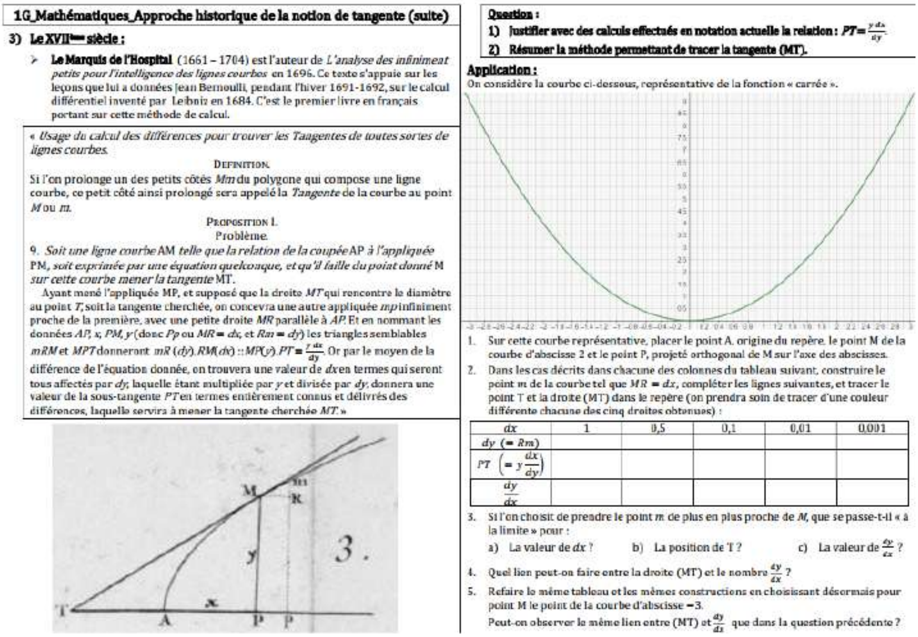



**Figure 6.** Méthode des tangentes appliquée au folium dans le cas $M\,(1\,;\,2)$ et $a = 3{,}5$, puis dans le cas $M\,(2\,;\,-1)$ et $a = 4{,}5$

En classe, au contraire, la construction des tangentes a été effectuée sur une courbe bien connue des élèves: la parabole représentative de la fonction carré ( figure 7).



**1G_Mathématiques_Approche historique de la notion de tangente (suite)**

**3) Le XVIIème siècle :**

- **Le Marquis de l'Hospital** (1661 – 1704) est l'auteur de *L'analyse des infiniment petits pour l'intelligence des lignes courbes* en 1696. Ce texte s'appuie sur les leçons que lui a données Jean Bernoulli, pendant l'hiver 1691-1692, sur le calcul différentiel inventé par Leibniz en 1684. C'est le premier livre en français portant sur cette méthode de calcul.

« *Usage du calcul des différences pour trouver les Tangentes de toutes sortes de lignes courbes.*

DEFINITION.

Si l'on prolonge un des petits côtés *Mm* du polygone qui compose une ligne courbe, ce petit côté ainsi prolongé sera appelé la *Tangente* de la courbe au point *M* ou *m*.

PROPOSITION I.

Problème.

9. *Soit une ligne courbe* AM *telle que la relation de la coupée* AP *à l'appliquée* PM, *soit exprimée par une équation quelconque, et qu'il faille du point donné* M *sur cette courbe mener la tangente* MT.

Ayant mené l'appliquée MP, et supposé que la droite *MT* qui rencontre le diamètre au point *T*, soit la tangente cherchée, on concevra une autre appliquée *mp* infiniment proche de la première, avec une petite droite *MR* parallèle à *AP*. Et en nommant les données *AP*, *x*, *PM*, *y* (donc *Pp* ou *MR* = *dx*, et *Rm* = *dy*) les triangles semblables *mRM* et *MPT* donneront *mR* (*dy*).*RM*(*dx*) :: *MP*(*y*).$PT = \frac{y\,dx}{dy}$. Or par le moyen de la différence de l'équation donnée, on trouvera une valeur de *dx* en termes qui seront tous affectés par *dy*, laquelle étant multipliée par *y* et divisée par *dy*, donnera une valeur de la sous-tangente *PT* en termes entièrement connus et délivrés des différences, laquelle servira à mener la tangente cherchée *MT*. »

**Question :**

1) **Justifier avec des calculs effectués en notation actuelle la relation : $PT = \frac{y\,dx}{dy}$**
2) **Résumer la méthode permettant de tracer la tangente (MT).**

**Application :**

On considère la courbe ci-dessous, représentative de la fonction « carrée ».

1. Sur cette courbe représentative, placer le point A, origine du repère, le point M de la courbe d'abscisse 2 et le point P, projeté orthogonal de M sur l'axe des abscisses.
2. Dans les cas décrits dans chacune des colonnes du tableau suivant, construire le point *m* de la courbe tel que $MR = dx$, compléter les lignes suivantes, et tracer le point T et la droite (MT) dans le repère (on prendra soin de tracer d'une couleur différente chacune des cinq droites obtenues) :

| $dx$ | 1 | 0,5 | 0,1 | 0,01 | 0,001 |
|---|---|---|---|---|---|
| $dy\ (= Rm)$ | | | | | |
| $PT\ \left(= y\frac{dx}{dy}\right)$ | | | | | |
| $\frac{dy}{dx}$ | | | | | |

3. Si l'on choisit de prendre le point *m* de plus en plus proche de *M*, que se passe-t-il « à la limite » pour :
   a) La valeur de *dx* ? b) La position de T ? c) La valeur de $\frac{dy}{dx}$ ?
4. Quel lien peut-on faire entre la droite (MT) et le nombre $\frac{dy}{dx}$ ?
5. Refaire le même tableau et les mêmes constructions en choisissant désormais pour point M le point de la courbe d'abscisse −3.
   Peut-on observer le même lien entre (MT) et $\frac{dy}{dx}$ que dans la question précédente ?

**Figure 7.** Fiche d'activité élève basée sur la méthode des tangentes du marquis de l'Hospital et application à la parabole

Dans la discussion avec les participants, la force didactique de cette approche a été soulignée. En utilisant la similitude du triangle caractéristique, difficile à appréhender du fait que ses côtés tendent simultanément vers 0, et d'un triangle plus tangible, dont un côté est la sous-tangente, l'image abstraite de ce triangle évanouissant est avantageusement remplacée par le triangle *PTM*, ancré sur l'axe des abscisses et dont une dimension *PM* est fixe. Si l'on prend les point *m* et *M* de plus en plus proches l'un de l'autre, soit *dx* de plus en plus proche de 0, la suite des points *T* tend vers une position limite bien identifiable par les élèves, et avec un coefficient directeur qu'ils sont en mesure d'estimer.

## 2.4 Bilan de cette approche par la sous-tangente

Nous avons partagé avec les participants notre analyse et notre bilan de cette séquence d'enseignement. La perspective historique choisie pour cette activité a porté ses fruits : elle a dépaysé les élèves en les immergeant dans une pensée d'une autre époque et un formalisme qu'ils ne connaissaient pas. Elle a également rempli son objectif en permettant de surmonter l'obstacle épistémologique et didactique que représente le nombre dérivé en tant que coefficient directeur d'une tangente.

## *Conclusion*

Les participants à l'atelier se sont trouvés immergés dans nos dispositifs pédagogiques et ont été à même de les comparer et discuter de leurs avantages respectifs. Les échanges ont permis de relever ce que nos deux approches ont en commun : l'appui sur des objets géométriques, tangentes et sous-tangentes. Ces approches permettent, l'une comme l'autre, d'ancrer la compréhension d'un concept abstrait, le nombre dérivé, dans des configurations plus concrètes pour les élèves. L'objectif de faciliter leur entrée dans le calcul différentiel est donc atteint.

Le débat peut alors porter sur le choix pour l'enseignant de suivre l'une ou l'autre de ces deux voies, choix dépendant essentiellement de la progression dans laquelle il souhaite l'inscrire, plus algébrique avec la méthode de Descartes, et plus analytique avec celle du Marquis de l'Hospital, celle-ci induisant également un autre regard sur les courbes, « polygones d'une infinité de côtés infiniment petits ». Un travail complémentaire peut ainsi être envisagé sur les équations et la géométrie analytique dans le premier cas, et déboucher sur une introduction de la fonction exponentielle par la sous-tangente dans l'autre.

## RÉFÉRENCES


Descartes, R., (1637). La Géométrie , *Discours de la méthode,* Leyde, Jan Maire.

Euclide, (1804). *Les éléments de géométrie d'Euclide, traduits littéralement et suivis d'un Traité du cercle, du cylindre, du cône et de la sphère, de la mesure des surfaces et des solides, avec des notes, par F. Peyrard.*, Paris, F. Louis.

L'Hospital (de), G. F. A., (1696). *Analyse des infiniment petits, pour l'intelligence des lignes courbes*, Paris: Imprimerie Royale.

Schneider-Gilot M., (1991). Quelques difficultés d'apprentissage du concept de tangente, *Repères-IREM,* 5, 65- 82.

Descartes, R., (1898). Charles Adam et Paul Tannery (Eds), *Correspondances*, tome II, Paris, Léopold Cerf, 170-172.

# MATHEMATICAL ARTISTRY FOR JUSTICE:

## A workshop on bringing episodes in the history of mathematics to life in the classroom by means of theatre, incorporating a short play set in ancient China

**Gavin HITCHCOCK**
aghitchcock@gmail.com

## ABSTRACT

This paper describes the theoretical motivation and practical mounting of a workshop in which participants could experience and reflect on the value of theatre in bringing mathematical ideas and history to life in the classroom. All participants were involved in the production and enactment of a short pre-scripted play, requiring minimal preparation. This was followed by discussion, small group work, and feedback. The play, given in full below, evokes the cultural context within which a Chinese mathematical practitioner may have operated about two millennia ago, and suggests how the Rule of Excess and Deficiency for equation solving could have been inspired and recorded, as Problem 17 in Chapter 7 of the *Jiu zhang suan shu (The Nine Chapters)*. Nine representative scribes/mathematicians from different cultures make cameo appearances, giving the names for 'the unknown' used in equation-solving in their time. The dialogue is accessible and lively, laced with humour and emotions, portraying the excitement of mathematics-making. Based upon primary sources, the play can inspire interest in authentic contextual history of algebra while motivating the learning of algebra.

### *1. Introduction*

Using theatrical devices as communication tools is an art as old as humanity. Dialogue form has been used for a long time to communicate philosophical, mathematical and scientific ideas, for example, by Plato, Galileo, Alfréd Rényi, Donald E. Knuth, Imre Lakatos, and many more (Rényi, 1967; Knuth, 1974; Lakatos, 1976). However, I believe the power of dialogue and live theatre is neglected in current curriculum-driven, time-constrained educational systems.

Videos and films with mathematical, historical and biographical themes might be excellent, but to really win the attention of young people and engage their hearts and their minds it is hard to beat personal involvement in the enactment of theatre. To watch children involved in plays, or live theatre games, or improvisational drama, is to see them come alive! This paper aims to encourage the creation and use of such devices to awaken a love and appreciation for mathematics and its story, and to counter the boredom felt by so many learners.

Live theatre can encourage imaginative entry into the minds and emotions of historical characters, sharing their concerns and their excitement. This is not easily achieved by merely reading their writings. Indeed, the process of generating a dialogue based closely on primary source material and correspondence can be a tool for uncovering and displaying hidden motives, inspirations and emotions, for the researcher as well as the educator (Hitchcock, 2012). Creating dialogues to dramatize the history of mathematics can be approached in various ways. Rényi's *Dialogues in Mathematics* features Socrates, Archimedes and Galileo, and sets out to be 'lively and vivid' and 'comprehensible to non-specialists', while presenting problems 'in their full complexity.' Rényi felt this aim was achievable only by means of the dialogue form. To preserve historical fidelity, he sought to emulate the writing styles of his three protagonists, while assuming some poetic license and 'avoiding anachronism as far as possible.' In contrast, Lakatos's *Proofs and Refutations* deliberately dissociates his characters in dialogue from any historical counterparts, naming them 'Alpha', 'Beta', 'Gamma', etc. He thus contrives a degree of separation between the 'concept story' and the 'human story', indicating in footnotes the historical counterpoint to the classroom dialectic. Designing scripts for live theatre brings its own challenges. Using original sources ensures some authenticity, and there is a natural fascination in different cultural contexts. But this has to be carefully balanced with the requirements of good theatre and accessibility. The tensions are not easily negotiated, and the balance must be adjusted for different levels of education. Scripting a play, one has to take liberties and make guesses. What's recorded and preserved in primary sources is usually the final product of prolonged thought, informal correspondence and debate. At best, it captures the bare bones of the human story behind the mathematics. By 'filling in' some of the missing conversation – the vast tracts of unrecorded communication – we can aim to reconstruct or 're-incarnate' the flow and ancestry of ideas in communities over time. The challenge is to weave together, by the art of the dramatist, the primary source fragments and try to complete the puzzle – to capture with the natural flow of a good story-line something of the elusive and complex historical dialectic behind the reported fragments. For more on the tensions and challenges of achieving this, see Hitchcock, 1992, 1996, 2000, 2023; the last has further references.

The aim of the workshop reported here was to experience and reflect on the ways that the devices of story, narrative, dialogue, drama and theatre can enliven mathematical ideas and history in the classroom, with minimal resources and preparation. Participants were challenged to mount a pre-scripted play from scratch – with casting, rehearsing and production all taking place in about an hour. In the second hour, plenary discussion took place, and then discussion in small groups followed by feedback. Sections 2-4 of this paper give the detailed structure of the workshop and indicate how the play was introduced to the participants. To convey the flavour of the live workshop, and to show how the production can be managed with students in practice, the text gives an almost verbatim account of some of what was said to the workshop participants.

Section 5 consists of the script of the play, precisely as given to the participants. It is designed to evoke the context and flavour of an ancient culture in which mathematics emerged in response to everyday problems but was gradually becoming an art in own right. The action is introduced by a narrator and a mathematics teacher, who discuss the emergence of ancient techniques for equation solving. The scene is set in the Office of Community Advice and Arbitration in a Chinese town two thousand years ago. The administrator receives an agitated client, and responds to her anxiety about the purchase of a piece of land of mixed quality. The play is based upon the posing and solving of Problem 17 in Chapter 7 of the *Jiu zhang suan shu (The Nine Chapters)*; some of the dialogue is based on the *Sunzi suan jing (Mathematical Classic of Master Sun)* (references are given in footnotes to the play). This play also showcases the rich variety of cultures that contributed to the development over thousands of years of the art of 'finding the unknown.' In this way it can provide background and colour for what the modern curriculum calls 'solving equations', where the standard techniques are the fruit of a long multi-cultural adventure of human thought, but are commonly encountered by learners with no associated historical narrative. The words and the mathematics are elucidated by the narrator and the mathematics teacher, with the aid of slides in which equations are transcribed into modern notation.

This play is designed for use with teacher trainees and well-motivated secondary school learners; simpler scripts are needed for younger learners. Similar plays of mine have been enacted, with warm response, in many summer-school and conference settings, and published individually in various places (references may be found in Hitchcock, 2023). The mathematics education market seems to be focused on rigidly curriculum-oriented texts; it is hard to find or to publish theatrical resource collections. But I believe a great opportunity is being lost. Therefore, this paper urges the need for more theatrical resources for enriching the learning of mathematics, and provides a model script, while describing how such plays can be easily produced to great effect.

## *2. Welcome, synopsis and aim of the workshop*

Welcome! Here's an overview of our 2-hour workshop:

- Welcome & introduction – 5 minutes
- Allocation of production parts & casting – 15 minutes
- Rehearsing in corners – 15 minutes
- Performance of the play – 30 minutes
- Reactions & critique – 15 minutes
- Discussion in groups – 20 minutes
- General feedback and conclusion – 15 minutes

In this workshop we will see how theatre can bring mathematical ideas and history to life in the classroom. The workshop will make the case and I hope be inspirational too, by demonstrating theatre in action, involving all of you in the production. Then we will reflect together on what we have been part of, critique the play, and share any similar experiences. After that, we will discuss in smaller groups how to use biographical and primary source material to generate different plays and dialogues for various classroom contexts, and how you might exploit the power of theatre in your own teaching. So, workshop aim in summary: *observe, enact, critique, create!*

A personal disclaimer: I have had plays like this one enacted, in whole or in part, at a number of ESUs, HPM meetings, and other conferences, and some of my plays have been used by others, in classrooms or maths-camps. But I am an academic and, though passionate about the idea of using history and theatre for mathematical enrichment in schools, I have had limited opportunity to mount plays in real classrooms. Therefore, I am not the expert here! I hope some of you may be, or may become so. My challenge to you is to become co-creators of an exciting educational tool and art-form! I want you to run with the ball, catch a vision, translate your own love of mathematics and its history into magic theatrical moments in your own classrooms. It's not so daunting – to demonstrate that is a major aim of this workshop. We'll put this play on now with minimal fuss, minimal props and rehearsal, and (I hope) have lots of fun. Pretend you are a child again, with no inhibitions!

### 3. *Introduction to the play*

The play is designed to motivate the learning of algebra as well as inspire interest in authentic contextual history of algebra. To achieve this, it needs, above all, to be entertaining! So, the dialogue should be as lively as possible, with all appropriate emotions and humour, bringing out the drama, the anxiety and the excitement. The play is introduced by a narrator, MARIA, and a modern Western mathematics teacher, EMMY. The curtain goes up on a scene in the Office of Community Advice and Arbitration in a Chinese town, two thousand years ago. The administrator YU-LIN is at his desk, and an agitated client, YUNG CHANG, enters. She fears that her husband may have made an unwise purchase of land for growing rice and millet, and seeks possible redress. In addition to these four major parts, there are nine small parts to play: representative scribes and mathematicians from different cultures speak briefly, giving the names for 'the unknown' used in equation-solving in their time. The actors may use some artistic aids – dress, hats, beards, moustaches, mannerisms – towards achieving some cultural authenticity, but it is important to be respectful to the various cultures and not to caricature them in potentially offensive ways.

### 4. *Production and casting*

We will now allocate parts, to be displayed on slides, colour-coded. Don't be shy to volunteer or to nominate somebody, so we can move on. We will need stage-crew, light & sound crew, five directors, four main parts, and nine smaller parts. Colour-coded name cards are handed out as we go, to hang around necks. Nobody needs to memorise their part – scripts will be provided.

Actors! As soon as you have been cast, go to different corners (according to colour coding) with your directors, and start rehearsing. Directors! Help the actors focus on projection of voices, use of body language, emotional colour, entries and exits. At the same time, consider what simple costume or head-wear might be effective for artistic purposes, for distinguishing characters and for enhancing authenticity. A few items are laid out on a table, and some props may be handed to people as they are cast. Remember: while we want to celebrate the diversity of cultures involved in the development of mathematics, it is important to portray your characters respectfully. Meanwhile, the stage-crew prepares the set, and sound crew selects and plans music Actors and directors can talk to the stage crew about where to come on, where to sit/stand, etc. This should all be achieved in 15-20 minutes.

**STAGE CREW**: Set up chairs, desks, tables, anything you can think of.
**SOUND & LIGHT CREW**: Work out how and when to adjust lights and play some Chinese music at beginning and end of Chinese scene.
**FIVE DIRECTORS**: Coach actors, using directions in the script and rehearsing selected passages.

- For MARIA & EMMY (full script)
- For YU-LIN & YUNG CHANG (full script)
- For Egyptian Scribe, Babylonian Scribe, Diophantus
- For Arabic, Indian & Chinese mathematicians
- For Italian abbacist, European mathematician, German cossist

**FOUR MAJOR PARTS** (full script)

- MARIA, narrator, historian of mathematics
- EMMY, co-narrator, mathematics teacher
- YU-LIN, mathematician-administrator in the Office of Community Advice and Arbitration in a Chinese town, some two thousand years ago
- YUNG CHANG, a female client

**NINE SMALLER PARTS** (partial scripts)

- EGYPTIAN SCRIBE (2000 BCE)
- BABYLONIAN SCRIBE (1800 BCE)
- DIOPHANTUS OF ALEXANDRIA (3rd century)
- ARABIC MATHEMATICIAN (9th century)
- INDIAN MATHEMATICIAN (12th century)
- CHINESE MATHEMATICIAN (13th century)
- ITALIAN ABBACIST (14th century)
- EUROPEAN MATHEMATICIAN (15th century)

- GERMAN COSSIST (16th century)

## 5. ***THE PLAY: Mathematical Artistry for Justice***

**PROLOGUE**

[*The narrator MARIA introduces the scene, assisted by EMMY, a twenty-first century mathematics teacher. They are seated at a small table front stage, to one side of the stage, where they remain throughout the play. The 'curtain' will rise upon a scene in the Office of Community Advice and Arbitration in a Chinese town, two millennia ago, with administrator YU-LIN seated at a desk, and his counting board beside him.*]

MARIA: Hello! My name is Maria, and I am your storyteller. I want to invite you to come with me to China about 2000 years ago. And this is Emmy, a mathematics teacher, who will accompany us.

EMMY: Hello! I will try to explain what the Chinese mathematician is doing, and relate it what we might call doing algebra today. Algebra has had a very long history, and changed character greatly over time.

MARIA: Let's start with the big picture. The subject we call algebra emerged in ancient Egypt and Mesopotamia, more than four thousand years ago. It began with procedures for solving problems about finding proportions, quantities, lengths and areas –

EMMY: – That'll be what we now call solving first degree equations and second degree equations.

MARIA: Right – except that their methods were often what we would call geometrical! The ancient methods have been preserved in texts – a few papyrus rolls in Egypt, lots of baked clay tablets in Babylonia – that were mostly used for training students in the Scribal Schools. These methods arose first out of quite practical considerations, but they became in the end objects of study for their own sake.

EMMY: Then there are even more complicated equations that arise when you are interested in volumes!

MARIA: That's right! Like: how should a granary be designed in order to hold enough grain, or rice, for the village until the next harvest? How big should a cistern be, or a dam or a canal, to carry sufficient water for the community? And what if you wanted to double or triple the capacity? What dimensions should those bricks be to ensure one man could carry a brick? And how big could a column, or a column base, be before transportation became impossible?

EMMY: And what if you wanted to build a massive wall, like the Chinese did, to protect your country from invaders?

MARIA: Good example! To build the Great Wall of China over a given piece of terrain, how much stone and how much mortar would be needed? What would it cost to transport it?

EMMY: How many people would be needed for transportation of materials and building the wall? How would you feed them all?

MARIA: How much should they be paid? And how quickly could it be done?

EMMY: The kinds of equations that emerged in the context of volumes are now called third degree equations, or cubic equations. Solving them involves taking cube roots at some stage. In theory, an equation could have any degree, and our modern family name for all of these equations people might want to solve is ‘polynomial equations'. So – did anyone in the ancient world come up with a higher degree equation, fourth degree or fifth degree?

MARIA: Not very often! But the Chinese eventually developed a numerical method for solving cubic equations and then extended it to deal with polynomial equations of any degree, using successive approximations, getting more and more accurate.

EMMY: So, for thousands of years, algebra was mostly about the solving of polynomial equations, with one unknown to be found. But there is another important kind of algebraic problem, isn't there? Finding two or more unknowns which simultaneously satisfy some equations. When were such problems encountered?

MARIA: Over 2000 years ago, the Chinese exponents of the art of *suan* (that means mathematics) had methods for solving *systems of linear equations*, like two equations involving two unknowns, where there are no squares or higher powers of the unknowns. One useful technique was to insert trial solutions. This involved calculating with actual numbers, rather than trying to juggle mentally with unknowns, and at least gave them a very good idea of the size of the true solutions.

EMMY: So, did they not know how to find exact solutions for their unknowns?

MARIA: Oh yes! – they went further and used something they called the Rule of Excess and Deficiency, and it gave them exact values. And all their calculating was done in a brilliant, quick fashion, using a large counting board marked with columns, on which they moved their red and black counting rods.

EMMY: How did they write down and record their calculations?

MARIA: They would write on bamboo tablets with a brush made of rat hair, dipped in ink. Let's go and see how they did it. We go back 2000 years, to a Chinese town and visit the Office of Community Advice and Arbitration.

[*Chinese music plays*]

**CURTAIN RISES**

[*YU-LIN is seated at a table, writing on bamboo tablets with a brush, which he dips into a small container of ink. Beside him on another table is a large counting board marked with columns, on which are some red and black counting rods about the size of a man's little finger. A woman, JUNG CHANG knocks and enters. She is agitated, and is waving a piece of paper.*]

YUNG CHANG: Good morning, Master Yu-lin! Excuse me! May I consult you, please?

YU-LIN: Good morning, Yung Chang! You are welcome. How are you and your family in these difficult times?

YUNG CHANG: We are well, thank you, all things considered. But there is something that I am very anxious about – it's about a financial transaction ... if I may …?

YU-LIN: Speak on, dear lady! For such matters am I here. Be seated in this chair. Tell me all.

YUNG CHANG: *[sitting down]* Thank you, Master Yu-Lin! It's about some land my husband bought yesterday, for growing millet and rice, and we all went to look at it today – and I am very worried about the quality of the land – *[she breaks down, and covers her face with her hands]*

YU-LIN: Here, take this. *[he hands her a paper handkerchief, and she blows her nose]* Now, my dear, have courage and stay calm, for when the red and black rods on my counting board begin to dance *[he motions towards the board on his desk]*, they may have a happier tale to tell! What can you tell me about this land you have bought?

YUNG CHANG: It is mostly very dry and rocky, and has poor soil. It was very expensive. We had taken out a loan ... I think my husband may have been robbed! He is a very gullible man ... he should have consulted me first. Do we have any legal redress from the justice court?

YU-LIN: Tell me, Yung Chang, did your husband make sure the seller wrote down on the bill his terms? Did he give any details about the quality of land he was offering?

YUNG CHANG: He did say he was asking less on account of the unterraced and non-arable parts of the land. It's down here somewhere ... *[she holds out the piece of paper]* But he didn't say how much was good grain-growing land, and it seems to me that most of it is bad land! I shall not be able to feed my family … we shall have nothing to sell to pay back the loan we have taken. [*she breaks down again and sobs into the handkerchief*]

YU-LIN: Hush, now, it may not be so bad. My calculation art will uncover the truth. Let me see that bill of sale ... *[she hands it to him]* Hmmm ... He states clearly his terms: for 300 coins you get one acre of good land. For 500 coins you get 7 acres of stony land. It says you have purchased altogether 100 acres. What did you pay?

YUNG CHANG: The price was exorbitant – the seller is a wicked, greedy man! He demanded 10,000 coins!

*[YU-LIN scratches chin and twirls moustache, pondering the problem and making notes, while EMMY explains the slide]*

**SLIDE**:

Price of one acre of good land is 300 coins; price of 7 acres of stony land is 500 coins.

Purchased altogether 100 acres. Paid: 10,000 coins.
Unknowns: amount of good land, and amount of poor land.

In modern terms: Let there be x acres of good land and y acres of poor land. Then we know

$$x + y = 100$$
$$300\,x + \frac{500}{7}y = 10{,}000$$

EMMY: [*can improvise if desired*] Here's a modern expression of the problem. We would call it solving a pair of simultaneous linear equations in the two unknowns, x and y, whose sum is the total amount of land. Can you see how the figures on the bill are expressed in terms of those two equations? … Anybody have questions?

YUNG CHANG: *[waits until there are no more questions and EMMY gives her a signal]* Master Yu-lin, can your art really tell us anything about the kind of land we have bought?

YU-LIN: Hmmmm ... Yes, certainly!

YUNG CHANG: That would seem like a miracle to me!

YU-LIN: *[laughs]* Yes, it continues to seem like a miracle of the darkest arts to some of my students! But, Yung Chang, teach your children this: *suan* – mathematics – is a noble art! *[strikes a pose and recites:]*

Mathematics rules the length and breadth of the heavens and the earth; it affects the lives of all creatures; it forms the beginning and the end of the five constant virtues – benevolence, righteousness, propriety, knowledge, sincerity; it brings to birth the yin and the yang; establishes the symbols for the stars and the constellations; manifests the dimensions of the three luminous bodies; maintains the balance of the five phases – metal, wood, water, fire, earth; regulates the beginning and the end of the four seasons; formulates the origins of myriad things.[53]

[53] This paragraph is based closely on the preface to the *Sunzi suan jing (Mathematical Classic of Master Sun)*, trans. in Lam and Ang, 1992, quoted, with Joseph Dauben's commen-

YUNG CHANG: *[clearing her throat]* Excuse me, Master Yu-Lin, I don't want to take too much of your valuable time ...

YU-LIN: *[recollects himself, coughs, and concentrates once more on the bill]* – erm, yes, I am perhaps getting carried away, but what I am saying, Yung Chang, is that mathematics can shed its pure light on many things. I understand you are eager to see mathematics bring forth children close to the *yin* and the *yang* of your own heart right now – the unknown elements of your problem today *[gestures toward the bill in his hand]* ... The unknowns are the number of acres of good, arable land, and the number of acres of poor land. I let those unknown numbers be called by the names *tian*, and *di*. What you desire to know are *tian* and *di*, and that is what my mathematical art will discover.

*[He begins to place rods on the counting board and move them about, while MARIA and EMMY speak. YUNG CHANG fidgets during the interlude, looking worried and impatient, and blowing her nose, but also showing interest in what YU-LIN is doing.]*

MARIA: Those words mean 'heaven' and 'earth'. So, the Chinese name for the basic method of finding one unknown is *tian yuan*, 'Method of the Celestial Element'. When there is a second unknown involved, it is called *di*, meaning 'earth.' There is an old Chinese book on algebra called 'Heaven and Earth in a Bag'.

EMMY: Our name for the art of finding unknowns is 'algebra' – derived from an Arabic word.

MARIA: Algebra has been developing for a long time, with contributions from many cultures. Whatever culture a scribe came from, the unknown would be given a special name. Let's hear from a few!

*[Cameo Appearances; actors should be pre-seated in order of appearance, each getting up and walking across the stage, stopping in the middle to speak;*

---

tary, in Katz, 2007, 297. Although this work is believed to date from some centuries later, it is likely to be drawn from much earlier treatises, and the preface gives a good picture of the reverence in which the ancient Chinese held mathematics.

*then (keeping their order) in a couple of minutes they will walk back across the stage to take a bow, one by one, returning to their seats]*

EGYPTIAN SCRIBE (2000 BCE): Let the unknown be called *Aha*.

BABYLONIAN SCRIBE (1800 BCE): The unknown shall be called *Line*.

DIOPHANTUS OF ALEXANDRIA (3rd century): I say let the unknown number be called *Arithmos*.

ARABIC MATHEMATICIAN (9th century): The unknown shall be called *shei*, and the unknown square shall be called *Māl [pronounced to rhyme with first syllable of 'starling']*.

INDIAN MATHEMATICIAN (12th century): In calculation the unknowns are the *bīja*. I call the measure of an unknown quantity, *yāvattāvat*; or, if there is more than one unknown number to be found, I call them by the names of colours, and I write a letter, for short.

CHINESE MATHEMATICIAN (13th century): The name of the unknown element is *tian*; or if four unknowns are present we call them *tian, di, ren* and *wu*.

ITALIAN ABBACIST (14th century): The unknown is called *Cosa!*

EUROPEAN MATHEMATICIAN (15th century): That which we seek shall be called *res*, or *radix*.

GERMAN COSSIST (16th century): We shall call the unknown *Coss*!

SLIDE:

| **Egyptian** | **2000 BCE** | ***Aha*** | **Heap, pile** |
|---|---|---|---|
| **Babylonian** | 1800 BCE | *Line* | Geometric length |
| **Diophantus** | 3rd century | *Arithmos* | The Number |
| **Arab** | 9th century | *Shei, Māl* | Treasure |
| **Indian** | 12th century | *Bīja,*<br>*Yāvattāvat* | Seeds<br>As much as so much |
| **Chinese** | 13th century | *Tian,*<br>*Di, Ren,* W*u* | Heaven,<br>Earth, Man, Matter |
| **Italian Abbacist** | 14th century | *Cosa* | The Thing |
| **European** | 15th century | *Res*, R*adix* | Thing, Root |
| **German Cossist** | 16th century | *Coss* | borrowed from the Ita- |

MARIA: In the language of Yu-lin and Yung Chang, *tian* means ‘heaven’ – and stands for an unknown number of whatever – people, coins, acres of land.

*[each scribe-mathematician walks back across the stage and bows, as MARIA mentions them]*

The ancient Egyptians’ word *Aha* means a heap or pile, [waits for scribe to bow] and we guess their calculations were often about grains, loaves, fruits or other food that is stored in piles.

The Babylonians worked a lot with geometrical diagrams, so their Sumerian word for unknown means ‘line’.

Diophantus of Alexandria uses a Greek word, *Arithmos*, which simply means ‘the number’.

The Arabic mathematician’s word *Māl* means ‘treasure’, or valued property.

The Indian’s word *bīja* means seeds – things yet to germinate in the dark earth and emerge into the light! His word *yāvattāvat* means ‘as much as so much’; the letters he spoke of stand for three colours.

The medieval Chinese mathematician still uses Yu-lin’s old word *tian* for an unknown. His names for four unknowns, which I would rather not pronounce, mean ‘Heaven, Earth, Man, Matter’.

The Italians’ word *Cosa* means ‘The Thing’, and the fourteenth century Italian experts at the art of calculating with Hindu-Arabic numerals were called *maestri d’ abbaco*, or abbacists, meaning ‘master-calculators’.

European scholars wrote in Latin for centuries, and the words *res* and *radix* mean ‘thing’, and ‘root’.

The Germans’ word *Coss* is derived from the Italians’ word, and in sixteenth century Germany, algebra was called the ‘Art of the Coss’, and so its practitioners were known as ‘Cossists.’

What do we call the unknown today, Emmy?

EMMY: Well, algebra is full of symbols now. This helps for economy and elegance, but, just like a foreign language, it can be tough at first, for learners.

The unknown can be represented by any letter or symbol. We usually use letters from the end of the alphabet – x, y, z.

MARIA: The Indian way was similar – to use the first letters from three colour names, like r, b, and y, for red, blue, yellow. Of course, they would use letters from their own script and colour names from their own language.

EMMY: If there's only one unknown, we usually call it x, ever since René Descartes, the 17th century French mathematician.

MARIA: It is possible that there are ancient origins in various x-words. The medieval Arabic word, *shei*, for an unknown thing, was translated into Greek as *xei*, spelt X, E, I. And the Greek word for an unknown person is *xenos*, from which we get 'xenophobia' – an irrational fear of outsiders, or foreigners; both begin with an x.

EMMY: I like the fact that when the German physicist Wilhelm Röntgen discovered a wonderful, mysterious, and unknown form of radiation which could help us 'see' inside the human body, he naturally named the rays *X-strahlen*, which we still call X-rays, though Germans call them *Röntgenstrahlen*. And then there is the X-factor – made famous by a certain television programme seeking talented people with that mysterious, undefinable thing that makes celebrities out of unknowns ... *[both laugh]*

MARIA: The quest to discover unknowns is everywhere! Let's go back now to ancient China, and see how Yu-lin is doing with calculating the unknowns for Yung Chang.

YU-LIN: From these numbers you have given me today, Yung Chang, I am working out how much good land your seller claimed to be including in the sale, and how much bad land.[54]

---

[54] This is Problem 17 in Chapter 7 of the *Jiu zhang suan shu* (*The Nine Chapters)*. For details, see Katz, 2009, 209–10 and Katz, 2007, 270–72. For detailed working of similar problems using this same method of 'excess and deficit' – a kind of 'double false position', from both the *Suan shu shu* and the seventh chapter of *The Nine Chapters*, see Katz & Parshall, 2014, 83–87; and Cullen, 2004, 79.

YUNG CHANG: I still cannot believe it – those numbers leave me in thick darkness! How can these unknown quantities emerge from the darkness into the light? But I will be very grateful if you can perform the miracle –

YU-LIN: The method we use is not magic, it appeals not to gods and demons but to the science of reason! It is called 'The Method of Excess and Deficiency.' It leads us as we seek the two unknowns of your problem – by reducing the problem to finding one unknown, which we can discover, and thence we can find the other unknown also. Thus, we unify the *yin* and the *yang*, to turn the two opposites into one. We must begin with a supposed solution. Suppose there are 20 acres of good farmland, 80 acres of poor farmland.[55]

YUNG CHANG: I don't believe there is that much good land!

**SLIDE:**

Method of Excess and Deficiency in modern notation:

Trial solution: Suppose 20 acres of good land, 80 of bad.
Then the total price would be:

$$300 \times 20 + \frac{500}{7} \times 80 = 6{,}000 + 5{,}714\,\frac{2}{7} = 11{,}714\,\frac{2}{7}$$

Thus, the excess over 10,000 is $1{,}714\,\frac{2}{7}$.

YU-LIN: Then that would cost you, hmmm ..., twenty acres at three hundred each, that's six thousand coins; and eighty acres at, hmmm ... [*mutters and surveys the counting board, writing down the answer with his brush*]. Altogether it is 11,714 and 2/7, so there is an excess over your buying price, and it is 1,714 and 2/7.

YUNG CHANG: You mean two-sevenths of a coin? – but that's silly! And you say *excess*? You mean there *isn't* that much good land?

YU-LIN: That's right. Now, if we suppose instead that there are only 10 acres of good land –

[55] Yu-Lin's words and calculations are based closely on the solution in the *Jiu zhang suan shu*.

YUNG CHANG: – Oh no! That's 90 acres of bad land! Surely there cannot be that much bad!

**SLIDE:**

In modern notation:
Trial solution: Suppose 10 acres of good land, 90 of bad.
Then the total price would be

$$300 \times 10 + \frac{500}{7} \times 90 = 3{,}000 + 6{,}428\,\frac{4}{7} = 9{,}428\,\frac{4}{7}$$

Thus the deficiency under 10,000 is $571\,\frac{3}{7}$.

YU-LIN: Hmmm ... *[mutters and moves rods around on counting board, writing down the answer]*t would cost you 9,428 and 4/7. This time there is a *deficiency* of 571 and 3/7.

The Rule of Excess and Deficit says: Display the assumed rates of good land, and lay down the corresponding excess and deficiency below:

**SLIDE:**

In modern notation:

| | | | |
|---|---|---|---|
| amounts of good land: | 20 | 10 | $10\left(1{,}714\frac{2}{7}\right) + 20\left(571\frac{3}{7}\right)$ |
| excess and deficit: | $1{,}714\,\frac{2}{7}$ | $571\frac{3}{7}$ | $1{,}714\,\frac{2}{7} + 571\,\frac{3}{7}$ |

YU-LIN: Now cross-multiply by the rates, and add the results to get the numerator. Also add the excess and the deficiency to get the denominator. Next, divide the numerator by the denominator ... if there are fractions, reduce them ... *[more muttering and moving rods around counting board]*

**SLIDE:**

Solution in modern notation: There are x acres of good land and y acres of bad land.

x is : 10 × excess and 20 × deficiency
divided by excess added to deficiency

$$x = \frac{10\left(1{,}714\ \frac{2}{7}\right) + 20\left(571\ \frac{3}{7}\right)}{1{,}714\ \frac{2}{7}\ +\ 571\ \frac{3}{7}} = 12\frac{1}{2},$$

$$y = 100 - 12\frac{1}{2} = 87\frac{1}{2}.$$

YUNG CHANG: What's it saying? What do the rods say?

YU-LIN: The answer is ... [*mutters and keeps moving rods around counting board*] ... Hmmm, this is quite a challenging calculation ... Ah! The answer is twelve-and-a-half – that is how many acres of good land. And so there must be eighty-seven-and-a-half acres of bad land. [*writes answer down*]

YUNG CHANG: Eighty-seven and a half acres of bad land! Oh-oh!!

YU-LIN: Unfortunately, as you suspected, most of your land is not arable. But that was reflected in the price, so I think you may have no grounds for appeal. Even by the seller's own calculations, he claims to be selling you only a small amount of good land. It is likely that you will find it is an honest price. Here is my calculation, with my signature. *[he hands her a bamboo tablet; and then gives her back the bill]* And here is your bill of sale.

YUNG CHANG: Thank you!

YU-LIN: But if you are not satisfied, would you like me to send a surveyor?

YUNG CHANG: No, no, thank you! I know how much *they* cost. We have very little money left, and a big loan to repay.

YU-LIN: But remember, dear lady, that this problem had two unknowns – and the other one, the number of acres of good land, is the one you should be most interested in. Twelve and a half acres – that's something!

YUNG CHANG: Yes, you are right. It is not as bad as I feared it might be. And at least my husband has not been cheated. I will try to rent some of the rocky land out to my neighbours for their cows and pigs. Eighty-seven and a half acres, hrrumph! Thank you for your assistance. How much do I owe you?

YU-LIN: Consider it a favour from the sweet goddess Mathematics! – Just bring me a bag of grain at harvest time from your twelve-and-a-half acres of good land!

*[they bow to each other]*

YUNG CHANG: Thank you, Master Yu-Lin! May your *suan* art prosper! May you discover many a hidden *tian* for the people who come to consult you! And may your wife and your land be fertile … *[EXITS]*

YU-LIN: Farewell! *[smiling as he turns to audience]* A little mathematics goes a long way – what would we be without it? I suppose I should really not be so generous with my skills – my wife would kill me if she found out. *[laughs]* I sincerely hope she does not turn out to be more fertile! But I will please her by bringing home the money the Minister has promised when we finish compiling the manual on mathematics. May my *suan* – my mathematical art – prosper, indeed! Good land and bad land – now, that's an excellent problem to put into the book! I will write it up while it is fresh in my mind ... *[he dips his brush-pen in the ink and begins to write]*

**CURTAIN FALLS** *while Chinese music plays*

**CAST LINES UP AND TAKES A BOW**

### *6.* ***Discussion***

After the play, there was plenary discussion and feedback, reflecting on the play and how it might be used in classrooms. Participants were asked for their 'gut responses' to the play, and critique and questions were invited. A lively conversation took place, prompted by such questions as these:

- What were the best moments? What didn't work well?
- What improvements can you suggest, in staging, action, emotion, voice projection?
- Authenticity – how close to stay to the primary source material?
- Can any of you share experiences of using theatre? Or being inspired by theatre? Or seeing young people inspired by theatre/improv/drama?
- How might this play be used for educational purposes at various levels?
- Could you write something similar for different groups or levels of communication; for different balances of personal, historical, mathematical?
- Can personal involvement – actually enacting a character from another world – bring a new dimension into our learning and motivation?

- How can such experiences be encouraged under the pressures of the curriculum?}

The discussion was wide-ranging and positive. There is no space here to attempt more than a very brief summary. As usual, surprise was expressed at how little effort and preparation was needed to produce an effective piece of theatre, and how much fun was to be had in the process. The script was agreed to be appropriate and entertaining. Discussion focused on the issue of making readily available for teachers more resources that are relevant and accessible to learners. The difficulty was faced and discussed of balancing the requirements of art and authenticity, and the importance was stressed of insisting that participants show respect for diverse cultures. It was agreed that physical participation in a play, enacting dialogue between historical characters, is an excellent way of 'getting into the heads' of old mathematicians and feeling something of their emotions.
The participants then broke into small groups, and discussed one or more of the following assigned questions, with each group reporting back briefly afterwards.

1. Think of an episode in the history of mathematics, perhaps set in your own country, that could be enlivened by a play and used in a classroom. Brainstorm ideas for the play you could write.
2. Think of parts of the standard mathematics curriculum that are hard to motivate, and brainstorm ideas for a introducing such a topic with a play.
3. How can we win educators/administrators/heads over to the cause? How can we motivate them to invest time & resources in mathematical drama?

## *7. Concluding remarks*

An important feature of this workshop was the opportunity for each participant to act, help direct and rehearse the actors, or set up stage props, lighting, or sound. The discussions, both plenary and in small groups, showed enthusiasm for the idea of incorporating such experiences in the classroom, as a means of firing up interest in both mathematics and its history. The workshop closed by issuing a challenge: We need more theatre in mathematics education, and we need more resources. Consider creating your own, preferably based on primary sources and staying as true as possible to the history and context.


### Acknowledgements

I would like to record here my gratitude to the conference organizers, and to the participants for their creative energy in engaging with their roles and for their feedback during lively discussions. I would also like to acknowledge David Milner's kind assistance in making the Chinese counting board and rods used at the workshop.


## REFERENCES


Cullen, C. (2004). Writings on Reckoning, a translation of the *Suan shu shu*. Needham Research Institute Working Papers (Cambridge). http://www.nri.cam.ac.uk/suanshushu.html Accessed 4 November 2022.

Dauben, J. W. (2008). 算数書 *Suan Shu Shu,* A Book on Numbers and Computations. English Translation with Commentary. *Archive for History of Exact Sciences*, 62(2), pp. 91-178.

Hitchcock, A. G. (1992). The 'Grand Entertainment': Dramatizing the Birth and Development of Mathematical Concepts. *For the Learning of Mathematics,* 12(1), pp. 21-27.

Hitchcock, A. G. (1996). Dramatizing the Formation of Mathematical Concepts: Two Dialogues. In: Calinger R. (Ed.). *Vita Mathematica: Historical Research and Integration with Teaching*. Mathematical Association of America Notes 40, pp. 27-41.

Hitchcock, A. G. (2000). A Window on the World of Mathematics 1870: Reminiscences of Augustus De Morgan. In: Katz V. J. (Ed.). *Using History to Teach Mathematics: An International Perspective*. Mathematical Association of America Notes, 51, pp. 225-230.

Hitchcock, A. G. (2012). Discovering History by Dialogue. In: Archibald T. (Ed.) *Proceedings of the 37th Annual Meeting of the CSHPM and 5th Joint Conference with the BSHM*, Trinity College Dublin, pp. 47-67.

Hitchcock, A. G. (2023). 'Entrance into All Obscure Secrets': A workshop on bringing episodes in the history of mathematics to life in the classroom by means of theatre, incorporating a short play set in an ancient Egyptian scribal school. In: M. Zack, & D. Waszek (Eds.), *Research in History and Philosophy of Mathematics: The CSHPM 2021 Volume* (pp. 277-286). (Annals of the Canadian Society for History and Philosophy of Mathematics.) Birkhäuser (to appear June 3 2023).

Katz, V. J. (2009). *A History of Mathematics: An Introduction*. Boston: Addison-Wesley.

Katz, V. J., & Parshall, K. H. (2014). *Taming the Unknown: A History of Algebra from Antiquity to the Early Twentieth Century*. Princeton and Oxford: Princeton University Press.

Knuth, D. E. (1974). *Surreal Numbers*. Reading, Massachusetts: Addison-Wesley.

Lakatos, I. (1976). *Proofs and Refutations: the Logic of Mathematical Discovery*. Cambridge: Cambridge University Press.

Lam, L. Y. & Ang, T. (1992, 2004). *Fleeting Footsteps: Tracing the Conception of Arithmetic and Algebra in Ancient China* (including a translation of *Master Sun's Mathematical Manual*). Singapore: World Scientific.

Rényi, A. (1967). *Dialogues in Mathematics*. San Francisco: Holden-Day

# PEDRO NÚÑEZ: ALGEBRA IS ALSO FOR GEOMETRICAL PROBLEM SOLVING

**Fàtima ROMERO-VALLHONESTA[1], Iolanda GUEVARA-CASANOVA[2], Carles PUIG-PLA[3]**

[1]Centre de Recerca per a la Història de la Tècnica, Av. Diagonal 647, 08028 Barcelona, Spain

[2]Departament d'Educació, Generalitat de Catalunya, Via Augusta 202-226, 08021, Barcelona, Spain. and Departament de Didàctica de les Matemàtiques, Universitat Autònoma de Barcelona, 08913, Cerdanyola del Vallès, Spain

[3]Departament de Matemàtica Aplicada 1, ETSEIB, Universitat Politècnica de Catalunya, Av. Diagonal 647, 08028 Barcelona Spain,

[1]fatima.romerovallhonesta@gmail.com, [2]iolanda.guevara@uab.cat, [3]carles.puig@upc.edu

**ABSTRACT**

In this workshop, we will show how a 16th century historical text can be used to learn mathematics. We will present some problems from the *Libro de Algebra en Arithmetica y Geometria* by Pedro Núñez, about squares, rectangles, and triangles and how to use them in the classroom. Specifically, they can be used in the curriculum of Catalonia in the third or fourth year of compulsory secondary education (14 to16 year-old students), when the subject of study is the resolution of squares, rectangles, and triangles[56].

## 1 Introduction

One of the undertakings of the ABEAM[57] history group, to which we belong, is to design math activities to implement in the classroom, for students to learn mathematical concepts or procedures from original sources. Since it is about students learning mathematics through history, and not just the history of mathematics, not all historical episodes or sources are appropriate for designing activities to implement in the classroom (Romero-Vallhonesta & Massa-Esteve, 2016). In this sense, we consider the *Libro de Algebra en Arithmetica y Geometria* (Book of Algebra in Arithmetic and Geometry) by Pedro Núñez, a relevant text to implement some mathematical concepts in the class. We presented it to the students in the original Spanish version that they can read and understand without great difficulty.

---

[56]This research is included in the project: PID2020-113702RB-I00.

[57] Barcelona Association for the Teaching and Learning of Mathematics.

## 2 Pedro Núñez and his *Libro de Algebra en Arithmetica y Geometria*

First of all, we presented the author, Pedro Núñez, a mathematician considered by some historians as the most important figure in Portuguese science, focusing on his *Algebra*, his main mathematical treatise (Leitão, 2010).

Pedro Núñez (1502-1578) was born in Alcácer do Sal (Portugal). It is for this reason that he added Salaciense to his name in most of his printed texts. He studied at the University of Salamanca (ca. 1521-1522) and at the University of Lisbon where he obtained a degree in medicine (1525). He also learned astronomy and mathematics, and is probably best known for his contributions to the nautical sciences, which he approached, for the first time, in a mathematical way.

By 1534, he may have begun writing the *Libro de Algebra en Arithmetica y Geometria* probably during his stay in Salamanca, but which was not published until 1567 (Núñez, 1567). This work contains the problems that we will analyse in this session. It is one of the first works with algebraic content to be published in the Iberian Peninsula.

This discipline was developed from the so-called practical arithmetic, where equations begin to be written with abbreviations and symbols, in contrast to the totally rhetorical way of expressing them in most previous texts. An example is the *Suma de la art de arismetica* by Francesc Santcliment, a work printed in Barcelona in 1482.

The *Libro de Algebra* by Pedro Núñez is divided into three parts, which the author calls main parts and consists of 710 pages. The first part deals with the purpose of algebra, its rules and *conjugations*, which is how the author refers to the equations. Also it demonstrates the rules of simple and compound *conjugations* and particular cases. In the second main part, Núñez declares the unknowns which he calls *dignities* and explains how to operate with them. There are 12 chapters dedicated to operations with roots. The author devotes 15 chapters to the theory of proportions, the last of which to the roots of binomials.

The third main part of the work is the most extensive and develops the solution of equations that he had addressed in the first part, and it does so in a much more complete way. It consists of 7 chapters, 5 have a rather theoretical character and 2 of which exemplify the application of algebra to arithmetic and geometry. The work ends with a letter from the author to the readers in which Núñez cites authors who have written algebra texts such as Pacioli, Cardano, and Tartaglia.

### *3. Two problems implemented with students (16 years old)*

During the 2021-2022 academic year we gave two problems from the *Libro de Algebra en Arithmetica y Geometria* (Massa-Esteve, 2010) to a group of 16-year-old students. The first one was Problem 19 of Chapter 7, of the Third Part of the book, and the second was the Demonstration of the Third Rule of Compounds, First Part of the book (p.10v).

In this section we present Problem 19 of Chapter 7, as it was introduced to students, and an explanation of how they solved it. We divided the original text into different parts and asked the students questions to guide them in understanding the text. They were given an explanation of some keywords.

We started by telling the students that solving a problem generally requires several stages as follows:

a) Interpret the problem presented and understand what is required
b) Apply known solution strategies or explore new ones
c) Answer what the problem asks
d) Argue the justification of the solutions obtained
e) Ask new questions related to a problem solved.

Dealing with the first step, interpreting the problem presented and understanding what is required, can be complicated because we are starting from a formulation written in 16th century Spanish. In addition, at the time of the author, symbolic language was quite different from what we use nowadays. The relationship between known quantities and unknown quantities was written using some abbreviations.

These are some of the keywords and their abbreviations:

The unknown was called the *cosa* (thing), abbreviated to *co*, nowadays known as x. The unknown multiplied by itself, what we now call the unknown squared ($x^2$)was called the *census,* abbreviated as *ce.*

To make it easier for students to interpret the problem presented and understand what is required, we suggest reading the text, sentence by sentence, to make the corresponding interpretation as well as drawing and writing what is needed to solve the problem. After each part of the problem, different steps are proposed to the students, eight in total.

This is the first part of the problem given to the students:

> 19. Si la area fuere conoſcida, y la sũma de los dos lados tambien fuere conoſcida, cadavno de los lados por ſi ſera conoſcido.

That means: If the area were known, and the sum of the two sides were also known, each side would be known.

First step. How would you interpret this formulation? What figure is the author referring to?

Second step. Can you reformulate it in your own words?

Third step. Draw the figure.

> Sea la area del rectangulo de 12. braças quadradas, y la sũma de los dos lados ſea vna linea de 8 braças.

The students had no problem with these three questions, everybody could do them, although some didn't answer the second question.

That means: If the area of a rectangle is 12 *fathoms squared* and the sum of the two sides is a line of 8 *fathoms*.

Fourth step. Construct your drawing according to these conditions, with the symbols you think are appropriate.

They placed the measurements in the drawing, *x* & *8-x*, and had no problem using the sign "-". They mainly used *x*. One student used *b* and *h* (base and height). Two students referred to the unknown sides as *a* and *b*.

Fifth step. What do you think a *fathom* is? Do you need to know exactly what it is to solve the problem?

They all deduced that this is a unit of measurement, and they didn't need to know the equivalence with the measures we use nowadays.

> Pornemos pues que vno de los lados ſea 1 co. y ſera luego el otro 8.m̃.1 co. y multiplicãdo vno por otro, haremos 8 co.m̃.1 ce. que ſeran yguales a 12. porque la area ſe haze por la multiplicacion de vn lado por el otro, que con el haze angulo recto.

That means: Let's say one of the sides be 1 *co* and the other 8 *m*∼ 1*co* and multiplying one by the other we get 8 *co m∼ 1ce which* will be equal to 12, because the area is made by multiplying one side by the other, that produces a right angle with it.

Remember that *co* is the abbreviation of *cosa* and *ce* is the abbreviation of *census* that Núñez used for the unknown and its square respectively.

Sixth step. Write this part of the problem symbolically and place the data in your drawn representation. What do you think *m* ∼ represents?

Most of the students wrote x(8-x); 8x - $x^2$ =12;  8x = $x^2$+12 or the same with a, instead of x.

In some cases, they managed with xy=12 and y=8-x

Or even ab=12; a+b=8. In general, the sign "-" was not an obstacle for them.

Ygualando hallaremos que 8 co. ſon yguales a 1 ce. p̃.12. que es la tercera delas compueſtas,

That means: Equalling, it will be found 8 *co* as equal to *1 ce p* 12, which is the third of the compounds.

Seventh step. Write this equality (it will be found that 8 *co* equals *1 ce p* 12) symbolically, that is, the corresponding equation, in current notation.

In general, they didn't separate the 6th and 7th steps, they did it all on step 6 and after writing the final equation again in the 7th step.

Perhaps it would be better to present the problem using a sheet with two columns, with the wording of the different parts on the left and the right column left blank because some students do not need such structured guidance.

y obrando por ella, hallaremos que vno de los lados es 6. y el otro es 2.

That means: Applying it (that means following the rule for the third case of the compounds) we will find that one of the sides is 6 and the other is 2.

Eighth step. Using your chosen method, find the values of the sides and explain how you have achieved the solution.

The students mainly solved the problem but with no explanation.

Three of them factorized the equation $x^2$-8x+12=0, into $(x - 2)(x - 6) = 0$ but only one of them arrived at the end concluding that $x = 2$ and 6.

We explained that Núñez avoids writing expressions with negative quantities. Negative numbers were not used at that time. To solve the equation, the author refers to “the third of the compounds” which is one of the cases that Núñez considers at the beginning of his book.

Núñez classifies the equations (conjugations) into simple and compound. The simple ones have only two terms (regardless of their degree) and the compound ones, three terms. One formula is enough for us to solve the three compound cases because we accept negative coefficients, but at that time, negative coefficients made no sense, so they had to consider three cases.

The classification of Núñez:

Cōjugaciones ſimples:
1. Cenſos yguales a Coſas.
2. Cenſos yguales a Numero.
3. Coſas yguales a Numero.

Cōjugaciones compueſtas:
4. Cenſo y coſas yguales a numero.
5. Coſas y numero yguales a cenſo.
6. Cenſo y numero yguales a coſas.

The problem we have shown led to one equation corresponding to the third of the compounds which, when Núñez refers to all conjugations, in general he calls it the sixth of the conjugations.

The “third of the compounds” which Núñez refers to in this problem, is the case in which the linear term equals the square plus the independent term and that we currently write as $x^2 + c = bx$, being $b$ and $c$ positive.

After the explanation in class and because of the students' interests, we proposed that they demonstrate the Third Rule of Compounds. We work in the same manner as with Problem 19. We divided the original text and asked the students questions to guide them in understanding every fragment and completing the demonstration.

### *4. Two more problems to discuss with the attendees of the workshop*

As we also find in Euclid's *Elements* (Heath, 1956), Núñez sometimes solves problems, the results of which he will use later. One of them is problem 358[3] that we gave to the attendees so that they could understand Núñez's meticu-

[58] Núñez, 1567, 281v-284r.

lous way of working that leaves no unanswered details and often checks what he has done using other methods. In addition, it is a good example in which the power of algebra is shown to solve the problem in a simpler way.

The wording is as follows:

If we know the length of the sides of a triangle, the height (cathetus in Núñez's words) that falls on the side adjacent to two acute angles is known, and also the parts it divides the side on which it falls.

First of all, Núñez says that it has to be taken into account that if the triangle is acute (oxygonio in Núñez's words) there will be three heights within it (that is, the three heights will fall inside the triangle), but in the right triangle and in the obtuse triangle there will be no more than one height within.

The author considers the cases of equilateral and isosceles triangles. In the first, any height bisects the side on which it falls. In the second case, if it is considered the height that starts from the angle that forms the two equal sides, it also divides the opposite side in half. Both cases are proved by applying the Pythagorean theorem.

Then, Núñez considers the case of a scalene triangle and in the example he gives, of sides 13, 14 and 15, he finds the height that falls on the longest side and the segments into which it divides, applying Euclid (II,13)[59]. Since the square of the longest side is smaller than the sum of the squares of the other two sides, he deduces that it is an acute triangle and can apply the same procedure to all three sides.

The author adds that it is not necessary, however, to repeat the whole procedure because once one perpendicular is known, the others can be easily calculated, since the proportion of the sides is reciprocal to the proportion of the perpendiculars that fall on the sides. He calculates the perpendicular that falls on the side measuring 13 from the one already calculated on the side 15, using the reciprocal proportion. Núñez also directly used the formula of Euclid (II, 13), and checked that it gave $12\frac{12}{13}$, the same result[60].

The author uses an identical method when one of the perpendiculars falls inwards and the other outwards, as in the case of obtuse triangles. Also, if one

[59] In Euclid (II, 13) there is the procedure for finding the square of the opposite side at an acute angle.

[60] Núñez gives the results in the form of a mixed number, which we would currently express as 156/13.

perpendicular is known, the others can be calculated by the rule of three, according to Núñez. That means, using the reciprocal proportion he mentioned before. However, the perpendicular that falls outside also has its own rule for Euclid (II, 12)[61]. He says that it can also be found by algebra considering 1*co*[62]. one of the parts in which the side is divided. Applying the Pythagorean theorem, he obtains an equation of 1st degree (simple conjugation) and solves it. Finally, he says that the "oxygonium" case can also be solved by algebra.

After having commented on what Núñez intends with problem 33, problem 61 was given to the attendees to solve in their own way and think about how they would implement it in their classrooms and what guidelines they would give to their students, along the lines of those we had shown with problem 19. Then Núñez's way of solving it was discussed.

The wording of the problem 61[63] is as follows:

*If the side of the equilateral triangle is known, the side of the largest square that fits in it will also be known.*

Núñez begins the problem by supposing that the side of the equilateral triangle is 10 and that in the attached figure, *edfg* is the largest square that can be inscribed in the triangle.

Núñez then gives the following rule to solve the problem:

i) multiply the side of the triangle by itself obtaining 100

ii) multiply this 100 by 12 and it will be 1200

iii) from the root of 1200 remove the triple from the side of the triangle which is 30.

iv) the reminder is the side of the square, that is, in current notation:

The author, then, justifies the rule geometrically.

First, he proves that the triangle *aef* is equilateral and, therefore, its side is the side of the square.

The proof of this, is based on the parallelism of *ef* and *dg* (Núñez says that they are *equidistant*), which makes the angles *aef* and *abc* equal as are the angles *afe* and *acb*. Since the angle in "a" is common, the two triangles *aef* and

[61] In Euclid (II, 12) there is the procedure for finding the square of the opposite side at an obtuse angle.

[62] x in current language.

[63] Núñez, 1567, $242^v$-$247^r$.

*abc* have the same angles and therefore their sides must be proportional. Since *abc* is equilateral, so is *aef*.

After that, Núñez calculates the angle *bed*, which is half of *ebd* and it is deduced that *bd* is half of *be*. To find the angle *bed*, it is known that *aef* is 2/3 of a right angle (it is an angle of an equilateral triangle). Since the angle in *fed* is right, *bed* is the difference to reach a flat angle, which will be 1/3 of a right angle. An alternative method is applied also, as is usual in Núnez's explanations. The author considers the triangle *ebd* which has a right angle and an angle which is 2/3 of a right angle. The third angle, therefore, must be 1/3 of the right angle.

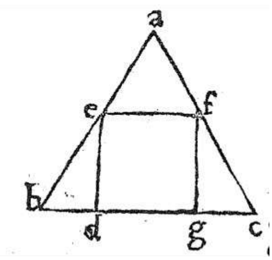
Next, the author supposes that *eb* is 2 and applies Pythagoras to triangle *ebd*. As *bd* is 1, *ed* will be $\sqrt{3}$. Then $ab = 2 + \sqrt{3}$ .

But at the beginning of the resolution, Núñez had supposed that *ab* was 10, then:

$$\frac{2+\sqrt{3}}{3} = \frac{10}{?} \Rightarrow ae = \frac{10+\sqrt{3}}{2+\sqrt{3}} = \sqrt{1200} - 30$$

Here the author explains step by step how to rationalize (i.e., multiply by the conjugate, which the author expresses as *residue* or *reciso*). Finally, it is not necessary to divide because the denominator is the unity.

Núñez now summarizes the procedure and continues with an explanation of the operations he has done, of multiplying by the *residue* and summarizing again. In the midst of these explanations, he also says that if the side of the triangle were 20, this 20 would be multiplied by itself equalling 400 and then by 12 equalling 4800, and the root of this number, $\sqrt{4800}$, is the first part of the *residue*. The second part is 60, the triple of 20. He concludes that the side of the square is $\sqrt{4800} - 60$.




He now says that he will solve the problem by algebra and that he will use the same figure:

Let the side of the square be 1 *co*. (we will use x). The sum of *bd* and *gc* is 10 - x.

And since *ae* equals *ef*, and since *aef* is equilateral, it will be that *be* also equals 10 - x and so *bd* equals $5 - \frac{1}{2}$x

Now the author applies Pythagoras to the triangle *bed*:

$$\frac{3}{4}x^2 - 15x + 75 = (de)^2$$

$$100 - 2 \qquad = (de)^2$$

$$(10-x)^2-\left(5-\frac{1}{2}x\right)^2=(de)^2$$

But *de* = *dg* = x and, therefore,

$$\frac{3}{4}x^2-15x+75=x^2 \Rightarrow 300=x^2+60x$$

This equation is of the first type of *compound conjugations* and the author says that the value of the thing is $\sqrt{1200}-30$ .

As we have already said in the introduction, our aim with the implementation in the classroom of this type of activity designed from relevant historical texts, is the learning of mathematical techniques and concepts in their context.

The history of mathematics is important, not only for the memory of our heritage, but also for educators to return to sources, and therefore a knowledge of these is very useful (Romero-Vallhonesta & Massa-Esteve, 2016).

The workshop was very interesting because not everyone chose the algebraic method. Some sought a more creative solution based on the symmetry of the figure. The importance of discussing alternative methods was recognised, and a creative approach to solve the problems was highlighted. In the workshop it was also noted that the Catalan students have the opportunity to read a text from the original 16th century version.

## REFERENCES


Heath, T. (ed.) (1956). *Euclid. The Thirteen Books of Euclid's Elements*, Nova York, Dover.

Leitão, H. (2010). 'Pedro Nunes e o libro de Algebra', *Quaderns d'Història de l'Enginyeria*, vol. XI, 9-18.

Massa-Esteve, M. R, (2010). 'Àlgebra i Geometria al *Libro de Algebra en Arithmetica y Geometria* (1567) de Pedro Núñez', *Quaderns d'Història de l'Enginyeria*, vol. XI, 101-125.

Núñez, P. (1567). *Libro de Algebra en Arithmetica y Geometria*. Anvers: En casa de los Herederos de Arnoldo Birckman a la Gallina Gorda. http://alfama.sim.ucm.es/dioscorides/consulta_libro.asp?ref=B22138456&idioma=0.

Romero-Vallhonesta, F. & Massa-Esteve, M.R. (2016). 'Història de les Matemàtiques per a l'ensenyament de les matemàtiques. Analitzant les fonts'. *Actes de la XIII*

*Jornada sobre la Història de la Ciència i l'Ensenyament.* Barcelona: Societat Catalana d'Història de la Ciència i de la Tècnica, Filial de l'Institut d'Estudis Catalans, 59-70.

# ANALYTIC NARRATIVE OF STRATEGIC INTERACTIONS THROUGHOUT CONSTRUCTIVIST APPROACH

**Giovanna BIMONTE, Ilaria VERONESI**
University of Salerno, via Giovanni Paolo II n. 132, FISCIANO (SA) ITALY
gbimonte@unisa.it, iveronesi@unisa.it

## ABSTRACT

Mathematical theories based on the concept of rational choice - individual decision theory, game theory, social choice theory - have spread rapidly in many fields: economic, social, political. The mathematically and logically systematic formulation of Game Theory arose as a necessity to explain situations of conflict or interaction between several individuals dating back to earlier times. Past events, in fact, can be reinterpreted through Game Theory models. History becomes, therefore, an "analytical narrative" of events, in which mathematics is the main tool for the formalisation of theoretical models that become universal keys for analysing relationships between agents.

## *1 The background*

Mathematical theories based on the concept of rational choice, such as individual decision theory, game theory, social choice theory, have been widely diffused in the economic, social and political fields. Game Theory received its first general mathematical formulation by John von Neuman and Oskar Morgenstern in 1944. In the last 40 years, Game Theory has become the most important and useful tool when comparing situations where the best action of one agent depends on expectations of what one or more other agents will do, and vice versa. The mathematically and logically systematic formulation of Game Theory arose as a necessity to explain situations of conflict or interaction between several individuals dating back to earlier times. Past events, in fact, can be re-interpreted through Game Theory models, in its classical formulation (games in normal form) or using a tree representation (games in extensive form). History becomes, therefore, an "analytical narrative" (Mongin 2018) of events, in which mathematics is the main tool for the formalisation of theoretical models that become universal keys for analysing relationships between agents.

## *2 The National Guidelines and the MHS project*

The globally integrated laboratory project has been developed in full compliance with the national indications issued by the Ministry of Education, which emphasises that *"[the student] will be able to frame the various mathematical theories studied in the historical context within which they developed and will understand their conceptual meaning. The student will have acquired a historical-critical vision of the relationships between the main themes of mathematical thought and the philosophical, scientific and technological context."*

The Department of Mathematics of the University of Salerno projected an educational path named Liceo Matematico, translated with Mathematical High School Project (MHS) developed in extracurricular hours, whose aim is to recover the interdisciplinary dimension of mathematics in teaching to stimulate reflection on fundamental human and social problems and promote the integration of knowledge.

Researchers designed and tested educational paths and laboratories, among them the one of Mathematics, History and Economics described in the present paper, aimed at restoring to culture that overall vision and to understand how mathematics has influenced historical events or has sometimes been guided by them.

## *3 The Globally Integrated Laboratory Project*

In our work we describe an interdisciplinary activity developed in classes of "Liceo Matematico" project working with a constructivist approach in which students move using their natural learning skills and in which they personally and actively build the appropriate knowledge, while the teacher assumes the role of consultant, assistant and guide (Vygotsky, 1962).

Thanks to the mediation of the teacher, all processes of laboratory activities are conducted by students in first person. They analyse some important historical events, recognizing the essential elements of history to reconstruct the analytical narrative and identify the equilibrium of the model. With the use of narrative analytics, students examine the game of interaction in order to evaluate the outcomes of the history. Students identify the agents: some are individuals, but others are collective actors, such as power groups or electorates. Students reconstruct the preferences of the actors, through the analysis of possible alternatives and the information they possessed at the time of the choice of strategy. Students examine the expectations they formed and the strategies they adopted, taking into account all the constraints on their actions. In this way we model the processes that led to an outcome. Finally, in a collective discussion, students analyse the equilibrium of the model and that of the actual history.

The activities are developed with various levels of difficulty and different use of the mathematical tools acquired. In particular, from a constructivist perspective, students approached geometry problems solved with the use of experimental geometry software. Students considered the succession of choices and calculated all possible outcomes and associated payoffs. Without having any knowledge of optimisation ('unconstrained' and 'constrained') for functions in several variables, the students constructed solutions to the proposed tasks by analysing the historical and strategic elements of the examined context.

The activities are elaborated with various levels of interaction, a transdisciplinary vision of knowledge for the contextualization of ideas and mathematical discoveries in the historical period, the interaction of knowledge for a metacognitive reflection and the anecdotal description to involve and interest the students (immediate and motivating approach, with little impact on the training framework).

## *4 The Analytics narrative*

Our analysis focuses on some historical events using the analytics narrative approach. Bates and collegues (1998), proposed an original reconciliation of the history with the modelling allowed by mathematical theories of rational choice. With narrative analysis, authors argued that certain historical events raising interpretative and explanatory problems cannot be solved by narratives in classical form and require the use of well-formalised models.

Using the Analytics Narrative approaches students learned research methods, starting with understanding the fundamentals of selecting cases for study. We traced the historical context in which the events occurred, identifying the actors, the decision points they dealt with, the choices they made, the possible strategies they adopted and those they avoided, and how their choices generated events and outcomes. The construction of the analytical narrative in the class work led us to highlight and focus on the logic of the processes that generate the phenomena we study, in terms of the rational choice theory. Indeed,

Game Theory considers the sequence of choices and emphasizes their importance for outcomes, which reflect the influence of history, the importance of uncertainty, and the ability of people to manipulate and strategize.
Mongin (2017) highlighted that some historical events that raise interpretive and explanatory problems cannot be solved by the narratives in classical form, and thus require the import of models. The *Analytical narrative* thus refers to the change in narrative type that emerges from the use of models that fit the historical events. All economic and social theory was transformed by Nash's ideas, which formalised something that was already recognizable in the historical events. Analytical narratives use qualitative data, such as changes in institutional patterns and decisions made by individuals or groups. What is really surprising for students is the construction of individual and collective preferences behind Economic modelling: People have consistent and stable preferences, and they choose the alternative from the set of feasible alternatives that is ranked highest under these preferences. As pointed out by Brams (2011), disciplines in the humanities (religion, law, history, literature...) represent a world we do not normally associate with mathematical calculations of strategic interaction and rational choice.

## 5 *Activities, some examples*

### 5.1 The war strategies of Napoleon

Among the various activities carried out by the students, the workshop on Napoleon and the search for an optimal point were also presented. The laboratory started from a historical narrative, then dealt with mathematical analysis to explore the repercussions in the economic field.

In the Renaissance period, many attempts were made to design an instrument that facilitates arithmetic calculations and geometric operations, especially to respond to a military need in which firearm technology required more and more precise mathematical notions. Among these instruments we certainly find Galileo's compass. The great French general Napoleon was passionate about mathematics and his mathematical knowledge helped him in his military career. He was convinced that all Euclidean geometry could be described through the compass without the aid of the ruler and put this hypothesis to Lorenzo Mascheroni who answered affirmatively and described it in the volume "The geometry of the compass" that he dedicated to "Bonaparte the Italic" comparing him to a geometrician. Napoleon in battle used field-compasses to find the central position for attacks. That point was extremely important for organizing assault strategies but, on the battlefield, it was difficult to find through angular triangulation. Instead, it was much easier to rely on field-compasses and obtain it graphically thanks to the "Napoleon's theorem" which states that

*The barycenters of equilateral triangles, built externally on the sides of any triangle, form an equilateral triangle.*

The proof of the theorem leads to the determination of the geometric position of the Fermat-Torricelli point.

The optimal position of Napoleon can be determined by solving a constrained optimization problem or with analytical geometry and properties of triangles barycenters. The great French general Napoleon was also a great mathematician and it was mathematics that helped him in his military career. One of his youthful passions was the compass. He was convinced that all Euclidean geometry could be described by this simple instrument. In battle, Napoleon used 'field compasses' to find the central position. The determination of this point was crucial for managing assault strategies.

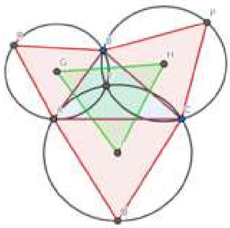


**Figure 1.** Napoleone's theorem

### 5.2 Democratic stability under rational political choices after the Civil war in the USA

A person's behaviour is rational if it is in his best advantage, given his information. On March 4, 1865, in the speech of his second presidential inauguration, Abram Lincoln exposed some significant, surprising and uncomfortable positions even for his own political party. He underlined a position far from the rhetoric of warmongering nationalisms, highlighting that God's plans were inscrutable and it was not sure that He would support them and support them in war. He recognized and affirmed that the cause of conflict was slavery and that the result, initially unintended, has been its abolition. In particular, Lincoln said: “Both parties deplored war; but one would make war rather than let the nation survive; and the other would accept war rather than let it perish. And the war came”.

In the same way, Aumann (2006) recognized that “Wars and other conflicts are among the main sources of human misery”. According to Auman an economic system, a social or individual choice can be viewed as a game, and the incentives of players to make one choice over another when interacting with each other has a degree of complexity and often leads to surprising, sometimes counterintuitive results. Aumann continued:” You want to prevent war. To do that, obviously you should disarm, lower the level of armaments. Right? No, wrong. You might want to do the exact opposite. In the long years of the cold war between the US and the Soviet Union, what prevented “hot” war was that bombers carrying nuclear weapons were in the air 24 hours a day, 365 days a year. Disarming would have led to war.”

## *6 Conclusion*

In the “Liceo Matematico” project, working with a constructivist approach, we developed an interdisciplinary activity to analyse historical events through mathematical models. We examined some important historical events, identifying the essential elements of history in order to reconstruct the analytical narrative and identify the equilibrium of the model. The students were asked to identify the agents (individuals, groups, voters) and reconstruct the actors' preferences by analysing the possible alternatives and the information they possessed when choosing a strategy. Taking into account all the constraints on the players' actions, we examined the possible outcomes of the game, using both techniques from Game Theory and elements of analytical geometry. In this way, we modelled the processes that led to historical events. Finally, we analysed the equilibrium of the model and that of real history.

## REFERENCES


Aumann, R. J. (2006). War and peace. *Proceedings of the National Academy of Sciences*, 103(46), 17075-17078.

Bates RH, Greif A, Levi M, Rosenthal JL, Weingast B (1998) *Analytic narratives*.

Princeton University Press, Princeton
Bimonte G., Tortoriello S.F., Veronesi I., “Game Theory Lab: A Gamification Laboratory for High School Students”, *Handbook of Research on International Approaches and Practices for Gamifying Mathematics*, 2022, DOI: 10.4018/978-1-7998-9660-9
Bimonte G., Tortoriello F.S., Veronesi I., “Economics and mathematics: a transdisciplinary path using geodynamical models and computational software”, pagg. 9128-9131, *Proceedings of INTED2021 Conference* 8th-9th March 2021, ISBN: 978-84-09-27666-0
Brams S (2011) *Game theory and the humanities: bridging two worlds*. MIT Press, Cambridge
Greif A (2002) Game theory and history. In: Aumann RJ, Hart S (eds) *Handbook of game theory*, ch. 52, vol 3. Amsterdam, Elsevier, pp 1989–2024
Israel, G., & Gasca, A. M. (2008) *Il mondo come gioco matematico: la vita e le idee di John von Neumann.* Bollati Boringhieri.
Mongin, P. (2010) Analytic narrative. In: Kurian GT, Alt JE, Chambers S, Garrett G, Levi M, McClain P. (eds) *The encyclopaedia of political science*, vol 1. CQ Press, Washington, DC, pp 48–51
Mongin, P. (2016) What are analytic narratives? *HEC Paris research* paper no. ECO/SCD-2016-1155.
Mongin, P. (2018). A game-theoretic analysis of the Waterloo campaign and some comments on the analytic narrative project. *Cliometrica*, 12(3), 451-480
Rogora E., Tortoriello F.S., (2018), Matematica e cultura umanistica. *Archimede* vol. 2, p. 82-88, Firenze
Rogora, E. & Tortoriello, F.S. (2021). Interdisciplinarity for learning/teaching mathematics, *Bolema: Boletim de Educacao Matematica*
Tortoriello, F.S., Veronesi, I. (2021) Historical path in mathematical games: a didactic laboratory activity dedicated to upper secondary school students, *Proceedings The 25th World Multi-Conference on Systemics, Cybernetics and Informatics: WMSCI*
Tortoriello F.S., Veronesi I., “Internet of things to protect the environment: a technological transdisciplinary project to develop mathematics with ethical effects”, 2021, *Soft Computing*, https://doi.org/10.1007/s00500-021-05737-x
Vygotsky, L. S. (1962), *Language and thought*, Ontario, Massachusetts Institute of Technology Press

# A POSSIBLE TRANSITION FROM GEOMETRY TO SYMBOLIC ALGEBRA THROUGH THE HISTORY OF MATHEMATICS

**Emilia FLORIO[1], Giuseppina FENAROLI[2]**

[1]Università della Calabria, Rende (CS), Italy

emilia.florio@unical.it

[2]Università di Genova, Italy

fenaroli@dima.unige.it

## ABSTRACT

In this paper we present the experimentation of an educational path, designed for a secondary school class, aimed at highlighting a connection between geometric and algebraic knowledge starting from passages taken from the *Kitāb al-jabr wa al-muqābala* of Al-Khwārizmī and Abū Kāmil's *Kitāb fi al-jabr wa al-muqābala*. The motivation for such activity is twofold: the knowledge of the historical development of mathematics can increase its disciplinary knowledge; moreover, focusing attention on the role of the semiotic registers favors understanding of mathematics by the students. The main moments of the experience were: administration of a questionnaire to students, to detect any discontinuity in learning Euclidean geometry and algebra; linguistic-literary reading of passages chosen from the texts of Al-Khwārizmī and Abū Kāmil; identification of the mathematical elements in the chosen passages; identification of different semiotic registers, through which mathematics expresses itself, and of their role in the search for the solution of a problem and its parameterization; conclusions regarding the effects of the proposed educational path. Among the effects of the presented didactic path, we highlight the students' awareness of how the transition process from geometry to symbolic algebra improves understanding of the historical evolution of mathematics; the generalization of a problem and its solution through parameterization and the acquisition of operational skills in the modeling process.

## 1 *Introduction*

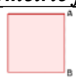
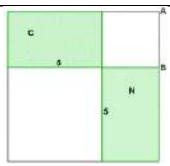
In this work we present the phases of a didactic experience carried out in Italy. The context of the experimentation, lasting 8 curricular hours, was a second class with 28 students of a Technical Institute in Genoa. The research question posed was to study the lack of connection, often perceived by students, between geometry and algebra. This experience was based on the didactic path described in Florio (2020) and Florio et al. (2020), in which students are shown, through the analysis of some problems, how algebraic language is grafted onto geometric language, allowing them to translate a problem from spoken algebra to symbolic algebra. The choice of particular propositions was made on the basis of the observation that they constitute a simple and effective environment for exploring the potential of some forms and registers with which mathematics expresses itself, to ascertain how the changes between these registers can happen in a way trustable and effective, and to evaluate the types of responses that such changes can produce in solving a problem. They also favored the development of students' ability to generalize a problem through the use of a parameter and to move towards mathematical modeling.

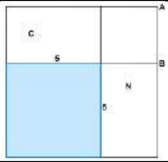
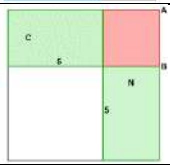
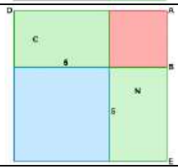
## 2 *Materials and methods*

The theoretical framework that supported the experimentation has taken into account the relationships between the built of mathematical knowledge and its historical construction (Menghini, 1994; Furinghetti, 2020). Furthermore, the semiotic and epistemological reflections, within which the proposed propositions can be seen as a valid support material for the teaching of mathematics (D'Amore & Radford, 2017; Duval, 2018), have been considered. In particular, Abū Kāmil's proposition comes at a crucial moment in the semiotic evolution of the language of mathematics and constitutes one of the first written testimonies. In it, a simple problem is presented and solved using spoken algebra, which gradually leads from the Euclidean narrative to the Cartesian one. In this step, the geometric constructions play an important role in promoting the recognition of the object we are talking about and which is fundamental for the realization of learning (Barbin, 2015). In the same passage, in which the register change takes place, it is possible to grasp essential prodromes to the modeling process and crucial for the development of problem solving skills. The conducting and results of the experiment found their main methodological tools following a "modular" approach (Jankvist, 2009) to mathematical concepts through the use of original sources or by considering "historical packages" (Tzanakis et al., 2002). After some historical notes on the two works by Al-Khwārizmī (Rashed, 2007) and Abū Kāmil (Rashed, 2012) under consideration, the students were given a questionnaire to test their opinions on the link between algebra and geometry and then, specifically, they were asked to solve an equation of second degree, to write the solution formula and to give a justification. We continued by presenting the geometric justification of a particular second degree equation taken from the work of Al-Khwārizmī, comparing it with the algebraic one present in their textbook. Work on the first proposition of Abū Kāmil's treatise followed. It was then proposed the reading of the Italian translation, made by us, of the original text and, subsequently, the translation into the language of spoken algebra and finally into the language of symbolic algebra was carried out. Finally, a questionnaire was proposed to the students to test the effects of the lived experience.

## *3 Results*

The students responded to the first questionnaire by attributing greater complexity to algebra than geometry and greater learning difficulty as it does not allow a visual confirmation of the objects being treated. 57% of them did not see any link between algebra and geometry, 43% perceived some relationship. 86% of students solved the proposed second degree equation by correctly applying the solution formula, 79% were unable to give any justification for it. Starting from the fact that no student had been able to justify the solution formula, we gave them photocopies of the text, which we translated into Italian, taken from Al-Khwārizmī's *Kitāb al-jabr wa al-muqābala*, which shows the procedure indicated by the author to solve the equation $x^2 + 10x = 39$ and its geometric justification. After a careful and commented reading of the text, together with the students we built a first correspondence table between the passages described in words by Al-Khwārizmī and their translation into geometric form and a second correspondence table between the geometric passages and their translation into algebraic form. The two tables are summarized in the following:

| ***Solution described by Al-Khwārizmī*** | ***Geometric form*** | ***Algebraic form*** |
|---|---|---|
| Let $AB$ be a square, to which we add ten of its roots. |  | $x^2$ |
| Let us divide ten into two halves and draw two surfaces $C$ and $N$ on both sides of $AB$. The length of each of the two surfaces will be one half of ten roots, and its width is the side of $AB$. |  | $10 : 2 = 5$<br>$5x$<br><br>$(5x + 5x = 10x)$ |

| | | |
|---|---|---|
| We complete the figure with a square with a side equal to 5, starting from one of the angles of $AB$, which is half of the ten roots we added to it. |  | $5 \cdot 5 = 25$<br>$\mathbf{25}$ |
| We therefore know that the first surface is the square, that the two surfaces on either side of this are ten roots and that the sum is thirty-nine |  | $\boldsymbol{x^2 + 10x = 39}$ |
| and that, to complete the larger surface $DE$, we have to add the square of side 5 - that is 25, which we add to 39, obtaining 64; |  | $\mathbf{25 + 39 = 64}$ |
| we take its root, which is eight and which is one of the sides of the larger surface; |  | $\sqrt{64} = 8$ |
| if we subtract from it an amount equal to the one we added to it, that is 5, there remains 3, which is the side of the surface $AB$. |  | $x = 8 - 5 = 3$ |

To highlight the actuality of the procedure followed by Al-Khwārizmī and the fact that the solution formula they studied is obtained, in its algebraic form, with the same procedure, we have considered, instead of the particular equation $x^2 + 10x = 39$, the more general equation of the same type $ax^2 + bx = c$ and retraced the passages of Al-Khwārizmī, associating each algebraic expression with its corresponding geometric expression. We then built a table of correspondence between the algebraic and geometric passages that led to the determination and justification of the positive solution of the equation $x^2 + 10x = 39$. Some students were struck by the fact that a result presented to them only with formulas had been justified with geometric figures, thus making natural and sharable those passages that the formulas had made difficult and of which it was difficult to identify how they could have come to mind. At this point the students could have followed what Abū Kāmil did to solve the problem of finding the length of the side of a regular pentagon inscribed in a circle with a diameter equal to 10. We therefore proposed to them the reading of the translation made by us of the passage taken from the *Kitāb fi al-jabr* *wa al-muqābala* concerning the considered problem. Subsequently, we compiled together with the students Table 1 shown in Florio (2020), where in the left part is reported the problem expressed in words by Abū Kāmil and, in the right part, its translation into symbolic algebra. The students observed the use of the properties of the considered figure, described geometrically by Euclid in the Elements, to write the equation that solves the problem and then express it with the signs of our current mathematical culture. They were thus able to observe how Abū Kāmil started from those results obtained by Euclid (Elements IV, 11) which allowed him to think of the pentagon as already built. This made sense of his quest to determine its side. We pointed out to the students how the drawn figure and the beginning of Abū Kāmil's speech are in the Euclidean register, as can be seen from the terms introduced, from the use of capital letters to indicate points and segments and from the proposal to "construct" the line $CLD$. The subsequent observation concerned the change of register by Abū Kāmil. In fact, by placing $ED$ equal to "a thing", he passed from the geometric register to the algebraic register. At this point we asked the students how they would solve the problem of finding the side of the regular pentagon inscribed in any circle of diameter $2r$. Some of them replied that it was possible to replace 10, that is the diameter of the particular circle considered by Abū Kāmil, with $2r$, that is, the diameter of any circle. We then invited them to retrace the steps contained in the right part of the previous table. We have thus come to determine the acceptable solution to the problem: $x = r\sqrt{\frac{5-\sqrt{5}}{2}}$. This new formulation has allowed us to observe that the ratio between the side $x$ of the regular pentagon and the radius $r$ of the circle circumscribed to it is constant and independent of the pentagon considered: $\frac{x}{r} = \sqrt{\frac{5-\sqrt{5}}{2}}$. In this last part of our path the students were able to observe and learn, on an elementary example, how the change of register generated the prodromes of a new process, the one related to parameterization and, therefore, to the subsequent *modeling* in mathematics. As a conclusion of the experience, students were given a second questionnaire to test which impressions and what effects had produced the collective experience we have had. The students declared: to have grasped an evident link between algebra and geometry and intuited how a synergistic use of these two mathematical fields can increase the possibility of producing ideas to solve problems, favoring imagination and personal creativity ( 93%); to have acquired the awareness of how the same mathematical object can be described using different languages (30%); to be able to use some mathematical tools related to previous years of study with greater certainty (21%); to have acquired a less abstract and algorithmic vision of mathematics and to feel more encouraged in learning (39%). With regard to the history of mathematics, the students felt that: it can in general facilitate the learning of mathematics (43%); having read it, some contents are current (29%); stimulate curiosity and interest in mathematics also for the human and social component often hidden in the presentation of the sentences of school texts (21%). For the students it was comforting to hear that a definition of a few lines or an agile calculation tool, presented "casually" in textbooks, sometimes took hundreds of years to be understood and formulated as we know it today.

## REFERENCES


Barbin, É. (2015). *Les Constructions Mathématiques avec des Instruments et des Gestes*. Paris: Ellipses.

D'Amore, B., & Radford, L. (2017). Enseñanza y Aprendizaje de Las Matemáticas: Problemas Semióticos, Epistemológicos y Prácticos. Bogotá: Univ. Distr. F. José de Caldas.

Duval, R. (2018). *Understanding the Mathematical Way of Thinking: The Registers of Semiotic Representations*. Cham: Springer.

Florio E. (2020). A synergy between history of mathematics and mathematics education: A possible path from geometry to symbolic algebra. *Education Sciences*, *10*, 243/1-16.

Florio E., Maierù L., & Fenaroli G. (2020). The construction of regular polygons between geometry and algebra: A didactic-historical example. *For the learning of mathematics*, *40*(2), 23-24.

Furinghetti, F. (2020). Rethinking history and epistemology in mathematics education. *International Journal of Mathematical Education in Science and Technology*, *51*, 967–994.

Menghini, M. (1994). Form in Algebra: Reflecting, with Peacock, on Upper Secondary School Teaching. *For the Learning of Mathematics*, *14*, 9-14.

Jankvist, U.T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics*, *71*, 235–261.

Rashed, R. (2007). *Al-Khwarizmi. Le commencement de l'algèbre*. Paris: Blanchard.

Rashed, R. (2012). *Abū Kāmil: Algèbre et Analyse Diophantienne*. Berlin: de Gruyter.

Tzanakis, C., Arcavi, A., de Sa, C. C., Isoda, M., Lit, C. K., Niss, M., Pitombeira de Carvalho, J., Rodriguez, M., & Siu, M. K. (2002). Integrating history of mathematics in the classroom: An analytic survey. In *History in Mathematics Education: The ICMI Study*, Fauvel, J., van Maanen, J. (Eds.). Dordrecht: Kluver, *6*, 201–240.

# UN ESEMPIO DI PROBLEM POSING NEI CORSI UNIVERSITARI DI MATEMATICA

**Nicla PALLADINO, Roberta PAOLETTI, Emanuela UGHI**
Università di Perugia, via Vanvitelli 1, 06123 Perugia, Italy
nicla.palladino@unina.it

## ABSTRACT

Il “Divina Proportione” di Luca Pacioli contiene descrizioni ed immagini di nuovi poliedri concavi che durante il Rinascimento furono rappresentati e studiati. Pacioli mostra come costruire due nuove tipologie di solidi, che chiama “abscissi” (troncati) ed “elevati”. Gli abscissi si ottengono tagliando via gli angoli solidi dai poliedri regolari convessi; gli elevati sono ottenuti da poliedri regolari o troncati aggiungendo su ogni faccia piramidi aventi come base le facce del poliedro. Pacioli descrive, infine, poliedri prima troncati e poi elevati tra cui il dodecaedro troncato elevato, dichiarando che sei vertici delle piramidi del poliedro giacciono su uno stesso piano. Abbiamo sottoposto lo studio e l’interpretazione del testo originale agli studenti del corso di Storia della Matematica per la laurea magistrale in Matematica. Esponiamo le fasi del laboratorio ma soprattutto vogliamo evidenziare come i risultati sono stati interessanti didatticamente, sia da un punto di vista storico sia per l’attivazione di abilità e competenze in altri ambiti (geometrico, analitico, informatico). Riteniamo infatti che un’attività del genere sia da ritenersi un buon esempio di utilizzo della storia della matematica in didattica.

## *1 Il corso di Storia delle Matematiche*

Nell’anno accademico 2020/21, le due autrici docenti del corso di Storia della Matematica della laurea magistrale in Matematica all’Università di Perugia hanno scelto di trattare l’evoluzione dei concetti di poligono e poliedro, concavi e convessi, ponendo particolare attenzione agli “stellati”, a partire dal Medioevo fino a Keplero. L’obbiettivo propostoci per il corso è stato favorire l’acquisizione di una visione storica di determinanti momenti significativi nello sviluppo della matematica mostrando l’evoluzione di alcuni dei principali concetti, metodi e teorie. Il corso è stato frequentato da studenti che hanno seguito anche il corso di Didattica della Matematica e ci siamo quindi soffermati sulla individuazione e comprensione di ostacoli epistemologici emersi nella sistemazione di alcuni concetti matematici nel corso dei secoli. Inoltre, abbiamo mostrato un possibile utilizzo della storia della Matematica nella didattica (in questo caso universitaria ma facilmente riproducibile in classi di scuola secondaria superiore, in quanto la maggior parte dei concetti matematici messi in gioco sono già conosciuti a tale livello scolastico). Il corso è consistito quindi di due parti: nella prima, abbiamo mostrato l'evoluzione dei concetti di poligono e poliedro, soprattutto stellati, studiando la storia dei problemi e delle idee matematiche collegati ad essi; ci siamo soffermati su matematici e artisti, studiandone le biografie, i lavori, contestualizzando nella loro epoca la geometria che presentano, provando a ricostruire rapporti ed influenze reciproche. Nella seconda parte, abbiamo tratto spunto dalle teorie, dai problemi e dalle idee esposti, per sviluppare attività di laboratorio. La lunga storia delle figure stellate inizia, infatti, come scoperta da parte di artisti di figure che essi adoperano nei propri decori; nel medioevo notiamo un primo approccio allo studio di singoli oggetti da parte dei matematici. Con il Rinascimento, lo studio diventa sistematico e apre le porte ad una teoria matematica. Nella cultura occidentale, Adelard of Bath (1080-1152) (Günther 1873) e Thomas Bradwardine (c. 1290-1349) (Bradwardine1989) sembrano essere stati i primi a dare alcune nozioni riguardanti le figure stellate. Durante il Rinascimento, Piero della Francesca (c. 1412-1492) fu uno dei protagonisti della graduale transizione della visione di solidi concavi da figure artistiche a oggetti matematici. Il suo "Libellus" (Davis 1977) fu da Luca Pacioli (c.1445-1514) incorporato in (Pacioli 1509). Il suo contributo è legato al matematico e pittore tedesco Albrecht Dürer (1471-1528) che in (Dürer 1525) intendeva fornire strumenti contemporaneamente pratici e teorici utili per artisti ed artigiani. Egli accenna a corpi ottenibili sovrapponendo piramidi alle facce di corpi originari e poi procede alle descrizioni di sette solidi semiregolari che si ottengono per troncamento dai poliedri regolari. Influenzato da Dürer fu Simon Stevin (1548-1620) che pubblicò (Stevin 1583). Il suo approccio allo studio dei poliedri dà un forte contributo al processo di "matematizzazione" di quelle figure ancora prevalentemente utilizzate in ambito artistico. Affronta lo studio dei solidi regolari e semiregolari e determina procedimenti per costruire poliedri a partire da quelli di Dürer. Le idee di Dürer vengono elaborate dall'italiano Daniele Barbaro (1513-1570) che pubblica (Barbaro 1569). Il processo che conduce ad una esaustiva matematizzazione delle figure stellate può considerarsi giunto ad un punto decisivo con Johannes von Kepler (1571-1630). In (Kepler 1619) troviamo una definizione precisa di "stella", classificata come poligono regolare e la teoria sui poliedri stellati.

L'opera, matematicamente compiuta, conclude la fase di "scoperta", dando definizioni, classificazioni complete, dimostrazioni (si veda Brigaglia *et al.*

2018). Il percorso storico illustrato nella prima parte del corso ha fornito elementi per contestualizzare pensiero, tecniche, procedure, idee su cui sono nate le riflessioni per la fase di laboratorio; questi infatti sono stati, nella seconda parte, elementi su cui ragionare per sviluppare e poi risolvere alcuni problemi. La contestualizzazione storica è stata elemento fondamentale perché abbiamo trattato problemi emersi dallo studio delle conoscenze degli autori presentati e che solo in tale ottica possono essere formulati e affrontati. In generale, molti sono gli studi sull'efficacia in termini di apprendimento, nell'utilizzare la storia della matematica nel suo insegnamento: la storia della matematica riguarda anche il modo in cui si è evoluto un concetto, i diversi approcci dei matematici del passato riguardo ad esso, le loro difficoltà, la loro creatività, le loro idee, fino alla fase di costruzione e formalizzazione. Potremmo allora utilizzare la storia per una sorta di laboratorio epistemologico in cui esplorare lo sviluppo delle conoscenze matematiche (Radford 1997). Dal punto di vista dell'apprendimento, Skemp (1969) afferma che un approccio che fornisce solo il prodotto finale della scoperta matematica, e non i processi attraverso i quali esse vengono raggiunte, non favorisce l'apprendimento. La realizzazione di attività didattiche sotto forma di laboratori facilita invece dialogo, riflessione e apprendimento. Abbiamo portato nell'aula universitaria un tipo di attività laboratoriale; abbiamo esaminato alcuni aspetti e alcuni momenti dello sviluppo dei concetti di poligono e poliedro, lavorando sui testi originali, soffermandoci sulle difficoltà degli autori, sul passaggio graduale avvenuto dalle idee alla formalizzazione matematica, come attività di studio e riflessione in cui abbiamo provato a formulare dei problemi e poi anche ad affrontarli e risolverli. Stoyanova e Ellerton (1996) classificano le attività di problem posing in tre diverse categorie: *free problem posing situations*, *semi-structured problem posing situations* e *structured problem posing situations*. Una situazione di problem posing è considerata "semi-structured" se agli studenti vengono presentati dei problemi aperti o delle situazioni non strutturate e sono invitati ad esplorare la situazione e a completarla applicando conoscenze, abilità e concetti derivati dalle loro precedenti esperienze matematiche. Tra i problemi semi-strutturati, gli autori (rifacendosi a Krutetskii 1976) fanno rientrare anche quelli derivanti dall'interpretazione di figure. Secondo (Bonotto & Dal Santo, 2015) il problem posing corrisponde ad un processo secondo il quale gli studenti, in base alle loro conoscenze, costruiscono delle interpretazioni personali di situazioni concrete e le formulano come problemi matematici significativi. Per gli studenti questo processo diventa perciò una opportunità di interpretazione e di analisi critica che favorisce il pensiero critico, la creatività, l'apprendimento. Abbiamo allora predisposto situazioni di semi-structured problem posing nella didattica universitaria: con gli studenti abbiamo riflettuto, a partire da alcuni spunti sorti dalla lettura di testi originali, sulle possibili interpretazioni dei brani, alla luce delle conoscenze dell'epoca e delle conoscenze attuali sull'argomento; essi sono stati sollecitati a porsi problemi, a esplorarli utilizzando conoscenze, abilità, concetti matematici già acquisiti, nonché le nozioni e le opportune contestualizzazioni storiche presentate nella prima parte del corso. Il laboratorio che qui illustriamo è stato sviluppato a partire dal "Divina Proportione" di Pacioli e dalle figure disegnate da Leonardo da Vinci.[64]

## 2 *Un esempio nella didattica universitaria*

Pacioli scrisse il "Divina Proportione" tra il 1496 e il 1498; in esso mostra come costruire due nuove tipologie di poliedri, che chiama "abscissi" (o troncati) ed "elevati". Gli abscissi si ottengono tagliando via gli angoli solidi dai cinque poliedri regolari; gli elevati sono ottenuti dai regolari o troncati, aggiungendo su ogni faccia delle piramidi che abbiano come basi le facce del poliedro. Descrive anche poliedri prima troncati e poi elevati tra cui, alla sezione "LII", il "dodecaedro troncato elevato" raffigurato nella Tav. XXXIIII (Fig. 1.). La prima parte della sezione LII riguarda la costruzione del poliedro: si crea a partire dal dodecaedro solido regolare costituito da 12 facce pentagonali regolari congruenti, 20 angoli solidi e 30 spigoli. Il dodecaedro troncato deriva dal dodecaedro a cui vengono tagliati gli spigoli; si crea così un solido con 32 facce (20 triangolari e 12 pentagonali regolari), 60 spigoli e 30 vertici. Infine, Pacioli eleva il solido precedente (riportiamo di seguito la nostra "traduzione" al fine di chiarire la costruzione del poliedro generato): il corpo che si crea in questo modo è composto dal dodecaedro troncato, all'interno, che si mostra alla mente solo attraverso l'immaginazione, e da 32 piramidi, di cui 12 pentagonali, tutte di uguale altezza e di cui le altre 20 sono triangolari, tutte di

[64] La curiosità a questa particolare questione, da parte delle due docenti, è nata dalla lettura di Huylebrouck (2013).

uguale altezza. Le basi delle piramidi sono le facce del suddetto dodecaedro e corrispondono puntualmente, cioè i triangoli alle piramidi triangolari ed i pentagoni alle piramidi pentagonali.

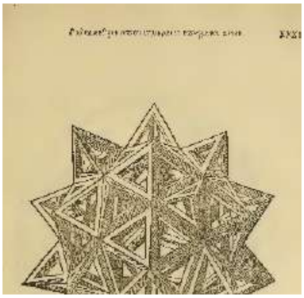


**Figure 1.** Il dodecaedro troncato elevato di Leonardo

Alla costruzione, Pacioli fa seguire un assunto: proiettando su un piano, questo corpo riposerà sempre su 6 cime delle piramidi, una delle quali è una piramide pentagonale, le altre cinque triangolari. Possiamo pensare l'attività di problem posing suddivisa in diverse fasi. In un primo momento, la studentessa (la seconda coautrice del presente lavoro) ha avuto il compito di leggere e interpretare la sezione LII e gli altri brani a cui fa riferimento. La costruzione è dal punto di vista matematico poco dettagliata, anche perché Pacioli basa le sue osservazioni dai disegni e dai modelli dell'oggetto e ciò riflette l'approccio del periodo medievale e rinascimentale. Si possono allora individuare parole e procedimenti diversamente interpretabili, ognuno dei quali può dar luogo a realizzazioni differenti. A partire dalla lettura e dall'esame del testo, la studentessa ha quindi formulato il problema su cui lavorare: "In quali circostanze l'assunto di Pacioli può considerarsi corretto?". Nella seconda fase, la studentessa ha preso in esame la situazione dal punto di vista storico ed epistemologico, confrontando la costruzione di Pacioli con i metodi conosciuti dai matematici contemporanei (alla luce della trattazione affrontata nella prima parte del corso).

Tutte le costruzioni del periodo sono approssimative, seppur con approcci e metodi differenti (Dürer e Barbaro provano a chiarire i poliedri di Pacioli); costruzioni matematicamente corrette si avranno solo con Stevin, che descriverà con perizia i diversi metodi di troncamento che conducono a diversi poliedri. Sono state quindi formulate delle congetture sulla base delle riflessioni:

abbiamo prima di tutto assunto che le piramidi di cui si avvale Pacioli siano rette, sicché le facce laterali sono triangoli isosceli (o equilateri). Fissando l'altezza di uno dei triangoli, le altezze delle 32 piramidi rimangono determinate (quella delle piramidi pentagonali sarà diversa da quelle triangolari); viceversa, fissando l'altezza delle piramidi a base pentagonale, i triangoli isosceli rimangono determinati. La fase seguente è stata la modellizzazione matematica, ma per fare ciò la studentessa ha preferito preventivamente costruire un modello in cartoncino del solido. Manipolando ed aprendo a metà il modello, ha realizzato che l'enunciato di Pacioli è reso più semplice limitandosi ad osservare il comportamento del solido ottenuto unendo al dodecaedro iniziale interno una sola piramide pentagonale ed una sua adiacente piramide triangolare (Fig. 2.): l'enunciato sulla complanarità dei sei punti di Pacioli corrisponde al fatto che i vertici delle due piramidi si trovano alla stessa altezza rispetto alla base pentagonale. Il problema è stato poi ricondotto ad un problema piano sulla sezione del solido.

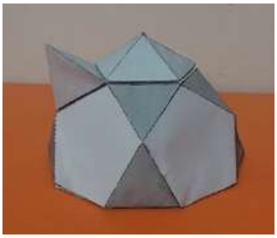


**Figure 2.** Parte del dodecaedro troncato elevato in cartoncino

La studentessa ha dovuto quindi individuare gli strumenti matematici per risolverlo. Ha scelto di avvalersi di un software di geometria dinamica, riportando in esso le misure fisse note e ponendo variabile l'altezza corrispondente alla piramide pentagonale in modo da osservare diverse configurazioni corrispondenti a differenti altezze di tale piramide (Fig. 3.). L'enunciato risulterà corretto solo quando la retta che unisce i vertici delle due piramidi sarà orizzontale. Quando si fissa l'altezza corrispondente a facce che sono triangoli equilateri, la retta non risulta mai orizzontale. La risposta al problema è che l'enunciato di Pacioli può essere vero solo per un opportuno valore dell'altezza dei triangoli isosceli in questione. L'esistenza di tale soluzione è stata ricavata osservando che l'inclinazione della retta (identificata con la tangente dell'angolo) è una funzione continua dell'altezza rappresentata dallo slider (uno slider è la rappresentazione grafica di un numero che può essere variabile) e che tale tangente passa da valori negativi a valori positivi, per cui deve esistere un punto in cui si annulla. L'altezza della piramide pentagonale corrispondente a tale punto dello slider e i corrispondenti triangoli isosceli forniranno dunque la soluzione per cui l'enunciato di Pacioli risulterà corretto.

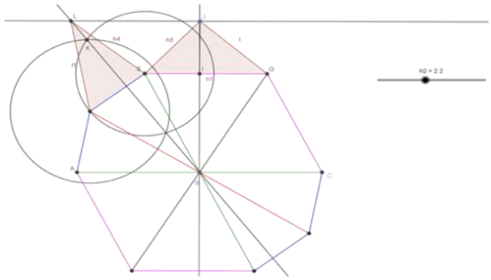


**Figure 3.** Sezione del dodecaedro troncato elevato

## *3 Conclusioni*

Abbiamo descritto una delle attività di problem posing degli studenti del corso universitario (per la laurea magistrale) di Storia, frequentanti anche il corso di Didattica della Matematica (formativo per i futuri docenti di scuola superiore), elaborate nell'anno accademico 2020/21. Tutti i laboratori hanno avuto origine dallo studio e dall'interpretazione di brani originali (in italiano, latino o tedesco) di autori affrontati durante le lezioni e dalla ricostruzione dei concetti matematici e geometrici descritti. La valutazione si è basata su un colloquio orale su tutti gli argomenti oggetto del corso, compresi quelli relativi al progetto, e nella presentazione del percorso di laboratorio. La metodologia adottata ha consentito agli studenti di raggiungere, secondo noi, un doppio obiettivo: un apprendimento più significativo dei concetti storici e geometrici direttamente utilizzati nel laboratorio, rispetto a quelli presentati dai docenti mediante un approccio più tradizionale. Gli studenti hanno inoltre avuto l'opportunità di ampliare ed applicare conoscenze, competenze e abilità riguardanti concetti e uso di strumenti acquisiti in altri corsi universitari, come algebra lineare, calcolo infinitesimale, uso di software geometrico, geometria proiettiva, etc. Soffermandoci sugli scopi che c'eravamo prefissi, gli studenti hanno scoperto e potuto direttamente lavorare con una nuova metodologia di insegnamento. Essi hanno potuto sperimentare uno dei possibili utilizzi della storia della matematica nell'insegnamento della disciplina, usando testi originali e producendo attività interdisciplinari. Dalle “Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento” per i percorsi liceali (Ministero dell’istruzione, università e ricerca 2010), si legge che è indispensabile saper “collocare il pensiero scientifico, la storia delle sue scoperte e lo sviluppo delle invenzioni tecnologiche nell’ambito più vasto della storia delle idee” (p. 7). Riguardo poi agli obiettivi per la Lingua italiana, leggiamo che per poterli raggiungere il “percorso utilizzerà le opportunità offerte da tutte le discipline con i loro specifici linguaggi per facilitare l’arricchimento del lessico e sviluppare le capacità di interazione con diversi tipi di testo, compreso quello scientifico: la trasversalità dell’insegnamento della Lingua italiana impone che la collaborazione con le altre discipline sia effettiva e programmata” (p. 257). Nel documento sulle Raccomandazioni del Consiglio dell’Unione europea del 22 maggio 2018 relative alle competenze chiave per l’apprendimento permanente (Gazzetta ufficiale dell'Unione europea del 4 giugno), si sollecita a “promuovere l’acquisizione di competenze in scienza, tecnologia, ingegneria e matematica, tenendo conto dei collegamenti con le arti, la creatività e l’innovazione” (p. 4). Leggiamo poi che le “metodologie di apprendimento quali l’apprendimento basato sull’indagine e sui progetti, misto, basato sulle arti e sui giochi, possono accrescere la motivazione e l’impegno ad apprendere. Analogamente, metodi di apprendimento sperimentali, l’apprendimento basato sul lavoro e su metodi scientifici in scienza, tecnologia, ingegneria e matematica possono promuovere lo sviluppo di varie competenze” (p. 12). Infine, gli studenti hanno avuto modo di riflettere su alcuni ostacoli epistemologici incontrati storicamente nella sistemazione matematica delle definizioni, proprietà, teoremi riguardanti argomenti connessi ai poligoni e poliedri stellati. Ci sembra che i risultati siano stati interessanti didatticamente, sia per l’apprendimento significativo di concetti da un punto di vista storico sia per l’attivazione di abilità e competenze in altri ambiti, come quello geometrico, analitico ed informatico. I risultati sono stati interessanti nell’ottica della formazione di docenti per la scuola secondaria: riteniamo infatti che un’attività del genere sia da ritenersi un buon esempio di utilizzo della storia della matematica in didattica.

## REFERENCES

Barbaro, D. (1569). *La pratica della perspettiva*. Borgominieri, Venezia.

Bonotto, C., Dal Santo L. (2015). On the relationship between problem posing, problem solving and creativity in primary school. Singer, F.M., Ellerton, N., Cai, J. (Eds) *Mathematical Problem Posing*. New York: Springer.

Bradwardine, T. (1989). *Geometria speculativa*. Steiner Verlag Wiesbaden.

Brigaglia, A., Palladino, N., Vaccaro, M.A. (2018). Historical notes on star geometries in mathematics, art and nature. Michele, E., Marco, A. (Eds.). *Imagine Math 6 Between Culture and Mathematics*. Springer International Publishing, pp. 197-211.

Davis, M.D. (1977). *Piero della Francesca's mathematical treatises*. Longo, Ravenna.

Dürer, A. (1525). *Underweysung der Messung, mit dem Zirckel und Richtscheyt, in Linien, Ebenen unnd gantzen corporen*. Nuremberg.

Günther, S. (1873). Lo sviluppo storico della teoria dei poligoni stellati nell'antichità e nel medioevo. *Bullettino di Bibliografia e di Storia*, Tomo VI, Roma, Tipografia delle Scienze matematiche e fisiche, pp. 313-340.

Huylebrouck, D. (2013). *De Divino Errore.* www.researchgate.net/publication/260692086_De_Divino_Errore

Kepler, J. (1619). *Harmonices Mundi*. Frankfurt.

Krutetskii, V. A. (1976). The psychology of mathematics abilities in schoolchildren. Chicago: University of Chicago Press.

Pacioli, L. (1509). *Divina proporzione*. Venezia.

Radford, L. (1997). On Psychology, Historical Epistemology and the Teaching of Mathematics. *For the Learning of Mathematics*, 17(1), 26-33.

Skemp, R. (1969). *The psychology of learning mathematics*. Harmondsworth. UK: Penguin.

Stevin, S. (1583). *Problemata Geometrica Libri 5*. Anversa.

Stoyanova, E., Ellerton, N.F. (1996). A framework for research into students' problem posing in school mathematics. Clarkson P. C. (Ed), *Technology in mathematics education* (pp. 518-525), MERGA: The University of Melbourn

# MATHEMATICS, ELOQUENCE AND POLITICS: the deductive hypothetical model in the political discourses that changed the course of history over the last two centuries

**Francesco Saverio TORTORIELLO, Ilaria VERONESI**
University of Salerno, via Giovanni Paolo II n. 132, FISCIANO (SA) ITALY
fstortoriello@unisa.it, iveronesi@unisa.it

## ABSTRACT

In the present paper we illustrate an interdisciplinary didactic path of mathematics and history developed in the Mathematical High School Project (MHS). The MHS is an experimental research project developed by the research group in didactics of the Department of Mathematics of the University of Salerno (Italy). It aims to build a network of knowledge that is strictly interconnected and goes beyond the individual disciplinary themes; mathematics is "the" substratum and "the" language that creates links between the two cultures, the humanistic and the scientific ones. We analyzed the speeches of various undisputed protagonists of the last two centuries, such as Jefferson, Lincoln, Kennedy, Martin Luther King, through the lens of hypothetical-deductive logical reasoning, also making references to other areas (literary, political...) with constant links to Euclidean geometry.

## *1 Introduction*

In the National Guidelines issued by the Ministry of Education[65], i.e. the guidelines for the development of educational paths with reference to the learning outcomes of the scientific high schools, we read "The path of the scientific high school aims at studying the connection between scientific culture and the humanistic tradition. It favors the acquisition of knowledge and methods of mathematics (...). It guides the student to deepen and develop the knowledge and skills necessary to (...) identify the interactions between the different forms of knowledge, ensuring the mastery of the languages, techniques and related methodologies, also through laboratory practice ". Among other objectives, it is emphasized that students need to grasp the relationships

[65] https://www.gazzettaufficiale.it/eli/id/2010/06/15/010G0111/sg, (D.P.R. 15-03-2010, n. 89, Art. 8 comma 1),

between scientific thought and philosophical reflection and have to understand the supporting structures of the argumentative and demonstrative procedures of mathematics, also through the mastery of the logical-formal language and, specifically, they have to use them in identifying and solving problems of various typologies. In this perspective, mathematics becomes the language for interpreting reality and the tool for building and strengthening critical thinking. The Mathematical High School Research Project adopts the objectives indicated by the Ministry of Education and develops thematic paths and laboratory activities that are developed in an interdisciplinary key and have mathematics as a cognitive substrate and as a universal model and language.

## 2 *Methodologies*

The development of educational activities in a laboratory setting in MHS referrers to Vygotsky's constructivist approach to teaching and learning (Vygotsky, 1962). This approach prioritizes the acquisition of new information through experiences of interaction among students, teachers and researchers. Through this teaching-learning activities, teachers play the crucial role of semiotic mediators of knowledge, guiding students towards a deeper understanding of the subject matter.

In order to facilitate this process, educators must follow a research-action path that encourages learning through hands-on experiences, collaboration, and reflection and the suggestion of stimuli that capture students' passion. In the activities of this project the students worked in small groups in a peer collaboration in an action-research strategy and explored the texts of numerous discourses, recognizing the essential scaffolding of the hypothetical deductive approach.

## 3 *The laboratory*

In the course of "mathematics and history" for fourth grade students of the MHS we decided to deepen a less evident aspect of the use of the mathematical method, not aimed at implementing the knowledge of technologies but as a theoretical model of the development of critical thinking. The methodological choice adopted in this laboratory is consistent with the approach adopted in the various paths of the MHS, see for example (Bimonte et al., 2021), (Bimonte et al., 2022), (Alfano et al., 2021), (Tortoriello & Veronesi, 2021). The students were divided into groups and the activities were coordinated by the teachers and researchers who proposed them the speeches of various prominent historical figures, undisputed protagonists of the last two centuries. The articles by Bischi (2013) and Hirsch D., Van Haften (2010) had already analysed Lincoln speeches, from a logical-mathematical point of view; the original contribution by the authors of the present paper is to have created an educational laboratory starting from this idea and expanding it to search for the same protocol in other authors.

The students were provided with the texts of the speeches of Thomas Jefferson (1743-1826), Abraham Lincoln (1809,1865), John F. Kennedy (1917-1963), Martin Luther King (1929-1968), both in English and Italian, and they were asked to analyze them in order to recognize in the model of oratory chosen by the authors the same axiomatic protocol expounded by Euclid in the Elements for the demonstrations of geometry and mathematics i.e. the hypothetical-deductive logical reasoning, also making references to other areas (literary, political...) with constant links to Euclidean geometry.

We chose speeches that characterized and profoundly marked the historical-political path of each author and students had to divide them into the fundamental modules of the demonstration of Euclidean geometry in its completest form, that is composed by six parts: the enunciation, the setting out, specification, construction, proof, conclusion (Heath, T., 1921, pp. 370-373), (Bischi, 2013, p. 34). This form is richer than the most frequent one that is found in many geometrical demonstrations of theorems, composed by hypothesis, thesis and demonstration where some of the parts are omitted. And this greater articulation of the hypothetical deductive process fits well to applications in political discourses in which argumentation is essential for the affirmation of messages.

In the analysis of the texts, other demonstrative techniques also emerge, such as the "reductio ad absurdum", or the use of previously carried out demonstrations as well as rhetorical figures and references to mathematics and logic.

Among other objectives of the laboratory activities, students had to grasp the relationships between scientific thought and philosophical reflection and had to understand the supporting structures of the argumentative and demonstrative procedures of mathematics, also through the mastery of the logical-formal language and, specifically, had to use them in identifying and solving problems of various kinds. In this perspective, mathematics becomes the language for interpreting reality and the tool for building and strengthening critical thinking.

## *4 Lincoln's Mathematical speeches – some examples*

To provide a clear representation of the parallelism between Euclidean demonstration and Lincolnian rhetoric, we show some of the passages that were given to students and were the subject of text analysis.

### 4.1 An "if-then-else" speech

For example, in the following discourse Fragment on slavery of July 1854 (https://papersofabrahamlincoln.org/documents/D200785), the assumption of the thesis is reached through a demonstrative construction with a hypothetical deductive approach typical of geometric demonstrations: *"If A can prove, however conclusively, that he may, of right, enslave B. why may not B snatch the same argument, and prove equally, that he may enslave A? You say A is white, and B is black. It is color, then; the lighter, having the right to enslave the darker? Take care. By this rule, you are to be slave to the first man you meet, with a fairer skin than your own. You do not mean color exactly? You mean the whites are intellectually the superiors of the blacks, and, therefore have the right to enslave them? Take care again. By this rule, you are to be slave to the first man you meet, with an intellect superior to your own. But, say you, it is a question of interest; and, if you can make it your interest, you have the right to enslave another. Very well. And if he can make it his interest, he has the right to enslave you. "*

### 4.2 A "reductio ad absurdum" proof

Another example is found in the 1859 speech in Columbus, Ohio in which Lincoln uses the proof for absurd (Hirsch D., Van Haften D., 2010, p.518): *"It is as impudent and absurd as if a prosecuting attorney should stand up before a jury, and ask them to convict A as the murderer of B, while B was walking alive before them."*

### 4.3 A Euclidean proof

The very short Gettysburg speech of 19 November 1863 (Hirsch D., Van Haften D., 2010, p. 396) that sanctioned the success of the Union in the Civil War, has been defined as "the greatest political speech in history". In only 272 words Lincoln shifts the focus from war to freedom, to an egalitarian social model with a masterful use of Euclidean demonstration. This discourse is built in a systematic and rigorous way on the model of Euclidean proof and the six steps can be observed.

*"Four score and seven years ago our fathers brought forth on this continent, a new nation, conceived in Liberty, and dedicated to the proposition that all men are created equal. Now we are engaged in a great civil war, testing whether that nation, or any nation so conceived and so dedicated, can long endure.* (enunciation)

*We are met on a great battle-field of that war.* (setting out)

*We have come to dedicate a portion of that field, as a final resting place for those who here gave their lives that that nation might live. It is altogether fitting and proper that we should do this. But, in a larger sense, we can not dedicatee — we can not consecrate — we can not hallow — this ground.* (specification)

*The brave men, living and dead, who struggled here, have consecrated it, far above our poor power to add or detract. The world will little note, nor long remember what we say here, but it can never forget what they did here.* (construction)

*It is for us the living, rather, to be dedicated here to the unfinished work which they who fought here have thus far so nobly advanced. It is rather for us to be here dedicated to the great task remaining before us—that from these honored dead we take increased devotion to that cause for which they gave the last full measure of devotion—that we here highly resolve that these dead shall not have died in vain—* (proof)

*that this nation, under God, shall have a new birth of freedom—and that government of the people, by the people, for the people, shall not perish from the earth.* (conclusion)".

## 5 Conclusions

The research path that analyzes with the typical codes of mathematics the speeches of historical protagonists of various historical periods, allows to deepen formal models, that are characteristics for the study of mathematics, with a transdisciplinary vision in which the laws and rules that describe a phenomenon are first recognized heuristically and subsequently demonstrated with the rigor that is proper to the discipline. The request for formalism, decontextualized, allows in the "ars oratoria" to reach assertions that are convincing because they are obtained through hypothetical-deductive reasoning. The exploration in search of other speeches that have the same structure as those of Lincoln allowed us to recognize, in the plots of American history and the defense of civil rights, other speakers closely connected to each other, even if distant in time.

President Jefferson's inauguration speech, for example, is considered one of the reference texts of the American libertarian tradition. Jefferson was Lincoln's predecessor and biographers say that when he retired from politics, he left his copy of Euclid's Elements in the Library of Congress and it was in those rooms that Lincoln approached the study of geometry (Hirsch D., Van Haften D., 2010, p.306).

Another example is President Kennedy's speech in West Berlin on June 26, 1968, "Ich bin ein Berliner," in which the same concepts of "future," "peace," and "freedom" as the Gettysburg Address are invoked.

Martin Luther King Jr.'s speech on August 28, 1963, "I Have a Dream", in which he expressed the hope that one day the African-American population would enjoy the same rights as whites. This speech, which has become a symbol of the fight against racism, was given in front of the Lincoln Memorial in Washington, like a symbolic circle that closes itself.

The speeches of historical figures from different eras and contexts allowed students to verify that what they had explored about Lincoln's oratory art is not a sporadic case, the result of the author's stylistic choices, but is the affirmation that the methods of mathematics in their indisputable demonstrative power, are therefore assertive and effective for consistently carrying out their ideas and positions.

The interdisciplinary path of geometry, history and literature intrigued students from the beginning. They felt involved in an exploratory activity through the reading of historical texts that have characterized topical moments at the international political and institutional level. They discovered the fascination of rereading history from the point of view of the protagonists of the events, understanding the strategic importance that a speech may have for the evolution of a wider context. The laboratory activity has therefore allowed the students to better and actively understand the importance of communication for the conveyance of contents.

## REFERENCES


Alfano, E. A., Amirante, A., Veronesi, I. (2021). Matematica, eloquenza e politica: il modello ipotetico deduttivo di Euclide nei discorsi di Abraham Lincoln. *G.R.I.M. "Quaderni di Ricerca in Didattica" N.4*

Bernardini, C., De Mauro T. (2005). *Contare e raccontare. Dialogo sulle due culture*. Bari, Laterza.

Bimonte G., Tortoriello F.S., Veronesi I., (2021) "Economics and mathematics: a transdisciplinary path using geodynamical models and computational software", pagg. 9128-9131, Proceedings of INTED2021 Conference 8th-9th March 2021, ISBN: 978-84-09-27666-0

Bimonte G., Tortoriello S.F., Veronesi I., (2022) "Game Theory Lab: A Gamification Laboratory for High School Students", Handbook of Research on International Approaches and Practices for Gamifying Mathematics, DOI: 10.4018/978-1-7998-9660-9

Bischi I., (2013), L'oratoria matematica del Presidente Lincoln, *Alice & Bob* n. 36-37.

De Bono E. (2000), Il pensiero laterale, *Bur edizioni*.

Hirsch D., Van Haften D. (2010), Abraham Lincoln and the structure of reason.

Heath, T., (1921) A history of Greek mathematics. Vol. 1: From Thales to Euclid. Vol. 2: From Aristarchus to Diophantus. Oxford: Clarendon Press.

Maroscia P. (2010), Matematica e racconto, *La Matematica nella Società e nella Cultura, Rivista dell'Unione Matematica Italiana,* Serie 1, Vol. 3, p. 375–397.

Rogora E., Tortoriello F.S. (2018), Matematica e cultura umanistica. *Archimede* vol. 2, p. 82-88, Firenze

Rogora, E., Tortoriello, F.S. (2021). Interdisciplinarity for learning/teaching mathematics, Bolema: Boletim de Educacao Matematica

Tortoriello, F.S., Veronesi, I. (2021) Historical path in mathematical games: a didactic laboratory activity dedicated to upper secondary school students, *Proceedings The 25th World Multi-Conference on Systemics, Cybernetics and Informatics: WMSCI.*

Vygotsky, L. S. (1962), Language and thought, Ontario, *Massachusetts Institute of Technology Press.*

https://papersofabrahamlincoln.org/documents/

https://www.gazzettaufficiale.it/eli/id/2010/06/15/010G0111/sg

# SPATIAL THINKING IN ANCIENT GEOMETRY. THE CASE OF SUBCONTRARY SECTION

**Luis Carlos VARGAS-ZAMBRANO, Gisela MONTIEL-ESPINOSA**
Center for Research and Advanced Studies of the National Polytechnic Institute, Mexico City, Mexico
luis.vargas@cinvestav.mx, gmontiele@cinvestav.mx

## ABSTRACT

Conics are not the exception into an ever-growing list of scientific findings with spatial thinking influence. Under this scenario, a proposition of Apollonius of Perga Conics treatise was analyzed for identifying and extracting processes of reasoning related to the space as a frame of reference, and representations of geometric objects; thus, explaining what the spatial thinking role in the construction of conic sections was.

## *1. Spatial thinking in the history of science and mathematics*

The knowledge construction has been in a closed relation with spatial thinking because the scientific thinking nature is spatial, even non-spatial knowledge is communicated through diagrams, maps, and schemas (Newcombe, 2016). The history of science is full of examples where spatial thinking is the protagonist: the double helix of DNA was developed as a three-dimensional spatial model; the periodic table organizes into columns and rows the relations among elements (National Research Council, 2006; Newcombe, 2010; 2013); Dandelin spheres are a spatial mechanism to find foci and directrix of conic sections into the cone (Salinas and Pulido, 2017). Also, different disciplines, such as geoscience, engineering, and neuroscience have implemented spatial thinking to visualize the processes that affect the earth's formation, anticipate how a set of forces may affect the design of a structure, and visualize specific parts of the brain for surgical procedures, respectively (Newcome, 2010).

Other discoveries in history such as the atom, cell, the solar system, and the universe have different spatial representations which come from micro-space and macro-space, useful in the teaching and learning process, moreover, mathematics concepts are not the exception. "Geometry […] in particular is a mathematical area concerned with the space around us, with the shapes in the space, their properties, and different 'patterns' and 'thinking patterns' for which they serve as trigger and basis" (Hershkowitz, 2020, p. 774); therefore, the historical and epistemological construction of geometric concepts would have an implicit spatial component susceptible of being investigated.

### 1.1 Conics are not the exception, are an opportunity

We focused on conics because these notions are perennial in mathematics teaching as a result of their algebraic and geometric approaches, relations between plane and space geometry (Barbin, 2008), and different contexts related to the art, mathematics, astronomy, and architecture (Mancini and Menghini, 1984; Berger, 2010). However, this variety of contexts has not been considered totally in the mathematics curriculum and the geometric approach has disappeared or has been reduced as an illustrative tool, furthermore, spatial skills do not take a relevant place when these are fundamental in conics learning (Vargas-Zambrano and Montiel-Espinosa, 2020); although there is not an important open problem concerning the conics, there are still few circles remain in the curriculum, the other conic sections are gone, nevertheless, conics are an integral part of our lives (Berger, 2010).

Basically, there are at least three ways to think about the conics: via the cone, via quadratic equations, and via transformations; the first one keeps a spatial nature in conics' genesis which has mixed with different meanings throughout history (Bartolini Bussi, 2005). Although circles, parabolas, and sometimes ellipses appear in school as cuts of the cone, it seems that finding the foci and directrix of the conics as of the equation and vice-versa is enough in the teaching and learning processes; even the procedure for cutting a cone introduces the analytic treatment without any apparent relation between them (Salinas and Pulido, 2017), because foci and directrix are the shared geometric elements among the conics but irrelevant and skipped in their construction as cuts of the cone.

## *2. Research issue*

Our research approach to the History of Mathematics and Education is epistemological: we recognize that mathematics are human activity product; history enriches the knowledge we impart in classrooms; interesting problems and original meanings have disappeared (Buendía and Montiel, 2011), and even ways of thinking too. For example, our literature review (see Vargas-Zambrano and Montiel-Espinosa, 2020) showed the procedure for cutting a cone has a high component of spatial thinking which has not taken advantage in the conics' teaching, the genesis of conics more exactly in Ancient Greece, considers essential spatial thinking to construct, communicate and understand these notions, consequently we pose two questions: what *processes of reasoning* allow the construction of conics as cuts of the cone? And what is the *spatial thinking* role? So, we will answer and discuss the circle as a conic section due to the classical discussion about the circle as *loci* or *solid loci* in Ancient Greece, specifically in *Conics* written by Apollonius of Perga.

## *3. Conceptual and methodological framework*

"Spatial thinking concerns the locations of objects, their shapes, their relations to each other, and the paths they take as they move" (Newcombe, 2013, p. 28). Kind of thinking is based on a constructive amalgam of three concepts: *space, representation*, and *processes of reasoning* (National Research Council, 2006). Firstly, and in our case, *space* will be *Euclidean space;* it refers to a container made up of a network of positions where the geometric objects are located when they are mobile or stationary (Clements and Battista, 1992). Secondly, the *representation* corresponds to the set of *primitives* or geometrical objects (point, line, plane, 2D-figure, surface, 3D-figure) and their properties. Thirdly, *processes of reasoning* are the set of *dynamic relations, static relations*, and *transformations* between *primitives* (National Research Council, 2006). These conceptual elements can be visible when rationality is contextualized, owing to reasoning coming from human activity (Cantoral, 2020). Mathematical activity such as human activity organizes *actions:* direct interactions of the subject (individual, collective or historical) over the object, in a specific environment (Torres-Corrales and Montiel, 2019). In our case, direct interactions into *Euclidean space* between subject and geometric objects for social knowledge construction.

## *4. Results*

The unit of analysis for this paper is proposition 5, book I of *Apollonius of Perga Conics.* We adapted from Torres-Corrales and Montiel (2019) and Cantoral (2020) the methodological questions "what does the subject do?" and "how does she/he do it?" for identifying *actions* and mathematical activities as will be seen below; therefore we recognize that propositions text from

*Apollonius of Perga Conics* —at least the first fourteen— are a clear description of an implicit diagram, which were developed through historical direct interactions between subjects and concrete material.

### 4.1. What did Apollonius do?

According to the proposition statement, Apollonius pretended to prove that section GHK is a circle; basically, circle GHK comes from a perpendicular section to the plane ABC (see figure 1, a). Also, Apollonius (ca. 200 B.C.E./2013) specified three necessary geometric objects —for his proof— that act as a set of *primitives* because these were defined or constructed in past definitions and propositions: oblique cone with vertex A, plane GHK and axial triangle ABC (Apol. I. 3). "The set of primitives is a way of capturing our encounters with a world full of objects (occurrences of phenomena): objects are the things that we are trying to understand" (Golledge, 1995; 2002 as cited in National Research Council, 2006, p. 36). Oblique cone appears in definition 3 of book 1, it is constructed by three *primitives:* points A and B; straight line AB and curved line BLC, these are related through a *dynamic relation (rotating)* because B moves in the circumference. On proposition 5 of book I, oblique cone be converted into a *primitive*, due to these geometrical objects, points, straight line and curved line keep intrinsic *static relations* like size, location, and orientation, because they are subparts with relations among them into a new object (Sinclair, Cirillo, and de Villiers, 2017; Newcombe, 2016).

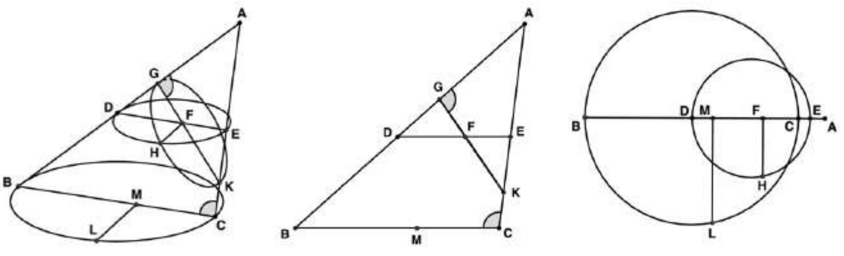


**Figure 3.** a) panoramic view, b) side view, and c) top view of proposition 5 of book I.

### 4.2. How did Apollonius do it?

Exposition of the proposition details the *primitives* and their properties. Into *Euclidean space* Apollonius got the triangle ABC, circle DHE, and circle GHK as sections, combining two *processes of reasoning: direction of move-*

*ment* and *cross-sectioning*. Apollonius cut an oblique cone with a plane respectively: through the axis and perpendicular to the circle BC; parallel to the circle BC; and subcontrariwise to the circle BC by side A of the axial triangle. Both *processes of reasoning* are *dynamic relations* between entities, their features are evaluated with respect to other entities or a frame of reference (National Research Council, 2006). Although *cross-sectioning* is a relation intrinsic and dynamic only (Sinclair et al., 2017; Newcombe, 2016), it depends on the orientation of the cutting plane and the geometrical structure (Cohen and Hegarty, 2012). For instance, when Apollonius generalized the procedure for cutting a cone, he used two geometrical structures: the right cone and the oblique cone, each one is cut by two orthogonal cutting planes to get the axial triangle and the circle, and one oblique cutting plane to get the conic section, however, the *orientation* as *static relation* is not enough, because in addition *direction of movement* underlies the geometric reasoning, planes that cut the oblique cone satisfy a condition with respect to other entities like the base of the cone or the axial triangle.

Exposition continues and Apollonius claimed: "[...] the triangle AKG similar to the triangle ABC and lying subcontrariwise, that is, so that the angle AKG is equal to the angle ABC. And let it make as a section on the surface, the line GHK" (Apollonius, ca. 200 B.C.E./2013, p. 9). After *cross-sectioning,* Apollonius recognized similar triangles on the plane ABC or axial triangle; his affirmation involves two *processes of reasoning: changing perspective* and *comparing shape and length* (figure 1, b). When Apollonius uses plane geometry in his proof, the point of view changes from a panoramic view (oblique cone) to a side view (axial triangle); this spatial *transformation* is elemental to scientific reasoning for comprehending and testing ideas (figure 1, a and b) (National Research Council, 2006). Consequently, Apollonius developed his ideas into the 2D-space, on circles BLC and DHE (Apol. I. 4) and axial triangle (figure 1, b and c). To illustrate: Apollonius (ca. 200 B.C.E./2013) specified that GHK is a circle again. Thus, in the plane DHE, he starts *comparing shapes,* $\text{rectangle}\,\mathrm{DF},\mathrm{FE} = \text{square}\,\mathrm{FH}$. This equality of areas depends on DE segment, which comes from *cross-sectioning,* exactly a common section between the axial triangle ABC and the circle DHE (Eucl. XI. Def. 4). Segment FH is not a common section, but it is parallel to LM (Eucl. XI. 6), then FH and LM are lines with *location* and *direction of movement;* furthermore, DHE is a right triangle, therefore $\mathrm{DF} : \mathrm{FH} :: \mathrm{FH} : \mathrm{FE}$

(Eucl. III. 31, VI. 8). If DF, FH and FE are proportional, then the rectangle contained by the extremes DF and FE equals the square on the mean FH (Eucl. VI. 17). Hence, mean proportional is linked to a *process of reasoning* into *spatial thinking: comparing length.*

The end of exposition focused on the side view and *comparing size,* angles AKG and ABC are equal, therefore AKG and ADE too. And the opposite angles at the point F in the plane ABC are also equal. Hence, *comparing shape and length,* triangles DFG and KFE are similar (figure 1, b). If EF, FK, GF and FD are proportional (Eucl. VI. 4), then the rectangle contained by the extremes EF, FD equals the rectangle contained by the means FK and GF (Eucl. VI. 16).

### 4.3. What was Apollonius doing this for?

Apollonius (ca. 200 B.C.E./2013) proved that there is another procedure for cutting an oblique cone and getting a circle. From the orthogonal cutting plane, he found the circle DHE and argued that $\text{square FH} = \text{rectangle EF, FD}$; therefore, from the oblique cutting is getting a circle because $\text{rectangle KF, FG} = \text{square FH}$.

## *5. Conclusions*

We identified two *spatial thinking roles:* epistemological and communicative. The first of them refers to explicit *processes of reasoning* into *Euclidean space* in direct relation with the geometric objects' construction such as *rotating* for oblique cone*; direction of movement* and *cross-sectioning* for conic sections; and *comparing shape and length* for mean proportional. The second of them refers to implicit *processes of reasoning* into the *Euclidean space* which helps to understand and follow geometric reasonings, such as *changing perspective, location* and *orientation.*

**REFERENCES**

Apollonius. (2013). *Conics: Books I-IV* (R. C. Taliaferro & M. N. Fried, Trans.). Santa Fe, United States: Green Lion Press. (Original work published ca. 200 B.C.E.)

Barbin, É. (2008). Perennial notions and their teaching. In E. Barbin, N. Stehliková & C. Tzanakis (Eds.), *History and Epistemology in Mathematics Education: Proceedings of* *the fifth European Summer University* (pp. 157-161). Pilzen: Vydavatelský servis.

Bartolini Bussi, M. (2005). The meaning of conics: historical and didactical dimensions. In J. Kilpatrick, C. Hoyles, O. Skovsmose & P. Valero. (Eds.), *Meaning in Mathematics Education* (pp. 39-60). New York, United States: Springer. https://doi.org/10.1007/0-387-24040-3_4

Berger, M. (2010). Conics and quadrics. In M. Berger, (Ed.), *Geometry Revealed a Jacob's Ladder to Modern Higher Geometry* (pp. 181-247). New York: Springer. https://doi.org/10.1007/978-3-540-70997-8_4

Buendía, G., & Montiel, G. (2011). From History to Research in Mathematics Education: Socio-Epistemological elements for Trigonometric Functions. In V. Katz, & C. Tzanakis (Eds.), *Recent Developments on Introducing a Historical Dimension in Mathematics Education* (pp. 67-82). Washington, DC: The Mathematical Association of America. https://doi.org/10.5948/UPO9781614443001.008

Cantoral, R. (2020). Socioepistemology in Mathematics Education. In S. Lerman. (Ed), *Encyclopedia of Mathematics Education* (pp. 790-797). Springer, Cham. https://doi.org/10.1007/978-3-030-15789-0_100041

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). New York: National Council of Teachers of Mathematics, Macmillan Publishing Co.

Cohen, C., & Hegarty, M. (2012). Inferring cross sections of 3D objects: A new spatial thinking test. *Learning and Individual Differences,* 22(6), 868-874. https://doi.org/10.1016/j.lindif.2012.05.007

Hershkowitz, R. (2020). *Shape and Space: Geometry Teaching and Learning.* In S. Lerman. (Ed), *Encyclopedia of Mathematics Education* (pp. 774-779). Springer, Cham. https://doi.org/10.1007/978-3-030-15789-0_138

Mancini, L., & Menghini, M. (1984). Conic sections in the sky and on the earth. *Educational Studies in Mathematics,* 15, 191-210. https://doi.org/10.1007/BF00305896

National Research Council. (2006). *Learning to Think Spatially.* Washington, DC: The National Academies Press. https://doi.org/10.17226/11019

Newcombe, N. (2010). Picture this: Increasing math and science learning by improving spatial thinking. *American Educator,* 34(2), 29-35.

Newcombe, N. (2013). Seeing relationships: Using spatial thinking to teach science, mathematics, and social studies. *American Educator,* 37(1), 26-31.

Newcombe, N. (2016). Thinking spatially in the science classroom. *Current Opinion in Behavioral Sciences,* 10, 1-6. https://doi.org/10.1016/j.cobeha.2016.04.010

Salinas, P., & Pulido, R. (2017). Understanding the Conics through Augmented Reality. *EURASIA Journal of Mathematics Science and Technology Education,* 13(2), 341-354. https://doi.org/10.12973/eurasia.2017.00620a

Sinclair, N., Cirillo, M., & de Villiers, M. (2017). The learning and teaching of Geometry. In J. Cai (Ed.), *Compendium for Research in Mathematics Education,* 457-489.

Torres-Corrales, D., & Montiel, G. (2019). Characterization of uses of trigonometric notions in Mechatronics Engineering from Mathematics Education. *ECORFAN Journal Spain,* 6(10), 9-21. http://doi:10.35429/EJS.2019.10.6.9.21

Vargas-Zambrano, L. C., & Montiel-Espinosa, G. (2020). A research approach on the role of space in the construction of conic sections. In A. I. Sacristán, J. C. Cortés-Zavala & P. M. Ruiz-Arias (Eds.), *Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 737-738). Mexico: PME-NA. https://doi.org/10.51272/pmena.42.2020-111

# FIRST SIGNS OF EXTENSION FROM THE PLANE TO THE SPACE: CONTRIBUTIONS BY JOHANN HUDDE AND PHILIPPE DE LA HIRE TO 3-DIMENSIONAL ANALYTICAL GEOMETRY

**Pedro L. GARCÍA [a], María T. GONZÁLEZ [b]**
[a]Pontifical University of Salamanca, C. Henry Collet, 90, 37007, Salamanca, Spain plgarciali@upsa.es

[b]University of Salamanca, Paseo de Canalejas, 169, 37008, Salamanca, Spain maite@usal.es

## ABSTRACT

The goal of this paper is to present a historical approach to the first indications of several variables' functions space graphical representations from the generalization of plane representations, which may be key to facilitate the teaching of this type of curves. The two representations found and mentioned in this paper were developed by Hudde and La Hire. We show the way they introduced this representation as a transition from a geometrical to an algebraical interpretacion.

## 1 *By way of introduction*

The study of curves in the field of geometry is one of the axes of the current development of the concept of the function of several variables. It is for this reason that the present study presents the first efforts made by mathematicians to establish graphical representations of a type of curve in space with a certain type of equation. It is worth noting that this initial connection between geometry and algebra coincides with the birth period of an area of mathematics known as Analytic Geometry.

Descartes (1596-1650) points out that his method for dealing with curves associated with an algebraic equation could be extended from the plane to space although according to Anfossi (2004) "he mentioned three-dimensional geometry but wrote nothing about it" (p. 25). It is important to note that the work of Fermat and Descartes during the seventeenth century caused a real revolution in the field of geometry but this new discipline, as they conceived it, was from a didactic point of view not very effective and difficult to understand.

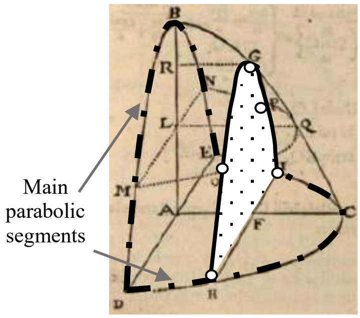
## 2 *First representation as plane sections of a surface*

Hudde (1628-1704), presented a foretaste of the use of spatial coordinates in his work on plane sections of a surface. Although he did not develop his own notation for 3 dimensions, he emphasized the novelty of manipulating curves of degree greater than two as plane sections of a surface (Boyer, 1967). Hudde manages to generate a network of curves in space that were expressed by means of analytical equations of degree greater than 2, taking as bases curves of degree 2. Although the method is not sophisticated, he manages to surprise that they are generated precisely in space and not in the plane as they had been presented since Descartes (figure 1).

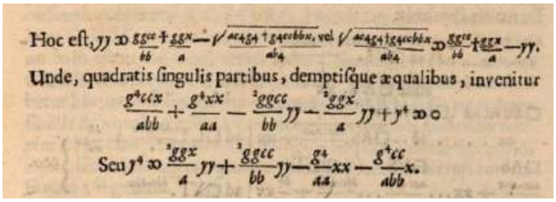
"Exercitationes mathematicae libri quinque". Frans Van Schooten (1657).
Fifth book; thirty miscellaneous sections.

| **Representation of the sections** | **Notation used** |
|---|---|

i)




ii)



Hoc eſt, $yy \propto \frac{ggcc}{bb} + \frac{ggx}{a} - \sqrt{\frac{aag^4 + g^4ccbbx}{ab^4}}$, vel $\sqrt{\frac{aag^4 + g^4ccbbx}{ab^4}} \propto \frac{ggcc}{bb} + \frac{ggx}{a} - yy$.
Unde, quadratis ſingulis partibus, demptiſque æqualibus, invenitur

$$\frac{g^4ccx}{abb} + \frac{g^4xx}{aa} - \frac{2ggcc}{bb}yy - \frac{2ggx}{a}yy + y^4 \propto 0$$

Seu $y^4 \propto \frac{2ggx}{a}yy + \frac{2ggcc}{bb}yy - \frac{g^4}{aa}xx - \frac{g^4cc}{abb}x$.

Equation for the curve defined by the HOGPI points

$$\left\{ y^4 = \frac{2ggx}{a}yy + \frac{2ggcc}{bb}yy - \frac{g^4}{aa}xx - \frac{g^4cc}{abb}x. \right.$$

- Note that it expresses the equations of degree 4 that define the curves of the sections of the parabola in space.
- But there is no proper notation for the 3 dimensions of the generated surface.

**Source:** Images reproduced courtesy of the Bavarian National Library. Germany.
i.- In: http://reader.digitale-sammlungen.de/en/fs1/object/display/bsb10525757_00491.html
i.- In: http://reader.digitale-sammlungen.de/en/fs1/object/display/bsb10525757_00493.html

**Figure 1.** Representation of flat sections of a fourth-degree surface proposed by Johann Hudde

As shown in Figure 1, Hudde, starting from a parabolic segment in the plane, constructs a new parabolic segment perpendicular to it. From these principal curves, other segments perpendicular to each other are generated. Forming a surface in space; where each section of the surface is a power curve of fourth, octave, and so on. As can be intuited in the description and the visualization of figure 1 this network of curves generated in the space, Hudde expresses them in analytical equations of degree higher than 2; taking as base curves of degree 2 that he initially defines in the perpendicular planes. It is surprising how he generates these equations in space, and not in the plane as they had been presented.

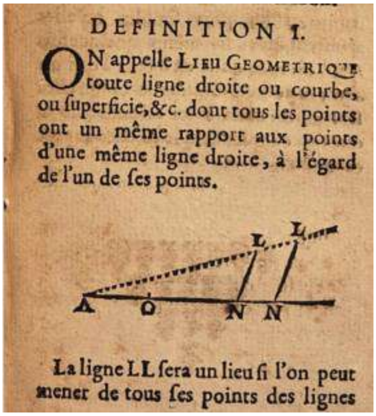
This anticipation of using spatial coordinates presented in a paper on plane sections of a surface in the year 1657, does not generate a specific notation for curves in space but innovates in the method for manipulating curves of degree greater than two, as plane sections of a surface. This fact shows us how complex it can be to approach a new creative experience of mathematical knowledge starting from the basis of a known or existing model.

## 3 *First analytical extension of the plane to space*

La Hire (1640-1719), simultaneously, presented important and significant advances to this new discipline. His work published in 1679 entitled: "Nouveaux Éléments des Sections Coniques: Les Lieux Géométriques: Les Constructions ou Effections des equation" deserves special attention. The author presented the use of Cartesian methods to solve geometric problems and their translation into equations, established conditions for a geometric locus, as well as the equations for the construction of those locus. It is precisely through this exceptional mastery of analytic geometry that he presented the first hint of how this condition can be extrapolated to three dimensions. La Hire defined the equations of the geometric locus for the plane and described, with its respective figure, how it should be approached in the case of a surface.

| *"Chapitre II: De la nature des lieux et de la réduction des équations pour la construction des lieux". Philippe de La Hire (1679).* Second book;<br>The Geometric Places. |
|---|
| **For the two-dimensional case** |

i)


DEFINITION I.

ON appelle Lieu Geometrique toute ligne droite ou courbe, ou ſuperficie,&c. dont tous les points ont un même rapport aux points d'une même ligne droite, à l'égard de l'un de ſes points.

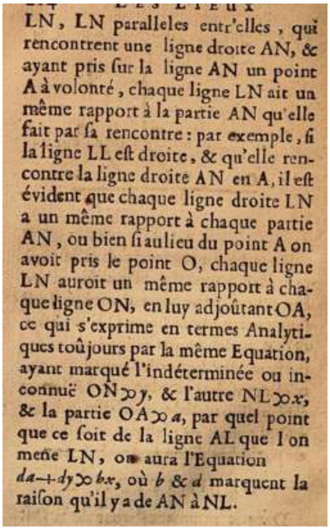


La ligne LL ſera un lieu ſi l'on peut mener de tous ſes points des lignes

*pg.213*

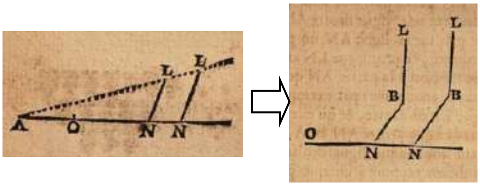
Note that it defines what a locus is and proposes in analytical terms equations that correspond to the type of place.

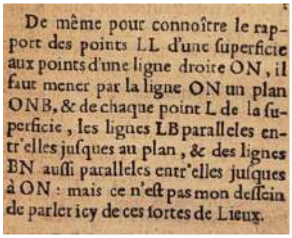
ii)


LN, LN paralleles entr'elles, qui rencontrent une ligne droite AN, & ayant pris ſur la ligne AN un point A à volonté, chaque ligne LN ait un même rapport à la partie AN qu'elle fait par ſa rencontre: par exemple, ſi la ligne LL eſt droite, & qu'elle rencontre la ligne droite AN en A, il eſt évident que chaque ligne droite LN a un même rapport à chaque partie AN, ou bien ſi au lieu du point A on avoit pris le point O, chaque ligne LN auroit un même rapport à chaque ligne ON, en luy adjoûtant OA, ce qui s'exprime en termes Analytiques toûjours par la même Equation, ayant marqué l'indéterminée ou inconnuë ON $\infty$ $y$, & l'autre NL $\infty$ $x$, & la partie OA $\infty$ $a$, par quel point que ce ſoit de la ligne AL que l'on mene LN, on aura l'Equation $da+dy \infty bx$, où $b$ & $d$ marquent la raiſon qu'il y a de AN à NL.

*pg.214*

**Extends the analytic description of a place with three dimensions**





De même pour connoître le rapport des points LL d'une ſuperficie aux points d'une ligne droite ON, il faut mener par la ligne ON un plan ONB, & de chaque point L de la ſuperficie, les lignes LB paralleles entr'elles juſques au plan, & des lignes BN auſſi paralleles entr'elles juſques à ON: mais ce n'eſt pas mon deſſein de parler icy de ces ſortes de Lieux.

- Note that although he extends the analytic description to a locus of a surface in space, he expressly states that it is not his intention to speak openly of such loci.
- Although in an enunciative way it is the first time that it is alluded to directly.

**Source:** Images reproduced courtesy of the Lyon Public Library. France.
i.- In: http://reader.digitale-sammlungen.de/en/fs1/object/display/bsb10525757_00491.html
ii.- In: http://reader.digitale-sammlungen.de/en/fs1/object/display/bsb10525757_00493.html
https://books.google.be/books?id=_13EahsLFVQC&printsec=frontcover&hl=es&source=gbs_ge_summary_r&cad=0#v=onepage&q&f=false

**Figure 2.** Analytical descriptive extension on a 3-dimensional place proposed by Philippe de La Hire

As shown in figure 2; La Hire in chapter 2 of his second book, describes how to extend or pass from a condition of a geometric place of 2 dimensions to 3 dimensions, and although he points out that it is not his intention to speak openly of this type of places, he describes and represents explicitly how to approach a geometric place of a surface in space. Thus showing the first example of how to express a surface analytically through an equation with three unknowns.

## *4 In summary*

These are the first two signs of the transitional stage that occurred at the end of the 17th century between a geometrical interpretation and its corresponding algebraical translation. Mathematicians, although they do not develop their own systematic notation for surfaces, manifested the need to use three-dimensional auxiliary constructions to perform plane demonstrations.

**REFERENCES**


Anfossi, P. (2004). *Geometría Analítica*. (18a. Ed.). México: Editorial Progreso S. A. de C. V.

Boyer, C. B. (1965). Johann Hudde and space coordinates. *The Mathematics Teacher*, 58(1), 33-36.

# THEME 4

# MATHEMATICS AND ITS RELATION TO SCIENCE, TECHNOLOGY, AND THE ARTS: HISTORICAL ISSUES AND SOCIO-CULTURAL ASPECTS IN RELATION TO INTERDISCIPLINARY TEACHING AND LEARNING

# THE 18TH CENT. CONTROVERSY ABOUT THE SHAPE OF THE EARTH: TEXTUAL AND TRANSLATION ISSUES

**Alain BERNARD (1), Catherine DARLEY (2)**
(1) University of Paris Est Créteil, INSPE de Créteil, IREM de Paris Nord and Centre A. Koyré, Paris, alain.bernard@u-pec.fr.
(2) Collège Alviset, IREM de Paris Nord, Paris, catherine.darley@ac-paris.fr

## ABSTRACT

The controversy about the shape of the Earth that occurred in France at the beginning of the18th century is an interesting and thought-provoking case of scientific controversy which might interest students of various ages and school levels. The first part contains a quick reminder about the controversy, its context, the main questions at stake and its pedagogical interest. With these ideas in mind, one of us conceived a pedagogical scenario meant for a historical course at middle school level. The workshop was intended to go through the historical material we used for it, for reasons we explain in the second part. In the course of this research, we find out several interesting texts, for which we needed English translations. Looking for the latter confronted us with two unexpected but interrelated challenges: one related to the quality of the translation, the other to the choice of our texts. We present this material and explain the issue in the next part. We conclude with the new ideas that came out of this discussion.

## *1 The controversy about the Shape of the Earth and its pedagogical interest*

This controversy occurred in France during the first decades of the 18th century and reached its climax in the 1730's. On the scientific side, it was part of a long-lasting scientific debate beginning within the Royal Academy of Science. It originated in two conflicting theories and experimental data. According to Newton's and Huygens's theories, the Earth would be flattened on the poles with an equatorial bulge —that is an oblate spheroid or "melon". This were essentially astronomical theories presented with great mathematical rigor and buttressed by Richer's pendulum's observations (1672). On the other hand, according to the French astronomer Jacques Cassini, also director of Paris Observatory, the Earth would be bulged to the poles and flattened at the level of the equator — a prolate spheroid or "lemon". The Cassini (father and son) thought they have demonstrated this fact after several reliable campaigns of survey and measure of the meridian arc in 1700, 1718, and 173. Their surveying measures were reputed to be made with the greatest rigor and seemed to invalidate Newton's mathematical theory. Furthermore, Jacques Cassini had thought he could justify these results with a cartesian like interpretation recalling the vortex theory, a theory that became mainstream among several prominent academicians and mainly Fontenelle.

On the mundane and "philosophical" side, the debate evolved in the 1730's into a heated dispute opposing famous figures of the enlightenment movement, mainly Voltaire and Maupertuis. Maupertuis' defense of Newton's theory published in 1732 launched the controversy for it directly contradicted the Cassini's measurements and theories. At stake was also the very conception of the universe: Newton's principle of attraction and an empty universe on the one hand, Descartes' mechanistic theory of vortices accounting for a matter-packed universe, on the other. Maupertuis was by then recognized as a brilliant scientist but was also an ambitious intellectual looking for glory and public recognition. In this regard, he succeeded in gaining the support of Voltaire (and conversely) in those years and generally convinced the French public opinion (with its strong tendency to Anglomania), up to the high aristocracy – including the ministers. Partly due to his complex and egocentric personality, Maupertuis did a lot to make this scientific controversy into a public event, discussed much beyond the walls of Académie des Sciences (Voltaire 1738, 1752).

What strongly contributed both to the scientific and mundane debate was the decision, taken by French minister Maurepas (and largely inspired by Maupertuis, to whom he was very close) to finance two expeditions, one to the equator lead by Godin, La Condamine and Bouguer and the other led by Maupertuis himself to the pole in 1737. One of the purposes was to check whether the length of one degree of meridian significantly decreased (in the "lemon" case) or increased (in the "melon" view) when going from the equator to the north.

The northern expedition, of which two detailed accounts were made with all the calculations, one by Maupertuis (1738) and the other by Abbé Outhier (1744), was highly publicized and served as a powerful means to advance Maupertuis's ambitions. But this did not put an end to the scientific controversy. The use of instruments was disputed: for example, the use of an English sector (more precise according to all the members of the expedition) rather than a French instrument (of the type used by the Cassinis) outraged some academicians. Most of all, the incoherency of all measurements made of arcs of meridians in various parts of the world soon became a matter of perplexity.

Moreover, many academicians as well as the general public, were actually unable to follow the complex astronomical calculations and sophisticated theories developed, for example, by Clairaut in his 1743 masterpiece. As a result, the success of the "Newtonian" point of view, buttressed by the successes of the analytical mathematical approach of which Maupertuis, Clairaut, or, later, D'Alembert were representative, took several years to become dominant at the Academy of Sciences, partly for ideological reasons and partly for scientific ones. Ironically, the incoherency of measurements remained a subject of concern for many scholars, like D'Alembert, and it was only at the end of the 18th century that it was discovered that the measurements made in the North were indeed highly problematics and miscalculated owing to a careless use of the sector.

This interesting affair with its many facets is one of the first cases of scientific controversy reaching a relatively large public audience. It soon became part of an ideological fight to defend Newtonian theories against others, as part of the French Enlightenment movement. For the same reason, it constitutes one of the major steps in the penetration of Newtonian ideas in France in this period (Badinter 1999, Schank 2008).

All this make it an excellent subject for both historical and scientific activities for students, leading them to understand the role played by mathematicians in one of the most spectacular changes in modern worldviews, and also developing their critical thinking. But this depends on one crucial condition: for the mere authority of luminaries of the French enlightenment like Voltaire, Maupertuis, or D'Alembert, is not enough to convince anyone of the value of what happened then. Studying this episode requires gaining a minimum understanding of what was at stake, why it was controversial or disputed, and what were the concrete means used to tackle the issue.

This, in turn, brings with it important challenges, given the complexity of the controversy in question. To take a single but significant example, the alternative theories above crucially depend on the understanding that the shape of the Earth can be detected through observations such as (O1) the changing period of pendulums or (O2) the increasing length of one degree of meridian. Both the mode of measuring these phenomena and (C) their exact connection to the shape of the earth are not at all obvious. This subject, on the other hand, could thus be put at the core of an interdisciplinary approach to the question, as we have tried to imagine and implement.

## *2 Getting students to understand the controversy: looking for the "right" textual material*

One of us (Catherine) construed a pedagogical scenario for middle school students in the framework of a history course whose subject was *The Enlightenment and its relation to sciences*. Students were given an overview of the intellectual and technical progresses characteristic of the 18th century, with a focus on the new instruments and calculation techniques that progressively permitted a systematic topographical survey of France. At one point they were given a basic explanation about the nature of the controversy (an adapted version of the explanation given in §1) and documents about the Maupertuis expedition to the North, some maps as well as images of various instruments. The key idea was to submit them the following question: *how did the scholars in the Age of Enlightenment manage to establish with certitude who was right about the form of the Earth, Newton or the Cassinis?*

We do not need to enter here the details of the scenario of the interesting attempts of the students, which are explained elsewhere (Darley 2018). The important point for us is that in order for the scenario to achieve its goals, the contribution of a science or mathematics teacher is essential. We should note, by the way, that this perspective basically amounts to say that we have a case of a history teacher looking for mathematics and a mathematics teacher, not mathematics teachers looking for history and historians, which is the usual premise of ESU encounters.

While the circumstances have not hitherto permitted such cooperation, we decided to prepare for it by looking for historical material that would sustain such a pedagogical construct. The Salerno workshop presented preliminary results of this search, leading us to focus on two series of texts:

1. Excerpts from one of the diaries of Maupertuis's voyage to the North, namely Abbé Outhier's journal (1737). While the French text is easily available and readable to nowadays students, several excerpts of key episodes were chosen from the English version published in (Pinkerton 1808).

2. Excerpts from two important articles written by D'Alembert for the French Encyclopedia edited several decades later: one on the notion of degree

(in any sense: degree or arc, of angle, and of meridian), and the other, D'Alembert's famous article on the shape of the earth.

The first series was meant to help students explore the concrete procedure used to determinate the length of a degree of meridian near the North pole (observation O1) and understand the difficulties implied as well as the human means and intelligence involved. The second series was meant to tackle the difficult connection between the variation of the length of one meridian degree at various latitudes, and the focus problem about the shape of the earth. As in the workshop, we will only pay attention here to the challenges arising from the exploration and use of the first series from Outhier's journal.

## *3 Lost in translation: what the English version of Outhier's travel led us to understand*

### 3.1 Maupertuis' and Outhier's diaries

The two narratives of the expedition to the North, Maupertuis' and Outhier's, share a basic purpose: to represent the reader with the basic details of the expedition, showing all measurements were made with enough care and seriousness so that the result could be trusted. They implied a complex procedure of triangulation between two points having latitudes differing from approximately one degree. The two points in questions are the location of Kittis and Tornea in the two illustrating maps (Fig.1 and 2) taken from Maupertuis (1) and Outhier (2). The two locations differ from nearly 1° of latitude, as shown in Fig 1.

Fig.2 shows in outline what is figured out in Fig.1 with the cartographic details: the latter gives an idea of the difficulty of the operation. The two locations are some 130km apart from each other, and the ten locations marked by letters (K for Kittis, T for Tornea, etc.) correspond to submits that were very difficult to access, especially under very hard climatic conditions. This ten locations approximately form a heptagon KPACT*k*N, which, added to the reference length B*B* measured on Torneas' river during the winter, could enable the scientists to determinate the length of the arc of meridian KM, separating the positions of Kittis and Tornea.

Maupertuis gives a quick account of the travel conditions and mainly insists on the great difficulties of each operation; he gives, above all, many details about the calculations and verification of measurements. Outhier, by contrast, gives less explanation on the overall purpose of the expedition and a lot more details about the concrete conditions of the various steps of the whole procedures. In particular, he explains in some details the decisions that had to be made at every stage of the trip, and the difficulties related to the use of the embarked instruments, especially the zenith sector constructed for them by Graham. The latter is needed to determinate the difference of latitude between two locations by measuring zenith distances and was used by Maupertuis and his colleagues to evaluate the difference of latitude between Kittis and Tornea. It is a huge instrument, the transportation and installation of which is difficult and needs a whole construction, which is detailed in Outhier. The famous illustration (Fig 3) taken from Cassini de Thury later book on the Parisian meridian line gives an idea both of the size of such an instrument and the difficulty of its use.

The various excerpts chosen for the workshop refer to several key stages in Outhier's diary: (1) on the 2nd of July, Outhier explains the intensive discussion among the group after their arrival in Tornea ten days before: where are they to find the right locations for the requested sequel of triangles? (p.51-52); (2) one month and a half later, on the 21st of Aug, they have to reconstruct one signal left in Horrakilero and that had been entirely burnt (p.90); (3) on the 2nd of Sept, Outhier describes the trick they used to make observations in Tornea in the bell tower of the church, just after the office, taking opportunity of the clear weather (p.96); (4) finally on the 9th of sept (p.102-103) the construction of the observatory in Pello, near the northern point of Kittis. This episode is nicely illustrated by a detailed map of the impressive constructions made on this occasion. Generally speaking, Outhier's journal abounds in nice maps of the region, representations of buildings, and vivid scenes capturing special episodes.

We will not enter into any more detail here, rather we need to focus now onissues connected with the translations.

## 3.2 The problems with our choice of English translations

As explained above, the main purpose of the two diaries was to account for the scientific reliability of the expedition. This explains why, beyond the narrative itself, they contain a host of figures and calculations for each station and observation either of distances or elevation angles. These figures and maps are contained in the thirty last pages of Outhier (203-234, around one tenth of the entire work); as for Maupertuis's book, the narrative is contained in a lengthy preface of 80 pages while the main part of the book is devoted to observations (100 pages) completed by nine maps and schemas of the triangles.

By contrast, Pinkerton's 1808 edition kept the two narratives only, with no mention of calculations and charts. The reader, therefore, has no way of grasping the general situation of the explored region nor the scientific strategy used for charting the region and determining the relevant distances and latitudes. The same neglect for the scientific dimension of Maupertuis' enterprise is evident in the mistranslation for the word "sextant", standing for the huge zenith-sector constructed by Graham, as mentioned above.

All this might appear somewhat surprising given that Pinkerton himself was a geographer and had announced in the preface to his gigantic edition of travel narratives that he would follow the ordering adopted for his Geography. While we do not know exactly who translated the text (his collection mainly consist in a newly edited compilation of already published narratives in Green or elsewhere, but Maupertuis's and Outhier's texts are presented as "newly translated"), Pinkerton for sure arranged the texts and, together with the publishers, was responsible for the main choices of presentation. His choice might be explained by the general meaning that geography writing had at the beginning of the 19th century: according to Sitwell's analysis (1972), geography then emphasized the comparison of nations according to their level of power and idiosyncrasies rather than universal physical characteristics like their topographical situation.

Whatever the case, we were progressively and led to recognize that this translation was quite inadequate for our purpose. This leads us back to the main conclusions of our workshop.

## *4 Main conclusions*

Putting together the various elements needed to fully understand the scientific nature of the 18th cent. controversy about the shape of the Earth, and also to make it clear to our audience, we came to several conclusions.

The first is that explaining both observations O1 and O2 (from Outhier's journal) and the connection to the issue of the shape of the earth (C, from D'Alembert's *a posteriori* reflections) is not only overambitious at middle school level but unnecessary for our most basic purpose. Indeed, getting into the meaning and purpose of the expedition is far enough to understand not only the difficulties of such an enterprise, but also, and above all, its scientific character: Outhier is very clear about the fact that big and small questions appeared all along the trip and were constantly motivated by the overall purpose[66].

The second conclusion bears on the perception of this purpose: contrarily to what Pinkerton's edition seems to suggest, it is not really possible to understand the special tension inherent in Outhier's narrative, without getting a sense of the overall strategic schemas they had in mind from the outset: what are the means "to form a sequence of triangles" is the obsessive idea that guides the travelers and help to clarify their basic purpose. And this idea is made clear through the maps and observation data consigned to the end of the French version. In other words, the textual material needed should include, at least in part, such maps. Reading the excerpts could then become an exercise of making sense of their questions, by taking into account the overall purpose, which can be done by simultaneously looking at the narrative and the underlying schemas.

## REFERENCES

### *18th century texts*[67]


D'Alembert (1754), article « DEGRE » in *Encyclopédie*, IV, p.761-763. [E]

D'Alembert (1756), article « FIGURE DE LA TERRE» in *Encyclopédie*, VI, p.749-761. [E]

Maupertuis (1732) *Discours sur les différentes figures des astres*. Imprimerie Royale.

Maupertuis (1738) *La figure de la terre*. Imprimerie Royale. [*]

Outhier (1744) *Journal d'un voyage au nord en 1736 e t1737*. Paris : Piget et Durand. [*]

Pinkerton (1808) *A general collection of the best and most interesting voyages and travels in all parts of the world*: vol.1. London: Longman. *Maupertuis' 1737 Discourse : 231-258; Outhier'sdiary : 259 ff.* [A]


[66] As for problem C, it should really be the subject of an in-depth reflection on the very notions of angle and curvature, that should be in principle the purpose of the mathematics course and not reducible to an isolated activity.

[67] [*] signals texts that can be downloaded on the web site of Bibliothèque Nationale de France (Gallica), [A] at the site archives.org, and [E] on the ENCCRE website for the collaborartive edition of Diderot and D'Alembert *Encyclopédie*, cf. . http://enccre.academie-sciences.fr/encyclopedie/

Voltaire (1734). *Lettres Philosophiques.* Amsterdam: Lucas. See letter.14 “On Descartes And Sir Isaac Newton”, 139ff. [*]

Voltaire (1752) *Micromegas.* Berlin. *See ch. V.* [*]

### *Modern studies (selective choice)*

Badinter, E. (1999). *Les passions intellectuelles. I. Désirs de gloire, 1735-1751*. Paris: Fayard.

Boyé A., Lefort X. (1993) *Mesurer aussi bien la Terre que le ciel*. IREM des Pays de la Loire, Nantes, 1993. *https://publimath.univ-irem.fr/numerisation/NA/INA93004/INA93004.pdf*

Darley, C. (2018) « Une séquence en cours d’histoire de 4ème autour du Voyage en Laponie de Clairaut », Academic blog *Mathématiques, histoire, citoyenneté*. https://mathistcit.hypotheses.org/509

Greenberg, J. L. (2010). *The problem of the Earth's shape from Newton to Clairaut: The rise of mathematical science in 18th century Paris and the fall of 'normal' science*. CUP.

Passeron, I. (1996)« Savoir attendre et douter » : l'article Figure de la Terre », *Recherches sur Diderot et sur l'Encyclopédie*, 21/1, 131-144

Passeron, I. et Chambat, F. (2006) La figure de la Terre au XVIIIème siècle, un problème géodésique ou astronomique ? Recorded conference [in French] on *Planet Terre.* https://planet-terre.ens-lyon.fr/ressource/figure-terre-2006-conf.xml

Shank, J. B. (2008). *The Newton wars and the beginning of the French Enlightenment*. Chicago: University of Chicago Press.

Sitwell, O.F.G. (1972) “John Pinkerton: An Armchair Geographer of the Early Nineteenth Century”. *The Geographical Journal* 138-4, 470-479.

# DU COMPAS DE FABRIZIO MORDENTE DE SALERNE AU COMPAS DE PROPORTION

**Jean Michel DELIRE**
Institut des Hautes Études (Université Libre de Bruxelles), Belgique
jeanmicheldelire@gmail.com

**ABSTRACT**

Between Fabrizio Mordente of Salerno (1532-1608) and Nicolas Bion (*ca.*1652-1733), *via* Michel Coignet (1549-1623), the compasses (with multiple points, pantometers, proportional compass) quickly evolved, but the geometrical problems proposed by the treatises remained similar. This workshop recounts the early history (second half XVI$^{th}$ – beginning XVII$^{th}$ centuries) of these various compasses in the frame of the numerous travels and encounters of their inventors, especially Fabrizio Mordente, who has been a mathematician at the service of several rulers (Rudolph II, the Duke of Guise, Alexander Farnese), travelled and met other mathematicians, such as Coignet in Antwerp, or Clavius and Grienberger in Rome. As a young man, he also crossed the Mediterranean Sea and the Indian Ocean up to Goa (India). After comparing the peculiarities of the different compasses, we proposed to the participants some mathematical problems to be solved with the help of paper and wood models of compasses.

## 1 *Évolution du compas, de Mordente à Coignet*

L'établissement des éléments des triangles est la base de la *géodésie*, *via* une méthode nommée aujourd'hui *triangulation*. Cette méthode a été explicitée par Gemma Frisius, né en 1508 en Frise (îles de Wadden). Gemma étudia les mathématiques et la médecine à l'Université de Leuven, et édita en 1529 une version française de la *Cosmographia seu descriptio totius orbis* (Lanshut, 1524) de Petrus Apianus (1495-1552). Dans une seconde édition, publiée à Anvers en 1533, il ajouta son *Libellus de locorum describendorum ratione* **( …)**, dans lequel il propose d'utiliser la triangulation pour établir correctement les cartes.

Par la suite, Gemma commença à produire ses propres instruments, dans un atelier qu'il avait créé avec Gaspard Van der Heyde, un graveur et joaillier. En 1534, l'un de ses étudiants fut Gérard Mercator (de Kremer) Rupelmondanus (1512-1594), avec lequel il construisit un globe céleste et un globe terrestre, sur lequel ils voulaient représenter toutes les découvertes récentes.

Mercator devint célèbre grâce à son invention d’une nouvelle méthode de représentation plane du globe terrestre.

Bien entendu, le développement des méthodes de triangulation entraîna l’invention de nouveaux instruments, comme le *compas de proportion* ou son prédécesseur, le *compas à pointes*, par Fabrizio Mordente, et les *règles pantomètres*, par Michel Coignet.

Fabrizio Mordente (1532-1608) est né à Salerne. Après ses études à l’Université de Naples, dès 1552, il voyagea, visitant les pays méditerranéens, allant jusqu’en Mésopotamie et, depuis le Golfe persique, jusqu’à Goa (Inde, territoire conquis par le Portugal après la traversée de Vasco de Gama en 1498), où il resta trois ans. De là, il revint à Lisbonne, par les bateaux régulièrement affrétés pour l’Inde et retour[68]. Mordente visita encore Londres, Calais, Paris, les Flandres et Anvers, Bruxelles, Namur, Liège, Cologne, plusieurs villes allemandes, Prague, la Hongrie et, enfin, Venise, Florence, Rome, pour revenir à Naples

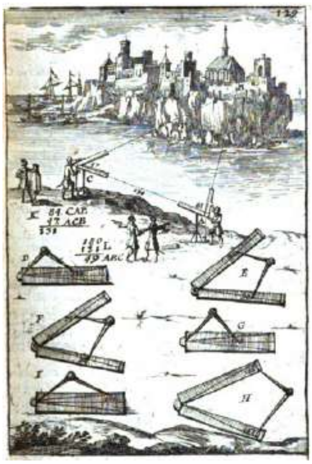


**Figure 1.** Nous commençons par une illustration, tirée d’un livre ancien de géométrie pratique (Manesson-Mallet, 1702), qui présente différents compas et leur utilisation en vue de fixer les éléments d’un triangle et la distance d’une île au continent.

[68] D’après Camerota (2000, p. 20), qui résume la description détaillée du voyage ‘épique’ décrit par Mordente dans la préface de son ouvrage *Le Proposizioni*, 1598. Nous devons aussi à ce livre de Camerota les images des quatre pages suivantes, à l’exception de la figure 8.

Mordente ne précise pas la durée de chacun de ses séjours, à l'exception de Goa et de certains de ses voyages. Il parle de dix ans en tout, et sa première publication paraît à Venise en 1567. C'est une explication en une feuille (figure 2) de l'utilisation d'un compas.

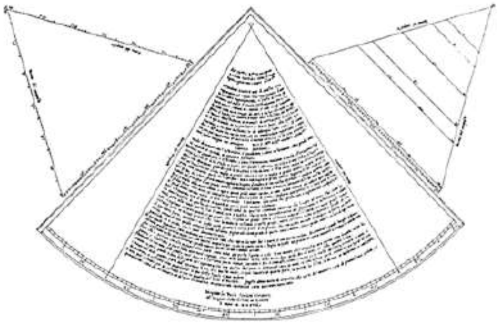


**Figure 2.** Explication de l'utilisation d'un compas.

Entre 1568 et 1570, il fit faire un compas similaire à Urbino, avec des curseurs placés dans des glissières taillées dans les jambes du compas (figure 3).
Entre 1568 et 1570, il fit faire un compas similaire à Urbino, avec des curseurs placés dans des glissières taillées dans les jambes du compas (figure 3).

**Figure 3.** Le compas construit a Urbino.

Le but du premier compas (Venise, 1567) était de mesurer – en minutes – de combien un certain arc dépasse un nombre entier de degrés en multipliant ce dépassement par 60 à l'aide d'un compas de réduction de quotient 1/60 (figure 4) et un rapporteur.

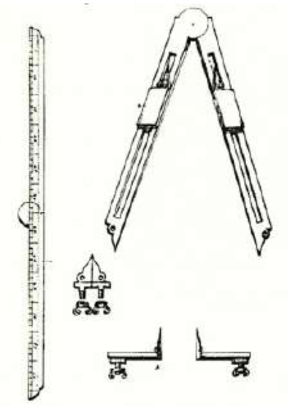


**Figure 4.** Compas de réduction de quotient 1/60.

Placé entre les jambes du compas, ce rapporteur mesurera le nombre de minutes lorsque les deux jambes seront ouvertes conformément au dépassement.

Par la suite, Mordente améliora son compas, non plus pour mesurer seulement des angles, mais aussi pour le calcul proportionnel. Les jambes, devenues plates, sont munies d'une règle (figure 5), probablement du type indiqué par Giacomo Contarini (1536 – 1595) dans une collection qu'il décrit dans un manuscrit, aujourd'hui conservé à l'Oxford Bodleian Library ( Canon.Ital.145). Contarini était un sénateur vénitien, ami de Galilée et de Palladio, qui avait accumulé une collection artistique et scientifique, dont une partie est encore visible à Venise.

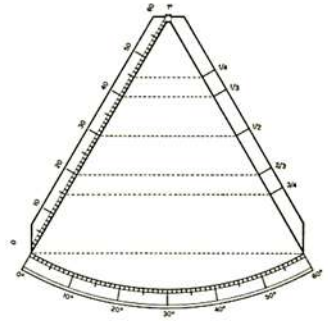


**Figure 5.** Jambes du compas munies d'une règle.

Fabrizio Mordente aspirait à être engagé comme mathématicien de cour et voyagea beaucoup pour cela, arrivant à Vienne en 1572 avec son frère Gasparo. Il y présenta son nouveau compas à l'empereur Maximilien II. Pour le couronnement de son fils, Rodolphe II (règne 1575 – 1612), il présenta encore d'autres instruments. Rodolphe lui-même suggéra à Fabrizio d'ajouter une nouvelle pointe sur l'axe de son compas. C'est durant cette période viennoise que Fabrizio, qui y rencontra des mathématiciens particulièrement intéressés par les fortifications, partagea ses idées concernant d'autres lignes à ajouter au compas et envisagea de publier un livre. Mais c'est son frère qui décrivitt, à la demande de l'empereur, les recherches de son frère, dans un livre publié à Anvers: *Il Compasso del S. Fabritio Mordente con altri Instrumenti Mathematici ritrovati da Gasparo suo fratello*, Anversa, Christophoro Plantini, 1584. Ce traité prit la forme typique des livres de géométrie pratique montrant comment le compas pouvait être utilisé pour résoudre divers problèmes euclidiens.

En 1585, Fabrizio Mordente est à Paris où il espère une pension de la reine mère Catherine de Médicis. Il y publie une estampe, imprimée le 20 mars par Jean le Clerc, rue Frementel à l'Estoile d'or. La figure placée sous le compas évoque trois manières de transformer une fraction en une fraction de 12. (Figure 6 )

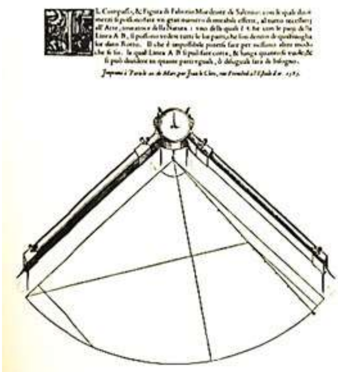


**Figure 6.** Sous le compas sont representées trois manières de transformer une fraction en une fraction de 12.

Fabrizio est ensuite ingénieur au service du Duc de Guise, qui appuyait Philippe II et la Ligue Catholique contre Henri III et l'hérétique Henri de Navarre. Mais après la mort du Duc de Guise, assassiné (sur ordre d'Henri III) à la fin de 1588, il entre au service d'Alexandre Farnèse (1545 – 1592), Duc de Parme, nommé par Philippe II gouverneur des Pays-Bas en 1578. En 1589, Fabrizio et son frère Gasparo sont à la cour de Bruxelles où ils obtiennent le privilège royal pour la publication, deux ans plus tard, de *La Quadratura del* *cerchio, la Scienza de' residui, il Compasso et riga di Fabritio, et di Gasparo Mordente fratelli salernitani*, à Anvers (chez P.Bellerus). Le livre est évidemment dédié à Alexandre Farnèse qui est représenté (figure 7) entouré des allégories de la géométrie et de l'arithmétique.


**Figure 7.** Frontispice du livre dédié à Alexandre Farnèse.

Dans le haut, on aperçoit une évocation de la prise d'Anvers par Alexandre, avec le pont de barques sur l'Escaut pour empêcher le ravitaillement de la ville assiégée.

À cette époque, vivait à Anvers Michel Coignet, fils de Gillis (1515-1562), constructeur d'instruments et bijoutier. Gillis ayant disparu alors que Michel était trop jeune pour lui succéder, ce dernier enseigna les mathématiques et le français dans l'une des nombreuses écoles où étaient formés les enfants des familles de commerçants, très nombreux à Anvers. Admis dans la guilde des maîtres d'école en 1568, il construisit son premier astrolabe en 1572, puis devint jaugeur de vin (figure 8) après avoir réussi un concours qui exigeait de pouvoir graver un bâton permettant de mesurer la quantité de vin restant dans un tonneau. Il édita en 1580 un petit opuscule sur le sujet : *Pratycke om lichtelyk te leeren visieren alle vatten metter wisselroede* [Méthode pour aisément apprendre à évaluer tous les tonneaux à l'aide du bâton de jauge].


PRACTYCKE
Om lichtelijc te leeren visieren alle Vaten metter Visselroede:
Door VVillem Raedts, Maestrichter.
Van nieuws ouersien, ende in veel plaetsen verbetert, ende met seer profijtelijcke aenwysinghen vermeerdert, ende verciert: Door Michiel Coignet.
T'hantwerpen,
By Hendrick Hendricksen. 1580.
Met Privilegie van X. Jaren.

**Figure 8.** Livre de Coignet sur le métier de jaugeur de vin.

Michel Coignet est surtout connu pour avoir publié en 1580 *Nieuwe Onderwijsinghe op de principaelste puncten der Zeeuaert*, comme appendice à la traduction de l'*Arte de Navegar* de Pedro de Medina[69]. La traduction française est publiée seule en 1581, ce qui fait connaître Coignet dans toute l'Europe et lui vaut d'entrer, quelques années plus tard, au service de l'archiduc Albert (1559-1621) en tant que mathématicien et ingénieur. Michel Coignet obtenait ainsi un poste comparable à celui qu'avait occupé Fabrizio Mordente auprès d'Alexandre Farnèse, puisque Philippe II (1527-1598) avait cédé, peu avant sa mort, le gouvernement des Pays-Bas à sa fille Isabella (1566-1633) et à son mari l'archiduc Albert. On n'est pas sûr que Mordente ait rencontré Coignet, mais c'est hautement probable, vu que Mordente a visité Anvers à plusieurs reprises. De plus, peut-être suite à une telle rencontre, Coignet s'est intéressé aux compas de proportion, ou plutôt aux échelles qu'ils portent. Il a exprimé ses idées sur le sujet dans de nombreux manuscrits et en différentes langues : Bruxelles (latin et français), Anvers (espagnol), Paris (espagnol), Modène (italien), Madrid (italien), Prague (latin), etc., en les illustrant de figures et de problèmes résolus.

[69] Traduction en néerlandais faite par Merten Everaert de Bruges.

Nous allons maintenant envisager quelques-uns de ces problèmes, en comparaison de problèmes proposés par Mordente et transmis dans un livre édité en français en 1626 (source 1° en fin d'article).

## *2 Quelques problèmes résolus à l'aide des compas de Mordente et Coignet*

### 2.1 Tracer une droite perpendiculaire à une extrémité d'un segment

a) Mordente, d'après la source 1° (Figure 9)

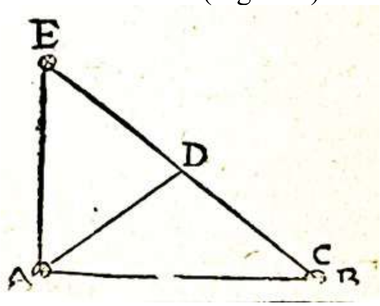



**Figure 9.** Tracer une perpendiculaire.

p.11 : « V. Proposition. *Lever une perpendiculaire sur l'extrémité d'une ligne droicte*. Soit la ligne AB, & que les coursaires du Compas soient bien mis en égale distance, puis ouvrez le Compas, & mettez l'une des poinctes des coursaires en A, & l'autre en C, puis la poincte centrale denottera (sic) la marque D, parce que l'ouverture du Compas est fait (sic) à discretion : cela fait, ouvrez le Compas de telle sorte que les trois poinctes soient en ligne droicte : & ayant mis la poincte centrale en D, & une des poinctes des coursaires en C, & l'autre poincte du coursaire vous marquera le poinct E, duquel vous menez une ligne sur A, qui sera perpendiculaire. »[70]

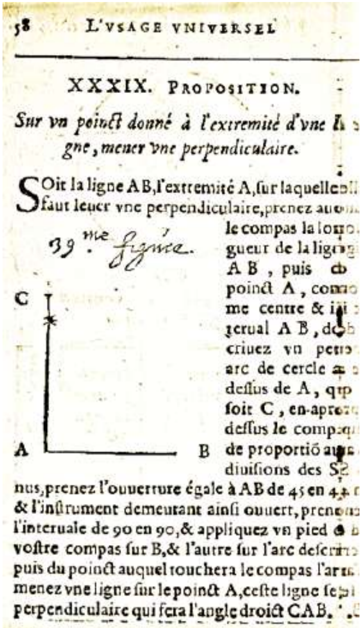
La preuve (que $\widehat{\mathrm{CAD}} + \widehat{\mathrm{DAE}} = 90°$, par exemple) est laissée aux lecteurs.

p.58 : « XXXIX. Proposition. *Sur un poinct donné à l'extrémité d'une ligne, mener une perpendiculaire.* »

La ligne est AB et l'on commence par dessiner l'arc de cercle de centre A et de rayon AB. On applique ensuite le compas de proportion avec AB sur la ligne des sinus de 45 en 45. Le compas ainsi ouvert, on prend l'ouverture de

[70] En comparant ceci à la construction classique, on comprend l'intérêt du compas de Mordente.

90 en 90 avec un compas commun et, sa pointe sèche étant placée en B, on porte un trait sur l’arc de cercle, ce qui donne le point C à la perpendiculaire de A. (Figure 10)



58 L’VSAGE VNIVERSEL

XXXIX. PROPOSITION.

*Sur vn poinct donné à l’extremité d’vne li-gne, mener vne perpendiculaire.*

SOit la ligne AB, l’extremité A, sur laquelle il faut leuer vne perpendiculaire, prenez auec le compas la longueur de la ligne AB, puis du poinct A, comme centre & interual AB, descriuez vn petit arc de cercle au dessus de A, qui soit C, en-apres dessus le compas de proportiõ aux diuisions des Sinus, prenez l’ouuerture égale à AB de 45 en 45 & l’instrument demeurant ainsi ouuert, prenez l’interuale de 90 en 90, & appliquez vn pied de vostre compas sur B, & l’autre sur l’arc descrit, puis du poinct auquel touchera le compas l’arc, menez vne ligne sur le poinct A, ceste ligne sera perpendiculaire qui fera l’angle droict CAB.



**Figure 10.** Explication détaillée.

Pour comprendre cette construction, basée sur la trigonométrie, il faut se reporter à la figure du compas de proportion donnée dans le livre de 1626 (voir Annexe, face C). En effet, Coignet propose de nombreuses lignes (déjà dans ses « règles pantomètres » du ms. KBR, II-769 (Bruxelles) (Figure 11), qu’il répartit sur les quatre faces de deux compas.

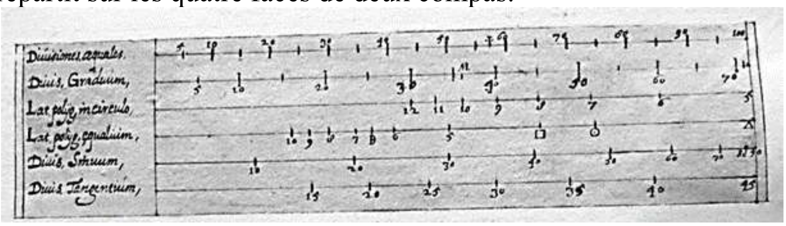



**Figure 11.** Règles pantomètres de Coignet.

c) on notera que, dans le même ms. KBR, II-769 (Bruxelles), Coignet annonce cette proposition comme la première, après quarante autres utilisant les lignes du compas autres que celles des sinus : « S’ensuyvent les propositions qui se resoulvent par Divisionum Sinuum, 41 D'un poinct donne a l'extrémité d'une ligne droicte eslever une perpendiculaire ».

Il en est de même dans le ms. Erfgoedbibliotheek Hendrik Conscience B264708 (Anvers) ainsi que dans le ms. BnF, Espagnol 351 (Paris) : « Siguen algunas proposiciones lasquales se resolven por las Divisiones del Sinus y de las Tangentes, Proposicion 39 Como se sacara una linea perpendicular del punto extremo de otra linea que se propone ».

## 2.2 Construire deux moyennes proportionnelles entre deux lignes données

L'importance de ce problème réside, depuis l'Antiquité, dans le fait qu'il permet de résoudre la duplication du cube, dit problème de Délos. En voici l'histoire légendaire racontée par Ératosthène (c. 284-194), le directeur de la Bibliothèque d'Alexandrie, rapportée par Eutocius.

« *Ératosthène au Roi Ptolémée, Salut !*

*On rapporte qu'un des anciens poètes tragiques avait mis à la scène Minos qui, faisant préparer un tombeau à Glaucos, et ayant remarqué qu'il avait cent pieds de long de tous côtés, disait : « Tu as choisi la chambre sépulcrale du roi petit, qu'elle soit doublée ; ne te méprends pas sur ce qui convient, et double aussitôt chaque partie du tombeau. » Or, il semble bien que Minos se soit trompé ; car, lorsqu'on double les côtés, un plan devient quadruple et un solide huit fois plus grand. Chez les géomètres aussi on a cherché la manière de doubler un solide donné tout en lui conservant la même forme, et le problème de cette espèce fut appelé la duplication du cube, car, s'étant proposé un cube, ces géomètres s'efforcèrent de le doubler. Or, après avoir été tous et longtemps embarrassés, c'est Hippocrate de Chio qui fut le premier à s'apercevoir qu'un cube serait doublé si l'on parvenait à trouver deux moyennes proportionnelles en proportion continue entre deux lignes droites dont la plus grande est le double de la plus petite ; en sorte que l'embarras fut changé pour lui en un autre et non moindre embarras. On dit que plus tard, des Déliens, chargés par un oracle de doubler un de leurs autels, et tombés dans le même embarras, furent envoyés chez Platon, et demandèrent aux géomètres qui résidaient à l'Académie, de leur trouver ce qu'ils cherchaient*. » (Ver Eecke, 1960)

a) dans le ms. KBR, II-769 (Bruxelles), Coignet énonce le problème « 19[71] Chercher entre deux lignes droictes donnees, deux autres lignes entremoyennes en proportion continue », puis prend un exemple « Explication, Soit la première ligne donnée A de 54 parties égales, et D la 4e de 16 parties. [Il] convient [de] trouver les deux lignes entremoyennes B et C, etc. ». Il résout ce problème de manière classique, comme le fait, par exemple, Bion avec un compas de proportion. Mais, à la fin (Fol. 60v) du même manuscrit, Coignet donne une autre méthode de construction de deux moyennes proportionnelles :

« Apendices à la 19 proposition. Pour agrandir ou augmenter tout corps Géométricq$^{\beta}$[72] laquelle praticq$^{\beta}$ l'on apellela Duplication du Cube. (…) Or, pour doubler ou tripler ou bieng augmenter tout corps à tant qu'on voudra, les Anciens cõme Platon, Architas de Tarente, et autres ont practiqué divers Instruments pour le faire, lequel estoit fort pénible à faire. L'invention de Coignet[73] sur ce Problème (en marge : est facille) par sa reigle pantomètre, soit en l'ensuyvante figure AB, le diamètre d'une boulle d'une libvre de fer et l'on demande celuy de deux libvres…Pour ce faire, l'opération sera telle, faictes sur AB. Un quadrangulum rectangulum A.B.C.D/ dont le coste A.D sera double à la ligne AB, les lignes DC et AD prolongerez vers E et F. Après ouvrirez la pantomètre iusques à tant que les deux branches soÿent bien à droict angle, le coing droit I mettrez sur la ligne perpendiculaire D.F. et son coste sur le point A, en haulsant ou abaissant ledict angle I, sur la ligne D.C.F. iusques à tant que vous trouverez avecq un simple compas, que la ligne B.H. sera egalle avecq D.G, coupez par les deux branches de la pantomètre. Or aÿant trouvé cela dittes que B.H. est le diamètre d'une boulle de deux libvres qu'on cherchoit etc. (…) Démonstration : Les 4 lignes A.D, DI, H.B, B.A sont en proportion continuelle dont il s'ensuite que cõme la ligne AD se tient à la

[71] Bizarrement, le Ms de Bruxelles (KBR) saute de la proposition 14 à la 19. Ce n'est pas le cas dans les manuscrits d'Anvers et de Paris, où cette proposition est la 16e (dans les deux cas).

[72] Le scribe remplace souvent la finale –ue par un signe ressemblant à $\beta$ ou à $^3$.

[73] Bizarrement, Coignet parle ici de lui à la troisième personne. Ce n'est pas le cas dans les Mss d'Anvers et de Paris, où il utilise « esto por un nuestra (mot biffé) Invencion » (Anvers) et « esto por un nuestra nueva Invencion » (Paris). Il s'agit donc d'une invention de Coignet, qu'il ne considèrerait plus comme nouvelle à l'époque du Ms d'Anvers. Celui-ci étant daté de 1618, on pourrait en déduire que celui de Paris est antérieur à cette date et postérieur à la date du Ms de Bruxelles (1610-1612), où cette nouvelle construction est rajoutée à la fin.

ligne BA, ainsÿ le corps faict sur la ligne H.B. au corps faict (semblable au premier) sur la ligne BA. Mais AD est double à la ligne B.A, dont s'ensuit que le corps ou boulle sur HB sera double à la boulle sur AB, et ainsÿ des autres … »

(Figure 12 : provenant des manuscrits de Bruxelles et de Paris)

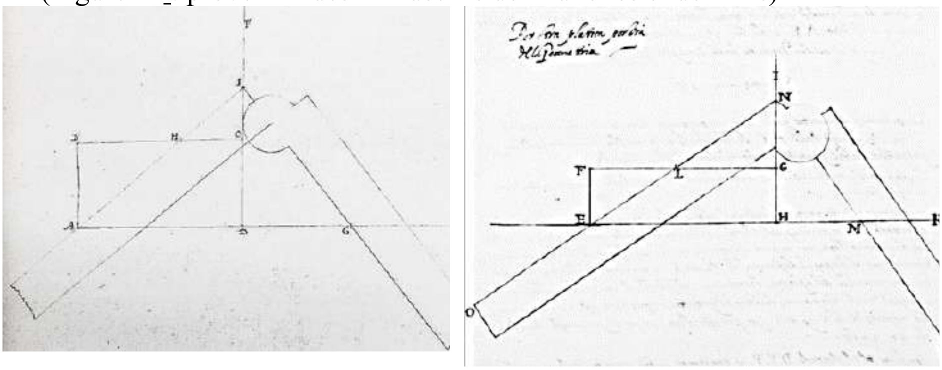
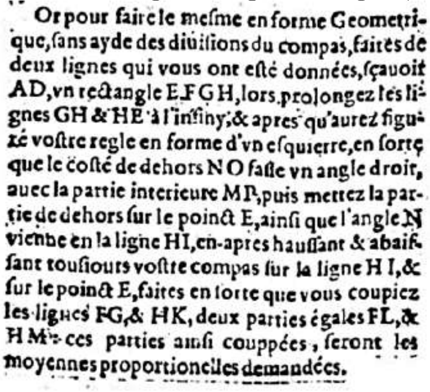



**Figure 12.** Construction de deux moyennes proportionnelles à l'aide du compas.

b) Coignet d'après la source 1°

« XVI. Proposition. *Trouver deux moyennes proportioneles entre deux lignes données.* » Comme dans les manuscrits, le texte commence par l'exemple 54 et 16, qu'il résout, puis, immédiatement après, l'auteur donne une solution, qu'il dit géométrique (Figure 13)


Or pour faire le mesme en forme Geometrique, sans ayde des diuisions du compas, faites de deux lignes qui vous ont esté données, sçauoit AD, vn rectangle EFGH, lors prolongez les lignes GH & HE à l'infiny; & apres qu'aurez figuré vostre regle en forme d'vn esquierre, en sorte que le costé de dehors NO fasse vn angle droit, auec la partie interieure MP, puis mettez la partie de dehors sur le poinct E, ainsi que l'angle N vienne en la ligne HI, en-apres haussant & abaissant tousiours vostre compas sur la ligne HI, & sur le poinct E, faites en sorte que vous coupiez les lignes FG, & HK, deux parties égales FL, & HM: ces parties ainsi couppées, seront les moyennes proportionelles demandées.

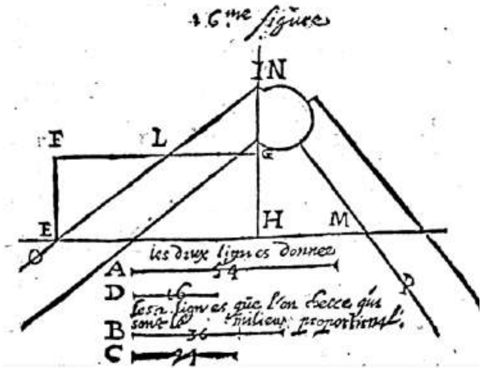
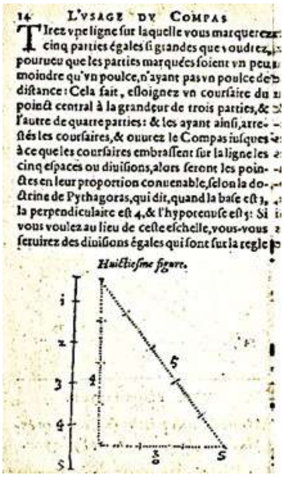



**Figure 13.** Explication détaillée.

c) pour Mordente, d'après la source 1°, nous nous limiterons à sa construction d'un angle droit car il ne donne pas explicitement la construction d'une double moyenne proportionnelle (Figure 14)


14 L'VSAGE DV COMPAS

TIrez vne ligne ſur laquelle vous marquerez cinq parties égales ſi grandes que voudrez, pourueu que les parties marquées ſoient vn peu moindre qu'vn poulce, n'ayant pas vn poulce de diſtance : Cela fait, eſloignez vn courſaire du poinct central à la grandeur de trois parties, & l'autre de quatre parties : & les ayant ainſi, arreſtés les courſaires, & ouurez le Compas iuſques à ce que les courſaires embraſſent ſur la ligne les cinq eſpaces ou diuiſions, alors ſeront les poinctes en leur proportion conuenable, ſelon la doctrine de Pythagoras, qui dit, quand la baſe eſt 3, la perpendiculaire eſt 4, & l'hypotenuſe eſt 5 : Si vous voulez au lieu de ceſte eſchelle, vous-vous ſeruirez des diuiſions égales qui ſont ſur la regle

*Huictieſme figure.*



**Figure 14.** Construction d'un angle droit.

« VIII. Proposition. *Mettre les trois poinctes du Compas en angles droicts, selon l'intention de Pythagoras.* »

Bien entendu, cette construction est basée sur le triangle 3-4-5. On trace une ligne de 5 parties égales au choix, puis on éloigne les curseurs de 3 et 4 telles parties et on ouvre le compas de sorte que les curseurs touchent les extrémités de la ligne. Le compas est alors ouvert à angle droit.

## 2.3 Construire un polygone régulier, en particulier un pentagone

Pour déterminer la corde de 36°, c'est-à-dire le côté du décagone, Ptolémée procède comme suit dans l'*Almageste* : il rappelle *Éléments*, XIII.9 : *Si le côté de l'hexagone et celui du décagone (...) sont composés, la droite entière est coupée en extrême et moyenne raison (...)*, soit : $r^2 = c_{10}.(c_{10} + r)$. Cela peut être aisément démontré à l'aide de la partie gauche de la (Figure 15) $\widehat{GOH} = 36° \Rightarrow \widehat{OHG} = 72°$ et $\widehat{FHO} = 108° \Rightarrow \widehat{OFH} = 36°$. Les triangles GHO et GOF sont donc semblables $\Rightarrow \frac{GO}{GH} = \frac{FG}{GO}$ ou $\frac{r}{c_{10}} = \frac{c_{10}+r}{r}$. On en déduit $c_{10} = \frac{\sqrt{5}-1}{2}.r$

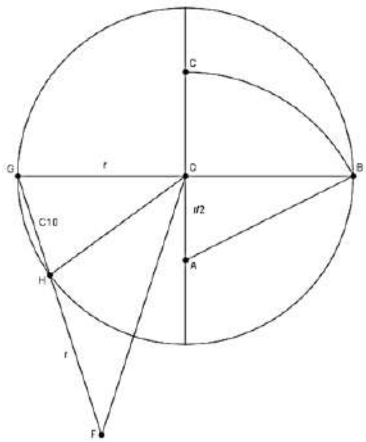
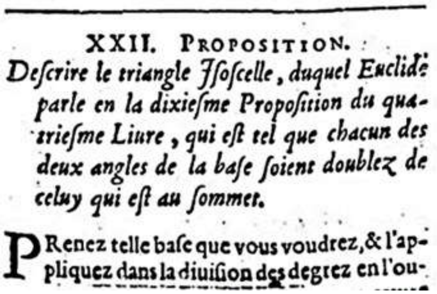



**Figure 15.** Détermination de la corde de 36°.

Sachant cela, le côté $c_{10}$ du décagone peut être construit de la manière suivante : les points A et B sont sur des diamètres perpendiculaires de sorte que $|OA| = r/2$ et $|OB| = r$. On tourne le segment AB autour de A de manière à amener B en C sur le diamètre AO. $|OC|$ est donc la longueur du côté $c_{10}$ cherché. On peut calculer la longueur de $c_{10}$ en fonction du rayon $r = 60$, en l'exprimant en parties, minutes et secondes de parties comme l'a fait Ptolémée dans sa table de cordes.

a) Coignet d'après la source 1° (Figure 16) « XXII. Proposition. *Descrire le triangle Isoscelle, duquel Euclide parle en la dixiesme Proposition du quatriesme Livre, qui est tel que chacun des deux angles de la base soient doublez de celuy qui est au sommet.* »

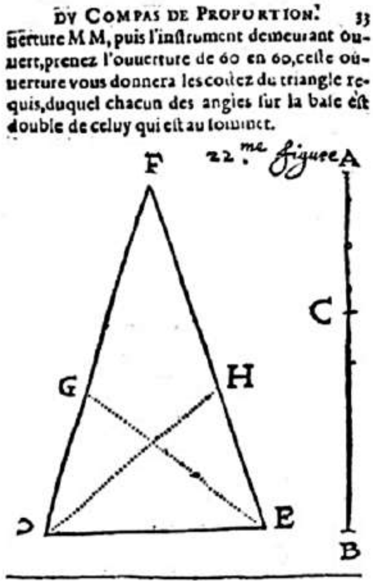
On place la base du triangle cherché entre les divisions M et M (de la face B du compas de proportion en Annexe) et l'ouverture entre 60 et 60 en donnera les deux côtés égaux. (Figure 16 suite)



XXII. PROPOSITION.

*Descrire le triangle Isoscelle, duquel Euclide parle en la dixiesme Proposition du quatriesme Liure, qui est tel que chacun des deux angles de la base soient doublez de celuy qui est au sommet.*

PRenez telle base que vous voudrez, & l'appliquez dans la diuision des degrez en l'ou-



DV COMPAS DE PROPORTION. 33

uerture M M, puis l'instrument demeurant ouuert, prenez l'ouuerture de 60 en 60, celle ouuerture vous donnera les costez du triangle requis, duquel chacun des angles sur la base est double de celuy qui est au sommet.



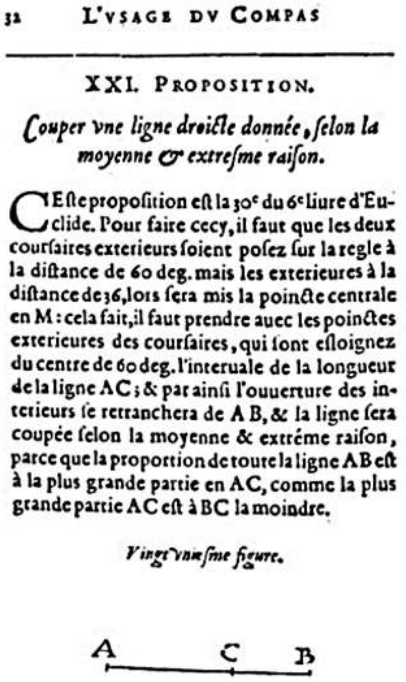
**Figure 16.** Construction du triangle d'or. **Figure 16** suite.

Justification : le triangle isocèle dont les deux angles à la base sont doubles du troisième est dit *triangle d'or* parce que le rapport de sa base sur l'un de ses autres côtés égale le nombre d'or et c'est l'inverse de ce rapport (car $1/\varphi = \frac{\sqrt{5}+1}{2}$) que l'on construit ici en prenant, entre 60 et 60, le rayon d'un cercle dans lequel on peut inscrire un décagone de côté égal à la distance entre M et M. En effet, l'angle au sommet du triangle est 36° et le rapport du côté sur sa base est $\frac{\sqrt{5}+1}{2}$.

La solution de cette proposition utilise la division M des règles pantomètres. Il en est de même de la suivante :

b) Mordente, d'après la source 1°

« XXI. Proposition. *Couper une ligne donnée proportionellement* (…) en sorte que toute [la] ligne soit à la grande partie, comme la plus grande portion est à la moindre ». (Figure 17)



32 L'VSAGE DV COMPAS

XXI. PROPOSITION.

*Couper vne ligne droicte donnée, ſelon la moyenne & extreſme raiſon.*

CEſte propoſition eſt la 30e du 6e liure d'Euclide. Pour faire cecy, il faut que les deux courſaires exterieurs ſoient poſez ſur la regle à la diſtance de 60 deg. mais les exterieures à la diſtance de 36, lors ſera mis la poincte centrale en M: cela fait, il faut prendre auec les poinctes exterieures des courſaires, qui ſont eſloignez du centre de 60 deg. l'interuale de la longueur de la ligne AC; & par ainſi l'ouuerture des interieurs ſe retranchera de A B, & la ligne ſera coupée ſelon la moyenne & extréme raiſon, parce que la proportion de toute la ligne AB eſt à la plus grande partie en AC, comme la plus grande partie AC eſt à BC la moindre.

*Vingt-vneſme figure.*



**Figure 17.** Partage en moyenne et extrême raison à l'aide du compas.

Cela revient, algébriquement, à résoudre $1/x = x/(1-x)$, si x est la « grande partie » et 1 la longueur de « toute [la] ligne ». Cette équation est équivalente à $x^2 = 1 - x$, qui définit le nombre d'or $\varphi = \frac{\sqrt{5}-1}{2}$ (voir ci-dessus). Cette construction revient donc à résoudre la *section en moyenne et extrême raison*.

Il faut comprendre 'intérieures' dans le texte. On utilise donc la ligne centrale de la face B des règles pantomètres (Figure 18), qui est en fait une ligne des cordes (puisque 72° correspond au 5, c'est-à-dire au pentagone de la ligne des polygones inscrits dans un même cercle). En plaçant les extérieures à la graduation 60°, on les place à une distance de la pointe égale au rayon du cercle (corde (crd) 60° = r), et les intérieures seront distantes l'une de l'autre de crd 36° (car M est placé à la graduation 36°).

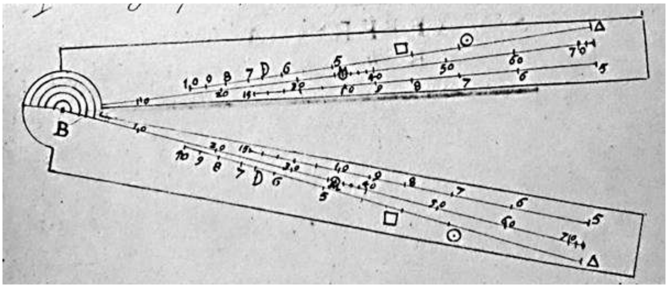
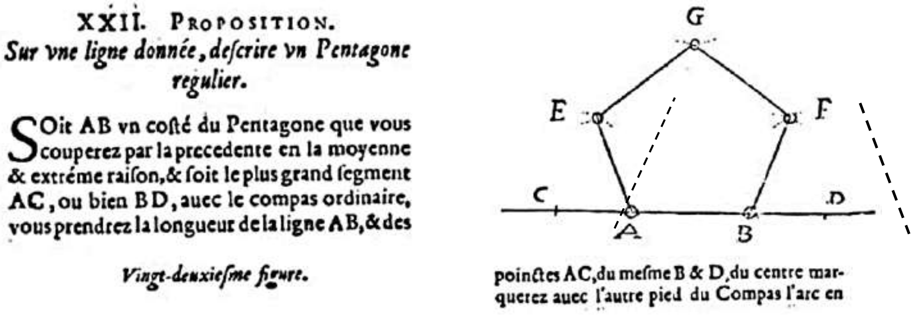


**Figure 18.** La face B des règles pantomètres.

Or, crd 36° est le côté $c_{10}$ du décagone, qui a la propriété que $\frac{r}{c_{10}} = \frac{c_{10}+r}{r}$. En retranchant l'écart des intérieures de AB, on aura bien $\frac{AB}{AC} = \frac{AC}{CB}$ car $\frac{AC}{CB} = \frac{r}{r.\frac{\sqrt{5}-1}{2}} = \frac{r.\frac{\sqrt{5}+1}{2}}{r} = \frac{AB}{AC}$, en interprétant « il faut prendre avec les poinctes extérieures des coursaires, qui sont esloignez du centre de 60 deg. l'intervalle de la longueur de la ligne AC » par AC = r.

On voit plus clairement M, ainsi que la correspondance cordes-côtés des polygones, sur les dessins des règles pantomètres du ms. KBR, II-769 (Bruxelles) de Coignet (voir, ci-dessus, la figure 11).

« XXII. Proposition. *Sur une ligne donnée, descrire un Pentagone regulier* » (Figure 19)



XXII. PROPOSITION.
*Sur vne ligne donnée, descrire vn Pentagone regulier.*

SOit AB vn costé du Pentagone que vous couperez par la precedente en la moyenne & extréme raison, & soit le plus grand segment AC, ou bien BD, auec le compas ordinaire, vous prendrez la longueur de la ligne AB, & des

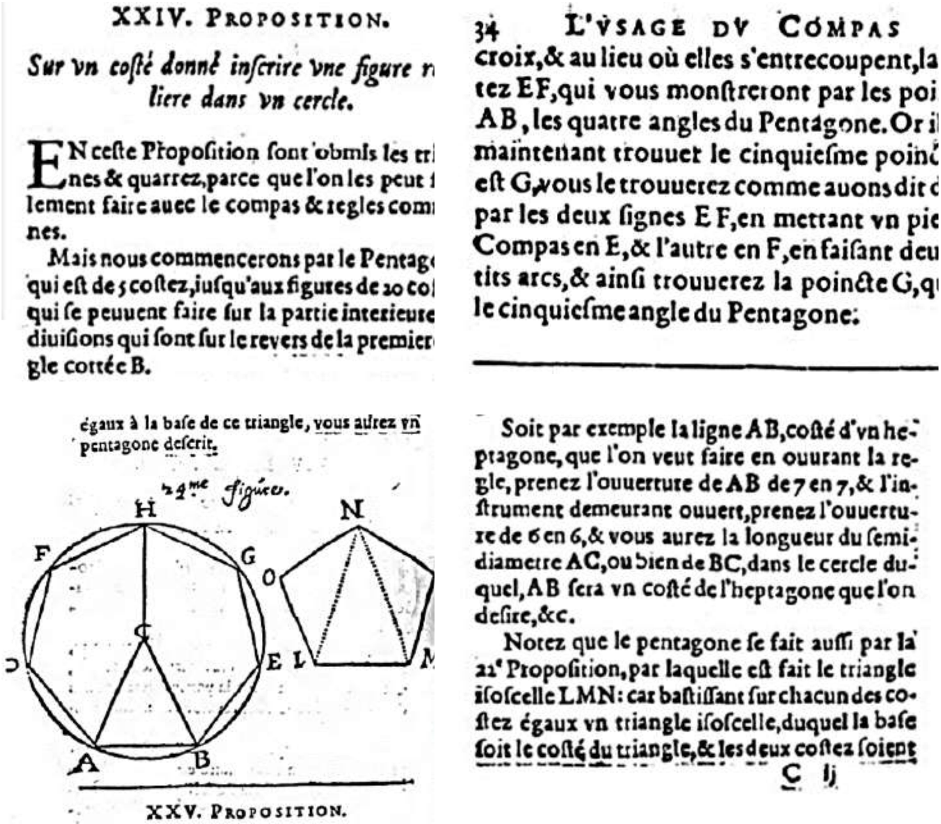
*Vingt-deuxiesme figure.*



poinctes AC, du mesme B & D, du centre marquerez auec l'autre pied du Compas l'arc en

**Figure 19.** Construction du pentagone régulier.

c) Coignet, d'après la source 1° (Figure 20). On coupe AB en moyenne et extrême raison et o porte sur la droite AB le plus grand segment, depuis A (AC) ou depuis B (BD). On reporte la longueur |AB| depuis C et A, puis depuis B et D, cela donne les sommets E et F. On fait de même depuis E et F, ce qui donne G. Pour la clarté, nous avons ajouté à la figure de Mordente les segments CE et DF, ce qui fait apparaître les triangles d’or CAE et BDF.


XXIV. Proposition.

*Sur vn costé donné inscrire vne figure r... liere dans vn cercle.*

En ceste Proposition sont obmis les tri... nes & quarrez, parce que l'on les peut f... lement faire auec le compas & regles comm... nes.

Mais nous commencerons par le Pentag... qui est de 5 costez, iusqu'aux figures de 10 co... qui se peuuent faire sur la partie interieure... diuisions qui sont sur le revers de la premier... gle cottée B.

égaux à la base de ce triangle, vous aurez vn pentagone descrit.

XXV. Proposition.

34 L'VSAGE DV COMPAS

croix, & au lieu où elles s'entrecoupent, la... tez EF, qui vous monstreront par les poi... AB, les quatre angles du Pentagone. Or il... maintenant trouuer le cinquiesme poinc... est G, vous le trouuerez comme auons dit c... par les deux signes EF, en mettant vn pie... Compas en E, & l'autre en F, en faisant deu... tits arcs, & ainsi trouuerez la poincte G, qu... le cinquiesme angle du Pentagone.

Soit par exemple la ligne AB, costé d'vn heptagone, que l'on veut faire en ouurant la regle, prenez l'ouuerture de AB de 7 en 7, & l'instrument demeurant ouuert, prenez l'ouuerture de 6 en 6, & vous aurez la longueur du semidiametre AC, ou bien de BC, dans le cercle duquel, AB sera vn costé de l'heptagone que l'on desire, &c.

Notez que le pentagone se fait aussi par la 21e Proposition, par laquelle est fait le triangle isoscelle LMN: car bastissant sur chacun des costez égaux vn triangle isoscelle, duquel la base soit le costé du triangle, & les deux costez soient

C ij

**Figure 20.** Construction de l’heptagone régulier.

## RÉFÉRENCES


Camerota, F. (2000). *Il compasso di Fabrizio Mordente: per la storia del compasso di proporzione*. Olschki.

Manesson-Mallet, A. (1702). *La géométrie pratique* (Vol. 2). Paris: chez Anisson.

Ver Eecke, P. (Ed). (1960). *Les œuvres complètes d'Archimède suivies des commentaires d'Eutocius d'Ascalon* (Vol. 2). Vaillant-Carmanne.


### Sources primaires

1° *La géométrie réduite en une facile et briefve practique, par deux excellens instrumens, dont l'un est le pantomètre ou compas de proportion de Michel Connette, Ingenieur du feu Serenissime Archiduc Albert, enrichy de huict divisions pardessus le commun & ordinaire : L'autre est l'usage du compas à huict poinctes, inventé par Fabrice Mordente, Mathematicien de feu Alexandre Farnese, Duc de Parme & de Plaisance, &c. Et composé en italien, par Michel Connette. Œuvre tres-utile pour tous curieux des Mathematiques qui desirent estre soulagez de la longue & penible description des figures Geometriques. Traduits en François par P.G.S. Mathématicien*. A Paris, chez Charles Hulpeau, ruë Daulphine, à l'Escharpe Royale, & en sa boutique sur le Pont neuf, proche les Augustins, MDCXXVI (1626)

2° ms. KBR, II-769 (Bruxelles) : *USUS Duodecim Divisionum geometricarum, per quas (et ope Unius circini Vulgaris) fere omnia Mathematicorum Problemata facili negotio resolvunutur* – Opera et Studio Michaëlis Coigneti Antverpiani Seren[m] Belgij Principium Mathematici, ex ipsa numerorum occulta parti (que Algebra Vulgo dicitur) maxima ex parte excogitata ettc[a], 1610

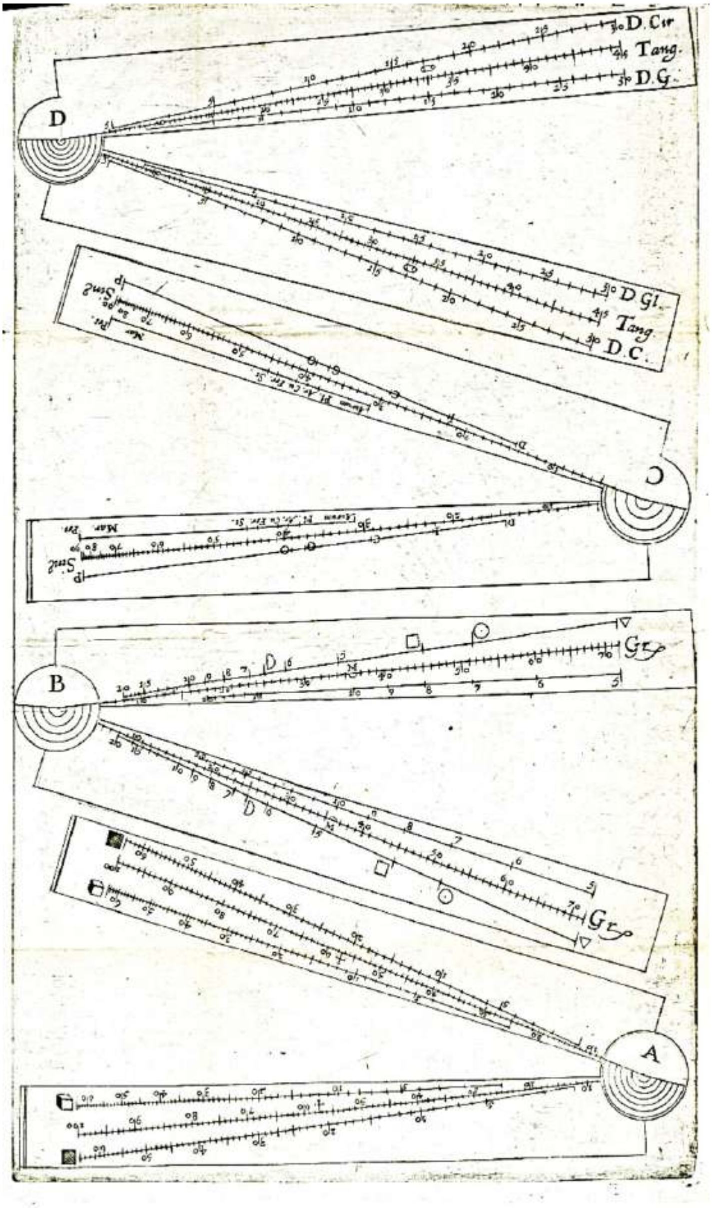
3° ms. Erfgoedbibliotheek Hendrik Conscience B264708 (Anvers) : *De la composicion y uso de las dos Reglas pantometras*, (…) Platicado y compuesto por Miguel Coñietto, natural de la ciudad de Anvers, Mathematico del Serenissimo, muy alto, y poderoso Señor, ALBERTO, Archiduque de Austria, Duque de Borgoña, Brabante, &cs. Conde de Habsburg y Flandes &c., sans date (probablement 1618)

4° ms. BnF, Espagnol 351 (Paris) : *El uso Delas doze divisiones Geometricas puestas en las dos Reglas Pantometras* (…) Platicado y compuesto por Miguel Coñieto, natural de la Ciudad de Anvers y Mathematico del Serenissimo Señor Archiduque Alberto, sans date

**Annexe**

Les compas de proportion de Coignet, selon la source 1° (entre les p.4 et 5).

Les manuscrits d'Anvers (fol.3v et 4r) et de Paris (fol.2v et 3r) ont des figures semblables.



# INTERDISCIPLINARITY IN SPECIAL RELATIVITY: DEVELOPMENT OF ACTIVITY FOR PRE-SERVICE TEACHER EDUCATION

**Lorenzo MIANI[(1)], Lucia MODICA[(2)], Olivia LEVRINI[(1)]**

(1) Department of Physics and Astronomy "A. Righi", Alma Mater Studiorum, University of Bologna, Via Zamboni 33 - Bologna, Italy

(2) Department of Mathematics, Alma Mater Studiorum, University of Bologna, Via Zamboni 33 - Bologna, Italy

lorenzo.miani2@unibo.it

lucia.modica@studio.unibo.it

olivia.levrini2@unibo.it

## ABSTRACT

From a historical point of view, mathematics and physics are closely related and interconnected fields of knowledge. However, the division present in academic and scholastic worlds leads to a separation of knowledge that limits the ability to develop a comprehensive view that grasps the important commonalities and boundary aspects. Special relativity can be taken as an example to highlight the deep connections between the two disciplines. This paper presents an activity, reserved for pre-service mathematics and physics teachers, based on an analysis of the mechanisms of boundary crossing between the two disciplines within the theory. This work highlights how specific and well-detailed attention to these mechanisms can shed light on nuances that are very important in enabling learning about the multifaceted nature of Lorentz transformations.

## 1 *The IDENTITIES project*

The society in which we live is deeply marked by a profound interconnection between different areas of knowledge. Thinking about issues we face daily, such as artificial intelligence, bioengineering or climate change, we realise how these problems cannot be addressed from a unique point of view (NSC, 2013). A sectoral and mono-disciplinary education cannot prepare one for the challenges of today and tomorrow. For this reason, for a couple of decades now, several international research groups have been focusing on interdisciplinary research areas, creating opportunities and providing funds to invest in research that goes beyond the delineated boundaries of individual disciplines (EC, 2021; NRC, 2013).

The work we present in this paper is related to one of these projects, the IDENTITIES project (Integrate Disciplines to Elaborate Novel Teaching Approaches to InTerdisciplinarity and Innovate pre-service Teacher Education for STEM challenges, https://identitiesproject.eu/), in which it is assumed that the search for the meaning of interdisciplinarity cannot do without the meaning of disciplines and their epistemological identities. The project aims to create interdisciplinary, innovative and transferable teaching modules and courses to be tested and used in teaching training contexts, with a focus on the connections between physics, mathematics and computer science.

## 2 *Interdisciplinarity and Special Relativity in Science Education*

In the research area of physics education, many studies have focused on the connection between the disciplines of mathematics and physics, as they have always been deeply interconnected (Ataide & Greca 2013; Branchetti, Cattabriga, and Levrini, 2019; Pospiech, 2014). However, the disciplinary separation that occurred around the last century has led to epistemological, conceptual and practical issues that are reflected in the outcomes and knowledge learned by students. Indeed, many works in the literature focus on the difficulties students have in combining concepts and tools from the two disciplines and the difficulties teachers have in preparing courses or lessons that touch on both (Levrini & De Ambrosis, 2010; Margot & Ketler, 2019), or how the underlying reasoning belonging to the two disciplines leads to the creation of profound differences between the two (Rédei, 2020).

To address these issues, some authors have used a historical perspective to highlight the deep ties that bind the disciplines, e.g., Galili (2018) or Tzanakis (2002; 2016). Using a history-pedagogy-mathematics/physics (HPM/Ph) perspective, Tzanakis has analysed and explored the links between the two disciplines from a historical and epistemological point of view, highlighting how the use of practical examples of exchange and collaboration that occurred at the historical level can bring out the deep interconnection present between mathematics and physics. Using the Special Theory of Relativity (STR) as one of these examples, he focused on the differences in the approaches and backgrounds between Lorentz, Poincaré, Einstein, and Minkowski (Tzanakis, 2016).

In the physics education literature, STR has been studied from many different aspects, from conceptual change (Hewson, 1982; Posner et al., 1982; Scherr et al, 2001, 2002) to historical reconstruction (Levrini, 2002; Darrigol, 2006; Jankvist & Kjeldsen, 2011) or students' difficulties (Alstein, 2020; Guisasola, 2009; Tanel, 2014,). One aspect lacking from the literature, however, is an analysis that demonstrates the potential for learning at the boundary between disciplines, that is, one that focuses on the learning potential that an interdisciplinary view can bring for students.

Our work builds on other works already done in analysing original papers on the Special Theory of Relativity in which, through the use of a framework to highlight the mechanisms for crossing the boundary between disciplines (Akkerman & Bakker, 2011), the deep interconnections between the two and the different perspectives of the authors themselves are highlighted (Miani, 2021; Miani, 2022). These works were then carried forward through the construction of an analysis grid that allowed, through an iterative process of analysis and comparison of results, to highlight approaches peculiar to the individual disciplines to the general theme, and how these approaches succeed in highlighting different aspects of the same concept or tool, in particular Lorentz Transformations (Modica, 2022; Miani, Modica, Levrini, in progress).

Here we present the design and the implementation of an activity realised to focus on these approaches and the theoretical procedures used by the original authors (Lorentz, 1904; Poincaré, 1906; Einstein, 1905; Minkowski, 1908) to demonstrate their aims.

## 2 *Activity design*

The activity is based on a work of analysis and critical reading of 4 texts fundamental to the emergence of STR (Miani, 2021; Modica, 2022). The analysis of the texts aims to highlight the boundary-crossing mechanisms between disciplines applied by the authors. Each author has a very different background, approach and method of explaining and using mathematical tools. The lens developed in Modica (2022) allowed us to emphasize these mechanisms, highlight patterns of boundary crossing, and mark each article according to a different reasoning process. These results have been then used to guide the activity proposed to the students.

For the lesson, excerpts concerning the topic of Lorentz transformations were selected. The reason these were chosen is that, from the analyses conducted, they can be seen as a boundary object, as they provide different points of view and take on different facets depending on the perspective (or disciplinary approach) from which they are viewed. To contextualize the excerpts, Lucia prepared presentations of 5 minutes each to introduce the different papers and the intent of each of the 4 authors.

38 students took part in the activity, of which 9 had a mathematical background and 29 a physics one.

In the previous lectures, the class had already approached STR from different perspectives (e.g., its historical evolution, the way it has been proposed in school, and the different teaching approaches with which it can be taught within the field of physics education).

### *3 Activity development*

The activity is divided into two parts (two lessons of 3 hours each): in the first lesson, a presentation was made on the concept of boundary, presenting it from both an etymological and a social/historical point of view. after that, students were asked, through the use of a Wooclap, to present what they thought were the reasoning procedures peculiar to the disciplines of mathematics and physics:

- In your opinion, which proceedings do you recognise as being proper to or characterising disciplines such as mathematics or physics?

After everyone discussed these together, they moved on to the general presentation of the articles, on which the students then worked for the rest of the first lesson in 9 groups of 4/5 students each. At the end of the first lesson and the beginning of the second, the groups returned the answers given to the questions that had been posed as the delivery of the analysis activity, namely:

- What is the procedure (or procedures) that distinguishes each author?
- In your opinion, does such a procedure belong to mathematics, physics or both? and why?

Once all the opinions had been collected, we presented to the students the methods and results of our analysis conducted through the implementation of the grid based on Akkerman and Bakker's boundary-crossing mechanisms. following this presentation, we then carried on a discussion of about an hour in which we discussed the potential that a study such as the one just presented could bring compared to a general analysis without lenses such as the one carried on by the students themselves in the first lesson.

## 4 Results and discussion

As a result of the first question, 89 responses were received (Fig. 1). The responses can be viewed in detail in Modica (2022, p. 123). In general, what is evident is the presence of a general and at times stereotypical view of what mathematics and physics can be and their character traits.


**Figure 4.** Wordcloud representing the main procedures in mathematics and physics (obtained through wooclap.com)

About the analysis of the texts, it is noticeable how the students' analysis brought out the different processes peculiar to each author, but without being able to find distinct features among the four. Instead, the lens developed through the framework on boundaries succeeds in naming these processes, and distinguishing and defining them. The ability to name these procedures allows one to develop a clearer view of the approaches and intents of the different authors, thus enabling a deeper understanding of their reasoning mechanisms.

## 5 Conclusions

Special relativity plays a key transitional role between classical physics and modern physics. The construction of interdisciplinary activities such as the one presented can help to first understand the dynamics of the theory's development among different disciplines and the roles they can play in providing insights into the theory's content. In addition, this kind of activity allows one to extrapolate and name those mechanisms intrinsic to the disciplines that are often applied unconsciously but encapsulate the very epistemological essence of the disciplines, through which deeper and more multifaceted knowledge can be achieved.

### SELECTED REFERENCES


Akkerman, S. F., Bakker, A. (2011). Boundary Crossing and Boundary Objects. *Review of Educational Research, 81*(2), 132-169. DOI:10.3102/0034654311404435

Einstein, A. (1905). On the electrodynamics of moving bodies. Original paper “Zur Elektrodynamik bewegter Körper”, Annalen der Physik. 17. In Lorentz, H. A., Einstein, A., Minkowski, H., Weyl, H. (1952). *The Principle of Relativity: a Collection of Original Memoirs on the Special and General Theory of Relativity. With Notes by A. Sommerfeld*. Translated by W. Perrett and G. B. Jeffery. London: Methuen and Co., Ltd., 35-65.

Lorentz, H. A. (1904). Electromagnetic phenomena in a system moving with any velocity smaller than that of light. Huygens Institute - Royal Netherlands Academy of Arts and Sciences (KNAW), Proceedings, 6, 1903-1904, Amsterdam, 1904, 809-831. In Lorentz, H. A., Einstein, A., Minkowski, H., Weyl, H. (1952). *The Principle of Relativity: a Collection of Original Memoirs on the Special and General Theory of Relativity. With Notes by A. Sommerfeld.* Translated by W. Perrett and G. B. Jeffery. London: Methuen and Co., Ltd., 9-34.

Miani, L. (2021). *Highlighting Interdisciplinarity between Physics and Mathematics in Historical Papers on Special Relativity: Design of Blended Activities for Pre-Service Teacher Education*, Master Thesis (Alma Mater Studiorum, University of Bologna) 2021, https://amslaurea.unibo.it/23544/

Minkowski, H. (1908). Space and Time. Original paper: "Raum und Zeit", Physikalische Zeitschrift, 10: 75– 88. Translated by Lewertoff, F., Petkov, V. (2012). *Space and Time: Minkowski's Papers on Relativity.* Minkowski Institute Press, 39-55.

Modica, L. (2022). *Highlighting Interdisciplinarity between Physics and Mathematics in Historical Papers on Special Relativity: Development of an Analytical Tool for Characterising Boundary Crossing Mechanisms.* Master Thesis (Alma Mater Studiorum, University of Bologna).

Poincaré, H. (1906). Sur la dynamique de l'électron. Rendiconti del circolo matematico di Palermo, 21, 129– 176. Session of July 23, 1905. Translated by B. D. Popp in Popp, B. D. (2020). *Henri Poincaré: Electrons to Special Relativity,* 45-102. Springer Nature Switzerland AG 2020. DOI:10.1007/978-3-030- 48039-4_5

Tzanakis, C. (2016). *Mathematics & physics: an innermost relationship. Didactical implications for their teaching & learning.* History and Pedagogy of Mathematics, Montpellier, France. https://hal.archivesouvertes.fr/hal-01349231.

# AN INTERDISCIPLINARY APPROACH TO THE SECOND QUANTUM REVOLUTION FOR EDUCATING TO PROBABILISTIC THINKING: THE RANDOM WALK CASE.

**Sara SATANASSI, Olivia LEVRINI**

Department of Physics and Astronomy, University of Bologna, via Irnerio 46, Bologna, Italy

sara.satanassi3@unibo.it, olivia.levrini2@unibo.it

## ABSTRACT

Today more than ever we live in an accelerated society in continuous change (Rosa, 2013), an uncertain society in which we have to understand how to manage risk and contingencies. It is therefore pivotal to find new words, a new vocabulary and new competencies that can better help us to grapple with the contemporary society and provide the new generations with them. The physics of the '900, like quantum physics and science of complex systems, has proved particularly rich in this perspective as well as the technological revolutions in progress such as the Second Quantum Revolution at the heart of many investments today. Quantum technologies, exploiting the capabilities of isolating, controlling, and manipulating the single quantum object and its properties, give to quantum physics a new perspective and provide new teaching and learning possibilities. In this contribution, we present an approach to the Second Quantum Revolution and, in particular, an activity on classical and quantum random walk designed to shed light on the ongoing scientific and technological advancement and in its making as well as to touch on some of the most important foundational and epistemological debates such as the differences between the epistemic and ontological probability, and the true randomness that characterize quantum physics.

## *1 Introduction*

The contemporary society, more than ever, is requiring us to face global challenges (i.e., climate change, COVID pandemic) and it is pointing out how important educating to the “logic of uncertainty” is. As De Finetti argues, the main problem lies in the scholastic claim of reducing the whole process of thinking to the trivial logic of Yes or No (De Finetti, 1989). The science of the twentieth century is a rich source of concepts and theories that could promote the shift from deterministic thinking, namely the one that allows us to predict with certainty the evolution of a system given the initial conditions, to probabilistic thinking and the logic of the uncertainty. Together with science of complex systems, chaos theory, and theory of probability, quantum physics provide a new vocabulary and new words (like ontological probability, space of possibilities, uncertainty) that can help to better grapple with the contemporary society.

In particular, among the many societal, scientific and technological challenges, there is the research and development of very new technologies whose functioning is based on the law of quantum physics. As Dowling and Milburn stated, "*We are currently in the midst of a second quantum revolution*" (2003, p.1655). As other revolutions, this Quantum Revolution is not only challenging scientific research but also science education research, requiring it to help to expand the workforce in the field, to create new ecosystems and synergies between universities, enterprises, and schools, to convince new generations to choose STEM careers, and to promote a citizenship's quantum literacy[74].

To contribute to this challenge, within the projects I SEE[75] and IDENTITIES[76], we designed an approach to Second Quantum Revolution and developed a course for secondary school students to value the ongoing revolution as, first, a cultural revolution and its intrinsic interdisciplinarity as a potential way to include students' different kinds of reasoning, tastes and identities. We focused on the interdisciplinarity between physics, mathematics, and computer science with the aims of i. fostering the understanding of basic concepts of quantum physics (quantum state, superposition principle, state manipulation/evolution, measurement, and entanglement), ii. reflecting on differences between Boolean and quantum logic underlying the functioning of computers (e.g., Feynman, 1981; Deutsch, 1985; Wilce, 2002) and iii. exploring the role of ontological probability to promote the development of probabilistic thinking.

In the course, we introduce students to the pillars of quantum physics as a theory (the postulates and the basic concepts), to some relevant quantum technologies and algorithms such as quantum cryptography, quantum teleportation, and the quantum random walk. Emphasis is given to protocols and algorithms since they provide a glance at contemporary challenges and, at the same time, are addressable with a bunch of concepts that embody a paradigm shift (Kuhn, 1969), which unhinged the way we looked at and investigated nature as well as our conception of object: from classical to quantum physics. Furthermore, during the course students are asked to reflect also on the main implication of the Second Quantum Revolution on many

---

[74] https://qt.eu/app/uploads/2020/04/Strategic_Research-_Agenda_d_FINAL.pdf

[75] https://iseeproject.eu/

[76] https://identitiesproject.eu/

dimensions such as politics, economy, society, environment, education and so on.

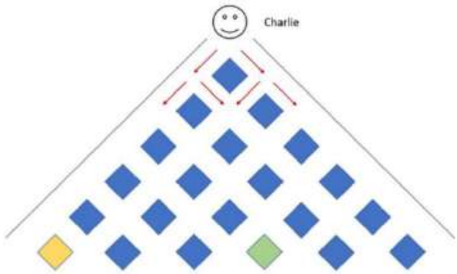
In the following, the random walk activity is presented as a context to reflect with high school students on the intertwining between mathematics, physics, and computer science and how the probability in the quantum case is intrinsic and rooted in the "main ingredients" of the quantum algorithm.

## *2 The quantum random walk: unpacking the ontological status of probability in the interplay between physics, mathematics, and computer science.*

The random walk activity was designed to explore the aspect carried out in the previous section. To pursue the first aim, we kept the quantum technicalities as simple and clear as possible to foster a deep understanding of the essential physical concepts: the concepts of quantum state and superposition principle, state manipulation and evolution, measurement, and entanglement. To pursue the second one, we compared the classical and quantum random walk algorithms in terms of the logic underlying their functioning. To reach the third one, we fostered students to reflect on the differences in terms of nature of probabilities in the classical and quantum case. The activity, that lasts about 2h, consists of a teamwork activity on the classical version of drunkard's walk problem (Pearson, 1905) and discussion; an introduction to the model of the random walk to scaffold the comparison between the classical and the quantum cases; a collective exploration of the classical and quantum random walk through an interdisciplinary lens (mathematic, physics, computer science) and the introduction of some application.

We start by posing the drunkard's problem:

*Charlie, after drinking too much wine, returns to the city of Eve. As soon as he crosses the city gates, a problem arises: he no longer remembers where he lives or the way back. He then begins to walk between the blocks, proceeding randomly and never going back, hoping to find the right way. What is the probability that Charlie will reach his house (green square) at random? Is the probability that he will reach, at random, his friend's house (yellow square) the same? How would you model the problem? How do you model the "randomly proceeding"?*




After students solve the problem in a group and collectively discuss the results, we build the model starting from students' solutions. In the building of the model, we mainly focus on the "randomly proceeding", which is modelled by *the coin flipping*, and the Charlie's moving through the city "never going back", which is modelled by choosing a *shift operator* according to which "if the outcome is head Charlie moves one step to the left, if the outcome is tail Charlie moves one step to the right". We build the algorithm as a sequence of coins flipping and application of the shift operator.

We then start to scaffold the comparison between classical and quantum random case by asking students what they expect if Charlie should follow quantum physics and by stressing the different logic of the coin. In fact, the classical coin can assume only two possible values "head OR tail". In the quantum case, we can design a coin that creates a superposition state, namely that transforms a state ($|head>$ or $|tail>$) in a "linear combination of head and tail": $|head> \rightarrow a|head>+b|tail>$, where the square of $a$ and $b$ give the probability of measuring respectively $|head>$ OR $|tail>$. The shift operator remains conceptually the same in the classical and quantum case. Following Kempe's treatment (2003), we introduce to student the quantum random walk algorithm and calculate Charlie's quantum probabilities to reach the yellow or the green house. The mathematical, physical, and computer science dimensions are intertwined, in the classical and in the quantum case, by focusing first on the model and the calculation to obtain the solution and the probability distribution (mathematics).

We pass then to physical examples and applications of the random walk (physics), and to the code in python showing what modelling the problem with a computer and a simulator means. Operationally, students are introduced to different kinds of representations (algebraic, circuital/logical, physical and the coding) both to stress some conceptual revolutionary aspects and to include different kinds of students' understanding and reasoning. By showing the mathematical and computational representations, we pave the way to reflect on the nature of the coefficients *a* and *b* that are incorporated in the mathematical description, intrinsic to the quantum object and we introduce the concept of ontological probability. The physics highlights the physical interpretation of the coefficients, namely the interference phenomenon that leads to an asymmetrical probability distribution in the quantum case. Furthermore, through the computational perspective, we introduce students to another pivotal debate: the problem of generating random numbers. The comparison with the classical case and the impossibility to generate "truly" random numbers allow us also to reflect on the intrinsic determinism that characterizes the standard classical computation and the intrinsic non-determinism of the quantum one. We conclude the activity by showing some application fields of the random walk algorithm, such as research algorithms (e.g., Shenvi, Kempe, Whaley, 2003), decision-making and optimization algorithms, econophysics (e.g., Orùs, Mugel, Lizaso, 2019) and art.

## *3 Final remark*

This activity is emblematic of our approach to the Second Quantum Revolution. The random walk activity touches on and show to students some of the contemporary challenges like research and optimization problems. Furthermore, it proved to be a context to take a glance at pivotal epistemological debates that can support students to embrace the uncertainty and probabilistic thinking, promote the development of the logic of uncertainty providing them with thinking tools to navigate the complexity of the present and orient themselves toward a more collectively and individually sustainable future (OECD, 2019).

## REFERENCES

De Finetti, B., & Mondadori, M. (1989). *La logica dell'incerto*. Il Saggiatore.

Deutsch, D. (1985). Quantum theory, the Church–Turing principle and the universal quantum computer. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, *400*(1818), 97-117.

Dowling, J. P., & Milburn, G. J. (2003). Quantum technology: the second quantum revolution. *Philosophical Transactions of the Royal Society of London. Series A:*

*Mathematical, Physical and Engineering Sciences*, 361(1809), 1655-1674.
Feynman, R. P. (1981). Simulating physics with computers, 1981. International Journal of Theoretical Physics, 21(6/7).
Kempe, J. (2003). Quantum random walks: an introductory overview. *Contemporary Physics*, 44(4), 307-327.
Kuhn, T. S., & Carugo, A. (1969). *La struttura delle rivoluzioni scientifiche* (p. 251). Torino: Einaudi.
OECD (2019). OECD Future of Education and Skills. OECD Learning Compass 2030: A Series of Concept Notes. Retrieved: https://www.oecd.org/education/2030-project/contact/OECD_Learning_Compass_2030_Concept_Note_Series.pdf
Orús, R., Mugel, S., & Lizaso, E. (2019). Quantum computing for finance: Overview and prospects. *Reviews in Physics*, 4, 100028.
Pearson, K. (1905). The problem of the random walk. Nature, 72(1865), 294-294
Shenvi, N., Kempe, J., & Whaley, K. B. (2003). Quantum random-walk search algorithm. *Physical Review A*, 67(5), 052307.
Wilce, A. (2002). Quantum logic and probability theory.

# MATHEMATIZATION OF FLUIDS MOTION

## An example from Hydrology

**Selvin GALO ALVARENGA, Ricardo CANTORAL†, Gisela MONTIEL-ESPINOSA**
Center for Research and Advanced Studies of the National Polytechnic Institute, Mexico City, Mexico
selvin.galo@cinvestav.mx, gmontiele@cinvestav.mx

## ABSTRACT

Recognizing the contributions in research on teaching and learning of mathematics of change, we present a first approximation to the process of the mathematization of fluid motion from a practice-centered approach. To study this process, we carried out an analysis of original sources in the area of Hydrology, from where we infer some variational practices.

## *1 Introduction*

In research on teaching and learning of mathematics of change (see Kaput, 1994) we find at least two approaches focused on variational aspects. On the one hand, the Covariational Reasoning framework (Carlson et al., 2002) developed on the notion of mental action, and on the other, the Variational Thinking and Language research line adopts a social point of view and it's the basis of the Socioepistemological Theory (ST) (Cantoral, 2020; Cantoral & Farfán, 2004). Our project arises within the last approach, where specific mathematical practices have been identified as fundamental in the mathematics of change, such as *comparison, seriation, estimation*, and *prediction* (Cantoral et al., 2018). Recognizing in the literature that most contributions in this research approach are being developed around the mathematization of a particle motion (see Cantoral et al., 2018), our project seeks to extend it by studying the process of the mathematization of fluid motion and begins analyzing original sources searching practices that accompany this process. In a later phase of the project, the results of this one will base a design research that promotes variational practices in university students.

## *2 Theoretical and methodological considerations*

The ST prioritizes practices that accompany work with mathematical objects, interested in their role in the construction of mathematical meaning (Cantoral, 2020). For this, a sociocultural posture is adopted, recognizing different forms of mathematical knowledge as valid, among them, *scientific*, *popular,* and *technical* (Cantoral et al, 2018). For this reason, studies are carried out in various scenarios (school, historical, and professional, among others).

According to a practice approach, practice is conceived as organized nexuses of activity composed of actions, which are executions of bodily doing and saying (Schatzki, 2001). In the ST focusing on their mathematical character, these practices are organized in a nested model composed at the first levels by *actions, activities*, and *socially shared practices* (see Cantoral, 2020). In the methodological phase, these practices are identified with analytical questions: *what is done and said*?, *how is it done and said*?, and *why is it done*?, (see Cantoral et al., 2023).

Here, we present a synthesis of the analysis of technical knowledge (Hydrology) in a historical setting, to identify variational practices that accompany the mathematization process of groundwater motion.

### *3 Preliminary results*

As a first approximation to the mathematization of fluid motion, relied on Freeze & Cherry (1979), we analyze mathematical practices on the Darcy's experiments developed for the water supply of Dijon (reported in 1856).

First, we identify the consideration of a straight circular cylinder filled with sand with a cross-section with area $A$, hermetically closed at each end with tubes allowing water inlet and outlet and equipped with a pair of manometers at a distance $\Delta l$ from each other (see Fig. 1a). We interpret [what is done?] *geometrize* the phenomenon and *recognize variables* involved; [how is it done?] *using geometric* shapes and *establishing a reference system*. Immediately, a *specific flow rate* as $v = Q/A$ (where Q is the volume of water per time unit) and $\Delta h = h_2 - h_1$ (difference of heights at the manometers) are defined. We assume [what is done?] *constructing new comparison* ways; [how is it done?] *establishing other variables* composed of more than one magnitude.

Darcy's experiments showed that $v$ is directly proportional to $\Delta h$ when $\Delta l$ is held constant, and inversely proportional to $\Delta l$ when $\Delta h$ is held constant. This relationship known as Darcy's Law describes groundwater flow in porous media and can be written as $v = -K\, dh/dl$ (where $K$ is the hydraulic conductivity, a soil property). We interpret [what is done?] *establishing a rela-*

*tionship between variables*; [how is it done?] *measuring and comparing the change of variables*.

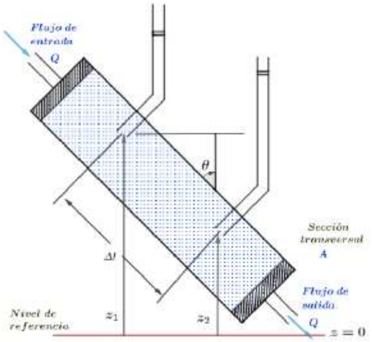



**Figure 1a.** Darcy's Law Experiment

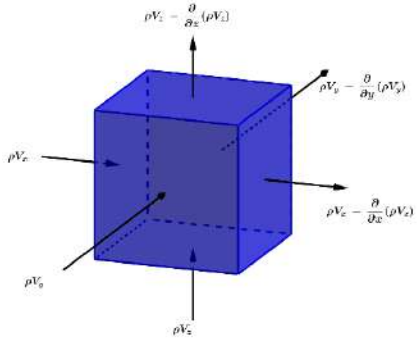


**Figure 1b.** Elemental control volume

This law is used to establish the equations for a steady-state flow in an isotropic homogeneous medium (the conductivity $K$ is constant and independent of the measuring direction). Consider an elemental control volume (see fig. 1b) and establish the continuity equation $-\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z} = 0$. We identify [what is done?] *geometrizing* the phenomenon; [how is it done?] *using* geometric shapes, *establishing* a reference system, and *using* physical principles. Then, incompressible fluid is considered where density is constant, $-\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} - \frac{\partial v_z}{\partial z} = 0$, and replacing $v_x$, $v_y$, y $v_z$ by its corresponding Darcy's Law we obtain the steady-state flow equation through an anisotropic saturated porous medium: $\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = 0$. Later, for an *isotropic homogeneous medium* $K_x = K_y = K_z$ y $K(x, y, z) = C$, the equation is reduced to the Laplace equation: $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0$. Here, [what is done?] *Simplifying the differential equation*; [how is it done?] *using* physical principles to *constantifying variables*].

In synthesis, a *geometrization of the phenomenon* is made by considering a straight circular cylinder (rectangle in fig. 1a) and a cube (fig. 1b), a *reference system* is constructed for *comparing magnitudes*: in the first case (fig. 1a), between heights at the manometers and distance between them; and second case (fig. 1b), between amounts of mass entering and leaving the elementary control volume. Also, *physical principles* are used to *express the relationships between magnitudes* in way of *differential equations*: in the first case, the proportional relationships are expressed in the form of Darcy's Law; and in the second, the Laplace equation. Finally, variables are *constantified* using physical principles, i.e., *mathematical expressions are reduced* by considering certain variables as constants, for example, by limiting the flow to a steady state or a homogeneous and isotropic medium.

## *4 Conclusion*

In this process of mathematizing of fluids motion in the case of groundwater, based on the nested model (Cantoral, 2020) we recognize as actions: *geometrizing of movement phenomena*, and *constructing a reference system*; as activities: *measuring and comparing magnitudes*; as socially shared practices: *constantifying* variables using of physical principles to arrive to differential equations that *relating* variables and describe the behavior of phenomena.

## REFERENCES


Cantoral, R. (2020). Socioepistemology in mathematics education. In S. Lerman (ed.). *Encyclopedia of Mathematics Education*. 790-797. https://doi.org/10.1007/978-3-030-15789-0

Cantoral, R., Espinoza, L. & Gaete-Peralta, C. (2023). Exponential behaviour and variational practices in Chilean newscasts: a socioepistemological view. *ZDM* 55, 147–161. https://doi.org/10.1007/s11858-022-01458-z

Cantoral, R. & Farfán, R. M. (2004). Thought and variational language in the introduction to Analysis. In W. Yang, S. Chu, T., Alwis & K. Ang (Eds) *Proceedings of the Ninth ATCM*. https://atcm.mathandtech.org/EP/2004/2004S341/fullpaper.pdf

Cantoral, R., Moreno-Durazo, A. & Caballero-Pérez, M. (2018). Socio-epistemological research on mathematical modelling: an empirical approach to teaching and learning. *ZDM Mathematics Education 50*, 77–89. https://doi.org/10.1007/s11858-018-0922-8

Carlson, M., Jacobs, S., Coe, E., Larsen, S. & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378. https://doi.org/10.2307/4149958

Cruz-Márquez, G. & Montiel-Espinosa, G. (2022). Indirect distance measurement and geometric work in the construction of trigonometric notions. *Acta Scientiae, 24*(4), 81–108. https://doi.org/10.17648/acta.scientiae.6911

Darcy, H. (1856). *Les Fontaines Publiques de la ville de Dijon*. Dalmont.

Freeze, A. & Cherry, J. (1979). *Groundwater*. Prentice-Hall.

Kaput, J. (1994). Democratizing access to Calculus: new routes to old roots. In A. H. Schoenfeld (Ed.), *Mathematical Thinking and Problem Solving* (pp. 77–192). Routledge.

https://doi.org/10.4324/9781315044613

Schatzki, T. (2001). Practice mind-ed orders. In T. Schatzki, K. Knorr Cetina & E. von Savigny (Eds.), *The practice turn in contemporary theory*. (pp. 50-63). Routledge. https://doi.org/10.4324/9780203977453

# THEME 5:
# TOPICS IN THE HISTORY OF MATHEMATICS EDUCATION

# BOARD GAMES IN MATHS EDUCATION

## comparing the *Intiateur Mathématique* (1910, Hachette) and Polyminix (2019, Creativamente)

**Maria Cristina MIGLIUCCI, Vanessa PANICHELLI**
independent scholars, Department of Education (Primary School Mathematics Research Unit) Roma Tre University Rome, Italy
mariacristina.migliucci@gmail.com, van.panichelli@stud.uniroma3.it

## ABSTRACT

The workshop focuses on educational materials (board games involving composition/decomposition) addressed to children, presenting an outstanding historical case, the 1910 "jeu de petits cubes" by Jacques Camescasse(1869-1941), named *Initiateur Mathématique*: inspired by the Charles-Ange Laisant's (1841-1920) book Initiation Mathématique and produced by the French publishing house Hachette and in the 1930's by CEL Coopérative de l'enseignement laic (on Célestin Freinet's (1896-1966) choice).We will connect its pedagogical and epistemological substratum with that of a recent educational board game produced in Italy, Polyminix, a 2D version of a game inspired by Solomon W. Golomb's (1932-2016)'s polyominoes. The tradition of toys and boxes for elementary learning started in the 18th century, including decomposition and composition and ratios (as in Friedrich Fröbel's (1782-1852) gifts and the learning of the decimal numeration system). Camescasse combined both aspects – also thanks to the cutting-edge technical design and materials – and had a clear outlook on mathematics education connecting arithmetic to geometry and the metric system: «La mathématique est une et… peu divisible», he wrote in the Notice containing the directions for the *Initiateur*, included in the box. The workshop will be divided into two parts: participants will be involved in hands on activities in both.

### *1. Jacques Camescasse and his Initiateur Mathématique*

The first part of the workshop focuses on the *Initiateur Mathématique*, that was presented using a copy "reengineered" and built by Alberto Tretta (Falegnameria Tretta, Gallese near Viterbo, Italy). We recommend beginning with some informations about his author: Jacques Camescasse (1869-1941).

Camescasse's contribution is placed at the turn of the 20th century, when many educational innovation proposals flourished, motivated by the spread of public education and inspired by the interest for the political and cultural value of the school and the renewed trust in the educational value of elementary mathematics[77]. From the point of view of the teaching of mathematics to children, the practices in use up to that time were mainly related to numerical training; on the contrary, the new initiatives also aimed to spread among the popular classes an educational mathematics that also includes geometry, and an approach that sees the pupils as protagonists, promoting their initiative, understanding, and enjoying mathematics. Jacques Camescasse, in the wake of the pedagogical vision of Charles-Ange Laisant (1841-1920) in his work Initiation Mathématique (1906), develops and patent produced with Hachette the educational material contained in a box, called *Initiateur Mathématique* (1910). He complies to Laisant's call "to the friends of childhood": his game is all about the pleasure of the child and he understands it as a factor of change, an educational aid capable to evolve a fossilized practice, around a few procedures. Laisant, a mathematician and parliamentarian of the radical left who then approached the anarchist movement, had long denounced this situation (Auvinet 2017), referring to Johann Pestalozzi (1746-1827) and Jean Macé (1815-1894), the latter author of the best-seller also translated abroad the *Arithmétique du Grand Papa* (1863). Camescasse was a freemason and esperantist, follower in youth of the French pedagogist Paul Robin (1837-1912) with whom he collaborated in the Prévost orphanage in Cempuis (Oise, northern France). His experience with students in the carpentry workshop is strongly connected to the conception of the game of small cubes of wood that can be assembled thanks to metal slats. Camescasse tried to implement tangibly the advice that Macé gave in the already cited book, "using sacks, boxes, beans and peas" as a first attempt to arrive at what he called "objective teaching": through objects that are seen, touched, moved; as Laisant also wrote -enthusiastic about producing a material more powerful than drawing or beans- *entre les doigts* (Laisant in Camescasse 1916, p. 5).

[77] Camescasse's *Initiateur Mathématique* and his educational contribution was studied by the second author in her master thesis (Panichelli 2021)

## *2. The workshop*

The workshop is recommended for 25 participants, to be held in a class with tables; it is organized in two main phases: first, the participants will put their hands on the reproductions of Camescasse cubes, and then they will move on to the Polyminix game. During both phases, the educational opportunities of these games will be discussed, commenting on experiences that have already occurred with both games, in the classroom with the pupils as well as with teachers in training.

### 2.1 The game box *Initiateur Mathématique* (1910)

The game box *Initiateur Mathématique* contained 1200 cubes, 600 red and 600 white and 144 steel bars. The box was accompanied by a Notice of the author: “Pour être sainement éducatif, tout enseignement doit être objectif, surtout au début. L'abstraction systématique introduite dans l'enseignement, sans préparation objective, est nuisible ». (Camescasse 1916, p. 10)


**Figure 1.** Left: Camescasse’s box, steel rods, and wooden cubes. Source: Jobbé-Duval 2020. Right: the illustrative booklet on the Initateur Mathématique, 1st edition, 1910. Source: Boutin 2019, p. 155 (possibly photograph of the specimen kept at CNAM)

Camescasse envisaged a first use for very young children to use the cube as building bricks. The game received much appreciation: Alain, a famous philosopher of the time,[78] about Camescasse's cubes :« Je me gardai bien de leur dire quelle était la solution juste car jeter la vérité toute trouvée à ceux qui la cherchent, c'est une mauvaise action et je les renvoyai au jeu de cubes. » (Camescasse 1916, p. 30), and the pedagogue Céléstin Freinet (1896-1966) who, in 1931, decided to bring back the game through his co-operative because he was convinced about the educational value of the game. Despite this since the mid-twentieth century this game has been used less and less and today is kept at the Musée des Arts et Métiers in Paris.


**Figure 2.** The reproduction of *Initiateur Mathématique* by Alberto Tretta's carpentry. Left: the cubes, made of two different types of wood, and aluminum rods; each cube contains two grooves that allow it to fit inside the bars. Right: the box containing the game. Picture by V. Panichelli.

It is important to handle the cubes with both hands, on a table (Fig. 3).

[78] Pseudonym of Émile-auguste Chartier (1868-1951)

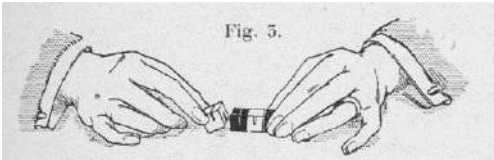


**Figure 3.** Example of manipulation of cubes for the construction of a row of ten. Source: Notice sur *l'Initiateur Mathématique* 1916, p. 17

At the beginning each participant will form a bar made by 10 cubes to represent the number ten; then participants will form the hundreds, combining ten bars of ten, working in groups (Fig. 4). We point out how in this teaching material the three components of mathematics are always altogether: numbers, geometry, and measurement; through the small cubes it is possible to introduce tens, hundreds and thousands as well as cm, $cm^2$ and $cm^3$. Camescasse himself stated that « La mathématique est une et… peu divisible» (Camescasse 1916, p. 36 )

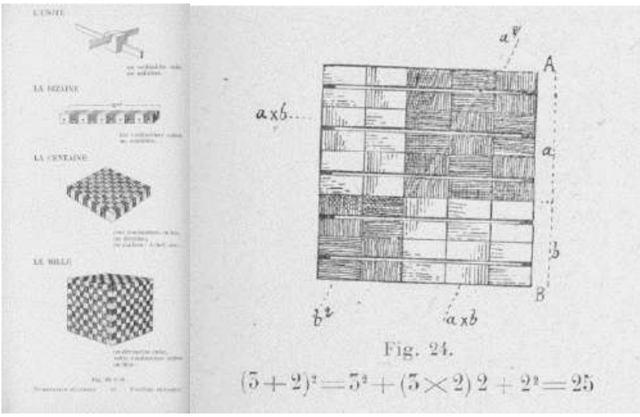



**Figure 4.** left: the ten, the one hundred and the thousand by cubes. Source: Camescasse 1916, p. 26; right: the square of the binomial represented with Camescasse' cubes. Source: Camescasse 1916, p. 31.

With his cubes we can explore the decimal numbering system, the operations but also the arithmetic theorems and measure, implementing an objective education through a tangible material. The main difference between these cubes and the more traditional blocks used to teach base ten arithmetics, is that the cubes can be assembled and disassembled, avoiding pushing too early children towards abstraction. The most important value of this game is its manipulative dimension that allows us to touch the numbers and the relationships between them. Following Camescasse's words: "Les jeux des enfants guident souvent l'éducateur attentive" (Camescasse 1916, p. 8), our role as educators is to follow our pupil's game.

### 2.2 The Polyminix board game (2019)

The second part of the workshop, as already mentioned, is devoted to Polyminix board game. Every participant is given a single bag with pieces and cards. The game Polyminix, inspired by the world of geometry with continuous forays into arithmetic, was born in the summer of 2019[79]. The educational potential of the game for geometry or measurement - key aspects of the introduction to scientific thought - is closely linked to its power of attracting students. It can be used from kindergarten (with appropriate simplifications) up to second grade secondary school. It is part of the materials proposed in a national competition for mathematics "Matematica per tutti" designed by ToKalon Association and is included in the teaching guide produced for the Erasmus+ ANFoMAM project[80] (see references).

Polyminix is based on a choice of 15 polyominoes and playing cards (Fig. 5). There are three types of cards: green (entry level), yellow (intermediate), red (difficult). Each card has a central white squared region, representing the playing area, that is the surface to be covered; the number written on the white region refers to its extension (how many unitary squares, see Millán Gasca 2016, Millán Gasca and Spagnoletti Zeuli, 2015). The white surface must be covered with polyominoes, according to precise indications, contained in the six black rectangles on top and bottom of each card.

[79] Plyminix was designed by the first author of the present paper, with Anna Mazzitelli, a primary school teacher, together with the gaming Italian company CreativaMente.

[80] For additional information about the national competition "Matematica per tutti" see https://matematicapertutti.it/; Anfomam (*Aprender de los niños para formar at los maestros en el área de matemáticas*) project, number 2018-1-ES01-KA203-050986.

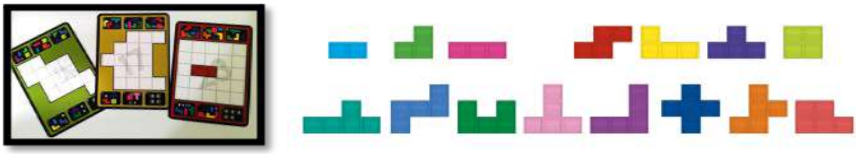


**Figure 5 left**:  Polyminix playing cards; **right:** Polyminix's polyominoes

The workshop consists in proposing some of the game modes represented in each card.

The first game mode "Cover the figure!". A choice of polyominoes which surely covers the white checked part is assigned, the right combination has to be found. In Figure 6, 3 possible solutions to the same problem. Sometimes it happens, during classroom activity, that the solution is hard to find, so someone tries a "partially correct solution" (Fig. 7): the solution involved a polyomino which was not among the allowed ones.

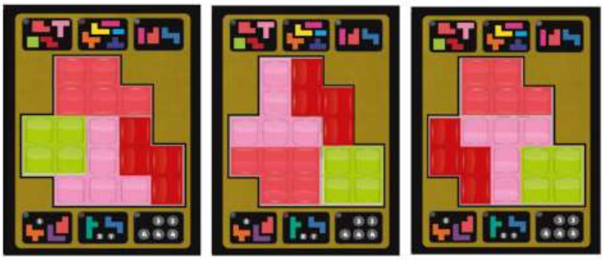


**Figure 6.** Three possible solutions to the same problem situation.

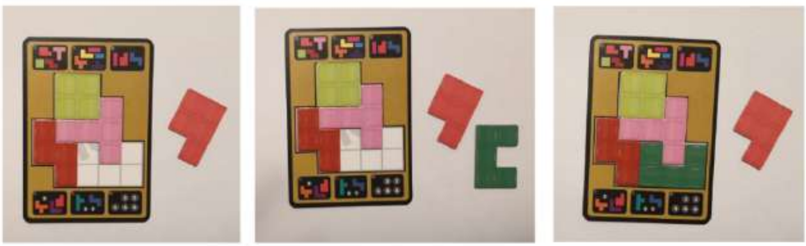


**Figure 7.** A "partially correct" solution

At this point the participants are offered what was proposed in the classroom by the teacher: a new problem, resulting from the previous partially correct solution. Cover a surface, with the only rule to use the dark green pentamino (the “C”). In Fig. 8 some possible combinations. This random problem born from an error has more than one solution, a good observation to do with the students: it happens very often that right solutions are not unique.

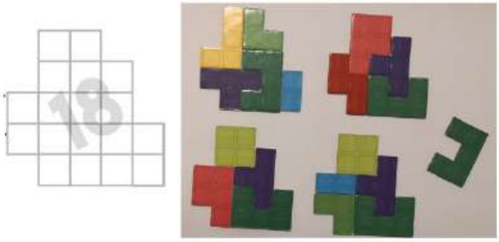



**Figure 8.** Some possible tessellations of the white squared area

We point out that we can explore a connection between geometry, measure, arithmetic, through composition and decomposition of polyominoes: to cover a figure with different polyominoes corresponds to obtain the same number adding different integers; a clear and tangible application, for example, of the commutative property in arithmetic and of geometric decomposition (Fig. 9)

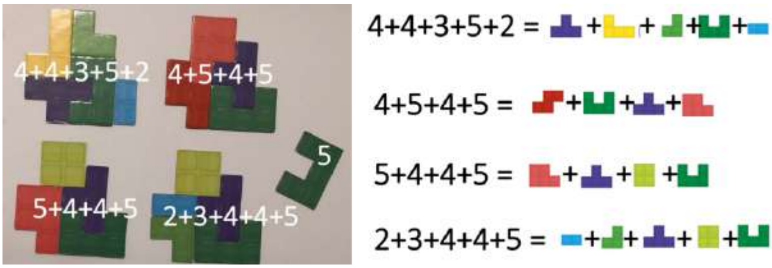



**Figure 9**. Geometrical decomposition and arithmetic decompositions of the number 18

The second game mode “find the missing piece”. Only part of the needed polyominoes is showed: to cover the figure the player must also find the missing one/ones. The missing polyominoes are indicated with letters X and Y, the unknowns, a first approach to the use of variables. After a first free exploration towards the solution, the leader encourages to proceed by reasoned exploration, possibly going into the world of arithmetics. If the area measures 18, the given polyominoes add up to 15, the missing polyomino has to be a tromino. (Fig. 10)

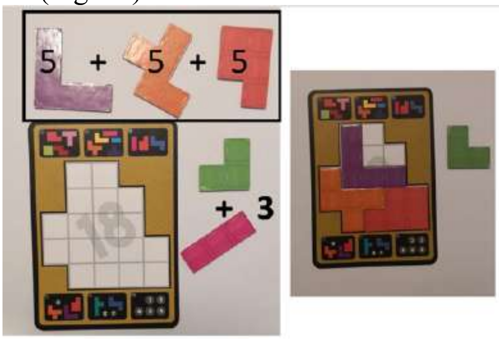



**Figure 10.** An arithmetic solution, but not a geometric one

Third game mode: “Find the right pieces whose measure is...”. The only clue given to the player is a set of numbers, referring to the size of the pieces, the player must choose which pieces, to cover the area.

### *3. Final remarks*

At the end of the workshop, starting from the comparison of the ancient and modern material, there will be a discussion with the participants on what unites both experiences that have been proposed; this discussion aims at bringing out first of all, the principle of composition and decomposition: both games are made of little bricks/cubes, that build and break up. Children can have the vivid perception of assembling and disassembling with both games, forming tens, hundreds and thousands (and back to the single cubes), touching the multiplication, with the cubes, as well as forming numbers and areas with the polyominoes.; this leads naturally to the question of connecting, in composing and decomposing, figures with numbers, geometry and arithmetic. Finally, a discussion about the power of game and playing is very appropriate, starting from some considerations extrapolated from the work of Johan Huizinga (1872-1945) *“Homo ludens”*:

- The game is mainly a free action; it is not real life; it is the awareness of being different from ordinary life.
- The game is a serious thing, but it is not just serious nor amusing, it belongs to an independent, unique and extraordinary category.

- The game is tension, suspense, excited expectation, the chance of a good or bad outcome.

Furthermore, we recall the point of view of Roger Caillois (1913-1978) who, before others, treated the theme of the play from a sociological point of view and after that, play has no longer been interpreted only as a mere entertainment or just a children's activity. Caillois stated that playing is something circumscribed, a space separate from common reality, a magical place where rules and ways of everyday life are suspended; he classified games relying on player attitude: *agon* (competition), *alea* (fortune), *mimicry* (simulation), *ilinx* (vertigo) (see Caillois 1958, Regoliosi 2022). Participants can be asked which categories of attitude they identify with, which ones were brought up during the session.

During game sessions, learning is encouraged and made easier since children develop a real *mimesis* (the aptitude of human beings to imitate, to become like anyone or anything, the capacity to look at the world outside oneself and learn by means of imitation, see Scaramuzzo 2016) within the object and the game, forgetting everything around, immersing completely in the fiction of the game. The small cubes were born in the wake of the battle undertaken by Laisant in favor of the education for children (see Millán Gasca 2015), but they are also born to instruct while having fun, so we can consider Camescasse's cubes as an educational aid halfway between games and didactic material, useful to introduce mathematical concepts and to physically represent mathematical theorems and also to make constructions; Polyminix, born in 2019, was designed with the same purpose.

## REFERENCES

Auvinet, J. (2017), De l'usage des récréations pour une Initiation mathématique selon Laisant C.A., *Dossier:Récréations mathématiques APMEP* 523:172-182.

Caillois, R. (1958/2001), *Man, play and games*. Champaign: University of Illinois Press.

Camescasse, J. (1916), *Notice sur l'Initiateur Mathématique. Jeu de petits cubes assemblables aidant à appliquer dans la famille et à l'école L'Initiation Mathématique de C.-A. Laisant*. 3° ed. Paris: Hachette.

Boutin., M. (2019), Les jeux dans les collections du conservatoire National des arts et métiers de Paris, 7 – L'Initiateur Mathématique, *Le vieux papier* 434: 154-163.

Golomb, S. (1954), Checkerboards and polyominoes, *The American Mathematical Monthly* 61 (10): 675-682.

Huizinga, J. (1938/1980), *Homo ludens.* Boston: Beacon Press.

Millán Gasca, A. (2015), Mathematics and children's mind: the role of geometry in the European tradition from Pestalozzi to Laisant. *Archives Internationales d'Histoire des Sciences*, 65(2): 759-775.

Millán Gasca, A., Spagnoletti Zeuli, E. (2015), La geometria nei materiali e nelle immagini per apprendere il sistema di numerazione posizionale decimale. Dalla storia alla scuola di oggi, *Periodico di matematiche* serie XI, CXXV (3): 23-40.

Millán Gasca, A. (2016), *Numeri e forme*. Bologna: Zanichelli.

Panichelli, V. (2021), *L'innovazione didattica a scuola come questione culturale: il caso di Jacques Camescasse e della scatola-gioco Initiateur Mathématique (1910),* unpublished Master Degree dissertation. Rome: Roma Tre University.

Regoliosi, L. (2022), *Un approccio pedagogico attraverso il gioco alle matematiche elementari tra gli 11 e i 15 anni: analisi di una proposta di innovazione didattica, il con-corso «matematica per tutti»,* [A pedagogical approach through games to elementary mathematics between 11 and 15 years old: analysis of the innovative didactical proposal the "Matematica per tutti" competition], unpublished PhD thesis. Rome: Roma Tre University.

Scaramuzzo, G. (2016), Aristotle's 'homo mimeticus' as an educational paradigm for human coexistence, *Journal of Philosophy of Education* 50(2): 246–260.

Schiopetti, V. (2015). *Charles-Age Laisant nella storia dell'istruzione matematica*, unpublished Master degree dissertation. Rome: Roma Tre University

# THE DEFINITIONS OF THE CONCEPTS “CIRCLE” AND “RHOMBUS” IN GREEK MATHEMATICS TEXTBOOKS USED IN PRIMARY EDUCATION FROM 1830 TO THE PRESENT DAY

**Garyfallia Alexandridou[1], Yannis Thomaidis[2], Konstantinos Nikolantonakis[3]**

1. Primary School of Chortiatis, Thessaloniki, Greece, g_alexandridou@hotmail.com
2. School Advisor (retired), Thessaloniki, Greece, gthom54@gmail.com
3. University of Western Macedonia, Florina, Greece, knikolantonakis@uowm.gr

## ABSTRACT

In this paper, the definitions of the concepts *“circle”* and *“rhombus”* are studied and recorded, as they appear in Greek mathematics textbooks used in primary education from the beginning of the 19th century to the present day. The study concludes that the traditional teaching method in the past was based on the approach of geometric concepts through the teaching of stereometry. The formal definitions of the concepts were included in the older textbooks, while today there is an absence of definitions of the basic geometric concepts, as teaching is done in an empirical way, emphasizing on manipulative and pictorial material. A correlation is also made between these definitions and those recorded in the course of the evolution of mathematics for the above notions, highlighting aspects of their historical development, with an ultimate aim to their further usage in teaching and learning.

## 1. *Introduction*

Mathematical definitions are fundamental to the axiomatic structure that characterizes geometry and play a key role in the development of deductive reasoning and proof skills (Mariotti & Fischbein, 1997). They are the most important means of conveying the meaning of mathematical concepts, the basic tools of mathematical language used for written and oral communication in teaching and learning mathematics (Shir & Zaslavsky, 2001). In addition, definitions help to distinguish between critical and redundant elements so that one can clearly see a mathematical situation and can distinguish examples from counterexamples of a concept. In this way, deeper conceptual understanding is achieved and the use of correct mathematical terminology is reinforced (Morgan, 2005).

School textbooks are a fundamental teaching tool that is used in the educational process, being an important resource for students, as they are among the most important factors influencing their learning opportunities in mathematics, while at the same time serving as a means of guiding the teacher. Moreover, mathematics textbooks as a theme of research have continued to receive rapidly growing attention internationally. Investigations on the development of the exposition of a given concept over a certain period contributes to understanding the meaning attributed to the concept in a given community in that period (Schubring & Fan, 2018).

Therefore, recognizing the importance of the role of definitions in mathematics education and given that textbooks are the main source of educational opportunities for students to teach and learn mathematics, this study presents the historical development of the definitions of the concepts of the circle and the rhombus in Greek mathematics textbooks from the beginning of the 19th century to the present day. Our points of interest are the way of approaching the above primary concepts of geometry, including the definitions used, in terms of their nature and type. The study takes into consideration whether the definitions presented are "genetic", as they show how the concept is constructed, or whether they are given on the basis of the main characteristics of the concepts. Furthermore, it is examined whether the definitions are "hierarchical" or "partitional". According to DeVilliers (1994), the classification of any set of concepts depends on the process of creating a definition of the corresponding concepts. A "hierarchical" definition allows the inclusion of more particular concepts as subsets of the more general concept. In a "partitional" definition on the other hand, the concepts involved are considered disjoint from each other (i.e. squares are not considered rectangles).

From the study of school textbooks, the definitions of the above concepts from five (5) textbooks from the period 1830-1982 were recorded and selected to be presented in this paper, as well as the definitions contained in the mathematics textbooks used from 1982 to 2006. Moreover, this paper also presents the way of teaching the geometric concepts of the circle and the rhombus as suggested by the current Curriculum, mathematics teaching guidelines and teacher's books.

## 2. *School mathematics textbooks from 1830 to 1982*

### 2.1 Fatseas, A. (1870). Compendium of geometry A.M. Legendre

This Geometry begins like Euclid's first book, first introducing the "definitions" of the basic concepts and then proving the "propositions" that follow.

The circle is defined as: "the shape produced by the rotation of a line about one of its ends in a plane" (p. 18). This definition is genetic, as it shows the 'birth' of the circle from the rotation of a straight-line segment, which remains in the same plane, around one of its ends.

The way a circle is constructed is also contained in Heron's definition of the circle, where it is explicitly mentioned: “Circle is the figure described when a straight line, always remaining in one plane, moves about one extremity, as a fixed point until it returns to its first position" (Kipouros, 1995, p. 33).

Also, according to Spinoza, the most appropriate definition of a circle is "a figure constructed from a line by holding one end of the line in place while rotating the other end" (Goudeli, 2015, pp. 7-8).

The rhombus is defined as the quadrilateral "having sides equal and angles unequal" (p. 4). It is noted here that the rhombus is not part of the family of parallelograms but is separated from the other concepts as a subcategory of quadrilaterals. Therefore, the definition is partition and is given based on equality of sides and angles of quadrilaterals.

### 2.2 Karagiannidis, A. (1906). Practical geometry for the 4th, 5th and 6th grades

The book is divided into three parts. In the first part, the rhombus is defined as “the parallelogram which has all sides equal to each other” (p. 14). This definition is hierarchical, as it includes the rhombus in the family of parallelograms. It is also stated that all squares are both rhombuses and rectangles because they have both sides equal to each other and right angles. This classification is consistent with the modern hierarchical classification of quadrilaterals.

The second part begins with the study of the sphere. In this section there is a definition of the circle as the "flat surface, one point of which is equidistant from all the points of the line to which it ends. This point is called the center of the circle” (p. 24). This definition is in accordance with Euclid's definitions (15) and (16), where it is stated that “a circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another. And this point is called the center of the circle” (Exarchakos et al., 2001, p. 91). Thus, the term "circle" here also means the circular disc, while as stated a little further on, the "line on which the circle ends is a single curve and is called the circumference" (p.24).

### 2.3 Michaelides, E. (1946). The Little Geometer - Practical Geometry

In this book, in the section on rectangular parallelepipeds the rhombus is defined as "the parallelogram that has all four sides equal" (p. 27). This is followed immediately by instructions for constructing a rhombus and it is clearly stated that a square is not a rhombus. It is worth mentioning that, of all the geometry books studied for this paper, this book is the only one that contains the term "rhomboid" by which is meant the oblique parallelogram, just as in Euclid's Elements. Specifically, it is stated that "the oblique parallelogram is a rhomboidal shape (the shape that pieces of baklava pastry are usually shaped like)" (p. 28).

The circle is studied in the section of the cone. It is approached empirically, as students are asked to copy the base of a cone and the resulting shape is then called a circle. "The circle is bounded by a closed and regular curved line. This closed and regular line is called circumference" (p. 51).

### 2.4 Papadopoulos, P. D. (1952). Practical geometry for the use of pupils of the 5th and 6th grade of primary schools

In Papadopoulos' Geometry, the rhombus is studied in the section on the oblique parallelogram and is defined as the oblique parallelogram that has all four sides equal (p. 43).

In the next section, and after a preliminary study of the cylinder, students are intuitively introduced to the concept of the circle. Here "circle is called a flat surface enclosed by a curved line, all the points of which are equally distant from a point on the surface, which is situated in the middle and is called the center of the circle" (p. 64). It appears that by "circle" is also here understood the circular disc.

### 2.5 Kyriazopoulos, A. & Alexopoulos, V. A. (1969). Arithmetic – Geometry of the 5th grade of primary school

In this textbook the study of the rhombus is done in the section of parallelograms where the hierarchical definition given is "a rhombus is an oblique parallelogram which has all its sides equal" (p. 145). The circle is studied in the next section. As in the previous textbooks, the concept is first approached intuitively and then it is defined. Thus, “circle is a flat segment enclosed by a closed curved line, all the points of which are equally distant from a point called the center” (p. 152).

### 3. *School mathematics textbooks used in the period 1982-2005*

In 4th grade, students are asked to construct a circle to lead them intuitively to the conclusion that a circle is the curve formed after a complete rotation of the end of a paper strip around a fixed point.

In the 5th grade, geometric solids are studied first and later flat shapes. The rhombus is included in the family of parallelograms and is then said to have all four sides equal, although this is not a formal definition of the concept. In the next chapter, the concept of the circle and the circular disk is taught. An introductory activity helps students to distinguish the concept of a circle from that of a circular disc.

In grade 6, the lesson concerning the circle begins with some review questions and then the definition of the circle is given: "A circle is a closed curved line whose points are all equally distant from the center, while a circular disk is the flat surface enclosed by the circle" (p.126).

### 4. *School mathematics textbooks used from 2006 to the present day*

In the current mathematics textbooks for all grades of primary school, there are minimum formal definitions of basic geometric concepts. It is also observed that formal definitions of the concepts circle and rhombus are not mentioned in the textbooks. Concerning the teaching of these concepts, it is concluded, that the circle is a concept that is studied very often in all primary school grades, while on the other hand, there is a lack of involvement of pupils with the concept of the rhombus, which appears for the first time as a teaching subject in the third grade. An attempt to emphasize the hierarchical relationships between quadrilaterals is evident from the proposed 4th grade activities, with particular emphasis on students' understanding that the square is a special case of the rhombus and the rectangle. In grade 5, no reference is made to the concept of the rhombus, but a deeper approach to the concept of the circle is attempted, as pupils are introduced for the first time to the terms relating to the main elements of the circle (center, radius, diameter, length, number π). In grade 6, some geometric definitions and formulas are given, but these do not include definitions of basic geometric concepts.

## 5. *Conclusions*

From the study of the textbooks, it became evident that the approach to basic geometric concepts, was usually through the teaching of stereometry. It was also observed that while the older textbooks included formal definitions of concepts, today geometry in primary school is approached intuitively.

Regarding the definitions of the concept of the circle, it appears that in most textbooks the definitions have common elements with Euclid's definition. The only textbook that contains a genetic definition of the circle is Fatsea's Compendium of Legendre's Geometry (1870), which describes how the circle is constructed (like Heron's and Spinoza's definition).

It is also observed that the meaning attributed to the term "circle" changes over the years. Euclid's definition establishes the circle as the surface delimited by its circumference, as is similarly observed in school geometry textbooks until 1982. Today, the term "circle" corresponds to the circumference of the circle.

Regarding the concept of the rhombus, it is worth mentioning that out of the 23 older textbooks studied, the concept of the rhombus is mentioned in only 9 of them, which suggests that the rhombus was not a priority subject in the teaching of geometric concepts at primary school level. The most common definition that appears is "a parallelogram that has all four sides equal", which is hierarchical as it places the rhombus in the family of parallelograms. In two cases, however, there are two partition definitions based on the equality of sides and angles of quadrilaterals, which are identical to the definition of the Elements. The term "rhomboid" appears only in one textbook, referring to the oblique parallelogram, as in Euclid's Elements.

In concluding this paper, it is considered necessary to point out that the role of definition in mathematical thinking is somehow neglected in today's primary school mathematics textbooks and curricula. It is not clear whether this is because it is taken for granted or simply because it is overlooked. We must, however, remember that there are contents in which reference to a formal definition is essential for the correct performance of exercises.

**REFERENCES**

Goudeli, K. (2015). *Introduction to Spinoza's philosophy.* Association of Greek Academic Libraries, National Metsovio Technical University of Athens, Athens.

De Villiers, M. (1994). The Role and Function of a Hierarchical Classification of Quadrilaterals. *For the learning of mathematics*, *14*(1), 11-18.

Exarchakos, Th., Kontogiannis, D., Ntziachristos V., & Tsigoni T. (2001). *Euclidean ¨Elements¨. Modern rendering with introduction, explanations and commentary. Volume I: The geometry of the plane, books I,II,III,IV,V,VI,* Athens.

Fatseas, A. (1970). *Compendium of geometry by A. M. Legendre*, Athens. Retrieved from http://e-library.iep.edu.gr/iep/collection/browse/item.html?code=86-84215&tab=01

Karagiannidis, A. (1906). *Practical geometry for the 4th, 5th and 6th grades*, Ilia N. Dikaiou Eds, Athens. Retrieved from. http://e-library.iep.edu.gr/iep/collection/browse/item.html?code=01-18796&tab=01

Kipouros, Ch. (1995). *Heron Alexander's - Names of Geometric Terms - Geometric*. Retrieved from http://www.hms.gr/apothema/?s=ss&i=2

Kyriazopoulos, A., & Alexopoulos, V. (1969). *Arithmetic - Geometry of the 5th grade of Primary School, (1st Edition),* Athens: OEDB. Retrieved from. http://e-library.iep.edu.gr/iep/collection/browse/item.html?code=03-20555977&tab=01

Mariotti, M. A., & Fischbein, E. (1997). Defining in classroom activities. *Educational Studies in Mathematics*, *34*(3), 219-248.

Michaelides, E. (1946). *The Little Geometer - Practical Geometry. (6th Edition),* I.D. Kollaros & Co. (Eds), Athens. Retrieved from http://e-library.iep.edu.gr/iep/collection/browse/item.html?code=03-20560573&tab=01

Ministry of Education & Pedagogical Institute (2004). *My mathematics 4th grade. Part One,* Athens: O.E.D.B. Retrieved from http://users.sch.gr/vaskitsios/katsba/dim/biblia/index.htm#6

Ministry of Education & Pedagogical Institute (2004). *My mathematics 5th grade. Part Two,* Athens: O.E.D.B. Retrieved from http://users.sch.gr/vaskitsios/katsba/dim/biblia/index.htm#6

Ministry of Education & Pedagogical Institute (2004). *My mathematics 6th grade. Part One,* Athens: O.E.D.B. Retrieved from http://users.sch.gr/vaskitsios/katsba/dim/biblia/index.htm#6

Morgan, C. (2005). Word, Definitions and Concepts in Discourses of Mathematics, Teaching and Learning, *Language and Education*, *19* (2), 102-116.

Pedagogical Institute (2003). *Interdisciplinary Unified Framework of Curricula - Analytical Curricula of Compulsory Education*, Volume A, Athens.

Papadopoulos, P. D. (1952). *Practical geometry for the use of pupils of the 5th and 6th grade of primary schools,* Athens: P.Dimitrakos. Retrieved from http://e-library.iep.edu.gr/iep/collection/browse/item.html?code=20-01513&tab=01

Schubring, G., Fan, L. (2018). Recent advances in mathematics textbook research and development: an overview. *ZDM Mathematics Education* **50**, 765–771. https://doi.org/10.1007/s11858-018-0979-4

Shir, K., & Zaslavsky, O. (2001). What constitutes a (good) definition? The case of square. In M. van den Heuvel -Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education, 4*, 161–168. Utrecht, Netherlands: Utrecht University.

# TEXTBOOKS DEVELOPED IN THE PATH OF MODERN MATHEMATICS (1976-1980): A COMPARISON

**Mária Cristina ALMEIDA**
Centro Interdisciplinar de Ciências Sociais (CICS.NOVA), UNL
malmeida@fcsh.unl.pt

## ABSTRACT

From 1968 to 1975 the centralized Portuguese school system began with six years of mandatory education (6-11 years old). Those that wanted to pursue education should choose between secondary schools (liceus) and technical schools. From 1976, a unification of the remaining secondary school years was implemented, and the secondary education comprised a General Course – grades 7, 8 and 9 – and a Complementary Course – grade 10 and 11. The establishing of the Unified Secondary Course required curricular changes, and new mathematics programs are issued. As a consequence, three mathematics textbook collections for the General Course gradually appeared on the market. In the period of our study the ideas of modern mathematics were being applied in Portuguese schools. This text intends to understand different ways to interpret the mathematics programs for Unified Secondary Course, by comparing first editions of these collections of textbooks. We looked at the general structure of the texts, the intended relationship with the reader and the ways in which exercises were proposed. The present text focus on the introduction of the grade 7's topic 'Numerical equations of the first degree in Q' and identifies differences and similarities in the approach to this concept. Results show that two of the collections were organized to promote the active participation of students. Observing the introduction of the topic, there are similarities in the approaches of two of the textbooks that used inverse operations to solve equations whereas the other used scales. It stands out that only one textbook refers to the equivalence of equations. Regarding the exercises, two of the textbooks have an Exercise Book and a Worksheet Book, respectively, to support students' study. The third includes a list of exercises on solving an equation as activities for the students. The study was based on a documental analysis of a descriptive and interpretative nature, with a historical perspective. Our sources were legislation, programs, and textbooks.

## *1 Introduction*

From 1968 to 1974, the centralized Portuguese school system began with six years of mandatory education (6-11 years old), followed by parallel branches for secondary education: the liceus and the technical schools. The 25th of April Revolution that occurred in 1974, overthrew the dictatorial regime and restored democracy in Portugal. Later, the structure of the education system began to change. A major alteration was the elimination of the two educational tracks that existed for secondary education and the creation of the Unified Secondary Course, in 1976. The unification was considered a means to balance educational opportunities for all students. The immediate result was the alteration of designation, the Liceus and the Technical Schools became Secondary Schools. The secondary schooling encompassed two parts: the lower secondary (7th – 9th grade) and the upper secondary (10th – 11th grade).

In Portugal, the textbook approval system established one single officially approved textbook for all school disciplines since 1947. After the 1974 revolution, the teachers at each school decided on the adoption of textbooks. They chose among textbooks produced by private publishers (Almeida, 2007).

In the panorama of mathematics education, the reform of Modern Mathematics attempted major changes in content and methods for teaching mathematics, in many countries. This curricular movement brought to the agenda a proposal that gains different interpretations in the countries that participate in this renewal. There are many journals and books that circulate and are translated, disseminating the new modern mathematics. In each country the movement is received and incorporated according to its culture and its specificities (Moon, 1986). The reform took place in all levels of education from primary to higher education in most countries of the world. From the mid-1970s onwards, other curriculum options were developed internationally, and reform was declining (Furinghetti, Matos, & Menghini, 2013).

The Modern Mathematics movement affected Portuguese school mathematics culture in two major ways, paralleling international trends. Firstly, there were major changes of the representation of what constitutes appropriate mathematical content. Secondly, mathematics was believed to be a major driver for social and economic development (Matos, 2009). In Portugal, the events concerning this reform can be divided into three intertwined periods: the beginnings, from 1957 until 1963, in which the flow of new ideas can be detected; experimentation, from 1963 to 1968, during which the new ideas were implemented in classrooms; and dissemination, from 1968, that saw the gradual generalization of the reform to all students. From 1968 until 1974, programs were sequentially modified to incorporate the new ideas as these pupils progressed through the cycles. After gradual changes, the new math programs had been replaced in early 1990 (Almeida e Matos, in press).

These initial considerations give an insight on the context of our study. The end of the single officially approved textbook, allowed the publishing of textbooks that provide their author(s) interpretation(s) of the mathematics programs for secondary education established in 1976. We traced three mathematics textbook collections developed for the lower secondary (7th, 8th and 9th grade), the name of each collection is *Compêndio de Matemática*, *Eu e a Matemática* and *M*. It is the purpose of this text to understand the authors' perspectives presented in these collections of mathematics textbooks by looking at the general structure of the textbooks, the intended relationship with the reader and the ways in which exercises were proposed. Here we will focus on the textbooks for the 7th grade and on the introduction of the topic 'Numerical equations of the first degree in Q'. Data analysis followed a proposal by Maz (2005) adapted to the approach of first degree equations. In it, we looked at the historical context, the author, the structure of the work, and the initial approach of first degree equations proposed by the textbooks.

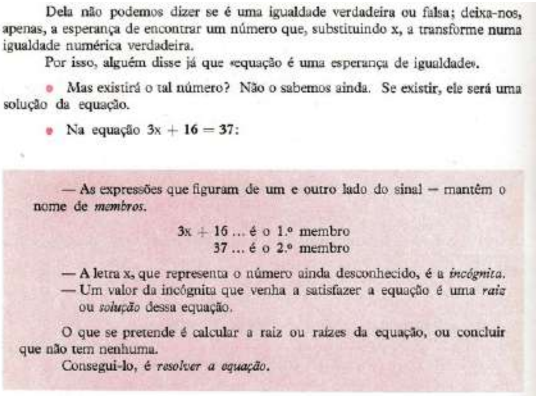
We based our research on a documental analysis of a descriptive and interpretative nature with a historical perspective. The documentary corpus consists of mathematics textbooks for secondary education. We analysed the first edition of the textbooks within the period of our study. Other sources were legislation and mathematics programs.

## *2 The collections*

### 2.1. Collection '*Compêndio de Matemátic*a'

The authors António Almeida Costa, Alfredo Osório dos Anjos e António Augusto Lopes were teacher trainers and had been active participants in the reform of Modern Mathematics in the liceus. They were authors of the single official approved mathematics textbook for the liceus, published in 1971.

In the preface the authors state that the textbook is their interpretation of the program, responding to a personal stand on the prosecution of the learning objectives for the 7th grade. The program for the 7th grade has seven topics, but the Index of the textbook show only six chapters, that correspond to six of the programs' topics, the topic 'geometric transformations' is part of the chapter corresponding the topic function. The colour of the text and of the titles of sections is black and red, respectively.



Dela não podemos dizer se é uma igualdade verdadeira ou falsa; deixa-nos, apenas, a esperança de encontrar um número que, substituindo x, a transforme numa igualdade numérica verdadeira.

Por isso, alguém disse já que «equação é uma esperança de igualdade».

- Mas existirá o tal número? Não o sabemos ainda. Se existir, ele será uma solução da equação.

- Na equação $3x + 16 = 37$:

— As expressões que figuram de um e outro lado do sinal — mantêm o nome de *membros*.

$3x + 16$ ... é o 1.º membro
37 ... é o 2.º membro

— A letra x, que representa o número ainda desconhecido, é a *incógnita*.
— Um valor da incógnita que venha a satisfazer a equação é uma *raiz* ou *solução* dessa equação.

O que se pretende é calcular a raiz ou raízes da equação, ou concluir que não tem nenhuma.

Consegui-lo, é *resolver a equação*.

**Figure 1.** Definitions included in a box
Source: Costa e Anjos (1976, p. 82)

There aren't workbooks associated with this collection. There is usually a selection of application exercises and their solution at the end of each section of the chapters. There are no references or indication of other sources related to the mathematics topics addressed in the textbook. The presentation of contents, though expository, tries to motivate the student to reflect and guide him or her in reaching conclusions, leaving, for example, blank spaces to complete, and posing questions to the student. The use of boxes helps to stress parts of the text of the various sections (Figure 1).

## 2.2. Collection '*Eu e a Matemática*'

The first author Maria Engrácia Domingos was a certified teacher of liceus, the other authors Mário Cerqueira Correia and Télio T. Fernandes were authors of textbooks for technical schools.

These books are written only in black and do not have bibliographic references or indication of other sources related to mathematical subjects. A glance at the Index show that the books follow the 7th grade programs' sequence of topics, but they also reveal that the authors had some autonomy, in adding extra contents, regarding pre-requisites.

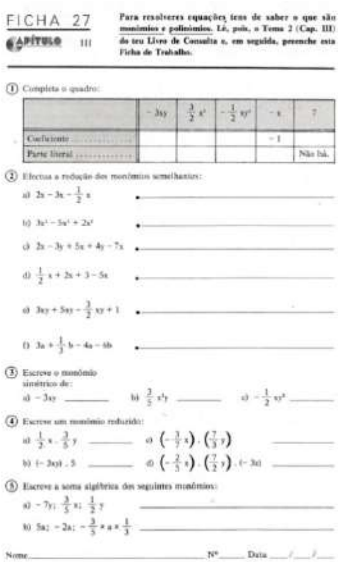

FICHA 27
CAPÍTULO III
Para resolveres equações tens de saber o que são monómios e polinómios. Lê, pois, o Tema 2 (Cap. III) do teu Livro de Consulta e, em seguida, preenche esta Ficha de Trabalho.
1 Completa o quadro:
Coeficiente
Parte literal
Não há.
2 Efectua a redução dos monómios semelhantes:
3 Escreve o monómio simétrico de:
4 Escreve um monómio reduzido:
5 Escreve a soma algébrica dos seguintes monómios:
Nome
Nº
Data

**Figure 2**. Example of a Worksheet
Source: Domingos, Correia e Fernandes (1976)

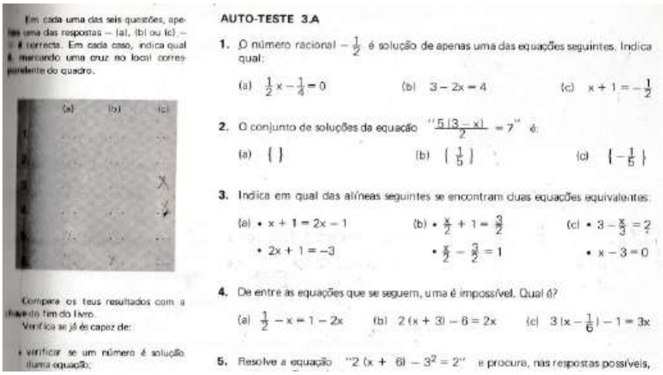
In a foreword, the authors describe the collection and provide recommendations to students and teachers. They explain that there is a Reference Book and a Guidance Book, outlining the purpose of these books. The Reference Book is assumed as a source of information, following all topics of the program for the 7th grade. The Guidance Book' function consists of facilitating the acquisition of knowledge, through Worksheets (Figure 2), which correspond to all the topics presented in the Reference Book. There are references to the pedagogical principles in which the work stands that are those of personalized teaching, which is based on adapting teaching to individual differences. The authors leave proposals to support the teacher's choice of classroom practices. There were Worksheets at the end of each topic that allowed students to assess understanding before moving on.

## 2.3. Collection '*M*'

The authors Paulo Abrantes and Raúl Fernando Carvalho were members of a group of mathematics teachers that advocate changes in mathematics teaching and learning. They participated in the foundation of the Mathematics Teachers Association, in 1986, and had an active role in teacher education and in mathematics education.

Textbooks are printed in two colours, black and red, and do not present bibliographic references or indication of other sources related to mathematical subjects. Exercises Books are printed in black only. The books are structured according to the 7th grade program.

In an introduction to their work, the authors assume it was conceived for the student. So, the caution in preparation included: the language used; the examples chosen; the form of initial approach to the concepts; the form of active participation in the study that is proposed to the student ("activities", notes, ...).


Em cada uma das seis questões, apenas uma das respostas — (a), (b) ou (c) — é correcta. Em cada caso, indica qual é, marcando uma cruz no local correspondente do quadro.

| | (a) | (b) | (c) |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | X |
| 4 | | | X |
| 5 | | | |
| 6 | | X | |

Compara os teus resultados com a chave do fim do livro.

Verifica se já és capaz de:

• verificar se um número é solução duma equação;

**AUTO-TESTE 3.A**

**1.** O número racional $-\frac{1}{2}$ é solução de apenas uma das equações seguintes. Indica qual:

(a) $\frac{1}{2}x - \frac{1}{4} = 0$ (b) $3 - 2x = 4$ (c) $x + 1 = -\frac{1}{2}$

**2.** O conjunto de soluções da equação "$\frac{5(3-x)}{2} = 7$" é:

(a) $\{\ \}$ (b) $\{\frac{1}{5}\}$ (c) $\{-\frac{1}{5}\}$

**3.** Indica em qual das alíneas seguintes se encontram duas equações equivalentes:

(a) • $x + 1 = 2x - 1$ • $2x + 1 = -3$

(b) • $\frac{x}{2} + 1 = \frac{3}{2}$ • $\frac{x}{2} - \frac{3}{2} = 1$

(c) • $3 - \frac{x}{3} = 2$ • $x - 3 = 0$

**4.** De entre as equações que se seguem, uma é impossível. Qual é?

(a) $\frac{1}{2} - x = 1 - 2x$ (b) $2(x + 3) - 6 = 2x$ (c) $3(x - \frac{1}{6}) - 1 = 3x$

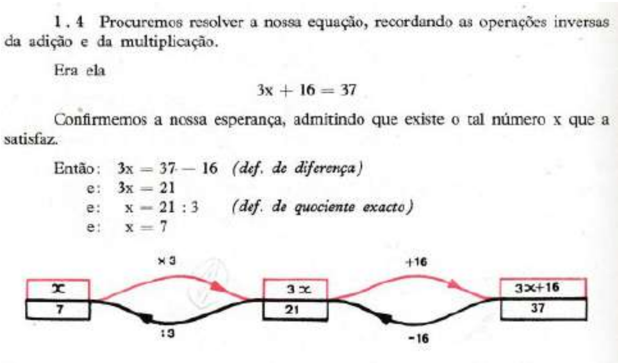
**5.** Resolve a equação "$2(x + 6) - 3^2 = 2$" e procura, nas respostas possíveis,

**Figure 3.** Example of a self-assessment test
Source: Abrantes e Carvalho (1980, p. 49)

There are two books for the 7th grade, a Textbook, which is a study book for students, and an Exercises Book, was aimed at helping the pupils in the study of mathematics. The Exercises Book includes, at the end of each subchapter, a self-assessment test, through which the students can control their learning (Figure 3). The two books are instruments for the student, which allow them to structure, acquire and evaluate their knowledge.

### *3 Numerical equations of the first degree in Q*

The topic equations represented for students the beginning of a new phase in their study of mathematics since this is a basic concept of Algebra. Furthering the study of numerical expressions, involving numbers and operations with which they contacted previously, now, other expressions involving new symbols and manipulation rules arise, which require a higher level of abstraction. This led me to question the introduction of the topic 'Numerical equations of the first degree in Q’, to identify aspects that change in the approach to this concept.


1.4 Procuremos resolver a nossa equação, recordando as operações inversas da adição e da multiplicação.

Era ela

$$3x + 16 = 37$$

Confirmemos a nossa esperança, admitindo que existe o tal número x que a satisfaz.

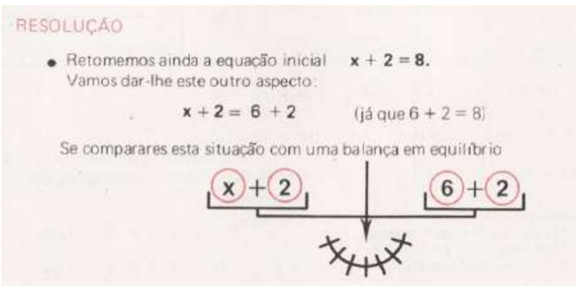
Então: $3x = 37 - 16$ *(def. de diferença)*
e: $3x = 21$
e: $x = 21 : 3$ *(def. de quociente exacto)*
e: $x = 7$



**Figure 4.** A procedural scheme

Source: Costa e Anjos (1976, p.82)

All the collections start with a motivation problem and address at the very beginning of the chapter the language associated to equations (notion of equation, variable, solution of an equation, ...). The ‘*Compêndio de Matemática*’ introduces the solving of equations by using inverse operations of addition and multiplication (Figure 4). And the collection ‘*Eu e a Matemática*’ uses an approach based on a procedural scheme like the one above to find the value of the unknown variable that gives meaning to equality. These collection present procedures that can be performed on both sides of the equation, that will lead to practical rules to resolve an equation. The collection ‘*M*’ introduces the solving of equations using scales method (“balanças”) (Figure 5) to find the solution. It is the only collection which refers to equivalent equations and its rules.


RESOLUÇÃO

- Retomemos ainda a equação inicial $x + 2 = 8$.
  Vamos dar-lhe este outro aspecto:

$$x + 2 = 6 + 2 \qquad \text{(já que } 6 + 2 = 8\text{)}$$

Se comparares esta situação com uma balança em equilíbrio

**Figure 5.** The use of a scale to illustrate the situation
Source: Abrantes e Carvalho (1980, p. 58)

## *4 Final remarks*

Our textbook analysis show that the books are structured according to all the knowledge that a student is supposed acquire on the topics proposed in the 7th grade program.

Though we can trace, in all collections, an understanding that students construct their own knowledge, only in two of the collections the authors assume that their work is organized to facilitate and promote the active participation of students in their learning. Regarding the way in which the three works introduce the topic 'Numerical equations of the first degree in Q', there are two different approaches to make equations and their solving comprehensible for students. One the one hand, there is a schematic style that uses inverse operations. On the other hand, there is the use of scales.

Research becomes very difficult when investigating past school practices, because usually only indirect evidence can be obtained, as is the textbooks. So, it would be interesting to study in which extent these collections were used and for what purpose.

**Acknowledgement**

This work was partially supported by Portuguese funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., in the context of the project PTDC/CED-EDG/32422/2017.

## REFERENCES

Abrantes, P., & Carvalho, R. F. (1980a). *M7. 7º ano de escolaridade*. Lisbon: Texto.

Abrantes, P., & Carvalho, R. F. (1980b). *M7 – Exercícios de Matemática. 7º ano de escolaridade*. Lisbon: Texto.

Almeida, M. (2007). *A sombra da Matemática... Um contributo para a compreensão desta disciplina no 3º ciclo liceal (1947-1974)*. (Tese de Mestrado). Universidade Lusófona de Humanidades e Tecnologias. Lisbon.

Almeida, M. C. & Matos, J. M. (2023). The Distinct Facets of Modern Mathematics in Portugal. In *Modern Mathematics. An International Movement?* Springer. DOI 10.1007/987-3-031-11166-2.

Costa, A. A., & Anjos, A. Ó. (1976) *Compêndio de Matemática. 7º ano de escolaridade*. Porto: Porto Ed.

Domingos, M. E; Correia M. C., & Fernandes, T. T. (1976a) *Eu e a Matemática. 1º ano do curso Secundário Unificado. Livro de consulta 1*. Porto: Edições ASA.

Domingos, M. E; Correia M. C., & Fernandes, T. T. (1976b). *Eu e a Matemática. 1º ano do curso Secundário Unificado. Livro Guia 1 – Fichas de trabalho*. Porto: Edições ASA.

Furinghetti, F., Matos, J. M., & Menghini, M. (2013). From mathematics and education, to mathematics education. In A. B. M. Clements, C. Keitel, J. Kilpatrick, & F. Leung (Ed.), *Third International Handbook of Mathematics Education* (pp. 273-302). New York: Springer.

Matos, J. M. (2009). Changing representations and practices in school mathematics: the case of Modern Math in Portugal. In K. Bjarnadóttir, F. Furinguetti, & G. Schubring (Eds.), *"Dig where you stand" Proceedings of a Conference on On- going Research in the History of Mathematics Education,* 123-137. Reikyavik: University of Iceland.

Maz, A. (2005). *Los números negativos en España en los siglos XVIII y XIX* (Doctoral dissertation, Granada University, Granada). Retrived february 28, 2022 from http://digibug.ugr.es/handle/10481/556#.WSkoRWjyvIU

Moon, B. (1986). *The "NewMaths" curriculum controversy. An international story*. London: Falmer Press.

# History of Mathematics in Brazilian Secondary School Textbooks

**Marcello AMADEO, Cleber HAUBRICHS, Aline BERNARDES, Bruna MOUSTAPHA-CORRÊA**
Universidade Federal do Estado do Rio de Janeiro (UNIRIO), Rio de Janeiro, Brasil.
Instituto Federal de Educação Ciência e Tecnologia do Rio de Janeiro (IFRJ), Nilópolis, Brasil.
coletivo.chemat@gmail.com

**ABSTRACT**

This paper presents some results about the integration between the History of Mathematics and the teaching of mathematics in Brazilian secondary education textbooks.

## 1 *Introduction*

The integration between the history of mathematics (HM) and the teaching of mathematics has been widely discussed in recent decades. Several works (e.g. Furinghetti, 2020; Saito, 2018; Fried, 2014) exemplify how this discussion has developed, the conceptions of educators who are dedicated to this integration and the intrinsic tensions between these two areas, history and education, which have different objects, objectives and methodologies.

Our research group CHEMat (*Coletivo de História no Ensino de Matemática/Collective of History in Mathematics Teaching*) studies the integration between HM and mathematics teaching. CHEMat was created in 2017 and is composed of university and school teachers, and undergraduate and graduate students. This paper emerges as a result of these discussions, on understanding which HM reaches Brazilian schools.

In Brazil, basic education is a governmental responsibility. Therefore 82.6% of primary and secondary students are enrolled in public schools (INEP, 2022). The *Ensino Médio* is the Brazilian secondary education level. It consists of three years and it attends students from 14 to 18 years old. It is the last stage of Brazilian school education.

Considering that usually textbooks are the main reference for teachers and students, they play a fundamental role in Brazil due to the PNLD (National Book and Teaching Material Program). The program ensures that all public school students have access to textbooks. Taking these factors into account, CHEMat has engaged in answering the following question: which HM reaches teachers and students of basic education from PNLD 2018 textbooks?

## *2 Metodology*

To map the presence of HM found in textbooks, we developed a data collection instrument as an online form implemented with the *Google Forms* web application. Each appearance of HM in textbooks is considered an *historical insertion*. Each insertion was analyzed through 60 questions, organized into 12 groups. The questions identify the insertion; indicate its position and layout in the book; identify historical periods and territories, historical characters, documents and research/teaching institutions; describe its iconography; the contents of the chapters in which it is placed, as well as the contents mentioned therein; identify whether or not mathematics and history may be separated in the historical narrative; indicate the proposition of math exercises, historical investigations or other types of activity; identify the references; and investigate the types of historical narrative, the didactic function of insertion (Carlini & Cavalari, 2017), and themes about the uses of history in teaching (Fried, 2014).

After a pilot study carried out between August and September 2020, the collection instrument was adjusted, allowing the group to start collecting data from 8 textbooks collections approved in the PNLD 2018 in March 2021. In this paper, we present the partial results of the study carried out until May 2022 from the collection of data of 4 collections: Luiz Roberto Dante's *Matemática: Contexto e Aplicações*; Eduardo Chavante & Diego Prestes' *Quadrante Matemática*; *Matemática:* Iezzi et al.'s *Ciência e Aplicações*; Joamir Souza & Jacqueline Garcia's *Contato Matemático*.

## *3 Data analysis*

When we went through the 4 collections indicated, each of them with 3 volumes, thus totaling 12 books, we collected 215 historical insertions. In the next paragraphs, we present some partial results regarding the historical periods, territories and historical characters mentioned in the insertions, the themes of Fried (2014) and the didactic function of Carlini and Cavalari (2017).

We begin our analysis by highlighting the *when*, *where,* and *who* elements. It is important to emphasize that a historical insertion can tell episodes from different periods and regions. For example, a single entry might begin by quoting Babylonian tablets from Antiquity, then an Arabic mathematical practice in the Middle Ages, and end with a work in Modern Europe. Thus, the numbers presented account for all these occurrences.

### 3.1 Whens and wheres

It is quite revealing that the HM of textbooks highly values Antiquity (111) and Modernity (146), but little Medieval (16) and Contemporaneity (27). And even within this scope, the emphasis on events that occurred in Ancient Greece (60) or in Modern Europe (126) is overwhelmingly greater than all the others *when* and *where* identified.

We also note that the territories/periods *Spain, Portugal and other countries in Iberian Europe (from the 16th to the 19th century)* and *Colonial America (from the 15th to the 18th century)* did not receive any mention. This absence is intriguing given that the historical formation of Brazil is directly associated with Portugal. And even advancing to the 19th or 20th centuries, there is no mention of episodes or characters from the HM from Brazil. This is an alarming absence when it comes to textbooks written for Brazilian students, since it can lead the student to think that Brazil does not produce mathematics. We suppose that it is due that the European HM from Modernity to the present is more abundant in the historiography than the history of other territories and periods, including Brazil. Consequently, this excess on one side and absence on the other are not a choice of textbook authors alone.

### 3.2 Whos

In the next item, we counted 145 characters in a total of 372 mentions. Of these, we have 74 characters, next to half of the total, mentioned only once. As for the most quoted characters, we found 12 names, listed in Table 1. Considering what was observed above, it was to be expected an absence of names of Brazilian characters. Another deserving highlight is the absence of women, reaching all territories and all times.

| Frequency | Name | Frequency | Name |
|---|---|---|---|
| 16 | Gauss, Carl Friedrich | 9 | Archimedes |

| | | | |
|---|---|---|---|
| 15 | Euler, Leonhard | 9 | Euclid (from Alexandria) |
| 13 | Galilei, Galileu | 9 | Fermat, Pierre de |
| 12 | Cardano, Girolamo | 8 | Descartes, René |
| 10 | Leibniz, Gottfried Wilhelm | 8 | Pascal, Blaise |
| 10 | Pythagoras | 8 | Thales |

**Table 1** – Most quoted mathematicians

We resent that Brazilian history is still not recognized by Brazilian textbooks. We also regret how the textbooks still portray mathematics as an exclusively male activity. The presence in textbooks of Brazilians and women could contribute to making our books more appropriate to the history of our country, more inclusive and more plural in terms of gender equality (Haubrichs & Amadeo, 2021).

The number of people quoted (145) in textbooks is much higher than that of books, works, primary sources, contexts or institutions (53). This shows how much the textbooks still focus the HM on the people and how little valued is the mathematical production itself. It leads us to think that instead of telling episodes of *the history of mathematics* what is being told are episodes of *the history of mathematicians*, or even *the history of male and European mathematicians*.

### 3.3 Didactic roles

Let's move on to a more qualitative analysis that considers the didactic role of insertion. Considering the work of Carlini and Cavalari (2017), we tried to classify the insertions according to their didactic functions (Table 2). Such categories are not mutually exclusive. It is possible that a historical insertion can be used as a didactic strategy and, at the same time, seek to elucidate why a mathematical property was developed in such a way.

| Category | # insertions | Category description |
|---|---|---|
| HM and general cultural formation | 121 | When historical information brings general knowledge related to mathematics, which does not contribute to the learning of mathematical content. |
| HM and the elucidation of the 'why' | 53 | When history shows how, why and under what circumstances certain mathematical knowledge emerged. |
| HM and didactic | 51 | When history makes it possible to develop |

| | | |
|---|---|---|
| strategy | | some mathematical reasoning |
| HM and the elucidation of the 'for what' | 35 | When history shows the usefulness or applications of certain mathematical content (in mathematics itself or in other areas) over time or in a specific period. |

**Table 2** – Didactic function of the insertions. Source: from the authors and adapted from Carlini and Cavalari (2017, pp. 77-78).

Our attention was drawn to the huge number of didactic function insertions dedicated to the general cultural formation of students (121). It is more than double that of each of the others, including the role that prescribes HM as a teaching strategy (51). It concerns us since insertions of this nature do not have any intention of contributing to meaningful learning. The history is presented as a curiosity, an isolated or simple informative fact.

### 3.4 Historical orientation

The last set of analysis categories is inspired by Michael Fried's (2014) discussion and his look at the possible uses of history in teaching (Table 3). In the use of our data collection instrument, the three themes were not proposed as mutually exclusive.

| Category | # insertions |
|---|---|
| HM as Motivacional Theme | 139 |
| HM as Curricular Theme | 59 |
| HM as Cultural Theme | 7 |

**Table 3** – Fried's themes classification.

The large number of insertions classified in the motivational theme draws attention. According to Fried (2014, p. 682), this perspective of using HM by educators is quite problematic. The motivational theme does not do mathematics justice, as it assumes that mathematics by itself is not interesting to students; they do not do justice to history either, as they do not recognize it as a field of research, but as a collection of curiosities and anecdotes.

It is interesting to note that 90% of *HM and general cultural formation* insertions are also *HM as a motivational theme*. On the other hand, 78% of the *HM as a Motivational Theme* insertions are also of the *HM and general cultural formation*, and 51% of all inserts are simultaneously of both types. This indicates that there may be a close relationship between Fried's *motivational theme* and Carlini and Cavalari's *general cultural formation*.

Different from the *motivational theme*, the *curricular theme* and the *cultural theme* take more seriously the task of integrating the mathematical contents, the didactic exposition and the historical context. Between these two themes, there is a predilection for the *curricular theme*. This result is consistent with the fact that the *curricular theme* is usually associated with an anachronistic view of mathematical concepts. This conception of the history of textbooks seems to us to be associated with the fact that the authors rely on HM books based on a traditional historiography, mainly on Carl B. Boyer's *History of mathematics* and Howard Eves's *An introduction to the history of mathematics*. Both of them are very popular in Brazil and have several editions in portuguese. Consequently, we have an insignificant number of insertions as a *cultural theme*, since this theme presupposes a perspective on history that is more aligned with current historiography, which still has very little circulation between the textbooks authors.

## *4 Conclusions*

The results presented here reveal that the integration between history and mathematics teaching in school textbooks is still very superficial. We positively acknowledge the authors' initiative to include HM in their books. This is revealed by the significant number of historical insertions. However, most of the insertions present a naive way of how history could be integrated into teaching: It prefers an anecdotal history or mere curiosity instead of conceiving history as a didactic or significant strategy of the student's cultural formation and mathematical learning.

We believe that this portrayal of HM in textbooks may be associated, on the one hand, with the historical sources that the authors use and with the way mathematical contents are traditionally presented in textbooks. Inserting history as a motivation to present the mathematical content as it is done in textbooks or inserting mere historical curiosities in isolated boxes throughout the chapters is a way of appropriating the HM, but without significantly affecting the exposure of the content. Integrating HM into teaching could demand a reformulation in the exposition of contents, in the resignification of mathematical concepts, in the nature of the exercises, and ultimately in the mathematics curriculum itself.

## REFERENCES


INEP Instituto Nacional de Estudos e Pesquisas Educacionais Anísio Teixeira. (2021). *Resumo Técnico: Censo Escolar da Educação Básica 2021*. https://www.gov.br/inep/pt-br/areas-de-atuacao/pesquisas-estatisticas-e-indicadores/censo-escolar/resultados

Carlini, E. M. P., & Cavalari, M. F. (2017). As funções didáticas da história da matemática nos livros didáticos de matemática do ensino médio. *Hipátia – Revista Brasileira de História, Educação e Matemática*, *2*(2), 73–88.

Fried, M. N. (2014). History of mathematics in mathematics education. In M. R. Matthews (Ed.), *International Handbook of Research in History, Philosophy and Science Teaching* (pp. 669–703). Springer.

Furinghetti, F. (2020). Rethinking history and epistemology in mathematics education. *International Journal of Mathematical Education in Science and Technology*, *51*(6), 967–994.

Haubrichs, C., & Amadeo, M. (2021). História da Matemática nas Coleções do PNLD 2018. *HIPÁTIA – Revista Brasileira de História, Educação e Matemática*, *6*(2), 199–214.

Haubrichs, C., & Bernardes, A. (2020). Inserções de história em livros didáticos de matemática: elaborando um instrumento de coleta de dados. In G. Sanglard et al. (Eds.) *Anais Eletrônicos do 17º SNHCT* (pp. 1–14). Sociedade Brasileira de História das Ciências.

Saito, F. (2018). A pesquisa histórica e filosófica na educação matemática. *Eventos Pedagógicos*, *9*(2), 604–618.

# FRENCH MATHEMATICAL INFLUENCES IN THE FIRST BRAZILIAN GEOLOGICAL SCHOOL IN THE LATE 19$^{TH}$ CENTURY

**Davidson Paulo AZEVEDO OLIVEIRA**
CEFET-MG, Amazonas Avenue, 7675, Belo Horizonte, Minas Gerais, Brazil
davidson@cefetmg.br

## ABSTRACT

The aim of this paper is to present some French influences in Brazil at the end of the 19$^{th}$ century specially concerning the mathematics at the mineralogy and geology school: Escola de Minas de Ouro Preto (EMOP). We discuss about the Brazilian context in which this institution was created as well as his founder, the French geologist Claude-Henry Gorceix. We also compare entrance exams and preparatory courses with French schools. Finally, it will be presented the difficulties of the adoption of the metric system in Brazilian schools. In mathematics Gorceix implemented a study of higher mathematics content such as Calculus of Derivatives before entering higher education with a teaching that did not resort to memory but the reasoning.

## *1 Introduction*

The aim of this paper is to present some French influences in Brazilian teaching and research at the end of the 19$^{th}$ century in mathematics at the Mining School of Ouro Preto (EMOP – Escola de Minas de Ouro Preto). Brazil was a Portuguese colony until 1822. The status changed, especially in science, with the arrival of the Portuguese royal family in 1808 when Portugal was about to be invaded by Napoleon Bonaparte's troops. The intention of coming to Brazil, supported by the British, was to ensure Portugal's independence (Schwarcz & Starling, 2015). Before that time, books, in general, could not be printed in Brazil, it could circulate in the colony, as long as it was produced abroad and were approved by a censorship screening, in order to keep Portual's control over the colony. After the independence (1822), the Emperor was Dom Pedro I (1798-1834) until 1831, when he renounced the throne and returned to Portugal. His son, Dom Pedro II (1825-1891), was only six years old and could not be the Emperor In 1840, after his emancipation at the age of 15, he became Emperor and ruled Brazil until 1889, when the Republic was declared.

Between 1864 and 1870, the hardest war took place in South America: the "Guerra da Tríplice Aliança", in which Brazil, Argentina and Uruguay fought against Paraguay, killing a large part of that country's population (Schwarcz & Starling, 2015). At that time, according to Schwarcz & Starling (2015, p. 303) "It is certain that in 1871, in the midst of a political crisis, the emperor, who seemed bored in his own country, prepared to travel around the world, the emperor, who seemed bored in his own country, was preparing to travel around the world which he knew only from books"(translated by the author). This is how the history of the creation of the School of Mines of Ouro Preto – Escola de Minas de Ouro Preto (EMOP) begins.

The table below presents a summary of the main historical and scientific events that will be discussed throughout this paper. It is intended to be a panoramic view to direct the reader chronologically.

| Historical Events | Scientific Events |
|---|---|
| 1808 – Royal Family came to Brazil | |
| 1822 – Brazilian Independence | |
| 1831 – Dom Pedro I return to Portugal | |
| | 1832 – Creation of a School of Mines |
| | 1862 – Prohibition of Metric System |
| 1864-1870 - War of Triple Alliance | |
| 1871 – Dom Pedro II goes to Europe | |
| 1874 – Gorceix comes to Brazil | |
| | 1876 – Beginnig of the EMOP |
| | 1877 – Preparatory Course |
| | 1880 – Changing in the Preparatory Coure |
| | 1885 – Most important changing in the Preparatory Course |
| | 1889 – Brazilian becames a Republic |
| 1891- Gorceix goes back to France<br>Death of Dom Pedro II | |
| 1919 – Death of Gorceix in France | |

## 2.5 Creation of a mineralogy and geology school

In 1823, the Constituent Assembly discussed the opening of schools of mineralogy in Brazil. In 1832, a project was approved and transformed into law by the General Legislative Assembly concerning the creation of a

School of Mines in the province of Minas Gerais. Only in 1875 there was the approval of the Laws and Regulations of one institution, the EMOP, with the start of the activities on October 12th 1876.

The creator was the french geologist Claude-Henry Gorceix (1849-1919), invited by Dom Pedro II during his travel (his escape from the country because of his low popularity after the war against Paraguay, according to Schwarcz and Starling, 2015) to Europe in 1871. The french teacher studied at the Lycée de Limoges and at the École Normale Supérieure in Paris (1863-1866). Later, he was sent on a scientific mission to the French school in Athens. According to the organization report available at the Brazilian National Archive (Brasil, 1875), the EMOP was destined to provide managers for the exploitation of mines, for the metallurgical establishments and engineers employed by the State in the provincial Mines of the Empyre.

The regulation, created by Gorceix, represented a revolution in Brazilian schools (Carvalho, 2002; Silva, Thiengo, 2003) especially when considering the inclusion of an entrance exam, full-time work for teachers and students, a ten-month school year followed by two months of practical work, good pay for teachers, free education, scholarships and even prizes of trips abroad for the best students.

The regulation required entrance examinations (oral and written) for access to the institution, a practice that was usual in France, but not in Brazil (Silva & Thiengo, 2003). In this respect, the politician José Maria da Silva Paranhos, known as Visconde de Rio Branco, acting director of the Polytechnic School of Rio de Janeiro and one of the responsible for analyzing the Regulation proposed by Gorceix, spoke out against the entrance exams. According to him, admission should be made as in other Brazilian higher education institutions and accept all candidates approved in the secondary school concluding examinations, called at that time preparatory examinations.

The tests included elementary geometry, analytical geometry (of lines, circumference, 2nd degree curves); algebra up to 2nd degree equations, use of logarithm tables; rectilinear trigonometry; descriptive geometry of lines and planes; elementary physics; notions of chemistry concerning methaloids; notions of botany and zoology; straight and imitation drawing; the French or English or German language (Brasil, 1875).

The first written exam of Trigonometry required of the candidates to determine tan (98º) in the first question. And, in the second one, was given one side of the triangle and one angle and asked to find the other two sides and the angle formed by them. This questions where similar to the questions presented in Lacroix's book (1863, p. 47).

After the results, Gorceix sent a letter to the Brazilian Emperor Dom Pedro II, with complaints about the mistakes the students committed, "the trigonometric calculation, numerical resolution of one triangle, left much to be desired" (Gorceix, 1876). He also highlighted the necessity of introducing better methods into secondary and higher education in Brazil and suggested the usefulness of a preparatory course, similar to those that existed in Paris institutions.

## 2.6 Preparatory Course

Gorceix decided to set up a one-year preparatory course, but not to exclude entrance examinations, as suggested by teachers from Polytechnique School of Rio de Janeiro [Escola Politécnica do Rio de Janeiro]. It is worth clarifying that in the documents one can find other names such as preparatory, annex and provisional course. This is due to the fact that the course was preparatory for the entrance exam and, initially, was thought of as provisional. The adjective annex is due to the fact that it was taught by EMOP's own professors and was considered to be part of EMOP.

The Decree-law of 1832 that created a School of Mines in Brazil, which nevertheless was not implemented, had a similar course (Brasil, 1832). The one that Gorceix created was based not only on the 1832 course, but also on the Preparatory course of the École des Mines de Paris (Oliveira, 2020).

This one-year preparatory course started in 1877 (Brasil, 1877) with three subjects, most of which had a strong mathematical component. Despite this course, Gorceix was still not satisfied and decided to extend it for another year in 1880 (Brasil, 1880). Another more important change was made in 1885, with implementation of the two-year preparatory course (Brasil. 1885).

As far as we find evidence, the first Preparatory Course was created due to the mistakes made by the candidates in the entrance examinations, especially in the questions of trigonometry, the logarithm table, calculus and descriptive geometry. The aim was to complete the scientific education of candidates wishing to enroll at the École des Mines, being, at the beginning, a provisory course.

The Decree-law of 1880 stipulated that there would be a professor of mathematics and mechanics; then, the Brazilian Professor Archias Medrado was hired. Although we have not yet analyzed his proof, he might have done tests of trigonometric calculation, solve a transcendental equation or other practical calculations in two hours, according the Brasil (1880).

This Decree-law increases also the Preparatory Course by one year with three subjects, one in the first year and two in the second. It is important to emphasize the amount of content concerning mathematics and to highlight the calculation of derivatives. There was, at the entrance examinations from 1879 to 1882, questions about finding the minimum or maximum value of an algebraic fraction. The content of the mathematics and derivative calculation must be taken into consideration in all the exams and preparatory courses.

The preparatory course was extinguished in 1897, when the capital was transferred from Ouro Preto to Belo Horizonte, although the school remained in town with less students. Four years later Brazil became a Republic and Gorceix did not have the same support he had from Dom Pedro II for the conduction of the institution. So he decided to left the direction of the school and went back to France, Bujaleuf, until his death in 1919.

## 2.7 Teachers

For the selection of teachers, Gorceix preferred those who were French (Armand de Bovet, Arthur Thiré, Paul Ferrand – all of them ex-students from École Polytéchnique and École de Mines from Paris), because, in his opinion, there were no people in Brazil with the required level, although he has hired Mr. Medrado. For the selection examinations, he abolished the writing and defense of a thesis, unlike in France. According to him, it would require time which the candidates could no longer afford, cost money, and prove little from the point of view of teaching. Meanwhile, he kept a lesson after 4 hours of preparation without books or handwritten notes, as was the practice in his country (Carvalho, 2002).

It is also important to point out Gorceix's opinion on Brazilian secondary education: deplorable, according him. He was especially critical of the mistakes made by the teachers who passed them on to the students. Some teachers made up the students say that the logarithm of 0 is -1, or writes that the square of a - b is $a^2 - b^2$ (Lima, 1977). That was why he thought it was necessary to teach basic mathematics and also differential calculus before entering at the school of mines, a practice that was not common in other countries at that time such as, for instance, Germany.

## 2.8 Metric System

Another French influence in Brazil was the adoption of the metric system. There was a law in Brazil, dating from 1862, saying that in 10 years the use of any other metric system would be forbidden. Those who used any other metric system would have to pay a fine or even go to jail. However, in the following is an excerpt from a proof dated 1888 in which the student Belarmino Martins de Menezes did not use the French system, but the traditional Brazilian system.

> *"To get the surface area of a triangle expressed in **square arms** it would be enough to divide the number of square meters by 4.84, which is the number of meters a square arm has, but there is no time"*

Gorceix was worried about the implementation of the French metric system in Brazil. He went to Paris in 1882, due to personal reasons, and bought there a small box containing the various measures of the metric system and send then to the Brazilian Ministry. In his opinion, it would be useful if every school had one of these boxes in order to show students how to use it (Lima, 1977, p.194-195).

In this regard, Zuin (2017) points out that system for weights and measures are cultural products and the breaking of cultural codes and traditions does not happen without conflicts at the social level. In the 1870s there was an increase in taxes, the increase in products, economic issues, allied to the military recruitment that was imposed, erupted, in several locations in Brazil so the population was dissatisfied. At the same time, many suspicions revolved around the new standards proposed by the law, French metric system there were then groups rebelling, leading to a movement, especially in the north of the country, aimed at destroying the decimal standards, triggering a movement that became known as Revolta dos Quebra-Quilos (Revolt of the kilo-breakers), between 1872 and 1877. There were those who condemned the French metric system for its attachment to tradition; others renounced it for associating it with the French Revolution and Enlightenment ideals that could compromise the monarchical regime in Brazil (Zuin, 2017). Besides that, until the middle of the 20$^{th}$ century, the Brazilian system was used in school books.

## *Final Considerations*

Gorceix implanted in Brazil a different education both in content, methods, and spirit. In mathematics one can perceive a study of calculus of derivatives before entering higher education with a teaching that did not resort to memory but the reasoning. Besides that, he brought French scholars (such as Armand de Bovet, Arthur Thiré, Paul Ferrand) and teaching methodologies to the country in order to develop Brazil education, and also spread French culture.

The French influence also occurred in other institutions and scientific knowledge in Brazilian territory, specially at the end of 19$^{th}$ century and beginning of 20$^{th}$. Although, the teaching of calculus in secondary schools and also the use of metric system demanded more effort in EMOP than in other places. Besides that, the preparatory courses started in São Paulo in 1892, probably influenced by Gorceix.

## REFERENCES


Brasil (1832). Decreto de 3 de outubro de 1832 – Crêa um Curso de Estudos Mineralogicos na Província de Minas Geraes. Coleção de Leis do Império do Brasil de 1832. Rio de Janeiro. V.1. parte 1. p.98.

Brasil. (1875). Decreto 6 026 de 6 e novembro de 1875. Cria uma Escola de Mi*nas a Provín*cia de Minas Gerais e dá-lhe *Regulamento.* Coleção de Leis do Império do Brasil de 1875. Rio de Janeiro, v.2, parte 2, p.701.

Brasil (1877). Portaria número 372 de 12 de setembro de 1877. Coleção de Leis do Império do Brasil de 1877. Rio de Janeiro, v. 3. p.304.

Brasil (1880). Decreto 7628 de 14 de fevereiro de 1880 – Altera diversas disposições relativas à escola de Minas de Ouro Preto. Coleção de Leis do Império do Brasil de 1880. Rio de Janeiro. v.2. p. 61.

Brasil (1885). *Decreto 9448 de 27 de junho de 1885. Dá novo Regulamento à Escola de Minas*. Coleção de Leis do Império do Brasil de 1885. Rio de Janeiro. v.2. p.484.

Carvalho, J. M. (2002). *A Escola de Minas de Ouro Preto: o peso da glória*. 2 ed. Editora UFMG.

Gorceix, C. H. (1876). *Letter to Dom Pedro II on September 14* $^{th}$: Museu Imperial/Ibram/MinC/nº785/2017 maço186, doc 8455

Lacroix, S. (1863). *Traité Élémentaire de Trigonométrie Rectiligne et Sphérique, et D'Application de L'Algébre a la Géométrie.*

Lima, M. R. (1977). *D. Pedro II e Gorceix: a fundação da Escola de Minas de Ouro Preto*.

Fundação Gorceix.

Oliveira, D. P. A. (2020). Um estudo de avaliações de matemática na Escola de Minas de Ouro Preto de 1876 a 1891. Tese de Doutorado. Unesp – Rio Claro.

Schwarcz, M. L. & Starling, M. H. (2015*). Brasil: Uma biografia*. Companhia das Letras.

Silva, C. M. S. & Thiengo, E. R. (2003). *Claude-Henri Gorceix: Trabalho e Competência na criação de uma escola e na formação de discípulos.* Episteme.

Zuin, E. de S. L. (2017). *Sistema Métrico Decimal como um saberescolar no Brasil: alteração das práticas escolares na segunda metade do Oitocentos*. Revista De História Da Educação Matemática, v.3. n.2. p. 169-194.

# "HANDS OFF EUCLID!": THE AMBIGUOUS RECEPTION OF THE "MODERN MATHEMATICS" REFORM IN GREECE

**Dionysios BALTZIS**
baltzisdio@gmail.com

## ABSTRACT

The modern mathematics reforms concerned a broad reorganization of school mathematics curricula. Their reception in the West was not homogenous and was defined by the various national contexts. The Greek case is highly interesting because of the distinctive place that mathematics holds in the Greek mentality. The initial active promotion of the reforms gave way to the underground dispute in terms of defending the national interests and protecting Euclid's system. The configurations which took place, with special regard to geometry, point out the contradictions of the Greek mathematical community. In particular, the first, experimental phase of the reforms was very carefully formulated by Greek authorities in a sequence spread over four years, while the second, generalized phase was abruptly set in motion through a highly ambiguous course. The context of this discrepancy lies in the instable social – political climate of the 1960s. The combination of a nationalistic framework of the 1960s and an efficient (albeit controversial) promotion at the outset of the reforms resulted in a compromising arrangement of counterbalance: introduction of modern mathematics side by side to the existing curricula. This process led to mixing of epistemological elements, reform fatigue of students and teachers and immoderate expansion of syllabi. The regime change, dubbed "Metapolitefsi" cut the path to a more alleviated approach of the reform, but did not result in a "back to basics" process.

## *1. Introduction*

As is well known, the modern mathematics reforms are one of the most well documented topics in the history of mathematics education (Karp & Furinghetti, 2018). Thus, the literature on the reforms is already massive and constantly enriching. The Greek case, on the other hand, has been virtually left out of the research scope. My purpose here is to examine the modalities of the reform's reception in the Greek context of the 1960s, to describe its peculiarities and to propose a interpretative framework for this complex process.

## 2. *The “modern mathematics” reform in Greece*

At first glance, the trajectory of the reform in Greece is not much different to those of Western Europe. A brief overview of the reform’s phases can be as follows: after a very warm reception of the reform’s objectives, the Ministry of Education set up an experimental procedure, which spread over the course of four years, and was evaluated as successful. Political instability that followed for the next three years, though, cancelled the educational reform underway as well as the modern mathematics reform. In 1967, the ultra-conservative dictatorial regime reset education to its previous features. The modern mathematics reform was an exception: it spread to all high school curricula, with no differentiation regarding the students. The aftershock was felt by everyone around mathematics education. The 1974 regime change set the path to more moderate curricula.

Despite its analogies to the other western countries’ trajectories of the reform, the peculiarities of the Greek case are striking and concern the specific features of the Greek mathematical community and their articulation with the national context of the Sixties.

### *2.1. Aspects of national and educational context*

In the 1950s, Greece entered a period of overall reconstruction and healing from the wounds of the second World War and the Civil War that followed it.

Until the end of the 1950s, mathematics education in Greece was defined as "Traditional Mathematics" (Toumasis, 1990). Similarly, the introduction of elements of modern mathematics in undergraduate curricula did not form a generalized shift to the latest developments in mathematics.

### *2.2. The Royaumont Seminar & Greek Mathematics executives: embrace & misunderstanding*

Among the participants of the Royaumont seminar we find two Greeks with very different profiles: Nikolaos Sotirakis and Kanellos Georgontelis. We know very little about the second one. Sotirakis, on the other hand, was a well-known actor within the Greek mathematical community of the post war era.

Until 1961, the implementation of the reform in Greece was unanimous. This was due to the enthusiasm about the rapid scientific progress, combined with the more moderate spirit of the seminar than that represented in *"New Thinking in School Mathematics"* (De Bock & Vanpaemel, 2015). Thus, the Greek mathematical community embraced the Royaumont seminar conclusions. In what followed, the activity of two people had a particularly significant impact. Sotirakis, just mentioned, was the first.

The second was Nikolaos Michalopoulos, who at that point filled several key positions, among which the Presidency of the Hellenic Mathematical Society (HMS). Talking to the audience of the 1st Panhellenic Mathematical Conference, and having described the ongoing "revolution" in mathematics and its imminent introduction to school curricula, he linked the implementation of these changes with national duty and presented set theory as a Platonic discovery exploited by Cantor. This paradoxical association defined the public discussion throughout the 1960s. There is no indication that this argument was merely a ploy to justify the introduction of modern mathematics in school mathematics curricula. Therefore, it must be interpreted as an ideological choice, one that would impose or simply accelerate the reforms.

These perceptions were challenged by the fact that the modern mathematics reform was not merely irrelevant to Ancient Greek Mathematics, but also in direct contradiction to traditional Euclidean Geometry as a school subject.

### *2.3. The first phase of the reform: experimental implementation*

In November 1961, a committee was set up aiming at the experimental procedure regarding the reform of school mathematics curricula. The experimental procedure was designed in stages by the Ministry of Education and the OECD. The committee completed its tasks in the autumn of 1965 after four years of methodical work which composed of the writing of new textbooks, the supervision of the "pilot" classes, and the evaluation of the experimental procedure. (Ministry of Education, 1963)

Few were internationally involved in developing an experimental mathematics pedagogy (Schubring, 2014). Such an international norm is confirmed by the Greek case. A striking exception to this norm is N. Sotirakis, who regularly wrote articles and lectured on the subject.

### *2.4. Criticism of the modern mathematics reform*

The reforms in Europe were not accompanied by significant reactions (Kilpatrick, 2012). In Greece, arguments based on "national duty" combined with the conservative political climate of the 1960s made foreseeable criticisms difficult.

The only substantial objection was expressed by the greatest contemporary Greek historian of ancient Greek mathematics, Evangelos Stamatis. His reasoning was controversial; however, the reaction against his views was ad hominem and excluded him from the public debate (Thomaidis, 1991).

It is worth noting that from the outset of the reforms, it was the main arguments regarding the implementation of the reforms concerned their relation to Ancient Greek mathematics.

### *2.5. The discontinuation of the reform*

In 1964, the conditions were met for a new modernization effort, which focused on adapting education to the financial demands. After the first (experimental) phase of the "modern mathematics" reform was completed, the gradual publication of the new curricula began. What the Ministry of Education intended was to stabilize "modern mathematics" in the lower secondary education and to gradually expand it (Kritikos, 1980). However, political instability caused both reform procedures to halt.

### *2.6. The second phase of the reform: general implementation*

The military dictatorship promptly disrupted the 1964 education reform, restoring its previous orientation. Meanwhile, they promoted the abrupt general implementation of "Modern Mathematics" in both lower and upper secondary education, regardless of the students’ prospects and without substantial provision for the retraining of teachers.

In the following year, new curricula were published for all grades of secondary education. The proposed solution was a compromise: the inclusion of modern mathematics in the existing curricula. Nevertheless, this configuration created more problems than it actually solved, since it led to the mixing of opposing epistemological elements. In addition, it was not accompanied by an increase in the teaching hours.

### *2.7. The renaming of the "Appendix of the Bulletin of the HMS"*

In January 1968, the HMS periodical "Appendix of the Bulletin" was renamed to "Euclid". This is highly interesting in terms of a semiotic framework, since its audience consisted primarily of students.

Later that year, “Euclid” accommodates the speeches delivered during the awards ceremony for the 1968 nationwide student mathematics competition. In his speech the HMS general secretary, Aristides Pallas (1968) argued that:

> *“Euclid’s Elements were and still are considered to be the most perfect book of Geometry [...] endured and is still enduring absurd attacks, but it remains and will remain an impregnable fortress. So, our Motto is «Hands off Euclid»”.*

This is particularly revealing of the perceptions that prevailed in the HMS leadership and in the Greek mathematical community in general. Of course, a comparison with Dieudonné's slogan “Euclid must go!” cannot be avoided.

The renaming of the journal to "Euclid" coincides with the general implementation of the reform and illuminates the hegemony of a detached attitude towards "modern mathematics". In fact, this configuration corresponded to the implementation of the reform in the national context of the sixties.

### *2.8. The aftermath of the second phase of the reform*

In any case, it quickly became apparent that the results were unsatisfactory. In 1969, a more moderate curriculum was published, which remained in force for 5 years. However, the problems persisted. (Toumasis, 1990)

Furthermore, in stark contrast to the OECD directions, Euclidean geometry was rapidly growing in relation to the university entrance exams, which, set the pace for the developments in the secondary education. The educational reality revolved around exercise methodology in the spirit of these exams and therefore, the curricula’ new content was mined. (Thomaidis, 1991)

## *3. Conclusion*

Apparently, the unfolding of the "modern mathematics" reform in Greece involves important peculiarities in relation to the international movement. The foreseeable strong reactions were expressed in an "underground" manner, and the public debate was distinctively limited.

After the general implementation of the reform, the desired results were far from close, and some key aspects of the reform were gradually withdrawn. This process coincided with the restoration of parliamentary democracy. “Metapolitefsi” consisted of major reconfigurations, and "modern mathematics" was no exception. This process resulted in the transition to the configuration of moderated "modern mathematics". In other words, there was no back-to-basics phase. However, these developments were fragmentary and resulted in an impasse.

## REFERENCES


De Bock, D., & Vanpaemel, G. (2015). Modern mathematics at the 1959 OEEC Seminar at Royaumont. In Bjarnadóttir, K. et al. (Eds.). *Dig where you stand 3. Proceedings of the Third International Conference on the History of Mathematics Education*. Uppsala: Uppsala University – Department of Education.

Karp, A., & Furinghetti, F. (2018). The Hamburg score (In lieu of an Introduction). In Furinghetti, F. & Karp, A. (Eds.). *Researching the history of mathematics education* (pp. 241-259). Springer, Cham.

Kilpatrick, J. (2012). The new math as an international phenomenon. *ZDM (ZenrtrallBlattfür Didaktik der Mathemtik),* 44(4), 563-571.

Kritikos, N. (1980). Nikos Sotirakis, mathematician and teacher. *Εκπαίδευση & Επιστήμη [Education & Science],* 2, 11 – 15 (in Greek).

Ministry of Education. (1963). *Experimental instruction of mathematics in Lower Secondary Education. Book II, Mathematics for the 2nd grade*. Athens: ΟΕΔΒ [Organization for School Textbooks Publishing] (in Greek).

Pallas, A. (1968). Speech at the award ceremony of the HMS Panhellenic Competition. *Ο Ευκλείδης [Euclid],* Β1, 12 – 17 (in Greek).

Schubring, G. (2014). The road not taken—The failure of experimental pedagogy at the Royaumont Seminar 1959. *Journal für Mathematik-Didaktik*, 35(1), 159-171.

Thomaidis, Y. (1991). The coordinates of school Geometry in Greece (1960 – 1990). *Σύγχρονη Εκπαίδευση [Modern Education],* 61, 27-38. (in Greek).

Toumasis, C. (1990). The epos of Euclidean geometry in Greek secondary education (1836–1985). *Educational studies in Mathematics*, 21(6), 491-508.

# EUCLID'S PARALLEL POSTULATE AND A NORWEGIAN TEXTBOOK IN GEOMETRY FROM EARLY 19TH CENTURY

**Andreas CHRISTIANSEN**
University of Bergen, Norway
andreas.christiansen@uib.no / andreas.christiansen.bergen@gmail.com

## ABSTRACT

Bernt Michael Holmboe (1795–1850) was teacher at Christiania Kathedralskole in Norway from 1818 until 1826, after that he was professor at the University of Christiania. Holmboe wrote textbooks in mathematics, and he was very influential in the development of school mathematics in the first half of the 19th century in Norway. Holmboe's presentation of the subject matter was traditional and "Euclidean". The focus of this paper will be to elaborate on Holmboe's approach to the problematic topic of parallel lines, and the parallel postulate. Holmboe attempts to give a proof of the parallel postulate, and this proof is presented in detail.

## *1 Introduction*

This paper will show how the parallel postulate from Euclid's *Elements* (Euclid, 1956) was presented in a textbook in geometry written by Bernt Michael Holmboe from Norway (Holmboe, 1827). The study goes backwards in theorems, step by step, to see how Holmboe's argumentation leads up to a theorem which is to be understood as the parallel postulate.

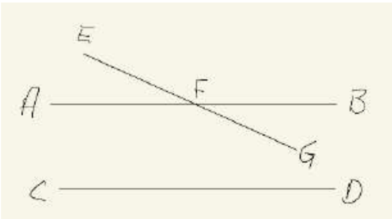
There will first a brief background of Euclid and his well-known parallel postulate, and a short presentation of the Norwegian mathematician and textbook author Bernt Michael Holmboe. The main focus will be to elaborate on Holmboe's approach to parallel lines, and the parallel postulate. Holmboe attempts to give a proof of the parallel postulate, which will be presented in detail. Issues addressed in this paper, and earlier research about the textbook in geometry by Bernt Michael Holmboe, may be found in Christiansen (2009, 2010, 2012a,b, 2015). All translations from Norwegian and Danish-Norwegian to English are done by the author.

## *2 Euclid's parallel postulate*

The Elements by Euclid (Euclid 1956) has for over two thousand years been the exemplar for classical geometry, and for the axiomatic and deductive structure of pure mathematics. The impact of the Elements on modern textbooks in plane geometry is obvious, and it was common for some countries to use books 1 through 6 as textbooks before making their own textbooks.

Among Euclid's postulates, the *parallel postulate* occupies a special position, because it does not have the same intuitively obvious character as the others. Therefore, there was long disagreement as to whether this was an axiom or a theorem. Many attempts were made to prove it, but only through the construction of the non-Euclidean geometries in the beginning of the 19th century was it definitively established that such evidence could not be made.

Proclos lived 410–485 AD and wrote a commentary to Book 1 of the Elements. Before giving his proof of the parallel postulate, he examined an argument which is similar to the one about Achilles and the tortoise where he is continouisly halfing the distance between the two lines to




show that it is impossible for them to ever meet. In his proof, the lines AB and CD are parallel, and the line EFG cuts through AB, then EFG will also cut through CD. His argument is that FB and FG are two straight lines from the point F, and the distance between them will increase, which is correct, but without the parallel postulate we cannot assume that the distance between AB and CD does not also increase (Euclid, 1956; Gray, 2008).

## 3 Holmboe as author of textbooks

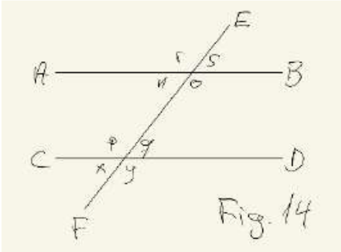

Bernt Michael Holmboe (1795–1850) was teacher at Christiania Kathedralskole in Norway from 1818 until 1826, and after that he was professor at the University of Christiania until his premature death in 1850. As a young man, he became the teacher of Niels Henrik Abel, and Holmboe is today best known as the teacher who discovered Abel's genius and became his first benefactor. Holmboe wrote most of the textbooks in mathematics that were used in the learned schools in Norway between 1825 and 1860, and he was a very influential person in the development of school mathematics in Norway in this period. He wrote textbooks in arithmetic, geometry, stereometry, trigonometry, and higher mathematics.

The textbooks in geometry came in four editions, but only the first two in Holmboe's lifetime. The differences between these two editions are not large, some corrections of errors, and a few re-arrangements of statements. His textbook in geometry were – with one exception – used in all the learned schools in Norway in this period. Holmboe's presentation of the subject matter was in many ways very traditional and "Euclidean".

## *4 Holmboe and the parallel postulate*

It is the belief of this author that Holmboe is aiming to give his students an intuitive proof of a corollary in his textbook that corresponds to the parallel postulate. This chapter will present the line of theorems and corollaries needed for the proof of this, to show Holmboe's reasoning. Holmboe does nowhere in his textbook mention Euclid or the parallel postulate. All quotations are from Holmboe (1827).

**§41** — Theorem 41 is a theorem saying that "[w]hen two parallel lines are intersected by a straight line, then all outside angles are equal to their corre-

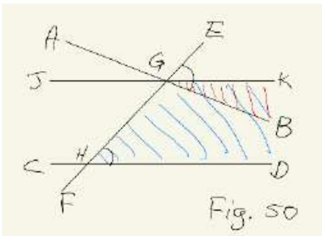



sponding inside angles". In this figure r = p, and so on. This is proved by assuming the opposite and showing that you then will get a contradiction. Theorem 41 in Holmboe's textbook corresponds to Proposition 29 in the Elements (Euclid 1956, vol. I, p. 311). Proposition 29 is also proved by assuming the opposite and using the parallel postulate to prove the contradiction.

Five corollaries follow from Theorem 41, showing all the different relations between the sizes of the angles and let's look at two of them in more detail. Corollary 1 state, among other things that "[w]hen two parallel lines are intersected by a straight line, then inside opposite angles are equal", n = q and o = p, and it is proved by using theorem 41. Corollary 2 states, also among other things that "[t]wo straight lines that are intersected by a third straight line are not parallel in the following situations" ... and one of those situations is "[w]hen the sum of a pair of inside angles is not equal to 2R [two right angles]". That is $n + p \neq 2R$ or $o + q \neq 2R$.

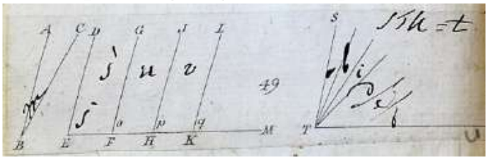
This is the parallel postulate, and it is also proved by using the conditions stated in Theorem 41.

**§40** — Theorem 41 is proved by using Theorem 40 which states that "[w]hen two parallel lines are intersected by a third in such a way that an out-



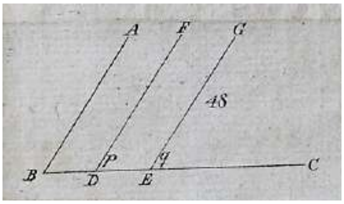

side angle is greater than the opposite inside angle, then the lines are not parallel". If EGB > EHD then AB is not parallel to CD. This proved by a result from Theorem 39 that states that if all these lines are prolonged indefinitely, then will the plane KGB be larger than the plane KGHD, and therefore must AB cut CD.

**§39** — Theorem 39 simply states that "[t]he plane between the legs of an angle is always larger than the plane between two straight lines on one side of a transversal when corresponding angles are equal and when all lines are pro-



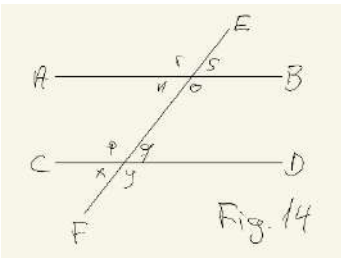

longed indefinitely". This is proved by constructing an angle that contains the angle ABC so many times that it becomes larger than the angle DEM. We have STU > DEM, and <STU = n ·ABC. On the line EM you then make a number of congruent copies of DEFG as shown, and very briefly, n · ABC > DEM > n ·DEFG, which entails that the plane ABC is greater than the plane DEFG.

**§38** — Theorem 38 is a necessary theorem simply stating that when BD = DE and the angles B, p, q is equal, then the planes ABDF and FDEG are con-




gruent. Paragraph 37 is just an exercise showing how to construct a parallel to a given line through a given point.

**§36** — That brings us to Theorem 36 which says that "[w]hen two straight lines are intersected by a third such that one outside angle is equal to its corre-



sponding inside angle (r = p), then these two lines will not intersect no matter how long they are prolonged at both sides". The proof for this is that if for instance A and C meets, then these lines together with the transversal EF will form a triangle and then will r > p which contradicts the condition.

After this theorem Holmboe presents the definition of parallel lines which is practically the same as Euclid's – "Two straight lines in the same plane which does not meet when prolonged infinitely to both sides, are said to be parallel to each other, or the one is parallel to the other".

Holmboe's backward line of theorems end in the fact that the sum of angles in a triangle equals 2R, which is equivalent to Euclid's parallel postulate.

## *5 Conclusion*

There is no focus on parallel lines as a topic in the Norwegian primary school today. The definition used is that two lines that never meet are parallel to each other. One of the learning objectives in 4th grade is to "explore, describe and compare properties of two- and three-dimensional figures using angles, edges and corners", and one of the learning objectives in 6th grade is to "describe properties of and minimum definitions of two- and three-dimensional figures and explain which properties the figures have in common, and which properties distinguish them from each other". In 9th grade, two of the learning objectives are to "explore the properties of different polygons and explain the concepts of formality and congruence" and to "explore, describe and argue for relationships between the side lengths of triangles". (Kunnskapsdepartementet, 2019)

Pupils today need of course understanding of parallel lines for these objectives, and for using digital tools, but the topic of parallel lines is not emphasized as it was almost 200 years ago. Modern pupils understand what parallel lines are, but they may be lacking the awareness that the concept has tremendous difficulties. In Holmboe's textbook (Holmboe, 1827) the second chapter is called "About two straight lines intersected by a third", 35 pages long, and the third chapter is called "About parallel lines", 29 pages long. Later there is a chapter five called "About straight lines' relationship to each other" which is 13 pages long. The entire textbook is 158 pages.

Holmboe's textbook in plane geometry was a typical textbook for his time, but this way of presenting the subject matter was challenged (Christiansen, 2012a). Textbooks of today represents a much more pedagogical and didactical correct way to present the subject matter, rather than strict demands on rigour in concepts and methods that characterised textbooks from the 19th century and earlier.

## REFERENCES


Christiansen, A. (2009). Bernt Michael Holmboe (1795–1850) and his mathematics textbooks. *BSHM Bulletin: Journal of the British Society for the History of Mathematics*, 24, 105–113.

Christiansen, A. (2012a). A controversy about geometry textbooks in Norway 1835–36. In Bjarnadóttir, K., Furinghetti, F., Matos, J. M., and Schubring, G. (Eds.), *"Dig where you stand" 2. Proceedings on the second "International Conference on the History of Mathematics Education",* 117–128. New University of Lisbon, Portugal: UIED.

Christiansen, A. (2012b). Geometry textbooks in Norway in the first half of the 19th century. In *HPM2012. The HPM Satellite Meeting of ICME–12*, 683–692, Daejeon, Korea.

Christiansen, A. (2015). An analysis of two Norwegian geometry books from 1827 and 1835, and the reactions they caused. *Nordic Studies in Mathematics Education*, 20(3–4).

Euclid (1956). *The Thirteen Books of the Elements*, volume I–III. New York: Dover Publications, Inc. Translated with Introduction and Commentary by Sir Thomas L. Heath. Unabridged and unaltered republication of the Second Edition from 1925. (First edition 1908).

Gray, J. (2008). Geometry. In *The Princeton Companion to Mathematics*, chapter II.2, 83–95. Princeton University Press.

Holmboe, B. M. (1827). *Lærebog i Mathematiken. Anden Deel, Inneholdende Begyndelsesgrundene til Geometrien*. (1st ed.). Christiania : Jacob Lehman.

Kunnskapsdepartementet (2019). Læreplan i matematikk (MAT01–05). Laid down as regulations. Læreplanverket for Kunnskapsløftet 2020. https://www.udir.no/lk20/mat01-05

# IN SEARCH FOR THE EARLY ROOTS OF THE EUROPEAN MODERN MATHEMATICS MOVEMENT

**Dirk DE BOCK**
KU Leuven, FEB, Campus Brussels, Warmoesberg 26, 1000 Brussels, Belgium
dirk.debock@kuleuven.be

## ABSTRACT

The origins of the European modern mathematics movement are to be situated in the early 1950s within the International Commission for the Study and Improvement of Mathematics Teaching. In 1952, Caleb Gattegno, the actual animator of the Commission, managed to bring the Bourbakists together with the Swiss psychologist Jean Piaget on the theme of "mathematical and mental structures." A connection was established between the structures in the work of Bourbaki and the structures of children's cognitive development as revealed by Piaget. For that reason, it was argued that the didactics of mathematics needed to be rethought from the Bourbaki perspective on mathematical structures. A model for the science of mathematics gradually became a model for mathematics education.

## *1 Introduction*

The 1959 Royaumont Seminar is commonly seen as the official launch of the modern mathematics movement in Europe (Schubring, 2014). The Royaumont Seminar was indeed a decisive meeting: For the first time in history, European mathematicians such as Jean Dieudonné, Gustave Choquet, and André Lichnerowich, who were members of or had a strong connection to Bourbaki, as well as American reformers such as Edward G. Begle, director of the governmentally strongly supported School Mathematics Study Group, were brought together. However, the claim that the New Math originated in the US and crossed the Atlantic in 1959 (Nadimi Amiri, 2017) is unjustified: Already in the early 1950s, ideas were launched in Europe to reorganize the teaching of mathematics according to the model that Bourbaki had developed for the science of mathematics from the end of the 1930s onwards.

The organization that initiated the European reform movement was the International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM), formally established in 1952 after Caleb Gattegno, an Egyptian-born mathematician and psychologist, had paved the way for it in the previous two years. We reconstruct the debates at the first meetings of this group, particularly the meeting in 1952 where the Bourbakists met the Swiss developmental psychologist Jean Piaget. Debates resulted in the assumption of an alignment of mathematical and mental structures, which became a main argument for the reform of mathematics education in Europe.

## *2 The debates within CIEAEM in the early 1950s*

In April 1950, Gattegno brought together an international group of experts in mathematics, psychology, and education, including some experienced mathematics teachers, in Debden (UK). Although the number of participants was limited, the range of competences had to allow for "a thorough reconsideration of the whole problem of the child and mathematics" (Gattegno, 1947, p. 220). This meeting, followed by two similar meetings in 1951, one in Keerbergen (Belgium) and one in Herzberg (Switzerland), led to the official founding of the CIEAEM in La Rochette sur Melun (France) in April 1952.

In La Rochette, the foundations were laid for the "modern mathematics" movement in Europe. The meeting was not an accidental encounter between top mathematicians and top psychologists/epistemologists of that time; on the contrary, the meeting was carefully prepared by Gattegno who, as a holder of doctorates in mathematics and in psychology, was familiar with the recent developments in these fields, in this case particularly the work of Bourbaki and Piaget. Already at the Debden meeting, Gattegno would have announced: "I will have the Bourbakists, I will have Piaget, I will have Gonseth" (Félix, 1986, p. 26).

The debate was initiated by Dieudonné who outlined Bourbaki's points of view, paying particular attention to the origin and essence of structures in modern mathematical science (Félix, 1986). He argued that structures are by no means artificial constructs that appear out of nowhere; they are "explicitations" of ideas that were already present in the work of great mathematicians of the past, implicitly and under different guises, but which were not yet recognized as such. Their role in mathematical research was clarified by Lichnerowicz: "A structure is a tool that we search for in the arsenal we have at our disposal. It is not at this stage that it is created" (quoted from Félix, 2005, p. 82). Choquet and Lichnerowicz also testified about how they actually used structures in their research (Félix, 1985, 1986, 2005). For the teachers in La Rochette, the vision of the Bourbakists and the way they practiced mathematics was nothing less than a revelation. After World War II, Bourbaki's work was known to research mathematicians, but most secondary school teachers, even those who had graduated in mathematics, were completely ignorant of this "modern" evolution within mathematics.

During the discussion, Dieudonné emphasized that the Bourbakists were not dealing with questions of a philosophical or metaphysical level, only common logic was used (Félix, 1986). Associations of mathematical structures with extra-mathematical constructs were not suggested by Bourbaki, but they were established by Piaget, who explicitly related Bourbaki's structures to the mental operations through which a child interacts with the world (Piaget, 1955)[81]. More specifically, Piaget identified the fundamental structures and stages of early mathematical thinking, as revealed by (his) psychological research, with the mother structures in the work of Bourbaki:

> *Now, it is of the highest interest to ascertain that, if we retrace to its roots the psychological development of the arithmetic and geometric operations of the child, and in particular the logical operations which constitute its necessary preconditions, we find, at every stage, a fundamental tendency to organize wholes or systems, outside of which the elements have no meaning or even existence, and then a partitioning of these general systems according to three kinds of properties which precisely correspond to those of algebraic structures, order structures, and topological structures.* (Piaget, 1955, pp. 14–15)

Piaget's identification of Bourbaki's mother structures with the basic structures of thinking, implying a harmony between the structures of "contemporary" mathematics and the way in which a child constructs mathematical knowledge, had a straightforward pedagogical implication: The learning of mathematics takes place through the mother structures of Bourbaki, the structures with which 20th-century mathematicians had founded and built their science. Correspondingly, Piaget (1955) asserted that "if the building of mathematics is based on 'structures', which moreover correspond to the structures of intelligence, then it is on the gradual organization of these operational structures that the didactics of mathematics must be based" (p. 32). In other words: A model for the science of mathematics was promoted as a model for mathematics education. Some teachers who participated in the 1952 meeting

[81] Piaget (1955) was the summary of his presentation in La Rochette (1952), as mentioned in a footnote to that book chapter.

in La Rochette immediately adapted their teaching practice to what they had learned (Félix, 1986).

Bourbaki's reconstruction of mathematics from a limited number of basic structures, connecting different branches of this science and underlining its fundamental unity, "supported" by Piaget's theory of cognitive development, would further stimulate the debate within CIEAEM. For the 1954 meeting in Oosterbeek (The Netherlands), the theme "The modern mathematics at school" was chosen. One of the questions concerned the adaptation of curricula "in the light of what we know about modern mathematics and the thinking of the child and the adolescent" (Lenger, 1954–1955, p. 58). According to Frédérique Lenger, it was primarily up to the teacher to introduce something of the spirit of modern mathematics into their teaching.

> *Modern mathematics [...] is the result of an awareness of structures and the relationships between these structures. The thinking of the modern mathematician is relational. And it seems to me that relational mathematical thinking can be recreated by a child or adolescent if the teacher is aware of it and if he presents the appropriate situations.* (Lenger, 1954–1955, p. 58)

## *3 UNESCO enters the scene*

In the mid-1950s, a systematic effort to map national developments in the teaching of mathematics at the secondary level was undertaken by UNESCO, in cooperation with the International Bureau of Education led by Piaget (UNESCO, 1956). In an extensive survey of secondary school mathematics, in which 62 countries participated, one of the questions reflected a growing interest in modern mathematics: "to what extent does the evolution of modern mathematics affect secondary education?" (p. 10). The review of the answers of the 62 countries stated:

> *The question as to what precise extent modern developments in the mathematical field have affected the secondary teaching of mathematics was answered by some twenty of the countries. In some cases the reply states merely that those developments were taken account of in the formulation of the secondary mathematics syllabuses. In other cases details are given of definite modifications made or impending in those syllabuses, including the introduction of infinitesimal calculus, coordinate geometry, statistics,*

*etc., and added stress on functions, vectors, the calculation of probability, differential and integral calculus, and applied mathematics.* (pp. 26–27)

Mathematical structures were not included in this list. Willy Servais, who was a delegate for Belgium at the UNESCO conference in Geneva in July 1956, at which the results of the survey were discussed, did make explicit mention of modern mathematical structures in his report:

*To what extent can the more abstract [...] mathematical structures, discovered within classical mathematics and developed worldwide by today's mathematicians, have a beneficial impact on secondary education? This is a very recent question to which pioneers in many countries are seeking an answer. The results obtained so far hold the promise of a pedagogical innovation.* (Servais, 1956–1957, p. 40)

In general, survey responses revealed an awareness of the need for changes in secondary school mathematics curricula, but little mention of implemented reforms. At the national level in Europe, discussion about a modernization of secondary mathematics curricula emerged from the mid-1950s, particularly in France and Belgium, and from the end of the 1950s (but before the 1959 Royaumont Seminar), some concrete reform initiatives were taken (see, for example, Barbazo & Pombourcq, 2010; De Bock & Vanpaemel, 2018, 2019).

## *4 Conclusion*

At the meetings of the CIEAEM in the early 1950s, especially at the 1952 meeting in La Rochette, the seeds were sown for a structural approach to the teaching of mathematics, i.e., modern mathematics. Bourbaki offered the mathematical rationale and Piaget provided the psychological justification. In the subsequent years, the CIEAEM would continue to play an important role in refining its modern view on mathematics education.

The European modern mathematics movement was certainly not an "import product" of the American New Math. On the contrary, our findings provide evidence for Bob Moon's claim that "a 'wave' of development in the USA 'crossed over' to Europe, although it is oft repeated, may be too simplistic a picture […] a pattern of 'parallel' innovation would be a more appropriate characterization" (Moon, 1986, pp. 46–47).

## REFERENCES


Barbazo, E., & Pombourcq, P. (2010). *Cent ans d'APMEP* (Brochure APMEP n° 192). Paris, France: APMEP.

De Bock, D., & Vanpaemel, G. (2018). Early experiments with modern mathematics in Belgium. Advanced mathematics taught from childhood? In F. Furinghetti & A. Karp (Eds.), *Researching the history of mathematics education: An international overview* (pp. 61–77). Cham, Switzerland: Springer.

De Bock, D., & Vanpaemel, G. (2019). *Rods, sets and arrows. The rise and fall of modern mathematics in Belgium.* Cham, Switzerland: Springer.

Félix, L. (1985). Souvenirs d'une époque archaïque. *Bulletin de l'Association des Professeurs de Mathématiques de l'Enseignement Public, 352,* 5–13.

Félix, L. (1986). *Aperçu historique (1950–1984) sur la Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques (CIEAEM). 2e édition revue et augmentée*. Bordeaux, France: l'IREM de Bordeaux.

Félix, L. (2005). *Réflexions d'une agrégée de mathématiques au XX$^{e}$ siècle*. Paris, France: l'Harmattan.

Gattegno, C. (1947). Mathematics and the child. *The Mathematical Gazette, 31*(296), 219–223.

Lenger, F. (1954–1955). Rencontre internationale de professeurs de mathématiques. *Mathematica & Paedagogia, 4,* 57–59.

Moon, B. (1986). *The "New Maths" curriculum controversy: An international story*. Barcombe, United Kingdom: Falmer Press.

Nadimi Amiri, S. (2017). *Historicizing the journey of the New Math reform from the United States to Luxembourg in the 1960s and 1970s* (Unpublished doctoral dissertation). The Faculty of Language and Literature, Humanities, Arts and Education, University of Luxembourg, Luxembourg.

Piaget, J. (1955). Les structures mathématiques et les structures opératoires de l'intelligence. In J. Piaget, E. W. Beth, J. Dieudonné, A. Lichnerowicz, G. Choquet, & C. Gattegno, *L'enseignement des mathématiques* (pp. 11–33). Neuchâtel, Switzerland: Delachaux et Niestlé.

Schubring, G. (2014). The road not taken—The failure of experimental pedagogy at the Royaumont Seminar 1959. *Journal für Mathematik-Didaktik, 35*(1), 159–171.

Servais, W. (1956–1957). L'enseignement des Mathématiques dans les écoles secondaires. *Mathematica & Paedagogia, 10,* 34–43.

UNESCO. (1956). *Teaching of mathematics in secondary schools* (Publication No. 172 of the International Bureau of Education, Geneva). Paris, France: UNESCO

# PORTUGUESE TEACHERS' CONCEPTIONS AND PRACTICES ON THE HISTORY OF MATHEMATICS IN TEACHING ($7^{TH}$ TO $12^{TH}$ GRADES)

**Ana P. MARTINS, Hélder PINTO, Helena GOMES, Luís MENEZES**
ESE Viseu and CIUHCT; Piaget Institute, RECI and CIDMA;
ESE Viseu and CIDMA; ESE Viseu and CI&DETS
anapatmartins@gmail.com, helder.pinto@ipiaget.pt,
hgomes@esev.ipv.pt, menezes@esev.ipv.pt

## ABSTRACT

This study, based on what are the international recommendations for the introduction of the History of Mathematics (HM) in the teaching of this discipline, seeks to describe the Portuguese reality. Thus, the study aims to know: (i) teachers' training in HM; and (ii) their conceptions and practices about the use of HM in the classroom. An online questionnaire was answered by a group of 432 Portuguese Mathematics teachers of secondary education ($7^{th}$-$12^{th}$ grades). Approximately half of the teachers considered their training in HM obtained in their higher education as frequent or solid. Almost all categorized as non-existent or reduced their continuous education in HM. Regarding the teachers' practices, an overwhelming majority reports using HM as a didactic resource in Mathematics class and only about a third do it rarely. Teachers resorted more frequently to textbooks and problems with a historical context and favored the use of HM in the introduction to mathematical content. As for the potential of using HM in teaching, teachers evaluated the didactic potential of HM very positively. The biggest constraints were related to the extension of the official curricula, the difficulty of evaluation and the scarcity of support materials.

## 1 *Introduction*

The topic of using HM as a resource for teaching Mathematics is already quite old in research (Clark, 2019; Martins et al., 2021a). Some of these studies focus on the characterization of what teachers think and do with HM in Mathematics classes. Other studies are dedicated to developing and evaluating didactic proposals that consistently incorporate HM in Mathematics classes.

Although in Portugal the interest in HM comes from the last two decades of the 20th century (GTHEM, 1997), knowledge about the reality of schools, mathematics teachers' perspectives and teachers' training are limited. Therefore, within the scope of the project "(H)ISTO é Matemática: História da Matemática no ensino da Matemática" (History of Mathematics in teaching), we developed this study with the aim of knowing: (i) secondary (7th-12th grades) teachers' training in HM; and (ii) their conceptions and practices about the use of HM in the classroom. Notice that the subject of HM it's not mandatory in the initial training of these teachers, although some of them may have had topics or some curricular units about it.

## *2 State of the art*

The field of HM in mathematics education is a domain where the potentialities of using HM are analysed to improve the teaching of Mathematics at all education levels. This field was formalized inside the ICMI with the creation of a thematic affiliated organization in 1976 (HPM). This field is still active today; for instance, in Clark (2019), it is possible to find current examples from the utilization of HM in teaching, in Brazil, Denmark and USA.

Portugal, since the late 20$^{th}$ century, was aware of the latest developments in this field: Portugal hosted, in 1996, in Braga, the *ESU2*, an important event for the HPM community; the Portuguese Association of Mathematics Teachers published in 1997 the translation of texts of relevant authors such as D. Struik, J. Fauvel and F. Swetz (GTHEM, 1997); a long paper was published in 2001 in one of the journals of this association presenting the "benefits from integrating History of Mathematics into teaching" (Ferreira & Rich, 2001).

One of the first studies about teachers' conceptions on HM was done, in 2004, in Hong Kong. This investigation was managed by M.-K. Siu and included 608 participants, with the main conclusion: "the value of history of mathematics is highly regarded by schoolteachers, but the degree of initiative on actually using history of mathematics in the classroom is very low!" (Siu, 2007). Siu appointed the main reasons for the teachers do not use HM in the classroom: "I have no time for it in class!" (67% agreed with this statement), "there is a lack of resource material on it!" (64%) and "there is a lack of teacher training in it!" (83%). More recently, a similar study was conducted in France by M. Moyon (646 participants) with similar conclusions: for instance, "71% of the teachers are interested in HM and would like to introduce it in their teaching", but many "do not know how to do it" (Moyon, 2021).

The present work is part of the project "(H)ISTO é Matemática: History of Mathematics in teaching; in (Martins *et al*, 2021a, 2021b, 2021c) and (Costa *et al*, 2021) it's possible to find more results of this project.

## 3 Methodology

An online questionnaire was answered by 432 Portuguese Mathematics teachers of secondary education ($7^{th}$-$12^{th}$ grades; generally, students 12-18 years old).

The teachers surveyed were mostly female (78%), with an average age of 52 years old, an average service time of 24 years. Nearly one quarter have a master/doctor degree, most in Mathematics Education.

The survey was organized into three sections according to the themes and categories of analysis defined (Table 1):

| Themes | Categories |
|---|---|
| Training in HM<br>*(2 questions)* | • Initial training<br>• Continuous Training |
| Conceptions on the use of HM for teaching<br>*(2 questions)* | • Potential for the use of HM<br>• Constraints to the use of HM (student, training, curriculum guidelines and resources, nature of the subject) |
| Practices of using HM for teaching<br>*(7 questions)* | • Ways in which HM can be used in the classroom<br>• HM resources for the classroom<br>• Impact of the use of HM on student learning |

**Table 1.** Themes and categoris of analysis.

For the answers, we adopted a four-level agreement scale, 1 being the minimum and 4 the maximum levels of agreement. Quantitative data analysis methods were used, namely descriptive statistics.

## 4 Results

The study's findings are categorized into three categories: (i) training in HM; (ii) teachers' conceptions on the HM; and (iii) teachers' practices related to the use of HM in Mathematics teaching.

### 4.1. Training in History of Mathematics

In this section, we talk about the initial and continuous training in HM of the interviewed teachers.

Most of them (49%) considered their higher education's HM training as non-existent or minimal and 25% said it was solid (obtained, for example, in subjects of HM). 50% of the teachers specified their training: most of them (95%) referred it was obtained in specific subjects of HM. Continuous training in HM was said to be non-existent by 46% of the teachers, reduced by 50%, and frequent by 4%. Only a minority (20%) responded to the request for examples and most of them specified short-time training courses.

### 4.2. Conceptions of teachers on the use of History of Mathematics in Mathematics teaching

In this section, we describe the conceptions on the use of HM in Mathematics teaching of the 432 inquired teachers. Those conceptions were studied in two categories: potential of the use of HM and constraints to the use of HM.

Teachers appreciated in a very evident way the potential of using HM in Mathematics teaching. The survey results showed that, in all items, the levels of agreement 3 or 4 points appear between 75.2% and 88.6% of answers and the average of the scores (arithmetic mean) is high (between 3 and 3.3). Also, the answers highlight the understanding that HM allows to illustrate the usefulness and importance of Mathematics (with 88.6% of answers at levels 3 and 4), allows also to illustrate relationships between different mathematical domains (84.7% in the same levels) and it favours the demystification of mathematics as a finished product, showing that doubt and error are part of the human (mathematical) activity (82.47% in the same levels). The item with the least frequency of response at level 4 (and at levels 3 and 4 together) is the one relative to which HM enables the development of skills beyond mathematical knowledge, such as documenting, analysing, and discussing mathematical subjects (only with 27.5% in level 4).

On the results on the constraints to the use of HM, teachers' opinions were more dispersed since the CV is higher than in the previous data (CV is a measure of dispersion that allows evaluating the representativeness of the mean by the proportion of the standard deviation in relation to the mean value; if CV > 50%, the mean is not representative of the data set and the smaller the CV, the greater the representativeness of the mean). Still, we consider that the mean is representative in all cases. Teachers recognize as major constraints aspects related to the curriculum guidelines, namely the fact that the course syllabus is extensive and makes it difficult, or even prevents, the inclusion of this didactic approach (94.5% in levels 3 or 4). They also identify the difficulty of evaluation because the integration of the HM in the assessment is not hampered by curricular guidelines (average of 2.9). The scarcity of support materials is also chosen (average of 2.8 and 70.6% in levels 3 and 4). The nature of the discipline is the aspect least identified as a constraint (the lowest average, 1.6).

### 4.3. Practices of teachers on the use of History of Mathematics in the Mathematics classroom

Most teachers (82%) said they use/already used HM as a didactic resource. 11% do/has done it several times and 60% do/have done it occasionally.

On the results of teachers' ways of using HM, the most frequent use was the introduction of mathematical content (87% levels 3 or 4, average 3.4). Also rated positively was the allusion to the history of mathematical symbology (average 2.9). The less rated items were teachers' presentations and resolution of mathematical tasks (31% levels 3 or 4, averages 2.1). (All the averages respecting the topic ways of using HM are representative.)

Teachers' levels of agreement about the use of resources in teaching practices with HM shows that scholar textbooks predominate (66% levels 3 or 4, average 2.8), followed by problems of mathematical context (more than 62% levels 3 or 4, average 2.7), text/video/websites (more than 57% levels 3 or 4, average 2.6). The least used resources were primary sources; teaching materials developed by teachers and ancient instruments (averages 2.0, 1.8, and 1.8; not representative averages).

An overwhelming majority (97%) responded when asked to describe one of their teaching methods with HM. Such examples show strong links with scholar textbooks: Pythagorean theorem, construction of sets of numbers, Pi number, Thales' theorem and mathematical biographical notes.

Regarding the perceived effects of HM use on students learning, teachers did not differ significantly in their ratings (CV values are low), with positive results (53% to 72% levels 3 or 4, averages between 2.5 and 2.9 and representative). The items with the highest levels of agreement were facilitating the establishment of connections between Mathematics and reality; increased motivation for solving the proposed tasks and increased enjoyment of students (over 60% at levels 3 or 4, averages 2.9, 2.8 and 2.8). The lowest score was facilitating the understanding of mathematical content (average 2.5 and near 50% levels 3 or 4).

## *5. Discussion and conclusions*

The results presented for the Portuguese teachers of secondary education (7th-12th grades) are very similar to the results of studies for other countries (Siu, 2007; Moyon, 2021): there is a gap between the conceptions (the potential of HM is highly regarded by teachers) and practice (the effective use of HM in classroom is very intermittent). The study also allows us to conclude that half of the teachers said that their initial training in HM is solid; the majority reported that their continuous training in HM is reduced or non-existent. Regarding their conceptions on the use of HM in Mathematics teaching, teachers recognize the potential of using the HM to learn Mathematics, particularly because it allows to illustrate the usefulness and importance of Mathematics, it also favours the demystification of mathematics as a finished product, and allows to illustrate relationships between mathematics and other disciplines. The biggest constraints are related to official curricula, the difficulty of evaluation and the scarcity of support material. Most teachers reported using or having used HM as a teaching resource in mathematics class (and many do this regularly).

Teachers used more often scholar textbooks, as well as problems of mathematical context and text/video/websites in their practices and preferred the use of HM in the introduction to new mathematical content. Teachers evaluated the impact of using HM on students' learning very positively. There is an enormous consensus among teachers from the 7$^{th}$ up to the 12$^{th}$ grades on the need for more training in HM.

Finally, it should be highlighted that these results are aligned with the conclusions presented in (Martins, 2021c), where a similar survey was applied to 259 Portuguese teachers of the 2$^{nd}$ Cycle of Basic Education (5$^{th}$-6$^{th}$ grades).

**REFERENCES**

Clark, K. (2019). History and Pedagogy of mathematics in mathematics education: History of the field, the potential of current examples, and directions for the future. In U. T. Jankvist, M. Heuvel-Panhuizen & M. Veldhuis (Eds.), *Eleventh Congress of the European Society for Research in Mathematics Education* (pp.

29-55). Utrecht University.

Costa, C., Menezes, L., Ribeiro, A., Pinto, H., Martins, A. P., Clain, T., Gomes, H. & Aires, A. P. (2021). Perspectives of Future Teachers of Early Years from two Portuguese Institutions on the History of Mathematics. In L. Chova, A. Martinez, I. Torres (Orgs.), *Proceedings of ICERI2021 Conference* (pp. 8222-8230). IATED.

Ferreira, R. and Rich, B. (2001). Integrating history of mathematics into the mathematics classroom, *Quadrante*, *10*(2), 67–96.

GTHEM (1997). *Relevância da História no Ensino da Matemática*. APM.

Martins, A. P., Clain, T. C., Costa, C., Pinto, H., Ribeiro, A., Gomes, H. & Menezes, L. (2021a). Conceções e Práticas de Professores sobre a História no Ensino da Matemática num Contexto de Formação, *Millenium*, *2*(9), 249–259.

Martins, A. P., Gomes, H., Menezes, L., Costa, C., Pinto, H., Ribeiro, A. & Clain, T. C. (2021b). A História da Matemática no percurso formativo de futuros professores dos anos iniciais. In B. Silva, L. Ameida, A. Barca, M. Peralbo & R. Alves (Orgs.), *Atas do XVI Congresso Internacional Galego-Português de Psicopedagogia* (pp. 1399-1412). Universidade do Minho.

Martins, A. P., Gomes, H., Pinto, H., Ribeiro, A., Clain, T. C., Costa, C. & Menezes, L. (2021c). Teachers' Conceptions and Practices on the History of Mathematics in Teaching at 5th and 6th Grades in Portugal. In L. Chova, A. Martinez, I. Torres (Orgs.), *Proceedings of ICERI2021 Conference* (pp. 7078-7084). IATED.

Moyon, M. (2021). I would like to introduce history in my mathematics lessons but I do not know how to do it! *HPM Lecture*, 16 July. Retrieved from https://www.youtube.com/watch?v=NTIi44eCMLM

Siu, M. K. (2007). No, I don't use history of mathematics in my class. Why? In F. Furinghetti, S. Kaijser, & C. Tzanakis (Eds.), *Proceedings HPM2004 & ESU4* (pp. 268–277). Uppsala Universitet.

# SPANISH SCHOOL ENCYCLOPEDIAS (1901-1965)

## An unexplored source for the History of Mathematics Education in Spain

**José M. MUÑOZ-ESCOLANO[1], Antonio M. OLLER-MARCÉN[2], María SANTÁGUEDA-VILLANUEVA[3]**

[1]Universidad de Zaragoza, C. Pedro Cerbuna 12, Zaragoza, Spain, [2]Centro Universitario de la Defensa de Zaragoza, Ctra. de Huesca s/n, Zaragoza, Spain, [3]Universitat Jaume I, Av. Vicent Sos Baynat s/n, Castelló de la Plana, Spain

[1]jmescola@unizar.es, [2]oller@unizar.es, [3]santague@uji.es

Since the 19th century, and until the second half of the 20th century, Spanish primary education was organized in grades. The name of such grades, as well as their number, varied over time. However, in general terms, we can affirm that, for over a century, Spanish primary education consisted of four grades: 6-8 years (usually called first or elementary grade), 8-10 years (second or medium grade), 10-12 years (upper or improvement grade), and 12-14 years (extension or professional initiation grade). Between 1901 and 1965, mandatory education was usually restricted to the period 6-12 years (Galera Pérez, 2018). By a Royal Decree from October 26, 1901 the cyclic nature of Spanish primary education was officially established: "Each of the three grades into which this education is divided will embrace all the subjects indicated, distinguished only by the breadth of the program and by the pedagogical character and duration of its exercises" (*Gaceta de Madrid*, year CCXL. Madrid, October 30, 1901. No. 303, p. 497). This Royal Decree also established the different subjects that had to be covered during primary education, in particular, Arithmetic and Geometry. However, no official syllabi for this level were published until 1953. According to different authors (Escolano Benito, 1997; Viñao, 2001), the newly established cyclic nature of primary education was one of the main factors that led to the flourishing and popularization of the so-called *school encyclopedias*. These particular textbooks (see Figure 1) were defined by two main characteristics: each book corresponded to one grade rather than to one school year (different letter size was often used within a book in order to organize the contents into school years), and they contained all the subjects in one single volume.


**Figure 1.** The four encyclopedias by the same publisher.

This type of textbooks already existed in Spain since the last quarter of the 19th century. They were cheap, they were adapted to the context of unitary schools in which students of different ages shared the same classroom and, in the absence of official syllabi, they provided the teachers with a ready-made, easy to implement lesson plan. All these factors also explain the great popularity of school encyclopedias in Spain during the first half of the 20th century. In spite of their popularity and their extensive use during almost a century, very little research on school encyclopedias has been carried out from the point of view of History of Mathematics Education in Spain. Nevertheless, works such as (Santágueda-Villanueva & Goméz, 2021) clearly illustrate their interest as a source material. We now mention various possible approaches that take into account some of the elements pointed out by Schubring (1987), such as the necessity of considering the contemporaneous context of the textbooks, or the interest of analyzing the role of the textbook author.


**Figure 2.** *Nueva Enciclopedia Escolar* (second grade). Editions from 1931, 1940, 1944, and 1955.

First, we can take into consideration the fact that there usually exist many different editions of a given encyclopedia (see Figure 2). This allows us to carry out longitudinal case studies in which we can take into consideration contextual elements, not only educational (publication of official syllabi in

1953, for example), but also social (the Spanish Civil War from 1936 to 1939, for instance). This type of approach can be used to analyze the possible evolution in the treatment of particular topics, or to describe how the evolution in the society was reflected (or not) in aspects like the context of the proposed problems.

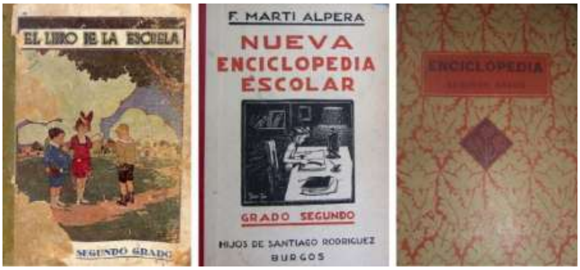



**Figure 3.** School encyclopedias with different author profile. Teachers association (left), individual (center), and religious order (right).

A second possible approach takes into account that the popularity of school encyclopedias entailed the publication of this form of textbooks by many different profiles of authors and publishers. We can find collaborative works promoted by teachers' associations, more traditional publications written by a single author, or religious orders that published textbooks initially, but not only, to be used in the context of their own schools (Figure 3). In this way, it might be interesting to research about the possible dissemination of novel pedagogical ideas depending on the background and ideology of the authors and editors. Finally a third interesting approach is related to the fact that some publishers had in their catalogue both encyclopedias and textbooks for the different subjects. As we mentioned before, one of the defining characteristics of school encyclopedias was that they contained all the subjects in the same volume. However, we must note that they were not just the result of putting the different textbooks together. In Figure 4, for instance, the encyclopedia devoted 20 pages to Arithmetic and 8 pages to Geometry. However, the textbooks had 134 and 66 pages, respectively. It is quite possible that these books were intended to be used in different socio-economical contexts. This means that, even if education was mandatory for those ages (grade one corresponded to ages 6-8), the training given to the students could greatly differ according to the type of textbook that was used which, in turn, could depend on several variables (type of school, demographic group, being in a rural or urban area, etc.). This was true also after the publication of the official syllabi in 1953. Consequently, it seems interesting to compare the treatment of different mathematical topics both in encyclopedias and in regular textbooks (Villanueva Baena, 2015).


**Figure 4.** School encyclopedia, Arithmetic and Geometry textbooks (grade 1) from the same publishing house.

We think that these ideas illustrate the potential of school encyclopedias as a fruitful source for the History of Mathematics Education in Spain. In the near future, we intend to explore some of them.

## REFERENCES

Escolano Benito, A. (1997). Libros escolares para programas cíclicos: epítomes, compendios y tratados: las primeras enciclopedias. In A. Escolano Benito (Dir.), *Historia ilustrada del libro escolar en España: del Antiguo Régimen a la Segunda República* (pp. 425–448). Fundación Germán Sánchez Ruipérez.

Galera Pérez, A.D. (2018). Escuela pública durante la I Restauración (1875-1931): Aspectos administrativos y curriculares. *Cabás*, *19*, 17–42.

Santágueda-Villanueva, M., & Gómez, B. (2021). Los modelos de enseñanza de los problemas de aligación en las enciclopedias escolares españolas. *Boletim de Educação Matemática*, *69*(35), 365–388.

Schubring, G. (1987). On the methodology of analyzing historical textbooks: Lacroix as textbook author. *For the Learning of Mathematics*, *7*(3), 41–51.

Villanueva Baena, M.C. (2015). *La editorial Luis Vives, una empresa de mediación cultural. Valores, modelos de socialización y contenidos en los manuales de lectura (1890-1975)*. [Doctoral dissertation, Universidad de Zaragoza].

Viñao, A. (2001). El libro escolar. In: J. A. Martínez Martín (Dir.), *Historia de la Edición en España (1836-1936)* (pp. 309–336). Marcial Pons.

# THEME 6

# HISTORY OF MATHEMATICS IN ITALY

# STARTING FROM THE HISTORY OF MATHEMATICS IN EARLY MODERN ITALY: FROM PRIMARY SOURCES TO MATHEMATICAL CONCEPTS
## Abacus Mathematics and Archimedean Tradition

**Veronica GAVAGNA**
University of Florence, Department of Mathematics and Computer Science,
Viale Morgagni 67/a, Florence, Italy
veronica.gavagna@unifi.it

## ABSTRACT

The proposed activity consists of the analysis of the Archimedean technique used by Piero della Francesca to determine the volume of the double vault, supported by figures drawn in dynamic geometry enivronment. The study of the problem, adequately contextualized from a historical point of view, provides interesting insights into the language of (Renaissance) mathematics and the evolution of the concept of demonstration.

## 1 *The proposed activity*

The following activity is related to two fundamental issues in medieval and early Renaissance mathematics: the abacus tradition [see Gamba and Montebelli 1987] and the Archimedean tradition [see Napolitani 1998 and 2010]. In the Middle Ages, these traditions were basically expressions of two culturally worlds apart – on one side the world of practitioners and on the other side the world of humanistic circles– but their fates began to intertwine in the second half of the fifteenth century by the work of figures such as Piero della Francesca (1412-1492) and Luca Pacioli (1445-1517), when they attempted to use Archimedean techniques for the determination of volumes of some solids. The case we examine is that of the volume of the double vault (or rib vault), described by Piero della Francesca in his *Libellus de quinque corporibus regularibus*.

The activity has not yet been tested with students, so this contribution is only a proposal: given its complexity, it is suitable for students in the final year of high school or for teachers in training. In particular, I believe it could be a suitable activity to be developed in the so-called "Liceo Matematico" (https://www.liceomatematico.it/), an experimental project spread throughout Italy that has been tried and tested for some years now, geared toward promoting interdisciplinary laboratory activities in high school. In this activity, in fact, mathematical, historical-mathematical and linguistic skills are involved as well as the ability to interact with dynamic geometry environments.

The path necessarily begins with the narration, given in lecture-style by the teacher, of the historical-mathematical context in which Piero's treatise is set. In the next paragraph I present a short *excursus* highlighting the most significant aspects of the historical framework, in order to define the relevance of the example proposed here.

The second phase of the activity consists of the analysis of Piero's text, to be conducted in groups of 4 or 5 students after a collective reading led by the teacher: students will become researchers and try to interpret a historical text by re-constructing its mathematical meaning. The various interpretations, resulting from the group work, will then be compared in a collective mathematical discussion, in which a shared interpretation of Piero's text will be negotiated. It is important that the teacher will coordinate the discussion in order to emphasize the critical points of the demonstration, as well as the linguistic-mathematical aspects, such as the lack of symbolic expressions.

The activity could then continue with a comparison between the determination of the volume of the double vault presented by Piero and by Archimedes in his *Method* and discussed in [Archimedes 2013] and [Napolitani & Saito 2013 and 2014], but we do not have the opportunity here to explore this in depth.

## *2 The Abacus tradition and the Archimedean tradition*

The so-called "abacus mathematics" began to spread in Italy in the first half of the thirteenth century, following the economic revival and the restart of trade in the Mediterranean area. This economic and commercial development made an increasing need for a new mathematics, based on a system of numeration and computational algorithms more efficient than the Roman ones still in use in the Latin West.

One of the main instruments, but certainly not the only one, of this cultural revolution was the work of Leonardo Pisano, also known as Fibonacci. In particular, his *Liber Abbaci*, written in 1202 and revised in 1228, presented the positional decimal numbering system with Indo-Arabic numerals and new efficient algorithms for calculating. Besides, *Liber Abbaci* dealt with topics and problems typical of mercantile mathematics, and devoted the last chapter to the algebra of first and second degree equations. Topics in practical geometry, such as the area of plane figures and the volume of solid ones, were later treated by Leonardo in his *Practica Geometriae*. The dissemination of the *Liber Abbaci* and *Practica geometriae*, written in Latin, was made possible mainly by two factors: the birth of the "abacus schools" and the process of vulgarization of the texts. In the abacus schools students could receive either basic training, limited to elementary notions and knowledge of systems of weights and measures, or more advanced training, which opened up to the great mercantile trade and required complete mastery of mercantile mathematics. These schools also trained craftsmen, painters, sculptors, or what has been called the "middle cultural class" ("ceto culturale intermedio").

The approximately three hundred extant abacus treatises are collections of solved problems, generally written in vernacular language with a discursive style, rich in locutions borrowed from speech: they are very interesting both from a mathematical and linguistic point of view, but rarely known to high school students.

As for practical geometry, it did not always appear in abacus treatises, and when it did, it appeared in various forms: from short sections relating to plane geometry enriched by some notion of the volume of solids, to more extensive treatises. The writings on practical geometry also were a collection of solved geometrical problems: besides, they were not purely speculative problems as in Euclid's *Elements*, but they always contained numerical data and were solved using arithmetical or algebraic techniques.

At the time when abacus mathematics was beginning to spread, the Flemish Dominican William of Moerbeke arrived at the papal court of Viterbo, one of the most prestigious intellectual centers of the 13th-century Mediterranean. In Viterbo he found a circle of scholars – among them Campanus of Novara, John Peckam and Witelo – interested in science and translations. Even if Moerbeke is first known for his translations and revisions of Aristotle's complete work and commentaries on it, which became for about two centuries the standard text for university teaching, a very remarkable scientific achievement was his translation of Archimedes, Eutocius, and Ptolemy. In particular, he translated the Archimedean *corpus* from Greek into Latin using the so-called Codex A and Codex B drawn up around the 9th century in the Byzantine environment. Archimedes' work was particularly difficult to understand, both because of the content itself and because of the needed skills, which were not limited to Euclidean geometry alone but, for example, also included the theory of conics. Appropriating the entire Archimedean work therefore required not only tremendous intellectual effort but also the restoration of Greek mathematics as a whole. This project, however, was not on the horizon of 13th-century scholars, and thus William of Moerbeke's translation had little circulation in the scholarly community, so that Archimedes' geometrical knowledge and demonstrative techniques remained essentially unknown. Only a few results of *Measurement of a Circle* and *On the Sphere and Cylindre* - coming from the arabic tradition, filtered into the Latin world (and in the abacus schools) because they were useful in practical geometry.

The fifteenth century restoration of the Greek Classics, did not improve the situation: Moerbeke's translation continued to have limited circulation and at some point it apparently disappeared. It was rediscovered by the German philologist Valentin Rose only in 1881 in the Vatican Library in Rome (it is the codex Ott. Lat. 1850) and became a key witness for the Danish philologist Ludwig Heiberg who set up the critical edition of the Archimedean *corpus*. The fate of the Greek codices A and B was different: while Codex B disappeared in the 14th century, Codex A had better luck. The translation of Codex A was commissioned by Pope Nicholas V, the passionate bibliophile and humanist founder of the Vatican Library, to Iacopo di San Cassiano. All Renaissance editions of Archimedes' works, both manuscript and printed, including the *editio princeps*, published in 1544 in Basel, were based on Iacopo's translation.

Some documents of the Vatican Library Archives prove that the translation of Iacopo was loaned in 1458 to Francesco del Borgo, cousin of the famous painter Piero della Francesca, one of the greatest artists of the Italian Renaissance and also the author of mathematical works, such as an abacus treatise and the work on Platonic solids *Libellus de quinque corporibus regularibus*. Piero's works testify how abachistic mathematics began to broaden its horizons, expressing curiosity and interest in speculative and not just in practical works [Daly Davis 1977, Napolitani 2007]. The translation made by Iacopo was in fact lent by Francesco del Borgo to his cousin Piero, because he wanted to set up his own copy of the Archimedean *corpus*, correcting mistakes and remaking drawings by his own hand where necessary. Piero's redaction has been identified by James Banker in codex 106 in the Biblioteca Riccardiana in

Florence [Banker 2010] and was probably prepared by Piero while he was in Rome working on the Vatican Rooms (1458-1459). Piero then exploited his study of Archimedes to obtain some of the results illustrated in the *Libellus*, including the calculation of the volume of the double vault, that's *Casus X* of the fourth Chapter. The manuscript of the *Libellus*, now in the Vatican Library (Vat. Urb. Lat. 632), remained unpublished, but in 1509 Luca Pacioli published, without indicating the author, the entire *Libellus* translated into vernacular as the third part of his *Divina proportione*, under the title *Libellus in tres partiales tractatus divisus quinque corporum regularium et dependentium active perscrutationis*. The calculation of the volume of the double vault is *Casus 10* of this part.

## 3 *The historical sources*

Piero's calculation of the volume of the double vault presents multiple elements of interest.

From a historical point of view, this case represents one of the earliest evideces of a dialogue between the humanistic mathematics and the practical one. The most intriguing aspect, however, is the following one. As is well known, the so-called Codex C, discovered in 1906, contained the (unknown) *Method of Mechanical Theorems*. In this treatise, Archimedes explained to Eratosthenes a series of heuristic techniques he had used to find areas and volumes of various figures, and finally he showed a new technique to determine the volume of a cylindrical hoof and of the solid obtained as the intersection of two cylinders of equal radius at right angles, namely the double vault. However, Proposition 15 (which proves that the double vault is two-third of the cube circumscribed about the intersection of cylinders) is mutilated and its reconstruction is only conjectural [Napolitani & Saito 2013]. In light of this news, it is surprising and fascinating that Piero, without knowing of the existence of the *Method* (as far as we know), tackled the same problem faced by Archimedes and tried to solve it using Archimedean techniques.

From a linguistic point of view, "Case 10" is very lucky and fruitful, since we have at our disposal:

- the direct Latin source, namely the manuscript of Piero della Francesca's *Libellus*, digitized and available on the Vatican Library website (https://digi.vatlib.it/view/MSS_Urb.lat.632 );

- the critical edition of the *Libellus*, which can replace the manuscript text (which is not easy to read) or serve as a guide to decipher it [Piero della Francesca 1995];
- Luca Pacioli's 16th-century vernacular translation published in *Divina proportione* (1509) and available at websites such as Gallica (https://gallica.bnf.fr/) or the digital library of the Museo Galileo in Florence (https://www.museogalileo.it/it/)

Together with to these primary sources, the English translation published by Marshall Clagett in his *Archimedes in the Middle Ages* [Clagett 1978] is available. It is useful for students who do not read Latin or Italian.

From a didactic point of view, this proposal provides the opportunity to study an original source: students will become researchers who have to interpret a historical text. The interpretation of this text will also require considerable ability to visualize three-dimensional objects: therefore they will construct and study, in small groups, the intersection at right angle of two cylinders of equal radius in a dynamic geometry environment (DGE) to observe from various points of view what the double vault looks like. It would also be interesting for the students exploring the use of the rib vault in architecture, starting with the porch of the Palazzo Ducale in Urbino, probably built at the time Piero was in town.

## 4 The analysis of the historical source

In this paper, I will necessarily use Clagett's translation of Piero's text (in italics) [Clagett 1978, pp.408-410]; of course in an Italian classroom I would use a direct source, i.e. Pacioli's vernacular translation and/or Piero's text.

As we have said, it is recommended that the first reading of the text will be guided by the teacher; later students, in groups, will carefully read the text aided by worksheets, trying to draw the suitable figures in a DGE in order to explore them.

Let us begin by reading the statement.

*There is a certain cylinder whose diameter is 4 brachia – the diameter of each of its bases – and another cylinder of the same size pierces it orthogonally. We seek the quantity that is removed from the first cylinder by means of this hole.*

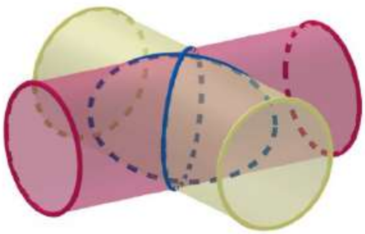


**Figure 1.** The intersection of two cylinders of equal radius at right angles

Piero seeks the volume of the intersection of the two cylinders using a very colloquial linguistic register, closer to the world of practitioners rather than the world of speculative geometry. As in all abacus treatises, the author deals with a specific object (the length of the diameter is given), thus turning the general problem into a generic example. In the following passage Piero describes "the cavity", that is, the double vault.

*You ought to know that the perforated cylinder is perforated in a straight line both at the beginning and the end of the cavity, that is, where the hole begins and ends and the axis of the piercing cylinder crosses through the axes of the pierced cylinder at right angles in their cavity and the lines of these form a square* [and, in fact, the intersecting lines in all the planes above and below and parallel with the plane of the axes form squares except] *at the top and the bottom* [where single lines only intersect] *and* [there] *they touch each other in two points, one at the top and one at the bottom.*

A description of the particular double vault – generated by two cylinders of diameter 4 "brachia" – and the procedure to determine its volume follow the general description. Note that Piero's style is completely prescriptive, as required for an abacus teacher.

*Example. Let the pierced cylinder be H and the piercing cylinder be G and let the hole be ABCD, and let touching points in their cavity be E and F and we seek the volume of the hole. We have said that the width of each cylinder was 4 brachia. Therefore, the square ABCD, is 4 brachia on each side. These sides multiplied together make 16 and EF, which is the width of a cylinder, is 4, and when multiplied by the surface of the base, i.e. by 16, makes 64. This you divide by 3 and 21 1/3 is the result. This doubled becomes 42 2/3 and so* *much is removed from cylinder H as the result* [of the formation of the] *said hole, i.e. 42 2/3.*

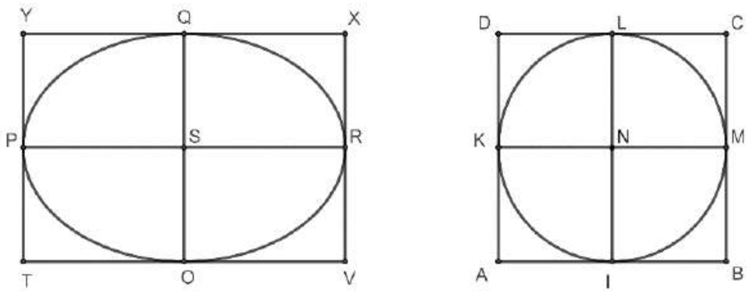



**Figure 2.** Plane sections: ellipse (inscribed in the rectangle) and circle (inscribed in the square)

From this point on, however, Piero abandons the abacus approach preferring to use an Archimedean technique, working first on plane sections and then on solids. He considers the square section ABCD and inscribes therein the circle IKLM, and considers the rectangular section passing through the diagonal of ABCD – represented by the rectangle TVXY, of side YT and TV, respectively equal to the square side and its diagonal – and inscribes therein an ellipse. He has now to determine what relationship holds between these figures.

*This is proved as follows. You know that the said cylinders make a square in the hole, which square is ABCD. Therefore, you may draw a square hole of the same size which we let be ABCD and in it you inscribe circle IKLM with center N. Then you draw another* [rectangular] *surface TVXY, each of whose opposite sides is equal to the diagonal AC of the said hole, while each of the other two sides is equal to AB. In this you describe a proportional circle* [i.e. an ellipse] *tangent to each side of the said rectangle in points O, P, Q and R. Let its center be S. I say that the ratio of square ABCD to rectangle TVXY is as circle IKLM to ellipse OPQR, and the ratio of circle IKLM is to its square ABCD as ellipse OPQR is to its rectangle TVXY as is demonstrated by the fifth* [proposition] *of the third* [work] *of Archimedes,* On Conoids.

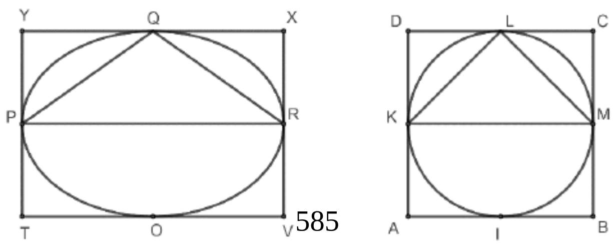


Figure 3. Triangles inscribed in the half-ellipse and in the semicircle

Piero recalls the fifth proposition of the third book of Archimedes' *Conoids* but in fact the *Conoids* was handed down in only one book. Piero's mistake is probably due to the fact that the *Conoids* is the third work transcribed in his manuscript. Proposition 5 states: “If AA’, BB’ be the major and minor axis of an ellipse respectively, and if be the diameter of any circle, then (area of ellipse) : (area of circle) = $AA' \ x\ BB' : \ d^2$” [Heath 2002, 115]. After establishing a proportion between the square, the inscribed circle, the rectangle and the inscribed ellipse, Piero moves on to consider the triangles inscribed in the semicircle and the half-ellipse.

*Now you divide square ABCD into equal parts by line KM. Then you draw lines KL and LM and △KLM will be formed; and you divide rectangle TVXY into equal parts by line PR. Then you draw lines PQ and QR, forming △PQR. I say that*

△ KLM : △PQR = square ABCD : rect TVXY

*And*

△KLM : square ABCD = △PQR : rect TVXY

*And it was said above that*

Circle IKLM : square ABCD = ellipse OPQR : rect. TVXY

*And so it follows from common knowledge* [viz. The axiom: quantities equal to the same quantity are equal to each other] *that*

△KLM : circle IKLM = △PQR : ellipse OPQR

After some arithmetic manipulation, Piero comes to determine that the ratio of the circle to the isosceles triangle inscribed in the semicircle is equal to the ratio of the ellipse to the isosceles triangle inscribed in the half-ellipse. Having determined this relationship between the sections, it is time to return to the solids

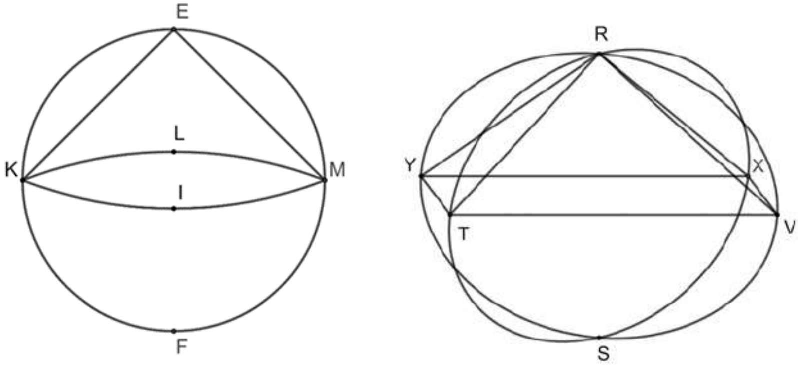
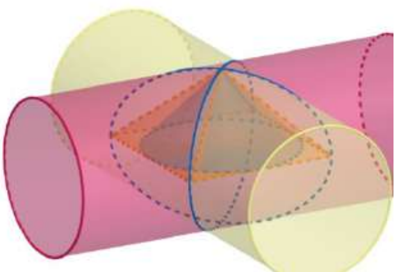



**Figure 4.** The cone inscribed in a sphere and the square-based pyramid inscribed in the vault

*And with this understood, let us make solid figures. The first will be spherical and designated EKMF with axis EF and the other which encloses square TVXY*[82] *by means of two ellipses. One is TRXS and the other is YRVS and they intersect each other in point R and in point S. In each of these two* [solid] *figures I shall produce a pyramid. In the sphere EKMF I shall delineate EM circularly. Then I shall draw lines KE and EM and produce pyramid KLMI on the round base* [i.e. cone KLMI]. *Then I shall produce another pyramid in the other corporeal figure, which will be TR, YR, XR, VR.*

The previous figures are remakes of Piero's figures, but the three-dimensional view below – which students can draw independently by trying to "translate" the text into a suitable graphical representation – can better illustrate how the cone and square-based pyramid described above are constructed.



[82] It will be useful to remark that the square TVXY is the the square before named ABCD.

**Figure 5.** The square-based pyramid and the cone inscribed in the intersection of two cylinders

*These pyramids* [i.e. the cone and the pyramid] *are in the same ratio as their parents, i.e. as the corporeal figures in which they are constructed, as is demostrated above in the plane figures, since circle TRXS is equal to circle OPQR*[83] *in surface TVXY and the sides of the pyramid TR, RX are equal* [respectively] *to the two sides of △PQR, i.e. PQ and QR. And the sides KE and EM of the cone in the sphere are equal* [respectively] *to the sides KL and LM of △KLM of circle IKLM. Let us conclude then that the ratio of the pyramid TR, YR, XR, VR to its* [parent] *solid TRXS* [i.e. to the common segment of the two cylinders] *is as the ratio of cone KEM whose base circle is IKLM to its* [parent] *spherical solid KEMF.*
In modern terms, Piero proved

$$\triangle \text{KLM} : \text{circle IKLM} = \triangle \text{PQR} : \text{ellipse OPQR}$$

or

$$\triangle \text{PQR} : \text{ellipse OPQR} = \triangle \text{KLM} : \text{circle IKLM}$$

and from that proportion he deduced that
Volume (pyramid) : volume (double vault) = volume (cone) : volume (sphere)
This is a very crucial passage but it is not well justified in Piero's text. Indeed, the reconstruction of Piero's whole argumentation is an open problem from the historical viewpoint, that could led to a very interesting discussion among the students, invited to formulate conjectures and suggestions. Students can also discuss the conjecture expressed in [Gamba, Montebelli, Piccinetti 2006].

To conclude the proof, it is easy to note that the previous proportion allows to find the volume of the double vault, since the ratio of the sphere to the inscribed cone with base the maximum circle is known thanks to Archimedes (*On Sphere and Cylinder*) and the volume of the pyramid is also easily determined.

[83] Questions for the students: are TRXS and OPQR really circles? Why?

*Therefore by I.33 of* On the Sphere and the Cone (!) *of Archimedes, where he says that any sphere is quadruple the cone whose base is equal to a greater circle of the sphere and whose axis is equal to the radius* [of the sphere]*, sphere KEMF is quadruple cone KEM and thus the parent solid TRXS* [which is the common segment of the two cylinders] *is quadruple pyramid TR, YR, XR, VR. And so you take the base TVXY which is 4 brachia on each side; multiply the sides together and the result is 16. This you multiply by the axis which is 2 and the result is 32. This you divide by 3 and 10 2/3 is the result* [as the volume of the pyramid]. *Its* [parent] *solid TRXS* [i.e. the common segment of the cylinders] *is 4 times as great. Therefore, mutiply 10 2/3 by 4 and the result is 42 2/3 as was said before. And thus you have what is removed from cylinder H by that hole* [namely] *42 2/3 brachia.*

To conclude, besides all the various interesting aspects mentioned above, the study of Piero's text could offer also a frutiful chance to reflect on the meaning of the term “demonstration” in practical geometry, where it usually means “to show by means of a good example”: discussing the epistemological value of this approach also allows students' beliefs about the concept of demonstration in mathematics to emerge.

## REFERENCES


Acerbi F., Fontanari C., Guardini M. (eds) (2013), *Archimede. Metodo. Nel laboratorio del genio*, Torino, Bollati Boringhieri.

Banker, J. (2010), Luca Pacioli e Piero della Francesca, in Giusti E., Martelli M., (Eds.) *Pacioli 500 anni dopo*, Aboca, 205–220.

Clagett, M. (1978), *Archimedes in the Middle Ages*, vol. III Part III *The Medieval Archimedes in the Renaissance 1450-1565*, The American Philosophical Society 1978.

Daly Davis, M. (1977), *Piero Della Francesca's Mathematical Treatises: The* Trattato d'abaco *and* Libellus de Quinque Corporibus Regularibus, Longo.

Gamba E., Montebelli V. (1987), *La matematica abachistica tra ricupero della tradizione e rinnovamento scientifico*, in “Cultura, scienze e tecniche nella Venezia del Cinquecento, Istituto Veneto di scienze, lettere e arti, Venezia 1987, 169-202

Gamba E., Montebelli V. (1996), *La geometria nel* Trattato d’abaco *e nel* Libellus de quinque corporibus regularibus *di Piero della Francesca: raffronto critico*, in Dalai Emiliani M., Curzi V. (Eds.), *Piero della Francesca tra arte e scienza*, Marsilio Padova 1996, 253-268.

Gamba E., Montebelli V., Piccinetti P. (2006) La matematica di Piero della Francesca, *Lettera matematica Pristem,* 59, 49-59

Heath T.L., *The Works of Archimedes,* Dover 2002.

Napolitani P.D. (1998), La tradizione archimedea, in Giusti E. (Ed.), Luca Pacioli e la matematica del Rinascimento, Petruzzi 1998, 77- 102.

Napolitani, P. D. (2007), *Piero e la tradizione del testo di Archimede nel Quattrocento*, in Manescalchi R., Martelli M. (Eds), *L'Archimede di Piero. Contributi di presentazione alla realizzazione facsimilare del Riccardiano 106*, Sansepolcro, Grafica European Center of fine Arts.

Napolitani P.D. (2010), *Archimede nella tradizione dell'abaco e nell'Umanesimo*, in Giusti E., Martelli M. (Eds.) *Pacioli 500 anni dopo*, Aboca, 221-246.

Napolitani P.D., Saito K. (2013), *Archimedes and the Baths: not only one Eureka*, in Lucore S.K., Trümper M., *Greek Baths and bathing culture: new discoveries and approaches*, Walpole, Leuven, 181-188.

Napolitani P.D., Saito K. (2014), *Reading the Lost Folia of the Archimedean Palimpsest: The last proposition of the* Method, in Sidoli, N., Van Brummelen, G. (Eds) *From Alexandria, Through Baghdad*, Springer, Berlin, Heidelberg, 199-225.

Pacioli L. (1509), *Divina proportione*, Venetiis, Paganius Paganinus 1509.

Piero della Francesca, *De quinque corporibus regularibus*, Codice Urb. Lat. 632 https://digi.vatlib.it/view/MSS_Urb.lat.632

Piero della Francesca (1995), *Libellus de quinque corporibus regularibus, corredato della versione volgare di Luca Pacioli,* Edizione Nazionale degli scritti di Piero della Francesca, Giunti.

# STARTING FROM THE HISTORY OF MATHEMATICS IN LATE MODERN ITALY (XVIII-XX CENTURIES): FROM PRIMARY SOURCES TO MATHEMATICAL CONCEPTS

**Elena LAZZARI[84], Maria Giulia LUGARESI[85], Paola MAGRONE[86], Elena SCALAMBRO[87]**

lzzlne@unife.it lgrmgl@unife.it paola.magrone@uniroma3.it
elena.scalambro@unito.it

**ABSTRACT**

The workshop intends to propose didactic paths that use materials derived from primary historical sources, accessible to secondary school students, integrated with laboratory activities. Its focus is the history of mathematics in Italy, from the eighteenth to the twentieth centuries. The workshop is divided into four parts: 1. The dissemination of infinitesimal calculus in Italy, by comparing two different approaches, the Leibnizian paradigm of the "differentials" and the Newtonian conceptualization of the "ultimate ratios of vanishing quantities". 2. The science of waters as the main field of applied mathematics. 3. Re-launching Italian education and research after political unification. 4. The geometry of paper folding and the resolution of problems of third degree. All the materials related to this workshop, including slides and some supplementary worksheets, can be found on the website of Mathesis Ferrara at the link: http://dmi.unife.it/it/ricerca-dmi/mathesis/materiali-esu-9

## *Introduction*

History of mathematics represents a useful tool for learning mathematics and in the last decades the researches in the field of mathematical education have had a great development both nationally and internationally. As regards Italy, from a legislative point of view, the *Indicazioni Nazionali* for upper secondary school (2010) underline the importance of "connecting different mathematical theories with historical problems that had originated them". The aim of this paper is to present to the teachers a variety of approaches and examples of activities that can be used in their classroom. All these approaches share the

[84] University of Ferrara, Via Machiavelli 30, Ferrara, Italy.
[85] University of Ferrara, Via Machiavelli 30, Ferrara, Italy.
[86] University of Roma Tre, Via della Madonna dei Monti 40, Roma, Italy.
[87] University of Torino, Via Carlo Alberto 10, Torino, Italy.

same general idea, that is to use materials derived from primary historical sources in mathematics. The focus is the history of mathematics in Italy, from the eighteenth to the twentieth centuries. We present topics which can be a starting point for reflection and activities aimed mostly at high school students, in an interdisciplinary context. We suggest supporting our educational proposals with practical activities that students can easily reproduced in the classroom, like geometric constructions with dynamic softwares as GeoGebra, or hands-on activities, such as some simple examples of paper folding.

## 1 *The dissemination of infinitesimal calculus in Italy*

The searching for the tangent to a given curve and the squaring of figures were the main problems, to which the invention of infinitesimal calculus was able to give an answer. Between the sixties and the eighties of the 17$^{\text{th}}$ century, both Leibniz and Newton came to the invention of infinitesimal calculus through two different methods. Starting from the definition of "differential" Leibniz gave the rules of the homonymous calculus and its main applications. It corresponds to taking the difference between two infinitely close values of the variables. These "differences" (or "differentials") represent the fundamental parameters for the description of the curves. Starting from the problem of the squaring of figures, Newton suggested a kinematic method and used quantities, named "fluxions", for which he gave specific rules of calculus, absolutely similar to Leibniz' differentials. According to a mechanical vision of geometry, Newton considers the variables, that he called "fluents", as quantities whose value increases or decreases with continuity. The instantaneous speeds are called "fluxions".

These two different approaches to infinitesimal calculus can provide interesting didactic suggestions. Starting from the reading of selected extracts from the works of two Italian mathematicians that followed respectively Leibnizian and Newtonian approaches, i.e. Maria Gaetana Agnesi and Joseph Louis Lagrange, students can be guided to the discovering of the main ideas at the base of differential calculus that they studied during their schooling.

## 4.3 A Leibnizian approach in the first Italian treatise on Analysis: Maria Gaetana Agnesi's *Instituzioni Analitiche*

Maria Gaetana Agnesi's treatise *Instituzioni Analitiche ad uso della gioventù italiana* (1748) represents a significant example of the circulation of Leibniz's approach in Italy.

Born in Milan in 1718, the author is one of the first female mathematicians in history. She devoted the second volume of her *Instituzioni* to differential and integral calculus, conceived as the analysis of "differences in variable quantities, of whatever order those differences may be" (p. 341).

The "versiera" is a special cubic curve associated with Agnesi's name thanks to her description within the *Instituzioni* (problem III). This curve has several teaching potentials and it represents a privileged starting point to introduce the use of original sources in the classroom. The didactic path – suitable to different grades of secondary school – can be articulated in the following phases.

1. Historical introduction to Agnesi and her time.
2. Construction of the versiera by points starting from Agnesi's original problem, where it is required to find the locus of points in the plane that satisfy a certain proportion (upper secondary schools), or rhyming instructions (lower secondary schools).
3. Dynamical construction of the versiera, with GeoGebra.
4. Derivation of the Cartesian equation of the versiera and reflection on the cognitive pivot of dependent/independent variables (only for upper secondary schools).
5. Guided reading of the preface of the *Instituzioni* and debate about the gender gap in science and the role of women in society relying on Agnesi's words.
6. Production of an artwork which includes the shape of the versiera, starting with the doodle that Google dedicated to Agnesi on the occasion of the 296$^{th}$ anniversary of her birth.

From the point of view of gender differences, Agnesi appears to be a forerunner of the times: this aspect can be significant for educational guidance, making Agnesi's scientific studies and mathematical commitment an example in the eyes of the students. Furthermore, such an activity includes several factors of didactic effectiveness: historical framing of a mathematical theory; reading of original sources; passages between different domains of maths (from discrete to continuous, from Euclidean to Cartesian plane, from geometric properties to the analytical equation, …); interdisciplinary links (literature, art, …).

### 4.4 Newtonian conceptualization of the "ultimate ratios of vanishing quantities"

In his studies on the foundations of infinitesimal calculus, Lagrange gives great importance to the calculus of differences. In his treatise, *Principj di Analisi Sublime* (~1754) before exposing the differential calculus, Lagrange gives an algebraic exposition of the calculus of finite increases. A crucial point is the passage from "finite increases" to "infinitesimals". In the second part of the *Principj*, the author develops the algebraic calculus of finite differences. The differential calculus determines "the ultimate ratios of the difference dy/dx, i.e. the ultimate terms to which the general ratios of the differences continuously approach, while these continuously decrease". Lagrange's notation is that of Leibniz, but the approach is Newtonian. Students can be guided to the concept of "limit" through the reading of Lagrange's words.

*The ratios of the differences are named ultimate ratios of the differences, considering them in the point where they are going to vanish. Actually these ratios are not ratios of any real differences, since it's supposed that each of them has become equal to zero. They only express the ultimate terms, to which the general ratios of differences continuously approach, while they continuously decrease. These ratios are also named first ratios of the differences, because they can be seen as limits from which the general ratios of the differences, considered as rising to receive continuous increases, begin* (see Borgato 1987, p. 129, author's translation).

Browsing the pages of the *Principj* students can also find observations about geometrical constructions, that can be easily reproduced in the classroom. Starting from the reading of original source, students can know and repeat the passages through which Lagrange determines the equation of a particular curve (for instance, the so-called "Conchoid of Nicomedes") and try to reproduce it with the help of GeoGebra.

## *2 The science of waters as the main field of applied mathematics*

According to the classical tradition hydrostatics and science of waters represent two different disciplines. The foundations of hydrostatics can be found in Archimedes' treatise *On floating bodies*, while the science of waters was mainly studied in an empirical way. Italy boasts a long hydraulic tradition and the contribution given by mathematicians in this field has been quite relevant until the 18$^{th}$ century.

We suggest a possible educational activity for students of the first two years of high school in order to present the topic related with mathematics applied to the science of waters from an historical point of view. As regards mathematics, the *Indicazioni Nazionali* for Scientific High School underline the importance to learn at the end of their schooling "a historical-critical point of view of the relationships between the main items of mathematical thinking and the philosophical, scientific and technological background". Following this general idea and taking into account some relevant original historical sources, the proposed educational activity will deal with problems about equilibrium and motion of fluids. The involved subjects are Maths, Physics, History, Italian and Latin (if provided), so the activity has an interdisciplinary nature. It can be divided into the following parts, each of them corresponds to about two hours of lesson:

- The concept of fluid; Physical quantities and their definitions: density, pressure, flow rate, speed.
- Statics of fluids (Hydrostatics): Stevin's law; Pascal's principle; Archimedes' principle.
- Dynamics of fluids (Hydrodynamics): Continuity equation; Bernoulli's theorem and its applications; Torricelli's law.

The activity starts defining some useful physical quantities (absolute and relative density, specific weight, pressure), then we can enunciate the so-called "Stevin's law", that was firstly published in 1586. The law expresses the pressure applied by a fluid inside an immersed body in function of the deepness *h* of the body, of the gravitational acceleration *g* and of the density $\rho$ of the fluid. A simple consequence of the Stevin's law is the principle of Pascal: "The pressure at a point in a fluid at rest is the same in all directions", that was discovered by the French mathematician Blaise Pascal and published posthumously in 1663 (*Traitez de l'équilibre des liqueurs*). The chapter about Hydrostatics can be further studied from an historical perspective through the reading of the propositions 3-7 (Book I) of Archimedes' treatise *On the floating bodies*, that express the conditions for which a body balances, floats or sinks. In particular, "a solid lighter than a fluid will be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced". Two original sources, respectively in Latin and in Italian language, can be presented in the classroom to present Archimedes' discovery. Vitruvius in his treatise *De architectura* (I century b.C.) quoted the problem of the crown, as a possible origin of the principle of Archimedes. Galileo, instead, proposes to use the so-called "bilancetta", an instrument that allows to directly weigh the bodies in the water, using the Archimedean concept of specific weight (*La bilancetta*, 1656).

The chapter about Hydrodynamics will focus on two main themes: the continuity equation; the theorem of Bernoulli and some important consequences.

The continuity equation states that for an ideal and steady motion fluid $\rho A v = \text{constant}$, where $\rho$ is the density, while the quantity $Av$ is the (volumetric) flow rate and represents the volume of fluid that crosses the section in a unit of time. From this equation we can obtain the so-called "law of Castelli", which was formulated by the Italian mathematician Benedetto Castelli (*Della misura dell'acque correnti*, 1628): "the sections of the same river discharge the same quantity of water in the same times, even if the sections are unequal", i.e. the flow rate is constant.

Bernoulli's theorem, which owes its name to the Swiss mathematician Daniel Bernoulli, expresses the principle of conservation of energy. Many consequences can be deduced from this theorem. From an historical point of view, we can quote the law of the efflux. Thanks to the analogy with free fall, Evangelista Torricelli finds a useful model to explain the relationship between speed, pressure and depth (*De motu aquarum*, 1644).

To conclude our activity, we can give an overview of some particular devices for measuring the speed of water at different depths, distinguishing between "fixed water gauges" and "moving water gauges". We can take inspiration from the rich iconographic apparatus present in some of the most important eighteenth-century Italian treatises: F. D. Michelotti (*Sperimenti idraulici*, 1767-71); T. Bonati (*Delle Aste Ritrometriche e di un nuovo Pendolo per trovare la Scala delle Velocità di un'Acqua corrente*, 1799).

## 3 Re-launching Italian education and research after political unification

After political unity in Italy, in 1861, there was a strong resumption of mathematical research and education. On the one hand, a connection with the most advanced sectors of European mathematical research, on the other, a colossal commitment to the creation of a national education system. As for the teaching of mathematics at secondary school level, there was, as in the rest of Europe, a return to synthetic geometry, without the admixture of algebra. The original Euclidean text was initially revived but later new texts were produced and a new didactic proposal emerged to blend plane and solid geometry and introduce new results in elementary geometry, the fusionism (Borgato 2016).

The proposed educational activity consists of a moment of in-depth study and reflection on the root axis theory, with the aim of highlighting the potential of fusionism. The discussion of this theory by the fusionist approach turns out to be particularly effective and gives the opportunity to show some of the strengths of the movement, such as the simultaneous discussion of plane and solid geometry topics and the simplification of some proofs of plane geometry theorems by means of stereometric considerations. The reference text for the development of the curriculum is the original historical source *Elementi di Geometria*, by Lazzeri and Bassani, in the second edition of 1898. The work is available in a digital version: http://mathematica.sns.it/opere/169/; special reference will be made to Chapter III (Systems of Circles and Spheres) of Book III.

The educational activity is designed for students in the second two years of high school and is expected to last 4/6 hours. During each lesson, students, divided into small groups, will read and analyze the texts of selected theorems directly from the original source. The activity will be guided by a worksheet designed to encourage discussion among group members and help them identify the hypothesis and thesis of the proposition. They will also be asked to elaborate the statement again using simpler language. The discussion will extend to the whole class, under the guidance of the teacher, and will continue with the analysis of the proof supported by previously prepared GeoGebra constructions. The most significant constructions will be made by the students themselves in the lab, under the guidance of the teacher or in a discovery activity in pairs.

All the activities included in the educational pathway (selected theorems, worksheets, GeoGebra constructions) are available in Italian on the University of Ferrara Department of Mathematics website at the following link: http://dm.unife.it/matematicainsieme/fusion/index.html. As an example, one of the theorems that will be analyzed in the pathway is given below.

*Theorem* - Given three circles in a plane, such that their centers are not in a straight line, the three radical axes of these circles, taken two by two, pass through the same point.

*Proof* - Let $c_1$, $c_2$ and $c_3$ be the three given circles in an α-plane, so that the three centers are not in a straight line. Let three equal spherical surfaces $S_1$, $S_2$ and $S_3$ of radius greater than the radii of the three given circles pass through them, what is always possible, and let $O_1'$, $O_2'$ and $O_3'$be the centers of these spheres. The planes perpendicular to the segments $O_1'O_2'$, $O_2'O_3'$ and $O_1'O_3'$ at their midpoints, cut the alpha plane according to the three radical axes of the pairs of circles $c_1c_2$, $c_2c_3$ and $c_1c_3$, and moreover pass through the same straight line $r$ (Theorem 161, Cor.) which is the geometric locus of the points equidistant from $O_1'$, $O_2'$ and $O_3'$. Obviously, the intersection cannot be parallel to the alpha plane because, if it were, the three radical axes would have to be parallel to each other and thus, contrary to the hypothesis, the centers of the three given circles would be in a straight line. So the line r meets the alpha plane at a point, which is common to all three radical axes.

Figure 1 shows the scan from the original source and at the following link 216_Teorema - GeoGebra you can open the GeoGebra construction associated with the proof.

**216. Teorema.** — *Dati tre circoli in un piano, tali che i loro centri non sieno in linea retta, i tre assi radicali di questi circoli, presi due a due, passano per uno stesso punto.*

Essendo $c_1$, $c_2$, $c_3$ i tre circoli dati in un piano $\alpha$, in modo che i tre centri non sieno in linea retta, si facciano passare per essi, ciò che è sempre possibile, tre superficie sferiche eguali $S_1$, $S_2$, $S_3$ di raggio maggiore dei raggi dei tre circoli dati, e sieno $O_1'$, $O_2'$, $O_3'$ i centri di queste sfere. I piani perpendicolari ai segmenti $O_1'O_2'$, $O_1'O_3'$, $O_2'O_3'$ nei loro punti di mezzo, tagliano il piano $\alpha$ secondo i tre assi radicali delle coppie di circoli $c_1, c_2$; $c_1, c_3$; $c_2, c_3$, ed inoltre passano per una medesima retta $r$ (§ 161, Cor. 1°), che è il luogo geometrico dei punti equidistanti da $O_1'$, $O_2'$, $O_3'$. Evidentemente la intersezione $r$ non può essere parallela al piano $\alpha$, perchè, se lo fosse, i tre assi radicali dovrebbero essere paralleli fra loro (§ 40, Teor.), e quindi, contrariamente all'ipotesi, i centri dei tre circoli dati sarebbero in linea retta. Dunque la retta $r$ incontra il piano $\alpha$ in un punto, il quale è comune a tutti e tre gli assi radicali.

**Figure 1.** Theorem 216, Lazzeri and Bassani (1898, p. 188)

## *4 The geometry of paper folding and the resolution of problems of third degree*

This workshop refers to the use of paper folding to construct the solutions of third-degree problems, introduced by T. Sundara Row in 1893 and then completed by Margherita Beloch (1879-1976); she dealt with this topic in the courses for prospective high school mathematics teachers (courses of “complementary mathematics”) at the University of Ferrara in the first half of the 20th century.

As is well known, some elementary geometric problems (trisection of the angle, duplication of the cube, squaring of the circle) are impossible to solve with the classic instruments ruler and compass, but they can instead be solved with paper folding techniques. The necessary material to realize this workshop is: sheets of paper and a pen to mark points and the folds, when needed. The paper, therefore, is the tool with which the demonstrations are carried out and also the physical place where the demonstration takes place. This makes paper folding workshops even more engaging and challenging.

The activity is addressed to high school students who know what a parabola is (focus, directrix), what a polynomial is and what it means to look for its roots; it will be up to the teacher to decide how much to go deep and develope the formal demonstrations. The workshop can also be offered to university students of different study courses (perhaps going more into the detail of the demonstrations), and, vice versa, to younger students, restricting to the visualization of the parabola with first activity.

The method of resolution is illustrated by Beloch very briefly (Beloch 1934); in the essays (Borgato & Salmi 2018, Magrone & Talamanca 2018, downloadable on the web page indicated in the abstract) a lot of details can be found.

To start the workshop, we recommend to give some news about Margherita Beloch (see, for example, Magrone 2021): she was a leading exponent of the Roman school of mathematics led by Guido Castelnuovo (1865–1952) and Federigo Enriques (1871–1946) as well as founder of the school of mathematics at the University of Ferrara - where she was full professor from 1927 to 1954 - and one of the first Italian women mathematicians to become a university professor. Her research interests ranged from algebraic geometry to more applied and pioneristic topics such as photogrammetry and röntgenfotogrammetry. Then, to enter in the realm of paper folding geometry, project on a screen the pictures and definition of the basic folds of origami geometry, observing possible similarities with Euclid's axioms.

The first hands-on exercise consists in implementing the fold O5 repeatedly; this leads to folding the envelope of the tangents of a parabola. At this point, start a “mathematical conversation” with the participants about why the displayed curve is an actual parabola (see Magrone & Spreafico 2022). Next step consists in realizing the fold O6, also called “Beloch’s fold” (Borgato & Salmi 2018, Magrone & Talamanca 2018): the line obtained is the common tangent to two parabolas (Fig. 2, right).

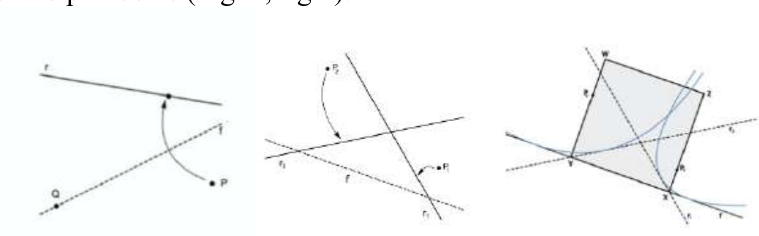


**Figure 2.** From left to right, Fold O5, fold O6 or Beloch's Fold, Beloch’s square and the two parabolas with their common tangent. Pictures are taken from Magrone & Talamanca 2018

This fold leads to the solution of third-degree problems by the application of the method invented by Edward Lill (1830-1900), a graphical way to find the real roots of polynomials. Beloch’s fold enables a graphical construction, which leads to find a “resolvent path” envisaged in Lill’s method (for details see again the above mentioned papers). The solution is physically constructed with paper, in particular a certain angle is found, and the trigonometric tangent of this angle gives the numerical value of the desired root.

### *Conclusions and feedbacks from students*

In this paper, we have presented a variety of activities, focused on the use of primary sources of history of mathematics in teaching mathematics. Some of these activities have been already tested in the classroom and have had interesting feedbacks.

A part of the parabola folding workshop was tested by P. Magrone in 2022 in a pilot experiment together with M. L. Spreafico, with two parallel groups of students (two groups of 75 students, for details see Magrone & Spreafico 2022). The activity included both the visualization of a parabola as an envelope of tangents, obtained with paper folding, and further exercises on focal properties and coordinates (not considered in the present paper). A well-known topic as the parabola curve, widely studied in high school, is "rediscovered" through an activity such as origami, and looked at with new eyes;

conic curves are proposed at school almost exclusively from an algebraic point of view, seen almost exclusively as equations and the geometric definition is often neglected, while it is essential from both an historical and epistemological point of view. The action of folding embodies the geometric definition of the parabola, so the attention is on the geometric locus. As soon as each student has ended the folding, the teacher asks to prove that the curve that appears is actually a parabola. This question puzzles the students, they are not used to solve this kind of problems, where they have almost nothing but a sheet of paper marked with folds. The search for the answer forces them to think deeply on the geometric definition of parabola and to recall the gestures that lead to the creation of the curve. The (anonymous) tests on the satisfaction of the activity that we administered at the end showed how much the hands-on session was appreciated, so much so that many students suggested us to propose it again to their colleagues for the following year.

Starting from the 2019-2020 school year, the activity on Agnesi's Witch has been experimented on multiple occasions, both at lower and upper secondary school levels. In total, ten classes and over 200 students were involved (Scalambro 2022). From the feedback of people who took part in the activity, it emerged that the use of historical sources contributes to promoting a narrative approach even in an “abstract” discipline like mathematics, encouraging students’ engagement. Some peculiar aspects surfaced from the experimentation of this laboratory in lower secondary school. Firstly, the importance of developing curiosity and interest towards the subject, also addressing topics that may seem too “high” or complex. Secondly, there is a risk often encountered in daily teaching practice of spurring students to study mathematics exclusively in relation to its usefulness. Teachers involved instead emphasized how such historical approach contributed to contextualize the topics of study and to dismantle – at least in part – the idea of mathematics primarily linked to its practical implications. Finally, embracing the conception of mathematics as an activity that can be described as a long conversation over the millennia, the aspect of identification with the presented character is essential. In this case, it is important to keep in mind that Agnesi is a woman and a precursor of her time, thus representing a significant example to students regarding the love and dedication for mathematics, as well evidenced by the excerpts from the *Instituzioni Analitiche* presented during the activity.

Other activities have been designed, preparing both a selection of primary sources and some supplementary worksheets, that can be a useful guide for teachers.

All the materials prepared for our workshop can be freely consulted online on the website of Mathesis Ferrara at the link: http://dmi.unife.it/it/ricerca-dmi/mathesis/materiali-esu-9

**REFERENCES**

Beloch, M. (1934). Alcune applicazioni del metodo del ripiegamento della carta di Sundara Row, in *Atti dell'Accademia delle Scienze Mediche, Naturali e Fisico-Matematiche di Ferrara*, 2, Vol. XI, 186- 189.

Borgato, M.T. (1987). Giuseppe Luigi Lagrange. Principj di analisi sublime, *Bollettino di Storia delle Scienze Matematiche*, 7 (2), 45-198.

Borgato, M.T. (2016). Il fusionismo: moda didattica o riflessione sui fondamenti della geometria?, *Periodico di Matematiche*, s. XII 8(2), 45-65.

Borgato, M.T., Salmy, R. (2018). La geometria degli origami e la risoluzione delle equazioni algebriche, *Periodico di Matematica*, 3, 57–71.

Capone, R. (2022). Interdisciplinarity in Mathematics Education: From Semiotic to Educational Processes. *EURASIA Journal of Mathematics, Science and Technology Education*, 18 (2). https://doi.org/10.29333/ejmste/11508

Doig, B., Williams, J., Swanson, D., Borromeo Ferri, R., Drake, P. (2019). *Interdisciplinary Mathematics Education- The State of the Art and Beyond*. Springer.

Lazzeri, G., Bassani, A. (1898). *Elementi di geometria: libro di testo per la R. Accademia Navale*. Giusti.

Lugaresi, M.G. (2015). Le raccolte italiane sul moto delle acque, *Bollettino di Storia delle Scienze Matematiche*, 35 (2), 201-304. ISSN 0392-4432.

Lugaresi, M.G. (2021). Mathematics and hydraulics between Turin and Ferrara in the 18th century: the works by F. D. Michelotti and T. Bonati, *Revue d'histoire des mathématiques*, 27 (2), 127-169. DOI:10.24033/rhm.233

Magrone, P., Talamanca, V. (2018). *Folding cubic roots: Margherita Piazzolla Beloch's contribution to elementary geometric constructions*. In L. Balko et al (eds.), *Proceedings, 16th Conference on Applied Mathematics Aplimat 2017*, Slovak University of Technology in Bratislava, Publishing house Spektrum STU, Bratislava, pp. 971-984.

Magrone, P. (2021). Margherita Piazzolla Beloch in the Italian Mathematics education tradition, in P. Magnaghi et al. (ed.) *Faces of Geometry 2*, pp. 209-222, Springer.

Magrone, P., Spreafico, S. (2022). Nuovi approcci nei corsi di Matematica per l'Architettura: connettere forme e formule in geometria attraverso esperienze laboratoriali, *Annali online della Didattica e della Formazione Docente*, 24, pp. 99-122.

Pepe, L. (1981). Il calcolo infinitesimale in Italia agli inizi del secolo XVIII, *Bollettino*

*di Storia delle Scienze Matematiche*, 1 (2), 43-101.
Roero, C.S. (2016). Giornali, Accademie e Traduzioni: il successo europeo delle *Institu-zioni Analitiche* di Maria Gaetana Agnesi. *Physis*, 51, 145–162.
Scalambro, E. (2022). Maria Gaetana Agnesi e la versiera. Una proposta laboratoriale per la scuola secondaria di primo grado. *Archimede*, 4/2022, pp. 202-216.
Sundara Row, T. (1893). *Geometric exercises in paper folding,* Addison and Co.

**MULTIMEDIAL SOURCES**

Website of Mathesis Ferrara, http://dmi.unife.it/it/ricerca-dmi/mathesi

# LA RINASCITA DELLA LOGICA IN ITALIA NELLA SECONDA METÀ DEL '900 E LA SUA INTRODUZIONE NEI PROGRAMMI SCOLASTICI DI MATEMATICA

**Antonio VEREDICE**
Sapienza Università di Roma, Piazzale Aldo Moro, 5, 00185, Rome, Italy
antonio.veredice@uniroma1.it

**ABSTRACT**

Dopo la Scuola di Peano, gli studi di Logica Matematica vengono abbandonati in Italia per riprendere solo negli anni '60 grazie a Ludovico Geyomonat e ad un gruppo di ricerca pluridisciplinare di matematici e filosofi. Dopo aver rievocato i passaggi fondamentali di questo fenomeno, vedremo come – anche grazie a questa rinata comunità di logici italiani – la Logica sia entrata a far parte dei curricula di Matematica nei diversi livelli scolari.

## *1 La rinascita della Logica in Italia*

### 1.1 La Scuola di Peano

Giuseppe Peano si occupa di Logica in una serie di scritti tra il 1888 e il 1903. Fra di essi, un ruolo di primo piano è rivestito dal *Formulario*, opera di sistematizzazione della conoscenza matematica espressa in forma simbolica. È un progetto che coinvolge molti giovani studiosi di quella che viene definita la Scuola di Peano: Giovanni Vailati, Filiberto Castellano, Cesare Burali-Forti, Giovanni Vacca, Alessandro Padoa, Mario Pieri. L'ambizione è realizzare il sogno di Leibniz di costruire quella *"characteristica universalis", in grado di formalizzare i processi mentali, attraverso l'individuazione delle idee primitive e l'ideazione di simboli appropriati "che quasi dipingano l'intima natura dei concetti"* (Luciano-Roero, 2008).

L'influenza di Peano e della sua Scuola sulla comunità scientifica internazionale è ben nota. Russell, nella sua autobiografia afferma di essere rimasto colpito - durante il Congresso Internazionale di Filosofia tenutosi a Parigi nel 1900 - dalla chiarezza di idee e dalla capacità di argomentazione di Peano. Hans Freudenthal, riguardo al convegno di Parigi del 1900, ha affermato che *nel campo della filosofia della scienza la falange italiana fu eccelsa: Peano, Burali-Forti, Padoa, Pieri dominarono la discussione nel modo più assoluto.*

Peano, al pari di Frege, ha dato una spinta propulsiva alla Logica moderna creando un linguaggio interamente simbolico per la matematica: la *pasigrafia*. Secondo Lolli (Lolli 2020), Peano costruisce la pasigrafia con due obiettivi: garantire rigore e assenza di ambiguità alle espressioni matematiche e pervenire ad una sintesi, una specie di *compressione a fini enciclopedici* che ha la sua realizzazione nel *Formulario.* La naturale evoluzione di questo percorso porta Peano alle successive ricerche linguistiche. Infatti, come sostenuto da Cassina (Peano 1958) e ribadito da Geymonat, la Logica Matematica e le ricerche sull'*interlingua* costituiscono due momenti di uno stesso percorso, un *medesimo e grandioso programma volto a realizzare, in forma moderna, alcuni fra i più caratteristici temi dell'insegnamento leibniziano* (Geymonat, 1959).

### 1.2 La *decadenza di questo genere di studi*

Lo spostamento di interessi dalla Logica all'Interlingua allontana Peano dal dibattito sui fondamenti e, in qualche modo, lo isola rispetto alla ricerca internazionale in Logica.

Ma l'isolamento di Peano non avviene solo a livello internazionale. Anche in Italia le sue idee non hanno diffusione al di là della sua Scuola, né in ambiente matematico né in ambiente filosofico. Fra i matematici italiani, per un atteggiamento scettico e sospettoso nei confronti della Logica e della crisi dei fondamenti che viene concepita come un problema inesistente da chi, *convinto della piena solidità della propria scienza, non è disposto ad ammettere questa crisi* (Geymonat, 1959); fra i filosofi per l'ostilità dell'Idealismo nei confronti della Logica.

Le cose non cambiano molto negli anni del dopoguerra, quando la Logica, identificata con il neo-positivismo, viene *rifiutata il linea di principio anche dai marxisti di influenza sia crociana sia gentiliana, come pure lukácsiana* (Lolli, 2020).

In questo contesto non meraviglia la quasi completa sparizione degli studi di Logica in Italia nella prima metà del '900; proprio nel momento in cui il fermento nato dalla crisi dei fondamenti stava generando, in Europa e negli Stati Uniti, una varietà di temi di ricerca (Teoria della Dimostrazione, Calcolabilità, Teoremi di Gödel, etc.).

Alla situazione della Logica italiana in questo periodo storico fa riferimento Ludovico Geymonat quando parla di *decadenza - in Italia - di questo gene-*

*re di studi che, proprio nel secolo XX ha trovato invece così rigoglioso sviluppo al di là dei nostri confini.* (Geymonat, 1959).

### 1.3 La *rinascita* con Geymonat

Gli interessi di Geymonat per la Logica Matematica nascono con ogni probabilità durante la frequentazione del Circolo di Vienna nel 1934. Tornato in Italia egli inizia, nel '36, un'opera di divulgazione su temi di Logica e Filosofia della Scienza e si impegna per il rinnovamento della cultura scientifica italiana. Risalgono a questo periodo i primi incontri del Centro di Studi Metodologici (CSM), una *comunità di liberi ricercatori,* provenienti da diverse aree disciplinari: Abbagnano, Buzano, Frola, Geymonat, Nuvoli e Persico. Geymonat è l'elemento unificatore fra la componente filosofica e quella matematico-scientifica del CSM ed è convinto che, per riformare la cultura scientifica italiana, si debba progredire nello studio della Logica Matematica.

In questo senso è cruciale l'incontro fra Geymonat e Ettore Casari, giovane filosofo che nel 1955 trascorre un periodo di studio a Münster dove approfondisce la conoscenza della Logica sotto la guida di Hermes, Ackermann e Hasenjaeger. Nel 1957 Casari tiene a Torino un corso di Logica su iniziativa del CSM e, a partire dalle note di questo corso, sviluppa il testo *Lineamenti di Logica Matematica* pubblicato nel 1960 nella collana di Filosofia della Scienza di Feltrinelli diretta da Geymonat.

Nell'anno accademico '60/'61, oltre a Casari, anche altri giovani studiosi vengono in contatto con Geymonat e si interessano alla Logica Matematica: Evandro Agazzi, Corrado Mangione e Maria Luisa Dalla Chiara. Sono loro a spingere Geymonat, nel 1962, a chiedere al CNR l'istituzione del Gruppo di ricerca per la Logica Matematica.

Le riunioni del Gruppo si tengono alla Statale di Milano, in via Festa del Perdono, con cadenza quindicinale il sabato pomeriggio. Dal 1963 ne prendono parte anche i matematici fiorentini Roberto Magari, Piero Mangani e Mario Servi, il torinese Flavio Previale, e il padovano Bruno Busulini.

L'anno successivo si unisce anche Carlo Cellucci, studente di Filosofia alla Sapienza di Roma che, nel 1964, si laurea con una tesi di Logica con Geymonat relatore e Casari correlatore; la prima tesi di Logica Matematica discussa nella Facoltà di Lettere e Filosofia della Statale di Milano.

Uno degli aspetti peculiari del gruppo di Logica del CNR è la composizione eterogenea che vede insieme filosofi (il gruppo milanese) e matematici (i fiorentini e i torinesi). Da un lato ciò ha costituito una ricchezza culturale per il gruppo, dall'altro ha fatto emergere un contrasto tra due differenti visioni. I logici di formazione filosofica, primo fra tutti Casari, erano dell'avviso che fosse necessario recuperare il divario di conoscenze rispetto ai progressi della Logica internazionale prima di fare ricerca; i matematici invece - e soprattutto Magari – erano convinti che si potesse dare da subito un contributo alla ricerca. Nonostante le divergenze, le attività del Gruppo proseguono per anni con passione ed entusiasmo. Nel 1966, grazie alle borse di studio del CNR, si aggiungono Annalisa Marcja, Paolo Pagli e Gabriele Lolli.

Alla fine degli anni '60, due eventi segnano l'uscita della nuova Logica italiana, da quella che Casari ha definito *fase catacombale*: La Scuola di Specializzazione di Logica Matematica tenutasi a Varenna nel 1968 e il primo Seminario di Teoria dei Modelli all'Istituto di Alta Matematica di Roma nel 1969. In queste occasioni i logici italiani entrano in contatto con studiosi di importanza internazionale (Hermes, Mostowski, Robinson, Chang, Morley).

### 1.4 L'organizzazione e la struttura della Logica italiana.

Tra la fine degli anni Sessanta e l'inizio degli anni Settanta, i logici italiani formatisi nel Gruppo del CNR, organizzano, nelle varie sedi universitarie, dei centri di ricerca.

A Siena nel 1972, Roberto Magari diventa il primo direttore del Dipartimento di Matematica. Attorno alla sua figura carismatica si costituisce un nucleo di giovani studiosi: La Scuola di Siena (Laura Toti Rigatelli, Raffaellla Franci, Aldo Ursini, Claudio Bernardi, Giovanni Sambin, Franco Montagna).

Nel 1982 nasce a Siena la Scuola di Specializzazione in Logica Matematica con lo scopo di avviare i giovani ricercatori alla ricerca nel campo. Contemporaneamente iniziano gli Incontri di Logica: convegni che spaziano su vari temi (Logica Algebrica, Teoria degli Insiemi, Ricorsività, Categorie, Teoria dei Modelli, Teoria della Dimostrazione, Logica e Informatica, Logica e Didattica) e che diventano, negli anni '70 e '80, un punto di riferimento e di incontro per la comunità logica italiana.

Sempre nel 1982 inizia la pubblicazione del periodico Notizie di Logica (NdL) a cura della Scuola di Specializzazione di Siena. Si tratta di un notiziario informativo su vari aspetti che riguardano la Logica in Italia (pubblicazioni, convegni, corsi universitari, concorsi etc.).

Nel 1987, su proposta di Giovanni Sambin, e dopo un dibattito sulle pagine di NdL, nasce l'AILA (Associazione Italiana di Logica e sue Applicazioni). La fondazione dell'AILA è un atto ufficiale di identità per la comunità logica italiana che riconosce al suo interno, oltre ai protagonisti del periodo della rinascita (Casari, Dalla Chiara, Lolli…) anche personalità provenienti dal mondo dell'informatica teorica (Corrado Böhm, Mariangiola Dezani, Giuseppe Longo). L'altra associazione di riferimento per la Logica – soprattutto per il versante filosofico – è la SILFS (Società Italiana di Logica e Filosofia della Scienza), fondata nel 1950 e, dopo un periodo di decadenza fra il 1965 e il 1972, *riattivata* grazie ad Evandro Agazzi.

Oltre a Siena, anche altre sedi universitarie diventano, nel corso degli anni '70, centri di insegnamento e di ricerca in Logica.

A Firenze sia nell'Istituto di Filosofia grazie a Casari e ai suoi allievi (Pierluigi Minari, Andrea Cantini, Sergio Bernini, Michele Abrusci, Giovanna Corsi, Gisèle Fischer Servi, Francesco Paoli e Stefania Centrone) e a Maria Luisa Dalla Chiara; sia a Matematica grazie a Piero Mangani (ordinario di Algebra dal 1973) con il quale lavorano Annalisa Marcja, Sauro Tulipani, e poi Francesco La Cava, Donato Saeli e Carlo Toffalori. In un secondo momento si unisce Daniele Mundici personalità di rilievo per la Logica Italiana. Dopo aver prodotto importanti lavori nel gruppo di Logica di Firenze, nel 1987 Mundici si sposta a Milano come professore ordinario.

A Milano, prima del trasferimento di Mundici, l'eredità del Gruppo di Geymonat è tenuta viva, sul versante filosofico, da Corrado Mangione con Edoardo Ballo, Silvio Bozzi e Giulio Giorello.

A Torino, fra gli anni '60 e '70, Flavio Previale avvia allo studio della Logica Gabriele Lolli, Franco Parlamento e Piergiorgio Odifreddi. Inoltre, negli anni '70, grazie alla presenza di Corrado Böhm, si forma un gruppo di studiosi che danno importanti contributi in Informatica Teorica (Mariangiola Dezani, Mario Coppo, Ines Margaria, Simona Ronchi della Rocca, Maddalena Zacchi).

A Pisa le ricerche in Logica seguono un percorso molto diverso dal resto dell'Italia. Ennio de Giorgi, pur non essendo un logico, ha forti interessi per le tematiche fondazionali e promuove discussioni su tali argomenti nei seminari che si tengono presso la Scuola Normale. Da queste discussioni nasce la ricerca sulle teorie degli insiemi prive dell'assioma di fondazione a cui collaborano Giuseppe Longo, Marco Forti e Furio Honsell i quali avranno un ruolo fondamentale nell'evoluzione dei rapporti fra Logica Matematica e Informatica Teorica.

## *2 La Logica entra nella Scuola*

Nel periodo della cosiddetta *matematica moderna* (anni '60 e '70) la Logica e la teoria ingenua degli insiemi sono state introdotte nei curricula di matematica pre-universitari in alcuni paesi (Durand-Guerrier, 2020). A partire dagli anni '80 la questione del ruolo della Logica nell'insegnamento della Matematica è diventata molto controversa. Alcuni insegnanti e anche molti matematici provenienti dall'*Accademia*, hanno sostenuto che le competenze logiche richieste dalla Matematica si acquisiscono naturalmente *facendo matematica* e non necessitano di una istruzione specifica in Logica.

### 2.1 I logici e la didattica della Matematica

La comunità dei logici italiani, formatasi negli anni della *rinascita*, è stata coinvolta nel dibattito sull'introduzione della Logica nei curricula di Matematica. Molti logici (Ferdinando Arzarello, Claudio Bernardi, Cinzia Bonotto, Ruggero Ferro, Giangiacomo Gerla, Carlo Marchini, Giuseppe Rosolini, Carlo Toffalori e Roberto Tortora) si sono fatti promotori, tra la fine degli anni '80 e l'inizio degli anni '90, di iniziative per approfondire il tema dell'insegnamento della Logica nella Scuola.

Nel 1988 l'Incontro di Logica di Roma è dedicato alla Logica Matematica nella Didattica. Dal 1990, sulla rivista NdL, inizia la rubrica *Logica e Didattica* a cura di Ferro e Gerla con l'idea che il rapporto Logica-Didattica non debba *esaurirsi nella mera divulgazione o, peggio, nella solita tripletta "connettivi logici-tavole di verità-circuiti elettrici"*. Si propone invece, per la parola *Logica*, un'accezione più ampia che riesca ad abbracciare temi importanti come *l'intelligenza artificiale, l'epistemologia* e i *fondamenti*.

Nel 1992 l'AILA fonda un gruppo di lavoro che organizza una settimana di formazione in Logica Matematica per gli insegnanti delle scuole superiori; con la collaborazione di Lucia Ciarrapico, Ispettrice del Ministero della Pubblica Istruzione. Da tale esperienza nasce, il volume sull'*Insegnamento della Logica* (Ciarrapico-Mundici, 1994).

Queste iniziative vogliono rispondere alla diffusa richiesta di Logica nelle attività di aggiornamento per gli insegnanti, in seguito all'introduzione di argomenti di Logica nei programmi ministeriali.

## 2.2 La Logica nei programmi scolastici.

L'ingresso della Logica nei programmi scolastici rientra nel solco di una tradizione che risale agli anni '60, quando, in seguito alla spinta innovatrice del movimento bourbakista, si è cominciato a riflettere sull'introduzione di argomenti di matematica moderna nei curricula pre-universitari.

Un esempio di tale tendenza è l'idea, che si fa spazio tra alcuni insegnanti e in molti libri di testo, che i numeri debbano essere introdotti, a livello di scuola elementare, a partire dagli insiemi. Alla base di questa idea c'è la definizione di Cantor di numero ordinale come classe di equivalenza - di insiemi ben ordinati – rispetto alla relazione di isomorfismo. I limiti pedagogici di questo approccio vengono messi in evidenza in (Pellerey, 1989).

L'*ondata bourbakista* degli anni '60 viene parzialmente superata fra gli anni '70 e '80, nella fase di elaborazione dei programmi delle scuole elementari, medie e PNI per i licei.

Nei programmi del 1985 per le scuole elementari, la Logica è uno dei cinque temi fondanti per l'ambito matematico. Ha dunque una notevole importanza e viene vista non come contenuto matematico da aggiungere a quelli tradizionalmente insegnati ma come abilità trasversale: *l'educazione logica, più che oggetto di un insegnamento esplicito e formalizzato, deve essere argomento di riflessione e di cura continua dell'insegnante*. Questa riflessione continua deve essere perseguita ponendo attenzione alla *precisione e completezza del linguaggio* e proponendo attività *concrete* che siano ricche di *potenzialità logiche: classificazioni mediante attributi, inclusioni, seriazioni*. Si suggerisce di porre attenzione alla consapevolezza nell'uso delle rappresentazioni simboliche allargandone l'uso a contesti diversi: la *rappresentazione logico-insiemistica* può essere estesa *anche ad aritmetica, geometria, scienze e lingua* ... e a *elementari questioni di tipo combinatorio che forniscono un campo di problemi di forte valenza logica.* Si chiarisce inoltre che *la simbolizzazione formale di operazioni logico-insiemistiche non è necessaria, in via preliminare, per l'introduzione degli interi naturali e delle operazioni aritmetiche.*

Nelle scuole medie, i programmi del '79 - elaborati, per la matematica, da Emma Castelnuovo e Giovanni Prodi - portano molte innovazioni, sia nei contenuti sia nelle metodologie. Uno dei temi in cui si articolano i programmi, *matematica del certo e del probabile*, include argomenti di matematica moderna, fra cui Logica, Probabilità e Statistica. In questo tema si fa esplicito riferimento ad *affermazioni del tipo vero o falso…uso corretto dei connettivi logici (e, o, non): loro interpretazione come operazioni su insiemi e applicazioni ai circuiti elettrici* e anche alla possibilità di *presentare equazioni e disequazioni in forma unificata utilizzando il concetto di "frase aperta".*

Emerge un'idea di Logica come disciplina portatrice di competenze trasversali. Tuttavia la modernità e la profondità di questi programmi hanno rappresentato anche un limite per la loro realizzabilità. Come notato in (Ciarrapico, 2001) si tratta di un *programma forse troppo ambizioso per trovare reale applicazione nelle scuole….nella scuola media, tranne lodevoli eccezioni, si continua per lo più ad insegnare algebra, più di quanto previsto dal programma stesso, e geometria, con una didattica del tutto tradizionale.*.

Nella scuola secondaria di secondo grado il dibattito per il rinnovamento dell'insegnamento della matematica dà origine, negli anni '70, a una serie di sperimentazioni portate avanti dai Nuclei di Ricerca Didattica attivati nei Dipartimenti di Matematica di alcune università italiane. Queste esperienze vengono raccolte, nel 1985, dal Piano Nazionale per l'Informatica (PNI), grande progetto di riforma finalizzato all'introduzione dell'informatica nella Scuola in modo estensivo. L'obiettivo del progetto è da un lato introdurre gli studenti ai concetti, ai linguaggi e ai metodi dell'informatica (*insegnare l'informatica*), dall'altro utilizzare gli strumenti informatici per rinnovare metodologicamente il processo di insegnamento-apprendimento (*insegnare con l'informatica*). Per l'attuazione del progetto, il Ministro dell'Istruzione, Franca Falcucci, nomina un Comitato Scientifico di cui fa parte Giovanni Prodi. Uno dei primi problemi che si presenta riguarda come inserire l'insegnamento dell'informatica, se come disciplina autonoma o nell'ambito di altre discipline. Il problema viene risolto introducendo l'Informatica nel programma di Matematica. Il ruolo cruciale di mediatore fra la Matematica e l'Informatica viene attribuito alla Logica che, nelle parole di Prodi, ha il *compito di fare da ponte fra l'informatica e la matematica tradizionale*.

Ciò determina, per la prima volta in Italia, l'introduzione di argomenti di Logica, in maniera importante e sistematica, nei programmi della scuola secondaria di secondo rado. Si fa riferimento a: logica delle proposizioni, regole di inferenza, variabili, predicati, quantificatori, sintassi e semantica, coerenza, indipendenza e completezza di un sistema di assiomi. In continuità con quanto già detto per i programmi delle elementari dell'85, gli estensori dei programmi PNI affermano che *gli elementi di logica non devono essere visti come una premessa metodologica all'attività dimostrativa ma come una riflessione che si sviluppa man mano che matura l'esperienza matematica dell'allievo.*

I programmi PNI possono apparire oggi troppo ambiziosi e in alcune parti, soprattutto quelle relative all'Informatica, inevitabilmente datati. Tuttavia emerge da essi la volontà di far cogliere allo studente il valore culturale della matematica sotto due punti di vista differenti ma coesistenti: come strumento per interpretare e prevedere la realtà e come strumento di riflessione epistemologica. In entrambi questi aspetti alla Logica viene riconosciuto un ruolo fondamentale.

Cosa è rimasto di ciò nelle attuali Indicazioni Nazionali? Sicuramente il valore culturale della matematica e l'importanza di contestualizzarla in senso storico e filosofico. Tuttavia la Logica viene completamente esautorata dal ruolo che aveva nel PNI; di più, nelle Indicazioni Nazionali per la Matematica, la parola *Logica* non è affatto presente. I temi più vicini alla Logica – e già presenti nel PNI come temi di Logica e Informatica – sono: l'*approccio assiomatico nella sua forma moderna*, il *principio di induzione* e il *concetto di algoritmo*. Questi temi hanno però perso la loro origine unitaria che, in ultima analisi, è riconducibile alla tradizione della Logica Matematica. Anche il riconoscimento di tale tradizione sembra venire meno nelle Indicazioni Nazionali quando si afferma che *i momenti che caratterizzano la formazione del pensiero matematico sono: la matematica nella civiltà greca, il calcolo infinitesimale…che porta alla matematizzazione del mondo fisico, la svolta che prende le mosse dal razionalismo illuministico e che conduce alla formazione della matematica moderna e a un nuovo processo di matematizzazione che investe nuovi campi (tecnologia, scienze sociali, economiche, biologiche) e che ha cambiato il volto della conoscenza scientifica.* Non si fa cenno al periodo della Crisi dei Fondamenti (Cantor, Gödel…) e a tutto ciò che ne è scaturito, al confine tra matematica e filosofia, in termini di riflessioni sulla matematica e sulle sue applicazioni in campo tecnologico.

È possibile che il termine *Logica* sia sparito dalle tematiche da trattare nelle scuole perché ritenuto anacronistico o ambiguo. Ambiguo perché, negli ultimi decenni il termine Logica viene spesso associato a quesiti e test più che a una vasta area di studio. Anacronistico perché la visione di una Logica come ponte fra Matematica e Informatica, tanto radicata negli anni ’80, è quasi sparita dall’opinione pubblica odierna.

**REFERENCES**


Ciarrapico, L. - Mundici, D. (1994). *L’insegnamento della Logica.* Ministero della Pubblica Istruzione, in collaborazione con AILA.

Ciarrapico, L. (2013). *L’insegnamento della Matematica, dal passato recente all’attualità.* Relazione svolta a Cattolica il 7 ottobre 2011, in occasione del Congresso ATD (associazione per la Didattica e la Tecnologia).

Durand-Guerrier, V. (2020). *Logic in Mathematics Education.* Encyclopedia of Mathematics Education. S. Lerman, Springer.

Frege, G. (2020). *Alle origini della nuova logica: carteggio scientifico con Hilbert, Husserl, Peano, Russell, Vailati e altri.* A cura di G. Gabriel, H. Hermes, F. Kambartel, C. Thiel, A. Veraart. Edizione italiana a cura di C. Mangione. Prefazione di G. Lolli. Bollati Boringhieri.

Geymonat, L. (1959). *Peano e le sorti della Logica in Italia.* Bollettino dell’UMI, Serie 3, Vol. 14, n.1, p.109-118.

Lolli, G. (2020). *Ettore Casari e la rinascita della logica in Italia.* Accademia delle Scienze di Torino, Classe di Scienze Morali, Storiche e Filologiche, Volume 154.

Luciano, E. – Roero, C. S. (2008). (a cura di) *Giuseppe Peano. Matematico e Maestro.* Dipartimento di Matematica dell’Università di Torino.

Peano, G. (1958). *Opere scelte.* A cura dell’UMI, Vol. II: “Logica Matematica, interlingua ed algebra della grammatica”. Edizioni Cremonese, Roma.

Pellerey, M. (1989). *Oltre gli insiemi: nascita, crescita e crisi dell’insiemistica, nuovi orientamenti nella didattica dell’aritmetica.* Tecnodid.

# THE HISTORICAL FAGNANO'S PROBLEM: TEACHING MATERIALS AS ARTIFACTS TO EXPERIMENT MATHEMATICAL AND PHYSICAL TASKS IN ITALIAN HIGH SCHOOL

**Maria Giuseppina ADESSO[(1)], Roberto CAPONE[(2)], Oriana FIORE[(1)]**

(3) High School "G. Da Procida", Via Gaetano De Falco n.2 - Salerno, Italy

(4) Department of Mathematics, University of Bari "Aldo Moro", Via Amendola, 4 - Bari, Italy

(5) High School "P.E. Imbriani" via Pescatori – Avellino, Italy

mapinadesso@gmail.com, roberto.capone@uniba.it, orianafio@gmail.com

## ABSTRACT

The starting point of this research is Fagnano's problem: "*For a given acute triangle, determine the inscribed triangle of the minimal perimeter.*" This problem has been investigated using methodologies both from mathematical analysis and synthetic geometry, and many demonstrative strategies have been provided. The *orthic triangle* is the problem's solution. This problem has been recently extended to convex quadrilaterals. In particular, Fagnano's problem is relevant in billiard physics: the orthic triangle is the minimum periodic orbit of an acute triangular billiard. An open question still remains referring to quadrilaterals: "Are the Orthic Quadrilaterals the minimum periodic orbits in a real billiard?" This communication aims to describe experimental teaching in an Italian high school, focused on Mathematical and Physical tasks, from a historical perspective. Starting from a historical and epistemological context, an interdisciplinary learning path has been planned, which has been experimented with about sixty fifteen-year-old students. The applicability of some geometric theorems to different contexts of reality has been tested through the realization and use of specific artifacts as teaching materials.

## *1 Rationale*

This work draws on other works by the same authors on the history of geometry teaching in Italy in the early 20th century (Furinghetti, 1997; Clark, 2012; Adesso et al. 2018; Adesso et al. 2019a; Adesso et al., 2019b; Adesso et al., 2020). In the previous works, the history of geometry teaching in Italy, in the period from the final years of the 19th century to the first half of the 20th century, was analyzed, taking into account the influence of both school reforms and the "New geometry of the triangle", first introduced in France in 1873. In the last works, we referred to some theorems about Cevian and orthic triangles, which may be included in the "New geometry of the triangle", although they were discovered in Italy before 1873. Some Italian booklets and textbooks have been analyzed to show the influence of these factors on geometric teaching.

## 2 *Fagnano's problem: a historical overview*

The starting point of this research is Fagnano's problem: *For a given acute triangle, determine the inscribed triangle of the minimal perimeter*.

This problem has been investigated using methodologies both for mathematical analysis and synthetic geometry, and many demonstrative strategies have been provided.

Coxeter & Greitzer (1967) quoted that this problem "*was proposed in 1775 by Fagnano, who solved it by calculus*." Nevertheless, Coxeter showed a geometrical proof. "The proof shown here is due to H. A. Schwarz." The Schwarz proof was based on triangle reflections.

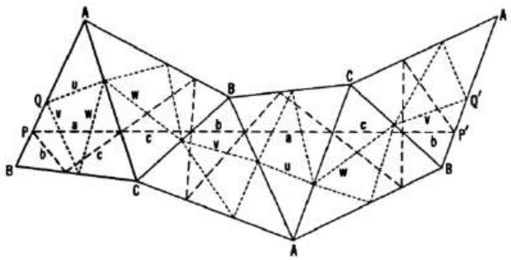



**Figure 1.** Schwartz proof to the Fagnano problem, as figure 4.5A in Coxeter-Greitzer

Fagnano's problem was first published by Giovanni Francesco Fagnano dei Toschi (1715-1797)in Nova Acta Eruditorum, 1775, p. 281-303, «*Problemata quaedam ad methodum maximorum et minimorum spectantia*». Here, Fagnano's calculus solution was not shown (and it seems unpublished), whereas the relevance of geometry as a means to both elegantly and simply solve certain problems was outlined.

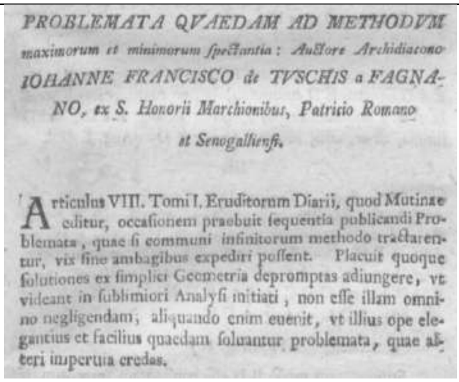
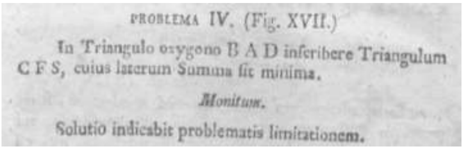
| | |
|---|---|
| <br>PROBLEMATA QVAEDAM AD METHODVM maximorum et minimorum spectantia: Auctore Archidiacono IOHANNE FRANCISCO de TVSCHIS a FAGNANO, ex S. Honorii Marchionibus, Patricio Romano et Senogalliensi.<br><br>Articulus VIII. Tomi I. Eruditorum Diarii, quod Mutinae editur, occasionem praebuit sequentia publicandi Problemata, quae si communi infinitorum methodo tractarentur, vix sine ambagibus expediri possent. Placuit quoque solutiones ex simplici Geometria depromptas adiungere, vt videant in sublimiori Analysi initiati, non esse illam omnino negligendam; aliquando enim euenit, vt illius ope elegantius et facilius quaedam soluantur problemata, quae alteri impervia credas. | *Article VIII of a tome I of the Journal of Scholars, which was published in Modena, offered the opportunity to publish the following problems, which, if treated with the calculus method, could hardly be solved without uncertainty. It seemed opportune also to add solutions derived from simple geometry so that those initiated to higher analysis may realize that this (i.e., geometry) is not to be neglected/despised altogether; indeed, it sometimes happens that thanks to it, one can more easily and elegantly solve certain problems that one might otherwise consider impervious.* |

**Figure 2.** The Geometry relevance to solve some problems, in Problemata quaedam ad methodum maximorum et minimorum spectantia

Problem IV was the well-known Fagnano problem, and he solved it by using circle properties.



PROBLEMA IV. (Fig. XVII.)
In Triangulo oxygono B A D inscribere Triangulum C F S, cuius laterum Summa sit minima.
*Monitum.*
Solutio indicabit problematis limitationem.

*In the BAD triangle, inscribe CFS triangle, having the addition of the minimum side.*

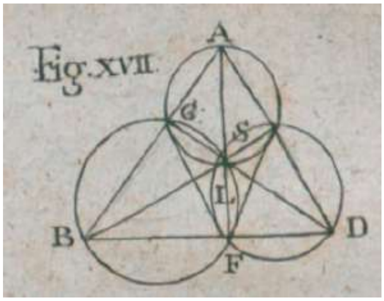



**Figure 3.** Fagnano's problem in Problemata quaedam ad methodum maximorum et minimorum spectantia

Nevertheless, it seems that Giulio Carlo Fagnano dei Toschi (1682-1766), Giovanni's father, first introduced this problem, giving a part of the solution: In the book When Least is Best (Nahin, 2021), it is shown that "*This problem has its origin with the Italian mathematician Giulio Carlo Toschi di Fagnano, who showed the existence part, and his priest-mathematician son Giovanni Francesco Fagnano, who completed the minimization argument in 1775. The* *father's contribution was to show, given any acute-angled triangle ABC and any given point U on one of the sides, how to construct the inscribed triangle of the minimum perimeter with a vertex on BC side.*"

The orthic triangle (Fagnano, 1775) is the problem's solution. An orthic triangle of a given ABC triangle is defined as the figure obtained by drawing the three segments joining the feet of the three heights of the ABC triangle. In the acutangle triangle, the orthic triangle is inside the ABC triangle (fig. 4a), in the right-angled triangle, the orthic triangle is the height relative to the hypotenuse, whereas, in the octusangle triangle, it is outside the ABC triangle (fig. 4b).

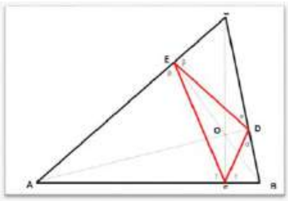



**Fig. 4a.** The orthic triangle in an acutangle triangle

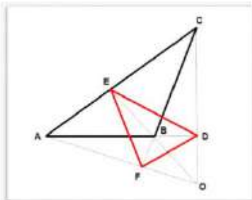



**Fig. 4b.** The orthic triangle in an the octusangle triangle

This problem has been recently extended to convex quadrilaterals (Mammana et al., 2010). Specifically, they proved the following theorem:

*If Q is cyclic and orthodiagonal, the orthotic quadrilaterals of Q inscribed in Q have the same perimeter. They also have the minimum perimeter with respect to each quadrilateral inscribed in Q.*

The given proof was similar to the Schwarz ones, starting from reflection properties.

## 3 Activity design

Fagnano's problem is also relevant in billiard Physics: the orthic triangle is the minimum periodic orbit of an acute triangular billiard (Gutkin, 1997). Nevertheless, in Gutkin's work, there is an inaccuracy. Instead of the term, "orthic triangle" is used "pedal triangle", actually the orthic triangle is a particular pedal triangle, it is the pedal triangle relative to the orthocentre. Our activities focused about the following question:

"Are the Orthic Quadrilaterals the minimum periodic orbits in a quadrilateral billiard?". Actually, no proofs of this question could be found in the literature. Nevertheless, the authors verified this hypothesis by using some artifacts. An interdisciplinary unit has been planned focused on the historical and epistemological context of Fagnano's problem: starting from its geometrical proofs to the Physics applications. The interdisciplinary learning unit included History, Latin, Literature, Maths, Physics (Capone, 2022). This learning unit has been experimented with about sixty students (three classrooms), attending the second year of a Scientific High School in the South of Italy. Three Mathematics and Physics teachers carried out the laboratorial activities, planned with two researchers, one in Mathematics education and other one in Physics education. In each classroom, the students were grouped in five groups (about four students in each group). The activities have been planned in the following phases:

1) Discovering the triangle with minimum perimeter must be inscribed in a triangle using GeoGebra, a geometrical dynamical software. Our students still don't know calculus, so they followed the Giulio-Giovanni Fagnano proof.
2) Discovering quadrilaterals with minimum perimeter: here, we just planned to verify that the orthic quadrilateral was the one with minimum perimeter to be inscribed in a quadrilateral, always by using GeoGebra.
3) Creating an orthic triangle or quadrilateral on real triangular and quadrilateral billiards, to verify the previous phases.
4) In the real world, the Physics billiard is affected by friction, so an innovative artifact was built to verify that the orthic triangle and the orthic quadrilateral are the minimum orbits in triangular and quadrilateral billiards: authors defined them as "light billiards", because the laser light was used instead of a typical billiard ball.

## *4 Activity development*

### 4.1 Phases 1 – 2 Historical and geometrical analysis of Fagnano's problem

The historical context was first analyzed. Students were involved in a translation from the medieval latin of the *«Problemata quaedam ad methodum maximorum et minimorum spectantia».* The students also analyzed all the historical documents about Fagnano's problem, as we summarised in the previous paragraphs (Coxeter & Greitzer, 1967), and they organized them in a text.

The Maths teacher guided them through analyzing and reproducing (by using GeoGebra) different geometrical proofs of the Fagnano problem (L. Fej´er and While H. A. Schwarz). Group 3 shows the existence (similar to Giulio’s proof), and group 4 shows the unicity (similar to Fe’ier proof).

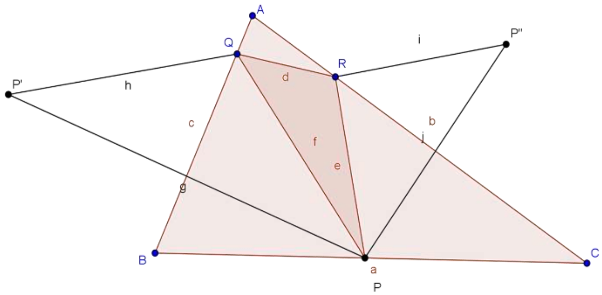



**Figure 5.** The students (Classroom 1, group 3) proof that exist a triangle, inscribed in another one, which has a minimum perimeter.

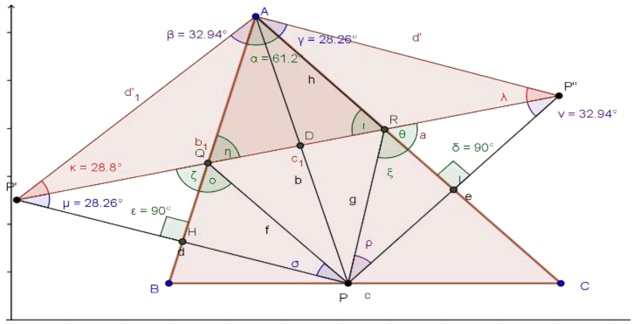



**Figure 6.** The students (Classroom 1, group 4) verification of the unicity, following Fe’ier proof.

Here we also show the students translated proof text about the existence: *P’ is the symmetric point of P with respect to AB side and P’’ with respect to AC side. P’Q is congruent to PQ and P’’R is congruent to QR. So, the perimeter of PQR triangle is the same of P’Q+QR+RP’’. It will be minimum if this broken line is a single line. For each position of P on BC there exists and is unique a triangle of minimum perimeter corresponding to this condition. The solution to Fagnano's problem is therefore to be found in the family of triangles with this characteristic, obtained by varying P on BC.*

### 4.2 Phase 3 Orthic quadrilateral on billiard.

In the next activity phase, students learned the orthic quadrilateral; in Figure 7. Group 2 in Classroom 3 showed different orthic quadrilaterals inscribed in a quadrilateral Q, also including the principal orthic quadrilateral, with respect to the Varignon parallelogram (1731).

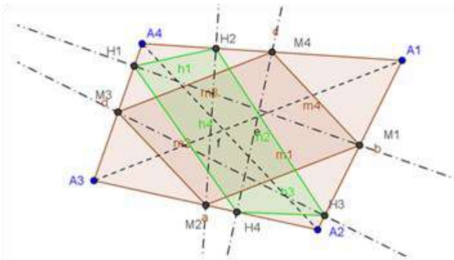



**Figure 7.** Main orthic quadrilateral, with respect to the Varignon parallelogram, MiHi = M-altitudes as shown by group 2 in Classroom 3.

Students were involved in the creation of an orthic quadrilateral by using coloured ribbons on a rectangular billiard (see Fig. 8):

1) Particular case: a) Orange ribbons: diagonal of the billiard; Blue ribbons: Varignon parallelogram (joining the midpoints of each sides), which is also the main orthic quadrilateral in the rectangular billiard
2) General case: a) Orange ribbons: diagonals (fig. 8a); b) Yellow ribbons: starting from a point P, students draw parallel to the diagonals, such as obtaining the parallelogram $V_1V_2V_3V_4$. (fig. 8b) c) Blue ribbons: from each vertex of the parallelograms, a perpendicular to the opposite side starts: the V-altitudes $H_1$, $H_2$, $H_3$, $H_4$. (fig. 8c) d) Red ribbon: joining the foots of the V-altitudes we have the orthic quadrilateral (fig. 8d).

Students also built small modelling of real quadrilateral billiard with different shapes (see for example Fig. 9, built by group 4 in Classroom 1)

**Figure 8a**. Orange ribbons: diagonals

**Figure 8b.** Yellow ribbons: parallelogram $V_1V_2V_3V_4$

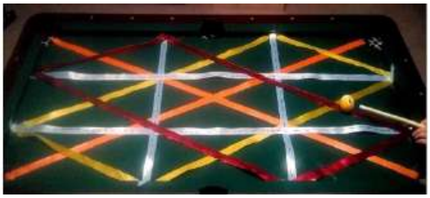


**Figure 8c.** Blue ribbons: V-altitudes $H_1$, $H_2$, $H_3$, $H_4$.

**Figure 8d.** Red ribbon: orthic quadrilateral on a rectangular billard

**Figure 9.** Trapezoidal billiard and an orthic quadrilateral

After they built the orthic quadrilateral on quadrilateral billiards, they tried to verify that the ball's trajectory was the same as the orthic quadrilateral, thus showing that the family of the orthic quadrilateral (having minimum perimeter) was also the closed periodic orbit for the main quadrilateral.
The results were that, in each billiard, the trajectory followed the perimeter of the orthic quadrilateral, but the orbit was never definitely closed.

### 4.3 "Light Billiard"

The trajectory of the billiard ball was analyzed by using the Tracker software. It was observed that the Snell law was not completely satisfied: as it is shown in Fig. 10 the motion was uniform but the velocity before and after the collision with the billiard side was different. The friction caused it.

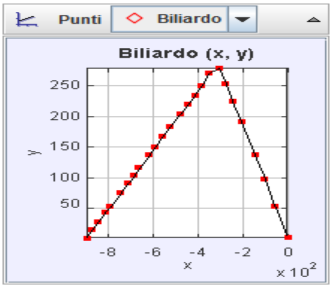



**Figure 10**. Tracker analysis: the friction caused a motion with different velocities, contrarily at the Snell law.

In order to avoid friction and to verify that the orthic quadrilaterals was the closed orbit with a minimum perimeter, a "light billiard" was built by the students, supported by the Maths and Physics teacher. In Fig. 11 a triangular and a quadrilateral light billiard are shown, where the laser light was the trajectory of the orthic quadrilateral, as supposed.

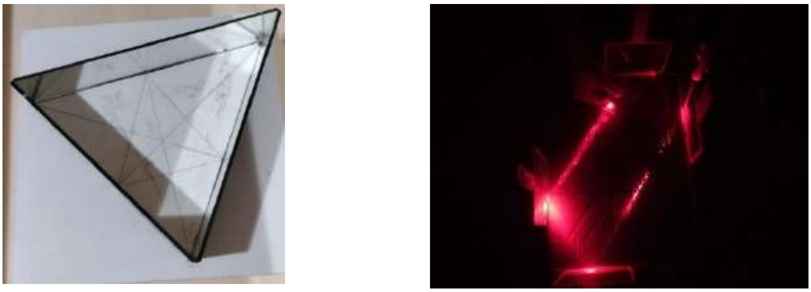


**Figure 11.** Triangular and quadrilateral light billiards

## 5 *Conclusions*

About sixty students were involved in an interdisciplinary learning unit including History, Latin, Literature, Maths, Physics. They analyzed and translated original Fagnano paper and critically studied the different geometrical proofs to it, using dynamical geometrical software to verify the different approachs. Nevertheless, some artifacts have been realized to verify a «new idea»: a correlation between the billiard Physics, the Geometric Optic and the Fagnano's problem. The purpose of these activities carried out with the students had several educational aims: to highlight the importance of contextualising a problem historically; to motivate students to study mathematics and physics; to show how interdisciplinary learning can lead to a broader view of a problem. This approach allows students to learn mathematics in their own way and develop mathematical ideas without the textbook and beyond the classroom. The collaborative learning framework involving heterogeneous group members provides an opportunity to learn mathematics concepts, even at a higher level. It is concluded that students' knowledge constructability requires an apprenticeship in culturally specific cognitive and social practices. Furthermore, the history of mathematics makes up an important component of learning mathematics. Its integration into mathematics curricula helps the students to understand that mathematics is "...a discipline that has undergone an evolution and not something that has arisen out of thin air." (Jankvist, 2009, p.239)

## SELECTED REFERENCES


Adesso M. G., Capone R., Fiore O., & Tortoriello, F. S. (2018). Dai quadrilateri ortici alla fisica del tavolo da biliardo. In *Quaderni di Ricerca in Didattica, Proceedings of GiMat*. (pp. 127-128). Palermo: G.R.I.M.

Adesso M. G., Capone R., Fiore O., & Tortoriello F. S. (2019a). Discovering Neglected Synthetic Geometry on Social Networks: Learning Maths as in the Historical Italian Academies. In *Proceedings of the Eighth European Summer University on History and Epistemology in Mathematics Education* (ESU 8), Vol. 1 (pp. 561-573). Oslo: OsloMet Skriftserie.

Adesso M. G., Capone R., Fiore O., & Tortoriello, F. S. (2019b). Mathematics learning and teaching in an interdisciplinary framework simulating ancient academy. In M. Shelley & S.A. Kiray (Eds.). *Education Research Highlights in Mathematics, Science and Technology 2019* (pp. 63-78). Ames: ISRES Publishing.

Adesso, M. G., Capone R., Fiore O. & Tortoriello, F. S. (2020). Walking through the history of geometry teaching by Cevian and orthic triangles and quadrilaterals. In *"Dig where you stand" 6. Proceedings of the sixth International Conference on the History of Mathematics Education* (pp. 355-368). WTM-Verlag.

Alasia, Cristoforo (1900). *La recente geometria del triangolo*. Città di Castello: S. Lapi.

Capone, R. (2022). Interdisciplinarity in Mathematics Education: From Semiotic to Educational Processes. *EURASIA Journal of Mathematics, Science and Technology Education*, *18*(2).

Clark, K. M. (2012). History of mathematics: Illuminating understanding of school mathematics concepts for prospective mathematics teachers. *Educational Studies in Mathematics*, *81*, 67-84.

Coxeter, H. S. M., & Greitzer, S. L. (1967). *Geometry revisited* (Vol. 19). Maa.

Fagnano, G. F. (1775). Problemata quaedam ad methodum maximorum et minimorum spectantia. *Nova Acta Eruditorum*, 281-303.

Furinghetti, F. (1997). History of mathematics, mathematics education, school practice: Case studies in linking different domains. *For the learning of mathematics*, *17*(1), 55-61.

Gutkin, E. (1997). Two applications of calculus to triangular billiards. *The American mathematical monthly*, *104*(7), 618-622.

Jankvist, U. T. (2009). A Categorization of the "Whys" and "Hows" of Using. History in Mathematics. *Education. Educational Studies in Mathematics*, 71(3).

Mammana, Maria Flavia, Micale Biagio, Pennisi Mario (2010). Ortic Quadrilaterals of a Convex Quadrilateral, *Forum Geometricorum*, *10*, 79-91.

Nahin, P. J. (2021). *When Least is Best: how mathematicians discovered many clever ways to make things as small (or as large) as possible* (Vol. 114). Princeton University Press.

Varignon P. *Elemens de Mathematique de Monsieur Varignon*. Paris: Pierre-Michel Brunet, Fils, 1731, 1972, New Canaan, Conn.: Readex Microprint Landmarks of Sci

# THE "UGO MORIN" DIDACTIC RESEARCH CENTRE: LINKING RESEARCH IN MATHEMATICS EDUCATION AND TEACHING IN ITALIAN SCHOOL

**Adriano DEMATTÈ, Luigi TOMASI**
Centro Ricerche Didattiche "Ugo Morin"
Via S. Giacomo 4, Pieve del Grappa (TV), Italy
adrdematte@gmail.com, luigi.tomasi@unipd.it

## ABSTRACT

This article presents the "Morin" Centre, an association founded on 1968 as a service to the schools in Italy. Since 1970, it publishes a journal: "The Teaching of Mathematics and Integrated Sciences". In over half a century of activity, the Centre has focused on some fundamental issues of mathematics education, such as: role of problems, use of new technologies, value of the history of mathematics. The attention of teachers for the activities of the Centre is still alive. Collaboration with Universities and schools appears to be an important resource for the future.

## *Introduction*

The "Ugo Morin" Educational Research Centre is an independent cultural association the members of which are mainly mathematics teachers, university professors, people interested in the study and teaching of mathematics and integrated sciences; other institutions, such as schools and universities, can also be members of the Centre. The Members' Assembly is summoned every year by the President on the occasion of the annual Seminar which is traditionally held at the end of the summer. A Presidency Council is made up of the President and six Advisers elected by the Members' Assembly. A Scientific Commission is appointed by the Presidency Council.

## *1 The story*

The "Morin" Centre was founded on December 27, 1968, in Paderno del Grappa, now Pieve del Grappa (Treviso, Italy), with the denomination of "Group of Pedagogy of Mathematics" (Sitia, 1994; Ferrari, 2020). The founders conceived the Centre as a service to the Italian school – to renew it in favour of young people – with the following aims: teachers' training, participation in conferences in Italy and abroad, bibliographic information, exchange of experiences actually carried out (Tomasi, 2022).

In the report of the two intense days of work of 27-28 December 1968 (Sitia, 1994), motivations and ideals of the association appear linked to the personal history of each of the participants and to that period of great renewal in international society and in mathematics education. Teachers made speeches proposing to form working groups to elaborate possible curricular changes in reference to the contents of *modern mathematics* as a resource for the renewal of the traditional contents – see (Tomasi & Demattè, these *Proceedings*). Teachers and school directors feel an urgent need for the assistance of professional mathematicians. The hope of collaboration between primary and secondary school teachers is highlighted. The denomination "Group of Pedagogy of Mathematics" underlines the fact that the founders do not intend to work only with reference to mathematics, but above all in pedagogical and didactical perspective. The renewal cannot be imposed from above but must be the fruit of the work of teams of specialists in mathematics, psychology, pedagogy, and didactics. Some statements in the report are in line with the spirit of contestation of that period: it is stated that it is not possible to accept without serious reservations the programs proposed by the Instruction Minister, and the wish has been expressed for the "abandonment of a teaching method that became sterile in the boredom of empty algorithmic exercises". It is said that the renewal of mathematics teaching means above all a pedagogical revolution. A provisional statute is reported, discussed, and approved unanimously by the Assembly (Sitia, 1969).

In 1970 the journal *L'insegnamento della matematica* [The Teaching of Mathematics] was founded; in 1971 it got an "Advisory Committee". In 1978 the journal took on its current name: *L'Insegnamento della Matematica e delle Scienze Integrate* [The Teaching of Mathematics and Integrated Sciences] (Ferrari, 2020).

After the founding Assembly, the first "National Seminar on Mathematics Teaching" was held on 26-27 September 1970 and, after that, every subsequent year. Now, it is flanked by other initiatives such as "Sunday training courses" in mathematics didactics for primary and lower secondary school teachers and meetings in schools and universities, especially in northern Italy (Ferrari, 1994). In over half a century of activity, the Centre has focused – often in a pioneering way – on some fundamental issues of mathematics education, such as: the role of problems, the use of new technologies, the value of the history of mathematics.

## 2 Important people in the history of the Centre

Ugo Morin (1901-1968) was born in Trieste (Triest, in German) at that time part of the Austro-Hungarian Empire, now Italy. After a period as captain of the merchant marine, he enrolled at the University of Padua and graduated in mathematics in 1926. He immediately began his academic career, as an assistant and then as a professor of Geometry at the Universities of Padua and Trieste. He participated in the struggle for the liberation of northern Italy from Nazi-Fascism. In the 1960s Morin was one of the protagonists of the debate about introduction of "modern mathematics" in the Italian secondary school; he was also author of innovative textbooks (Sitia, 2018; Tomasi, 2018; Guerraggio, 2022). His first textbook was devoted to junior high school geometry; he chose this school as his first commitment, because he thought it to be the most difficult to tackle. This textbook was followed by the *Elements of geometry for upper secondary school*, part I (1958), part II and part III (1959), also written with his assistant Franca Busulini. It is a work reprinted several times, with subsequent updates and improvements, until 1976. It includes the language of sets, introduces concepts and methods of abstract algebra and proposes a study of geometry based also on the geometric transformations of the plane, following the approach of Klein's “Erlangen Programme” (Tomasi, 2018).

Since its foundation, the main animator of the Centre was Candido Sitia − fr. Roberto (1922-2002), a de La Salle Brother of the Christian Schools, graduate in physics, teacher, and principal of secondary school. He participated in international conferences and had contacts with the most eminent scholars of mathematics education in Italy and in Europe, whom he often invited to the seminars at the Centre. He was a member of the CIIM [Italian Commission for Teaching of Mathematics] a Commission of UMI [Italian Mathematical Union] (Ferrari, 2002).

Among the members of the Advisory Committee of *L’insegnamento della Matematica e delle scienze* integrate there were: Frédérique Papy-Lenger (1921-2005), Belgian, who ran an experimental training program for kindergarten teachers based on a new curriculum; Georges Léopold Anatole Papy

(1920-2011), who influenced French mathematicians in search of an evolution of the pedagogy of mathematics;. Hans Freudenthal (1905-1990) who made substantial contributions to algebraic topology and, in 1968, founded the journal *Educational Studies in Mathematics*; Tamás Varga (1919-1987) who was one of the major personalities of Hungarian and European mathematics education – in the 1960s and '70s he proposed a reform project; Vittorio Checcucci (1918-1991), Italian, who conducted his scientific activity in the sector of Geometry and Mathematics education; Salvatore Ciampa (1930-1973) who studied in Italy and at the Columbia University and held, from 1972, the position of President of CIIM; Angelo Pescarini (1919-2000) who was a secondary school teacher – also professor in charge of the University of Ferrara − and planned projects for primary school. The current president of the Centre is Cinzia Bonotto of the University of Padua. Honorary president is Mario Ferrari of the University of Pavia; he alsoserved as president until 2017. A fundamental characteristic is that the collaborators – university teachers and primary or secondary school teachers – are all volunteers.

## *3 Publications and website*

The journal includes educational research articles, reports of experiences in the classroom, proceedings of the annual Conference, articles by the AIRDM (Italian Association for Research in Mathematics Didactics) collected in special issues. It is divided into A-edition dedicated to primary and lower secondary school (the referent is Cinzia Bonotto) and B-edition dedicated to upper secondary school (the referent is Pier Luigi Ferrari of the University of Eastern Piedmont). All issues of the journal are available by using a password provided to the subscribers. In addition, monographs on in-depth topics are published: the *Quaderni* [Booklets] divided into three series (didactic, work, research) and the *Formazione Professionale* [Professional Training (of mathematics teachers)] series by the Didactic Research Unit of the University of Pavia.

For more information about the Centre, please see http://www.centromorin.it/. There, among other things, you can find the list of topics addressed in the annual seminars, the issues of the journal, the library file. In fact, the Centre has a well-stocked library, richer than the libraries of other Italian associations and even richer than the mathematics education sector of university libraries. It contains about 7500 volumes (almost all registered and presented online) and receives or exchanges 78 Italian and international journals oriented to teaching of mathematics and science, to didactic and pedagogical research, and to epistemology. The library is open to all Members of the Centre, to teachers and to all those who make a request. In addition, from the site it is possible to download a copy of the 1478 original of *Treviso Arithmetic*, the first ever printed mathematics book.

## *4 Conclusion*

We asked ourselves the question: in what terms was the intention of linking research in mathematics education and teaching in Italian school − according to the founders' aims (quoted in section 1) − achieved? To prepare this contribution, we retraced the story of the association − to highlight the role of people, obstacles, and material resources − through discussions with other members of the Centre and with the honorary President. We have identified three crucial periods: 1) the first two years from the foundation, in order to focus on how the activities of the Centre could have started; 2) the long intermediate period during which most of the aims of the founders had the possibility of realization; 3) the last five years for a reflection on future perspectives.

1) The founders all had a more or less long experience in teaching. They manifested love for their profession as educators − as evidenced by their willingness to face two days of intense work during the Christmas holidays. They demonstrated the desire for renewal − as shown in the report of the two days. Their attitude of service to colleagues was realized above all in collaboration with the reference figure of Candido Sitia. We consider his long-lasting dedication to the Centre in some sense as a consequence of his choice to dedicate his life to the service of education among the Brothers of the Christian Schools. He deserves the credit for having built a network of relationships between the founders and with other possible collaborators. The Centre found its headquarters in a building owned by the Brothers of the Christian Schools.

2) A group of volunteers dedicated their work to the activities of the Centre. The journal has had an ever-increasing circulation. New publications were created (the previously cited *Booklets*). Alongside the annual Seminar, “Sunday training courses” have started. All this happened even if external funding was limited to short periods.

3) Previous activities continued and collaborations with universities and national associations (UMI-CIIM, AIRDM) have been strengthened. However, the activities promoted by the Centre are attended by fewer young teachers and the number of subscribers to the journal (i.e., members of the Centre) is decreasing. It seems essential to make the Centre better known even among those scholars and teachers who, due to the geographical distance, find it difficult to participate in face-to-face meetings.

In summary, we can say that the aim to involve teachers and maintain links with researchers in mathematics education has been achieved over the years. Volunteers support the aims of the Centre. As a critical point, we fear a shift away of educational research from the aspects that teachers consider central to classroom work. The debate within the Centre and the issues addressed during the annual Seminars concern mathematical content and reporting of class experiences; a specific reflection aimed at the pedagogical aspects remains as a future requirement.

**REFERENCES**


Ferrari, M. (1994). Ricordando una “nascita”. *L’Insegnamento della Matematica e delle Scienze Integrate*, vol. 17 A + B N.5, 406–408.

Ferrari, M. (2002). Fratel Roberto Sitia. *L’Insegnamento della Matematica e delle Scienze Integrate*, vol. 25 A N.4, 322–332.

Ferrari, M. (2020). La rivista del Centro Morin, cenni di storia. *L’Insegnamento della Matematica e delle Scienze Integrate*, vol. 43 A+B N.3, 208–244.

Guerraggio, A. (2022). La Resistenza gentile. *Prisma*, N.40, 32–35.

Sitia, C. (1969), Educazione e Insegnamento della Matematica. *Rivista Lassalliana;* republished in *L’Insegnamento della Matematica e delle Scienze Integrate*, vol. 31 A N.3, 209-236.

Sitia, C. (1994). Lo spirito del solaio – Ricordi di un’avventura. *L’Insegnamento della Matematica e delle Scienze Integrate*, vol. 17 A + B N.5, 409–424.

Sitia, C. (2018). L'impegno di Ugo Morin per la didattica della matematica. *L’Insegnamento della Matematica e delle Scienze Integrate*, vol. 41 B N.4, 467–478.

Tomasi, L. (2018). Un ritratto di Ugo Morin (1901-1968) nel cinquantesimo anniversario della sua scomparsa. *L’Insegnamento della Matematica e delle Scienze Integrate*, vol. 41 B N.4, 403–431.

Tomasi, L. (2022). Il Centro Ricerche Didattiche “Ugo Morin”. *Prisma*, N.40, 35.

# HISTORY OF MATHEMATICS AS A TOOL TO EDUCATE FOR ANTI-RACISM

**Elena SCALAMBRO**[1], **Erika LUCIANO**[2]
[1] Dip. di Matematica "G. Peano", Via Carlo Alberto 10, 10123 Torino, Italy
[2] Dip. di Filosofia e Scienze dell'Educazione, Via Verdi 8, 10124 Torino, Italy
elena.scalambro@unito.it, erika.luciano@unito.it

## ABSTRACT

In the present paper, we analyze how the history of mathematics – and that of STEM disciplines in general – can become an effective tool to educate and heighten awareness about the Memory of the Shoah, with the aim of promoting an antiracist attitude. Several actions to achieve this goal have always been conducted in schools within the humanities. However, we would claim the important role assumed by the history of science in this perspective, starting from the analysis of some educational activities proposed in schools within the project *'Mathematics knows no races or frontiers'. Ideas for historical-mathematical reflection for an antiracist education*.

## *1 Towards an antiracist education*

While the awareness towards the Shoah had always been prerogative of humanities in educational context, only in recent years it emerged as unspoken potential of scientific disciplines. Hence, we would discuss the opportunity and the impact of introducing these issues in schools starting from a different perspective, the one of the history of mathematics and the history of science. The need to promote antiracist attitudes among the youngest stems also from the increasing episodes of intolerance. The growth of migration flows and the persistent economic difficulties have risen concerns at all levels regarding the return of racist and xenophobic attitudes among the Italian population. In the last few years, the national monitoring centre denounced the spread of acts of intolerance against those who are regarded as 'different' for reasons of ethnicity, race, religion, gender, sexual orientation, physical or mental disability. A substantial percentage of these acts takes place in schools and, having the younger generation as protagonist, is amplified by social networks. The task of educating to inclusion and antiracism, also included in the goals of the United Nations' action program "Agenda 2030" in connection with the development of global citizenship, is one of the most compelling challenges of our educational system. Although there are several initiatives in this direction, they are often piecemeal and fragmented and there is the risk of 'losing' their true meaning.

The best practices of didactics of the Shoah stress that racist preconceptions essentially spring from ignorance, and only thanks to the knowledge of what happened in the past it is possible to acquire the tools needed to identify and combat the symptoms of intolerance. The national guidelines *Per una didattica della Shoah a scuola*[88] identify the interdisciplinary and transcultural perspective as the root cause that makes the anti-racism education one of the most complex educational challenges, since it involves different skills and specializations. Conversely, the cross-cultural dimension invoked is typical of the multidisciplinary approach of the history of science and mathematics. Because of their universal language, which transcends every race and frontier, STEM disciplines lend themselves well to develop these themes. A good teaching of the history of scientific disciplines can contribute to counter the cognitive-essentialist bias which underlies the racist views of new generations (Corbellini, 2020; Rutherford, 2020). Historical-scientific research may become an educational tool and a carrier of public engagement aimed at 'disarming' false arguments, stereotypes and slogans which are often puerile but simple, and therefore effective (Castelnuovo, 1997; Segre, 2018). The plots of the internal history of STEM, can make the future generations aware of the instrumental use that was made of such disciplines, which are usually considered – by their very nature – impervious to ideological conditioning. In this regard, it is hardly necessary to recall that state racism in the German Reich and in the Fascist Italy, justified by the false myth of 'racial purity', broadly benefitted from the scientific support and the complicity of some men of science to promote forms of discrimination and persecution, besides to plan and perpetrate eugenic and/or extermination programs. However, the horizons of historical research regarding the correlation between mathematical, statistical, biological, medical, psychiatric sciences and racisms did not end with the Short Century and have been progressively extended to the short and long term, in a perspective of connected history, taking into account social Darwinism, the relationships racism-colonialism, science-apartheid, etc.

[88] i.e. *For a didactics of the Shoah in schools.*

## 2 *A response to local needs*

Our project fits in a well-defined context, that of Piedmont, whose chronicle has recently recorded many serious acts of racism and anti-Semitism: writings on walls, drawings of swastikas, insults of despicable violence, which bring back in vogue the most forbidden Nazi-fascist stereotypes. These are intolerable facts for a society that wants to be inclusive, free, and democratic, and even more for a city like Turin, Gold Medal of Resistance. The perception that these are not isolated episodes and the consequent concern about the surfacing of forms of hatred that make the past dramatically close, are reflected in the statistical data: from January 1, 2018, to the end of February 2019, 118 acts of racism were reported in Piedmont, 36% of which in Turin and its metropolitan area. In 8% of cases, they occurred in a school context. On the other hand, according to *IRES Piemonte* annual report for 2020, only 13.67% of the population considers racism and other forms of discrimination to be of concern and, due to Covid-19 emergency, this percentage dropped to 9.5% in 2021. In the same year, however, there was a 13% increase in acts of intolerance compared to the previous year. In our region – but also in Italy in general – it is regrettable to note the scarcity of proposals and materials concerning STEM disciplines, whose ancient and recent history can make a valid contribution to a deeper understanding of this issue. This is even more regrettable considering that the Piedmontese scientific community was among those most affected by the racial laws of 1938 (Capristo, 2014; Luciano, 2018). Hence, came about the idea of the Research Group in History of Mathematics at the University of Turin (coord. by prof. Luciano) to design educational activities and training courses to sensitize teachers and students to Memory and awareness, starting from the critical re-reading of some aspects and moments of research and teaching of mathematics and science during the fascist dictatorship and motivating the direct involvement in research of sources and investigation of facts. Delving into the historical past, studying and analysing it with the typical lucidity of scientific thought and logical-deductive argumentation is a significant operation also in relation to our present and to the multi-identity and multi-ethnic society in which we live.

### 3 *Educational activities*

In the school years 2020-21 and 2021-22 three schools were involved in the project. During the first year, students (8th degree) of the lower secondary school “U. Foscolo” in Turin delved into the personal and professional trajectories of scientists who were victims of racial persecution and experienced the effects of state anti-Semitism, reading and commenting on the correspondence of that ‘dark’ period. In the following year, they focused on the impact of racial laws in Italy on the lives of female mathematicians and scientists. The fates of women examined were divided into three macro-scenarios: emigration, permanence in Italy in hiding, and deportation to concentration camps. In both cases, to make their research accessible to a wide public, students created a freely navigable multimedia content which was published on the school website on the Holocaust Memorial Day.[89]

Two different paths were taken by the two high schools that took part in the project. At the IIS “Santorre di Santarosa” in Turin, three classes (11th-12th degree) with biochemistry study address opted to focus on the eighteenth-century debate on polygenism and racism and on the instrumental use of mathematics (statistics, demography, ...) and science (biology, anthropology, medicine, ...) to justify colonialism, racism and anti-Semitism, in modern and contemporary times.[90] In addition, some students were involved in deepening the figure of three scientists who lived during the Nazi-fascist period, chosen in relation to their address of study: J. Mengele, E. Segrè and R. Levi Montalcini. Another class began with the history of the school: its building is one of the finest examples of Fascist-era architecture in the city of Turin, with a Littoria Tower identical to that of the city’s central square. Starting from the local context, they analysed the effects of the racial laws on the city of Turin with special attention to scientific faculties.

At the Liceo “G. Peano” of Cuneo, the Mission Memory path was pursued: teachers and students rediscovered personal and professional trajectories of their colleagues who were persecuted for racial reasons, with particular focus on the partisans struggle in the Langhe. After having listened to the testimony

---

[89] The final products are available online at the links below.
https://www.icfoscolo.org/wp-content/uploads/2021/01/MOOC_Giornata-della-Memoria.pdf.
https://www.icfoscolo.org/wp-content/uploads/2022/01/Scienziate-ebree_MOOC.pdf

[90] To consult their work, see https://www.dropbox.com/sh/17xeqssf4dfgvse/AABPWWTa-HqxMl63gyUkGzuHa?dl=0.

of the Montanari family, one of the oldest Jewish families in Cuneo, and having discussed with the scholar of Judaism A. Cavaglion, they privileged the historical events linked to the territory. Older students (13$^{th}$ degree) wanted to pay tribute to two of the many scientists of Jewish origin who brought fundamental developments to their disciplines and were affected by the consequences of racial laws: V. Volterra and T. Levi-Civita. Starting with archival research, younger students (11$^{th}$ degree) focused on the reconstruction of the biography of Ugo Levi, who taught mathematics and physics in Saluzzo in the historical period considered. They also interviewed some of his former students (B. Segre, an engineer, and A. Bosi, in turn then professor of history and philosophy at the same school) to learn more about his personality as a teacher and how he was affected by the anti-Semitic wave that swept through Fascist Italy. At the end of this path, a web page was created, containing the research and reflections of the students, with the aim of disseminating the results and to give back to the territory a piece of its history (https://liceocuneo.it/progetto-pls/). This project was likewise an opportunity for a rapprochement between retired and current teachers and students and for a rediscovery of their roots.

## *4 Conclusive remarks*

Despite the limits of a temporally limited intervention, positive results in terms of appreciation and involvement of schools indicate the usefulness of continuing the path of Shoah education also from the perspective of history of STEM. According to the feedbacks of teachers and students, some aspects are particularly significant. Firstly, this kind of awareness-raising to the Memory of the Shoah can ensure a more inclusive and interdisciplinary didactic approach and can be declined in many ways and at different school levels. At the same time, it contributes to promote social integration and cultural inclusion of students from fragile backgrounds: indeed, reflecting on past forms of discrimination can help to avoid repeating the same mistakes in the present.

Secondly, these experiences reveal the effectiveness of dealing with historical sources: beyond their intrinsic importance, they captivate students, thus facilitating a major level of commitment. The analysis of historical documents was appreciated by the students as they 'got closer' and empathized with those who experienced first-hand the effects of state anti-Semitism and, from these readings, developed some reflections on today's context, actualizing what they learned.[91] Moreover, such critical reading helps to make mathematics and science living subjects in students' eyes.

Lastly, educating for tolerance is notably significant relating to civic education: learning to appreciate other people and their differences is one of the key skills of the rising generations of citizens. To this end, history of STEM becomes a resource, a vehicle for improving students' consciousness: "history is something that can make us aware of who we are, and how we have come to be the individuals that we are" (Radford, 2014, p. 89).

The history of mathematics and STEM disciplines, hitherto little involved in the field of antiracist education, can therefore provide an important means of combating ignorance, with the goal of defeating racial discrimination and intolerance, and can contribute to identify appropriate antiracist resources to incorporate into school curricula. We hope that the ideas, insights and experiences illustrated can form a basis for new educational actions in this direction, which are especially needed in our society, crossed by currents of racial hatred and contempt for diversity.

## REFERENCES


Capristo, A. (2014). *Gather What You Can and Flee. Jewish Intellectual Emigration From Fascist Italy*. New York: CPL Editions.

Castelnuovo, E. (1997). Federigo Enriques e Guido Castelnuovo nel ricordo di Emma Castelnuovo. *Bollettino dell'Unione Matematica Italiana*, s. 7, *11*-A, 227–235.

Corbellini, G. (2020). Serve la genetica per prevenire il razzismo a scuola. *Scienza in rete*. https://www.scienzainrete.it/articolo/serve-genetica-prevenire-razzismo-scuola/gilberto-corbellini/2020-02-05. Accessed 22 September 2022.

Israel, G., & Nastasi, P. (1998), *Scienza e razza nell'Italia fascista*. Bologna: Il Mulino.


[91] In this regard, some student comments are given here: "In the letters you can see that Jewish scientists tried to find work even in a place far away from where they worked, taking their family with them, because they were thinking about their children and their future."; "It is important to understand that it doesn't matter if you follow a different religion or speak a different language from others because we are all equal."; "What happened to the Jews certainly must not happen again because we have to think about what they had to suffer and that there were too many victims who died unjustly.".

Luciano, E. (2018). From Emancipation to Persecution: Aspects and Moments of the Jewish Mathematical Milieu in Turin (1848-1938). *Bollettino di Storia delle Scienze Matematiche*, *38*(1), 127–166. DOI:10.19272/201809201005.

Radford, L. (2014). Reflections on History of Mathematics. In M. N. Fried & T. Dreyfus (Eds.), *Mathematics & Mathematics Education: Searching for Common Ground* (pp. 89–109). New York: Springer. DOI:10.1007/978-94-007-7473-5_7.

Rutherford, A. (2020). *How to Argue with a Racist: History, Science, Race and Reality*. London: Weidenfeld & Nicolson.

Segre, L. (2018, February 19). *Incontro con Liliana Segre al Teatro Ponchielli di Cremona* [video]. YouTube. https://www.youtube.com/watch?v=bX5BA2-V5lQ. Accessed 22 September 2022.

# “MODERN MATHEMATICS” IN ITALY

**Luigi TOMASI, Adriano DEMATTÈ**
Centro Ricerche Didattiche “Ugo Morin”
Via S. Giacomo 4, Pieve del Grappa (TV), Italy
luigi.tomasi@unipd.it, adrdematte@gmail.com

## ABSTRACT

This contribution deals with the history of the attempts to introduce the so-called "Modern Mathematics" into teaching in the 1960s. After a short presentation of the global situation, we focus on the case of Italy. We will present the development of the phenomenon and focus on the opinion of teachers and mathematicians who were involved in the debate. We will mention some of them and we will focus on why "Modern Mathematics" in Italy has had a limited diffusion, except for some aspects.

## *Introduction*

The term "Modern Mathematics" is intended to indicate the new approach to mathematics based on set theory and algebraic, order and topological structures. In the intentions of the proponents, mathematics, even at school, had to be transformed from a study on entities to a study of the properties common to these entities, according to the proposals coming from the "Bourbaki Group", a collective of mainly French mathematicians who, starting from 1930s, proposed to found the whole mathematics on algebraic, order and topological structures. Their main work was *Éléments de mathématique* [Elements of mathematics] (in eleven volumes, 1939-2016, about 7000 pgs.). In other countries, primarily the United States, similar, albeit different, movements are labelled "New Math", see (Furinghetti & Menghini, 2023a).

## *1. “Modern mathematics”*

The OECD (Organisation for Economic Cooperation and Development) Royaumont (France) Seminar, November 1959, is a milestone of the reform of European “Modern Mathematics". As reported in (Furinghetti & Menghini, 2023a, p. 59) in short, “the theme of the Seminar was not only the need for new thinking in both mathematics and mathematics education − including changes in curricula and teacher training − but also the development of appropriate follow-up action (OEEC, 1961). Conviction of some participants was that introduction of new topics should facilitate the study of abstract algebra, analysis, applications of physics and other sciences in university courses.

In his talk delivered at the Royaumont Seminar, Jean Dieudonné - a leading exponent of the “Bourbaki Group” – launched his famous motto "*À bas Euclide! Mort aux triangles!*". Dieudonné's provocation is emblematic, meaning the outdatedness of traditional teaching and in particular of Euclidean geometry.

Some members of the CIEAEM - Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques [International Commission for the Study and Improvement of Mathematics Teaching] took part in this conference. After his Royaumont lecture, in 1964 Gustave Choquet published *L’enseignement de la géométrie* (Italian translation in 1967).

At Royaumont only the general lines of a clear separation from tradition were laid. The following year (August 1960) in Dubrovnik (now Croatia) the concrete indications for the teaching of mathematics in lower and upper secondary schools were drawn up. In October 1961, the OECD published the "Dubrovnik Program" with the title *Un programme moderne de mathématiques pour l’enseignement secondaire* [A modern mathematics curriculum for secondary education].

In the same years, Jean Piaget's theories on the close correlations between the construction of the child's mental structures and mathematical structures were disclosed. This very concise reference to Piaget’s thought is to remind his participation in the debate stated, for instance, by his vice-presidency of CIEAEM.

The new programs of maths were applied in the secondary schools, first in France and Belgium. Almost immediately, the movement of modern mathematics was criticized for the excessive formalism and the claim to replace "traditional" mathematics, especially Euclidean geometry, with "modern" mathematics since the earliest grades of elementary school. Famous mathematicians, such as René Thom (1923-2002) and Hans Freudenthal (1905-1990), have expressed criticisms. Thom (1970) argued that we shouldn’t privilege the formal rigor, but the meaning of the mathematical objects involved, because children do not base their thinking on abstract structures that pre-exist in their mind, but rather on concrete experiences.

## 2 *The teaching of mathematics at the beginning of the 1960s in Italy*

At the beginning of the 1960s, universities in Italy began to update their courses according with the new proposal from ‘beyond the Alps’ introducing, for instance, abstract Algebra in the first year of the degree course in mathematics. On the contrary, primary and secondary school education remained rather traditional, despite the various proposals for renewal after institution of Unified Middle School (for pupils aged 11-14) in 1962; see (Linati, 2016).

Several Italian mathematicians and teachers participated in international seminars. Luigi Campedelli (Florence University) and Emma Castelnuovo (Middle School "Tasso", Rome) were the Italian delegates at the OECD Royaumont Conference (1959). Mario Villa (Bologna University) and Ugo Morin (Padua University) were present at the OECD Dubrovnik meeting, 1960.

The International Commission on Mathematical Instruction (ICMI) and the Commissione Italiana per l’Insegnamento Matematico (CIIM) [Italian commission for mathematical teaching] organized an international conference in Bologna from 4 to 8 October 1961 on the teaching of modern mathematics in secondary schools. Following the proposals that emerged at the Bologna Conference, a course for teachers was organized, and 40 "pilot classes" were set up in 1962-63. It was an experimentation of a "Bourbakist" type of teaching: see (Villa, 1962). For this experimentation, a group of mathematicians wrote the school textbooks: Mario Baldassarri, Ugo Morin, Mario Villa, Luigi Campedelli, Tullio Viola among others. We would like to highlight (Morin & Busulini, 1962). The 40 "pilot classes" took place in upper secondary schools (lyceums and institutes for prospective primary teachers). The experimentation generally involved the penultimate grades of secondary school, even if up to that point the students had already been trained on completely different topics and methods (Ciarrapico & Berni, 2017). Vita (1986) notes that the outcomes of the "pilot classes" were never collected and publicized. He claims that the failure of that experience was due to the lack of a global plan.

## 3 *‘‘Modern mathematics“ and renovation of Italian teaching*

Also Italian mathematicians and teachers criticized the pedagogical proposal of the "modernists", assuming a position that was in line with the most part of opinions at international level (Fabris, 2021; Furinghetti & Menghini, 2023b). Some mathematicians and teachers elaborated their alternative proposals.

Of the two Italian participants in Royaumont Seminar, Luigi Campedelli (1903-1978), president of CIIM in 1964-1972, showed a certain detachment, or at least caution, regarding modern mathematics (Furinghetti, 2019).

Emma Castelnuovo (1913-2014) tried to follow the new guidelines on teaching modern mathematics, adapting the contents to the skills and competences of middle school students. She believed it necessary, however, to introduce the concepts of modern mathematics in a natural way as soon as the opportunity arose. She opted for a dynamic and active didactic, based on movement, on borderline cases, on intuitive infinitesimal reasoning, on simple models and on the presentation of counterexamples. She was author of school textbooks, promoter of mathematics exhibitions, and author of publications for teachers' training. Bruno de Finetti (1906-1986) proposed a genetic method, antithetical to the axiomatic one of modern mathematics. He agreed about a radical simplification and revision of the mathematical tools and advocated a unified vision of different mathematical topics. He considered "Euclid's geometry so unnatural and heavy because of the lack of distinction between affine and metric properties" (de Finetti, 1965, p. 124).

The attempt to introduce modern mathematics into primary and secondary schools, although essentially unsuccessful, has had the positive effect over time of stimulating a rethinking of mathematics teaching and the subsequent development of projects. We would like to mention, in particular, the "Prodi Project" (proposed and coordinated by Giovanni Prodi, 1925-2010) and his volumes of *Matematica come scoperta* [Mathematics as a discovery] (published between 1975 and 1982). Other important textbooks: Lucio Lombardo Radice and Lina Mancini Proia, *Il metodo matematico* [The mathematical method], 1977; Francesco Speranza and Alba Rossi Dell'Acqua, *Il linguaggio della matematica* [The language of mathematics], 1979; Walter Maraschini and Mauro Palma, *Problemi e modelli della matematica* [Problems and models of mathematics], 1981; Bruno Spotorno and Vinicio Villani, *Matematica: idee e metodi* [Mathematics: ideas and methods], 1982. About these textbooks, see (Tomasi, 2012). We recall the theme on "Correspondences and structural analogies" in the ministerial programs for the middle school of 1979. It was introduced with the following words: "[It] will not give rise to a separate discussion. During the three years, whenever the opportunity arises, similarities and differences between different situations will be recognized" (our translation). This statement well summarizes the orientation in Italian mathematics education in the seventies and eighties. In the primary school, it became usual to start by introducing logical concepts and operations on sets (including intersection, union, complement, …) and only later to introduce the concept of natural number. A project coordinated by the University of Pavia included activities linked to logic and set theory for the introduction of connectives, or activities on relations between sets. Some teachers attempted integration between different approaches, also by using materials such as Dienes logic blocks (Furinghetti & Menghini, 2023b).

## *4 Conclusion: "Modern Mathematics", what remains?*

What is left of "Modern Mathematics" in the 1970s and in subsequent years, in Italy? It can be observed that today there is a large gap between mathematics presented in secondary school and that proposed in university teaching. Provocatively, we can say that mathematics taught in secondary school is still stuck, for some aspects, to a century ago! Instead, in the universities the new topics find a lot of space.

The most widespread Italian textbooks report specific parts dedicated to geometric transformations. There is also a partial reorganization of the proposal of traditional themes: e.g., the tracing of curves obtained with translations and symmetries of elementary curves with respect to the Cartesian axes. Some schools promote supplementary courses on the topics such as algebraic structures and topology. We recall the experience of the "Mathematical Lyceums" in which the teaching of mathematics is enhanced also through the contents proposed by the modern mathematics.

Main merit of the debate on the modern mathematics is to have questioned the tradition. In the programs for Primary School (1985), it is stated that "logic and theory of sets are no longer the foundation of mathematics, but a means to analyse the mathematical discourse and guide its development" (Pellerey, 1989, p. 176). Giovanni Prodi recognises that "Undoubtedly today there would not be such a widespread awareness of the unity of mathematics if Bourbakism had not existed" (Prodi, 1995, p. 416). Summing up, the phenomenon of "Modern Mathematics" appears to be a booster of the change that has taken place in the Italian school and a promoter of international openings.

## REFERENCES


Ciarrapico, L. & Berni, M. (2017). *I curricoli di matematica, gli ordinamenti scolastici e le riforme dal 1940 al 2015*, Edizioni UMI.

De Finetti, B. (1965). Programmi e criteri per l'insegnamento della matematica alla luce delle diverse esigenze. *Periodico di Matematiche*, s. 4, 43, pp. 119–143.

Fabris, S. (2021), *L'introduzione delle cosiddette «matematiche moderne» in classe*, Thesis (supervisor A. Fiocca), Ferrara University.

Furinghetti, F. (2019). Riforme internazionali e innovazioni nazionali: Luigi Campedelli quarto presidente della CIIM dal 1964 al 1972. *Matematica, Cultura e Società. Rivista dell'Unione Matematica Italiana*, s.1, 4, pp. 269-288.

Furinghetti, F. & Menghini, M. (2023a). The Royaumont Seminar as a booster of communication and internationalization in the world of mathematics education. In D. De Bock (Ed.), *Modern mathematics: An international movement?* (pp. 55-78). Cham: Springer.

Furinghetti, F. & Menghini, M. (2023b). Modern Mathematics in Italy: A difficult challenge between rooted tradition and need for innovation. In D. De Bock (Ed.), *Modern mathematics: An international movement?* (pp. 147-168). Cham: Springer.

Linati, P. (2016). *L'algoritmo delle occasioni perdute*. *La matematica nella scuola della seconda metà del Novecento*. Trento: Erickson.

Morin U. & Busulini F. (1962). *Elementi di Geometria*. Padua: Cedam.

OEEC (1961). *New thinking in school mathematics*. Paris, France: OEEC. French edition: *Mathématiques nouvelles*. Paris, France: OECE.

Pellerey, M. (1989). *Oltre gli insiemi*. Naples: Tecnodid.

Thom, R. (1970). Les Mathématiques "modernes"; une erreur pédagogique et philosophique? in *L'âge de la science*, vol. III, n# 3 (pp. 225-242), Paris: Dunod.

Tomasi, L. (2012). Alcune linee di storia dell'insegnamento della matematica nella scuola secondaria in Italia dal 1945 ad oggi, *Annali online della Didattica e della Formazione Docente,* vol.3, n. 3 (pp. 106−121), Università di Ferrara.

Villa, M. (1962). *Per un insegnamento moderno nelle scuole secondarie*. Bologna: Pàtron.

Vita, V. (1986). *I programmi di matematica per le scuole secondarie dall'unità d'Italia al 1986 (rilettura storico critica)*. Bologna: Pitagora Editrice.

# ISSUES RELATED TO THE TEACHING OF MATHEMATICS AT THE "STUDIO PADOVANO" IN MID-1800S

**Giuseppe CANEPA**
I. T. I. Galileo Galilei, Genova
canepa.giuseppe@ecg-genova.edu.it

## ABSTRACT

The developments in mathematics in the decades around mid-19th century led some professors of mathematics and science from the University of Padua to ask themselves significant questions about the teaching and learning methods of their disciplines. The scholars of the "Studio Padovano", name of the faculty of philosophy and mathematics at the University of Padua, to which we will refer, are: Giovanni Santini, Giusto Bellavitis, Domenico Turazza, and some others. They worked on developing ideas and hypotheses on a new legislation related to teaching and on finding the most appropriate ways to influence decisions about those matters at the institutional level. They also worked on drafting educational projects for original approaches to their teaching subjects and their actual implementation during lectures with students. From the pedagogical-educational point of view, the steps to obtain a good transmission of knowledge to the new generations were the subject of a debate (through an exchange of correspondence) and also were the motivation for the production of appropriate publications on the subject. The analysis of this material, still partly handwritten and unpublished, illustrates a complex moment in which these scholars already acted as if they belonged to the very same state, although still under the Austro-Hungarian legislative sphere. The study of this material can also provide interesting points of comparison with the problems that characterized the scientific activities of subsequent periods.

## *Introduction*

This talk refers to a set of historical materials and its intention is to offer insights that could be useful for teacher education and cultural enrichment (Boero&Guala, 2008). The material that has led us to an in-depth study, related to these issues, comes from a long historical tradition. It represents both an interesting reference for teacher training and an input for a more articulated conception of teacher's role within educational institutions.

In the material presented, historical and educational issues of the 19th century are linked to highlight aspects that are still very relevant today: The views of the three mathematicians examined present interesting observations and explicit comments related to their teaching practice. All of this clearly resonates with today's issues and meanings. The material can therefore be a useful comparison gymnasium for today's teachers; for example, for the following content: studies on the level of abstraction of the content of the subjects taught with reference to the age of the children; the teacher's choice of whether to propose topics to the pupil in a synthetic or analytical manner, whether to start from the particular or the general, whether from practical problems to arrive at theoretical abstraction or vice versa.

## *1 The protagonists and their teaching texts*

The development of the University of Padua is closely related to that of the Astronomical Observatory, the Istituto Veneto di Scienze, Lettere e Arti, and the Accademia Patavina di Scienze Lettere e Arti. Simultaneously these active institutions lived off the work of the same protagonists who exchanged roles and assignments within the various structures. A number of texts, including recent ones, have been published on the subject (Borgato&Pepe, 2011; Casallato&Pigatto, 1996), and there is a considerable number of manuscripts rich in information in: the Bellavitis Fund of the Istituto Veneto di Scienze, Lettere e Arti, Venice; “Cassetta Loria”, Genoa University Library; “Istituto Mazziniano”, Genoa; other Libraries in Piacenza, Naples, Bologna. A detailed study of the texts in adoption by the following professors: G. Santini, C. Conti, P. Maggi, S. R. Minich, D. Turazza, G. Bellavitis, showed a gradual transition from texts by other universities to handwritten handouts of the professors to end with texts published by the professors themselves, rich in topics that took into account European developments in mathematics and science.

The three figures, interesting for their sensitivity and study on teaching, that we take into consideration in this paper are: Giovanni Santini, Domenico Turazza, and Giusto Bellavitis.

They all owed a lot to Santini (1787-1877) who transferred to them the style of teaching, esteem and trust with the students. He was also a rare school leader for all of them because of his style in writing treatises for study. Santini had been student in Pisa and at the Brera observatory from 1806, he became Rector of the University of Padua twice. He was a member of 21 academies, both Italian and foreign and published works, mainly on astronomy, known and praised throughout Europe. The work we are interested in, highly praised when it appeared and still an example of scientific literature, is *Elementi di*

*Astronomia* (*Elements of Astronomy)* (Santini, 1819). The first edition dates 1819, the second is from 1830, and was conceived by Santini when he was given the chair of astronomy. He wrote:

*I meditated for a long time about which was the most convenient treatise for the education of young people [...] I therefore applied myself to a course of lessons which, neither too elementary nor too sublime, allow to easily read the […] astronomical works published in the Ephemerides (Santini 1830, p.III).*

As a general purpose he states:

*It has seemed to me expedient to stick to the plan of going back from combined astronomical observations and discussed with each other to the cognitions of the laws, which the celestial bodies obey with such regularity (Santini 1830, p.V).*

Domenico Turazza (1813-1892) graduated in mathematics in Padua, taught Descriptive Geometry in Pavia and then Geodesy and Hydrometry in Padua, director for life of the School of Application for Engineers and finally Senator of the Kingdom. He was a member of more than thirty academies and scientific societies. He published 90 works mainly scientific but also literary.

Turazza wrote eight texts dedicated to teaching, two of them for high school education. We consider here the text *Trattato di idrometria ad uso degli ingegneri* (*Treatise on Hydrometry for the Use of Engineers)*. In the preface he states:

*The desire to place in the hands of my young students a book, which, beside serving them as a guide in the ordinary course of my lessons, would also return to them as a benefit in the practical exercise of this difficult part of their profession, was an incitement to me (Turazza 1835, p.3).*

But what was his idea of hydraulics?

*Anyone who has set his mind to the study of hydraulics, will have of light noticed that that part of it, which more properly belongs to practice, receives almost no support from general theories, (Turazza 1835, p.3).*

He goes on to explain that all the theoretical researches made, in which he took part, are but mere hypotheses, from which practice can derive no benefits. The dozens of authors covered in this text, both Italian and foreign, make this text a comprehensive compendium for knowledge on the subject at the time. Turazza was one of the leading European hydraulic engineers of the nineteenth century. Antonio Favaro, one of the prosecutors of the Paduan school of which we speak here, stated on his Master:

*In all the scientific works to which I have alluded and in the others I will still touch, he [Turazza] appears mainly and above all a Master: teaching is always his supreme aim, and if at times he seems to deviate and depart from it, he returns to it very soon and for the deviation he shows discontent and repentance (Favaro 1892, p.7).*

The third and final figure we discuss is Giusto Bellavitis (1803-1880). Regarding this mathematician we have at our disposal a great deal of manuscript material in addition to his numerous publications. Although he did not follow a regular course of study, in fact he was self-taught, he dedicated much of his life teaching and studying how to teach best. He was inspector of the Scuola Reale Superiore in Venice, a member of the Society of XL and of Accademia dei Lincei. From 1866 he was a senator of the Kingdom of Italy. He published 223 papers, many devoted to education in general and many to the specific teaching of various subjects; he published the *Rivista di Giornali* containing many articles on school and university. Among the publications, the *Lectures on Descriptive Geometry with Notes* of 1851 represent an important example of the application of Bellavitis' theories on teaching. The preface proposes:

*Of every problem I first gave the general solution in words only and without the aid of figures; with this I tried to exercise the mind and imagination of the scholar, and also wanted to avoid that kind of teaching for particular cases (Bellavitis 1851, p. VII).*

For the author, writing has drawing as its big sister, the purpose of Descriptive Geometry is to give rules and principles to drawing, solving problems of the geometry of space or even representing a three-dimensional object in a plane. This is the first didactic work in Italian to deal with topics that for the time were recent, set forth in German and French texts; thus the author had to create neologisms. Topics range from definitions of geometric entities to intersections of surfaces, from contacts (tangential, etc.) to curvatures of lines and surfaces. Of these lectures we also have the handwritten version, which the mathematician used in his lessons.

Bellavitis describes the design, the way to make it, and only later proposes its algebraic-analytic solution. We recall that the manuscripts binder also contains documentation on exam questions, student papers, evaluations and considerations of the results obtained by the teacher.

## *2 A 100-year-long article*

Bellavitis in the 1853 work *Riflessioni sull'istruzione pubblica* (*Thoughts on Public Education)* (Bellavitis 1853) clearly shows us his position towards learning in general, the problems of the time and the solutions he would implement. He deals with education from the earliest age to university studies. Before we analyze some essential points, still strongly relevant today, we report here an emblematic sentence by Bellavitis in a letter dated June 22, 1860, (Mazziniano Museum in Genoa, Cremona-Cozzolini Fund) to the young Luigi Cremona who would shortly start the teaching of geometry, "you will need to remember that profit is measured not by the much taught but by the little that is well learned".

With the principle that the path of all teaching should go from what is already known to what has to be learned, the author proposes a division of studies by age: 6-10 years old, compulsory general studies where one learns to read, write, count; the main subjects are: arithmetic and Italian language; from age 10 to 15, general higher studies (principles of letters and science to which no one of civilized condition can be ignorant); from age 15 to 18, studies for start-up to university (with freedom of final choice in university career) or studies for ecclesiastical status or for the various professions. A futuristic vision, since in the mid-nineteenth century a high percentage of the population of Lombardo-Veneto and the rest of Italy was illiterate. Regarding textbooks, especially for the first cycle of studies, Bellavitis sets out his concerns:

*It cannot be hoped that in elementary education teachers will know the best way to teach, nor can this be learned with some pedagogical lessons; therefore it will not infrequently happen that a teacher, taking the narrowness of his ideas as the norm, judges pedantry the convenient development of the subjects of a text established for the instruction of young minds (Bellavitis 1853, p.127).*

In the 1853 work, there are also considerations of examinations to be held each year with the own teacher and at the end of each course examinations also with teachers of the next course; it is important establishing a merit ranking and exams for admission to new courses; to introduce possible honorary rewards as a mean to achieve results. In that period, for the teaching of geometry, there was a great debate concerning the effectiveness of the Euclidean method. The discussions led to numerous articles and school texts that went in this direction. Bellavitis exposed his thinking in these terms:

*On the subject of the theoretical teaching of geometry I do not share the opinion, on the other hand very respectable, that the method of Euclid must be followed, and that it is useful to exercise the scholar in a series of linked arguments; I would prefer to make the road as flat and easy as ever possible (Ibid., p.155).*

This work is a clear representation of the school situation of the time and Bellavitis' plans for education. It would take Italy more than a century to implement certain ideas. The mathematician was also asked by the University of Padua to draw up a plan for restructuring the teachings of the Athenaeum. He set it out in a lengthy manuscript also found in the papers of the Veneto Institute's archives, where all the files that go into the merits of the contents of all the teachings are also kept.

## *Conclusions*

We can speak of a Paduan school because we identify a group of mathematicians who were university professors in Padua, friends, members of institutes and academies in Veneto, in continuous correspondence. The studies and proposals for education, of the discussed authors, showed up in part in the laws of united Italy (the exchange of letters with L. Cremona, F. Brioschi, E. Betti and others is interesting in this area). The dimensions of the dynamics related to the teaching of mathematics and in particular geometry, at that time in Padua, are on a European level. Regarding this, we recall that much of the mathematics taught today in high schools and universities took on its current appearance during that historical period. Bellavitis asks himself which is the best way to recruit teachers, what characteristics they should have and how to identify them. It is interesting to note how there is a cyclicality that becomes fashionable, in the various historical periods, concerning the curvature that the teachings should take; for example, Euclidean geometry has popped up three times in the last two centuries, as well as the importance of the ancient languages Greek and Latin as the basis for everything else in knowledge. In a less obvious way this occurs in all subjects. This depends on the alternating sensitivity of human groups compared to the perception of the concrete world and its mental representations.

## REFERENCES

Bellavitis, G. (1851). *Lezioni di geometria descrittiva con note*. Padova: Seminario.

Bellavitis, G. (1853). Pensieri sull'istruzione pubblica.*Atti Istituto Veneto di Scienze Lettere ed Arti*, 119-161.

Boero, P., &Guala, E. (2008). Development of mathematical knowledge and beliefs of teachers: The role of cultural 400 analysis of the content to be taught. In *The International Handbook of Mathematics Teacher Education*, Sullivan P., Wood T. (Eds.). Rotterdam: Sense, *1*, 223-244.

Borgato, M.T., & Pepe, L. (2010-2011). Accademie, Istituti, Società scientifiche e ricerca matematica in Italia nel XIX secolo.*Atti Istituto Veneto di Scienze Lettere e Arti*, *169*, 107-124.

Casellato, S., & Pigatto, L.(1996). *Professori di materie scientifiche all'Università di Padova nell'ottocento*. Trieste:Lint.

Favaro, A. (1892). *Della vita e delle opere del senatore Domenico Turazza*. Padova:Randi.

Santini, G. (1830). *Elementi di astronomia*. Padova: Seminario.

Turazza, D. (1845). *Trattato di idrometria ad uso degli ingegneri*. Padova:Seminario

Finito di stampare nel mese di giugno 2023
con tecnologia *print on demand*
presso la tipografia The Factory Srl
per conto di Edizioni Nuova Cultura
p.le Aldo Moro, 5 - 00185 Roma
www.nuovacultura.it

per ordini: ordini@nuovacultura.it
[Int_ 9788833656014_17x24bn_BM02]