History and Epistemology in Mathematics Education

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PREFACE

This volume contains texts and/or abstracts of all contributions to the scientific programme of the 6th *European Summer University* (ESU 6) on the *History and Epistemology in Mathematics Education*, which took place in Vienna, from 19 to 23 July 2010. This was the sixth meeting of this kind since July 1993, when, on the initiative of the French IREMs[1] the first *European Summer University on the History and Epistemology in Mathematics Education* took place in Montpellier, France. The next ESU took place in Braga, Portugal in 1996, conjointly with the *HPM*[2] Satellite Meeting of ICME 8), the 3rd in Louvain-la-Neuve and Leuven, Belgium in 1999, the 4th in Uppsala, Sweden in 2004, conjointly with the *HPM* Satellite meeting of ICME 10 and the 5th in Prague, Czech Republic in 2007.

Since its original conception and realization, ESU has been developed and established into one of the major activities of the HPM Group. Its purpose is not only to stress the multifarious role that history and epistemology can play in the teaching and learning of mathematics, in the sense of a technical tool for instruction, but also to reveal that mathematics should be conceived as a living science, a science with a long history, a vivid present and an as yet unforeseen future.

This conception of mathematics and its teaching and learning is reflected into the main themes along which the scientific program of each ESU is structured. This time, they were as follows:

1. Theoretical and/or conceptual frameworks for integrating history in mathematics education
2. History and epistemology implemented in mathematics education: classroom experiments & teaching materials, considered from either the cognitive or/and affective points of view; surveys of curricula and textbooks
3. Original sources in the classroom, and their educational effects
4. History and epistemology as tools for an interdisciplinary approach in the teaching and learning of mathematics and the sciences
5. Cultures and mathematics
6. Topics in the history of mathematics education

Publishing the Proceedings of the ESUs has always been a major task, since in all cases they have become standard references in this domain[3]. In addition, it has been decided that the Proceedings is published after ESU 6, so that authors are given the opportunity to enrich their text as a result of the feedback they would gain during this European Summer University. As a consequence, this volume is divided into six parts that correspond to the six main themes mentioned above. It includes full texts and/or abstracts of the 91 contributions to the scientific program of ESU 6. In particular, full texts have been submitted for 61 out of the 91 contributions to the ESU 6 program and 52 of them were finally accepted, including 6 plenary lectures and 2 panel discussions. Each submitted full text for a workshop, or an oral presentation has been reviewed by one or two members of

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[1] *Institut de Recherche sur l'Enseignement des Mathématiques*.

[2]The International Study Group *on the Relations between the History and Pedagogy of Mathematics*, affiliated to ICMI.

[3]The proceedings are available online from the HPM websites http://www.clab.edc.uoc.gr/hpm/ and http://grouphpm.wordpress.com

the Scientific Program Committee at the usual international standards. In most cases authors were asked to amend their papers. Papers that have been finally accepted are included here. In all other cases in which either the text was not accepted, or no full text has been submitted, only an abstract of the corresponding contribution appears. In addition, abstracts for poster contributions and short communications are also included.

For each main theme, one plenary lecture was delivered and its text appears in the corresponding section. The same holds for the two panel discussions, which were also delivered in plenary sessions. There are also papers coming from workshops, which are a type of activity of special interest, making focus on studying a specific subject and having a follow-up discussion. The role of the workshop organizer was to prepare, present and distribute the historical/epistemological (3-hour workshops) or pedagogical/didactical material (2-hour workshops), which motivated and oriented the exchange of ideas and the discussion among the participants. Participants read and worked on the basis of this material (e.g. original historical texts, didactical material, students' worksheets etc). The reader of these Proceedings will find here historical resources, like abstracts of original texts, and pedagogical resources for all levels of mathematics education, from elementary school to the university. Finally, there are texts and abstracts based on 30-minute oral presentations, short communications and poster contributions.

There were 152 contributors and participants from 28 different countries worldwide. They were secondary school teachers, university teachers and graduate students, historians of mathematics, and mathematicians, all interested in the relations between mathematics, its history and epistemology, its teaching, and its role at present and in the past. We thank all of them. Special thanks go to the 29 members of the International Scientific Program Committee, (see p. 699), who willingly reviewed the submitted papers, thus contributing essentially to the scientific quality of this volume, and all members of the Local Organizing Committee (see p. 700), who succeeded to make ESU 6 an insightful and interesting scientific event that took place in a warm and friendly atmosphere. We also thank the personnel of the Vienna University of Technology for their help and kindness. Finally, we thank all institutions which, in one way or another supported the organization of ESU 6: The Institute of Discrete Mathematics and Geometry of the Vienna University of Technology, Vienna, Austria, for hosting ESU 6, the Austrian Federal Ministry of Science and Research (BMWF), the Government of the City of Vienna (Wien Kultur), the Vienna Convention Bureau, Casio Europe and Texas Instruments for financial support of the meeting and the publication of its proceedings.

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Theoretical and/or conceptual frameworks for integrating history in mathematics education

HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION:

Problems and Prospects

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ABSTRACT

Where educators are committed to goals determined by the need to teach students modern mathematics or mathematics as it is used in modern scientific or technological contexts, history of mathematics may be forced to serve aims not only foreign to its own but even antithetical to them. This "conflict of interest" must be confronted if one wishes to embrace the history of mathematics as something more than another tool in the mathematics educator's arsenal, but as an inquiry important in its own right. This paper will suggest that this may involve redefining our general goals for mathematics education more than the specific concrete ways in which we bring history of mathematics into the classroom. To this end, the paper suggests looking at the teaching of literature as a model.

1 Introduction

All of us who attend HPM conferences, receive its newsletter, and, generally support the organization, must, at some level at least, believe that history and pedagogy of mathematics do go together, that there is some deep harmony between them. I myself believe this no less than anyone else faithful to the HPM. Yet, for several years now I have been trying to show that the alignment between history of mathematics and mathematics education is neither obvious nor unproblematic (e.g. Fried, 2001, 2007). In particular, this is the case if one takes history of mathematics seriously, and not only as a tool to be used or cast away whenever it is found to be convenient or inconvenient. In the first—and longest—part of my talk, then, I shall give the general outline of where I have found difficulties in the incorporation of history of mathematics in mathematics education. It is this part that the word “problems” refers to in the title. But my intention here is not criticism. Rather it is to find the ground from which we may explore a deeper relationship between history of mathematics and mathematics education. For this reason, I shall give particular weight to the nature of the historical enterprise in general. In the second part, the part to which the word “prospects” in my title refers, I shall try not so much to solve the problems of the first part, but to see what it means to find a solution. And I might as well say here at the outset, that I think this means redefining our very goals in mathematics education. So my message will, in some ways, be a radical one.

2 Problems — A Quasi-Dilemma

I have used the word problems in the plural because there are, naturally, a variety of problems confronting mathematics educators who wish to bring history of mathematics into the classroom. Many of these problems are practical problems similar to those that burden anyone wanting to do something new or non-routine in mathematics education, or, any other kind of education. The point I wish to make in this first part of my talk is that these practical problems mask deeper, more fundamental difficulties, ones that have to do with the very identity of mathematics education and history of mathematics.

One immediately has a hint of this even with the expression “non-routine” just used: is the history of mathematics non-routine? In a factual way, of course, it is non-routine in the sense that it is not pursued in the classroom in the same way as algebra is. But is it non-routine in the sense of being non-essential? What about the very practical problem of time? This is generally high up in the list of practical problems: where can time be found for history of mathematics in an already very full curriculum? If history of mathematics is non-essential, then, indeed, making time for history must be justified. If it is somehow essential, essentially part of an ideal routine of mathematics, then time must be made for it. If the time given a subject in the curriculum is a measure of the priority given the subject, then whether or not time is given to the history of mathematics, or how much time is given it, reflects its priority in mathematics education—one might say its very legitimacy as a part of the routine. But if it is decided that it should be part of the routine, that, in turn, forces us to consider what it is the students will learn when they study the history of mathematics in their mathematics classes.

These kinds of questions, therefore, which begin as practical questions, leads us to ask about the nature of mathematics education, its aims and its priorities, and no less about the nature of the history of mathematics itself. I will begin then with history of mathematics; however, my remarks will mostly concern history itself. This is based on the assumption that to understand the nature of the history of mathematics we need to grasp what it means for the history of mathematics to be history. And it is taking the history of mathematics as history, what Uffe Jankvist calls “history as a goal” (e.g. Jankvist, 2009), that clashes with the history of mathematics as a tool for teaching mathematics.

2.1 History

It seems part and parcel of any intellectual pursuit that one thinks about the nature of the pursuit almost as much and as seriously as one pursues the actual business of it. This is plainly true regarding historians. So besides purely historical investigations, an historian such as G. R. Elton found time to write a book called *The Practice of History*; E. H. Carr wrote *What is History?*; R. G. Collingwood, *The Idea of History*; Marc Bloch, *The Historian's Craft*. Bloch's work, I might add, has a sharp poignancy about it, because Bloch wrote it as a prisoner of the Gestapo, and he was executed by them before he could finish it. This underlines the importance such works about history has for the historians who write them, which is not to say there is complete agreement among them on what history is and what is at the heart of the historian's craft. Still, there are some commonalities.

Chief among the commonalities is an acute awareness of the tension between past and present or, at very least, the need to confront the question of past and present. It would be hard to think of history with no reference to the past of course. But is it just about the past? One must refer to the present to a certain extent since historians' materials, their objects of study, are things that have made their way into the present. For this reason Elton (1967) defines history as being "...concerned with all those human sayings, thoughts, deeds and sufferings which occurred in the past and have left present deposit; and it deals with them from the point of view of happening, change, and the particular" (p.23). The last part of Elton's statement makes it clear also that it is not just past or present that is essential but how these are treated, namely, "from the point of view of happening, change, and the particular." The historical mode of thinking demands treating these "survivals" from the past, as Michael Oakeshott calls them (see Oakeshott, 1999), precisely as survivals. A survival is a survival from another world. One interrogates survivors to understand where they came from—a world not conditioned by the existence of ours, yet one out of which ours grew.

Since I have mentioned Oakeshott, let me follow him a little further. To experience the past in the present *as history* — and for Oakeshott history is a mode of experience — one must view the past unconditionally. To describe a relationship to the past that depends on the present, in other words, that sees the past in terms of present values, needs, and ideas, Oakeshott uses the term "practical past." The historical past, accordingly, is defined in opposition to the practical past; it is a past understood in terms of its separateness from the present. Thus in his chapter on historical experience in *Experience and Its Modes* (Oakeshott, 1933), Oakeshott sets out the historian's task as follows:

> What the historian is interested in is a dead past; a past unlike the present. The *differentia* [emphasis in the original] of the historical past lies in its very disparity from what is contemporary. The historian does not set out to discover a past where the same beliefs, the same actions, the same intentions obtain as those which occupy his own world. His business is to elucidate a past independent of the present, and he is never (as an historian) tempted to subsume past events under general rules. He is concerned with a particular past. It is true, of course, that the historian postulates a general similarity between the historical past and the present, because he assumes the possibility of understanding what belongs to the historical past. But his particular business lies, not with this bare and general similarity, but with the detailed dissimilarity of past and present. He is concerned with the past as past, and with each moment of the past in so far as it is unlike any other moment" (p.106).

Historical experience is naturally an experience in the present and one belonging to a living and breathing historian. Still, that does not preclude the desideratum to view the past in its particularity, even though the past enters and informs the present in this way.

This desideratum takes its concrete form in historians' rule to avoid anachronism. It is not an easy rule to obey, since we are being who live in the present and whose immediate experience is not that of the historical subjects we study. The struggle with anachronism is at the heart of the tension between past and present, with which I began this section. It might be said, indeed, that the historical art is one that aims to keep that struggle alive. The dangers of submitting to anachronism and the subtle ways in which can subvert history was discussed most trenchantly and colorfully by Herbert Butterfield in his classic, *The Whig Interpretation of History* (Butterfield, 1931/1951). Although Butterfield's polemic was immediately directed towards specific historians such as Macaulay and Trevelyan, it was in fact a critical view, along the lines which we have been discussing, of all historiography. A definition, more or less, of Whig history is given by Butterfield right at the start of his book:

> What is discussed is the tendency in many historians to write on the side of Protestants and Whigs, to praise revolutions provided they have been successful, to emphasise certain principles of progress in the past and to produce a story which is the ratification if not the glorification of the present (p.v)

Whiggism creates a distortion of the past not only by reading modern intentions and conceptions into the doings and writings of thinkers in the past, which is anachronism in its most direct form, but also by forcing the past through a sieve keeping out ideas foreign to a modern way of looking at things and letting through those that can be related to modern interests. For example, in reading Proclus' *Commentary on Book I of Euclid's Elements*, a Whig historian would leave out Proclus' arguments in the "first prologue" about the nature of mathematical being and role of mathematics in the moral education of the soul and emphasize Proclus' comments relating to logical difficulties, missing cases, alternative proofs connected to the familiar geometrical propositions in the *Elements*. These things are truly to be found in Proclus, but a Whig historian would give the impression that these are the only things in Proclus, or the only things of any worth in Proclus.

Whig historians treat the past, almost by definition, as a "practical past," adopting Oakeshott's term: they seek in the past what is useful for the present. The problem with this is that by using the present to determine what is useful for the present, one finally forfeits learning something from the past. More specifically, as Butterfield tells us:

> If we turn our present into an absolute to which all other generations are merely relative, we are in any case losing the truer vision of ourselves which history is able to give; we fail to realise those things in which we too are merely relative, and we lose a chance of discovering where, in the stream of the centuries, we ourselves, and our ideas and prejudices, stand. In other words we fail to see how we ourselves are, in our turn, not quite autonomous or unconditioned, but a part of the great historical process; not pioneers merely, but also passengers in the movement of things" (p.63)

In short, if the project of the Whig historians is to provide an enlightening view of the present, then their approach to this, according to Butterfield, is self-defeating.

2.2 Whig and non-Whig history of Mathematics

It can be argued that the problem of Whig history is particularly acute in the case of the history of mathematics. This is because mathematics enjoys a status of being a constant component of thought not only in the modern world, but also everywhere else and at all other times — even beyond the planet! Thus, when the question of how we should try and communicate to intelligent extraterrestrials was first seriously taken up, the widely accepted view "...was that pictorial messages based on science and mathematics would be universally comprehensible by technologically advanced civilizations" (Vakoch, 1998, p.698).[1] But to the extent that mathematics is continuous over time and place, a universal body of content, it is really a-historical and non-cultural, or, at best, its peculiarly historical and cultural aspects involve only trivial matters of form. With such an assumption in the background, it would be hard for a historian not to be Whiggish: present mathematical knowledge, short of logical errors, is mathematical knowledge *tout court*; past mathematical knowledge, to be understood, has merely to be translated into a modern idiom. What one learns from the history of mathematics, in its Whiggish form, is, in short, mathematics. And one feels fully justified in treating mathematicians of the past as Littlewood famously said of the Greek mathematicians, namely, as "Fellows of another

[1] Vakoch, the director of interstellar message composition at the famous SETI Institute, casts doubt on this still prevalent assumption.

college" (quoted in Hardy, 1992, p.81); mathematician's of the past, like one's colleagues, are useful for gaining insights into one's present mathematical research. Clifford Truesdell (1919-2000), who was a good example of this kind of historian, wrote unabashedly of this "practical past" as the goal of the history of mathematics:

> One of the main functions [the history of mathematical science] should fulfill is to help scientists understand some aspects of specific areas of mathematics about which they still don't fully know. What's more important, it helps them too. By satisfying their natural curiosity, typically present in everybody towards his or her own forefathers, it helps them indeed to get acquainted with their ancestors in spirit. As a consequence, they become able to put their efforts into perspective and, in the end, also able to give those efforts a more complete meaning" (in Giusti, 2003, p.21)

Truesdell brought to his own historical work immense and exacting mathematical and scientific insight, but his approach was generally to show where Euler, Lagrange, the Bernoullis, and the others he studied got it right and where they got it wrong—and right and wrong, in his view, were to be taken as absolutes: the same today as yesterday.

This is not to say that the history of mathematics can never make a judgment and never pronounce something right or wrong. There is certainly no lack of instances where mathematicians of the past pass judgment on their own contemporaries or on mathematicians of their own past: surely, historians must say something about that! But by what criteria can one pronounce something right or wrong? Except in the simplest cases, such criteria themselves must be taken as subjects for the history of mathematics. To remove them from history, to leave unquestioned the appropriateness of modern criteria, like those used to referee a mathematics research paper, for judging past mathematics is precisely the Whig position. A more sound approach, from an historical point of view, that is, a non-Whiggish one, is to show why writers of texts, which have made it to the present, thought their work or others' work right or wrong, to try and tease out their own presuppositions, not only regarding their criteria for correctness and incorrectness, but even more so for their way of conceiving a mathematical object or idea.

2.3 An Example: Similarity in Greek Mathematics

To give this last point some body and to begin my considerations of mathematics education, I would like to consider an example from Greek mathematics. I could have chosen one of several topics. For example, I could have chosen the word "mathematics" itself; its origin in words such as *mathein,* "to learn" (aorist infinitive) and *mathēma*, "a thing learned" or "a lesson," and its use in these senses by Plato, among others, makes one realize that even "mathematics" may not have had exactly the same meaning for Greeks as it does for us: the ancient *mathēmatikos* might have been a somewhat different creature than the modern mathematician. I could have chosen the problem of "geometric algebra." This was the focus of Sabetai Unguru's famous 1975 paper, which put on the table the issues I have been discussing in the specific context of Greek mathematics and its historiography. Instead of these, I will say a few words (too few, for sure, but time forbids more...) about the idea of similarity.[2] This is a good example of a mathematical idea that one might expect to be the same for us and Euclid if anything were. It is also a central idea

[2] See Fried & Unguru, 2001 and Fried, 2009 for a more detailed discussion.

in almost every school geometry curriculum, and many of the theorems students study can be found in Euclid.

The modern notion of similarity is colored by the idea of a transformation. For example, one speaks of a "dilation" or "contraction" which is a transformation taking a point (x,y) to a point (x',y') where $x'/x=y'/y=k$, k being the "magnification factor." This view of similarity has great power. For one, congruence can be thought of as a special case, namely, when the magnification factor is equal to 1. But more than this, by conceiving similarity in terms of transformations, one relates similarity to the entire space containing geometrical objects. This priority of space to objects is also, of course, what a coordinate system establishes. The effect of this is that similar figures will be similar in virtue of being in a space transformed in a certain way so that similarity becomes one thing for all figures. We still say that similar figures are figures having the same shape though perhaps differing in size; however, because it is the space rather than the object that is the immediate referent for the transformation, we do not have to trouble ourselves too much about what it means say a figure has a certain shape that is the same as another.

Assuming the modern notion of similarity, one can easily show how statements concerning similarity in Euclid, Archimedes or Apollonius are consistent with it, statements such as these:

> Euclid, *Elem*.VI.8: "If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another."
>
> Euclid, *Elem*.III.24: "Similar segments of circles on equal straight lines are equal to one another."
>
> Apollonius, *Conics* VI.11: "Every parabola is similar to every parabola."
>
> Archimedes, *On Plane Equilibriums*, Postulate 5: "In unequal but similar figures, the centers of gravity will be similarly situated. By points similarly situated in relation to similar figures, I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides."

Indeed, it is the power of the modern notion of similarity that allows one to think of these statements in a perfectly unified way.

But the Greek approach to similarity was not unified. This was because, in that geometry, to be similar, *homoios*, meant *primarily*, and not derivatively, to be of the same shape. With that, the question of what it means to be the same shape becomes central. And since there are different kinds of shapes, there are, accordingly, different definitions of being similar, different answers to the question what makes two figures have the same shape: similarity of rectilinear figures is one thing; similarity of circular segments is another; similarity of conic sections, yet another. Thus we have from Euclid alone:

> *Elem. III (def. 11), Similar circular segments*: "Similar segments of circles are those which contain equal angles or in which there are angles equal to one another."
>
> *Elem. VI (def. 1), Similar rectilineal figures*: "Similar rectilineal figures are such that they have each of their angles equal and sides about the equal angles proportional."
>
> *Elem. IX (def. 9), Similar solid figures*: "Similar solid figures are those contained by similar plane areas (*epipedōn*) equal in number."

Elem. IX (def. 24), Similar cones and cylinders: "Similar cones and cylinders are those of which the axes and diameters of the bases are proportional."

Add to these, other definitions from Apollonius and Archimedes:

Apollonius, Conica VI (def. 2), Similar conic sections: "...similar [conic section] are such that, when ordinates are drawn in them to fall on the axes, the ratios of the ordinates to the lengths they cut off from the axes from the vertex of the section are equal to one another, while the ratios to each other of the portions which the ordinates cut off from the axes are equal ratios."

Archimedes, Conoids and Spheroids (Introduction), Similar obtuse-angled conoids (i.e. hyperbolas of revolution): "Obtuse-angled conoids are called similar when the cones containing the conoids are similar."

One can see that even the notion of ratio and proportion, though generally a component in the various definitions is not essential: specifically, it plays no part in the definition of similar circular segments. In fact, Euclid's book on ratio and proportion is Book V of the *Elements*, whereas the definition of similar circular segments appears in Book III.

The connection between congruence and similarity that is so natural for us was much more subtle in Greek mathematics. For one, there is no mathematical term "congruence" in Euclid or anywhere else in the mathematics of his time. There is "equality," *isos*, and figures are equal if they are congruent, that is, if one can be made to coincide *epharmozein* with the other. And so we have Euclid's common notion 4: "*Things which coincide* (*ta epharmozonta*) with one another are equal to one another." "To be congruent" is a good translation for *epharmozein*, but the latter is not a basic relation in Euclid but a basic criterion for equality. For this reason, our congruence theorems for triangles are theorems about equality in the *Elements* where equality is proven by proving congruence. On the other hand, for Apollonius, conic sections are equal if and only if they are congruent, that is, if and only if they can fit on one another: two ellipses of the same area are not called equal. Far from being a special case of similarity, equality and similarity a kept apart in Apollonius' *Conics*. Thus, in Book VI of the *Conics*, where Apollonius speaks about similarity and equality, we have pairs of propositions such as *Conics*, VI.2, which says that ellipses and hyperbolas are equal whenever the "figures" on their axes are equal and similar,[3] and *Conics*, VI.12, which says that hyperbolas and ellipses are similar when the same "figure" is similar. From a modern perspective, VI.2 ought to be a trivial corollary of VI.12, which, needless to say, does not refer to or require VI.2 in its demonstration.

I could say more about the curious role the phrase "equal and similar" has in the semiotics of Greek mathematical discourse (see Fried, 2009), but I think what has been said suffices to show the radical difference between the modern conceptualization of similarity, which flows from the idea of transformations, and that of Euclid and Apollonius, which derives directly from the consideration of the nature of particular mathematical objects. I have tried to argue that, regardless of one's interpretation of the matter, as an historian, one is obliged to confront a conceptual difference such as this; one is obliged to face the divide between past and present. This is what I have called the historian's commitment. It is a commitment that, I believe, is ultimately at odds with the Whig perspective, but at very least it is one that requires the historian to look intently at

[3] The "figure" or *eidos* of an ellipse or hyperbola is the rectangle whose sides are the diameter and latus rectum of the section.

the past as the past and to adopt, even as a sort of null hypothesis, the position that the past is different than the present. This brings us to mathematics education and the nature of its own commitments.

2.4 Mathematics Education

So what kinds of considerations must mathematics educators bring to bear on an historical discussion like the one above concerning similarity in Euclid and Apollonius? First of all, while the history of mathematics can bracket the present in order to understand the past, mathematics education typically justifies itself by the power and necessity of mathematics in modern contexts, in science, engineering, economics and industry. This is certainly consistent with the spirit of the American *Principles and Standards for School Mathematics* (NCTM, 2000). There, we read:

> The level of mathematical thinking and problem solving needed in the workplace has increased dramatically.
>
> In such a world, those who understand and can do mathematics will have opportunities that others do not. Mathematical competence opens doors to productive futures. A lack of mathematical competence closes those doors.
>
> ...More students pursue educational paths that prepare them for lifelong work as mathematicians, statisticians, engineers, and scientists.
>
> ...Today, many students are not learning the mathematics they need. In some instances, students do not have the opportunity to learn significant mathematics. In others, students lack commitment or are not engaged by existing curricula. (NCTM, 2000, Introduction)

Regarding the specific question of similarity, the emphasis of the NCTM, as one might expect, is the modern one of transformations. Starting with preschool (!), instructional programs are supposed to enable students to “Apply transformations and use symmetry to analyze mathematical situations” (NCTM, 2000, Overview of the geometry standard for Pre-K-2). And by the middle school years, this standard is matched by the expectation that students be able to:

- Describe sizes, positions, and orientations of shapes under informal transformations such as flips, turns, slides, and scaling;
- Examine the congruence, similarity, and line or rotational symmetry of objects using transformations. (NCTM, 2000, Overview of the geometry standard for Grades 6-8)

This emphasis on modern mathematics in the school program—and in the case of geometry, on transformations—is hardly unique to the NCTM program.[4] Nor does the emphasis on the societal and scientific needs of modern mathematics belong only to the

[4] See for example the Italian program for mathematics as presented in the European Mathematical Society (EMS, 2001) document, Reference Levels in School Mathematics Education in Europe, Italy, where again students in the first years of high school (ages 14-16), according to a “widespread experimental curriculum,” the Brocca curriculum, are to learn:

- Euclidean plane and space and its isometric transformations. Geometric figures and their properties. Equidecomposable polygons; Pythagorean theorem.
- Dilatations and similarities. Thales theorem (i.e.: In a parallel projection between two lines, the lengths of corresponding segments are proportional) (p.3).

Needless to say, the idea of transformations can be found in the national curricula of many of the countries described in the EMS document.

NCTM, although it is perhaps more explicit there than elsewhere. This emphasis is not at all unreasonable: the ideas and methods of modern mathematics are, as I have already said, truly powerful and deep.

But accepting this kind of emphasis also means that mathematics educators cannot bracket the present, as historians can and must. When mathematics educators — even those with real historical sensitivity and knowledge — confront a chapter in the history of mathematics, like ours on similarity in geometry, they must heed, to some extent at least, the counterweight of their obligation to teach mathematics in a modern spirit. They must consider how relevant the chapter is to the modern mathematical ideas they need to convey, how well it fits the subjects required by their curriculum. Their considerations of time and scheduling, as I mentioned in the introduction, are only signs that history of mathematics in the classroom must be subordinated to such standards as I have described in the example of the NCTM. There may be some historical topics for which a happy medium can be found, some cases where chapter in history of mathematics fits snuggly in the curriculum without requiring too great a compromise as to its historical character. This may be; but it is not the point. The point is that when mathematics education emphasizes mathematics as it is understood and practiced today, as it is needed in science and engineering, it will be predisposed to treat the history of mathematics in a Whiggish spirit, it will have that sieve in hand which separates relevant from irrelevant ideas.[5] This predisposition is not an injunction to be Whiggish; it is, rather, a kind of internal pressure at work in any attempt to introduce history of mathematics into mathematics education, where the latter is directed, as it generally is, towards modern mathematics.

One might be tempted to compare this situation with that of the history of mathematics itself. After all, it too, as remarked above, struggles with the problem of anachronism. However, engaging in that struggle is part of what it means to do the history mathematics: historians are derelict of their duty if they not live in the tension between past and present. But in the case of mathematics education the problem is one of conflicting demands, a kind of dilemma: keep modern mathematics as one's main end and thus make history serve modern mathematics, that is, adopt a Whig version of history of mathematics, or keep history of mathematics as history and put aside the perfectly legitimate emphases of programs that train students to use and understand the modern mathematics essential for all the pure and applied sciences.

This is a dilemma; but it is a quasi-dilemma because the force of the dilemma derives from accepting ends like those described in the example of the NCTM. Those ends built as they are on the power of modern mathematics to address societal and scientific needs are legitimate and not easily dismissed; however, they are not absolute. One can entertain other ideas about what it means to teach mathematics or what it means to be considered mathematically educated. With this we can begin to consider the prospects of a mathematics education shaped by history of mathematics as a form of knowledge rather than a mathematics education that only uses history of mathematics as a tool to promote ends not necessarily in line with those of history.

[5] A student of mine, Abdelrachman Affan, will soon be examining how presuppositions of mathematics teachers weighing the inclusion of history in mathematics education differ from those of history teachers weighing the inclusion of mathematics in history education.

3. Prospects

3.1 History as a goal

In considering the prospects for making history of mathematics part of mathematics education, I ought to reiterate what I mean by this. Again, to use Uffe Jankvist's phrase, the problem is how history of mathematics can become a *goal* in mathematics education. When we teach our students the idea of a derivative of a function or the meaning of similar figures or the Pythagorean theorem, we may have in mind some applications of these things: in teaching the Pythagorean theorem, for example, we may have in mind the application of that theorem in deriving the equation of a circle in analytic geometry. I doubt any mathematics educator, however, would seriously weigh the option of not teaching the Pythagorean theorem if some other way of deriving the equation of a circle were shown or if it happened that the equation of a circle were dropped from the curriculum. We teach the Pythagorean theorem because we believe that ignorance of it is unforgivable in a person deemed mathematically educated. This, obviously, is what we mean first of all when we say that we teach something as a goal. But it also means more than that. For knowing the Pythagorean theorem cannot mean the mere ability to recite it. It must be understood in relation to other things one learns. Therefore, one is taught, for example, in what ways the Pythagorean theorem is equivalent to the vanishing of the scalar product of orthogonal vectors—and, one hopes, one is told also how it differs from this. One might say that knowing the Pythagorean theorem is learning to see it as part of one's mathematical landscape.

Asking what things ought to be focal points in a mathematical landscape is a way of framing the entire challenge of forming a curriculum. Thus, one might ask — and this, minus the metaphor, is generally the way the question is put — where does history of mathematics fall in this mathematical landscape of ours or where should it be placed? One asks, for example, in conjunction with what topics should historical highlights be brought in, what historical episodes are to be presented to students and when? These questions have a seeming reasonableness, yet I believe they hide a categorical error. For asking where history of mathematics is located, or should be located, in the mathematics curriculum, just as one might ask about the Pythagorean theorem, is a little like asking where among the buildings here — the library, the administration offices, the laboratories — should we place the Vienna University of Technology? Inquiring where topics and ideas are placed in the mathematical curriculum assumes, to use a term from Saussurean semiotics, a synchronic view of mathematics: one sees mathematical concepts, techniques, theorems as part of a coherent harmonious whole where everything has a definite place and immutable relationship with everything else. In such a view it makes sense to say that the Pythagorean theorem truly is equivalent to $\mathbf{u}\cdot\mathbf{v}=0$ whenever $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. History, on the other hand, is not just another *component* in this harmonious whole; rather it is another *view* of the landscape of mathematical concepts, techniques, and theorems. Again appealing to Saussurean terms, this other view is a diachronic one, which Saussure himself said is as incomparable with the synchronic view as a longitudinal section of a plant is with its cross section: both explain the whole but from different perspectives and according to different principles of order.

So how one understands history as a goal for mathematics education must be different from how one understands, say, the Pythagorean theorem as a goal. Both are understood as being valuable in themselves as knowledge, that is, not as a tool only for something else more important; but while the latter fits into a kind of system as a key component, part of the puzzle, as it were, the former involves changing how we look at mathematics education altogether. I recognize this is a strong statement, but it is, I think, an inescapable one in light of what was said in the first part about the nature of the historical outlook. Indeed, it is inescapable in light of the simple fact that history *has* a definite outlook; as Sabetai Unguru has said, “The history of mathematics is history not mathematics” (Unguru, 1979, p.563): history of mathematics is not just another mathematical topic.

3.2 What kind of curriculum would be consistent with history of mathematics as a goal?

I put this as a question and put it this way because the moment one reorients oneself towards mathematics viewed historically I believe that more than one way of realizing an educational program for it will present itself, that there is no single definite curriculum answering the requirements of history of mathematics as a goal. That said, one aspect of such a program is surely unavoidable. I have in mind the presence of original mathematical texts in one form or another, abridged or unabridged, translated or un-translated, singly or conjointly with other texts. In this connection, it is worth recalling the remark by G. R. Elton quoted above about history being "...concerned with all those human sayings, thoughts, deeds and sufferings which occurred in the past and have left present deposit...” (Elton, 1967, p.23). For the general historian, that “present deposit” includes written texts; for the intellectual historian, like the historian of mathematics, it is *primarily* texts. It is true that the written texts, as has been so often pointed out, are only end products and hide a long, often private and tacit process of thought; however, more than anything else in our possession, mathematicians’ own texts nevertheless provide hints of that process and do show how the mathematicians themselves sought to present their thought. For this reason Eva T. H. Brann (1979) has contrasted original texts and modern textbooks in the following way:

> Textbooks, then, are opposed to works that are original in both senses of the term, in being the discoveries or reflection of the writer himself, and in taking a study to its intellectual origins, using the original language of discovery. In sum, textbooks follow primarily a scheme of presentation; texts convey the order of inquiry (p.100).

In an obvious way, original mathematical texts also reflect how mathematicians have sought to engage other mathematicians in their thought; they represent communicated thought. And that places original texts at the center of what we should call mathematical culture and tradition. Tradition, which in some sense is nothing more than a collection of mathematical texts,[6] has, accordingly, come into the arguments of those who promote the use of original sources in teaching mathematics (and here I should underline that I am quite far from the first to stress the importance of original works: indeed, we shall hear more about this with Michael Glaubitz’s plenary). For example, Laubenbacher, Pengelley, & Siddoway (1994) write this in defense of using original texts

> For a novelist, poet, painter or philosopher such observations would be old news, since their disciplines have long recognized the importance of studying the original work, techniques and perspectives of classical masters. And in so doing, they are never removed from an

[6] I am paraphrasing Brann (1979, p.64) here.

> understanding of how people have struggled, and have created works of art. Young artists thus see themselves as part of a creative tradition. Unfortunately, we have lost this sense of tradition in our discipline, and, ironically, we can perhaps blame much of this loss on the dazzling explosion of mathematics in this century. It is time we step back from our accomplishments and recapture a historical perspective.

Becoming absorbed in a tradition, however, is not a matter of nostalgia, nor of slavish loyalty, nor yet of chauvinism. A tradition always involves a kind of tension that, as Brann (1979) has pointed out, is built into the very etymology of the word. For *tradere,* from which "tradition" is derived, means both "to pass on" and "to betray." This doubleness in the meaning of the word captures a doubleness in our attention when we read texts seriously: we strain to understand the meaning and intention of our authors, what questions they are asking and how they are responding to them; however, we are also attentive to our own thoughts and our own desire to move beyond those whom we read, our betrayal of them. In this way, reading historical mathematical texts, on the one hand, forces us to try and understand past mathematicians in their own voice without imposing our own conceptions, while, on the other hand, our own self is kept in view; we see our mathematical selves in the act of confronting alternatives to them. Becoming involved in a tradition, in this corpus of mathematical works, brings us, therefore, to a true historical consciousness, which, as argued at the outset, is not so much living in the past as it is living the tension between the past and the present.

So, what I am proposing is that we view mathematics education, or rather becoming educated mathematically, not so much as the mastery of certain techniques in mathematics or even certain concepts in mathematics such as a function or derivative, but as reading and learning to read a collection of mathematical texts. Although certain texts, such as Euclid's *Elements*, would always be included, the particular choice of texts is precisely one of the things that will distinguish those different ways of realizing a program making history as a goal, which I alluded to at the beginning of this section. The reading of texts, the care given to authors' modes of presentation and their points of attention, the cultural context of works, and so on, would make mathematics education into a kind of literary education.

The comparison between a mathematics education informed by historical mathematical texts and literature education seems to me natural and potentially fruitful. To start, there are connections between mathematics and literature despite the modern habit of placing them on two sides of an impassable fence: Paul Valéry, for example, found endless inspiration in mathematics, and the great literary critic Northrop Frye said explicitly that, "The pure mathematician proceeds by making postulates and assumptions and seeing what comes out of them, and what the poet or novelist does is rather similar" (Frye, 1964, p.126). More directly relevant to mathematics education, the comparison between literature and mathematics forces us to turn away from justifying mathematics education on the basis of utility and towards justifying on the basis of culture and our own human identity as makers of mathematics.

But what I would like to stress most of all in this connection is the way literature sees the study of literature as a matter of dwelling in a world of texts, just as I would like to suggest for mathematics education in the light of history of mathematics. It is a different kind of landscape than the synchronous one described above in which there was a harmony of universal and immutable mathematical ideas; it is a landscape of texts whose place can be understood only in terms of other texts, and, as texts, these are creations of the human imagination. This, among other places, was stressed in Frye's book *The Educated Imagination* (Frye, 1964), which he addressed to teachers. Frye also makes the point that this kind of landscape is a landscape also for writers, not just learners:

> This allusiveness in literature is significant, because it shows...that in literature you don't just read one poem or novel after another, but enter into a complete world of which every work of literature forms a part. This affects the writer as much as it does the reader. Many people think that the original writer is always directly inspired by life, and that only commonplace or derivative writers get inspired by books (p.69).

In a little essay called "The Prerequisites," the poet Robert Frost writes something very similar to Frye:

> A poem is best read in the light of all the other poems ever written. We read A the better to read B (we have to start somewhere; we may get very little out of A). We read B the better to read C, C the better to read D, D the better to go back and get something more out of A. Progress is not the aim, but circulation. The thing is to get among the poems where they hold each other apart in their places as the stars do (Cox & Lathem, 1968, p.97).

Returning to history of mathematics and mathematics education, what these comparisons with literature point to is the prospect of a mathematics education in which students engage in a mathematical version of what Robert Hutchins called "the great conversation" (Hutchins, 1952). Because this means students must read authors carefully and try to understand what those authors were trying to communicate and why, because they must place themselves in another time and try to reenact the voice of another, to use Collingwood's phrase, because they must pay attention to the nuances of the text, its form, its particular way of putting things, participation in this great mathematical conversation is truly an historical enterprise. Being mathematically educated, in this light, means becoming at home in this conversation. Mathematics education conceived this way would make history a goal. But because these texts are indeed mathematical, students are also doing mathematics in precisely the way Frye says that writers, by reading writers, are involved in something essential to writing. This of course is a very different view of a mathematics education than one is used to — certainly it is different than the mathematics education suggested by the quotations above from the NCTM Principles and Standards — yet it is a mathematics education that does not neglect mathematical thinking and, more, it is one that brings one into a greater mathematical world. So, with that, I should perhaps end by repeating what Butterfield said about what we lose without a truly historical approach to things, or, put in the present terms, without being a part of this mathematical conversation, without knowing the diverse mathematical voices of the past and only seeing things through the present:

> ...we lose a chance of discovering where, in the stream of the centuries, we ourselves, and our ideas and prejudices, stand. In other words we fail to see how we ourselves are, in our turn, not quite autonomous or unconditioned, but a part of the great historical process; not pioneers merely, but also passengers in the movement of things" (Butterfield, 1931/1951, p.63)

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THE ROLE OF THE HISTORY AND EPISTEMOLOGY OF MATHEMATICS IN TEACHERS TRAINING

PANEL DISCUSSION

INTRODUCTION

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The theme of “History of mathematics in pre and in-service teacher training” has already been established as one of the six topics of the 1st European Summer University in Montpellier (France) where one panel and many workshops were held. The Proceedings of this meeting contain 7 papers on this theme (IREM, 1993, pp.451-496). Five years later, it was also the subject of chapter 4 of the ICMI Study (Fauvel & van Maanen, 2000). The current contribution will offer the possibility to study evolution of ideas on the theme, as well as some institutional changes. If we are not able to give a complete outline of the situation in each country from which our members come, it is clear that the possibility of communication among the participants in our different meetings is important for both encouragement, and as a way of communication, in order to compare and exchange thoughts and information on projects we are involved with.

The panel was organized to enable 4 communications concerning 4 countries: Italy, Norway, England and France. In order to have a more fruitful interaction, the four participants had to answer the same 6 questions in total, divided into three groups. The aim of the first group was to situate the purpose of investigation in an institutional frame for each country:

1. How is teacher training conceived in your country (pre and in-service teacher training)?
2. What is the official place of the history and epistemology of mathematics in this training?

The following group concerned the theoretical approach of each participant:

1. What are your conceptions about the role played by history and epistemology of mathematics in teacher training? At what level? What goals can you identify?
2. What purposes and what kind of (theoretical or methodological) approaches and practices can you tell us about, that you have witnessed in your country?

The last group was devoted to practices and products with one or two concrete examples where each participant was involved:

1. Were you successful in putting into practice your conceptions and goals?
2. Have you new perspectives for future?

The four participants of the panel were:

-Fulvia Furinghetti (Italy) teaches didactics of mathematics to students in mathematics. She is a member of the Italian Commission of Mathematical Instruction (sub-commission of the Italian Mathematical Union). Fulvia was the HPM chair during 2000-2004.

-Bjørn Smestad (Norway) teaches mathematics to future teachers in primary and lower secondary schools and is the editor of the HPM Newsletter.

-Snezana Lawrence (England) teaches mathematics education to the students preparing to teach in secondary schools. She is the Education Officer for the British Society for the History of Mathematics.

-Evelyne Barbin (France) teaches history of mathematics to future professors of primary schools and she is co-president of the inter-IREM Committee “Epistemology and History of Mathematics” which organize teachers’ trainings for all the levels.

I - TEACHERS AND HISTORY OF MATHEMATICS IN ITALY

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1 Pre-service and in-service mathematics teacher education in Italy

1.1 Pre-service teacher education

The Italian system of education is centralized. In the recent past there have been a few changes in respect to the situation described in Furinghetti (1998). At present teachers of all school subjects must have a university degree (*Laurea*) obtained according to the 3+2 schema, with significant differences in the curriculum, which depend on the school levels where prospective teachers will teach.

Primary teachers teach mathematics together with other subjects. They obtain the degree in educational departments. In their curriculum there are no courses in the history of mathematics, but in some universities elements of the history of mathematics are introduced in courses on mathematics education.

Secondary teachers obtain their degree in departments as follows. For mathematics the degree for lower secondary teachers (students aged 11-14) has to be in science, or in mathematics, physics, or chemistry. The majority of mathematics teachers in lower secondary school have the degree in science; their university curriculum encompasses at most two mathematics courses and two physics courses. The degree of upper secondary mathematics teachers (students aged 14-19) has to be in mathematics or physics; degrees in other fields such as engineering, statistic, computer science or economics are allowed in certain cases.

In this paper I focus on some features of the curriculum aimed at undergraduate studies in mathematics: this is the only curriculum which may encompass the history of mathematics, while the curriculum aimed at undergraduate studies in physics, science etc. does not encompass courses on the history of mathematics. The mathematics teacher education carried out in mathematics departments may be considered a happy exception in the general panorama of teacher education in Italy, because there is a tradition of interest for primary and secondary teaching inside the community of mathematicians. This tradition, together with others factors, has fostered the development of ‘mathematics education’ as an academic discipline. As a matter of fact, for about the last 50 years, the curriculum of mathematics in Italian universities encompasses special courses addressed to prospective teachers (different in various universities). Possible contents of these courses are:

- critical presentation of mathematical topics linked to mathematics teaching
- foundation of mathematics
- didactics of mathematics
- mathematical laboratories
- history of mathematics.

It is possible that a graduate is appointed as a teacher without a preliminary period of practice in the classroom; this may happen for any school subject. It is then, a valid concern of mathematics departments to provide courses aimed at these future professionals.

History of mathematics is taught in most universities. Drawing from my experience I would say that usually students like history and sometimes ask to write their dissertation of Laurea on this subject.

1.2 In-service teacher education

There is no regulation or a national plan for in-service teacher education in mathematics, as is indeed the case for other school subjects. There are scattered national initiatives by the Ministry of Instruction and local initiatives organized by groups of teachers, schools, and mathematics teacher associations. In these activities there are sometimes slots dedicated to the history of mathematics.

Because the university courses on the history of mathematics are not compulsory, there are teachers who never encountered the subject in their educational path. Nevertheless, I often met teachers who are attracted by history. At present teachers' interest in history comes not only because, perhaps, they have a personal interest, but because also the guidelines of the national mathematics programs issued by the Ministry of Instruction in 2010 mention explicitly the historical perspective in mathematics teaching and learning. As a consequence, the problem of having some knowledge on the history of mathematics is not only cultural but also professional, as teachers must have the knowledge suitable to cope with the requests of the Ministry.

The ways in which those teachers who did not attended university courses on the history can become acquainted with historical facts are:

- from the historical notes in the textbooks (see Demattè's contribution to the panel *The history of mathematics in school textbooks* in this volume)
- through the local initiatives of re-training courses
- and by reading the books on popularization of the subject.

It happens that after a rather casual approach teachers may feel encouraged to deepen their historical knowledge. Until a few years ago the access to original sources was a problem, because in Italy only a very few 'readers' have been published and original volumes are accessible with difficulties. The availability of important works on the web has made possible the contact with original sources.

2 History of mathematics in teacher education: theory and practice

My approach to the use of history in teacher education has been described in (Furinghetti, 2007; Jahnke et al., 2000). Here I recall the main points and afterwards I briefly describe an example of realization.

2.1 Theoretical frame

My students are students in mathematics whose university curriculum is aimed at teaching. They have beliefs that have been elaborated mainly on the ground of school experience. Among other issues, the beliefs that are of concern are:

- the nature of mathematics
- self as a learner
- self as a teacher
- and the process of teaching and learning.

The importance of prospective teachers' beliefs is due to the influence they have on the future teachers' choices. In this context Megan Frank (1990, p. 12) wrote: "Teachers teach the way they have been taught". This opinion is an old one. Similar ideas are present in Klein's Introduction (1911). According to Klein, when novice teachers have to decide what to do in the classroom they put aside what they learnt at university and go back to their mathematical culture as it was built during secondary school: this fact originates a "doppelte Diskontinuität (p. 2)" (double discontinuity). In the same vein Émile Borel (1907) wrote (p. 657) "[…] une des raisons pour lesquelles l'enseignement secondaire se perfectionne lentement, c'est que l'enseignement que l'on donne ne peut pas différer beaucoup de celui qu'on a reçu" [one of the reasons that make changes in secondary teaching so slow is that what we teach can not differ so much from what we have received]. Gino Loria (1933) put forwards similar ideas in his report on the ICMI inquiry about teacher education around the world.

In my courses for prospective teachers I make my students challenge their existing beliefs for fostering flexibility and openness to different positions. I try to make them reflective practitioners that are able to learn from their practice, as advocated by Jaworski (2006). History, if appropriately used, creates an environment suitable to reflection, because, as (Jahnke et al., 2000) put it:

- Integrating history in mathematics challenges one's perceptions through making the familiar unfamiliar
- The history of mathematics has the virtue of 'astonishing with what comes of itself'
- To walk in the foreign and unknown landscape provided by history forces us to look around in a different manner and brings to light elements which otherwise would escape.

2.2 An example of realization

Algebra is a typical topic in which the gap between what is done in a secondary school and at a university, and the consequent phenomenon of the double discontinuity, are evident. To challenge my students' beliefs about algebra and its teaching I use medieval arithmetic problems. One of them is the following problem 47 in the Medieval treatise *Trattato d'Aritmetica* by Paolo Dell'Abbaco:

> A gentleman asked his servant to bring him seven apples from the garden. He said: "You will meet three doorkeepers and each of them will ask you for half of all apples plus two taken from the remaining apples." How many apples must the servant pick if he wishes to have seven apples left?

The students are invited to solve this problem and to write carefully their solution process. Afterwards, they analyze the written process of a colleague. At the end of the session, the students discuss the findings. In the following I present two examples of processes produced by my students that evidence the possibility of two ways of solving the problem:

> Case A) The student starts from the apples required before passing through the last door. Since the doorkeeper asks half plus 2 apples, the 7 apples are half of the amount less 2. Then, before the last door, the gentleman has 18 apples. The student observes that 18 is (7•2)+4, then, deduces that before the second door, the gentleman has (18•2)+4=40 apples and thus he must pick (40•2)+4=84 apples.

> Case B) The student names *y* the apples picked by the servant. Then apples left after meeting the first doorkeeper are (*y*-*y*/2)-2. Repeating this reasoning after the meeting with the second and the first doorkeeper the student writes the algebraic model (a first degree equation in the unknown *y*) of the problem.

We see that two solving paths may be followed, which may be put in relation with the analytic and synthetic methods:

- arithmetic path: from the known (left apples) to the unknown (apples to be picked)
- algebraic path: from the unknown (apples to be picked) to the known (left apples)

The algebraic path is based on the analytic method. To illustrate the different nature of arithmetic and algebraic problems I propose further exercises such as the two reported in the Appendix taken from *Propositiones ad acuendos juvenes* (*Problems to sharpen the young*) by Alcuin of York. In this way I lead my students to reflect on the fact that algebra is not only generalization, not only abstraction, not only using symbols, not only an extension of arithmetic: algebra is a method and the analytic method is its core. Then François Viète's innovation doesn't appearing to come out of the blue, but begins to be seen as a consequence of this way of looking at algebra.

The use of original sources is functional to my aims. Presentation of medieval problems does not have the aim or pretence to be put to students as being "real problems", but the distance from the time from which they originated makes students perceive them not so artificial as, sometimes, real problems appear. The prospective teachers may use medieval problems in classroom and exploit other potentialities, such as the reflection on language, the links with literature and with general history.

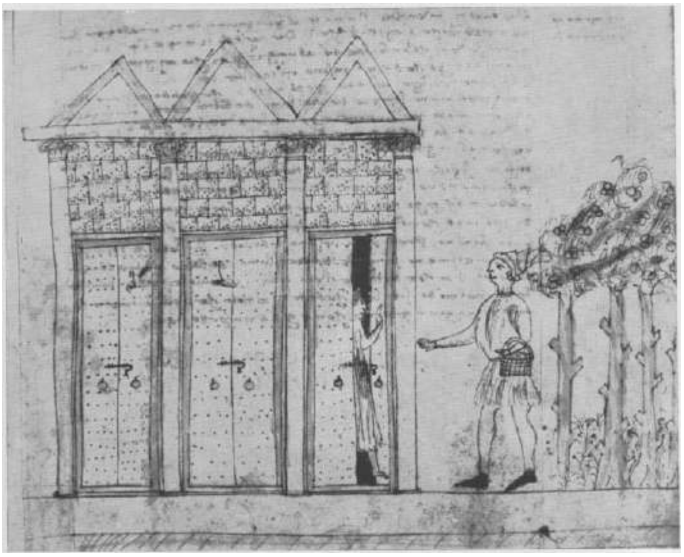

Fig. 1. Illustration of Problem 47 in *Trattato d'Aritmetica* by Paolo Dell'Abbaco

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– For information on the history of teacher education in Italy visit the website designed by Livia Giacardi: http://www.subalpinamathesis.unito.it/storiains/uk/training.php

Appendix. Two problems by Alcuin

1. A snail was invited by a swallow to lunch a league away. However, it could not walk further than one inch per day. Let him say, he who wishes, "How many [years and] days did it take for the snail to walk to that lunch?"
4. A certain man saw some horses grazing in a field and said longingly: "O that you were mine, and that you were double in number, and then a half of half of this [were added]. Surely, I might boast about 100 horses." Let him discern, he who wishes, how many horses did the man originally see grazing?

II - HISTORY AND EPISTEMOLOGY OF MATHEMATICS IN TEACHERS TRAINING IN NORWAY

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1 Institutional Context

There are basically four different ways of becoming a mathematics teacher in Norway (see table). Formerly, there was only one path for teacher training for the whole grade span from 1 to 10, but this has, from 2010, been split into two: 1–7 and 5–10. Both of these pathways last four years and integrate the subjects, pedagogy and didactics into the training. In the 1–7 path, half a year of mathematics (including didactics) is obligatory; in the 5–10 path, one year of mathematics (including didactics) is obligatory for students planning to become mathematics teachers.

Grades (Ages)	**1-7 (age 6-13)**	**5-10 (age 9-16)**	**(5-) 8-13 ((9-) 12-19)**	
Length	4 years integrated subject matter and pedagogy/didactics	4 years integrated subject matter and pedagogy/didactics	4 or 5 years integrated subject matter and pedagogy/didactics	3-5 years of subject matter + 1 year of pedagogy/didactics
Math.	Half year compulsory for *all* teachers	One year (for mathematics teachers)	One year (for mathematics teachers)	One year (for mathematics teachers)
National guide-lines	Yes	Yes	No	No

For the 8–13 education (in theory 5–13), a teacher traditionally completes a bachelor's or master's degree in mathematics, with one year of pedagogy and didactics added on. This has now been partly replaced by a four or five year education where pedagogy, didactics and the subject are integrated. In both of these variants, at least one year of mathematics is obligatory.

For all the different pathways which teachers can undertake to train as mathematics teachers, only the 1–7 and 5–10 pathways have national guidelines. (Kunnskapsdepartementet, 2010). In the national guidelines, we find the goals of training are connected to "the historical development of mathematics", "the importance of mathematics as a formative subject and its interaction with culture, philosophy and social development" and "how knowledge of mathematics is developed" (my translations).

There is no national system for in-service training. This is left to the schools. Currently, I know of no history of mathematics courses available for teachers.

2 Theoretical Approach

The education of mathematics teachers has at least two goals (Christensen & Nordberg, 2007):

1. to develop students' knowledge of mathematics and their attitudes towards it
2. to develop their ability to *teach* mathematics

Given that many students enter teacher education with misconceptions and a lack of knowledge of key areas in mathematics (Rasch-Halvorsen & Johnsbråten, 2007), there is a danger that goal 1, and in particular developing mathematical skills, will overshadow goal 2 in the minds of the students. On the other hand, that may be all the more of a reason for offering a new approach to learning mathematics.

Concerning the goal 1, all the usual reasons for including the history of mathematics with any group of students/pupils hold. For goal 2, a closer look at teacher knowledge is needed.

I would like to use the model for mathematical knowledge for teaching, developed by Deborah Ball and her colleagues, as a starting point for looking at teacher knowledge (Ball, Thames, & Phelps, 2008). Ball et al are following up on Shulman (1986), and are trying to come to grips with what knowledge teachers use in their teaching. I would like to take part in a discussion on how history of mathematics can contribute to each of these "domains".

I would argue that the history of mathematics can play a part in all of these domains, but in different ways. For instance, in the area of subject matter knowledge:

- *Common content knowledge* is mathematical knowledge that is not special for teachers, such as being able to calculate correctly. In the teaching of teacher students, history of mathematics can play a role just as with anyone else learning mathematics.
- *Specialized content knowledge* is mathematical knowledge primarily necessary for teachers. Ball et al (2008) mention as an example the ability to see a new algorithm and decide whether it is sound. Work on the history of mathematics can help develop such skills, because in history we see mathematics done in very different ways all the time. One example is the work on historical algorithms, see Nikolantonakis and Smestad's paper at this conference.
- *Horizon content knowledge* is knowledge of how the mathematics that pupils are learning now is connected to mathematics that they will learn later and to what they learned earlier. History of mathematics can contribute to seeing the connections between topics, for instance between statistics and probability, and be (as mentioned by Evelyne) a "therapy against scattering".

In the area of pedagogical content knowledge, history also has a part to play:

- *Knowledge of content and students* includes typical misconceptions. History of mathematics sheds light on some of these by showing historical obstacles. The early discussions on probability are examples of this.
- *Knowledge of content and teaching* involves different ways of representing mathematics. Of course, knowing the history of quadratic equations helps to see the different approaches, as mentioned by Michael Glaubitz in this volume.
- *Knowledge of content and curriculum* benefits from a historical perspective of how the mathematics curriculum has developed. For instance, the rise and fall of the "New Math" is an intriguing example of how curricula are influenced by external factors.

In my opinion, the history can play a part in the whole spectrum of teacher knowledge. We could discuss what the different ways of working on history of mathematics can contribute to in the light of these (or other) frameworks for teacher knowledge.

But we should not forget to ask the question "What is 'mathematics' in this context?" If we include history of mathematics in what pupils should learn (that is, we see it as a goal, not just as a tool), subject matter knowledge will include history of mathematics, KCS (knowledge of content and students) will include knowledge of which parts of history of mathematics are suitable for students and so on. That puts high demands on us as teacher educators.

This may not be the way Ball et all (2008) were thinking – the model of Ball is based on classroom experiences, but maybe not on classrooms where the history of mathematics was included in practice. Following up on Uffe Jankvist's comments at the start of this conference: this may be an area in which our perspectives may enrich theories on mathematics education.

Or, instead of clinging to Ball, we could make four goals out of the two we started with:

1a. to develop students' knowledge of mathematics and their attitudes towards it

1b. to develop students' knowledge of the history of mathematics (and their attitudes towards it)

2a. to develop their ability to teach mathematics (also using the history of mathematics)

2b. to develop their ability to teach the history of mathematics.

Most people will agree that 1b is a prerequisite for teaching mathematics using history of mathematics, as well as for teaching history of mathematics, of course. But as Chun-Ip Fung pointed out (Fauvel & Van Maanen, 2000), there is often a question of where to put the emphasis. Torkel Heiede (Fauvel & Van Maanen, 2000) mentions a course of 33 three-hour sessions, which could give an overview of the whole history of mathematics and James Kiernan described a similar one at the previous ESU (Kiernan, 2007). Within the Norwegian half-year course, however, which is trying to cover all the relevant mathematics and math education in perhaps only 40 three-hour sessions, I think we can do no more on 1b than give a glimpse of the history and give the students motivation for further work. That is, we must hope that students will continue learning after becoming teachers. So, what will be their basis for doing that?

Michael Glaubitz' talk in this conference gives reason for hope, in that it may be possible to do useful things without a solid and comprehensive knowledge of the history of mathematics to begin with.

3 Work to be done

In Norwegian textbooks for teacher training, the history of mathematics is presented through the texts for the students to read, with a few exercises here and there. (Smestad, 2010) With one notable exception, there are no ideas in these textbooks of other ways to include the history of mathematics in teaching. The one exception, the classic “Matematikk for lærere” (“Mathematics for teachers”) (Breiteig & Venheim, 1998), includes a list of options, but does not give examples of them. This is a striking contrast to the way most of the textbooks are generally written, with activities and suggestions for teaching.

So I believe that the main problem in Norway is that there are so few resources available to teachers. Teacher education can, sadly, only be a start of a learning process, but this process ends abruptly if teachers have nowhere to look afterwards. There is a tremendous gap between what we hear at conferences like HPM and ESU and what Norwegian teachers have access to. Improving on that situation is important. After doing that, at least it will make some sense to spend energy in teacher education to get teachers interested in the history of mathematics.

Therefore I am working on a wiki (eleviki) to make available materials on history of mathematics and ideas for teaching it. As I discussed in my talk on Monday, I want to make available concrete ideas for teaching, based on ideas from conferences like these and the literature (i.e. Demattè, 2006; Katz & Michalowicz, 2004; Pinto, 2009). In doing this, I also want to take into consideration weaknesses I’ve found in the Norwegian literature up to now. Thus, I want to emphasize the use of mathematics, not only its development, I want to emphasize mathematics suitable for teaching in primary schools, include mathematicians’ motivations, present more original sources, and to link all of these to the easily available Norwegian sources, making them easier to find by teachers. (Smestad, 2010)

In conclusion: in the present situation, Norwegian primary school teachers are not equipped to start including history of mathematics as they finish their teacher education. My goal is that some of them become interested in the subject, and have the means to gradually learn more. In the Norwegian situation, only by gradually improving the situation will the good examples be developed that will help bring more significant change in the future. A successful implementation in schools will be difficult unless there is a certain number of enthusiastic teachers who have already started this process.

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III - THE ROLE OF THE HISTORY AND EPISTEMOLOGY OF MATHEMATICS IN TEACHERS TRAINING – THE CASE OF ENGLAND

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1 Pre-service and in-service mathematics teacher education in England

1.1 Pre-service teacher training in England

There are three routes into teaching that are available to the entrants to the teaching profession in England, all requiring a completed first degree and a grade C in Mathematics and English at GCSE level[1]. These are:

- Postgraduate Certificate in Education, given at the successful completion of the course based at a university for around thirteen weeks, with the remainder of around 120 days spent in two schools in partnership with the university
- School-centered Initial Teacher Training, delivered by a consortium of schools and colleges, where the greatest part of instruction comes from the 'lead' school, and practice is organized in more than one of the schools from the consortium
- Graduate Teacher Training, based in a school (employment-based route) with a university as a school's partner to provide around ten days of training in professional studies (and almost no subject-related content).

All three of the possible routes into teaching last for one academic year; the first two being heavily supported/subsidized by the government through grants and bursaries, especially in such shortage subject as mathematics.

Whilst the teacher education programmes are fairly loosely prescribed by the government, for the universities and schools providing the courses leading to the Qualified Teacher Status, the outcome is deemed to be satisfactory evidence of covering the 33 Qualified Teacher Standards, as defined by the Teacher Development Agency.[2] The pursuit and inclusion of the history of mathematics in teacher training can, in the view of flexibility of designing the training courses, fall under various such standards.[3] Whilst it

[1] GCSE is a basic qualification, granted at 16.

[2] Qualified Teacher Standards for the Teachers in Education are given at the Teacher Development Agency's website – TDA (1), and the same exemplified for the training providers are listed in bibliography under TDA (2).

[3] Some of the Standards which can be possibly seen as offering opportunities for the introduction of the historical element into mathematics education are:

- **Standard 8:** Have a creative and constructively critical approach towards innovation, being prepared to adapt their practice where benefits and improvements are identified
- **Standard 10:** Q14 Have a secure knowledge and understanding of their subjects/curriculum areas and related pedagogy to enable them to teach effectively across the age and ability range for which they are trained.
- **Standard 18:** Understand how children and young people develop and that the progress and well-being of learners are affected by a range of developmental, social, religious, ethnic, cultural and linguistic influences

is, perhaps, easier to identify how one of the standards (standard 14 to be precise), dealing with the 'subject knowledge' can easily be linked with the history of mathematics in both terms of 'history as a tool' and 'history as a goal' approaches (Jankvist, 2009) it can also be used in teacher training courses in multiple ways, for example through 'historical-genetic-principle' (Schubring 1977), or by using the history to enable teachers in education to gain a greater repertoire of pedagogical approaches[4] (Van Maanen, 1997, Jahnke 1996). Barbin (1995, 1996, 1997) argues for the construction of knowledge through problem solving, while Furinghetti (2007) offers the alternative to use the history of mathematics as a 'reorientation' tool. Both of the latter mentioned are possible approaches to build on the students' existent subject knowledge in mathematics as the entry qualifications to the teaching profession include the 'considerable' element of mathematics in a degree (defining the 'considerable' is not always consistent between the institutions). The standards 18 and 19 further offer opportunities for introducing the history of mathematics through a cross-curricular and cross-cultural dimension (Lawrence, 2009; Barta, 1995; Katz, 1997; Grugnetti, 1994; and Proia & Menghini, 1984).

1.2 In-service teacher education

There is no unified view of the in-service education, or as we prefer to call it, continuing professional development, in England but the practice has developed over time to encourage the teachers individual engagement with academic research which, in turn, relies on their practice.[5] As of 2008, the 'historical and cultural roots of mathematics'[6] are part of the entitlement for every child's experience of mathematics; in practice however, the lack of in-service training means that this is usually not a matter of principle but rather a matter of a few individual cases as described in 2.2.

Between 1998 and 2008 the Gatsby Teacher Education Projects, of which the Gatsby Teacher Fellowship was a part, (funded by the Sainsbury Trust and affiliated with the Royal Society), as well as the Royal Society Small Grants for links between schools and industry, offered a small variety of opportunities for teachers to engage with research and include the history of mathematics into their continuing professional development. In fact only one project, by the author, was directly linked to the history of mathematics in the classroom (resources for 11-16 year olds, resulting in the website mathsisgoodforyou.com).[7]

The Royal Institution,[8] although not funding directly teacher development, has for

- **Standard 19:** Know how to make effective personalised provision for those they teach, including those for whom English is an additional language or who have special educational needs or disabilities, and how to take practical account of diversity and promote equality and inclusion in their teaching.

[4] As reported by Gulikers (2001), 227.

[5] For example, through the foundation of the National Teacher Research Panel, supported by the DCSF (Department for Children, Schools, and Families) during the Labour era.

[6] The National Curriculum, Mathematics KS3 programme of study (11-14 year olds): http://curriculum.qcda.gov.uk/key-stages-3-and-4/subjects/key-stage-3/mathematics/index.aspx and KS4 programme of study (14-16 year olds): http://curriculum.qcda.gov.uk/key-stages-3-and-4/subjects/key-stage-4/mathematics/index.aspx. Accessed 16th October 2010.

[7] All of the Gatsby Teacher Fellowship projects are listed on their website, see GTP.

[8] The Royal Institution of Great Britain was founded in 1799 by the 'Society for Bettering the Conditions and Improving the Comforts of the Poor' (as opposed to the prestigious Royal Society who, some perceived, favoured the well-to-do and established scientists). From 1978, under the guidance of Sir Professor Christopher Zeeman, a mathematician, the Mathematics Master Classes for 13-14 year olds were first

several decades supported the network of teachers who organize the Saturday master classes for the very able pupils. This has, for no explainable reason, been the 'hot-bed' of the development of ideas among teachers on how to use the historical context for their master-classes, and a number of prominent UK-based teachers have, through this involvement, developed a number of historical resources for their classes.[9]

Since Smith's (2004) Review[10] into the teaching and learning of mathematics, and the founding of the National Centre for Excellence in the Teaching of Mathematics (NCETM, founded in 2006) which followed the recommendations of the report, the consensus has been established[11] that the collaboration, networking, awareness of a variety of practices and resources, and the self-study and research in peer groups, are the most valid forms of CPD in mathematics education. To support the development of such a model of practice, the NCETM, immediately upon their foundation, issued a call for proposals for small-scale projects which fit the criteria of collaboration and research, and the project described at the end of this section was one which was supported through the first round of grants between 2006/7 and 2007/8 academic years.

In 2007, the Open University developed a 10-credit course for teachers on the history of mathematics, under the title 'The Story of Maths'. The course's four one-hour programmes were aired in 2008 on BBC4 and later BBC1, with tremendous success,[12] propelling the history of mathematics into an orbit of popular culture, and Professor Marcus du Sautoy, the lead presenter, into something closely resembling celebrity among mathematics teachers. 'The Story of Maths' course now runs twice a year, attracting around 200 students a year, and the resources that were part of the series are sold by popular retailers such as Amazon.

In the summer of 2009, Mathematics became part of the Prince's Teaching Institute programme. The Prince of Wales founded his Teaching Institute in 2001 with English and History as the first subjects, with the underlying aims '…to generate discussion about the specific contribution to education made by English Literature and History and about what constitutes an education in these subjects'[13]. After the interest of teachers expressed at the first Summer School in Mathematics held at the Queen's College, Cambridge,[14] the first continuing professional development event was organized by the Prince's Teaching Institute. This was a day on the History of Mathematics, held on 19th March 2010 at the London Mathematical Society. The oversubscription of the event, and the reports and demands of the teachers on the Mathematics Summer School Programme at the events afterwards, meant that the Prince's Teaching Institute plans to have a similar in-service

established, out of which grew a network of Master Classes throughout the country (now registering around 600 of the teachers/schools that provide these).

[9] To name but a few Martin Perkins and Peter Ransom have been active members of this network. The current list of schools, organizers, and network groups is available from http://www.rigb.org/contentControl?action=displayContent&id=00000001857.

[10] The UK Government commissioned a review into the teaching of mathematics into post-14 mathematics education, the results of which were summarized by professor Adrian Smith in the mentioned report.

[11] In particular see the aims of the NCETM (National Centre for Excellence in the Teaching of Mathematics at https://www.ncetm.org.uk/ncetm/about. Accessed 16th October 2010.

[12] If we are to judge by the reviews in the popular press; for example see http://entertainment.timesonline.co.uk/tol/arts_and_entertainment/tv_and_radio/article4893458.ece. Accessed 16th October 2010.

[13] See 'About us' on the Prince's Teaching Institute website.

[14] June 28th – July 1st 2009.

training day perhaps once a year in the future.[15]

Fig. 1. The participants of the first Mathematics Summer School of the Prince's Teaching Institute, Mathematical Bridge in Cambridge, July 2009.

Teachers who do not undertake either of the above in-service opportunities, may use the two major mathematics education websites, the NRich[16] and Plus Magazine[17], which give a good selection of articles, classroom ideas, and resources for the classroom on the history of mathematics. These stretch all ability levels and all stages of the secondary education, and the University of Cambridge's Mathematical Institute, with the NRich, organize usually three in-service training days for teachers on the various problem-solving approaches and work with the gifted and talented mathematicians. Although not directly linked to the history of mathematics, often the topics of problems and examples are 'rich' in history, giving another opportunity for teachers to get engaged with the subject.

While the above therefore, does not perhaps paint a picture of a unified development of a continuing professional development practice and the place history of mathematics may have in it, it does show an increased trend to favour the topic. It is also evident that there is an increased interest in the subject-matter and an increased awareness of its importance, from the change of the curriculum in 2008 to incorporate the historical and cultural roots of mathematics, to the commitment shown by the national bodies (the NCETM and the Prince's Teaching Institute) to support the history of mathematics as a valid and valuable contribution to the development of teachers at in-service level.

[15] For the short history of the PTA, see http://www.princes-ti.org.uk/AboutUs/Ourhistory/.

[16] See http://nrich.maths.org/public/

[17] See http://plus.maths.org/content/

2 History of mathematics in teacher education: theory and practice

2.1 Theoretical frame

As discussion on the history of mathematics ensued at the National Centre for Excellence in the Teaching of Mathematics[18] in 2007, the theoretical frameworks for incorporating the history of mathematics into mathematics education were examined.[19] On the national level, the introduction of the 'historical and cultural roots of mathematics' in 2008, through the (still valid and current in October 2010) National Curriculum meant that social constructivism as a theoretical framework would or could, to a certain degree, become an element of mathematics education and be placed at the heart of practice in mathematics classrooms across the country. However, there was an obvious lack of training opportunities for teachers in this regard. To bridge this gap, identified both by the Open University and the Prince's Teaching Institute (see the previous section), the two institutions offered courses and one-day events for teachers from across the country. The current state of affairs is therefore that the history of mathematics is a recognized and valid part of in-service training for teachers.

2.2 An example from practice

As already mentioned, the Smith Report (2004) identified collaborative practice as the most crucial and missing link in the pre-2004 provision of continuing professional development in England. The Lesson Study model, as exemplified by the Japanese, American, and Hungarian teachers (Burghes, 2008) was offered as a possible approach.[20] The History of Mathematics and Collaborative Teaching Practice, a project supported by the NCETM through 2006-8, linked the collaboration between teachers to establish a practice of enquiry and research into both the aims and motivations for using the history of mathematics into teaching.[21] In this project, two major benefits for teacher in-service training through the engagement with the history of mathematics have been identified:

- The reorientation – especially of non-specialist and primary teachers (Furinghetti, 2007)
- The mapping of the professional development landscape by teachers to identify their position in this landscape and their possible routes through it (Lawrence, 2008, 2010).

In summary, the history of mathematics in the classroom is becoming a reality through a number of initiatives, and the interest that teachers have in the subject is alive and well. An increased demand for the short courses for the practicing teachers is an encouraging

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[18] See in particular the article 'History of Mathematics' on the NCETM portal at https://www.ncetm.org.uk/resources/3245, and the History of Mathematics on-line community on the same portal at https://www.ncetm.org.uk/community/1324 (you must be a member to contribute to this community).

[19] See Lawrence, (2008), (2010), and Kaye (2008).

[20] Although the Burghes wrote the mentioned report in 2008, he has been a leading figure in promoting the collaborative teaching practice (and the Lesson Study) nationally; Burghes was also the first director of the National Centre for Excellence in the Teaching of Mathematics (NCETM).

[21] See the project report at https://www.ncetm.org.uk/files/397060/Final_Report_G002_Simon_Langton_Grammar_School.pdf and the project website at http://www.mathsisgoodforyou.com/lessonstudy/home.html.

trend. The curriculum change, I believe, will not be a change for the worse in this respect if teachers' voices are listened to.

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IV - HISTORY AND EPISTEMOLOGY OF MATHEMATICS IN TEACHERS TRAINING IN FRANCE

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It is difficult to give a complete institutional situation of teachers training in France, specially because things have been changing for some years, and also because the institutional structures (universities, "rectorats") are autonomous (not only for finance). Recently, a question of financial problems also arose, because of which for instance, the summer national universities disappeared some ten years ago.

Until 2010, the pre-service teachers training depended on each IUFM (University Institute for Teachers Training), which exists in each "Academy". These institutes provided training of one year for beginner teachers. When I taught in IUFM of Creteil, the course included 30 hours of teaching on the history of mathematics, but nothing on the subject was evident in some others IUFM. Since 2010, the IUFM discontinued the one-year model of teacher training. The future teachers now have to obtain a master's degree at the universities, where, at the same time, the epistemology and history of sciences appeared in the curricula of such courses. The importance attributed to the teaching of these subjects however, depends on the universities.

Until around twenty years ago, the in-service teacher training depended on the academic structures named "rectorats", and especially on the financial situation of these structures. We learned at the beginning of October 2010 (we did not know that when we were in Vienna) that many rectorats discontinued all teacher training (for secondary teachers) for financial reasons.

1 History and Epistemology in teachers training: new approachs of mathematical thinking and teaching

The history gives a cultural approach to mathematics, by placing mathematics into the context of the histories of sciences, technology, ideas, and societies. But it gives also an epistemological approach of mathematical activity, by emphasizing the part played by the problems, conjectures, experiences, by the rigor, analogy, error and modeling (CII-EHM, 2007).

The theme of the 17th Colloquium organized in Nancy by the Commission Inter-IREM (Epistemology and History of Mathematics) CII-EHM was "La figure et la lettre" (CII-EHM, 2010). Indeed, the history of mathematics permits to analyze the parts played by writing, figures, calculus, symbols, tables or diagrams in the activities of reasoning and proving. From this point of view, the Colloquium organized an event on "Explanation and Proof in Mathematics" some years ago, which was very interesting (Hanna, 2010).

The historical and epistemological approach raise issues related to a reflection on methods and contents of the teaching, and those related to the constructivist conceptions of notions and theories. Specifically, the history is seen here as a tool to study the processes of rectification of mathematical concepts.

History of sciences can also be used as a tool to enable the multidisciplinary approach to teaching. It was the theme of the last French Summer University organized by the CII-EHM, which took place in IREM of Poitiers in 2000 (CII-EHM, 2003). A good example is the study of motion by Galileo (Barbin, 1997). Of course, it concerns the physics because Galileo researched the law of falling bodies. But this law is also a rupture in the philosophy of nature. Indeed, the *Physics* of Aristotle was a research on the causes of the phenomena of nature, while Galileo was only interested in the effects of the motion, more precisely he studied the relations between the effects, like time, distance and speed, in a given motion. The reason for this interest turns out to had been a technical problem asked by the gunners, on the relation between the angle of the canon and the place where the cannon-ball falls, and this led to the investigation of the trajectory of a canon ball.

The contribution of the history of mathematics in relation to the new curricula is a subject developed in the IREMs. For example, the introduction of a new conception of probability in teaching linked with statistical ideas led the IREMs to new studies on historical relations between probabilities and statistics (CII-EHM, 2004). More recently, the introduction of algorithms in teaching, in relation to the use of computers in the classroom, was an opportunity to analyze the history of algorithms, their role and their importance in the history of mathematics.

The history of mathematics can, therefore, become a tool of criticism about the successive "new curriculums" in mathematics. I would like to mention the scattering of knowledge at both secondary and university levels, the confused part played by the assertions in secondary schools and the problems arising when the mathematics is seen only as a discipline to serve the other disciplines. Today, we can consider that the history of mathematics is a kind of "therapy against scattering", for instance it can be used to understand the relations between algebra, geometry and calculus, to study the "architecture" of mathematics, and to reflect on the "durability" of some mathematical knowledge along the centuries.

2 Purposes and themes for historical and epistemological training: four examples

I would like to give four examples of historical themes studied in the pre-service or in-service teacher training and in courses to mathematics students in Nantes.

The first theme concerns the reciprocal transformations of problems and concepts. The example of the tangent to a curve is especially interesting. We find geometrical propositions in Euclid's or Apollonius' writings, where the curves are conceived as geometrical objects, where the tangents touch the curves in one point (they did not cut them) and where the proofs are developed by absurdum (suppose they cut etc.). In the XVIIth century, the problem became a "method of invention" as related to the study of tangents. For Roberval, the problem became a kinematic one: a curve is the trajectory of a point in motion and the tangent is the direction of the motion. For Descartes, the problem was an algebraic one: the curve and the tangent circle are defined by two equations which must have only one common solution. So the problem of relationship between tangent and its curve is transformed and, at the same time, the notion of a curve to which a tangent is sought is transformed also.

The notion of rectification of concepts can be a way to introduce an idea of a long-term interest to the teaching of mathematics, and to anticipate the future development of the student teachers.

The second theme is “the proofs and the mathematical methods”, taken by the students of mathematics who want to become teachers. We begin with the axiomatic and deductive proof by Euclid, then we oppose this kind of proof with the algebraic method of Descartes to solve geometrical problems, the infinitesimal methods of Leibniz and Newton, the projective method of XIXth century and the method of “equipollence” of Bellavitis. This is used as a way of making a distinction between the context of proof and the context of invention, which are often confused in mathematical teaching.

The study of epistemological obstacles is dealt with through the extension of the concept of number: rational and irrational numbers, negative quantities and numbers, imaginary quantities and complex numbers, and the construction of real numbers. It is an opportunity to engage and reflect on the positivity of errors in mathematical activity (CII-HEM, 2007).

The last theme is “proofs and algorithms”, which took place at the in-service teacher training proposed by the IREM of Nantes. Firstly we examined the notion of algorithm in history, algorithms on numbers but also algorithms on geometrical figures. Secondly, we read many kinds of written algorithms to understand their difficulty. Thirdly, we arrived to the concepts of algorithms and machines with the example of the machine of Post.

3 Reading and working with original texts

When the French IREM began to organize in-service teacher training at the end of the 1970s, not many original texts were available. As a consequence, the collaboration between IREMs was organized with a purpose of choosing and editing texts which seemed interesting. I can mention some works which came out of this: those on the analysis of Euler, Lagrange and Cauchy, works on geometry texts by Arnauld, Clairaut, Monge, Poncelet and Chasles. The first anthology of these ancient texts appeared in 1987 under the title *Mathématiques au fil des âges*. In the same period, some IREMs edited anthologies on particular domains, like the IREM of Toulouse which edited three booklets on the equations of the first degree, on equations of the second degree, and on equations on the third and fourth degrees.

From 1999, the CII-EHM edited a collection of anthologies of ancient texts on more focused themes, like *La construction des nombres réels dans le mouvement d’arithmétisation* (Boniface, 1999). The title of the last anthology is *L’espérance du hollandais ou le premier traité de calcul du hasard* (IREM de Caen, 2006). It contains original texts of Christian Huygens, and his readers, de Montmort, Jacques Bernoulli, de Moivre and Euler. For each text, there is a historical introduction and commentaries on problems solved by mathematicians. It is therefore, possible to compare solutions, but also make comparison between the different conceptions of the probabilities and their calculus.

The French historian Paul Veyne wrote that “history [has] the virtue of astonishing with what comes of itself”: indeed, the reading of original sources can produce astonishment and give a possibility of a “dépaysement” (Barbin, 1997). Of course, the ancient texts must not be immediately interpreted in modern terms.

I give a course to future teachers of primary schools on the historical relations between numbers and figures. In the practical part of the teaching, students read original texts and the questions lead them to read them with eight purposes:

- to interpret (in historical context)
- to translate (example : Egyptian text)
- to compare translations of an original text

- to compare original texts
- to write in the same manner as somebody
- to pass from one manner to another
- to interpret (through the modern language)
- to compare two interpretations (in historical context or in modern language).

Interpreting an original text inside its historical context is very interesting for teachers or future teachers. Indeed, this process offers a possibility to reflect on epistemological obstacles, on the appropriateness between problems and methods, on the meanings of different solutions of a problem and so, on the meaning of the underlying theories.

4 To form but not to standardize: the introduction of an historical perspective in teaching mathematics

I think that the major role of the history and epistemology of mathematics is to lead the teachers to think, to think mathematically, and to think that their pupils are also persons with an ability to think. From this point of view, any model of introduction of the history of mathematics into the classroom will be harmful, especially if "lessons" of this model replace the historical study and the epistemological analysis.

In 1988, on the occasion of the meetings HPM in Florence and ICME in Budapest, the CII-EHM published a blue book titled in its English version *A case for an introduction of a historical perspective in the teaching of mathematics* (CII-IREM, 1988). This book is also edited by John Fauvel with the title *History in the classroom, IREM's papers* (Fauvel, 1990). This book contains 21 experiences of introducing the history of mathematics into the classroom. I wrote in the foreword to this book, that the reader should not consider these experiences as models of practice; these stories are the facts about practice of teachers who are in a situation which allow them to do research.

What does it mean to introduce a historical perspective into the teaching of mathematics? We do this neither because we can imagine a teaching as an imitation of a historical process, nor because we want to create a new school discipline completely independent of mathematical practice. The reason is that all the historical and epistemological reflections that a teacher comes across through the history of mathematics he/she can integrate in his or her teaching. This can be in order to situate the knowledge in historical context, to explain how and why a concept or a theory was invented, to read original sources, to solve historical problems, or to construct interdisciplinary activities with her or his colleagues.

To propose historical problems to pupils or students is very appropriate, because the problems give the meanings of concepts and theories. The book of the CII-EHM on the history of problems was translated into English by Chris Weeks, under the title *History of Mathematics, Histories of problems* (CII-EHM, 1997).

In 2009, the CII-IREM published a new blue book (which is black and red) tilled *Des défis mathématiques d'Euclide à Condorcet*. There are 9 experiences describing the introduction of the history of mathematics into teaching from colleges (students aged 11-14) to the university level. The authors are Anne Boyé, Renaud Chorlay, Jean-Paul Guichard, Patrick Guyot, Gérard Hamon, Frédéric Laurent, Loïc Le Corre, Dominique Tournés and myself (CII-EHM, 2010). Here also, we made sure to say, that these experiences should not to be considered as models. This is the reason for which each authors tells her or his experience by telling the story in first person, which is not common in French texts on teaching.

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ON DIOPHANTUS' *ARITHMETICA*, THE EXACT NATURE OF HIS PROJECT & ITS INTEREST FOR MATHEMATICS TEACHING TODAY

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ABSTRACT

Diophantus's *Arithmetica* is one of the most puzzling texts we kept from ancient Greek mathematics. Besides the fact that nothing is known of his author, not even the precise period in which he lived, the very nature of the project underlying this work is unclear. It has often been interpreted as an early work in algebra, beginning with some of the Arabic mathematicians that identified Diophantus's techniques as algebraic ones and therefore viewed him (eventually) as an algebraist. Indeed, Diophantus's work looks like algebra in many respects, most notably the use of a scale of unknowns (with their special, abbreviated designation) and the techniques used to solve the equations obtained within the treatment of the problems. But it is also unlike what is familiar to modern algebraists, for Diophantus puts very little emphasis on the equation within the treatments of problems he proposes and gives much more importance to the arithmetical problems themselves or to the numbers satisfying them. These questions are made still more complicated by the fact that we have no contextual interpretation of this work that would be really convincing, in spite of notorious attempts to do so like Tannery's or Jakob Klein's; therefore, the interpretation of Diophantus is inevitably distorted by the various layers of historical interpretations that have made this work a part of the progressive constitution of modern algebra.

In the workshop, I shall present a new interpretation of Diophantus's work, the product of an intensive cooperation between myself and Prof. Jean Christianidis (univ. of Athens). The result is an interpretation of the coherency of Diophantus's project taken as a whole that makes reasonable sense of it without any appeal to the later tradition(s) of algebra. For this purpose, we have developed an accurate and contextualized characterisation of key concepts in Diophantus (like 'problem', 'solution', 'positions', 'invention') as well as special analytical tools for the analysis of particular problems that I shall present in the course of the workshop. I hope to show that they are useful to enter the 'spirit' of Diophantus's approach to invention in arithmetical problems.

I will put much emphasis on one key aspect of this approach, namely the fact that it amounts to a clever and systematic process of *translation* of the indeterminate terms of a given problem within the terms of the arithmetical theory presented in the preface. While this process is of central importance in Diophantus, my personal feeling is that it tends to be neglected, if not completely ignored, in modern mathematical teaching. Most often, the problems available for study are given in such a way, that only simplistic choices for the unknown are suggested, when they are not just imposed on the students: this only suggests to them that this choice is of no real importance or even that there is no choice at all. I hope, therefore, that the confrontation with Diophantus will lead the participants to raise for themselves important questions about the modern practice of (algebraic) problem-solving.

PROOFS IN PRESYMBOLIC ALGEBRA

A preliminary account with implications for education

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ABSTRACT

In a survey of pre-symbolic algebraic texts, we have found proofs which roughly speaking could be classified in three types that we call “naïve”, “geometric”, and “algebraic”.

We call “naïve”, proofs whose argument relies on a lettered geometrical figure and a discourse that refers to what is seen in the figure, to actions of cutting and pasting on the figure, and to relations between parts of the figure. The warrant for the truth of what is said is what is seen on the figure, without casting any doubt on the sight.

This way of working with geometrical figures corresponds in some sense with the Iamblichus account of pre-euclidean geometry as a *historiē* (ἱστορίη) of shorts, an empirical investigation of the properties of geometric figures based on sight. However, the geometric figures that could appear in the text of the *Elements* are no longer the object of the study of Euclidean geometry, but rather signs that stand for the geometric objects whose possibility of construction is postulated. Indeed, Euclides defines the objects of geometry by separating them from the visible properties of geometric figures drawn on the ground, as means of organizing them (“A point is that which has no part”, I, 155; “A line is breadthless length”, I, 158) Hence, these definitions have to be accompanied by a postulate of the very conditions of the discourse within which the reader must dialogue. Furthermore, the properties of these objects can not be examined by cutting and pasting the geometric figures, unless the critique of this empirical procedure has been established. The properties of these objects are not revealed by plain sight, but rather each demonstrated proposition builds a new sense for the objects involved; each proposition accepted by the community of mathematicians institutionalises this sense as meaning, as a unit of content culturally established.

Consequently, we used the name “geometric” for proofs in algebra whose warrant no longer is what is seen in the figure, but theorems from Euclid’s *Elements*.

In this sense, al-Khwārizmī’s proofs of the algorithms for solving the composed second degree equations are “naïve”, as well as Ibn Turk’s proofs, and Thābit ibn Qurra, Abū Kāmil or ^{c}Umar Al-Khayyām are “geometric”.

However al-Khwārizmī does not use any figure to prove the algorithms for solving the simple ones. He just says: “I have shown its inference and its necessity”. And he further resort to this kind of explanation when dealing with operations on polynomia, explicitly stating that the semiotic means used to warrant the truth are the (algebraic) expressions: “As regards its necessity, it is clear by words (*al-laf*, expression)”. This kind of proofs is found in Al-Karajī and ^{c}Umar Al-Khayyām, along with the “geometric” ones, and is fully treated by As-Samaw’al and Ibn Al-Ha’im. We call it “algebraic” proofs, and they are characterised by the fact that geometrical figures are no longer involved in the proofs, which use the pre-symbolic algebraic expressions as the only semiotic mean.

DOES HISTORY HAVE A SIGNIFICANT ROLE TO PLAY FOR THE LEARNING OF MATHEMATICS?

Multiple perspective approach to history, and the learning of meta level rules of mathematical discourse

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ABSTRACT

In the present paper it will be argued that and proposed how the history of mathematics can play a significant role in mathematics education for the learning of meta rules of mathematical discourse. The theoretical argument is based on Sfard's theory of thinking as communicating. A multiple perspective approach to history of mathematics from the practice of mathematics will be introduced along with the notions of epistemic objects and techniques. It will be argued that by having students read and analyse mathematical texts from the past within this methodology, the texts can function as "interlocutors". In such learning situations the sources can assist in revealing meta rules of (past) mathematical discourses, making them explicit objects for students' reflections. The proposed methodology and the potential of history for the learning of meta-discursive rules of mathematical discourse is exemplified by analyses of four sources from the 17th century by Fermat and Newton belonging to the calculus, and it is demonstrated how meta level rules can be made objects of students' reflections. The paper ends with a proposal for a matrix-organised design for how the introduced approach to history of mathematics for elucidating meta-discursive rules might be implemented in upper secondary mathematics education.

1 Introduction

One can think of several purposes for using history in mathematics education: (1) For pedagogical reasons; it is often argued that history motivates students to learn mathematics by bringing in a human aspect. (2) As a didactical method for the learning and teaching of the subject matter of mathematics. (3) For the development of students' historical awareness and knowledge about the development of mathematics and its driving forces. (4) For general educational goals, with respect to which the so called cultural argument makes the strongest case for history, but history can also serve general educational goals in mathematics education of developing interdisciplinary competences as a counterpart to specialisation (Beckmann 2009). These purposes are not necessarily mutually independent. In carefully designed teaching sessions all four of the above mentioned purposes can be realized in varying degrees.[1]

Regarding the question whether history promotes students' learning of mathematics I have argued in (Kjeldsen 2011), that by adopting a multiple perspective approach to history from the practice of mathematics, history has potentials in developing students' mathematical competence while providing them with genuine historical insights. In the present paper, I will go a step further and suggest that history might have a much more

[1] See (Kjeldsen 2010) where it is shown how all these four purposes can be accomplished in problem oriented and student directed project work. In (Jankvist and Kjeldsen 2011) two avenues for integrating history in mathematics education are discussed with respect to the development of students' mathematical competence and historical awareness anchored in the subject matter of mathematics, respectively, both within a scholarly approach to history. In (Kjeldsen forthcoming) a didactical transposition of history from the academic research subject to history in mathematics education is proposed for developing a framework for integrating history of mathematics in mathematics education.

profound role to play for the learning of mathematics. This suggestion is based on Sfard's (2008) theory of commognition.

In the following it will be argued that, and proposed how, the history of mathematics can play a significant role in the teaching and learning of mathematics. The theoretical argument is outlined in section 2. In section 3, the multiple perspective approach to history of mathematics from its practice is presented along with some tools of historians'. The adaptation for mathematics education is discussed in section 4. The potential of history for the learning of meta-discursive rules of mathematical discourse is exemplified in section 5 through analyses of four sources from the 17th century by Fermat and Newton belonging to the calculus. In section 6 a proposal is outlined for a so called matrix-organised design for how such an approach to history of mathematics for elucidating meta-discursive rules might be implemented in upper secondary school. The paper ends with a concluding section 7.

2 The theoretical argument for the significance of history

In Sfard's (2008, 129) theory of *Thinking as Communicating* mathematics is seen as a discourse that is regulated by discursive rules, and where the objects of mathematics are discursive constructs. There are two kinds of discursive rules both of which are important for the learning of mathematics: object-level rules and meta-discursive rules.

The object-level rules have the content of the discourse as object. In mathematics they regard the properties of mathematical objects. The meta-discursive rules have the discourse itself as object. They govern proper communicative actions shaping the discourse. The meta-discursive rules are often tacit. They are implicitly present in discursive actions when we e.g. judge if a solution or proof of a mathematical problem or statement can count as a proper solution or proof (Sfard 2000, 167). The meta-discursive rules are not necessary; they are given historically.

The meta-discursive rules are connected to the object-level of the discourse and have an impact on how participants in the discourse interpret its content. As a consequence, developing proper meta-discursive rules are indispensable for the learning of mathematics (Sfard 2008, 202). This means that designing learning situations where meta-discursive rules are elucidated is an important aspect of mathematics education. History of mathematics is an obvious method for illuminating meta-discursive rules. Because of the contingency of these rules, they can be treated at the *object level* of history discourse, and thereby be made into explicit objects of reflection. Hence, history might have a significant role to play for the learning of mathematics, precisely because meta-discursive rules can be treated as objects of historical investigations. By reading historical sources students can be acquainted with episodes of past mathematics where other meta-discursive rules governed the discourse. If students study original sources in their historical context, and try to understand the work of past mathematicians, their views on mathematics, the way they formulated and argued for mathematical statements etc. the historical texts can play the role as "interlocutors", as discussants acting according to meta rules that are different than the ones that govern the discourse of our days mathematics and (maybe) of the students. By identifying meta rules that governed past mathematics and comparing them with the rules that govern e.g. their textbook, students can be engaged in learning processes where they can become aware of their own meta rules. In case a student is acting according to non-proper meta rules he or she might experience what Sfard calls a commognitive conflict, which is "a situation in which different discursants are acting according to different metarules" (Sfard 2008, 256). Such situations can initiate a metalevel change in the learner's discourse.

This, of course, presupposes a genuine approach to history. In section 3 and 4 it will be argued that within a multiple perspective approach to the history of the practice of mathematics, and by using historian of mathematics' tools such as the idea of epistemic objects and techniques, original sources can be used in mathematics education to have students investigate and reflect upon meta-discursive rules. For further discussion of this see (Kjeldsen and Blomhøj 2011), where also some student directed problem oriented project work performed by students at degree level mathematics are analysed with respect to students' reflections about meta-discursive rules to provide empirical evidence for the theoretical claim. These projects will not be presented here. Instead I will present a proposal (see section 6) for a so called matrix-organised design for how such an approach to history of mathematics for investigating meta-discursive rules might be implemented in upper secondary school.

3 A multiple perspective approach to history

The so called whig interpretation of history has been debated at length in the historiography of mathematics.[2] In mathematics education Schubring (2008) has pointed out how translations of sources, due to an underlying whig interpretation of history, have changed the mathematics of the source. In the whig interpretation history is written from the point of view of the present, as explained by the British historian Herbert Butterfield, who coined the term in the 1930s:

> It is part and parcel of the whig interpretation of history that it studies the past with reference to the present … The whig historian stand on the summit of the twentieth century and organises his scheme of history from the point of view of his own day. (Butterfield 1931, 13)

If we want to use history to throw light on changes in meta rules from episodes of past mathematics to our days mathematics whig interpretations of history poses a problem, because, as it has been pointed out by Wilson and Ashplant (1988, 11) history then becomes "constrained by the perceptual and conceptual categories of the present, bound within the framework of the present, deploying a perceptual 'set' derived from the present". In this quote, Wilson and Ashplant emphasis exactly why one cannot design learning and teaching situations that focus on bringing out differences in meta rules of past episodes in the history of mathematics and modern ones within a whig interpretation of history. Historical sources cannot function as "interlocutors" that can be used to clarify differences in meta rules if the sources is interpreted within the framework of how mathematics is conceptualized and perceived of today.

The trap of whiggism can be avoided by investigating past mathematics as a historical product from its practice. This implies to study the sources in their proper historical context with respect to the intellectual workshop[3] of their authors, the particular mathematicians, to ask questions such as: how was mathematics viewed at the time? How did the mathematician, who wrote the source, view mathematics? What was his/hers

[2] Discussions of whig interpretations in the historiography of mathematics can be followed e.g. in the following papers (Unguru 1975), (van der Waerden 1976), (Freudenthal 1977), (Unguru and Rowe, 1981/82), (Grattan-Guiness 2004).
[3] See (Epple 2004).

intention? Why and how did mathematicians introduce certain concepts? How did they use them and for what purposes? Why and how did they work on the problems they did? Which kinds of tools were available for the mathematician (group of mathematicians)? Why and how did they employ certain strategies of proofs? Such questions can reveal underlying meta rules of the discourse at the time and place of the sources. By posing and answering such questions to the sources, possibilities for identifying meta rules that governed the mathematics of the source can emerge, and hereby also opportunities for turning meta rules into explicit objects of reflection in a teaching and learning situation.

As explained by Kjeldsen (2009b, 2011) one way of answering such questions and to provide explanations for historical processes of change is to adopt a multiple perspective approach to the history of the practice of mathematics. I have taken the term "a multiple perspective" approach from the Danish historian Jensen (2003). It signifies that episodes of the past can be studied from several perspectives, several points of observation, depending on which kind of insights into, or from, the past, we are searching for. Episodes in the history of mathematics can e.g. be studied from the perspective of sub-disciplines within mathematics to understand if, and if so, how other fields in mathematics have influenced the emergence and/or the development of the episode under consideration. They can be studied from an applied point of view to understand e.g. dynamics between pure and applied mathematics, or the role of mathematical modelling in the production of mathematical and/or scientific knowledge. They can be studied from a sociological perspective to understand the institutionalization of mathematics, its funding etc. They can be studied from a gender perspective, from a philosophical perspective and so on.

4 Adaptation for mathematics education

In mathematics education the above approach can be implemented on a small scale, by focusing on a limited amount of perspectives that address the intended learning. In the present context the purpose is to use past mathematics and history of mathematics as a means for elucidating meta discursive rules and make them into explicit objects of students' reflections. Hence, students should study the sources to answer clearly formulated historical questions that concern the underlying meta rules of the mathematics in the source.

Theoretical constructs that have been developed by historians of mathematics and/or science to investigate the history of scientific practices can be used to "open" the sources. With respect to the purpose of the present paper of uses of history to reveal meta rules of a (past) mathematical discourse by studying the history of mathematics from its practice, the notions of epistemic objects and techniques are promising tools. The term epistemic object refers to mathematical objects that are treated in a source, i.e. the object about which mathematicians were searching for new knowledge or were trying to grasp. The term epistemic technique refers to the methods employed in the source by the mathematicians to investigate the epistemic objects.[4] These theoretical constructs can give insights into the dynamics of concrete productions of pieces of mathematical knowledge, since they are constructed to distinguish between elements of the source that provide answers and elements that generate mathematical questions.[5]

[4] These notions have been adapted into the historiography of mathematics by Epple (2004) from Rheinberger's (1997) study of experimental science.
[5] For examples of uses of this methodological tool see (Epple 2004) and (Kjeldsen 2009a).

The question is whether history dealt with in this way, where students study episodes from the history of mathematics from perspectives that pertain to meta rules of (past) discourses, ask historians' questions to the sources concerning the practice of mathematics, and answer them using theoretical constructs such as epistemic objects and techniques, can facilitate meta level learning in mathematics education. In the following section four texts from the 1600s will be analyzed to provide some answers to this question.

5 Analysis of four sources within the proposed methodology

Four texts from the 1600s will be used in the following; two by Pierre de Fermat (Fermat I and Fermat II) and two by Isaac Newton (Newton I and Newton II). Fermat I is Fermat's text on maxima and minima taken from Struik's (1969) *A Source Book in Mathematics, 1200-1800*, whereas Fermat II is called "A second method for finding maxima and minima", which is published in Fauvel's and Gray's (1988) reader in the history of mathematics. Newton I is Newton's demonstration of how he found a relation between the fluxions of some fluent quantities from a given relation between these. This text is the one prepared by Baron and Bos (1974), whereas Newton II is Newton's method of tangent taken from Whiteside's (1967) *The Mathematical Works of Isaac Newton*. The quality of these translations of sources can be criticised, and investigated for degrees of whiggism (Schubring 2008), but this will not be done in the present paper. In a teaching situation the students should work with the four texts, but in order to give the reader an impression of the texts, summaries of the four texts are inserted here:

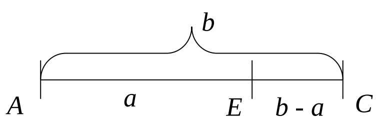
In Fermat I, Fermat stated a rule for the evaluation of maxima and minima and gave an example. The text is summarised below in Box 1.

Fermat I: On a method for the evaluation of max. and min.

Rule: let a be any unknown of the problem

- Indicate the max or min in terms of a
- Replace the unknown a by $a+e$ – express max./min. in terms of a and e
- "adequate" the two expressions for max./min. and remove common terms
- Both sides will contain terms with e – divide all terms by (powers of) e
- Suppress all terms in which e will still appear – and equate the others
- The solution of this equation will yield the value of a leading to max./min.

Example: To divide the segment AC at E so that AE x EC may be a maximum



Max: $a(b\text{-}a) = ab\text{-}aa$

$(a+e)b\text{-}(a+e)(a+e) = ab+eb\text{-}aa\text{-}2ae\text{-}ee$

$ab+eb\text{-}aa\text{-}2ae\text{-}ee \sim ab\text{-}aa$ *"adequate"*

$eb \sim 2ae + ee$ remove common terms

$b \sim 2a + e$; $b=2a$; $a=½b$; divide, suppress, solve

Box 1

If the above procedure is translated into modern mathematics using functions and the derivative it can be explained why Fermat reached the correct solution. But this does not explain how Fermat was thinking, since he knew neither our concept of a function nor our concept of derivatives. In Fermat II we can get a glimpse of how Fermat was thinking.

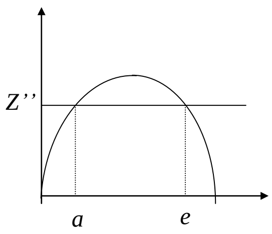
The text is summarised below in Box 2.

Fermat II: A second method for finding maxima and minima

- Here he explained why his "rule" leads to max./min.: correlative equations – Viete
- Resolving all the difficulties concerning limiting conditions

Example: To divide the line *b* such that the product of the segments shall be a max.

> If one proposes to divide the line *b* in such a way that the product of the segments [*a* and (*b*-*a*)] shall equal *z''* … there will be two points answering the question, and they will be found situated on one side and the other of the point corresponding to the max.



$ba\text{-}aa = z''$ and $be\text{-}ee = z''$

$ba\text{-}aa = be\text{-}ee$; $ba\text{-}be = aa\text{-}ee$

Divide by $a\text{-}e$

$b = a + e$

At the point of maximum we will have $a = e$, then

$b = a + a = 2a$, hence as before $a = \frac{1}{2}b$.

If we call the roots *a* and *a*+*e* (instead of *a* and *e*) the procedure follows the rule from text I.

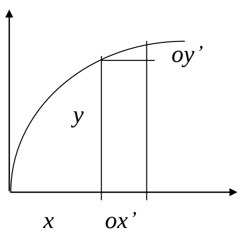
Box 2

In Newton I, Newton explained through an example, how, given a relation between fluent quantities, a relation between the fluxions of these quantities can be found. In Box 3 his procedure is summarised and illustrated with an example of a second degree equation instead of the third degree equation that Newton used in the text.

Newton I: Find relation between fluxions from fluents

Newton's fluxions and fluents

- Curves are trajectories (paths) for motions
- Variables are entities that change with time – fluents *x* , *y*
- The speed with which fluents change – fluxions *x'* , *y'* (*Newton: dots!*)
- Newton: All problems relating to curves can be reduced to two problems:
  1. Find the relation between the fluxions given the relation between the fluents.
  2. The opposite.



Example: $axx+bx+c\text{-}y=0$ substitute *x*, *y* with $x+x'o$, $y+y'o$

$a(x+x'o)(x+x'o)+b(x+x'o)+c\text{-}y\text{-}y'o=0$

$\underline{axx}+a2xx'o+ax'x'oo+\underline{bx}+bx'o+\underline{c\text{-}y}\text{-}y'o=0$

$a2xx'o+ax'x'oo+bx'o\text{-}y'o=0$

$a2xx'+\cancel{ax'x'o}+bx'\text{-}y'=0$ divided by *o*; cast out terms with *o*

$a2xx'+bx'\text{-}y'=0$ hence $y'/x'=2ax+b$

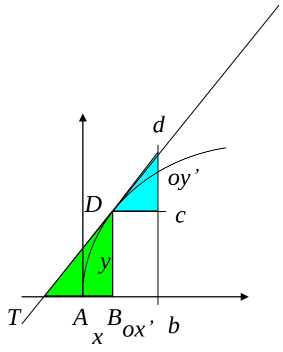
Box 3

In Newtons's terminology *o* denotes an infinitely small period of time, so *ox'* [Newton used a dot over *x* instead of *x'* to designate the fluxions] is the infinitely small addition by which *x* increases during the infinitely small interval of time.

Finally, in Newton II, Newton showed how to draw tangents to curves and illustrated it with the same example as he used in the first text. In Box 4 below the example is illustrated with reference to the example used in Box 3.

Newton II: To draw Tangents to Curves

Example:



Similar triangles: *dcD* and *DBT*

TB:BD = Dc:cd "infinitesimal triangle"

BT/y = x'o/y'o =x'/y'

x'/y' can be found by the method from Newton I

Box 4

The suggestion made in this paper is that these four sources can be used to exhibit changes in meta rules of mathematical discourse, if students read the sources from the perspective of rigor, and focus on entities and arguments. The following worksheet (Box 5) can be used to guide the students work. It consists of two sets of questions. The first set concerns questions that help the students to identify the epistemic objects and techniques of the two texts. The students are asked to compare and contrast the answers they get from studying Fermat, Newton, and their textbook, respectively.

Perspective

Rigor – entities, arguments

Worksheet: History from the practice of math. Compare/contrast Fermat and Newton

Questions:

What mathematical objects are Fermat/Newton dealing with? Compare/contrast

How do they perceive them? – compare with your textbook

What are the problems they are trying to solve?

What techniques are they using? – what do we do today?

How do they argue for their claims? – how do we argue today?

Epistemic objects and techniques

Can you find any changes in understandings of the involved mathematical concepts from Fermat over Newton to today? Explain

Can you find any changes in the way of argumentation from Fermat over Newton to today? Explain

What kind of objections do you think your math teacher would have to Fermat's and Newton's texts?

Meta-rules – explicit object of reflection
Opportunities provided by history

Box 5

The second set of questions refers directly to meta rules of the involved mathematical discourses.

Regarding the first set of questions, an analysis of the four texts and the comparison between the objects that Fermat and Newton investigated, how they perceived them, the problems they tried to solve, the techniques they used and the arguments they employed might be summarised in the following scheme (Box 6):

Fermat:	**Newton:**
Objects: curves - algebraic expressions ex.: multiplication of line segments	Objects: any curve variables that change in time
Perceive: Area; geometrical problems treated by algebraic methods	Perceive: trajectories for moving particles
Problem: evaluate max/min	Problem: relations between fluxions (velocities) given relations between the fluents
Techniques: equations, roots, algebraic mani.	Techniques: algebraic mani; physics, geometry
Argue: Text 1: shows the method works on an example Text 2: heuristic arguments with roots in equations given by an example	Argue: Physical arguments about distance and velocity, algebraic arguments, infinitesimal triangle, *o*-infinitely small

Box 6

Regarding the second set of questions, which refers to meta rules of the discourse, the following changes can be discussed (se Box 7):

Changes in understanding:

Fermat: curves; algebraic expressions

Newton: curves, traced by a moving point, variables change in time

Today: functions, correspondence between variables in domains

Changes in the way of argumentation:

Fermat: ad hoc; “it works – its true”; heuristic argument, no infinitely small quantities

Newton: more general procedure, physical arguments, infinitesimal triangle, infinitely small quantities (*o*)

Today: limit, the real numbers, epsilon-delta proofs

Box 7

In Kjeldsen and Blomhøj (2011) we have analysed some student directed problem oriented project work conducted by students in a degree level university mathematics programme. Here we were able to demonstrate that history, used within the framework of a multiple perspective approach to the history of mathematics from its practice, can be used in mathematics education to give students insights into how meta rules of a mathematical discourse are established and why/how they change. These projects were made in a rather unique educational setting and the question is whether this methodology can be implemented in more traditional educational settings. The analyses of the sources guided by the worksheet (Box 5) and presented in Box 6 and Box 7 suggest that this approach can elucidate meta rules and turn them into explicit objects for students reflections. In the following section I present an outline for a so called matrix-organised design for how such a multiple perspective approach to history of mathematics from its practice might be implemented in upper secondary mathematics education.

6 Implementation in upper secondary school: A proposal

In the Danish upper secondary school system history of mathematics is part of the mathematics curriculum. The curriculum is comprised of a core curriculum which is mandatory and is tested in the national final, and a supplementary part, which should take up 1/3 of the teaching. History is mentioned explicitly in the supplementary part, which means that all upper secondary students should be taught some aspects of history of mathematics. The supplementary part of the curriculum is tested in an oral examination together with the core curriculum. In Box 8 below an outline is presented for a matrix organised design for how history could be (but has not yet been) implemented in a Danish upper secondary school for elucidating meta rules within the theoretical framework of section 2, 3 and 4, using the sources and the worksheet presented in section 5.

Implementation in a Danish high school: a proposal

Step 1: Six groups – basic groups (worksheets would have to be prepared for each group with respect to the intended learning)

1. The mathematical community in the 17th century
2. The standard history of analysis
3. Who were Fermat and Newton?
4. The two texts of Fermat - the questions of the worksheet of Box 5
5. The two texts of Newton - the questions of the worksheet of Box 5
6. Berkeley's critique of Newton

Step 2: Six groups – expert groups (each group consists of at least one member from each of the basic groups)
The experts teach the other group members of what they learned in their basic group. Each expert group write a common report/prepare an oral presentation of the collected work from all six basic groups as it was discussed in their expert groups

Step 3: A plenary discussion lead by the teacher focuses on methods of argumentation, the development/changes in the perception of objects and techniques, compared with the standards of today.

Box 8

This design follows a three step implementation. First six groups (so called basic groups) are formed who look into some aspects of the historical episode in question. In Box 8 it is suggested e.g. that group 1 investigates what the mathematical community of the 17th century looked like. Guided by a worksheet with questions relevant for the intended learning, the work in this group will provide the students with a sociological perspective on mathematics and its development. In step 2 new groups (so called expert groups) are formed. They consist of at least one member from each of the six basic groups. In this way each new group consists of individual experts. Each expert now teaches the other members of the new group what he/she learned in his/hers basic group, and based on their shared knowledge provided by the various experts they answer the second set of questions of the worksheet in Box 5. The design is referred to as being matrix organised because it can be illustrated with a matrix, where the members of basic group 1 is listed in column 1, the members of basic group in column 2, etc. In step 2 the expert groups are formed by taking the students in the rows, i.e. expert group 1 consists of the students listed in row 1; expert group 2 of the students listed in row 2, etc. In this way all expert groups consists of at least one member from each basic group. In such a set up it is possible to create complex teaching and learning situations where students work independently and autonomously in an inquire-like environment, developing general educational skills as well.[6]

7 Discussion and conclusion

The main question in the present paper is whether working with sources in the spirit of the worksheet of Box 5 within the methodology outlined in section 3 may give rise to situations where meta rules of (past) mathematical discourses are made into explicit objects of students' reflections, and whether this can assist the development of students' proper meta rules of mathematical discourse. As pointed out above, the analyses of the sources guided by the questions of the worksheet in Box 5, and the suggestions for answers outlined in Box 6 and 7, suggest that history and historical sources can be used within the methodological framework of section 2, 3 and 4 to elucidate meta rules and make them explicit objects for students reflections.

Regarding the second part of the question, whether such an approach to the use of history and historical sources in mathematics education also can assist the development of students' proper meta rules of our days mathematics is a complex question which is much more difficult to document. The framework and methodology outlined in this paper provide a theoretical argument for the claim that history has the potential for playing such a profound role for the learning of mathematics, but in order to realize this in practice more research needs to be done, and methodological tools for detecting students' meta rules and for monitoring any changes towards developing proper meta rules need to be developed.

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[6] Such a matrix organised design for using history in mathematics education to elucidate meta rules of past and present mathematics, using sources from the history of the development of the concept of a function, to have students reflect upon those, to develop students' mathematical competence, and general educational skills of independence and autonomy is being tried out in a pilot study in a Danish upper secondary class at the moment. Preliminary results from this study indicate that some of the students act according to meta discursive rules that coincide with Euler's; and that reading some of Dirichlet's text created obstacles for the students, that can be referenced to the differences in meta discursive rules. Results from the study will be published in forthcoming papers.

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THE TEACHER AS A RESEARCHER IN THE HISTORY OF MATHEMATICS

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ABSTRACT

A mathematics teacher has the potential to be a good researcher in the history of mathematics. A researcher should have a broad knowledge in mathematics, and should be interested in the various ways in which mathematical problems can be solved and mathematical theory can be developed. It would be even better if the researcher has, next to the mother tongue, one or more other languages. Knowledge of the classical cultural languages would be even better, but with translated editions of sources one can already do important historical research.

The other way round, the same kind of claim holds: someone who does, or has done research in the history of mathematics, could make a good mathematics teacher. This person has an open eye for the strange routes along which mathematics grows, and can apply this attitude also in fostering the growing young mathematician. Moreover this person has a broad cultural knowledge of mathematics, knowledge which supports motivation to learn mathematics. Mathematicians had a reason to create new mathematics. New theory solved the problem on which some of them had worked for years. Knowing the problem behind the theory can help students understand why it is important to learn the theory. It can also elucidate the internal structure of the theory.

The claim behind the workshop is therefore:

Research in the history of mathematics is a valuable and attractive activity for mathematics teachers, and it has a strong professional development effect on them.

The workshop consists of several elements:

1. a short introduction about the rationale behind this claim
2. presentation of results of research done by teachers
3. plenary discussion, with the purpose to strengthen the rationale and to refine the ‘picture’ by have input about the situation in various countries
4. brainstorm in small groups about historical topics that would make a good research subject for teachers
5. plenary inventory of research topics and making up some general conclusions.

As background material, some (passages from) research publications from the Netherlands will be provided, which will support the presentation listed under 2.

HOW CAN WE IMPROVE OUR REASONING?

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ABSTRACT

Normally mathematicians and mathematics teachers think that it is sufficient to know logical rules to improve our reasoning abilities, but this doesn't explain why we can observe the occurrence of systematic errors in reasoning. Peter Wason and Philip Johnston-Laird from Princeton University stated that the *content* of a claim has an influence on the way people reason. This conjecture is contrary to Piaget's claim that we master logic in childhood.

After several experiments, they conclude that when we reason, we construct *mental models* of situations, and we use those models to represent possibilities. Following this new theory Logic is an essential tool for all sciences, but is not a psychological theory of reasoning.

The explanation for the systematic errors in our reasoning is, following the *Model Theory*, that we aren't so able to represent the false, i.e. we tend to mentally represent only the truth.

My students and myself, after having explored different logical questions, have invented the "L'ocalogik" game to let other students reach awareness about difficulties occurring in reasoning having fun at the same time. This game was proposed at the Scientific Communication Festival "Scienza Under 18" that took place in Monfalcone (Gorizia) from the 6th to the 8th of May 2010.

THE USE OF PERIODICITY THROUGH HISTORY

Elements for a social epistemology of mathematical knowledge

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ABSTRACT

This is a historical review related to the use of periodicity in order to form a significant basis that will broaden the current educational schemes for the periodic property. We intend to exhibit the *periodic aspect* as part of an epistemology of practices so the meaning for periodicity can be redefined and its teaching in school could be differentiated from the sole need for acquiring, recognizing or handling the periodic property.

1 Introduction: educational problems in regard to periodicity

Periodicity is a concept present in the development of scientific thought. Starting with pattern observation, human beings are capable of abstracting this property in order to generate scientific knowledge. Examples of its use while developing scientific knowledge are numerous: Pannekoek (1961) identifies the systematic observation of celestial bodies' periodic behaviour as the origin of astronomy as a scientific activity; Whitehead (1983) points out periodicity as the property which favours an analysis of the analogies between different physical phenomena. It turns out to be a property of different kinds of objects that begins in the everyday individual experiences (the year seasons, night and day), it enters into school's mathematics from the very beginning (periodic decimal numbers) and goes through several school's disciplines (phenomena in physics, functions as in calculus) all of which form part of the students' scientific culture.

In view of this, periodicity – as a quality of that which is repeated at certain intervals – should be part of a functional mathematical knowledge that would allow the student to travel between different areas of scientific knowledge. However, its current treatment in school limits its recognition and use as a result of the narrow analytic framework within which it is dealt. We give examples that illustrate two school situations related to the treatment of periodicity; the first is about a mathematical object belonging to the first school levels and the second is about objects belonging to higher educational levels.

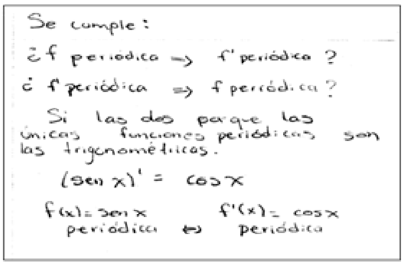
Example 1: Periodic Fractions

The initial learning in school for periodicity is related to patterns with special emphasis upon the observation of order. The tasks, aiding this, refer to completing sequences of drawings; nevertheless, these are not enough for the student to recognize the periodic behavior of a mathematical object, nor for him to take advantage of it, in order to perform some other type of tasks. As an example, figure 1 shows the procedure followed by a 21 years old college student when she was asked to find the number that ranks 120 in the periodic fraction 7/22: she began writing explicitly all numbers trying to reach the 120^{th} place (Buendía, 2006a).

Figure 1. Which is the number that ranks 120?

Example 2: Periodic Functions

At higher school levels, periodicity is treated as a property of functions, particularly trigonometric ones. This property is presented by using the equation $f(x) = f(x+p)$ placing x in the domain of the function and period p, which is institutionalized as *the* definition of periodicity and in many cases, is bounded to $\sin(x) = \sin(x+2\pi)$. Thus, the only reference that students often have, are trigonometric functions and especially, the sine one. Figure 2 shows the answer of a mathematics teacher at the double implication *f periodic* ↔ *f' periodic* (Buendía & Ordoñez 2009).



Se cumple:
¿f periodica ⇒ f' periódica?
¿f' periódica ⇒ f periódica?
Si las dos porque las únicas funciones periódicas son las trigonométricas.
$(\text{sen } x)' = \cos x$
$f(x) = \text{sen } x$ $f'(x) = \cos x$
periódica ↔ periódica

Yes, both (are true) because the only periodic functions are the trigonometric ones.

(sin x)'= cos x

f(x) = sin x f'(x) = cos x
periodic ↔ periodic

Figure 2. *f periodic* ↔ *f' periodic*

Faced with this type of didactic phenomena, we tried to broaden the current educational schemes for the periodic, so that this property can be understood not only as the acquisition or application of the equation $f(x) = f(x+p)$, but as everything that is related to periodicity: the particular repetitive behavior of the mathematical object and how it can be significantly recognized. We use the history of mathematics as a source for information regarding the way men did what they did while developing mathematics related to the periodicity. In that sense, we will talk about the *periodic aspect*, as an expression that will allow us to characterize the periodic quality in different mathematical objects that live in school and to make it meaningful for each of them.

2 Theoretical and methodological aspects

According to Bell (1949), the mathematics of periodicity, as opposed to the mysticism with which natural periodic phenomena were treated, was originated in 1748 when Euler determined the values of circular functions. Whitehead (1983) mentions this property as a concrete example of the effect upon the abstract development of mathematics over science and even establishes that the birth of modern physics was based upon the application of the abstract of periodicity to a wide variety of concrete examples, and "when it became completely abstract, it was useful" (p. 334). The tasks performed by those people are totally dependent on their socio-cultural paradigms; this is the type of data that make up our significant basis for the periodic quality.

Socioepistemology, as a theoretical approach, recognizes mathematics as a human, cultural and historically determined activity; it points out that didactic phenomena cannot be understood or analyzed without reviewing the evolution of the object to be taught, (…) which leads us to question the contents and meanings proposed in the study plans (Ferrari and Farfán, 2008; p. 310). A socio-espistemological research is based upon the recognition of didactic phenomena and educational problems (figure 3) and undertakes a historical review, in those communities that build and use mathematical knowledge, evidencing those circumstances that are socio-culturally situated, which surround the scientific tasks of man (Buendía & Montiel, 2011). It is a search for practices, which, without being necessarily explicit, does foster, guide or even govern the generation of certain mathematical knowledge. As a product – never ending, always in constant configuration – of these reviews, it is feasible to propose an epistemology of practices that supports the role of these in the construction of mathematical knowledge.

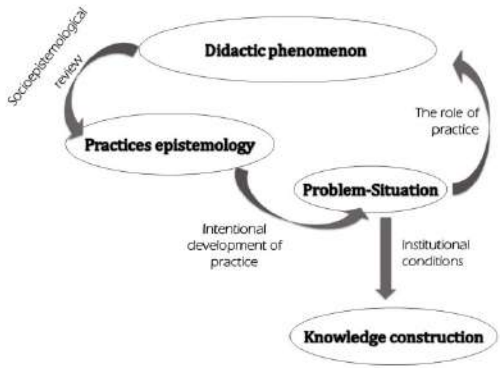


Figure 3. Methodology

The epistemology of practices is the base of the didactic designs for the mathematics' classroom; however, in order to achieve this, the practices must be re-interpreted (middle of the diagram figure 3). As a first step, the practices would have to be intentionally developed in a context based on the students' socio-cultural and institutional reality, because it is not the purpose to repeat activities or challenges in their historical version. Thus, Socioepistemology does not turn to history in order to incorporate it to the teaching of mathematics, but rather looks for what man did to generate knowledge, reinterprets those practices and develops them intentionally in didactic designs.

These didactic designs can be quite different: activities sequences, laboratory practices, proposed methods for class work. But what they all have in common is the previous work: a *socioepistemology* that supports them. And it is through them that a positive impact is achieved in the classroom.

The main objective of this paper is to show some of the results of the socio-espistemological review in its historical aspect to generate a significance base for the periodic quality; in order to do this, here are the methodological tools guiding this review.

2.1 The use given to periodicity

Under a socio-epistemological point of view, Espinoza and Cantoral (2010) state that scientific work must be understood as a production with history; belonging to an era, to a human being who has his own germinal ideas and his own means of significance. Additionally, it should be recognized that every mathematical work is different in regard to its intention to divulge, because there are differences between a didactic intention and the intention for publishing scientific knowledge.

What we have done is to try to distinguish the different *uses of periodicity* in various mathematical works which we felt were emblematic. This epistemological construction for *use* (Cordero, 2008; Cordero and Flores, 2007) refers to the different functions and forms that periodicity can take in mathematical works dealing with periodic situations, phenomena, movements or mathematical objects. Analyzing these different forms and functions of periodicity will allow us to evidence practices related to the significance of this property, or, to put it another way, practices related to the constitution of periodicity.

The chosen mathematical works are presented in three moments whose central point is the XVIII century. The reason for this selection is that during that era, trigonometric functions formally are included in the analysis and periodicity became a property of them. We consider then that the use of periodicity shows significant changes during this time. Therefore, analyzing what happens before and after the XVIII century is useful for the discussion we are proposing.

3 First Moment: Periodicity as a shared property that may be generalized

The works of Barbin (2006), Arnol'd (1990), among others, point out the scientific interest existing in the XVIII century related to time measuring, especially, the time having as its pragmatic purpose the creation of increasingly accurate clocks. At this time mechanistic thinking governs scientific work in such a way that science is the bearer of new developments, inventions, progress and break-ups. Techniques (inventions, machinery, and mechanical arts) are the tools that provide an insight into how nature works; phenomena become artificial issues that can be studied.

How to achieve a scientific relationship between the observable – let's say the swing of a pendulum - and the *know-how* sought by science — such as manufacturing increasingly accurate clocks? This was what society requires from scientists. In this paradigm, periodicity takes on the form of an internal quality characterizing different behaviours, something that is taken as given, intrinsic to the action and that allows the abstraction of the concept, in order to generate similarities between all those different behaviors.

Seemingly unrelated situations such as measuring the time, the nature of musical sounds, or the internal composition of bodies were addressed as special cases of vibratory motion. Hence then the relevance of studying and analyzing basic motions as the spring or pendulum and that meant finding relationships between components and laws. Hooke is a concrete example which illustrates the use of periodicity under this paradigm. His scientific work was guided by the search of laws, especially those governing elastic phenomena and materials, which make him go down in history (Cross, 1994).

Above all, Hooke was a “man-laboratory”[1], so his interest in scientific issues had to be diverse; he addressed issues such as the composition of bodies, free fall and planetary motion. This *know-how* could surround that which, in our days, school succinctly presents as *Hook’s Law;* he had to begin by understanding the relationship between the force exerted on the mass of a chord and its position x because this relation would allow him to go further in all his other areas of interest. Periodicity was taken as the common property that allowed him to generate a more abstract way of thinking and to apply the main principles of the periodic quality on many other behaviors.

For example, he established that the internal movements of a solid body are vibratory and, therefore, those bodies whose particles oscillate harmonically will have a relatively stable form and volume (Gal, 2002). As for the fall of bodies, Hooke established that it was necessary to take into account whether the body was outside or inside the Earth; in this last case, the law is different since the layers traversed by the body will attract it in different directions. Therefore the law of motion inside is apparently similar to that observed in elastic oscillations (Arnol’d, 1990).

In the study of springs, school mathematics recognizes the historical fact of Hooke’s law but, in creating an epistemology of practices for periodicity, we are more interested on how Hooke used that property. In this first moment, periodicity was not yet assigned to a certain function, nevertheless the periodic quality favoured the development of scientific thinking. From the geometric and mechanic study of springs, periodicity allowed to study other vibrating situations. It was a shared quality that could be generalized.

4 Second Moment: Periodicity as a property of trigonometric functions

The XVIII century is characterized by the process of converting mathematical analysis into an autonomous scientific discipline: all the initial calculus’ concepts are gradually losing their geometric and mechanic shell, moving toward a geometric and algebraic formulation (Youschkevitech, 1976). The dominant paradigm of that time focuses on the mathematization of movement, and therefore, properties such as periodicity are being questioned from this angle.

The problem of a vibrating chord[2] perfectly reflects this new paradigm as it causes a strong reflection on the definitions given to several of the concepts addressed until then, among them that of function, its properties and representations. In fact, Grattan-Guiness (1970) states that most of these scientific controversies came to light only at the time when there was established a requirement for periodicity, as a condition for the solution of the problem. It seems that this property is the focus of discussion on the problem and the controversy lies on what was being understood as periodic at that time: in the XVIII century, this attribute is

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[1] During the 40 years in which he was curator for the Royal Society (the English Academy for Science), he had to present between three and four experiments each week in order to demonstrate natural laws (either his own experiments or the experiments of other scientists). This characterizes his production – and his problems with other scientists of his time – because he did not always have time to mathematically support his experiments or findings or even to write them down (Arnol’d, 1990).

[2] The problem of the chord can be described in the following way: suppose that a flexible chord is pulled taut and its ends are fixed at 0 and a of the abscissa. Then the chord is slackened until it takes the form of a curve $y = f(x)$ and is then let go. The question is: what is the movement described by the curve?

intrinsically contained within an analytic expression, without reference to an independent domain; the analytic expression is the function itself (Farfán, 1997).

Euler, for example, accepted as a periodic function a parabola $f(x) = hx(a-x)$, since it can be made periodic by reflecting the arcs in relation to the straight lines $x = \pm na$. This seems to imply a return to geometric aspects so we can talk about a "geometric periodicity". Although this idea of assigning different intervals of definitions for a function was not consolidated at this time, for us is important to notice that periodicity is characterizing the repetitive behavior of the curve and not necessarily the function being analyzed.

In *Introductio in analysin infinitorum,* periodicity is a property that allowed Euler to generalize the properties in his newly raised geometric function, and, in his own words, *unravel many questions the use of which will be of advantage for the solution of the most difficult issues* (Durán, 2000). For a fuller discussion of the use of periodicity in *Introductio* see Buendía and Montiel 2011. Various authors as Durán (2000) consider the – implicit – recognition of periodic property for the sine and cosine.

In this second moment, periodicity takes sense in the mathematization of movement, especially as a tool for predicting. This use necessarily influences upon its formalization, as in Euler's *Introduction,* taking forms as $\sin(2\pi + z) = +\sin z$ or $\cos(2\pi \pm \varphi) = \cos\varphi$ which operate within an increasingly analytical framework for doing mathematics. These forms are the ones which in today's schools have become institutionalized.

In our socio-epistemological search for meaningful aspects of periodicity, this property can be seen now as qualifying the behavior of any given function, particularly in its graphic form. It also acquires meanings in the mathematical manipulation of periodic phenomena, for example, to predict on them.

5 Third Moment: Periodicity and its variations

During the XIX century, the practical usefulness – understood as a scientific thought encompassing various areas – quickly gave way to purely mathematical interest. In this context, the use of periodicity becomes interesting as a property that can be exploited for the development of scientific thinking. This can be found in the work developed by Poincaré, who is considered as the last universal man. By broadening his studies on differential equations and on elliptic functions that have double periodicity[3], periodicity is a part of *invariance* under transformations forming a group. This property assumed a more complex form because, starting from the premise that a function is periodic with period p if $f(z) = f(z+p)$, it now emphasizes that for a complex function, what happens is that the function *f does not change* when z undergoes a transformation of the $z' = z + kp$ (k integer) group. Thus, the elliptic functions do not change when z is substituted by the group of transformations $z' = z + kp + k'p'$ (Collete, 1986).

Based upon this approach, he developed works such as the one that merited the prize offered by the King of Sweden, Oscar III, during the mathematical competition trying to determine the stability of the Solar System, as a variation of the three-body problem. This

[3] An elliptic function has two different periods p_1 and p_2 thus verifying that $E(z + p_1) = E(z)$ and $E(z + p_2) = E(z)$.

problem consists in determining at any given time, the positions and velocities of three bodies, of any mass, subjected to their mutual attraction and starting from given positions and velocities. The equations involved cannot be resolved in terms of known functions and since the problem even has countless solutions, Poincare concentrated on the relationship between these periodic solutions (Collette, 1986). Thus, he developed a new approach for finding solutions to differential equations governing periodic movements in which the positions and velocities are simultaneously involved.

According to Aluja (2005), the periodicity of movement is determined not only when the body goes through the same point, but at the same speed and in the same direction at a given time. The periodic quality is used in the so called Poincaré section: instead of following with a telescope a body's entire trajectory around the Earth, one focuses on a plane going from north to south, from one horizon to the other, and which is aligned with the center of our planet. One must take note of the place where it goes by for the first time, its speed and its direction, and one lays in wait, focusing only upon the plane. Periodicity causes the body to appear at the same point, with the same speed and direction, so the periodic quality and its variations can become a tool for prediction.

Thus, periodicity would seem to be made up by increasingly abstract generalizations, trying to understand and take advantage of the complexities of periodic movements. These descriptions must take into account the interrelations, but not only how between variables such as time and distance, but also in their variations. In this context, periodicity works as an important predictive tool because of its variations characteristics: how does a periodic function varies and how do their variations change.

Now is evident the poor treatment that the school gives to periodicity. In first place, periodicity is use to qualify a function – merely, a trigonometric one - but its teaching do not favor the recognition of the function behavior as the one that causes the periodic adjective. So, any sinusoidal function, including its graph or the motion modeled by it, inherited the periodic property; periodicity ends as “anything that repeats”. This can cause didactic phenomena as shown in the initial examples. The third moment enriches this meaning by pointing out not only the repetitive character of a mathematical object, but also “how does it repeats itself”.

6 Discussion: periodicity's uses

In the historical data we have presented, periodicity has been continually appearing in the development of the scientific thinking of humanity through different uses. To speak about the use of periodic property in the light of the – implicit – exercise of practices provides mathematics and especially periodicity with a different epistemological character.

Periodicity evolves from being an intrinsic property of several phenomena which cannot be questioned, because it does not make sense, but which allowed the generalization and finding similarities. This is, perhaps, a handling similar to the one found in school, especially in lessons on physics and differential equations that deduce laws and properties. However, recognizing the role of practices broadens our horizon.

Although there are moments where there is a formalization and generalization of scientific knowledge in which the periodic property takes its standard form as equality, it is important to recognize that the related tasks, as predicting in motions, give meaning and significance to those formalizations and generalizations. There is an implicit practice that gives meaning to what nowadays school presents as a pre-existing property which only has to be *applied.*

It should be pointed out that the scientific and meaningful use of periodicity describes not only the repetition of a movement, but also its successive variations. Or, it graphically describes the behavior of any type of curve: a translation in one direction.

On this basis of significance for that which is periodic, visible activities such as *modeling, graphing, experimenting, measuring* stand out, all of which are performed in different situations and under very different mathematical tools. These activities could seem to be articulated, motivated or even regulated by practices such as *prediction* or *formalization.* The proposal is that these activities and practices, epistemologically related to the significant recognition of periodic property, sustain the didactic proposals.

7 Toward the mathematics classroom

7.1 Recognizing periodicity

Buendía (2006b, Buendía and Cordero, 2005) offers evidence that the intentional prediction on repetitive graphics supports the distinction between *it is repeated* and *how it is repeated.* In order to do this she presents a set of 8 diagrams (figure 5), asks for a description of how any type of mobile is moving in each of them, and asks for a prediction of the mobile's position in the 231 time (the diagrams show up to t = 12) and then to tell which of them are periodic. This promotes a distinction between the type of repetition present at each graph, because in order to be able to predict, it is necessary to distinguish the repetitive behavior of time as well as the repetitive behavior of distance. Thus, periodicals are those which comply, as was answered by an engineering student at the end of the interview, *with the condition of equal times and equal distances*.

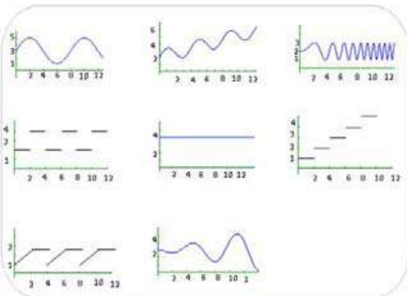


Figure 5. Predicting on diagrams

Vázquez (2008) states that the significant recognition of what is periodic requires two actions enhanced by the fact of predicting upon a repetitive object; first, identifying upon the object that part which will contain sufficient information to be able to predict (analysis unit) and, then, to be able to apply some procedure in order to perform said prediction. It is important to point out that these actions are performed using the mathematical tools that students acquire according to their age and educational level. The example in figure 6 shows these two actions performed by Dulce, a 7 year old student, who at the beginning of the interview said that she knew how to add from 10 to 10, even though she had not yet been taught this in school.

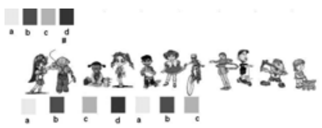
To end the week, I want to give different flavored chewing gum to my friends in school, following that order: (a) mint, (b) tutti-frutti, (c) peppermint, and (d) cinnamon. My friends are 23. I want to know, what flavor will correspond to the last person in line? 	First, she matched with her fingers the printed images of the chewing gums with the children standing in line, but then the researcher gave her several boxes of chewing gum. There were yellow boxes for the mint flavor, pink boxes for tutti-frutti, green for peppermint and red for cinnamon. Dulce realized that, in order to be able to predict, the easiest thing would be to first do her *units of analysis* in the form of small piles of colors, which would make it easier for her to take chewing gums from each group.  Once she had identified the four flavors, she discovered that with 8 chewing gums she was able to repeat them by adding from 8 to 8. She added three times 8 and then took away one chewing gum saying that the mint flavor would correspond to him.

Figure 6. Prediction activity for children

These researches suggest that the significant recognition of the periodic aspect is linked not only to the completion of sequences, but also to activities involving long term predictions; they do not depend on whether a student can be able to predict or not, because it is in the use of his own mathematical tools that the way in which an object is repeated comes into play. This poses a much more significant backdrop for the discussion of periodicity.

7.2 Periodicity and its variations

Buendía and Ordoñez (2009) performed an analysis on Miranda’s (2003) didactic material, in which, faced with the need to make a prediction, it becomes necessary to analyze not only the repetition of a movement, but also its variations. This material is about cam design (figure 6); the analysis of how the movement changes, according to the cam’s form, is a key point that must be taken into account to ensure that the mechanism will not cause any problems.

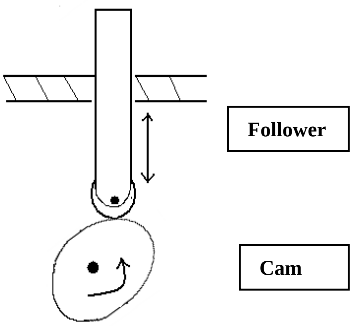


Figure 6. The follower regularly pushes the cam

Miranda points out that, although there are cams that cause seemingly soft movements, such as the parabolic (see figure 7), its derivatives may actually not be thus. Being that derivative is a measurement of the speed with which the cam's movement changes, it will help to check that the follower's movement is *soft* and maintains its periodic behavior. Therefore it is important that, for high speeds, we check that the cam profiles do not show abrupt changes in speed or acceleration, so the diagrams must illustrate that there are no abrupt changes in slope or discontinuities.

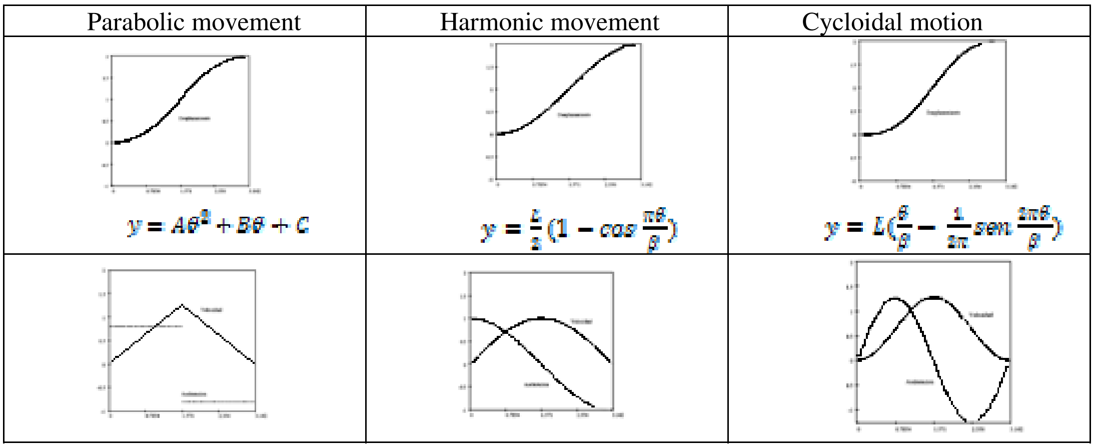


Figure 7. Displacement, speed and accelerations of possible movements

In this treatment, the mathematical concepts of successive derivatives and the physical concepts of speed and acceleration acquire meaning in the mathematization of the periodic variation.

7.3 The periodicity in graphics

The most common presentation of periodicity in text books compares point by point two states of one function by means of the equation $f(t+k) = f(t)$. However, we have discussed how the periodic quality, in the behaviour of graphics, can be identified comparing *equal pieces*. This is often present in visual characterizations involving mathematical terms such as the displacement (Courant and John, 1982):

> *Geometrically interpreted, f(t) has period p (p = b-a) if a displacement of its diagram in p units toward the right hand side again leads us to the same graphic.* (p. 337)

Viewing periodicity through graphics may be done point by point or by pieces; both are present in the didactic system and the teacher's explanations should take into account these different uses for that which is periodic.

8 Final comments

School deals with a finished mathematical knowledge and its goal is to find a way for students to acquire that knowledge. By proposing a socio-epistemology of the mathematical knowledge, we are questioning that mathematical knowledge. From being a finished mathematical result that we have to learn, it turns out to be the product of human activity while developing scientific knowledge; its nature is recognized as totally situated and dependant of the context.

In each of the epistemological moments presented, there is a set of historical data about scientific studies in which the periodicity was somehow involved. The basis of meanings for periodic property has considered not only those historical facts, but also why and how the men got involved. We can recognized that the mathematics developed has been guided or even governed by underlying practices such as predicting; from them, periodicity gets meaning and significance.

We are problematizing the periodic property knowledge confronting school mathematics with the use of knowledge in different historical settings. The purpose is to recognize those significations that belong to knowledge but usually become diluted, altered or lost when setting up a school discourse.

Periodicity turns out to be so much more than the application or the testing of equality. Through history, periodicity was used as an argument that qualifies a certain repetitive behavior and from there, under a more analytic use, was a property for functions that modeled oscillatory movements. School mathematics usually recognizes this last moment, but we think the very nature of periodic property is shaped also by its other socio-epistemological moments shape. All of them should have space in the mathematical curricula in the form of didactic interventions that intentionally develop the practices that constitute the socio-epistemology for periodicity.

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A BRIEF STUDY OF GEORGE BOOLE'S PAPER:

"Exposition of a general theory of linear transformations"

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ABSTRACT

In the History of Mathematics if we seek the origins for English Algebra in the nineteenth century, certainly we would find at least two names and a theory: Joseph Sylvester and Arthur Cayley and the Invariant Theory. Historians of mathematics argue that this theory has expanded in the English scenery after the publication of the "Exposition of the General Theory of Linear Transformations" by George Boole, in 1841, providing the basis for both Cayley and Sylvester's studies. We made a brief essay about Boole's work (1841) encouraged by the thesis that this work can be considered by historians of Mathematics the beginning of the Invariant Theory. However Boole's paper does not point out to an argument plausible enough to support a casual relationship between both studies.

L'HISTOIRE DE LA REPRÉSENTATION GÉOMÉTRIQUE DES NOMBRES COMPLEXES ET L'ENSEIGNEMENT DE LA GÉOMÉTRIE.

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ABSTRACT

Dans cet article, nous essayerons d'abord de souligner la diversité des conceptions qui ont mené des mathématiciens vers la fin de la XVIIIème et le commencement du XIXème à développer une représentation géométrique des nombres complexes. La présente partie sera basée, sur les travaux récents et sur une lecture personnelle des sources historiques. Nous discuterons dans une deuxième partie de la manière dont ces diverses conceptions ont été diffusées et pratiquées par les mathématiciens et l'importance que la représentation géométrique du complexe a prise dans la seconde moitié du XIXème siècle dans certaines disciplines des mathématiques. Une troisième partie est une tentative de déterminer la place que peut avoir la géométrie des nombres complexes dans la formation des professeurs de mathématiques de l'enseignement secondaire au Brésil.

1 Introduction

Ce travail est issue d'une réflexion visant à déterminer le contenu d'un enseignement centré sur la géométrie des nombres complexes destiné à la formation des professeurs de l'enseignement secondaire, à l'Institut de Mathématiques de l'Université fédérale de Rio de Janeiro. Avant d'évoquer ce problème, il est bon de voir comment cette vision géométrique des nombres complexes est apparue, et comment elle est devenue dans bien des recherches un outil incomparable.

2 L'émergence d'une vision géométrique des nombres complexes

Ce que nous entendons dans l'histoire des mathématiques comme la représentation géométrique des nombres complexes recouvre en fait différentes conceptions élaborées par différents mathématiciens à la fin du XVIIIème et dans la première moitié du XIXème siècle. L'article de Study traduit par Cartan en 1908 dans *l'Encyclopédie des Sciences mathématiques* cite presque toutes les références des ouvrages et articles où figurent ces différents points de vue (Cartan & Study 1908, pp.339-354). Notre intention n'est pas de faire un exposé exhaustif, mais seulement de tenter de classer ces différentes représentations suivant leur problématique.

Wessel, Warren et Hamilton:

Le travail Wessel n'a pas été connu de la communauté des mathématiciens avant la fin du 19ème siècle quand il fut découvert et traduit en Français en 1897 (Wessel 1897). Dans son mémoire, Wessel tente de déterminer comment on peut concevoir un calcul de segments orientés. Il définit ainsi les opérations sur ces segments, et démontre alors que ces opérations sur ces objets géométriques peuvent êtres réinterprétés en associant à chaque segment un nombre complexe (Flament 2003 p.110 and sq.). Avec Wessel nous avons donc plutôt une représentation analytique de grandeurs géométriques par les nombres complexes. Wessel tente par ailleurs de déterminer des nombres qui correspondraient aux segments orientés de l'espace, mais il échoue dans sa tentative. La problématique de Warren est analogue (Andersen 1999, p.83), son point de départ est aussi un calcul géométrique, et il s'agit de représenter les objets géométriques et leurs opérations au moyens des nombres complexes. Son originalité la plus marquable est le lien qu'il établit entre les rotations du plan et le produit de deux nombres complexes.

La démarche d'Hamilton (1837) est plus générale et son support intuitif n'est pas l'espace mais le temps. A partir d'une notion intuitive du temps, il construit tous les nombres, partant des nombres entiers jusqu'aux irrationnels qu'il conçoit comme limite de nombres rationnels. Pour introduire les nombres imaginaires, il a recourt à des couples de moments, définit les opérations sur ces couples et montre qu'elles ont les mêmes propriétés que celles des opérations sur les quantités imaginaires. En fait il reconnait dans les complexes un plan de dimension deux, puisque que le problème qu'il se pose dans les années qui suivent est de savoir si l'espace possède lui aussi un calcul analogue à celui des complexes, ainsi qu'il le raconte (Hamilton 1853, p.31):

> «il y avait toutefois un motif qui me poussait à attacher une importance particulière à la considération de triplets, séparés d'ensembles plus généraux, desquels on a rendu compte. C'était le désir de connecter, suivant une méthode utile et nouvelle (ou pour le moins intéressante), calcul et géométrie, au travers de quelque extension non découverte, à l'espace de trois dimensions d'une méthode de construction ou de représentation, qui a été employée avec succès par Mr. Warren et en fait par bien d'autres auteurs» (notre traduction).

Mais en ce sens la démarche d'Hamilton rejoint celle de Wessel, il s'agit de représenter analytiquement les mouvements du plan et de l'espace au moyen d'un calcul algébrique.

Argand et Cauchy:

La démarche d'Argand (1806) et le but de son travail témoignent dune perspective différente. Il s'agit de donner une légitimé des nombres négatifs et imaginaires au moyen d'une représentation géométrique. Il fait correspondre à chaque nombre imaginaire une ligne orientée, et explique la signification géométrique que l'on peut tirer des opérations sur les grandeurs imaginaires (pour une analyse précise de ce travail, Flamment 2003, pp. 165-195). On sait la succession d'articles dans les *Annales* de Gergonne que l'ouvrage suscitera, après la divulgation de ses principaux résultats par Français dans ce journal. Il est particulièrement significatif que ces articles n'aient eu de développements immédiats dans les écrits des principaux mathématiciens de l'époque, même si Legendre avait été impressioné par le travail de d'Argand (Schubring 2001, p. 128-129). Cauchy attendra 1847 pour en venir explicitement à une représentation géométrique des quantités imaginaires, rappelant

> «que dans mon Analyse algébrique publiée en 1821, je m'étais contenté de faire voir qu'on peut rendre rigoureuse la thérorie des expressions et des équations imaginaires, en considérant ces expressions et ces équations comme symboliques» (Cauchy 1847, pp.175-176).

Suivant Bourbaki 1984, p.203, il y a un écart entre la définition à caractère formel que donne Cauchy et la pratique dans la mesure où dans beaucoup de ses travaux il y a un lien entre expressions imaginaires et points du plan. Nous pouvons difficilement imaginer aujourd'hui les méthodes du calcul des résidus sans une vision géométrique sous-jacente des nombres imaginaires. Dahan Dalmenico a analysé en détail cet écart et l'explique par une tension entre différents programmes de recherches dans lesquels s'est engagé Cauchy. Elle oppose le programme fondationnel de Cauchy, 1821 (Dahan Dalmenico 1997, pp.30-32) où les quantités imaginaires sont définies formellement, et participent d'une construction et d'une définition du champs de l'analyse, aux différentes méthodes et pratiques qu'il développe dans l'analyse de la variable complexe dans les mémoires des années 1820 et 1830 (Dahan Dalmenico, 1997, pp.33-36), où elle discerne, certes, une évolution dans les formulations, mais qui, selon elle, ne permettent pas d'affirmer que Cauchy utilise une vision géométrique (Dahan 1997, p.35). Toujours est-il qu'aprés son Mémoire sur les quantités géométriques de 1847 (Cauchy 1847), il est difficile de nier le caractère géométrique dans ses travaux postérieurs sur les fonctions de la variable complexe, puisque dans ce mémoire, il montre comment appliquer cette conception géométrique à la démonstration du théorème fondamental de l'algèbre (Gilain 1897, p. 68).

Gauss

Avec son mémoire *Theoria residuorum biquadraticorum*, K. F. Gauss (1831a) développe une nouvelle perspective de recherches pour l'arithmétique et l'algèbre. Afin de démontrer dans toute sa généralité la loi de réciprocité quadratique, il réalise une extension du domaine de l'arithmétique aux nombres qu'il qualifie de complexes, une nouvelle terminologie qui sera vite adoptée par la communauté des mathématiciens, et associe à tout nombre complexe un point du plan. Il revient dans le compte-rendu de son travail sur cette présentation:

> «Par contre l'arithmétique des nombres complexes est susceptible de la plus compréhensible saisie par les sens (anschaulichsten Versinnlichung), et même si l'auteur dans son exposé a observé cette fois un traitement arithmétique pur, néanmoins pour rendre ainsi cette compréhension plus vivante, et pour cette saisie par les sens fortement recommandée (empfehlende Versinnlichung), il a donné aussi les indications nécessaires, lesquelles seront suffisantes pour le lecteur qui pense par lui-même» (Gauss 1831b, p.174, notre traduction).

Gauss ne cherche pas à donner un statut ontologique aux nombres complexes par le biais de cette présentation sensible. En évoquant le traitement arithmétique pur, Gauss suggère que l'existence des nombres complexes vient de l'algèbre pure. Il est d'ailleurs remarquable qu'il utilise le terme Versinnlichung qui signifie l'acte de rendre sensible, plutôt que Vorstellung (terme qui correspond à représentation). La nécessité d'une telle présentation sensible est plutôt liée à l'art de l'invention. Comme il l'indique (ibid. p.175),

> «L'auteur a abordé il y a bien des années cette partie importante de la mathématique avec un point de vue différent, suivant lequel un objet pouvait être attribué aux grandeurs imaginaires tout aussi bien qu'aux négatives: mais il a manqué jusqu'ici une occasion d'exprimer de manière précise celui-ci publiquement, même si le lecteur attentif en retrouverait facilement les traces dans l'écrit de 1799 sur les équations et dans l'écrit du Prix sur les transformations des surfaces» (notre traduction).

Il est clair que cette vision des nombres complexes l'a souvent guidé dans ses recherches, comme en témoigne, au-delà des textes qu'il cite dans son compte-rendu, la fameuse lettre à Bessel de 1811 (Gauss W. B. X, p.367) où il explique à propos de l'intégrale d'une fonction de la variable $x = a + bi$, qu'il faut associer à ce nombre un point du plan dont les coordonnées sont prises à partir d'un axe réel et d'un axe imaginaire. Ce point de vue accompagne ainsi ses raisonnements dans tous les domaines où intervient la variable complexe.

La démarche de Gauss est ainsi bien différente de celle de Cauchy. En effet, il n'y a pas de tension entre vision géométrique et caractère symbolique des nombres complexes. Cette vision géométrique accompagne ses recherches et est mise à contribution dans ses découvertes.

3 Diffusion et pratiques liées à la représentation géométrique des nombres complexes

Il est aussi intéressant de noter l'impressionnante suite d'articles et d'ouvrages en langue allemande portant sur une conception géométrique des nombres complexes qui suit ce travail de Gauss. W. Matzca ne dénombre pas moins de dix publications s'étalant entre 1834 et 1847 (Matzca 1850, pp.150-151). Si ces travaux n'apportent pas d'éléments essentiellement nouveaux, ils ont permis, comme le remarque (Flament 2003, pp.272-273) une divulgation rapide de cette conception.

La fécondité de cette approche des nombres complexes sera encore confirmée par les travaux de Riemann. Dans les Principes fondamentaux pour une théorie générale des fonctions d'une variable complexe (Riemann 1851), il considère dès les premières pages les fonctions complexes comme une relation entre deux points de deux plans complexes, puis presqu'immédiatement (p.6) comme des fonctions du plan dans lui-même. Tout son travail verse sur des considérations géométriques, la dérivée d'une fonction de la variable complexe est conçue comme une transformation linéaire conservant les angles, les représentations des fonctions multiformes utilisent des plans multiples reliés par les ramifications. L'ouvrage s'achève sur le problème des représentations d'une surface sur une autre avec en conclusion une référence aux deux travaux les plus importants de Gauss sur ce dernier sujet.

Riemann situe donc sa recherche dans la continuité de l'oeuvre de Gauss. C'est aussi le cas de sa thèse d'habilitation (Riemann 1854), où l'on peut lire (pp.281-182): « ... à l'exception de quelques brèves indications données par M. Gauss dans son second Mémoire sur les résidus quadratiques, dans les Gelehrte Anzeigen de Göttingen et dans son Mémoire de Jubilé, et quelques recherches philosophiques de Herbart, je n'ai pu m'aider d'aucun travail antérieur ». La source d'inspiration est encore ici le compte-rendu de Gauss de 1831 où, en conclusion, il évoque une extension du plan complexe à des variétés de dimension supérieure (Gauss 1831b, p. 178).

Un autre aspect qui apparaît dans les années qui suivent est l'étude des propriétés des transformations géométriques planes au moyen des nombres complexes. Siebeck semble être le premier qui ait développé ce type d'étude dans un article *Über die Graphische Darstellung imaginär Functionen* (Cartan & Study, n. p.364). Il expose les propriétés des complexes et quelques fonctions élémentaires comme les similitudes à l'aide des nombres complexes. L'exemple peut-être le plus significatif est l'étude de quelques transformations circulaires (Siebeck 1858, pp.243-244), où il retrouve les résultats trouvés par Möbius (1855) qu'il cite explicitement. Möbius avait étudié ces transformations du point de vue de la géométrie synthétique. Ici l'analyse complexe s'en approprie le contenu.

Ces transformations circulaires, joue aussi un rôle déterminant dans la géométrie projective (homographies) ainsi que le souligne F. Klein dans le programme d'Erlangen de 1872 (Klein 1974, pp.18-21) à propos de ce qu'il appelle « la géométrie des rayons vecteurs réciproques » (nom que l'on donne alors à ce type de transformation). Ces transformations jouent aussi un rôle dans la construction de modèles de la géométrie hyperbolique comme l'a mis en évidence Poincaré dans ses mémoires sur les fonctions Fuchsiennes (Gray & Walter 1997).

4 L'enseignement des nombres complexes et la géométrie

Ce qui apparaît aussi nettement dans la conception de Klein est que l'approche analytique et synthétique fusionnent lorsqu'on parvient à caractériser les géométries au travers des groupes de transformations et de leurs invariants. Or cette opposition (analytique/synthétique) est demeurée longtemps dans le domaine de l'enseignement secondaire et supérieur. C'est encore le cas dans les programmes brésiliens de l'enseignement secondaire, où géométrie synthétique et analytique sont souvent enseignées de manière séparée au cours des trois années du lycée, cette division se reproduisant dans les cours de Licence qui forment les professeurs de l'enseignement secondaire. Pour changer ce cadre, il faut d'abord poser le problème de la formation des professeurs, et fournir des outils afin de concevoir une application des programmes qui ne soit pas aussi dogmatique. Nous pensons que la géométrie des nombres complexes peut contribuer à changer ce cadre.

Nous avons souligné l'importance des homographies et antihomographies du plan complexe, qui ont joué un rôle important dans la compréhension des différentes géométrie et dans leur classification.

C'est aussi ce que souligne (entre autres considérations) le rapport d'étape de la commission Kahane (2000), rédigé pour l'essentiel par Daniel Perrin, (puisque le contenu de ce rapport se retrouve déjà dans Perrin (1999). Dans sa lecture du Programme d'Erlangen et les perspectives qu'il trace à l'enseignement de la géométrie, il préconise l'enseignement de géométrie «riche», en particulier la géométrie de l'inversion. Daniel Perrin définit une géométrie riche comme une géométrie donnant lieu à une double lecture des propriétés par le biais d'un isomorphisme (Kahane 2000, p.370 et aussi Perrin 1999, pp.11-12):

> « L'exemple le plus spectaculaire de tel isomorphisme concerne la géométrie anallag-matique plane cf. §i). Algébriquement l'isomorphisme s'exprime ainsi: si q désigne la forme de Lorentz sur R en 4 variables: $q(X, Y, Z, T) = X^2 + Y^2 + Z^2 - T^2$ on a un isomorphisme de son groupe orthogonal direct avec le groupe des homographies à coefficients complexes: $O^+(q) \cong \mathrm{PGL}(2, \mathbb{C})$. On peut penser que ces isomorphismes jouent un grand rôle dans la géométrie élémentaire et que les théories dans lesquelles on rencontre un isomorphisme de ce type sont particulièrement riches. Dite de manière grossière, l'idée est la suivante: le fait que le groupe admette ainsi deux variantes fait que la géométrie en question cumule les deux types d'invariants "naturels" correspondant à ces variantes, dans l'exemple précédent l'invariant birapport de PGL(2,C) et l'invariant φ de $O(q)$ (la forme polaire de q qui donne les notions d'orthogonalité ou de contact des cercles-droites) et donc produit deux fois plus de théorèmes "intéressants". Cet isomorphisme explique sans doute la richesse de la géométrie anallagmatique (que l'on vérifie en parcourant les vieux manuels)».

Ce point de vue éclaire l'importance essentielle que revêt l'enseignement de la géométrie anallagmatique, du fait de la richesse des configurations qu'elle permet d'étudier.

Nous pouvons ajouter à cet aspect, ce qui partiellement en découle, c'est-à-dire le fait qu'elle représente le pont d'une part entre la géométrie du plan euclidien et la géométrie projective et aussi, un élément essentiel dans la construction de modèles euclidiens des géométries non-euclidiennes. Pour donner un exemple, on peut citer l'ouvrage de Lyon (Lyon [2001]) qui propose dans ses premiers chapitres, un parallèle en géométrie pré-euclidienne (sans l'axiome des parallèles) entre le modèle habituel de représentation et le demi-plan de Poincaré où les isométries sont les homographies.

Si nous considérons maintenant les homographies et anti-homographies simplement comme transformations du plan complexe, elles ont comme sous-groupes les similitudes et les isométries du plan, ce qui permet en fait de traiter aussi bien la géométrie de l'inversion que la géométrie euclidienne. Ces groupes donnent par ailleurs des exemples assez riches d'applications conformes, notion destinée à être développée au niveau infinitésimal dans le cours d'analyse complexe. Ils permettent aussi de sortir du tout linéaire dans la mesure où l'objet qui est préservé est un cercle-droite.

Comment organiser un contenu centré sur la géométrie des complexes susceptible d'être enseigné ? Nous pensons que la perspective suivie dans cet article est un des éléments qui permettent de le définir. En effet, on a trop souvent, dans la conception des programmes, situé l'enjeu de l'enseignement autour d'une énumération de notions qui doivent être enseignées dans un certain ordre (en partant souvent du plus élémentaire au plus élaboré.) de l'affine au projectif, du linéaire au non-linéaire. Nous pensons qu'il faut d'abord réfléchir aux notions décisives qui sont au carrefour des théories et qui devraient être au cœur de notre enseignement, tout d'abord à l'université et dans la formation des futurs professeurs de l'enseignement secondaire, afin d'avoir ensuite dans un deuxième temps, les moyens de promouvoir un véritable changement des programmes dans l'enseignement secondaire.

Il faut sans doute aussi se livrer à quelques expériences didactiques au niveau de la formation des professeurs, et privilégier par exemple certaines notions décisives quitte à ne pas en rendre compte de manière exhaustive. Pourquoi ne pourrait-on, après avoir présenté les propriétés géométriques du plan complexe, initier l'étude de la géométrie des complexes par le groupe des homographies et leurs propriétés générales, privilégiant dans cette étude certaines configurations liées à des problèmes ? C'est d'ailleurs la démarche choisie par Eiden (2009), dans son chapitre V.

5 Pour ne pas conclure

L'histoire de la représentation géométrique des complexes permet de comprendre l'importance de certains concepts qui doivent être l'enjeu d'un enseignement de la géométrie des nombres complexes. Elle raconte aussi que cette vision géométrique est directement liée à la pratique des mathématiques dans différentes disciplines. Ce qui est en question dans la formation des professeurs, comme plus généralement dans l'enseignement, ne consiste pas seulement dans les notions que l'on enseigne, mais surtout dans les connections qui existent entre elles, la longue chaîne des raisons que constitue la pensée mathématique. En ce qui concerne les homographies, c'est moins la notion en elle-même qui importe que le fait qu'elle constitue un lien entre divers domaines de l'activité géométrique. Cela ne préjuge en rien de la forme dans laquelle elle doit être enseignée, car il est nécessaire ensuite de chercher les formes didactiques propres à transmettre cette notion et ses connections, en somme, à communiquer la vision géométrique des nombres complexes.

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MATHEMATICAL CONNECTIONS AT SCHOOL

Understanding and facilitating connections in mathematics

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ABSTRACT

For nearly 20 years the National Council of Teachers of Mathematics (1989, 2000) has recommended that teachers enable pupils to recognise and to use connections among mathematical ideas (Presmeg 2006). This statement is consistent with developments in mathematical education globally. Combinations of epistemological and sociological approaches like those of Heintz (2002) to describe connections in mathematics as a science, provide useful tools for examining ways in which teachers support the making of connections, for instance between school mathematics and everyday life of pupils or between different fields of mathematics: geometry, algebra and stochastics.

In order to deepen the understanding of connections in the school environment the author starts with presenting some modern approaches to mathematics as a science. The author then goes on to draw conclusions for enabling connections in the mathematical classroom by constructing problem-nets as a special learning environment. On the basis of diverse examples of application at school the transfer to the practice of mathematical education is established.

1 Connections in mathematics as a science

Educational standards all over the world (for example NSC in South Africa, NTCM in USA, diverse curricula in Germany) recommend that teachers enable pupils to recognise and to make connections among mathematical ideas. Therefore the main question of this paper is: *How can teachers support the discovery of connections between different fields of mathematics: geometry, algebra and stochastics?* To understand the nature of mathematical connections it could be helpful to look at connections in mathematics as a science. Therefore, the author will first look at how the work of mathematicians from different fields is often necessary in order to produce new results. Then, the author will give an example from a learning environment which will illustrate how interdependence of epistemic and sociological aspects of mathematical connections can be taken into account when teaching mathematics.

In recent studies epistemological aspects of mathematics are seen in their interdependence with sociological aspects (Heintz 2000, Prediger 2002). According to that, interconnections in mathematics refer not only to mathematical objects and scientific topics but also to the cooperation among mathematicians. Thereby high theoretical coherence as epistemic component and wide spread of social consensus is recognised as characteristic qualities of mathematics.

Partly seriously and partly joking, mathematicians use the *Erdős number* to describe social structure of the mathematics. Paul Erdős has written over 1400 papers with over 500 co-authors. His productivity inspired the concept of the Erdős number. Jerry Grossman (see "The Erdős Number Project" 2009) and his colleges define Erdős number in the following way:

"In graph-theoretic terms, the mathematics research ***collaboration graph* C** *has all mathematicians as its vertices; the vertex* ***p*** *is Paul Erdős. There is an edge between vertices* ***u*** *and* ***v*** *if* ***u*** *and* ***v*** *have published at least one mathematics article together. We will*

usually adopt the most liberal interpretation here, and allow any number of other co-authors to be involved; for example, a six-author paper is responsible for 15 edges in this graph, one for each pair of authors. Other approaches would include using only two-author papers (we do consider this as well), or dealing with hypergraphs or multigraphs or multihypergraphs. The Erdős number of $\boldsymbol{v}$*, then, is the distance (length, in edges, of the shortest path) in* C *from* $\boldsymbol{v}$ *to* $\boldsymbol{p}$*. The set of all mathematicians with a finite Erdős number is called the Erdős component of* C*. It has been conjectured that the* ***Erdős component*** *contains almost all present-day publishing mathematicians (and has a not very large diameter), but perhaps not some famous names from the past, such as Gauss. Clearly, any two people with a finite Erdős number can be connected by a string of co-authorships, of length at most the sum of their Erdős numbers* (http://www.oakland.edu/enp/readme/ 10.12.2010).*"*

Based on information in the database of the American Mathematical Society's Mathematical Reviews, an automatic collaborations distance calculator was created to determine Erdős numbers. Referring to that, e.g. Andrew Wiles has Erdős number 3. Furthermore, the automatic distance calculator can help to find the distance between two different mathematicians in accordance to their co-authorships (see Hischer 2010). Even if Grossman sees Erdős numbers just as a silly game, which have nothing to do with a mathematician's status or the quality of his work, such games illustrate the importance of cooperation on a social level and show how intensely mathematicians are working with each other. However, it does not provide more detailed information about the content or topics of their communication. These are represented in the variety of mathematical publications listed for example in Zentralblatt MATH (Z-MATH). This is an online Database, which contains about 2.9 million entries drawn from about 3500 journals and 1100 serials starting from 1868 and even earlier to the present. To categorise items in the mathematical science literature, Z-MATH uses the Mathematics Subject Classification (MSC). The MSC is a hierarchical scheme with three levels of structure. At the top level there are 64 mathematical disciplines. Examples for such disciplines are Number theory (11-XX), Algebraic geometry (14-XX), Probability theory and stochastic processes (60-XX). As a tree structure MSC itself is not networked. The network qualities of Z-MATH come from the papers itself. Consequently, every paper is related to chosen disciplines and connects them by these means. Since every paper was written by one or more authors it connects people too. In this sense, we can see mathematical papers as representations of mathematical connections at the epistemic and social level at the same time. Furthermore, Z-MATH makes it possible to observe the development of mathematical inquiries from outside across time. An example for this idea is given below (see Figure 1). This entry from Z-MATH shows a paper that connects mathematicians and mathematical disciplines.

Zbl 0823.11030
Taylor, Richard; Wiles, Andrew
Ring-theoretic properties of certain Hecke algebras. (English)
Ann. Math. (2) 141, No. 3, 553-572 (1995). ISSN 0003-486X
Classification :
11G05 Elliptic curves over global fields
11F11 Modular forms, one variable
11D41 Higher degree Diophantine equations
13C40 Linkage, complete intersections and determinantal ideals
14M10 Complete intersections
14H52 Elliptic curves

Figure 1: Entry from Z-Math

The paper of 1995 contains last steps to the proof of Fermat's Last Theorem and marks an end of a long mathematical adventure, which started in the 17th century with the conjecture of Fermat's Last Theorem. The conjecture states that no positive integers *a*, *b*, and *c* can satisfy the equation $a^n + b^n = c^n$ for any integer value of *n* greater than two. The 1670 edition of Fermat's *Diophantus' Arithmetica* includes his handwritten commentary, particularly his "Last Theorem".

This commentary was republished in 1932 and is listed in Z-MATH as well. Since older documents listed in Z-MATH are not classified according to MSC, it is not possible to say which fields of mathematics connect the originally formulated conjecture. Nevertheless the subsequent proof of this conjecture connects different fields of mathematics like the modular and the elliptic worlds. Furthermore, it connects mathematicians from different countries and from different historical periods. Some steps of this process are shown in the diagram below (see Figure 2). To create the diagram I used illustrations of elliptic curves and modular curves, mathematical formula and also historical facts from Kramer's short presentation of the proof in 1995.

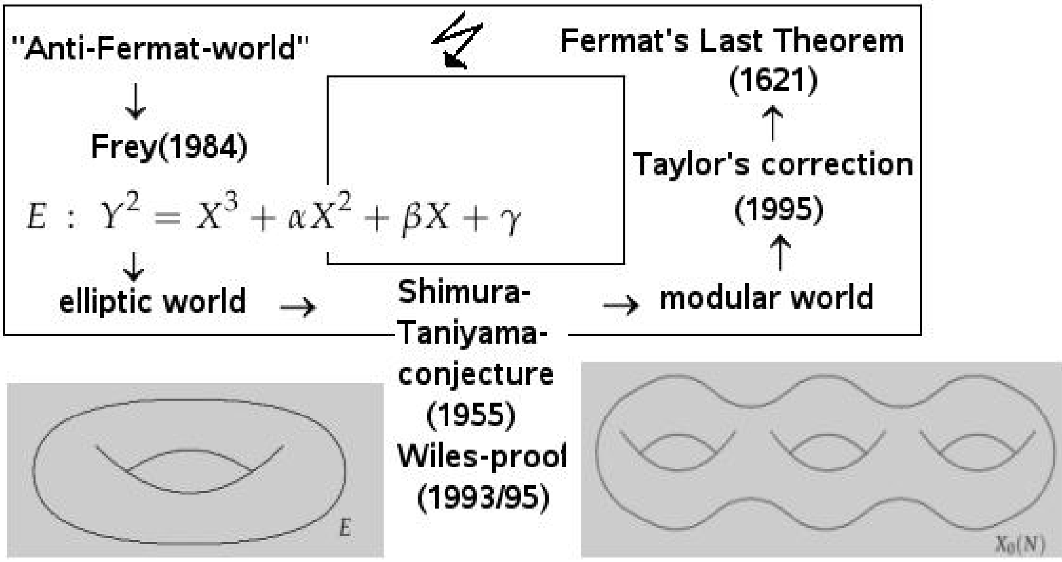

Figure 2: Fermat's Last Theorem

Consequently as a part of the proof, in 1984 Frey explored what would happen if Fermat's Last Theorem was false and there is at least one solution. Starting on the base of this hypothetical solution he found an elliptic curve and connected Fermat's Last Theorem with the Shimura-Taniyama-Conjecture (1955), which was not proven at that time. The Shimura-Taniyama-Conjecture was posed in Japan and connected the elliptic and the modular worlds. It was proved by Wiles in 1993 and the proof had to be corrected by Wiles and Taylor two years later. In this manner, different mathematicians shared posing and proving the main theorem and its parts.

In addition to the analyses and visualisation of social networks of mathematicians in Germany, Z-MATH was connected with Organizational Risk Analyser (ORA). ORA is a dynamic meta-network assessment and analysis tool with distance based, algorithmic, and statistical procedures for studying and visualising networks. To visualise social networks of mathematicians in Germany from 1990 until now, not only mathematicians but also mathematical topics according to the MSC as well were modelled with graph theory like vertices and publications as edges between these vertices (see Figure 3). This figure was created in cooperation with two students of informatics and Olaf Teschke from ZMATH, who helped me to link ORA with data from ZMATH. In order not to expand the document length unnecessarily, only the titles of selected MSC-disciplines are listed in here. Nevertheless, Figure 3 helps us to imagine mathematical interconnections not only as connections of scientific topics, but as connections within the scientific community of mathematicians too.

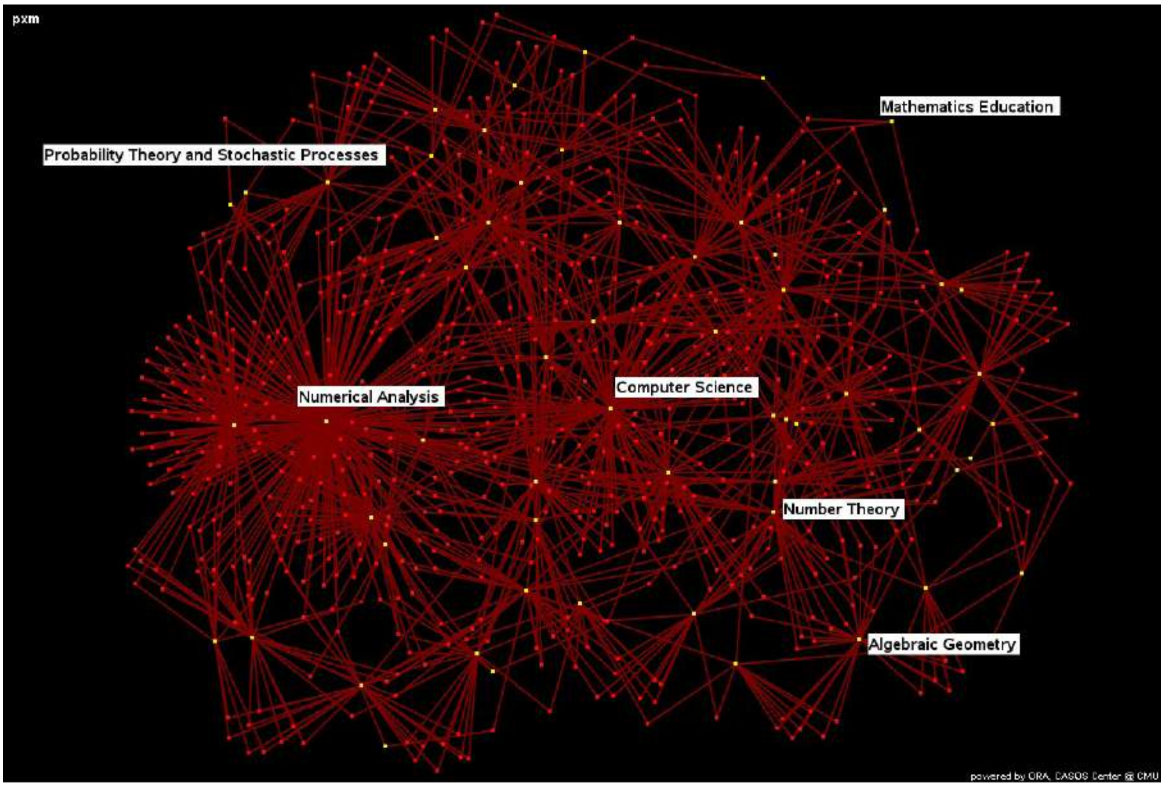

Figure 3: Mathematical connections in Germany (1990-2010)

Since every publication in ZMATH is linked to one or more mathematical discipline and one or more authors we modelled mathematicians (red) and disciplines (yellow) as vertices of the graph. The edges emerged through co-authorships and were modelled with help of ORA-algorithms. Since it is not the main subject of the paper I can not give here the explanation of the algorithms, which are used to visualise social networks. Very accessible introduction applied into theory of networks is the book of Horst Hischer. You will find it the list of references.

What can we learn from this about the nature of mathematical connections for teaching mathematics? Firstly, it shows those disciplines which have more connections to other fields like numerical analysis and computer science. If mathematicians cannot find classical solutions for the problems posed in their own discipline, they cooperate with their colleagues who are studying e.g. Numerical Methods or dealing with Computer Science. Secondly, even the titles of the disciplines like Algebraic Geometry reveal to school teachers, that many connections should be possible between algebra and geometry. Thirdly we see, that connecting the fairly recent discipline of probability theory with other disciplines can challenge not only pupils but experienced mathematicians as well. In addition to all of this, we see that segmenting mathematics as a scientific field in disciplines and topics is very important in order to classify mathematical objects on an epistemological level. This conclusion facilitates the segmenting of school mathematics in fields like geometry, algebra and stochastics as a premise for making and understanding connections. Hence, segmenting teaching material and the student's mathematical knowledge into categories as in the usual way in curricula, schoolbooks and teacher preparations should be maintained. In addition to segmenting school mathematics it is important to create situations where students can combine their knowledge from different segments of school mathematics. This is already being done when teachers deal with connections among geometrical and algebraic aspects for example, by using functions to describe geometrical concepts or calculating areas. Nevertheless, incorporating stochastics would be the most innovative step forward.

That is the reason why problems, which refer to different mathematical fields, should be given to the students. They should recognise and establish connections among mathematical ideas. Problems like those could be presented to the students for example at the end of the school year. It is with thanks to the cooperation of mathematicians that mathematical connections eventually emerge. Students, therefore, should be encouraged to cooperate by working on such problems. Segmenting school mathematics on one hand, and creating learning environments (where it is possible to weaken the borders between these segments) on the other are two interdependent factors of making connections at school.

2 Making connections with PYTHAGORAS-TREE

To review different "segments" of school-mathematics students could be divided into five or six small groups for solving problems together and sharing their solutions with the whole group. As the history of Fermat's Last Theorem teaches us, it is necessary to find mathematical problems or contexts, which would connect different segments of school mathematics (algebra, geometry and stochastics) on the one hand, and motivate students to cooperate, on the other hand. An example for a mathematical context, which could fulfil the

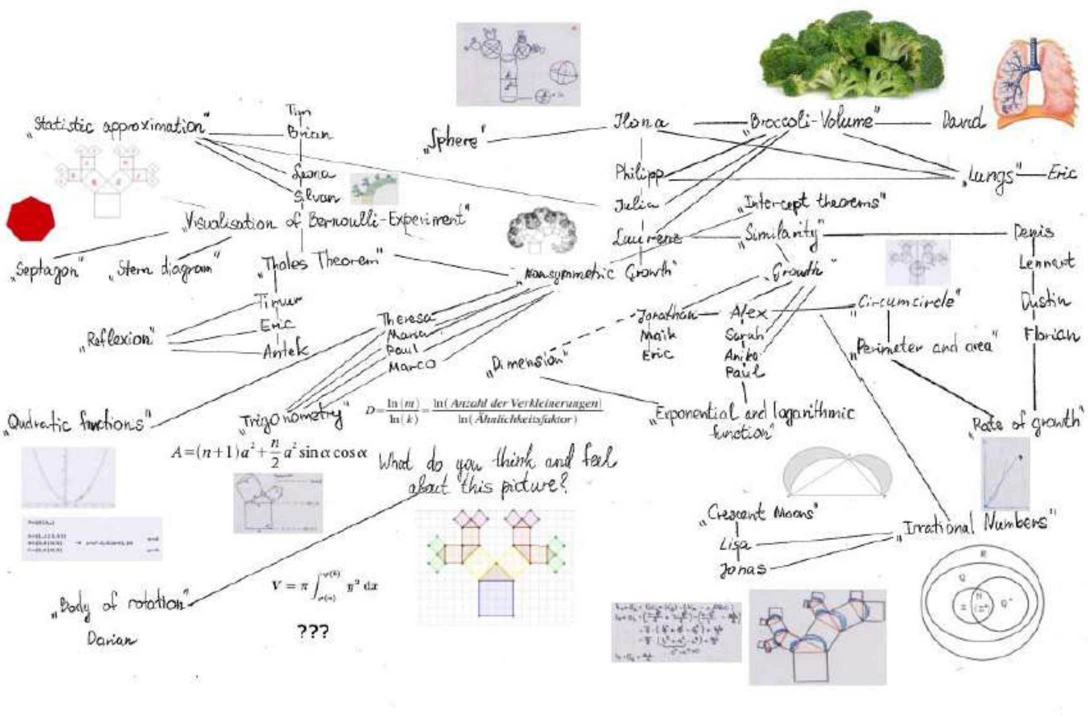

Figure 4: Pupils connecting mathematics with Pythagoras-Tree

above described pedagogical hopes, will be given in the next chapter. Figure 4 shows a

diagram which illustrates how students of 9th and 10th grade at a grammar school in Berlin worked on the given problems in the context of the Pythagoras-tree.
The students accomplished the work in two sessions lasting between two and four hours. Similar to the visualised networks of mathematicians, we can see “students-networks” around the Pythagoras-Tree. We can also see the names of students as “mathematicians”, mathematical topics and titles of the problems illustrated with some pictures. To give an idea of how the context of Pythagoras-Tree could be used in the mathematical classroom in order to enable students to recognise connections by cooperating, a learning environment, which was tested at school, will be described in the following. The learning environment included a short text (see Figure 5) with general information on the Pythagoras-Tree and initial problems, which are related to major topics of school mathematics. The students were asked to work in jigsaw-puzzle.

A Pythagoras tree is generated by adding to the top of a square (the 'trunk') a right-angled triangle sitting on its hypotenuse (branches). The 'twigs' are further squares added to the two sides adjacent to the hypotenuse. On the opposite sides, rectangular triangles are added again. These triangles are similar to the first one. And so it goes on. All growing branches end in squares (leaves). The picture on the work sheet stating “initial problem” shows a symmetrical Pythagoras tree with three levels (see Figure 6). Pick one of the following problems and solve it in small work groups. Do not forget to complete the table of competence for your chosen problem. Further squares can be added to each of the sides adjacent to the hypotenuse. These are the 'twigs'.

Figure 5: General Information on the Pythagoras-Tree

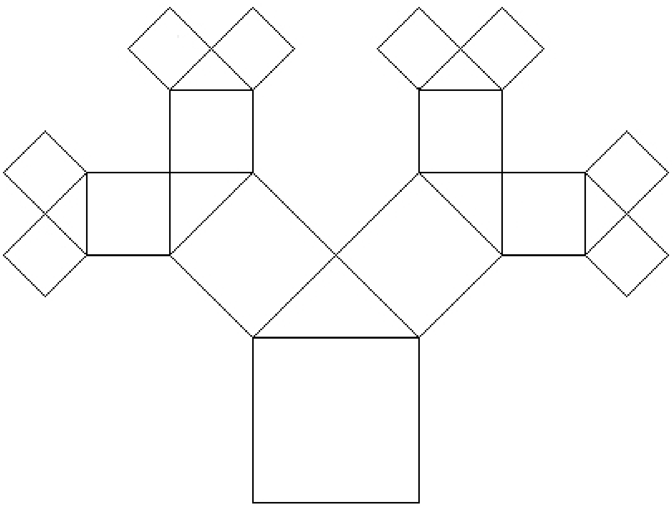

Figure 6: 3-Level-Pythagoras-Tree

Students were introduced to all the initial problems. They were then asked to choose one problem, to work on it in small groups (with a maximum number of five students) and to prepare a presentation of the solution for the whole class. First seven problems are presented without solution. To the last three problems are given students’ own solutions. These solutions should illustrate my intentions in the area of making mathematical connections - in the same way as professional mathematicians. I hope therefore to offer new ideas to teachers about what they can expect from their students or how they can adapt the environment to their own classroom.

1. Figures and Lengths

Explore whether rational numbers are sufficient for the description of the possible side length. How do these lengths relate? Give reasons for your guess.

2. Dependencies

Look at the picture. Which functions describe the dependence of the area/perimeter on its trunk width? Use all possible ways of presenting a function.

3. Similarity

Which similar shapes do you recognise in the picture? Why are those figures similar? Give the parameters of similarity. It is said that Pythagoras trees have a similar appearance to broccoli. What do you think?

4. Estimate

Ask your classmates and teacher to estimate the area of the drawn figure. Find the mode, median, mean and the span of your sample. Present your results in a boxplot. Work out the area by using the width 'a' of the trunk (use your ruler to measure it). Compare the estimate with the calculated value. What do you see?

5. Continuation

The symmetrical Pythagoras tree can be expanded with no limitations. Which function would you use to describe the relationship between the area of the leaf and the number of levels? Find a function to describe this equation and draw a graph. Find the inverse function. Starting at the trunk, how many levels of a symmetrical Pythagoras tree do you have to add, in order to end up with a leaf size (area) of 1/128 of the area of the trunk? Find the number of levels of a Pythagoras tree which possesses more than one million leaves.

6. General formula for calculating the area

Find the formula for calculating the area of non-symmetrical Pythagoras trees. Satisfy yourself that this formula is correct. Note: 'a' is the width of the trunk, 'n' the number of levels and 'α' stands for the base angle.

Additionally: Find the correct formulae for the Pythagoras trees with base angles of 30°, 45° and 60°. Is there anything particular here? How would you explain this?

7. General formula for calculating the circumference

How could you calculate the perimeter of non-symmetrical Pythagoras trees, when the width of the trunk, the number of levels, and one base angle of the biggest possible triangle are given? Find a general formula.

Solutions of selected problems

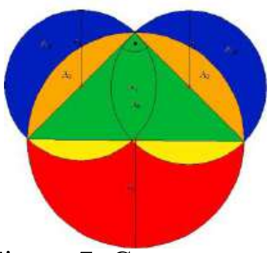

Figure 7: Crescent moons

9. Crescent moons

Small crescent moons appear (blue coloured) due to overlapping of the circles of the two sides adjacent to the hypotenuse and the hypotenuse of a right-angled triangle. Draw the corresponding crescent moons for all the levels of the shown picture. In which way can you calculate very quickly the total area of all the drawn crescent moons?

What did students make from this problem? To solve this problem Jonas and Lisa worked together. Their solution presented contains algebraic and geometrical elements (see Figure 8). Jonas shows his strong point in geometry. Lisa is doing better in algebra.

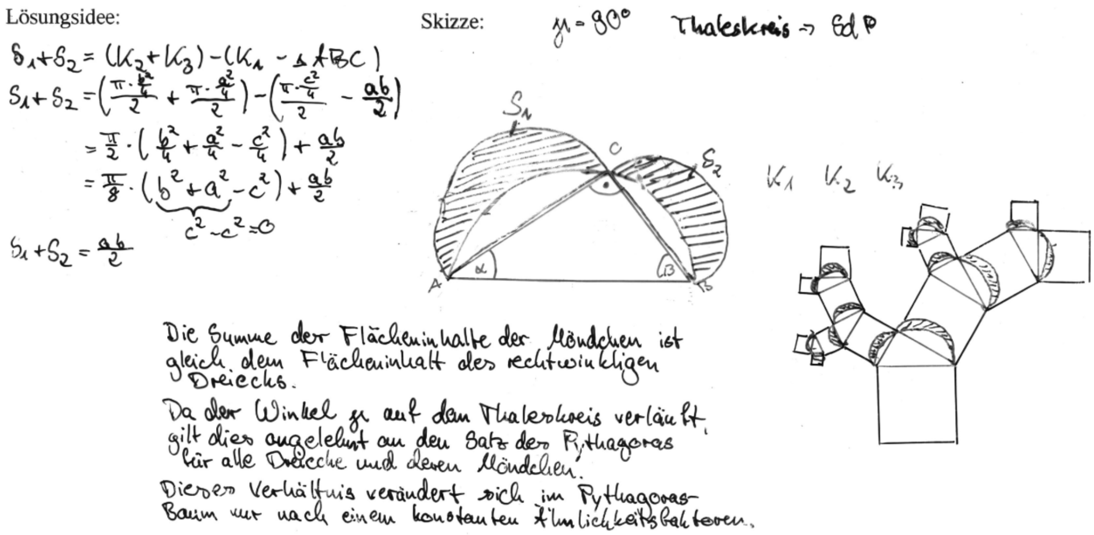

Figure 8: Pupils connect algebra and geometry

In order to solve the problem the students have drawn a picture first. Jonas then translated different areas of the figure into algebraic terms, but he could not simplify them. Lisa transformed the formulas with algebraic tools. She worked on the formal level and lost connection to the geometrical meaning of the formula. At the end of the algebraic transformations, the students asked the teacher to help them. The teacher reminded them of the Pythagorian Theorem, which helped them to translate variables a, b and c into the lengths of the sides of the rectangular triangle and they simplified the expression using the equation from the Pythagorian Theorem. As the result they wrote down the equation in the fifth line. Afterwards they translated the results into words, where they said that the area of the crescent moons is equal to the area of the triangle. They extended this observation to the following steps of the Pythagoras-Tree. Furthermore the students noticed similarity of the parts of Pythagoras-Tree and drawn the appropriate picture. This example shows how students complemented each other to solve the problem combining their geometrical and algebraic knowledge.

10. Multiplication rules

Possible outcomes of a Bernoulli-Experiment can be visualised with the Pythagoras tree, very similar to the probability tree diagram. The Pythagoras tree has to be read from the bottom to the top in order to follow the time line. As at the start of a Pythagoras tree, there is a square available for the visualisation of an experiment at a higher level.

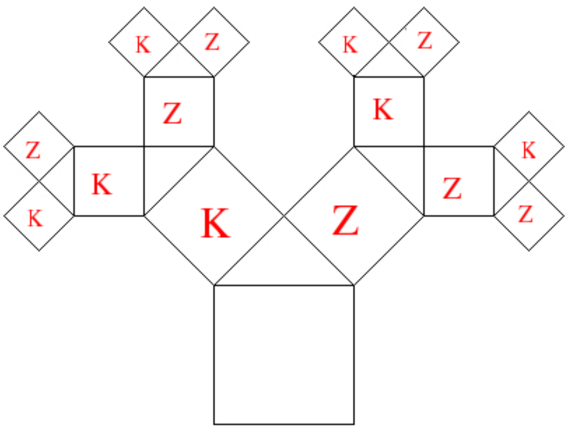

Figure 9: Bernoulli-Experiment

On each level, the area of each of the newly generated squares will be half that of the original square. The total of the square area stays the same on each level. The drawing shows a threefold toss of a coin presented in a Pythagoras tree. What are the benefits and disadvantages of this way of visualisation in comparison to using a tree diagram?

What did students make from this problem?

After they had discussed the problem above, Eric, Antek and Timur were engaged in solving the following question: We throw a dice four times in a row and want to know how often the number six comes up. However, we are unable to show this experiment in a symmetrical Pythagoras tree. What would a Pythagoras tree look like according to this experiment? Using Thales' Theorem they have replaced the isosceles right-angled triangle with a non-isosceles right-angled triangle. This has resulted in the generation of two squares in level two with a ratio of 1:5. Is this model transferable to other Bernoulli-Experiments? Find further examples.

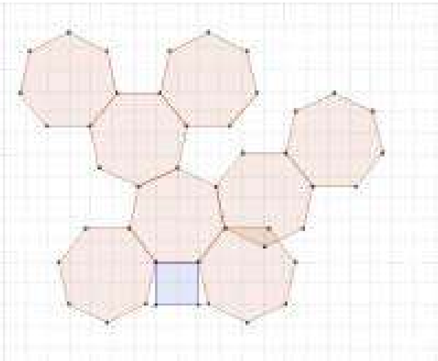

Figure 10: Heptagon

Results of Eric, Antek and Timur were presented to Leona, Silvan, Tim and Brian. To find a representation for the experiment with the dice they tried to find an analogy to the triangle in the coin-experiment. They suggested a construction from heptagon to visualise the dice-experiment. By producing heptagons with computer tools like GeoGebra (http://www.geogebra.org/cms/10.12.210) they found very quickly, that it was not an appropriate model. The conjecture students produced was proven false.

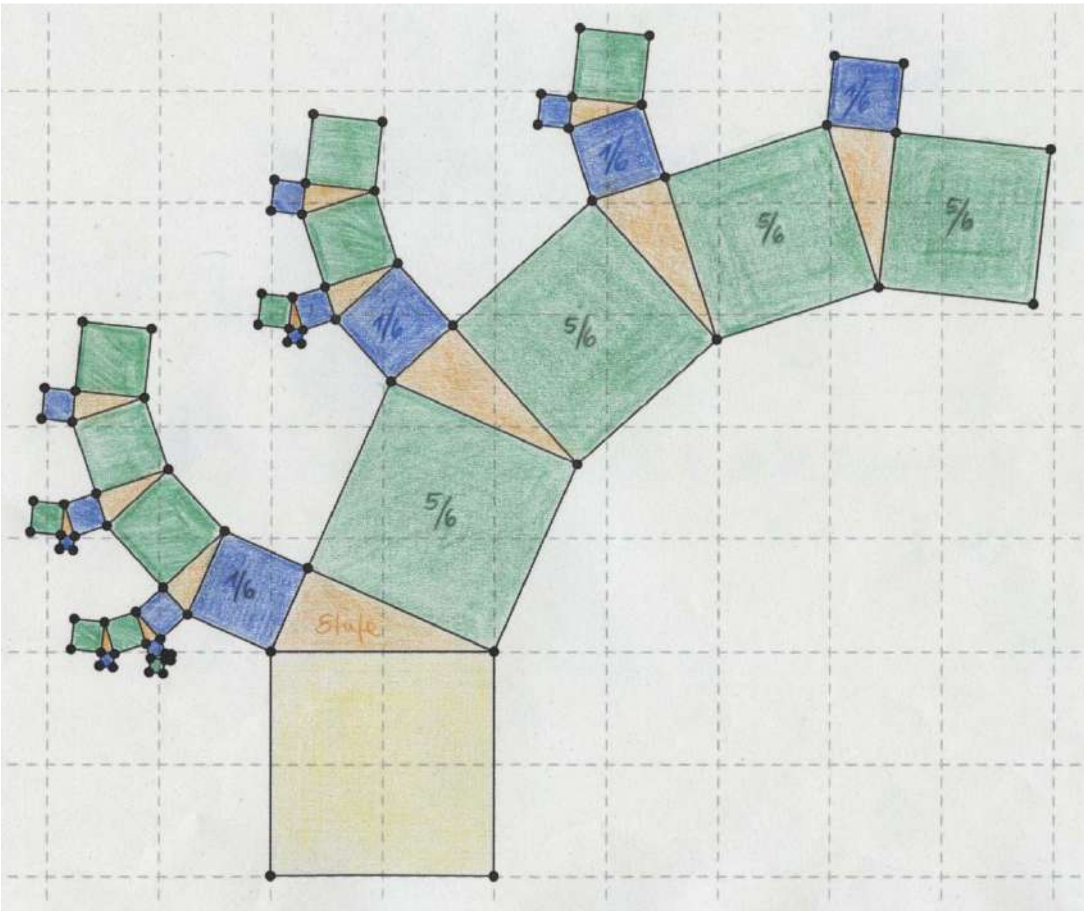

Figure 11: Visualisation of dice-experiment

Leona, Silvan, Tim and Brian constructed according to suggestions from Eric, Antek and Timur, a non-symmetrical Pythagoras-Tree to visualise the dice-experiment (see Figure 11). As an advantage of this model in comparison to the well known tree-diagrams in stochastics Leo said: “You can see that the probability to throw the number six, four times in a row, is very small. In this picture it is almost a point, but it is not a point. As a result the Pythagoras-Tree is not very useful, if you have many more steps”. Students then made their generalisation and concluded that the Pythagoras-Tree is very suitable to visualise the Bernoulli-experiments, since it has two branches and the Bernoulli-Experiment has two possibilities: “success” and “failure”. However, this generalisation is valid only for experiments with few steps. We can see that the students connected important ideas of geometry and stochastics. They used equality of the areas of figures to visualise equality of probabilities in every step of the Bernoulli-experiment. Students shared their ideas, produced and tested their own conjectures and studied the statements of other students. Starting with the given problem, students somehow cooperated like professional mathematicians to connect different “worlds” of school mathematics.

Broccoli

People say that Pythagoras trees look similar to broccoli. Draw a broccoli using known geometrical shapes. Find a possible function, which describes the volume of a broccoli in terms of the diameter of its trunk.

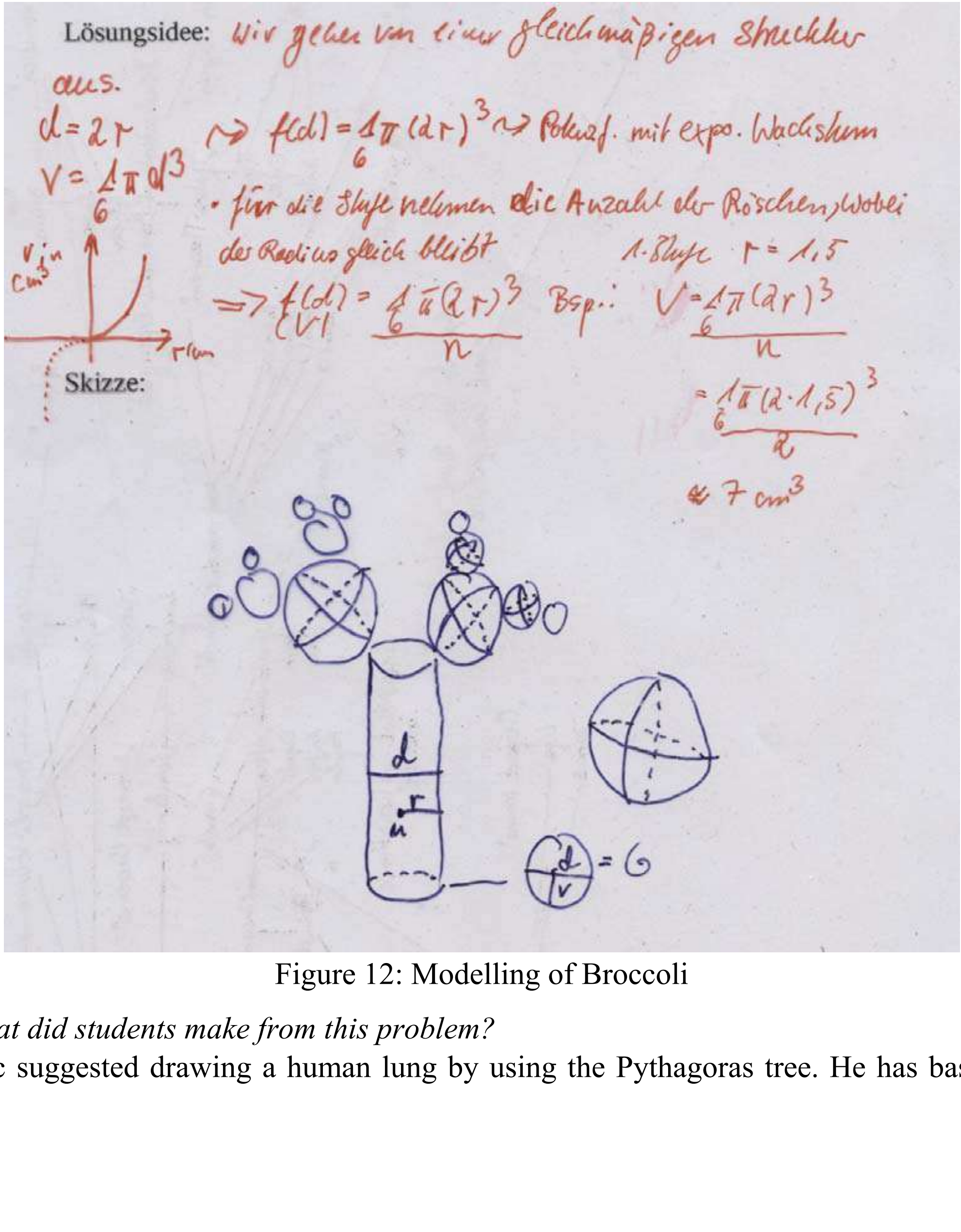

Figure 12: Modelling of Broccoli

What did students make from this problem?

Eric suggested drawing a human lung by using the Pythagoras tree. He has based his idea on the similarity between broccoli and a lung. Ilona said, that Eric's suggestion is more suitable e.g. for the tumour, because lungs only have finite space to grow. She suggested using other fractals like the Serpienski-Triangle to model lunges. Since the teacher made short remark about fractals as self-similar structures and gave an example from Serpinski-Triangle helping out in the small groups.

Ilona, Laurenz, Philip and Julia were modelling broccoli with geometrical solids (spheres). They then translated them into functions and used various representations of Functions (graph, table, equation). The students noticed that the sum of the volumes is equal at every step of the broccoli. But if you look at the solution carefully, you will notice a mistake. The students divided the volume of the spheres by n (amount of the steps) instead of multiplying it by n. Another mistake concerns the notes, where students say that power functions have exponential growth. However, for their presentation of the results to the whole class they received some critical comments. To describe concepts from biology it was necessary to combine ideas from geometry, like those from the area of spheres and from algebra and functions like the power-function. Likewise professional mathematicians' results were presented to the "scientific community" and mistakes were found and discussed. A similar case took place in the story of Fermat's Last Theorem, where a mistake appeared even after the presentation of the proof and had to be corrected afterwards. Mistakes of mathematicians can help teachers to allow students to make mistakes by making connections and correcting them.

3 Conclusions

The students' results commented above showed that they applied their knowledge of different mathematical segments to study the Pythagoras-Tree. They varied different mathematical representations, such as the number of steps of the Tree, length of the side, angles of the tree. They co-operated by proving, producing or falsifying conjectures. The students did not only solve mathematical problems referring to different areas of school mathematics and connecting them, but posed their own questions. Therefore, the example of a "Pythagoras-Tree" as a learning environment gives an idea of how teachers can put their students in situations similar to those of mathematicians'.

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CONCEPT MAPS AS VISUALISATION

Their role as an epistemological device for introducing and implementing History of Mathematics in the classroom

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ABSTRACT

A Concept Map is a graphical multi-layered and flexible metacognitive tool for organising and representing knowledge. Various forms of concept map have been developed for use in teaching (Novak and Canas 2008) The ability to construct such maps, particularly in a virtual environment, allows the juxtaposition of ideas and the creation of new connections. Visualisation of such possibilities has a powerful epistemic facility capitalising on the creative pedagogical potential of the teacher and the active engagement of the learner (Arcavi, 2003; Giaquinto, 2007). This paper describes the use of such maps in the mathematics classroom and offers a theoretical basis for practical intervention using Design Research (Swan 2006) allowing many points of entry into historical material that supports pupils' cognitive, affective and operative engagement with mathematical learning.

1 The UK Curriculum 2008

In 1999 the education system for England and Wales[1] was centralised by a government that produced a curriculum enshrined in traditional beliefs about 'levels' of knowledge, and a collection of disparate activities rarely connected in any sensible way producing little serious engagement with mathematical thinking. An Inspectors' report on secondary schools showed that too many pupils were taught formulas they did not understand, and could not apply:

> "The fundamental issue for teachers is how better to develop pupils' mathematical understanding. Too often, pupils are expected to remember methods, rules and facts without grasping the underpinning concepts, making connections with earlier learning and other topics, and making sense of the mathematics so that they can use it independently." (Ofsted 2008, p.5)

Since 2008 the government organisation QCDA[2] has worked with schools on a revised curriculum to publish guidance[3] complemented by video case studies where 'enrichment and enhancement' activities are used with a range of learners[4].

Recognising the Historical and Cultural Roots of Mathematics is now one of the Key Concepts in the new Programmes of Study for Secondary Schools (ages 11 – 18) (QCDA 2008). Consequently, it may now become easier to incorporate the teaching of this Key Concept in such a way as to enable the history to emerge from the discussion of *canonical situations* (be they images, texts, or conceptual problems) introduced by the teacher. This approach has the advantage of being able to link different areas of the curriculum by

[1] Different systems apply in Scotland and the Republic of Ireland.
[2] The Qualifications and Curriculum Development Authority, the Government sponsored body set up to maintain and develop the national curriculum and associated assessments, tests and examinations. This organisation is now being disbanded by our coalition government.
[3] QCDA (2009) *Engaging mathematics for all learners.*
[4] The revised Programme of Study for Secondary Mathematics can be found at http://curriculum.qcda.gov.uk/key-stages-3-and-4/subjects/key-stage-3/mathematics/index.aspx

offering situations derived from historical sources, thereby enabling pupils to appreciate connections between parts of mathematics that have hitherto been concealed or ignored.

2. Heritage not History

In a series of papers, Radford (from 1997 to 2006) has demonstrated that mathematical knowledge is deeply embedded in a given culture, and that each new cultural phase re-interprets and changes the conceptions of earlier thinkers so that new ideas become possible. Furthermore, in our attempts to understand the history, we inevitably bring our modern socio-cultural conceptions of what the past was like with us. Marwick (2001) recognises that history, being about 'what happened in the past', and says that the best we can do is to make clear our basic assumptions and, acting on the evidence available, make as honest a story of it as we can. We confess our ignorance where we do not know, and make clear our speculations where evidence is sparse. History is a cumulative activity and accounts should always subjected to debate, qualification, and correction.

Concerning the use of history in education, Grattan-Guiness (2004) makes a distinction between the History and the *Heritage* of mathematics. History focuses on the detail, cultural context, negative influences, anomalies, and so on, in order to provide evidence, so far as we are able to tell, of what happened and how it happened. *Heritage,* on the other hand, address the question "How did we get here?" where previous ideas are seen in terms of contemporary explanations, and similarities with present ideas are sought. He says,

> "The distinction between the history and the heritage of [an idea] clearly involves its relation to its prehistory and its posthistory. The historian may well try to spot the historical *foresight* - or maybe lack of foresight - of his historical figures,…. By contrast, the inheritor may seek historical *perspective* and hindsight about the ways notions actually seemed to have developed."(2004:168)

and

> "…heritage suggests that the foundations of a mathematical theory are laid down as the platform upon which it is built, whereas history shows foundations are dug down, and not necessarily into firm territory." (2004, p.171)

Referring to the apparent development of mathematics, the interpretation of Euclid's work as 'geometrical algebra' (as in Heath and van der Waerden for example) has been shown to be quite misguided5 as history, but as heritage is quite legitimate because it is the form in which Arab mathematicians of the 10th century justified the logic of their creation of algebra (Rashed 2007, p.18-24).

Each culture had its own ways of defining the objects of their inquiry and recent historiography (Chemla 2004, Robson 2008, Plofker 2009) has shown that research from a historical-epistemological point of view can provide us with considerable information about the development of mathematical knowledge. This information is principally specific, but also has implications for wider theoretical developments. Radford and others have already suggested that:

> "The way in which an ancient idea was forged may help us to find old meanings that, through an adaptive didactic work, may probably be redesigned and made compatible with modern curricula in the context of the elaboration of teaching

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[5] Typically, this is done with Euclid II,4 and described as 'completing the square', see the examples in Katz (1998: Section 2.4.3) for a more nuanced interpretation. For a critique of Heath, see Netz (1999).

sequences …" (Radford 1997, p.32)

Nevertheless, even though locked into our own cultural contexts, with the sensitive use of modern historiography, we can work on the heritage we discover to develop materials that give new insights into learning in a modern curriculum.

3. Maps, Narratives and Orientations

A concept map is a graphical multi-layered metacognitive tool for organizing and representing knowledge. In view of a variety of experiences the original form has now been considerably modified by Novak & Canas (2008) and other users into a series of flexible ways of working, appropriate for a number of different disciplines. Concept Maps have considerable advantages over linear text. They can be used to support the collaborative development of knowledge, the sharing of vision and understanding, the transfer of expert knowledge, and the enhancement of metacognition. Burke & Papadimitriou (2002) have proposed that when faced with non-linear text, the *Map*, a user would need two more elements; a *Narrative*, that provides information on the general context and background, and an *Orientation* that describes the activities provided for pupils to 'find their bearings' in the map of the ideas presented.

This idea gives us the freedom to consider a map in a *virtual environment* where the arrangement of concepts, objects, events, propositions and actions may be partially ordered and even multi-layered, crucially breaking up the linear sequence and juxtaposing different ideas. No map is ever 'complete'; what may be chosen to be the principal concept(s) at one stage can be rearranged according to the needs of the learning process, and of the individuals involved. In contrast, most curriculum activities are presented to teachers as a *linear narrative* of topics, restricted to some imagined age-related 'levels of competence' of the pupils. Clearly, to be relevant and useful, Maps have to be developed collaboratively where, for example, a group of teachers, or a teacher and an 'expert' share knowledge and combine their vision. In this way we can present both pupils and teachers with Maps to be explored and interpreted. However, the *Map* needs some background *Narrative* and some suggestions for *Orientations* to help pupils address problems arising from the situation[6].

Maps clearly have both a metacognitive and an epistemological function. By organising ideas, concepts and events in a particular way, and examining the possible links between them in a visual display, maps can be used as plans for teaching and scaffolding for learning, leading us to new connections between ideas. Indeed, Barbin (1996, p.18) states that:

> "The history of mathematics shows that mathematical concepts are indeed constructed, modified and extended in order to solve problems. Problems come, as much as into the birth of concepts as into the different meanings attached to concepts as tools for the resolution of problems. This emphasis on the role of problems in the historical construction of knowledge can lead to a new way of conceiving history."

and, I would add, to a new way of finding problems suitable for the classroom.

The importance of visualisation in these activities is clear; from the representation of

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[6] In the case of the relationship between teachers and pupils the use of Narrative and Orientation would vary according to the circumstances and these ideas were discussed in the Workshop accompanying this presentation.

objects, to manipulating them physically and learning to do so in the mind to bring out hidden properties. Furthermore, visualisation has linguistic and semiotic connections, and Barthes (1977) writing on the rhetoric of the image maintains that there are many different ways we read an image, (linguistically, iconically, coded and uncoded), and this idea is no less relevant to the structures and maps we design.

> "The image, in its connotation is constituted by an architecture of signs from a variable depth of lexicons …. The variability of readings … is no threat to the 'language' of the image if it be admitted that language is composed of idiolects, lexicons and sub-codes. The image is penetrated through and through by the system of meaning, in exactly the same way as man is articulated to the very depth of his being in distinct languages." (Barthes 1977, p.47)

Adapting a Map to explore links through the curriculum to historical contexts can act as part of a developable knowledge structure to be offered for integrating aspects of our mathematical heritage into a teaching programme, where the history then becomes integral to the exploration of the mathematics. A Map is there *to enable teachers to have the freedom to develop their own Narrative,* it can throw light on certain problems, suggest different approaches to teaching, and generate didactical questions. It is thus possible to offer ways in which teachers, starting from a particular point in the standard curriculum, could incorporate the teaching of 'Key Concepts' (QCA 2008) to link with some important developments in the history of mathematics.

4. Visualisation and Epistemology

Visualisation is an important function in mathematics teaching and learning, even more so now we have available a greater variety of media than hitherto. Arcavi (2003) regards visualisation as a vital aspect of our classroom communication of mathematical ideas and concept building. This ability is fundamental in our interpretation, creation and reflection upon a variety of images and diagrams that depict and communicate information, aid thinking, and help to advance our understandings. Arcavi's paper is important in describing the facilities of the human mind and their importance in the cognitive and affective aspects of learning and teaching. However, Giaquinto (2007) explores to what extent we can rely on our visual abilities to develop deeper cognitive awareness and justifications for epistemological advances. In particular, he explores the nature of our cognitive grasp of structures that may give us clear insights and develop new knowledge. While a large part of our knowledge comes from direct experience, and vicariously from someone else's experience, we can also derive knowledge from theory. Since theoretical structures can be depicted, he argues that *we can know structures by means of our visual capacities* and he forms the idea of 'visual templates' demonstrating the way we can appreciate them and manipulate them. A visual display can suggest connections that inspire theory. Once we have perceived the elements of a particular configuration, we have the ability to perceive configurations of other visual types as structured in the same way.

> "Thus, I suggest, we can have a kind of visual grasp of structure that does not depend on the particular configuration we first used as a template for the structure. … Once we have stored a visual category pattern for a structure, we have no need to remember any particular configuration as a means of fixing the structure in mind. …. There is no need to make an association or comparison. So this is more direct

than grasp of structure via a visual template." (Giaquinto 2007, p.221)
As a philosopher, Giaquinto is cautious of the word 'intuition' because he is interested in carefully exploring the cognitive aspects of our appreciation of structures, but in my view, ideas that come to us 'suddenly' (often after some months of incubation) could well be stimulated by 'metacognitive manipulation' of a (public or private) Concept Map.

5. A Practical Approach through Design Research

In response to the requirement of history in the new curriculum, the perceived lack of knowledge and the absence of appropriate resources among teachers, in 2009 a Working Group on History in the Mathematics Classroom was set up to "select, share, trial, evaluate and modify appropriate material in the light of teachers' experience so that together we may discover sensible ways of introducing the rich historical and cultural roots of mathematics to our pupils."7 The approach adopted by the Working Group has evolved from experiences of colleagues in presenting 'episodes' from the history of mathematics in workshop form, so that interesting and worthwhile problems can arise from the historical contexts8.

In order to address the question of teachers' lack of knowledge and experience, the principal focus lies in providing secondary teachers with professional development materials that start from the fundamental ideas that they are required to teach, and open up the possibilities of developing the concepts involved by finding *historical antecedents* to support the connections between and motivations for these ideas, and the possible links to be made between the mathematics and other subject areas. The project has two aspects; developing practical materials for teachers, and providing a context that offers qualitative research opportunities. This involves regular liaison with teachers[9] who wish to try out these ideas in their classrooms.

The methodology of Design Research (Swan 2006) being used in this project involves a systematic series of interventions to transform a classroom situation. It is a collaborative and iterative approach to research and development in which theoretical arguments and reviews of existing studies are brought together in the design of new teaching approaches. Proposals for intervention are evaluated in classrooms using standard methods (Robson 1993)[10] where outcomes lead to further refinement of the theories and approaches. Revised plans are further tested and the emerging results reveal ways in which teaching methods may be designed to become more effective. This methodology involves discussions on approaches, different for different classrooms, including draft lesson plans with notes, copies of historical materials, and links to appropriate resources that offer flexibility and allow for modifications. In this way, a 'lesson' can be seen as a series of 'micro tasks' each offering serious epistemological challenges both mathematically and pedagogically. This historico-epistemological research demonstrates the potential for providing data on cognitive, affective and operative aspects of

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[7] This is an official Working Group of the British Society for Research in Learning Mathematics (BSRLM). Members are practicing teachers, teacher trainers, research students and trainee teachers. See http://www.bsrlm.org.uk/

[8] Some of these problems were used in the Workshop accompanying this presentation.

[9] The teachers involved are volunteers who usually work in schools local to members of the Working Group who access schools through their job as teacher –trainers.

[10] For example, forms of action research such as interviews, questionnaires, case studies, participant observation, etc.

learning thereby offering a range of affordances (Greeno 1998)[11].

6 Negotiating Meanings in the Mathematics Classroom

Living in the world is a constant process of negotiation of meaning (Wenger, 1999). Meaning does not lie in the individual nor in the world but in the dynamic interaction between the two. Participation in meaning is an active process of taking part in activities as a member of a community. It therefore has both social and personal dimensions and thus shapes the individual and the community in which the individual participates (Rogers, 2002).

These principles were applied to the mathematics education community by Adler (2000). However, Douady (1986, 1991) had already provided an application of the principle of negotiation of meaning to mathematical learning as a ‘dialectique’, between pupil and teacher. Douady discussed the interface between teaching and learning in mathematics, and examined the role of the teacher in providing opportunities for learning. She considered the teacher's role principally to set up conditions for learners to make connections between various aspects of mathematics, and then to mediate the learning of new concepts by drawing on learners' ideas and negotiating these in discussions in the classroom.

In the formation of concepts, there is a *dialectic* between a mathematical concept as a tool (*outil*) for use in solving a problem, and as an object (*objet*) when it has the status of an independent mathematical entity. This idea of moving between two states is central and is included as part of the recognised process of building mathematical knowledge. Douady sees learners using objects as tools, and also working with tools to build objects. Her model is cyclic, where concepts, used as tools, give rise to new tools which in turn become objects, and the role of the teacher is central in providing learners with a task that requires them to make use of their existing knowledge. In their writings to teachers, Watson & Mason (2005) show how cognitive conflict, negotiation and concept building, together with encouraging justification and the extension of ideas, can be given clear practical guidance for operating in the classroom.

Many episodes from the history of mathematics show how the protagonists are offering an idea for discussion and different positions are taken up, according to the understandings of the correspondents. Recently, a dialogic approach by Barbin (2008) illustrates the negotiations to be found in many historical materials.

7 Working with Teachers and Pupils

The implementation of the Design Research approach first requires participants who are going to be sympathetic to the general aims of the project. A number of colleagues in the Working Group have access to schools through their work as teacher-trainers, so much cooperation is already established.

Here I outline the stages of evolution of the planning and general procedures in the context of one particular class of 13/14 year olds in an English state secondary school, who have an experienced teacher who worked with a researcher to introduce some historical elements into her mathematics teaching. However, this teacher was constrained by the demands of the curriculum, as well as having to maintain elements of school policy,

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[11] Greeno sees *affordances* as “qualities of systems that can support interactions and therefore present possible interactions for an individual to participate in” (1998, p.9)

and run her teaching programme parallel with another class. The pupils were near the end of their school year, and about to pass on to the next class.[12]

Initial discussions took place between the teacher and researcher about the curriculum and some relevant historical contexts that finally focussed on the general idea of 'completing the square' (a subject that is part of the pupils' normal programme of study). The teacher's awareness of the pupils' background knowledge and skills enabled particular pedagogical strategies to be considered.

For this group of pupils, their background knowledge and skills were fairly well known (though not consistently nor entirely secure), and so some revision was arranged to cover the use of items such as brackets and multiplication of binomials. For many pupils moving too quickly into generalisation with symbols can lead to confusion, and the expansion of expressions like (a + b)(c + d) can be very awkward and confusing. The 'Grid Method' (Fig. 1) is a useful device commonly taught in English Primary schools.

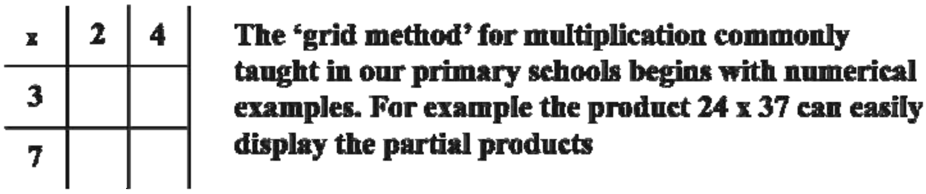

Fig. 1. Grid Method for Multiplication

Pupils can also have practice in manipulation of templates representing squares and rectangles (Rogers 2009), and visualisation comes into play here where metacognitive possibilities are exposed:

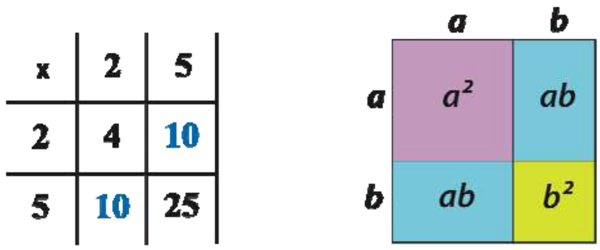

Fig. 2. Grid and Euclid II, 4.

From the grid multiplication display it was decided to progress to identical binomial products such as (2 + 5)(2 + 5) that produce an arithmetical square [4 + (2x10) + 25] = 49, and contemplating a geometrical square with two squares and two identical rectangles symmetrically placed. During discussion with the pupils, the geometrical square can be labelled with algebraic expressions showing products (Fig. 2).

The initial curriculum discussions involve making concept link sketches and noting possible situations for the introduction of references to historical contexts. The use of the historical Narrative in conjunction with the teachers' knowledge of the curriculum in writing the Orientation is essential. Choosing what to do and deciding on prerequisites, to a large part determines the style of the Orientation. Preliminary discussions took place in

- choosing a general topic area from the current teaching programme

[12] Pupils in the UK system progress through the school according to their age, so Year 9 are 14-15 years (Key Stage 3) and Year 10 are 15-16 years old (Key stage 4).

- discovering what historical links with the central concepts might be possible (The Map)
- considering the contextual and pedagogical approach including pupils’ assumed knowledge background and skills pre-requisites
- understanding and establishing something of the historical background (The Narrative)
- planning a sequence of lessons (The Orientation)
- providing for feedback and modifications

The general area of the curriculum chosen was section ‘3.1 Number and Algebra’ from QCDA (2009). In particular:

“rules of arithmetic applied to calculations and manipulations with rational numbers (KS3)”

“linear, quadratic and other expressions and equations (KS4)”

As well as considering the pupils’ knowledge, provision needs to be made for the Key Concepts and Processes (QCDA 2009), and it is now left entirely to the teacher to decide how to interpret these statements in terms of classroom tasks and activities.

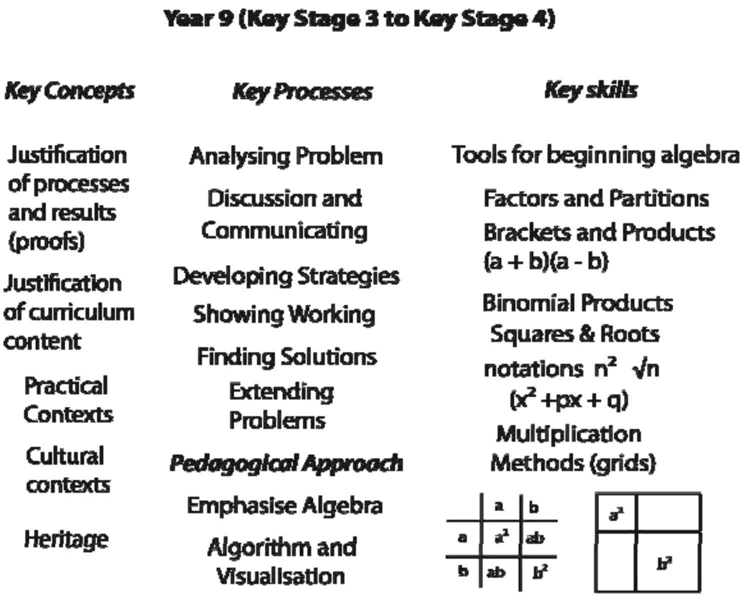

Fig. 3. Transcription of Syllabus Content Map Discussion

The transcribed Sketch Map above (Fig. 3) represents the results of an early discussion with the teacher on the three Key areas of the curriculum; Concepts, Processes and Skills in the transition from KS3 to KS4. The heading “Year 9” refers to the class of 13/14 year olds, and the ‘transition’ in the curriculum (in terms of algebraic skills) involves a progression from solving linear equations to developing techniques for solving quadratic equations. The ideas expressed here intend to build on the pupils’ current knowledge and use their abilities of visualization and representation to work on tasks that will assist that progression. Below, in the ideas for developing the lessons (Fig.4), ‘Number and Algebra’ can be seen in the centre of the picture, while on the left appear the activities potentially linking to some historical contexts.

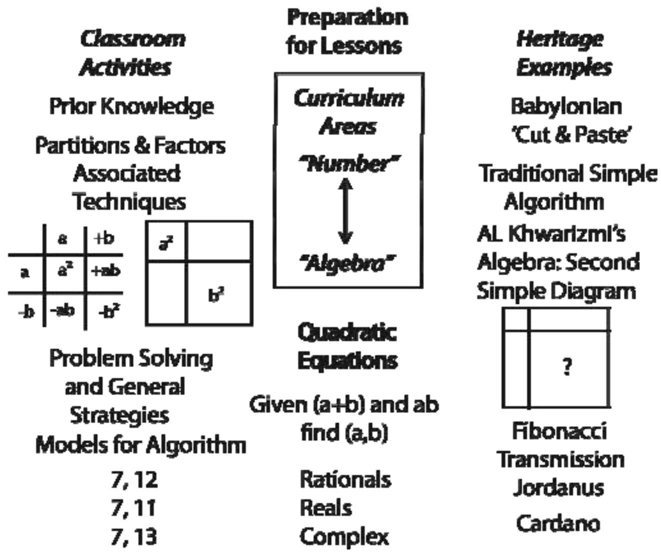

Fig.4. Transcription of Lesson Preparation Map

It is commonly the case that school algebra has been presented as generalised arithmetic, thereby missing the most important aspect of algebra as powerful structural tool for identifying and operating with relationships. In this case having studied linear equations, a possible progression to quadratics shows how two approaches, one from geometrical structure and the other from an arithmetic algorithm can combine to produce general ways of solving the problem. *the general strategy here is to develop the idea that algorithms derived from arithmetical practices can be transformed into geometrical and wider structural concepts.*

A draft lesson sequence was then created that consisted of a series of tasks each allowing opportunities for discussion and contemplation of different aspects of these ideas. Two of these tasks are shown in Appendix A (Jordanus de Nemore's Solution[13]) and Appendix B (Thinking of Two Numbers).

8 Pedagogical Freedom and Curriculum Progression

Typically, when good teachers work mathematically with pupils, they elicit their pupils' knowledge through interactions that prompt pupils to express their own ideas and lead them towards improved and more efficient methods and clearer definitions. The use of Concept Maps however initially sketchy and tentative, can enable us to organise material in a way that allows originality, flexibility, and potential for developing new pedagogical avenues. Viewing mathematics lessons as series of questions, prompts and 'micro tasks' rather than a collection of procedures to be practiced, opens new possibilities and allows connections to be made between curriculum topics as well as with mathematical history and the wider culture.

Asking questions like "What if?" and "Then what?" and encouraging pupils to visualise, seek patterns, compare and classify, explore variations in structures, test

[13] The diagram for completing the square often used in History of Mathematics texts is found in (Rashed 2009: 110) but for this case Al-Khwarizmi drew another diagram a page later showing a gnomon with a space to be filled (Rashed 2009: 112). This procedure is shown in (Hughes 1981: 16)

conjectures, identify properties and relations and make meaning from involvement in a particular task, requires focussing attention on the details of contexts and developing understandings and structural relations that are generic and transferable.

The Design Research approach does not have to be undertaken on a grand scale, these methods are available to anyone who by evaluating their own pedagogy engages in action research, and by taking small steps can achieve pedagogical freedom while at the same time can satisfy the demands of the school curriculum.

What is equally important is the knowledge that there is a community of teachers and researchers working towards similar aims, and that combined with a sensitive and purposeful pedagogical approach, the history of mathematics can provide teachers and pupils at all levels with contexts that can open new visions of the nature of mathematical activity.

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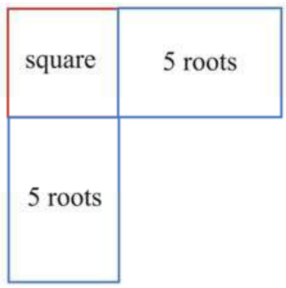
Appendix A.
Jordanus de Nemore’s solution for Al-Khwarizmi’s problem:

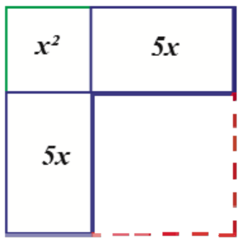
“The Mal plus ten roots are equal to thirty-nine dirhams.”

1. Read this through carefully, and explain Jordanus’ method to your neighbour.



Solution method according to Jordanus de Nemore (1225 - 1260).

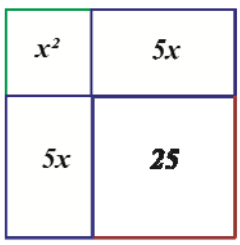
Since the square and the ten roots are equal to 39, and use the two equal halves of the roots to make a gnomon.
(Euclid II, 4).



Our Algebraic Representation
$[x^2 + 10x] = 39$
The roots are shown as:
$x^2 + [5x + 5x] = 39$

What is the size of the missing square?

Completing the square:



$[x^2 + 5x + 5x] + 25 = 39 + 25$

$x^2 + 10x + 25 = 64$
$(x + 5)(x + 5) = 64$
$(x + 5)^2 = 64$

$x + 5 = 8$ $\qquad x = 3$

2. Try these problems using the method above:
$x^2 + 6x = 45$, $\quad x^2 + 14x = 15$, $\quad 4x^2 + 7x = 15$, $\quad x^2 + 2/3x = 35/60$.

Make up some more equations of your own and solve them using Jordanus’ method. What problems arise and what questions could you ask to improve this situation?

(The ‘Mal’ is a word still used today for the unknown and is sometimes called ‘the treasure’. A dirham is a unit of money.)

Appendix B: Thinking of Two Numbers

This is a class activity, managed by the teacher.

"I am thinking of two numbers; their sum is seven and their product is twelve, what are the numbers?"
The first part can be done orally, calling out the number pairs, writing them on the board, or using a visual display:

Sum	7	9	15	14	17	20
Product	12	20	56	33	60	64

The first pairs of numbers are usually easy, if pupils know their tables.
Even so, pupils need time to think and it is important not to lose anyone at this stage as the numbers get bigger, and they can be asked to write down the solution instead of calling out.
Pupils need to experiment like this for a few more numbers.

Sum	26	32	36 …	31	40	53	50 ...
Product	168	252	323 …	234	384	696	609 …

It is up to the teacher to decide at what stage whether to allow calculators.

The point of this game is to help pupils to develop a simple arithmetic strategy for finding the two numbers.
With the early examples, finding the Partitions of the Sum number gives a clue to the Factors of the Product number
7 = (3 + 4) and 3 x 4 = 12 etc.
26 = (13 + 13) but 13x13 = 169 however (12 + 14) gives 12x14 = 168
Partitions of 32: (16,16), (15,17), (14,18) and 14x18 = 252
Partitions of 31: (15 ½ , 15 ½), (15,16), (14,17), (13,18) and 13x18 = 234

So we have a general strategy

Take half the Sum number, then add and subtract in stages to find pairs for testing the Product.
If we think of the sum as a semi-perimeter of a rectangle, and the product as an area, we have a version of a 'geometrical algebra'.

MODELLING IN CLASSROOM

'Classical Models' (in Mathematics Education) and recent developments

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ABSTRACT

The influence of modelling in mathematics education is gaining in importance since applied mathematics has become popular in classrooms again. Today it is one of the most challenging topics for teachers and students in school. The extraordinary part of such modelling tasks in classroom is traversing the modelling cycle: therefore questions have to be formulated; data has to be found and connected to an appropriate model, so that conclusions could be drawn by certain interpretations.

In particular by taking 'classical models' into account in education, e.g. models in / of financial mathematics or applications in trigonometry, the idea of modelling in classroom can be shown as a traditional aspect in mathematics education with a long history.

Considering this idea mathematics can be shown as a useful and applicable instrument in daily life to students, like Freudenthal, Krygovska, Pollak or Steiner stated it in 1968 at the conference "How To Teach Mathematics so as to be Useful".

1 Interests for mathematical modelling

Arguments in the discussion of mathematics education for learning mathematical modelling are outlined and listed by Barbosa (2003). Modelling tasks should …

- motivate the learners / students,
- facilitate the learning of mathematical contexts,
- prepare the learners to apply mathematical knowledge in different areas,
- contribute to the development of general and mathematical competencies,
- guide the learners to understand the socio-cultural role of mathematics which is gaining importance in the community.

The arguments are comprehensible in the way mathematics educators are acting. But the specific advantages of modelling-tasks are not that obvious. The author does not offer any theoretical suggestions or empirical results to justify his arguments.

A possible motivation of the listed arguments can be found in Blum (2003). He argues that mathematical modelling is one of the reasons why people are learning mathematics and simultaneously describes the manner how people are learning mathematics. Hence mathematical modelling provides arguments for the importance of learning mathematics, it even supplies arguments for the whys and wherefores.

As a first conclusion it is possible to find two different perspectives combined in one idea: Mathematics is helpful to learn mathematical modelling on the one hand, and on the other hand mathematical modelling is helpful to learn mathematics (cf. Ottesen, 2001). Reasons for this perception are brought by mathematical history.

2 Modelling and its history

The importance of mathematical modelling founded through applied science, in particular applied mathematics. This discipline was strongly connected to its neighbouring disciplines like physics, astronomy or engineering sciences (cf. Blum, Galbraith, Henn & Niss, 2007). A symbiosis between real and applied mathematics can be detected since the 19th century. From there on trends preferring the real mathematics or the applied mathematics can be located (cf. Kilpatrick, 1992).

Such a trend can be identified in the 1960s to the 1970s, the ‘New Math Reform’. Representatives of this movement claimed that children should learn to think logically and abstract very early. As a consequence the theory of sets was introduced to primary schools. At the same time contrary views were brought to public, in particular of persons working in tertiary education. They claimed to consider more applications in mathematics. The reason for it can be found in the UK for example. Troubles got obvious, because alumni of mathematical studies were not able to take their knowledge for solving real problems despite excellent certificates.

The initial point for observing applications in mathematics (education) and mathematical modelling can be found in the conference of Freudenthal (1968) "WHY TO TEACH MATHEMATICS SO AS TO BE USEFUL". In the first edition of "Educational Studies in Mathematics" (1968) articels of participants presented at that conference can be found. The papers of Freudenthal (1968), Pollak (1968) and Klamkin (1968) should be mentioned eminently.

Pollak (1968) quotes – in opposition to the impacts of that time – the importance of applied mathematics: "To be against applications of mathematics in teaching is like being against motherhood and in favor of sin." Klamkin (1968) quotes the importance of mathematics for applications: "… being useful means not the teaching of applied mathematics but the teaching of mathematics such that it can be applied. Nevertheless, the two must be intertwined for each to be more meaningful."

The interdependence of mathematics and applications in German speaking countries is discussed not until the conference "Anwendungsorientierte Mathematik" (cf. Dörfler & Fischer, 1977) took place. Aspects of applied mathematics for education are included and discussed from a scientific perspective the first time. A next milestone for introducing mathematical modelling can be found in Pollak (1979). In his chapter XII, *The interaction of mathematics and other school subjects*, Pollak defines the field of applied mathematics that should be integrated to a modern mathematics classroom situation:

1. "Applied mathematics means classical applied mathematics.
2. Applied mathematics means all mathematics that has significant practical application.
3. Applied mathematics means beginning with a situation in some other field or in real life, making a mathematical interpretation or model, doing mathematical work within that model, and applying the results to the original situation.
4. Applied mathematics means what people who apply mathematics in their livelihood actually do. This is like (3) but usually involves going around the loop between the rest of the world and the mathematics many times".

Through this characterizations of applied mathematics it is possible to describe this proceeding exchange from the "rest of the world" and "(applied) mathematics" the first time. A first modelling cycle is defined and can be described very simple by a little graphic:

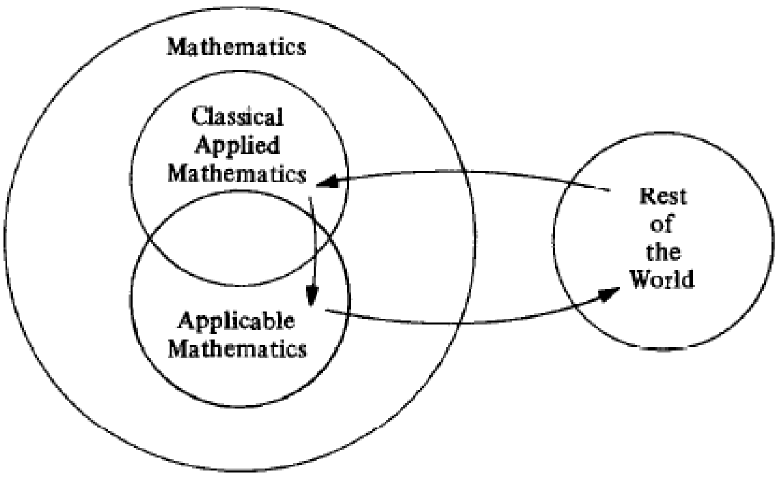

Figure 1. Modelling cycle of Pollak (1979)

Pollak (1979) illustrates the figure as follows: "In this picture the left-hand side shows mathematics as a whole, which contains two intersecting subsets we have called classical applied mathematics and applicable mathematics. Classical applied mathematics represents definition (1) and applicable mathematics, definition (2). Why doesn’t (2) contain all of (l)? The overlap between these is great, but it is not true that all of classical applied mathematics is currently applicable mathematics. There is much work in the theory of ordinary and partial differential equations, for example, which is of great theoretical interest but has no applications which are visible at the moment. Such work is included in definition (1) as classical applied mathematics, since this contains all work in differential equations; on the other hand, if it is not currently applicable, it does not belong in definition (2).

The rest of the world includes all other disciplines of human endeavour as well as everyday life. An effort beginning in the rest of the world, going into mathematics and coming back again to the outside discipline belongs in definition (3). Definition (4) involves, as will be seen, going around the loop many times."

3 Modelling and its visualization – modelling cycles

The discussion about modelling in mathematics education was established by the accomplishments of Pollak, in particular the introduction of real-world-problems in mathematics education was favored by mathematics educators. This basic idea (cf. Klika, 2003) was discussed intensively afterwards. A lot of suggestions for implementing mathematical modelling in education were created and can be found in literature (cf. ISTRON – www.istron-gruppe.de; cf. MUED – www.mued.de).

On the basis of the shown considerations different modelling cycles were created in addition. In 1985 two further modelling cycles were presented by Fischer and Malle (1985) as well as by Blum (1985). The last one is the one which is taken into account in its main features since today.

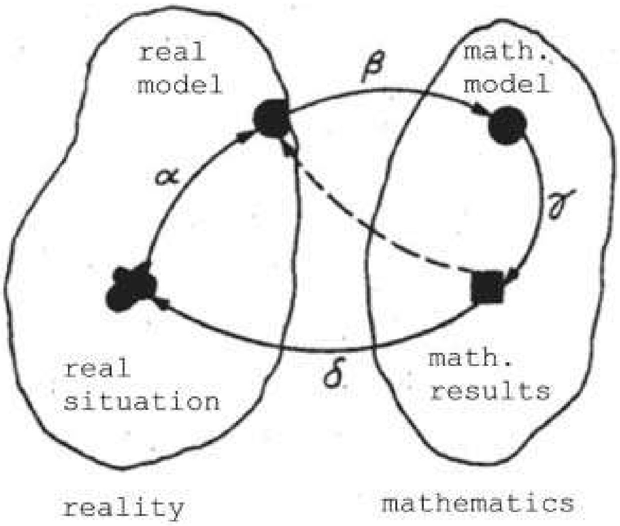

Figure 2. Modelling cycle of Blum (1985)

Fischer and Malle (1985) emphasize, that applying mathematics is a kind of process, which can be divided into several modules, which can also be passed through several times, so that the model itself can be advanced. The development of modules does not have to follow the specific schedule – sometimes some modules have to be observed, sometimes they are strongly connected to others, so that a differentiation is complicated, sometimes some module can be missed.

Following those instructions it is getting obvious that mathematical modelling is a complex activity for mathematics education. Students have to have a lot of different competencies in different (mathematical) areas (e.g. competencies in communication, mathematical basic competencies, competencies in reflection (mathematical) facts), especially if one wants to consider the complexity of this idea.

In particular Fischer and Malle are anticipating with the model of Blum and Leiss (2005, p. 19), which was created on the basis of Blum's shown modelling cycle. This modelling cycle is the basement for all modelling activities and modifications on modelling cycles nowadays.

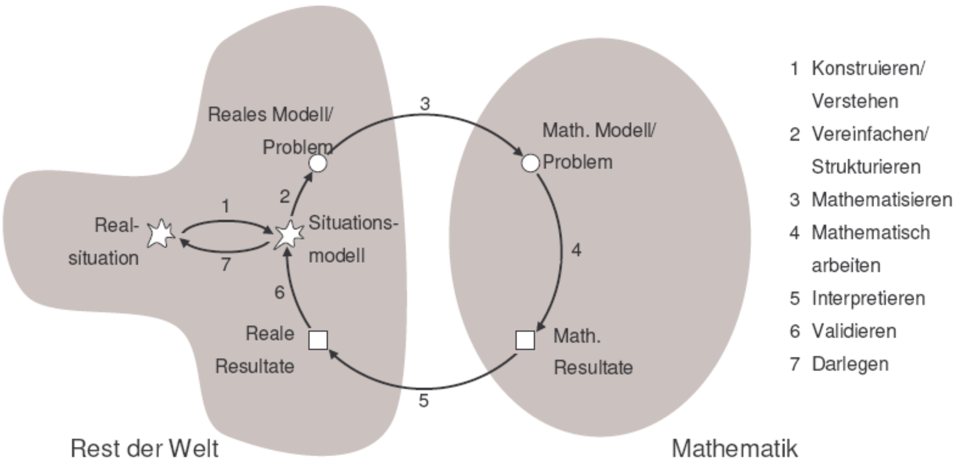

Figure 3. Modelling cycle of Blum & Leiss (2005)

In literature a lot of further modelling cycles can be found, e.g. those of Müller and Wittmann (1984), Schupp (1987) or Burhardt (1988). The last one seems to be worth mentioning because this modelling cycle is designed like a flow-chart-diagram. In particular it is possible to draw more attention to processes of system-dynamics with its help as well as the combination to other disciplines, e.g. computer-science (-education), can succeed.

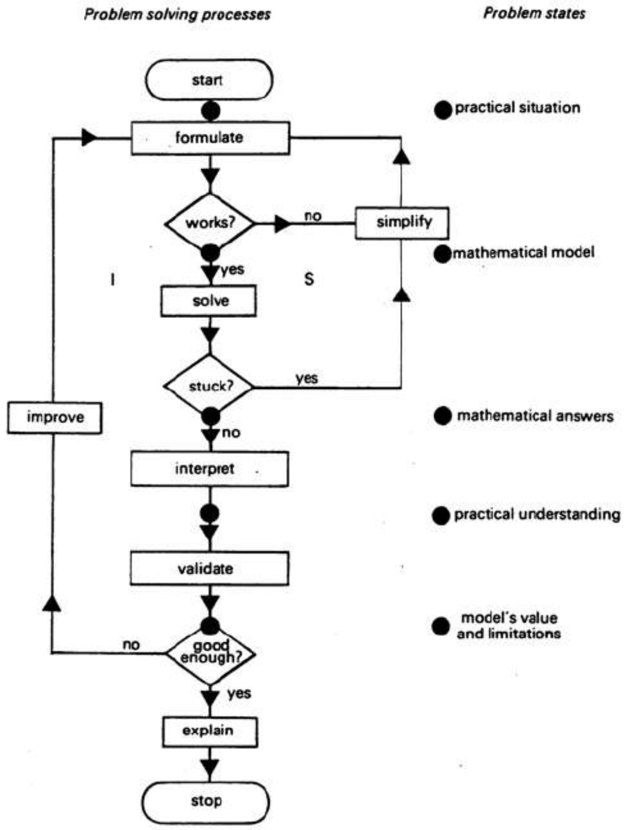

Figure 4. Modelling cycle of Burkhardt (1988)

In contrast to the mentioned modelling cycles Burkhardt's starting point is an arbitrarily one. That can also imply that the starting point cannot be found in a practical situation; it even can be situated in an inner-mathematical problem, e.g. modelling with the help of linear regression, in a list of measurement readings, with a certain amount of measuring points. It is not necessary that a real-life situation in the environment of the problem must be found. I do not concentrate on such a situation in the following part, although the chosen situations could be interpreted like this.

4 A showcase for modelling with historical material

If we have a closer look at an egg, we will see that its shape is very harmonic and impressive. Considering a hen's egg it is obvious that the shape of all those eggs is the same. Because of the fascinating shape of eggs I tried to think about a method to describe the shape of such an egg with mathematical methods. Searching the literature I found some material from Münger (1894), Schmidt (1907), Wieleitner (1908), Loria (1911), Timmerding (1928) and Hortsch (1990). The book of Hortsch is a very interesting summary about the most important results of 'egg-curves'. He also finds a new way for describing egg-curves by experimenting with known parts of 'egg-curves'. The modality how the authors are getting 'egg-curves' is very fascinating. But none of them has thought about a way to create a curve by using elementary mathematical methods. The way how the authors describe such curves are not suitable for mathematics education in schools. So I thought about a way to find such curves with the help of well known concepts in education. My first starting point is a quotation of Hortsch (1990): "The located ovals were the (astonishing) results of analytical-geometrical problems inside of circles." The second point of origin are the definitions of ‘egg-curves’ found by Schmidt (1907) and presented in Hortsch (1990).

Definition 1 (cf. Schmidt, 1907):

Schmidt quotes: "An ‚egg-curve‘ can be found as the geometrical position of the base point of all perpendiculars to secants, cut from the intersection-points of the abscissa with the bisectrix, which divide (obtuse) angles between the secants and parallel lines in the intersection-points of secants with the circle circumference in halves. The calculated formula is $r = 2 \cdot a \cdot \cos^2\varphi$ or $(x^2+y^2)^3 = 4 \cdot a^2 \cdot x^4$".

In education the role of modelling is gaining in importance. Different systems, like computer-algebra-systems (CAS), dynamical-geometry-software (DGS) or spreadsheets, are used in education. With the help of technology it is possible to design a picture of the given definition immediately. In the first part I use a DGS because with its help it is possible to draw a dynamical picture of the given definition. The DGS I am using is GeoGebra. It is free of charge and very suitable in education because of its handling.

First of all the one point P of such an egg as it is given in the definition has to be constructed.

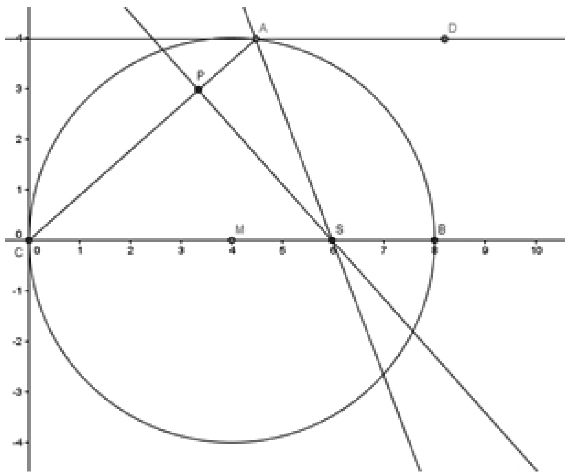

Figure 5. Construction by obtaining the definition of Schmidt (1907)

According to the instruction a circle (center and radius arbitrarily) is constructed first, then a secant from C to A (points arbitrarily). After that a parallel line to the x-axis through the point A (= intersection point secant-circle) is drawn and the bisecting line CAD is determined, which is cut with the x-axis. So the point S is achieved. Now the perpendicular to the secant through S can be drawn. The intersection point of the secant and the perpendicular is called P and is a point of the ‘egg-curve’. By activating the "Trace on" function and using the dynamical aspect of the construction the point A moves towards the circle and the ‘egg-curve’ is drawn as Schmidt (1907) has described it. This can be seen in the following figure:

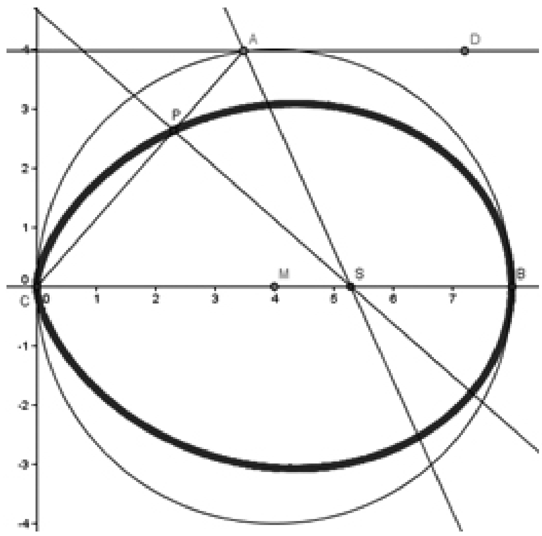

Figure 6. An ‘egg-curve’

Now a way to calculate the formulas $r = 2 \cdot a \cdot \cos^2\varphi$ or $(x^2+y^2)^3 = 4 \cdot a^2 \cdot x^4$ as mentioned above has to be found.

Let us start with the following figure:

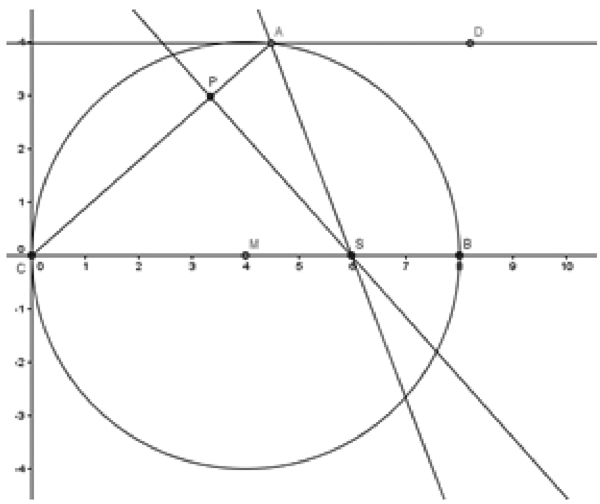

Figure 7. Initial situation for calculating the equations of the egg-curve

Because of the construction the triangle CPS is right-angled. Furthermore it can be recognized that the distance CP and PS is the same and that the triangle CAB is also rectangular, because it is situated in a semicircle. This can be seen in the following figure, where I have also drawn the real ‘egg-curve’ as dashed line.

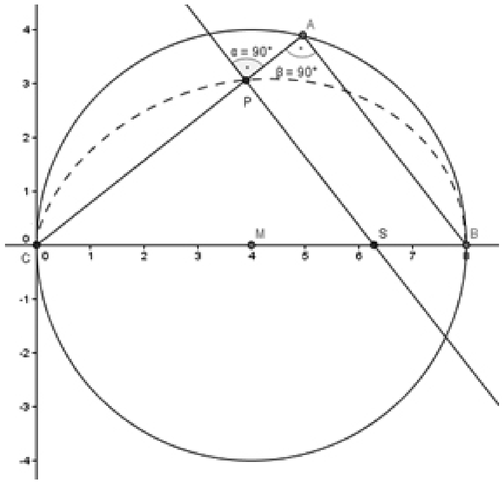

Figure 8. Observing the triangles

Because of the position of the points C (0, 0), A (x, y), B (2·r, 0) and the construction instruction the coordinates of point S and P can be calculated. Therefore only a little bit of vector analysis is necessary. The calculation can be done in the CAS Mathematica. First of all the points and the direction vector of the bisecting line w need to be defined:

{c = {0, 0}, a = {x, y}};

a - c

{x, y}

1/Norm[a - c] (a - c)

$$\left\{\frac{x}{\sqrt{\mathrm{Abs}[x]^2 + \mathrm{Abs}[y]^2}}, \frac{y}{\sqrt{\mathrm{Abs}[x]^2 + \mathrm{Abs}[y]^2}}\right\}$$

w = -1 / √((x)^2 + y^2) {x, y} + {1, 0}

$$\left\{1 - \frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}}\right\}$$

$$w = \left\{1 - \frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}}\right\};$$

Now the equation of the normal form of the bisection line can be calculated. It is cut with the x-axis and defines the intersection-point S.

```
vn = {y, √((x)^2 + y^2) - x};
vn.{u, v} == vn.a
```

$$u\,y + v\left(-x+\sqrt{x^2+y^2}\right) == x\,y + y\left(-x+\sqrt{x^2+y^2}\right)$$

```
vn.{u, v} == vn.a /. v → 0
```

$$u\,y == x\,y + y\left(-x+\sqrt{x^2+y^2}\right)$$

```
First[%] / y == Last[%] / y
```

$$u == \frac{x\,y + y\left(-x+\sqrt{x^2+y^2}\right)}{y}$$

```
Simplify[%]
```

$$\sqrt{x^2+y^2} == u$$

```
s = {√(x² + y²), 0};
```

Now the intersection point P of the secant and the perpendicular through S is calculated.

```
Solve[{x u + y v == (x) (√((x)^2 + y^2)), -y u + (x) v == 0}, {u, v}]
```

$$\left\{\left\{u \to \frac{x^2}{\sqrt{x^2+y^2}},\ v \to \frac{x\,y}{\sqrt{x^2+y^2}}\right\}\right\}$$

$$p = \left\{\frac{x^2}{\sqrt{x^2+y^2}},\ \frac{x\,y}{\sqrt{x^2+y^2}}\right\};$$

All important parts for finding the ‘egg-curve‘ are calculated. Let us have a closer look at figure 8. It is easy to recognize that there are two similar triangles – triangle CPS and triangle CAB. The distance CP shall be called r and the radius of the circle shall be called a. The distance CB has now the length 2·a. The other two distances which are needed CA and CB have the length $\sqrt{x^2+y^2}$. The similarity of the triangles is applied:

$$\frac{CA}{r} = \frac{2\cdot a}{CS} \Leftrightarrow \frac{\sqrt{x^2+y^2}}{r} = \frac{2\cdot a}{\sqrt{x^2+y^2}}$$

Transforming this equation gives:

$$x^2 + y^2 = 2\cdot a\cdot r$$

By using the characteristic of the right-angled triangle CAB and calling the angle ACB φ the cosine of this angle is obtained:

$$\cos\varphi = \frac{\sqrt{x^2+y^2}}{2\cdot a} \Leftrightarrow$$

$$4\cdot a^2\cdot \cos^2\varphi = x^2+y^2$$

Inserting this connection in the equation gives:

$$4\cdot a^2\cdot \cos^2\varphi = 2\cdot a\cdot r$$

Shortening this equation shows:

$$r = 2\cdot a\cdot \cos^2\varphi$$

By substituting r and cosφ it is possible to get the implicit Cartesian form, mentioned in the definition:

$$\sqrt{x^2+y^2} = 2\cdot a\cdot \cos^2\varphi$$

$$x^2+y^2 = 4\cdot a^2\cdot \frac{x^4}{r^4}$$

$$x^2+y^2 = 4\cdot a^2\cdot \frac{x^4}{(x^2+y^2)^2}$$

respectively

$$(x^2+y^2)^3 = 4\cdot a^2\cdot x^4$$

As it can be seen the ‘egg-curve’ has been modelled by elementary mathematical methods. Through using technology teachers and students get the chance to explore such calculations by using the pivotal of modelling. Through such calculations the necessity of polar-coordinates can get obvious.

Definition 2 (cf. Münger, 1894):

Another construction instruction is formulated by Münger (1894). He quotes:

> "Given is a circle with radius a and a point C on the circumference. CP_1 is an arbitrarily position vector, P_1Q_1 the perpendicular to the x-axis, Q_1P the perpendicular to the vector. While rotating the position vector around C point P is describing an egg-curve. The equation of this curve is $r = a\cdot\cos^2\varphi$ in Cartesian form $(x^2+y^2)^3 = a^2\cdot x^4$."

As it is given in the construction instruction a circle (radius arbitrarily) and a point C on the circumference of the circle is constructed. Then an arbitrarily point P_1 on the circumference of the circle can be construed. The perpendicular to the x-axis is construed through P_1 which is cut with the x-axis and delivers Q_1. After that the perpendicular to the secant CP_1 through Q_1 is construed. All these facts can be seen in the following picture:

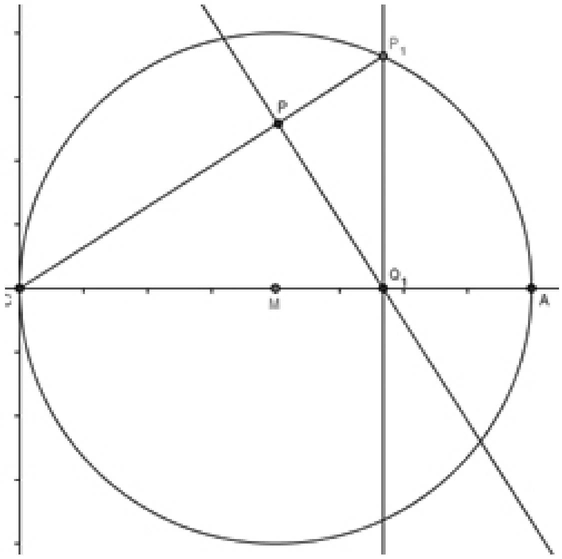

Figure 9. Construction by obtaining the definition of Münger (1897)

If the point P_1 is moved toward the circle, P will move along the ‘egg-curve’. It will be easier to see if the "Trace on" option is activated.

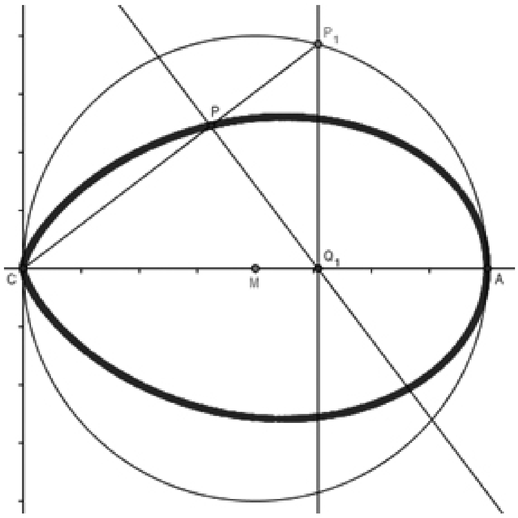

Figure 10. Another ‘egg-curve’

The formula given by Münger (1894) can be found in a similar way as the other formula was found. The most important fact which has to be seen here is that in this picture two rectangular triangles CPQ_1 and CP_1A exist. Those triangles are similar.

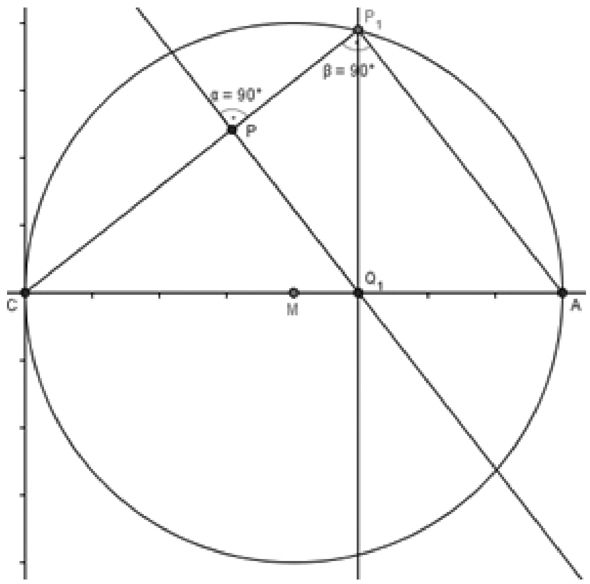

Figure 11. Observing the triangles

The coordinates of the points can be found mentally – without any calculation:
C (0, 0), P_1 (x, y), A (2·a, 0), Q_1 (x, 0)
If the distance CP is called r, then the coordinates of P will not be used. Otherwise they can be calculated analytically. For the sake of completeness I write down the coordinates of P:

$$P\left(\frac{x\cdot y^2}{x^2+y^2}, \frac{y^3}{x^2+y^2}\right)$$

By using the similarity of the both triangles the following equation will be obvious:

$$\frac{\sqrt{x^2+y^2}}{r} = \frac{2\cdot a}{x}$$

Through elementary transformation, because of the mathematical fact $\cos\varphi = \frac{\sqrt{x^2+y^2}}{2a}$ in the triangle CP_1A and substitution of the term $\sqrt{x^2+y^2}$ by 2·a·cosφ the following equation is calculated:

$$2\cdot a\cdot x\cdot\cos\varphi = 2\cdot a\cdot r$$

The result is:

$$r = x\cdot\cos\varphi$$

Because of the fact (assumption in the calculation) that the x is part of our circle – it is the x-coordinate of P_1 – x can be substituted by x = a·cosφ, with a as the radius of the starting circle. So the formula of Münger is found with polar-coordinates:

$$r = a\cdot\cos^2\varphi$$

If the implicit cartesian form should be stated, another substitution has to be done. The result is:

$$(x^2+y^2)^3 = a^2\cdot x^4$$

Epilogue

As it can be seen mathematical modelling is – more or less – an important part in mathematics and mathematics education (cf. Pollak, 2003, 2007). The importance of application-oriented mathematical classroom situation is unquestioned. In combination with mathematical modelling any use of mathematical concepts, methods for describing real-life-situations with the help of mathematics and solving of them is intended. One goal of mathematics education should be a demand for the development of modelling-skills of students. These skills can be met if one is able to recognize the modelling-capacities of mathematical contents and features and if one is able to apply heuristical strategies for supporting the mathematization of real contexts and the application of mathematical contents and features. But such skills can only be developed successfully if students are working active, independently on their own. The usage of historical material respectivelly historical ideas of mathematicians or historical techniques in mathematics can support this.

By bringing ‘classical models’, e.g. Bateman-function(s) (cf. Siller, 2010), Pell’s equation or the Easter formula, into classrooms two important aspects of curricula are met: the role of history in mathematics education and mathematics is increased in students‘ minds as well as the importance of mathematical modelling documented by the history of mathematics itself. Questions students have to deal with should always be answered and discussed by considering (scientific) findings of a connated reference discipline and a strong focus on the practical implementation by observing mathematics education in reality. So students are able to recognize that the history of mathematics education is important

- for mathematics itself, because it would be hard to answer the question "What ist mathematics?" without knowing anything about its history.
- for the learning of mathematics itself, because students are able to recognize the usefullness and acceptability of this subject by recognizing mathematics as a historical phenomenon.

Summing up the idea of modelling under respect to historical developments in mathematics (education) allows to find interesting problems and a methodological plurality so that one is gaining a deep and serious insight into the discussed topic(s) and problem(s) instead of using standard methods.

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CLASSIFYING THE ARGUMENTS AND METHODS TO INTEGRATE HISTORY IN MATHEMATICS EDUCATION: AN EXAMPLE

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ABSTRACT

The ICMI Study volume "History in Mathematics Education", published in 2000, includes a comprehensive list of arguments for integrating history in Mathematics Education (ME) and methodological schemes of how this can be accomplished. To classify the above arguments, Jankvist distinguished between using "history-as-a-goal" and using "history-as-a-tool" in ME. Independently, Grattan-Guinness distinguished between "history" and "heritage" aiming –among other things - to help understanding better which history could be helpful and meaningful in ME. In a recent paper by the authors, these pairs of concepts are used to classify more finely and deeply these arguments and methodological schemes. The present paper aims to illustrate this 2X2 classification scheme by means of a specific example, namely logarithms and related concepts as taught to upper high school students.

1 Introduction

In the last decades, there has been a growing interest in integrating the history of mathematics (HM) in mathematics education (ME). Arguments for this integration have been put forward to refute possible objections and/or enhance the interest of the ME community, educational material has been produced, empirical research has been conducted and methodological schemes have been described & implemented.

For a long time, there were no coherent theoretical ideas and framework to place, see and compare all these activities. A serious attempt in this direction is the comprehensive ICMI Study volume (Fauvel & van Maanen 2000), but further important work followed:

Jankvist (2009a,b,c) reconsidered the general arguments for HM in ME (the *whys* in his terminology) and methodological schemes (the *hows*), introducing two interesting criteria:

- To classify the *whys* according to whether history appears as a ***goal***, or as a ***tool***, with emphasis on "*meta-perspective*" issues, or "*inner*" issues, respectively.
- To clarify the *hows* according to a 3-level distinction of the possible types of implementations: *illumination approaches*, *modules approaches* and *history-based approaches*.

Independently, Grattan-Guinness (2004a,b) introduced the distinction between "***history***" and "***heritage***" to interpret mathematical activities and their products. This is an important conceptual tool to revisit the issue of "which history is appropriate to ME?" (Barbin 1997), attempting to clarify existing conflicts and tensions between a mathematician's and a historian's approach to mathematical knowledge. Grattan-Guiness (2004b) gives several examples by contrasting the general characteristics of the two concepts.

History and *Heritage* should be seen as ***complementary*** ways to approach and understand mathematics as a human activity, in the sense that none of them, taken alone, can lead to a sufficiently wide and deep enough understanding of what (a specific piece of) mathematics is. Similarly, Jankvist's distinction between the use of *history-as-a-tool* and *history-as-a-goal* in ME should be seen as complementary ways to classify the arguments and the methodological approaches to introduce a historical dimension in ME. The term "complementary" is used closely to that used by N. Bohr to describe the microphysical reality and subsequently raised to a general conceptual tool to understand reality (Bohr 1934, 1958).

Recently, these ideas have been described in more detail and an attempt has been made to classify the *whys* and *hows* more finely, by projecting them onto the 2X2 grid formed by the two

dipoles, namely (*history-as-a-tool*, *history-as-a-goal*) and (*history*, *heritage*) (Tzanakis & Thomaidis, to appear). The term "dipole" is used here to emphasize the interconnections between concepts, thus reflecting better their complementary character mentioned above. This paper, which **complements** the one above, aims to illustrate these ideas by means of an example. Therefore, a brief description of the conceptual dipoles above is given in the next section, together with a list of the ICMI *whys & hows* and Jankvist's distinction of possible implementations. Their corresponding classification in terms of these dipoles is given in section 3. Finally, these theoretical ideas are illustrated in section 4 by means of an example; logarithms and related concepts to be taught in high school.

2 The two conceptual *dipoles* and the *whys* & *hows* of integrating history in mathematics education

2.1 The two conceptual dipoles

Jankvist introduced two broad ways in which HM could be helpful and relevant to ME: ***History-as-a-tool*** and ***History-as-a-goal***, which are intimately connected with issues within mathematics (*inner issues*) and issues that concern mathematics itself (*meta issues*):

> "***History-as-a-tool*** concerns the use of history as an assisting means, or an ***aid***, in the learning [or teaching] of mathematics…. in this sense, history may be an aid both …"[1] "as a motivational or affective tool, and … as a cognitive tool …"[2] "[It] concerns … inner issues, or ***in-issues***, of mathematics [that is] issues related to mathematical concepts, theories, disciplines, methods, etc.— the internal mathematics"[3].
>
> "***History-as-a-goal*** does not serve the primary purpose of being an aid, but rather that of being an ***aim*** in itself … posing and suggesting answers to questions about the evolution and development of mathematics, … about the inner and outer driving forces of this evolution, or the cultural and societal aspects of mathematics and its history" (Jankvist 2009b §1.1). "[It] concerns … learning something about the meta-aspects or ***meta-issues*** of mathematics … [that is] issues involving looking at the entire discipline of mathematics from a meta perspective level" (Jankvist 2009c, pp239-240).

These two ways are mutually exclusive, in the sense that the emphasis put in each case is different and to a large extent incompatible with each other. But although "…history-as-a-goal 'in itself' does not refer to teaching history of mathematics ***per se***, but using history to surface meta-aspects of the discipline…in specific teaching situations, [it] may have the positive side effect of offering to students insight into mathematical in-issues of a specific history" (Jankvist 2009d, p.8). Conversely, using "history-as-a-tool" to teach and learn specific mathematics may stimulate reflections at a meta-perspective level, extrapolated from the particular subject considered; that is, an ***anchoring*** of *meta-issues* into the *in-issues* that constitute the study of the subject may result (Jankvist 2009b, §§5.3, 5.4, 6.1, 6.3). These are important interrelations, stressing the indispensability of both "history-as-a-tool" and "history-as-a-goal", which thus constitute what we call a coherent ***conceptual dipole***.

Independently, Grattan-Guinness distinguished between ***History*** and ***Heritage***:

> The ***History*** (***Hi***) of a mathematical subject *N* refers to "…the development of *N* during a particular period: its launch and early forms, its impact [in the immediately following years and decades], and applications in and/or outside mathematics. It addresses the question '*What happened in the past?*' by offering descriptions. Maybe some kinds of

[1] Jankvist 2009b §1.1.
[2] Jankvist 2009d, p8.
[3] Jankvist 2009c, p240.

explanation will also be attempted to answer the companion question '*Why did it happen?*''[4]. "[It] should also address the dual questions "*what did not happen in the past?*" and "*why not*?"; false starts, missed opportunities…, sleepers, and repeats are noted and maybe explained. The (near-)absence of later notions from *N* is registered, as well as their eventual arrival; *differences* between *N* and seemingly similar more modern notions are likely to be emphasized"[5].

The ***Heritage*** (***He***) of a mathematical subject *N* refers "…. to the impact of *N* upon later work, both at the time and afterward, especially the **forms** which it may take, or be **embodied**, in later contexts. Some modern form of *N* is usually the main focus, with attention paid to the course of its development. Here the mathematical relationships will be noted, but historical ones…will hold much less interest. [It] addresses the question "*how did we get here?*" and often the answer reads like "the royal road to me." The modern notions are inserted into *N* when appropriate, and thereby *N* is unveiled... *similarities* between *N* and its more modern notions are likely to be emphasized; the present is *photocopied* onto the past" (Grattan-Guiness, 2004a, p.165).

Grattan-Guinness argues that ME can profit equally well from **both** *Hi* and *He* and gives a detailed list of the differences between them (Grattan-Guinness 2004b, §1.3), showing their **incompatibility**:

"The distinction between history and heritage is often sensed by people who study some mathematics of the past, and feel that there are fundamentally different ways of doing so. Hence the disagreements can arise; one man's reading is another man's anachronism, and his reading is the first one's irrelevance. The discords often exhibit the differences between the approaches to history usually adopted by historians and those often taken by mathematicians." (Grattan-Guinness 2004b, p.8).

But, their **indispensability** in understanding the development of mathematics is clearly emphasized:

"The claim put forward here is that *both history and heritage are legitimate ways of handling the mathematics of the past; but muddling the two together, or asserting that one is subordinate to the other, is not*." (Grattan-Guinness 2004b, p.8)

Hence, the two concepts are **complementary** in the sense of section 1, constituting the poles of another *conceptual dipole*.

As far as the introduction of a historical dimension in ME is concerned, the distinction between *history* and *heritage* is close to similar distinctions between pairs of methodological approaches; *explicit* & *implicit* use of history, *direct* & *indirect* genetic approach, *forward* & *backward heuristics* (Fauvel & van Maanen 2000, ch.7, pp.209-210). Hence, this distinction is potentially of great relevance to ME (Rogers 2009), serving -among other things- to contribute towards answering the recurrent question "Why and which history is appropriate to be used for educational purposes?" (Barbin 1997).

2.2 A concise list of the *whys* & *hows*

In this section the *whys* & *hows* are listed according to the ICMI Study volume (Fauvel & van Maanen 2000, §§7.2,7.3) and Jankvist (2009c, §6), noting that the *whys* correspond to didactical *tasks* to be attempted and the *hows* correspond to *methodological approaches* to be followed.

2.2.1 The "ICMI Study *whys*"

The areas in which the HM is beneficial for the teaching and learning of mathematics are

[4]Grattan-Guiness, 2004b, p.7.
[5]Grattan-Guiness, 2004a, p.164.

listed below.

A. *The learning of Mathematics*

1. *Historical development vs. polished mathematics*: To uncover/unveil concepts, methods, theories etc.

2. *History as a re-source*: To motivate, to raise the interest, to engage the learner by linking present knowledge and learning process to knowledge and problems in the past.

3. *History as a bridge between mathematics and other disciplines/domains*: From where and how did a great part of mathematics emerged? To bring-in new aspects, subjects and methods.

4. *The more general educational value of history*: To develop personal growth and skills, not necessarily connected to mathematics.

B. *The nature of mathematics and mathematical activity*

1. *Content*: To get insights into concepts, conjectures & proofs, by looking from a different viewpoint; to appreciate "failure" as part of mathematics in the making; to make visible the evolutionary nature of meta-concepts.

2. *Form*: To compare old and modern; to motivate learning by stressing clarity, conciseness and logical completeness.

C. *The didactical background of teachers*

1. *Identifying motivations*: To see the rationale for introducing new knowledge and progress.

2. *Awareness of difficulties & obstacles*: To become aware of possible didactical difficulties and analogies between the classroom & the historical evolution.

3. *Getting involved and/or becoming aware of the creative process of "doing mathematics"*: To tackle problems in historical context; to enrich mathematical literacy; to appreciate the nature of mathematics.

4. *Enriching the didactical repertoire*: To increase the ability to explain, approach, understand specific pieces of mathematics and on mathematics.

5. *Deciphering and understanding idiosyncratic and/or non-conventional approaches to mathematics*: To learn how to work on known mathematics in a different (old) context; hence to increase sensitivity and tolerance towards non-conventional, or "wrong" mathematics.

D. *The affective predisposition towards mathematics*

1. *Understanding mathematics as a human endeavour*: To show and/or understand evolutionary steps.

2. *Persisting with ideas, attempting lines of inquiry, posing questions*: To look in detail at similar examples in the past.

3. *Not getting discouraged by failures, mistakes, uncertainties, misunderstandings*: To look in detail at similar examples in the past.

E. *The appreciation of mathematics as a cultural endeavour*

1. *Appreciating that mathematics evolves under the influence of factors intrinsic to it*: To identify and appreciate the role of internal factors.

2. *Appreciating that mathematics evolves under the influence of factors extrinsic to it*: To identify and appreciate the role of external factors.

3. *Appreciating that mathematics form part of local cultures*: To understand a specific piece of mathematics through approaches belonging to different cultures.

2.2.2 The "ICMI Study *hows*"

The *hows* for integrating HM in ME according to the ICMI Study volume are:

A. *Learning* **history** *by providing* ***direct historical information***

Isolated factual information; historical snippets; separate historical sections; whole books and courses on history etc.

B. *Learning* ***mathematical topics*** *by following a teaching* ***approach inspired by history***
Teaching modules inspired by history; worksheets based on original sources; historical-genetic approach; modernised reconstructions of a piece of mathematics etc.

C. *Developing*
1. *Awareness of the intrinsic nature of mathematical activity* (***intrinsic awareness***) and
(i) The role of general conceptual frameworks
(ii) The evolutionary nature of all aspects of mathematics
(iii) The importance of the mathematical activity itself (doubts, paradoxes, contradictions, heuristics, intuitions, dead ends etc);
2. *Awareness of the extrinsic nature of mathematical activity* (***extrinsic awareness***)
(i) Relations to philosophy, arts and social sciences
(ii) The influence of the social and cultural contexts
(iii) Mathematics as part of (local) culture and product of different civilizations and traditions
(iv) Influence on ME through ME history.

2.2.3 "Jankvist's *hows*"

A similar account of Jankvist's possible types of implementations is:

A. ***Illumination*** *approaches*: Teaching and learning of mathematics, in the classroom or the textbooks used, is supplemented by historical information of varying size and emphasis.

B. ***Modules*** *approaches*: Instructional units devoted to history, and often based on the detailed study of specific cases. History appears more or less directly.

C. ***History-based*** *approaches*: Directly inspired by, or based on the HM. Not dealing with studying the HM directly, but rather indirectly; the historical development not necessarily discussed in the open, but often sets the agenda for the order and way in which mathematical topics is presented.

Both types of *hows* correspond to possible implementations of the HM into ME, of a different character, however; the ICMI Study *hows* focus on different emphases, whereas, Jankvist's focus strictly on the adopted methodologies.

3 The conceptual dipoles as a means to classify the *whys* & *hows*

Taking into account the description of the conceptual dipoles in §2.1, a 2X2 table results, composed by the elements of each dipole. Then, according to the description in §§2.2, 2.3, each of the *whys* and *hows* can be placed in at least one cell, depending on how sharply and clearly it has been described (Fauvel & van Maanen 2000, §§7.2, 7.3; Jankvist 2009c, §6). Thus, the two dipoles act as a "magnifying lens", either requesting a more complete description of each *why* and *how*, or/and providing a clearer orientation of the way each *why* and *how* could be implemented. Items appearing more than once in Table 1 are shaded and those placed with reserve appear with an interrogation mark[6], suggesting that the *whys* are not **irreducible** with respect to the two dipoles, but consist of **simpler** elements, as explained in more detail in Tzanakis & Thomaidis (to appear). Hence, they should be further analysed, so that they fall into only one cell of Table 1. But this remains to be shown and further work is needed (A.4 is not in the table; it should be analysed further).

[6]This convention is applied to all subsequent tables.

Table 1: The classification of the ICMI ***whys***

	History	***Heritage***
***History as a goal* (emphasis on *meta-issues*)**[7]	C.2, C.3(?) E.1, E.2, E.3	A.3; B.1, B.2 D.1
***History as a tool* (emphasis on *inner-issues*)**[7]	A.3; C.1, C.3, C4, C.5 D.2, D.3	A.1, A.2 B.1, B.2(?) C.2, C.3 E.3

This classification of the *whys* is **finer** to the extent that the dipoles have been determined as sharply as possible, which presupposes the detailed study of the *whys* and each *conceptual dipole* in the context of specific examples. In addition, this and the following tables can be considered in relation to the **target population** to whom they are addressed, specifying which entries are better suited to whom: mathematics teachers, curriculum designers, producers of didactical material, mathematics teachers' trainers and advisors.

Table 2: The classification of the ICMI ***hows*** (cf. §2.2.2)

	History	***Heritage***
***History as a goal* (emphasis on *meta-issues*)**[7]	Direct historical information: A Intrinsic awareness: C.1(ii) Extrinsic awareness: C.2(ii)	Direct historical information: A Extrinsic awareness: C.2(i) (iii) (iv)
***History as a tool* (emphasis on *inner-issues*)**[7]	Intrinsic awareness: C.1(i) (iii) Learning mathematical topics (explicit use of history): B	Learning mathematical topics (implicit use of history): B

Labels (i)-(iv) in this table refer to the sub-items in §2.2.2.C and provide an example of the "irreducibility" idea mentioned above; the development of mathematical awareness has been described clearly in the ICMI Study volume, allowing for a clearer classification of its various aspects[8]. The same holds for learning mathematical topics by following an approach explicitly, or implicitly inspired by history (cf. §2.1, last paragraph), but not so for learning history by providing direct historical information.

Table 3: The classification of Jankvist's ***hows*** (cf. §2.2.3)

	History	***Heritage***
***History as a goal* (emphasis on *meta-issues*)**[7]	*Modules* approaches: B	*Illumination* approaches: B *History-based* approaches(?): C
***History as a tool* (emphasis on *inner-issues*)**[7]	*Illumination* approaches: A *Modules* approaches: B	*History-based* approaches: C

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[7]Relating *History-as-a-goal* and *History-as-a-tool* with *inner-issues* and *meta-issues*, respectively is done keeping in mind the possible cross-interrelations mentioned in §2.1!

[8]E.g. items 3.2.3(b) definitely concern *meta-issues*; (ii) requires to consider issues in their historical context of a particular period; (i) (iii) (iv) touch upon issues that connect the present to the past and are likely to be based on a *heritage*-like approach, though this is clearer for (ii) & (iv) than for (i). On the other hand, 3.2.3(a)(i), (iii) mainly concern specific historical examples whose mathematical content should be explored in a way that awareness of *meta-issues* may be developed (or be *anchored* there, in the sense of §2.1).

This table suggests that Jankvist's *hows* constitute a promising identification of broad categories of approaches to be analysed into more sharply described ones (this is already apparent in Jankvist 2009c, §6).

4 An example

To illustrate the general classification of section 3, we consider a specific example, namely, the network of interrelated notions

Power of numbers–exponent–logarithm & (logarithmic) base–exponential function–logarithmic function

In teaching and learning these concepts, or any other mathematical subject, questions are often raised, whose answer presuppose both teacher's knowledge of the historical development and (meta)knowledge of how to deal with this historical knowledge in the classroom. The range and depth of this knowledge is closely related not only to the subject itself and the questions raised, but also to the learners' age, the level of instruction, the didactical aims and how the subject fits into the curriculum. These factors determine to a large extent the relation between HM and ME in the specific subject to be taught and learnt.

We approach such possible questions as specific didactical *tasks* to be accomplished didactically, thus reflecting the *whys*[9] and corresponding answers as an outline of possible *approaches* to do that, thus reflecting the *hows* in section 3[8]. In this way we will illustrate the fitting of the *whys & hows* into the 2X2 classification scheme of section 4. However, we would like to emphasize that the formulation of the questions below and the outline of their answer constitute only one **choice** among various possibilities and that different questions and/or different answers could be provided, depending on the factors mentioned in the previous paragraph. We simply aim to indicate how the suggested classification is applied in a particular case, **once** we have specified the questions raised and their possible answers.

For the reader's convenience, we indicate *questions* by • and *answers* by ► and we refer directly to their relation with the numbering of *whys & hows* in section 2, respectively. We note that the questions and answers as they appear below have been formulated based on the literature of the last 25 years or so, on a historically motivated teaching of logarithms (e.g. Katz 1986, 1995; Thomaidis 1987; Toumasis 1993; Fauvel 1995; van Maanen 1997; Clark 2006; Stein 2006; Barbin et al. (2006); Panagiotou 2011).

The concept of logarithm, which is usually, introduced for the first time to high school students (16-17 years old) in the context of teaching of the exponential and logarithmic functions, raises several questions, already from its definition.

• **q_1**: Why the exponent of a number's power is called the *logarithm* of that power relative to that number as a base? (A1, B2)

Any attempt to answer this question cannot avoid direct, or indirect reference to the HM:

►**a_1**: When the concept of logarithm was invented in the early 17th century, the modern exponential notation of powers did not exist yet, hence the logarithm was **not** defined as the exponent of a power relative to a base. (2.2.2.A, 2.2.3.A)

This answer immediately leads to new questions:

• **$q_{2\text{-}1}$**: Why was the concept of logarithm introduced at all? (C1)
• **$q_{2\text{-}2}$**: How was the logarithm defined originally? (A2)
• **$q_{2\text{-}3}$**: Why was the **term** "logarithm" used/introduced? (A2)

[9]Cf. the first paragraph of §2.2.

To answer such questions, sufficiently deep historical knowledge is required, hence the choice and use of appropriate historical sources is raised (a book on the HM, a treatise on the history of logarithms, relevant original sources etc) and whether the answers to be used will be accompanied by historical references or not, e.g.

►**a_2**: The answer could be limited to a modern, strictly mathematical framework, if the correspondence between an arithmetical and geometrical progression is used, from which it can be easily explained the usefulness of the logarithm as a tool to simplify numerical calculation and the etymology of the word "logarithm"[10]. History is used implicitly; no traces are left in the classroom. (2.2.2.B-implicit use of HM; 2.2.3.C)

This answer raises new questions:

- **q_3**: How can this model be useful in the classroom? (C4)

►**a_3**: The answer to this question too can also be restricted in a modern, strictly mathematical context, explaining technically how to construct a 'dense" geometrical progression and a corresponding "dense" arithmetical one. (2.2.2.B-implicit use of HM, 2.2.3.C)

Though it may be explained that at that era numerical calculations with many-digit numbers was time-consuming, tedious job – hence there was vivid interest to exploit any idea on their simplification, like the correspondence of arithmetic and geometric progressions -, it is readily appreciated that the same holds for the construction of two practically useful "dense" progressions. Hence, new questions arise naturally:

- **$q_{4\text{-}1}$**: What was the motivation to get involved in the construction of two practically useful "dense" progressions? (C1)
- **$q_{4\text{-}2}$**: Is it really true that this problem is of such great mathematical interest that it is worthwhile to get involved in it, ignoring the practical difficulties inherent to its solution? (C3)

►**a_4**: The problem of simplifying tedious numerical calculations was posed in the context of dealing with economic relations (e.g. using tables of interests) and astronomical measurements (use of trigonometric tables). Therefore, answering these questions leads outside mathematics, and refers directly to the relation of Mathematics to other disciplines in that historical period, which evidently, cannot be considered solely in mathematical terms. (2.2.2.A; 2.2.3.B)

This leads to new questions.

- **q_5**: How did people cope with elaborated calculations in various disciplines, including Mathematics? (A3, E1, E2)

►**a_5**: The answer will definitely refer to the reciprocal relation between Mathematics and other disciplines which are based/using Mathematics, elaborating on the use of groups of calculators paid by financial institutions or observatories, the development of new methods to construct numerical tables of higher accuracy, the invention of tricky methods to simplify calculation (like the trick of "prosthaphairesis"[11]) etc. (2.2.2.C.2(ii), 2.2.3.B)

- **q_6**: Who was the first to construct logarithmic tables, and how did he achieve to that? (C4)

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[10] *Logarithm*: From the Greek *logos* (ratio) and *arithmos* (number); every term of the arithmetic progression shows the number (multitude) of the ratios of successive terms of the geometrical progression up to the corresponding term of this progression. E.g., 6 in the arithmetic progression 0,1,2,3,4,5,6 (which equals $log_2 64$) indicates that to get the term 64 of the geometric progression 1,2,4,8,16,32,64 starting from its first term, six ratios are inserted, namely, 2:1, 4:2, 8:4, 16:8, 32:16, 64:32.

[11] From the Greek *Prosthesis* = addition and *aphaeresis* = subtraction; a trick based on trigonometric relations to transform the product of two trigonometric numbers into sums of such numbers (Smith 1959, pp.455-472; Barbin et al 2006, ch.II; Thomaidis 1987, §3).

► **a_6** This is a complex question the answer to which includes references to the not rare fact of "independently made similar or identical discoveries/inventions" in mathematics; the arithmetical background of Bürgi's "*red numbers*" and the kinematical-trigonometric background of Napier's "*logarithms*". A historically complete answer should stress the essential differences between these two approaches; that is, the fact that they essentially concern different conceptions of "logarithmic" notion, though of course they have as a common starting point the correspondence between arithmetic and geometric progression and serve the same purpose. In addition, it is important from a didactical point of view to study biographical elements of the scientists involved in the invention of logarithms. (2.2.2.A; 2.2.3B)

Though the logarithmic tables solved the crucial – at that time - problem faced by those involved in complex arithmetical calculations and their use was adopted with enthusiasm, nowadays their use has banished and what remains is the concept of logarithmic function relative to a given base. Given that at the time of the invention of logarithms both the idea of a base and the function concept were nonexistent, the following questions naturally arise:

- **q_{7-1}**: What was the reason and/or questions that led to the connection of the concept of logarithm with those of exponent and base? (B1)
- **q_{7-2}**: What was the reason and/or questions that led to the connection of the concept of logarithm with the concept of a "logarithmic function"? (B1, C1)

► **a_7:** Answers to these questions could be given in a strictly mathematical context with no, or only limited reference to the historical development. However, given that this historical development is closer to contemporary Mathematics, many of the original texts are suitable for reading and discussion in the classroom. In this way, the integration of the HM could be more direct, efficient and demanding, contributing to the development of a classroom discourse, resembling a "community of researchers" which explores mathematical questions and problems and deepens its understanding of Mathematics by studying historical texts. (2.2.2.B-explicit use of HM; 2.2.3.B)

The above questions and answers can be rearranged and seen in the context of the classification scheme of section 4 in the following table.

Table 4: The classification of questions and answers on logarithms

	History	***Heritage***
***History as a goal* (emphasis on *meta-issues*)**	a_4 (2.2.2.A, 2.2.3.B) q_5 (E1, E2) a_5 (2.2.2.C, 2.2.3.B) a_6 (2.2.2.A, 2.2.3.B)	a_1 (2.2.2.A, 2.2.3.A) q_{7-1} (B1) q_{7-2} (B1)
***History as a tool* (emphasis on *inner-issues*)**	q_{2-1} (C1) q_3 (C4) q_{4-1} (C1) q_{4-2} (C3) q_5 (A3) a_5(2.2.2.C, 2.2.3.B) q_6 (C4) a_6(2.2.2.A, 2.2.3.B) q_{7-2} (C1) a_7(2.2.2.B, 2.2.3.B)	q_1 (A1, B2) q_{2-2} (A2) a_2 (2.2.2.B, 2.2.3.C) q_{2-3} (A2) a_3(2.2.2.B, 2.2.3.C) q_6 (A2)

Remarks: (a) If the questions and answers were formulated more sharply and in more detail, we expect that they would not appear in more than one cell each.

(b) The arrows link questions, which refer to meta-issues with answers connected to inner issues; maybe this illustrates the *anchoring* process mentioned in section 3. This is an idea on which to elaborate more.

5. Concluding remarks

This paper is theoretical and much work remains to be done by analysing specific examples to check the validity of the basic ideas, their usefulness in actual implementation and their efficiency to better understand different aspects of which and how HM could be integrated in ME and for what purpose. A preliminary illustration of the classification schemes by means of a specific example was given in section 4, which should be considered only as a first step towards a better understanding and sharpening of this classification. We believe this is a promising line of inquiry that will sharpen the arguments for and approaches of integrating HM into ME and will better reveal possible interrelations of the conceptual dipoles introduced here.

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History and epistemology implemented in mathematics education: classroom experiments & teaching materials, considered from either the cognitive or/and affective points of view; surveys of curricula and textbooks

AN IMPLEMENTATION OF TWO HISTORICAL TEACHING MODULES

Outcomes and perspectives

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ABSTRACT

Distinguishing between two different uses of history of mathematics in mathematics education (history as a tool to enhance the inner issues – in-issues – of mathematical topics, concepts, methods, etc.; and history as a sort of goal, in the sense that it is considered a goal to teach students something about the historical development of mathematics, its influence on society and culture as well as the other way round, and the fact that mathematics is a human endeavour and the development of mathematics therefore likewise – that is to say some of the more meta-perspective issues – meta-issues – of mathematics) the present presentation concerns an empirical investigation of the use of history as a goal. More precisely two longer historical teaching modules were designed and implemented in a Danish upper secondary mathematics class in order to answer the following research questions:

1. In what sense, to what extent, and on what conditions is it possible to have upper secondary students engage in meta-issue discussions and reflections of mathematics and its history in terms of 'history as a goal'?
2. In what sense and on what levels may an anchoring of the meta-issue discussions and reflections in the taught and learned subject matter (in-issues) be reached and 'ensured' through a 'modules approach'?
3. In what way may teaching modules focusing on the use of 'history as a goal' give rise to changes in students' beliefs about (the discipline of) mathematics, or the development of new beliefs?

In the presentation I shall mainly focus on the two first questions, displaying data from the implementation of the teaching modules to support possible answers to these. This involves also, for example, a discussion of certain design principles for using history in mathematics education. The third question shall only be touched upon briefly, outlining the main results.

The above mentioned research will be related to the body of empirical research already available in the field of using history in mathematics education, thereby also discussing the status of this field from an empirical point of view as well as the role and perspectives for future empirical research within the field.

I am honored to have been invited to give this opening plenary lecture at the 6th European Summer University in Vienna and I thank the organizers and the scientific program committee for this opportunity.

Three years ago I attended my first ESU, in Prague, and there I gave a short presentation which resulted in an even smaller abstract in the proceedings. Today I plan to 'expand' a little on that abstract. More precisely, I shall talk about the research and outcomes of my doctoral thesis (Jankvist, 2009d), which I handed in and defended last year, and related publications. At the end of the talk I shall present some of my own views in terms of perspectives for further research on history in mathematics education, in particular regarding the empirical dimension of this.

I shall structure my talk into four parts: first, an introduction; second, a discussion of students' meta-issue discussions and their anchoring; third, a brief discussion of students' beliefs about mathematics as a discipline; and finally, some perspectives for future research.

1 Introduction

When talking about history in mathematics education, we may generally distinguish between two different kinds of arguments for or purposes with using (or integrating) history in the teaching and learning of mathematics. I have previously referred to these as history as a tool and history as a goal (Jankvist, 2009a), and I shall do that here as well. When using history as a tool, history acts as an assisting means, or aid, for the teaching and learning of mathematics. We may distinguish between different roles that history may play when acting as a tool: it may be used as motivating or affective tool; it may be used as a cognitive tool; a kind of pedagogical tool; or as an evolutionary one, the latter referring to the arguments for using history that say that the learning trajectory of the students should more or less follow that of the historical development of some topic. Common for all of these, however, is the intention that students should come away having learned something about a selection of inner issues – in-issues – of mathematics, e.g. mathematical ideas, concepts, theories, methods, algorithms, ways of argumentation and proof, etc. When using history as a goal the in-issues are not necessarily of first priority, instead the meta-perspective issues – or meta-issues – are. The term 'goal' must not be misinterpreted in the way that it is a matter of teaching history per se. Rather it refers to the situation where it is considered a goal in itself to teach the students something about the historical meta-issues of mathematics: e.g. the historical development of mathematics; societal and cultural influences on this development and the other way around; mathematics' interplay with other areas of practice through history; that mathematics is a human endeavor that takes place in, and to some extent therefore also dependant of, time and space; etc.

In a similar manner as the 'whys' of using history may be split into two different categories, so may the 'hows', i.e. the approaches to using history, be split into three different categories. The first of these comprises the illumination approaches, where the ordinary teaching is somehow supplemented by historical information – in a way these are 'spices' added to the 'mathematical casserole' that is being served to the students. The second consists of the modules approaches which are instructional units devoted to history, often a specific case. These can be either small curriculum-tied modules or longer more free modules, as for example the student projects done at Roskilde University (Kjeldsen & Bjomhøj, 2009). Third we have the history-based approaches. These are approaches directly based on or inspired by the historical development of some mathematics, in a sense we can say that history sets the agenda for the presentation of some topic. Of course, each of these types of approaches may be scaled according to size and scope. For a longer and deeper discussion of history as a tool versus history as a goal, the three broad approaches to using history as well as a survey of the related literature, see Jankvist (2009a).

Now, integrating history in mathematics education is of course not a new idea and examples of older texts about this are laCour (1881), Poincaré (1899), Klein (1908), and Toeplitz (1927). But despite it being an old topic, only very few empirical studies have been made. When conducting a survey as part of my thesis, I was able to identify around eighty somewhat or clear-cut empirical studies. The vast majority of these concern a use of history as a tool. Only about ten percentages or so may be said to primarily address a use of history as a goal. Why this distribution one could ask, in particular when so much of the literature on history in mathematics education seems to value arguments of history as a goal (Jankvist, 2009a)? Well, maybe one reason is that mathematics education researchers concern themselves with the teaching and learning of mathematics and no some 'fuzzy' meta-issues of the subject and its history! Nevertheless, when we look into curricula, syllabi, and teaching plans we often find an emphasis on matters that address elements of history as a goal, e.g. in terms of students' view of mathematics etc.

One example of such can be found in the mathematics program for Danish upper secondary school, where students now are to "demonstrate knowledge about the evolution of mathematics and its interaction with the historical, scientific, and cultural development" (UVM, 2007). This is to come about through some "teaching modules on the history of mathematics" (ibid.), which are included in the so-called supplemental curriculum that takes up 1/3 of the total teaching time.[1] The rhetoric in these new Danish regulations follows those of the Danish so-called KOM-report – 'KOM' is a Danish abbreviation for 'competencies and mathematics learning' (Niss & Jensen, 2002) – which also focus on history as a goal. More precisely the KOM-report lists eight first order mathematical competencies, which include for example mathematical thinking, reasoning, problem solving, etc., and three second order competencies, so-called types of overview and judgment.[2] These are (i) The actual application of mathematics in other practice and subject areas; (ii) the nature of mathematics as a subject (discipline); and (iii) the historical evolvement of mathematics, internally as well as in a societal context. It is of course the latter which is of our main interest, and about this the KOM-report says:

> The central forces in the historical evolution must be discussed including the influence from different areas of application. [...] The type of overview and judgment should not be confused with knowledge of 'the history of mathematics' viewed as an independent topic. The focus is on the actual fact that mathematics has evolved in culturally and socially determined environments, and on the driving forces and mechanisms which are responsible for this evolution. On the other hand, it is obvious that if overview and judgment regarding this evolution is to have *solidness* they must rest on concrete examples from the history of mathematics." (Niss & Jensen, 2002, p. 168, 68; my translation and emphasis)

The KOM-report's mentioning of 'solidness' (or solidity) suggests some kind of anchoring of the students' meta-issue reflections in the related mathematical in-issues. By anchoring, I am referring to something that substantiates discussions and reflections about meta-issues on a basis of knowledge and understanding of the related in-issues, e.g. by revealing insights about the meta-issues that could not have been accessed or uncovered without knowing about the in-issues, or by providing in-issue evidence for meta-issue claims or viewpoints (Jankvist, forthcoming(a)).

In the wake of the new Danish regulations for the upper secondary mathematics program, new textbook systems came out from different publishes. An analysis of three of these systems first and foremost revealed that the treatment of history in the new generation of textbooks often is quite anecdotical, that it is most often detached from the related mathematics (in-issues), and that when it appears it often does so in a manner that seems 'pasted on', e.g. in special colored boxes etc. (Jankvist, 2008a) – putting it on the edge: on the right hand page there will be the regular presentation of core curriculum mathematics, and on the left hand page there will be a colored box telling some anecdote about a old mathematician, perhaps mentioning some mathematical results due to him or her, but these

[1] The supplemental curriculum is tested on a local oral exam, whereas the 2/3 core curriculum are tested on a national written exam.

[2] No English translation of the KOM-report is available yet, but a presentation and discussion of the eight mathematical competencies and the three types of overview and judgment may be found in Jankvist & Kjeldsen (preprint).

will have nothing to do with the curriculum mathematics on the opposite page.[3] This observation together with that of the distribution of empirical studies and the KOM-report's mentioning of 'solidness' led me to ask the following three research questions in my doctoral thesis:

1. In what sense, to what extent, and on what conditions is it possible to have upper secondary students engage in meta-issue discussions and reflections of mathematics and its history in terms of 'history as a goal'?
2. In what sense and on what levels may an anchoring of the meta-issue discussions and reflections in the taught and learned subject matter (in-issues) be reached and 'ensured' through a 'modules approach'?
3. In what way may 'history as a goal' modules give rise to changes in students' beliefs about (the discipline of) mathematics, or the development of new beliefs?

2 Students' meta-issue discussions and their anchoring

The way of trying to provide a basis for answering these questions was by designing two historical teaching modules to be implemented in a Danish upper secondary level mathematics class. The first of these was on the early history of error-correcting codes and was based on the works of Shannon (1948), Hamming (1950), and Golay (1949). The second module was on the history of public-key cryptography (Diffie & Hellman, 1976) and RSA (Rivest, Shamir & Adleman, 1977). The historical case of the first module may be considered a story of modern applied mathematics, whereas the case of the second may be considered a story of a modern application of old mathematics proper (number theory). And an important element in the selection of these cases is that they both concern applications which are present in students' everyday life (Jankvist, 2009b). Another important element concerns one of the design features of the modules, namely what I shall refer to as general topics and issues in the history and historiography of mathematics. Some examples are e.g. inner and outer driving forces in the development of mathematics, multiple discoveries, the discussion of pure versus applied mathematics (Jankvist, 2009a). Yet an example, one from the historiography of mathematics, is the so-called epistemic objects and epistemic techniques (Rheinberger 1997; Epple, 2000, Kjeldsen, 2009). For example, when Hamming was developing his error-correcting codes at the Bell Laboratories around 1946-47 then these codes were the epistemic objects under investigation, but in order to develop them he relied on a lot of already well-established epistemic techniques such as the notion of metric due to Frechet and elements of linear algebra which we may ascribe to Grassman. These general topics and issues may be – and in this case were – used to identify exemplary 'local' cases in the history illustrating more general 'global' features in the historical development of mathematics. Other design features included a use of translated excerpts from original sources and a strong focus on both meta-issues and in-issues in the material that the students were to work with, the latter came about for example with mathematical tasks on relevant in-issues of the historical case as well as a use of so-called essay assignments. I shall exemplify the idea of essay assignments later, but for now it shall suffice to say that these dealt explicitly with the meta-issues of the historical case in question.

Regarding the experimental setup, written textbooks was prepared for both modules

[3]One exception is in the book system by the publishing house Gyldendal, where they have invited professional historians of mathematics, Tinne Hoff Kjeldsen and Jesper Lützen, to write a chapter each on a historical topic somehow related to the curriculum. But unfortunately such elements in the new book systems are a rarity.

(Jankvist, 2008b, 2008c). The first module was implemented in the class' second year of upper secondary level (students age 17-18) and the second module in their third and final year. The modules were taught by the class' regular mathematics teacher and each module had a duration of approximately fifteen double lessons (one double lesson being 90 minutes). An essential element of the implementation (and design) was that the students worked in groups and handed in their essays in groups as well. The generated data from the implementation (to be used in answering the three research questions) consisted of: (i) questionnaires and interviews with both students and teacher; (ii) students' written mathematical tasks and essay assignments; and (iii) videos of the teaching and in particular of one focus group of students.[4]

In my talk today I shall mainly focus on the second historical teaching module, because it provided the basis for an answering of research question 2 about anchoring. However, a few comments about the first teaching module are in order. In terms of research question 1, the first module provided an existence proof that it is possible to have students discuss and reflect upon meta-issues of mathematics and its historical development. But in order for this to happen a setting must be provided. The use of essay assignments and general topics and issues seems promising elements for such a setting (Jankvist, 2010). Also, if general topics and issues are introduced probably they may assist in anchoring the students' meta-issue discussions in the in-issues (Jankvist, 2009d) – and thereby help preventing the integration of history becoming anecdotical and anachronical.[5] But in terms of research question 2, what the first module did do was that it showed the existence of anchoring being present – the second module made this existence proof more constructive.

In order for us to be able to follow the students' discussions of the meta-issues regarding the history of public-key cryptography and RSA and to provide an answering of research question 2, a brief introduction to this historical case is required.[6] Our story begins at Stanford around 1975 where Whitfield Diffie, Martin Hellman, and Ralph Merkle as a team are driven by a desire to solve the so-called key-distribution problem. This is a problem of private-key cryptography that deals with the distribution of the private encryption/decryption keys. One way of phrasing it is to say that in order for two parties to share a secret (the message) they must already share a secret (the key). During post World War II the world became much more international which resulted in increasing demands for secure communication, first with banking and later with the beginning of electronic mail, the Internet, etc. Diffie was one of the first to truly realize this and as a result he came up with the idea for public-key cryptography: a cryptosystem where the encryption-keys were made publically accessible, but where the person to receive the encrypted message held a personal private decryption-key. The system was based on an idea of a mathematical one-way function; that is a function for which calculating f is a straightforward operation, but for which it for all practical purposes is impossible to calculate f – 1. Of course, the phrase 'for all practical purposes' is not really something to be used in a mathematical definition, but what it means is that it may take a computer a second to calculate f whereas it may take it an eon to calculate f – 1 – and in, say, a million years who is going to care about the secret message sent anyway. So this is not the same as saying that f – 1 does not

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[4]For an elaborated discussion of how the focus group students were selected, see Jankvist (2009d).
[5]The latter is also referred to as 'Whig' history. For a discussion of this in relation to history in mathematics education, see Fried, 2001.
[6]A longer version may be found in Jankvist (2009d). And of course a much more elaborated account may be found in the book by Singh (1999) which is dedicated to the subject of cryptography.

exist, because it does; only that it is difficult to find. In fact, even coming up with a one-way function in the first place for which f and $f-1$ satisfied the system requirements proved so difficult that Diffie and Hellman eventually gave up and published their idea for public-key cryptography in 1976 without having a concrete implementation of it. Fortunately three other researchers got hooked on the idea after reading Diffie and Hellman's paper. These were Ronald Rivest, Adi Shamir, and Leonard Adleman from MIT. Rivest and Shamir, being mainly computer scientists, would come up with ideas for one-way functions, and Adleman, begin a mathematician, would then put them to the test. After some 42 attempts Rivest finally came up with something that would work, after he had deciding looking into number theory. The idea, as many of you will know, is that it is a straightforward operation to take to very large prime numbers, say 2-300 digits each, and multiply these together to get a larger integer. However, being given this large integer and then finding its prime factorization is for all practical purposes impossible. Hence, the one-way function. The three researchers patented their solution in 1977, which got named RSA after their initials, and they eventually became millionaires, but that is a different story. What is interesting from our point of view is the mathematics that Rivest, Shamir, and Adleman used in their solution. One thing is finding the one-way function, but around this a cryptosystem must be built up, one that generates the public encryption and private decryption keys. This system builds on old, even ancient, number theory. I will not give a detailed description of the system here, since many of you will already have seen it before, and if not you may find a description in Jankvist (2009d), but I shall mention the three historical theorems. These are: the Chinese remainder theorem, which is ascribed to Sun Zi around the year 400;[7] Fermat's little theorem from 1640;[8] and Euler's generalization of this, known as Euler's theorem, from 1735-36.[9]

The essay assignments, as mentioned previously, consisted of a main essay assignment where the students were to provide two different accounts of the historical case; one focusing on 'who and when' and another on 'why and how'. In a way, this was of course meant as a comment to the new textbook systems' approaches, since these usually include history from a 'who and when' point of view and rarely touches upon the 'why and how'. In order to prepare the students for their answering of the main essay they were to do three supportive essay assignments first and use these in their answers of the main essay. The first of these concerned inner and outer driving forces in the history of public-key cryptography and RSA; the second concerned the fact that public-key cryptography and RSA are actually multiple discoveries, both were discovered more or less simultaneously within the British Government Communication Headquarters (see Singh, 1999); and finally, the third was concerned with the discussion of pure and applied mathematics, taking as its departure point the students' reading of 2/3 of the English edition of G.H. Hardy's A Mathematician's Apology from 1940. For now, I shall only consider the third supportive essay assignment (for elaborations of both the main essay and the supportive essay, see Jankvist, 2009d). In this essay the students were, among other things, asked to do the following: Discuss Hardy's view on pure and applied mathematics and relate this to the case of RSA. For the focus group this resulted in the following discussion.[10]

[7]In a modern formulation the Chinese remainder theorem may be formulated as: let $m_1, m_2, \ldots, m_n$ be positive integers relatively prime by pairs. The system $x \equiv a_1 \pmod{m_1}$; …; $x \equiv a_n \pmod{m_n}$ then has a unique solution x modulo $m = m_1 m_2 \cdots m_n$, meaning there is a solution x for $0 \leq x < m$ and all other solutions are congruent to this solution modulo m.

[8] Fermat's little theorem may be formulated as: if p is a prime, then for every n: $n^p \equiv n \bmod p$.

[9] Euler's theorem may be formulated as: $\gcd(n,m)=1$, then $n^{\varphi(m)} \equiv 1 \bmod m$, where $\varphi(m)$ is Euler's φ-function.

[10]The students' names have been changed, but the sexes of the students remain the same.

Harry: It says [in the teaching material] that one of the problems in number theory is to decide if a number is a prime or not. And in this book [the *Apology*], he [Hardy] says that finding primes is pure mathematics, because when you are a mathematician then you already have the frame for the area you are working within, you know what a prime number is, in contrast to a physicist or a chemist who works with some applied mathematics, they have to work with things relatively, I would say: Here is a table [points to the table], but to you... for some other person this might not be a table, or for some other thing in a different universe, this might not be a table. But a prime number will always be a prime number.

Andrew: So the pure mathematics cannot be discussed, you might as well say.

Lola: That depends on how you conceive that the history of mathematics has been developed, because if someone else had been sitting and thinking about a number and had found some other connections, well then a prime might suddenly not have the same meaning.

Harry: But now they have this frame...

Andrew: You could say that, what we are kind of working with... it is our frame, it is the numbers we have, it is our frame, and then within these there is a lot of pure mathematics, for example prime numbers.

Lola: Yes, primes can never be different, if you look at how we look at mathematics.

Andrew: No, if you stick within our frame.

Lola: But if we imagine that the numbers had some different values or whatever you'd say, then...

Andrew: Yes, but then you are changing to a new frame, and then there is a connection within this frame.

Lola: Well... but then you could also say...

Harry: In that way it might be kind of relative, I can see that. But what he says is that you can't discuss it. He says that in this world we live in here there can be two different physicists who tell you what this is [points to the table] while two different mathematicians [equally] can tell you what a prime number is.

Lola: Yes, and that is what Hardy he says, right? Do you want me to write that?

Andrew: But then you could also say that the pure mathematics is as objective as anything can ever become, right, because it isn't colored by anything.

Harry: But we have to see it in connection to RSA.

Before entering into the analysis of the above discussion, a bit of explanation regarding the students' references to Hardy's Apology will have to be given. One of the passages that the students' refer to and base their discussion upon is the following:

> A chair or a star is not in the least like what it seems to be; the more we think of it, the fuzzier its outlines become in the haze of sensation which surrounds it; but '2' or '317' has nothing to do with sensation, and its properties stand out the more clearly the more closely we scrutinize it. […] Pure mathematics […] seems to me a rock on which all idealism founders: 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but *because it is*, because mathematical reality is built that way. (Hardy, 1992, p. 130)

Of course, one of the more famous quotes from Hardy, which is often referred to when discussing number theory in relation to its application in cryptography is the following (ibid.):

> Real mathematics has no effects on war. No one has yet discovered any warlike purpose

to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years.

In my analysis of the students' discussions in relation to an answering of research question 2, I adopted and adapted Anna Sfard's discursive approach to learning (Sfard, 2008). Roughly speaking, Sfard begins with the assumption that understanding/learning is a form for change. What changes she asks, and based on a survey of literature on philosophy and learning psychology she concludes that it is one's thinking that changes. Thinking, she argues, is a type of (inner) communication. She then defines her term commognition, as a contraction of communication and cognition. Discourse is defined as "the different types of communication, and thus of commognition, that draw some individuals together while excluding some others" (Sfard, 2008, p. 91). Mathematics is a discourse – and from our point of view, we may of course equally argue that so is history, philosophy, and history of mathematics. This line of thought by Sfard leads to the conclusion that understanding is change in discourse. By drawing the parallel that reflection is also change in discourse, we may then be able to identify what I shall refer to as potential anchoring points in the students' discussions, the term 'point' referring to a point in time in the discussion. The students' discussions may follow two different discourses: meta-issue discourses, which are those that address history, philosophy, sociology, psychology, etc.; and in-issue discourses, which are the mathematical discourses. The potential anchoring points are those where the students' discussion change from a meta-issue discourse to an in-issue discourse – i.e. places where the meta-issue discussions and reflections may be anchored in the related mathematical in-issues.

So, what discourses may we observe in the discussion of the focus group students as displayed above? Well, there is a historical discourse present in which the students are trying to link Hardy's statements to RSA. There is also a mathematical discourse with their talk related to number theory and in particular prime numbers. And there are elements of a somewhat philosophical discourse with their talk of 'frames'. The potential anchoring point(s) of this discussion are the place(s) where they change from either the historical or the philosophical discourse to the mathematical one and begin reasoning by means of their knowledge of prime numbers. It is important to stress again that the way of identifying anchoring as illustrated above only reveals potential anchoring points. Once the potential points have been identified they will have to be either verified or rejected by means of methodological triangulation with other data sources. In the case above, we could for example look into the students' work on mathematical tasks, test or interview questions, etc. related to prime numbers (or related concepts such as relatively prime, Euler's φ-function, etc.) to get an idea of Harry's, Andrew's, and Lola's understanding of the in-issues of the mathematical discourse. Thus, identifying anchoring points, and hence anchoring, is a two step process.

Carrying out such investigations for both teaching modules (see Jankvist, 2009d, also for the analysis of the excerpt above) revealed the presence of at least four different 'levels' regarding anchoring: (1) the non-anchored; (2) anchored comments, which are comments that can be verified as being anchored by triangulation, but which give rise to no further discussion; (3) anchored arguments, which is where an in-issue discourse is used to underpin a meta-issue point (of view) or argument; and finally (4) anchored discussions, where anchored comments or anchored arguments are taken up by other group members and eventually come to make up the basis for a meta-issue discussion. The latter (4) is exemplified by the excerpt above of the focus group's discussion of pure and applied mathematics with reference to Hardy, number theory, and in particular prime numbers, the in-issue that comes to make up a point of reference for them in their treatment of the meta-issues.

3 Students' beliefs about 'mathematics as a discipline'

This brings me to part 3 of my talk, which is concerned with students' beliefs about mathematics as an academic discipline. Beliefs are generally considered to be a kind of lenses through which one looks when interpreting the world; psychologically held understandings; and beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual (Philipp, 2007). Lester, Jr. (2002, pp. 352-353) asked the question: "Do students know what they believe?" Of course, one of Lester's reasons for asking this question in the first place may be to provide the answer himself, which he does stating that: "I do not think most students really think about what they believe about mathematics and as a result are not very aware of their beliefs" (ibid.). Furinghetti and Pehkonen (2002) argue that one should take into consideration beliefs students hold consciously as well as unconsciously.[11]

The question, of course, when trying to study students' beliefs and any changes that may occur in these becomes how to take things like the above into account. What I will do is that I will tell you the approach I took in doing so, and then it will be up to you to decide if you find this to be appropriate. The methods I used to try and access students' beliefs consisted of questionnaires, follow-up interviews, and videos. The class of 23 students did four rounds of questionnaires: one before the first module; one in between the modules; one immediately after the second module; and one sometime after this. After each round of questionnaires, 12 students, selected in such a way that their questionnaire answers represented the answers of the class as was best possible, was interviewed about their questionnaire answers, essay assignments, etc. Besides being a method for accessing the students' beliefs, these rounds of questionnaires and follow-up interviews to some degree also resulted in the students' awareness of their (consciously held) beliefs being developed some. The focus group was video filmed whenever they worked together on tasks, either mathematical tasks or essay assignments, and these films came to function as a way of trying to uncover some of these particular students' more unconsciously held beliefs. All the four rounds of questionnaires and interviews, and the implementation of the modules, took place over a period of one year.

Allow me to give you some examples of the questions, and type of questions, that the students were asked in their questionnaires, thus also implicitly illustrating what elements I consider to be part of 'mathematics as a discipline' (for a more clear definition see Jankvist (2009d), where you may also find the questionnaires in full length). The questions the students were asked may be divided, for matters of clarity, into to three groups – three groups not completely unrelated to the KOM-report's three types of overview and judgment. The first of these I shall refer to as the historically oriented questions, and as examples of such I provide the following two:

- When, how, and why do you think the mathematics in your textbook came into being?
- What do you think a researcher in mathematics does?

The second group is the sociologically oriented questions, illustrated by:

- Do you think mathematics has a greater or lesser impact in society today than 100 years ago?
- Where is mathematics applied in society and (your) everyday life?

[11] A long discussion of the beliefs literature in relation to the present study may be found in Jankvist (2009d) and Jankvist (forthcoming(b)).

And finally the third group, the philosophically oriented questions, of which I mention:

- Do you believe mathematics, or parts of it, can become obsolete?
- Do you believe mathematics is something you discover or invent?

Of course this last question stems from an old, long, and still ongoing discussion, but it actually turned out to be one of the more interesting questions to ask students at this level, because it puzzled them and the more they thought about it, the more it confused them. But it also made them reflect about, sometimes change, their view of things, and it made them try to justify and/or exemplify their beliefs, also by referring to elements of the two historical cases of the teaching modules. And it would not only be so that they settled on one or the other – discovery or invention – some students would by themselves argue that one would precede the other, which is even along the lines of what Hersh (1997) argues. So indeed it seems to me that such open-ended questions may play an important role in the clarification and/or development of students' beliefs.

Another, to some degree also open-ended, question that we will have to ask is whether the beliefs as I could detect them, with the methods described above, are then actually beliefs. One of the practicalities you are faced with in a study like this is that you have to start somewhere: I took the students' answers to the first round of questionnaires and follow-up interviews as their virginal beliefs. Yes, perhaps one year may not be enough time to see actual changes in these beliefs – because in the literature beliefs are considered to be entities somewhat persistent. Thus, it may be more correct to speak about changes in views – that is if we take 'views' to be something less persistent than beliefs. Such views, however, may have the potential to develop into beliefs over time. We may also speak about students' images of mathematics. In my study I have taken students' images of mathematics to consist of their beliefs as well as their views.

For a longer, more deep and elaborated discussion and display of the students' beliefs as I found them, you are referred to Jankvist (2009d, forthcoming(b)). Here I just summarize the main findings in order to provide you with an answer to research question 3. So, on a basis of explicitness – if students are not explicit about their beliefs/views, we cannot say anything about them – I was able to identify three dimensions of change in the students' beliefs/views: (1) a growth in consistency in students' beliefs/views regarding related questions and issues were noticed; (2) an increase in students' justification of the beliefs/views they were expressing; and (3) students provided a larger amount of exemplifications in support of their beliefs/views, e.g. by referring to the two historical cases of the modules.

4 Perspectives: Further research

When being given the chance to give a plenary lecture like this, I think it is only fair that I provide you with my personal opinion on possible further (empirical) research in this area. Although the study, I have reported on above is concerned with history as a goal, I shall begin by making some observations regarding studies on the use of history as a tool.

When history does play the role as a tool, it is most often in the sense of stimulating students' learning of specific mathematical concepts, etc. or in terms of trying to motivate the students through meta-issue matters to try and learn the mathematical in-issues dictated by curriculum. That is to say, most empirical studies on history as a tool in mathematics education seem to focus on history as either a cognitive tool or history as a motivational or affective tool. But perhaps the development of students' mathematical competencies is a more 'natural setting' for the use of history as a tool. Kjeldsen (2010) has discussed how history may be used (as a tool) in this way through a study of original sources relating to curriculum topics, and furthermore how to do so in a non-Whig manner (cf. Fried, 2001). The KOM-report's eight mathematical competencies in relation to the use of history as a tool are also discussed at length in Jankvist & Kjeldsen (preprint).

When history (as a goal) is part of a curriculum, as in the Danish case, the problem becomes to avoid its treatment being anecdotical, on the one hand, and to ensure some kind of anchoring in the in-issues, on the other hand. The study presented in this paper shows that students can indeed deal with meta-issues related to the history of mathematics in a reflective manner and that they, on different levels, are able to anchor their treatment of these in the related mathematical in-issues. Furthermore, the study also shows that historical modules like the two presented here did have as an outcome that some students ended up with more balanced, multifaceted, reflected and profound images of mathematics. Now, it is not always important whether students believe one or the other, for instance whether they believe in invention or discovery of mathematics. What is important is that they reflect upon their beliefs, try to accommodate in case of conflicts, and hold their beliefs as evidentially as possible. The students were invited to do this in the modules. But in order for this to happen a scene needs to be set: the students must be provided with (meta-issue) aids (e.g. the 'general topics and issues') and some evidence (the two historical cases) before these things can take place. Because, as I have phrased it in my thesis,

> ... we cannot expect students' (core) beliefs to change in any way are they not confronted with some 'evidence'. Not until students have access to evidence – or counter-evidence – are they likely to criticize rationally, reason about, and reflect upon their beliefs, and possibly accommodate and change them. (Jankvist, 2009d, p. 257)

Regarding further research related to the study presented here, the results for research question 3 could perhaps serve as a model for what we will consider to constitute students' reflected images of mathematics as a discipline. This is illustrated in figure 1. By considering the three dimensions of consistency, justification, and exemplification as components in a model for students' reflected images (beliefs and views), we may to a larger degree be able to assess the development of such images – a development which is implicitly called for in many curriculum descriptions when talking about enlarging students' appreciation, awareness, etc. of mathematics. Possibly such a model could also serve as input to operationalize the KOM-report's three types of overview and judgment – thus, not only the one of them regarding history.

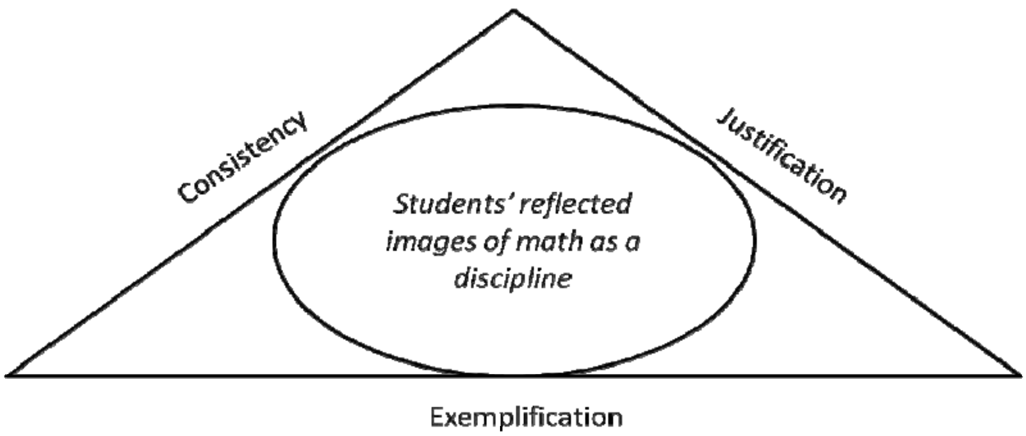

Figure 1. A model of students' reflected images of 'mathematics as a discipline' as made up by the three dimensions of consistency, justification, and exemplification.

I shall round my talk off with some statements regarding research on history in mathematics education. First of all, we need much more empirical research within the History and Pedagogy of Mathematics (HPM) community. Surely, I am not the first and only to state this (see also the discussion in Jankvist, 2009c), but I think that it needs to be repeated, in particular if we want to spread the message of history in mathematics education. And I hope that we do! Secondly, if one scans the relatively few empirical studies that do exist, one will observe that only rarely do these relate to general mathematics education research (MER). This is a least a pity and a most a major flaw on our behalf, I think, because MER has a lot to offer HPM, in particular in terms of theoretical constructs, conceptual frameworks, methodology, etc. that we can use and benefit from when conducting our studies. And HPM would have something to offer MER in return(!) – at any rate, more than just the usual statements regarding epistemological obstacles and historical parallelism, which is what one most frequently finds whenever history is mentioned in the MER literature. One example, although only speculative, could perhaps turn out to be the model for students' reflected beliefs (figure 1). If this model, which is based on findings from using history in mathematics education, i.e. HPM related research, can be used to assess students' development of the KOM-report's two other types of overview and judgment (the actual application of mathematics and the nature of mathematics), then that would be an example of HPM contributing to MER. But in order for something like this to happen and for mathematics education researchers to realize what HPM has to offer MER, we need to link the two tighter together – beginning with our own research![12]

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[12]One such initiative has been taken by the CERME (Congress of the European Society for Research in Mathematics Education) working group on history in mathematics education, where the following theme has been added to the call for papers for CERME-7 to take place in Rzeszów, Poland, February 2011: "Relationships between (frameworks and empirical studies on) history in mathematics education and theories and frameworks in other parts of mathematics education." See: http://www.cerme7.univ.rzeszow.pl/

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THE HISTORY OF MATHEMATICS IN SCHOOL TEXTBOOKS

Panel Discussion[1]

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ABSTRACT

Given that on an international scale, there is a growing research interest in the introduction of a historical dimension in mathematics education and actual implementations in the classroom, textbooks and curricula, this 90-minute panel discussion made focus on the following issues:
(a) *The current situation*: What is the international experience on the inclusion of the History of Mathematics (HM) in school textbooks?
(b) *Classification of possible ways to include the HM in school textbooks*: What are the pros and cons, the aims and the methods of the different ways this could be done, or has actually been done?
(c) *Quality criteria, prerequisites and aims of integrating the HM in school textbooks*. Do they exist? Have they been taken into account in writing textbooks that have a historical dimension?

1 Introduction

There is an increasing interest to integrate history into Mathematics Education (ME): Curricula, classroom, books, didactic material, homework, teachers' training. The panel discussion was confined to the introduction of a historical dimension in **textbooks**, this term being understood in a wide sense to include non-conventional forms; e.g. based on ICT. In many countries, there are specific regulations in the mathematics curricula on what this integration really means and how it should be implemented. Some examples were presented, focusing on primary and secondary education (but the discussion touched upon university education and teachers' training, as well).

A short presentation of typical examples in four different countries was given (France, Italy, Poland, Greece; see section 2) and participants were welcome to provide further information based on their own experience, or the situation in their own country, either beforehand and/or on the spot; their contribution was distributed via e-mail to all participants before the Summer University (cf. sections 2 & 3). This helped very much those participants who wished to actively contribute to the discussion.

After a brief introduction to the subject by the coordinator, a short presentation by each panelist along the panel's main themes was given, based on the case of the panelist's country. The follow-up discussion was based on questions among the panelists and the audience and comments made by participants with experience on the subject, who were invited to provide further information (some of them have already sent their comments in advance, as mentioned above). The session ended with the coordinator's summary of the information, opinions and suggestions put forward during the discussion.

[1] Contributions from other participants are explicitly acknowledged at the appropriate place in the text.
[2] Coordinator.

2. The history of mathematics in school textbooks: A short presentation

Each panelist presented the situation in his/her country briefly, illustrating the main points by means of specific examples. In response, to this, as well as, to the documents circulated to all participants, further short presentations came in, illustrating the situation in other countries. Due to space limitations, this section summarizes all these contributions, without details on specific examples. The interested reader is referred to the literature below.

2.1 France[3]

Currently, there is a new math curriculum from the beginning of the primary school, to the end of the " lycée", implemented in primary and middle school (11-14) and the first year of the "lycée" (15-16 year old students). It implies important changes in the math content, especially in secondary school. For instance, an important new item is on "algorithms and programming". School mathematics now has to be useful in modern everyday life and be linked to other scientific disciplines, such as physics, biology, economics, technology, or even the humanities (such as the teaching of history of art in the middle school).

However, there is only a brief reference to the introduction of the HM in the middle school's curriculum: " *it's an opportunity to approach the history of sciences, especially when introducing the notions of number π, square roots, ... and measures*"; and there is nothing about the HM in the new curriculum for the first year of the lycée. Very recently, a few comments have been added about the interest in introducing a historical perspective, in the mathematics curriculum for the last two years of the lycée.

Nevertheless, more and more math teachers have recently begun to be interested in using HM, mainly through the work of the IREMs (see however Barbin 2010). But, from now on, the new math teachers will have a compulsory initiation to the history and epistemology of mathematics at the University. Meanwhile, there is a sort of popular infatuation with the HM (even fictionalised), found in films, novels etc. However, it seems that among algorithms, programming, working with other subjects etc, there is no more time for the HM in the French school curriculum. In conclusion, there is no obligation to introduce the HM into teaching; it is left to the teacher's discretion, even if the HM may appear as an "added value". What is encouraged, is to give a historical perspective in special interdisciplinary optional modules, which is a very different point, of course.

In any case, however, one may seriously doubt that the HM is the first criterion of a team of teachers to choose a textbook for their school. Hence, the role of the HM in textbooks depends on the authors; they do as they wish. We have studied most of the secondary math textbooks that are currently used in math classes in France.

In many cases, you can find nothing on history, or only one, or two "pseudo-historical exercises", e.g.: "*Archimedes is a famous ancient Greek mathematician. He was born in Syracuse in 287 BC. and died in 212 BC. How many years did he live?* This is pompously entitled: math and history! The only interest in this exercise is mentioning Archimedes' name. Some other exercises, or activities, have only a superficial historical covering and can be misleading if the teacher is not well informed. But fortunately, in most recent math textbooks there is, usually, more correct historical information, than it used to be; mainly historical notes, introductions, or activities for further study in mathematics. There are also suggestions for homework, especially in middle school. Sometimes, some very interesting

[3] By A. Boyé.

activities can be found, based on original sources. But in fact, they are difficult for a pupil to study alone, and for a teacher with no knowledge of this history to supervise his students.

So, investigating the way the HM is included in school textbooks, the following questions arise in most cases: How have these history-based elements to be used? Just left to the pupils' discretion to read only if they are interested in? To be worked with the teacher in the classroom? In that case, is there any additional resource material to support the teacher's intervention? Are these elements always pertinent to the introduction of a historical and cultural dimension in teaching mathematics? Maybe, they enlighten math teaching, but do they modify the pupils' perception of mathematics? In particular, do they humanize mathematics?

Most teachers will need additional resource material to support their courses, which is often unavailable. Still, if there was no history in math textbooks (as it is often the case), it is likely that the HM would have almost completely banished from math courses in France. At least, teachers and pupils currently have opportunities to ask questions and try to get more information.

Finally, the new technologies provide new opportunities to improve the use of the HM in textbooks. The recent school textbooks propose additional material on the web, for teachers and pupils. Usually, it consists of the solutions of exercises and nothing on the HM. But in one exception, there is a historical timeline from ancient times to nowadays[4], with mathematicians' names all along the timeline, linked to short biographies, the chapters concerned in the textbook, and more information. It is certainly an incitement for using HM, directly linked to the textbook used in the classroom. It is neither closed, nor rigid, so that, even if someone is not well informed on the history of a specific subject one wants to approach, may be, he will find here what he needs. This possibility has to be further explored more thoroughly.

Due to lack of space, we cannot present in detail our analysis of the issues raised above. It has been based on several French contemporary textbooks for primary and secondary school, with an additional brief comparison with the 1960-1970 textbooks, and some of the beginning of the XXth century and made focus on how the existence of historical elements in the mathematics textbooks is influenced by recent fashionable ideas, the curricula's incentive, and in another way, by the IREMs' work.

2.2 Italy[5]

Analyzing Italian textbooks, we can find two fundamental ways of using the HM:

- doing mathematics; i.e. operating for developing mathematical skills, abilities, competencies etc
- reading, in order to know something about mathematics in the past

With respect to the HM, the point of view of teachers and publishers are often the same. In my opinion, this fact is important to understand why the HM appears in school textbooks seldom. Several objections can be raised (summarized in Siu 2006), though I would like to emphasize that teachers consider the HM as a valuable resource; e.g., my colleagues consider the HM an interesting topic, "a good stuff". Some of them read books, collect articles from magazines, and sometimes use the HM with their students: narrations about evolution of a concept (e.g. π), further reading (e.g. biographies), or visiting exhibitions (e.g. in city museums). Publishers sometimes enrich the book with colours, insert images (historical pictures, which are unusual in mathematics texts), historical snippets, etc. to make the

[4] www.repere-prof.hachette-education.com
[5] By A. Demattè.

textbooks more attractive. In summary, both teachers and publishers consider the HM "something more", a way to attract students' attention. Through their agents, Italian publishers contact teachers, who either alone, or in groups choose textbooks for their classes, which almost certainly are approved by the teachers' assembly. Of course, publishers are interested in taking into account teachers' preferences. Hence, although history is "feebly present" both in professional choices of mathematics teachers and in textbooks, it is a potential resource to improve pedagogy of mathematics and enrich textbooks.

Not long ago, some Italian textbooks for upper secondary school, authored by very good mathematicians and researchers in pedagogy of mathematics, included a lot of history. Currently, one of the most popular textbooks is Bergamini et al 2008. Its last edition contains some pages named EXPLORATION. These pages finish with ACTIVITIES, requesting for instance, that students search in the web the history of the *Arithmetic triangle* by using keywords like: Tartaglia's triangle, Pascal's triangle, arithmetic triangle, binomial coefficients. This search in the web is present in every EXPLORATION (see Appendix). However, not all secondary school mathematics textbook contain HM. Some just mention a few words at the beginning of some chapters.

HM is only for reading? In teachers and publishers' view the answer seems to be in the affirmative, which, in my opinion, raises an obstacle for making the HM part of the class: Reading (or speaking) about the HM requires additional time and the teacher either has to subtract it from activities for learning new mathematical concepts or, require from the students more time for homework. Both alternatives are difficult indeed!

On May, 26th 2010 the new "*National directions about specific learning aims*" by the Ministry of Education. For each school subject of the *Scientific Lyceum* (upper secondary school; 14-19 year-old students), "General suggestions and competencies" have been published[6]. It is remarkable that, with respect to mathematics, by means of "copy and paste" of some parts, the program of other types of Lyceums is obtained. Other "pedagogical recommendations" are in the ministerial document; e.g.

> "[the student] is expected to be able to insert different mathematical theories in the historical context, where they have been developed, and he is expected to understand their meaning", or
>
> "The most important moments must be remembered: Greek mathematics, Calculus in 1600, Enlightenment, rationalism and modern mathematics, […] the meaning of postulates, axioms, definitions, theorems, proof […], Euclid's *Elements* in Western Mathematics […], the Euclidean approach should not be only an axiomatic system […] Ruler and compass: their historical meaning in Euclidean geometry […]. The concepts of continuity, differentiability, integrability and historical problems regarding instantaneous speed, tangent straight line, areas, and volumes".

In summary, we can say that although the inclusion of history in Italian mathematics textbooks has a long tradition, it is not common, any longer. The recent national regulations contain suggestions that could serve as important starting points to develop activities focused on the HM as a powerful pedagogical resource and an integral part of textbooks, too.

2.3 Poland[7]

In the last decade, there has been an important change in Poland, in the system of general

[6] http://www.indire.it/lucabas/lkmw_file/licei2010///decreto_Indicazioni_nazionali%20_26_05.pdf
[7] By E. Lakoma.

education for pupils of 7 to 19 years old. The old system “8+4” was progressively replaced by the system “6+3+3”: 6 grades in elementary school, followed by 3 grades in gymnasium and finally 3 grades in lyceum. These changes are accompanied by a new system of assessment, applied at the end of each level of education, in the form of a general examination for all students at a given level of education, the evaluation being done by an external institution, according to the regulations of the ministry of education.

The main idea of the new educational changes was to place general education within the framework of the so-called key competencies, which are developed through the realization of the curriculum basis for general education, established by the ministry of education. This curriculum basis includes for each educational level a list of essential mathematical skills to develop and mathematical notions to form, which are considered to be necessary from the point of view of general education and developing the key competencies.

As far as ME is concerned, in this document the HM is absent. However, the HM in ME can be present at the level of the implementation of the curriculum basis.

Looking at the contemporary curriculum proposals for school mathematics in Poland, one can notice that most of them include a fairly superficial attention to history. Similarly, the textbooks proposed to support these curricula tend to include at most a few biographical notes and some rather basic historical information. However, among many proposals for the mathematics curriculum, there is a project called Mathematics 2001, which includes textbooks and other didactical material, with relatively more on mathematics history than other proposals. This project uses the HM as a starting point for didactical situations which can be interesting for pupils and a source of original simple reasoning, readily understandable and helpful for today’s pupils.

By presenting some examples taken from textbooks and other didactical material associated with them, the most important issues concerning the role of the HM in mathematics learning at all levels of education, can be discussed from the point of view of both the learner and the teacher.

2.4 Greece[8]

It is the first time in Greece that the (new) mathematics curriculum for compulsory education (officially announced in 2002 and implemented via newly written textbooks since 2007 (Greek Pedagogical Institute 2002) includes many and extensive references to the didactical integration of the HM into the teaching of mathematics. These references vary from the specific teaching objectives, to the didactical methodology and the textbook content. Besides the usual historical snippets, there are many activities of historical content, aiming to provide teaching tools for better understanding the textbooks’ mathematical content. However, a critical reading of this historical material reveals the existence of serious errors, obscurities or omissions in the historical comments, which make their use very questionable. Indicative examples can be given to support this conclusion (see Thomaidis & Tzanakis 2010).

In addition, although the official guidelines given by the Greek Ministry of Education follow what didactical research seems to suggest nowadays and emphasize the important role HM can play in ME, their actual classroom implementation is far from being satisfactory. Besides the weaknesses mentioned above, teachers feel unready to follow the guidelines, given that there is no (pre-service, or in-service) training for that and no

[8] By C. Tzanakis.

appropriate accessible resources.

2.5 Denmark[9]

The new reform for the Danish High School system includes history in the mathematics curriculum. To implement the new curriculum, different groups of math teachers write textbook sets that cover the curriculum; HM is treated differently in each set. Each school buys the textbook set(s) it likes, or can afford. In one case, "guest authors" were invited as specialists to contribute to different issues; one of these contributions was a small history chapter[10]. These contributions by “guest authors” are printed in colour to indicate that they are not part of the ongoing presentation of the mathematical subjects treated in the book. In such chapters it is possible to give an account of the development of pieces of mathematics at a historically fairly high level. Associated with a chapter on geometry and trigonometry, there is a chapter on the history of surveying and mapping, treating also the history of the mapping of Denmark in the 18th century and how this led to Wessel’s geometrical interpretation of complex numbers. The interaction between practical problem solving, the mathematical tools, and insights into new mathematics are discussed and presented in this 6-page chapter, which is also used by the textbook authors as a suggestion for project work, enriched with mathematical and historical tasks. In this way, it is possible to present a few case studies that in some sense are representative for different aspects of the HM, in a way that is both historically sound and connected to the mathematics of the textbook.

2.6 Israel[11]

An interesting non-conventional new idea is implemented in Israel; namely, to connect the curriculum to some contemporary pieces of mathematics and associated problems dealt with by the mathematics community, and their historical background. A study in the format of an action research focusing on interweaving contemporary mathematical news with their historical background in secondary school curriculum has been conducted over the past 3 years. Parts of the study and some of its results were presented in a workshop (see ch.2.6, this volume).

The key idea is to provide teachers with an on-line access to PowerPoint presentations, each of 15 minutes duration of some *math-news-snapshot* to be implemented once a fortnight in the ordinary teaching. This is in line with the tendency to adapt textbooks to on-line books and the use of ICT in general.

2.7 United Kingdom[12]

In the Curriculum for England and Wales (published September 2008), one of the “Key Concepts” is the *Applications and Implications of Mathematics* and here we find that school pupils should be “Recognizing the rich historical and cultural roots of mathematics.” This statement is *statutory* (i.e. it is the law) but no rules or guidance are given in this curriculum document as to how this statement may be put into practice. A similar situation exists in Scotland.

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[9] By T.H. Kjeldsen (Roskilde University, Denmark)

[10] The same was done with other mathematical subjects as well; e.g. critique and meta reflections on the use of statistics etc. Professionals are invited to write these chapters, i.e. the history chapter was written by a historian of mathematics, the statistics chapter by a statistician and so on.

[11] By Nitsa Movshovitz-Hadar & Batya Amit (Israel Institute of Technology, Haifa, Israel).

[12] By L. Rogers (University of Oxford, UK).

The integration of aspects of the HM into any curriculum requires careful consideration of the national characteristics and 'millieu' of the educational system, the philosophy (or lack of it) of the curriculum-builders, the political and economic contexts prevailing, and of the system of both pre-service and in-service teacher education and training. Borrowing 'ready-made' (or translated) materials from other nationalities is often counter-productive, and pays no attention either to national characteristics, opportunities, needs, local heroes (or villains), or to the particular social opportunities that may present themselves.

New media are available over which governments, official curriculum builders and textbook writers have little or no control, and this provides many opportunities for good (and bad) material to be made available. In the classroom, the teacher is finally responsible and can choose whether or not to pay attention to historical material in printed books. Hence, it is possible to use the new media to make available to teachers appropriately resourced materials that have a clear link with items in their school curriculum, and materials for pupils that show how mathematics is the foundation of, is integral to, and inspires many aspects of our cultural history. Maybe, such "non-conventional" forms of "textbook" material is becoming the 'norm', being more directly accessible to the pupils themselves!

Indicative examples can be given:

- '*Episodes*' with historical notes, references and pedagogical notes and questions for teachers on the NRICH website - a part of the Millennium Mathematics Project (based at Cambridge University); e.g.:

The Development of Algebra 1, http://nrich.maths.org/6485

The Development of Algebra 2, http://nrich.maths.org/6546

- The "*History Corner*" talks on the interactive website of the Association of Teachers of Mathematics (ATM)

For the most recent see:

'Root-two' and irrational numbers, http://www.atm.org.uk/mti/218/root-two.html

3 The main themes in more detail

Both the panelists and several participants were involved in the discussion during the panel session and beforehand via e-mail, and raised several issues pertinent to the main themes of the discussion (see Abstract above). This section summarizes their contributions[13], which have been codified for the sake of brevity and easy reference.

3.1 Classification of possible ways to include the HM in school textbooks

There are many different ways to conceive the integration of the HM in textbooks, some of which have been implemented in several countries, worldwide. Below is a list, divided into groups, which is not exhaustive of course; a similar list analyzed in detail can be found in El Idrissi 2006.

(a_1) Separate chapters, disconnected from the main text: Introductions, Epilogues, Appendices, List of historical remarks etc.

(a_2) History appears as a natural, integral and explicit part of the textbook (see e.g. Hairer & Wanner 1996, Toeplitz 1963)[14].

[13]We acknowledge valuable contributions to this section by (in alphabetical order): B. Amit & N. Movshovitz (Israel), F. Bevilacqua (Italy), T. Kjeldsen (Denmark), L. Rogers (UK), B. Smestad (Norway).

[14]Another tentative attempt by B. Smestad (written in Norwegian) can be accessed at http://home.hio.no/~bjorsme/Sannsynlighetsregning3.pdf

(a_3) History permeates the textbooks, usually integrated implicitly (see e.g. Stillwell 1989).

(b_1) Historical "*snippets*" (see Fauvel & van Maanen 2000, §7.4.1), loosely connected to the main text & course objectives: historical notes with factual information and associated activities, usually in a different format & colors from the main text, possibly with pictures and inserted in a box.

(b_2) Presentation, comments & interpretation of images from books, or original paintings, photographs etc[15].

(c_1) (Short) excerpts from original documents, to introduce a topic, possibly with comments that would help the reader to grasp better the mathematical content, both in its modern form and in its original context.

(c_2) (Worksheets with) exercises, recreational problems and games based on original texts; suggestions for well-defined, optional short projects for further work at home and/or the classroom (including e-sources & the web).

(d_1) Suggestions for further reading at the end of each chapter possibly with historical notes and/or annotated bibliography.

(d_2) A guide to the literature (books, journals, websites) for further reading, research and insights.

(e_1) Connections with some contemporary piece of mathematics and its history: Hints to recent developments, related to outstanding old issues/problems (e.g. recently proved theorems, conjectures, unsolved for a long time and the efforts to answer, solve, or prove them that stimulated important developments), or constituting good examples that can fascinate students, at the same time making clear that mathematics is a continuously evolving science.

(e_2) Presentation, discussion and analysis of errors made in the past and led, or motivated new developments.

(e_3) Links to other sciences (especially those that rely heavily on mathematics) and the arts.

3.2 Quality criteria, prerequisites and aims of integrating the HM in school textbooks

Clearly, the possibilities mentioned above, should serve specific aims, satisfy some minimum quality criteria and have certain prerequisites. Otherwise they produce no positive result. Below is a summary of all these that emerged through individual contributions to and during the panel discussion.

3.2.1 Aims

Historical material in textbooks could serve a variety of different aims, both on inner issues of mathematics and meta-issues (cf. El Idrissi 2006, §1):

(A) *Inner issues of mathematics*

(A_1) To contextualize mathematical concepts.

(A_2) To motivate students go deeper into mathematics and look for more details, through the study of sense-making situations of specific mathematical concepts and methods.

(A_3) To serve as recreational activities.

(A_4) To link school mathematics to mathematics in-the-making, or current new findings & trends in research.

(B) *Meta-issues of mathematics*

[15] In this connection one could add the cover of the text book and/or its title; see El Idrissi 2006 §§2.1, 2.2)

(B_1) To stress and highlight the evolutionary nature of mathematics.

(B_2) To modify students conception of mathematics.

(B_3) To portray mathematicians as creative human beings and more generally, to humanize the mathematics courses.

(B_4) To connect mathematics to other disciplines and the general historical & cultural milieu, hence to help students develop an interdisciplinary understanding of mathematics.

(B_5) To stimulate collaboration of students and teachers in interdisciplinary activities.

3.2.2 Quality criteria

Any implementation of HM in textbooks should satisfy some reasonable, necessary quality requirements:

(a) It should be based on mathematically & historically correct information.

(b) It should serve the objectives of the teaching unit, where the historical material is incorporated.

(c) It should be supported by additional material to help the teacher: extensive bibliography; resource material; specific hints on what the historical material is didactically good for etc.

(d) It should definitely be in harmony with students' capability.

(e) It should avoid "pseudo historical" exercises, or problems.

3.2.3 Prerequisites

Historical elements should constitute hints to motivate and stimulate students & teachers' interest. Hence, it is necessary:

(a) To develop teachers' training on how to profit from the historical elements and to use original texts.

(b) To provide extensive bibliography for further reading by the teacher, suggestions for further optional reading by the student, to look for more details in case he/she wishes so.

(c) To have access to the resource material.

4 Some critical points raised in the discussion

In the previous two sections, examples of the current situation in some countries were presented and concise lists related to the main themes of the panel discussion were given. However, additional points were raised during the discussion, or were communicated beforehand, in the form of *observations*, *claims*, *judgments* and *suggestions of how to make a historical dimension effective and attractive*. They are summarized in this section. We thank all participants, who contributed to enrich the discussion and we explicitly acknowledge their contribution. However, because of space limitations, they are presented as a list, in the hope that they provide insights, which could inspire further work.

4.1 *Observations*

(1) It seems that no real differences among countries are apparent. This concerns mathematics textbooks and their use by schoolteachers, not official regulations (A. Demattè, Italy).

(2) When there are official regulations in which HM is integrated, then it is not possible to avoid it in ME (T. Kjeldsen, Denmark).

(3) It seems that the two most serious and persisting obstacles against the integration of the HM in teaching, and in textbooks in particular are the lack of time in the classroom and examination constraints; their influence depends on the school system of course (C. Tzanakis, Greece)

4.2 ***Claims***:

(1) Many teachers and educators are unsatisfied by math history pills dropped here and there in the textbooks. It seems that a main reason for that is that in this way, the teacher and student are put in the position of a **spectator**, which gets them bored, "….*because math is not a sport that you can follow as a spectator. If you want to appreciate it, you need to practice it*"[16] (F. Bevilacqua, Italy)

(2) The original sources are difficult to understand and the personal stories of great mathematicians are not mathematics. It seems that the HM is something only teachers can turn into a didactic success. Therefore what we really need are separate texts for teachers and appropriate training, showing them what a powerful tool HM can be in the development of various precious goals (F. Bevilacqua, Italy)

(3) Not all historical examples and cases are suitable to develop didactical material for the classroom. Appropriate choice is crucial (F. Metin, France).

4.3 ***Judgments & critical issues***

(1) A more in depth analysis of teachers' educational aims and their arguments against using the HM in classroom is necessary (cf. Siu 2006), before writing historically oriented (parts of) textbooks and designing appropriate training. (A. Demattè, Italy).

(2) It is important that teachers should be **motivated** to integrate the historical aspects of the textbooks and HM in general in the classroom; otherwise, they will not do anything in this direction (I. Mavrommati, Greece).

(3) It is important that secondary school teachers are competent in the HM. Therefore, this should form a compulsory part of their undergraduate studies, or in-service training. Unfortunately, this is not the case; university mathematics programs usually do not include history and epistemology courses; pure mathematicians do not consider this as a valuable ingredient of a mathematician's undergraduate studies (R. Bebbouchi, Algeria).

(4) Very often, epistemological issues are implicit or explicit to any historical material. Therefore, it is necessary that the historical material in a textbook is accompanied by epistemological comments for the teacher, for which teachers are usually unaware. This raises the important issue of teachers' training in this area. In general, HM in textbooks should also make focus on epistemological issues as well; it should be conceptual history as well (C. Vicentini, Italy).

4.4 ***Suggestions***

(1) What initiatives could the HPM Group undertake to promote the inclusion of the HM in mathematics textbooks? Is it possible that at the end of each meeting, a "manifesto" be translated into different languages and be sent to publishers (and/or universities, research institutions, regional school administrations etcl)? During a Conference, could a panel discussion be devoted to a very specific subject as means to share ideas on writing a hypothetical chapter of a textbook (e.g. on the "Introduction to algebraic manipulation through the HM")? (A. Demattè, Italy).

(2) It necessary to suggest concrete changes in the mathematics curricula and struggle for that (I. Dias, Portugal).

(3) Distinguishing and presenting some appropriately chosen cases as generic examples of how the HM in textbooks could be used in school practice, would be helpful for the teachers in their class work (E. Lakoma, Poland).

(4) Teachers' training in using the historical aspects of textbooks is essential. In this

[16]Gaurav Suri & Hartosh Singh Bal, *A Certain Ambiguity: A Mathematical Novel*, (quoted from the Italian edition, Ponte alle Grazie, 2007, pp.25-26.

connection, small “pieces” of historical material should be used first. (A. Boyé, France)

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APPENDIX[17]

Recently, I wrote an e-mail to the authors of one of the above mentioned books (Bergamini et al 2008). I introduced myself and this panel discussion and asked for “…. some **motivations** of this choice, as well as the pedagogical **values** you assign to the HM. It could be a valuable contribution, as author of the book, to let us know the virtues HM could have for school textbooks”. Prof. Bergamini answered me, listing some pedagogical “opportunities” and highlighting “methodology”:

“*Opportunities*

-Mathematics contributes in human thought evolution. For instance: Mathematics and democracy (deductive method, proof and birth of democracy in Greek “polis”)

-Mathematics is a cultural building, which is therefore slow, laborious, but powerful, e.g.: Syncopated algebra; from words to algebraic symbols

-Mathematics is not only western. e.g.: The father of polynomials – Al-Karaji’s contribution to algebra

-Mathematics is made by men, with their passions, rivalries, and will to prevail, e.g.: Tartaglia, Cardan and their challenges

-Mathematics contributes to technological growth and is connected with other subjects, also from the historical point of view, e.g.: Triangles on doors, “download triangles” in architecture, Lions’ door in Mycenae……

[Methodology] *We propose a very short introduction, which give essential information and, after this, hints for deepening and researching; the Internet is a starting point. In our opinion, this approach is better than writing long historical pages to attempt complete explanations. Research and interpretation of information look like correct ways to motivate, and understand. This methodology inspires all our boxes named ‘Explorations’*”.

[17] By A. Demattè.

STUDENTS' META-ISSUE DISCUSSIONS OF HISTORY OF MATHEMATICS

Looking for anchoring

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ABSTRACT

The idea of this workshop was to have a closer look at the video transcripts providing the empirical basis for the plenary lecture on theme 2 (please refer to the paper for this).

The transcripts used were from the implementations of two teaching modules (Jankvist, 2008a; 2008b) that took place in a Danish upper secondary mathematics class in the fall of 2007: the first module was on the early history of error-correcting codes (Shannon, Hamming, and Golay); and the second was on the history of public-key cryptography, RSA, and the related number theory (in particular the Chinese remainder theorem, Fermat's little theorem, and Euler's theorem).[1]

During the workshop the participants were first given a brief outline of the historical cases as well as an introduction to the didactical tools (elements of Sfard's (2008) theory of commognition) which were to be used in the analysis of the transcripts of students' discussions. The tasks of the workshop consisted in having the participants – in small groups – do their own analysis of the transcripts provided to them. More precisely, the participants were asked to try to identify potential anchoring points in two longer excerpts, and then perform an analysis of these based on Sfard's word use and discuss this based on a comparison with an analysis based on methodological triangulation with other data sources. The purpose of this was to illustrate to the participants – and to have them discuss among themselves – how these two different analytic tools can complement each other, but also, how in this case Sfard's word use does not seem able to stand completely alone.

Descriptions of both historical cases, design of the teaching modules, the students' discussions in full transcripts, and a use of both methodological triangulation and Sfard's word use to analyse these may be found in Jankvist (2009a). A discussion of the first module on the early history of error-correcting codes may also be found in Jankvist (2010). The second module on the history of public-key cryptography, RSA, and related number theory is discussed in Jankvist (forthcoming) along with an in-depth analysis of identified potential anchoring points and verification/rejection of these based on the analysis methods discussed in this workshop.

Finally, the importance of anchoring when using history as a goal (Jankvist, 2009b) was discussed briefly.[2] The question of whether or to what extent we can ensure anchoring being present in our teaching activities was raised. Conclusions seemed to be that it is doubtful if we will ever be able to point to a complete list of necessary and sufficient criteria for anchoring always being present (and in particular in terms of 'anchored discussions'). However, if we can point to some of the necessary criteria, then surely we can increase the likelihood of this taking place!

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[1] For further references, see the paper from the plenary lecture (ch.2.1 this volume).
[2] For further discussion, see also Jankvist & Kjeldsen (preprint).

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MAPS, NARRATIVES AND ORIENTATIONS:

The use of Concept Maps for exploring our Mathematical Heritage in the Classroom

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ABSTRACT

This workshop reported on the use and implementation of concept maps for promoting the use of historical material in the classroom that is clearly linked to areas of the common mathematics curriculum. The idea of heritage as the form in which material and ideas are handed on in the course of the development of a mathematical concept was explained. The map is principally a pedagogical device for enabling teachers to connect items in the curriculum to events and circumstances in the cultural contexts and the history of mathematics thereby enabling the creation of their own narratives.

A selection of maps that have been used in the classroom was introduced, and participants were encouraged to use them to develop their own lesson plans and potential tasks for pupils that are relevant to their own curriculum needs.

The material is principally designed for teachers of pupils aged 11 - 18, but can also be used for teachers' professional development.

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- An example of some background materials as narrative can be found at: http://nrich.maths.org/public/index.php
- See 'Past Issues' for April and May 2009 'The Development of Algebra Parts 1 & 2'

ANCIENT NOMOGRAMS FOR MODERN CLASSROOM ACTIVITIES

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ABSTRACT

Nomograms (or graphical charts) were among the calculating instruments most used by the engineers during period 1850-1970, before appearance of the electronic calculators. They are still in use today in certain areas, like medicine.

We think that it is possible to put these tools up to date and to exploit them pedagogically to work certain points of present school programs: graphical resolution of equations, intersection of lines and alignment of points, functions of two variables, contour lines, logarithmic scales. Indeed, nomograms provide a suitable context for the changes of register (graphical, numerical, algebraic, geometrical), what is likely to promote the acquisition of mathematical skills. We could check it in several experiments in classroom with students of 15-17 years.

During the workshop, we proposed the following activities:

- Reading of engineers' original texts describing the design of nomograms to solve simple algebraic equations (texts by L. Lalanne, J. Mandl, W.H. Chippindall, M. d'Ocagne, J. Clark...).
- Manipulation of large graphical tables to solve equations.
- Simulation of nomograms with dynamic geometry software.
- Report of experiments realized in classroom and exam of some student works.

DESIGN AND HIGH-SCHOOL IMPLEMENTATION OF MATHEMATICAL-NEWS-SNAPSHOTS

An Action-Research into *Today's News is Tomorrow's History*

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ABSTRACT

This paper presents lessons from a three year study in the format of an action-research that focused on interweaving, in high-school curriculum, mathematical-news-snapshots (MNSs) which include their historical background. It follows the ESU5 presentation of the rationale for such a study. Parts of the study, in particular a reverse-engineering analysis tracking-back the design principles upon which 10 MNSs were developed for the study, and some of the implementation study results related to the impact on students' perception of mathematics as a living discipline, were presented at ESU6 workshop. They are elaborated in this paper. A sample MNS focusing on Kepler Conjecture is also presented.

1 Introduction

At ESU5, Prague 2007, we discussed the rationale for integrating mathematical news in high-school mathematics teaching, in order to decrease the gap between contemporary mathematics and school-mathematics curriculum (Movshovitz-Hadar, 2008). The proposed pedagogy was interweaving 10-15 minutes especially designed Mathematical-News-Snapshots (abbr. MNSs), based upon their historical background, in the regular teaching of high-school mathematics. We advocated then an empirical study to examine the feasibility of this proposed pedagogy, and to explore its impact on studens' perception of mathematics as a living discipline. This paper presents such a study in the format of a 3 years action-research in which the 1st author acted as a researcher-teacher in her own classes and the 2nd author acted as a design advisor and co-researcher. Parts of this study and some of its results were presented at ESU6 workshop, Vienna 2010, and are elaborated in this paper.

2 The study

2.1 Study design and population

The study reported here took place in three age-group classes at one high-school in Israel. The first author acted as a teacher-researcher in the experimental classes and as an observer-researcher in a few other classes. Three parallel classes, one in each age-group, that learn mathematics at the same level as the experimental classes and their population is of comparable qualities, served as control groups for some of the comparison measurements.

2.2 MNSs design

Movshovitz-Hadar (2008) suggested a classification of mathematical news into five categories with examples many of which can be made accessible to high-school students:

(i) A recently presented problem of particular interest and possibly its solution.
(ii) Long-term open problems recently solved.
(iii) A recently revisited problem.
(iv) A mathematical concept recently introduced or broadened, including concepts that evolved into new areas in mathematics.
(v) A new application to an already known piece of mathematics.

For this study a mathematical result was designated as *suitable* for designing a MNS iff it meets all the following criteria: (i) It was published in the professional mathematics journals in the past 30 years. (ii) This result or its historical background has some connection to high-school curriculum or to another aspect of high-school students' life. (iii) It seems likely that this result can be made accessible to high-school students through a snapshot about this result to be developed based upon the design principles specified later.

Following a literature survey for suitable pieces of news, a series of 10 PowerPoint presentations for a periodical-MNS-interweaving-program was prepared especially for the study. Their contents were based upon the following news:

- The proof of Kepler's Conjecture; (See details below.)
- The discovery of large prime numbers; (See details in Amit et al. 2011.)
- News about the mathematics of Sudoku; (See details in Movshovitz-Hadar 2008.)
- The proof of Fermat's Last Theorem;
- The mapping of the E_8-Group;
- Solution of the Four-Colour problem;
- The linear algebra behind Google search engine;
- The Digits of π ;
- Benford's Law applied to tax-frauds and to election frauds;
- The Fundamental Theorem of Algebra applied to astrophysics.

The design of this series of MNSs was based upon design-principles (abbr. DPs) which were predetermined according to the researchers' math-education perception of best-practice. They came in three types: (i) Mathematics-content related DPs; (ii) About-mathematics-content related DP; (iii) Pedagogy-related DPs.

(i) <u>The mathematics-content related DPs were</u>: Each MNS presentation should expose students to:

1. A new mathematical result (of one of the 5 categories mentioned earlier) published in the past 30 years, and possibly yet unsolved related problem(s) as well;
2. Basics in mathematics from the curriculum, needed to make the news accessible and related to student's presumable state of knowledge;
3. Some advanced and/or extra-curricular mathematics relevant to the news possibly including new concepts and vocabulary as relevant and necessary for basic understanding of the matter.

(ii) <u>The about-mathematics-content related DPs were</u>: Each MNS presentation should include in as much as it is relevant:

4. Elements reflecting the nature of mathematics, such as the nature of definition, proof, and truth; the nature of problem posing and problem solving; search for patterns; generalization;
5. Historical background relevant to the news such as long-term conjectures, failures and success in proving them, and cultural background if relevant;

6. Details about mathematicians:
   a. Details about the special contribution of mathematicians who were involved in the piece of news or in its history;
   b. Personal details about such mathematicians, such as their life time, their relationship with the professional community and with the public;
7. Connections:
   a. Connections to some area within mathematics or to some topic in the curriculum;
   b. Connections between mathematics and other disciplines or to everyday life or to any other students' common interests;

(iii) The pedagogy related DPs were:

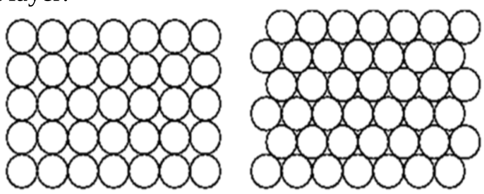
8. Each MNS will be in the form of a PowerPoint presentation consisting of both verbal and visual parts (Photos, diagrams and animated illustrations) to demonstrate, illustrate or represent mathematical ideas, concepts and other elements;
9. The verbal part will take the form of expository statements as well as a Socratic dialogs (Q & A) inviting discourse and student's involvement;
10. The language should be simple and friendly, concrete and non-formal as much as possible, bridging between students' level of mathematics and the advanced level of the mathematical news presented;
11. Duration will be limited to about 15 minutes so as not to take up too much of the ordinary instruction time.

2.3 A sample snapshot: The Kepler Conjecture

The PowerPoint presentation of The Kepler Conjecture MNS consists of 17 slides. Table 1 presents their verbal contents, expected answers to short questions, and some parenthesized notes about the visuals.

Table 1: A Snapshot on: The Kepler Conjecture

Slide No.	A Condensed Version of The Text on the Slide
1	**The Kepler Conjecture** A Mathematical-News-Snapshot For High-School
2	**The Cannonball Packing Problem** • In 1590 the British sailor, Sir Walter Raleigh, loading his ship for a voyage, wished to figure out the number of cannonballs a box of a given height could contain. • He assigned the task to his assistant Thomas Harriot (The slide contains a photo of a cannon besides a pile of 3 cannonballs)
3	**Thomas Harriot** • A British astronomer, navigation instructor, and mathematician, who established the first Algebra School. • His mathematical work focused on solving equations having negative and/or complex roots - a field in which he preceded his generation. (The slide contains a photo of Thomas Harriot)

4	**Let us join him in the calculations** The first layer:  **Square Packing** **Hexagon Packing** What is the difference between the two modes of packing?
5	**Let us compare** • Assuming each sphere has a 1 unit radius, and the base of the packing is 10 by 10 square units: • What portion of the base area do the spheres occupy? • We reduce the consideration to the circular cut of the Spheres: • In the square packing? - $25\pi/100$ • In the hexagon packing? - $23\pi/100$ (The Illustrations from slide no. 4 are repeated here as a reminder with frames defining the packing borders)
6	**And in Space?** • How is the second layer placed in each case? – • Consider a sphere in the center of the box, how many spheres does it touch in each case? (12 in both) • Which packing has a better use of the box volume? – Mathematicians consider infinite space. They found that in both cases the spheres take about 74% of the volume (~3/4). (The Illustrations are repeated to demonstrate the 3 different layers, an animated demonstration of counting the number of spheres each central sphere touches, is added to each mode of packing.)
7	**Back to Harriot** • After reaching this solution, Harriot, wondered about a more general problem, an act typical to mathematicians: • Is there a better sphere-packing? i.e., occupying a higher percentage of space? • With this problem he approached the mathematician and astronomer Johannes Kepler. (The slide contains a photo of the young astronomer Johannes Kepler)
8	**Kepler the Astronomer - The Dream** • Kepler was not only an astronomer and a mathematician but also a science fiction writer. • Between the years 1620-1630, Kepler wrote a fantasy called *Somnium* (Latin for The Dream). • It was published by Kepler's son Ludovico, after Kepler's death. (The slide contains a photo of *Somnium*'s front cover)
9	**Kepler's Dream-Story** • In the narrative, a student is transported to the moon by mysterious forces. • It presents a detailed imaginative description of how the earth might look when viewed from the moon. • It is considered the first serious scientific treatise on lunar astronomy. (The slide contains a photo of Johannes Kepler in his later years).

10	**In 1611 Kepler argued that:** • Harriot's solution is the best sphere packing (in the 3-d space). It became known as the fruit seller's packing. • Kepler left no proof to his claim. Therefore it became known as: "The Kepler Conjecture". • Although it seems intuitively true, it remained an unproved conjecture for almost 400 years! (The slide contains a photo of a box of fruit packed in 2 layers)
11	**Kepler's Conjecture- Timetable** • 1611: Kepler claims that: No packing of congruent balls in Euclidean three-space has density greater than that of the face-centered cubic packing. • According to Kepler Conjecture, the best packing density is about 74% (~ π/sqrt 18). • 1611-1900: Many mathematicians attempt to prove the conjecture, and fail. • 1900: Kepler Conjecture appears as Problem no.18 in a very important list of problems that will occupy the mathematics community in the 20th century. (The slide contains a photo of a spinning pyramid)
12	**1900: A major mathematical event** • Every 4 years there is an international congress of world mathematicians. In 1900, it took place in Paris. • The opening address was given by David Hilbert, a prominent mathematician of the time. Hilbert presented, a list of 23 mathematical problems he proposed as the most challenging towards the 20th century. • To-date, many problems were solved while others are awaiting their turn… • What was the fate of Kepler Conjecture? (The slide contains a photo of David Hilbert)
13	**1998: Kepler Conjecture proved** • 1998: Thomas Hales a U.S. mathematician claims he has proved Kepler Conjecture, by extraction of all possible packing using super-computers. • He submitted the paper for publication in the most reputable journal: Annals of Mathematics. (The slide contains a photo of Thomas Hales)
14	**1998-2005: The struggle for publication** • The editor nominated a team of 12 notable mathematicians to go through Hales' proof. • In 2003, Fejes-Toth, the chairman of the team, created a historical precedent: • Summing up 5 years during which the team examined Hales' proof, Fejes-Toth wrote that the team did not succeed in proving the correctness of the proof nor would they be able to do it in the future – Due to exhaustion … (The slide contains a photo of Fejes-Toth)
15	**The Approval- The Compromise** • 2005 : On the 14th of November, 7 years after Hales submitted his paper (!), The mathematical-logical part of the proof was accepted and published in 120 pages of Annals of Mathematics, Vol. 162 (3) pp. 1063-1183
16	**A new beginning: FLYSPECK!** • Hales gathered all the other parts including the computational parts and the software involved as part 2 and: • Since 2003: A World-Wide project is on-going aiming at the verification and approval of this part. • The project's goal is to produce. The Formal Proof of Kepler, known by the acronym FPK, nicknamed: Flyspeck, http://code.google.com/p/flyspeck/
17	**The end** but keep updated! Check for more news! (The slide contains a photo of a spinning pyramid)

2.4 Main research question

The main research question was: What is the impact of exposing high-school students to contemporary mathematics employing MNSs, on students' perceptions about the nature of mathematics as a living discipline?

2.5 Cycles of the action-research

It is typical for an action research to consist of various cycles/spirals (Kemmis, 1988; Coghlan & Brannick, 2001, 2005). Our action research took place in two phases, each consisting of 2 cycles. Phase 1 was a design study. (i) The first cycle in it was the MNSs development cycle. The initial step in this cycle was a literature search for a *suitable* piece of news as defined above. The design step itself was based upon the set of *predetermined* principles mentioned above. Following a *field-trial* of each snapshot in one class, a revision step took place according to the observations. Then updates were added if more news had been published meanwhile[1]. (ii) The second cycle took place at the end of the development cycle of all 10 MNSs. It took the form of a reverse engineering study. (See details in section 2.6 below). Phase 2 was an implementation study. It also took place in 2 cycles: (i) Implementation of each MNS in various classes; (ii) Implementation of various MNSs in each class. Qualitative and quantitative data were collected in both phases as will be elaborated below.

Diagram 1 illustrates the relationship among the various cycles

2.6 MNS reverse engineering analysis based upon the Design Principles

Having developed all the PowerPoint presentations based upon the above mentioned Design Principles, a validation process was conducted by assigning 5 experts a reverse engineering task. Their task was to track-back the various DPs in the PowerPoint presentation of each MNS. The purpose was to analyse the outcome of the first cycle of Phase 1, and get a typology of the set of MNSs. (ESU6 workshop participants went through a similar process focusing on Kepler Conjecture MNS, after being exposed to the PowerPoint presentation.)

Each expert received a PPS version of the PowerPoint presentation and screenshots of all the slides of each MNS. Adjacent to each screenshot was a table in which s/he were to insert the design principles of contents and of pedagogy as s/he attributed to each bullet in that slide. Then they were to summarize the occurence frequencies of each DP in each MNS and get the percentage of the total occurence of every DP per MNS, yielding the relative frequency distribution of DPs per MNS. A negotiation session took place to bring the 5 experts into agreement in case discrepancies were found among their results. Following the validation of these experts' analyses, it was possible to compare the inner composition of the MNSs to one another and relate the results to other data collected during Phase 2 of the study.

[1] For example, in 2008, after the first MNS was developed about Sudoko mathematics, some more news were published: Elser Bauschke and his graduate student Jason Schaad created a computer program that solves Sudoku puzzles using the projection algorithm method.(See http://www.schaad.ca/hpr.html) (Rehmeyer 2008).

Diagram 1: The cycles of the action research

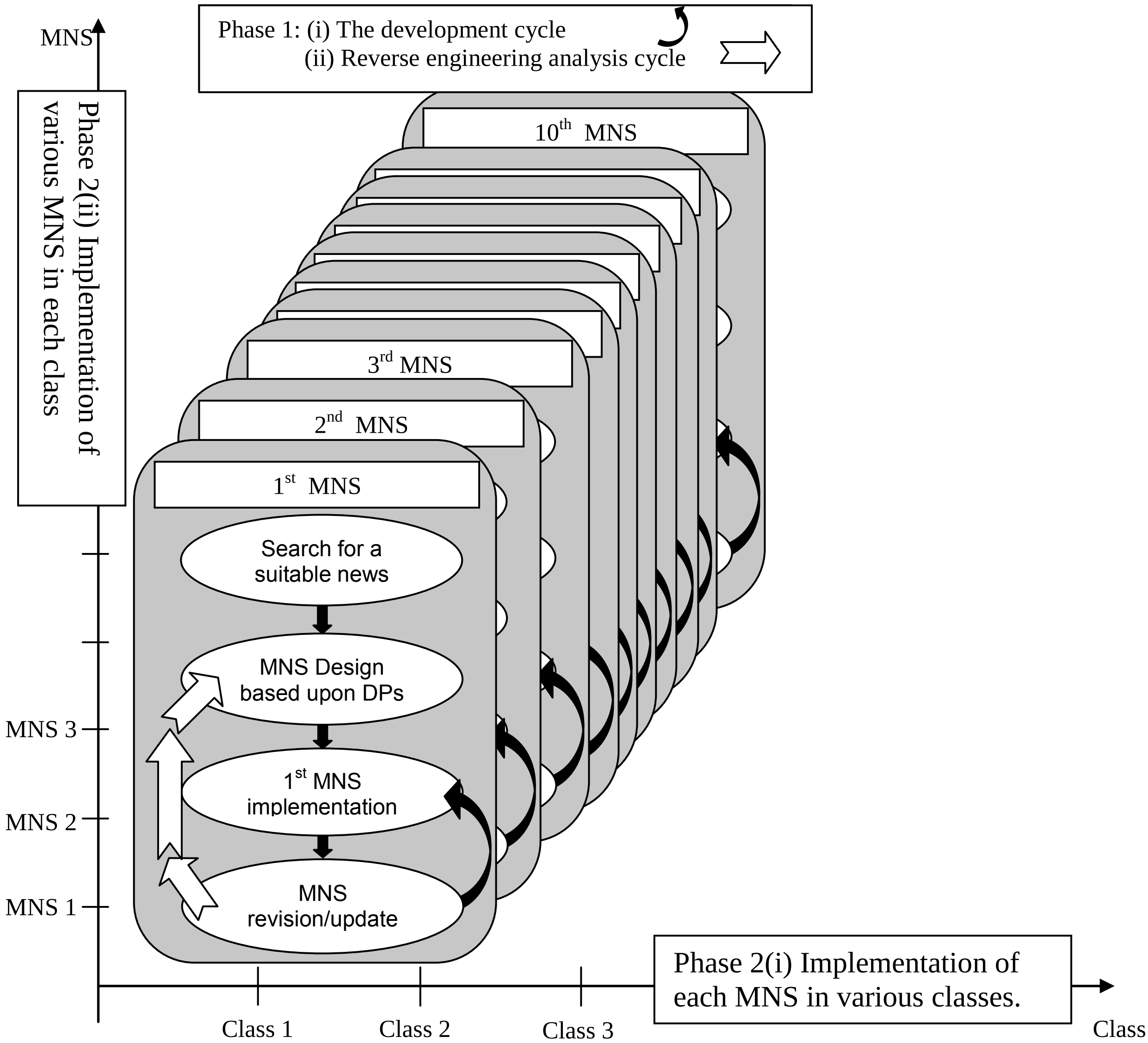


Diagram 2 presents an overall comparison among the 10 MNSs with respect to the relative frequency occurence (%) in each MNS of the mathematics-content related DPs (1, 2, 3), the about-mathematics-content related DPs (4, 5, 6, 7), and the pedagogy related DPs (8, 9). Diagram 3 presents the relative frequency occurrence (%) in the various MNSs of DP 1, 5, 6, namely, of mathematical news, historical background and details about mathematicians.

From Diagram 2 one can see that: All MNSs are pedagogically rich. In all MNSs but one (Sudoko), as the mathematics-content relative occurence decrease the about-mathematics-content relative occurence increase. The about mathematics-content and the mathematics-content DPs split the MNSs into two subgroups: (i) MNSs in which the mathematics-content related DPs are dominant (E.g., E8, PageRank, Sudoku) (ii) MNSs in which the about-mathematics-content related DPs are dominant (E.g., Kepler, Fermat, Primes, 4 Colour).

From Diagram 3, it becomes clear that in each MNS the news is linked to the history of mathematics.

Diagram : An overall comparison among the 10 MNSs with respect to the relative frequency occurence (%) in each MNS of the mathematics-content related DPs (1, 2, 3), the about-mathematics-content related DPs (4, 5, 6, 7), and the pedagogy related DPs (8, 9)

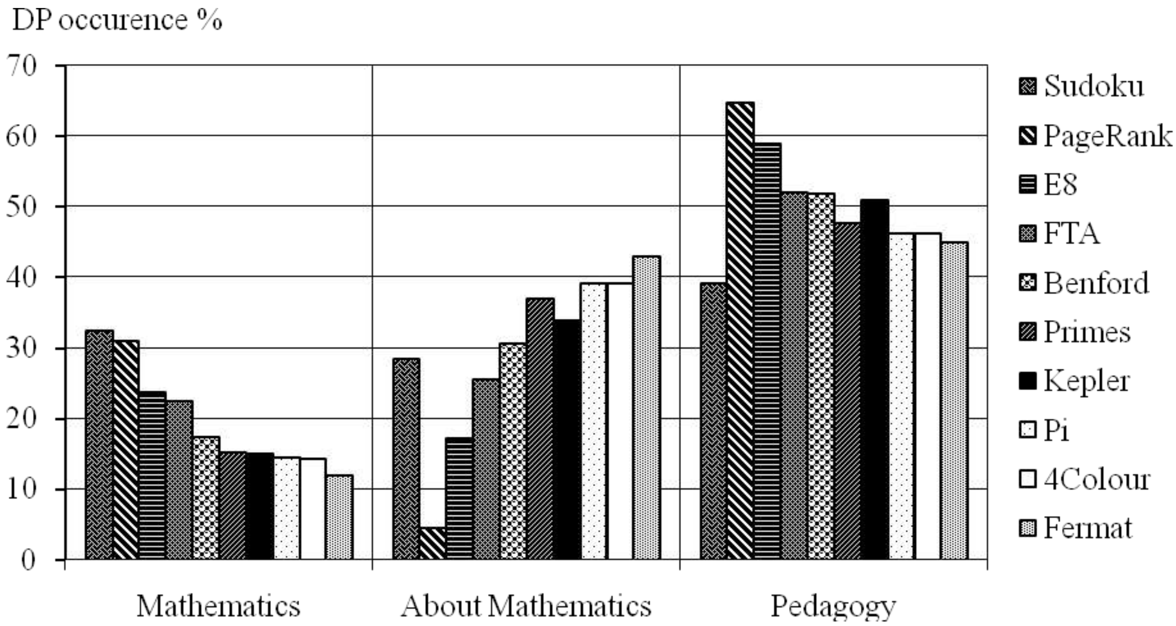
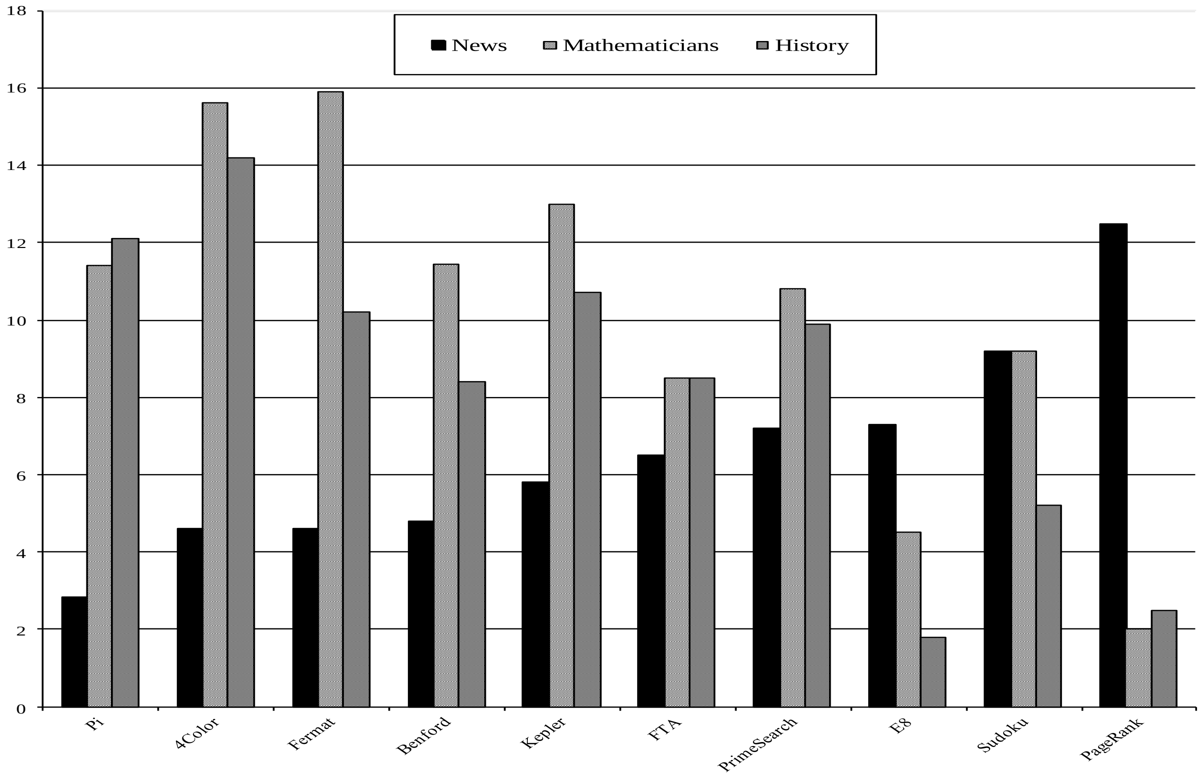


Diagram 3: Relative Frequency Occurrence of: Mathematical News (DP 1), Historical Background (DP 5) and Details About Mathematicians (DP 6), in the MNSs



2.7 Data collection instruments

In each cycle of the implementation study (Phase 2), the following data collection Instruments were used:

(i) Immediately following the implementation of each MNS:

- Students' feedback-questionnaire;
- Group-interview ;
- Teacher's Field Diary documenting special events during the MNS exposition.

The feedback-questionnaire administered after the exposition of each MNS consisted of the following questions:

- If there was anything you found impressive, please state what it was and why.
- If there was anything new you learned from the MNS, please state it.
- If there was anything you found difficult to follow, please state it.
- If there is anything you intend to share with anyone who was not there, please state with who and what.
- To what extent did you enjoy the MNS? Please state why.
- How do you compare this MNS with others you were exposed to?
- Any other comment?

(ii) Before and after the exposition of the whole series of MNSs in 3 intervention classes and in 3 non-intervention control classes:

- Attitudes Questionnaire (28 open-ended items);
- Pre and post individual interviews with 3 students per intervention class.

In the Attitudes Questionnaire students described their interest, attitudes, feelings and opinions about mathematics and learning-mathematics. For example here are a 2 of the 28 items:

- Who of the mathematicians you heard about is your favourite and why? What did s/he do?
- Do you think that school mathematics you learn is different from your parents' school mathematics? If yes - provide details. If no – explain why.

Verbal responses given to the various open-ended items in the Attitude-Questionnaire were analysed qualitatively, and coded for each item separately as follows: A response that expressed a conception of mathematics as unchanging, stagnated, dead-end discipline was marked "Negative". A response that expressed a conception of mathematics as a vivid, active and constantly developing was marked "Positive". If there was no answer or an answer stating more or less that "I do not know", it was marked Neutral. When in the response a student entered a universal generalisation or a strong statement in either direction it was marked "Strongly negative" or "Strongly positive". The five marks per item were gathered for each group and a comparison within groups was performed for the pre-test and the post-test responses.

3 Classroom Implementation – The challenge and some results

3.1 Bridging between school mathematics, contemporary mathematics and its history as a challenge to the teacher

The challenge to the teacher who wishes to implement the MNSs method is threefold: (i) curriculum issues; (ii) system issues; (iii) budget issues. (ESU6 workshop participants discussed these issues).

(i) Curriculum issues

- Teachers are expected to "cover" the curriculum under time constraints. Hence every deviation from the curriculum and any extra-curricular activity is a threat to reaching this goal.
- Many students find school mathematics difficult to cope with. Teachers are responsible for students' success in it. Any extra-curricular activity is a threat to reaching this goal.

(ii) System issues

Teachers and their classroom practice are subject to inspection by the superintendent, the principal, peer teachers, students and last but not least important, by parents. A deviation from the mandatory curriculum and the common mode of teaching invites severe criticism. It puts the teacher in a defensive position.

(iii) Budget issues

- Many schools are not yet equipped with the technical-equipment (computers and projectors) that are required for the implementation, nor are they able or willing to invest efforts in obtaining such equipments. This puts the motivated teacher in a problematic situation. A partial solution to it is adapting the PowerPoint presentation to a frontal exposition using the blackboard.
- Implementation is highly demanding on the part of the teachers. Without an appropriate compensation they are unlikely to take this challenge upon them.

Nevertheless, based upon our action-research teacher's experience, teachers who overcome all those obstacles are likely to find themselves becoming long-life-learners, constantly gaining new knowledge and possibly changing their perceptions regarding contemporary mathematics, its history and their integration in mathematics education. Moreover, students' benefit, as we have seen in our study, and as indicated by the partial results presented in the following section, proves these efforts worthwhile.

3.2 Partial Results

Due to space limitation we confine ourselves to results related to Kepler Conjecture MNS followed by some more general results.

Here are a few selected quotes from the responses 9th grade students gave to the feedback-questionnaire administered following the exposition of Kepler Conjecture MNS.

Expressions of Motivation:

- "The story is interesting, I am curious to hear more about what follows …"
- "I was very interested and enjoyed the thrill and the tension in this snapshot…"
- "I would like to know who will finally complete the FPK project! Really I would like to be informed"

Expressions of feeling of understanding:

- "Some parts were quite complicated but following the explanations it became clear- Now I can say I really understood it!"
- "Quite unusual for me to admit I understood and enjoyed maths… even loved it."
- "The 3-D animations and the pyramid helped me understand everything- - in mathematics it is important to see what happens - - it helps to follow the explanations."

Expressions of being impressed:

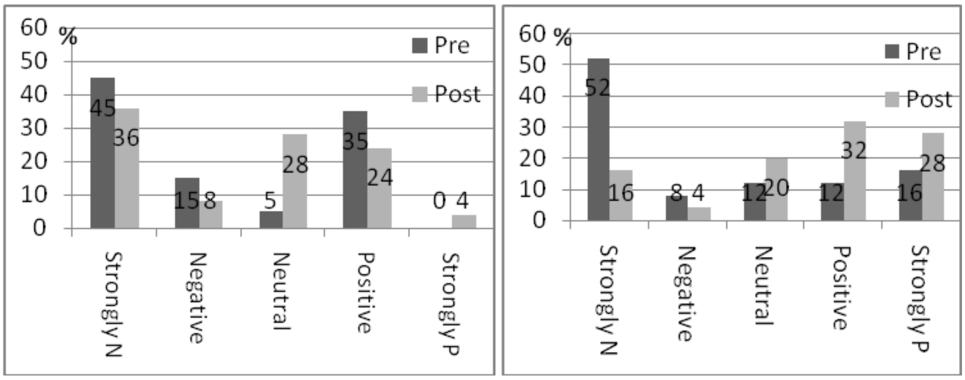
- "One mathematician asks for help from another one and so on: This is new for me that a mathematician needs help from a friend (just like us…)"
- "Harriot seems a clever and interesting man."
- "I was very excited to hear that the proof was accepted."
- "People from all over the world working together – This is some Revolution!"

In general, in all 28 items of the Attitude Questionnaire, a minimal or no change was found in the distribution of the 5 mark between the two administrations of the test in the control group. In the experimental intervention classes, a noticeable shift to the positive side appeared in mostly all the items.

Due to space limitations we confine this paper to results obtained for the three following items only:

(a) Please relate in details to the following statement "Mathematics is a creative profession". Results are summarised in Diagram 4.

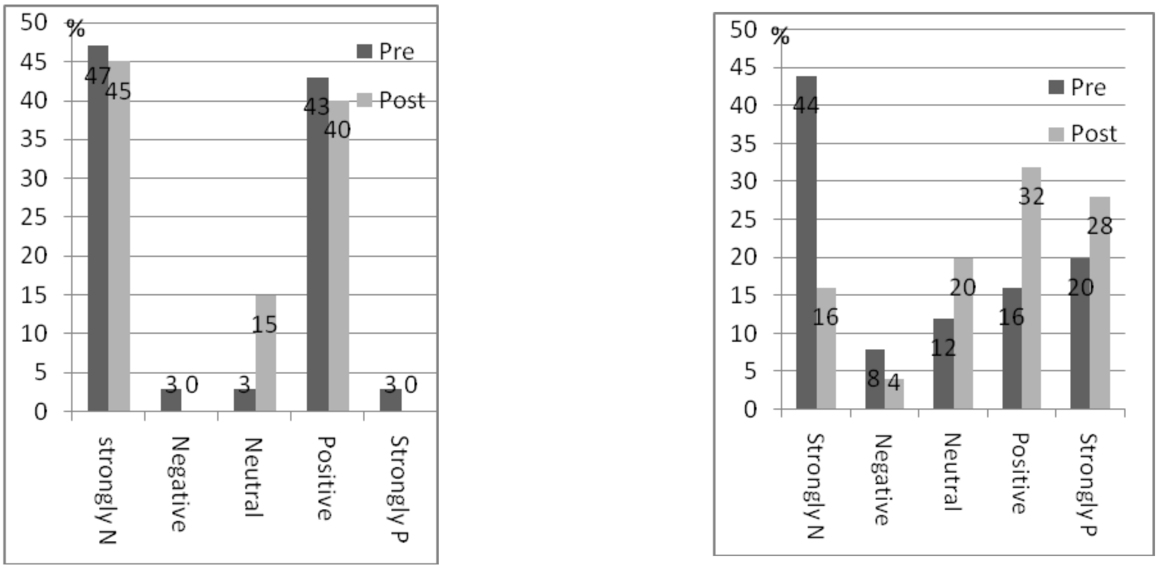
Diagram 4: Distribution of marks for pre-test and post-test responses to item (a) in the control group (left, N=20) and the experimental group (right, N=25)



As can be seen from Diagram 4, it so happened that the experimental group started on this item with more strongly negative attitudes (52%) than the control group (45%), and ended up the other way around (16% vs. 36% respectively). On the other end, the experimental group started with more strongly positive attitudes (16%) and this number increased to 28% at the end of the experimental implementation of MNSs, while the strongly positive attitudes on this item of the control group started low (0%) and remained low though a slight increase occurs there too (4%).

(b) What do you find most interesting in school mathematics? Responses are summarised in Diagram 5.

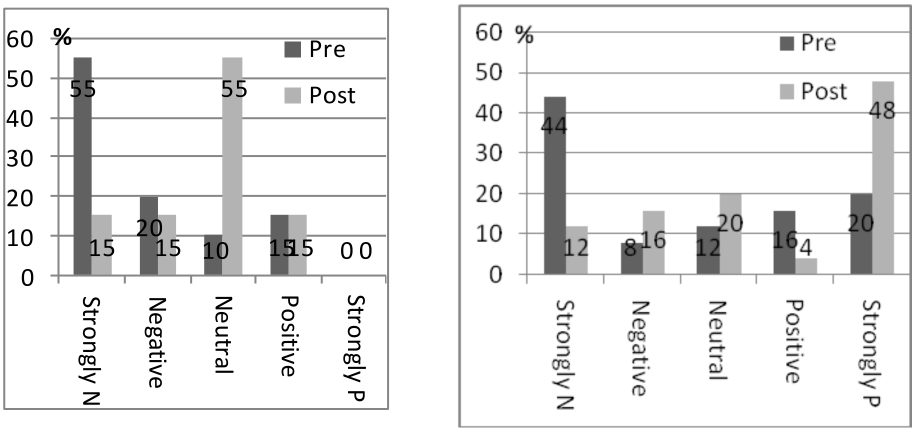
Diagram 5: Distribution of marks for pre-test and post-test responses to item (b) in the control group (left, N=20) and the experimental group (right, N=25)



As can be seen from Diagram 5, on this item the two groups started high on strongly negative attitudes but the control group remained almost the same (47% to 45%) while the experimental group decreased largely (44% to 16%). In the rest of the marks there appear also to be very small change in the control group while the change inclined towards the positive side in the experimental group is remarkable.

(c) Would it surprise you to hear that there is nothing new in the field of mathematics? Provide detailed reasons to your answer. Responses are summarised in Diagram 6.

Diagram : Distribution of marks for pre-test and post-test responses to item (c) in the control group (left, N=20) and the experimental group (right, N=25)



As can be seen from Diagram 6, on this item the experimental group started off with 20% students possessing a strongly positive perception. This percent increased to 48% at the end of the experiment. The control group started and ended with no student with a strongly positive oriention and a small percent of positive ones (15%) in both case as well. It is worth noting that the neutral mark in all three items in both groups increased between the pre and the post-tests. This can be explained by the fact that students omitted this question probably as they were tired of the whole questionnaire repeating itself the second time.

We close this section with a few quotes from experimental group students' responses to item (c).

Pre-test	Post-test
• No news because no one is pursuing studies in the field these days. • It is obvious that the entire field is a group of old formulas and drills – so there cannot be any innovations. • In mathematics everything is known - it is the same mathematics all the years. • Mathematics is stagnated in a closed frame and cannot and will not change. • All the answers are known to everything.	• Mathematics is developing all the time. There are many open problems not yet solved, as we saw in the snapshots. • The technology helped many mathematicians to complete proofs and publish new results. Recall the snapshots. • Mathematicians are searching for new solutions- ALL THE TIME • We learnt about many new findings this year. • It is not at all simple yet mathematicians succeed in discover in more and more…

4 Closing Remarks

The proposed pedagogy for introducing mathematics news in high-school so as to decrease the gap between school mathematics and contemporary mathematics assumes that a snapshot is a short intermezzo, taking a small portion of the weekly class-time, and is preferably linked to the curriculum, so that it does not break the flow of teaching the ordinary curriculum. The MNSs developed for this study were not meant to present an in-depth theory, but rather to guide ‘a wander-about’ in the world of contemporary mathematics.

We accept an earlier viewpoint (Movshovitz-Hadar, 1993, 1988, 2006) regarding mathematics education as a meta-mathematical discipline. “...As such, statements about mathematics, with relevance to its learning, its understanding, its use, its teaching and its communication belong to mathematics education” (1993, p. 267). The proof for such meta-mathematical statements is similar to the proof provided by the empirical sciences rather than the common deductive proof in mathematics. In this paper we shared parts of the evidence resulted by our study which support the following general meta-mathematical statement: Employing the pedagogy of interweaving Mathematical-News-Snapshots in high-school mathematics teaching is an efficient strategy for bridging the gap between contemporary mathematics and school mathemtics curriculum. This ‘wander-about’ in the world of mathematics is intriguing enough to impress students, detailed enough to motivate them to do mathematics, and mind-opening enough to yield a desired image of mathematics as a vivid creative and ever-growing domain.

The reverse engineering analysis of the various MNSs provided eveidence as to a sequel to the former statement: Mathematical-News-Snapshots are deeply rooted in the relevant history of mathematics, the two go hand in hand in leading students to the perception of mathematics as a rich living discipline.

The two assertions are related to a concern expressed during ESU6 by several researchers from various countries, about introducing a historical dimension in mathematics education, and about neglecting the history of mathematics in high school curricula in general and in mathematics textbooks in particular, despite its potential for improving the quality of mathematics education (Tzanakis 2010). Our study contributes towards a solution for this concern. After all, today's news is tomorrow's history.

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ONE TEACHING EXPERIENCES BASED ON THE *NONIUS* OF PEDRO NUNES AND THE *ICOSIAN GAME* OF HAMILTON

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ABSTRACT

In the last summer school, promoted by our university, the first author lectured two 6 hours free courses concerning the history and the production of:
- the nonius of the Portuguese mathematician Pedro Nuñes (1502-1578);
- the icosian game invented in 1857 by the well known mathematician Sir William Rowan Hamilton (1805-1865).

The main aims of these courses were to integrate the history of mathematics in an enjoyable way, to show applications of the mathematics in games and to improve the knowledge about important mathematicians.
We describe the tasks developed and we report their implementation in the referred courses dedicated to young students (16-17 years old). Finally, based in this experiences, we discuss some advantages and disadvantages of perform this kind of classroom experiments.

HISTORY AND IMAGE OF MATHEMATICS
AN EXPERIMENT

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ABSTRACT

Is the history of mathematics an educational way for acting on mathematics view? To answer this question in our contribution we discuss the philosophy underlying an Italian book based on historical images. The book is primarily addressed to secondary school students, but also teachers and ordinary people may be potential readers. In the presentation some parts of the book are translated into English. Our discussion is an occasion for investigating the view of mathematics held by people. To this aim we present a questionnaire centred on this subject and analyze some findings coming from the answers we gathered. This analysis allows outlining a project for producing new teaching materials based on the history of mathematics, which encompasses original documents and historical images.

1 Introduction

In (Demattè & Furinghetti, 1999; Furinghetti, 2007) it is discussed the way in which students and teachers consider the development of mathematics. Their views are compared to the socio-cultural orientation, for which the evolution of mathematical concepts is determined by factors inside and outside the discipline, as shown in (Radford, 2006). In this perspective, mathematics is seen as a historical building connected to the context: different forms of mathematics are born, mathematics is used in various professions, is present in everyday life, has relationships with other disciplines, its learning is based on aspects of communication. In (Demattè & Furinghetti, 1999), in particular, some popular conceptions are analyzed: in order to succeed in math, it is better to remember the rules; creativity is not required in the mathematical reasoning; the approximate results in mathematics are not acceptable; symbols +, ×, -, : have been used since before the birth of Christ; etc. Jankvist (2010) illustrates an empirical study on the use of history in a Danish upper secondary class in order to study the students' capabilities at engaging in reflections on mathematics and its history. Liu (2009) analyses connections between the history of mathematics and the students' conceptions. The author describes an annual university course on Calculus which was presented in its historical context. A questionnaire on epistemological aspects of mathematics was administered to students at the beginning and at the end of the course; a similar questionnaire was also used with a group of students who did not attend the course. Results indicate significant influence on the epistemological beliefs of the discipline in the first group but not in the control group.

2 A book

To discuss how to act on the image of mathematics held by students by means of the history, I have written a book addressed to students of the final years of secondary school (16 years old onward), teachers from various disciplines (including history, philosophy, art), or readers who are interested in the popularisation of mathematics (Demattè, 2010).

The book is based on pictures taken from historical sources: pictures have been largely used in history for communicating mathematical ideas, see (Mazzolini, 1993). Words accompany pictures in order to create a unitary discourse and to focus on some aspects. Pictures strengthen what the verbal part say, like in a natural history museum where things and words, verbal and non-verbal communication coexist. Knowledge required for using the book in classroom (or elsewhere) is confined to elementary mathematics.

The book focuses on some relevant ideas of mathematics. Every chapter ends with a discussion ("History for us") about personal beliefs which are connected to the aspects treated in it. This discussion is an opportunity for the reader to reflect on his image of mathematics and it deals with factors that are not always made explicit in the classroom, but influence the personal relation with mathematics. I think it is important to stimulate students' awareness on these factors. Teachers may suggest students they should use the book to start personal insights or to read during holiday periods. It is also proposed to ordinary people to reflect on their vision of mathematics.

The basic reason which has motivated the book is the reflection on pedagogical choices for students who finish secondary school and do not meet mathematics in their study or profession. In any case, for them mathematics could be a cultural stuff and a tool for analysing reality.

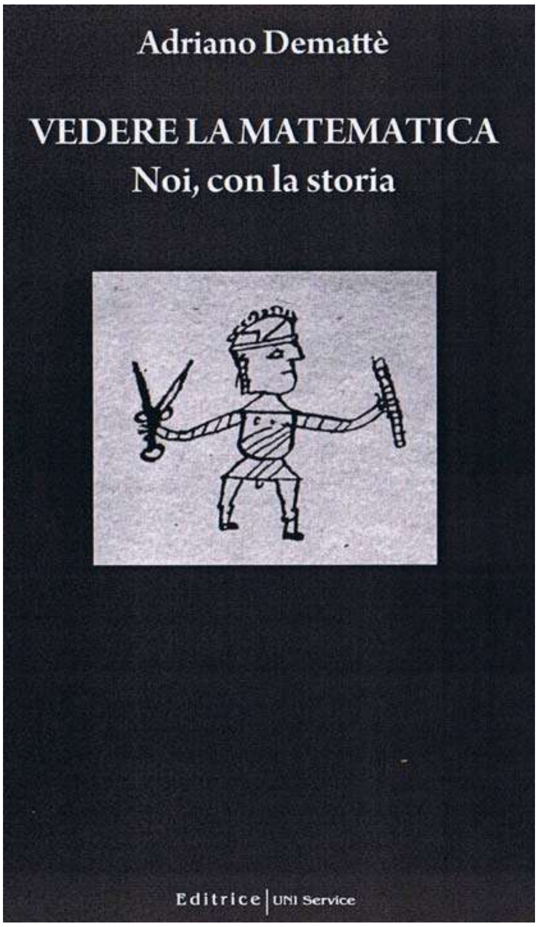

Figure 1. Image on the cover from: Giovanni Agostino Abate, *Giometria de figure quadre*, manuscript, 16th century.

VEDERE LA MATEMATICA
Noi, con la storia
[SEEING THE MATHEMATICS
We, with the history]

CONTENTS (keywords)

Presentation: Fulvia Furinghetti - University of Genoa (aims of the book, method, how to read it)

Image and images…

1. **General view** (images for illustrating relevant ideas in the history of mathematics, m. for everyone)
2. **Role of images in mathematics** (words and things, semiotic systems, geometric figures and definitions, originals, transposition: seeing historical events today)
3. **Suggestions and information in historical images** (people, instruments, diagrams, and drawings)
4. **Mathematics view** (reader's beliefs:
   – When I solve a mathematical problem I know that there is only one exact solution
   – M. I learn at school is not useful for real life
   – Different school subjects have to be learnt separately because they haven't any links
   – In my opinion, approximated result of a problem is not acceptable because of mathematical rigor
   – When I study m. I always worry not to make mistakes
   – I better learn m. if I study alone
   – M. I use in everyday life is ancient
   – M. and culture are separated)
5. **Something else about the history of mathematics** (history of science: why?, how to select historical

events, m. in every culture, epistemology, hermeneutics)

Figure 2. John of Holywood, image from a 15th c. manuscript
(4. How to write a number).

In the History

1. **The first files of data** (*ishango* bone, objects of clay; **history for us**: numbering is difficult, more and more complex m., connecting with simpler knowledge)
2. **Mathematics for administering a State** (rope, measures, right angle, Pythagorean terns, Herodotus, taxes, *groma*-Roman instrument; m. for a few dominant people, m. as fearing school subject, m. for doing order)
3. **Is mathematics we learn at school ancient?** (Ahmes papyrus, Moscow papyrus, geometry, Babylonia, square root, *Juzhang Suanshu*; ancient m., recent symbols, different manners for doing m.)
4. **How to write a number** (different numeral systems: Egyptian, Babylonian, Maya, Hindu-Arabic, binary, Sacrobosco-John of Holywood; one problem-many solutions, open problems in class)
5. **Does it depend on material we have?** (abacus, quipu, Napier sticks; different instruments-different learning, embodied mind)
6. **Algebra begins** (scribe Jsma-Ja, clay tablet TM 75 G 1693; tablets and modern drafts, contents in tablets and in modern exercise-books)
7. **Mathematics is full of errors** (Ahmes papyrus, “Euclidis Megarensis”, to draw an equilateral triangle, *Proofs and Refutations*; errors, inadequacy, to accept own errors)
8. **Pythagoras in China** (*Chou Pei Suan Ching*, bamboo/tree problem; Pythagorean theorem, Pythagorean triplets, arithmetic and geometry, mathematical aesthetics)
9. **A model to imitate** (Euclid’s *Elements*, Adelard of Bath, Clavio, Commandino, Spinoza, *Ethica ordine geometrico demonstrata*, fifth postulate; explanation, proof)

Figure 3. Archimedes, Italian stamp
(10. What is geniality?).

10. **What is genius?** (Archimedes, Gauss, Abel, Galois; mathematician-prototype of genius, biographies and mathematics view)
11. **Mathematical knowledge doesn't “accumulate in layers”** (Diophantus, Al-Khuwarizmi, symbolism; historical evolution, easy-difficult, progress-retreat)
12. **Recreational problems** (Alcuin of York, Fibonacci, Calandri, Peano, Carroll, Gardner; problems: m. and narrative, challenge in problem solving)
13. **Authority and knowledge** (reckoning masters, people in images and differences of rank; authority and democracy at school)
14. **Mathematics is culture** (*Margarita Philosophica*, research, specialisation, interdisciplinary themes; ignorance, cultural relevance of m.)
15. **Masters of abacus** (*Margarita Philosophica*, algorists, abacists, reckoning schools; oral and written communication, manipulation of symbols vs. meaningful learning)
16. **Mathematics and trade** (Paolo dell’Abbaco’s *Trattato di Aritmetica*, *Treviso Arithmetic*; images and real life, causes of m. evolution)
17. **Geometry for builders** (applications, constructing a rectangle; theorems in action, trial and error, goal of a mathematical reasoning)
18. **Mathematics and politics** (Copernicus, Cardan, Tartaglia, Carnot, Leibniz; necessity of tools for doing m., sources and laws)

19. More recent than we think (symbols for arithmetic and algebra; operations, algorithms, pedagogical choices)

Figure 4. From Paolo dell'Abbaco's Arithmetic, 14th c. (16. Mathematics and trade).

20. Is mathematics the same everywhere? (circle, Archimedes, Liu Hui, Kepler, arithmetic triangle; not only Europe)
21. Problems of priority (Oresme, Fermat, Descartes; misconceptions and diagrams, acknowledging priority)
22. Mathematics and war (problem of ladder, quadrant, measuring for shooting; war and m. growth, ethics)
23. Let's bet everything (gamble and God's will, probability, Laplace, de Finetti; certainty, random, points of view in m.)
24. Calculus (Leibniz, Newton; evanescent quantities in drawings, Calculus as relevant mathematical idea)
25. Mathematics and other sciences (sciences in ancient time, music, golden ratio; interpreting, understanding, explaining)
26. Geometry of position (Euler, topology, Königsberg bridges, Möbius; ontogenesis, phylogenesis, topology in primary school)
27. Beyond infinity (representing infinity, Escher, Peano's curve, Cantor; Hilbert: dignity of human mind, m. and philosophy)

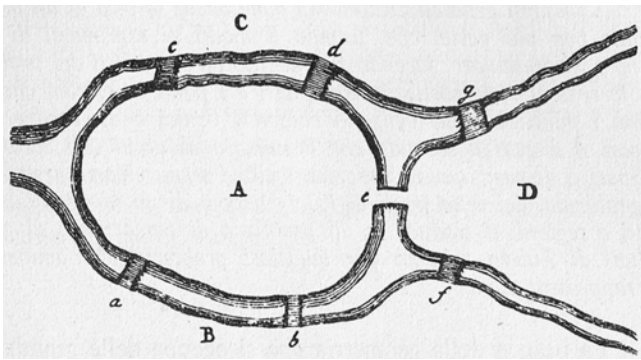

Figure 5. Euler, Königsberg bridges' problem (26. Geometry of position).

28. Etnomathematics (Indian *Sulbasutras*, D'Ambrosio, Gerdes; multiculturalism, m. inside and outside school)
29. Past, present and future (Mandelbrot, fractals, holograms, Morin, where is going m.?; is somebody discovering something in m.?)
30. Imagine a mathematician ([space for drawing], Roman bas-relief, portrait by Beham, Pacioli, Carroll; which m. view in portraits?)

Conclusions (images, history, narrative, mathematical ideas)
References (sites and books)

3 The questionnaire

After reading the book, a questionnaire is proposed (see below): one version for teachers, one for students, one for "common citizens". That one for student is essentially a test of profit which refers to the contents of the volume, although it also includes the requirement of personal opinions. It becomes a "board-book": after reading, students can submit their answers to the teacher and get an assessment on the completeness and relevance of references. In the questionnaire for the teacher, personal opinions about the image of mathematics and the use of history in interdisciplinary perspective are asked. Reflections are also requested to citizens who can, this way, recall their school experience: some parts of the book regard school mathematics and it is assumed that the reader had, right there, his/her most significant experience.

The idea to submit to the reader a questionnaire with open answers will be a way to bring his/her thoughts to specific points, but offering, at the same time, the opportunity to broaden the discourse. The aim is not to collect response data with a statistical value and it is assumed that the way in which the questions are worded can influence the respondents. The book explicitly shows a specific image of mathematics. I am aware that, because of the characteristics of the questionnaire, the reader could be even excessively appeasing. The goal, however, is to raise a debate on the image of mathematics and to suggest that history may be a subject for acting on it. We would like to collect personal opinions from teachers, students, and other readers, who can send their answers (anonymously, if they want).

GREMG – Gruppo Ricerca Educazione Matematica Genova
Project: **"History for discussing the mathematics view"**
[Download of the following questionnaire from
http://www.uni-service.it/vedere-la-matematica.html]
After you read the book *Vedere la matematica*, we ask you some questions. We are interested in knowing your opinion about ideas you have found in the book.
Answer and send the following questionnaire to the e-mail address **storia_e_immagine@dima.unige.it** ('**di**partimento di **ma**tematica **uni**versità di **ge**nova'). If you want, write your name and your personal address.
We ask the teachers who use the book for class to collect students' answers and to send files to the previous e-mail address or to send paper sheets to:

Prof.ssa Fulvia Furinghetti
Dipartimento di Matematica dell'Università
Via Dodecaneso, 35
16146 Genova

We guarantee that personal data will be used only inside the research Project.
Here is the questionnaire for you. Choose:

σ, if you are secondary school student

ι, if you are teacher

χ, if you are neither secondary school student nor teacher

Thank you very much for attention and for collaboration

QUESTIONNAIRE

σ

For **SECONDARY SCHOOL STUDENTS**

1. Kind of school...
2. You read the book:
□ after an internet search
□ thanks to a suggestion of a friend
□ because of a request of your teacher
□ others ....
3. Remember mathematics you have learned at school: do you think that most of it was known before Christ? Choose at least three examples in the book.
4. In the book you don't find an explicit answer but only some hints... Before Christ, could there be somebody who was able to use percentage? Why?
5. About the historical origin of symbols you use for arithmetic, algebra,... before reading the book, what was your opinion regarding the period in which mankind started using them? Was it the same you found in the book?

6. With respect to the content of different chapters, how would you describe human knowledge before Renaissance, if you compare it to that after Renaissance? Did you find something new in the book?
7. "Development of mathematics has been due to the need of applications": do you agree? Why?
8. In your opinion, what is described in the book could help changing school contents? How?
9. Let's do a little "mathematics geography": in your opinion, what is really interesting about mathematics outside Europe?
10. "Nowadays, nobody discovers anything in mathematics": do you agree with this statement? Why?
11. Knowing other school subjects helps better learning mathematics. Why?
12. In your opinion, what *mathematics view* appears in the book? Compare it to your personal *mathematics view*.
13. Briefly describe the three contents which are the most interesting, in your opinion, and explain why you have chosen them.
14. Express your opinion (is the book interesting for you? which contents are more remarkable? which ones less? Have you found some contents you want to deepen? etc.)
15. If you want, write other remarks.

QUESTIONNAIRE

ι

For **TEACHERS**

1. You read the book:
☐ after an internet search
☐ thanks to a suggestion of a friend/colleague/acquaintance
☐ others ....
2. Kind of school where you teach and your school subject
3. In your opinion, is *mathematics view* a relevant pedagogical focus? Why? How can it influence students' involvement? What about other school subjects?
4. Do you consider significant that students know historical evolution of mathematics? Which aspects, first of all?
Do you agree about the mathematics view which appears in the book? Why?
5. In your opinion, what has been or what could be the interest of your students who read or will read the book?
6. Among ideas described in the book, which ones can students better learn?
7. Which ideas described in the book do you agree with? Which ones do you disagree with?
8. Have you found contents you might deepen? If so, which ones?
9. Does the book contain hints for classroom activities? Why?
10. Do you use the book as a pedagogical tool? If so, in which way? If not, why?
11. About 'sensing use' of mathematical symbols, do you think that history can provide teachers pedagogical hints and stimulate students' reflection?
12. Do you think the book has a pedagogical value for teachers teaching other school subjects?
13. If you want, write other remarks.

QUESTIONNAIRE

χ

For readers who are **NEITHER SECONDARY SCHOOL STUDENTS NOR TEACHERS**

1. You read the book:
☐ after an internet search
☐ thanks to a suggestion of a friend/colleague/acquaintance
☐ others ....
2. How do you use mathematics (in your job or in your everyday life or as a topic of interest)
3. What are the most interesting aspects you have found in the book?
4. In your opinion, is knowing historical evolution of mathematics important for one's personal culture? Why?
5. Did you know that some researches about the history of mathematics or about the mathematics view are in progress? What is their relevance, in your opinion?

6. In your opinion, what mathematics view appears in the book? Do you agree or not? Why?
7. The book often reports remarks about school life: compare them to your personal experience.
8. Have you found some contents you want to deepen? Which ones?
9. Have you found some contents which are, in your opinion, uninteresting at all? Which ones?
10. If you want, write other remarks.

4 The workshop

Materials for working groups

- Copy of the book (in Italian)
- Translated contents with keywords (in English)
- Translated questionnaire for readers (in English)
- Following questions for the Workshop
- Answer sheets

Questions for the Workshop

a) In your opinion, can a reader establish meaningful links between mathematics she / he have learned at school and mathematics shown in the book?
b) Can the learning of the history of mathematics change mathematical beliefs? What role could have the book?
c) What's the pedagogical value of historical images?
d) After secondary school (18), many students don't learn mathematics anymore: how can history be a starting point for their autonomous interest in mathematics?
e) Could some historical topics enter class for treating the history of mathematics[1], not only for building new mathematical concepts?
f) Other remarks...

Answers to the questions for the Workshop

a) All participants in the workshop showed almost the same opinion, i.e.: it is necessary that students get more assistance by the teacher and supplementary material (more specific questions, worksheets, exercises, websites); these links can grow if the teacher points out the connection specially in the teaching of a general topic in school and the reading of a linked topic in the book takes place at the same time.

These observations confirm, in my opinion, that it is not easy to establish links between school mathematics and mathematics as shown in the book. In response to the questionnaire, a student wrote: "This book is a very positive book, because it makes math more like the reality that surrounds us". In this statement I see a double value. 1) It shows that the student recognizes that school mathematics has a potential role with respect to reality ("like the reality" could mean "which also refers to aspects outside the discipline," or "which uses a language less specialized, does not include the systematic use of rigorous deduction, refers to global images, etc."). 2) It highlights that student's view of school mathematics and the view suggested in the book are different, so not all that can be said about the former can also be said about the latter, and vice versa. Moreover, note that in the entire questionnaire, students have not used specific examples taken from math class.

[1] "Strong role" of the history of mathematics (Demattè, 2007) or "History as a 'goal'" (Jankvist, 2010).

b) A participant declared he expects it could change math beliefs but under the right circumstances (such as the presence of a well informed teacher), depending on the specific beliefs; yes for: “M. and culture are separated”, “Different school subjects have to be learnt separately”. “M. and culture are separated” (if culture is taken in a broad sense) can be changed because there are many examples in the book that show a connection; chapter 7 can change opinion 5 (“When I study m. I always worry not to make mistakes”). Images might strengthen the idea that mathematics we use is ancient. Therefore not every listed belief can be tackled by reading the book but those which are inherent to mathematics and culture. If we look at the previously pointed out believes, it is doubtful that so many of them can be changed by teaching the history of mathematics, for example the second one (“M. I learn at school is not useful for real life”). Concerning the second, pupils could think it only changed real life some centuries ago.

Every participant made some distinctions: the study of history, and the use of images as well, could act only on some beliefs. It is necessary that the teacher assumes a mediating role.

I agree that some students may believe that mathematics was a useful tool to act on the truth only a few centuries ago. It emerges the educational problem to convince the students that past and present are connected (this way, they could understand the real meaning of the history of mathematics). A student of mine asked me: why do we study history, if we must understand the importance of mathematics we learn today? A question with a suggestion inside.

c) [In the book] There are different kinds of images: 1) There are those meant to illustrate mathematical ideas or problems (Koenigsberg bridges), 2) Pictures of mathematics (*Margarita Philosophica*), portraits, pictures of math activity, 3) Suggestion pictures, not necessarily mathematical (*Divina Proportione*), suggestion of mathematical ideas (basket patterns), 4) Produced pictures (draw the mathematician). Here it is a list of expressions which has been used by participants about the pedagogical value of historical images: attracting students, motivating, enter new situations, answering questions, showing that mathematics is ancient but evolves, giving new perspectives and frames for discussion, starting explorations, suggesting ideas for practical demonstrations, visualizing concepts, stimulating conjectures, supporting or replacing texts, anchoring an idea, starting thinking or talking about a problem, offering a ‘picture of the time’.

Figure 6.

During the work, a participant pointed out some analogies about postures between the image on the cover of the book and some rock drawings found in the Negev Desert (in figure 6, a similar petroglyph from Valcamonica-North Italy).

I would like to highlight another suggestion created by historical pictures, i.e. their metaphorical value with respect to mathematical reasoning. It is quite easy to imagine that persons in previous pictures (people who are using mathematics in everyday life, like in figure 2 or 4) are acting with a goal. When our students make mathematics, do they work with a goal in mind (a doubt which has to be clarified, a thesis which has to be proved,...), or do they work only for implementing assigned procedures?

d) Some participants agreed that history could be a starting point for autonomous students' involvement in mathematics because, in their opinion, history seems to be interesting for many students and a book like this one could make the reader more susceptible for the history of mathematics in general. Others were skeptical: I don't think it could be very easy; for involving student it could be better to start closer to present-day use of mathematics and from there to go backwards (how did we arrive at the mathematics we use at present? where does school mathematics fit in?...); I think if someone is not really interested in mathematics, history of mathematics can't convince him to start autonomous work in mathematics.

It is really rare that, when they don't learn at school anymore, students are interested in mathematics. It's a pity. How could teachers stimulate their interest? The first step could be to present them resources such as guides for using the web, multimedia, digital or in hard copy books, etc. In my opinion, the problem to involve them is difficult but must be considered.

e) Yes, but in interdisciplinary work or projects, in connection with general history, for instance, in France the revolution in connection with Monge and Condorcet, in England Newton and his time when England became an important country, in the Netherlands in the Seventeenth century when de Witt was not only an important statesman but also a mathematician, in Germany Adam Riese and his time, etc. Biographies of mathematicians could be very encouraging to students. It probably could be an optional subject. Students could use this book to select a topic and study it more in depth.

f) The book could be a good additional schoolbook especially for pupils that do not really like mathematics or for those ones that would like to know more about history of mathematics. A nice thing is that every chapter is like a small story. The teacher can choose the chapter and doesn't need to teach the whole book. Links between actuality and history are very interesting and provide net perspectives. "La storia per noi" gives new ways of looking at pictures and at mathematics. Some of the topics (for example chapter 11) are very useful to change or even to destroy old views, because of provoking. The book looks nice and it could benefit from some activating material. Regarding not previously listed beliefs, another participant says: the book can suggest it is only about man, not woman; I miss some 20^{th} century applications and the danger is that the reader will think mathematics as an old stuff. A popular belief about math is that it is 'finished', that there is nothing left to discover. Then such a book could be even dangerous!

Answers f) complete answers e) and propose a range of ways to use the history of mathematics in itself. The suggestion for interdisciplinary works is really meaningful and expresses the core of the history of mathematics in pedagogy. In my opinion, biographies could also show that the most important mathematicians could be sometimes weak or even could make errors. The opportunity to learn this or other kinds of optional subjects is a way for deepening and for involving less motivated students. The book was just meant to be used for activities in the classroom, but the teacher could also require the student to read it on their own. The questionnaire for students can also be seen as a tool to direct students' attention and to focus some relevant aspects.

In a previous work I have analysed the belief "nobody is discovering new things in mathematics" (Demattè, 2004). Some students of upper secondary school (aged 15) answered oral interview concerning their views of the development of mathematics.

Almost unanimously they declared to believe that mathematics is evolving today, but the examples they produced were certain possible applications to experimental sciences or technology. They didn't refer to the current mathematical research and to the idea that mathematics has developed for internal needs, independent of the demands of applications in other sciences or industry. I agree that students have to know that mathematics is also a modern science. The problem is that mathematical research is often highly specialized. In ESU6, Batya Amit and Nitsa Movshovitz-Hadar have presented examples of activities with students having the aim to bridge the gap between contemporary mathematics and mathematics education, by means of activities linked to the story. This type of educational research should be supported and it could be part of a reflection on mathematics aimed at students, teachers, and people who do not work at school.

Regarding the statement "Mathematical knowledge doesn't 'accumulate in layers'" (#11 in the Contents), a participant pointed out the distinction between the fact that, in a certain moment, there could be 1) ignorance of a previous mathematical topic, 2) conscious choice not to take this topic into account. Depending on the true alternative, interpretation of the statement could differ.

Acknowledgments

So many thanks to all workshop participants, first of all to those who gave their written contributions: Maria Correia de Almeida, Michael Fried, Hannah Hoffmann, Jenneke Krüger, Swetlana Nordheimer, Anne Michel Pajus, Harm Jan Smid. I hope that the ideas of every one of us have been faithfully reported herein. I also wish that my comments could be the subject of further debate.

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DEFINING DERIVATIVES, INTEGRALS AND CONTINUITY IN SECONDARY SCHOOL

A phased approach inspired by history

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ABSTRACT

Historically, the concepts of analysis have been developed in the opposite order compared to the deductive order. In the 17th century, Newton and Leibniz introduced derivatives and integrals in order to solve physical and geometrical problems. In the 19th and 20th century, mathematicians have provided foundations for this theory, by giving a precise definition of the limit concept. Not only were foundation problems typical of the 19th century, but also these precise definitions had become necessary by the more general concept of a function. 'Strange' functions had appeared, whose continuity, differentiability and integrability could not be decided without refined definitions.

In many courses and textbooks, formal definitions are introduced from the beginning. For pupils, these definitions seem to complicate things unnecessarily. Later on, in these courses and textbooks, one makes "cheese with holes" (by replacing too difficult proofs by the words "one can prove that…"). Our proposal is to use "visual" concepts of derivative (slope of the tangent), integral (area) and continuity (connected curve) in a consequent way, at least in a first phase. At the end of the course, we propose to confront pupils with some oscillating functions and functions with an infinite number of discontinuities, in order to motivate refined definitions. The last phase, only for the most mathematically oriented pupils, consists of using these refined definitions in a few examples of proofs. For this proposal, we used historical inspiration without copying history.

1 Introduction

1.1 Cheese with holes

In many textbooks for secondary schools the concepts of calculus have at one hand complicated definitions and at the other hand graphical interpretations, which are much easier. The pupils get the impression that a mathematical definition should be complicated and formal in order to merit the status of a mathematical definition. For them, a derivative "is" simply the slope of the tangent and an integral "is" simply an area; limits of differential quotients and Riemann sums are in their view a way of saying the same thing in an needlessly complicated way. Similarly, a continuous function is for the pupils an uninterrupted graph and a limit is nothing more than the value obtained by going "closer and closer" to a point (or further and further away to infinity); they do not understand the usefulness of expressing this with ε and δ.

Is the rest of the calculus course based upon the "visual" ideas or upon the formal definitions? This is not always clear. For example, there is the *mean value theorem*, saying that the integral of a continuous function f in an interval $[a, b]$ is equal to $(b - a) f(c)$ for a c in $[a, b]$. From a visual point of view, this theorem is obvious (you "see" it in figure 1!). It does not need to be formulated nor proved.

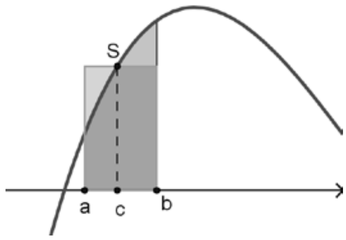

Figure 1

However, the textbook provides a proof, based upon the "intermediate values theorem" (a continuous function in [a, b] attains all values between $f(a)$ and $f(b)$) and Weierstrass' theorem (a continuous function in [a, b] attains a minimum and a maximum). These two theorems, visually equally obvious as the mean value theorem itself, are not proved. The textbook says that *one can prove them* but that the proof is too difficult. This is cheese with holes! The pupil feels disoriented in this world the rules of which he does not understand.

1.2 Phases

We propose to replace the situation described in 1.1 by a phased approach. In a first *visual phase*, we start with derivatives and integrals, motivated by applications and defined using the visual ideas of tangent lines and areas, and using an intuitive idea of limit. A large part of the calculus course in secondary school can be done in this visual phase. In a second phase, we confront the pupils with some strange functions for which the visual definitions do not work anymore. This establishes a motivation to look for more refined mathematical definitions in a third phase. The fourth and last phase consists of using the refined definitions in proofs.

Our approach inverts the logical order, in which firstly the limit concept is defined and then derivatives and integrals are defined as limits. We rather follow the historical evolution: derivatives and integrals are from the 17th century (Newton, Leibniz) whereas the formal definition of a limit only appeared in the 19th century (Cauchy, Weierstrass), when the concepts of numbers and functions had evolved and "strange" functions had become possible, for which the visual ideas did not work anymore.

2 Visual phase

2.1 Derivative

We suggest introducing the derivative by using several contexts: skis following the slope of a (two dimensional) mountain, a tangent to a circle and the instantaneous velocity. In these contexts, pupils get familiarized at the same time with the notion of a tangent line and of a derivative. The common idea is a chord connecting two points $P(a, f(a))$ and $Q(x, f(x))$, which becomes a tangent "in the limit" when Q approaches P more and more. This can be shown with dynamic geometry software such as GeoGebra (figures 2a-d). The point Q can approach the point P from both sides, but when it coincides with P, there is no line PQ any more. So, the points have to get nearer and nearer to each other without ever coinciding. The limit idea is introduced here in an informal way; there has been no preliminary course about limits.

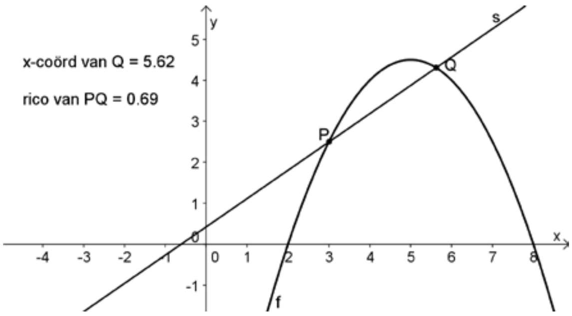

Figure 2a

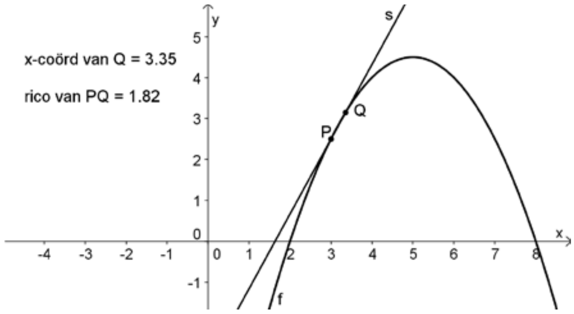

Figure 2b

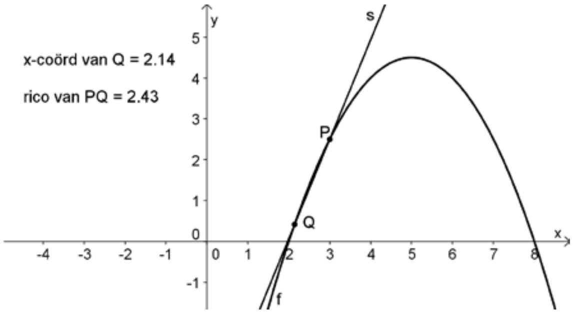

Figure 2c

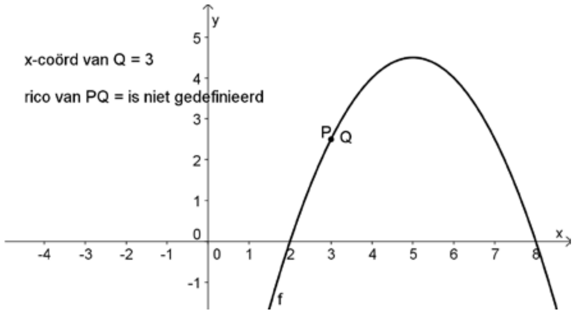

Figure 2d

The slope of the tangent line to the graph of f in the point $P(a, f(a))$ is called the *derivative* of f at a and is written down $f'(a)$. In the visual phase, this is the *definition* of the derivative and not its graphical interpretation.

In order to calculate the derivative by means of the formula of the function f, we calculate the slope $\dfrac{f(x)-f(a)}{x-a}$ of the chord PQ and then we determine the *limit* value when x approaches a more and more (without coinciding with a). This limit is written $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$. This is a way of calculating the derivative, not its definition!

In the example of the derivative of $f(x) = x^2$ at 3, we have $\dfrac{x^2-9}{x-3} = x+3$ (for $x \neq 3$). When x approaches 3, $x + 3$ approaches $3 + 3 = 6$, of course. We write:

$$f'(3) = \lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 6.$$

The non-differentiability cases at a point where there are two semi-tangents (figure 3a) or a vertical tangent (figure 3b), are also dealt with in this visual phase, as well as derivative functions, rules to calculate derivatives, asymptotes, limit calculations, l'Hospital's rule, the study of functions and several applications.

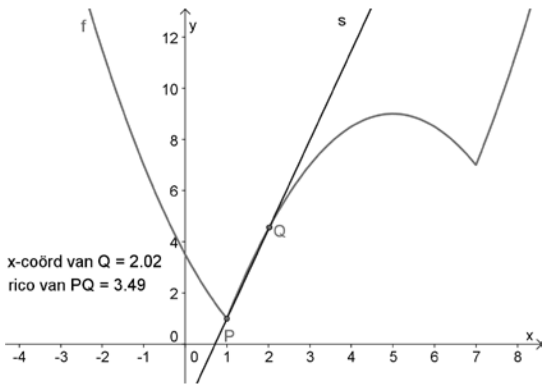

Figure 3a

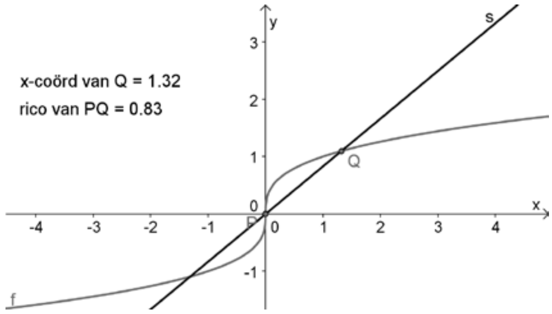

Figure 3b

2.2 Integral

Integrals are also introduced by way of contexts: the distance moved from the speed, the water volume in a bath from the flow rate... These contexts show that the oriented area between the graph and the horizontal axis can have different meanings depending upon the quantities represented on the axes. The integral of a function f from a to b is defined as the oriented area between the graph and the horizontal axis from $x = a$ to $x = b$. In the visual phase, this is the *definition* of the integral and not its geometrical interpretation. Riemann sums (lower sums, upper sums and others) are a method to calculate the integral, not its definition.

Let us see how we can prove the fundamental theorem in this visual phase. We define the *integral function* $F_a(x) = \int_a^x f(t)dt$. We want to prove that $F_a'(x) = f(x)$. We have

$$F_a'(x) = \lim_{\Delta x \to 0} \frac{F_a(x+\Delta x) - F_a(x)}{\Delta x}.$$

The numerator is the area from a to $x + \Delta x$ minus the area from a to x, hence the area from x to $x + \Delta x$ (figure 4a). In order to divide this area by Δx, we can transform it into the area of a rectangle with basis Δx (figure 4b). The upper side of this rectangle cuts the graph in (at least) a point $(c, f(c))$, and $f(c)$ is the height of this rectangle (figure 4c). Because the number c is “squeezed” between x and $x + \Delta x$, it will approach x more and more when Δx diminishes towards 0. In formulae:

$$F_a'(x) = \lim_{\Delta x \to 0} \frac{F_a(x+\Delta x) - F_a(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} f(c) = f(x).$$

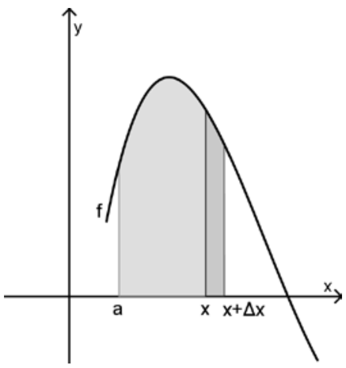

Figure 4a

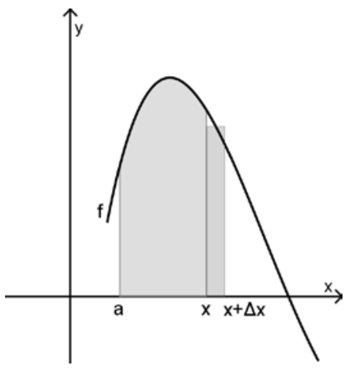

Figure 4b

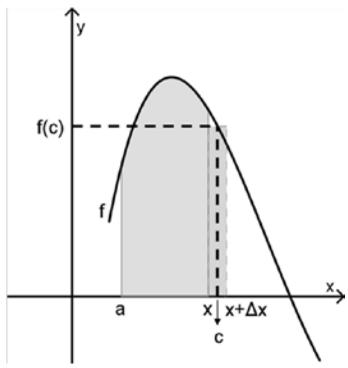

Figure 4c

We could specify that we supposed f to be continuous by saying that the upper side of the rectangle cuts the graph. However, a "visual" idea of continuity (an uninterrupted graph) is sufficient. We could also verify that the proof remains valid when f or Δx are negative. But in spite of this, the proof is satisfying in this visual phase.

The continuation is well known: primitive functions, integral calculation by means of primitives, integral calculation techniques, geometrical applications, applications in physics and economy. All this is situated in the visual phase.

3 Problems with the visual definitions

In a second phase, we confront the pupils with some *strange* functions. These functions show that the *visual* definitions of the concepts do not suffice any more. Moreover, these functions will cast doubt on certain properties that were "obvious" in the visual phase. For example, in the visual phase the pupils were convinced that a function is always piecewise continuous (continuous except at the points where the graph makes a "jump"), that the only non-differentiability cases were those of the figures 3a and 3b (different left and right derivatives; vertical tangent) and that the integral of a bounded function always exists.

We show the following graphs to the pupils, preferably with computer software as GeoGebra so that they can zoom in on interesting parts (figure 5a-c).

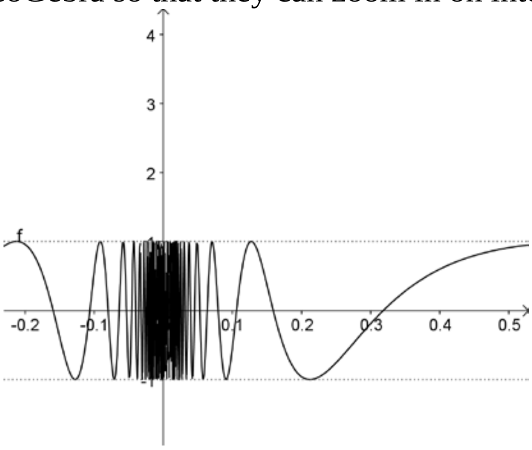

Figure 5a: $f(x)=\begin{cases}\sin\dfrac{1}{x} & (x\neq 0)\\ 0 & (x=0)\end{cases}$

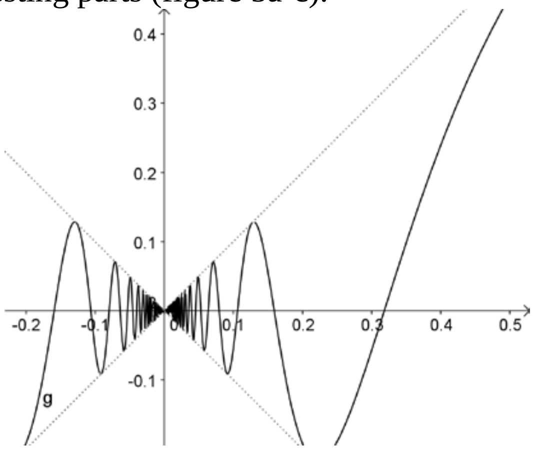

Figure 5b: $g(x)=\begin{cases}x\sin\dfrac{1}{x} & (x\neq 0)\\ 0 & (x=0)\end{cases}$

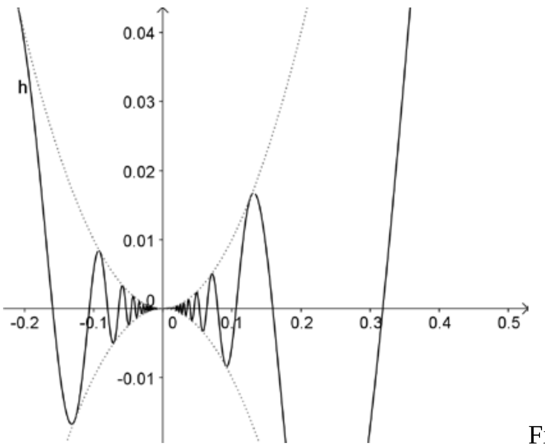

Figure 5c: $h(x)=\begin{cases}x^2\sin\dfrac{1}{x} & (x\neq 0)\\ 0 & (x=0)\end{cases}$

Are these functions continuous? In a way, none of these graphs are interrupted, but they are very different from graphs drawn “in one stroke”. Does the function f have a limit in 0? When x goes to 0, $f(x)$ continues to oscillate between -1 and 1; should we say therefore that there is an infinity of limits at 0, i.e. all the numbers of the interval $[-1, 1]$? Do these functions have a derivative at 0? It is very difficult to imagine skis following the slope of the “mountain” towards $x = 0$. What could be the value of the integral, e.g. from -1 to 1? The oriented area is composed of an infinite number of pieces above and below the horizontal axis; the integral would thus be an infinite series... The pupils realize that the visual definitions do not suffice in order to decide on these questions.

It can be even worse. The Dirichlet function (figure 6) seems to have as a graph a set of two uninterrupted lines, but we could also say that the function jumps “all the time”. Is it a function? Newton would have said *no*; the functions in the 17th century were determined by algebraic formulae or power series. But in recent mathematics, it suffices that each number x in the domain has a unique image in order to speak about a function[1].

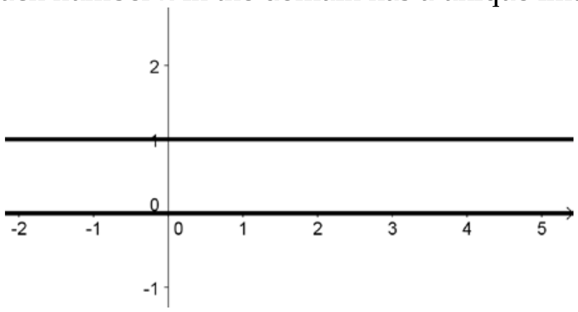

Figure 6: $d(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

The last function, our favourite, is the Thomae function (figure 7). Does it have limits? Is it continuous? Integrable? This function will bring us some surprises. Surely, we are very far away from the “visual” concepts of paragraph 2.

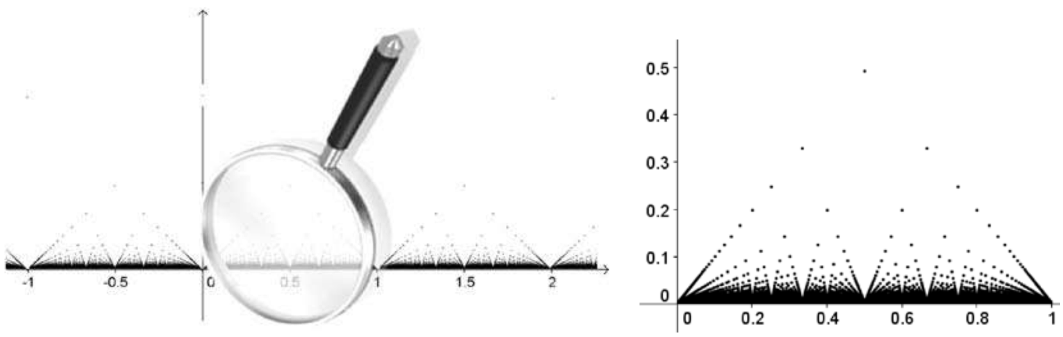

Figure 7: $t(x) = \begin{cases} \dfrac{1}{q} & \text{if } x \text{ is rational and equals } \dfrac{p}{q} \text{ after simplification} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

1 The formulae fans will be pleased to know that there is indeed a formula for the Dirichlet function:

$$d(x) = \lim_{m\to\infty} \lim_{n\to\infty} \cos^{2n}(m!\pi x).$$

4 Refined definitions

4.1 Limit and continuity

The refined definitions of derivatives and integrals will be based upon the limit concept. Hence, we should first of all find a definition of limit that is essentially not based upon visual ideas. Of course, in order to find this definition we will accept inspiration from visual ideas.

Let us restrict ourselves to the case of a limit in a number equal to a number. In an analogous way, we can treat the cases of a limit at infinity and/or equal to infinity, as well as left or right limits and limits of sequences.

Compare the graphs of the functions f_1 and f_2 (figure 8). The first one is the graph of the function f of paragraph 3 (figure 5a) translated by the vector (3, 0). The graph of f_2 is the graph of the same function but "squeezed" between the lines $y = \pm\, 2(x - 3)$. (The numbers 3 and 2 have been inserted merely in order to create a more general example.)

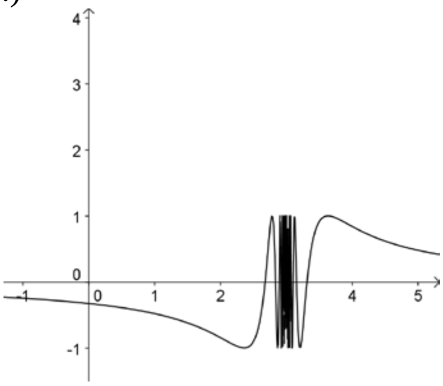

Figure 8a: $f_1(x) = \sin \frac{1}{x-3}$

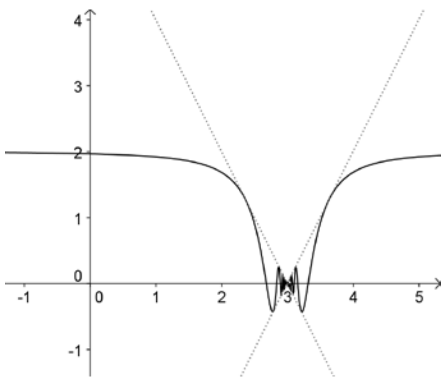

Figure 8b: $f_2(x) = 2\,(x - 3) \sin \frac{1}{x-3}$

These functions have an interesting behaviour around $x = 3$. Let us have a closer look (figure 9).

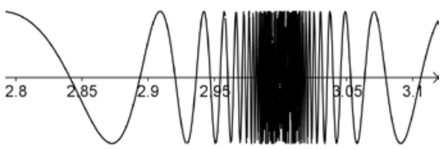

Figure 9a: $f_1(x) = \sin \frac{1}{x-3}$

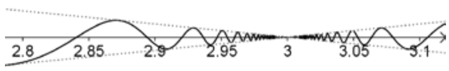

Figure 9b: $f_2(x) = 2\,(x - 3) \sin \frac{1}{x-3}$

In each interval around $x = 3$, f_1 continues to oscillate between -1 and 1. It does not "go to a limit". We say that the limit of f_1 at 3 does not exist. However, f_2 becomes "very small" when x approaches 3. The function "goes to 0", even if it continues to oscillate and does not get nearer to 0 in a monotonous way. The difference between $f_2(x)$ and 0 can be made "as small as we want". For this purpose, it is sufficient to take x "close enough" to 3. If we decide, for instance, that the difference between $f_2(x)$ and 0 should be smaller than 0.1, it suffices to take x at a distance 0.05 to 3. Indeed (figure 10), if $|x - 3| < 0.05$, then

$$|f_2(x) - 0| = |2(x - 3) \sin \tfrac{1}{x-3}| = 2|x - 3| \cdot |\sin \tfrac{1}{x-3}| < 2 \cdot 0.05 = 0.1.$$

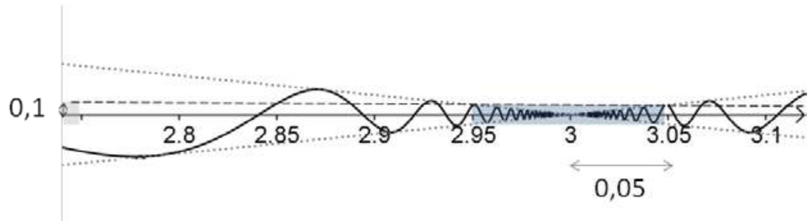

Figure 10

After another example where the limit is not 0, we can generalize this idea in a definition: $\lim_{x\to a} f(x) = b$ means :

$\forall\varepsilon > 0:$	$\exists\delta > 0:$	$0 < \lvert x-a\rvert < \delta$	$\Rightarrow$	$\lvert f(x)-b\rvert < \varepsilon$
For ε as small as you want,	if you take *x* close enough to *a* (but not equal to *a*),			*f(x)* will be in the interval]*b* – ε, *b* + ε[.

Hence, we adapt δ to ε. As our function f_2 is squeezed between two lines with slopes ±2, it suffices to take δ equal to the half of ε.

For the function f_1, the definition is fulfilled for no value of *b*. As soon as we take ε equal to or smaller than 1, there will always be some *f*(*x*) outside the interval]*b* – ε, *b* + ε[, whatever close *x* is taken to 3. Hence there is no limit at 3.

Let us specify that one does never take limits at points that are not accessible within the domain of the function. The point has to be an *accumulation point* of the domain, i.e. each interval]*a* – δ, *a* +δ[(for any δ) should contain points of the domain unequal to *a* itself.

Essentially, the definition of limit is based merely upon numbers, inequalities and logic. The visual and dynamic idea (*x* getting closer and closer to *a*) has been eliminated indeed.

A limit is not very interesting at a point where the function is continuous. There, the limit is, of course, equal to the function value. This gives us the possibility to *define* continuity now that we dispose of a refined definition of limit: *f* is *continuous* at *a* if $\lim_{x\to a} f(x) = f(a)$.

Let us verify that the new definitions of limit and continuity enable us to decide on the continuity of the functions of paragraph 3.

The functions *f* and *g* (figure 5) are analogous to the functions f_1 and f_2 used here to introduce the refined definition of limit. So the function *f* (figure 5a) has no limit at 0 and is therefore not continuous at 0. The limit of the function *g* (figure 5b) is 0 and coincides with *g*(0), so *g* is continuous at 0. The function *h* is also continuous at 0.

For the function *d* of Dirichlet (figure 6), no limits exist. Indeed, whatever *a* and *b*, for ε = 0.5 (or smaller) the images of numbers in an interval]*a* – δ, *a* + δ[are both 0 and 1, so they cannot be all in]*b* – ε, *b* + ε[. Hence the function is nowhere continuous.

The Thomae function (figure 7) is even more startling. Take a number *a* between 0 and 1, rational or not. We will prove that the limit at *a* is 0. Take any ε > 0. Except a finite number of points, all points of the graph between *x* = 0 and *x* = 1 lie under the line *y* = ε (figure 11). The set of the points *u* in [0, 1] such that *t*(*u*) ≥ ε is finite. Take δ equal to the distance between *a* and the element of this set closest to *a*. If 0 < |*x* – *a*| < δ, then we have *t*(*x*) < ε and thus |*t*(*x*) – 0| < ε.

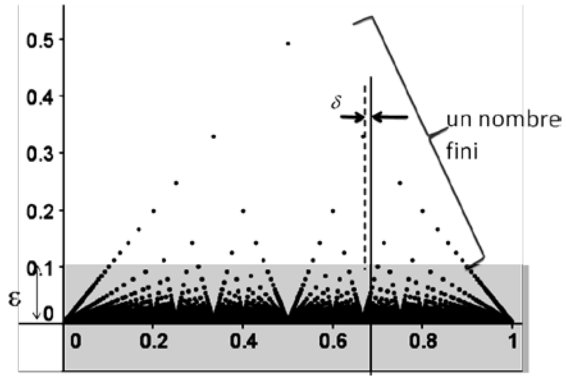

Figure 11

As the function is periodical, this result is also valid for *a* elsewhere than between 0 and 1.

At the rational numbers, the limit is 0 and not equal to the image. At the irrational numbers, the limit is 0 and equal to the image. So the function is continuous at each irrational number and discontinuous at each rational number! The visual idea of continuity is far away!

4.2 Derivative

In the visual phase, the derivative was defined as the slope of the tangent; the limit of the differential quotient was not the definition but merely a method to calculate the derivative from the formula of the function. Now that we have at our disposal a precise definition of limit, we can inverse things. We define the derivative as the limit of the differential quotient: if $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ exists and equals a number, we call *f* *differentiable* at *a* and we call that number the *derivative* of *f* at *a*, notated $f'(a)$. The *tangent* at the point $(a, f(a))$ is, by definition, the line $y - f(a) = f'(a)(x - a)$.

With this definition, we can verify, for instance, that the oscillating function *g* (figure 5b) is not differentiable at 0 and that the function *h* (figure 5c) is indeed differentiable at 0.

$$\begin{aligned} g'(0) &= \lim_{x\to 0}\frac{g(x)-g(0)}{x-0} \\ &= \lim_{x\to 0}\frac{x\sin\frac{1}{x}}{x} \\ &= \lim_{x\to 0}\sin\frac{1}{x} \end{aligned} \qquad\qquad \begin{aligned} h'(0) &= \lim_{x\to 0}\frac{h(x)-h(0)}{x-0} \\ &= \lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{x} \\ &= \lim_{x\to 0} x\sin\frac{1}{x} \end{aligned}$$

As we saw in paragraph 4.1, the first limit does not exist and the second one is equal to 0. So $g'(0)$ does not exist, whereas $h'(0) = 0$.

The sign of the derivative of *h* changes an infinite number of times in each interval around 0. The sign of h' around 0 does not correspond to one of the “known” cases of the visual phase ($\boxed{+\ 0\ -}$, $\boxed{-\ 0\ +}$, $\boxed{+\ 0\ +}$, $\boxed{-\ 0\ -}$).

The pupils can construct their own “beautiful” oscillating functions with another derivative as 0. The graph of figure 12, for instance.

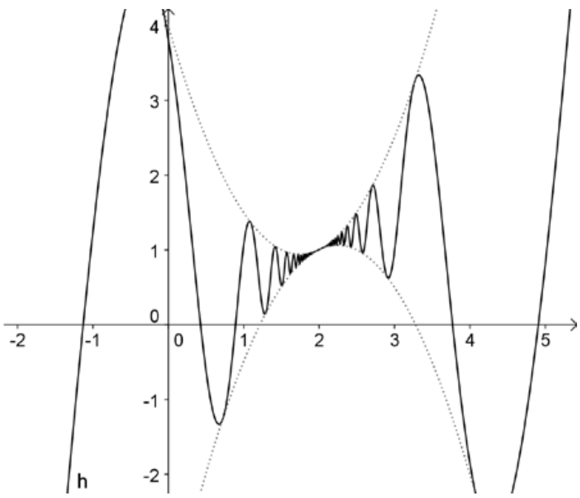

Figure 12: $y = (x-2)^2 \sin\dfrac{10}{x-2} + \dfrac{x}{2}$

The numerator 10 instead of 1 makes the oscillations more visible; the substitution of x by $x - 2$ translates the interesting point to the right and the addition of the term $\frac{x}{2}$ makes the derivative at 2 to be $\frac{1}{2}$ instead of 0. Note that the derivative at 2 is positive but that there is no interval around 2 wherein the function is increasing.

4.3 Integral

As for the derivative, the calculation method of the visual phase, based upon limits of Riemann sums, becomes the definition.

Given a bounded function f and $[a, b]$ an interval in the domain of f. We divide $[a, b]$ into n equal parts $[x_{k-1}, x_k]$ ($k = 1, \ldots, n$) of width $\Delta x = \dfrac{b-a}{n}$. We consider the *lower sum* $s_n = \sum_{k=1}^{n} m_k \Delta x$ and the *upper sum* $S_n = \sum_{k=1}^{n} M_k \Delta x$, wherein $m_k = \inf f[x_{k-1}, x_k]$ and $M_k = \sup f[x_{k-1}, x_k]$. We say that f is *integrable* if $\lim s_n$ and $\lim S_n$ are equal to the same number. This number is *the integral* $\int_a^b f(x)\,\mathrm{d}x$.

This definition is equivalent to the definition of Riemann (19th century), who divided the interval into parts that were not necessarily equal.

The definition is once again based upon the concept of a limit (of sequences). In this “refined” world, we define the area by means of an integral and not the other way round.

Let us examine whether the function d of Dirichlet (figure 6) is integrable, for instance in the interval [0, 1]. It is clear that for all n, the lower sum s_n is 0 and the upper sum S_n is 1. So $\lim s_n = 0 \neq 1 = \lim S_n$. The function d is thus not (Riemann-)integrable.

What can be said about the function t of Thomae (figure 7), for instance in the interval [0, 1]? Again, all the lower sums s_n are 0 and thus $\lim s_n = 0$. We will prove that $\lim S_n = 0$, so that the integral exists and equals 0.

In order to prove that $\lim S_n = 0$, we have to establish that

$$\forall \varepsilon > 0 \colon \exists\, p \colon n > p \Rightarrow S_n < \varepsilon.$$

Suppose $\varepsilon > 0$ is given, as small as you want. Draw the line $y = \frac{\varepsilon}{2}$. For each upper sum S_n, the sum of the areas of the rectangles that remain below this line is certainly less than $\frac{\varepsilon}{2}$, the area of the rectangle of length 1 and height $\frac{\varepsilon}{2}$ (figure 13). (More exactly, the sum of the terms $M_k\Delta x$ for which $M_k < \frac{\varepsilon}{2}$ is less than $\frac{\varepsilon}{2}$.)

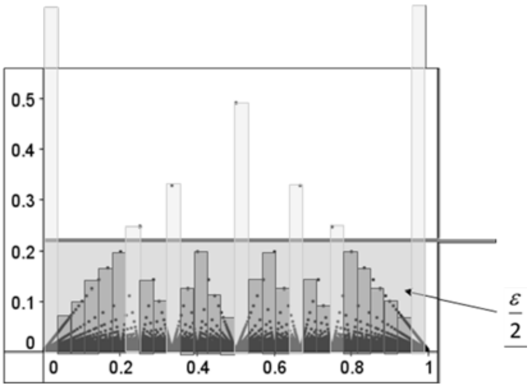

Figure 13

We can take n such that the sum of the areas of the other rectangles, those exceeding the line $y = \frac{\varepsilon}{2}$, is also less than $\frac{\varepsilon}{2}$. Indeed, there is only a finite number of such rectangles, say m (figure 14). The sum of the areas of the rectangles that exceed the line $y = \frac{\varepsilon}{2}$ is less than $m \cdot \frac{1}{n}$. Therefore, it suffices to divide the interval [0, 1] into a number n of parts such that $m \cdot \frac{1}{n} < \frac{\varepsilon}{2}$, in other words: to take $n > \frac{2m}{\varepsilon}$. For $p = \frac{2m}{\varepsilon}$, we have $n > p \Rightarrow S_n < \varepsilon$, what we had to prove.

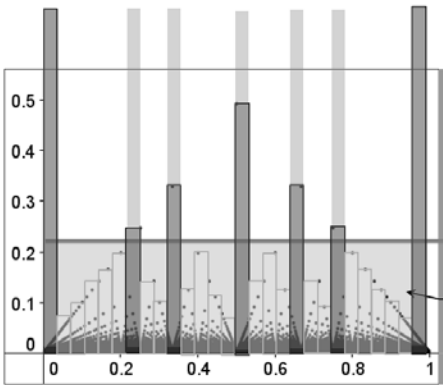

Figure 14

5 Proofs with the refined definitions

What remains valid of the visual phase? In principle, everything should be done again! Of course, this is not what we propose for the secondary school. Only to some mathematically gifted pupils, we could show one or two examples of proofs in this "new world" governed by the refined definitions. This would prepare these pupils for a university analysis course, which starts immediately with the fine definitions. In this text, we restrict ourselves to one example of a demonstration (paragraph 5.1) and a few descriptions of results (paragraphs 5.2-4). For more details, we refer the reader to our article in Uitwiskeling (Eggermont & Roelens, 2009).

5.1 The limit of a sum

The theorem "the limit of a sum is the sum of the limits" is an example of a property that did not need a proof in the visual phase: if $f(x)$ "goes to" b and $g(x)$ "goes to" c, then of course $f(x) + g(x)$ goes to $b + c$. However, in the world of the fine definitions, a limit replaces a complicated expression with quantifiers, an implication and inequalities. As we cannot trust visual intuition any more, we need to give a proof by ε-δ.

It is given that $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$. We have to prove that $\lim_{x\to a}\big(f(x)+g(x)\big) = b+c$. This means:

$$\forall\ \varepsilon > 0 : \exists\ \delta > 0 : 0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - b - c| < \varepsilon.$$

Take an arbitrary $\varepsilon > 0$. We have to find a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - b - c| < \varepsilon.$$

The given limits can be written like this:

$$\forall\ \varepsilon_1 > 0 : \exists\ \delta_1 > 0 : 0 < |x - a| < \delta_1 \Rightarrow |f(x) - b| < \varepsilon_1,$$
$$\forall\ \varepsilon_2 > 0 : \exists\ \delta_2 > 0 : 0 < |x - a| < \delta_2 \Rightarrow |g(x) - c| < \varepsilon_2.$$

The triangle inequality provides a relationship between $|f(x) + g(x) - b - c|$, $|f(x) - b|$ and $|g(x) - c|$:

$$|f(x) + g(x) - b - c| \le |f(x) - b| + |g(x) - c|.$$

If we can prove that $|f(x) - b| + |g(x) - c| < \varepsilon$, we will have attained our goal. Let us apply the given expressions with $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$:

$$\exists\ \delta_1 > 0 : 0 < |x - a| < \delta_1 \Rightarrow |f(x) - b| < \tfrac{\varepsilon}{2},\ \exists\ \delta_2 > 0 : 0 < |x - a| < \delta_2 \Rightarrow |g(x) - c| < \tfrac{\varepsilon}{2}.$$

In order to have at the same time $|f(x) - b| < \frac{\varepsilon}{2}$ and $|g(x) - c| < \frac{\varepsilon}{2}$, we should take x close enough to a so that, at the same time, $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$. Therefore, take $\delta = \min\{\delta_1, \delta_2\}$. Then we have:

$$0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - b - c| \le |f(x) - b| + |g(x) - c| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon,$$

what we had to prove.

5.2 Continuity

In the visual phase, continuity was a global property of the graph in an interval. However, the fine definition is about continuity at individual points. In the examples of strange functions, we met a function that is continuous in certain points without being continuous in an interval (the Thomae function in the irrational points). But if a function is continuous in an interval (that is: at each point of the interval), does it indeed have an uninterrupted graph in that interval? We should specify the idea of "uninterrupted". In fact, the following theorems express the intuitive idea of an uninterrupted curve.

Intermediate values theorem: a continuous function f in an interval $[a, b]$ attains all values between $f(a)$ and $f(b)$.

Theorem of Weierstrass: a continuous function f in an interval $[a, b]$ attains an absolute minimum and an absolute maximum in this interval, in other words:

$$\exists\, c, d \in [a, b] : \forall x \in [a, b] : f(c) \leq f(x) \leq f(d).$$

The proofs of these theorems are not easy. The first one is proved in Eggermont & Roelens, 2009.

5.3 Derivative

The strange functions have broken some simple ideas about derivatives. Some properties that were considered obvious in the visual phase are true, other are false. We specify that a function is called *increasing* in an interval if for all x_1, x_2 in that interval $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$, in other words if a point of the graph that is more “to the right” than another point, is always “higher” than that other point.

- If the derivative at a of a function is positive, f is increasing in an interval around a. This is *false*. A counterexample has been given in figure 12 (paragraph 4.2).
- If the derivative at a of a function f is positive, there is an interval around a where $f(x) < f(a)$ for $x < a$ in this interval and $f(x) > f(a)$ for $x > a$ in this interval. This is *true*. The difference with “increasing” is that here we do not compare two arbitrary points but each time one arbitrary point with the fixed point $(a, f(a))$.
- If a function f has a positive derivative everywhere in an interval $[a, b]$, it is increasing in $[a, b]$. This is *true*, but not easy to prove. The difficulty is that the derivative is defined in a point whereas the definition of “increasing” involves the comparison of two points. The proof goes through the theorems of Rolle and Lagrange. We think this is rather university material.
- If the sign of the derivative around a is $\boxed{+\ 0\ -}$, then the function has a maximum at a. This is *true*.
- If a function has a maximum at a, the sign of the derivative around a is $\boxed{+\ 0\ -}$. This is *false*. A counterexample can be constructed by squeezing an oscillating function between two parabolas, as in figure 15.

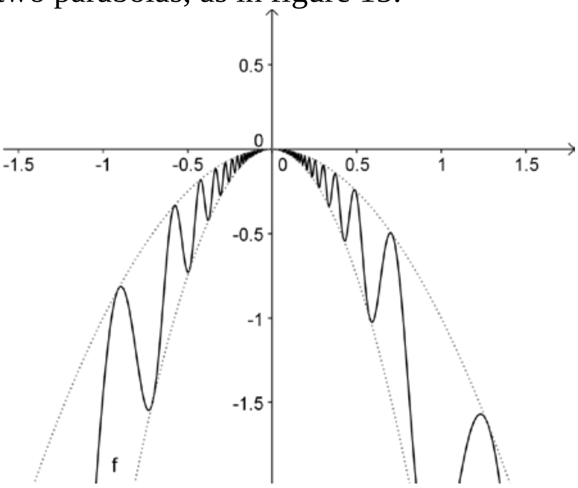

Figure 15: $y = x^2 \sin\dfrac{10}{x} - 2x^2$

5.4 Integral

We could discuss the proof of the fundamental theorem (paragraph 2) again. When the integral from x to $x + \Delta x$ has been replaced by $f(c)\ \Delta x$ (the area of a rectangle), we made use of the *mean value theorem*, which did not need to be formulated nor proved in the visual phase. Let us recall this theorem: "the integral of a continuous function f on an interval $[a, b]$ is equal to $(b - a)\ f(c)$, for a c in $[a, b]$".

If we decide to prove this theorem in the refined phase, we have to use Weierstrass' theorem (see 5.1), which gives us two numbers m and M so that $m \le f(x) \le M$ for all x in $[a, b]$. This implies, by the use of Riemann sums,

$$m(b - a) \le \int_a^b f(x)\,\mathrm{d}\,x \le \mathrm{M}(b - a),$$

or, after dividing by $b - a$: $m \le \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,\mathrm{d}\,x \le \mathrm{M}.$

By the intermediate values theorem, this implies that there is a c in $[a, b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x = f(c),$$

which proofs the mean value theorem.

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GEOMETRIC PROBABILITY APPLICATIONS THROUGH HISTORICAL EXCURSION

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ABSTRACT

The aim of the contribution is to show how the discussion of the history of geometric probability can reveal its simplest application possibilities and its numerous interdisciplinary relations, and how it can attract a deserved interest of students in this fascinating branch of science. The paper summarizes the workshop held by the author at the conference ESU-6; materials provided to participants for the experiments are available at the Internet address "http://euler.fd.cvut.cz/∼hyksova/probability".

1 Introduction

*For most students and even for many teachers, probability theory belongs to the least favorite parts of school mathematics, although it relates to our lives more than anything else; in fact, we are surrounded by randomness and constantly face situations where real outcomes are not completely certain, so that probability estimations come into play. The key problem is perhaps the fact that school probability calculus is usually restricted to tossing coins and throwing dices, i.e., to problems that do not a bit seem to relate our everyday lives and to illustrate the importance and the wide applicability of probability theory. The aim of this contribution is to show how the theory of geometric probability – in addition to other probabilistic topics – can help to persuade students that probability theory plays a key role in many fields closely related to our health, safety, wealth etc.

Similarly as the standard probability, the theory of geometric probability originated as a tool of spatial games understanding and for a detailed elaboration of the rules ensuring their fairness. Later on, it turned out to be a substantial tool for extracting quantitative information on spatial objects (e.g., the percentage composition of a rock, the average volume of the crystalline mineral grains, the area of grain boundary surfaces or the length of grain edges per unit volume of a rock etc.; similarly it is used to get information on cells in a tissue, tumors or lesions in organs, vessels etc.) from probes of a lower dimension (sections or microscope images, linear or point probes). Recall that when we pass from the investigation of properties of finite populations to geometric properties (e.g., volume, area, length, shape) of geometrically describable objects (e.g., human beings and their organs and blood vessels, animals, plants, cells in a tissue, rivers, rocks etc.), replacing random population samples by probes of a lower dimension mentioned above, and instead of "classical" probability we base our inferences on geometric probability, we pass from statistics to the domain of *stereology*. Since we are not only surrounded but even formed by such structures, stereology and

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hence also geometric probability are of great importance to our lives and represent a substantial tool for exploring and understanding the world around us.

Mathematically, geometric probability was introduced as the extension of the classical definition of probability to situations with uncountable number of cases: instead of the number of favourable and all possible cases it is now necessary to work with measures of relevant sets. For example, probability that a point X lying in a set C lies also in a subset $B \subseteq C$ is defined as the ratio

$$P(X \uparrow B|X \uparrow C) = \frac{m(B)}{m(C)}\,, \tag{1}$$

where m denotes a convenient measure of the given set (e.g., area or volume for a set in 2D or 3D, respectively; see also footnote 3). In the following sections we will see that from this simple definition, useful applications immediately follow.

2 Points in plane – area estimation

The first known discussion of a problem concerning geometric probability appeared in the private manuscript of Isaac Newton (1643–1727) written between 1664 and 1666 (Newton, 1967): consider a ball of negligible size falling perpendicularly upon the centre of a horizontal circle divided into two unequal sectors B_1, B_2 (Fig. 1 left); suppose that the ratio of areas of these sectors is $2 : \sqrt{5}$ and the prospective win for a player is equal to b_1 in the case that the ball falls into the sector B_1, and b_2 if it falls into B_2. Newton claims that the "hopes" of the player worth

$$\frac{2b_1 + \sqrt{5}\,b_2}{2 + \sqrt{5}}\,.$$

His aim was to show that the ratio of chances can be irrational; nevertheless, from today point of view we can say that he provided the first area fraction estimation: let us set $b_1 = 1$, $b_2 = 0$ (i.e., the player wins 1 unit if the ball hits the sector B_1, nothing otherwise); the hopes $2/(2 + \sqrt{5})$ express directly the probability of hitting B_1. On one hand, it is equal to the ratio of the area of the sector B_1 and the whole circle $C = B_1 \cup B_2$; on the other hand, the same probability can be expressed as the ratio of the mean number N_{hits}, how many times the ball hits the sector B_1, provided it is thrown N_{total} times on the circle C, and the total number of throws N_{total} :

$$P(X \uparrow B_1 \,|\, X \uparrow C) = \overline{\frac{N_{hits}}{N_{total}}} = \frac{A(B_1)}{A(C)}\,. \tag{2}$$

If the area $A(C)$ is known, the equation (2) can be used for the estimation of the area $A(B_1)$ by counting the number of throws and hits. Denoting the average fraction of hits by the symbol P_P, and the area fraction by A_A, we immediately obtain one of the fundamental formulas of stereology:

$$P_P = A_A\,. \tag{3}$$

Instead of "throwing" isolated random points, it is possible to place a point grid randomly on C and calculate the ratio of points hitting any set of interest $B \subseteq C$ and

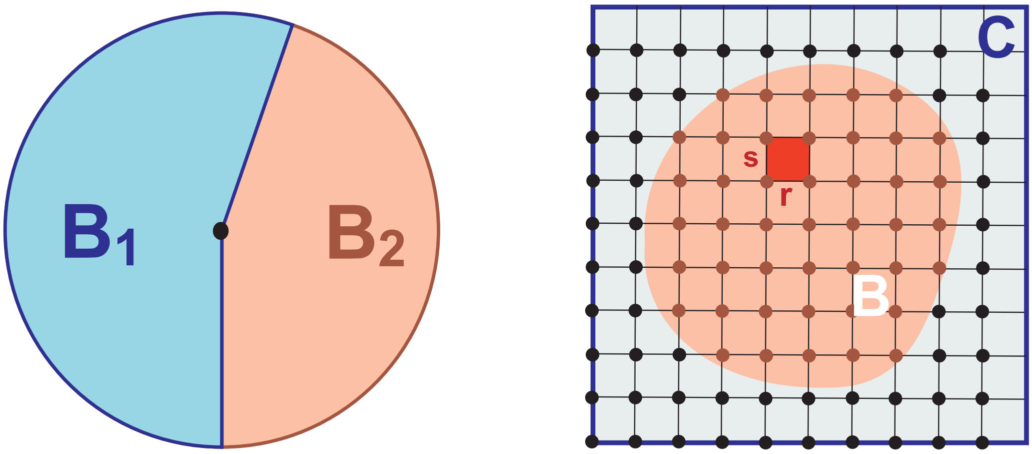

Figure 1: Circle sectors from Newton's example (left), planar point grid (right)

C. Finally, considering C to be the rectangle of the area $rs \cdot N_{total}$ formed by the grid (see Fig. 1 right), we obtain the direct estimation of the absolute size of the area $A(B)$:

$$A(B) = rs \cdot \overline{N_{hits}}\,. \tag{4}$$

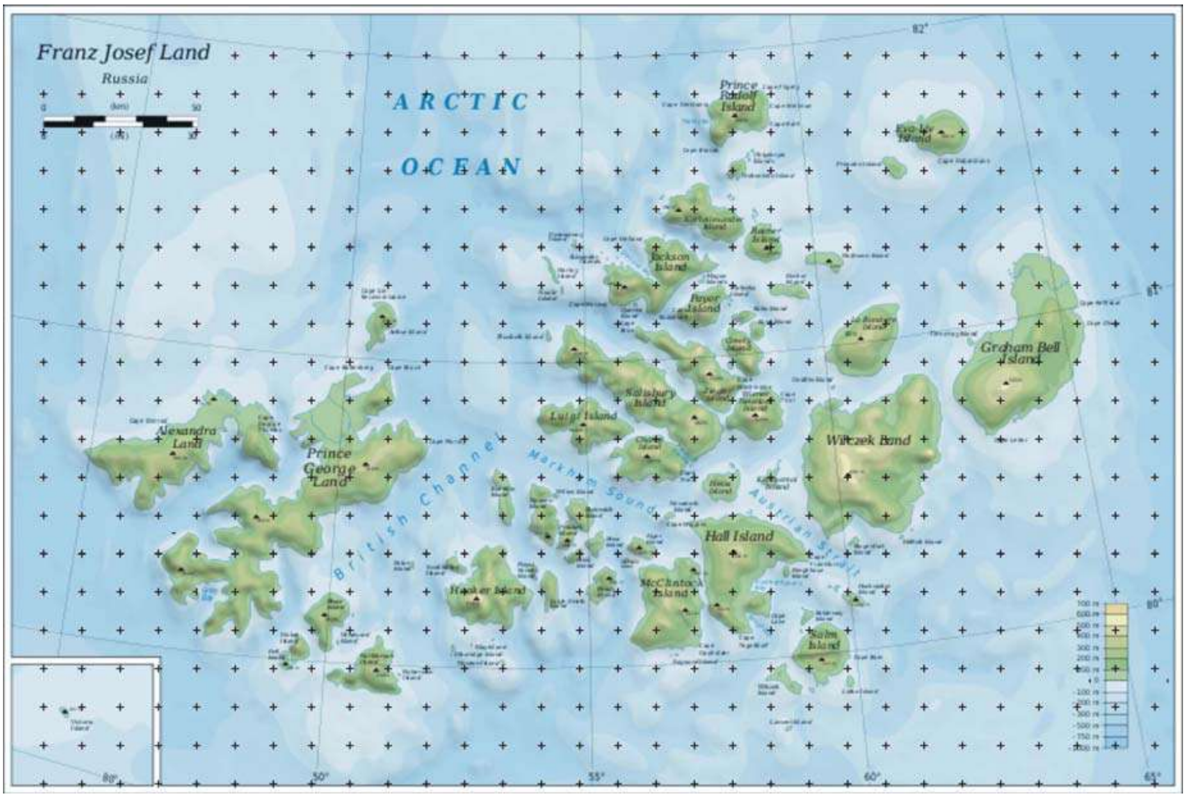

Figure 2: Simple classroom application of the point-counting method[1]

In the classroom, it is possible to test the described "point-counting method" in simple experiments and to compare it with some other possibilities how to estimate an

[1]Map adapted from: http://en.wikipedia.org/wiki/File:Map_of_Franz_Josef_Land-en.svg

unknown area of some complicated region, that can be suggested by students themselves (e.g., using a millimeter graph-paper). It is sufficient to print a point grid on a plastic sheet and place it randomly on the investigated area – for example, a map of a country, lake, group of lakes or islands etc. Several such materials were presented in the workshop and are available at the above mentioned Internet address.

Figure 2 illustrates the estimation of total area of islands named Franz Josef Land (their official land area is 16 134 km^2) by the point-counting method.[2]

3 Points in space – volume estimation

Similar ideas can be repeated in 3D. Probability that a randomly selected point X that hits a set C hits also a subset $B \subseteq C$ was introduced as the ratio of volumes of these sets:[3]

$$P(X \uparrow B \,|\, X \uparrow C) = \frac{\overline{N_{hits}}}{N_{total}} = \frac{V(B)}{V(C)} \,. \tag{5}$$

Consider C to be a prism with the volume $V(C) = rst \cdot N_{total}$ formed by the spatial point grid depicted in Fig. 3 (left) and randomly placed on the set B. Then the volume $V(B)$ can be estimated by

$$V(B) = rst \cdot \overline{N_{hits}} \,. \tag{6}$$

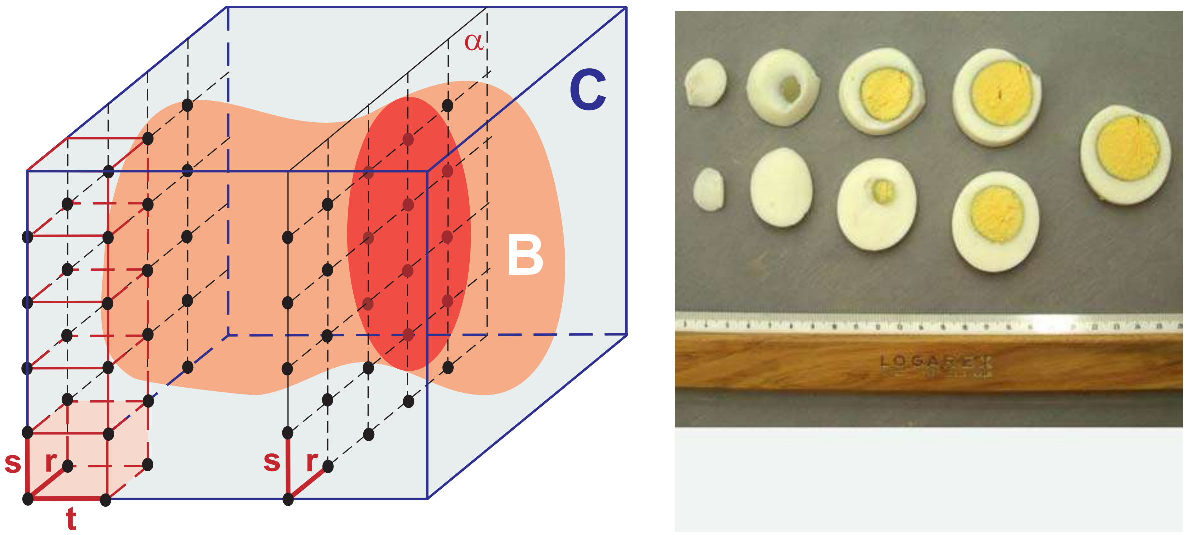

Figure 3: Spatial point grid (left), volume estimation of an egg (right)

In geology (but also in metallurgy or biomedicine) we often search the volume fraction V_V of some phase in the sample. If the phase is homogeneously distributed in the sample, the restriction of the points considered in equation (5) to any of parallel planes

[2]From Fig. 2, we obtain an estimation 16,250 km^2. For accuracy of the point-counting procedure, one of the simplest formulas gives the following total number of lattice points P_T necessary to achieve the desired variance: $P_T = P_P(1 - P_P)/\sigma^2(P_P)$. For more details see e.g. (Underwood, 1970).

[3]Hadwiger theorem proved in (Hadwiger, 1957) implies that the space of "convenient measures" in the formula (1), i.e., measures that are invariant under the group of Euclidean motions, is of dimension $n+1$ in an Euclidean space of dimension n, and volume is only one of them. An accessible exposition can be found in (Klain and Rota, 1997).

formed by the points of the spatial grid (e.g., plane α in Fig. 3 left) gives the same average fraction as the whole grid. That is, the average areal fraction A_A determined on this section is equal to the average volume fraction V_V of the investigated phase in the sample and it is equal to the average fraction of hitting points:[4]

$$P_P = V_V = A_A. \tag{7}$$

The idea of determination of the volume of some object or volume fraction of its specific phase from areal estimations in sections by systematically random parallel planes[5] can easily be illustrated by the estimation of the volume of an egg. Sections by systematically random system of parallel planes are represented by slices made by an egg slicer, into which an egg is randomly placed (see Fig. 3 right). Planar point grids in sections form together a spatial point grid. Volume of an egg can thus be estimated using equation (6), where N_{hits} is calculated in individual selections and added together. The result can be compared with another volume measurement – e.g., by placing the egg into a graduated cylinder with water. Similarly, the volume fraction of a yolk in the egg can be estimated from the estimation of its areal fraction in particular slices. As it was mentioned above, in the case of the spatial homogeneous distribution of the investigated phase, the volume fraction estimation can be based on one or only several sections as a random sample, using the equation (7). It should be stressed that an egg is used only for a better understanding the core of the method, which is especially effective for dealing with finely dispersed phases such as small particles in a rock or a metal, cells in a tissue etc. Formerly the grid was inserted in the eyepiece of the microscope or superimposed on a photomicrograph; in the age of computers, estimations can be done using a convenient image processing software.

At the beginning of the previous section, the example formulated by Isaac Newton was discussed. Another – but far more famous and influential – example is connected with the name of Georges-Louis Leclerc, later Comte de Buffon (1707 – 1788), and it will be discussed in the next section. The systematic development of geometric probability started with the treatises published by Morgan William Crofton (1826 – 1915) who derived the key theorems concerning straight lines in a plane and briefly outlined possible generalization to lines in space (Crofton, 1868 and 1885); this generalization was done in full details by Emanuel Czuber (1851 – 1925), who published the first monograph entirely devoted to geometric probability (Czuber, 1884). Czuber payed a great attention to the definition of geometric probability as a content ratio in $\mathbb{R}^n$; in the special case of $\mathbb{R}^2$ and $\mathbb{R}^3$ in the sense of (2) and (5).[6]

It is rather surprising that even though the above mentioned point-counting method for volume estimation follows straightforward from the definition available in the 1880's, it took half a century before it was discovered by geologists. Let us remark that long before geometric probability was systematically developed, Achille Ernest Oscar Joseph Delesse (1817 – 1881) introduced a mechanical method for the estimation of the volume density of a mineral homogeneously distributed in a rock, based on the measurement

[4] For an exact proof see e.g. (Underwood, 1970) or (Baddeley and Vedel Jensen, 2005).

[5] The first plane is placed randomly into the investigated object and the others are parallel with the constant distance.

[6] For more details on the history of geometric probability see e.g. (Czuber, 1899), (Kalousová, 2008), (Saxl and Hykšová, 2009), (Seneta at al., 2001) and (Todhunter, 1865).

of the area density of the investigated mineral in a random plane section (Delesse, 1847 and 1848). The idea that 3D reconstruction of an object is not necessary for its volume estimation, was indeed revolutionary. Nevertheless, practical implementation of the method was still rather laborious: Delesse proposed to cover the polished, enough large random section with an oiled (and thus transparent) paper, trace exposed portions of particular minerals, using different colors for different phases, glue the paper on a tin foil, cut the mosaic and weight particular groups of the same color; areal fractions of minerals correspond (for a homogeneous foil of constant width) to weight fractions, and as mentioned above, also to desired average volume fractions.

Practical methods for quantitative analysis of rocks continued to develop independently of the theory of geometric probability. From various modifications of Delesse's method, let us mention the contribution of August Rosiwal (1860 – 1923), published half a century later (Rosiwal, 1898). Inspired by Delesse, Rosiwal continued in simplification even further and showed that instead of measuring areas of planar regions, it was sufficient to measure lengths of line segments on a line system superposed with the section. In other words, he proved that the average volume fraction of a mineral is equal to its average area fraction and it is equal to its average linear fraction:

$$V_V = A_A = L_L \,. \tag{8}$$

As it was already mentioned, it took more than another 30 years before the point-counting method was discovered and started to be used by geologists: it was described by Andrej Aleksandrovich Glagolev (1894 – 1969) who proved what can be in today terminology expressed by the formula (see (Glagolev, 1933)):

$$V_V = A_A = P_P = L_L \,. \tag{9}$$

4 Curves in plane – length estimation

Perhaps the most famous example from the field of geometric probability (and often also the only one that secondary school students meet) is Buffon's needle problem presented for the first time at the session of French Académie Royal des Sciences in 1733 (see the report in (Fontenelle, 1735)) and published in full details in (Buffon, 1777). Recall the formulation of the problem: *A slender rod is thrown at random down on a large plane area ruled with equidistant parallel straight lines; one of the players bets that the rod will not cross any of the lines while the other bets that the rod will cross several of them. The odds for these two players are required.* In other words, we search the probability that the rod hits some line.

Denote by d the distance between parallels and ℓ the length of the rod, $\ell < d$. Buffon solved this problem with the help of integral calculus and showed that the hitting probability equals $2\ell/\pi d$. Barbier (1860) provided the solution without the use of integral calculus, which can be loosely described in the following way (for more details, see (Kalousová, 2009) and (Klain and Rota, 1997)). Let N be the number of intersections of a randomly dropped needle of length ℓ with any of the parallel straight lines. If $\ell < d$, it can take only the values 0 and 1, otherwise it can take several integer values. Let p_n denote the probability that the needle hits exactly n parallels, and $E(N)$ the mean number of intersections (or the expectation of the random variable N). We can write

$$E(N) = \sum_{n \geq 0} n p_n \,. \tag{10}$$

For $\ell < d$ we have $E(N) = 0 \cdot p_0 + 1 \cdot p_1 = p_1$, which is the probability we search.

Now consider two needles of lengths ℓ_1 , ℓ_2 , let N_1 , N_2 denote the numbers of their intersections with given parallels. Provided the needles are not bound together, the random variables N_1 and N_2 are independent and the mean value of the total number of intersections can be written as the sum

$$E(N_1 + N_2) = E(N_1) + E(N_2)\,. \tag{11}$$

If the needles are welded together at one of their endpoints, N_1 and N_2 are no more independent; nevertheless, the expectation remains additive and the equation (11) holds again. Similarly for m needles bound together to form a polygonal line of arbitrary shape: $E(N_1 + N_2 + \cdots + N_m) = E(N_1) + E(N_2) + \cdots + E(N_m)$.

Since the mean number of intersections $E(N_i)$ evidently depends on the needle length ℓ_i, we can write $E(N_i) = f(\ell_i)$. Equation (11) gives

$$f(\ell_1 + \ell_2) = f(\ell_1) + f(\ell_2) \tag{12}$$

for any $\ell_1, \ell_2 \in \mathbb{R}$. The function $f(\ell)$ is thus linear,[7] i.e., there exists a constant $r \in \mathbb{R}$ such that

$$f(\ell) = r\ell \quad \text{for all } \ell \in \mathbb{R}\,. \tag{13}$$

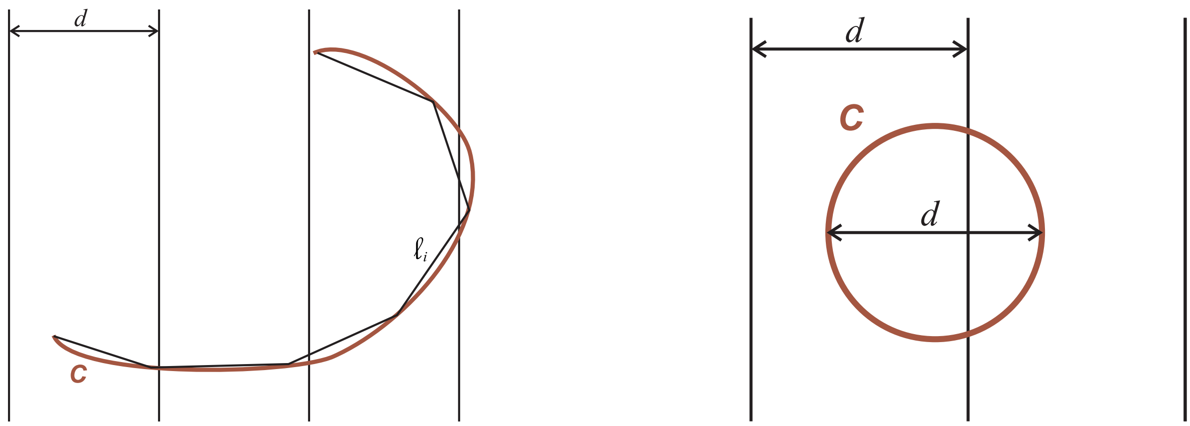

Figure 4: Buffon's needle problem

Now any planar curve C of the length ℓ can be approximated by polygons (Fig. 4 left); passing to the limit, we obtain the mean number of intersections to be again $E(X) = r\ell$. Thus, to determine the value of the constant r, it is sufficient to consider one curve of a convenient shape. Let us simply choose a circle of diameter d (Fig. 4 right) which intersects the system of parallels always in two points, so that $2 = r\pi d$, i.e., $r = 2/(\pi d)$. For the curve of any length ℓ we thus have

$$E(N) = \frac{2\ell}{\pi d} \tag{14}$$

[7]Equation (13) implies $f(0) = f(0{+}0) = f(0){+}f(0)$, so that $f(0) = 0$; $f(2) = f(1{+}1) = f(1){+}f(1)$, similarly for any natural $s, t : f(s) = s \cdot f(1)$, $f(1) = t \cdot f(1/t)$, thus $f(s/t) = f(s)/f(t)$. In other words, f restricted to rational values of ℓ is linear. Since f is a monotonically increasing function with respect to ℓ, we infer that the equation (13) holds for all $\ell \in \mathbb{R}$.

and for a needle with the length $\ell < d$ this equation gives directly the searched hitting probability:

$$E(N) = p_1 = \frac{2\ell}{\pi d}\,. \tag{15}$$

In the school mathematics, the only "application" of Buffon's needle problem is usually the estimation of the number π. Nevertheless, even a non-excellent student can realize that more effective and more exact methods exist for this aim. What is much more important, is another application: in many cases, it is too complicated to measure exactly the length of some curve or a system of curves (e.g., a vascular bundle, a watercourse, cell boundaries in a tissue section etc.); we can "throw" a system of parallels on its picture, calculate the number of intersections N and use (14) for the estimation of the unknown length:

$$[\ell] = \frac{\pi d}{2} \cdot N\,. \tag{16}$$

In today terminology, the considered system of parallels represents a test system of lines with the length intensity (the mean length in a unit area) $L_A = 1/d$ (see Fig. 5 left). Using this notation, we can write (16) in the form:

$$[\ell] = \frac{\pi}{2} \cdot \frac{1}{L_A} \cdot N\,. \tag{17}$$

Dividing both sides by the area A of the region covering the investigated curve, and denoting by $N_L = N/(AL_A)$ the number of intersections for a unit length of the test system, we obtain one of the fundamental formulas of stereology, which provides an unbiased estimator of the length intensity $\ell_A = \ell/A$:

$$[\ell_A] = \frac{\pi}{2} \cdot N_L\,. \tag{18}$$

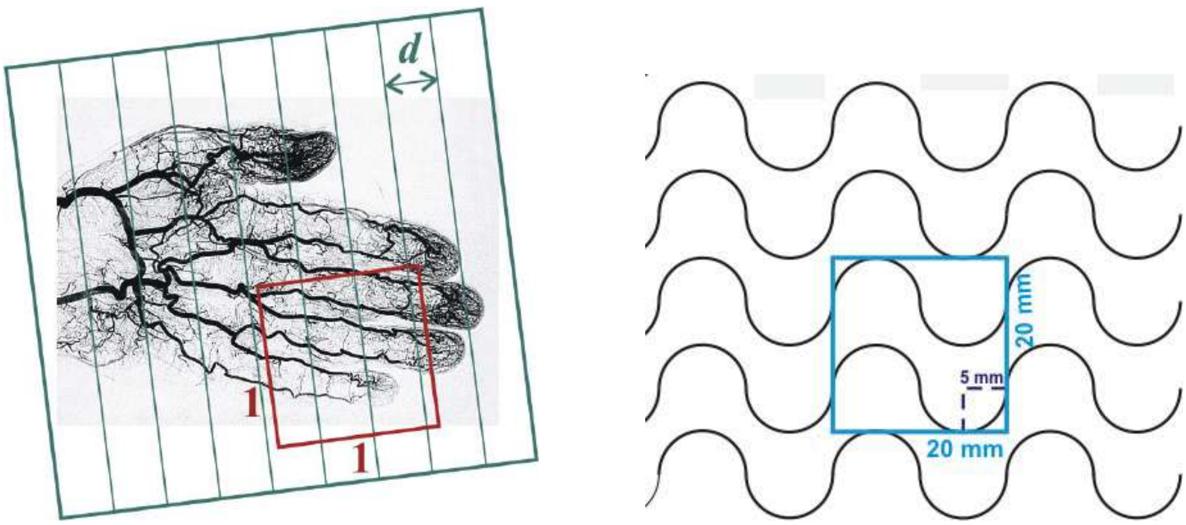

Figure 5: Length estimation based on the needle problem

The equation (18) holds even in the case that the test system is formed not by parallels, but by a system of curves (Fig. 5 right) or a unique curve of constant length intensity.

For students, length estimation is a much more convincing and real application of the needle problem, and they can try to implement it themselves, using for example a map of a river or a highway, as shown in Fig. 6. They can use plastic sheets with parallels and with the system of isotropic curves (Fig. 5 right) and discuss which one is more convenient in this case and why.

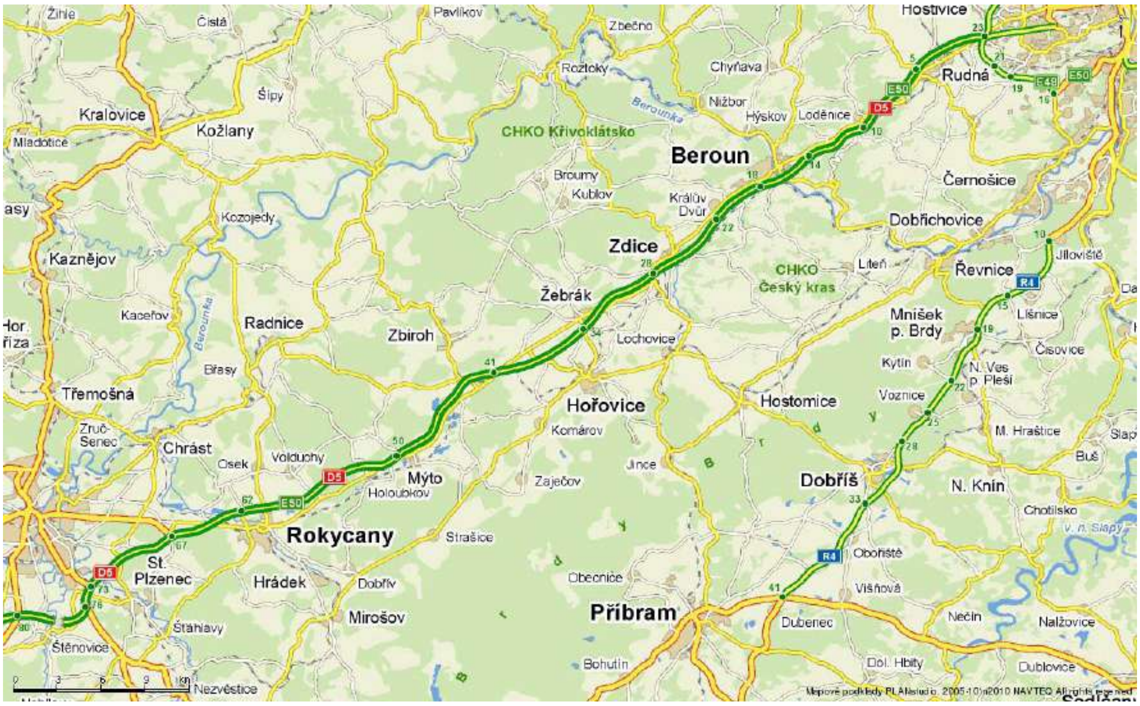

Figure 6: Estimation of the length of River Berounka[8]

After its publication, Buffon's problem remained unnoticed till the beginning of the 19$^{\text{th}}$ century. Without a reference to its author, the needle problem was mentioned by Pierre-Simon de Laplace (1749 – 1827), who proposed to use it for estimating the length of curves and area of surfaces; nevertheless, he gave only one example, namely the circumference of a unit circle (Laplace, 1812). Since then, other mathematicians introduced some generalizations; for example, Isaac Todhunter (1820 – 1884) considered a rod of the length rd, a cube and an ellipse (Todhunter, 1857), later also a closed curve without singular points (Todhunter, 1862). Gabriel Lamé (1795 – 1870) included Buffon's needle problem and its generalizations to a circle, an ellipse and regular polygons in his lectures held at École normale supérieure. Inspired by these lectures, Joseph-Émile Barbier (1839 – 1889) published the general theorem concerning the mean number of intersections of an arbitrary curve with the system of parallels (14), and what is indeed remarkable, he replaced equidistant parallels by an arbitrary system of lines and even by a unique curve of constant length intensity (in his words, the total length of the line system in each square metre of the plain was the same) and came to the estimator (18). And what is even more remarkable, he continued in generalizing further to 3D and formulated three more theorems that express other today fundamental stereological formulas used for the estimation of surface area or curve length; for more details and Barbier's original formulations see (Kalousová, 2009).

[8] Map source: © PLANstudio, spol. s r.o.

5 Lines in plane – size estimation

Let us finally mention Crofton-Cauchy formula explicitly derived in (Crofton, 1868): *The mean breadth of any convex area is equal to the diameter of a circle whose circumference equals the length of the boundary.* In other words: let b denote the breadth of a convex figure X in $\mathbb{R}^2$, that is, the projection of X into the normal to the given direction (see Fig. 7 left), and let L denote its perimeter; using Cauchy theorem, Crofton proved the following formula for the mean projection length of X into the isotropic bundle of directions:

$$w = \frac{1}{\pi}\int_0^\pi b d\varphi = \frac{L}{\pi}, \qquad \text{thus} \qquad L = \pi w\,. \tag{19}$$

Notice that for a circle, the breadth in any direction is $2r$, so that (19) gives the usual formula for the circle perimeter $L = 2\pi r$. What is remarkable is the fact that also for any other convex figure, the perimeter can be simply computed by multiplying the mean breadth by the number π. On the other hand, we can calculate for example the mean breadth of a square with the side a, which is $w = 4a/\pi$.

Now consider a system of parallels with $d > w$ "thrown" on X. The hitting probability is equal to $L/(\pi d)$ [9] and using (19), we can write $p = w/d$. Repeated throwing of test lines on the given figure and counting the fraction of successful hits thus provides the estimation of the mean breadth w. For example, students can try to check whether the Emmenthal cheese indeed contains the eyes of a proper size (Fig. 7 right).

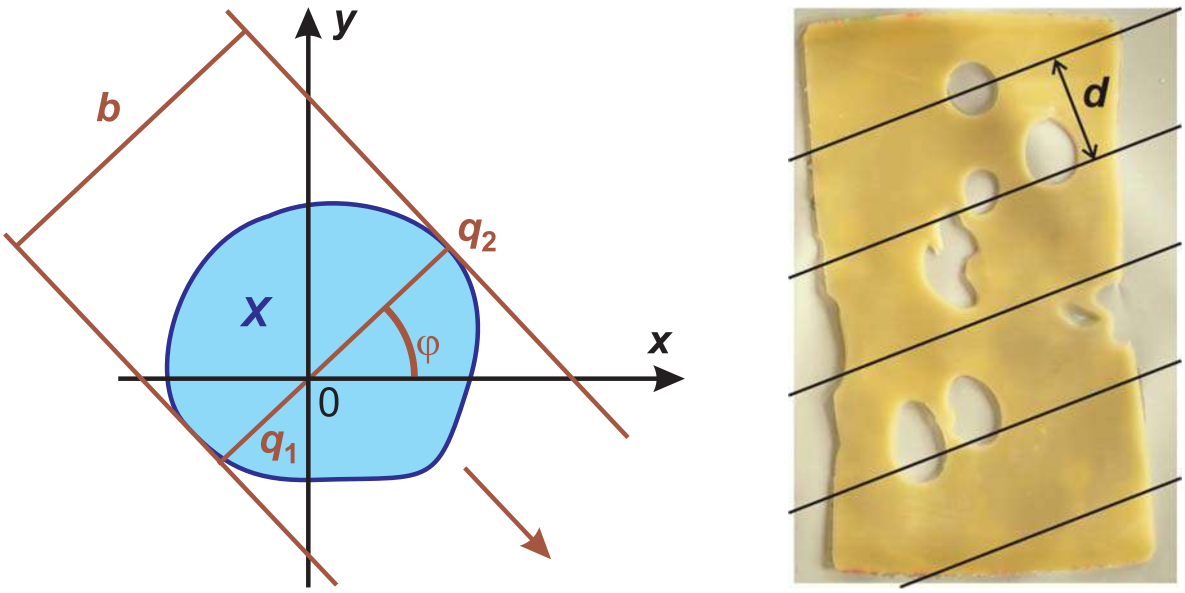

Figure 7: Mean breadth estimation

[9]For a circle with the radius $r < d/2$, the hitting probability equals

$$\frac{2r}{d} = \frac{2\pi r}{\pi d} = \frac{L}{\pi d}\,.$$

Using (14), we can derive the same formula for any convex figure with the perimeter L.

Let us finally remark that Czuber (1884) proved the spatial analogy of Crofton-Cauchy formula and proved that for any convex body, the quadruple of the mean projection area is equal to the surface of the body. Thus, for a sphere we have the usual formula $S = 4\pi r^2$; for a cube we can determine the mean projection area to be $6a^2/4$.

6 Conclusion

The aim of the paper was to awaken an interest in the part of probability theory that is usually neglected in the school mathematics, although its importance and the number of applications has substantially increased since the beginning of the 20$^{\text{th}}$ century when examples from this field were even included among exercises for the leaving examination. Geometric probability can help teachers to answer convincingly the usual students' question (whether in the connection with probability or mathematics in general): "What is it for?" Here they can observe important applications in biomedicine (dermatology, nephrology, oncology, cardiology etc.), material engineering (quantitative analysis of metals, composites, concrete, ceramic), geology etc.

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THE TEACHING OF MATHEMATICS MEDIATED BY THE HISTORY OF MATHEMATICS

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ABSTRACT

The aim of this paper is to present issues related to a teaching exploitation methodology based on the History of Mathematics. These issues are related to the apprehension of mathematical concepts involved in a teaching sequence designed from the History of Mathematics. We explored the teaching sequence with in-service teachers and future mathematics teachers. They were challenged to find a general formula for quadratic equation solution using the related sequence. Upon short courses completion we verified participants' responsiveness and availability to work with such sequence. We found difficulties in associating an irrational number with a measure of one square side and also difficulties in mathematical problem proposing. We confirmed the need to complement the referred sequence with extra reading material, specific recommendations, as well as proposing tasks to support teachers in their work with the sequence in their own classrooms. This study is part of an ongoing work and aims to bring contributions to the debate about the effectiveness of the History of Mathematics in Mathematics Education.

1 Introduction

In related studies on the History of Mathematics and Mathematics Education there is a variety of approaches. According to what was observed by Siu (2007), most of these studies discuss the role and the importance of the History of Mathematics in teaching and learning process. However, few studies discuss that the History of Mathematics contributes, in fact, for mathematics teaching. This instigated us to investigate the History of Mathematics effectiveness in Mathematics teaching. In this study the term effectiveness refers to our intention to clarify if the history of mathematics is a good ally (or not) to mathematics teachers in the performance of his professional actions.

Some researchers (Kjeldsen, 2010; Kjeldsen and Blomhøj, 2009) have published articles that show ways to integrate the History of Mathematics in Mathematics Education in undergraduate courses. These articles present different integration aspects and show how it leads to an improvement in Mathematics learning. Our study, however, focuses on Mathematics teaching.

To innovate education, many teachers are looking for a special support. This is the case of introducing History of Mathematics in teaching process, as shown in the article written by Siu (2007). Based on obtained data from 360 mathematics teachers in 41 schools, it shows that teachers consider important to use the History of Mathematics in classroom. However, these teachers say they do not use this feature.

According to the data obtained by Siu (2007), from 608 issued responses by in-service math teachers and future teachers, there are at least 15 reasons for this. Among these reasons, some are stronger (and other less), justified by the respondents. In this article we focus on only two of them:

1. The lack of material resources, mentioned by 64.47% of respondents.
2. The lack of teacher's preparation, mentioned by 82.89% of respondents.

Our many years experience, working with in-service and future math teachers show that obtained data by Siu (2007) correspond to our country reality. We believe that the arguments used to justify non-using the History of Mathematics is presented as a problem that requires investigation.

These data reinforce the importance of continuing our study whose aim is to bring contributions to the debate about the effectiveness of the History of Mathematics in Mathematics Education. Our study is divided into five parts:

1. To investigate teacher's responsiveness and availability to deal with a *method based on history*;
2. To provide historical documentation for in-service teachers use in the classroom;
3. To predispose future teachers to use historical materials in the classroom;
4. To observe the use of historical materials in the classroom;
5. To verify the History of Mathematics efficacy in Mathematics Education.

The term *method based on history* is here used in the perspective pointed by Jankvist (2009). The *method* refers to a teaching approach inspired on the History of Mathematics.

The present article discusses the responsiveness and availability of the participants regarding to the use of the referred *method* which approach the History of Mathematics in an indirect way. Here we present some results that led us to a first reflection on the effectiveness of the History of Mathematics in Mathematics Education.

We have taken as reference a prepared material by Radford and Guérette (2000), whose title is "*Second Degree Equations in classroom: the Babylonian approach*". The authors develop and utilize a teaching sequence based on Babylon mathematics, whose purpose is to lead students to reinvent the formula that solves a general quadratic equation.

Analyzing the referred teaching sequence we asked ourselves: the in service teachers and future teachers are prepared to use this sequence in their classrooms or we need to modify it somehow? To answer this question, we came up to short courses specially designed for these teachers. In these courses we challenged participants to find the formula for quadratic equation solution by using the sequence. Our initial goal was to assess the responsiveness of the participants and their willingness to work with such sequence in the classroom.

2 Concerns about using the History of Mathematics

We considered relevant to refer to the theoretical approach that involves the discussion of *whys* and *hows* to use the History of Mathematics in teaching and learning processes. We also considered relevant to refer to the arguments that support this type of the History of Mathematics use in Mathematics Education.

A study developed by Jankvist (2007) highlights the importance of conducting empirical research involving the History of Mathematics, so that positive experiences reached by a teacher, a classroom or a school can be transferred and shared. Considering this, it is possible to be clear about the arguments that support such History of Mathematics use.

In another study Jankvist (2009) proposes a way to organize and structure the discussion of *why* and *how* to use the History of Mathematics in mathematics teaching and learning. To organize and structure this discussion he develops a deep and wide research, through which identifies two categories: the arguments for the use of the History of Mathematics (the *whys*) and the different ways to make use of history (the *hows*).

The first category (the *whys*) is subdivided into two forms: history as a *tool*, which involves mathematics internal learning issues and history as a *goal*, which involves history use as a purpose contained within it.

The second category (the *hows*) is subdivided into three different ways to make use of history: the *illumination* approaches, based on motivational and affective arguments; the *modules* approaches, which builds on cognitive arguments; and the *history-based* approaches to makes reference to arguments of their own evolutionary History of Mathematics.

Besides that, Jankvist (2009) states that interrelationships knowledge between the *whys* and *hows* may become easier to analyze education material that makes use of the History of Mathematics, to see if it meets certain requirements and goals.

Longing for a theoretical and general exposure Jankvist (2009) explores a central point in his analysis: he separates strictly, the categorizations of the *whys* and *hows*. This is due to the fact that by making a literature review this author finds an image "opaque" in *whys* and *hows* of discussion. Given that he seeks such categorization.

It is important to bear in mind that the categorizations of the *whys* and *hows* are not so absolute. They serve the following purposes: create conditions for a real analysis of the History of Mathematics in Mathematics Education.

Taking as reference this theoretical approach, we observed that the knowledge of the interrelationships among the *whys* and *hows* can be used in decision making about the content, presentation and organization form of the History of Mathematics uses in Mathematics Education, as well as the materials to be used by both teachers and students.

Ahead of this, we decided to start our own investigation with reference to this theoretical approach and teaching sequence, inspired by the History of Mathematics prepared by Radford and Guérette (2000). The choice was a function of its form and organization of the History of Mathematics and materials used by these authors.

Although this sequence was originally used by its authors, in basic education students, it did not present as an obstacle to our purposes. We worked with in-service teachers and future mathematics teachers, seeking to clarify issues related to their responsiveness and availability to work with a teaching methodology based on the History of Mathematics.

Below we present the result of the teaching sequence that came out to meet the requirements and purposes of our study.

3 The teaching sequence

Regarding the integration of history in mathematics teaching, there are studies indicating that the sequences inspired by the History of Mathematics can be used to teach some math concepts. This is the basic idea behind the article "*Second Degree Equations in classroom: the Babylonian approach*" Radford and Guérette (2000). These authors developed a teaching sequence aimed to lead students to reinvent the formula that solves a general quadratic equation.

The sequence takes as a reference a problems solving method that uses geometrical figures (represented on paper) that can be cut, moved and pasted. We call this *cut-and-paste geometry* method. Jens Høyrup (1990a, 1990b) was the first to suggest that Babylonians scribes devised this method to solve geometric problems and called it *naive geometry*.

It is not our intention to discuss here the method itself, since it is adequately addressed in Radford and Guérette paper (2000). We are interested in presenting a summary of the teaching sequence. Here, the sequence will be presented in five parts, as it appears in the original article.

Part 1. Naive geometry introduction

The goal of the first part of the sequence is to present *naive geometry* to students. To achieve this goal, students will work in cooperative groups to complete three tasks, which are discussed below by steps:

Step 1.1

The teacher asks the students to solve the problem below using any chosen method they want.

What are the dimensions of a rectangle whose semi-perimeter is 20 and whose area is 96 square units? (Problem 1A).

Students are encouraged to solve the problem using any method. After completing the task the teacher returns to geometric context and uses geometric figures represented on paper, placed on the blackboard. Doing this, he shows students the technique of the *naive geometry*.

Step 1.2

Once presented the technique, the teacher explores similar problems. For example, the following problem could be studied:

What are the dimensions of a rectangle whose semi-perimeter is 12 and whose area is 30 square units? (Problem 1B).

Step 1.3

To help students achieve a better understanding, the teacher asks them to bring a description (written for next day class) of the steps that must be followed in solving these problems. The description should be clear enough to be understood by any student from any other class, of the same year.

Part 2. Written text discussion and new problems formulation

Step 2.1

This part begins with a discussion of the requested descriptions. Working in cooperative groups, students should compare the descriptions given by them and reach an agreement on the points that could cause conflict or that could allow a better understanding. When all group members reach an agreement, the teacher chooses one student from each group to present their findings to the other groups.

Step 2.2

Next, students are encouraged to develop problems considering the following restriction: the both sides of the rectangle should be expressed by integers. Then, as a second exercise, the rectangle sides not necessarily need to be expressed by integers. Thus students are challenged to find answers containing rational numbers.

Part 3. A different use for *naive geometry*

Step 3.1

In Part 3 is presented to the students a problem that requires another *naive geometry* approach. The problem is the following:

> *The rectangle length is 10 units. Its width is unknown. We place a square on one side of the rectangle, as shown below. Together, the two figures have an area of 39 square units. What is the given rectangle width? (Problem 2).*

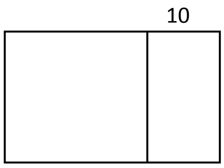

FIGURE 1

The teacher asks students to solve the problem 2 with similar ideas to those used to solve problem 1. If the students fail, the teacher can introduce a new solving problems method. Here the procedure used was "*complete the square*" which is detailed presented in Radford and Guérette (2000).

Briefly the procedure is as follows: using large cardboard figures, placed on the blackboard, the teacher cuts vertically the given rectangle (length 10, see Figure 1) by dividing it into two equal parts, then grabs one of the pieces and places it on the square base, represented in Figure 1. Students should realize that the geometric resulting shape is almost a square. Then the teacher points out that this geometric form can be completed in order to become a new square.

Step 3.2

Then, similar problems are presented to the students to be solved in group.

Step 3.3

As in Part 1, students are asked to prepare a written description containing the steps that must be followed in solving this kind of problem.

Part 4. Discussing procedures and proposing new problems

Step 4.1

As in Part 2, we start a discussion about the written descriptions prepared by the students that contains all necessary steps to solve problems explored in Part 3.

Step 4.2

Then the teacher asks them to propose some problems (analogous to problem 2) involving a specific condition to the rectangle sides:

(i) the rectangle sides must be expressed in integers numbers,
(ii) the rectangle sides must be expressed in rational numbers,
(iii) the rectangle sides must be expressed in irrational numbers.

Part 5. Finding the formula

In this part students think about problems similar to those discussed in Parts 3 and 4. However, it is not assigned a specific number for the rectangle base nor to the area of the two given figures. The aim is to encourage students to discover the formula that solves quadratic equations.

Step 5.1

The teacher explains to the students that the interest now is to find a formula that provides an answer to the problems explored in Parts 3 and 4. The teacher can suggest that they take as reference the written procedure in part 4 and that they use letters instead of words. To facilitate the formulas comparison developed by students, in a next phase, the teacher may suggest the use of the letter "***b***" to the rectangle's base and "***c***" to square and rectangle's area together (see Figure 2):

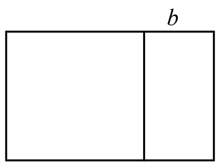

FIGURE 2

The obtained expressions are discussed in cooperative groups. The formula obtained is:

$$x = \sqrt{c + \left(\frac{b}{2}\right)^2} - \frac{b}{2}$$

Step 5.2

In this step the teacher can use procedures to implement geometrical problems to the algebraic language: if the side of the square is 'x', then its area is equal to x^2 and the rectangle area is bx, so the sum of both areas is equal 'c', then $x^2 + bx = c$. Now, in order to relate the equations with the formula, the teacher presents some real equations (such as $x^2 + 8x = 9$, $x^2 + 15x = 75$) and ask students to solve them using the obtained formula.

Step 5.3

In this step we consider the equation $ax^2 + bx = c$ and asks the students to find a formula to solve this equation. They should note that if this equation is divided by a (let's assume $a \neq 0$) we obtain the first term of the previous equation. You must then replace the 'b' for 'b/a' and '**c**' for 'c/a', in the previous formula, coming to the new formula:

$$x = \sqrt{\frac{c}{a} + \left(\frac{b}{2a}\right)^2} - \frac{b}{2a}.$$

Step 5.4

The last step is to consider the general equation $ax^2 + bx + c = 0$ and find the formula that solves it. The formal relationship with the previous equation $ax^2 + bx = c$ is clear: we can rewrite this equation as $ax^2 + bx - c = 0$. Thus, to obtain the equation $ax^2 + bx + c = 0$ we must substitute '***c***' for '**-** ***c***' and do the same in the formula. By doing this we get:

$$x = \sqrt{\frac{-c}{a} + \left(\frac{b}{2a}\right)^2} - 2a$$

Naturally, this formula is equivalent as to the well-known formula:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

For all numerical solutions we must consider the negative square root of $b^2 - 4ac$. This leads to the general formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

4 Working with math teachers

In search of evidence of responsiveness and availability from teachers in relation to the uses of a teaching sequence, based on the History of Mathematics, we offered short courses. The courses content took as reference the sequence developed by Radford and Guérette (2000). Besides that, we offered participants some basic historical information.

The first course was designed for in-service teachers and future teachers, all together. After that, courses were offered for in-service teachers and future teachers, separately, in order to observe any differences that might exist between the two groups. Each course lasted for a different period of time due to limitations in schedules of the participants. The courses were developed in periods of 4, 5 or 6 hours.

In all the courses we've attended in the classroom. During the course we observe the participants performance in attempt to overcome the challenges they were proposed to. Also took note on the teaching sequence steps that brought difficulties to the participants.

At the end of the course we asked them to answer several questions in order to consider how the participants assess the possibility of using the sequence in the classroom. The questions were:

a) Is it feasible to use this result at school?
b) Do you see yourself applying this sequence at school?
c) At what moment do you think the students would face difficulties?
d) What sequence part(s) would be more difficult for students?
e) It would require some adjustment? Which ones?
f) The History of Mathematics should or should not, appear explicitly when working with the teaching sequence?

The table below shows the offered courses. It makes focus on the date, duration and number of participants in each course:

Course	Date / Duration	Participants
1	October 2009 4 classes of 90 minuts duration	14 in-service mathematics teachers 23 future mathematics teacher
2	February 2010 2 classes of 3 hours duration	20 in-service mathematics teachers
3	April 2010 3 classes of 100 minuts duration	15 future mathematics teacher
4	April 2010 3 classes of 100 minuts duration	14 future mathematics teacher
5	April 2010 2 classes of 2 hours duration	4 in-service mathematics teacher in middle school, at the same time, post-graduation students
6	May 2010 2 classes of 2 hours and 30 minuts duration	14 in-service mathematics teachers

TABLE 1

5 Some results discussion

The collected data indicate the difficulties that the participants had to perform the proposed tasks and are related to how they judge the teaching sequence itself. We will highlight, in the next subsection, the main difficulties faced by teachers.

5.1 Difficulties in understanding the sequence during the course

Firstly we observed that the difficulties faced by in-service and future mathematics teachers are similar. The problems differ from one course to another only in intensity; therefore they are of the same nature. Below we discuss the most notable difficulties.

On the first part of the sequence the difficulties appeared in step 1.2. In seeking problem 1B solution, using the *naive geometry*, you must have a square whose side measure is equal to six and cut from this figure, another square whose side is equal to $\sqrt{6}$. In all courses the participants had some difficulties in obtaining the solution. In some cases, the researchers had to intervene by providing appropriate questions in order to instigate the solution searching. However, there were cases where this difficulty seems to have been more problematic than previously thought. One of the researchers, after provide the number 3 and 4 courses, wrote in his journal field:

> *"Most participants sought an integer to the square side. When they realized that they could start from a square with side equal to 6 and from this square they could remove a small square of area equal 6, they found out that the side of the square should be equal to*$\sqrt{6}$ *. However, some of them did not accept this as a truth value. Even after verifying that this result was consistent with problem data, some students did not accept this value to one square side, because they thought the square side should be linked only to an integer. At the beginning of activity at any of the groups, only one component tried to solve the problem using* $\sqrt{6}$*, but when*

encountered resistance from other group members, he stopped working with that number.

Another moment that special attention was demanded from the researchers was linked to step 2.2. At this time we asked the participants to prepare two problems. First: the rectangle sides should be expressed by integers. Second: the rectangle sides should *not* necessarily be expressed by integers.

They were instructed to write the problems on a paper sheet and hand over to the investigators. Upon receipt the problems the researchers mixed the sheets and distributed to the participants, so each group received the developed problem by another group. Then each group had to correct the statement (if necessary) and solve the proposed problem.

In accomplishing this task, it became clear that to propose a mathematical problem of this nature is not a simple task. We observed that many of the produced problems could not be solved without a proper correction in the statement. Furthermore, we observed that only some of the participants proposed problems whose answer would be an irrational number.

Another difficulty could be observed in the development of steps 5.2 and 5.3. In these steps the participants had to translate a problem of geometric field to algebraic field and solve it. Although it was possible to make use of geometric figures, we found that most participants had difficulty to perform that transposition. In all courses, the participants could only solve the proposed problems with the researchers support. It was observed that join the side of a geometric figure to a generic value (a symbol) is not immediate. Furthermore, the detachment of the geometric representation of a concept and its algebraic representation is not something that happens naturally, which requires special care by the teacher.

5.2 Teaching sequence assessment

In general, participants considered important to work with the teaching sequence. They emphasize that problems solution through geometry cut-and-past is important because it would allow students a better understanding of certain mathematical concepts. However, in-service teachers added that, for them to use this new approach in schools, it would be necessary to make some changes and overcome some difficulties, such as:

a) To work with cut-and-paste geometry n schools would be better to work with, scissors, paper or cardboard instead of animated slides as used in these courses;
b) before applying the sequence at school, they need to restore, in more detail, some basic concepts, to gain a deeper understanding of the most important points;
c) they must obtain school permission to use a new approach;
d) they must find a way to overcome students resistance, who consider such activities as not appropriate for a math class;
e) they have to find extra time to prepare lessons to use the new method;
f) they must find an extra reading material to help preparing the lessons;
g) they must find a way to overcome the student algebraic weaknesses.

In relation to "c item" above, was observed that this is a relevant issue for private schools teachers. There is evidence that these teachers have little autonomy to innovate mathematics teaching. However, it seems that teachers who work in public schools have more autonomy to do so.

Another issue that we considered important is the participant’s opinion on the History of Mathematics presentation in mathematics classes. Most participants felt that the Babylonian mathematics history should be fully displayed. However, we tried to make clear that teaching sequence was inspired by the history of Babylonian mathematics, but the uses of the sequence does not, necessarily, includes a full resumption of the Babylonian mathematics history.

With intent to observe the effectiveness of the History of Mathematics in Mathematics Education, we changed the order of the provided content for the courses. In course 1, we started with a discussion of Babylonian mathematics. After that, we explored the teaching sequence. In courses 2-6 we started with the sequence implementation and only introduced the discussion of Babylonian mathematics at the end.

At the end of each course we required an appraisal of the teachers on the following question: "The History of Mathematics should or should not appear explicitly in teaching sequence work?”. In the debate on this issue was clear that the use of the teaching sequence does not necessarily presuppose the introduction of historical issues. However, they insisted that the introduction of historical themes would be relevant to capture and maintain students' interest in math classes.

6 Implications in this investigation

Firstly we would like to emphasize that the future mathematics teachers, as well as in-service teachers, are our main partners in this research. So to continue our research project, we must take into account the views, wishes and recommendations by them expressed.

So our next step will be "*rewriting*" the teaching sequence, integrating the recommendations resulting from our previous work. We do not intend to modify the original sequence. However, those points, that brought difficulties to the participants, we intend to supplement with extra reading material, propose specific recommendations or tasks with the aim of supporting teachers in working with the sequence in their own classrooms.

For example, we intend to add to the original sequence, a text about the history of Babylonia mathematics, to be written especially for teachers. Our experience indicates that many teachers have not had the opportunity to attend any courses on the History of Mathematics during graduation. So if they are willing to explore the History of Mathematics related to this particular topic in their classrooms, we must provide them with appropriate reading materials.

Once concluded these adjustments we will come back to our work within-service mathematics teachers. We will invite them to take the course again. After that, we will ask them to apply the same sequence in their classrooms and evaluate the obtained data with us. Thus, we expect to have more elements to examine the question of effective use of the History of Mathematics in a classroom.

Moreover, considering theoretical reference, we note that in this study, participants conceive using history as an *illumination* approach, which relies on motivational and emotional arguments. This conception is linked to the *hows* category of using history. Thus, in reciprocity with what was proposed by Jankvist (2009) we believe that a discussion on history use can be beneficial for creating a new platform through which becomes possible the discussion of the History of Mathematics potential in Mathematics Education.

To these authors, the interest in the History of Mathematics use on teaching and learning should be based on a systematic and organized theoretical framework that takes the following questions into consideration:

- ✓ *Why can / should history be used for mathematics teaching and learning?*
- ✓ *How can and should history be used for mathematics teaching and learning?*
- ✓ *In what ways the arguments for history use of and the used approaches (different ways) are inter-related?*

With that in mind, on our study continuity, we intend to make a contribution in order to elucidate the following question: *is there any empirical evidence that teachers teach best when they make use of the History of Mathematics in the classroom*? We believe that this contribution may bring more elements to the debate about the effectiveness of the History of Mathematics in Mathematics Education.

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HISTORICAL METHODS FOR MULTIPLICATION

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ABSTRACT

This paper summarizes the contents of our workshop. In this workshop, we presented and discussed the "Greek" multiplication, given by Eutokios of Ascalon in his commentary on The Measurement of a Circle. We discussed part of the text from the treatise of Eutokios. Our basic thesis is that we think that this historical method for multiplication is part of the algorithms friendly to the user (based on the ideas that the children use in their informal mental strategies). The important idea is that the place value of numbers is maintained and the students act with quantities and not with isolated symbols as it happens with the classic algorithm. This helps students to control their thought at every stage of calculation. We also discussed the Russian method and the method by the cross (basically the same as "Casting out nines") to control the execution of the operations.

1 Theoretical basis

During the University studies for the future teachers of mathematics of Primary (but also of Secondary) education it is very important to develop a multidimensional scientific-mathematical culture (Kaldrymidou & als, 1991). The dimensions to work on could be:

- The knowledge, which includes the global apprenticing of mathematical notions and theories, one approach of school mathematics and in parallel one pedagogical and psychological approach of mathematics education,
- The knowledge about the knowledge, the understanding of the role, of the dynamic and the nature of mathematics,
- The knowledge for the action, the theoretical context of the organization and the approach of school mathematics and via this the evaluation of the work in the classes,
- The action, the praxis and the experience of mathematics education.

The set of these dimensions determine the base, which can help and train the future teachers. For training but also for teaching Mathematics everything should be constructed. In the next pages we should try to make an approach for the following question: Which shall be the initial and the continued training of teachers of primary (but also of Secondary) Education for supporting the introduction of a cultural, historical and epistemological dimension for the teaching of mathematics?

The main methodological issues of our work with our students were based in the following thesis (Arcavi & als, 1982, 1987, 2000; Bruckheimer & al, 2000):

- Active participation (learning should be achieved by doing and communicating),
- Conceptual history (evolution of a concept, different mathematical traditions, difficulties, etc.),
- Relevance (with the curriculum),
- Primary sources and secondary sources using primary sources,
- Using of worksheets
- Implementation

The purposes of this introduction of the historical and cultural dimension in the training and teaching of mathematics are various and are fixed more or less in the following (Fauvel & van Maanen, 2000):

- to humanize mathematics,
- to put mathematical knowledge in the context of a culture,
- to give students the opportunity to change their beliefs for the subject,
- to find and analyze epistemological obstacles and notions that are not very well understood by the teachers and consequently by the students,
- to show that mathematics has a history and was influenced by cultural and social parameters,
- to be another interdisciplinary project that could be studied with the students,
- to develop and enrich mathematical knowledge included in the curriculum.

The ways and strategies used for this introduction in the training and teaching of the subject are: histories, construction of activities, construction of exercises, reproduction of manuscripts, portraits, biographies, interdisciplinary projects, use of primary sources, use of new technologies etc.

2 The "Greek" Multiplication

In the following we are going to discuss the way we have worked the multiplication in an historical perspective by discussing the "Greek" multiplication.

For Biographical elements and descriptions about the treatises of Eutokios you can see Nikolantonakis, K., (2009), History of Mathematics for Primary School Teachers' Training, *Ganita Bharati*, Vol. 30, No 2, Page 181-194 but also Heath T. L., (1912). *History of Greek Mathematics*, Vol. I & II, Oxford.

To do the workshop you will need the following equivalences between the Greek alphabetical number system and our modern arithmetical system.

Units (1 to 9)	$\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\epsilon}, \bar{\varsigma}, \bar{\zeta}, \bar{\eta}, \bar{\theta}$
Tens (10 to 90)	$\bar{\iota}, \bar{\kappa}, \bar{\lambda}, \bar{\mu}, \bar{\nu}, \bar{\xi}, \bar{o}, \bar{\pi}$, ϙ̄
Hundreds (100 to 900)	$\bar{\rho}, \bar{\sigma}, \bar{\tau}, \bar{\upsilon}, \bar{\phi}, \bar{\chi}, \bar{\psi}, \bar{\omega}$, ϡ̄
Thousands (1000 to 9000)	͵α, ͵β, ͵γ, ͵δ, ͵ε, ͵ς, ͵ζ, ͵η, ͵θ

M = Myriad (10.000)

$\overset{\beta}{M}$ = 2 x 10.000 = 20.000, $\overset{\gamma}{M}$ = 3x10.000 = 30.000 etc.

L' = 1/2, δ' = 1/4 etc

We have given to the participants one by one the following operations and we asked them to transcribe them from the Greek alphabetical number system to our modern one. We have proposed them three examples, one with two-digit numbers, one with four digit-numbers and one with fractional numbers. Once they have done the transcriptions we asked them to explain us how Eutokios arrives to the result and which is the property behind his method. We have closed our presentation by making a comparison between this

algorithm and our modern one and by stressing the need for the teachers and afterward for the pupils to work on the Greek multiplication before attacking the modern one.
The first example is the multiplication 66 x 66

$\overline{\xi\varsigma}$
ἐπὶ $\overline{\xi\varsigma}$
$\overline{{,}\gamma\chi}$ $\tau\xi$
$\overline{\tau\xi}$ $\overline{\lambda\varsigma}$
ὁμοῦ $\overline{{,}\delta\tau\nu\varsigma}$

The above calculations could be seen with modern symbolism

	66
x	66
3600	360
360	36
Total	4 356

The mathematical analysis of the above mentioned calculations is:
(6 Tenths + 6 Units) (6 Tenths + 6 Units)=
36 Hundreds + 36 Tenths +
36 Tenths+ 36 Units = 4 356
The second example is the multiplication 1351 x 1351.

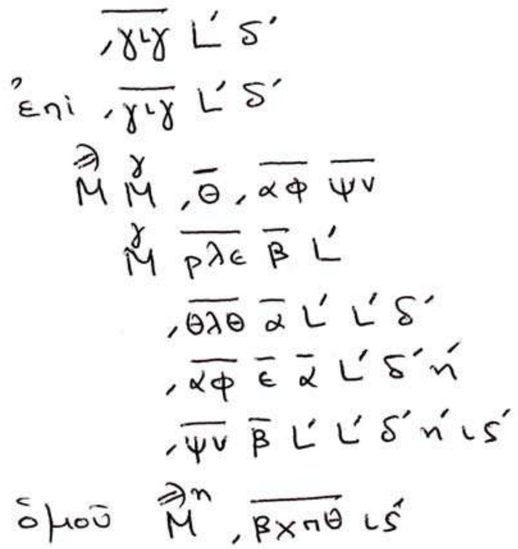
$\overline{{,}\alpha\tau\nu\alpha}$
ἐπὶ $\overline{{,}\alpha\tau\nu\alpha}$
$\overset{\rho}{M}\ \overset{\lambda}{M}\ \overset{\epsilon}{M}\ {,}\overline{\alpha}$
$\overset{\lambda}{M}\ \overset{\theta}{M}\ \overset{\alpha}{M}\ {,}\overline{\epsilon\tau}$
$\overset{\epsilon}{M}\ \overset{\alpha}{M}\ {,}\overline{\epsilon}\ {,}\overline{\beta\phi\nu}$
${,}\overline{\alpha\tau\nu\alpha}$
ὁμοῦ $\overset{\rho\pi\beta}{M}\ {,}\overline{\epsilon\sigma\alpha}$

With modern symbolism we have:

			1351			
		x	1351			
1 000 000	300 000	50 000	1 000			
300 000	90 000	15 000	300			
	50 000	15 000	2 500	50		
		1 000	300	50	1	
		Total	1 825 201			

The mathematical analysis of the above mentioned calculations is:
(1 Thousand + 3 Hundreds + 5 Tenths + 1 Unit)(1 Thousand + 3 Hundreds + 5 Tenths + 1 Unit) =
1 Million + 3 Hundred Thousands + 5 Myriads + 1 Thousand
3 Hundred Thousands + 9 Myriads + 15 Thousands + 3 Hundreds
5 Myriads + 15 Thousands + 25 Hundreds + 5 Tenths
1 Thousand + 3 Hundreds + 5 Tenths + 1 Unit = 1 825 201

The third example is with a fractional number.



With modern symbolism the multiplication is the following:

$$3013\ \frac{1}{2}\ \frac{1}{4}\ [= 3013\ ¾]$$

$$\times\quad 3013\ \frac{1}{2}\ \frac{1}{4}$$

9 000 000	30 000	9 000	1 500	750
30 000	100	30	5	$2\frac{1}{2}$

9 000	30	9	$1\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{4}$		
1 500	5	$1\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$		
750	$2\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	1/16	
Total	9 082 689 1/16					

The mathematical analysis of the above mentioned calculations is:

(3 Thousands + 1 Tenth + 3 Units $+\frac{1}{2}+\frac{1}{4}$)(3 Thousands + 1 Tenth + 3 Units + $\frac{1}{2}+\frac{1}{4}$)=
9 Hundreds Myriads + 30 Myriads + 9 Thousands + 15 Hundreds + 75 Tenths +
3 Myriads + 1 Hundred + 3 Tenths + 5 Units + 2 Units + $\frac{1}{2}$ +
9 Thousands + 3 Tenths + 9 Units + 1 Unit + $\frac{1}{2}+\frac{1}{2}+\frac{1}{4}$
15 Hundreds + 5 Units + 1 Unit $+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$
75 Tenths + 2 Units + $\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$+1/16

By the use of the Greek multiplication we can give explanations to the typical algorithm.

Greek Multiplication	Modern Algorithm
25	25
x17	x17
35	175
140	25
50	425
200	
425	

The goal is that pupils can understand the way of production of the partial products and the place-value of numerals of the factors of multiplication.

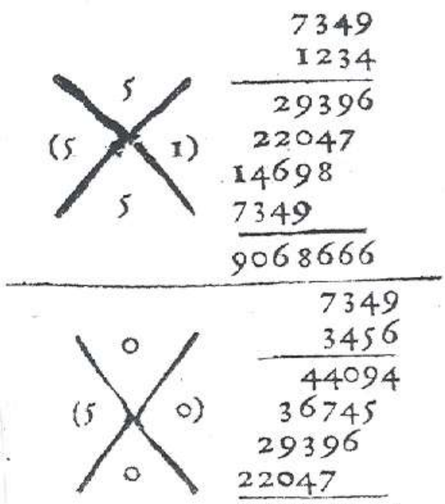
2 Multiplication in an old Norwegian textbook

The first textbook in what we today call mathematics in Norwegian was Tyge Hansøn's *Arithmetica Danica* from 1645. Geir Botten has recently written a book about it, based on the sole known copy. *Arithmetica Danica* shows how to use the numeral system, how to add, subtract, multiply, divide, use regula de tri, regula falsi, calculate square roots and do lots of "practical" calculations. It is believed that Hansøn partly based his book on earlier books in Nordic languages, although these connections have not been investigated. (Botten, 2009)

We are going to look at how multiplication is presented in this book, but first I'll point to some other aspects of the book that Botten finds interesting. I would love comments on them:

- The use of poems throughout. Example: "O Ungdom haff din Tid I act/ For Leddigang du dig vel vact/ Viltu I Regenkunst bestaa/ Ei nogn dag forgæffs lad gaa." ("O Youth, take notice of your time/ For idleness you must avoid / If you want to learn the art of calculation/ No day must pass in vain.") Also, some of the tasks are written as poems.
- The use of particular numbers with connection to monarchs etc: "I want to calculate the root of 2 486 929: On the basis of that I will learn in which year His Majesty my most gracious Lord and King Christianus 4th is born." (1577)
- Unrealistic answers. One task asks the age of a man, and the answer is 120.
- A special, explicit concern for female readers. Four pages are devoted to that, with topics such as buying fabrics, weaving etc.
- The prominence of alcohol in some exercises: "A man earns 15 shillings a day at the harbour when he is working, and drinks 9 when he is not. As the year passed, everything was spent drinking and he also owed 7 marks 8 shillings. How many days had he worked and how many had he not. (Facit 112 days worked, 200 days held sacred") (16 shilling = 1 mark)

Arithmetica Danica features this calculation:



Understanding the calculation is straightforward for modern students. But what are the crosses to the left? This was unknown notation for me when I first saw it, although it was not too hard to figure out. To my surprise, some Greek teacher students knew this when I asked them last year, as it is apparently still mentioned in some Greek classrooms.

The idea is basically the same as "casting out nines". We find the *repeated digit sum* of the first factor and put it to the left. Then we find the repeated digit sum of the other factor and put it to the right. The product of the digit sums we put on top. This should be equal to the digit sum of the product, and we put this below.

It should be noted that the term "casting out nines" suggests a process where we throw away nines as we go along, while "repeated digit sum" suggests that we calculate the full digit sum at first. In the workshop, we noted that both ways of doing it is still represented. We have no way of knowing what process Hansøn utilized.

Of course, the correctness of the method can be proved by use of *modular arithmetic***.** We could show this to students by looking at

$(a+9n)\cdot(b+9m) = ab+9bn+9am+81nm = ab+9(bn+am+9nm) = ab+9l.$

It can also be used for addition and subtractions, and even divisions.

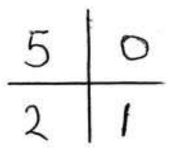
The method is not infallible – the main problem is that it is unable to spot a simple switching of two digits, for instance if you are calculating (successfully) that 9x4 = 36, and then are thinking "we put 3 down and carry the 6" instead of the other way around. There are also other common mistakes it does not detect, for instance in addition:

$$\begin{array}{rr} & 34 \\ + & 27 \\ = & 511 \end{array}$$

The history of "Casting out nines" is a bit uncertain. Florian Cajori (Cajori, 1991, p. 91) claims that it was known to the Roman bishop Hippolytos as early as the third century, but although the process of repeated digit sum (*pythmenes*) was known at that time (Iamblichus, 4^{th} c AD) (Heath, 1981, pp. 113-117), there seems to be consensus that the test was established in India or the Arab world. Avicenna (978-1036) supposedly referred to it as "the method of the Hindus". (Swetz & Smith, 1987, p. 189) and it is said to have appeared in the *Mahasiddhanta* of Aryabhata (II) (probably 10^{th} century). I have not checked this. However, it is simple to establish that it was included in Liber abaci (Fibonacci & Sigler, 2002) and in Maximus Planudes (1255-1305)'s *The Great Calculation According to the Indians* in late 1200s (Brown, 2006). Later, it was included in the Treviso Arithmetic in 1478 (Swetz & Smith, 1987).

In Liber Abaci, the problems of using 9 was discussed, and 7 and 11 were also used. Using 11 is almost as simple, and is slightly better.

The particular way of writing the check was used in several medieval schools (Flegg, 2002; Tattersall, 2005). Tattersall mentions that it was called the "cross bones check". This phrase is almost unknown by Google, for instance, so it doesn't seem to be widely known today. In the Treviso Arithmetic, this notation was used for division:



(Swetz & Smith, 1987, p. 87)[1]

The division in question is 7624:2 = 3812 rest 0. "If you wish to prove this by the best proof, multiply the quotient by the divisor, and if the result is the number divided the work is correct. // If you wish to prove it by casting out 9s, put the excess in the divisor, which is 2, in a little cross, underneath the left; then put the excess in the quotient, which is 5, above this 2; then place the excess of the remainder, which is 0, after the 5 on the other side. Then do as follows: multiply the excess of the divisor by that of the quotient, 2 times 5 making 10; add the 0 remainder, leaving 10; cancel the 0, cleaving [sic!] 1 for the

1 If anyone knows more about different notations used, we would be interested.

principal excess, and write this in the cross under the excess of the remainder. Then see if the excess of the number divided also equals 1, in which case the result is correct."

"Casting out nines" ("nierprøven") was included in Norwegian textbooks at least until 1985 (Viken, Karlsen, & Seeberg, 1985, p. 45). Here, only the method for multiplication was shown, and there was no proof.

Why is it no longer included in textbooks in Norway? Maybe because even this proof was considered too complex, and that a method without justification is unwanted. Moreover, there is anecdotal evidence that even teachers didn't understand that the method didn't find all errors. ("I received a note from an elementary teacher who asked why the method had been objected to in the texts above if it was a check that was taught in schools today. She was not aware that sometimes the method would confirm a false result. In particular if a digit reversal occurs in the answer, the method of casting out nines will not catch the error." (Ballew, 2010))

3 Egyptian-Russian method

In Norwegian textbooks for teacher education, you can find the following algorithm, called "Russian Peasant Multiplication":

Halfs	**Doubles**
49	183
~~24~~	~~366~~
~~12~~	~~732~~
~~6~~	~~1464~~
3	2928
1	5856
	8967

The students need a little time and more than one example just to realize how the algorithm works. They need to see that what we do is to halve the numbers in the left column while doubling the numbers in the right column (all the time leaving any fractions out) until we get to 1 in the left column. Then we cross out the lines having an even number in the left column, and find the product we wanted by adding the numbers that are not crossed out in the right column.

After that, they need quite a bit of time and help to understand *why* it works. We have found it helpful to let the students work on 16x23 and 17x23 as steps towards finding a general explanation. This makes it possible for them to see that while 16x23 = 8x46 = 4x92 = 2x184 = 1 x 368 = 368 is simply a matter of halving the one factor and doubling the other factor all the time, with 17x23 you get an additional 23 that you have to keep in mind while you go on with 8x46. Of course, such numbers that have to be "kept in mind" turn up whenever there is an odd number in the left column.

Of course, a similar – but not fully equal – way of doing it is the much older Egyptian one (as seen in the Ahmes Papyrus (aka Rhind Papyrus)):

X	1	183
	2	366
	4	732
	8	1464
X	16	2928
X	32	5256
	49	8967

The method also works the other way around. What is 8967:183?

X	1	X	183
	2		366
	4		732
	8		1464
X	16	X	2928
X	32	X	5256
	49		8967

It is exactly the same numbers, but the thought process is different.

What about 8970 : 183? We would end up with a remainder of 3, and the answer would be 49 3/183 or 49 1/61. (Here, of course, the Egyptians would only use unit fractions, while we could use any fraction)

4 Discussion

What is the use of the different methods of multiplication in teacher training? In the Department of Primary Education of the University of Western Macedonia in Greece we propose an optional course in the contents of which we are discussing the above mentioned methods but also many others and we are trying to give to the students – future primary school teachers – many alternative historical origin ways to have a deeper understanding of the modern multiplication algorithm. We would like also to stress the fact that the above “Greek multiplication” is part of the curriculum proposed to the third class of the Greek primary school (pupils of 9 years old) before the typical modern algorithm as a preparatory stage. In Norway, “Greek multiplication” is not a standard part of the textbooks, but it is discussed in teacher education.

For teachers, it is important to be able to understand different algorithms (for instance their students’ own efforts) and looking at historical algorithms can be a good way. This can be seen in connection with what Ball and others write about Special Content Knowledge for teaching (Loewenberg Ball, Thames, & Phelps, 2008). This entails an enrichment of their knowledge of multiplication.

For teachers, it is also important to see that the multiplication table is not necessary to do multiplications. These methods also show that history of mathematics has developed.

We hope that teachers will also become curious as to why ”our” algorithm works and why it has prevailed in school and society.

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THE HISTORY OF MATHEMATHICS IN THE CLASSROOM

Some Activities

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ABSTRACT

The importance of History of Mathematics in Portuguese school curricula is increasing but there is some difficulty to introduce this topic in the classroom. The material adapted to the level of the undergraduate students is still short and by that it is necessary the production of more useful material that teachers could easily apply to the classroom. This workshop presented various examples of contents of the History of Mathematics adapted to the classroom such as: Euclid's Elements; the Shadow Instrument of the Portuguese mathematician Pedro Nunes; the numeral system of ancient Egypt; the Napier's rods and the Genaille-Lucas rulers.

The importance of History of Mathematics in school curricula is increasing. For example, in Portugal, in the Mathematics program, the History of Mathematics appears as a crosscutting theme throughout the secondary school teaching and that should come along on several and different themes. The idea is not to present the History of Mathematics in a finite and condensate set of classes, but that this subject will be a good support and motivation for the study of other mathematical topics.

However, despite the growing importance of the History of Mathematics in school curricula, there is little literature on this theme applied to the context of the classroom, which difficult the work of teachers in its implementation. In the references at the end of this text, some works containing material related to the History of Mathematics suited to use in the classroom are displayed. One of the most important works in this area is the book "Learning Activities from the History of Mathematics" by Frank Swetz (1994). In this book, the author presents several topics of the History of Mathematics with its direct application to the school context through worksheets with questions, accompanied by texts - such as, for example, short biographies - accessible and appropriate to the skills and knowledge of younger students.

"*We are always looking for good problems to strengthen and broaden our student's knowledge of mathematics as well as to refine concepts taught in the class room. The history of mathematics supplies thousands of useful and interesting problems, problems that are mathematically and pedagogically sound and which, by their historical nature, possess an additional intellectual appeal for students.*"[Swetz, 1994, p.2]

This workshop tried to follow the same idea and presented some proposals relating to the History of Mathematics in the school context (in the classroom and beyond, such for example in an extra-curricular Math Club). We saw how to adapt some topics of mathematics at the school context using:

- the Euclid's Elements (examples of how to use the computer and some interactive content in classroom was presented in the approach to this book);
- the Shadow Instrument of the Portuguese mathematician Pedro Nunes (it can be easily constructed with paper and cardboard in the classroom and be used to measure the altitude of the Sun);

- the numeral system of ancient Egypt (system "visually more attractive" allowing the students to achieve a better understanding of the current numbering system by comparison and contrast);
- The Napier's rods and the Genaille-Lucas rulers that were used for the multiplication of natural numbers (they can also be easily constructed with paper and scissors in the classroom).

In this workshop, worksheets about these last three topics that could be used in the classroom were presented. These worksheets, with some adjustments, are reproduced in the following pages.

The first worksheet actually used in the context of the classroom was about Pedro Nunes and his Shadow Instrument in a 9th Grade class (Portugal, students with 14/15 years), which was very well received by all students. Some students have even constructed some Shadow Instruments of wood in a manual labor discipline, i.e., outside of math class. Also, all students had prepared a written report about the practical activity in order to consolidate the knowledge obtained from this activity. It was a lesson included in the theme of Trigonometry and served to show a possible practical application of this theme. Though simple, this example can show the actual importance of trigonometry in real life; is harder for a student to "see" the mathematics that exists, for example, in a car or in a computer, while on these historical examples, often using more basic math, is easier to see where the mathematics was used in the past to solve practical problems and to understand the role of mathematics as an important factor of the development and progress. From these assumptions, it is easy to believe that, with another level of complexity and sophistication, there are a lot of math in our everyday life.

The Napier rods and the Genaille-Lucas rulers are good examples to show the development of mathematics – though simple and rudimentary, it allows showing that technology, as the mathematics used on it, is/was a science under construction and in a gradual growth. The Napier rods were an instrument built to facilitate the calculation of multiplications. Although they were very helpful, these rods presented some flaws and defects (they required some calculations beyond what was observed directly on the instrument), which "caused" the need to create a better and more efficient instrument. The Genaille-Lucas rulers then arise to solve the flaws previously detected but they still presented some limitations too (not allowing direct multiplication by multipliers with more than one digit) and, because of that, later, these rulers also ceased to be used.

This kind of process in which certain instruments are gradually being replaced by others progressively better and more effective is a good example of how the technology and, in particular, the science associated with it, works. These rods are a good example of how math and science evolves, not only in a theoretical and abstract mode (that many times keeps so many students out from this discipline), but also in a practical way in order to solve concrete problems of real life. Also serves to show that the current calculators and computers were the result of an arduous and demanding intellectual process until reaching the current level of complexity and refinement. Realizing that current technology (such as mathematics itself), although very useful and powerful, is not final and is still evolving could be an important step to humanizing mathematics in the students minds. This multiplication worksheet was used with students aged between 12 and 16 years (in Portugal), under a Mathematics Club, having a very good acceptance.

The numerical system used in ancient Egypt, as in the previous case, can be very advantageous in the classroom to show that mathematics is not a static science. Moreover, using this and other numeric systems of ancient peoples, it is possible to make a parallel with the current decimal numerical system, highlighting their good qualities and the characteristics that made it the dominant system in most of the world. Many times the usual decimal system is presented as a given statement and the proper emphasis to the characteristics that make it very useful is not given. If there are other systems to compare it, many times more difficult to use and understand, it might be easier to motivate students to appreciate the actual number system used in their mathematics education. It seems to me that everything that allows renewing the basic tools of mathematics can be a good activity for several years of schooling (though, I have not yet had the opportunity to use it in the context of the classroom). This activity allows facing, once more, the math as a science in evolution.

In the Euclid's Elements, examples of how to use the computer to approach this book were shown. In particular, the website (in Portuguese)

http://wwmat.mat.fc.ul.pt/~jnsilva/elementos_livro_1/mat/elementos/index.htm

was shown, where interactive applets of all the propositions of the first book of the Elements are presented. These pages are different from the usual interactive approach to this book because all proofs are constructed from the beginning, i.e., the picture near the proof is done step by step following the explanation. This makes it easier for students to understand the statements which are being proofed in each step, what is being shown and what are the previous results that are being used. There exists already many interactive websites that use the Book I of Euclid's Elements, but all, as far as I know, present the figures that illustrate the proofs in its final form, making it extremely difficult to distinguish, for example, what are the initial data, what are the intermediate conclusions used in the proof and what is the result actually being proved.

This book of Euclid, given its structure and propositional logic, might be a good example to show how mathematical reasoning works. One of the cornerstones of mathematics, without question, is its cumulative nature and the accuracy of their statements and arguments. Showing how, starting from a few given assumptions (five postulates and five common notions), it is possible to construct a set of mathematical propositions of some scale, maybe leading the students to a better understanding of the power of the mathematical "building" which has been assembled over several generations. Furthermore, the brilliant way how the propositions and their proofs are displayed and linked to each other (usually, to proof a certain result, the exactly precedent proposition is used), is a great example of what is a text of mathematics and it is possible to find many similarities with the current math texts.

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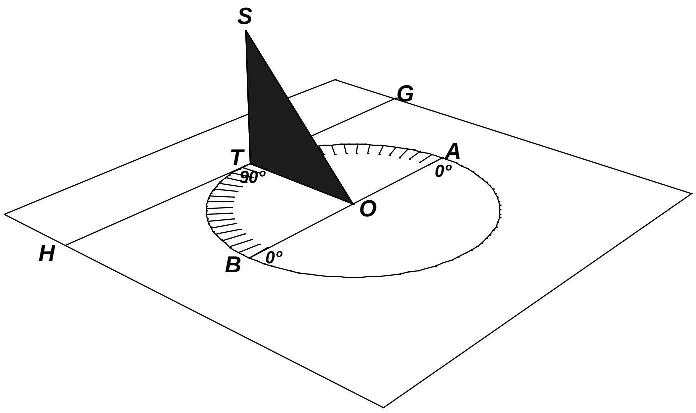
1 The Shadow Instrument (Pedro Nunes)

Contents: congruent triangles and trigonometry.
Material: cardboard, scissors and glue to construct the shadow instrument; measure tape for the practical activity presented at the end of this worksheet.

The Portuguese cosmographer and mathematician Pedro Nunes was born in 1502 in Alcácer do Sal and died in 1578 in the city of Coimbra. Among his scientific works Nunes presented several measuring instruments for use in celestial navigation. One of these was the Shadow Instrument that measures the angle that the rays of the sun make with the horizontal plan (“the altitude of the Sun”). This was a simple tool, which looked like a sundial, but it had a very ingenious innovation that gave directly the "height of the Sun" through the use of the shadow projected by itself. The measurement of the “height of the Sun” was very important for the ancient Portuguese mariners navigation but can also be used, with a little knowledge of trigonometry, to determine the height of objects (buildings, trees, street lamps, ...) when this is an inaccessible task by direct measurement – just using the measurement of the length of the shadows of those objects. After studying and understanding this instrument, you will make an activity where you practice this second use.

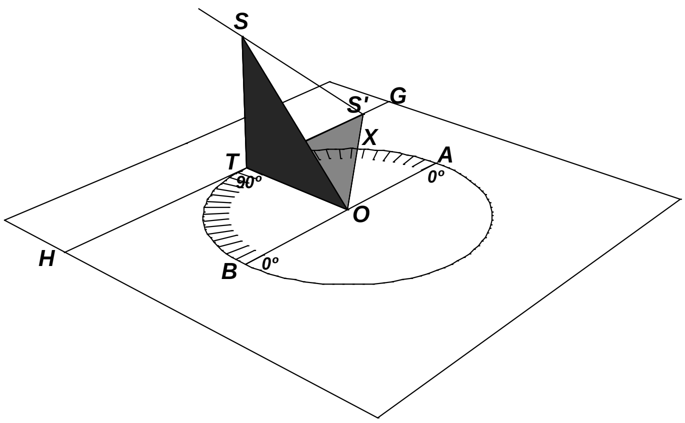
The Shadow Instrument was a plate, usually square, with a circle drawn and a isosceles right triangle set perpendicularly as shown in the next figure (the length of the leg of the triangle was equal to the radius of the circle; the tangent to the circle in T was also marked on the plate).



On the circle, it was still marked the diameter parallel to the tangent GH and the two quarters of the circle closest to the tangent graded from 0º to 90º from that diameter till the point of tangency T.

With this gradation, the angle that the sun makes with the horizontal plan is directly obtained. The Shadow Instrument is used as follows:

- Set the plate horizontally.
- Rotate the instrument until the shadow of the cathetus [*ST*] matches the line tangent to the circle; label *S’* the shadow of the point *S*.
- The intersection of the shadow that the hypotenuse of the triangle makes on the plate with the arc of the circumference between points *A* and *T* indicates the value of the angle that the sun makes with the horizontal plan; label that point as *X*.



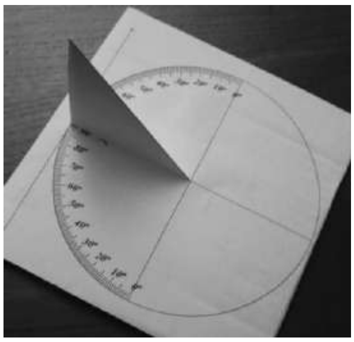
In order to understand and validate the functioning of this instrument, answer the following questions:

1. Show that the triangles [*S'TS*] and [*S'TO*] are congruent and that $\angle SS'T = \angle OS'T$.
2. Show that $\angle OS'T = \angle AOX$.
3. Show that the plan *SS'T* is perpendicular to the horizontal plan.
4. Show that the angle that the sun rays make with the horizontal plan is equal to $\angle AOX$, i.e., equal to the angle marked in the circle by the shadow of the hypotenuse of the triangle.

Practical activity:

Build a Shadow Instrument using the sheet provide at the end of this activity and by following these instructions:

1. Cut the two figures by the segments indicated by the "scissors";
2. Fold the bottom figure by the marks in order to construct an isosceles triangle;
3. Place the isosceles triangle by the opening in the circle;
4. Paste the last construction into a cardboard.



Determine the height of some objects (buildings, trees, street lamps,…) within the school using this instrument to measure the "altitude of the Sun" and a measure tape to evaluate the lengths of their shadows. Register your observations and calculations.

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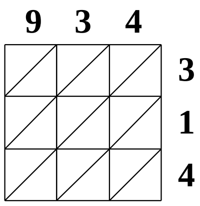
2 The Multiplication (The Napier’s rods and the Genaille-Lucas rulers)

Contents: lattice method of multiplication, the Napier’s rods and the Genaille-Lucas rulers.
Material: scissors, the Napier’s rods and the Genaille-Lucas rulers.

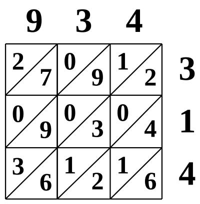
2.1 Lattice Method of Multiplication

At the beginning of the Renaissance, the emergence of various techniques such as, for example, the method of multiplication in lattice led to an increase of ease and speed with which the numerical calculations were made. Let’s see, then, how the method of lattice multiplication works.

Assume that you want to do the product 934 × 314. Start to build a table as shown in the figure below.



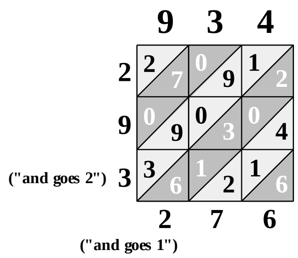
As 4 × 3 = 12, fill the two adjacent triangles that form the square of the column corresponding to the 4 and the row corresponding to the 3 with 1 and 2, respectively. In a similar way, fill up the rest until obtain the table that is presented below



Next, consider the six "diagonals" on the table. So we have that:

- the units digit of the product required is equal to the digit of "first diagonal": **6**;
- the tens digit is equal to the sum of the digits of the “second diagonal”: 4 + 1 + 2 = **7**;
- the hundreds digit is equal to the units digit of the sum of the digits of the “third diagonal”, while the tens digit of that sum goes to the next “diagonal”: 2 + 0 + 3 + 1 + 6 = 1**2**;
- the thousands digit is equal to the units digit of the sum of the digits of the “fourth diagonal” (don't forget to add the digit that comes from the previous “diagonal”), while the tens digit of that sum goes to the next “diagonal”: 1 + 1 + 9 + 0 + 9 + 3 = 2**3**;
- the ten thousands digit is equal to the digit of the sum of the digits of the “fifth diagonal” (don't forget, once more, to add the digit that come from the previous diagonal): 2 + 0 + 7 + 0 = **9**;
- the hundred thousands digit is equal to the digit of the sum of the digits of the “sixth diagonal” (note that nothing came from the previous "diagonal"): 2 = **2**.

Therefore, we have that 934 × 314 = 293276.

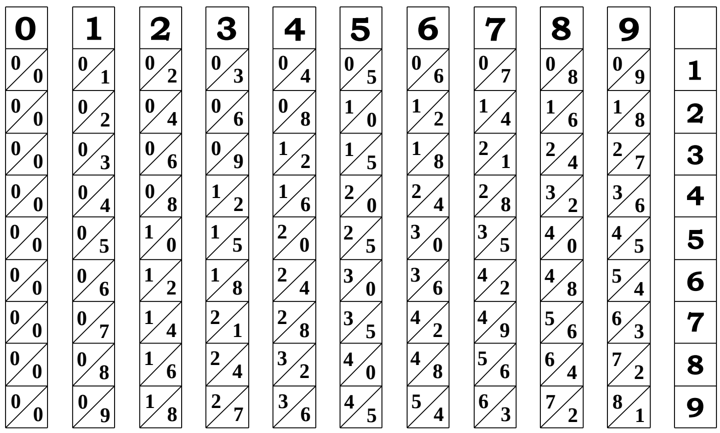


Exercise 1: Determine the numeric value, using the lattice method of multiplication described above, of the following expressions.

1.1 723×149; **1.2** 481×58; **1.3** 451^2.

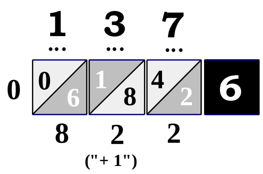
2.2 The Napier´s rods

John Napier (1550 - 1617), famous Scottish mathematician who devoted much of his life researching processes which allow the easiness of doing numerical calculations, noticed that the entries in columns used for lattice multiplication are always filled with multiples of the number that is on top of that column. From this finding, Napier constructed a collection of sticks with these ordered sets of multiples, thereby obtaining the desired lines in the lattice multiplication in a more fast and effective way. Thus, there is a practical tool, usually made of wood or bone, which facilitates the way of performing multiplications which often is designated by Napier's rods.



Let's see how this tool created by Napier works showing how to do, for example, the multiplication 137 × 6.

- ♦ Select the rods corresponding to the 1, 3 and 7 and put them side by side so that in the top appears the number 137. On the right of these rods should be placed the stick numbered from 1 to 9.
- ♦ To find the product you should "read" the line corresponding to the digit 6.
- ♦ Then add up the numbers for each "diagonal", using the same technique used in the lattice method of multiplication. Therefore, 137 × 6 = 0822.



To multiply two numbers with more than one digit, you have to make several partial products and then sum them to obtain the desired product. Take, for example, the multiplication 354 × 628. This product can be seen as 354 × (600 + 20 + 8), i.e., as **354 × 6** × 100 + **354 × 2** × 10 + **354 × 8**.

Therefore, the rods corresponding to 3, 5 and 4 should be placed side by side so that the number 354 appears on the top (again, on the right of these rods put the stick numbered from 1 to 9). To find the product you should "read" the lines corresponding to the digits 6, 2 and 8.

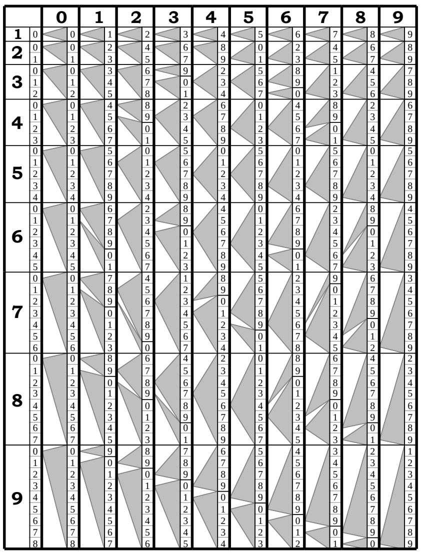
After performing these products with the Napier's rods (354 × 6; 354 × 2 e 354 × 8) is still necessary to "correct" these partial results with the respective powers of 10. So, in the example, we have

Calculations on the Napier's rods	*"Correction"*	*Total*
354 × 8 = 2832	2832 × 1	2832
354 × 2 = 708	708 × 10	7080
354 × 6 = 2124	2124 × 100	212400
		222312

Exercise 2: Using scissors cut the Napier's rods provided at the end of this activity. Use those sticks to determine the following products:

2. 1 305×9;
2. 2 127×83;
2. 3 5016×125;
2. 4 4328×56.7.

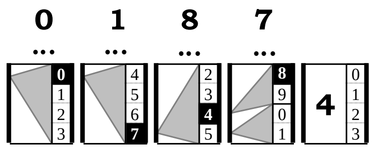
2.3 The Genaille-Lucas rulers



In the late nineteenth century a variation of the Napier's rods was invented where the need of transportation of digits for the next "diagonal" was eliminated (the instrument itself "does" such transportation). Another advantage is the fact that these rulers do not require the user to make any sum; it is possible to obtain the final result only by direct observation of the sticks. These rulers were invented by the French engineer Henri Genaille in reply to the problem posed by the mathematician Edouard Lucas to create an instrument where, indeed, this need to make any intermediate sums was eliminated. Therefore, these rulers are known as the Genaille-Lucas rulers.

Let's see an example of how the Genaille-Lucas rulers work. Think about the product 187 × 4.

- Select the rulers corresponding to 0, 1, 8 and 7 and put them side by side so that on the top appears the number 0187. On the right of these rulers should be the stick numbered from 1 to 9.
- To find the product 187 × 4, you should "read" the line corresponding to digit 4.
- Afterwards, just follow the "triangles" that exist in the rulers to find the value of the required product, beginning on the first digit (from the top) of the first column (from the right). So 187 × 4 = 0748.



Let’s see, through the example presented, some justification for the fact that these rulers work properly.

The number of units, 8, comes from 7 × 4 = 2**8**. Note, however, that we still need to carry 2 to the next column. This is done by the "gray triangle" which is near to the digit 8; in fact we have 8 × 4 + 2 = 32 + 2 = 3**4** which means that 4 is the ten digit. Note that the first digit of these column is 2 and that results from the fact that 8 × 4 = 32; the digits below are for the case of transportation from the previous column, just like in this case.

Then follow the "triangle" on the next column (carrying 3 from the tens digit column to hundreds digit column) until the digit 7 that correspond to 1 × 4 + 3 = 0**7** (hundreds digit). Note that the first digit from the hundreds column is 4, which come from 1 × 4 = 04; once more, the digits beneath are for the cases where exist transportation from the previous column. At this step we don’t carry anything for the next column and, therefore, the "triangle" at this column sends the user to the first digit, counting from the top, of the column of thousands, i.e., to **0**.

To multiply two numbers with more than one digit with these rulers, you have to perform various partial products and then sum them to obtain the final product using the same process that was indicated for the Napier’s rods.

Exercise 3: Using scissors cut the Genaille-Lucas rulers provided at the end of this activity. Use those sticks to determine the following products:

3.1 807×7; **3.2** 129×43; **3.3** 7608×317.

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3 Number System of Ancient Egypt

Contents: ancient number systems and arithmetic operations.

The need to count is very old and dates back to the time when man settled and began to practice agriculture and livestock (around 10000 BC). Initially, in order to record the information from those activities piles of stones or marks on sticks and bones were often used. Note that these characters were previous to the advent of writing itself (the oldest known form of writing belongs to the Sumerians, a people who lived between 3500 and 2000 BC), i.e., the "signs for numbers probably preceded the words for numbers, it is easier to make incisions on a stick than to establish a well-shaped phrase to identify a number." With time several number systems were created which allowed greater easiness in representing quantities of objects.

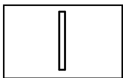
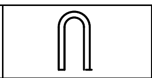
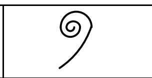
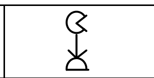
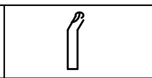
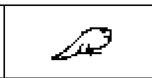
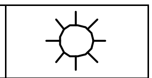
The number system standard today is the Hindu-Arabic system, being used basically all over the world. The name of this system comes from two people: the Hindu people who created it and to the Arabs who used and disseminated it. Although it was known by the Arabs at least since the eighth century, this number system arrived to Europe just in the thirteenth century, mainly by the action of Fibonacci (also known as Leonardo of Pisa) and his book *Liber Abaci* ("the book of the abacus" or "the book of calculation"). Recognizing that arithmetic with Hindu-Arabic numerals is simpler and more efficient than with Roman numerals, Fibonacci traveled throughout the Mediterranean world to study under the leading Arab mathematicians of the time and, in 1202, he published the referred work.

Homework assignment:

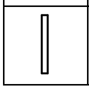
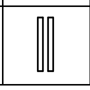
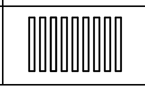
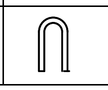
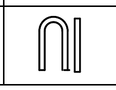
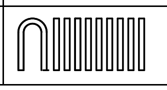
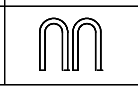
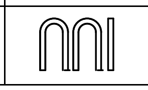
Research the Roman numeral system and write a small text explaining how this system worked (including the symbols used, how each number was represented and so on). Find also places and objects where even today we can find these numbers like, for example, on the kings and popes names, on watches and on the numbering of books chapters.

Other ancient peoples used different symbols to represent numbers and each number system had its own rules. On the next pages we will study the system used in Ancient Egypt as well as some of its essential characteristics.

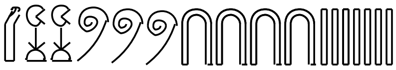
The number system used in Ancient Egypt is at least as old as the pyramids, dating back from 5000 years ago. This system was a repetitive one using special symbols, presented in the following table, for the different powers of 10, from 10^0 to 10^7.

							
$10^0 = 1$	**$10^1 = 10$**	**$10^2 = 100$**	**$10^3 = 1000$**	**10^4**	**10^5**	**10^6**	**10^7**
"stroke"	"heel bone"	"coil of rope"	"lotus flower"	"pointing finger"	"tadpole"	"kneeling man"	"sun"

To represent the other numbers they used repetitions of these special symbols but none should be repeated more than nine times, which means that it was a decimal system (set of 10 equal symbols were represented by a new special symbol). Thus, we had

1	2	…	9	10	11	…	19	20	21	…
		…				…				…

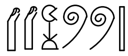
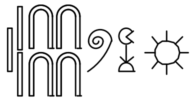
For example, the number 12347 was written as

 .

Remind that the Hindu-Arabic system used today is positional. For example, in the numbers 13, 237 e 351, the symbol "3" don't means always the same quantity: in 13=10+3 represents "3 units", in 237=200+30+7 represents "3 tens", while in 351=300+50+1 represents "3 hundreds".

Exercise 1.

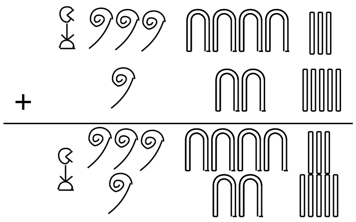
1.1 Complete the following table:

Hindu-Arabic system	*Ancient Egypt system*	*Hindu-Arabic system*	*Ancient Egypt system*
30		3261	
			
752		13125	
			

1.2 What is the largest number that can be written with the Egyptian number system?

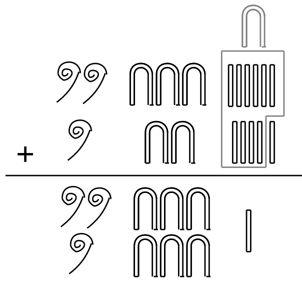
The **addition** was very simple in the Egyptian number system. It was enough to count how many symbols of each type existed in each of those numbers and write the final result with all the symbols of each replacing, if necessary, ten symbols of a given power of 10 by the next higher power. Let’s see the following examples:

♦ 1343+125



(1343)
(125)
(1468)

♦ 236+125



(236)
(125)
(361)

Exercise 2.

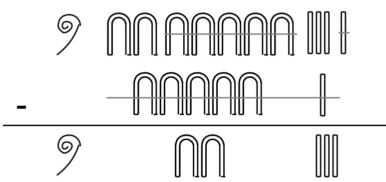
Using the Egyptian number system presented here, calculate the following sums:

2.1 2134+1201. **2.2** 1354+118. **2.3** 1643+357.

For the **subtraction** simply remove from the largest number, the symbols that form the lower number, "replacing" in the largest, if necessary, symbols of a given power of 10 by ten symbols of power immediately below.

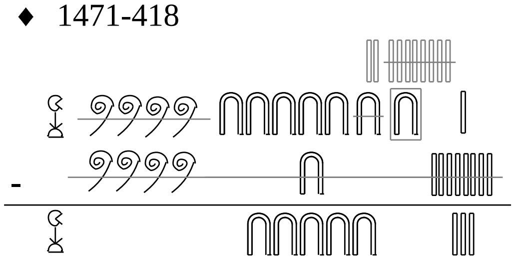
Note that in Ancient Egypt there wasn’t the concept of negative number or the concept of zero, and therefore in this context only makes sense considering the cases where the number to be subtract is less than the number to be subtracted. See, for example, the following subtractions:

♦ 174-51



(174)
(51)
(123)

♦ 1471-418



(1471)
(418)
(1053)

Exercise 3.
Using the Egyptian number system presented here, calculate the following subtractions:
3.1 293-122. **3.2** 1562-1303. **3.3** 14523-3128.

The **multiplication** by 10 in this number system is very simple because each symbol represents a power of 10, so each symbol in the representation of the number just needed to be replaced by the symbol of the next higher power.

Exercise 4. Complete the following table:

"n"	𓎆𓎆𓏺	𓍢𓎆𓏺𓏺𓏺		𓆼𓍢𓍢𓏺	
"n×10"	𓍢𓍢𓎆		𓍢𓎆𓎆𓎆		𓆼𓆼𓍢𓍢𓎆𓎆𓎆𓎆

The remaining multiplications were carried out, in general, by a process of duplation (successive doubling). To better understand from this point on, the usual Hindu-Arabic system will be used. Let's see an example, the multiplication 15×13. Firstly, it was necessary to double, successively, one of the factors (choose the number 15), recording the values as follows:

1	15 (=15×1)
2	30 (=15×2)
4	60 (=15×4)
8	120 (=15×8)
(16 > 13)	-

The process should stop when the successive doubling in the left column (successive values by which one multiplies the chosen factor) is equal or exceeds the value of the other factor. In the left column values whose sum gives 13 (the other factor) are now reported.

	***1**	**15**
	2	30
	***4**	**60**
	***8**	**120**
Total	13	<u>195</u>

Adding now the values of the right column corresponding to those indicated we obtain the desired final result of the multiplication: 15+60+120=195. This method of multiplying corresponds to the following:

15×13	=15×(1+4+8)	("left column")
	=15×1+15×4+15×8	(distributive property of multiplication)
	=15+60+120=195.	("right column")

Exercise 5.
Using the successive doubling method, calculate the following products:
5.1 25×8. **5.2** 30×27. **5.3** 312×11.

The method used in Ancient Egypt for the **division** also resorted to successive duplication, in a similar way to that used for multiplication. To understand this method study, for example, the division 91 ÷ 7. Firstly, it was necessary to double, successively, the number 7 (the divisor), recording the values as follows:

1	7 (=7×1)
2	14 (=7×2)
4	28 (=7×4)
8	56 (=7×8)
-	(112 > 91)

The process should stop when the successive doubling in the right column is equal or exceeds the value of the dividend. In the right column values whose sum gives 91 (the dividend) are now reported.

	1	***7**
	2	14
	4	***28**
	8	***56**
Total	13	91

Adding now the values of the left column corresponding to those indicated we obtain the desired final result of the division: 1+4+8=13. This method of division corresponds to the following:

91÷7	=(7+28+56)÷7	(“right column”)
	=7÷7+28÷7+56÷7	(distributive property of division)
	=1+4+8=13.	(“left column”)

Exercise 6.

Using the successive doubling method, calculate the following divisions:

6.1 304÷19. **6.2** 345÷15. **6.3** 1457÷47.

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A HISTORICAL APPROACH OF THE FUNDAMENTAL CONCEPT OF MEASUREMENT

Measuring "Time", in Portuguese Textbooks for 5th and 6th grades

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ABSTRACT

In this article we present a part of a PhD degree project on: "The fundamental concept of Measure: epistemological and pedagogical aspects related with the first six years of schooling".
It aims at analyzing the historical approach of the fundamental concept of Measure as presented in Portuguese mathematics textbooks for the 2nd Cycle of Basic Education. In particular, it intends: 1) to explain why do we look at the mathematical concept of Measurement as not only an elementary concept but, and above all, a fundamental concept in mathematics; 2) to identify a historical approach, in textbooks, in as much as it contributes for promotion and enrichment of learning the concept; 3) to analyze activities (problems and exercises) suggested in textbooks and requiring Measurement. Our specification of the concept of Measurement will be, in this article, "Time".

1 Introduction

Facing the persistent bad results in mathematics, achieved by Portuguese pupils, in the past years - in fact, Portugal is listed, in some international comparative tests (PISA 2000, 2003, 2006) in the last places of the analysed countries - the Portuguese Ministry of Education decided to take concrete measures, namely, implementing a national *Action Plan for Mathematics* on very precise areas: Mathematical Teacher Training, Mathematics Curricula and Mathematics Textbooks.

In what respects to Mathematical Teacher Training, a national in-service teacher training programme was implemented: it is designed for elementary school teachers (grades 1 to 6) [1] and it intends to deepen teachers' mathematical, curricular and pedagogical knowledge. In this programme, the importance of the History of Mathematics is cleared assumed.

As for the Mathematics Curricula, new Programmes for Basic Education were designed. The Mathematics Program for the 2nd cycle (used until 2008/2009) suggests the use of activities with some historical data, in order to *"help students to understand the relationship between some mathematics historical facts and some problems that man has sought to solve"*. It is also an aim of this programme to develop "interest in Mathematics historical facts". On the other hand, the new Mathematics Program for Basic Education (MPBE, that just started to be implemented in 2009/2010) strengthens the importance of History of Mathematics to understand Mathematics as an element of human culture and to realize that Mathematics is an alive science in continuous evolution.

[1] Portuguese Educational System has 3 Cycles of Basic Education and the Secondary Education. The 1st Cycle has 4 years of education; the 2nd one has 2 and the 3rd one has three years. The first and the second Cycles of Portuguese BE constitute the so-called Elementary Education, when children are 6 to 12 years old.

Concerning Mathematics textbooks, the Portuguese government established, in 2006, a system for textbooks accreditation: scientific commissions were formed having the responsibility to certificate textbooks (Despacho n.º 25190/2009); in addition, the assessment criteria have also been defined by law (Lei nº 47/2006 de 28 de Agosto). Therefore, all textbooks have to be certified by a group of specialists; then, in schools, teachers decide on the textbook that will be chosen to use in their schools for a period of six years.

Although the Mathematics Programme for the 2nd Cycle of Basic Education (particularly the new MPBE) suggests several didactical resources, the truth is that textbooks still play a significant role both inside and outside the classroom, and despite being, officially, recognised as “a tool for pupils’ use”, it is a fact that elementary teachers, as mediators of textbooks (Johnsen, 1993; Van Dormoien, 1986), tend to rely on them more than in any other curricular source (various Portuguese studies and reports reveal that 80% to 90% of teachers use it always or almost always). This situation in Portugal does not, therefore, differ from the situation in England, German or France (understood as, probably, the most influential educational systems in Europe). Haggarty and Pepin (2002) refer that teachers use textbooks regularly to prepare lessons or in classroom. In summary, mathematics textbooks seem to dominate what pupils learn, since they are mediators of the meaning and content of which is defined at the Mathematics Program (Chopin, 2004; Apple, 1993; Castell et al, 1989) and being the teachers’ primary aid organizing lessons and structuring the subject matter it may well turns out to be a substitute of the Mathematics Program itself.

Even though the Mathematics Programmes emphasize the importance of History of Mathematics in Mathematics Teaching, following, at least theoretically, some research suggestions, its implementation (in textbooks and in classroom) seems to be far from being accomplished and/or appropriated.

We shall begin this article by presenting our definitions about what we call elementary and fundamental concepts, in mathematics, explaining our reasons for establishing such a distinction.

We will take the specific mathematical concept of Measure, and in particular, the concept of “Time”. We will make a brief introduction to its history that will emphasize the fact of Measure emerging over time, its genesis being inherent to human activity and it being presented in many areas of mathematics as well as in several other areas of knowledge, both in the school context and in society.

Then we will present a set of criteria to analyze the most used mathematics textbooks for 5th and 6th grades, in Portugal, as well as the results of the analysis of activities (problems and exercises) suggested by textbooks and requiring measurement.

2 Elementary vs. fundamental concept

Concepts play a key role in the construction of mathematical knowledge. Fischbein (1993) defends that a mathematical concept expresses an idea, an ideal and a general representation of a class of objects with common characteristics. Furthermore, a definition of a mathematical concept, accurate and appropriate to the level of education is also the fundamental structure of mathematical knowledge (Wu, 2007).

In our opinion elementary and fundamental have different meanings, although a concept may be, at the same time, elementary and fundamental.

A mathematical concept taught and learnt at elementary level is an elementary concept.

For instance, the concepts of: number (excluding complex numbers); line; perimeter; area; mean; mode; angle; set or measure are, in our point of view, elementary concepts.

On the other hand, we say that a concept is fundamental if it emerges over time, its genesis is inherent to human activity and it is presented in many areas of mathematics as well as in several other areas of knowledge, both in the school context and in society, in general. For example, measure, count, limit or relation are fundamental concepts.

This meaning that we defend for fundamental concepts is clearly different from the one used by Ma (1999) when she states that Elementary Mathematics is fundamental mathematics, since it constitutes the foundation of this discipline and contains the rudiments of many important concepts in more advanced branches of this knowledge. In fact, in our opinion, there are fundamental concepts that are elementary but there are, certainly, others which are not. And, on the other hand, there are elementary concepts that are not fundamental.

For what was presented we consider Measure as an elementary concept, but above all, a fundamental one. Its history and relevance in mathematics teaching provide the evidence to make this assumption.

2.1. Why "Time"?

"Time", as a specification of Measurement is, in our opinion, considered a fundamental concept.

It is deeply connected to life - we hardly understand human life without time. Measuring time is a human necessity well documented by sources from Antiquity and is a multiple dimension concept: philosophical, scientific, human and scholar.

History, both of Mathematics, Science, Technology and Humanity shows us that the evolution of time measurement, of instruments for measuring time, of new units to measure time as well as the need to "democratize" the right time to implement different rhythms of life and to measure time accurately, have different reasons including religious, political, social and scientific.

The philosophic facet of "Time" highlights that the perception of this concept is different for each person; it is a subjective concept which perception is significantly affected by both actions and state of mind.

"Time" as a scientific concept is related to sequencing and is a physical magnitude present in the definition of other concepts and magnitudes (like velocity, angle and meter). Nevertheless, in what respects to the learning and teaching of "Time" it seems that it is reduced to "*read the clock. However, this teaching involves only social-conventional knowledge, which is often arbitrary"*(Kamii, 2003).

We acknowledge that "Time" is one of the most problematic magnitudes to teach since it is intangible and continuous and is difficult for children to perceive:

- the idea of *time interval* and of *recorded time*;
- the relations between the units to measure time (and its relations with more complex concepts);
- the different representation about time (circle or timeline);
- the range of ways of telling the same time;
- the estimation of time and operation with time units.

But we also acknowledge that these difficulties may prevail through years and influence the understanding of other mathematical concepts extremely related with

Measure and “Time”.

On the other hand, “Time” is also an elementary concept because it is taught at elementary level. For that reason it is important that elementary teachers, mathematics programmes, textbooks and other students’ resources are aware of these relations between history – development of Measurement – and pedagogical aspects to teach Measurement, and in particularly, “Time”. They must be conscious, for example, that measuring time (as any other magnitude) is based on approximations, estimations and should not be treated, as an exact concept.

3 Mathematics Textbooks

A textbook of Elementary Mathematics, being a resource to support pupil’s process of learning, is also a vehicle for mathematics culture, fundamental concepts and methods of mathematical knowledge, which, having a consistent content sequence, promotes thinking and integrates that knowledge in various human activities.

Textbooks, playing a contextualized view of pedagogic discourse, reveal the interpretation that the author has of the discipline program, reflecting the importance that he/she attaches to the content, knowledge and techniques, regarding the objectives defined by the educational system. They are, therefore, individual projects to an educational system for all. Choppin (2004) argues that a textbook should be the privileged support of educational content, the depository of knowledge, techniques or skills that a social group believes needs to be communicated to new generations.

Moreover, textbooks, as bridges between curriculums and educational actors, reflect and legitimize the traditions of national culture. Apple (1993) asserts that the curriculum is never simply a neutral assemblage of knowledge, somehow appearing in the texts and classrooms of a nation. It is always part of a selective tradition, someone’s selection, some group’s vision of legitimate knowledge.

Thus the mathematics textbooks seem to dominate what students learn, as we previously acknowledged, since they are mediators of the meaning and content of which is defined at the mathematics program and being the teachers’ primary aid to organizing lessons and structuring the subject matter.

3.1 History of Mathematics and Mathematics Teaching

Several well-known authors refer the importance of using the History of Mathematics in Mathematics Teaching and point several reasons to use it. Jones (1989) states that the History of Mathematics, combined with mathematical knowledge allows us to teach the "whys" in Mathematics ("Chronological whys," "logical whys" and "teaching whys"), to understand and to motivate students by the nature and role of mathematics. Klerk (2004) argues that the context of science and technology in the teaching of mathematics allows to integrate the History of Mathematics and to emphasize the close relationship with nature and human culture. This vision of Mathematics, intrinsic to the evolution of science and humanity, motivates students to a particular subject and demonstrates the evolution of this science and shows the human side of Mathematics. It softens mathematics and motivates students in their study (Klerk, 2007). *With such historical approach one can guide students to the point of realizing that there is a relationship between mathematical matters … and the wide field of reality … Mathematics does not stand in isolation, but* *forms part of a much bigger reality relating to different real world contexts*. (Klerk, 2004)

In Struik' (1997) opinion, History of Mathematics helps us to appreciate our cultural heritage, understand the trends in Mathematics Teaching and helps to increase students interest. Using History of Mathematics in Mathematics Teaching facilitate to understand the learning difficulties, it improves instruction and creates a classroom climate of search and investigation; it humanizes Mathematics (Avital, l995). Swetz (1995) defends that knowing History of Mathematics helps us to understand the Mathematics evolution.

One the other hand, Sui (2000) argues that

perhaps it can be added that not only does the appropriate use of History of Mathematics help in teaching the subject, but that in this age of "mathematics for all", History of Mathematics is all the more important as an integral part of the subject to afford perspective and to present a fuller picture of what mathematics is to the public community.

3.2 Criteria

As we said at the beginning of this article, we are presenting a small part of a PhD degree project. It is our purpose to analyse the most used textbooks in Portuguese public schools, for Elementary Mathematics. To do so, we defined a set of criteria based on different suggestions provided by Sá et al (1999), Portuguese Ministry of Education (PME)[2] (2006), Rezat (2006) and Ponte (2007), defining main domains (table 1) to analyze mathematics textbooks. Other authors provided some ideas to analyze textbooks, in general: Chopin (1992) and Gérard and Roegiers (1998).

Sá et al	PME	Rezat	Ponte
Scientific	Scientific; linguistic	Content	Scientific; didactical
Pedagogical	Curriculum suitability	Linguistic	Text; illustrations
Technical	Programme suitability	Visual	Citizenship construction
	Pedagogical/didactical	Pedagogical	Editorial
	Technical	Situational	

Table 1. Suggested categories to analyze mathematics textbooks

These are necessary criteria to analyse any mathematics textbook, yet not sufficient to analyse Elementary Mathematics textbooks and the approach of fundamental concept of Measurement. Therefore our set of criteria is divided into three main groups: Scientific; Methodological (Presentation of the Measure, The learning of Measure development, Illustrations) and Measure Purpose, based on didactical, historical and scientific issues about Measure.

Two of the aims of this article are two criteria from Methodological aspects and from Measure Purpose:

1) to identify a historical approach, in textbooks, in as much as it contributes to promote and enrich the learning of the concept;

2) to analyze activities (problems and exercises) suggested by textbooks and requiring Measurement.

We will also consider, to analyze the historical approach of Measure in textbooks three of the four categories defined by Siu (2000):

A for anecdotes

[2] We assumed the PME's suggestions because they are defined by law, although those criteria refer to textbooks in general, and not specifically to mathematical textbooks.

An authentic and amusing or instructive anecdote, which carries a massage can be used as a catalyst. Eves, quoted by Siu (2000), argues that

these stories and anecdotes have proved very useful in the classroom - as little interest-rousing atoms, to add spice and a touch of entertainment, to introduce a human element, to inspire the student, to instill respect and admiration for the great creators, to yank back flagging interest to forge some links of cultural history, or to underline some concept or idea.

B for broad outline,

Useful to overview a topic or even of the whole course at the beginning, or to give a review at the end. Siu (2000) refers that, ideas in the history of the subject

can provide motivation and perspective so that students know what they are heading for or what they have covered, and how that relates to knowledge previously gained.

C for content

This category establishes a relation between culture, mathematics and its history, showing the evolution of concepts and promoting an instructive discussion about the content to be taught.

Although the study of the History of Mathematics has an intrinsic appeal of its own, its chief raison d'être is surely the illumination of mathematics itself ... to promote a more mature appreciation of [theories]. (Edwards, in Siu, 2000)

4 Analysis

We analysed the most used textbooks in the 2nd Cycle of Basic Education in Portugal during 2009/2010 - official information, provided by the Portuguese Ministry of Education:

Cycle	Grade	Title	Publisher
2nd	5th	Matemática 5	Porto Editora
	6th	Matemática 6	Porto Editora

Table 2. Analyzed textbooks (not certificated)

4.1 Historical approach

We can find “Historical Notes” in the beginning of all topics in the analyzed textbooks. However, they are not more than simple historical information - indicating events, facts and curiosities. None of these notes have any kind of connection, at least not referenced throughout the text, with the approach of the content on each topic. There are some interesting “Historical Notes”, yet not pedagogically or mathematically explored. We can find, at the beginning of the *Direct Proportionality* topic in the 6th grade textbook, an interesting “Historical Note” relating this concept with society, Portuguese Mathematics History (Pedro Nunes – whose work is extremely connected with Measurement and “Time”) and his studies about Euclides. But, once again, there is no more references to this note in the topic (fig. 1).

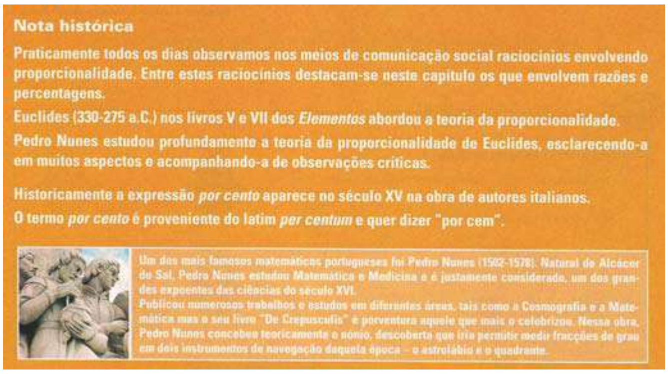

Figure 1. Historical note to introduce the *Direct Proportionality* topic
(in Neves, M.A. et al (2008). Matemática - 6º ano. Vol. II, p.38. Porto Editora)

Another example of a not explored historical approach, making no connection between its information and Measurement, in the 5th grade textbooks, is about numerical systems. Three different numerical systems (Egyptians, Babylonians and Chinese) are presented nevertheless with no further explanation or any purposed activity to find more about this theme, or to relate it with Measurement and "Time" (fig. 2).

Figure 2. Historical note to introduce the *Integers Numbers and decimal numbers* topic
(in Neves, M.A. et al (2008). Matemática - 5º ano. Vol. I, p.30. Porto Editora)

Other example of using History of Mathematics related with "Time" (fig. 3), is presented to introduce the chapter about *Angles*, in the 5th grade. This "Historical Note" induces student to a misconception: the division of a circle into 12 equal parts with 30º each (and later the idea of the accuracy of the length of a day and the concept of angle development – and the relation between hours and degrees) without contextualizing historically that conclusion.

Figure 3. Historical note to introduce the Angle chapter
(in Neves, M.A. et al (2008). Matemática - 5º ano. Vol. III, p.3. Porto Editora)

On the one hand, when considering a circle, with 360 º, divided into 12 equal parts (in this specific case relating months and years; but also with hours in the case of the clock) may raise some confusion related to:

- distinction between the true “Time” and the “Time” defined by Man;
- understanding the concept of angle (the relation between degrees and minutes; for example: when considering or representing a 360 degrees full turn of the minute hand, then 360 degrees are equal to 60 minutes, and typically 90 degrees, represented by a quarter-hour, 15 minutes will be associated. This is the idea that textbooks may develop in students when representing angles in watches. Later, students will learn that one degree is 60 minutes. This may lead to a misconception).

On the other hand, the signs of Zodiac, presented in the fig. 3 *no longer correspond to the actual positions of constellations* (Holford-Strevens, 2005) and here are historical indications that relate the sexagesimal system and the division of the day into 60 parts in their sub-multiples

The arithmetic of ancient Babylon was based on the number 60; accordingly, astronomers (despite the existence of double hours) divided the natural day into 60 parts, these parts in turn into 60^{ths}, and so on. (Holford-Strevens, 2005).

One other example, relating the concept of angle with clockwise movement, is presented at the beginning of the *Angle* topic (fig. 4): pupils are asked to look at a picture and notice that the balance pointers describe an angle.

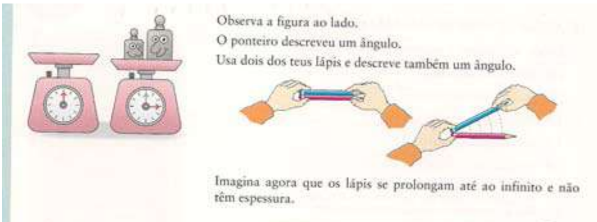

Figure 4. Introducing the concept of angle in the Angle chapter
(in Neves, M.A. et al (2008). Matemática - 5º ano. Vol. III, p.10. Porto Editora)

4.2 Activities

According to Mathematics Programme for 2nd Cycle of Basic Education, textbooks should contain numerical problems involving Measurement, and in particularly “Time”, demanding students to select the appropriate instrument and/or unit. However, pupils are not required to do so in context and there are no worked examples (only one exercise, in the 5th grade, asking the difference between two children’s ages in years and one more, in the 6th grade, asking “how many minutes are” – relating minutes, hours and fractions, see fig. 5). Once again it is implicit the connection between the position of the hands of a clock and the angle concept.

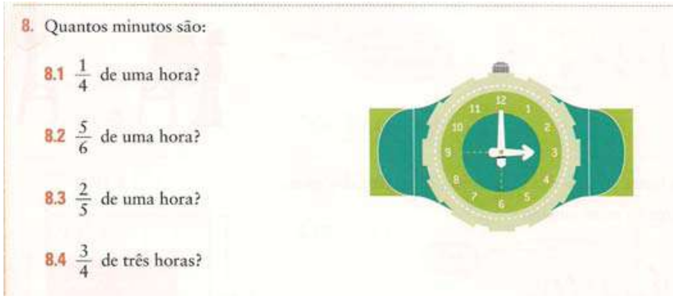

Figure 5. Exercise in Numbers and Calculus Topic
(in Neves, M.A. et al (2008). Matemática - 6º ano. Vol. II, p.11. Porto Editora).

Textbooks present some other exercises using Measure as a context (with different magnitudes) but in most of them the use of Measurement instruments is not necessary. Among a small number of exercises or activities requiring measurement we can find few soliciting the use of rulers, compasses or protractors (to construct geometric figures, to measure the length of objects and angles or the measures of an architectural project), but no other measurement instruments.

The 6th grade textbook introduces *Relative Numbers* with a linear representation of "Time", using negative numbers to represent years (fig. 6), without a reasonable explanation for that.
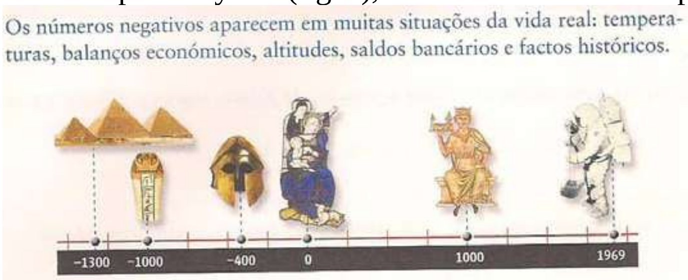
Figure 6. Example to represent negative numbers in real life
(in Neves, M.A. et al (2008). Matemática - 6º ano. Vol. III, p.74. Porto Editora)

In *Statistic* topic, textbooks for 5th and 6th grades, include several exercises with line charts for recording observations that evolve over "Time" (in hours or days of the week), as the Mathematics Programme recommends (Fig. 7). Though, those situations do not differ much from each other, in fact, activities/exercises (posing interpretation questions) are repetitive and there is no explanation of why using this type of graphic representation.

Figure 7. Line chart recording temperature over "Time" (hours)
(in Neves, M.A. et al (2008). Matemática - 5º ano. Vol. I, p.72. Porto Editora)

Some bar charts and pictograms are presented as well, recording observations evolving over days of the week. Still, some confusion, between discrete and continuous notions, may rise if to different types of graphic representation (discreet and continuous) of the same unit, day, are presented without explanation of this difference (Fig. 8).

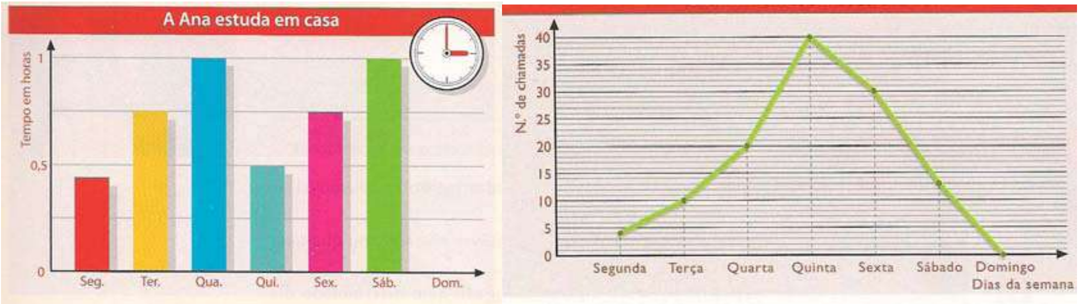

Figure 8. Bar chart recording "Time" of Ana's study (in hours) at different days and line chart recording the number of phone calls over days
(in Neves, M.A. et al (2008). Matemática - 6º ano. Vol. I, pp. 74 and 73. Porto Editora)

5 Some conclusions

The analyzed textbooks usually focus on telling "Time" rather than on developing the concept, which is not defined and is addressed in two topics: *Numbers and Calculus* and *Statistics*. It is clearly assumed by the textbooks' authors that "Time" is not considered a content, but an example, a context, a way to teach other contents (graphic representation, angles). Although this position may not be criticized, presenting certain situations related to the concepts of Measurement and "Time", whether as historical, as a content or context, should have a purpose.

The most used textbooks in 5th and 6th grades present a superficial view of the History of Mathematics, using it merely as an informal "wrapping", quite far from Siu's *B-broad outline* purpose, and there are examples of bad historical information, such as the example presented at fig. 3, that presents misconceptions.

It seems clear that Measurement is a powerful concept to use History of Mathematics and history of other fields (social, science, technology) – as we showed at section 3.2. – enlightening the evolution of this concept and "Time", promoting an instructive discussion and relating them with other concepts. However, the "Historical Notes" presented in the analyzed textbooks don't involve children in the concept construction and don't explore other dimensions of "Time", such as its subjectivity or its scientific facet. We continue therefore to attend to social-conventional knowledge of the time.

The exercises and activities presented in textbooks present "Time" in a context situation. However they are repetitive and don't require the selection of appropriate measuring instruments and its use (it brings up the clock as the sole instrument for measuring time). Estimation, in Measurement, is not considered in those exercises. It is clear that pupils are not involved in the construction of the concept. "Time" is given as the independent variable (fig. 7 and 8) and there are no textbooks that present activities where children could experience the transitivity and the unit iteration in the measurement of "Time".

These conclusions emphasize the need to use and develop other sources to prepare lessons and work in class. Pupils should be involved in activities with History of Mathematics related to the concept of Measure and, in particular, "Time", since elementary school. Also textbooks (and teachers) should present problems and exercises requiring the measurement of time from different perspectives and examples involving children in the construction of the concept and of the mathematical knowledge.

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MY TEACHING EXPERIMENTS IN THE HISTORY OF MATHEMATICS FOR THE *LICENCE* IN MATHEMATICS

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ABSTRACT

The new system LMD (licence, master, doctorate) ,in Algeria and specially in my university, gave an opportunity to introduce two courses in History of Mathematics during the second year of the licence in Mathematics.

As a teacher of those courses since three years, I discovered the sense, the difficulties, the motivation and the effects of this kind of courses, so different from a mathematic classical course.

How the Algerian students, completely closed concerning the history of mathematics because the inexistence of this education before the university and the poorness in books in this matter, perceived this course? What is the meaning I should give to this course making it attractive? What original ways I have to realize in the classroom for best educational effects

I try to answer these questions in this talk, enriched by my teaching experience.

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TEACHING HISTORY OF SCIENCE TO FUTURE TEACHER-TRAINERS: FIRST REPORT

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ABSTRACT

The academic year 2009-2010, a new course was integrated in the Master of Didactics at Paris 7 University. It was entitled "History of science in teaching and teacher-training".
Although this course can be seen as a natural development from the usual in-service teacher training which the M:A.T.H[1] group has been providing for thirty years, three features set it slightly apart. Firstly, the audience consists of in-service, experienced maths and physics/chemistry teachers, who aim at getting a Master's degree in order to become teacher-trainers (either part-time or full-time). Secondly, its size is significantly larger than that of standard 3-day teacher-training sessions. Thirdly, the may goals are to provide reflexive tools, and tools for self-training, by presenting the various issues related to the introduction of a historical perspective in the teaching of maths and the sciences.
In this talk, I described the schedule and outline of this course, discussed some of the works handed over by the "students", and presented a first assessment.

[1]M:A.T.H. is the history of maths group of the Paris 7 IREM (Institut de Recherche sur l'Enseignement des Mathématiques). This new course was created by Alain Bernard and Renaud Chorlay, with the assistance of Catherine Singh.

CLASSROOM EXPERIENCES WITH THE HISTORY OF MATHEMATICS

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ABSTRACT

Even though the history of mathematics is required in the lesson plan in Austria there are only rare possibilities to carry out historical projects or to discuss developments. By strenuous efforts as a grammar school teacher the author has realised some historical projects in the classroom, however. This paper will give a brief outline of the projects on the history of mathematics and focus on the problems and results. In doing so, classroom experiences with finished projects as tools for an approach to cultures and arts are introduced and an overview of the following topics is given:

- Adam Ries
- Austrian mathematicians
- Leonhard Euler
- Euclidian geometry
- Georg von Vega and logarithms
- School leaving examination - past, present, future

The paper will also present an excerpt of the pupils' work.

1 Introduction

The general part of the mathematics curriculum for secondary schools in Austria does call for historical considerations and developments in mathematics. The original text is shown below. The first part refers to the lower grades (ages 10 to 14), the second part concerns the higher grades (ages 15 to 18).

Historical considerations (lower grades):

Students are to gain insight into the development of mathematical terms and methods with the help of suitable topics. They should become acquainted with selected renowned historical mathematicians. Mathematics is to be presented as a dynamic science and its contribution to the development of western culture should be demonstrated. The significance of mathematics in the present should be stressed in the lessons.

Cultural-historical aspects (upper grades):

The essential role of mathematical findings and achievements in the development of European cultural and intellectual life makes mathematics an indispensable part of general education.[1]

These aspects are called for with regard to historical considerations: the development of terms and methods, famous mathematicians, mathematics as a dynamic science, and the importance of maths for the development of Western culture.

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[1] *„Historische Betrachtungen: (Unterstufe)*
Den Schülerinnen und Schülern ist an geeigneten Themen Einblick in die Entwicklung mathematischer Begriffe und Methoden zu geben. Sie sollen einige Persönlichkeiten der Mathematikgeschichte kennen lernen. Die Mathematik soll als dynamische Wissenschaft dargestellt und ihre Bedeutung bei der Entwicklung der abendländischen Kultur gezeigt werden. Die Bedeutung der Mathematik in der Gegenwart soll in den Unterricht einfließen.
Kulturell - historischer Aspekt: (Oberstufe)
Die maßgebliche Rolle mathematischer Erkenntnisse und Leistungen in der Entwicklung des europäischen Kultur- und Geisteslebens macht Mathematik zu einem unverzichtbaren Bestandteil der Allgemeinbildung.“ (http://www.bmukk.gv.at/medienpool/789/ahs14.pdf)

According to the Mathematics curriculum, teachers must deal with a great number of topics in a very short time, and they do feel the pressure of this lack of time. To make matters worse, several teaching hours have been cut during the last years. I remember my first years (30 years ago) as a teacher at the grammar school when I taught 34 maths lessons a week over the eight grades. Currently, mathematics is taught only in 28 lessons, a reduction of six valuable lessons. Despite this smaller numbers of lessons, there has been no reduction in the subject matter, however. In fact the contrary is the case. Many additional topics have been added, e. g. the use of CAS-systems, which is really up to date and increases the standard of maths-education, but it requires much more time for teaching. It is not surprising, therefore, that teachers resist the introduction of a program of history of mathematics. And most teachers may well ask: "Where do I find the time to teach history?" According to Fried: *"You do not need any extra time. Just give a historical problem directly related to the topic you are teaching, tell where it comes from; and send the students to read up its history. In this case the teacher is not forced to find extra time for extra material in an already overloaded program, and students are not forced to find extra time for extra homework"*.[2]

But for teachers it is not so easy to set historical problems for their pupils because of the way teachers are trained. At university, students are not required to listen to lectures on the history of mathematics or physics, and unfortunately there is not even a chair for the history of science in Vienna.

Fortunately, however, many young colleagues are really interested in the history of science and they are keen, for example, to connect science with ancient languages or arts. And it is really remarkable that the history of mathematics is offered to the students at the Vienna's UT.

I did my doctoral thesis on "Austrian mathematicians …" 20 years ago and I am quite enthusiastic about teaching the history of mathematics. According to Fried I like setting historical problems directly related to the topic, but during my 30 years of teaching I have come to realise that this is very difficult indeed. In fact, I did it partly by way of story-telling (e.g. fractions, number systems, trigonometry), and my experience is that story-telling is very motivating for pupils to enjoy mathematics. Even if the story-telling becomes more and more complex, the primary need for understanding the simple, archetypical stories remains strong nevertheless.

2 Projects

2.1 Adam Ries

The first modest "Adam Ries project" was carried out with 10-year-olds (1st grade) and it was directly related to the four basic arithmetical operations with natural numbers. The pupils learned "calculating on the lines". As an example, the addition of the numbers CCCVII, DCLXXVIII on the lines is shown below:

[2] Fried: Can Mathematics Education and History of Mathematics Coexist? In Science & Education 10 p. 394, 2001.

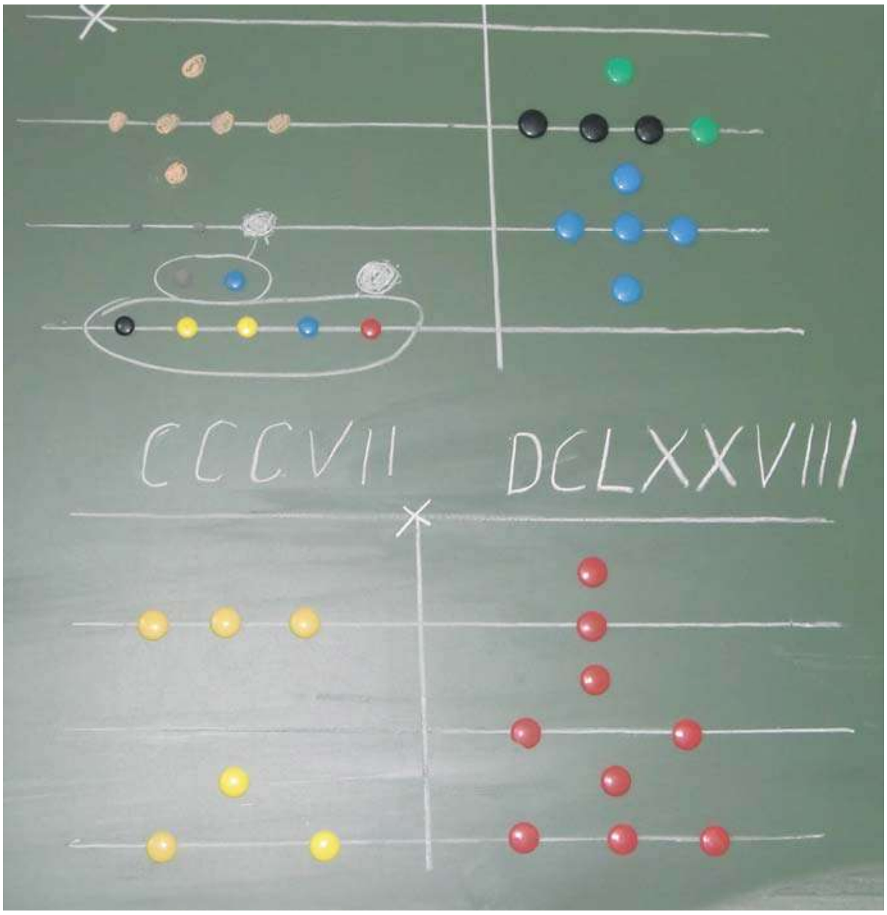


Figure 1: Calculating on the lines with beads (picture made by the author)

The lines show the place value from bottom to top. Beads in the interspace increase the value fivefold. As soon as there are five beads on the line, four are removed and one is moved into the interspace. Two beads in the interspace must be replaced by one bead on the next higher line.

Only 30 additional minutes were needed. As homework the pupils prepared a poster exhibition and short talks.

Another "Adam Ries project" was carried out with 13-year-olds (4th graders). A powerpoint presentation about the life and the reckoner books was done by the pupils on their own.

This project required only two lessons. One lesson was used to introduce the pupils to the school library's department of the history of maths, and the second lesson was dedicated to searching the internet for suitable sources for the preparation of a journey to the Adam Ries town of Annaberg-Buchholz in Sachsen-Anhalt, Germany. During the actual nine-hour bus ride a CD about Adam Ries' life and work was shown, and during their stay in Annaberg-Buchholz the pupils visited Adam Ries reckoner-school. There they were graduated to "Historian reckoner". The powerpoint-presentation was made, amongst other things, as a summary of the journey.

With this project mathematics was done outside of school, which helped to drum up interest in the history of mathematics. It was a great pleasure for me to see that the pupils were able to pass "Adam Ries" on to colleagues, friends and parents. Undoubtedly this project was a considerable contribution to the pupils' general education and even contained elements of social work.

2.2 Austrian mathematicians

With regard to Austria's millennium celebrations in 1996 this project was interdisciplinary. It concerned a group of 13-year-olds (4th graders) in the subjects mathematics, arts and technical drawing. The objective of the project was to find Austrian mathematicians and artists and to make a connection to the corresponding periods. The pupils worked together in groups to present the lives and works of famous Austrians. With great detail, an exhibition of the lives and works of Hermann von Kärnten, Johannes von Gmunden, Christoph Rudolff, Paul Guldin, Georg von Vega, Johann Blank, Kurt Gödel, Olga Tausky Todd was presented and a paper was published.

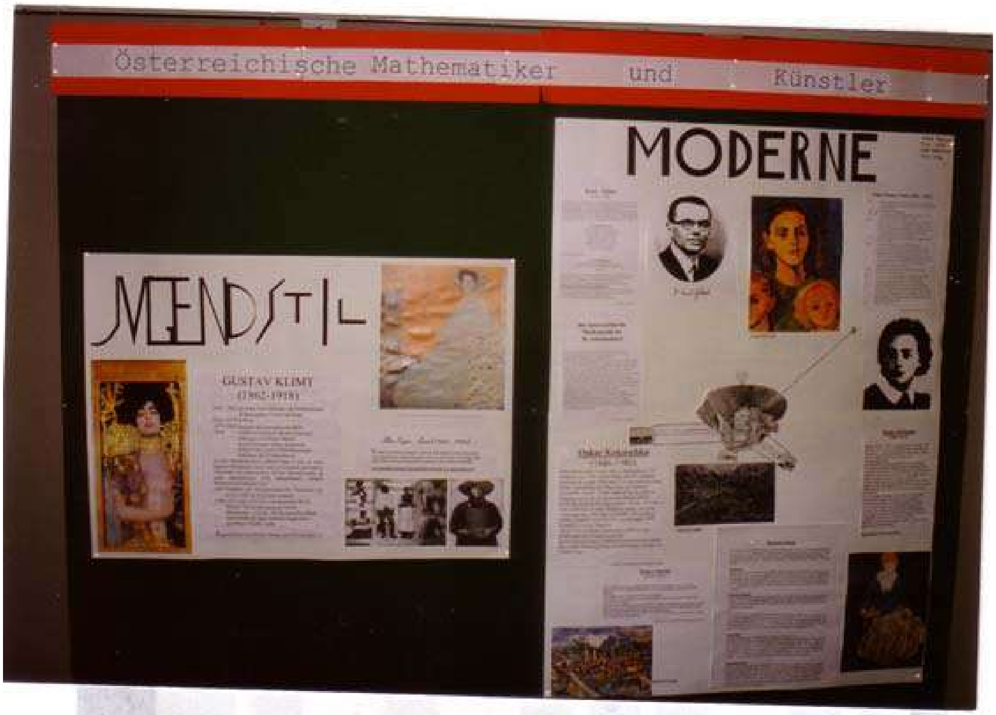


Figure 2: Picture from the exhibition (made by the author) "Austrian mathematicians and artists"

The project required about two weeks (16 lessons) in all three subjects. As sources books from the school library were used exclusively and considerable work was done as homework. This project made the pupils recognise the creative nature of mathematical enquiry. They gained an insight into the various techniques of research as well as the analysis and synthesis of mathematical history, and they learned to grapple with the study of biographies and the history of the subject.

2.3 Leonhard Euler

This project was conducted with 16-year-olds (6th graders) and it required ten additional lessons. On three afternoons the pupils worked together in groups of up to five people, researching "Euler" in the library and on the Internet.

They started with looking for "primary resource sites" like: Euler's Introductio and they found a number of "mutating sites" – which provided links to and advice on topics such as number theory and others. In addition, "accumulative and list sites" were very helpful as well. As a result of the project the pupils published a paper and a power-point presentation. An excerpt of the content of the paper is shown below:

1707	Born on April 15in Basel. Father: Paulus Euler – pastor. Mother: Margaretha Brucker
1713	Began the education at the Latin grammar school in Basel.
1720	Attended lectures at the University of Basel.
1722	School leaving exams. Attended mathematics lectures by Johann I. Bernoulli.
1723	Gained his master degree. (In his thesis he compared Newton's and Descartes's systems of natural philosophy.)
1726	Publication of his first mathematic paper in Leipzig.
1727	Appointment to the Academy of Sciences of St. Petersburg.
1731	Full member of the Academy and Professor of Physics.
1733	Professor of Mathematics .
1734	Collaborator within the scope of the Russian general map. January 17: marriage with Katharina Gsell, daughter of a painter originating from St. Petersburg.
1735	Collaboration in the administration of the Geographical Department of the St- Petersburg Academy.
1738	After suffering a nearly fatal disease he went blind on his right eye.
1741	Acceptance of the invitation of Frederick the Great of Prussia to join the Berlin Academy.
1745	Euler's father died in Riehen.
1746	Director of the mathematical class of the Berlin Academy.
1749	First personal meeting with Frederick the Great.
1750	The last of his 13 children was born. Euler took up his mother to Berlin.
1755	Euler becomes an elect member of the Paris Academy of Sciences.
1762	Invitation of Catherine the Great to return to St. Petersburg.
1766	Weak eyesight on the left eye. Euler returned to the St. Petersburg Academy and spent the rest of his life in St. Petersburg.
1771	Blindness.
1773	Death of Euler's wife.
1776	Marriage with the half-sister of his first wife.
1783	Death on September 18 due to an apoplectic stroke.

Figure 3: translated pupil's paper excerpt of Euler's life and work

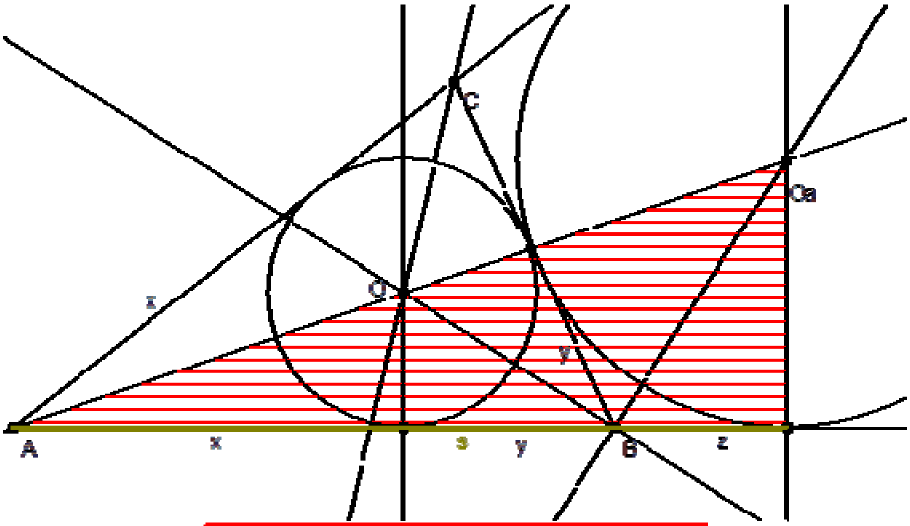
Furthermore Euler's Algebra, the seven bridges of Königsberg as well as Euler's autograph were studied among others.

The students learned that a line determined by any non-equilateral triangle which passes through the orthocenter, the circumcenter and the centroid is named after Euler. They proved this with congruent triangles by making use of the well known fact that the centroid divides the median line in a ratio of 2:1. Furthermore, they constructed the Feuerbach-circle and proved it.

I think that because of the project the pupils obtained a historically sensitive picture of Euler's work and an increased understanding and appreciation of constants, variables and functions. Furthermore, they received an impression of the number theory and of problems they had never heard before. The success of the "Euler-project" motivated the pupils to do a further project, and they chose the project described in the following paragraph.

2.4 Euclidian Geometry

As sources they used these well known works about geometry: Coxeter, Baptist as well as the newly published Scriba and an English translation of the "Elements". Elementary theorems were proved and the corresponding drawings were done with the "EUKLID" software. As an example, the geometric proof of Heron's formula for the area of a triangle is shown below:



$$\boxed{A = \sqrt{s(s-a)(s-b)(s-c)}}$$

Figure 4: Proof of Heron's formula constructed with "EUKLID": similar triangles, intersect theorems, s=(a+b+c)/2, x=s-a, y=s-b, z=s-c.

Plane geometry according to Euclid was not entirely new to the pupils, but as I watched them during the project I noticed their reactions and compared them with what Davis said in "The Education of a Mathematician" *"I loved the theorem proving portion of the textbook, and I loved to work the theorem proving problems that were set… If there was any one thing that hooked me on mathematics it was the approach to geometry of Euclid and the Greeks."*

Firstly, I noticed that the students were not able to offer geometrical proof. They were not aware that they had been doing Euclidian geometry ever since they entered grammar school. By studying the postulates and definitions in the "Elements" they did not see the necessity of definitions at first, but after studying some of the propositions repeatedly they were convinced of the basics of Euclidian geometry. Subsequently they took much pleasure in geometry, and even “fell in love” with it. They particularly liked drawing with the “EUKLID” software. They appreciated the accuracy of the software – as opposed to drawing by hand – and were happy to see that the sides of a triangle are really tangents to the inscribed circle. By constructing the Feuerbach-circle their happiness increased when they observed that the corresponding nine points are really on the periphery of the circle.

The students’ work exceeded the requirements of the curriculum, and they got a better insight into geometry than during standard lessons. They learned about the Nagel-point, the Gergonne-point, the Simpson-line and other phenomena. The standard of the students’ work was almost on a par with that of a “pre-scientific” paper. With a few corrections and only slight extensions it could be used as a project work for the school leaving certificate.

As this project was carried out with students from different classes, they embraced the opportunity to speak with their colleagues about their approach to geometry.

The success of the project is underlined by the fact that two years later one of the students chose this topic as the foundation of his “pre-scientific paper”.

Probably the following two projects will be realised in the future.

2.5 Georg von Vega and logarithms

VEGA was born in Slovenia and he spent most of his life in Vienna. He is well known in Slovenia, as his numerous memorials show, but he is not so famous in Austria, although a military barracks and a street in Vienna have been named after him. I think it is really worth studying VEGA’s calculation of logarithms and his calculation of 130 decimal places of π.

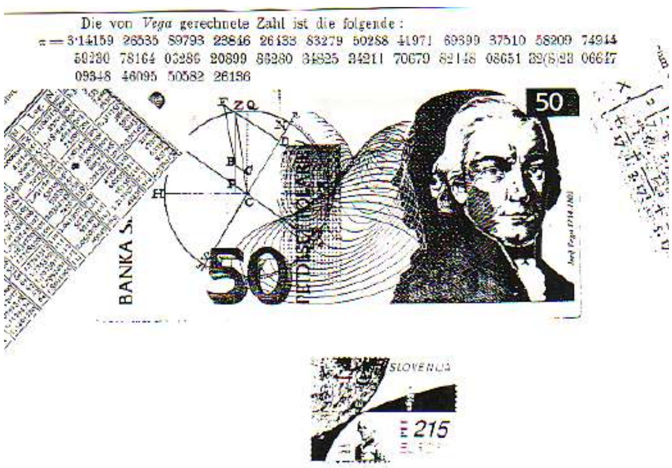


Figure 5: Slovenian Tolar bill, stamp dedicated to Vega, π, logarithmic table
(picture made by the author)

My idea is to hand out some original documents and books to 16-year-olds with the aim of making them work out a presentation of Vega's biography and especially about his logarithmic tables. Moreover, the pupils should be able to explain Vega’s method of the calculation of logarithms.

24. 3. 1754	- christening ceremony
1767 - 1775	- grammar school - Ljubljana
1775	- engineer
7. 4. 1780	- artillery
18. 11. 1780	- teachership of mathematics in the school for artillery
1782	- publication of his first book
1783	- publication: „Logarithmische, trigonometrische,und andere zum Gebrauche der Mathematik eingerichtete Tafeln und Formeln“
1. 4. 1784	- promotion
1. 3. 1787	- professor of mathematics ("Bombardierkorps")
1787	- marriage - Josepha Swoboda
1788	- war
1789	- siege of Belgrad - changed the mortar-batteries
1791	- son Heinrich was born
1793	- major
	- daughter Maria Theresia Regina was born
	- member of the academy of science in Erfurt
13. 10. 1793	- attacke - "Weißenburger Linien" - peace
10. 11. 1793	- conquest of Forts Louis - attest
1793	- publication: „Logarithmisch trigonometrisches Handbuch“
1794	- Stuttgart
	- writing member of the royal british society of science in Göttingen
1794	- publication: „Thesaurus“
1795	- Mannheim
11. 5. 1796	- knight of the Iron Cross
1796	- son Franz was born
7. 7. 1800	- wife Josepha died
22. 8. 1800	- baron
1802	- lieutenant colonel
26. 9. 1802	- Vega´s dead body was found in the Danube

Figure 6: Vega’s biography


LOGARITHMISCHE, TRIGONOMETRISCHE, und andere zum Gebrauche der Mathematik eingerichtete TAFELN UND FORMELN.

Herausgegeben von GEORG VEGA, Unterlieutenant und Lehrer der Mathematik bey dem K.K. zweyten Feld-Artillerie-Regiment.

WIEN, gedruckt bey Johann Thomas Edlen von Trattnern, kaiserl. königl. Hofbuchdruckern und Buchhändlern.

Figure 7: Vega’s logarithmic tables
(scan made by the author)

2.6 The school leaving exam – its past, present and future

This is the second project I hope I will be able to carry out. Pupils will be asked to study tasks of the school leaving exams of Einstein and of BRG Wiener Neustadt (from previous exams). These tasks will be provided to the pupils.

„Prüfungsarbeit in Geometrie: *19. September 1896 von 7:00 Uhr bis 11:00 Uhr*

Erste Aufgabe: *In einem Dreieck mit Umkreisradius r = 10 verhalten sich die Höhen wie 2:3:4. Berechne die Winkel und eine Seite.*

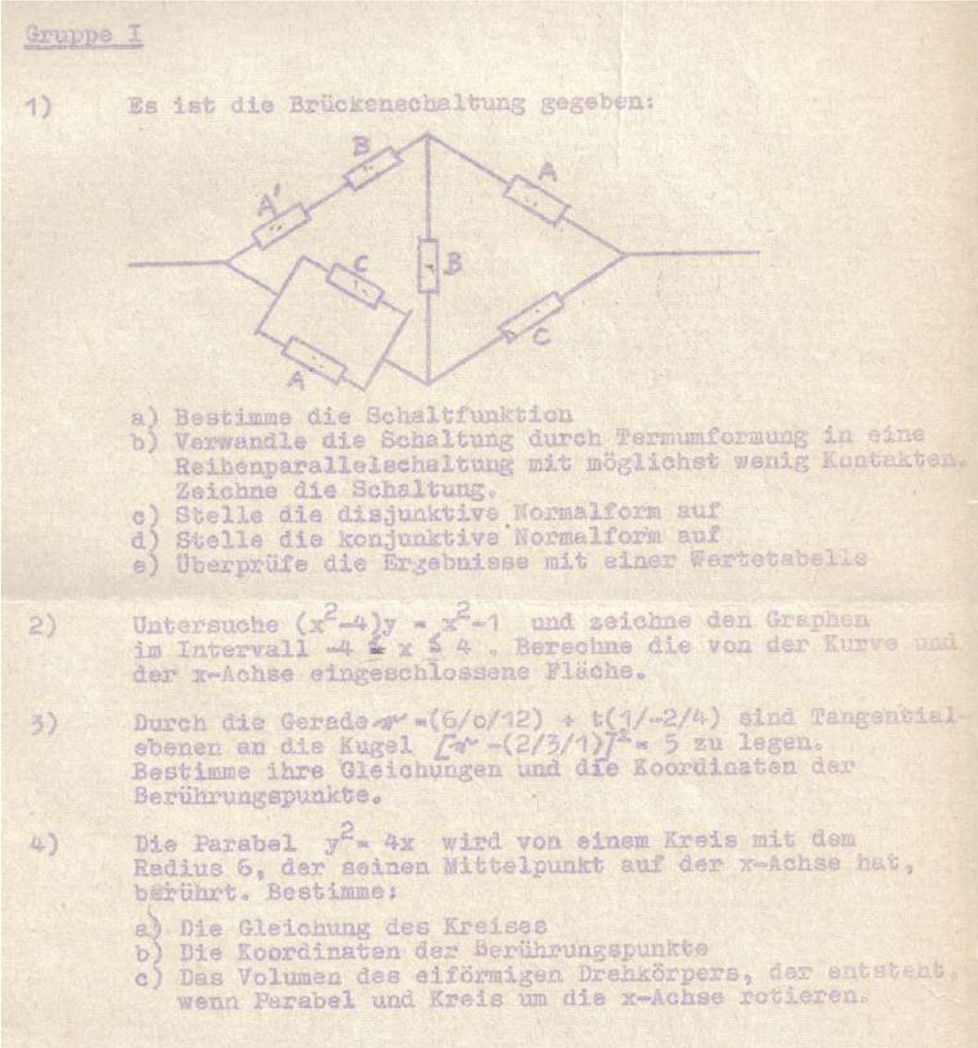
Zweite Aufgabe: *Gegeben ist ein Kreis mit Radius r dessen Mittelpunkt im Ursprung O eines rechtwinkligen Koordinatensystems liegt. Man zeichne senkrecht zur x-Achse Sehnen in diesen Kreis. Die Kreise, für die diese Sehnen Durchmesser sind, berühren die Ellipse mit den Halbachsen* $r\sqrt{2}$ *und r, und erst wenn der Abstand p ihrer Mittelpunkte von O einen gewissen maximalen Wert überschreitet, hört die Berührung auf. Man beweise diesen Satz und bestimme den Maximalwert von p.*

Prüfungsarbeit in Algebra: *21. September 1896 von 9:30 Uhr bis 11:30 Uhr*

Aufgabe: *Von einem Dreieck kennt man die Abstände l, m, n des Mittelpunktes des einbeschriebenen Kreises von den Ecken, man ermittle den Radius* ρ *des einbeschriebenen Kreises* $l = 1 \;\; m = \frac{1}{2} \;\; n = \frac{1}{3}$.“[3]

Figure 8: Einstein's school leaving exams

[3] Elemente der Mathematik Vol. 56 No. 2. , 2001, 45-54.



Gruppe I

1) Es ist die Brückenschaltung gegeben:

a) Bestimme die Schaltfunktion
b) Verwandle die Schaltung durch Termumformung in eine Reihenparallelschaltung mit möglichst wenig Kontakten. Zeichne die Schaltung.
c) Stelle die disjunktive Normalform auf
d) Stelle die konjunktive Normalform auf
e) Überprüfe die Ergebnisse mit einer Wertetabelle

2) Untersuche $(x^2-4)y = x^2-1$ und zeichne den Graphen im Intervall $-4 \leq x \leq 4$. Berechne die von der Kurve und der x-Achse eingeschlossene Fläche.

3) Durch die Gerade $\vec{x} = (6/0/12) + t(1/-2/4)$ sind Tangentialebenen an die Kugel $[\vec{x} - (2/3/1)]^2 = 5$ zu legen. Bestimme ihre Gleichungen und die Koordinaten der Berührungspunkte.

4) Die Parabel $y^2 = 4x$ wird von einem Kreis mit dem Radius 6, der seinen Mittelpunkt auf der x-Achse hat, berührt. Bestimme:

a) Die Gleichung des Kreises
b) Die Koordinaten der Berührungspunkte
c) Das Volumen des eiförmigen Drehkörpers, der entsteht, wenn Parabel und Kreis um die x-Achse rotieren.

Figure 9: BRG 1975 school leaving exams

It would make sense to do this project in the last class of grammar school but there is the problem of only three maths lessons a week, some of which will have to be cancelled due to tests and projects in other subjects. Experience has shown that the average number of weekly lessons is rather closer to two. So I am not very optimistic as far as this project is concerned, but I will keep it in mind. I am convinced that in view of the future centralised version of the leaving examination it is important that pupils "remember" the tasks set in the "old" version in order to see and judge the developments.

3 Conclusion

I think that with the help of the projects I have presented the pupils did not only learn about history from the different topics, but they were also confronted with different approaches to historical problems and they saw ways and methods that made more educational sense than modern ones.

The requirement in the national curriculum "to become acquainted with historical mathematicians" was no doubt fulfilled. By studying the biographies, the students learned more details about the spirit of the age and historical motivations. During the “Euclidian Geometry project” they came to understand the importance of the ancient Greek mathematicians in the development of European culture.

For triangle geometry, which is a part of the curriculum at nearly every level in Austrian schools, many relevant historical problems could be found. The “Adam Ries Project” showed that the history of mathematics is an essential part of general education. From Euler’s "Introductio” they got a good idea of the genius of this very important mathematician. While working with the “Introductio”, the students’ understanding and appreciation of functions increased significantly. Studying these historical mathematical texts can be compared with studying the great works of prose and poetry in literature. There is no doubt that this method, which – according to Fried (2001, p.401) - is called “radical accommodation”, helps to humanise mathematics.

Furthermore, the students learned to understand important developments from the Middle Ages to the present time as laid down by the “Austrian Mathematicians' Project”. The students had never heard about the biographies and works of some of the Austrian mathematicians before, and so this project contributed considerably to their appreciation of their native land and its people.

Finally, this project meets the demands for general education in the national curriculum. Simultaneously, the students learned that human beings thought differently in the past and that, by implication, they will think differently in the future as well.

This glimpse at my projects, which do not claim to be perfect, just offers a very basic framework for teaching the history of mathematics. I think a desirable objective for the future would be to analyse the development of mathematics in the context of its function in culture and politics.

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HISTORICAL CONFLICTS AND SUBTLETIES WITH THE $\sqrt{}$ SIGN IN TEXTBOOKS

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ABSTRACT

The ambiguity of the square root sign is a problem with historical origins. In this paper some of these historic origins are traced and discussed so as to enable us to give an account of the problem in two dimensions: current mathematical developments and the teaching proposal as it is shown in textbooks today. In conclusion, it can be said that although the problem does not exist from the point of view of current developments in mathematics, one cannot say the same thing from the point of view of mathematical education. On the contrary, in our study we have identified conceptual and operative approaches, and also omissions, with the √ sign, in modern and representative Spanish textbooks that may be the cause of deeply-rooted misunderstandings and conflicts.

1 Introduction

The ambiguity of the square root is a problem with historical origins, as we can see in important and influential textbooks published at about the time the school system began to be reorganized into a general system of education, with repercussions as regards the way of organizing present-day elementary school teaching of algebra.

Although the problem does not exist from the point of view of present-day developments in mathematics, one cannot say the same thing from the point of view of mathematical education.

On the contrary, the learning of the square root and the sense of use of the radical sign, √, present conflicts and subtleties as a consequence of the change in the sign's meaning when passing from arithmetic to algebra, and when going on to work with positive to negative numbers. This often goes unnoticed by teachers and textbooks.

This lack of perception may be the cause of conceptual and operative misunderstandings experienced by teachers and students. For example, students' opinion about the statement $\sqrt{\mathbf{25}} = \pm\ 25$ (Roach, Gibson & Weber, 2004), or $(-8)^{1/3}=-2$ (Goel & Robillard, 1997; Tirosh & Even, 1997), or the rule for multiplying square roots $\sqrt{a}\times\sqrt{b}$ when a and b are negative numbers: $\sqrt{-4}\times\sqrt{-9}=-6\neq\sqrt{-4\times-9}=6$ (Martínez, 2007), and the conflict with equality $\sqrt[6]{3^2}=\sqrt[3]{3}$ because on the left the index of the root is even, so that it has two opposing roots, whereas on the right the index is odd so it only one root (Gómez & Buhlea, 2009; Necula & Gómez 2009).

One way to give an account of the problem is "going back to history to study the evolution of algebraic ideas, analyzing historical texts as cognitions in the same way that we analyze students' productions, which in turn constitute mathematical texts" (Gallardo, 2008).

2 Historical traces of ambiguity in the square root operation and in the radical sign √

There are examples of ambiguity in the square root to be found in the teaching tradition that appears in such influential textbooks as like Euler's *Elements of algebra* (1822), Peacock's *Treatise on Algebra* (1845), De Morgan's *Elements of algebra* (1837) and Lacroix's *Elements of Algebra* (1831). Euler considered that "the square root of any number always has two values, one positive and the other negative; that √4 , for example, is both + 2 and - 2, and that, in general, we may take -√a as well as +√a for the square root of a" (Euler, 1822, p. 44).

In this text of Euler's, the square root operation and the radical symbol are ambiguous in their value, because they have two numerical values, in this case +2 and -2. Not only that, but the radical symbol √ is used with a duality of meaning and sense of use, which is different when it goes with a number √4 or with a letter √a. In √4 it is associated with the set of two numbers: ±2 and in +√a it is associated only with one number, an absolute value susceptible to the + or – sign. And in √4 it is used in order to indicate an operation in an abbreviated way, but in +√a it is used in order to express the result of this operation. In short:

1. Ambiguity of the operation
   - The square root of 4 is not one number, but two: +2 and -2.
   - √4 has two numerical values: ±2.
2. Duality of meaning of the symbol √
   - √4 is associated with the set of two numbers: ±2.
   - √a is associated only with one number, an absolute value susceptible to the + or - sign.
3. Duality in the sense of use of the symbol √:
   - √4 is used to indicate an operation.
   - √a is used to express the result of the operation.

On moving from arithmetic to algebra, the signs take on new sense in use

As we can see in Euler's text, on moving from arithmetic to algebra the square root symbol acquires new meaning and sense in use. In arithmetic, the square root of 4 can be found and it is unique, 2, which is written √4. Things change in algebra, since the square root of *a* cannot be calculated, so that to indicate its value the expression $\sqrt{a}$ is introduced, which not only represents an indicated operation but a result[1].

Traces of this change of sense in use can be found, for example, in the *Summa* (1494) by Pacioli who indicates the addition and subtraction of quantities of the same nature as 3x and 4x with words from the vernacular language: "with" and "from". On the other hand, addition and subtraction of non-homogenous quantities such as 3x and $4x^2$ are expressed with a symbol "p" or "m", which not only indicate the operation but also its result: "4co with 3co we shall say make 7co, and … 3co from 7co we shall say subtract 4co, because they are of the same nature … . if we wish to know 3co with 4ce, we shall say that they are 3co p 4ce" (op. cit. Distinctio octava. Tractatus Primus, p. 112).

Traces of conflicts with ambiguity in the square root operation

In the 19th century, when today's bases of mathematics were being set down, many of the

[1] The point of view of the psychology of the mathematics education supports this interpretation with the fact that a mathematical entity can be seen as an object and a process. Treating a mathematical notion as an object leads to a type of conception called structural, whereas interpreting a notion as a process implies a conception called operational (see Kieran (2006) and Sfard (1991), for example).

most influential mathematicians such as Peacock, De Morgan and Lacroix, recognised that the square root operation was ambiguous. For example, Peacock[2] says: "In passing from the square to the square roots, we shall always find two roots, which only differ from each in their sign" (Peacock, 1845, vol. II. p. 67).

And then, Peacock extends the ambiguity of the square root operation to two equivalent forms of symbolic notation: $a^{½}$ and $\sqrt{a}$. He says: "It follows, therefore, that $\left(a^2\right)^{\frac{1}{2}} = a^{\frac{2}{2}} = a = \sqrt{a^2}$ * (*The square root of a^2 may be –a as well as +a) (op. cit. p. 67). In this paragraph, Peacock seems to have us understand that: $\left(x^2\right)^{\frac{1}{2}} = \sqrt{x^2} = \pm x$.

Unlike Peacock, De Morgan proposes differentiating the two forms of symbolic notation for the square root: $\sqrt{a}$ and $a^{½}$: "Having two symbols to indicate the root of a, namely, $\sqrt[n]{a}$ and $a^{1/n}$, we shall employ the first in the simple arithmetical sense, and the second to denote any one of the algebraical roots, that is, any one we please, unless some particular root be specified. Thus $\sqrt{4}$ is 2, without any reference to sign; but $(4)^{½}$ may be either +2 or – 2" (De Morgan, 1837, p. 122-123). That is, De Morgan tells us: $(4)^{\frac{1}{2}} = \pm 2$ and $\sqrt{4} = 2$.

On the other hand, Lacroix recommends a general rule for the sign that affects the square root: "The double sign ± is to be considered as affecting the square root of every quantity whatever" (Lacroix, 1831, p. 122).

This rule is accompanied by a subtle question: "It may be here asked, why x, as it is the square root of x^2, is not also affected with the double sign ±?" (op. cit. p. 122).

This is the same as: why $\sqrt{x^2} = x$ and not $\pm x$? His answer was: "if in resolving the equation $x^2=b$, we write $\pm x = \pm\sqrt{b}$, and arrange these expressions in all the different ways, of which they are capable, namely: $+x = +\sqrt{b}$; $+x = -\sqrt{b}$; $-x = -\sqrt{b}$; $-x = +\sqrt{b}$, we come to no new result, since by transposing all the terms of the equations: $-x = -\sqrt{b}$, $-x = +\sqrt{b}$, or which is the same thing, by changing all the signs, these equations become identical with the first" (op. cit. p. 123).

Two epistemological conceptions emerge from the above. One is mentioned by Euler, where ambiguity affects the value of the radical sign and the square root operation (there is no difference between them and the sign and the operation are interchangeable):

$$\sqrt{4} = \pm 2 \quad \text{and the square root of } 4 = \pm 2$$

The other is seen in De Morgan's text, where ambiguity does not affect at the value of the radical sign:

$$\sqrt{4} = 2 \ \text{ but } \left(4^2\right)^{\frac{1}{2}} = \pm 2$$

Furthermore, a rule is emphasised in order to solve equations of the type $x^2=a$, for which the double sign ± does not affect the square root of x^2:

$$x^2 = 9 \rightarrow x = \pm\sqrt{9} \rightarrow x = \pm 3$$

3 A mathematical problem and an educational problem

These "epistemological / historical" conceptions reveal a mathematical problem and an educational problem.

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[2] A *Treatise on Algebra* by George Peacock is the result of Peacock's desire to draft a text with which his students could make sense of the emergence of the algebraic sign system (Gallardo, 2008)

The mathematical problem is that they violate formal requisites: one is the general requirements for any mathematical definitions and another is the requirements of the definitions of mathematical operations with real numbers (see Even & Tirosh, 1995).

The educational problem is that they create conflicts and misunderstandings in students, teachers, and even in school textbooks. For example, if $\sqrt{a^2} = \pm a$, then

$$\sqrt[6]{3^2} \neq \sqrt[3]{3} \Leftrightarrow 3^{\frac{2}{6}} \neq 3^{\frac{1}{3}}$$

as the index of the first radicand is an even number, two solutions exist (one being the opposite of the other) but in the second case, the index is an odd number and therefore there is a single root (see this conflict in the case of Patricia, a high school mathematics teacher in Gómez and Buhlea 2009, or Necula and Gómez, 2009).

Consequently, it can be said that $\sqrt[n]{a^m} \neq \sqrt[kn]{a^{km}}$, when *kn* is even and *n* is odd. But, this statement contradicts the definition of the rational exponent: a^r, $r \in Q$, since this must not depend on the representatives of the numbers involved in the operation.

$$\text{If } r = \frac{m}{n} = \frac{km}{kn} \text{ then it must follow that } a^r = a^{\frac{m}{n}} = \sqrt[n]{a^m} = a^{\frac{km}{kn}} = \sqrt[kn]{a^{km}}$$

Furthermore, $\sqrt{x^2} = \pm x$ violates the general requirements of the definitions of mathematical operations on real numbers. The basic arithmetic operations of addition and multiplication by a number different from zero establish univalent functions[3]:

$$x \to x + a, \quad x \to x \cdot a, \quad a \neq 0.$$

These functions have unique inverse functions corresponding to the inverse operations. However, the operation: $x \to x^2$ does not establish an univalent function; because $x^2 = (-x)^2$, and therefore it does not have an inverse function[4]. So, the operation: $x \to$ square root of x does not establish a function.

In order to solve the problem of the ambiguity of the radical symbol, mathematicians have decided to assign to the expression $\sqrt[2]{x}$, $x \geq 0$ only one value, one of the roots of x, the non-negative root, the one that they name principal root[5]. With this restriction, the correct thing is to write $\sqrt{4} = 2$, not ± 2.

But this does not signify that $\sqrt{x^2} = x$, because this gives rise to incoherencies that usually go unnoticed: $\sqrt{x^2} = x \to -2 = \sqrt{(-2)^2} = \sqrt{4} = \sqrt{(+2)^2} = +2$. The correct value from the point of view of current mathematics is $\sqrt{x^2} = |x|$, not $\sqrt{x^2} = x$.

With these two decisions, the mathematical problem of the ambiguity of the $\sqrt{}$ sign disappears, but not the educational problem.

The educational problem

The educational problem has to do with the way in which the decision that mathematicians have taken is transmitted and comes to the students, because students do not learn only what they are told; much of students' learning occurs when they attempt to make sense of the mathematical situations they encounter (Roach, Gibson & Weber. 1994). The

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[3] For each element in the domain there should be only one element image.

[4] To be inverted, the function $x \to x^2$, should fulfill the univalent requirement. So, it has to be confined to a positive or negative range.

[5] See, for example, Lang, 1971. p. 10

situations that students find in their experience, when learning, are the ones that appear in textbooks they use.

But what do the textbooks say? Spanish textbooks reflect the ambiguity observed in Euler's text Usually, the lesson about the square root begins with a generalisation of the arithmetic definition, followed by the denomination of new symbols introduced: "$\sqrt[n]{a} = b$ if $b^n = a$; $\sqrt[n]{a}$, is called radical; a, radicand, and n the root's index" (Anaya, 2004, p. 52). Following on from the definition, it is said that the square root and the power of two are inverse operations. So: $\sqrt{b^2} = b$

It is also then said that when the power is even for a positive number there are two numbers that agree with the definition, and therefore $\sqrt{36} = \pm 6$

But, the + sign can be omitted, such that √4 "only refers to the positive root: √4=2" (Anaya, 2004, p. 52). However, to distinguish negative roots, the – sign is written before the radical: $-\sqrt{a}$

In addition, in modern textbooks it is said that to solve equations of the type x^2=b there are two ways to focus on this. In the first one, x is directly cleared by finding its value, and in the second one, the square root is taken on both sides of the equals sign.

In both ways there are some textbooks that show traces of the historical and epistemological conceptions previously identified. These are:

1.a) Cleared by finding the value without writing the ± sign in front of the radical (ignoring Lacroix's rule: $x^2 = b \to x = \pm\sqrt{b}$) but putting it in front of its numerical value as in Euler's text.

Example: "$2x^2 - 8 = 0$ cleared $x: x^2 = \frac{8}{2} = 4 \to x = \sqrt{4} = \pm 2$" (Santillana, 1999, p. 64).

1.b) Finding the value and writing the ± sign in front of the radical, as in Lacroix's rule.

Example: "$3x^2 - 48 = 0 \to 3x^2 = 48 \to x^2 = 16 \to x = \pm\sqrt{16} = \pm 4$"(Anaya, 2004, p. 102).

2. Taking the square root on both sides of the equals sign without putting the ± sign in front of either of the radical signs that are obtained from it, in a duality sense of use, different when it goes with a number or with a letter.

Example: "$(x+3)^2 = 169$. We extract the square root of two members: $\sqrt{(x+3)^2} = \sqrt{169} \to x + 3 = \pm 13$" (Santillana, 2003, p. 47).

Although the answer is correct in the three options, one cannot say the same of the development shown, since they omit intermediate steps that do not take into account the current mathematical definitions: √4=2 and $\sqrt{x^2} = |x|$, which are necessary to avoid conceptual and operative misunderstandings experienced by students, something that often goes unnoticed by teachers and textbooks authors.

For example, the sequence 1.a, $x^2 = 4 \to x = \sqrt{4} = \pm 2$, reinforces the notion that $\sqrt{x^2} = x$ and $\sqrt{4} = \pm 2$, and does not take into account that $\sqrt{x^2} = |x|$ and √4=2. According to Roach, et al. (2004), the omitted steps are as follows: $x^2 = 4 \to \sqrt{x^2} = \sqrt{4} \to |x| = 2 \to x = \pm 2$

The sequence 1.b, $x^2 = 16 \to x = \pm\sqrt{16} = \pm 4$ also reinforces the notion that $\sqrt{x^2} = x$, and the omitted steps are $x^2 = 16 \to \sqrt{x^2} = \sqrt{16} \to |x| = 4 \to x = \pm 4$

The sequence 2, $(x+3)^2 = 169 \to \sqrt{(x+3)^2} = \sqrt{169} \to x + 3 = \pm 13$, again reinforces $\sqrt{x^2} = x$ and $\sqrt{169} = \pm 13$. In this case, the omitted steps are:

$$(x+3)^2 = 169 \to \sqrt{(x+3)^2} = \sqrt{169} \to |x+3| = 13 \to x + 3 = \pm 13$$

Conclusions

The square root symbol presents educational and mathematical problems with historic origins. Although the problem does not exist from the point of view of current developments in mathematics, one cannot say the same from the point of view of mathematical education.

On the contrary, in our study we have identified in current and representative Spanish textbooks misunderstandings related with historical and epistemological conceptions and sense in the use of the "radical" sign that are deeply rooted.

A review of Spanish texts shows that the teaching proposal reflects the ambiguity of the radical sign used in the expression $\sqrt{4} = \pm 2$, and does not take into account the current mathematical development $\sqrt{x^2} = |x|$ and √4=2. .

Finally, the important educational implication that should be pointed out is that in any educational proposal that aims to avoid misunderstandings such as the one evidenced here, the formal definition of radical must be considered, and it must be ensured that students understand the reasons for this definition.

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CONTRIBUTION OF CRYPTOGRAPHY TO MATHEMATICS TEACHING

A way to illustrate mathematics as a living science

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ABSTRACT

This paper examines the question to what extent cryptography and its history can be used to promote the idea of mathematics as a living science in mathematics education. Our investigations are substantiated in the context of the RSA cryptosystem and its mathematical background. Cryptography proves to be a suitable example for pointing out the development of the mathematical sciences triggered by development outside of mathematics – in this case the invention and usage of computers. The nature of problems solved and still open can broaden the students' view of mathematics as a living science that is still developing. The given considerations may be used as a framework for a teaching module about RSA as well as a subsequent occasion to reflect upon modern cryptography (and mathematics) for those who already know RSA.

1 Introduction

A project aimed to bring current mathematics into schools forms the background of the considerations presented in this paper. The teaching experiment corresponding to the project is an additional course in cryptography, students choose as part of their Abitur[1]. The course is a two-semester course consisting of weekly lessons a 3 x 45 min lessons, i.e., 135 minutes each week. The course was tested in two high-schools (gymnasium) in Berlin with 12 and 14 participants, respectively. The high-school students were about 18 years old and had no prior knowledge of cryptography, of number theory, or of congruences in particular. The teaching unit outlined in this paper is extracted and redesigned referring to the realisation and reflection of the original teaching experiment. It comprises 10 up to 12 units a 90 minutes and can be taught independently, e.g. as workshop.

Large parts of the German standard curriculum (e.g. [11]) do not extend beyond the scientific knowledge of the 19$^{\text{th}}$ century. Modern elements of mathematics are often limited to probability theory and some elective parts, that are not mandatory for all students. Teaching cryptography can contribute to this issue in different ways. Still "…mathematical education must follow, at least to some degree, what happens in mathematical research", as Lovász reasons in his article "Trends in mathematics: How they could change education?"[8]. He uses especially the term *algorithmic* mathematics (opposite to *structural* mathematics) to characterise the development in many branches of mathematics and applications through the use of computers: "it enriches several classical branches of mathematics with new insight, new kinds of problems, and new approaches to solve these". Mathematical activities in class that illustrate this, hence

[1] general German qualification for university entrance

give an insight into (the development of) modern mathematics in general. Thus, in this paper RSA is analysed in order to identify such possible ideas, techniques or algorithms.

As an example for modern cryptography, RSA requires only little mathematical background (predominantly elementary number theory) in order to be understood. That does not only refer to the cryptosystem itself, but to the process of the corresponding *development* as well. In contrast to the often found image of mathematics, the development of modern cryptography demonstrates that mathematics has no fixed structure or just needs some completion in order to give some helpful applications to other sciences. It is an example of how the development in mathematics is very often triggered by questions posed by other sciences, technological advances, or even social developments.[2]

In order to shift the focus on to the process of development as well as the RSA-algorithm itself, the following outlined teaching unit concentrates on the question *Why did it take more than 2000 years to invent RSA?* First RSA is motivated by a historical description of a classical problem of cryptography. Second we introduce RSA and answer the posed question. Third the influence of computers on the development of RSA is shown in more detail. Finally, aspects that may be helpful to broaden the students' view of mathematics are highlighted.[3]

2 Historical and practical background – Motivation

Cryptography has been used for several thousand years [1]. Some well known ciphers used in different centuries are named in fig. 2. They can be found in most popular and didactic literature about cryptography or cryptography in the classroom. However, all ciphers up to the 1970's are connected by one significant problem: the key exchange problem.

An illustration of the problem is a letter of an American soldier imprisoned by the Japanese sent to his family during the Second World War (fig. 1). The written text hides additional information that can be found by reading only the first two words in each line. But there was no way to transmit this information – the key – separately and securely without the knowledge of the Japanese. In this case the hidden message was puzzled out.

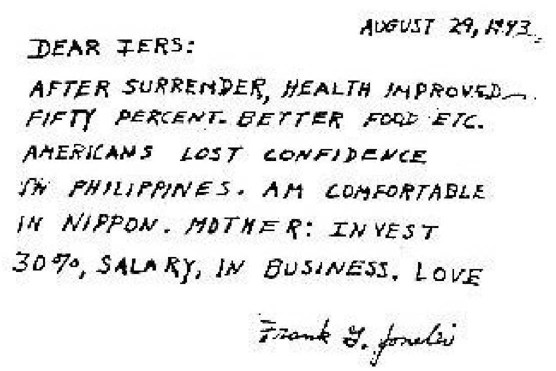
AUGUST 29, 1943.
DEAR IERS:
AFTER SURRENDER, HEALTH IMPROVED
FIFTY PERCENT. BETTER FOOD ETC.
AMERICANS LOST CONFIDENCE
IN PHILIPPINES. AM COMFORTABLE
IN NIPPON. MOTHER: INVEST
30%, SALARY, IN BUSINESS. LOVE
Frank G. Jonelis

Figure 1: Postcard of an American soldier [7]

Obviously, for every day use of cryptography, this is not an option. Therefore until 30 years ago, information was encrypted solely according to the following principle. A message M is encoded by an invertible

[2]An example is physics and its influence on the developments of calculus (mechanics) in the 18^{th} century, which also appears in teaching practise. Other examples are functional analysis (quantum mechanics) or differential geometry (general theory of relativity).

[3]In the opening plenary of ESU-6 Jankvist presented an empirical study about the use of "history as a goal" in mathematics teaching. One of his teaching units is about RSA. Therefore some of the results are quite related [5].

parametric function E into a cipher text $C = E_K(M)$. The parameter K is called the key and needs to be kept secret. The recipient decodes / decrypts the message with the inverse function D so that $D_K(C) = M$. The parameter K for the construction of E and D must be transmitted via a secure channel (personally in advance, by a messenger, etc.).

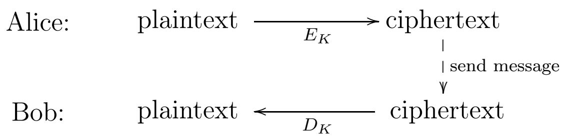

Figure 2: Data exchange with classic cryptography

Secure data exchange between strangers over an insecure communication channel is not possible this way. To overcome this problem, a cryptographic revolution was needed. With the development of the Internet this became of paramount importance as in principle all information sent over the Internet can be intercepted or monitored.

The revolution in cryptography that overcame the key exchange problem started in 1976. Diffie and Hellman published the idea of an algorithm including a pair of different keys instead of one, K_e for encryption and K_d for decryption [3]. From the knowledge of one of the keys, one cannot derive the other. Therefore, K_e can be transmitted via a public channel (hence the name: public key algorithm). An algorithm that implements this idea was published by Rivest, Shamir, and Adleman in 1978 [10], and is known as the RSA cryptosystem.[4] This leads to the following leading question for the teaching unit:

3 Why did it take more than 2000 years to invent RSA?

The development of public key cryptography can be seen as long term problem solving. To underline the similarities to problem solving in the classroom the answer is divided into three parts. We need to know, what is so revolutionary about Diffie's and Hellman's idea and later on how it was implemented. This leads to the principle of RSA, but covers less than 90% of the answer (considering the time interval of more than 2000 years). Practical aspects will complete the answer and illustrate how external driving forces influence development in mathematics.

It is less trivial than it seems to start thinking about a pair of keys instead of one key. For a function with the key parameter K it is

$$D_K(E_K(M) = M.$$

But how to find a suitable function, that offers different keys for encryption and description

$$D_{K_d}(E_{K_e}(M) = M?$$

Suitable means especially hard to invert even if one of the keys is known. To illustrate the difficulty, students were given the following exercise.

> *Exercise: Find candidates of suitable functions. Create a list of different kinds of functions and their inverses. Which of them fulfil the criteria of being difficult to invert and fast to calculate?*

[4]Independently from Rivest, Shamir, and Adleman the algorithm was found but not published for military reasons by Clifford Cocks in 1973. In contrast to the three authors above Cocks only needed one day to find a suitable algorithm, when he got to know the idea of the public key algorithm, James Ellis wrote down in 1969 (and Diffie and Hellman 1976) [12]. This fact illustrates that even in mathematical science problem solving is a very individual and not predictable process.

Some well-known functions and their inverses are subsequently listed.

$$\begin{array}{ll} f(x) = x + k & f^{-1}(x) = x + (-k) \\ f(x) = k \cdot x & f^{-1}(x) = \frac{1}{k} \cdot x \\ f(x) = x^k & f^{-1}(x) = x^{\frac{1}{k}} \\ f(x) = k^x & f^{-1}(x) = \log_k x \end{array}$$

Usually students define f over real numbers (or maximum possible intervals in $\mathbb{R}$). If students do so, they soon recognise that none of the known functions are useful candidates for cryptographic requirements.[5]

Additionally, the function needs to be easy to calculate and without rounding errors. Therefore, more complicated functions are not an option (e.g. including trigonometric functions). But what happens if you change the domain?

Comparing known possible domains will soon lead up to natural numbers or integers. Problems of calculability guide to residues modulo n. Thus, instead of determining functions within the domain of rational or real numbers, students investigated the same functions of residues modulo n. Residues behave with respect to addition and multiplication like integers, but differently concerning division and exponentiation. The second attribute gives the candidate for the function we are looking for. Compared to working with natural numbers it is quite different to extract the inverse of a residue which has been raised to a higher power, as the following example shows.[6]

Example: $19^7 \bmod 55 = 24$ and $24^{\frac{1}{7}} \bmod 55 \neq 19$, however $24^{23} \bmod 55 = 19$. As the example shows the inverse exponent 23 is not obvious from knowing the exponent 19. Background of finding exponents to invert exponentiation is *Euler's theorem*: For a and n relatively prime and φ the Euler function[7] it is

$$a^{\varphi(n)} \equiv 1 \bmod n.$$

By multiplication with a follows

$$a^{\varphi(n)+1} \equiv a \bmod n$$

and

$$a^{k\varphi(n)+1} \equiv a \bmod n.$$

for integers k. The last congruence can be used to find the pair of keys K_e and K_d we are looking for. If there are numbers e and d that give the exponent $k\varphi(n) + 1$ those are the pair of keys.[8] Because of

$$a \equiv a^{k\varphi(n)+1} \equiv a^{ed} \equiv (a^e)^d \bmod n.$$

Now e, n can be published, d remains secret. So everyone, e.g. Alice (sender) is able to encrypt a message a using the public key (e, n):

$$a^e \bmod n = c.$$

[5] Additionally the concept of functions is supplemented explicitly.

[6] A detailed teaching unit that follows the differences of operations between $\mathbb{N}$ and $\mathbb{Z}/n\mathbb{Z}$ to identify possible functions useful for cryptography is described in [9].

[7] $\varphi(n) = \#\{0 < t < n \mid \gcd(t, n) = 1\}$.

[8] Obviously, e (and d) is relatively prime to $\varphi(n)$. Therefore $\gcd(e, \varphi(n)) = 1$ and there exist $u, v \in \mathbb{Z}$ such that $e \cdot u + \varphi(n) \cdot v = 1$ (Bézout's Theorem). The integer u gives the factor d. Considering the example above it is $7 \cdot 23 \bmod \varphi(n) = 1$.

The cipher text c is decrypted by Bob (recipient) by using the private key (d, n):

$$c^d \bmod n = a.$$

Why is this secure? Why can't anybody calculate d with the knowledge of n and e? In practice e is chosen randomly with $\gcd(e, \varphi(n)) = 1$. The number d is calculated by using the extended Euclidean algorithm with the input values of $\varphi(n)$ and e.

That is possible for everyone, but only if $\varphi(n)$ is known.[9] To calculate the value of $\varphi(n) = n \prod_{p|n} (1 - 1/p)$ the factorisation is needed. That means, to make RSA secure the modulus n is chosen to be hard to factor. For RSA this is achieved by choosing n to be a product of two large primes.[10] The resulting procedure for the RSA key generation, encryption, and decryption is given in the box below together with an illustrating example.

Procedure of RSA	**Example**
Key generation	
Multiply large primes p and q	$p = 5, q = 11$
$n = p \cdot q$	$n = 55$
Compute $\varphi(n) = (p-1)(q-1)$	$n = 40$
Find integers[11] $e, d : ed \equiv 1 \bmod \varphi(n)$	$e = 7, d = 23$
Encryption of the message $a = 19$	
$c = a^e \bmod n$	$19^7 \bmod n = 24$
Decryption of the message c	
$a = c^d \bmod n$	$a = 24^{23} \bmod 55$

Figure 3: RSA – Key generation, encryption and decryption and illustrating example

Up to this point the mathematical background includes the Euclidean algorithm, modular arithmetic, and Euler's theorem (see fig. 4). Regarding the origin of the mathematical background the initial question can be changed to *Why did it take another 200 years to invent RSA?*[12]

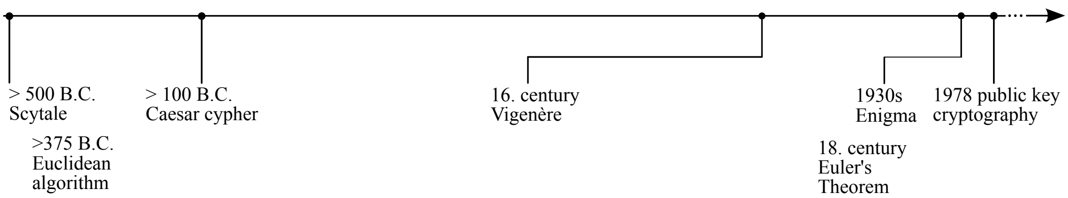

Figure 4: Timeline – History of cryptography and mathematical background of RSA

[9] A proof that the knowledge of the prime factorisation of n is equivalent to the knowledge of the private key can be found in ([2], p. 141).

[10] Obviously the factorisation is easy to calculate, e.g. by dividing n by all primes up to $\sqrt{n}$. Therefore *hard to factor* refers not to the existence but the practical way to find the factorisation, because of the number of steps and calculation time of the algorithm, which is executed by computers.

[11] While introducing RSA the Euclidean algorithm is not necessary. For small primes like this trial and error works too.

[12] Depending on the curriculum Fermat's little theorem can also be considered. For $n = p \cdot q$, p and q primes, the proof of RSA can be reduced to Fermat's little theorem. The elementary proof of Fermat's little theorem is more accessible to work with in class than Euler's theorem. The question to be posed would then be *Why did it take another 350 years to invent RSA*?

The answer can be found rather in the practical realisation of RSA than the formal description above.

The advances in technology and availability of computers were the crucial factors for the development of RSA for two reasons. First, the predominant historic use of cryptography had been the exchange of military secrets between *two parties*. In contrast, the use of computers increased, in particular, the exchange of sensitive data over *multi-party* communication networks, especially the Internet. To overcome the key exchange problem became a public concern, not only a military one.[13] Secondly, Euler's theorem, or the Euclidean algorithm were already valuable instruments in mathematics. But the use of computers made them practically applicable.
Thus, computers provided both the need for RSA as well as the means.

Similar developments can be found in different parts of mathematics. Computational mathematics, in particular numerical methods and discrete mathematics, has increased in importance in recent decades. A major contribution of computers consists in shifting problems of computability to the development and implementation of suitable (and in particular efficient) algorithms. Algorithm design has always been a classical activity in mathematics, but computers increased their visibility and respectability substantially [8]. On the other hand, computers introduced new problems, such as the need for data compression (information theory) and more philosophical questions, e.g. the nature of mathematical proofs [4].

4 Technology in Mathematics

Students use lots of algorithms in maths, e.g. multiplication algorithms and Gaussian elimination. Even procedures like identifing extreme values can be interpreted as an algorithm. Instruments like graphic calculators, spradsheed software, and CAS play an important role in school, as to visualise and perform individual calculations. Only a few algorithms are used to recognise the special values or chances (and limits) offered for mathematics and mathematics education by informatics. Therefore in the following we focus especially on these values, chances and limits when dealing with the algorithms that are integral part in the realisation of RSA.

The algorithms cover very old mathematics (Euclidean algorithms) as well as modern ideas (probabilistic algorithms). Developing and analysing the RSA related algorithms utilises classic problem solving strategies as well as computer oriented activities like analysis of running time and space. The old mathematics, its implementation, and practical limits especially of factorisation give students exemplary insight into mathematical development that has still not come to an end.

As an initial exercise students build their own example of RSA using primes as large as possible.[14] Depending on the used computer algebra system and knowledge about

[13] Because of more involved scientists this includes not only more chances to solve cryptographic problems but also more chances to verify the security of published ciphers as well. Today cryptography includes lots of applications for everyone's purpose, e.g. authentication of communication participants, the verification of the integrity of transmitted data and more – hidden in process of onlinebanking, software updates, mobiles etc.

[14] The term *large* depends on the CAS available. E.g the Voyage2000 only knows the *Mod(a,b)* command to calculate $a \bmod n$ but no *PowerMod(a,b,c)* to calculate $a^b \bmod c$ stepwise (accordingly to the square and multiply algorithm). Because of the number of digits for internal calculation the limits of calculability were found for primes with 3 digits and exponents higher than 200.

the corresponding/respective CAS commands problems arise by ...

(i) ... finding e and d (extended Euclidean algorithm),

(ii) ... rising a to the power of e and more often
... rising c to the power of d (square & multiply algorithm),

(iii) ... Factorisation of n to break RSA and

(iv) ... very interestingly: how to find primes (Miller Rabin primality test).

It is advisable to introduce the corresponding CAS commands later to motivate the algorithm development. But even if commands are known, the comparison between calculation by CAS and "by hand" leads to questions about the nature of the (black box) command (difference of calculation by hand and computer? limits of exact calculations?).

In the following it is outlined how the algorithms were introduced and which aspects were given priority.[15]

(i) Euclidean algorithm

This very old algorithm can be developed by the students and may be useful to introduce pseudo code.[16] Students have to describe the Euclidean algorithm in a general way and to prove it afterwards.

Exercise
How can we compute the gcd *of natural numbers a and b without knowing their factorisations?*
If $a > b$, then:

$$\gcd(a, b) = \gcd(a, a - b). \qquad (*)$$

a. Use this fact to compute gcd(322; 98) *in steps.*
b. Give a recipe for doing this in general.
c. Prove (*).

In the next step students get the pseudo code description and have to connect their written text results to corresponding parts in pseudo code. This illustrates the strong connection between textual solutions and programming solutions and calls attention to missing or mistakable parts (example, see fig. 5).

[15] According to the students previous knowledge and not to cover up the mathematical essence the algorithms are described in pseudo code only. As a consequence we don't distinguish between the mathematical object algorithm and its implementation as a computer program, which depends on the machine and/or on the programming language [8].

[16] Pseudo code is a compact and informal description of a computer programming algorithm, augmented with natural language descriptions of the details, where convenient, or with compact mathematical notation. It is easier for students to understand pseudo code than conventional programming language code and omits details that are not essential for the understanding of the algorithm. Pseudo code is commonly used to describe algorithms independently from individual programming language.

"The small number is subtracted from the greater one, thus $a - b$, this gives a natural number c. This step is repeated. So, it is calculated $c-b$ or $b-c$. This is repeated until one of the numbers is zero. The number different from zero is the gcd*. When a equals b it doesn't mind which one is subtracted from the other one."*

```
if a = 0
   return b
while b > 0
   if a > b
      a := a - b
   else
      b := b - a
return a
```

Figure 5: Textual description of the Euclidean algorithm (Natalie, 15 years) and pseudocodedescription

(ii) Square & multiply algorithm

The exercise for the square & multiply algorithm can be made the other way around. After testing alternatives to simple multiplication by using power laws, students can be given the pseudo code description of the square and multiply algorithm.

Exercise

a. Compute $4^{19} =: a^m$ using the given algorithm.

b. Try to find own examples that need less multiplications than the algorithm below.

```
Power := 1
while m > 0
   if m uneven
      Power := Power a
      m := m-1
   else
      a := aa
      m := m/2
return: Power
```

In doing so the students attention is forced to the different concepts of variables used in maths and computer sciences.

(iii) Factorisation of n

The issue of security of RSA is useful to confront the students with limitations of algorithms. The initial question is the following.

Exercise

Decipher the cipher text $c = 74273$, public key $n = 77057$, $e = 211$.

Obvious approach is to factorise the modulus n and follow the steps of the standard key generation using p and q (fig. 3). To study the limit of this approach, students can experiment with (their own) moduli like
$n = 9428761911366576583794774309332778301672 19$ and
$n = 213598703592091008239502318434121854383503406330042754837222169102 46\cdot\cdot$
$997011045373604390158396064 79$.

Using Mathematica it took 4 seconds to factorise the first modulus. The factorisation of the second one started during the lesson and was interrupted one week later (and would have needed more than 100 additional years on the same computer). After testing further examples the found exponential connection between factorisation time and the number of digits may indicate that it is not just a problem related to the computer equipment used.[17] It shows that even if there is an algorithm available (in this case to factorise natural numbers) it is not certain that it produces results efficiently.

A trickier didactic problem is to convince students of the security of an algorithm that depends on the non-existence of a practicable solution to a problem (in this case the factorisation of integers), especially if there is no proof of this non-existence.

There are other methods than simply dividing all integers from 2 up to square root n (e.g. Quadratic sieve ([2], chapter 10). But none of them are fast enough to be a satisfying alternative and to question the security of RSA.To illustrate this and to compare it with the process of simple division, the software CryptTool[18] was used.

But not to know a suitable algorithm is no proof that such an algorithm does not exist. Thus the students are confronted with an open question: is there an efficient way to find prime factors of large numbers? Similarly, is it possible to decipher RSA directly without factorisation?

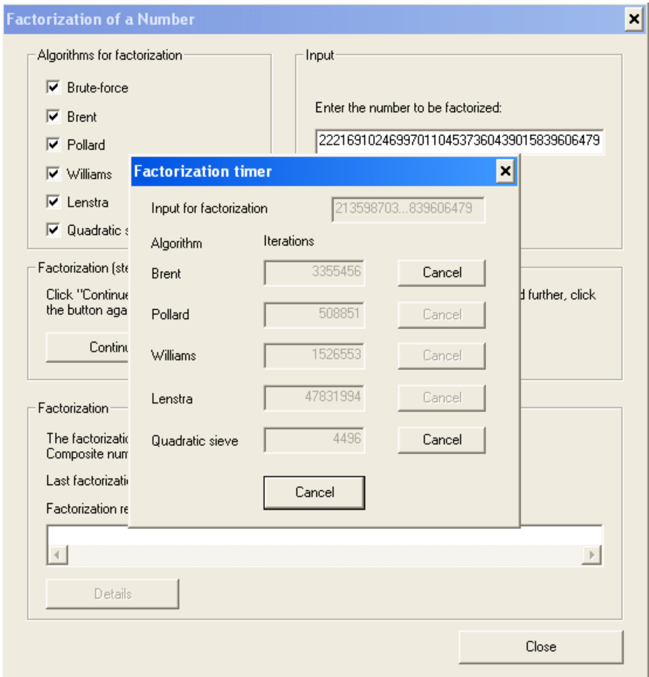

Figure 6: Comparing different algorithms to factorise large numbers using CrypTool.

(iv) Miller Rabin primality test

At the first glance the factorisation problem seems very similar to the problem of finding primes. But the latter can be solved very quickly in practice. This is at the cost of absolute certainty, because the algorithm of Miller Rabin is a probabilistic one. That is an interesting and rare connection between number theory and probability [6].

On the basis of the given literature the principle of the primality test was presented and complemented by exercises by a group of students. This was done in the context of six possible specialisations subsequent to the introduction of RSA.

[17]The key length of the modulus n used today is 1024 or 2048 bits. Numbers this large are secure against factorisation. Conceivable development in computer speed for the next 10-20 yearsThat is considered there. Therefore, its no option just to use a “better” computer (CAS instead of a calculator, 100 CAS instead of one CAS, super computer instead ...). Computers worthy of this task have not been constructed yet. Improvement that questions RSA (and other modern ciphers beyond extension of the key length where appropriate) is foreseen when the quantum computer is available. This is not yet in sight.
An impressive example of the different meaning of the term *large* when speaking about factorising large numbers can be found in [13].

[18]Free software to demonstrate and analyse classic and modern ciphers. Available in English, German, Polish, Spanish and Serbian.

5 Conclusion

The answer to *Why did it take more than 2000 years to invent RSA?* as presented above adds to knowledge and concepts from early school years including especially the key mathematical concept of numbers, functions and algorithm. Also it introduces actual development in mathematics. This actual development often gives contrast to the students' image of mathematics. It applies especially to the factorisation of natural numbers n, as the factorisation of n ...

– ...starts to be a problem, if n is large. In contrast to the simple problem solved in primary school using prime factors.[19]

– ...supports a sparsely used approach in students' school experience. Now the *lack of knowledge* can be utilised to solve a 2000 year old problem.

– ...is an integral part of an algorithm of which billions of people benefit every day without any proof of its security.

Or more from a student's point of view: Even the "well known" natural numbers are of scientific interest and far from being completely understood.[20] Even more amazing: the inability to solve a problem is useful.

The last aspect illustrates how mathematics develop much in the same way as natural sciences do, and that mathematical development does not take place in an enclosed environment but in mutual interaction with the surrounding world. In this case it was the practical problem of key exchange from outside of mathematics and the new perspective through the use of computers that made already valuable instruments (like the Euclidean algorithm or Euler's theorem) practically applicable and accessible. On the other hand, the present algorithm is inherently questionable, because its security is substantially based on practical reasons (length of the modulus n and some parameters of p and q to avoid possible attacks) and not on a mathematical proof. Therefore, teaching cryptography provides various links to possibly widen the students' view of mathematics and present mathematics as a living science. Summed up: mathematical history is an ongoing one, taking place today just like yesterday.

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[19] Similar problems to RSA and the factorisation arise for other ciphers that solve the key exchange problem in connection with the discrete logarithm (e.g. Diffie Hellman key agreement [2]). There exists no way to calculate the discrete logarithm.

[20] Open questions to underline this may arise in the context of the distribution of primes, which are constitutional for a RSA and other ciphers too. (Is there an infinite number of prime twins? Is there always a prime between n^2 and $(n+1)^2$?

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EVOLUTION OF COLLEGE STUDENTS' EPISTEMOLOGICAL VIEWS OF MATHEMATICS IN A HISTORY-BASED CLASS

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ABSTRACT

Among all potential approaches for developing students' understanding of the nature of mathematics, history is seen as one of significant means for achieving the goal. The present study aimed to investigate in what way and to what extent Taiwanese college students' epistemological views of mathematics had evolved in a history-based general education courses. A course, titled "When Liu Meets Archimedes", was designed to help college students comprehend the cultural features of the development of mathematics through highlighting the similarities and differences of the process formation of mathematics between Eastern and Western worlds. Through comparing and contrasting collected data, students demonstrated different epistemological views of mathematics in several aspects: (1) they were able to recognize diverse features of mathematics; (2) thy were more likely to realize how mathematics had interacted with cultural aspects; (3) they tended to comprehend the different mathematical cultures between the East and West. However, it was also found that they were unable to realize the intuition-based inductive approaches employed by ancient Chinese mathematicians may play an indispensable and complimentary role with deduction in mathematical thinking and less likely to acknowledge how creativity and imagination were involved in the process of mathematical thinking.

1 Introduction

Schoenfeld (1985, 1992) claimed that belief systems are one's mathematical worldview shaping the way one does mathematics. Its influence could be realized by that mathematics-related beliefs are the subjective conceptions that students implicitly or explicitly hold to be true, that influence their mathematical learning and problem solving. An individual's epistemology is the conception regarding the nature of knowledge and the processes of knowing, which has been identified as a significant factor in students' academic performance (Hofer, 1999; Schoenfeld, 1992; Whitemire, 2004). For instance, Whitemire (2004) found that college students at higher stages of epistemological development usually demonstrated a better performance at handling conflicting situations and showed more sophisticated information-seeking behavior. Therefore, the research on students' epistemology in mathematics (their beliefs about the nature of knowing and knowledge of mathematics) has increasingly received greater attention.

Among all potential approaches for developing students' understanding of the nature of mathematics, history is seen as one of significant means for achieving the goal. The essence of mathematics lies in its intellectual adventure, beauty of abstract form, and application in physical world. Kitcher (1984) stressed that any account of the growth of mathematical knowledge should be referred to its historical development. Therefore, to understand the epistemological order of mathematics, one must understand its historical order. Furthermore, mathematics is conventionally understood as a fixed and absolute knowledge in which social and cultural aspects play minor role. In this manner, knowledge base of mathematics is mistakenly viewed as unquestionably solid, and all human struggles and intellectual adventures in the development of mathematics are hidden (Lakatos, 1976). Actually, despite their different fashions, both Eastern and Western mathematics experienced a long period of uncertainty. As Kline's (1980) claim, in the early 1800s, no branch of mathematics was logically secure.

For revealing humanistic aspects of mathematical knowledge, relevant scholars have called for integrating history into school mathematics curriculum to demonstrate the dynamic, potentially fallible, and socio-cultural nature of mathematical knowledge and thinking (Barbin, 1996; Furinghetti, 1997; Horng, 2000; Radford, 1997 ; Siu, 1995). By referring to relevant studies, I listed five major reasons for using history in school curriculum (Liu, 2003):

1. History can help increase motivation and helps develop a positive attitude toward learning.
2. Past obstacles in the development of mathematics can help explain what today's students find difficult.
3. Historical problems can help develop students' mathematical thinking.
4. History reveals humanistic facets of mathematical knowledge.
5. History gives teachers a guide for teaching.

Following above claims, several studies were conducted to explore the effect of integrating history into mathematics curriculum. Results showed that history not only helped college students improve their attitudes toward mathematics, but also developed their own understanding of the nature of mathematical knowledge and thinking (Liu, 2006; 2007; 2009). These experimental studies were all carried out in calculus classes and content were tied to conventional topics. It was found that students' understanding of cultural and societal aspects of mathematics remained meager. By taking this concern into account, the present study aimed to investigate in what way and to what extent Taiwanese college students' epistemological views of mathematics had evolved in a history-based general education courses.

2 The Content and Context of the Course

The course, titled "When Liu Meets Archimedes--Development of Eastern and Western Mathematics", is a history-based general education course focusing on how the development of mathematics was related to Eastern and Western cultures. The course objective was to help students comprehend the cultural features of the development of mathematics through highlighting the similarities and differences of the process formation of mathematics between Eastern and Western worlds. A brief introduction of main topics is as follows:

2.1 Mesopotamia civilization and Babylonian mathematics

The course began with Mesopotamia civilization and Babylonian mathematic including the sexagesimal (base-60) numeral system and secret codes on the recovered clay tablets. For instance, students were shown a picture of a surviving clay tablet and asked to decode the numerical codes (Figure 1). They were reminded to reveal the code by transferring base-60 numeral system to base-10 numeral system. Then they got,

"1; 24, 51, 10 = 1+24/60+51/3600+10/216000 = 1.414212963 $\cong\sqrt{2}$ =1.414213562" and

"30×1.414212963 = 42.42638889 = 42+25/60+35/3600 = 42; 25, 35".

Above equalities imply that, in modern terms, the length of diagonal of a unit square is $\sqrt{2}$ and the length of diagonal of a square with side length of 30 is 42.42638889, an application of Pythagorean Theorem. Furthermore, higher Babylonian algebra such as solving system of equations was also introduced. Students were encouraged to compare the Babylonian style and modern fashion.

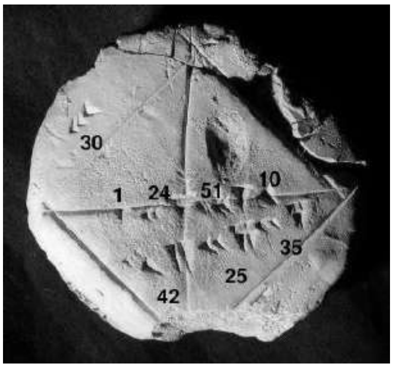

Figure 1 A Babylonian Clay Tablet

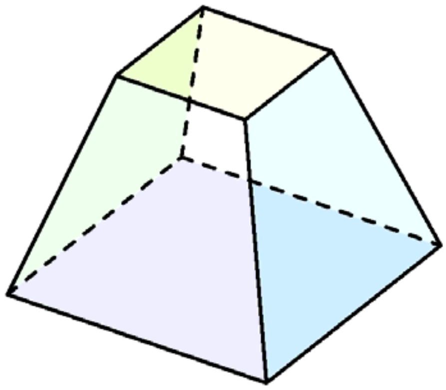
Figure 2 A truncated square pyramid

2.2 Egyptian mathematics

Hieroglyphic numeral system and unit fractions are two most particular features of Egyptian mathematics and a potential connection of Egyptian hieroglyph may be made to Sumerian script. Because Chinese characters are also hieroglyphic, students were attracted by this similar writing system but, on the other hand, were confused by the process of unit fractions. It is easy to factor 3/4 into 1/2+1/4 and 4/5 into 1/2+1/4+1/20. However, the factorization may not be unique. For instance, another representation of 4/5 is 1/3+1/4+1/5+1/60. The multiplicity of unit fraction representation brings an issue to the fore: How ancient Egyptian determined which is the simplest (in terms of mathematics) or best (in terms of practical needs) representation? Students were thus situated in a mathematical as well as cultural puzzle.

In addition, Moscow Mathematical Papyrus and Rhind Mathematical Papyrus not only reflect the achievement of ancient Egypt ca. 1800 B.C., but also reveal how mathematics was related to society at that time. A problem on Moscow Mathematical Papyrus receiving much attention is the volume of truncated square pyramid. For a truncated square pyramid whose top area is a square of length *a*, the bottom a square of length *b*, and the height *h*, the problem 14 actually asserted its volume V is:

$$V = \frac{1}{3}h(a^2 + ab + b^2)$$

Vetter (1933) had earlier indicated that how ancient Egyptian might reach the formula is a puzzle. The puzzle was then left to students as an assignment to work with, but yielded little response as a result.

2.3 Ancient Greece mathematics

Two particular features of ancient Greece mathematics were addressed in the class: the belief that mathematics is the key for understanding the universe and the relationship between philosophy and mathematics in general, geometry in particular. Both trends could be traced backed to Thales of Miletus for his rational thinking in explaining natural phenomena and deductive reasoning in establishing geometrical propositions, followed by Pythagorean thought of "*All are numbers*" and Platonic motto of "*Let no one ignorant of geometry enter here*". Students were encouraged to think about why mathematics was viewed by ancient Greek thinkers as an essential discipline for developing philosophical literacy and how it might be related to culture at the time.

Euclid and Archimedes' mathematical thoughts stand for two major trends of Western mathematics. The former established the logical foundation of mathematics in a deductive way, and the latter was complimentary with creative thinking and associated mathematics with physics. In addition to realizing how Euclid created a paradigm of modern mathematics on the basis of axiomatic systems, students were impressed by Archimedes' sophisticated use of his imagination in making connections between different mathematical objects. For instance, in what ways he had an insight of the area of a circle is equal to that of a right-angled triangle where the sides including the right angle are respectively equal to the radius and circumference of the circle. Further, under what circumstance he was able to think of employing the principle of leverage to derive the volume of a sphere. Students were learned that Archimedes' achievement not only signifies a landmark of ancient Greece mathematics, but also denotes a temporary rest of Western mathematics.

2.4 Scientific revolution and modern mathematics

Focus of the development of mathematics during 16 and 17 centuries was placed on the mathematical model of planetary motion and the quantification of physical motions. Using geometrical models to describe the pattern of planetary motion can be traced back to ancient Greek astronomer Hipparchus and Ptolemy of 2^{nd} century creating deferent-epicycle model for explaining the regression of planets. This well-designed but incorrect model was on the basis of geocentric theory and a mathematical belief of harmony, dominating Western scientific thought over a millenarian, later replaced by Copernicus' simpler heliocentric model and Kepler's three laws of planetary motion. On the other hand, students were aware of how Galileo designed experiments to measure the movement of falling bodies, not only inspiring Newton's discovery of gravity but establishing a paradigm for later scientific investigations.

Furthermore, Euler's strategy for solving Seven-Bridge Problem of Königsberg (Figure 3) was treated as an early typical instance of the beginning of western abstract mathematics, followed by the birth of group theory and invention of non-Euclidean geometry. Students were also reminded to keep an eye on the issue that, regardless of its impractical nature, why abstract mathematics eventually shows its unreasonable effectiveness of mathematics in the natural science, as Wigner (1960) claimed.

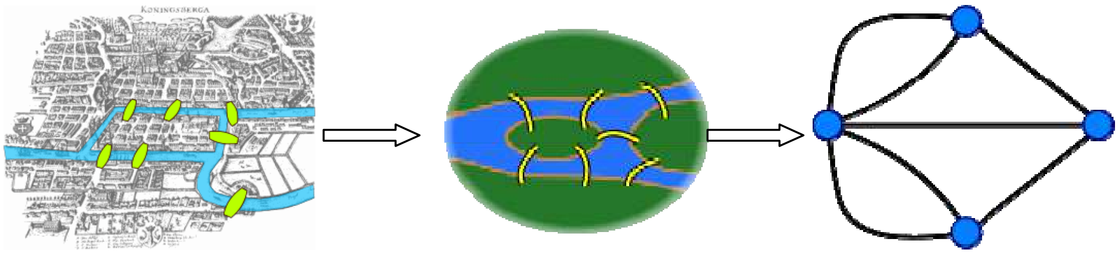

Figure 3 Euler's strategy for Seven-Bridge Problem of Königsberg

2.5 Ancient Chinese mathematics

Contrary to the deductive and abstract fashion of Western mathematics, the inductive and intuition-based style of ancient Chinese mathematics represents another paradigm. Students were told the mythical origin of Ho Tu (河圖), a diagram marked on the back of a dragon-horse rising from the Yellow River 5000 years ago, and Lo Shu (洛書), a figure scribed on the shell of a tortoise surfacing from Lo River 4000 years ago (Figure 4). Both Ho Tu and Lo Shu contain mathematical patterns (Lo Shu actually is a 3×3 magic square), but were then combined with Ba Gua (八卦) and Yin Yang (陰陽) to serve as fundamental elements of Feng Shui (風水) theory.

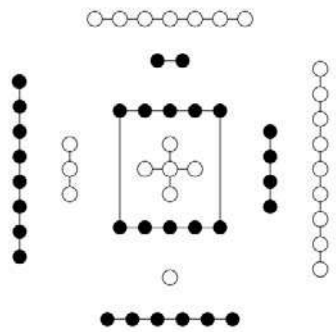

Figure 4 (a) Ho Tu

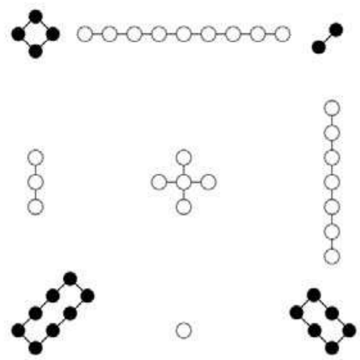

Figure 4(b) Lo Shu

Figure 4(c) Ba Gua & Yin Yang

Jiuzhang Suanshu (Nine Chapters on Mathematical Art , 九章算術) was the most important and representative mathematics book of ancient China. It distinguished itself from ancient Greece mathematical texts for its fashion of "solving without proof". A typical problem solved by the circular technique in *Jiuzhang Suanshu* can be seen as follows:

Question: Given a circle with area 300, what is the circumference?
Answer: 60
Solution: Multiply the given area by 12 and take the square root, it is the circumference.

It was not ready to know why the given area 300 should be multiplied by 12 followed by taking the square root. The puzzle was given as a problem-solving activity for students to figure it out. After a while of testing, they were soon realized that the answer was incorrect because the technique was based upon the experimental but wrong assumption that $\pi = 3$, which was widely accepted at that time.

The typical empirical nature of ancient Chinese mathematics was also manifested by Liu Hui's (劉徽) interpretation of *Jiuzhang Suanshu*. For example, he employed "Out-In Mutual Patching Technique" (出入相補) to "prove" the Pythagorean Theorem, which was called Gou-Gu (勾股) Theorem in Chinese. The "Out-In Mutual Patching Technique" not only reveals the empirical nature of ancient Chinese mathematics, but also lays the foundation of mathematical proof of ancient Chinese mathematics.

In addition, differing in Western algebraic system invented by Franciscus Vieta for solving equations during 16th century, Tian-Yuan Technique (天元術, Technique of the Celestial Unknown) is a Chinese system for resolving polynomial equations created in the 13th century. The technique is a positional system of rod numerals to represent equations and solving equations by means of oral pithy formulas and manipulation of rods. Students were aware of how ancient Chinese mathematics was built upon an empirical base.

2.6 A contrast between East and West

One of the major goals of the history-based course was to lead students to compare and contrast the diverse approaches between the East and West mathematics. Liu Hui and Archimedes are usually seen as respect representative figures among all. Liu Hui and Archimedes' ways for deriving the area of a circle and volume of a sphere were given as typical instances for students to generate their reflective thinking on various styles of mathematical thoughts. For instance, Liu Hui's circle-cutting method, rearranging infinite sectors of a circle to form a rectangular, heavily relied upon intuition. On the other hand, Archimedes equated the area of a circle to that of a right triangle and proved the result by *reductio ad adsurdum*, which was mostly deductive.

Furthermore, opposing to mathematical approach of Western astronomy, students also realized how ancient Chinese astronomers viewed the movement of celestial bodies in a mythical way. While observing the retrogression of Mars, instead of proposing potential theories, ancient Chinese astronomers saw the phenomenon as a misfortune for the emperor and altered observational information to please the emperor. The man-made data contributed to an acknowledgment of the pattern of celestial motion was lacking and a mathematical model was not available in ancient China, despite its more advanced observational instruments.

3 Results and Findings

For investigating in what way students responded to this history-based course and to what extent their views of mathematics evolved during the course, several instruments, including questionnaires, interviews, student journals and a web-based forum, were employed to achieve the goal. In the beginning of the course, all students' were invited to answer a questionnaire (Liu, 2010) and seven of them were randomly selected to participate in the follow-up interviews to document their initial views of the nature of knowing and knowledge of mathematics. At the end of the course, previous procedure was repeated to compare students' pre- and post-views. During the course, student journals and web-based forum collected students' responses to teaching material, important issues, and personal reflections upon the role of mathematics in diverse civilizations.

Through comparing and contrasting collected data, students demonstrated different epistemological views of mathematics in several aspects: (1) they were able to recognize diverse features of mathematics; (2) thy were more likely to realize how mathematics had interacted with cultural aspects; (3) they tended to comprehend the different mathematical cultures between the East and West.

3.1 Diverse features of mathematics

At the beginning of the semester, most students held a rigid view about mathematics in which the development of mathematical knowledge is mainly tied to daily needs and logical rules. After receiving this history-based course, though the previous two features remained fundamental in their minds, they were able to propose additional characteristics of mathematics. Chuan originally considered that, comparing to the rigid form of mathematics, art allows more space for personal expression. He whereas turned to hold that mathematics could involve personal style:

> *Interviewer*: In your [post-instruction] questionnaire, you said mathematics may contain several forms, such as numbers, graphs and computational equations. What do you mean by that?
>
> *Chuan*: In the first interview, I thought that mathematics is more formalized, like equations and formulas that we deal with. After taking this course, however, I felt mathematics could have a strong connection with art and science…Compare to mathematics, art is more abstract involving personal style. But mathematics may have personal style too, though it seems to be less visible than art. (Chuan, post-instruction interview)

For abstract mathematics, Chuan changed his initial instrumentalist views and expressed an understanding that though abstract mathematics appears to be irrelevant to daily life, it can be seen as a breakthrough of mathematicians themselves and its value could be justified in the future. Chuan's view was also endorsed by Yu's claim:

> [Abstract mathematics] seems meaningless to us, but that may be helpful to mathematicians. It could be a progress for himself/herself. We cannot assert that it is valueless at this moment just like some painters' works were not appreciated by contemporaries, yet turn out to be priceless. (Yu, post-instruction interview)

Furthermore, contrary to previous superficial understanding about the role of creativity and logic in mathematical thinking, they were able to recognize how creative thinking occurs. For instance, asked about how mathematicians think, Hui were impressed by Newton and Galilei's work and indicated that Newton and Galilei were firstly motivated by an idea and then made potential hypotheses, followed by logic-based verification. In her mind, proof and refutation constitute an alternating mode in mathematicians' thinking.

3.2 How mathematics had interacted with cultural aspects

Before entering this course, few of the students would associate the development of mathematics with culture. They acknowledged the difference existing among diverse countries, but usually referred the distinction to economic status. They were more likely to hold that highly developed countries can create more advanced mathematics than the lower developed regions do. As Hui indicated in the pre-instruction interview, "Europeans seemingly live with more leisure time to focus on philosophical aspects of mathematics". This misconnection of mathematics with economics was then replaced by the relationship between mathematics and culture, as Ling's claim on the post-instruction questionnaire:

> They [mathematics and social culture] are related. The focus of each society may vary. Some stress the exactness of computation to develop science, but others emphasize the use of diagram and ratio in architecture. This is why different mathematics was developed by different societies, making mathematics more colourful. Everything in our life is about mathematics. (Ling, post-instruction questionnaire)

Many attributed the different fashions of mathematics to life styles. Ko considered the birth of mathematics as being related to human's attempt to resolve living problems such as the overflow of Nile River in Egypt and Yellow River in China led ancient Egyptian and Chinese created diverse techniques to find the pattern. Hui also endorse the view that the content of mathematics was deeply influenced by social problem at the time and gave similar examples:

> *Interviewer*: Then, can you tell me why different societal cultures might develop different mathematics?
> *Hui*: In the West, like the overflow of Nile River, [Egyptians] needed to figure out a way to resolve the farming problem. In ancient China, laypersons were required to pay tax on the basis of the farmland area. They therefore had to create arithmetical rules. (Hui, post-instruction interview)

Hui's response was seemingly affected by the growth of geometry in ancient Egypt and arithmetic in ancient China. With the acknowledgment of the intimate relationship between mathematics and societal culture, students tended to comprehend the different mathematical cultures between the East and West.

3.3 Different Mathematical Cultures Between the East and West

By contrasting students' pre-instruction and post-instruction views, the most significant distinction was their awareness of different cultures between the East and West. In the beginning of the course, none of them recognized the mathematical culture may play a role in the development of mathematics. In the post-instruction interview, however, all interviewees demonstrated a moderate understanding of dissimilar fashion of the East and West mathematics. Horng indicated ancient Chinese mathematicians' interest in solving those problems with practical usage restricted the expansion of mathematics, and complicated rules of Tian-Yuan Technique hindered the progress of mathematical knowledge. He also noticed the different ways of interpretation of astronomical phenomena. For instance, ancient Chinese astronomers regarded the alignment of five planets as a bad sign of astrology, but Western scientists treated it as regular and made an effort to look for the pattern. Furthermore, another interviewee Chuan viewed dialectical nature as the major characteristics of Western mathematics:

Interviewer: Is there any significant characteristics for the development of mathematics? Did you learn anything from this course?
Chuan: Taking [ancient] China as an example, the development of mathematics in [ancient] China was quite different from that of the Western. I feel that the thoughts, forms, and styles in the Western were more personal and ancient China was more rigid without much breakthrough.
Interviewer: What factors caused the differences?
Chuan: In the Western culture, I feel that, any school of thought could be criticized by others if its doctrine or research was not persuasive. (Chuan, post-instruction interview)

Chuan's account was attributed to his conceptions that there were several competitive schools of scientific thoughts, such as Pythagorean *vs*. Eleatic schools and Platonism *vs*. Aristotelianism, in ancient Greece, whereas Confucianism played a dominated role in ancient China. This view was also endorsed by Yu's claim:

Interviewer: What factors made the difference between the development of Eastern and Western mathematics?
Yu: Confucianism did not place importance on mathematics. It emphasized the Six Arts…but I forget what they are.
Interviewer: They are rites, music, archery, charioteering, calligraphy, and mathematics.
Yu: Yes!
Interviewer: Mathematics was ranked at the last position.
Yu: Yes! The Western thought viewed mathematics as much more important, like quadrivium. They are……
Interviewer: Arithmetic, geometry, music, and astronomy.
Yu: Yes! (Yu, post-instruction interview)

Though students were unable to indicate the sophisticated relationship between ancient classic philosophy and the progress of mathematics, they appeared to realize the different characteristics of scientific thoughts may determine the paths and directions of later development.

4 Discussion and Conclusion

The present study aimed to investigate in what way and to what extent Taiwanese college students' epistemological views of mathematics had shifted in a history-based general education courses. Though several findings suggest the history-based course did direct students' attention to reflect upon humanistic and cultural issues of mathematics as aforementioned, it was found their conceptions remained at a superficial level at least on two aspects. First, they indicated the potential imitations of ancient Chinese culture in developing advanced abstract mathematics, but were unable to realize the intuition-based inductive approaches employed by ancient Chinese mathematicians may play an indispensable and complimentary role with deduction in mathematical thinking. They tended to judge the value of a mathematical approach in terms of the final outcome it brought about rather than the particular feature it involved. Second, they were less likely to acknowledge how creativity and imagination were involved in the process of mathematical thinking. Liu (2006) reported that, after experiencing a semester of historical approach problem-based calculus course, participating students' focus shifted from mathematics as a rigid product to mathematics as a dynamic and creative process, which were hardly seen in this study. Note that, the course in the present study was a 2-credit elective general education course, differing from that in Liu (2006) which was a 4-credit required course. The former stressed the cultural aspect whereas the latter emphasized problem-solving activities. Students in this history-based course might have an appropriate understanding of the extrinsic nature of mathematics (how mathematics interacts with societal culture), yet showed deficiencies in realizing the intrinsic nature of mathematics (how thinking is processed in mathematicians' minds). Could the two dimensions be well and equally taken into account in a single course? or Should we have to compromise? It might be a challenge for future curriculum development of this kind.

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NEWTON'S PRINCIPIA MATHEMATICA IN A TWENTY FIRST CENTURY CLASSROOM

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ABSTRACT

In this paper we present a didactic activity developed under the Project Based Learning approach. We wanted that our students experienced the way the seventeen-century scientists did calculations without using computers or electronic calculators. The didactic activity involved the use of Lemma V included in Newton's Principia Mathematica. We discuss our experience applying this activity to Engineering students and a group of in service teachers of high school and university. We also show some results of our students concluding that the main goals stated when we designed the activity were fulfilled.

1 Introduction

In our classrooms students live with electronic calculators and computers without any idea of how scientist made their discoveries in the past. Most students do not imagine how Archimedes, Pythagoras, Pascal, Newton, Euler or every famous mathematician and physicist did the calculations they needed. To be honest neither we did as students. Many of us do not imagine how ancient scientist could work without calculator or computers, or even without the Calculus and a gravitational theory like Newton's.

We designed our activity with three main goals; the first is to discover the way scientist worked in the past centuries. The second goal is to justify the need of numerical methods and the third one is to rediscover one of the most famous books in the History in a twenty first century classroom. In addition we want students to solve a problem that is not included in pre-calculus classes.

Project Based Learning (PjBL) approach was used to design the activity, we gave a short context and a set of real data to be used.

2 Origin of the activity

By 2005 two colleagues of us were working with Taylor's Series and its original source when they were told about some interpolation techniques proposed by Newton in his Lemma V included in "Principia Mathematica" (part of Proposition XL Theorem XXI, Book 3). After some time, one of the authors of this paper was researching about numerical series and its origins as part of his PhD thesis (Rosas, 2007) and Lemma V was mentioned again.

In Instituto Tecnológico y de Estudios Superiores de Monterrey (ITESM for short) Engineering students must learn mathematics based on "real world applications". This way teachers need to apply some didactic techniques like Problem Solving, Problem Based Learning, Cases Study, Project Based Learning and others. After some years of using these techniques students feel fine with this kind of learning (at least we think they feel fine). As a result of our technological era our students think that only computer solutions are good and sometimes they do not respect ancient methods.

In 2009 because of these reasons we decided to design a didactic activity that included Project Based Learning, an ancient technique and a “real world” situation. Our first idea was to give to our students some invented data of a car to approximate its position. Then we remembered that Newton’s Lemma V deals with asteroids and we thought that it would be a good idea to use real data of an asteroid. After a brief quest in diaries we found an asteroid (Ruiz, 2009) named 2009ST19, one kilometre in diameter and passing around 600,000 kilometres of the Earth, it was discovered by the Spaniard astronomer Josep Maria Bosh on September 16, 2009. We visited the Jet Laboratory Propulsion web page (Jet Laboratory Propulsion [JLP], n.d.) to find some data about it.

The Solar System Dynamics is a sub web page of JLP where we can find any data about celestial bodies in the section Small-Body Database Browser [SSD-SBDB] (Solar System Dynamics, n.d.). After writing the asteroid’s code in the search field we can find every known data about the asteroid. With the Ephemerides option we can choose the period of time where we want the system calculates asteroid’s data.

Some of the data we can get are number of observations, data-arc span, first observation used, last obs. used, elevation angle, perihelion, period, and Sun-Observer-Target (S-O-T). The called S-O-T is the angle between the Sun, object and observer; a short explanation of its different values is:

> Sun-Observer-Target angle; target's apparent solar elongation seen from observer location at print-time. If negative, the target center is behind the Sun. Angular units: DEGREES. (Solar System Dynamics, n.d.)

In our first design we choose the elevation angle of the asteroid, but we thought that its values were changing so much that our students could make mistakes. So we decided to use the S-O-T angle because its variations were more suitable to the example provided by Newton.

This way our design included an ancient method of interpolation used in a time of no computers and involved the use of a real object that could endanger the Earth’s life; this last thing was interesting for our students. To increase the feeling of the historical context we decided to use the original Latin version of Newton’s Lemma V.

3 Activity in detail

PjBL is a didactic technique built over real learning activities that catch students’ interest and also encourage students to search new knowledge. Activities under this approach are designed to solve a problem and generally show the many ways people learn and work in the real world.

Some characteristics of PjBL are: Environment focused on learning, Collaborative work, Real world tasks, Many ways of solving a problem, Time management and Different ways of evaluate students’ work. These characteristics let teachers to act like a guide instead of a “know-all” people. This guiding is important because in some projects students do not know how to solve the problem and neither the teacher.

The didactic activity was called “finding an asteroid”. In SSD-SBDB we selected 13 observations from July 18 to November 15 in 2009, with ten days of separation. We show the full table of these data in Image 1.

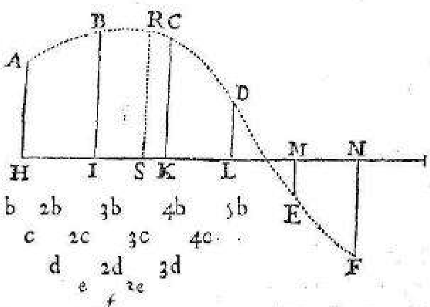
Date__(UT)__HR:MN		S-O-T
2009-Jul-18	00:00	110.6386
2009-Jul-28	00:00	98.2399
2009-Aug-07	00:00	85.0185
2009-Aug-17	00:00	68.5472
2009-Aug-27	00:00	48.2178
2009-Sep-06	00:00	57.0115
2009-Sep-16	00:00	80.3307
2009-Sep-26	00:00	97.0950
2009-Oct-06	00:00	111.1146
2009-Oct-16	00:00	124.4429
2009-Oct-26	00:00	137.5462
2009-Nov-05	00:00	149.7362
2009-Nov-15	00:00	159.0660

Image 1. Values of the elevation angle for asteroid 2009 ST19

In the activity we ask students to apply Lemma V from Newton's Principia Mathematica to the table in image 1 and to approximate the value of the S-O-T angle of the asteroid for two different dates. We choose September 10th because it is a date inside the given observations like Newton's example and Lemma V should give a good approximation of the real S-O-T angle, this was the first date. The second date was November 25th, a date outside the given observations. Lemma V then gives a wrong answer and we wanted to find what students would do when their calculated value were visible wrong. In image 2 we can see part of Lemma V schema and formulas.

[482]

2 c, 3 c, 4 c, &c. tertias d, 2 d, 3 d, &c. id eſt, ita ut ſit $HA - BI = b$, $BI - CK = 2b$, $CK - DL = 3b$, $DL + EM = 4b$, $-EM + FN = 5b$, &c. dein $b - 2b = c$ &c. Deinde erecta quacunque perpendiculari RS, quæ fuerit ordinatim applicata ad curvam quæſitam: ut inveniatur hujus longitudo, pone intervalla HI, IK, KL, LM, &c. unitates eſſe, & dic $AH = a$, $-HS = p$; $\frac{1}{2}p$ in $-IS = q$, $\frac{1}{3}q$ in $+SK = r$, $\frac{1}{4}r$ in $+SL = s$, $\frac{1}{5}s$ in $+SM = t$; pergendo videlicet ad uſque penultimum perpendiculum ME, & præponendo ſigna negativa terminis HS, IS, &c. qui jacent ad partes puncti S verſus A, & ſigna affirmativa terminis SK, SL, &c. qui jacent ad alteras partes puncti S. Et ſignis probe obſervatis erit $RS = a + bp, + cq + dr + es + ft$ &c.



Caſ. 2. Quod ſi punctorum H, I, K, L, &c. inæqualia ſint in

Image 2. Image of the schema and formula of Newton's Lemma V (Newton, 1667, p. 482).

The complete activity was available in a webpage format in the course's web page. Students had to open the activity and solve it in a one-week period. There was not a specific format for the final report, but it should include calculations, diagrams, graph or anything else they used to solve the activity. In the appendix we include the translated version of the activity.

4 Experiences applying the activity

So far we have used four times this activity. Three times were used with students in first year of Engineering and the fourth time the activity was solved by students in first semester of Master Degree in Mathematics Education.

Our first experience was with a group of approximately 25 first-year Engineering students. This time the activity included the original text written in Latin by Newton. Many students complained about the activity and asked for a translation in order to solve the problem. Because students could not apply the algorithm to the data we gave them, their teacher had to solve a short example in the classroom. After that students were able to repeat the process with the data set of the 2009ST19 asteroid. After so many complaints we had students' answers just like the example of the teacher. We decided not to include the answers as part of the research because all of them were almost the same and we did not observe any personal details. We think that the example drove the students too much.

As we wanted the students to discover how the seventeenth-century scientists calculated a planet's orbit without computers (and electronic calculators), we decided to include a short explanation of the algorithm in Spanish for the second time we applied this activity. After this modification, we gave the activity to the students and they focused on the algorithm and the involved mathematics. Two groups of Engineering with 30 and 32 students solved the activity.

Most students did the calculations by hand but they tested their results using calculators or a spreadsheet. This time they showed interest in the usefulness of basic arithmetic to solve something complicated as the orbit of an asteroid. A couple of students delivered solutions including remarks that we did not expect.

Last time we used this activity in a class was in an introductory course of first semester of an Online Master Degree in Mathematics Education. The students of this MD program need to be in-service teachers of high school or university (they usually are over 28 year old and we will refer to them as teacher-students in contrast to university students). Because of this characteristic we decided to use the Latin version of the activity. Teacher-students did not complain, they just asked if there were Spanish, or English versions of Newton's Principia Mathematica. But they did not wait for our answers; they searched Latin to Spanish dictionaries or automatic translators. Some teacher-students translated the Latin text; others found Spanish versions of the text. After a couple of days every teacher-student could read and follow Newton's text. We thought that the translations would have some mistakes but teacher-students shared their texts and fixed the mistakes of their translations.

The common experiences of this group were the spontaneous and cooperative work among teacher-students; they also liked the use of arithmetic to found the position of an asteroid. Some remarks of them were about use a Spanish version of the activity in their own classrooms.

5 Some students' answers

Now we analyse some of the answers that our students gave us. There were no restrictions on the final report, but many answers had common elements in technical details and format.

One of those common elements was a graph where students plotted the asteroid's data set as points. Some students included in their graph a point representing the unknown value but it was not a general characteristic. Almost half of these graphs were drawn with a continuous curve joining the points.

Most students included images of the paper sheets where they wrote their calculations. Divisions, products, sums and subtractions were included in the reports as an evidence of no using computers. The S-O-T angle of the asteroid calculated with the algorithm was quite near the real value.

On the left side of image 3 we can see the calculations of letters a, o, p, q, r, s, t, u, v, w, x, y and z following Newton's calculations. On the top middle of the image we can see a text inside a box titled "altura de A" ("the height of A") and below it the substitutions of the values found on the left side of the sheet. This is the case of the S-O-T angle for July 18. The last calculations are for the case of November 15, and as it can be seen in the last line the angle is negative (but it should be positive).

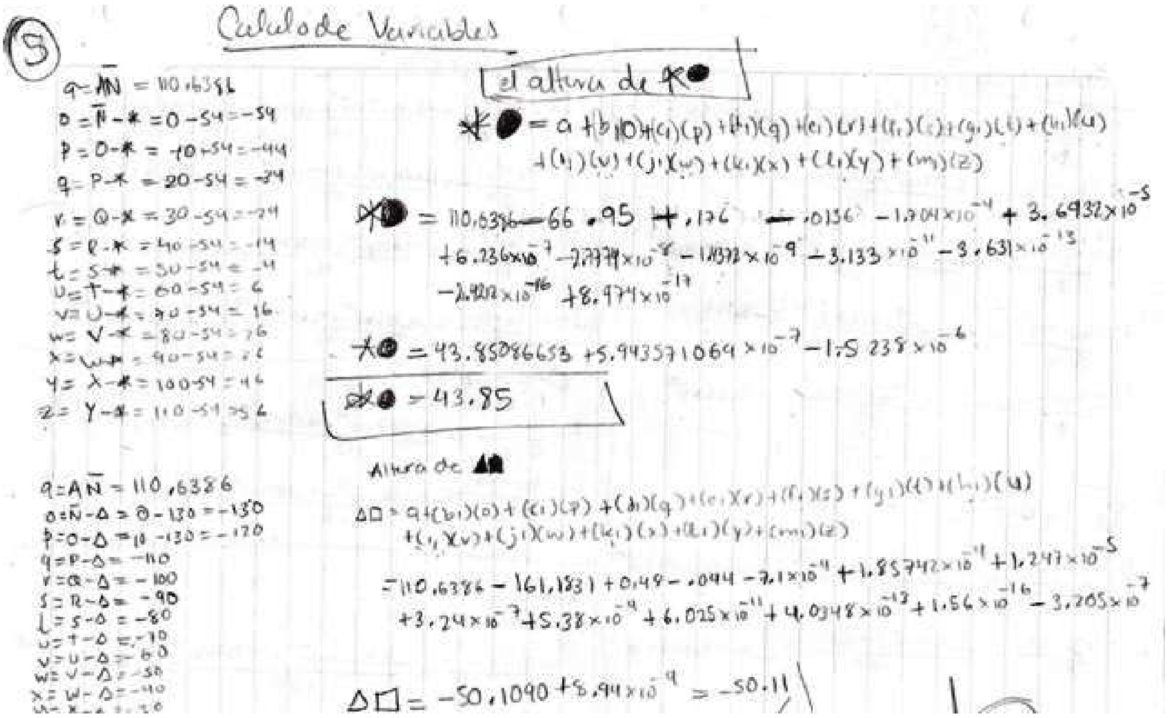


Image 3. Students' hand made calculations

One student's answer includes some remarks we think are interesting. He made his calculations by hand but he thought that he could be wrong. As a way of verification he used Stellarium (a free open source planetarium for the computer with 3D visualizations of the sky) (Stellarium, n.d.) and he found that his value of the elevation data calculated with Lemma V was a very good approximation. He said that with his telescope he took a photo of the asteroid but he gave us not details about the class of telescope and camera he used so we were not sure if he really took the photo. With the same kind of calculations another student commented that she used the software Mathematica to interpolate the value of the S-O-T angle and she found that her results were good.

Teacher-students' solutions were like students' solutions, the only difference were that teacher-students did not complain about the Latin version of Newton's Lemma. Most teacher-students delivered their calculations made by hand and a spreadsheet as a kind of validation of their results.

6 Conclusions

In every one of the three experiences we considered applying this activity we found the same situation: Students gave respect to ancient scientist and their work done without computers. We also found some students really involved in the activity; they felt very interested in Lemma V as a good way to approximate unknown values based in a few observations. We found students, who did not like the activity but they were no more than five and we think it is a good result. Every time we used "finding an asteroid" students were interested and we think it is because the problem is not a common problem like "find the equation of…"

We did not mention to our students that this technique involved an interpolation; our idea was to generate in our students' minds the need for this kind of methods. One student tested her results using interpolating software and this was unexpected. But it was only this case.

As a final remark we can say that we fulfilled our three goals in last three times we used this activity. Some teachers had given us suggestions to modify or strengthen the activity and we are working on that. We hope to get better results.

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APPENDIX

Didactic Activity: Finding an asteroid
Date of application: December 01, 2009.
Remarks: This activity was available in a web page and the images were of high quality, but in this appendix we include those images with low quality.

This activity is named "*Finding an asteroid*" because you will work with real data of the position of an asteroid, and you will use a method created by Newton in 1667.

The next two images show two pages of the book Philosophiae Naturalis Principia Mathematica written by Isaac Newton in 1687, as you can see it is written in Latin because at that time scientists wrote in Latin (actually we mostly write in English).

The first part of your job is to read Lemma V, yes in Latin.

xima ex parte ſupra Planetas verſantes, & eo nomine orbes axibus majoribus deſcribentes, tardius revolventur. Ut ſi axis orbis Cometæ ſit quadruplo major axe orbis Saturni, tempus revolutionis Cometæ erit ad tempus revolutionis Saturni, id eſt ad annos 30, ut $4\sqrt{4}$ (ſeu 8) ad 1, ideoque erit annorum 240.

Corol. 2. Orbes autem erunt Parabolis adeo finitimi, ut eorum vice Parabolæ abſque erroribus ſenſibilibus adhiberi poſſint.

Corol. 3. Et propterea, per Corol. 7. Prop. XVI. Lib. I. velocitas Cometæ omnis erit ſemper ad velocitatem Planetæ cujuſvis circa Solem in circulo revolventis, in dimidiata ratione duplicatæ diſtantiæ Planetæ ~~Cometæ~~ à centro Solis ad diſtantiam Cometæ ~~Planetæ~~ à centro Solis quamproximè. Ponamus radium orbis magni, ſeu Ellipſeos in qua Terra revolvitur ſemidiametrum tranſverſam, eſſe partium 100000000, & Terra motu ſuo diurno mediocri deſcribet partes 1720212, & motu horario partes $71675\frac{1}{2}$. Ideoque Cometa in eadem Telluris à Sole diſtantia mediocri, ea cum velocitate quæ ſit ad velocitatem Telluris ut $\sqrt{2}$ ad 1, deſcribet motu ſuo diurno partes 2432747, & motu horario partes $101364\frac{1}{2}$. In majoribus autem vel minoribus diſtantiis, motus tum diurnus tum horarius erit ad hunc motum diurnum & horarium in dimidiata ratione diſtantiarum reciprocè ~~reſpectivè~~, ideoque datur.

Lemma V.

Invenire lineam curvam generis Parabolici, quæ per data quotcunque puncta tranſibit.

Sunto puncta illa A, B, C, D, E, F, &c. & ab iiſdem ad rectam quamvis poſitione datam HN demitte perpendicula quotcunque AH, BI, CK, DL, EM, FN.

Caſ. 1. Si punctorum H, I, K, L, M, N æqualia ſunt intervalla HI, IK, KL, &c. collige perpendiculorum AH, BI, CK &c. differentias primas b, $2b$, $3b$, $4b$, $5b$, &c. ſecundas c,

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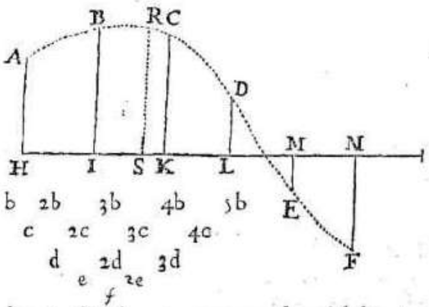
2 c, 3 c, 4 c, &c. tertias d, 2 d, 3 d, &c. id eſt, ita ut ſit $HA - BI = b$, $BI - CK = 2b$, $CK - DL = 3b$, $DL + EM = 4b$, $-EM + FN = 5b$, &c. dein $b - 2b = c$ &c. Deinde erecta quacunque perpendiculari RS, quæ fuerit ordinatim applicata ad curvam quæſitam: ut inveniatur hujus longitudo; pone intervalla HI, IK, KL, LM, &c. unitates eſſe, & dic $AH = a$, $-HS = p$, $\frac{1}{2}p$ in $-IS = q$, $\frac{1}{3}q$ in $+SK = r$, $\frac{1}{4}r$ in $+SL = s$, $\frac{1}{5}s$ in $+SM = t$; pergendo videlicet ad uſque penultimum perpendiculum ME, & præponendo ſigna negativa terminis HS, IS, &c. qui jacent ad partes puncti S verſus A, & ſigna affirmativa terminis SK, SL, &c. qui jacent ad alteras partes puncti S. Et ſignis probe obſervatis erit $RS = a + bp, + cq + dr + es + ft$ &c.

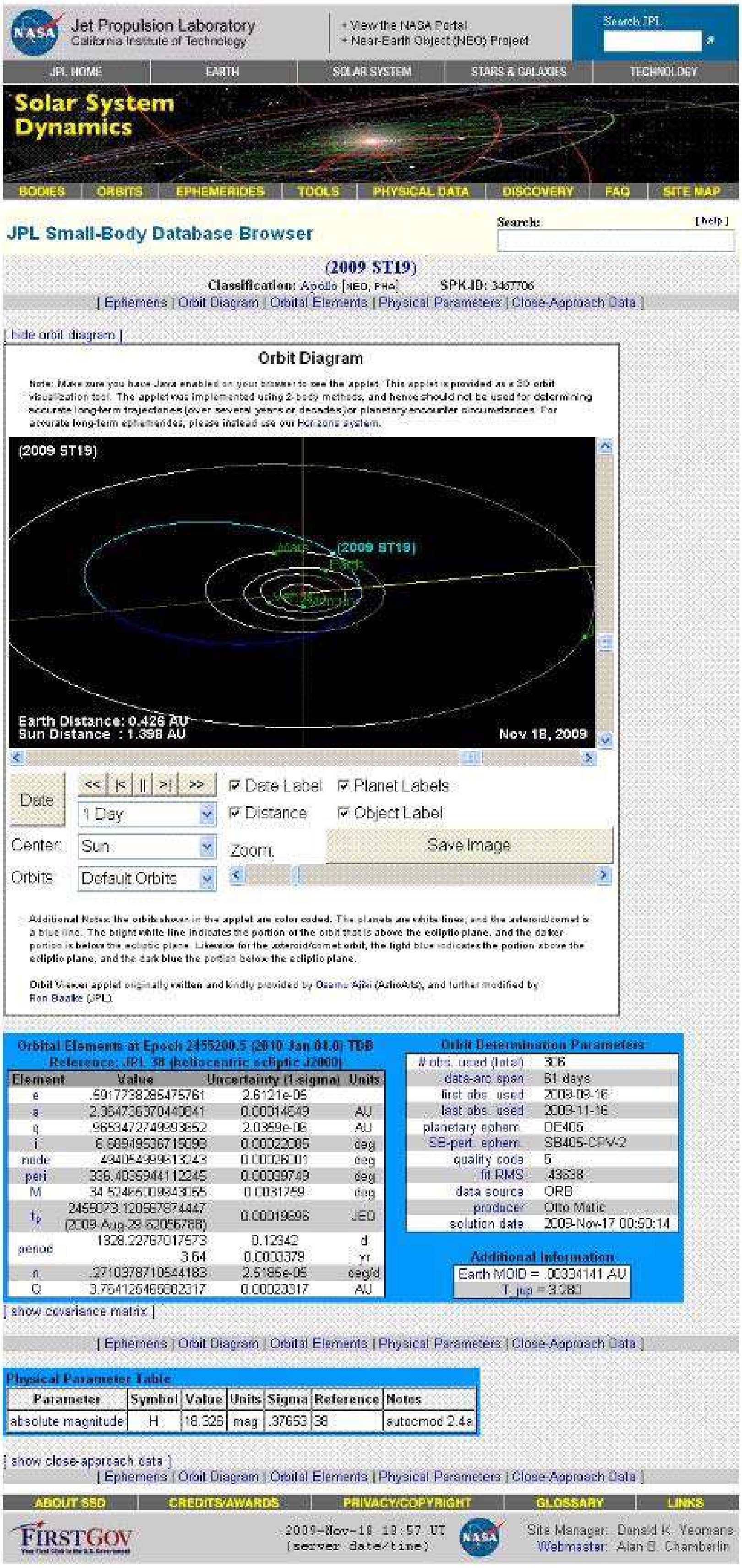


Caſ. 2. Quod ſi punctorum H, I, K, L, &c. inæqualia ſint intervalla HI, IK, &c. collige perpendiculorum AH, BI, CK, &c. differentias primas per intervalla perpendiculorum diviſas b, $2b$, $3b$, $4b$, $5b$; ſecundas per intervalla bina diviſas c, $2c$, $3c$, $4c$, &c. tertias per intervalla terna diviſas d, $2d$, $3d$, &c. quartas per intervalla quaterna diviſas e, $2e$, &c. & ſic deinceps; id eſt ita ut ſit $b = \frac{AH - BI}{HI}$, $2b = \frac{BI - CK}{IK}$, $3b = \frac{CK - DL}{KL}$ &c. dein $c = \frac{b - 2b}{HK}$, $2c = \frac{2b - 3b}{IL}$, $3c = \frac{3b - 4b}{KM}$ &c. Poſtea $d = \frac{c - 2c}{HL}$, $2d = \frac{2c - 3c}{IM}$ &c. Inventis differentiis, dic $AH = a$, $-HS = p$, p in $-IS = q$, q in $+SK = r$, r in $+SL = s$, s in $+SM = t$; pergendo ſcilicet ad uſque perpendiculum penultimum ME, & erit ordinatim applicata $RS = a + bp + cq + dr + es + ft$, &c.

Corol. Hinc areæ curvarum omnium inveniri poſſunt quamproximè. Nam ſi curvæ cujuſvis quadrandæ inveniantur puncta aliquot,

The second part of your job will be to apply the method described by Newton to a set of values of the angle called S-O-T. The red line shows the formula you can use in this activity.

The set of data belongs to the asteroid 2009ST19 that we got from the Jet Propulsion Laboratory web page:



Now we give you some values of the S-O-T angle from July 18, 2009 to November 15, 2009:

Date__(UT)	__HR:MN	S-O-T
2009-Jul-18	00:00	110.6386
2009-Jul-28	00:00	98.2399
2009-Aug-07	00:00	85.0185
2009-Aug-17	00:00	68.5472
2009-Aug-27	00:00	48.2178
2009-Sep-06	00:00	57.0115
2009-Sep-16	00:00	80.3307
2009-Sep-26	00:00	97.0950
2009-Oct-06	00:00	111.1146
2009-Oct-16	00:00	124.4429
2009-Oct-26	00:00	137.5462
2009-Nov-05	00:00	149.7362
2009-Nov-15	00:00	159.0660

Now you have to calculate to positions of 2009ST19 using Newton's formula:
1_ The angle on September 10.
2_ The angle on November 25.
The conditions of this activity are:

a) You cannot use computers
b) You cannot use calculators

This is because Newton had no any of them.

You have to deliver your report in DigitalDrop Box of our Blackboard's course; you must include an image of the calculations made by hand. Feel free to include any other data or image you think is important for your report.

Thanks…

HISTORY OF MATHEMATICS IN THE NORWEGIAN LANGUAGE

A survey of the literature

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ABSTRACT

Several studies have indicated that many mathematics teachers have a limited concept of what "history of mathematics" is and how it can enhance their teaching of mathematics. They are also unaware of where information on this may be found.
To help remedy this situation in Norway, I have built a Norwegian wiki for teacher education (http://eleviki.wikidot.com) and have made the inclusion of history of mathematics in the wiki a priority. I have done a survey of which resources are available to the teachers in Norway and will build the contents of the wiki based on the findings in this survey. This article will give an overview of the survey.

1 Introduction

There are many reasons to include history of mathematics in the teaching of mathematics, as has been argued persuasively for instance in the ICMI Study (Fauvel & Van Maanen, 2000). However, studies show that few teachers take this to heart (for instance Smestad, 2004). An interview study suggests that Norwegian teachers are unsure where to find information on history of mathematics in Norwegian suitable for classroom work (Smestad, 2008a).

Because of this, I have decided to try to make resources on history of mathematics, as well as pedagogical considerations in connection to history of mathematics, easily available for Norwegian teachers. My chosen tool is a wiki, a website that is editable by the readers, wherein I want to publish and link to texts in Norwegian on these topics. The most well-known wiki is the encyclopaedia Wikipedia, but I have created a wiki dedicated to teacher education (eleviki; http://eleviki.wikidot.com). In this way, I hope to reach many teachers that are currently eager to include history of mathematics in their teaching. In time, eleviki could be a collaboration tool for teachers wanting to develop further resources together.

Such a wiki may later be used in teacher education as well as by teachers in school, and I have ideas on how this should be done. In this article, however, I will not discuss such plans for the future. Moreover, I will not discuss alternative approaches, such as creating professional development courses for teachers, except to say that in the current political climate in Norway, it does not seem feasible to get sufficient numbers of participants for such courses on history of mathematics here.

As part of developing this wiki, I found it useful to start from a survey of what already exists in Norwegian on history of mathematics, to get a picture of what Norwegian mathematics teachers (or students) have access to in their native language. In this way, I could also get an idea of what I could simply refer to and in which areas the need for new texts were greatest. Therefore, I made a survey of the literature and analysed it using a number of criteria. The overview and the analysis is the subject of this article, and will in turn form the basis for the preparation of content of the wiki.

2 An overview of what is available in Norwegian on history of mathematics

My first draft of a list of literature on the history of mathematics in Norwegian was created by searching library databases, searching through the registers of the Norwegian journal for mathematics teachers, Tangenten, and by looking at my own bookshelves. When this list was as exhaustive as I could readily manage, I contacted all teacher educators of mathematics in Norway (via a mailing list called “the Notodden list”). This gave me a number of additions. I also used social media (Facebook and Twitter) to collect more ideas.

As I started to review the resources, I combed the reference lists for further additions. This has given the material that forms the basis for this analysis. This list will be kept up to date at the address http://eleviki.wikidot.com/ressurser-om-matematikkhistorie-pa-norsk

To decide what should be included in the list, I have used three criteria. Firstly, it has to be about history of mathematics, which I simply define as “the use or development of mathematics in the past”. Discussing Pythagoras’ theorem without any further connection to the past than the name, I will not regard as “history of mathematics”, while the use of mathematics ten years ago, will be included. Secondly, the history of mathematics must be a significant part of the resource. This is, of course, a personal judgment, but I will consider that an article should include more than a few biographical details about a mathematician to be “a resource on the history of mathematics”. Thirdly, the resource must be relevant to “grunnskolen” (primary and lower secondary school, years 1–10). This criterion is most easily met if “grunnskolen” is explicitly mentioned, but I will also include resources where I think that the mathematics or the subject does have relevance for “grunnskolen” or may be of interest to mathematics teachers there. I have interpreted this quite widely when collecting the material, while in the analysis of the material later, I have a somewhat narrower definition of relevance. Texts that I have written myself as part of the wiki project (eleviki) are not included in this survey.

The point of limiting myself to resources in the Norwegian language is partly that it is useful to have some limits to the survey, partly that most Norwegian teachers of mathematics can be assumed to prefer to read academic material in Norwegian (if at all). The threshold for using history of mathematics may be lower when the teacher does not have to translate anything.

3 Literature review

I know of no previous surveys of the literature on history of mathematics in the Norwegian language, nor in other languages. However, in the planning of the analysis, I have relied heavily on the literature on history of mathematics, in particular earlier attempts to categorize different parts of the literature. The most important of these has been the ICMI Study (Fauvel & Van Maanen, 2000), which has chapters on many key perspectives. Many other articles have also given important insights. Giving a full literature review here would mean giving a full literature review on the field of HPM. Instead, I will note articles of particular interest to the analysis in the next section.

It should be noted that the resources in my list are not necessarily written as pedagogical materials, and that criteria developed to categorise or evaluate pedagogical materials are therefore not automatically relevant to all resources.

4 Planning the analysis

The purpose of emphasizing history of mathematics in eleviki is to help remedy the situation described in my interview study of mathematics teachers (Smestad, 2008a). Teachers in that study called for readily available resources that could provide ideas for use in the classroom. The study also showed that teachers had different views on what history of mathematics is and how it can enrich mathematics teaching. From this and other literature, I have come up with several questions I would like the literature survey to answer:

– What is (considered to be) history of mathematics? History of mathematics can be considered both as the development of mathematics and the use of mathematics through history (Smestad, 2008a). A focus on development is well suited for presenting the dynamic character of mathematics, while focusing on its use more strongly shows the role of mathematics in society. Ideally, both perspectives should be included.
– Is history of mathematics reduced to pure biography? In the Norwegian textbooks for the 1997 curriculum, there was a tendency for this (Smestad, 2002). When teachers are "forced" to include history of mathematics, biography may be the easiest thing to turn to. That's why it is important that the teacher has resources available that show the breadth of possibilities, to avoid biographies becoming too dominant. In addition, I will look at whether the biographies included are used to illuminate what the motivation of the mathematicians to work with mathematics was.
– Which subjects from mathematics are treated? In advance of the analysis, I would guess that some topics, such as numeral systems, are well represented while others are less so.
– Which levels of schooling are the resources suitable for? In this context, I am particularly interested in the mathematics of "grunnskolen". I have previously had the impression that there may be fewer resources for primary school than for lower secondary school, both in the literature and at conferences on history of mathematics.
– Is history of mathematics considered as a tool or a goal (Jankvist, 2007)? That is: is history of mathematics just a vehicle for helping the students learn the mathematics, or are there additional reasons for learning history of mathematics?
– Is the history of mathematics "ready for use"? Can an interested mathematics teacher find ideas for his/her teaching, or must s/he develop the history of mathematics into a teaching sequence on his/her own? And to the degree that there are concrete ideas for teaching, what kinds of teaching do they promote? For instance, in the literature, there are examples of including history of mathematics in work with exercises (Smestad, 2008b), drama (Gonulates, 2007; Hitchcock, 1992) and so on. Chapter 7 in the ICMI Study (Fauvel & Van Maanen, 2000) gives an overview of different such ideas.
– To what degree can a mathematics teacher find original sources that are translated into Norwegian? Many believe that working on original sources give a particular output (O. Bekken, Barbin, El Idrissi, Métin, & Stein, 2004; Clark & Glaubitz, 2005; Thomaidis & Tzanakis, 2008), but then the sources must be available to the teacher.
– Are the resources easily available? Are they online, in easily accessible journals and

books, in textbooks for teacher education or perhaps only available by special loan from far-away libraries? Teachers will have more use of readily available resources, since teachers often have little time from they begin to plan a lesson to the lesson is supposed to be held.
– How much material is there? What number of pages?

For the purposes of this article, it’s not possible to assess the quality of the resources, for instance when it comes to historical correctness. I am aware that some of the literature, such as Ifrah, Birkeland, Jacobsen and Wiig (1997) (in its English version) has been criticised for having a low level of precision. (Dauben, 2002a, 2002b) A closer look at the quality will of course be needed before actually referring readers to particular resources.

To summarise: My question is: “Which sources on history of mathematics relevant to “grunnskolen” are available in Norwegian, and what aspects of history of mathematics are covered?”

5 Results

5.1 Quantity

The last question in the list above can be answered most easily. The books and articles on my list include more than 17,400 pages of history of mathematics. This figure is of course partly the result of various judgments, for instance I have not made an attempt to standardize the measuring unit “a page”. Still, the number says something about the size of the material. For a mathematics teacher interested in history of mathematics, there is more than enough reading material.

To get to that number, I have counted pages which I regard as relevant for history of mathematics. Every book and every article has thereafter been categorized as a whole. For instance, if a book which includes 200 pages of history of mathematics has some geometry, I have noted that, but not the exact number of pages of history of geometry. This is a methodological problem, but there was simply not enough time to make a detailed analysis of every single page.

The online encyclopaedia Wikipedia in bokmål (the most popular variant of Norwegian) had, at the time of the analysis, a few articles on history of mathematics and 279 articles on mathematicians. I have looked at the articles on history of mathematics and a sample (20 %) of the articles on mathematicians and have calculated a total based on this. In Wikipedia in nynorsk (the other main variant of Norwegian), there was almost nothing on history of mathematics at the time of the analysis.

5.2 Availability

Of the 17,400 pages, about 3,200 pages are available online. 11,400 pages are from books or journals that are available at most libraries at colleges and universities in Norway. Approximately 4,700 pages are from sources that are harder to obtain. (The sum is greater than 17,400 because there are sources that are both online and available from libraries.) I will use the term “easily available” about resources that are online or available from most of these libraries.

It should be added that the fact that a resource is online, does not necessarily mean that it is easy to find. This applies in particular to books that are digitalized and are available as free ebooks in the library system Bibsys. That is not necessarily a place where teachers will look and the books there are not necessarily linked to from anywhere else (until now).

5.3 Audience

Approximately 4,700 pages are written specifically for teachers. About 1,300 pages are George Ifrah's two volume work on numbers (Ifrah, et al., 1997), about 1,250 pages are in textbooks for teacher training, about 280 pages are articles (mainly from the journal Tangenten) and about 210 pages are in textbooks for children. The remaining pages are from various other books and websites.

Of these 4,700 pages, about 3,700 are easily available using the definition above. About 680 pages are available online.

5.4 View of history of mathematics

Different views of history of mathematics exist. One perspective is to look at the "first time…" – that is how the research front of mathematics has developed. The website "Earliest Uses of Various Mathematical Symbols" (Miller) is a well known example of this (in English). Another view is to look at how mathematics is an integral part of culture, and to look at how mathematics has been used in different periods. Ethnomathematics is an example of this. A third point of view is to look at the history of the school subject mathematics.

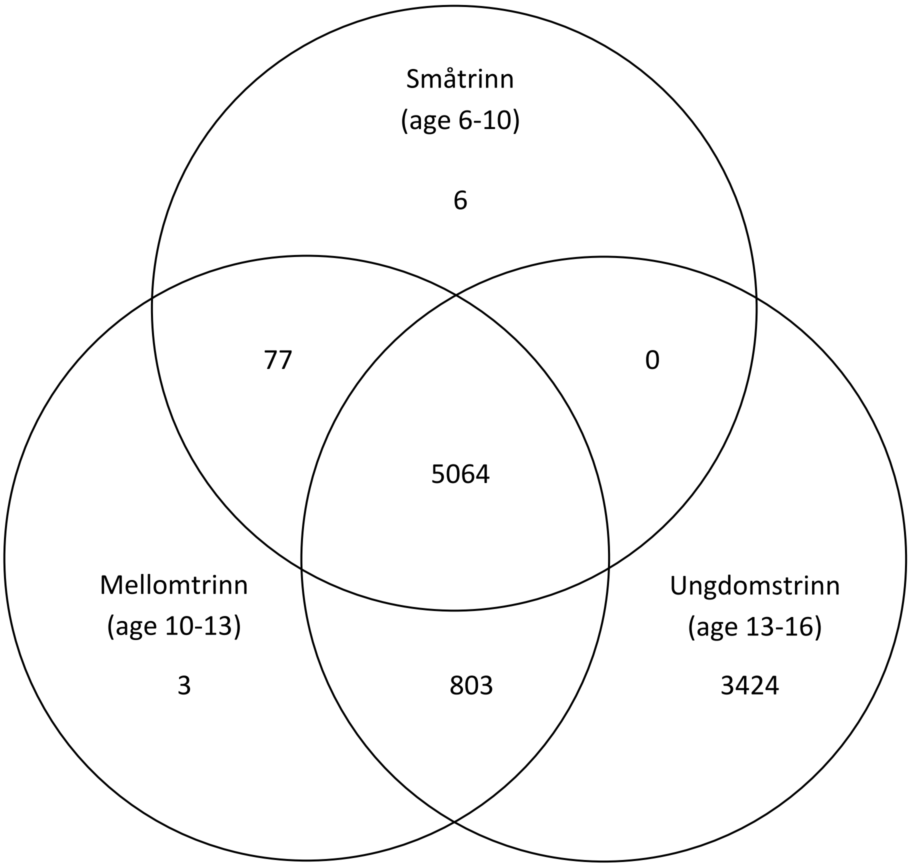
In the Norwegian material, the development of mathematics is dominant. Looking at the development is approximately twice as common as looking at the use of mathematics. Of what is connected to the use, almost one third relates to numbers or numeral systems, where it is more common to talk about who used the numeral systems than who developed them. The history of the school subject mathematics is mostly treated in separate books and articles, such as in Botten (2009b).

5.5 Level of pupils

My preconception was that there would be more literature on history of mathematics the higher the level in the school system. However, there is actually some literature of relevance to lower primary school as well.

By "relevance" to a level, I mean that the resource deals with history of mathematics that the pupils at the level are supposed to learn, according to the current curriculum (Utdanningsdirektoratet, 2006). This is a stricter requirement than the rather loose criterion that I used when compiling the list of relevant literature for the study.

We see that there is very little literature on history of mathematics in Norwegian for småtrinnet (year 1-4) only (6 pages), for mellomtrinnet (year 5-7) only (3 pages) or for primary school (year 1-7) only. (6 + 3 + 77 = 86 pages). But there is a whole body of literature that is relevant for these levels together with higher levels.



The bulk of the literature associated with the primary level treats numeral systems. Since numeral systems are included in most overview articles and books on the history of mathematics, these articles and books provide a large number of pages registered as relevant to primary school.

About 80 percent of all the literature relevant to "grunnskolen" is easily available. This percentage is relatively similar at all stages of "grunnskole". But only 15 percent is available online. Of the literature written exclusively for primary school, only 3–5 percent is available online. More than 40 percent of what I have considered relevant for "grunnskolen", is written explicitly for teachers.

The "use" aspect of history of mathematics can be found in about 60 percent of the literature, while the "development" aspect can be found in about 86 percent. Here, the variation between the levels is greater, and "use" is far more often found on the lower levels. The reason for this is, as mentioned before, that the numeral systems are so important at the lower levels.

5.6 Mathematical content

The mathematical topics range widely. I chose to divide the mathematical themes according to the headings of the current curriculum (Utdanningsdirektoratet, 2006). Based on an expectation that the subject "Number and algebra" is a large one, I decided to divide this in two, even though the curriculum treats it under one heading. Similarly, I divide the area of "Statistics, probability and combinatorics" into two. The division then gives the following table, where the numbers indicate the number of pages of the books and articles dealing with the topic. A single book can therefore be included in several of the numbers. Since I have not analysed how much of a book the various themes represented, these statistics must be interpreted with caution. It may still be a starting point for looking at parts of the material.

	Total	Easily available		Online	
Number	9275	7240	(78 %)	1230	(13 %)
Geometry	6870	5836	(85 %)	1090	(16 %)
Algebra	5426	4390	(81 %)	788	(15 %)
Functions	3405	2930	(86 %)	642	(19 %)
Measurement	3209	2589	(81 %)	647	(20 %)
Statistics	2051	1731	(84 %)	170	(8 %)
Probability and combinatorics	2017	1834	(91 %)	688	(34 %)

As mentioned earlier, numeral systems constitute a large portion of the material that is about numbers.

I believe it is an important finding that there is little literature on the history of probability and statistics.

For all of these topics, more than 90 percent of the literature emphasized the development of mathematics. 53-62 percent emphasized the use of mathematics, except for the theme of probability, where only 37 percent of the literature includes that aspect. This could possibly be due to the fact that probability theory is a relatively young discipline, which means that the early development is well known. The history of this development is also suitable for use in schools.

5.7 History as biography

History of mathematics can easily be reduced to pure biography (Smestad, 2002). In this material there are only 3,000 pages of articles and books that contain no (or only a negligible amount of) biography. However, the amount of biography varies from one kind of resource to another, from pure biographies (as some of the Wikipedia articles), via general histories of mathematics full of biography (such as Holme (2001, 2004)), all the way to articles on mathematics with just a little biography.

I have looked for instances where the biographical details have included anything about the mathematicians' motivation for doing mathematics. There are some examples of this. For instance, Florence Nightingale's motivation for working on statistics is mentioned (Flakstad, 1999), as is Andrew Wiles' motivation for working on Fermat's last theorem (Singh, 2004) and the motivation for working on code theory before and during the second world war (Singh, 2000). In general, I will still say that the motivation of mathematicians could have been better handled.

5.8 History as a tool or a goal

The question of whether history of mathematics is a tool to teach mathematics or a goal in itself, concerns only the part of the literature that is written with the teacher in mind. But even in these resources, the question of the point of including history of mathematics is rarely commented on. I have been looking for comments on why pupils should meet history of mathematics. The resources where such comments portray history of mathematics as a tool represent 1,900 pages. The resources where history of mathematics is considered as an end in itself represent around 600 pages (of which just over 400 pages have both aspects included). The idea of history of mathematics as a goal in itself is thus relatively rarely considered in the Norwegian language literature.

Earlier in this article I commented that the development of mathematics was about twice as prominent as the use of mathematics. If we look only at the resources where history of mathematics is presented as a tool, the relationship is not as skewed (98 percent development and 74 percent use). Of the resources where history of mathematics is presented as a goal, all includes the use of mathematics. It is tempting to interpret this as a sign that authors having such a conscious idea of the role of history of mathematics that they discuss it in their articles and books, also have a sufficiently nuanced view of history of mathematics to see that it concerns both the use and the development. But this is speculation.

5.9 Ready for use?

In the material I have examined, there are few resources that a mathematics teacher can use directly in his/her teaching. What I have found are:

– A number of lesson plans at the website matematikk.org (Celik, 2010) which the teacher can use directly.
– In the articles: only two ideas for lessons, but lots of exercises.
– In textbooks for teacher education: a few exercises, but mostly for teacher students, not pupils.
– In elementary school textbooks: some texts on history of mathematics for pupils and some exercises.
– In other books: a few exercises, but no lesson ideas.
– In some master theses, some exercises and lesson ideas are described. (These theses are not easily available for teachers.)

The fact that there are so few resources that are "ready for use" is probably the most striking result in this analysis. At conferences like HPM and ESU, a wide range of ways of including history of mathematics in teaching are described. Of these, only a very narrow and limited range reaches Norwegian teachers. The same applies to more theoretical descriptions of how history of mathematics can be helpful. Only one of the books (Breiteig & Venheim, 2005) discusses this to any degree.

And even when it comes to tasks, there is little to find. I have for instance not found a single proposal for a task or lesson in Norwegian where the history of statistics is exploited.

5.10 Original sources

There are very few original sources available in the Norwegian language which are at the same time suitable for use in "grunnskolen". Beyond short quotes that are spread throughout articles and books, I have found the following original sources:

– Some extended excerpts from Tyge Hansøn's Arithmetica Danica (Botten, 2009a, 2009b)
– The letter to Niels Henrik Abel offering him a position (Kirfel, 2003)
– Extract from the sagas of Snorre about probability (Onstad, 2000)
– A short clip from Marolois and a larger one from Stampioen (van Maanen, 2000)
– The complete Algorismus from Hauksbók (O. B. Bekken & Christoffersen, 1985)

– Some translated excerpts from the history of equations (Onstad, 1994)
– The beginning of Leonardo Pisano's Liber abbaci (Smestad, 2000)

This is a relatively meagre selection to choose from if you want to let your pupils work with original sources in the Norwegian language. Moreover, none of these are accompanied by advice for teachers on how they can be included in teaching in "grunnskolen". Only two of them are available online; they are, in total, 6 pages long. This probably means that the use of original sources in teaching will be impossible until the pupils are able to read them in other languages.

6 Conclusion

A Norwegian mathematics teacher who starts to navigate the literature on history of mathematics in Norwegian, with a goal of including it in his/her teaching, will have plenty to choose from. There is a lot of material written for teachers, relevant for all school levels and for all of school's main mathematics subjects, although there is little written specifically for primary school. Many of the resources are available online.

The development of mathematics is most prominent, while the use of mathematics is also present in many resources. The history of the school subject is mostly found in specialized sources. There are many biographies of mathematicians, but their motivation for working on mathematics is not that frequently discussed. Most often, there is no discussion of why history of mathematics is relevant for teachers, and where that is discussed, history of mathematics is mostly seen as a tool for teaching mathematics. Seeing it as a goal is more unusual.

The mathematics teacher must mostly find out by himself/herself how the history of mathematics can be transformed into meaningful mathematics lessons. There is little concrete advice on this and few examples of how it can be done. If the teacher wants to include original sources in Norwegian, there is little to choose from.

7 Future work

This study was done to obtain a basis for developing a Norwegian wiki with contents on history of mathematics. Based on what I have found in this study, I see several points that should be emphasized in future work. These are to

- emphasise the use of mathematics as much as the development of mathematics.
- develop (or translate) concrete ideas for lessons in history of mathematics that will make it easier for teachers to get going. (Books like Demattè (2006), Katz & Michalowicz (2004) and Pinto (2009) can be interesting sources.)
- make sure that primary school has relevant resources available. It is possible that cooperation with colleagues in other countries may be helpful.
- make sure that statistics and probability theory is included.
- emphasise the motivation mathematicians have had for working on mathematics.
- make more original sources available in the Norwegian language, for instance by translations from Danish and Swedish, if possible.
- include links to everything that is available online in Norwegian and to all other materials that are readily available.

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"CAUCHY IN GORIZIA" IN THE MEMORY OF GIORGIO BAGNI

A year after his passing away

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ABSTRACT

This publication is meant to be a tribute to Giorgio Bagni a year after his passing away and it deals with the work we did together in 2007 on the occasion of Augustin Louis Cauchy's 150th death anniversary. The starting point was giving to the local teachers and students information about the 2 years, from 1836 to 1838 that A.L. Cauchy spent in Gorizia. Moreover we aimed to afford the question of the inheritance this great mathematician left to nowadays students and teachers by means of an epistemological and hermeneutic approach to history of mathematics.

1 Introduction

Not many people know that Augustin-Louis Cauchy lived in Gorizia for two years, from 1836 to 1838. He arrived from Prague, helped by the Habsburgs, together with the Bourbon Court exiled from France after the revolution of July 1830. He was the mathematics and science tutor of Henry V , the grandson of Charles X, who was supposed to become the King of France if the Bourbons had managed to rule France again.

1.1 The project „Cauchy in Gorizia"

The research that Giorgio Bagni and I did in 2007 was financed by the project called "Progetto Lauree Scientifiche" promoted by the Italian Ministry of Education and Scientific Research together with the "Conferenza Nazionale dei Presidi delle Facoltà Scientifiche" and "Confindustria" (the national association of industrials), thorough a cooperation between the Mathematics Department of Udine University (thanks to prof. Fabio Zanolin), the Ufficio Scolastico Regionale del Friuli Venezia Giulia (thanks to dr. Luigi Torchio), the Istituto d'Arte "Max Fabiani" (thanks to the headmaster prof. Fabio Della Picca), the Comune di Gorizia (thanks to the Culture Counsellor dr. Claudio Cressati) and the Administration of the Grand Hotel Entourage (thanks to dr. Antonella Lovato).

On the occasion of Augustin Louis Cauchy's 150th death anniversary, Giorgio held two conferences at the Istituto d'Arte "Max Fabiani", open both to the students and the teachers, to present the work of Cauchy from the epistemological and hermeneutic point of view, while I held a public conference in collaboration with the "Comune di Gorizia" at the "Grand Hotel Entourage" more focusing on the life of Augustin Louis with special regard on the years he spent in Gorizia. At the end of my conference the Town Culture Counsellor unveiled a memorial marble plate, offered by the Administration of the Grand Hotel Entourage, at the entrance of the palace where Cauchy used to give his lessons to Henry V.

Hereafter you can find the historical content conveyed and the thorough examination of the foundations of Calculus under a cognitive as well as a meta-cognitive point of view in order to examine which can be considered the most valuable issues of the inheritance of

Cauchy for nowadays students and teachers.

2 Hard life of a Royalist mathematician

Augustin - Louis Cauchy was born in Paris on the 21 August 1789 , slightly over one month after the historical date, 14 July 1789, the fall of the Bastille, which marked the beginning of the French Revolution.

His father, Louis Francois Cauchy worked in very close collaboration with the police liutenant general of Louis XVI of Bourbon, who fled to England to survive the Revolution. Despite the hard times Augustin's family managed to live in Paris for a while, but at a certain moment was forced to seek refuge in Arcueil (set half way between the centre of Paris and Orly airport).

Augustin Louis's life was nothing but easy at the beginning. The Cauchys had to face a period of hard economic difficulties and fear for their personal safety. This would mark forever their young son. He not only suffered physically and remained rather weak and not in good health all his life long; but also, in a very stubborn way, never considered in his life any other revolutionary movement or idea which might cause to part him from the Bourbon Family. He had the same stubborn attitude with respect to his strong catholic beliefs.

In 1794 Roberspierre was beheaded and the Cauchys went back to Paris. Augustin's father showed great skills in adapting himself to different circumstances and political situations which offered him new opportunities: during Napoleon's consulate he was nominated file clerk at the Senate. He was able to keep his position until 1830 despite all the various political and institutional changes of that period.

Since his young age Augustin showed great learning skills (Belhoste, 1990 & 1991; Mahwin, 1995; Valson, 1968). He was bestowed all the prices for ancient languages by the Central Pantheon School which he had attended since 1802; but despite his literary success, he started, against his family tradition, an engineering course at the Ecole Polytechnique. At the Polytechnique he joined the Congregation de la Sainte Vierge, an association founded in 1801 by the Jesuit father Bourdier-Delpuits in order to organize prayer meetings. Later on, this association would have fought prevailing anti religious feelings from inside the most important institutions. After the Ecole Polytechnique Augustin-Louis was admitted to the School of Civil Engineering (l'Ecole de Ponts et Chaussées): everywhere he obtained brilliant results. In 1809 he became Engineer and the year after he was given a position at the harbour of Cherbourg where fortification works ordered by Napoleon were under way.

Thanks to the ardour and the zeal he carried in every task, Cauchy was quickly appreciated for his technical qualities, but his great interest soon turned towards pure science, which he initially cultivated in his free time. In 1813 he left the job in Cherbourg in order to return to Paris. Some intense activities on the polyhedrons, on the theory of the substitutions, on the double integrals, and a memory on the theory of the waves, rewarded by the Académie des Sciences, caught the attention of the experts on the young brilliant mathematician. Nevertheless, despite several attempts, he did not succeed in obtaining neither a chair, nor the election to the Académie des Sciences. Only after the restoration, during the reign of Louis XVIII of Bourbon, brother of Louis XVI, Cauchy was offered a post as teacher directly by the governor of the École Politechnique and than was named member of the Académie des Sciences by Royal Decree. In 1817 he was appointed teacher also by the Collége de France and later even at the Sorbonne.

His father arranged for Augustin-Louis the wedding with Louise de Bure which was celebrated in 1818, and signed by Louis XVIII in person. Louise was the descendant of a famous dynasty of publishers in Paris, and gave him two daughters, Alicia and Mathilde. In this period Cauchy applied his extraordinary energy to the mathematical research, to the Congregation and in a very committed way to the teaching at the École Polytechnique. In this period he took his time to rework the foundations of the mathematical Analysis and to base them systematically on the idea of limit of which he also specified the meaning. Unfortunately his efforts were little appreciated, so far very often the young university professor had to fight against the negative reactions of the Conseils de l'Instruction et de Perfectionnement of the École Polytechnique.

In 1821, at the end of a lesson which lasted far beyond the established timetable, Cauchy was hissed by five or six students. The Director made a report about the incident penalizing the university professor, but the Ministry considered the hisses mainly as a political manifestation against the ultrarealistic convictions of the teacher. The publication of the course of Analysis, several times required by the Conseil d' Instruction, was eventually assured by Augustin Louis. The *Cours d'Analyse algebrique* was issued in 1821, the *Résumés des leçons sur le calcul infinitésimal*, in 1823, the *Applications du Calcul Infinitesimal à la Géometrie*, between 1826 and 1828, and the *Leçons de Calcul Différentiel* in 1829, all published by the publisher De Bure, it has to be said. These works nowadays are considered the models of any modern treaty of analysis.

In 1824 Louis XVIII of Bourbon died and the throne passed to his brother Charles X, who would not be given the same high consideration and would be dethroned in July 1830 after the revolution. The Bourbon Family was forced to the exile and, before leaving to England, Charles X and his son, the Duke of Angouleme, abdicated in favour of their grandson, the Duke of Bordeaux, still under-age, orphan of father since his birth as he was the son of the assassinated second son of Charles X, and of Maria Carolina de Berry. After Charles X, Louis Philip of Orléans succeeded on the French throne and required from the public institutions an oath of fidelity to the crown: but Cauchy refused to swear and was consequently removed from the chair. He abandoned voluntarily France, seeking refugee in Friburg (Switzerland), where in 1831 he received the invitation from the king of Sardinia Charles Albert to move to Turin in order to hold the chair of Sublime Physics.

In 1833 Augustin Louis was called to Prague by Charles X of Bourbon in exile, with the task to provide for the scientific education of the Duke of Bordeaux, presumed heir to the throne of France that he would assume with the name of Henry V (but the thing never came true). The reason of this call must be connected to the disagreement between Charles X and his daughter in law Maria Carolina de Berry about the education of Henry V, disagreement that crossed all the party of the Bourbons' followers. Henry V's mother, from whom Charles X had withdrawn the regency in favour of himself, would wish for her son a kind of education more open to the libertarian ideas introduced by the French Revolution, because her dream was to be recognized one day as the mother of the effective king of France. On the other hand Charles X kept defending the principles of the Ancien Régime and trusting the Providence for the effective rescue of the throne by the Bourbon, so he claimed for his grandson an education more faithful to the old traditional principles. In this context it is clear why Cauchy, who did not have a good reputation as teacher and pedagogue, was offered the position of tutor for Henry V. The new tutor was asked not only to teach sciences, but to do it in a frame of deep catholic principles and not as an illuminated scientist. Who could ever have done it better than Augustin Louis?

Strongly faithful to his feelings of Bourbons follower, Cauchy did not hesitate to leave the residence of Turin, where he could dedicate himself entirely to science, in order to assume the new delicate task that would absorb a good part of his time. For five long years Cauchy followed his student in the peregrinations of the exile and tried to teach to the Duke of Bordeaux, who in the meantime had assumed also the title of Count of Chambord, the elements of mathematics and of other sciences. The pedagogical skills of the tutor were debatable, but his patience was heroic. Apparently the prince used to play him dirty tricks that went far beyond the limits of the simple joke. Cauchy was joined by his family in Prague and then he followed the Bourbon Court in exile to Teplitz, Budweitz, Kirchberg and Gorizia, where he arrived in the October of 1836 and where he witnessed the death of Charles X (Bader, 1995; Juznic, 2005). After the death of his grandfather, Henry V moved to Palazzo Strassoldo (the current Hotel Entourage) by his uncle and aunt, the Dukes of Angouleme. Cauchy was lodged at the hotel “Tre Corone” and he used to go to the Palace in order to give his tuitions to the Prince.

During his stay in Gorizia, Cauchy carried on his research. It has be estimated that 4% of his total production dates back to this period (Juznic, 2005). This means more or less 500 pages of articles. They were about the theory and the propagation of light and the complex functions. Besides this, during the period spent in Gorizia, Cauchy not only was the tutor of Henry V but he also followed and encouraged the studies of prof. Mocnik. He promoted his doctorate thesis and therefore helped him to leave the Normal School of Gorizia for the University of Graz (Juznic, 2005).

In October 1838 Henry V was 18 so Cauchy's task ended. Augustin Louis, who was almost 50, moved back to Paris with his family. He had been conferred in appreciation the title of baron by the Bourbons, title to which the mathematician attributed great importance. When back in Paris, he was offered different chairs and assignments, included a post by the Bureau des Longitudes, but his reluctance in taking the required oath held him far from every public office until the advent of the Second Republic in 1848, when the oath was suppressed. He attended only the Académie des Sciences, where every week he passed some communication. In 1849 Cauchy was named professor of Mathematical Astronomy at the Faculty of Sciences of Paris. In 1852 the emperor Napoleon III restored the oath, but after a while he exempted the royalist Cauchy and the republican Arago from swearing, so Cauchy was given the post of teacher of Mathematical Physics at the Faculty of Sciences and he held it till his death, on 23 May 1857. His last worlds pronounced in front to the priest of Sceaux were *“Les hommes passent, les oeuvres restent”* and, by good, he was right (Mawhin, 1995).

3 The mathematic work of Cauchy and the issue of the rigour

“Cauchy is insane” but he's the only one “who knows how to do mathematics” and “the only one that nowadays does pure mathematics” (Bottazzini, 1990). The young and enthusiastic Niels Henrik Abel (1802-1829) introduces in such an irreverent but stimulating way the thought of one of the main characters of the history of pure and applied mathematics.

Augustin-Louis Cauchy's work is huge and covers all the ranges of mathematics. Everywhere he was, he kept on writing books and memoirs and was second in publishing only to Euler.

His complete work started to be published by Gauthier-Villars in 1882 and was over only in 1974. It includes 27 volumes with more than 13,000 pages. Most important works about mathematics were *Cours d'Analyse algébrique* (Paris, 1821), *Leçons sur les applications du calcul infinitésimal à la géométrie* (Paris, 1826), *Exercices de mathématique* (Paris, 1826) and *Exercices d'Analyse et de Physique mathématique* (Paris, 1841-1844). He was involved in mechanics and optics too and was also keen on poetry, even though, as a poet, was not that great (Mawhin, 1995).

The importance of his analytical works is double: both substantial and formal. In them, we can find a lot of definitions and demonstrations which are considered, according to the XIX century sensibility, fully "rigorous".

Here follow some of Cauchy's words:

> «*Quant aux méthodes, j'ai cherché à leur donner toute la rigueur qu'on exige en géométrie, de manière à ne jamais recourir aux raisons tirées de l'algèbre. Les raisons de cette espèce, quoique assez communément admises […], ne peuvent être considérées, ce me semble, que comme des inductions propres à faire quelquefois pressentir la vérité, mais que peu s'accordent avec l'exactitude vantée des sciences mathématiques...*»

that can be translated:

> «*As for the methods, I have sought to give them all the rigour which one demands from geometry, so that one need never rely on arguments drawn from the generality of algebra. Arguments of this kind, although they are commonly accepted […], may be considered, it seems to me, only as examples serving to introduce the truth some of the time, but which are not in harmony with the exactness so vaunted in the mathematical sciences…*»

Cauchy's cultural project, therefore, appears to be very clear. He wanted to part the analysis from the procedures, currently used in his time, which had merely the aim to justify empirical results. These methods were almost never completely justified, and revealed in some cases to be clearly incorrect. They were deriving from an application of algebraic techniques to analytic situations without any real theoretical reason. Cauchy, instead, perceived the pressing need to base all the analytical concepts on precise foundations and was the first mathematician to do it.

In this perspective, what can be considered Cauchy's inheritance for mathematicians, and overall for nowadays teachers and students?

Let's consider the definitions of infinitesimal and limit given by Cauchy in the *Cours d'Analyse*:

> *"When the value successively attributed to a particular variable indefinitely approach a fixed value in such a way as to end up by differing from it as little as we wish, this fixed value is called the limit of all the other values. Thus, for example, an irrational number is the limit of the various fractions that give better and better approximations to it. In geometry, the area of a circle is the limit towards which the areas of the inscribed polygons converge when the number of their sides grows more and more, etc. When the successive numerical values of such a variable […] decrease indefinetely, in such a way as to fall below any given number, this variable becomes what we call an infinitesimal, or an infinetely small quantity. A variable of this kind as zero as its limit."* (Bottazzini, Freguglia & Toti Rigatelli 1992)

Could we define them as “absolutely rigorous”? Probably not, if examined on the light of the mathematical sensibility of our days. This answer, however, appears too simple to be considered satisfactory. The comparison with the previous mathematical literature certainly shows a clear radical change of trend: **Cauchy placed the notion of limit instead of that of derivative as a fundamental concept** for the first time in a treatise on Analysis.

The importance of analytical work of Augustin Louis is also due to a clear theoretical organisation of some geometric notions which were previously introduced only by intuition. From the end of XVII century on, the notions of area limited by a curve, of length of a curve, of volume limited by a surface and of area of a surface had been simply accepted as understood by everyone, and the fact that such quantities could be measured through integrals was considered one of the main realizations of calculus (Kline, 1991). Cauchy was the first considering **those quantities as defined by the integrals normally used only to calculate them.** This fact clearly represents an important and theoretically delicate change of perspective.

The question therefore is: what's Cauchy's inheritance for mathematicians, and above all for nowadays teachers and students?

In order to try to give an answer, we should introduce some considerations about the use of the history of mathematics with reference to teaching.

4 The inheritance of Cauchy: an hermeneutic approach

Let's take again in account Cauchy's definitions of limit and infinitesimal to discuss about their being or not “rigorous”. If we consider them in the light of today's mathematical sensibility, surely they are not, at least not completely. Whereas such an answer, as already said, seems too simple to be satisfactory.

There are some observations which are worth discussing.

From the point of view of a historical reconstruction, the comparison with the preceding mathematical literature highlights a clear reversal. In his *Traité élémentaire du Calcul Différentiel et du Calcul Intégral* published between 1810 and 1819, S.F. Lacroix hadn’t put the notion of limit at the heart of differential calculus, but the notion of derivative and differential (Lacroix, 1837).

The new treatment, which we owe Cauchy, reflects however the general lines which are often adopted by modern mathematical analysis, in which the limit itself is considered as the fundamental concept. (Boyer, 1982).

Therefore neither a historical reconstruction nor a today’s point of view has to be the exclusive horizon from which we would consider Cauchy's position.

The very question is: with reference to what notion of rigour, should we nowadays interpret the works of the great French mathematician?

According to Umberto Bottazzini, rigour in Mathematics is itself a historical concept and so it is in progress. Appealing to the requirement of rigour when explaining the development of mathematics seems actually a circular process. In fact, we come to the formulation of new *standards* of rigour when the old criteria don't allow a suitable reply to the questions that come from mathematical practice (Bottazzini, 1981).

We find ourselves in a difficult situation which presents two components which are quite different and for some aspects, at least apparently, contrasting. On one side an interpretation based on historical reconstruction carried out from the point of view of 19th century mathematical analysis would lead us to emphasize Cauchy's process of introduction of rigour, on the other side, a “modern” approach would draw us to consider critically the definitions and the concepts from our point of view and so with explicit reference to the epistemological statute of mathematics which is used, taught and learnt nowadays.

These two different interpretations are not anyway conflicting. A correct hermeneutic approach, referable to the ideas of Hans-Georg Gadamer (1900-2002) may help us considering the situation in all its complexity and richness, especially from the point of view of teaching (Bagni, 2006). We have to quote Wilhelm Dilthey (1833-1911) who underlined that the first condition of the possibility of a science of history consists of the awareness that everybody is a historical being even the person that studies and investigates history (Dilthey, 1962).

Gadamer (2000) moreover remarks that a historian often chooses the concepts with which he describes the characteristic historical physiognomy of its objects without explicitly caring for their origin and justification, omitting in this way to realize that the descriptive suitability he finds in the concepts he chooses might flatten what is historically distant on what is familiar.

To sum up, the risk of making something present-day without any awareness is concrete and is explicitly marked by different authors.

Should a more careful attitude be necessary for a historian?

Should it be advisable to model concepts on those which are characteristic of the period we are examining?

With reference to Cauchy's huge and important work of introduction of rigour, should we put ourselves in a mental attitude which is somehow close to the one of XIX century mathematicians?

Gadamer strongly claims that such a choice would be both illusory and groundless and that the naivety of an historian becomes an abyss when he starts realizing the problematic nature of his position and sets the principle that, in historic understanding, one has to put his own ideas aside, trying to think only according to the concepts of the period one wants to know (Gadamer, 2000).

Such naivety doesn't come simply from the inevitable failure to which this choice would be condemned, the main problem is different and much deeper: the historic conscience disregards itself if, in order to understand, leaves out just what makes understanding possible. To think historically actually means to bring to an end the transposition that the concepts of the past undergo when we try to think according to them. To think historically always involves mediation between those concepts and the actual way of thinking (Gadamer, 2000).

So, we are neither allowed to judge Cauchy's work as it had been done nowadays nor really able to evaluate it “from inside the XIX century”, so to speak. To interpret a historical event considering our point of view is somehow inevitable, but it mustn’t happen in direct and radical terms. *Mediation* is the key concept. It is not possible to appreciate the work of a great protagonist of the history of mathematics and more generally of the history of human thought, without considering its historicity; as well as at the same time, it is not possible give up completely our role of readers and interpreters.

The fact that we can't help duly considering a scholar's point of view while examining and interpreting the mathematical work of an author of the past, must not lead us either to consider any historical reconstruction useless or make us fear any form of arbitrariness when we use historical elements in our teaching.

We just want to recall that also very “hard” sciences have progressively loosened their hold as regards a demand of absolute objectivity. The epistemological statute of physics, for example, has had Heisenberg's uncertainty principle to reckon with. So it seems understandable the right, becoming at the same time a duty, to “mediate” inside a discipline like the history of mathematics which, since it is history, undoubtedly presents clear “humanistic” features.

In the light of these reflections, we can declare that Augustin-Louis Cauchy's lesson for the students and the teachers of the XXI century is alive, intense and involving. It is the great, rich attempt, historically placed, to give rigour to concepts and procedures of mathematical analysis. His correct interpretation, by nowadays readers, will give a lot of precious ideas for a wide cultural growth, both form the point of view of the true understanding of contents, and from a methodological approach.

In conclusion, it can be asserted that the work done in Gorizia in 2007 is an example of a possible collaboration between School, University, local authorities and private citizens in order to integrate the History of the Mathematics with regard to this hermeneutic approach not only in the Didactics and the increase of the epistemological awareness of teachers and students, but also in the increase of the historical knowledge of the citizens.

In short: the History of the Mathematics as an important, inseparable and unavoidable part not only of the teaching of Mathematics, but also of History as such.

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DEVELOPING THE 'GEOMETRICAL EYE' THROUGH MODEL BASED PROBLEM POSING

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1 Introduction

The poster describes an activity that has been applied at grade 8 at lower secondary school in Prague. Pupils cut a sheet of A4 paper (a model of a rectangle with the size aspect ratio $a : a\sqrt{2}$) to create a model of two congruent triangles and a small rectangle. They expressed the area of these two shapes. Then, they assembled the parts of the model to compose various geometrical objects. During this activity, pupils were encouraged to pose their own mathematical problems based on the model and to offer possible solutions.

2 Problems posed by the pupils and their solutions

Two concrete examples of the mathematical problems posed by the pupils and the solutions they offered will illustrate the activity and its impact on the development of pupils' visualisation and reasoning abilities. The problem settings and solutions are exact transcripts of original pupils' notes, while the pictures are photocopies of original pupils' drawings or photographs of modelled geometrical situations. More examples were provided in the poster.

Problem 1

Find the area of the geometrical figure in which the two triangles (figure 1) intersect.

figure1

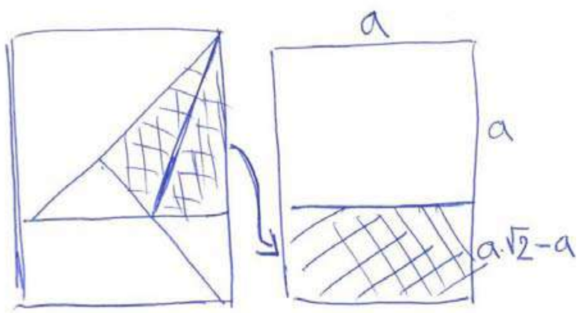

figure 2

Solution to Problem 1

The requested area is the area of two congruent right-angled triangles. These triangles can be rearranged to form a rectangle with side lengths a and $\left(a \cdot \sqrt{2} - a\right)$ respectively (figure2).

Problem 2

Determine the sizes of the two angles between the diameter of the small rectangle and each of its sides.

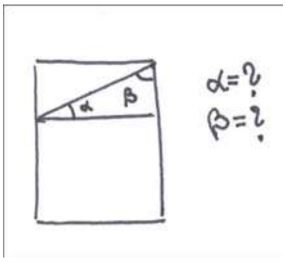

figure3

Solution to Problem 2

In analogy to the solution to Problem 1, we can rearrange the two congruent triangles forming the rectangle.

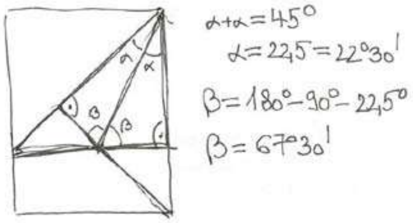

figure4

3 Conclusion

Models of geometrical figures are valuable in visualisation and understanding of geometrical concepts. However, using the models in mathematics teaching may cause the pupils to concentrate and rely upon figural rather than conceptual aspects of the geometrical objects being studied. The activity described by the poster combines its theoretical and practical parts to help integrate the conceptual aspects of the geometrical objects in pupils' cognitive structure. The practical part is represented by the experimental manipulation with the model, which enables the pupils to pose the problems. On the other hand, the necessity to communicate the problems and their solutions to other pupils creates a natural need for underlying theoretical concepts and their verbal or symbolic representations.

The activity is based on the approach to the geometry teaching by Godfrey and Siddons (1903), as it gives an actual example of connecting practice and theory in geometry teaching and learning.

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Original sources in the classroom

and their educational effects

THE USE OF ORIGINAL SOURCES IN THE CLASSROOM

Empirical Research Findings

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ABSTRACT

This article describes the theoretical framework and empirical results of a threefold comparative study in which original sources were used in the ordinary mathematics classroom according to a genetic, hermeneutic and conventional approach.

1 Introduction

In the last thirty years the use of history of mathematics has become a widespread and ever-growing approach in the teaching of mathematics. Its conjectured educational potential has been described extensively in many publications and conferences during that time. Very often, the proponents of teaching with history are also very passionate about the history of the subject itself. Obviously, this is no coincidence, and it surely applies to me. I remember that *my* own interest in the history of science began when I saw an image of Johannes Kepler's famous *Mysterium Cosmographicum.* It was printed on the cover of my physics textbook when I was a schoolboy. In fact, this was my first encounter with an original source and I was fascinated right away. I quickly learnt that the *Mysterium Cosmographicum* was meant to be a model of the solar system. It was very different from what I had ever seen before until then, although Kepler had already conceived it in the 16th century. And *because* it was so different, it piqued my interest. The words *Mysterium cosmographicum* seemed to indicate nothing less than that Kepler thought he had revealed God's geometrical plan for the universe. In fact, much of Kepler's passion for science stemmed from his belief that there were a link between the physical and the spiritual world. Of course, I had no idea about this when I saw the image for the first time. But my interest for the history of science and the history of thinking was aroused by this encounter with an original source.

My old physics textbook is out of use nowadays and has been replaced by newer editions, that I – a teacher myself by now – would use with *my* students. The newer editions still have the *mysterium* image on their covers, but not quite as large and eye-catching, and maybe not quite as thought-provoking as it had been before.

While those images have been shrinked by some layout artists, *my* interest in the history of science and mathematics has endured and become even greater. As a teacher I have always been eager to use history in the classroom whenever I think it is possible and helpful. And yet, what I could not be sure about, was the question if – apart from my subjective impression and maybe selective perception – there were any evidence that the history of mathematics really is as exciting and helpful for the majority of my students as it is for me? To say it in more general terms: That the history of mathematics is indeed a valuable resource for the learning and teaching of mathematics?

When I began to review the literature, I quickly realized – as has often been noted these days – that rather little is known empirically about this question, especially when it comes to teaching ordinary students in ordinary classrooms. So this is where my own commitment began. In order to find some answers I designed and conducted a theoretical and empirical study in which teaching with historical elements – ‘historical teaching’ – was compared with non-historical, ‘conventional’ teaching. My specific research questions were:

- How does the historical teaching affect the students’ mathematical skills compared to students who had conventional teaching?
- How does the historical teaching affect the students’ views on mathematics, teaching and their relationship to it?
- In what way and to what extent does the historical teaching allow for (meta-)reflection and discussion in the classroom?

There are, of course, many possible approaches to teaching with history. Personally I think, the study of original sources is the most demanding among them, both for teachers and for students. It is demanding in many ways: Of course, on part of the teacher it involves quite some preparation, but this is always the case when you are going to try something new. What is more, the study of original sources requires teachers and students to be prepared to dive into some strange and unknown realm of thinking, to appreciate cultural and historical contexts and – last but not least – to deal competently with written text that is more extensive than the word problems they are used to in mathematics. On the other hand, reading original sources does offer some exceptional opportunities (for a comprehensive account see [Jahnke et al. 2000, 292 ff.]). They might very well be worth the effort.

Original sources, however, can be used within at least two different theoretical frameworks, namely the genetic approach and the hermeneutic approach. Both of them were considered in this altogether threefold comparative study then. Its parts were

- teaching a certain topic with historical sources according to a genetic approach,
- teaching the same topic with historical sources according to a hermeneutic approach,
- and finally teaching the same topic with no history at all, that is to say ‘conventionally’.

In the next two chapters I am going to describe these different approaches and its effects respectively.

2 Teaching with historical sources according to a genetic approach

2.1 Theoretical considerations

Teaching with history according to a genetic approach is, indeed, an old idea. It was the great Felix Klein, who in his classic “Elementary Mathematics from an Advanced Standpoint”, argues for a genetic method of teaching wherever possible. This idea later has been adopted by Otto Toeplitz who explicitly proposed to use history in service of a genetic approach to modern mathematics. Within it history is used as a means of introducing new subjects in class or reconstructing a whole development. Toeplitz suggested that by doing so students could be guided to participate in the intriguing process of discovery that once captivated the heroes of the past. On the other hand he explicitly dissociated himself from the idea of doing history in class and said:

> I want to pick from history only the basics of those things that have stood the test of time and make use of them. Nothing is further from my thoughts than giving a lecture on history

> … as a student I have run away from a similar lecture. It is not the history of problems, theorems and proofs that is important to me, it is their genesis. [Toeplitz 1927, 95]

Thus, according to this view, history of mathematics is a tool in order to get the "real" mathematics, the mathematics of today, across to the students, and that in a genetic way.
Of course, Toeplitz' idea has been challenged by noted researchers. Walther Lietzmann for example objected that "if you wish to develop young people's mathematical skills you have to focus on the young people and not on the history of mathematics." [Lietzmann 1919, 135]. Wolfgang Klafki put it very trenchantly and said: "If you want to educate for the present, the present must be your starting point." [Klafki 1963, 127]. Serious objections against Toeplitz' way of using history can further be raised from an epistemological point of view. Toeplitz' approach in which he picks from history everything that is suitable in order to present, what he calls, "a great ascending line" from elementary to sophisticated knowledge advocates a continustic misconception of the history of science and mathematics. As Herbert Mehrtens puts it "the mathematician of today tends to declare all history the prehistory of the mathematics he knows" [Mehrtens 1992, 25]. Yet, since Kuhn's "Structure of scientific revolutions" we know that this perception is disputable. Of course, we find analogies in ancient and modern mathematics but as Niels Jahnke quite rightly points out: the things that are analogous are found in very different realms of experience and of thinking. [Jahnke 1991, 8]. For example, the Greek geometric algebra – as found in Euclid II – is no simple translation of Babylonian traditions, just as our own algebra is no cumulative continuation of Greek mathematics. Niels Jahnke concludes that

> A history of algebra does not exist in a strict sense. Instead we find a variety of related scientific work that has developed historically without forming a consistent and continuous history. [Jahnke 1991, 11]

much less a "great ascending line" that Toeplitz wishes to expose. We have to bear this in mind when we try to teach according to the genetic approach. So if for principal reasons we cannot find a great ascending line in history, we should at least just try to identify important stages. With this modifications and limitations in mind, we are prepared to do the first step of a comparative study in which historico-genetic teaching with original sources were to be compared with conventional teaching.

2.2 The teaching topic

The first thing you have to do is, find an appropriate topic to teach. After some consideration I chose 'quadratic equations and the quadratic formula', because this is really a classic, both in teaching with history *and* in the conventional secondary syllabus. The conventional goal of a unit on this topic is to have students learn the properties of quadratic equations, let them master the methods to solve them and give them some real-life problems to apply what they have learnt. In the genetic unit the goal should be the same, of course. History is just a tool then to achieve it in a different way and hopefully with some extra profit.

While it is never a problem to conceive a modern teaching unit on quadratic equations, it is different when it comes to developing a historical teaching unit to present, if not a "great ascending line" then at least some important stages in the history.

The first thing you have to do is look through the history to identify them. Depending on your demands this might become quite an effort. What I can do here is giving you a quick overview in the trust that some keywords will be sufficient for this audience. This overview is essentially based on the classic literature by Smith, Cantor and Tropfke.

The first known solution of a quadratic equation is the one given in the Berlin papyrus from the Middle Kingdom in Egypt. We have to note, of course, that Egyptian mathematics did not know equations and numbers like we do nowadays; it was instead descriptive and rhetorical. The problem I refer to is very well known and in our terminology reduces to a system of equations: $x^2 + y^2 = 100$ and $x = (3/4)\,y$. The solution that is given in the papyrus is a case of false position method.

On clay tablets the ancient Babylonians left another evidence of problems that, in our terminology, give rise to quadratic equations. The solving method is essentially one of completing the square. We've had a presentation on that this week.

"The Greeks were able to solve quadratic equations by geometric methods." Now this is a statement by David Eugene Smith, which of course can lead to some misunderstanding or even dispute. Let us agree that the Greeks dealt with geometrical problems that usually amounted to finding a *length* in a figure, which in our notion can be regarded as the root of a quadratic equation.

Hindu mathematicians took the Babylonian methods further so that Brahmagupta gives an almost modern method which admits negative quantities. He also used abbreviations for the unknown. Usually the initial letter of a color was used.

The Arabs did not know about the advances of the Hindus or at least they chose to ignore them, so they had neither negative quantities nor abbreviations for their unknowns. However Al-Khwarizmi gave a well known classification of different types of quadratics (although only numerical examples of each). The different types arise since Al-Khwarizmi had no zero or negatives. In his famous book algebra he has six chapters, each devoted to a different type of equation.

Al-Khwarizmi gives the rule for solving each type of equation, which essentially amount to the familiar quadratic formula given for a numerical example in each case, and then a proof for each example which is a geometrical *completing the square.* This particular derivation of the quadratic formula later was brought to Europe, where Al-Khwarizmi was given the name "father of algebra".

During the Renaissance in Europe, several mathematicians compiled the works related to the quadratic equations – Cardano for example blended Al-Khwarizmi's solution with the Euclidean geometry. At the end of the 16th Century the mathematical notation and symbolism was invented by François Viète. Viète was among the first to replace geometric methods of solution with analytic ones, although he apparently did not grasp the idea of a general quadratic equation. In 1637, when René Descartes published *La Géométrie*, modern mathematics was born, and the quadratic formula has adopted the form we know today.

And there it is – not a great ascending line, not a consistent history but a collection of work from the past that is related to quadratic equations and highlights some important stages.

2.3 The teaching units

Certainly, you have to reduce the material a little before you can make it a teaching unit, because otherwise it would be too much for the students and for the teachers. So I deleted Euclid, the Hindu mathematicians and the European renaissance from the list. Then I took the rest of the material and composed a historical teaching unit from it. It comprised nine lessons. In six of them the students read the original sources, in the remaining three lessons they learnt about the modern solution method by way of building upon Al-Khwarizmi's methods. They proceeded then with conventional exercises, problems and applications with modern methods.

The genetic teaching unit	The conventional teaching unit
1. Egyptian papyrus (false position method)	1. Introductory problem
2. Babylonian tablets (completion method)	2. Completing the square
3. Arab mathematics (eq. type 1)	3. The quadratic formula
4. … continued (eq. type 2)	4. Easy exercises
5. … continued (eq. type 3)	5. Word problems
6. … continued (the author's preface)	6. … continued
7. The modern formula	7. Easy applications
8. Word problems and applications	8. More difficult applications
9. Modeling problem	9. Modeling problem

Table 1: The genetic and the conventional (non-historic) teaching unit (outlines).

The conventional unit also comprised nine lessons. Its material was composed from standard textbooks in such a way that the students – from a technical point of view – were doing the same problems, but without any historical reference. An important difference was: In the conventional unit the students got an introduction into quadratic equations, completing the square and the quadratic formula right at the beginning, of course, with definition, theorem, proof and everything. After that they did the usual exercises, word and modeling problems.

The study was carried out with 175 students in 6 classes with 6 teachers from 3 different schools. 3 experimental classes studied the historical material, while 3 control classes pursued the 'conventional' way. Both groups then took an identical achievement test and a second one six weeks later in order to find out if one group had learnt the topic significantly better than the other.

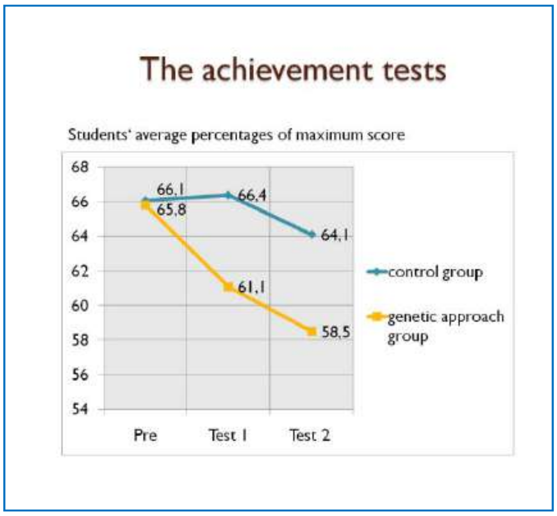
The participating teachers were offered special training. However, none of them made use of it, and evidently, I think, this is part of the reality in which historical teaching in schools would have to grow, at least in Germany. At any rate every teacher was supplied with a special accompanying booklet that explained the main ideas.

Additionally, the students were asked to fill in two questionnaires, one in advance of the experiment and the second one right after its conclusion. The questionnaires were to gather information about students' characteristics, their views on mathematics and, of course, on the teaching unit they had completed. By any means they were to account for the comparability of both groups in terms of similarity, which, in fact, they did.

All this amassed to huge amounts of data. Yet, the message was very clear: The historical teaching according to the genetic approach was a complete failure.

2.4 Some results of the teaching experiment

Let me begin with the achievement tests. They were identical in the experimental and in the control classes. Statistics show that the in-advance competence and performance levels were nearly the same in both groups. This was determined by an identical pre-test for all participants that was based on a standard competence test for German schools (figure 1; the numbers are percentages of the maximum score the students could achieve.)



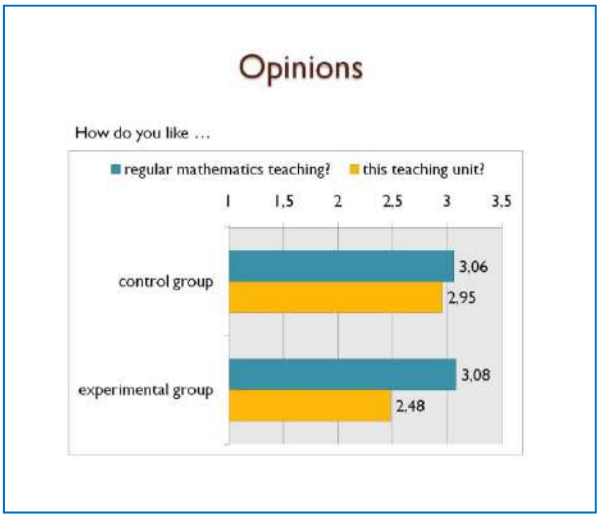


Figure 1: Results from the achievement tests   Figure 2: Opinions on the teaching units

Now look at the results from the first test right at the end of the teaching units. It consisted of problems of the type the students had worked with before. We observe nearly no change within the control group, but the students in the experimental group on average performed significantly worse. 66.4 versus 61.1; likewise in the second test, six weeks later; the performance was worse in both groups, but the difference has even increased and amounts to about half a grade. Thus we have to admit: the historical unit did not get the mathematics across in a satisfactory way.

When we have a look at the students' typical mistakes and problems, we find that they were particularly unobservant with plus- and minus-signs, that they often used a wrong formula which seemed to be a mixture from old and modern, that they in general quite often confused historical with modern approaches, that they had particular problems with applications and word problems (as opposed to the control group, which might be an indication of too little practice) and that some of them simply were not fast enough to finish the test in time.

Now let's have a look on some opinions (figure 2). How did the students like the teaching units? In both groups, Mathematics generally is a very popular subject. On a liking scale from 1 to 4 – with 4 meaning "I like it very much" –, regular mathematics teaching reached 3.06 in the control group, 3.08 in the experimental group. While this value dropped only insignificantly in the control group it has fallen sharply in the experimental group. 2.48 is a very mediocre value. A closer look at the experimental group reveals that even students who are rather interested in "normal" maths – this would be a three or a four on the liking scale – average only 2.86 when asked about the historical teaching. The others, who are rather not interested in mathematics only reach 1.83. If at all, the genetic approach is better suited for students who like mathematics very much.

What could be the reasons for these results? Let us have a look at some of the statements, the students have made.

"Although it was interesting, I think it was quite confusing at the same time. You never knew the important from the unimportant, as you did not know where it all should lead to."

"It was too much about history and too little about the modern methods."
"Although it was impressive to have a look at the long and winding way, I think the matter is rather attractive for our maths experts."
"Too much material, I did not know what to focus on."
"Interesting in principle, but too long."

I think this tells us, that the students didn't get any solid ground underneath their feet, but – so to speak – rather were swept away by the river of time. Or to put it less metaphorically: I think, the historical methods they were studying – basically the well-known Al-Khwarizimi methods – did not find any anchor in their previous knowledge. It appears that they were all too new and not connected to anything, maybe not even to each other. We have to keep in mind, that those students had no knowledge of quadratic equations before. When reading the three chapters from the Al-Khwarizimi algebra they possibly did not fully understand that they had something to do with each other. They did not have this kind of modern view on it, like we do. And when the history part came to a close the students would not connect the old methods to the modern ones although the teachers did spend some time on this issue. I think the majority of the students needed their mental capacities *now* to understand *the modern methods* in the first place, before they could think about linking them to Al-Khawarizmi. Note for example, that the modern methods would produce negative solutions, whereas the old ones would not.

I could show you more charts and results and talk about them, but let it suffice to say – this teaching did not work. The students rather got confused and appeared very displeased in the end. And of course they achieved none of the aforementioned extra goals, as it was an all-in-all too frustrating experience.

3 Teaching with historical sources according to a hermeneutic approach

3.1 Theoretical considerations

As stated above, this was a threefold study and I haven't yet said anything about its third part. So far we have covered only the genetic approach. There is an alternative to it, that is called the historico-hermeneutic approach or just the hermeneutic approach. It was proposed by Niels Jahnke. What does this thorny word 'hermeneutics' mean? If you look it up in a dictionary you will find, it is "the art or principles of interpretation". Hermeneutics, in fact, is involved when it comes to the reading of original sources. Traditionally these sources were the Holy Scriptures, and later the law, but by now, since Heidegger and Gadamer, hermeneutics has broadened its range.

In our context the big difference to the genetic approach is, that the students are not expected to trace a history of thoughts that leads them from past roots to the standards of today. Essentially that is, because the students should be very familiar with a topic before they even touch a historical text that deals with it. In the hermeneutic approach, students are asked to examine a source in close detail and explore its various contexts of historical, religious, scientific etc. nature. In hermeneutics this is called: to move within hermeneutic circles – and these are circles of ever new understanding.

Of course, historically this is a rather local and perhaps even modest approach compared to what you have in mind when you imagine a historically guided reconstruction of mathematics or parts of it. The hermeneutic approach would not give you an overview. But in this case I think: less might be more. As a matter of fact, the hermeneutic approach allows for some kind of deep-drilling and aims at very fundamental preconditions of learning by addressing and stiring the prior beliefs or views in the learners' minds. According to Niels Jahnke the historical material – which is in fact the original source – should

> show something, that is accessible to the ordinary student and at the same time strange and different from what he has known hitherto. [Jahnke 1991, 11]

This is noteworthy. In teaching contexts strangeness very often is associated with obstacles to learning. We saw how strangeness in fact led to failure in the genetic teaching unit. In this approach, however, strangeness is nothing negative. Quite the opposite, it is a lever to learn and, in fact, to understand in a broader sense. So how can this be? Because the strange elements do have some anchor point. The student who deals with something that he already knows but that is presented in a radically different, unfamiliar way, in a strange context, in an unknown guise and so on, can make the connections to this anchor point. In hermeneutics you would say: His horizon merges with the horizon of the past. *Horizon merger* is a term that was coined by Hans Georg Gadamer. In the horizon merger the student may begin to wonder and to reflect upon what he possibly had never thought about before. In essence he begins to develop deeper awareness. This is in fact an instance of broadening one's horizon. The hermeneutic approach puts great emphasis on this possibility. And it does so by utilizing a strategy of dissonance. According to this strategy you let students have experiences of dissonance in a moderate amount, which – in turn – arouse the students' epistemic curiosity. This term that has been introduced by the British psychologist Daniel Berlyne. Epistemic curiosity is aroused when a person is confronted with information that is partially incompatible with his prior knowledge, his beliefs or his expectations. It is also confirmed that this kind of incompatible information ensures greater retention and ease of retrieval from memory. The hermeneutic approach makes use of these psychological findings. It puts emphasis on strangeness. But to do so, there must be a familiar reference frame. Unlike the historico-genetic approach it is therefore applied only to subject-matters that students are already familiar with. Usually the structure is as follows: First, the students have a quite conventional introduction to the topic. No history is involved until the second step, in which the students read a historical source. In this source, the same topic is covered, but in a way, that is historically distant, different in its representation, used in strange contexts and so forth. This is the step where the students' epistemic curiosity is – hopefully – aroused. In the third and final step students are required to explore the source in even greater detail, perform a horizon merger and reflect upon questions that occur to them.

3.2 The teaching unit

With regard to our hermeneutic teaching unit, this should mean that we be content with even less historical material. We merely select one instance in history and work with it in greater detail. In this case I chose Al-Khwarizmis algebra, which is a classic choice. Here's the outline of the unit:

After a plain conventional three-lessons-introduction to quadratic equations that is taken right from the conventional unit the students read the corresponding excerpts and the author's preface to his book. Of course, they discuss the material with each other, they identify its strangeness – for example the rhetorical form, the absence of abbreviations, mathematical symbols and of negative numbers, its graphical clearness – and compare Al-Khwarizmi's methods to our modern ones. They do some exercises, the same as in the genetic unit, by the way. But above all, – they get into the hermeneutic circle and merge the horizons by doing some individual research on Al-Khwarizmi and his background, his scientific, social, cultural, religious and historical background. Ten years ago this might have been impossible. But nowadays with the internet at hand students can learn quite a lot about Arab mathematics, the Golden Age of the Islam, the House of Wisdom and so on.

You might ask: why should students put so much an effort into the context instead of better doing more exercises or applications? The answer is: because it helps them to make broader sense of what they have read and learnt. This is the core of hermeneutics – to achieve a balance between text and context. The context helps to clarify the reasons for a particular piece of work, it sheds light on the author's motivations and intentions. And we know, that is what people care about. To see other people as acting, thinking and suffering. Why is this? Because, as the philosopher and theologian Emilio Betti puts it,

> "There is nothing that man is so concerned about, as to understand his fellow man; nothing that appeals as temptingly to his mind as the lost trace of human being, that shines and speaks to him." [Betti 1962, 7]

The hermeneutic approach therefore puts an emphasis on understanding the trace of human beings. But – when you are with school students you have to provide a setting that helps them. So, in this teaching unit the students did not just do some internet reading on Al-Khwarizmi. They also were given the assignment to make use of their findings in a creative and productive way

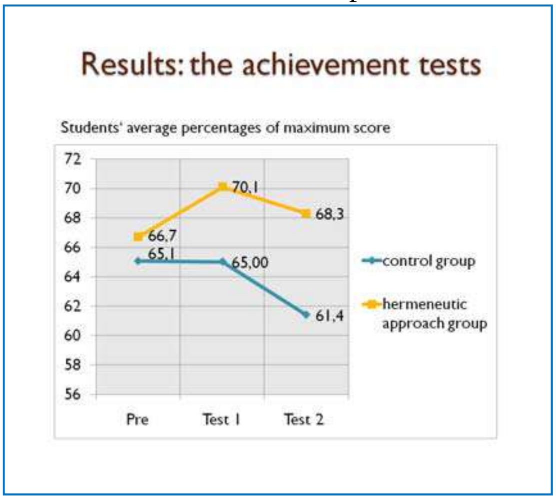
- by writing a fictitious interview with Al-Khwarizmi in which they let him give answers to who he was, why he had written the algebra, where his knowledge came from etc.;
- by writing a book review for Al-Khwarizmi's contemporaries, in which they appreciate his achievements, their reasons and their use for the society and for the Islam;
- by thinking up and performing a conversation between Al-Khwarizmi and the caliph who had asked him to write the algebra;
- even by writing some kind of science-fiction story in which Al-Khwarizmi meets a time traveler from today who tells him about his influence as "father of algebra" and who tries to persuade him to combine his three types of equations to get a unified approach.

The students worked in teams and could choose one of those assignments or suggest another. When they worked on it, they had to review again and again what they had read and learnt from the original source. The assignment, therefore, was also a means of knowledge consolidation. But it wasn't just consolidation, it was expansion as well. In hermeneutics you would say, they performed successive turns in the hermeneutic circle, and with each turn they were broadening their view and deepening their insight. – Insight into a world and a way of thinking that was very different from their own. And this is a crucial point. Hans Georg Gadamer, – I've mentioned him a couple of times by now, – puts it

> Every encounter with tradition that takes place within historical consciousness involves the experience of a tension between the text and the present. The hermeneutic task consists in not covering up this tension by attempting a naïve assimilation of the two but in consciously bringing it out. [Gadamer 1990, 311]

3.3 Some results of the teaching experiment

Where does this kind of hermeneutic movement in circles lead to? To answer this, I have done the second part of my comparative study, this time with 250 students, 10 classes in a 7-historical/3-conventional arrangement, 6 teachers and 6 schools. Please note, that the same achievement tests and pre-tests were used as before. Here are the results.



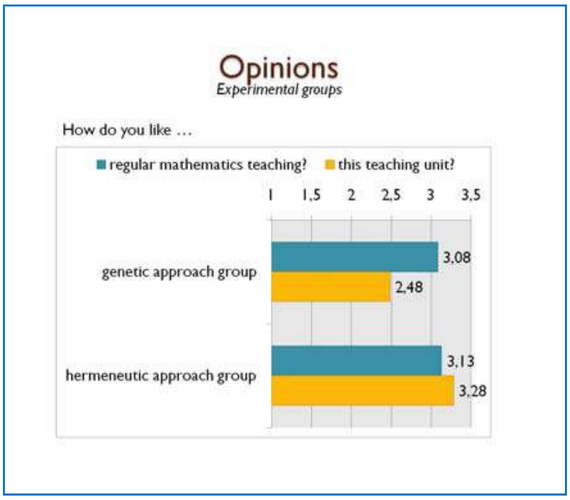


Figure 3: Results from the achievement tests. **Figure 4:** Opinions on the teaching units.

In the pretest the control group averaged on 65.1, in test 1 on 65.0 which is practically no change, and in test 2 on 61,4. The experimental group performed significantly better than the control group, especially with regards to the middle-term memory. I believe this has to do with the memory effects described by Daniel Berlyne in his strangeness-research.

Let us also have a look on the opinions. We compare the genetic and the hermeneutic approach groups right away. How did they like the teaching? Referring to the regular teaching, the genetic group reached 3.08 on the liking-scale, the hermeneutic group 3.13. This is no significant difference. But when we look at the values for the historical teaching, the genetic group only reached 2.48 whereas the hermeneutic group reaches 3.28.

On the one hand this difference can be explained by the students' impression of "too much and too confusing" in the genetic approach. On the other hand the hermeneutic approach led to a somewhat higher level of teaching. Students were given some freedom and responsibility when they were asked to do some research on Al-Khwarizmi and to make use of it in a creative way. I am sure that this fact played an important role because many students explicitly mentioned it. Also, it is my experience that the appreciation for the teaching unit drops, when a teacher does not make any use of these creative opportunities. Another observation is important. As we saw before, the genetic approach is better suited for students who like mathematics very much. With the hermeneutic approach this is not the case. The study shows that nearly half the students who like regular teaching very much did not like the hermeneutical teaching just the same. Rather their liking values dropped from 4 to 3 or even to 2. Nonetheless the overall average increased from 3.13 to 3.28. This is mainly due to the changes in the rest of the population. Quite a lot of students liked the hermeneutical teaching unit more than the regular teaching or, at any rate, they did not like it less. Thus we can say: the hermeneutic approach is suited for the great majority of students.

What about our intention to let the students reflect upon what they encounter? Did they seize the opportunity? Did they achieve some more of the extra goals of teaching with history? We can best answer this by looking at some of the statements they have made.

> *"I did not know that the Arabs have done so much for mathematics and science. Today, you would think that they were a little behind. Therefore I find it important to learn, what the people there have achieved so early."*
>
> *"Now I think in a different way about the Arabian countries and their culture, against which I have been prejudiced before."*
>
> *"Until now I thought that science and religion contradict each other and that scientists were not religious. But it may be different in their culture."*
>
> *"From visits to Spain with my family I knew that the Arabs had been superior in the Middle Ages. What I did not know was that this was true for mathematics as well."*
>
> *"I ask myself: Why have they lost their lead? Why did the Americans land on the moon and not the Arabs?"*

Statements like these are impressive, they are even touching. They show students on the trace of human beings instead of being on the trace of mere mathematical development.

4 Final remark

Let me finish this lecture with another personal remark. In the beginning, I told you about my first encounter with Kepler's *Mysterium Cosmographicum* and the impact it had on me. When I look at the results of this study, I begin to understand what it was that did captivate me so much then. Actually, it wasn't the model itself – of course, even then I knew it was wrong. Rather, it was the man's perplexingly original and unheard-of way of thinking, that was very strange and at the same time very fascinating to me. Betti calls it the lost trace of human being that shines and speaks to us. Elsewhere he writes that it should therefore be our finest duty to educate our youth in a way that – above all – they ask and learn about other people's thinking [Betti 1967, 203]. The students in our hermeneutic approach group eventually did learn at least as much about solving quadratic equations as they learned about a hitherto unknown, strange world, its people, their culture and their different ways of thinking and acting. Many of them even developed new appreciation for a culture they unconsciously had underestimated so far. Given this perspective, the hermeneutic approach in our teaching of mathematics could make a notable contribution to humanist, democratic and peace-promoting traditions in education.

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ICT AND HISTORY OF MATHEMATICS:
the case of the pedal curves from 17th-century to 19th-century

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ABSTRACT

Dynamic geometry softwares renew the teaching of geometry: geometrical construction becomes dynamic and it is possible to "visualize" the generation of curves. Historically this aspect of the movement (continuous or not) is natural and was well known to 17th-century mathematicians. Thus, during the 17th-century, the mechanical or organic description of curves was re-evaluated by scholars like Descartes or Newton.

In this article, we want to focus on a special class of curves: pedal curves. The definition of this kind of curves given, we will then briefly retrace their history between the 17th and the end of 19th-century. And finally, we present some activities which can be produced by a dynamic geometry software like geogebra.

1 Introduction

Before providing a short summary of the history of this kind of curve from the 17th to the 19th century, let's recall the standard definition of the pedal curve:

Definition 1. *(C) is a plane curve and O a point on the plane. We consider the foot P of the orthogonal straight line to the tangent from any point M on the curve. The pedal curve of (C) is the locus of P when M describes the curve.*

For instance, if we consider a circle (C), and O one point of the plane, then its pedal curve of centre O is the figure 1:

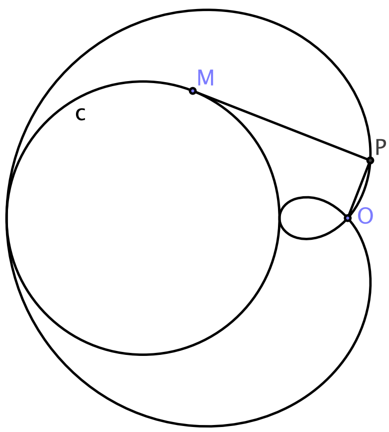


Figure 1: Circle pedal

2 The 17^{th}-century

2.1 Roberval

In fact, this curve (Figure 1) is the first pedal curve found in the 17th-century by Gilles Personne de Roberval (1602–1675). He spent his time as mathematics teacher in Paris. He was an active mathematician in Mersenne's circle. He was in relation with Blaise Pascal, Pierre de Fermat and Carcavi, but he was in conflict with René Descartes.

We find this notion in a text entilted *Observations sur la composition des mouvemens, et sur le moyen de trouver les touchantes des lignes courbes* included in a posthumous edition of *Divers ouvrages de Monsieur de Roberval* in 1693. It was reprinted in the journal *Mémoires de l'Académie Royale des Sciences de Paris* in 1730. In this text, one can find the parallelogrammical construction for composition of motion and several methods to construct tangents of some curves like conic sections and spirals.

In the part of the construction and description of the well-known Pascal spiral, he shows how to find this kind of curve with another method (Figure 2):

> Mais voici une des belles spéculations qui se puisse sur la description de cette ligne [the cochlea], et par le moïen de laquelle elle a été trouvée par le sieur de Roberval. Soit supposé le cercle CEB, & l'intervalle CD comme aux figures précédentes : du point C & de l'intervalle CD soit décrit le cercle DG* ; (...) aïant tiré des touchantes GF à ce cercle, & du point B tiré des lignes BF perpendiculaires à ces touchantes, que chacun des points F sera dans notre limaçon.

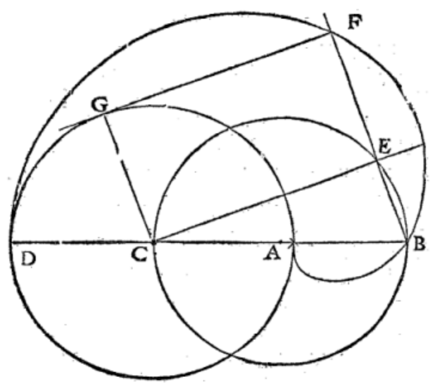


Figure 2: Roberval's construction

2.2 Newton

In the secondary literature, one finds that Newton knew pedal curves. According to B. Pourciau, in the 2nd edition of the *Principia Mathematica Philosophica Naturalis*, Newton uses pedal coordinates to show the relationship between the $\frac{1}{r^2}$ law and a trajectory. But, Newton is not explicit in this edition and Pourciau made the supposition that Newton has effectively considered this kind of coordinates. And, in the following editions this is the same thing. Moreover, in Newton's *Mathematical Papers* edited by Whiteside, there is no mention of this kind of pedal construction. But, at the end of nineteenth-century, it is usual to demonstrate this relationship with pedal coordinates.

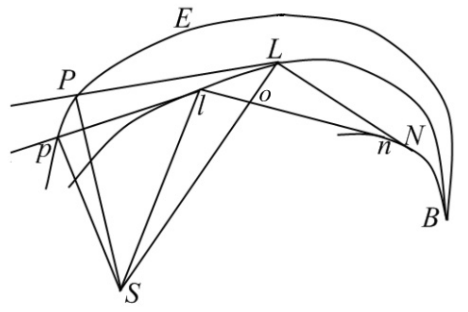


Figure 3: Maclaurin's construction

2.3 Which status for pedal curves?

One can say that there is no status for pedal curves in Roberval's works because he considers these *by the way*. They are not in a central place but are linked with his work on tangents, and he does not make any theory about it. The first major step about this curve is made by a Scottish mathematician, Colin Maclaurin.

3 The 18^{th}-century: the case of Maclaurin

Colin Maclaurin (1698–1746) become professor of mathematics in Marishall College (Aberdeen, Scotland) at the age of nineteen (in 1717). His two first papers sent at the Royal Society of London in 1719 are the first step of the recognition of his mathematical skills. One of them, entitled *Tractatus de curvarum constructione et mensura ; ubi plurimae series curvarum infinitae vel rectis mensurantur vel ad simpliciores curvas reducuntur*, is the first exposition of pedal curves. These two papers are the bases of his first main work, the *Geometria Organica: sive Descriptio Linearum Curvarum Universalis* (1720). All that follows is extracted from this one.

His definition is the same that we have seen before (Figure 3). But, in the *Geometria Organica*, he gives some properties of this kind of curve and proposes to classify the pedal curves for a family of special curves.

To do so, he introduces an orthogonal frame with the pole S as center. The coordinates of L are x and y. For any x and y, the quotient $\frac{\dot{x}}{\dot{y}} = \frac{m}{n}$ exists with m and n as finite quantities and $\dot{x}$ is the fluxion (*i.e.* the derivative) of x. So he can give two quantities, SP and SL as : $SP = \frac{my-nx}{\sqrt{m^2+n^2}}$ and $SL = \sqrt{x^2+y^2}$.

His main tool is what we call now the pedal equation, the *Æquatione radiali*, it is the quotient $\frac{SP}{SL}$.

Since a pedal curve of a given curve is also a curve, it is possible to draw the pedal curve of the pedal curve. So, he gives a construction of successive pedal curves, it is the sequence of positive pedal curves. But the main difficulty is to find the inductional relationship that enables us to define some pedal curves.

Nevertheless, from the radial equation of the first pedal curve, it is easy to deduce the radial equation to the second one by the same kind of quotient.

He introduces the construction of an antipedal as follows. The antipedal of a curve is a curve for which its pedal curve is the initial curve. He gives as we see below a relationship between a curve, its pedal and antipedal curves.

Let's give Maclaurin's property:

Proposition 3.1. *Given C' a pedal curve of centre S of the curve C. A simple geometrical construction produces the pedal curve of C of centre F from the pedal curve C'. To do so, it is sufficient from any point P of C' to have the perpendicular of PS and to F the perpendicular to PN. So the intersection point of these two perpendiculars, N, describes the pedal of F.*

After that, he proposes some examples. The first one is the pedal curve of a circle, he finds Pascal's spiral and announces that it is an epicycloid (found by Nicole in 1707) and it is a conchoid (circular base) too (by De la Hire in 1708). According to Loria, Cramer was the first one to link these three kind of curves, but the real one is Maclaurin. In fact, Cramer perused Maclaurin's book. The second examples are the pedal curve of conic sections (see below).

Maclaurin's main interest is curves whose radial equation is of the form $\frac{p}{r} = \frac{r^n}{a^n}$ where $p = SP$ and $r = SL$.
He proves the proposition:

Proposition 3.2. *The radial equation of its pedal (with same centre) is $\frac{p}{r} = \frac{r^{n/(n+1)}}{a^{n/(n+1)}}$, one of the m^{th} pedals is $\frac{p}{r} = \frac{r^{n/(mn+1)}}{a^{n/(mn+1)}}$ and its m^{th} negative pedal is $\frac{p}{r} = \frac{r^{n/(-mn+1)}}{a^{n/(-mn+1)}}$*

So, its possible to classify curves according to the radial equations:

Circle of radius a and the centre in the circonference	$p/r = r/2a$	$n = 1$
Right line with a distance a of the centre	$p/r = a/r$	$n = -1$
Parabola (the 1st antipedal of circle)	$p/r = (r/a)^{-1/2}$	$n = -1/2$
Equilateral Hyperbola (parameter a, centre is the origin)	$p/r = (a/r)^2$	$n = -2$
Cardioid (first pedal of circle)	$p/r = (r/2a)^{1/2}$	
Lemniscate (The first pedal of hyperbola)	$p/r = (r/a)^2$	

Two results for curves which are their radial equation as $p/r = (r/a)^n$:

Theorem 3.1. (The rectification)
If L describes the curve and B a (start) point of this curve, and given P and N respectively points corresponding in the pedal and antipedal curves. Then $\mathrm{Arc(BP)} = (\mathrm{n}+1)(\mathrm{Arc(BN)} + \mathrm{LN})$

And

Theorem 3.2. (Curvature radius)
The curvature is $\frac{a^n}{n+1} \times \frac{1}{r^{n-1}}$.

3.1 Which status for pedal curves?

These curves are central to his project. He proposes a systematic study and some properties : rectification, curvature,..., he tries to classify curves and applies them to a mechanical problem : Pourciau's indication is more for Maclaurin than Newton.

4 The 19^{th}-century

With the development of new geometries and several approaches for mathematical artefacts description, the pedal curves are used in many ways. And a lot of papers are entirely or partly devoted to pedal curves. During the first half of this century, this kind of curve is re-discovered by some German mathematicians, especially Jakob Steiner (1796–1863) who gave the German name : *Fusspuncktecurve*. Mostly during the second half of century, more than 400 articles concern pedal curves in different branches of mathematics : synthetic, differential geometry. And these curves are used as problems in the classroom and in entrance examinations for prestigious schools like the *École Polytechnique*.

4.1 In synthetic geometry

At the beginning of the nineteenth-century, following Poncelet's and Chasles' works, some mathematicians reconsidered the nature of the pedal curve. According to them, the pedal curve is the result of the composition of two transformations : After Poncelet's works, some mathematicians have found that the construction of pedal curve is equivalent to the composition of an inversion and a polar transformation.

During the first half of the nineteenth Century, some mathematicians like Jakob Steiner with his *Theorie der Kegelschnitte* and his two articles in Crelle's journal or Quételet in relationship with the caustics problem, work on pedal curves as a composition of transformations. But, the last one does not see it as a specific curve and does not extend it theoretically.

4.2 In differential geometry

Mostly during the second half of the nineteenth-Century, one can find some papers devoted to pedal curves. Some of them make the link with Maclaurin's construction like Haton de la Goupillière when he writes on curves with polar equation $\rho^n = A \sin n\omega$ (Haton, 1876). There are a lot of articles from Germans like R. Sturm with his paper, *Über Fusspunkts-Curven und -Flächen, Normalen und Normalebenen*, published in *Mathematische Annalen*.

At the end of nineteenth century, some mathematicians view pedal curves not only as curves but mostly as transformations. For instance, Sophus Lie considers pedal transformations as a part of his *Geometrie der Berührungstransformationen* (1896). Some years after, Gino Loria (1907) writes about the pedal transformation.

4.3 In teaching

Articles about this kind of curves are linked with teaching, and they are taught in France, Germany, Italy and Great-Britain as exercises. In the *Nouvelles Annales de Mathématiques* or *Educational Times*, one can see some exercises and problems focused specifically on pedal curves. For instance, the seventh problem of the 1847 entrance exam of École Polytechnique is "trouver le lieu des projections d'un sommet d'une section conique sur ses tangentes". It is in the commentary of the resolution of this problem given in the *Nouvelles annales de Mathématiques* that Terquem proposes to name pedal curve into french "podaire".

In textbooks devoted on geometry, one can find some problems in which pedal curves are important. For instance, at the end of the nineteenth-cetury, one can cite the fourth problem "Le sommet d'un angle constant c se meut sur une courbe directrice s pendant qu'un de ses côtés enveloppe une courbe donnée σ_1; trouver la courbe σ_2 enveloppe de l'autre côté." (Aoust, 1873, p. 171.). Aoust gives a relationship between caustic and pedal curves.

To conclude this part, one can say that pedal curves are present in all type of geometries without any central role, much like other tranformations. To summarize, pedal curves are significant enough to merit a specific field in the *Répertoire bibliographique des sciences mathématiques* (1894-1912): 02qα Podaires et podaires négatives.

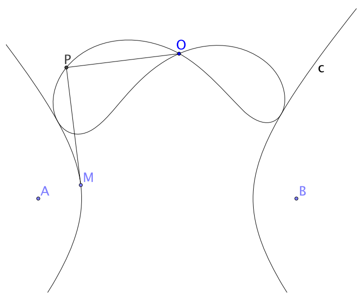
5 Pedal curves and Geogebra

We propose two short examples to show how to concile history of mathematics, old sources and dynamical geometrical software. The main interest of this kind of software is to integrate dynamic views with old problems and to open new interpretations in the creation process from mathematicians like Maclaurin or Newton.

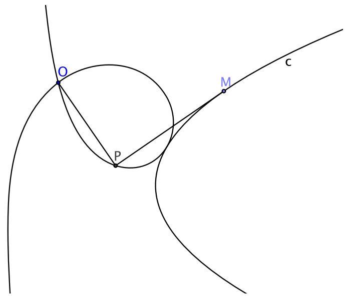
The first is to exploit an extract of Maclaurin's *Geometria Organica* in which he gives different cases of the pedal curve of conic sections. And the second one is from the *Nouvelles Annales* in which a method is given to use pedal curves to trisect angles.

5.1 Pedal curve of conic section

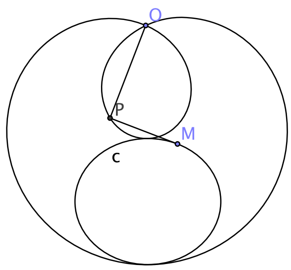
Let us follow Maclaurin in his examination of pedal curves of conic sections:



Pedal curve of hyperbola



Pedal curve of parabola



Pedal curve of ellipse

The three examples come from *Geometria Organica*, which Maclaurin split into three cases. In each case, he determines the situation with respect to the pole. With Geogebra, it is extremely easy to rediscover each case with a simple movement. For instance, for the pedal curve of the parabola, Maclaurin makes a relationship with the Newtonian description of some third order curves.

5.2 Angle trisection

The trisection of angles is an old problem: is it possible to trisect any angle only with compass and straightedge? From ancient times, mathematicians tried to resolve this one, but with only compass and straightedge, it is an impossible problem. This was demonstrated in the 19th-century with the works of Abel or Galois.

Here is an interesting use of pedal curve :

> M. le docteur Toscani, professeur de physique au lycée de Sienne (Toscane), fonde la trisection sur le lieu géométrique d'une *podaire* du cercle. Soit C le centre et V un point fixe, extrémité de l'arc à trisecter. Prolongeons le rayon CV d'une longueur $VP = CV$; projetons le point P sur toutes les tangentes au cercle; T étant le point de contact et P' la projection correspondante de P, lorsqu'on aura $VP' = VT$, alors $\widehat{VP'P} = \frac{1}{3}\widehat{CVP'}$. (Poudra, 1856)

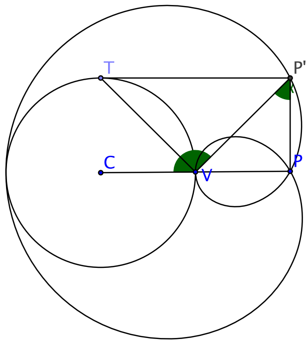


Figure 4: Trisection of angle

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JOOST BÜRGI'S CONCEPTION OF THE LOGARITHM (1620)

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ABSTRACT

In this workshop, participants were introduced to the text, *Arithmetische und Geometrische Progress Tabulen/sambt gründlichem unterricht/wie solche nützlich in allerley Rechnungen zu gebrauchen/und verstanden werden sol* (Bürgi, 1620), as well as its author, Joost Bürgi.

Early in the 17th century, John Napier and Joost Bürgi were responsible for articulating independent conceptions of the logarithm. Even parallel insights that occur simultaneously will produce proposals that are quite distinct. This was the case with the proposals of Napier and Bürgi. Both proposals for the logarithm were quite different and this is in part responsible for the different receptions each had in the mathematical community. The workshop began with a brief look into Bürgi's life and a description of his role in the scientific community of early-17th century Prague.

The remainder of the workshop was spent on reviewing excerpts from the *Arithmetische und Geometrische Progress Tabulen* (and working English translations), including Bürgi's stated reasons for developing his general tables for complex numerical calculations and several of the examples he presented in the "*Kurzer Bericht*" (or, short report) that accompanied his tables. Lastly, participants were engaged in a discussion about the diffierences between Napier's and Bürgi's conceptions of the logarithm and the influence of Bürgi's *Arithmetische und Geometrische Progress Tabulen*.

DETAILED REFERENCE

– Bürgi, Joost, 1620, *Arithmetische und Geometrische Progress Tabulen/sambt gründlichem unterricht/wie solche nützlich in allerley Rechnungen zu gebrauchen/und verstanden werden sol*, Paul Sessen, Prague.

TEACHING METHODS FOR THE USE OF ORIGINAL SOURCES IN THE CLASSROOM

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ABSTRACT

Any teacher who wishes to study original sources in the classroom is in need not only of adequate historical material, but also of some methodological preparation. While the former is rather easily accessible in pertinent books and publications, information and guidance on the latter is harder to find.

Yet, the use of original sources entails unfamiliar context elaborations and therefore requires non-standard studying methods that put more emphasis on hermeneutical, communication-accented and collaborative work, or otherwise its potentials are all too easily wasted. After a short introduction into profitable classroom strategies for working with original sources the focus is on practical method training. The provided material is suitable for the secondary level (age 14-18).

USING HISTORICAL INSTRUMENTS AND INTERACTIVE E-TEXTBOOK FOR EXPERIENCING THE INTERPRETATION OF HISTORICAL TEXTBOOKS

What prospective teachers learned through the lesson study

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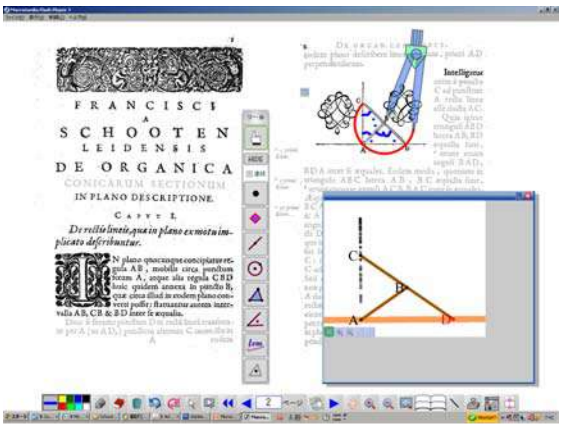
ABSTRACT

The history of mathematics for all project developed seventy seven materials for secondary school classroom (see: http://math-info.criced.tsukuba.ac.jp/Forall/project/history/). Each material provides three hours classes and includes the textbook as for the interpretation of original sources. Each material aimed secondary school students to interpret the textbook and learn mathematics as a human enterprise from the historical view of mathematics through their 'Aha' experience. For enabling 'Aha' experience, mathematical instruments and the files of dynamic software are provided. Each material was developed through Japanese lesson study methodology (Isoda. 2007).

During the first part of the workshop, participants experienced the exemplar of class using historical instruments and interactive e-textbook for getting acquainted with the aim of the material. At the same time, participants were able to experience the freeware 'dbook' developed by Masami Isoda, Univ. of Tsukuba. The freeware enables anybody to transform any of his/her printed textbooks such as historical textbook into digital-interactive materials. 'dbook' enables anybody to use his/her printed textbook as interactive e-textbook just like traditional instruments in his/her classroom. In this workshop, the historical textbook by Schooten (1646) was provided as an example.

All materials had been developed by the master program students (prospective teachers), at the University of Tsukuba, as part of the program. The project itself was done for a part of teacher education and math-educator education program. The second part of the workshop, what the master program students, prospective teachers, learned through a year was illustrated by the affection diagram.



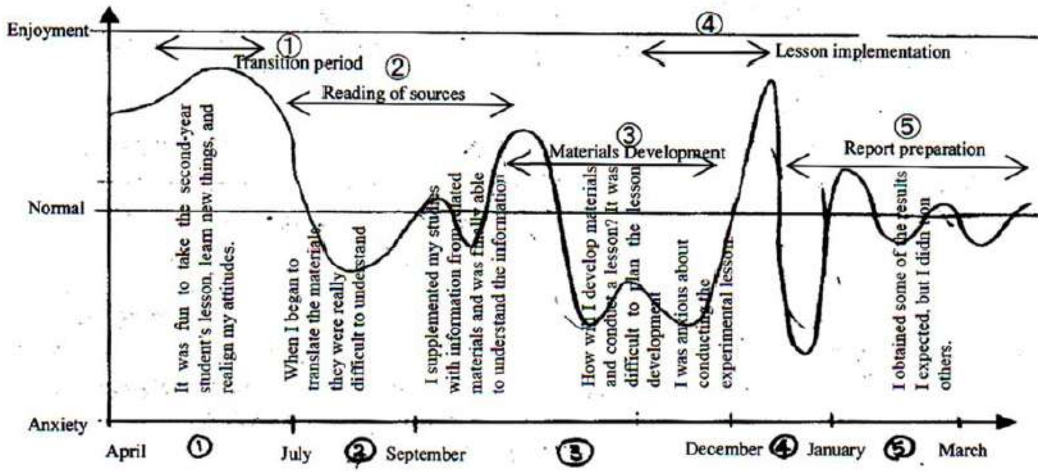


Figure 1. A Case of one prospective teacher's experiences in the project

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DIGITISING THE PAST MATHEMATICS BY THE FUTURE MATHEMATICIANS

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ABSTRACT

During the academic 2008/9 I started a project on digitising old mathematical textbooks with secondary school children. The idea was to engage students in analysing and comparing texts from different areas in order to bring about an understanding of the issues relating to motivation and engagement. The project is now running between Bath Spa University in England and a number of local partnership schools. Four types of participants are currently engaged on the project:

- University who provides support for trainee and established teachers
- Established teachers who are undertaking master modules based on action research projects related to digitisation and work with the original texts
- Trainee teachers who are investigating the role of the history of mathematics in mathematics classroom
- Pupils in secondary schools of varying ability levels.

The workshop presented the findings of the project in terms of quantitative and qualitative data obtained during the project, and attempted to engage the audience in working on analysis of a textbook from the 18th century with that of the 21st century. During the workshop we worked in identifying the potential source material for the development of 'rich' tasks and discussion start-points within a secondary classroom, based on the issues of:

- Language employed in setting a mathematical problem
- Method of proof, justification, and statement of mathematical facts
- Purposes and values as projected through the textbooks which represent a programme of mathematical study at secondary level
- Acquisition of meaning through dealing with different, or in some cases, original problem to which a mathematical solution is sought.

The workshop also showcased the trainee teachers' work on building maps of future continuing professional development landscapes as statements of intent through which they are able to identify their needs and desires for subject knowledge improvement. A number of initiatives to improve subject knowledge through engaging with the historical context in mathematics were discussed, with a view of gaining consensus on the major elements of such course for both future and existing mathematics teachers.

GOOD OLD ARITHMETIC

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ABSTRACT

Rediscovering what natural operations are is a weird experience for students, because they do not always realise that their modern way of counting is quite new (only several centuries).
What about multiplication by the checker? Or by the renaissance methods, apparently still ours but that the students cannot understand, as if the figures were written the opposite way? That because they are!
A variety of problems and arithmetical methods can be found in 16th century texts, and a majority of them avoid using algebra. Giving them to students allow to create a distance between their "mechanical" knowledge and what they really know.

The activities were used in the classroom with pupils aged 15 to students aged 20.
The texts include:

Juan de Ortega's *Arithmetic* (in Spanish, 1512 and in French, 1515)
Robert Record's *The Ground of Artes* (London, 1543 and further editions)
Simon Stevin's *Tenths* (1585)

and

Edouard Leon Mellema's *Arithmetic* (Anvers, 1582), in which we can find a method for solving equations without the Coss.

PEDAGOGICAL AND MATHEMATICAL GAMES THROUGHOUT THE TIMES

From Rithmomachia to Hex

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ABSTRACT

Rithmomachia, from the 11th century, is the oldest pedagogical game we know. It was created to teach the Pythagorean mathematics used in the Quadrivium. We gave a description of this game that, as a helper to teach arithmetic in the tradition of Boethius, was very famous for near five hundred years.
The relation between mathematics and board games runs deeper than the pedagogical games can tell.
Throughout the ages some board games were practiced which were, in a certain way, mathematical.
We focused our attention on how this happens, which varies from game to game and can show surprisingly deep relations between games and higher mathematics.
We saw some of the most important examples, ending with Hex, the connection game invented by Pit Hein and John Nash (independently) in the 20th century.

INQUIRY-BASED MATHEMATICS TEACHING, HISTORY OF MATHEMATICS AND NEW COMMUNICATION TOOLS : AN EXCITING CHALLENGE !

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ABSTRACT

Following a European project on the relationships between inquiry based science teaching (IBST), internet and communication tools (ICT) and a third topic, history of science, this article deals with some of the new questions raised by the recent introduction of epistemology, history of science and technology (EHST) in the school curricula. The definition of the inquiry and the role that can be played by epistemological thinking in science education are especially analysed in this text. Based on the analysis of historical examples and many on-line resources, as a first result, a new framework for the elaboration of new resources in IBST, ICT and EHST is given and some opening questions have been pointed out.

1 Context : IBST in Europe

All over Europe, the lack of student interest in science or in scientific careers has been noticed for years. This situation has led the European Community to launch a call for research projects in science education (the FP7 Science in Society program) and the publication of the Rocard Report about Science Education[1]. One of the main recommendations was to promote the advancing of teaching methods toward Inquiry Based Science Teaching (IBST) and the request for international comparisons. Engaged in 2008[2], the European research project Mind the Gap was one of the answers. Mind the Gap was a large didactic program in which about 15 European universities worked on inquiry based science teaching in order to develop useful scientific tools within this topic. Many historian of science wished to join in this program. In a first meaning, IBST consists in learnings based on an open problem in which the student has to propose experiment or use instruments in order to find a solution. Such inquiry based teaching can be divided into different aims : an understanding of the articulation between empirical evidences and concepts (e. g. testing hypothesis, modelling, results

[1] http://ec.europa.eu/research/science-society/index.cfm?fuseaction=public.topic&id=1100

[2] The program ended in 2010, website : http://www.uv.uio.no/english/research/projects/mindingthegap/index.html

evaluation), the practice of hands-on activities comprising an informations quest or not (experimentations), the introduction to a specific scientific language (argumentation, debate), the enhancement of students' autonomy ... Mind the Gap was a place wherein to think about the interaction between IBST, history of science and technology (HST) and internet and communication tools (ICT). In this article, we would like to render an account of one of these possible interactions and to show thereafter how historical on-line resources can constitute well-adapted references of authentic problems.

2 IBST in Mathematics : a historical answer

According to Pr. Martin Andler (University of Versailles – France), a contemporary mathematical research activity comprises : 45% devoted to observation, 45% to experiment and only 10% to demonstration[3]. In the field of mathematics learning, the inquiry-based style often claims to have been inspired by the "scholar at work". In mathematics as in the other fields of science, inquiry takes up a large part of the research process. There is no doubt about the fact that the job has changed throughout the ages. However, by rendering an account of these changes, the epistemology and history of mathematics can help to explain what this inquiry could be.

2.1 First example

In ancient times in favouring results to reasoning mathematicians rarely expressed themselves on their relationships to the experiments. However when they did so, they gave us the opportunity to see the complexity of the links between theory and the use of technical instruments. The history of science assures us that : mathematical theories never emerge from nothingness. The scientist describes, builds and explores multiple examples before proposing an analysis or a system. On this subject, the work of the Arabic scholar al-Sijzī is a model of such a process. Ahmad ibn Muhammad ibn `Adb al-Jalīl al-Sijzī was born and lived in Iran. Son of a mathematician, he worked between 969 and 998 and he wrote exclusively books on geometry. In all, he has written approximately fifty treatises and lots of letters to his contemporaries. Al-Sijzī was working within a very specific scientific context. Since the ninth century, the development of algebra in the Arabic world has created new types of questions on the fundaments of this field. For instance, the point-by-point construction of conics has been well known since Antiquity (see the Apollonius' book entitled the *Conics*, for example), and that method is efficient enough for the analysis of the main properties of those curves. But during the ninth century, as the major part of algebraic equations studied during that period can be solved by intersecting conics curves (ellipsis, parabola, and hyperbola), the necessary taking into account of these intersections creates new difficulties. Indeed this possibility is based on the continuity of the different curves which is difficult to "prove". The solution that has therefore been chosen is to associate the curve with a tool that enables a real construction. As the ruler and the compass allow straight lines or circles to be drawn and so justify their continuity, a new tool had to be invented to draw all the conics. Not only interesting from a mathematical point of view, such a technical development is also useful in technological areas such as the construction of

[3] Colloque Mathématiques, Sciences exprimentales et d'observation l'école primaire, Table ronde La démarche d'investigation en mathématiques , Chairman : P. Léna, École Normale Supérieure de Paris, September 28th 2005 : http://www.diffusion.ens.fr/index.php?res=conf&idconf=882

astrolabes and sundials where conics are essential.

Following his predecessors (Banū Mūsā brothers, Ibrāhīm ibn Sinān...) from whom he quoted in a precedent book on the description of the conic sections, al-Sijzī engages himself too in a treatise specifically on the *Construction of the perfect compass which is the compass of the cone*[4]. In this book, he studies a new tool, made-up a while ago by al-Qūhī : the perfect compass. As he announces in the first pages of his treatise, al-Sijzī wants to "build a compass which enables him to obtain the three sections of the cone." He first notes that all the conics can be obtained from the right cone (depending on the position of the cutting plane), and afterwards he proposes three possible structures for the perfect compass. The beginning of the study is technical, "We must now show how to shape a compass by which we can trace these sections. Shaping a stalk or AB. We put on the top tube, or NA. We link another tube to the end of it. [...]", and these instructions should enable the reader to build such a compass. But for al-Sijzī, the aim of his work on the perfect compass is not only to draw conics. The end of the text shows that this compass is also a theoretical tool and a tool for the discovery of new concepts. Al-Sijzī explains that the link between the circle and the ellipsis is quite obvious. Indeed, the construction of the ellipsis by orthogonal affinity and the formula for the area are both well known. But what are the links between the circle and the parabola or the hyperbola ? Now oriented towards the exploration and the solving of new problems, the practical tool becomes an instrument of discovery and as stated by al-Sijzī himself : "I always thought that there was a relationship between these two figures and the circle and their similarities and tried to get it but the knowledge of this has only become possible to me once I had learned how to turn the perfect compass following the positions of the plans."

In this example, the comings and goings between theory and practice appear clearly. Confronted with the theoretical problem of the continuity of curves, the scientist suggests the use of a new instrument. The experimentation with this instrument creates new theoretical results that create new questions and so on and so forth. In this text, al-Sijzī clarifies the role of mathematical instruments. They are objects as much as models and this dual status facilitates the theory-experiment passage.

2.2 Another example

Pierre de Fermat (1601-1665), the French lawyer who works during his free time on mathematics, is well known and his name is associated with the famous theorem which was only demonstrated in the 1990's. Nowadays, we can find [5] some parts of his mathematical works on the theory of numbers on the Web. Even if he has not published any treatise on this subject, some elements can be found in quotations of his Diophantus' book and included in his correspondence especially with Marin Mersenne, Huygens or Carcavi.

In a letter Fermat sent around august 1659 to Pierre de Carcavi (1600-1684), another amateur French mathematician, he tried to demonstrate some properties with, according to him, a new kind of demonstration he called "la descente infinie ou indefinie". This kind of demonstration is also used in the margin of his Diophantus' volume in

[4] The texts are available in a French translation in *Œuvre mathématique d'al-Sijzi. Volume 1 : Géométrie des coniques et théorie des nombres au Xe siècle*, Trad. R.Rashed, Les Cahiers du MIDEO, 3, Peeters, 2004.

[5] http://www.archive.org/details/oeuvresdefermat942ferm

order to prove that the area of rectangular triangle with integer sides (i.e. ones which are measured by integers) cannot be a square. Up to 1659, Fermat only uses this method to prove some negative results. He proposes a positive result, for the first time, in his letter to Carcavi, : any prime number such as $4k+1$ is the sum of two squares, and this form is unique. For instance, 5^2+1, $13 = 3^2+2^2, \cdots$. The demonstration is based on a reduction *per absurdum*. For instance, to prove the first above proposition, he presupposes that this kind of triangle exists. Then he shows that if it is true, we can find another triangle with shorter sides which validates the assertion, and so on. Since the sides must be integers he arrives at a contradiction through the method of infinite descent. It is quite easy to prove negative assertions, but its harder to show positive assertions. In his letter, Fermat announces that he only demonstrated that any prime number which could be written as $4k+1$ could be decomposed as the sum of two squares of integers and that this decomposition was unique but, unfortunately he did not give any demonstration. The first proof of this assertion seems to appear in Leonard Euler's (1707-1783) works under the Latin title of *Demonstratio theorematis Fermantiani omnem numerum primum formae* $4n+1$ *esse summam duorum quadratorum* [Demonstration of the Fermat's theorem "All prime number like $4n+1$ is sum of two squares"] [6], in which he strictly follows the Fermat's method.

The method of "infinite descent" has profoundly renewed the theory of numbers. Even if, as usual, Fermat does not demonstrate what he announces, he suggests to the other mathematicians that they can easily find the demonstration of the properties he gives. For instance, in the above mentioned letter, he does not give any demonstration, however, he asks his contemporaries to do so : "je serai bien aise que les Pascal et les Roberval et tant d'autres savants la cherchent sur mon indication" (*op. cit.* p. 432). In this letter, he shows two important aspects of mathematics. First, some methods of proof are not able to solve a problem and newer ones must be invented and then reused to solve other mathematical enigmas. Creativity can go through a new method and a new approach. Secondly, this letter suggests to the other mathematicians to deal with those kinds of problems. This demonstrates how mathematical progress can be transmitted.

2.3 Many types of inquiry

The reading of the ancient texts is always interesting. Both examples presented above are only a small part of the historical resources dealing with inquiry, but they are still meaningful. What the historical approach shows is that *inquiry* cannot be reduced to a single aspect of the research process. Depending on the situation, each scientist engages himself in a different type of inquiry. A theoretical question does not require the same method as an experimental one, etc. IBST is clearly inspired by the work of the scientist as a professional. Nonetheless, the way this one has been understood is sometimes a bit caricatural. The history of science can prevent us from simplifying to such an extent and can restore the wealth of the research process.

3 History of mathematics and on-line resources

Nowadays, on the web, anyone can find many websites, pages or documents related to the epistemology, history of science and technology (EHST). Some sites like "Internet

[6] http://math.dartmouth.edu/~euler/docs/originals/E241.pdf

Resources for History of Science and Technology"[7] have even been created to help users to find the right resources. More and more primary sources are also available on "Google books" or on the French project, "Gallica"[8]. It is relatively easy to read and to study ancient texts on science and technology. But in order to avoid mistakes or misinterpretations, it has been demonstrated that the texts must be contextualized. This is one of the main results of the research on the history of science. Now, due to the development of the new technology and the easiness of the on-line publication, some websites are devoted to IBST but they do not include any historical aspects. Is it possible to conciliate the two aspects : IBST with a historical approach ? In the Mind the Gap research program, the study of the relationships between inquiry and the history of science has shown that science and investigation are very close in different ways. In every scientific field, the scholars have to elaborate a way of questioning their object. In each case, new instruments should be built, and new experiments should be elaborated in order to be able to ask the right questions and finally to answer them. This first part of scientific activity is very similar to a second one which is dealing with theories and models. When their pertinence has been proved, the new concepts are applied to other situations or fields and they become a part of common knowledge. These newly born theories have still to be discussed and they constitute a third part of scientific activity. Communication between scholars often needs to create a suitable language and is essential in the building process of knowledge. An on-line publication should take into account all this wealth that gives many opportunities to the enlightening of historical documents and in making it suitable for IBST in multiple ways.

4 Digital document for EHST

According to Michael Shepherd and Livia Polanyi[9] quoted by Ioannis Kanellos[10], the genre of digital documents can be characterized through three constitutive elements :

- The *contents* (information, ...), organized following a *material* structure (disposition, page setting,...) which is often enough for a first and quick reading, and a *logical* structure (title, author, date, abstract,...) which brings some information on the intellectual organisation of the document.
- The *container* (support, medium), which determines the manner in accessing the information.
- The *context of production*, which relates the publication design. This context plays an essential role in the reading process of the document and it can be found as much in the content than in the container.

Contents as well containers, as such, do not enable the easy expressing of the context of production, and therefore the genre (historical or not), of a document. Only published information in the frame of the digital document identifies and determines the context

[7] http://www2.lib.udel.edu/subj/hsci/ (last accession 16th November 2009)

[8] http://books.google.com/ and http://gallica.bnf.fr/

[9] Michael Shepherd, Livia Polanyi, Genre in digital documents , 33rd Hawaii International Conference on System Sciences, Volume 3, 2000, pp. 3010, http://doi.ieeecomputersociety.org/10.1109/HICSS.2000.926693.

[10] See Ioannis Kanellos, Thomas Le Bras, Frédéric Miras, Ioana Suciu, « Le concept de genre comme point de départ pour une modélisation sémantique du document électronique », *Actes du colloque International sur le Document Électronique (CIDE'05)*/, Beyrouth, Liban, avril 2005, pp.201-216

of production and the genre. Sometimes the context of production is difficult to define and so the best way to enlighten it is to refer to a community of practices[11].

With this definition of the digital document, the context of production reveals the genre and shows the quality of the documents dedicated to EHST. We will thus give some criteria :

- The first and major one is the availability of primary sources. If there is no primary source, it becomes difficult to ensure that the document is within the frame of the history of science. The sources have to be contextualised and explained.
- The second one is the use of secondary sources. These documents may help to contextualise and to understand the scientific problem.
- The kind of media : texts, pictures, audio, video, ...
- The possibility to make a simulation, to create experiments
- The opening to a new view point and to other historical facts or problems (with hypertexts, links,...).

Then, these criteria can be connected with the aims of EHST digital documents. Is the document about the nature of science ? Does it refer to the macro or the micro history ? Is it based on the history of a concept or does it link science and society ? Does it deal with scientists' biographies or controversies, and so on and so forth.

5 Digital documents in HST for the use in IBST

All these questions opens interesting research ways on the elaboration of EHST documents suitable to a use in IBST. Some works have already been engaged, especially at the European level, and reader will find more detailed results (guidelines for digital documents building, content and form analysis grid, examples, ...) in our publications on this specific topic[12].

6 Conclusion

Nowadays, on-line publication is quickly increasing quantitatively and it gives many opportunities to create innovative learning sessions. Nonetheless, quantity is not quality and the new technologies such as web 3.0 already point out the risk of losing oneself in an ocean of data. Facing to this situation, historians of science should not stand back. The examples above show that a little vigilance enables us to make a document suitable for IBST with all its historical wealth. The task is not so heavy (3 or 4 paragraphs are often enough) and the community of historians of science should be aware of these questions. IBST is one of the main active topics in didactic research and, in this article, we have tried to show that historians of science have many things to say on this subject.

[11] A community of practices is defined through three aspects : the borders of its application field, its social existence, its language and the documents used and shared by the members of this community. It is also a group with interactions and learning that develop a feeling of belonging and a mutual engagement. See, for instance, Etienne Wenger, *Communities of Practice : Learning, Meaning, and Identity*, Cambridge University Press, 1998 or his website : http://www.ewenger.com/theory/

[12] See for instance the European collective book *The Usage of ICT and IBST by the History of Science and Technology*, O.Bruneau (Ed.),T.de Vittori Thomas (Ed.), P.Grapi (Ed.), P.Heering (Ed.), S.Laubé (Ed.), M.Massa (Ed.), Frank & Timme, Berlin, to be published in 2011.

Some new perspectives of collaboration have to be opened, and finally, as the situation does not concern only one country, a European network has to be enhanced in order to share all the experiences.

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THE CONSTRUCTION, BY EUCLID, OF THE REGULAR PENTAGON

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ABSTRACT

We present a modern account of Ptolemy's construction of the regular pentagon, as found in a well-known book on the history of ancient mathematics (Aaboe [1]), and discuss how anachronistic it is from a historical point of view. We then carefully present Euclid's original construction of the regular pentagon, which shows the power of the method of equivalence of areas. We also propose how to use the ideas of this paper in several contexts.

Key-words: Regular pentagon, regular constructible polygons, history of Greek mathematics, equivalence of areas in Greek mathematics.

1 Introduction

This paper presents Euclid's construction of the regular pentagon, a highlight of the *Elements*, comparing it with the widely known construction of Ptolemy, as presented by Aaboe [1]. This gives rise to a discussion on how to view Greek mathematics and shows the care on must have when adopting adapting ancient mathematics to modern styles of presentation, in order to preserve not only content but the very way ancient mathematicians thought and viewed mathematics. [1]

The material here presented can be used for several purposes. First of all, in courses for prospective teachers interested in using historical sources in their classrooms. In several places, for example Brazil, the history of mathematics is becoming commonplace in the curricula of courses for prospective teachers, and so one needs materials that will awaken awareness of the need to approach ancient mathematics as much as possible in its own terms, and not in some pasteurized downgraded versions. [2] As a matter of fact, this text has been used in a course for future secondary school mathematics teachers.

Secondly, it can also be used in secondary education. In many countries, most secondary education mathematics textbooks present just a series of results, some of them justified by examples, the use of analogies and heuristics, but do not show a single proof. So, using Frank Lester's words, many, many, students finish their schooling without ever "meeting the queen". [3] The construction of the pentagon is genuine mathematics, dealing with a result that arouses interest and at the level of secondary school students and it very suited to show the power of the deductive method in mathematics.

In the third place, it is important to show students or readers how mathematics changed along the centuries, how its tools and techniques evolved, and how to appreciate the mathematical accomplishments of past generations, what kind of problems they attacked, how their results were communicated. Discussions on these subjects would certainly enlarge the cultural awareness of students and readers.

[1] For a good discussion, see Schubring [11].
[2] I am definitely not implying that Aaboe's treatment, discussed in this paper, is such a version.
[3] An allusion to Bell's *Mathematics: Queen and servant of science.*

And last but not least, the construction deserves study because it is beautiful mathematics, done with very basic and simple tools. [4]

2 The regular polygons

Figures and configurations that show regularities have always been found interesting. The roses, friezes and tilings of the plane have been widely explored, over the centuries and in countless cultures, by artists and decorators, and studied mathematically, providing a fine example of group theory applied to geometry and crystallography.

The regular polygons, central in geometry, stand out among the important figures that show regularities. The first proposition of Euclid's *Elements* shows how to construct an equilateral triangle. The square also plays an important role in Greek mathematics, in which a major problem was to "square" a figure, that is, to build a square with area equal to the area of that figure. The regular pentagon was important to the Pythagoreans, starting from the sixth century B.C.E., since the pentagram (the regular star polygon of five sides, Figure 1) was the symbol of the Pythagorean brotherhood.

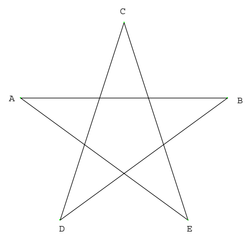

Figure 1: The regular pentagram

The Pythagoreans may have known the fact that the diagonal and the side of the pentagon are incommensurable. This can be seen taking into account that the intersection of the diagonals of the pentagon $ABCDE$ define a new pentagon, $EFGHK$, and so on (Figure 2). If the diagonal and the side of $ABCDE$ are commensurable, the same happens to the sides and diagonals of all the pentagons so obtained, and we arrive at a contradiction. This fact, together with the familiarity of the Pythagoreans with the pentagon, led von Fritz [13] to propose that the existence of incommensurable magnitudes was discovered using the pentagon and its diagonal, and not the square and its diagonal. However, von Fritz's opinion is not the favorite among historians of mathematics.

One of the highlights of the *Elements* of Euclid is the construction, in Book IV, of the regular pentagon. Euclid needs it in book XIII, in which he studies the five regular polyhedra, for the construction of the regular dodecahedron, whose sides are regular pentagons. He uses only the non-graduated ruler and the compass, as happens in all constructions in the *Elements*.

[4] We would like to thank the referees for the excellent suggestions, which helped to improve this paper.

The search for the construction of the regular polygons has been a recurring theme among mathematicians. When they could not find constructions that obey the Euclidean canons, they were able, at least, to find approximate constructions, or which require other resources besides the ruler and compass, as seen, for example, in the construction for the regular heptagon given by Archimedes.

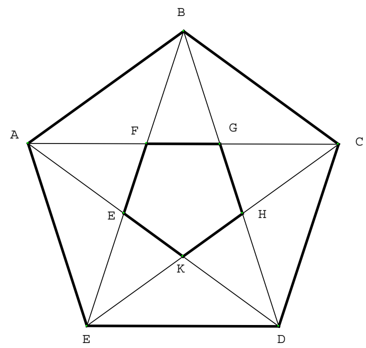

Figure 2: The regular pentagon and its diagonals

A complete answer for which regular polygons can be constructed with ruler and compass was given only in the nineteenth century, by Gauss and Wantzel. The first proved, in 1796, that it is possible to construct with ruler and compass the regular polygon of 17 sides. A little later, in his *Disquisitiones Arithmeticae* [4], he proved that a sufficient condition for the regular n-sided polygon to be constructible with ruler and compass is that n, the number of sides of the polygon, be of the form

$$n = 2^k p_1 p_2 \ldots p_s,$$

in which all p's are Fermat primes, that is, primes of the form $p_i = 2^{2^{n_i}} + 1$. Since the Fermat primes known presently are 3, 5, 17, 257 and 65537, it is possible, in principle, to construct the regular polygons with the number of sides equal to these primes or to products thereof. Thus, the regular polygons with less than 300 sides which can be constructed with ruler and compass are the ones with the following number of sides:

3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, 102, 120, 128, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, 102, 120, 128, 136, 160, 170, 192, 204, 240, 255, 256, 257, 272 136, 160, 170, 192, 204, 240, 255, 256, 257, 272.

Gauss stated that his condition is also necessary, but this was proved only by Pierre Wantzel, in 1836 ([14]), a French mathematician, in a work that proved the impossibility of solving the problems of angle trisection and of doubling the cube.

The criterion of Gauss does not provide an explicit construction of a regular polygon. The German mathematician Johannes Erchinger gave such a construction, with ruler and compass, for the polygon of 17 sides, in 1825, and Friedrich Julius Richelot proposed, in 1832, a method to construct the polygon of 257 sides, but there are doubts about its validity (see Reich [10]).

3 A well-known construction of the regular pentagon

There are many constructions of the regular pentagon, some of which use only the tools prescribed by Euclid. One, well known and particularly simple, due to Ptolemy, [5] proceeds as follows to inscribe a regular pentagon in the circle of center O and radius OB (Figure 3).

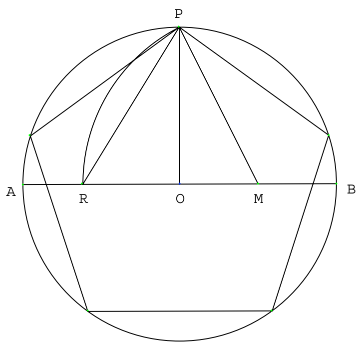

Figure 3: The construction of a regular pentagon inscribed in a circle

Let OP be the perpendicular to the diameter AB, starting from the center O of the circle, and M the midpoint of OB. With center M and radius MP draw the arc of a circle which cuts AB in R. Then RO will be the side of the regular decagon inscribed in the circle of radius OB. Once one knows the side of the regular decagon, it is easy to construct the pentagon, connecting the vertices of the decagon two by two.[6] Note that all the steps in this construction can be made with ruler and compass.

We shall initially justify this construction, following Aaboe ([1], pp. 54-56. See also Aaboe [2]).

Consider, in Figure 4, the triangle OKL, in which $KL = x$ is the side of the regular decagon inscribed in the circle of radius $OK = OL = r$. Thus, the angle $\widehat{LOK}$ measures 36°. Since LOK is an isosceles triangle, its base angles are equal, and so each one measures 72°. With center K and radius $KL = x$, draw a circle, and let T be the point at which it cuts OL. Then, because of the way it was drawn, KLT is an isosceles triangle, and therefore the angle $\widehat{KTL}$ also measures 72°. It follows that the angle $\widehat{TKL}$ measures 36°, and $\widehat{OKT}$ is equal to 36°, and therefore OKT is also an isosceles triangle. Then, $OT = KT = KL = x$.

Since the triangles OKL and KTL are similar, we have

$$\frac{OL}{LK} = \frac{KT}{TL},$$

[5] See Ptolemy, [9], I, 10, On the size of chords in a circle, pp. 14-15.

[6] Euclid shows, in Proposition XIII.10 of the *Elements*, that the sides of the regular dodecagon, pentagon and hexagon inscribed in a circle form a right-angle triangle. This implies that PR will be the side of the regular pentagon.

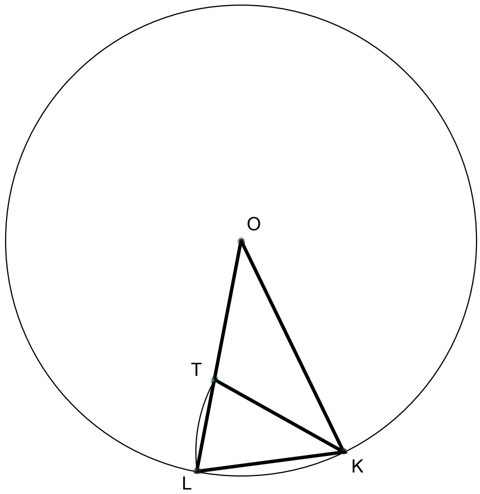

Figure 4: The triangles LKT and KOL are similar

that is,

$$\frac{r}{x} = \frac{x}{r-x} \Longrightarrow x^2 + rx - r^2 = 0.$$

The only positive root of this equation is

$$x = \frac{1}{2}r(\sqrt{5} - 1).$$

Let us return to Figure 3. The triangle POM has a right angle, and therefore

$$PM^2 = r^2 + \left(\frac{r}{2}\right)^2 \Longrightarrow PM = \left(\frac{r}{2}\right)\sqrt{5}.$$

So,

$$OR = RM - OM = PM - OM = \frac{r}{2}\sqrt{5} - \frac{1}{2}r = \frac{1}{2}r\left(\sqrt{5} - 1\right).$$

Thus, the segment OR is the side of the regular decagon. It is now easy to construct the regular pentagon: it is sufficient to join the vertices of the decagon, two by two.

We can make some comments about the treatment presented by Aaboe, which is mathematically correct, but deserves pedagogical and, equally important, historiographic remarks.

Firstly, as far as pedagogy is concerned, Aaboe says simply that it is "more convenient" to find the side of the regular decagon, without explaining why this is so. We will try to show why it is actually more convenient to work with the regular decagon.

Figure 5 shows a pentagon and a regular decagon inscribed in a circle. If we construct the decagon, we can construct the pentagon, joining the vertices of the decagon two by two. It is also true that starting with the pentagon we can construct the decagon. As a matter of fact, if you can construct with ruler and compass the regular polygon of n sides it is possible to construct the regular polygon of $2n$ sides, and conversely.[7]

[7]More generally, if r and s are co-prime natural numbers, and we know how to construct the polygons with respectively r and s sides, it is then possible to construct the polygon with rs sides. Euclid proves this for the case $r = 3$ and $s = 5$ in the last proposition of Book IV (IV.16).

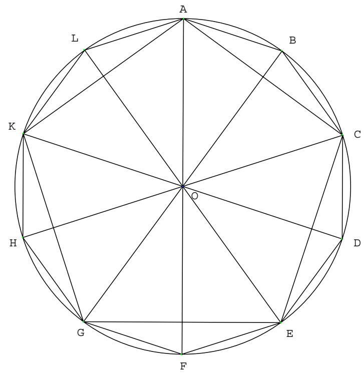

Figure 5: The regular pentagon and decagon inscribed in a circle

Why should one construct the regular decagon and not, directly, the regular pentagon? In Figure 5, the angles $\widehat{DOE}$, $\widehat{EOF}$, $\widehat{GOS}$ $\ldots$, $\widehat{COD}$ are equal, since they are the central angles of the regular decagon $ABCDEFGHKL$. Furthermore, the angle $\widehat{ODE}$ is equal to the angle $\widehat{KDE}$, and so to half the arc $KHGFE$. As this arc is equal to four times the arc DE, we see that the angle $\widehat{ODE}$ is equal to twice the angle $\widehat{DOE}$, that is

$$\widehat{ODE} = 2 \times \widehat{DOE}.$$

Thus, in the isosceles triangle ODE, each base angle is twice the vertex angle. Therefore, to construct the regular decagon inscribed in the circle of radius R, it is enough to construct an isosceles triangle ABC such that its base angles are both twice the vertex angle and then draw in the circle a triangle DEF equiangular with the triangle ABC.

The crucial point is that (See Figure 5) in the isosceles triangle OAB the vertex angle, $\widehat{AOB}$, is equal to half the base angles. As shown by Aaboe, the fact that the triangles OKL and KTL are similar (Figure 4) implies that the radius of the circle, OK, is divided by T in extreme and mean ratio.

Secondly, we have historiographic problems. Aaboe uses the fact that we are dealing with angles that measure 72^o and 36^o, respectively. However, to measure angles this way is entirely avoided in the *Elements*. In fact, it was unknown in classical Greek mathematics. The only thing necessary, as we explained above, is the relationship between the angles ODE and DOE (Figure 5), which follows from the results contained in the *Elements* about the angles inscribed in a circumference. Euclid never measures lengths, areas, volumes and angles. Since the Greeks did note have the real numbers, they did not measure magnitudes, they dealt directly with these, comparing them. For example, angles are compared with the right angle, polygons are transformed into squares (they are "squared") and the resulting squares are compared. Of course one can add segments, angles, areas and volumes.

Aaboe's presentation is mathematically correct. His mistake is to encase it in Greek dress, giving the reader the impression that the Greeks, in particular Euclid, worked in the manner shown by him, Aaboe.

More serious in Aaboe's presentation is his use of the so-called "Greek geometric algebra",[8] which has been the subject of heated discussion in the last decades. In particular, Sabetai Unguru has denounced, in several papers (see [3] and also [11] for full references), the anachronistic interpretation of Greek and Babylonian mathematics done by the proponents of the "Greek geometric algebra", in the light of modern developments, that is, the widespread algebraization of mathematics. [9]

The anachronism of Aaboe's text is shown particularly in the interpretation of Proposition II.6 of the *Elements* as a way to solve second degree equations, failing to understand its importance in the geometric context of the *Elements*.[10] *Elements* II.5 and II.6 certainly can be interpreted in an algebraic way, totally out of the context of classical Greek mathematics, but then one should not claim to be presenting what Euclid did. [11]

We now present Euclid's construction of the regular pentagon.

4 The construction of the pentagon by Euclid

The construction of the regular pentagon inscribed in a circle is the climax of Book IV of Euclid's *Elements*. It should be stressed that this construction does not use similarity. It is based entirely on equivalence of areas. Once discovered, it can easily be done. We emphasize, moreover, that the construction in Book IV clearly shows the strength of the method of the equivalence of areas, widely used by Euclid, until Book V, exclusive.

Since Euclid, in Book XIII of the *Elements* constructs, with ruler and compass, the five regular polyhedra, he could not avoid constructing, beforehand, the regular pentagon, because the faces of the regular dodecahedron are regular pentagons.

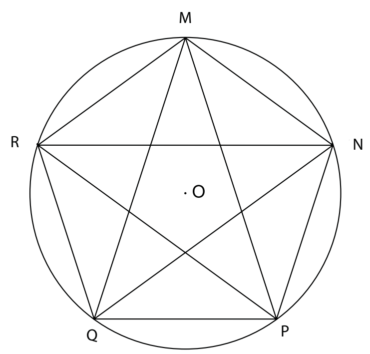

Figure 6: The angle at M is half the angle $\widehat{MQP}$

Euclid, like other classical Greek mathematicians, merely presents his results logically linked. Thus, we give, initially, a brief survey of his construction. We recommend

[8] "Greek geometrical algebra" was created, at the end of the 19th and beginning of the 20th centuries, by Neugebauer, Tannery and Zeuthen (See, for example, [15]). It became very generally accepted because of Heath's influential and well known translations of Greek mathematics. It was continued by van der Waerden, who agreed with this interpretation.

[9] A thoughtful and balanced view of this question can be found in Vitrac ([12], pp. 366-376)

[10] Of course, Ptolemy's original treatment is entirely in accordance with the methods and tools developed in the *Elements* of Euclid.

[11] A broad and deep discussion of this interpretation of Greek mathematics can be found in Fried and Unguru, [3].

returning to this initial description, after reading through the text. The construction itself is done in Propositions 10 and 11 of Book IV.

Euclid starts from the easily seen fact that in the regular pentagon, $MNPQR$ (Figure 6), the isosceles triangle MQP is such that its vertex angle is equal to half of each base angle. If we can inscribe such a triangle in a circle, we can construct the pentagon.

To perform this construction Euclid requires two lines of argumentation. The first, applying II.11 to the radius of the circle, KV, he finds the point Z such that the rectangle on KV and ZV is equal to the square on KZ (See Figure 7). [12] Let T, on the circumference, be such that $VT = KZ$. [13] Then Euclid draws an auxiliary circle, which passes through points K, Z and T [14] and using results about secants and chords in a circle, contained in II.36 and III.37, he shows, finally, that the triangle KVT satisfies the required property, i.e., the angle at its vertex is equal to half of each base angle.

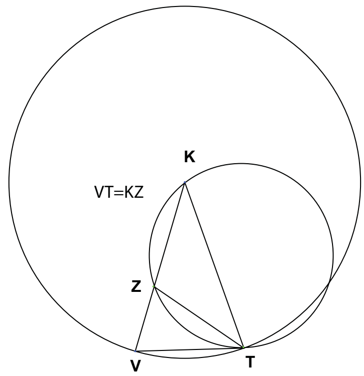

Figure 7: The triangles KVT and ZVT

After this, to construct the regular pentagon inscribed in a given circle, Euclid inscribes in this circle a triangle equiangular with the triangle just constructed, KVT.

We now turn our attention to the actual construction. We state, without proofs, the results from the *Elements* he needs, following Heath's versions ([6] and [7]).

We begin with Proposition II.6, very important in the *Elements*.

Proposition II.6: *If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.*[15]

In other words, the rectangle with sides AD, DB, together with the square on CB, is equal to the square on CD.

Euclid uses also the following result, which is a consruction problem

Proposition II.11:*To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.*

[12]Using the theory of proportions, expounded in Book V, and applied to figures in Book VI, this is equivalent to the division of KV in mean and extreme ratios.

[13]This can be done by proposition I.2.

[14]Euclid shows how to do this IV.5.

[15]We recall that, for Euclid, *straight line* can mean either a segment or the whole straight line.

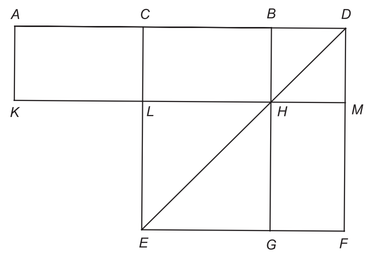

Figure 8: *Elements* II.6

That is, Euclid shows how to find, using only a straightedge and a compass, a point C on AB such that the rectangle of sides AB and CB is equal to the square on AC (Figure 9).

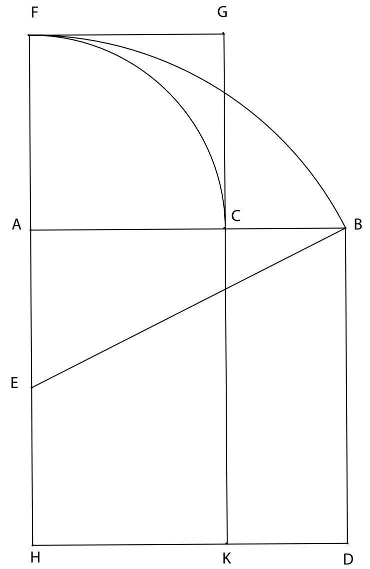

Figure 9: *Elements* II.11

Now we need some results about circles and their chords, from Book III of the *Elements*.

Proposition III.36: *If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.*

That is, the rectangle with sides PR and PS is equal to the square on PT (Figure 10).

Proposition III.37: *If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex*

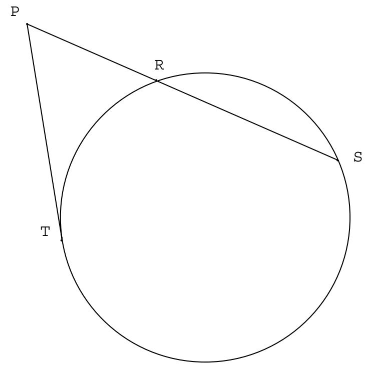

Figure 10: *Elements* III.36

circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.

This proposition is the converse of III.36.

For the last step before reaching the construction of the regular pentagon, we will use two additional results, also without proof:

Proposition III.22: *The opposite angles of quadrilaterals in circles are equal to two right angles.*

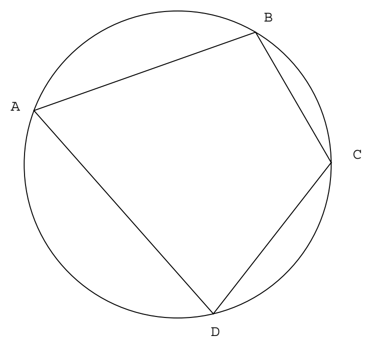

Figure 11: *Elements* III.22

That is, for the quadrilateral $ABCD$ of Figure 11, inscribed in a circle, the angles $\widehat{BAD}$ and $\widehat{BCD}$ are equal, taken together, to two right angles. The same is true of the angles $\widehat{ADC}$ and $\widehat{ABC}$.

Proposition III.31: *In a circle, the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; and further, the angle of the greater segment is greater than a right angle, and the angle of the less segment less than a right angle.*

This means that in the circle of Figure 12, the angle $\widehat{BAC}$ is right, the angle $\widehat{ABC}$ in the arc greater than the semicircle is less than a right angle, and the angle $\widehat{ADC}$ in the arc ADC, less than the semicircle, is greater than a right angle.

We need also the following result.

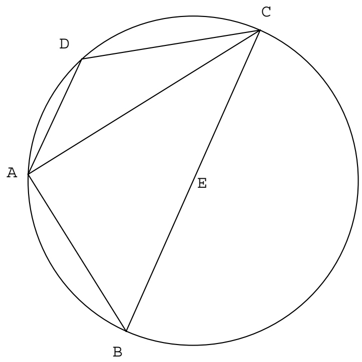

Figure 12: *Elements* III.31

Proposition III.32: *If a straight line touch a circle and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.*

This proposition states that the angle $\widehat{DBF}$ is equal to the angle $\widehat{BAD}$ (Figure 13).

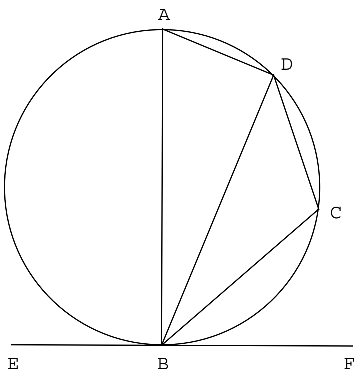

Figure 13: *Elements* III.32

What remains now is just to construct the regular pentagon. The essential step is the following construction problem. Euclid's text can be found in Heath ([7]).

Proposition IV.10: *Construct an isosceles triangle in which each base angle is double the vertex angle.*

Let a line segment AB be given (Figure 14). Using II.11, find the point C such that the rectangle of sides AB, CB is equal to the square on CA.

Draw the circle with center at A and radius AB. From B, draw BD equal to AC. Then, the triangle ABD has the required property, namely, that each of the base angles is equal to twice the vertex angle.

We summarize below, in a symbolic and compact way, the reasoning of Euclid. In what follows, ret (AB, BC) designates the rectangle of sides AB and BC, and quad (AC) represents the square on the segment AC.

The point C is such that ret(AB, BC) = quad(AC).

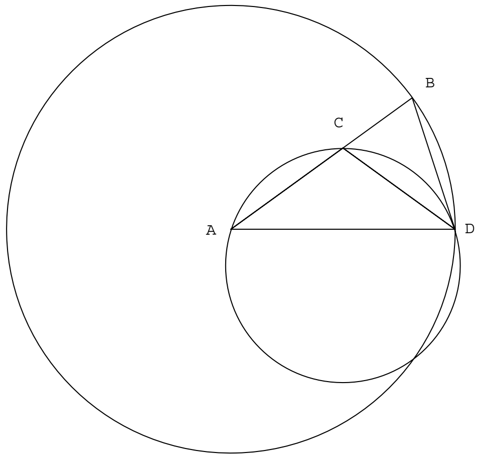

Figure 14: *Elements* IV.10

Draw AD and DC and construct the circle ACD circumscribed to the triangle ACD (This can be done by IV.5).

Since $AC = BD$, by construction, we have

$$\mathrm{ret}(AB,\, BC) = \mathrm{quad}(BD).$$

But then, it follows from III.37, that BD is tangent to the circle ACD.

By III.32, we have

$$\widehat{BDC} = \widehat{DAC} \Longrightarrow \widehat{BDC} + \widehat{CDA} = \widehat{DAC} + \widehat{CDA}.$$

Thus,

$$\widehat{BDA} = \widehat{DAC} + \widehat{CDA}.$$

Consider, in the triangle ACD, the external angle $\widehat{BCD}$. Then

$$\widehat{BCD} = \widehat{DAC} + \widehat{CDA} \Longrightarrow \widehat{BCD} = \widehat{BDA}.$$

Since $\widehat{BDA} = \widehat{DBA}$, we have

$$\widehat{BCD} = \widehat{BDA} = \widehat{DBA} \Longrightarrow \widehat{BCD} = \widehat{DBC} \Longrightarrow DB = DC.$$

Since $DB = CA$, it follows that $CA = CD$, and therefore $\widehat{CAD} = \widehat{CDA}$.

Thus,

$$\widehat{CAD} + \widehat{CDA} = 2\widehat{CAD} \Longrightarrow \widehat{BCD} = 2 \times \widehat{CAD},$$

and therefore

$$\widehat{BCD} = \widehat{BDA} = \widehat{DBA} = 2 \times \widehat{CAD} = 2 \times \widehat{BAD}.$$

Thus, in the triangle ABD, each base angle is twice the vertex angle. □

Before the final construction, we need

Proposition IV.2: *In a given circle to inscribe a triangle equiangular with a given triangle.*

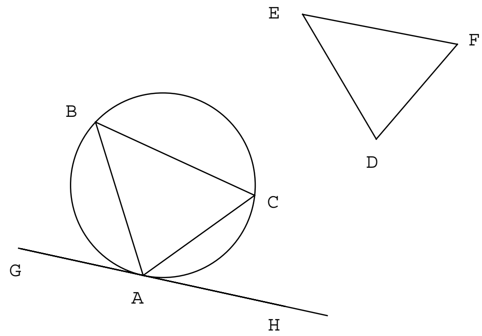

Figure 15: *Elements* IV.2

Let the circle and the triangle DEF be given (Figure 15). Let GH be the tangent to the circle, passing through the point A. Draw the angle $\widehat{HAC}$ equal to the angle $\widehat{DEF}$ and the angle $\widehat{GAB}$ equal to the angle $\widehat{DFE}$. Draw BC. Then, by III.32, the angle $\widehat{HAC}$ is equal to the angle $\widehat{ABC}$ and therefore the angle $\widehat{ABC}$ is equal to the angle $\widehat{DEF}$. Similarly, we can show that the angles $\widehat{ACB}$ and $\widehat{DFE}$ are equal. Therefore, the angle $\widehat{BAC}$ will be equal to the angle $\widehat{EDF}$.

Therefore, in the given circle there has been inscribed a triangle, with angles respectively equal to the angles of the given triangle.

□

We can finally present the result we set out to prove.

Proposition IV.11: *In a given circle to inscribe an equilateral and equiangular pentagon.*

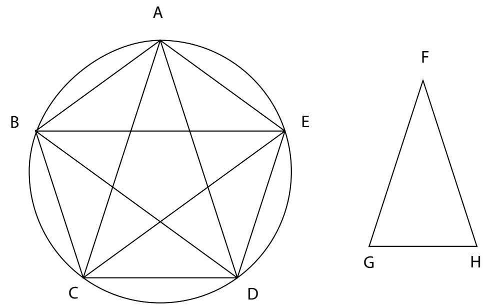

Figure 16: *Elements* IV.11

Construct the triangle FGH in which each base angle is twice the vertex angle. Inscribe the triangle ACD in the circle, with the angle $\widehat{CAD}$ equal to the angle at F and the angles $\widehat{ACD}$ and $\widehat{CDA}$ equal, respectively, to the angles at G and H (Figure 16).

Bisect the angles $\widehat{ACD}$ and $\widehat{ADC}$ by EC and DB, respectively. Draw the straight line segments AB, BC, DE and EA.

Since each one of the angles $\widehat{ACD}$ and $\widehat{CDA}$ is twice the angle $\widehat{CAD}$, and have been bisected by the lines CE and DB, respectively, it follows that the angles $\widehat{DAC}$, $\widehat{ACE}$,

$\widehat{ECD}$, $\widehat{CDB}$ and $\widehat{BDA}$ are equal to one another. But then the arcs AB, BC, CD, DE, EA are equal, and so the line segments AB, BC, CD, DE, EA are also equal.

Therefore the pentagon $ABCDE$ is equilateral.

Now, since the arc AB is equal to the arc DE, if we add the arc BCD to both, it follows that the arc $ABCD$ is equal to the arc $EDCB$. From this we see that the angle $\widehat{BAE}$ is equal to the angle $\widehat{AED}$.

For the same reason, each of the angles $\widehat{ABC}$, $\widehat{BCD}$, $\widehat{CDE}$ is also equal to each of the angles $\widehat{BAE}$ and $\widehat{AED}$.

Therefore the pentagon is equiangular, and so it is a regular pentagon.

□

Obviously, as noted by Hartshorne ([5], pp. 45-51), Euclid's construction can be modified, to become quicker and more efficient. It is enough to apply IV.10 directly to the circle in which we want to inscribe the regular pentagon.

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THE USE OF ORIGINAL SOURCES IN AN UNDERGRADUATE HISTORY OF MATHEMATICS CLASS

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ABSTRACT

When designing a course on the History of Mathematics, the instructor is faced with several questions. Certainly, the extent of the content covered is a primary problem. Once that had been decided, another primary concern becomes the approach, which is taken to the content. After years of telling second hand stories in similar courses, I have found it more interesting to let the material present itself. Why not let the student hear the original author speak for themselves?

Well, there are several problems with this approach. Much of what we need is neither accessible nor is in a language the student can understand. Much of what we have is unreliable, having been recopied many times over the years. Yet it is the basis for much of what we teach in such a course. If the instructor has done some research on the pitfalls of using original sources, the student will benefit from having been guided through them.

I would like to discuss resources, which are available in print form and on the web. I would also like to consider various approaches to the material including guided reading and assignments outside the classroom. The material might include works which are not strictly mathematical in content but demonstrate the place of mathematics in society.

It has been my great privilege for the past five semesters at Brooklyn College to teach MATH 41W: History of Mathematics. This is a class for undergraduates who are either mathematics majors or mathematics education majors. It is an elective course which is particularly recommended for prospective high school teachers. A typical group from Spring 2010 consisted of 20 students of whom 10 were mathematics majors, 9 were mathematics education majors, and one was a finance major, who proved to be one of the best. One catalog description of the course goes like this:

> "History of Mathematics. Development of mathematics from antiquity to recent times; emphasis on the inter-relationship of subject matter and on the rise of modern concepts. Prerequisite : Calculus II. 3 hours, 3 credits"

This particular description is from a 1962 catalogue when the instructor for the course was none other than the eminent mathematical historian Carl Boyer. He had taught the course from the beginnings of the college back in the 1930s until his demise in 1976. Since that time two changes have been made to the course description. The first is that the hyphen has been removed from the word "inter-relationship". The second is that a W has been appended to the course number. The latter change has been more formidable.

In an effort to equip the mathematics department with a writing-intensive course it was decided that MATH 41 would be the natural candidate for such a course. A prerequisite of English 2 was added to the course description. This is particularly important due to the number of non-native speakers in the class. Written work consisting of a minimum of ten pages would now be required. Some of this could be in the form of shorter exercises but the final result would naturally be some form of research paper. Students normally choose a research topic based on personal interest. I encourage them to write on topic not covered in class. As an additional incentive the more promising work can be expanded and submitted for the annual Boyer Essay competition.

One of the consequences of this writing intensive format is that students will have to do a significant amount of outside reading. The readings that they do for the class itself influence their choice of what they read for their papers. After years of telling stories like Archimedes at the battle of Syracuse, or the conflict over methods of solving the cubic, it became clear that these episodes are best told by the players themselves. It is instructive to let the students ruminate on the accuracy of these stories. In the words of Wardhaugh from his current work *How to Read Historical Mathematics*: “there’s always more to find out about the mathematical past” (Wardhaugh 2010, p.101). Students’ lack of confidence in their own abilities is diminished when they “see mathematicians struggling with problems they *can’t* solve” (*ibid* p.85). The evolution of the “problem of points” has always been a good example for me of how mathematicians grappled with a misunderstood concept. The ICMI study on *History in Mathematics Education* (Fauvel & van Maanen 2000) suggested reading the original sources will allow students to “appreciate the role representations play in the evolution of ideas” (*ibid* p.294). Freudenthal is quoted in that same study as suggesting that seeing mathematics in its original setting will give students “an opportunity to rediscover properties taken for granted” (*ibid* p.295). What students thought they understood will become truly clear.

So what exactly do we mean by an original source? First we need to distinguish between two types of sources: external and internal. External sources are those that deal with the backstory of the mathematics to be discussed. These can consist of biographies and general historical descriptions. They are not difficult to read but are important for clearing up historical misunderstandings. These readings can often be assigned as homework before they are discussed in class. Many teachers of the history of mathematics would not even consider including them in such a course. I think they are important for at least two reasons. First, the general historical preparation of the average mathematics student is poor and they will greatly benefit from such reading. Secondly, the readings are accessible and give the student motivation to try the more difficult readings. The mathematical content of the internal readings can frequently be challenging. For example, Apollonius cannot be read without the necessary prerequisites from Euclid. The ICMI Study has suggested it is always wise for the instructor to consider the level of difficulty presented by each original work and the background of the students (*ibid* p.317). In general, students will not be able to read these works in the original language. Since the translations are difficult enough, we use a students’ primary common language. In addition, they will be “required to integrate (both) the mathematical language of their lessons (and) their own way of expressing mathematics” (*ibid* p.299). Frequently, there is need to modify available translations but we need insure they don’t lose their meaning (*ibid* p.315). We always need to avoid “anachronistic thinking and evaluating mathematics out of context” (*ibid* p.296). It is therefore worthwhile for the instructor to be aware of Grattan-Guinness’ discussion on the difference between *history* and *heritage* (Grattan-Guinness 2009). While it is tempting to evaluate any mathematical result in terms of our present interpretation, we should not infer that that is what the original author intended. It is important that the student avoid coming to conclusions about the author's intent which are not contained in the work. This requires constant guidance from the instructor. Students should be made aware of the transmission process and modernization that occurs during translation.

Several approaches can be taken to the material. A direct approach will require no previous preparation. It has the benefit of "shock value". Students will work on a reading with a set of prepared questions. An alternative would use student generated questions. A more indirect approach could begin by introducing a "non-routine" problem to motivate the reading (Fauvel & van Maanen 2000, p.314). Michael Glaubitz recently has performed an empirical study comparing genetic and hermeneutic approaches. In the genetic approach a topic is first introduced in its original historical order while the hermeneutic approach assumes pre-knowledge of the topic. While the group using the genetic approach did worse than a similar group using a conventional approach, the group using the hermeneutic approach did better than either. Details can be found in this volume, ch.3.1.

I have divided MATH 41W chronologically into five distinct periods: Egyptian - Mesopotamian, Early Greek, Alexandrian Greek, Muslim-Indian-Chinese, and European (1000-1750). From the first period we examine the Rhind Papyrus, the Berlin Papyrus, and several cuneiform tablets from Mesopotamia. The Ishango Bone is presented as a possible topic for student research. Links are suggested where students may view some of these ancient artifacts. Some background history on Egypt can be found in Herodotus' *Histories* and Aristotle's *Metaphysics*. Several exercises from the Rhind Papyrus including unit fractions, ratio division, false position, and area of plane figures are discussed in class. Students discover the nature of the Mesopotamian sexagesimal system by deciphering a reproduction of a nine times table. The nature of Mesopotamian mathematics is further elucidated by examining a table of reciprocals, area problems (YBC6967), and the measure of the diagonal of a square (YBC7289). The Butler Library at Columbia University in Manhattan is the repository of an important artifact known as Plimpton 322. Students will read several explanations (Buck 1980, Robson 2002, etc.) of the nature of this tablet and be asked to compare them.

From the Early Greek period the class will read several fragments from writers spanning a period of a thousand years. Students will read further selections from Herodotus and Aristotle. Selections from Proclus, notably the Eudemian Summary, and fragments from Simplicius on Hippocrates of Chios are useful to understanding the beginnings of Greek mathematics. There are numerous writers who describe the Pythagoreans and their philosophy. These include Plato, Aristotle, Nichomachus, Plutarch, and the later Neo-Pythagoreans such as Proclus, Iamblichus, and Porphyry. The writings of Diogenes Laertius in the early 3rd Century are a large source of information. Unfortunately, much of it is untrustworthy. It is useful to have students compare several conflicting accounts.

From the Alexandrian Greek period we have the first notable mathematical text, Euclid's *Elements* (Heath 2007). By this time of the course students are equipped to understand that this is not an original work but the amalgamation of many earlier authors. Each instructor will need to determine which sections are most important for their class. Certainly, sections of

Books I, II, the number theory books (VII-IX) and the method of exhaustion (XII) are recommended. A few propositions from III and VI are vital for a later reading Apollonius (particularly I.11 on the parabola). Students can read up on the life and death of Archimedes in Plutarch's biography of the Roman general Marcellus. In class we discuss sections of Archimedes' *Measurement of the Circle*, *On Sphere and Cylinder*, and *Quadrature of the Parabola* (Heath 2002). Heron's discussion on the area of a triangle and square root methods from the *Metrica* along with Ptolemy's method for creating a table of chords from the *Almagest* are worthwhile readings. Problems from Diophantus' *Arithmetica* and Metrodorus' *Greek Anthology* (notably problem 126) are within the reach of most students.

From Asian mathematics (Muslim–Indian-Chinese) the availability of sources had been a major problem. While not completely solving this problem the recent sourcebook edited by Victor Katz has been a welcome addition (Katz 2007). It contains many sources from the Muslim world and yet I still find that I rely on the works of Al Khowarizmi and Omar Khayyam as they give a good overview of that culture's contributions. Indian mathematics has been more obscure. As for primary sources Bhaskara I's explanation of the work of Aryabatta (Keller, 2006) seems to be the most accessible for my students. The Colebrooke translations of the work of Brahmagupta and Bhaskara II are available online through Google Books (Colebrooke 1817). They are much more detailed but older. There is a recent translation in English of the Chinese *Nine Chapters of the Mathematical Arts* by Kangshen, Crossley & Lun (1999) but if your library cannot obtain it you may be out of luck. The price listed on Amazon was over $400. Several sections can be found in the Katz 2007 sourcebook. Libbrecht's translation of the work of Qin Jiu Shao is valuable (Libbrecht 2006). The diagram of the binomial triangle of Zhu Shiejie which can be found in most texts provides a lively discussion.

The last section of the course deals with mathematics in Europe. I like to begin with the nice round date of 1000 when a mathematician sat on the papal throne. Most of Gerbert of Aurillac's (Sylvester II) letters deal with church matters but a few have mathematical content. Leonardo of Pisa's *Liber Abaci* contains several kinds of problems worth discussing. The recent translation by Sigler (2002) has been slighted by some but is worth having. The works of Jordanus de Nemore and Nicole Oresme are available but can be challenging for most. Swetz' edition of the Treviso Arithmetic is very accessible. I am still waiting for a translation of Pacioli's *Summa*, the Euclid of his day. The first great mathematical work of the scientific revolution is Cardano's *Ars Magna* in 1545 (Cardano 2007). In this work he discusses solution of cubic and quartic equations, gives the background for the priority dispute, and also introduces imaginary quantities. Other worthwhile and accessible works from this period include Recorde's introduction of the equal sign, Stevin's La Disme (the tenth), and Viète's discussion on the nature of roots (Viète 2006). This last work can be found on the Gallica website in Latin but in a form which is readable by most students.

The Seventeenth and early Eighteenth Centuries offer a wide array of choices for original materials. Time constraints will limit the extent of what can be discussed in class. Perhaps, students can be encouraged to research mathematicians not talked about in class. My list of essentials includes Descartes, Fermat, Pascal, Newton, Leibniz, the Bernoullis, and Euler. Many of the works of the last author can be found at the Euler Archive: www.math.dartmouth.edu/~euler/ (see also Descartes 1954, Leibniz 2005). I believe it was

Laplace who recommended: “Lisez Euler, c'est notre maître à tous”. I have attached De Moivre to the above list. I’m sure that we all have our personal favorites. As background reading for this period I highly recommend Voltaire’s *Letters on England*. He gives a lively comparison of Descartes and Newton along with some insights on the mathematical community in general.

I have given some hints as to where to find these sources but I would now like to proceed in a more methodical fashion. I find that the sources fall into roughly four categories: individual translations, sourcebooks, instructor selected readers, and internet sources. Several of the individual translations can be found as inexpensive paperbacks published by Dover. These include Euclid, Archimedes, Cardano, Viete, Descartes, and Leibniz. There exists an out of print edition of Diophantus. Carl Boyer was nice enough to stock the Brooklyn College library with at least two copies of that work. My background in the Great Books Program has alerted me to the wealth of material in the *Great Books of the Western World* series. Titles include Herodotus (Vol.6), Plutarch (Vol.14), Plato (Vol.7), Aristotle (Vol.8), Euclid - Archimedes - Apollonius - Nichomachus (Vol.11), Ptolemy (Vol.16), Descartes (Vol.31), Pascal(Vol.33), and Newton (Vol.34). Although the translations of many of these works are significantly out of date, they do provide some possibilities.

Most courses of this nature that stress reading original sources include a sourcebook as a required text. I have never found one that completely suits my needs but have recently returned to using the one edited by John Fauvel and Jeremy Gray. It seems to work fine except for Asian mathematics. I have used Calinger’s *Classics of Mathematics* (Calinger 1995) in the past but found that it covered too much material and was cost prohibitive. A new choice has been offered by Jacqueline Stedall (2008) but mathematics before 1540 is given short shrift. Sourcebooks by D. E. Smith (1959) and Stephen Hawking (2005) just don’t contain enough of the material I have described. Two out of print sourcebooks which are excellent have been edited by D.J. Struik (1986) and Henrietta Midonick (1965). I wish they were available as texts. The latter contains selections on Indian and Chinese mathematics from Colebrooke (1817) and Wylie, respectively. The Katz sourcebook (2007) has been previously mentioned. A comprehensive sourcebook entitled *Early Greek Philosophy* has been edited by Jonathan Barnes (1987).

In an attempt to create a reader that would satisfy my need for external sources I contacted the good people at Pearson – Penguin. I found I could create my own custom reader online using their system. While I was able to locate many of my choices of authors, I found that the specific selections I wanted were not being offered. I could include some of these as “outside selections” (even though they were from the regular Penguin catalog) but I would have to pay a much higher price for them. In the end I decided to try their choices which included all of the following: Hammurabi’s Code, Herodotus, Aristotle, Plato, Plutarch, The Koran, Confucius, Hugh of St. Victor, Roger Bacon, Alberti, Vasari, Galileo, Descartes, Pascal, and Voltaire. In general, I would have to say that this reader did not satisfy our needs.

Lastly, many original sources (particularly external ones) can be found on the internet. One major problem I encountered with assigning sources from the internet is that the sources frequently do not have page numbers. This can lead to some confusion. Many Greek writers can be found at <classics.mit.edu>. Euclid’s *Elements* is available in a dynamic version at

<aleph0.clarku.edu/~djoyce/java/elements/elements.html>. I was duly impressed with Cambridge University's interactive page turner at <www.lib.cam.ac.uk/cgi-bin/PascalTriangle/ browse>. The Gallica website at Bibliothèque National contains a wealth of material. Google books can help locate many out of print works. As previously mentioned museum websites are a great source for mathematical artifacts.

The task of presenting original works of mathematics to a class of undergraduates may seem like a daunting task. It requires much preparation. How does one become confident that one can do this? Well, you must remember that you are not alone in this task. Attending any conference with themes on the history of mathematics will remind you that there are many others who have the same interest as you. Lately, I have found that regular attendance at reading groups such as the Euler Society or ARITHMOS have increased both my ability to deal with more difficult readings and my desire to read further. If you don't have such a group in your area, why not find some like-minded individuals and start one? You have nothing to lose but your pre-conceptions.

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THE FALSE POSITION IN THE PROPORTIONAL REASONING TEACHING

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ABSTRACT

This work presents the use of History of Mathematics in high school Math classes, involving 240 Brazilian students, aged between 14 and 17. The topic chosen was proportional reasoning through “false-position”. The false-position rule is a method to solve certain types of problems, giving an arbitrary value to the question; and, checking this value in the problem conditions, if it does not correspond to the solution, it is modified by proportionality. This method appears in the solution of several problems in the Egyptian papyrus (~1800 b.C.), was explored by Euclid on the “The Elements” (~300 b. C.), in the text “Arithmetic”, by Diofanto (~250 a.D.) and in the Hindu Bakhshali manuscript (~600 a.D.). The famous Arabic mathematician Al-Khowarizmi (~850 a.D.) uses the rule and also expands to more complex solutions. In 1202, Leonardo of Pisa (Fibonacci), publishes his famous book *“Liber Abbaci”*, in which, besides introducing to Europe how to calculate with the Hindu-Arabic numeral system, he solves several problems with this rule. In 1498, the German mathematician Johann Widman publishes his arithmetic book *“Rechenung auff Allen Kauffmanschafft*”, in which he solves several problems using the rule of the false position, which he calls “excess and deficiency”. In 1494, Luca Paccioli, in his book *“Summa de arithmetica, geométria, proportioni et proportionalita”,* discusses and applies this rule. In the Renaissance period many school-book authors already frequently use the false position rule, even in Spain and Portugal. In Brazil and in the Unites States of America it appears in some school-books in 1850. It is possible to find it in school-books in Brazil until the beginning of the 20th century, for example, the book *Aritmetica- Teoria e Pratica* (Arithimetics – Theory and Practice) by Andre Perez y Marin (1928). In the most used old school-books, it is cited as a rule without any explanation in the reason why it solves the proposed problem. It is believed that the teaching of the false-position rule should retake the methodological suggestions of the Mathematics education because, besides arousing in the student a desire to search for a solution, even if arbitrary, for a problem, it develops all the reasoning of error correction. We’ll report an experience made in Limeira, in São Paulo state – Brazil, in the beginning of the academic year in 2009 (from February to April). Several problems from the Rhind Papyrus, in Liber Abbaci, and others found in ancient Mathematics books from Brazilian and Spanish literature involving the false-position rule were selected and presented to students. Following teacher orientations, the students solved these problems using an arithmetic or geometric process and not an algebraic one, making it possible to check the fundamental property of the proportions shown by Euclid, as equality of two areas in solving problems.

It was observed that the students presented different strategies to solve problems that were about proportions, resorting to the use of the false-position rule instead of the rule of three. Some reflections and suggestions are highlighted in the development of this paper which relate to the teaching of Mathematics with the support of the History of Mathematics, which, in this case, specifically, promoted the development of proportional reasoning.

UNDERSTANDING MATHEMATICS USING ORIGINAL SOURCES. CRITERIA AND CONDITIONS

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ABSTRACT

As an explicit resource in the classroom, the History of Mathematics allows the improvement of the learning of mathematics. Through the analysis of significant texts from the historical evolution of mathematical concepts, the Group of History of Mathematics of the Association of Barcelona for the Study and Learning of Mathematics (ABEAM) has been developing historical materials for use in the classroom since 1998.

In this article we describe the situation in the Catalan curriculum of secondary school mathematics and analyze an example put into practice in the classroom. In addition we discuss the criteria for preparing the historical texts to be used in the classroom, as well as the conditions for using them as a powerful tool for understanding mathematics.

1 Introduction

We would like to recall the words of the Catalan mathematician Pere Puig Adam (1900-1960) on the importance of knowing the history of the concepts,

> "Science grows through a combined process of analysis and a synthesis of induction and deduction. Experience and observations are accumulated, facts recorded, analogies examined, concepts extracted, laws inductively developed and systems deductively constructed. The synthetic presentation of such an elaborate fabric undoubtedly provides solidity to the whole, but it fails to teach how such a fabric is woven, which is what education is about. Thus when used as an instrument for conveying knowledge, these factors have tended to accentuate the separation between two processes that should never have been separated: firstly, the process of the genesis of knowledge and secondly the process of its transmission".[1]

In accordance with this idea, we have prepared and experimented with some materials in the mathematics classroom during some courses. The aim of this article is to discuss through the analysis of an example implemented in the mathematics classroom the criteria and the conditions for transforming these activities that involved historical texts into a powerful tool for learning mathematics. First we present a survey of the evolution of this implementation the last few years in Catalonia.

In Spain, every autonomous community is in charge of their own secondary and graduate education, so we only focus on the implementation of history in the mathematics

[1]La ciencia crece por procesos mixtos de análisis y síntesis de inducción y deducción. Se acumulan experiencias y observaciones, se registran hechos, se examinan analogías, se abstraen conceptos, se inducen leyes y se tejen deductivamente sistemas. La presentación sintética de tales urdimbres da una indudable solidez al conjunto presentado; pero no enseña precisamente a urdir, que es lo educativo. Así, pues, usados como instrumento de transmisión de conocimientos han tendido a acentuar cada vez más la separación entre dos procesos que no debieron divorciarse nunca: el de la génesis de los conocimientos y el de su transmisión (Puig Adam, 1960, p. 95).

classroom in Catalonia.

The implementation of the history of mathematics has for twenty years inspired some individual actions between teachers. Thus, since the academic year 1990/91 financial aid for research work, granted every year to teachers by the Department of Education of the government of Catalonia, has been devoted to research into the relations between the history of science (including mathematics) and its teaching. This research work has resulted in the drawing up of memories that are now available to other teachers. Also in the high schools, workshops, centenaries, and conferences by teachers constitute further examples of activities in which history can be used to achieve a more comprehensive learning experience for students. For example, the workshop devoted to the study of the life and work of René Descartes (1596-1650), held in 1996 at the INS Carles Riba (a Catalan high school), provided students with additional background from a mathematical, philosophical, physical and historical perspective.

As a collective action we may mention that since 2003 to the present, every year Pere Grapi and Mª Rosa Massa have coordinated a workshop on the History of Science and Teaching organized by the Catalan Society of History of Science and Technology (SCHCT), and subsequently they have coordinated the publication of the proceedings. The aim of these workshops is to enable teachers to show their experiences in the classroom as well as to discuss the criteria and conditions for these implementations.

In the academic year 2007-2008, the Department of Education of the government of Catalonia introduced some contents of the history of science into the curriculum for secondary education, namely, the new Catalan mathematics curriculum for secondary schools, published in June 2007, contains notions of the historical genesis of relevant subjects into the syllabus.[2]

In the academic year 2009-2010 a new course has begun for training future teachers of mathematics. The syllabus of this Master's degree launched at the University includes a part on history of mathematics and its use in the classroom. For example, in Polytechnic University of Catalonia the title is: "Elements of history of mathematics for the classroom" and in Pompeu Fabra University the title is: "The history of mathematics and its use to teach math". Also in the academic year 2009-2010, an online pilot course on the history of science for science teacher training was put into practice. This course was produced by historians of science of the Catalan Society for the History of Science and Technology (SCHCT) under the name "Science and Technology through History"

The setting up of the Group of History of Mathematics of Barcelona (ABEAM) in 1998[3] was also a significant step. The aim of this group of teachers of Mathematics is to create History of Mathematics materials to be used in the classroom. The list of the texts implemented includes: *On the sizes and distances of the Sun and Moon* by Aristarchus of Samos (ca. 310-230 BC) (Massa Esteve, 2005b; Aristarco, 2007); Euclid's *Elements* (300

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[2]The list of these historical contexts includes: The origins of the numeration system; the introduction of zero and the systems of positional numeration; geometry in ancient civilizations (Egypt, Babylonia); the first approaches to the number π (Egypt, China and Greece); Pythagoras theorem in Euclid's *Elements* and in China; the origins of symbolic algebra (Arab world, Renaissance); the relationship between geometry and algebra and the introduction of Cartesian coordinates; the geometric resolution of equations (Greece, India, Arab World); the use of geometry to measure the distance Earth - Sun and Earth - Moon (Greece).

[3]The coordinator of the group is Mª Rosa Massa Esteve and the other members of group are**:** Mª Àngels Casals Puit (INS Joan Corominas), Iolanda Guevara Casanova (INS Badalona VII), Paco Moreno Rigall (INS XXV Olimpíada), Carles Puig Pla (UPC) and Fàtima Romero Vallhonesta (Inspecció d'Educació). The group is subsidized by the *Institute of Science of Education* (ICE) of the University of Barcelona.

BC) (Romero, Guevara, Massa, 2007); Menelaus' *Spheriques* (ca. 100) (Guevara, Massa, Romero, 2008a-2008b); *Almagest* by Ptolemy (ca. 85-165) (Romero, Massa, 2003); *The nine Chapters on the Mathematical Art* (s. I. AC) (Romero *et al.*, 2009); *Traité du quadrilatère* by Nassir-al-Tusi (1201-1274) (Romero, Massa, Casals, 2006) and *Triangulis Omnimodis* by Regiomontanus (1436-1476) (Guevara, Massa, 2005). The essential ideas on the implementation of history in the mathematics classroom of this group are reflected in the following sections.

2 Teaching mathematics using its history

The History of Mathematics in the mathematics classroom can be used in two ways: as an integral educational resource and as a didactic resource for understanding mathematics (Jahnke *et al.*, 1996; Barbin, 2000; Massa Esteve, 2003-2010; Dematté, 2006;).

In the first sense, history in the mathematics classroom can provide students with a conception of mathematics as a useful, dynamic, human, interdisciplinary and heuristic science.

- A useful science. Teachers should explain to students that mathematics has always been an essential tool in the development of different civilizations. It has been used since antiquity for solving problems of counting, for understanding the movements of the stars and for establishing a calendar. In this regard, there are many examples right down to the present day in which mathematics has proved to be fundamental in spheres as diverse as computer science, economics, biology, and in the building of models for explaining physical phenomena in the field of applied science, to mention just a few of the applications.
- A dynamic science. It will also be necessary to teach students whenever appropriate about problems that remained open in a particular period, how they have evolved and the situation they are in now, as well as showing that research is still being carried out and that changes are constantly taking place. History shows that societies progress as a result of the scientific activity undertaken by successive generations, and that mathematics is a fundamental part of this process.
- A human science. Teachers should reveal to students that behind the theorems and results there are remarkable people. It is not merely a question of recounting anecdotes, but rather that students should know something about the mathematical community; human beings whose work consisted in providing them with the theorems they use so frequently. Mathematics is a science that arises from human activity, and if students are able to see it in this way they will probably perceive it as something more accessible and closer to themselves.
- An interdisciplinary science. Wherever possible, teachers should show the historical connections of mathematics with other sciences (physics, biology, medicine, architecture, etc.) and other human activities (trade, politics, art, religion, etc.). It is also necessary to remember that a great number of important ideas in the development of science and mathematics itself have grown out of this interactive process.
- A heuristic science. Teachers should analyze with students the historical problems that have been solved by different methods, and thereby show them that the effort involved in solving problems has always been an exciting and enriching activity at a personal level. These methods can be used in teaching to encourage students to take an interest in research and to become budding researchers themselves.

In the second sense, the history of mathematics as a didactic resource can provide tools to enable students to grasp mathematical concepts successfully. The History of mathematics can be employed in the mathematics classroom as an implicit and explicit didactic resource.

The history of mathematics as an implicit resource can be employed in the design phase, by choosing contexts, by preparing activities (problems and auxiliary sources) and also by drawing up the teaching syllabus for a concept or an idea.

Nevertheless, it is necessary to bear in mind that the historical process of building up a body of knowledge is a collective task that depends on social factors. In the past, many mathematicians adopted the solution of particular problems as the aim of their research and were able to devote many years to their objectives. It is worth remembering that our students, while having the ground before them well-prepared, are addressing these notions for the very first time and often lack the motivation for solving mathematical problems.

Indeed, it is not history itself that is relevant for teaching, but rather the genesis of problems, the proofs that favored the development of an idea or a concept. The clarification of this development of ideas and notions can also act as a motivation for solving current problems. The evolution of a mathematical concept can thereby reveal the learning difficulties encountered by students, as well as pointing the way towards how the concept can be taught (Massa Esteve, 2005a).

In addition to its importance as an implicit tool for improving the learning of mathematics, the history of mathematics can also be used explicitly in the classroom for the teaching of mathematics. Although by no means an exhaustive list, we may mention four areas where the history of mathematics can be employed explicitly: 1) for proposing and directing research work at baccalaureate level using historical material[4]; 2) for designing and imparting elective subjects involving the history of mathematics; 3) for holding workshops, centenaries and conferences (Massa Esteve, Comas, Granados, 1996), and 4) for using significant historical texts in order to improve understanding of mathematical concepts (Massa Esteve, Romero, 2009). In this article we analyze the last point.

3 Using significant historical texts in the mathematics classroom: Criteria and conditions

The use of significant historical texts in the classroom to facilitate the understanding of mathematical concepts is the activity that can provide students with more valuable means for learning mathematics.

The main aims for its implementation in the mathematics classroom are to achieve that students a) know the original source on which the knowledge of mathematics in the past is based, b) recognize the socio-cultural relations of mathematics with the politics, religion, philosophy or culture in a certain period and, last but not least, c) improve mathematical thinking through the reflections on the development of mathematical thought and the transformations of natural philosophy.

What historical texts are suitable for use in the mathematics classroom? Not all historical texts are useful for the mathematics classroom. Initial selection could be historical texts related to the historical contexts in the new Catalan curriculum. Historical texts (e.g. proof or problems) have to be in some way anchored in the mathematical issue. Different types of

[4]The list of titles of these research works can be very long, for instance: Pythagoras and music, On Fermat's theorem, On Pascal's Arithmetic Triangle, On the beginning of algebraic language, Women and science, On the incommensurability problem, Scientific Revolution,…

historical texts should be used, depending on the step in the didactical sequence.

At what moment in the teaching process should we use historical texts in the mathematics classroom? Historical texts could be used to introduce a subject or a concept, to explore it more deeply, to explain the differences between two contexts, to motivate study of a particular type of problem or to clarify a process of reasoning.

How do we use historical texts in the mathematics classroom? We must keep in mind some points: It is necessary to clarify the relation between the historical text and the mathematical concept under study, so that the analysis of the text or significant proof should be integrated into the mathematical ideas one wishes to convey. The mathematical reasoning behind the proofs should be analyzed. Indeed, addressing the same result from different mathematical perspectives enriches students' knowledge of mathematical understanding. The proof should be contextualized within the mathematical syllabus by associating it with the mathematical ideas studied in the course so that students may see clearly that it forms an integral part of a body of knowledge, and it should also be situated within the history of mathematics to enable students to evaluate the historical development of the concept.

In order to use historical texts properly, teachers are required to present some features of the historical period and also to describe historical figures in context, both in terms of their own objectives and the concerns of their period. Situating authors chronologically enables us to enrich the training of students by showing them the different aspects of the science and culture of the period in question in an interdisciplinary way. It is important not to fall into the trap of the amusing anecdote or the biographical detail without any historical relevance. It is also a good idea to have a map available in the classroom to situate the text both geographically and historically.

As an example, we now provide a description of the implementation process of a significant historical text in the mathematics classroom.

3.1 Aristarchus of Samos in the mathematics classroom

The activity we are going to present deals with the work *On the Sizes and Distances of the Sun and Moon* (ca. 287 BC) by Aristarchus of Samos (ca. 310-230 BC). In order to implement the activity, we begin with a brief presentation of the epoch, Greek astronomy, and the person of Aristarchus himself. Then we situate his work in the history of trigonometry, analyze the aims of the author as well as the features of the work, and finally students are prompted to follow the reasoning of this work, Proposition 7, in order to arrive at new mathematical ideas and perspectives (see this material in annex). This classroom activity was implemented in the last cycle of compulsory education (14-16 year old).

3.1.1 The context: Greek astronomy

As many historians point out, the history of Greek astronomy probably began as part of the history of Greek philosophy, so that the first great philosophers were also the first astronomers. Thus we can quote, for example, Thales (ca. 624-547 BC), Pythagoras (ca. 572-497 BC), Eudoxus (ca. 408-355 BC), Aristotle (ca. 384-322 BC), etc.

According to Heath, Thales, known as an astronomer, predicted and explained the causes of an eclipse of the Sun; he understood the Moon and the Sun as disks or short cylinders that behaved as if floating in water (Aristarchus, 1981, pp. 137-138).

Other developments took place with Pythagoras and his followers, who recognized that the Earth was a sphere and that Venus, the evening star, was the same planet as Venus, the morning star. The motion of the Earth, the Sun, the Moon and the planets around a central fire was also a theory attributed to a disciple of Pythagoras, Philolaus of Crotona (ca. 470 BC).

Subsequently, Eudoxus proposed a theory of homocentric spheres to describe the motion of celestial bodies. He surmised that the Earth sat motionless at the center and the planets (including the Sun and the Moon) followed a circular course around it.

Aristotle, whose texts had great influence, analyzed observable realities and reconstructed universe theory in cosmology, integrating many of the ideas of his predecessors such as the geocentric, the structural framework of the universe of the two spheres, the Platonic principle of circular motion and even of the heavenly bodies, as well as the pre-Socratic theory of the four elements (Puig-Pla, 1996, pp. 41-55). He laid the groundwork for what we now call old physics and the basic lines of his doctrine were accepted as dogma for about sixty generations. Available sources that refer to the principles of natural philosophy of Aristotle are the eight books of *Physics*. Astronomical issues are discussed mainly in the four books of *De Caelo* and *Meteorology*. In fact almost all Greek, Arab and Christian astronomers accepted, implicitly or not, the fundamental premises of Aristotelian cosmology: the closed and finite nature of the cosmos, the immobility of the earth at the center of the universe and the essential difference between the two regions: the blue (superlunary) and terrestrial (sublunary).

Aristarchus of Samos, who appears between Euclid (ca. 300 BC) and Archimedes (287-212 BC), was one of the rare exceptions and put forward heliocentric ideas of the universe, as discussed later. However, Aristarchus, in his work *On the sizes and distances of the Sun and Moon*, used the geocentric theory and was one of the first to write a work that estimated the sizes of the Sun and Moon in relation to those of Earth and the distances from them to Earth.

3.1.2. The historical author: Aristarchus of Samos.

Aristarchus was born on the island of Samos in 310 BC. He was a pupil of Straton of Lampsacus, the third director of the Lyceum, the school founded by Aristotle. Little is known about his life. The limited information available is determined by the quotations found in later texts and the works he left behind. Thus, Ptolemy, in his *Almagest* (150), also called *Syntaxis Mathematics*, explains that Aristarchus observed the summer solstice in 280 BC. Ptolemy in the first paragraph of Book III of his work also mentions Aristarchus' procedures to determine the length of the solar year (Ptolemy, 1984, pp. 137-139). Later, Nicholas Copernicus (1473-1543) in his *De revolutionibus orbium coelestium* libri VI (1543) explains the observations of Aristarchus at Alexandria with Timocharis of Alexandria (approx. III BC) and Aristyllus (disciple of Timocharis).

Although he was recognized as an astronomer in his earlier works, in his time Aristarchus was called the "mathematical", and is quoted as one of the few men having a thorough knowledge of all branches of science, geometry, astronomy, music, ... For instance, Vitruvius (first century BC) mentions him in his work *De Architectura* (ca. 25 BC). On the other hand, we have no doubt that he was a very able mathematician, as is evident in the work of astronomy that he has bequeathed us. He also wrote about vision, light and colors. He said the colors were "forms imprinting the air with impressions of their own true nature."

However, Aristarchus is mostly known as the "old Copernicus." Historians are unanimous in stating that Aristarchus was the first to introduce the heliocentric hypothesis. Archimedes (Archimède, 1971, p. 135), his contemporary, affirms in a passage of his work *Sandreckoner* (216 BC) that it could be deduced that Aristarchus assumed that the spheres of the stars and the sun remained motionless in space and that the Earth revolved around the Sun. Aristarchus compared the sphere of fixed stars with the orbit of the Earth. In this sense, he is also quoted by Plutarch (ca. 46-125) in his work *Moralia*, which says that Cleanthes thought that Aristarchus should be attacked for moving the Earth to the center of the universe; that is to say, it was assumed that Aristarchus included the motion of the Earth in his theories. However, despite these references, Aristarchus in his work *On the sizes and distances of the Sun and the Moon* does not mention the heliocentric hypothesis. This hypothesis suggests that in his work he recognized that the Sun is much larger than the Earth and Moon and is much farther from Earth than the Moon. We now consider the content of this work in more detail.

3.1.3 The historical work: *On the sizes and distances of the Sun and Moon*.

This work according to Pappus was found in a collection of texts called *Small Astronomy* together with other works like: *On the sphere in mouvement* by Autólico of Pitania (ca. 320 BC), *Optic and Fenomena* by Euclid, *Sphaerics* and *De diebus et noctibus* by Teodosio (ca. 107-43 BC) and others.

From six hypotheses about the sizes and distances to the stars, and by means of eighteen propositions, Aristarchus demonstrates three theses. The contents of the first three hypotheses could be stated thus: the first says that the moon receives its light from the Sun; the second explains that the Earth is the center of the sphere in which the moon moves, and in the third he describes the maximum circle defining the parts of darkness and brightness of the moon that are in the direction of our eye. These scenarios therefore provide no angle and no measure, but rather describe the positions of the stars. The other three assumptions provide measurements probably obtained by observation. Thus, the fourth hypothesis means that when the moon is at right angles to the Sun and the Earth, the viewing angle of the Moon from Earth is 87 degrees, and the other angle of the triangle measured one-thirtieth (1/30) a quadrant (90 degrees), or 3°; the fifth gives that the size of the Earth's shadow is twice the Moon, and the sixth and final hypothesis explains that the moon as seen from Earth forms a cone, with an angle of 2°, which is one fifteenth of a zodiac sign (30 degrees).

The three theses found at the beginning of the book are: first, the distance of the Sun from the Earth is greater than eighteen times, but less than twenty times, the distance of the Moon from the Earth; the second, the diameter of the Sun has the same ratio as the diameter of the Moon, and the third, the Sun's diameter has to the diameter of the Earth a ratio greater than that which 19 has to 3, but less than that which 43 has to 6. The proof of Proposition No. 7, which is the first thesis, is the one we use to design the classroom activity.

Its rigor in reasoning was not accompanied by correct observations, and noted an angle of 87 degrees, when in fact it is almost 90 degrees. In fact Pappus says in his commentary that Aristarchus deduced previous relations, assumptions and comments, and said that changing the assumptions also changed the measurements obtained. So later Hipparchus and Ptolemy deduced other relationships approximating those we know today.

3.1.4 The significant text

A comprehensive evaluation of this work of astronomy must take into consideration the close relationship between the beginnings of astronomy and the origins of trigonometry, an aspect that contributes to a greater understanding. In the text, Aristarchus posed plane geometry problems, cutting the spheres of the Sun and Moon into maximum circles. To solve geometric problems, he relied on relationships, today regarded as trigonometric, between angles and sides of a triangle. The angles are expressed as fractions of the square and he wrote the trigonometric ratios as ratios of sides of triangles, so the upper and lower bounds of the value required can be determined.

Aristarchus' geometrical propositions are mainly based in Euclid's *Elements*. Eudoxus' theory of proportions in Book V of the *Elements* is used consistently, and its properties of inverting, alternating, composing and multiplying are implemented for equal proportions and also for inequality proportions. Aristarchus also bases these propositions implicitly on other relationships, which for us are trigonometric, as if he knew these relations or considered them to be trivial.

The text is a collection of propositions consistent with a sequential description of the ideas one wishes to display, bearing in mind their objectives, i.e., to calculate the sizes and distances of the stars. The propositions are mathematical exercises and operations between ratios and the singular construction of geometrical figures are evidence of the high quality of this mathematician. This is a rich, well structured text, and in our opinion the proofs are flawless in terms of rigor.

3.1.5. The significant proof: the proposition 7

In this activity, students are prompted to reproduce Proposition 7, which deals with the ratios between the distance of the Sun and the Moon from the Earth.

Aristarchus, *Proposition 7*

"The distance of the Sun from the Earth is greater than eighteen times, but less than twenty times the distance of the moon from the Earth" (Aristarchus, 1981, p. 377).

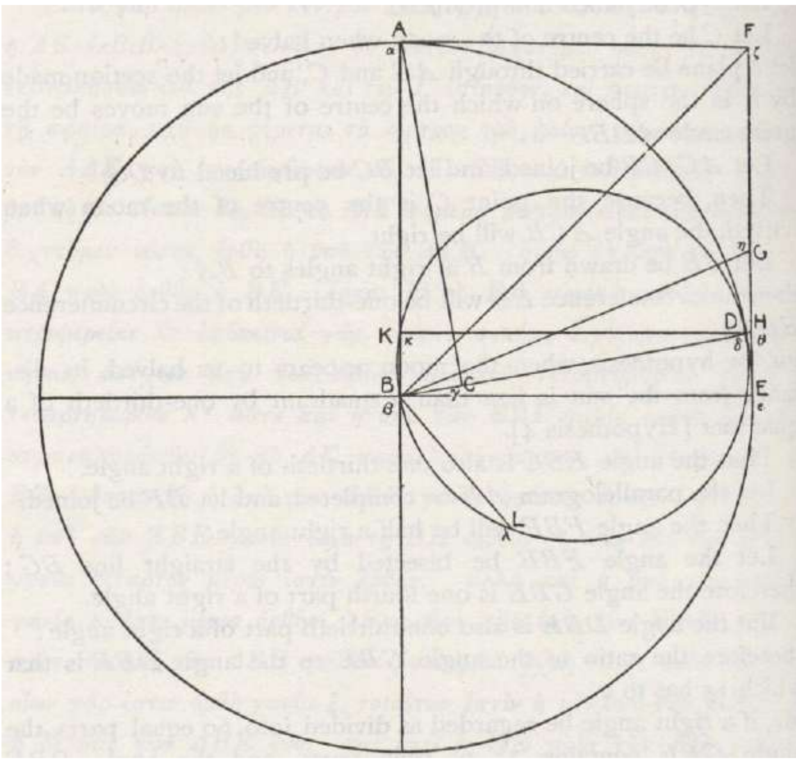

Figure 1. Aristarchus' Illustration (Aristarchus, 1981, p. 378)

In fact, B being the centre of the Earth, A the centre of the Sun and C the centre of the Moon (see Figure 1), Aristarchus showed:

$$1/18 > \sin 3^{\circ} = CB : AB > 1/20.$$

Aristarchus translated the problem from the triangle ABC to the constructed similar triangle BHE. He based his proof on the value of the angle ABC, which he stated as 87º and which he determined by observation (see the comment in the Section 3.1.3).

The angle FBE is 45º and the angle GBE the half, then the ratio between two angles GBE and DBE, which is 3º, is

$$GE : HE > (GBE) : (DBE) = 15 : 2$$

As $FB^2 = 2\ BE^2$, he used proportions in similar triangles and then $FG^2 = 2\ GE^2$. Using the inequality: $50 : 25 = 2 > 49 : 25$. So:

$$FG : GE > 7 : 5 \text{ and } FE : GE > 12 : 5 = 36 : 15.$$

Therefore by composing FE : GE with GE : HE we obtain FE > 18 HE and BE > 18 HE. Then, since the triangles ABC and BHE are similar, AB > 18 CB.

In the development of the implementation of the activity in the classroom we present Aristarchus in the context of Greek astronomy and the historical work: *On the sizes and distances of the Sun and Moon*. Then we analyze the text or significant proof by making a careful assessment of the characteristics and the mathematical reasoning behind the proofs. In this case, together with the students, after the implementation of the dossier (see annex), we should analyze the four mathematical strategies used by Aristarchus in this proof. The translation of the analysis of the triangle Sun-Earth-Moon to a similar triangle, the use of the relationship, as if it were trivial, between the tangents and angles (in actual notation $\text{tg}\,\alpha : \text{tg}\beta > \alpha : \beta$, with angles β, α of the first quadrant), the establishment of a ratio between the segments which determines the angle bisector and the sides of the triangle (using a proposition from the *Elements*), and lastly the approach of $\sqrt{2}$ by 7 : 5. At the end, students move the result obtained in the similar triangle to the initial triangle ABC, Sun-Earth-Moon and conclude that AB > 18 CB.

4 Remarks

We have designed activities related to several topics, in geometry, trigonometry and algebra, which may also contribute to improving the students' mathematical education.

According to the criteria and conditions mentioned above, we have created new teaching materials containing explanations that use a constructive learning method. This implementation in the mathematics classroom is designed for students to follow their own reasoning, just like the old mathematics did. The analysis of significant proofs reveals the different ways those students have of working and approaching problems, thereby enabling them to tackle new problems and to develop their mathematical thinking. The analysis of historical texts improves the students' overall formation, giving them additional knowledge about the social and scientific context of those periods.

Thus we may conclude that the use of the history of mathematics in our teacher training courses, both at the initial and permanent stages, should extend the knowledge of teachers, should enrich the learning of students and should improve the quality of mathematical training as a whole.

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On-line course: http://www.xtec.cat/formaciotic/dvdformacio/materials/tdcdec/index.html

ANNEX

DOSSIER ALUMNES ARISTARC

SOBRE LES MIDES I LES DISTÀNCIES DEL SOL I LA LLUNA (260 aC)

En aquesta obra, Aristarc de Samos (310 aC- 230 aC) parteix de sis hipòtesis sobre les mides i distàncies als astres i mitjançant divuit proposicions demostra tres tesis. Anticipant-se als mètodes trigonomètrics posteriors Aristarc va ser el primer en desenvolupar procediments geomètrics per aproximar els sinus d'angles petits.

Les hipòtesis en les que es basa són:

1.- La Lluna rep la llum del Sol.

2. - La Terra és com un punt al centre de l'esfera en la qual es mou la Lluna.

3.- Quan la Lluna se'ns mostra partida en dues parts, el gran cercle que separa la foscor i la claror de la Lluna s'inclina cap a la nostra visió.

4.- Quan la Lluna se'ns mostra partida per la meitat, llavors la mateixa Lluna s'allunya del Sol menys d'una quarta part (90º) en 1/30 part d'un quadrant (o sigui en 3º).

5.- L'amplada de l'ombra de la Terra es suposa com dues Llunes.
6. - La Lluna subtendeix una quinzena part d'un signe del zodíac (o sigui 1/15 part de 30º)
Diu Aristarc que amb aquestes hipòtesis pot provar la Proposició 7

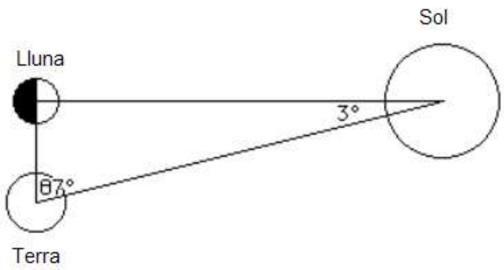
La distància al Sol des de la Terra és més gran que divuit vegades, però més petita que vint vegades, la distància a la Lluna des de la Terra.

És a dir:

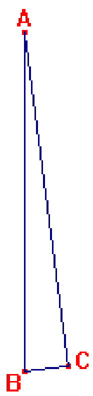
18 · distància Terra-Lluna < distància Terra-Sol < 20 · distància Terra-Lluna

Demostrarem la primera desigualtat :

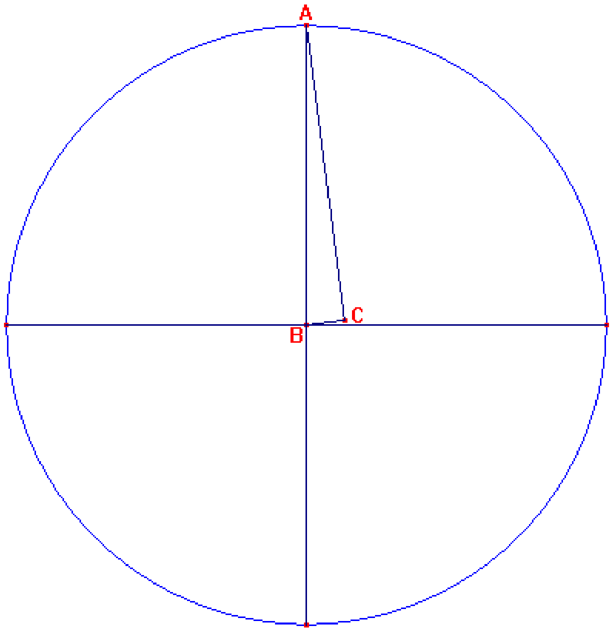
distància Terra-Sol >18 · distància Terra-Lluna



Demostració:
Sigui A el centre del Sol, B el centre de la Terra, i C el centre de la Lluna quan se'ns mostra partida per la meitat, llavors BC representa la distància Terra-Lluna i BA representa la distància Terra-Sol.
Reproduirem la demostració d'Aristarc pas a pas, construint progressivament el dibuix que ell mateix va utilitzar.
Considerem el triangle:



en el que A representa el Sol, B la Terra i C la Lluna quan se'ns mostra partida per la meitat.
El que volem demostrar és que BA >18·BC
Segons la hipòtesi 4, l'angle CAB és de 3º i l'angle BCA és recte.
Per tant l'angle ABC que mesura l'allunyament de la Lluna al Sol deduïm que és de …………
Construïu ara sobre el triangle següent la circumferència amb centre B i radi BA, prolongueu aquest radi per obtenir un diàmetre i traceu també el diàmetre perpendicular a aquest.



Anomeneu E l'extrem dret del diàmetre perpendicular al BA. Pel punt E aixequeu una perpendicular, prolongueu el costat BC i anomeneu H la intersecció d'aquestes darreres rectes.

Fixeu-vos ara en el triangle BEH que acabeu de construir i indiqueu el valor dels angles:

HBE: . . . . .

BEH: . . . . .

EHB: . . . .

Com són els triangles ACB i BEH? . . . .

Tot seguit Aristarc estudia el problema en aquest nou triangle BEH semblant a l'anterior i demostrarà que BH >18 · EH (1)

Completeu el dibuix construint el quadrat determinat pels costats AB i BE i anomeneu F el quart vèrtex. Traceu la diagonal BF del quadrat, llavors l'angle FBE val 45º. Traceu la bisectriu de l'angle FBE que tallarà al quadrat en G, llavors l'angle GBE val la meitat o sigui 90/4.

Fent la raó entre els dos angles GBE i HBE, s'obté: $\frac{GBE}{HBE} = \frac{90/4}{3}$ i simplificant: …… .

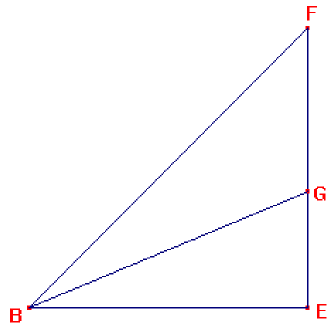
Diu Aristarc que, com que sabem que la raó entre els costats oposats a aquests angles és més gran que la raó entre ells, podem escriure que :

$$GE : HE > (GBE) : (HBE) = 15 : 2 \qquad (2)$$

Ara, aplicant el teorema més conegut referit als triangles rectangles, ... , *(indiqueu el nom d'aquest teorema)* al triangle rectangle isòsceles BEF, obtenim:

$$FB^2 = 2\,BE^2$$

Considereu ara el triangle BEF amb la bisectriu BG de l'angle FBE:



Traceu, des de G la perpendicular al costat BF i anomeneu E’ el peu d’aquesta perpendicular.
El triangle FE’G és rectangle i isòsceles.
Perquè l’angle GE’F és . . . . i els angles GFE’ i E’GF són tots dos de . . . . .
En aquest triangle isòsceles apliquem el teorema de Pitàgores, llavors

$$FG^2 = 2GE'^2$$

A més a més els dos triangles BEG i BE’G són iguals perquè tenen un costat comú . . . . i els tres angles iguals. O sigui que GE’=.... En aquestes condicions

$$FG^2 = 2GE^2$$

Aquesta darrera igualtat, llegida en forma de proporció diu: $FG^2 : GE^2 = 2$
i d’altra banda 2 = 50 : 25 > 49 : 25. Aristarc escriu *$FG^2 : GE^2 > 49 : 25$*. Tria aquests dos valors, 49 i 25 perquè són quadrats perfectes i traient l’arrel quadrada queda

FG : GE > 7 : 5.

D’altra banda com que FE = FG + GE , en lloc de treballar amb FG podem fer-ho amb FE, component la raó:
FE : GE = (FG + GE) : GE = FG: GE + GE : GE > 7 : 5 + 5 : 5 = 12 : 5 = 36 : 15
és a dir:

FE : GE > 36 : 15

Recuperem de (2) la raó GE : HE > 15 : 2 . Multiplicant-la per l’anterior obtenim

FE : HE > . . . .

És a dir que FE > 18 HE
Retornem ara al quadrat ABEF. FE = BE, i per tant, podem escriure BE > . . . . .
Com que BH és la hipotenusa i BE és un catet del triangle BEH, BH . . . . BE
i, per tant, ... i queda demostrada la desigualtat (1).
Per acabar, recordem que el triangle BEH era una construcció addicional per estudiar les desigualtats trobades però el nostre triangle original Sol-Terra-Lluna era BC.
Quina relació hi havia entre el triangle ACB i el BEH? . . . .
Quin seria el costat corresponent al BE en el triangle ACB? . . . .
Quin seria el costat corresponent al HE en el triangle ACB? . . . .
I ara la conclusió final.
En aquestes condicions la desigualtat BH > 18 HE del triangle BEH implica la desigualtat . . . . en el triangle ACB
Escriviu una frase final que, utilitzant la darrera desigualtat, expliqui la relació entre la distància del Sol a la Terra i la distància de la Lluna a la Terra
. . . .

USE OF THE HISTORY OF NEGATIVE NUMBERS IN EDUCATION

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ABSTRACT

It is well known that pupils have many conceptual difficulties with negative numbers. The notation of negative numbers used in ancient China employed two colors (red and black) to represent positive and negative numbers. In this way it was possible to give clear semantic interpretation of the various rules of calculation (especially the rules of multiplication). In the paper I describe the use of the notation of negative numbers by colored sticks to help children clarify the rules for calculation with negative numbers.

1 The problems with negative numbers

An important motivation for studying the history of negative numbers is the fact that many pupils have problems with them. There are common mistakes in calculations caused by formal learning of the rules of counting with negative numbers and such mistakes are very frequent not only at primary and secondary schools but also at colleges. Some of the possible reasons which leading to these problems are that pupils have just few opportunities to deal with negative numbers in their real life, teachers neglect the development of intuitive perception of negative numbers at primary schools, sometimes teachers tell the pupils who did not learn negative numbers that 2 minus 5 equals 0 and pupils learn the rules without comprehension of the concept of a negative number.

2 Literature survey

Our main question is how the use of the history of mathematics to avoid the problems pupils have with the understanding of negative numbers. Professor Milan Hejný in his Theory of the Mathematics Education (Hejný, 1989) describes formal knowledge as knowledge of rules learned by heart without real understanding. When we want to eliminate formalism we can search for inspiration in the history of mathematics. By an analysis of the history of mathematics we can receive useful ideas about the genesis of thinking which we can try to apply in teaching. (Hejný 1989, p.25)

Various answers to our question can be found in the book: *Mathematics Education, The ICMI Study,* edited by John Fauvel and Jan van Maanen (Fauel and van Maanen, 2000). Here we can find the results of experience with the use of history of mathematics in learning and teaching; different ways in which the history of mathematics might be useful; scientific studies of its effectiveness and many other information. We can find also passages on the use of the history of negative numbers.

Using the history of mathematics brings several other advantages. These advantages are mentioned by Tzanakis & Arcavi in the seventh chapter of that book:

> "An advantage of implementing history in the presentation and learning of mathematics is the opportunity it presents to appreciate and make explicit use of the constructive role of (i) errors, (ii) alternative conceptions, (iii) changes of perspective concerning a subject, (iv) paradoxes, controversies and revision of

> implicit assumptions and notions, (v) intuitive arguments, that appeared historically and may be put to beneficial use in the teaching and learning of mathematics, either directly or didactically reconstructed (Tzanakis & Arcavi and 2000, p.238)

Relations between epistemology, history of mathematics, and teaching and learning of mathematics we can find in several other papers as for instance: Historical Epistemology and the Teaching of Mathematics: Towards a Socio-Cultural History of Mathematics written by Luis Radford in 1997, Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics written by Fulvia Furinghetti and Luis Radford in 2002 and Historical conceptual developments and the teaching of mathematics: from phylogenesis and ontogenesis theory to classroom practice written by Fulvia Furinghetti and Luis Radford in 2008. The authors place the relation between historical development of mathematics and its education into a broad evolutionary and psychological perspective.

3 The significant points from the history of negative numbers

Negative numbers in ancient China

The oldest notations of numbers from China can be found on magical cubes from the 14th to the 11th century BC and on ceramic or bronze objects and coins from the 10th to the 3rd century BC. From the 4th century BC the Chinese used rods in their calculations.
In notation they expressed the value by means of real rods.

	1	2	3	4	5	6	7	8	9
ones, hundreds, ten thousands …	𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨
tens, thousands, hundred thousands…	𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱

Figure 1 Chinese numerals

The counting with these numbers had a positional character and this way of counting - counting with rods – is the oldest positional decimal system.

A very important mathematical work in China was Mathematics in Nine Chapters (see Hudeček 2008, and Kangshen et al., 1999). It is a collection of 246 problems with solutions which was prepared from older sources between the years 206-255 BC. In the third century AD Liu Hui added his commentary to this work. The eighth book of the Mathematics in Nine Chapters deals with solving systems of equations. For example:

> “Now sell 2 cattle [and] 5 sheep, to buy 13 pigs. Surplus 1000 cash. Sell 3 cattle [and] 3 pigs to buy 9 sheep. [There is] exactly enough cash. Sell 6 sheep, [and] 8

pigs. Then buy 5 cattle. [There is] 600 coins deficit. Tell: what is the price of a cow, a sheep and a pig, respectively?"

And below we can find the answer and the method of solution:

"Cattle price 1200, sheep price 500, pig price 300.
Method: Use the Array Rule rule: lay down2 cattle, 5 sheep, positive; 13 pigs, negative; surplus coins positive; next 3 cattle, positive; 8 pigs, positive. Deficit couns, negative. Calculate using the Sign Rule." (Kangshen et al., 1999, p.405)

Here, in the eighth book of the *Mathematics in Nine Chapters* we come across negative numbers for the first time in the history of mathematics. Very important is *the Sign Rule* which describes the rules for adding and subtracting of positive and negative numbers. For these numbers red and black rods were used. Thus the black rods meant negative and the red rods positive numbers. Positive numbers were called "cheng" and negative numbers "fu". Liu Hui adds his commentary to this method:

"(c) Liu: Subtracting one column from another depends on appropriate entries with the same signs. Opposite signs (entries) are from different classes. If from different classes they cannot be merged but are subtracted. So merging red by black is to subtract black, merging black by red is to subtract red. Red and black merge to the original colour this is "adding"; they are mutually eliminating. This is by eliminating (the top entries) using addition and subtraction to achieve the bottom constant (shi).

The prime purpose of the rule is to eliminate the first entry: magnitudes of entries in other position of no concern, either subtract or merge them. The reasoning is the same, not different.
(d) "Without extra" is "without merging". When nothing can be subtracted put the subtrahend in its place (with colour changed), subtracting the resulting constant (lu shi) from the bottom constant. This rule is also applicable to columns whose entries are mixed signs. In the rule, entries with same signs subtract their constant terms, entries with opposite signs add their constant terms. This is positive without extra, make it negative; negative without extra, make it positive." (Kangshen 1999, p.405)

Diophantos, 3rd century AD

Mathematicians encountered many difficulties in dealing with negative numbers. This can be seen for example in the case of Diophantos, for whom a negative number was just an operator. He chose coefficients in equations only in the way that he could get rational positive solutions. When an equation had two positive roots, he chose just one of them – the higher one and when there were two irrational or negative roots he declared that the equation had no solution.

India, 7th century AD

Indians dealt with negative numbers and they did not restrict themselves just to adding and subtracting. In their works we can find rules also for multiplication, division and second power. And so they moved debts to a more abstract one perception.

A quotation from ***Brahmagupta***´s work gives evidence of this:
"Negative, taken from cipher, becomes positive; and affirmative, becomes negative. Negative, less cipher is negative; positive, is positive; cipher nought. When

affirmative is to be subtracted from negative, and negative from affirmative, they must be thrown together" (Murray 1817, p. 340).

Cardano, 16th century AD

Cardano accepted also negative numbers as roots of equations but he called them fictious while positive solutions were called true.

"It will be remembered also that 9 is derivable [by squaring] equally from 3 and -3, since a minus times minus produces plus. But in the case the odd powers, each keeps its own nature: it is not plus unless it derives from a true number, and a cube whose value is minus or we call debitum, cannot be produced by any expansion of a true number."
"...For example: If
$x^2 + 4x = 21$
the true solution is 3 and the fictitious one -7" (Cardano 1545, p. 9-10)

Descartes, 17th ctr. AD

Descartes accepted negative numbers partially. He called negative roots false and showed that equations with negative roots can be transformed into equations with positive roots. Based on this fact he was willing to accept negative numbers.

Newton 17th and 18th century AD

A work of Issac Newton (1643-1727) *"Arithmetica universalis, sive de compositione et resolutione arithmetika liber"*- ***Universal Arithmetic or a Treatise of Arithmetical composition and resolution*** makes the issue of negative numbers clear. Newton says that quantities are either positive, or greater than nothing, or negative, or less than nothing. He took zero for nothing. He interpreted negative numbers as negative motion.

„IV. Quantities are either Affirmative, or grater than nothing, or Negative, or less than nothing. Thus in human Affairs, Professions or Stock may be called affirmative Goods, a Debts negative ones. And so in local motion, progression may be called affirmative motion, and Regression negative motion, because the first augments, and the other diminishes the Length of the Way made. And after the fame Manner in Geometry, if a line draw any certain Way be reckoned for affirmative then." (Newton 1720, p.3)

As we could see, negative numbers were only fictitious or false for mathematicians until the seventeenth century and many of them did not accept a negative number as a solution of an equations. This information could be helpful for teachers to understand that pupils can have problems with negative numbers despite the fact that they know the rules for counting with them. To help them to overcome these problems we can use some tasks from the original mathematical works in a class.

4 CHINESE NUMERALS IN A CLASS

I used the counting with Chinese rods as an activity in a class. Until the last year the pupils in Slovakia learnt all of the operations with negative numbers in the sixth grade of primary school. The new reform moved this topic to the beginning of the eighth grade.

For this activity – counting with Chinese rods – I chose two classes of the sixth grade at a primary school (12 years old children). These pupils had not dealt with negative numbers before.

I used the following aids:

- *Red and black rods*

They were made of hard paper or soft plastic and prepared by the teacher. If there is enough time for that, pupils can make them themselves.

- *Working sheet:*

The working sheet contained a brief history of Chinese mathematics with information needed for the work with rods together with a well-arranged table with Chinese numerals and some space for notes.

The phases of the activity were:

1. Narration about the history of Chinese mathematics

– A motivational narration about how Chinese used to count.

2. Explanation how numerals looked like in China.

– I wrote the Chinese numerals on the blackboard and, together with the pupils we wrote several numbers using these numerals. The pupils participated actively.

3. Create several numbers by means of the "Chinese rods".

– The pupils put together numbers by means of rods which they had on their desks.

4. Discussion about debts.

– I discussed with the pupils how to write down a debt, which colour symbolises a debt and I compared it with China.

5. Counting with red and black rods

– I will show specific assignments and their solutions in a while

6. Competition between four-member groups.

– After the pupils understood the principle of counting with coloured rods there was a competition between groups. These groups assigned tasks to each other. If a group found the right solution, it got a point. If the group with an assigned task did not find the right solution, the group which assigned the task could get the point if it found the right solution.

7. Feedback

Creating several numbers by means of the "Chinese rods":

We tried to express the number four by means of coloured rods in various ways. These are some of them:

4 red rods:

I I I I

5 red rods and 1 black rod:

I I I I I I

6 red rods and 2 black ones:

I I I I I I I I

We also dealt with the number minus 2

How can we express the number -2?

I I

I I I I

I I I I I I

I I I I I I I I I I I I

Then we have here an example of several assignments:

- 7 red rods plus 4 black rods 7+(-4)

- 7 black plus 13 black (-7)-(-13)

- 4 red minus 12 black 4-(-12)

At last I showed the pupils how the counting with a counting board look like:

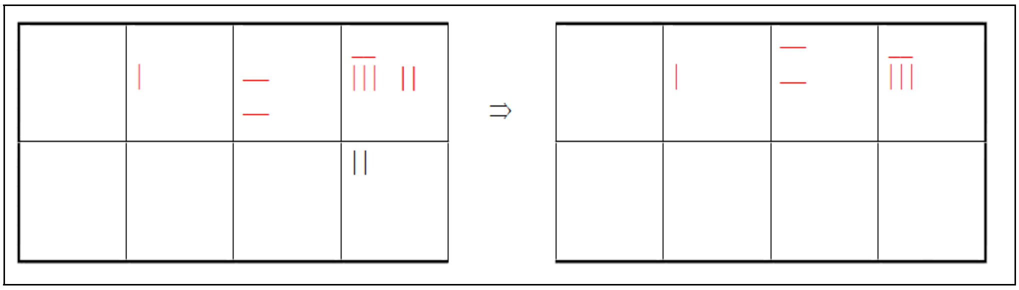

Figure 2

130+(- 2) = 128

Some facts from the feedback:

a. Motivational factor – the pupils were interested in the brief history, they asked for some more facts.

They were surprised that the Chinese could count already such a long time ago.

The pupils noticed that the zero was missing among the numerals.

b. Cultural context – the pupils associated the red colour with a debt but it was no problem for them to deal with the black colour, which symbolised a debt in China. It caused bigger problems for the teacher.

c. During the class the rules counting with negative numbers was not disclosed by the teacher of.

The pupils were able to get the right solutions quite fast also in the assignments which they had never seen before, for instance subtracting negative number from a positive one.

5 CONCLUSION

Looking at the historical examples we can see that for the understanding of negative numbers it was not enough to know rules for counting with them. I think this is one of the problems of today´s education. We teach pupil rules and then when they come across negative numbers for the next time, we assume that they already know how to count with them. It is questionable if it is a solution to move the whole thematic unit into a higher class as it was done in Slovakia. It would be better to focus on propaedeutic and use the fact that today except the debt we also know the thermometer, the elevator, the altitude above sea level and so on and this is our advantage over the mathematicians of the past.

Cooperation between subjects is important as well. A discussion of the connection between Newton´s decelerated motion and negative numbers in physics classes could bring more light into the ideas of pupils. It would be interesting to discuss with pupils what meaning the negative roots have according to them or whether they have thought about it and compare their views with the historical ideas.

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THE ALGORITHMS OF POINCARÉ, BRUN, AND SELMER

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ABSTRACT

The idea of continued fractions in several dimensions has at least two roots. One is the idea of generalizing J. L. Lagrange's characterization of quadratic irrational numbers as periodic continued fractions. This path was followed by C. G. Jacobi. The other idea is to provide approximations to an n-tuple of numbers by rational numbers with a common denominator. This problem is deeply rooted in history and related to musical theory. Several proposals related to the names of Jacobi, Poincaré, Brun, Selmer, and others have been made. Due to the elementary nature of posing the problem one can use some original publications even in school.

1 Jacobi's attempt

One of the oldest algorithms is the Euclidean algorithm. Most probably its subtractive form is the original version (Euclid uses the verb $\alpha\nu\theta\upsilon\phi\alpha\iota\rho\epsilon\iota\nu$ 'to subtract reciprocally' to describe this operation; see Fowler 1987). One starts with two real numbers $a_0 > 0$ and $a_1 > 0$ with $a_0 \geq a_1$. Then we form $\sigma(a_0, a_1) = (a_0 - a_1, a_1)$ and if necessary we reorder to obtain a new pair. If $a_0 - a_1 > a_1$, then we take $a'_0 = a_0 - a_1$ and $a'_1 = a_1$. We take $a'_0 = a_1$ and $a'_1 = a_0 - a_1$ in the other case. It is possible to speed up this algorithm by replacing subtraction with division. This means that we form the pair $\delta(a_0, a_1) = (a_0 - ka_1, a_1)$. Here we put $k \geq 1$ the greatest integer such that the equation $a_0 - ka_1 \geq 0$ holds. Then in all cases $a_1 \geq a_0 - ka_1$ and we have a cyclic reordering. From the present viewpoint it is preferable to use *matrices* and to describe the algorithm in the form

$$\begin{pmatrix} a'_0 \\ a'_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}.$$

All important recursion relations can be derived easily by using the product of matrices. Rational numbers $\frac{p_n}{q_n}$ which are called the convergents of the algorithm will be obtained as follows.

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}$$

It is easy to see that the algorithm stops if and only if $a : b$ is rational.
The algorithm is homogeneous in the following sense. The pair (x_0, x_1) and the pair $(\lambda x_0, \lambda x_1)$, $\lambda \neq 0$ lead to the same algorithm. This property suggests using the inhomogeneous version

$$Tx := \frac{1}{x} - k,\ x = \frac{a_1}{a_0},\ Tx = \frac{a'_1}{a'_0},\ k = k(x) = [\frac{1}{x}].$$

If $a : b$ is not rational then the rational numbers $\frac{p_n}{q_n}$ are good approximations to the irrational value $\frac{a}{b}$. It is possible to make a short digression to the relation between matrices and fractional linear maps. In ancient Greece it was well known that geometric ratios can lead to periodic algorithms. The most common examples are the Golden Ratio and the square root of 2. Translated into present day language we put $\lambda = \frac{-1+\sqrt{5}}{2}$ and we find the relation

$$\lambda \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}.$$

The idea of periodicity therefore is related to eigenvalues and eigenvectors. In inhomogeneous notation we find

$$\lambda = \frac{1}{1+\lambda}.$$

The connection to the Farey sequence $1, 1, 2, 3, 5, \ldots$ is immediate. If $F_{n+1} = F_n + F_{n-1}$ then one sees

$$\frac{F_n}{F_{n+1}} = \frac{F_n}{F_n + F_{n-1}}$$

which shows

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \lambda.$$

There are two ways to explore periodicity. One may start with given irrational numbers like $\sqrt{2}, \sqrt{3}, \ldots$ or one starts at the other end. The periodic expansions

$$x = \frac{1}{k+x}$$

or more generally

$$x = \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x}$$

lead to quadratic irrational numbers. Last but not least one can state the famous result of Lagrange.

There are at least two roots for multidimensional continued fractions (see e. g. Schweiger 2006). One root is the attempt to extend Lagrange's theorem to n-tuples of irrational numbers (this was the main point in Jacobi 1868). The other problem is the question about approximation of an n-tuple of real numbers by rational numbers with a common denominator. This question is related to musical theory (a good introduction is Wright 2009; a broader discussion can be found in Assayag et al. 2002). Jacobi, Poincaré, Brun, Selmer, and others made various proposals for multidimensional continued fractions. In what follows we will shortly describe their ideas.

In the year 1868 E. Heine published the paper „Allgemeine Theorie der kettenbruchähnlichen Algorithmen, in welchen jede Zahl aus drei vorhergehenden gebildet wird" which he found in the legacy of G. G. J. Jacobi (1804-1851). Due to its complexity this paper is hardly suitable for use in a classroom. Jacobi considers three real numbers („unbestimmte Zahlen") a, a_1, a_2 and a sequence of given quantities („gegebene Grössen") $l, m, l_1, m_1, l_2, m_2, \ldots$. Then he defines

$$\begin{array}{rcl} a_3 & = & a + la_1 + ma_2 \\ a_4 & = & a_1 + l_1 a_2 + m_1 a_3 \\ a_5 & = & a_2 + l_2 a_3 + m_2 a_4 \\ \dots & \dots & \dots\dots\dots \end{array}$$

Translated into matrix theory his algorithm is much more comprehensible in the form

$$(a_1, a_2, a_3) = (a_0, a_1, a_2) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & l \\ 0 & 1 & m \end{pmatrix}.$$

As before one sees that the triplets (a_0, a_1, a_2) and $\lambda(a_0, a_1, a_2)$ determine the same algorithmic course. Later in the paper Jacobi chooses a different approach. Let u_0, v_0, w_0 be three positive numbers and define $l_0 = [\frac{v_0}{u_0}], m_0 = [\frac{w_0}{u_0}]$. Then the recursion starts with $u_1 = v_0 - l_0 u_0, v_1 = w_0 - m_0 u_0, w_1 = u_0$. With the help of matrices one sees

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} -l_0 & 1 & 0 \\ -m_0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix}.$$

Naturally one proceeds by iteration.

In his examples Jacobi puts the first coordinates $u_0, u_1, u_2, \ldots$ equal to 1. Three examples are given which lead to periodic expansions.

i)

$$\begin{array}{rcl} (u_0, v_0, w_0) & = & (1, \sqrt[3]{2}, \sqrt[3]{4}) \\ (u_1, v_1, w_1) & = & (1, \sqrt[3]{2} + 1, \sqrt[3]{4} + \sqrt[3]{2} + 1) \\ (u_2, v_2, w_2) & = & (1, \sqrt[3]{2} + 2, \sqrt[3]{4} + \sqrt[3]{2} + 1) = (u_3, v_3, w_3) \end{array}$$

ii)

$$(u_0, v_0, w_0) = (1, \sqrt[3]{3}, \sqrt[3]{9})$$

$$(u_2, v_2, w_2) = (u_4, v_4, w_4).$$

iii)

$$(u_0, v_0, w_0) = (1, \sqrt[3]{5}, \sqrt[3]{25})$$

This is a very awkward example because after some lengthy calculations one finds $(u_7, v_7, w_7) = (u_{13}, v_{13}, w_{13})$. Note that the original publication contains two printing errors at this point.

But now we are confronted with a difficult problem. Up to now it is even not known if the triplet $(\sqrt[3]{2}, \sqrt[3]{4}, 1)$ leads to a periodic expansion. The generalization of Lagrange's theorem remains an open problem.

Matrices provide a good representation for convergents by using the equation

$$\begin{pmatrix} P_{i+1} & P_{i+2} & P_{i+3} \\ R_{i+1} & R_{i+2} & R_{i+3} \\ Q_{i+1} & Q_{i+2} & Q_{i+3} \end{pmatrix} = \begin{pmatrix} P_i & P_{i+1} & P_{i+2} \\ R_i & R_{i+1} & R_{i+2} \\ Q_i & Q_{i+1} & Q_{i+2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & l_i \\ 0 & 1 & m_i \end{pmatrix}.$$

The initial values are given by the unity matrix for $i = 0$. If we consider the first example then we get $l_0 = 1, m_0 = 1$, $l_1 = 2, m_1 = 3$, $l_2 = 3, m_2 = 3$ and then the algorithm continues periodically with $l_3 = 3, m_3 = 3$, $l_4 = 3, m_4 = 3$. In this way we obtain a sequence of approximating fractions with common denominator, the convergents, for the pair $(\sqrt[3]{2}, \sqrt[3]{4})$, namely the sequence $(1,1), (\frac{4}{3}, \frac{5}{3}), (\frac{15}{12}, \frac{19}{12}), (\frac{58}{46}, \frac{73}{46}), \ldots$.

2 Brun, Selmer and Poincaré

Much more easy to read are the papers by Viggo Brun “Algorithmes euclidiens pour trois et quatre nombres” (Brun 1957) and “Euclidean algorithms and musical theory” (Brun 1964). A revealing epilogue on Brun’s contributions has been written by Scriba 1985. Clearly Brun’s paper from 1957 requires some knowledge of French. In fact the oldest papers by Brun on this subject appeared in Norwegian. The starting point is a triple (a_0, a_1, a_2) with $a_0 \geq a_1 \geq a_2 > 0$. Then we form $\sigma(a_0, a_1, a_2) = (a_0 - a_1, a_1, a_2)$ and reorder. There are three possibilities.

$$a_0' = a_0 - a_1, a_1' = a_1, a_2' = a_2$$

$$a_0' = a_0, a_1' = a_0 - a_1, a_2' = a_2$$

$$a_0' = a_0, a_1' = a_1, a_2' = a_0 - a_1.$$

Again the use of matrices is recommended. For practical purposes the inverse matrices which correspond to the three types of reorder are helpful.

$$B(0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B(1) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B(2) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

But now it is time to reveal the connections with musical theory which are also described in Scriba 1985. In the just intonation scale the ratio of the frequencies of a a tone to its octave is $1:2$. We further find the ratio $2:3$ for the (just) fifth and $3:4$ for the (just) fourth. Now you want to construct a musical instrument which has a fixed number of strings within an octave. If you think of the keyboard of a pianoforte these should be twelve strings which will become shorter related to the inverse ratio of the pitch. This could be a good opportunity to make a digression to the history of musical scales or to the music of the various cultures. Now we request that the ratio of two subsequent pitches is a fixed number. We look for a number λ such that

$$\lambda^x \approx 2,\ \lambda^y \approx \frac{3}{2},\ \lambda^z \approx \frac{4}{3}.$$

This leads to a problem of Diophantine approximation, namely to approximate the triple $(\log 2, \log \frac{3}{2}, \log \frac{4}{3})$ by three whole numbers. Using some decimal approximations Brun algorithm leads to the sequence 122010100001.... The wanted triples of numbers (x, y, z) can be found as the columns of the matrices.The expansion 1220101 gives

$(12, 7, 5)$ and the longer expansion 12201010000 the values $(53, 31, 22)$. In a similar way we can also start with the ratios for the octave, the fifth and the major third $(5:4)$. This means to approximate $(\log 2, \log \frac{3}{2}, \log \frac{5}{4})$. One finds the triple $(12, 7, 4)$.

In the Western musical practice this approximation is of considerable importance. It corresponds to a partition of the octave into twelve steps such that subsequent pitches change by the factor $\lambda = \sqrt[12]{2}$. Therefore, a fifth is given in the equally tempered scale by $\lambda^7 \approx 1,498$ instead of by $\frac{3}{2}$. In the equally tempered scale the fifth is approximately two *cents* flat. On the other hand the fourth $\frac{4}{3}$ is replaced by $\lambda^5 \approx 1,335$ which is approximately two *cents* sharp. A *cent* corresponds to the division of an octave into 1200 microtones which gives 1 cent $\approx 1,0005778$. This factor can be derived in a quite different way. One requires that seven steps of an octave are the same as 12 steps of a fifth. In fact this is not true in the just scale, since 2^7 is different from $(\frac{3}{2})^{12}$. We find $2^{19} = 524288$ and $3^{12} = 531441$. The ratio $3^{12} : 2^{19} \approx 1,01365$ is called the Pythagorean comma. If we put the fifth equal to λ^7, then we obtain $2 = \lambda^{12}$.

Ernst Selmer's paper also appeared in Norwegian. He makes a very simple change in Brun's ideas. Let $a_0 \geq a_1 \geq a_2 \geq 0$. Then we put $\sigma(a_0, a_1, a_2) = (a_0 - a_2, a_1, a_2)$ and reorder. The corresponding (inverse) matrices are given as

$$D(0) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(2) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

However, if $a_0 > a_1 > a_2 > 0$ then the appearance of $D(1)$ or $D(2)$ does not permit a following $D(0)$. A genuine reordering of $\sigma(a_0, a_1, a_2)$ is necessary if $a_0 - a_2 < a_1$ which is equivalent to $a_0 < a_1 + a_2$. The new triple is either $(a_0', a_1', a_2') = (a_1, a_0 - a_2, a_2)$ or $(a_0', a_1', a_2') = (a_1, a_2, a_0 - a_2)$. However, in both cases we obtain $a_1 = a_0' < a_1' + a_2' = a_0$. Selmer had the hope that the question of periodicity would be easier to solve but his hope was in vain! He calculates the expansion of $(\log 2, \log \frac{3}{2}, \log \frac{5}{4})$ but the important denominator 12 does not appear.

In the year 1884 Henri Poincaré proposed a different approach. His paper is geometrically inspired and densely written but Arnaldo Nogueira (1995) has given a broad exposition including some figures. The arithmetic description is very simple. Let (a_0, a_1, a_2) be a triple of non-negative real numbers. Then there is a permutation π such that the condition $a_{\pi 0} \leq a_{\pi 1} \leq a_{\pi 2}$ is satisfied and we form

$$(a_0', a_1', a_2') = P(a_0, a_1, a_2) = (a_{\pi 0}, a_{\pi 1} - a_{\pi 0}, a_{\pi 2} - a_{\pi 1}).$$

This algorithm looks very innocent but in fact it leads to difficult problems. Contrary to Poincaré's hope it is not very useful for Diophantine approximation. After some calculations the triple $(\log 2, \log \frac{3}{2}, \log \frac{4}{3})$ leads to the approximation $(11, 5, 4)$ but again the important approximation $(12, 7, 5)$ is left out.

3 Conclusions

Multidimensional continued fractions could be a good topic for instruction at undergraduate level or even in school. The approach is elementary but leads to interesting problems including the up to now unsolved problem of finding a simple algorithm to generalize Lagrange's theorem. Clearly, some concepts of this paper preferably are thought for the background knowledge of the teacher. It would be also helpful to work with continued fractions before starting with their generalizations to higher dimensions. Hand-held calculators and computers can be used to calculate the very first approximations of given triples. It is possible to use matrices and if one goes into direction of periodicity to consider their eigenvalues. It is possible to use at least some parts of the original publications and to discuss mathematical language as expressed in different languages (German, English, French, and - very ambitious - Norwegian). This can connect independent working with the development of mathematics. Last but not least older papers can show that a good notation is very helpful.

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L´HOSPITAL'S CHALLANGE TO GEOMETERS FOR THE RECTIFICATION OF DE BEAUNE'S CURVE.

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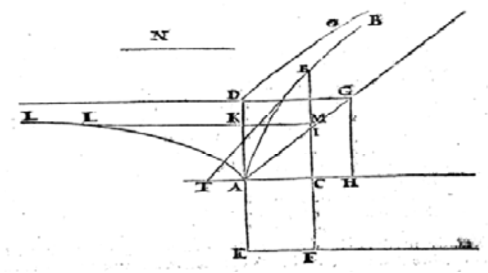
ABSTRACT

In 1692 l'Hospital wrote in Journal des Sçavans about the De Baune curve and cast a challenge for the calculation of its length, which was solved by Varignon 11 years later in the same journal. (L´Hospital, 1692, pp. 401-403, Varignon 1703, pp.117-121)
In this text I present Varignon´s calculation of the length of the De Beaune curve and an analysis of this solution. I show here that it was Bernoulli who solved the challenge of l'Hospital. Varignon used Bernoulli's solutions to write his article, without mentioning his name once.
The text is a good example for undergraduate students on the one hand, of how integration can be made simpler with the help of computer software and on the other hand, it exemplifies the importance of letters, which are reasonably easy to be read, as sources of research in the History of Mathematics. Nowadays, this kind of documentation is no longer being produced.

1 INTRODUCTION

I'll present l´Hospital's appeal for the rectification of the curve suggesting that he already knew it, due to the 4 properties he attributed to it. The interesting fact is that he does not make use of the logarithmic function.



L´Hospital, 1692, p.401

It is clear that the nature of this curve *AbB* depends on the quadrature of the hyperbola and also that it is mechanical in the sense of Descartes. Descartes was the first to consider the expression of curves by equations. This idea, upon which the application of Algebra to Geometry is founded, is very successful and fertile. Curves are divided into algebraic parts, which we call, following his ideas, geometric curves, and transcendental parts, which are the mechanical ones for him. The mechanical curves cannot be expressed by equations between *dx* and *dy*. There are two types of this kind of curves; they are:
1 The exponential curves, where one of the unknown, or both of them occur in exponents.
2 The intertranscendental curves in which the equations are expressed by means of radicals.

Let us consider now some of the properties of De Beaune`s curve, for L´Hôpistal:

1º Elle a pour asymptote la ligne DO parallele à AI.
2º Si l´on nomme AC x *, BC* y*, l´espace ABC compris par les droites AC, et par la portion AB de la courbe,* $= xy - \frac{1}{2}y^2 + nx$*. La distance du centre de gravité de l´espace ABC de la droite AC,* $= n + \frac{3xy^2 - 2y^2}{6xy - 3y^2 + 6nx}$ *et de AK* $= \frac{1}{2}n + \frac{3x^2y - y^2}{6xy - 3y^2 + 6nx}$*, Et l´on a par conséquent les solides, demi solides, formés par la révolution de cet espace, tant autour de AC que de AK ou BC.*
4º Il est facile de déterminer les centres de gravité de ces demi-solides. Mais comme on a besoin d´une adresse particulière pour rectifier cette courbe, en supposant la quadrature de l´hyperbole, je propose ce problème aux Geometres, les assurant qu´il mérite leur recherche. (L´Hospital, p.403)

One of the facts that have deeply puzzled me, is the link between the logarithmic function and the quadrature of the hyperbola, because of the analysis of historical books and documents that I have been doing for some time. This fact, although known since the 17th century, was not clear for the mathematicians. I quote Bourbaki:

Quoi qu´il en soit, J. Gregory, en 1667, donne, sans citer qui que ce soit,..., une règle pour calculer les aires des segments hyperboliques au moyen des logarithmes (décimaux): ce qui implique à la fois la connaissance théorique du lien entre la quadrature de l´hyperbole et les logarithmes, et la connaissance numérique du lien entre logarithmes "naturels" et "décimaux". Est-ce à ce dernier point seulement que s´applique la revendication de Huygens, que conteste aussitôt la nouveauté du résultat de Gregory....? Ce n´est pas plus clair pour nous que pour les contemporains ; ceux-ci en tout cas ont eu l´impression nette que l´existence d´un lien entre logarithmes et quadrature de l´hyperbole était chose connue depuis longtemps, sans qu´ils pussent là-dessus se référer qu´à des allusions épistolaires ou bien au livre de Grégoire de Saint-Vincent. (Bourbaki, 1969, p.214)

I started my research with De Beaune's letter to Roberval dated 16th October 1638 (Waard, pp.139-150), where he tries to solve the curve, which was named after him, under conditions on the tangent at any of its points. It is one of the first inverse tangent problems. Descartes' reply, correcting his proof, comes in a letter to Roberval dated 15th November 1638. Eventually, Leibniz, in a letter to Oldenburg in August 1676, demonstrates, using his differential calculus and his notation, that:

$$\int \frac{dy}{y} = -\frac{x}{c}$$

which gives: $y = -c \log \frac{y}{c}$

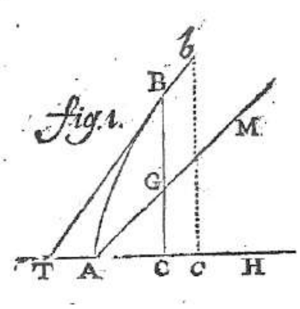
The elucidation came to me when l' Hospital wrote, in the *Journal des Sçavans*, about this curve and casted a challenge for the calculation of its length, which was solved by Varignon 11 years later in the same journal. (L´Hospital, 1692, p. 401-403 (Varignon. 1703, p. 111-121)

In this text, Varignon´s calculation of the length of the De Beaune's curve and an analysis of this solution will be presented.

2 Solution presented by Pierre Varignon in 1703

Varignon was a great friend to l'Hospital and defended him at the Academie des Sciences de Paris, against Rolle's attacks. It was he, who, eleven years later, accepted the challenge and demonstrated how to rectify DeBaune's curve, using differential calculus and knowledge of the rectification of the hyperbola. Such a solution was published in the Journal des Sçavans in 1703 (Varignon, 1703, pp.117-121):

SOLUTION DU PROBLÈME QUE M. LE MARQUIS DE L´HOSPITAL A PROPOSÉ DANS LE JOURNAL DES SÇAVANS DE 1692. PAG.598
(Journal de Sçavans, 1703, 2G, p.117) (Varignon)
Rectification of the curve from M. de Beaune
Considering *ABb* the curve from M. de Beaune, such that *BC* is divided at *G*, by the line *AGM*, which has a 45 degree angle with the axis *ACH*, *BC* is to the sub-tangent *CT* as given line *a* to *BG*, part of the ordinate comprehended between the line *AGM* and the curve



$$\frac{BC}{CT}=\frac{a}{BG}\ .$$

If *AC* is called *x* and *BC*, *y*, then we have *BG* = *y* – *x* (since the angle *GAC* is the same as the *AGC* angle) and this curve's property will be given by $a\,dx = y\,dy - x\,dy$ hence, $\frac{BC}{CT}=\frac{dy}{dx}=\frac{a}{y-x}$

It is known that the length of every curve is $\sqrt{dx^2+dy^2}$ (being *x* the abscissa, and *y* the ordinate).
Placing then, the value of dx determined by the equation of the curve in this expression, it is possible to find the small arc:

$$Bb = ds = \sqrt{\left(\frac{y\,dy - x\,dy}{a}\right)^2 + dy^2} = \sqrt{\frac{y^2dy^2 - 2xy\,dy^2 + x^2dy^2 + a^2dy^2}{a^2}} = \frac{dy}{a}\sqrt{y^2-2xy+x^2}$$

Now to reduce this amount to a single indeterminate, we put $z = y - x$; then, $dy = dz + dx$.
Inserting these values of *y* and *dy* in the element and multiplying the numerator and denominator by $a - y + x$ we get:

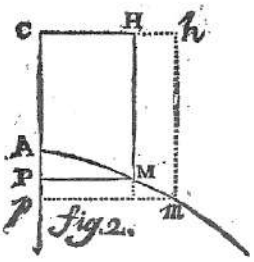
$$\frac{(a-y+x)}{(a-y+x)}\frac{dy}{a}\sqrt{y^2+a^2-2xy+x^2} =$$
$$\frac{a(dz+dx)+(z+x)(dz+dx)+x(dz+dx)}{(a-z)(dz+dx)a(a-z)}\sqrt{(z+x)^2+a^2-2x(z+x)+x^2} =$$
$$\frac{a\,dy - y\,dy + x\,dy}{a^2-ay+ax}\sqrt{y^2+a^2-2xy+x^2}$$

Since $a\,dx = y\,dy - x\,dy = z\,dy$, then, $a\,dx - z\,dz - z\,dx = z\,dy - z(dy-dx) - z\,dx =$ $z\,dy - z\,dy - z\,dx + z\,dx = 0$, which causes a change to this other quantity $\frac{dz}{a-z}\sqrt{a^2+z^2}$; and in order to have the radical sign in the denominator, we multiply the numerator and denominator by $\sqrt{a^2+z^2}$, which yields $\frac{dz}{a-z}\sqrt{a^2+z^2}=\frac{dz\left(a^2+z^2\right)}{(a-z)\sqrt{a^2+z^2}}=$ $\frac{a^2dz+z^2dz}{(a-z)\sqrt{a^2+z^2}}$; adding and decreasing a^2dz from the numerator, we get: $\frac{2a^2dz-a^2dz+z^2dz}{(a-z)\sqrt{a^2+z^2}}=\frac{2a^2dz}{(a-z)\sqrt{a^2+z^2}}-\frac{(a+z)dz}{\sqrt{a^2+z^2}}=\frac{2a^2dz}{(a-z)\sqrt{a^2+z^2}}-\frac{a\,dz}{\sqrt{a^2+z^2}}-\frac{z\,dz}{\sqrt{a^2+z^2}}$

The integral of the last term is $-\int_0^z \frac{z\,dz}{\sqrt{a^2+z^2}}=-\sqrt{a^2+z^2}\,\Big|_0^z=-\sqrt{a^2+z^2}+a$

Since at that time the trigonometric integrals weren't known, Varignon uses the following 'trick' to get the integral of the second term: multiplying both the numerator and the denominator by az and adding and subtracting the term $\frac{2dz\sqrt{a^2+z^2}}{a}$ gives $-\frac{a\,dz}{\sqrt{a^2+z^2}}=-\frac{a^2z\;dz}{a\sqrt{a^2z^2+z^4}}+\frac{2dz\sqrt{a^2+z^2}}{a}-\frac{2dz\sqrt{a^2+z^2}}{a}$. The integral of this term is $-\int_0^z\frac{a\,dz}{\sqrt{a^2+z^2}}=\frac{z}{a}\sqrt{a^2+z^2}-\int_0^z\frac{2\sqrt{a^2+z^2}}{a}dz$. This latter term is the integral of a hyperbola, that is, the quadrature of a hyperbola multiplied by 2 and divided by *a*, supposedly known to l'Hospital. Then, the integral is equal to the area of the rectangle HMP divided by *a*, the equation of the equilateral hyperbola *AMm* is $x^2-z^2=a^2$, so $AP=\sqrt{a^2+z^2}=x$. Therefore, the solution of the integral of this second term is minus the sum of the areas of the rectangle $CHPM=z\cdot\sqrt{a^2+z^2}$ and the space $CHAM=\frac{2}{a}\int\sqrt{a^2+z^2}dz$.

The integral of the first term $\int_0^z\frac{2a^2dz}{(a-z)\sqrt{a^2+z^2}}$ can be found by quadrature of an equilateral hyperbola whose apex is at the abscissa $\frac{a}{4}$.



[1]Varignon, 1703, p.120

Being the equilateral hyperbola *AMm*, whose semi-axis is *CA*, or be it $\frac{1}{4}a$, the abscissa *CP* will be $\frac{a\cdot\sqrt{\frac{1}{8}a^2+\frac{1}{8}z^2}}{a-z}$, the ordinate $PM=\frac{a^2+az}{4(a-z)}$.

$$CM=\frac{a\cdot\sqrt{\frac{3}{16}a^2+\frac{3}{16}z^2+\frac{1}{8}az}}{a-z}$$

Mapping the line *Cm* from point *C*, infinitely close to *CM* and from point *C*, as the center, describing a small arc *MN*, it is clear that the triangle *CMN* will be the hyperbolic triangle *CMA*. Then, in this case, Varignon considered the curve *MN*, as well as *Mm*, as being small segments.

To get the expression of this elementary triangle, the following are required: the analytical value of *CM*, finding *MN*, multiplying these two quantities for each other and dividing the result by 2. This means that the area of the right triangle *CMN*, right at *M*, is $\frac{CM\cdot MN}{2}$.

From this we get the value of *CM*. To find the value of $MN=\sqrt{Mm^2-mN^2}$. Since *Nm* is the differential of *CM*, then $d(CM)=\frac{a^3+a^2z}{8(a-z)^2\sqrt{\frac{1}{8}a^2+\frac{1}{8}z^2}}dz$.

The differential of *MP* will be $d(MP)=\frac{a^2}{2(a-z)^2}dz$.

Raising each of the expressions to the square, adding and extracting the square root and applying Pythagorean theorem: $Mm=\sqrt{\frac{\left(a^2z+a^3\right)^2}{64(a-z)^4\left(\frac{1}{8}a^2+\frac{1}{8}z^2\right)}+\frac{a^4}{4\left(a-z\right)^4}}dz$ Then, if we have the expression of the small lines *mM* and *mN*, we can find the segment $MN=\sqrt{Mm^2+Nm^2}$, so $MN=\frac{a^2dz}{\sqrt{8a^2+8z^2}\sqrt{3a^2+2az+3z^2}}$.

Calculating the area of the right triangle *CMN*, we get: $\text{area}\,CMN=\frac{CM\cdot MN}{2}=$ $\frac{a}{32\sqrt{2}}\frac{2a^2dz}{(a-z)\sqrt{a^2+z^2}}$, whose integral is the same as the hyperbolic domain *ACM*.

Then, the first integral is the hyperbolical space *ACM* divided by *a* and multiplied by $32\sqrt{2}$.

Then, the entire integral of the three first terms will be the same as 32 times the *ACM* space, multiplied by $\frac{\sqrt{2}}{a}$, minus twice the sum of the hyperbolical space *ACM* and the rectangle *CHPM*, also divided by *a* and minus the line given by the quantity $\sqrt{a^2+z^2}$ plus *a*, *which is what we looked for.*

3 Johann Bernoulli: His participation in this challenge

It is interesting to understand what happened in these eleven years between the challenge of l'Hospital and Varignon's response. Researching the letters Varignon and Bernoulli exchanged, we found the whole process of rectification of the De Beaune's curve.

I'll start with the letter Varignon sent to Bernoulli on May 24th, 1696, 4 years after l'Hospital's article had been published. In this letter Varignon writes:

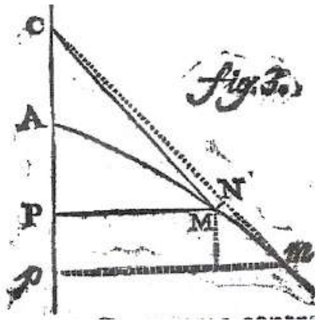
> *De grâce, Monsieur, dites moi comment vous trouvez les longueurs des logarithmiques. Je m´en suis démontré tout ce que M. Huguens en a dit*; mais je n´en saurais trouver les longueurs que par des progressions infinies, quoiqu´elles se puissent trouver autrement. Je suis dans la même peine par rapport aux longueurs de la courbe de M. de Beaune: je m´en suis aussi démontré l´une et l´autre des solutions que vous en avez données dans les Journaux de France et de Leipzig; mais pour les longueurs je ne les vois point encore; aussi dites vous dans le Journal de France qu´il faut une adresse particulière pour le trouver: c´est cette adresse que je vous prie aussi de me montrer. (*) (in Discours de la cause de la pesanteur,* which was attached to Traité de la Lumière, published in 1690; Costabel & Peiffer, 1988, p.98)

Bernoulli responds to Varignon on June 5th, 1696, but doesn't talk about the substitutions needed to solve the problem, which are again requested by Varignon in a letter on 18th June of the same year. In this letter Varignon writes:

> *Je savais bien que* $\frac{dy\sqrt{a^2+y^2}}{y}$ *est la différentielle de la courbe logarithmique; mais je ne vois pas, et je ne vois pas encore que* $\frac{dy\sqrt{a^2+y^2}}{y}$ *soit la différentielle d´un espace hyperbolique.* (Costabel & Peiffer, 1988, p.102)

On 11th August, 1696 Bernoulli writes back, showing Varignon how the integral of this differential can be calculated as a hyperbolic area, which is the area ABF, using the infinitesimal triangle as shown in the article by Varignon. First, Bernoulli demonstrates that: $\frac{dy\sqrt{a^2+y^2}}{y} = \frac{y^2+a^2}{y\sqrt{y^2+a^2}}dy = \frac{y\,dy}{\sqrt{y^2+a^2}} + \frac{a^2\,dy}{y\sqrt{y^2+a^2}}$.

The first integral is $\sqrt{a^2+y^2}$. Then, he works on the second, where he makes the coordinate changes, that is $y = \frac{a^2}{m}$, where $m \neq 0$, he then has the differential $\frac{-a\,dm}{\sqrt{a^2+m^2}}$. It is this integral which he claims to be the area of the region previously mentioned.



There is no need of such an artificial construction to show this; it is enough to calculate the area between the hyperbola and the line joining the point F to the origin *A*.

Taking the coordinates of *F* as *(x,y)*, which is also a point in the hyperbola $x=\sqrt{a^2+y^2}$, the line *AF* is given by the equation $x=\frac{y^2}{\sqrt{y^2+a^2}}$. As a result of the differential, we must integrate to find that the hyperbolic area will be: $\sqrt{y^2+a^2}\,dy-\frac{y^2}{\sqrt{y^2+a^2}}dy=\frac{a^2}{\sqrt{y^2+a^2}}dy$. Since the line is shown with the *y* squared, we have two lines, *AF* and *AF'*, therefore we will consider only half of this region.

Varignon responds Bernoulli's last letter thanking him for such a long answer on how to calculate the length of the logarithmic curve, but he still does not know how to deal with De Beaune's curve, for he writes:

> *…ayant cherché de même celle de la Courbe de Beaune, j´en suis demeuré à une différentielle que ne n´ai pu intégrée: la voici. Appelant à ordinaire les abscisses x, les ordonnés y, et le lieu de cette courbe étant* $=y\,dy-x\,dy$ *; je trouve son Elément* $\left(\sqrt{dx^2+dy^2}\right)=\frac{dy}{n}\sqrt{yy-2xy+xx+nn}$ *(en faisant* $y-x=z$*)* $=\frac{dz\sqrt{zz+nn}}{n-z}$
>
> *c´est de cette dernière différentielle que je ne savais trouver l´Intégrale: Je vous la demande en Grâce, et tout du long, s´il vous plaît* (Costabel & Peiffer, 1988, p.109)

Bernoulli's answer comes on July 27th, 1697. In this letter he shows Varignon how to solve the problem of the rectification of De Beaune's curve; Bernoulli writes:

> *Je suis bien aise que ma méthode de rectifier la logarithmique ait eu le Bonheur de vous plaire; mais j´en aurais eu plus de joie si elle vous avait donné assez de lumière pour faire l´imitation das la courbe de Beaune, comme je l´esperais: voici pour achever le calcul que vous avez seulement commencé, comment il s´y faut prendre: vous avez réduit l´élément de la courbe à cette quantité differentielle* $\frac{dz\sqrt{zz+nn}}{n-z}$ *dont vous ne sauriez trouver l´intégrale, que je trouve pourtant fort aisément, car* $\frac{dz\sqrt{zz+nn}}{n-z}=\frac{zz\,dz+nn\,dz}{(n-z)\sqrt{zz+nn}}=\frac{zz\,dz-nn\,dz}{(n-z)\sqrt{zz+nn}}\left(\frac{-z\,dz-n\,dz}{\sqrt{zz+nn}}\right)+\frac{2nn\,dz}{(n-z)\sqrt{zz+nn}}$, *ou l´intégrale de* $-\frac{z\,dz}{\sqrt{zz+nn}}$ *est* $-\sqrt{zz+nn}$ *; et l´intégrale de* $-\frac{n\,dz}{\sqrt{zz+nn}}$ *est comme je vous ai fait voir dans ma précédente un secteur hyperbolique; il reste*

donc encore $\frac{2nn\,dz}{(n-z)\sqrt{zz+nn}}$*, dont il faut trouver l´intégrale, ce que je fais ainsi: je pose* $n-z=s$*, ce qui donne* $\frac{2nn\,dz}{(n-z)\sqrt{zz+nn}}=$ *(en faisant* $s=\frac{nn}{t}$*)* $\frac{2nn\,dz}{(n-z)\sqrt{zz+nn}}=\frac{-2nn\,ds}{s\sqrt{2nn-2ns+ss}}=$ *(en mettant* $t=u+\frac{1}{2}n$*)* $\frac{n\,du\sqrt{2}}{\sqrt{uu-\frac{1}{4}nn}}$*, ou l´intégrale de* $\frac{n\,du\sqrt{2}}{\sqrt{uu-\frac{1}{4}nn}}$ *est aussi un secteur hyperbolique; ou si vous aimez mieux retenir une même formule, vous n´avez qu´à mettre* $\sqrt{uu-\frac{1}{4}nn}=t$*, et vous trouverez* $\frac{n\,du\sqrt{2}}{\sqrt{uu-\frac{1}{4}nn}}=\frac{r\,dr\sqrt{2}}{\sqrt{rr+\frac{1}{4}nn}}$ *Voilà donc l´intégrale de* $\frac{dz\sqrt{zz+nn}}{n-z}$ *réduite à* $-\sqrt{zz+nn}$ + *la différence de deux secteurs hyperboliques, que l´on peut par conséquent construire par la logarithmique; or puisque la courbe de Beaune se construit aussi par la logarithmique, il est évident que l´intégrale de* $\frac{dz\sqrt{zz+nn}}{n-z}$ *c´est-à-dire la rectification de la courbe de Beaune dépend de sa propre construction; et qu´ainsi la courbe étant une fois décrite, elle se mesure par des lignes droites.* (Costabel & Peiffer, 1988 pp.114-115)

Comparing the solution Varignon explores in his article, we can see which steps he took to come to a way to integrate this last differential $\frac{2n^2\,dz}{(n-z)\sqrt{z^2+n^2}}$. He considers the equilateral hyperbola *AMm* whose semi-axis *CA* is $\frac{1}{4}a$. The abscissa *CP* will be $\frac{a\cdot\sqrt{\frac{1}{8}a^2+\frac{1}{8}z^2}}{a-z}$. This abscissa is the result of all the substitutions Bernoulli executed. As it was put by Varignon, is seems to be a fixed value, or at best the function of a function, which changes the character of the curve.

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4 Conclusion

The challenge of the rectification of the De Beaune's curve was first cast by l'Hospital, in 1692, in the Journal des Sçavans, as a problem for geometers to solve. The solution came to light eleven years later, in the same journal, under the credit of Varignon. Little was known about how Varignon had solved the problem, since the solution he presented was achieved through the use of a variable substitution that was completely incomprehensible, which seemed to be a 'forced' substitution used to achieve a result he already expected to find. This was a dilemma for mathematicians, who could not find a sensible resolution for the problem. It was only after the publication of Johann Bernoulli's letter, in the 1940s, that we could have access to the correspondence between Varignon and Bernoulli, through which it was possible to understand Varignon's solution for the rectification of the De Beaune's curve, by way of the quadrature of the hyperbola that was known at the time, and which was solved by Bernoulli after a long correspondence between the two mathematicians.

All the steps found in the letters were here demonstrated, as well as Varignon's appeal and Bernoulli's respective responses, presenting the desired solution. Although Varignon was an important mathematician of his time and had a definite role in the history of the development of Differential and Integral Calculus, Bernoulli's merit in this field was far greater and his role in the support of important problems are both undeniable facts. I don't know why Varignon, on the occasion of the publication of his article solving l'Hospital's challenge, doesn't acknowledge Bernoulli, since it was him who in fact solved the challenge.

APPENDIX: Rectification of a curve

It seems that the first mathematician who rectified a curve was William Neil (1637-1670). His procedure to calculate the length of the segment of a "semi-cubical parabola" $(y^2 = x^3)$ over the interval $0 \le x \le a$,

> *was to subdivide this interval into an indefinitely large number n of infinitesimal subintervals, the ith one being* $\left[x_{i-1} - x_i\right]$. *If* S_i *denotes the length of the (almost straight) piece of the curve* $y = x^{\frac{3}{2}}$ *joining the corresponding points* (x_{i-1}, y_{i-1}) *and* (x_i, y_i), *then* $S_i = \left[(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2\right]^{1/2}$, *and the length of the curve will be given by* $S \cong \sum_{i=1}^{n} S_i$. (Edwards, 1973, p.118).

In "Methods of Series and Fluxions", Newton applies the basic fluxional technique for the computation of an arc length, which he describes as follows:

> *Problem 12: To determine the lengths of curves: In the previous problem we showed that the fluxion of a curve line is equal to the square root of the sum of the squares of fluxions of the base and the perpendicular ordinate. Consequently if we consider the fluxion of the base as a uniform, determinate measure (namely unity) to which the other fluxions shall be referred, and on top of this seek out the fluxion of the ordinate by means of the equation defining the curve, the fluxion of the curve line will be hand and from that its length should be elicited by Problem 2.* [(716)]
>
> [(716)] *Newton sketches the standard procedure for rectifying a curve: namely the arc-length t is the 'fluent' (taken between appropriate integration bounds) of the fluxion* $\dot{t} = \sqrt{\left[\dot{y}^2 + \dot{z}^2\right]}$ *, where the curve is defined with regard to perpendicular Cartesian coordinates* $AB = z, BD = y$. *The analogous procedure for the other types of coordinate systems is no discussed in any of the nine following examples, but we may conjecture that the simple polar at least would have been dealt with if Newton had ever completed his present tract.* (Whiteside 1967, p.315)

This is the known formula: $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

For Leibniz,

given a curve whose arc length is sought, letting denote the length of the tangent line intercepted between the x- axis and a vertical ordinate of (constant) length a. Then, from the similarity the triangle, it follows that $\frac{ds}{t} = \frac{dy}{a}$ *or* $a\,ds = t\,dy$. *Hence* $\int a\,ds = \int t\,dy$ *, so the rectification of the given curve reduces to a quadrature problem the calculation of the area of the region between the y-axis and a second curve whose abscissa x is the tangent t to the given curve.* (Edwards, 1973, pp.242-243).

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INTRODUCING COPERNICUS' *DE REVOLUTIONIBUS* TO TRAINEE MATH TEACHERS

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ABSTRACT

Mathematics teacher training demands a lot of care, considering the profession's complexity and the constant challenges teachers need to overcome in their daily work. Thus, there are several aspects that have to be taken into account to an efficient teacher training program: professional, pedagogical, psychological and reflexive aspects, amongst others. Consequently, there is other diverse knowledge required from a Math professor. In addition to the necessary conceptual and procedural mathematical knowledge, the Math teacher has to understand the nature of Mathematics to have a basic socio-cultural formation in general and specific educative questions. It is also needed that the teacher has such a cultural knowledge that allows her/him to be open minded to other non mathematical areas. So, we believe that the work with original Mathematics and Science History sources can provide the teacher training we want. Believing so, we took the original work of Nicolaus Copernicus, "De revolutionibus orbium coelestium". Copernicus' book was used to work with undergraduate students, future Math teachers. Our goal was to lead students to calculate themselves a sinus table. Following the schema used in "De revolutionibus", we constructed with ruler and compass all the geometric figures needed for the deductions of the theorems that allow us to calculate the entries of the chord table. Transforming a chord table into a sinus table was a simple task. Geographic maps, photos of referring old documents to the sprouting of trigonometry and texts on the theories of the universe were also carried through consultation. When giving more analysis, at great length, to the workmanship, we realized to be facing an important source of culture, able to provide to the Math teacher training something more than just the inserted mathematical knowledge in the construction of the table. The experience enabled us to discuss about the history of human kind, the development of sciences and of the society. Nevertheless, our research study, whose objective is the investigation of possibilities in using original historical sources of science (and/or mathematics) to provide a wider, deeper and multidisciplinary formation to the teacher training, is still in progress and many questions are open.

History and epistemology as tools for an interdisciplinary approach in the teaching and learning of mathematics and the sciences

PHYSICS–MATHEMATICS RELATIONSHIP
Historical and Epistemological Notes

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ABSTRACT

Nowadays it is unthinkable to learn and teach the scientific sense of physical and mathematical sciences without deepening its intellectual and cultural background, e.g. history and its foundations. In my talk several case–studies on the relationship between physics and mathematics were presented. Here, for brevity's sake, I will only discuss some of them.

1 Notes on Archimedean science and its heritage in *Opera geometrica*

1.1 On Archimedes

Archimedes (fl. 287–212 B.C.) was a deeply influential author for Renaissance mathematicians according (we can say) to the two main traditions: the humanistic tradition, adhering strictly to philological aspects, followed by Willem van Moerbeke (1215–1286), Regiomontanus (1436–1476) and Federigo Commandinus (1509–1575) and the purely mathematical tradition followed by Francesco Maurolico (1694–1575), Luca Valerio (1552–1618), Galileo Galilei (1564–1642) and Evangelista Torricelli (1608–1647). Nevertheless, Archimedean tradition represents an historical and epistemological component which has apparently already been solved (Heath 2002, XXXIX). One might ask: *how did the various cultures, practical and theoretical environments interact with the recovery of the Archimedean roots for the birth of modern science?* The response is not commonplace at all and requires more space (Capecchi and Pisano 2010b). Nevertheless, an hypothesis can be made regarding a frame of reference of the roots and of the Archimedean tradition and its Renaissance physical–mathematical relationship can be divided (as an outline) into a mixture of four epistemological interpretations: 1) a strictly mathematical Tartaglia root 2) a humanistic Moerbeke–Regiomontanus–Commandino root, 3) a mathematical–geometrical Maurolico root and 4) a modern Valerio–Galilei–Torricelli's root. (Capecchi and Pisano 2007). Based on previous works (Pisano 2007; Pisano 2008; Capecchi and Pisano 2010a). Here I present some epistemological reflections on the fourth. Knowledge of Archimedes' contribution (Clagett 1964–1984) is truly fundamental for a historical study of Evangelista Torricelli's (1608–1647) mechanics. Archimedes set mathematical *rational criteria* for determining physically centres of gravity and his work contains physical concepts formalized on mathematical–geometrical foundations. In *Book I* of *On Plane Equilibrium* (Heath 2002, 189–202; Heiberg 1881) Archimedes, besides studying the rule governing the law of the lever also finds the centre of gravity of various geometrical plane figures (Heath 2002). Moreover Archimedes does not develop all mechanics axiomatically, but sometimes he uses an approach for problems (the problematic approach). In that sense the relationship between physics and mathematics in the theory could produce novelties in its historical foundations: e.g., the epistemological status of the *Suppositio* is different. In fact, Archimedean suppositions are not (all of them) self–evident as are those of the Euclidean (and Aristotelian) tradition and may have an empirical nature. The use of *ad absurdum* proofs, due to the lack of reference to the first suppositions, does not allow for the assumption of a strictly mathematical–axiomatic structure (Bailly and Longo), e.g., typically deductive.

1.2 On Torricelli

In regard to Torricelli's works, I focus on *Opera geometrica* (Torricelli 1644; Capecchi and Pisano 2007; Pisano 2009). In particular, I focused on his discourses on the theory of the centre of gravity dealing with his famous principle in *Liber primis De motu gravium naturaliter descendentium*.[1] Evangelista Torricelli, in his theory on the centre of gravity[2], followed Archimedes' physical–mathematical approach using and the proofs can change, too: a) *Reductio ad absurdum* as a particular instrument for mathematical proof; b) Geometrical forms implicit in weightless beams and indirect reference in geometrical forms to establish the law of the lever; c) Empirical results to establish principles. Torricelli, e.g., conceived twenty–one different ways of squaring (Heath 2002, *Quadrature of the parabola–Propositio* 17 and 24, 246; 251) the parabola, which had already been studied by Archimedes: eleven times with exhaustion, ten with the indivisibles method. The *reductio ad absurdum* proof is very often present. Some results obtained via the indivisible technique were always checked by using different methods.

Torricelli presents problems that remain unsolved, according to him, in Galileo's dynamical theory. His main concern is to prove Galileo's supposition according to which velocity degrees for a body are directly proportional to the inclination of the plane over which they move (also called *Galileo's theorem*): "The speeds acquired by one and the same body moving down planes of different inclinations are equal when the heights of these planes are equal" (Galilei 1890–1909, Vol., VIII, 205). It is an attempt to prove Galilei's *supposition*. Torricelli seems to suggest that this supposition may be *physically proved* beginning with a "theorem" according to which "the momentum of equal bodies on unequally inclined planes are to each other like the perpendicular lines of equal parts of the same planes" (Torricelli 1644, 98). Moreover, Torricelli also assumes that Galilei's theorem has not yet been *mathematically* (in Archimedean sense) *proved*.[3]

1.3 On Archimedean influence in Torricelli's mechanics

In many parts, Torricelli explicitly declares his Archimedean background[4]. Like Archimedes, in the case of *quadratura parabolae*, he first obtains results via the mechanical approach and then reconsiders the discourse with the classical methods of geometry to confirm in a *rigorous* way the correctness of his results. Thus, Torricelli, with the help of a driving idea of a duplicate procedure, devotes many pages to proving a

[1]"*Praemittimus* [of equilibrium]. Duo gravia sumul coniucta ex se moveri non posse, nisi centrum commune gravitatis ipsorum discenda". (Torricelli 1644, 99). ("It is impossible for the centre of gravity of two bodies in a state of equilibrium to sink from any possible movement of the bodies" [my translation]).

[2]The *Opera geometrica* is organized into four parts. Particularly, Part 1, 2, 3, are divided into *books* and Part 4 is composed of an Appendix (Torricelli 1644; Capecchi and Pisano 2010a).

[3]In the first edition of Galilei's *Discorsi e dimostrazioni matematiche* (Galilei, 41–458) in 1638, there is no proof of the "theorem". It was added to the edition of Galilei's Works, Bologna 1656. (See some letters from Torricelli to Galilei regarding the "theorem" (Torricelli 1919–1944, vol. III, 48, 51, 55, 58, 61).

[4] E.g.: "Inter omnia opera Mathematics disciplinas pertinentia, iure optimo Principem sibi locum vindicare videntur Archimedis; quae quidem ipso subtilitatis miraculo terrent animos" (Torricelli 1644, 7).

certain theorem on the "parabolic segment", first by following the geometry of ancients[5] and then by proving the validity of the thesis also by means of the first "indivisibilium" (Heath 2002, 55–84*), idem* for the "solido iperbolico acuto"[6]. He states that the ancient geometers moved according to a method other than that followed in "in invenzione" suitable "ad occultandum artis arcanum". In this respect, it is interesting to note that he underlines the "concordantia"[7] of methods of different "rigours"[8]. Torricelli also seems to hold on to the idea for which the method of mathematical demonstration of ancients, as in Archimedes' method, has intentionally been kept hidden. But the Archimedean tradition in Torricelli's work goes further. In *De sphaera et solidis sphaearalibus* (Heath 2002) he presents an enlargement of the Archimedean proofs in books I–II of *On the sphere and cylinder* (*Ivi*). Moreover, Torricelli faces problems not yet solved by Archimedes or by the other mathematicians of antiquity. Adopting the same style as Archimedes, he does not try to obtain the first principles of the theory and does not limit himself to a single way of demonstrating. A prominent example is the *Quadratura parabolae pluris modis*:

> […] In quibus Archimedis Doctrina de sphaera & cylindro denuo componitur, latiùs promovetur, et omni specie Solidorum, quae vel circa, velintra, Sphaeram, ex conversione poligonorum regularium gigni possint, universalius Propagatur.[9]
> Veritatem praecedentis Theorematis satis per se claram, et per exempla ad initium libelli proposita confirmatam satis superque puto. Tamen ut in hac parte satisfaciam lectori etiam Indivisibilium parum amico, iterabo hanc ipsam demonstrationis in calce operis, per solitam veterum Geometrarum viam demonstrandi, longiorem quidem, sed non ideo mihi certiorem.[10]

Let's note that exposition of the mechanical argumentation present in Archimedes' *Method* was not known during Torricelli's time, because the *Method* was discovered by Johan Ludvig Heiberg (1854–1928) only in 1906. Therefore, in Archimedes' writing there were lines of reasoning which, due to lack of justification, were labelled as mysterious by most scholars. Then, to prove both methods it was necessary to assure the reader not only of the validity of thesis, but mainly to convince him of the strictness of Archimedes' exhaustion reasoning and *reductio ad absurdum*, by proving his results with some other technique. Archimedes's himself did not attribute the same certainty to his method, as he did to classical mathematical proofs[11]. His reasoning on *Quadratura parabola* (Heath 2002, Proposition XXIV, 251) is exemplary. Addressing Eratosthenes (276–196 a.C.), he wrote at the beginning of his *Method* (Heath 1912, 13). One of the characteristics of Torricelli's proofs was the syntactic recall to demonstrating the approach followed by the

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[5]"Quadratura parabolae pluris modis per duplicem positionem more antiquorum absoluta" (Torricelli 1644, 17–54). In *Opera geometrica* there are also some reference to Euclid's *Elements*, to Apollonius's *conic sections*, Archimedes, Galileo, Cavalieri's works.
[6]"De solido acuto hyperbolico problema alterum" (Torricelli 1644, 93–135). "De solido hyperbolico acuto problema secundum (*Ivi,* 112–135).
[7]"De solido acuto hyperbolico problema alterum" (Torricelli 1644, 103). "Concordantia praecedentis demonstrationis cum doctrina Archimedis" (*Ivi*).
[8]"Quadratura parabolae per novam indivisibilium Geometriam pluribus modis absoluta" (Torricelli 1644, 55).
[9]Torricelli 1644, *De sphaera et solidis sphaearalibus*, 2.
[10]Torricelli 1644, *De solido hyperbolico acuto problema secundum,* 116.
[11]It is well known that in the *Method* (Heath 1912; Id., 2002) Archimedes studied a given problem whose solution he anticipated by means of crucial propositions which were then proved by the *reductio ad absurdum* or exhaustion.

ancient Greeks with the explicit declaration of the technique of reasoning actually used. Besides the well known *Ad absurdum* there were also the *permutando* and *the ex aequo*. In *De proportionibus liber* he defines them explicitly:

> *Propositio* IX. Si quatuor magnitudines proportionales fuerint, et permutando proportionales erunt. Sint quatuor rectae lineae proportionales AB, BC, CD, DE. Nempe ut AB prima ad BC secundam, ita sit AD tertia ad DE quartam. Dico primam AB ad tertiam AD ita esse ut secunda BC ad quartam DE. Qui modus arguendi dicitur permutando.
>
> *Propositio* X. Si fuerint quotcumque, et aliae ipsis aequales numero, quae binae in eadem ratione sumantur, et ex aequo in eadem ratione erunt. Sint quotcumque magnitudines A, B, C, H, et aliae ipsis aequales numero D, E, F, I, quae in eadem ratione sint, si binae sumantur, nempe ut A ad B ita sit D ad E, et iterum ut B ad C, ita sit E ad F, et hoc modo procedatur semper. Dico ex equo ita esse primam A ad ultimam H, uti est prima D ad ultimam I. Qui modus arguendi dicitur ex aequo).[12]

Torricelli seems to neglect the algebra of his time and remains glued to the language of proportions. He dedicated a book to this language, *De Proportionibus liber* (Torricelli 1919–1944, 295–327), where he only deals with the theory of proportions to be used in geometry. In this way he avoids using the *plus* or *minus*, in place of which he utilizes the *composing* (*Ivi*, 316) and *dividing* (*Ivi*, 313). Such an approach allows him to always move with the ratio of segments. By following the ancients to sum up segments he imagines them as aligned and than translated and connected, making use of terms like “simul” “et” or “cum” (*Ivi*, Prop. XV, 318). In the following section, we present a table which summarizes the most interesting part of *Proportionibus liber* where Torricelli again proves theorems by referring to Archimedean style reasoning.

Table 1 The Archimedean mathematical–physical approaches–proofs in Torricelli’s *Quadratura parabolae*

Contents	**Kind of proofs**	**References (Torricelli 1644)**
		QUADRATURA PARABOLÆ PLURIS MODIS […] ANTIQUORUM
Lemma II,V,VI, X–XI,XII–XIII XVII – Propositio IV	*Ad absurdum* proof	“Quadratura parabolæ pluris modis per duplicem positionem more antiquorum absoluta”, 17–54. (Torricelli 1644, *Opera geometrica,* op. cit.)
Lemma XIV	*Ex æquo* et *dividendo* et *permutando*	(*Ibidem*)
Lemma XVI, XVIII	*Ex aequo*	(*Ibidem*)
Lemma XIX	*Ex æquo* et *Ad absurdum* proof	(*Ibidem*)
Propositio III	*Componendo*	(*Ibidem*)
Propositio V	*Ad absurdum* proof and *Componendo*	(*Ibidem*)
Propositio IX[13]	*Ex æquo* et *Ad absurdum* proof	(*Ibidem*)
		DE SPHAERA ET SOLIDIS SPHAEARALIBUS
Propositio XVIII	*Ad absurdum* proof	“De sphæra et solidis sphæralibus”, Libro Primo, 28–29 (*Ivi*)

[12] Torricelli 1919–1944, Proportionibus liber, pp. 313–314.

[13] *Lemma* e *Propositio* are proved by means of *componendo* and *conversione.*

Propositio XIX	*Ad absurdum* proof	"De sphæra et solidis sphæralibus", Libro Primo, 30–33 (*Ivi*)
Propositio XXII–XXIII	*Ex æquo*	"De sphæra et solidis sphæralibus", Libro Primo, 35–36 (*Ivi*)
Propositio IV	*Idem* proof powered by Archimedes	"De sphæra et solidis sphaearalibus", Libro secondo, 51–52 (*Ivi*)
		QUADRATURA PARABOLÆ PER NOVAM INDIVISIBILIUM
Lemma XXII,XXIX	*Ex æquo*	"Quadratura parabolæ per novam indivisibilium Geometriam pluribus modis absoluta", 61–72 (*Ivi*)
Lemma XXX Lemma XXXI[14]	Idem proof powered by Archimedes *Equiponderant*	"Quadratura parabolæ per novam indivisibilium Geometriam pluribus modis absoluta", 74–77 (*Ivi*)
		DE SOLIDO ACUTO HYPERBOLICO
		"De solido acuto hyperbolico problema alterum", 93–112 (*Ivi*)
Exemplum I–II, IV–V, X,XII–XIV Altri *Lemma* e *corollari*[15]	Proof *in concordantia præcedentis demonstrationis cum doctrina Archimedis* *Ex æquo* (Exemplum X)	"De solido acuto hyperbolico problema alterum", 95–108; (*Ivi*) (Ivi, 113–135)
Exemplum III,XI	Proof *in concordantia cum theoremata Euclidis*	"De solido acuto hyperbolico problema alterum", 97; 104–105 (*Ivi*)
		DE MOTU PROIECTORUM[16]
Lemma (Follow from Prop. XXXIII)	Proof of Archimedes' proposition on *Conoids and spheroids*[17]	"De motu proiectorum", Libro secondo, 187 (*Ivi*)
		DE PROPORTIONIBUS LIBER[18]
Propositio[19] III Propositio VIII Propositio IX Propositio X Propositio XI Propositio XII Propositio XIII–XIV Propositio XV Propositio XVI	*Ad absurdum* proof *Dividendo* *Permutando* *Permutando* et *Ex æquo* *Ex æquo* *Componendo* et *Conversone* *Permutando* *Permutando* et *Componendo* *Permutando* et *Ad absurdum* proof	"De Proportionibus liber" (Torricelli E. 1919–1944. *Opere di Evangelista Torricelli*, by Loria G. et Vassura G., Vol. I, Parte prima, Montanari, Faenza, 308; 313–317).

We notice that only proofs by means of indivisibles are not *reductio ad absurdum*. This is because these proofs are algebraic. Instead, in nearly all other proofs he uses the

[14] Torricelli cites "Antonio Roccha praestandi Geometra" (Giovannantonio Rocca (1607–?) as author of) *Lemma XXXI* in the style of the "Schola Cavaleriana" which he attempts to demonstrate. (Torricelli 1644, "Quadratura parabolæ per novam indivisibilium Geometriam pluribus modis absoluta", 76). Caverni also cites Rocca. (Caverni 1895, Vol. IV, 136, line 8).
[15] The others are not *ad absurdum* proofs. They are proved by geometrical construction.
[16] Torricelli 1644, "De motu proiectorum", Libro secondo, 154–190, line 15.
[17] Heath 2002, 99–150
[18]At the end of the book he wrote in *concordantia* to show the reader that he was in agreement with Euclid's demonstration (e.g., propositions XVII and XVIII, XXIV and XXV of *Elements, Book V.*).
[19]The others are not *ad absurdum* proofs. They are proved by geometrical construction.

mathematical technique typical of proportions, *Dividendo, Permutando* ed *Ex aequo*. In the end, Torricelli adopts the Archimedean method with some variations:

a) The use of Bonaventura Cavaliere (1598–1647)'s *method of the ancients* and *method of the indivisibles*[20] (Torricelli 1644), that is to say, differently from his predecessors, Cavaliere dealt with physical–geometrical matters using both methods, offering the reader judgment on the quality of the methods used.
b) This makes the analysis and interpretation of scientific thought and of the Torricelli *corpus* even more complex and therefore more interesting compared to the type of geometry applied, for example, to statics by his predecessors.
c) The later time period suggests a greater attention for those who, in 1644 on the threshold of the birth of the Leibneizian–Newtonian infinitesimal analysis and with theoretical knowledge of both the geometry of the Cartesian coordinates and of the indivisibles, sensed the necessity for a re–evaluation of the *method of the ancients* also attempting, for themselves, the comparison as can be seen clearly from Torricelli's correspondence with Cavalieri (Torricelli 1919–1944)[21].

2 Notes on the historical epistemology of science

2.1 The role played by logic and language in physics–mathematics contents

Crucial features in scientific language have to do with the paradox of the formalization of logic (Carnap 1943). In order to express the axioms and to construct a *meta–discourse* about them, we should use the natural language, which is not formalized; we cannot formalize it in advance, because we risk producing a regression to *infinitum*. Moreover, it is not natural to state the axioms of logics and then to consequently deduce all the rest from them (*Ibidem*).

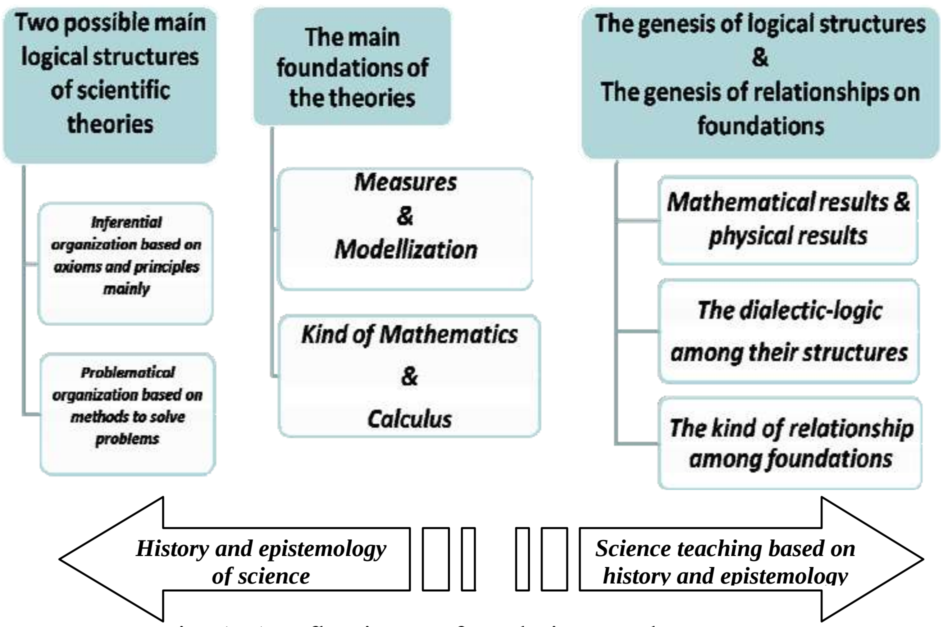

Fig. 1 A reflection on foundations and structures

In mathematical–classical logic, so–called *well–formed–statements* are assumed to be

[20]E.g.: *Quadratura parabola* (Torricelli 1644, 17–84).
[21]One could see letters between Cavalieri and Torricelli: "Racconto d'alcuni problemi, carteggio scientifico", by Giuseppe Vassura, Vol. III. See also: Capecchi and Pisano 2007.

either true or false, even if we do not have proof of either. In fact, from an inferential and classical logic system (e.g. a list of inferential propositions) one can only obtain a scientific dichotomy of hypothesis–these *free–from–self–contradiction* and among them, and to be *scientific,* a theory must be testable, e.g., subject to *falsification* (Popper 1963). Let's note that in that kind of system of reasoning, it is not possible to obtain *undecidable contents,* e.g. like (apparently) those generated by scientific DNSs belong to non classical logics. Particularly, if *undecidable* contents belong to a given principle of the theory, then we have an *out of the ordinary principle*.[22]

In the historical epistemology of science in relation to the logical structure of scientific theories, one can encounter both *Axiomatically Organised* theories (AO–theories) from which a few self–evident principles (or axioms) the whole theory is derived, as well as theories whose organization is based on solving given problems contained in the theories, which are thus *Problematically organized* (PO–theories). The assumed principles are often not as self–evident as the axioms are non–axiomatic principles. An AO theory is generally developed by means of advanced mathematics (e.g., mathematics–physics theory by differential equations, et al..) which starts its derivations directly from the axioms. A PO theory uses less advanced mathematics with the principles that only indicate a direction for the development of the theory and they may be «methodological» in nature (Kieseppä 2000). Moreover, it is characterized by the use of DNSs and most of the results are expressed by *reductio ad absurdum* statements. In previous studies it has been noted[23] that when in a scientific theory DNSs are largely used, a use of sophisticated mathematics is lacking, and the theory is based on declared problems to solve and without stating axioms or principles typically of an Aristotelian approach. In this sense, a formal characteristic of PO appears to be the occurrence of some DNS's that cannot be turned into equivalent positive sentences because the operative tools for proving them do not exist. In other words, as previously mentioned, in this kind of theoretical organization, the scientific contents of DNS «$\neg\neg A$» cannot be converted into an affirmative sentence corresponding to «A» because of the latter lack scientific proof. Therefore, DNSs, within the scientific theory, characterize a particular approach to science. Following that point of view, a borderline between classical logic and most non–classical logics is represented, not by the law of the excluded middle, but the double negation law. Generally speaking, when the double negation law fails, we are arguing outside classical logic and, *in–first–*

[22]Generally speaking, in non classical (or *constructive* or *intuitionistic*) logics, a statement is only true if there is proof that it is logically true, and only false if there is proof that it is logically false. In some previous papers (Drago and Pisano 2000; Id., 2004) on the history and foundations of physics, it has been shown that the discursive part of Sadi Carnot's *Réflexions sur la Puissance Motrice du Feu* of 1824 (Carnot S 1978; Id., 1986; Pisano 2001) does not include any principles (e.g, like in Newton's theory) but it presents more than 60 Doubly Negated Sentences (DNSs). A DNS (where $\neg\neg A$ = *A fails*) does not depend on an inferential scientific structure based on a classical logical dichotomy of theses, e.g., obtained by listed–deductive theorems (Popper 1959). Since scientific DNSs are not equivalent to the corresponding affirmative sentences, they belong to non–classical logic; typically the law of double negation, $\neg\neg A=A$. I also remark that DNSs as used by scientists cannot be dealt with by using classical first–order logic (Hodges 1983). In this sense, logical elements are considered as special categories (Pisano and Gaudiello 2009; Id., 2010) and not in the sense of a theory or of a new theory. Thus, one could think that *undecidable* contents are not logically adequate within an inferential system. Nevertheless, *what kind of logical organization can support DNSs in a scientific theory?* By means of that point of view, a discussion on the role played by investigation and methodology is science is possible.

[23]In previous papers of mine the reader can find more details on the use of logic and the organization of the theory in historical investigations (Drago and Pisano 2000; Pisano and Gaudiello 2009, 2010; Capecchi and Pisano 2010a: Id., 2010b).

approximation, we are arguing, within intuitionist logic.
Finally, two general ways to organize a scientific theory can be claimed: (1) a former one, e.g., based on Aristotle's argument, is organized through axioms (AO) and its logic is classical, (2) a *Problematic Organization* (PO), belongs to non–classical where a result could be also «fuzzy».

2.2 Logic in the history of mathematics

It could be said that if an object is shown by means of an *absurdum* proof, its existence is not soundly proved mathematically. While, in classical logic: «*A is true when $\neg\neg A$ is true*» because, within classical logic, one should only verify that a contradiction does not emerge. In *multi–valued logics*, however, more than two truth–values can exist. Let's see an example. Let *a* be a number in decimal form:

$$0, a_1 a_2 a_3 ... a_n$$

$$a_n = \begin{cases} 0 \Leftrightarrow 2n = p_1 + p_2 & \text{where } \{p_1, p_2\} \in \textit{primes numbers} \\ 9 \end{cases}$$

The property «$2n = p_1 + p_2$» is valid but we do not know if it is valid for every integer. In fact, in the XVIII century the mathematician Christian Goldbach (1690–1764) conjectured[24] that every even integer greater than 2 (*Goldbach number*) can be expressed as the sum of two primes numbers[25]. In this sense we can write, e.g, 10=7+3, 14=13+1, 18=13+5. As often occurs with conjectures in mathematics, one can read a large number of supposed proofs of the *Goldbach conjecture*, but they are not currently accepted by the mathematical community. To be brief, it is not possible to prove truth content in proofs of the *Goldbach conjecture*. Moreover, a counter–example is also impossible. Thus, generally speaking, one can write that *a = 0 (for all of its figures) cannot be claimed because scientific proof is lacking*. Let's note that its negated ($\neg(a = 0)$) sentence cannot be claimed. In fact, if the latter could be proved then that would mean that one was able to claim $a \neq 0$; but, it means that we should also be able to present a counter–example (e.g., negated of negated) of the *Goldbach conjecture*. Thus, one should conclude that: «*a* $\neg\neg$ (*a = 0*) *fails*».

2.3 Logic in the history of physics–mathematics

In the *Preface* (and in *Rules of Reasoning in Philosophy*) of *Philosophiae naturalis principia mathematica* Newton (Newton 1803) assumed his idea on relationship between physics and mathematics separating mechanics into two parts: practical and rational.

> Since the ancients (as we are told by *Pappus*) made great account of science of mechanics in the investigation of natural things; and the moderns lying aside substantial form and occult qualities, and endeavoured to subject to phaenomena of nature to the laws of mathematics, I have in this treatise cultivated mathematics so far as it regards philosophy. The ancients considered mechanics in a twofold respect: as rational which proceeds accurately by demonstration; and practical. To practical mechanics all the manual arts belong, from which mechanics took its name.[26] [...] rational Mechanics will be the science

[24]In 1742, the *Goldbach conjecture* was proposed in a letter addressed to the Swiss mathematician Leonhard Euler (1707–1783).
[25]Let's remark that it is a curious property because prime numbers cannot be deduced by division, while odd numbers and the sum of two odd numbers concern another operation.
[26]Newton 1803, "IX, line 4. (*Italic style* by the author).

> of motions resulting from any forces whatsoever, and of the forces required to produce any motions, accurately proposed and demonstrated [...] And therefore we offer this work as mathematical principles of philosophy. For all the difficulty of philosophy seems to consist in this—from the phenomena of motions to investigate the forces of Nature, and then from these forces to demonstrate the other phenomena [...][27].

Let's see the Newtonian principle of inertia (NPI):

> DEFINITION III. *The* vis insita, *or innate force of matter is a power of resisting, by which every body, as much as in it lies, endeavours to preserve in its present state, whether it be of rest, or of moving uniformly forward in a right line.*[28] (Newton 1803, I, 2; *Italic style* and capital letters belong to the author).
> Axioms; or Laws of Motion. Law I. *Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.*[29]

At present, one can read: *Every body will persist in its state of rest or of uniform motion (constant velocity) in a straight line unless it is compelled to change that state by forces impressed on it.* It has been remarked (Nagel 1961) that all physical laws can be expressed by means of a proposition preceded by two *universal* and *existential* quantifications[30]: ($\forall$) ("for all") and ($\exists$) ("there exists" or "for some"). A possible logical expression of the Newtonian principle of inertia can be:

$$A = \forall x \exists y : P(x, y) \tag{4}$$

A	= the proposition of the principle of inertia.
X	= a body.
y	= a complex system constituted by an inertial system, a closed system, and a clock.
P(x,y)	= a predicate concerning *x* and *y*: «if *x* is in *y*, than it is of its state of rest or of uniform motion».

Nevertheless, the aforementioned discussion belongs to classical mathematical logic which – by nature – is not interested in the *effective calculability* of its functions with respect to, operative–experimental– data. In this sense, the content of the first principle is lacking in experimental and *calculable evidence*. In particular:

The Newtonian principle of inertia claims that *y exists*, but it does not claim how one can find it.

The "whether" (or commonly "unless") contained in the proposition is not an *operative situation*. It only explains *a posteriori* the changes of state of motion occurred to the body.

A precise distinction when $v = 0$, and when $v \neq 0$, is required by NPI.

A precise knowledge when for $v = constant$ in orientation (direction and versus) and in scalar–magnitude for the entire path is required by NPI.

A precise knowledge of absence–forces or of a non–zero net force is required by NPI.

The Newtonian principle of inertia is valid subordinately to validation of $\sum F_i = 0$

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[27]Newton 1803, X, line 3.
[28]Newton 1803, I, 2. (*Italic style* and capital letters belong to the author).
[29]Newton 1803, I, 19. (*Italic style* belong to the author).
[30]At beginning of the last century, Thoralf Albert Skolem (1887–1963) suggested a technique to formalize the existential quantification on *y*–variable of a given predicate into a constructive mathematical function (Skolem [1920] 1967).

(for material–point and on the entire path).
Every physical variable should be subjected to its measurement. If the measurement cannot apply, the scientific content generates *uncertainties* in scientific knowledge. For that reason, the content of NPI as mentioned above, can be expressed by a DNS,
«$\neg\neg A$: *It is not true that* $v = 0$ *is not equal to* $v \neq 0$». Thus, all of the examined experimental–logical–*ambiguities* reported can be found in the Newtonian principle of inertia within a non–classical logic investigation.

2.4 A case study: principle of Inertia in Newton and in Lazare Carnot

Based on the previous section, if we consider an operative physics, to translate each *x* body in an effective procedure it is necessary to obtain an inertial system: an isolated system and a clock. Two centuries of unprofitable research demonstrate convincingly that with the sole knowledge of the body *x* it is not possible to operatively obtain many bodies. In order to make a physical–mathematical equation like (4) relatively operative, it might be obtained by forcing the predicate, which is by means of one of these three approaches.
1) Substitute the quantifier $\exists$ in (4) with a constant value y_0. This affirmation results in:

$$\forall x\ A(x, y_0) \tag{5}$$

That is, each *x* body at rest or in rectilinear uniform motion if placed in y_0 (that is if its motion is measured, with respect to a given inertial system, in a given closed system provided of a precise clock). Equation (5) corresponds to defining the clock and the reference system in two very different ways:

a. In the way followed by the physicists since the age of Galilei, that is with an empirical clock as a reference, with the closed system verified empirically and with the earth reference system; except changing those on a case by case basis in accordance with the following definitions).
b. In the idealistic way suggested by Newton (that is idealising this experimental method to the limit, transcending the same experience: introducing the idealised concepts of absolute space and time, that fix once and for all the clock and the inertial system and then implicitly suggest that we are always capable, as a matter of principle, of verifying if $F = 0$ or not and then knowing when a system is isolated or not).

2) To accept the fact that in general we ignore the generic function $\alpha(x)$ but to annul the problem of the existential quantifier saying: in specific circumstances experimental Physics can define: «$\forall x\ A(x, y)$», without further explanation as to what the experimental physicists should do. 3) In order to deny the physical importance of these problems,[31] qualifying them as metaphysical ones. We can only affirm that we can make experimental observations on an «*x*» body: the impossibility for a single body under observation to change on its own its status of motion when at rest or in a rectilinear uniform motion. No quantifiers, nor «$\exists$», nor then «$\forall$». This is what Lazare Carnot (1753–1823) did in *Principes fondamentaux de l'équilibre et du mouvement.* (Carnot L. 1803). In *Principes fondamentaux de l'équilibre et du movement,* Lazare Carnot offers his version of the principles of a PO type mechanics by the formulation of seven fundamental hypotheses (Gillispie and Pisano). I am only interested in the first one for this article:

[31]It has been demonstrated that it is possible to apply this method to the third principle of dynamics «for every action there is an equal and opposite reaction: that is the actions of two bodies are always equal to one another and directed towards opposite directions.

> *Notions préliminaires. Hypothèses admises comme lois générales de Équilibre et du mouvement. Conséquences déduites de ces hypothèses. 1° Hypothèse.* Un corps une fois mis en repos, ne suroît en sortir de lui–même, et une fois mis en mouvement, il ne suroît de lui– même changer ni sa vitesse, ni la direction de cette vitesse.[32]

One of the main differences between Lazare Carnot's mechanics and Newton's mechanics lies in the fact that the first speaks of every body in every time and in every place, while L. Carnot speaks of a restricted whole of situations: those where it is possible to affirm that a body is at rest or in motion. These situations are indicated by an intentional generic introduction "[…] once […]". It is thanks to these generic terms that Lazare Carnot's version avoids the problem implicit in Newton's terms, when we talk about rectilinear and uniform motion "[…] until [...]" that is. Lazare Carnot avoids the problem of deciding when it is $F \neq 0$ along the course (potentially infinite.) Therefore, in his principle of inertia L. Carnot correctly does not name the forces and does not ask for a verification of their absence $\sum_i F_i = 0$ along the entire course of the body. He says that it is not possible to evaluate in a definite way: *If a motion is absolute, or if there is a motion or a dragging force, […] and it has been very difficult to correct this error.* There is no verification of the absence of forces. Lazare Carnot, then, says deliberately "[…] once it is […]" then, in the condition where we can decide, due to specific circumstances, that a body is static or a rectilinear uniform motion. Therefore for the French scientist it is up to our judgment, empirical and occasional, to decide if a body is static or a rectilinear uniform motion. A problem equivalent to the previous one (establishing if $F = 0$ is exact) is the following: Newton would claim to establish exactly when a body is in a resting status as different from the motion status; this means deciding if $v = 0$ is exact (but not if $v < \varepsilon$). Lazare Carnot's definition "[…] once […]" avoids this problem. The definition of Lazare Carnot's first hypothesis does not claim to provide rules to verify the status of rest or motion. Generally these are impossible since they would be circular by the definition of an inertial reference system.

In the end, the principle of inertia states that rest and rectilinear uniform motion are equivalent. But what does *equivalent* mean? Newton's statement treats the two cases as if they were the same thing (…"at rest or in motion…"). Carnot's statement, however, is more cautious; it is broken up into two parallel but distinct affirmations: it does not take the passage from statics to dynamics for granted. So after this initial hypothesis, his other hypotheses articulate this equivalency in gradual passages. In fact, while his second hypothesis still concerns static situations, the third and the fourth hypotheses include dynamics. So we conclude that Carnot's hypotheses (after the first) are a precise strategy of passage from statics to dynamics. In addition we note that all of the aforementioned hypotheses are constructive since they are essentially experimental, except for the fourth which is considered by Carnot as a mathematical convention.

3 What is the role played by history in sciences teaching?

By focusing on mathematics and physics, the previously quoted aspects move towards a

[32]Carnot 1803, 49, line 3. (*Italic style* from the author). First hypotheses: "A body once at rest would not be able to move on its own, and when put in motion could not change its speed or direction by itself" (My synthetic translation).

larger base of analysis which includes not only disciplinary matters but also interdisciplinary issues in philosophy, epistemology, logic and the foundations of physical and mathematical sciences. We need a strong effort for an interdisciplinary approach to *teach* and *learn* the relationship physics–mathematics as a discipline of study (Martinez; Meltzoff et al.). It has been noted that teachers regularly have great difficulty teaching historical and philosophical knowledge about science in ways that their students find meaningful and motivating. Thus, *how is it possible to keep on teaching sciences being unaware of their origins, cultural reasons and eventual conflicts and values?* And *how is it possible to teach and comment on the contents and certainties of physics and mathematics as sciences without having first introduced sensible doubt about the inadequacy and fluidity of such sciences in particular contexts?* Education needs to revaluate scientific reasoning as an integral part of human (humanistic and scientific mixed) culture that could build up an autonomous scientific cultural trend in schools (Pisano 2009b). *In this sense, what about the importance of introducing the history of science as an integral part of the culture of teaching education to the extent of considering such a discipline – in its turn – as an indissoluble pedagogical element of history and culture?* (Pisano and Guerriero) "To foresee the future of mathematics, the true method is to study its history and present state".[33] It would be useful to pay particular attention to the elaboration of the *teaching–learning* process based on the reality observed by students (inductively), by a continuing critical reflection, e.g. by means of studying the historical foundations of modern physical and mathematical sciences. Therefore, turning from *teaching based on principles* to *teaching (also) based on broad and cultural themes* would be crucial. It would mean teaching scientific education as well, which is a kind of education that poses *problems* and as far as physics is concerned, introducing it through historical and philosophical criticism as well. It would be helpful to practically support processes on a multidisciplinary or even on co operational level, a kind of pedagogy able to re–consider, from this point of view, the relationship between theory and experience, history and foundations. Let's think about (1) the lack of a relationship between physics and logic (Pisano 2005)…. the organization of a scientific theory (axiomatic or problematic) and its pedagogical aspect based on planned and calculated processes, (2) when we use the term *mechanical* associated with a problem, model, law et al...; (3) the problems of foundations, for example in the teaching phase of the passage from mechanics to thermodynamics, is not yet completely solved; (4) teaching of the non–Euclidean geometries or of the planetary model as an introduction to the study of quantum mechanics, was born, as a matter of fact, only thanks to the fact that the old concept of trajectory was abandoned in favour of the probabilistic one; (5) the concept of infinite and infinitesimal in limits compared to measures in a laboratory… (6) et al… Through an educational offer enriched with the study of the foundations of physical and mathematical sciences, complete with the intelligent use of pedagogical computing technologies, a kind of teaching might be accomplished with the model of the prevailing method of the teaching–learning process mainly related to and coming from reality. It would be an attempt necessary to show how paths usually chosen have not been unique in the history of science but very often an alternative possibility has existed. For example: the statics in Jordanus de Nemore (ca. XIII century) and in Tartaglia (Tartaglia 1554, books VI–VIII), the physics–chemistry of Newton and Antoine–Laurent de Lavoisier (1743–1794), the

[33] Julius–Henri Poincaré (1854–19121) quoted in: Klein 1980, 3.

mechanics of Lazare–Nicolas–Marguerite Carnot (1753–1823) etc. More specifically: the second Newtonian principle is not strictly a physical law and it has just a little in common with physical laws by Galilei rather than showing that the historical foundations of thermodynamics which are based on five (Pisano 2010) epistemological principles, more than the classical ones read in a textbook. From cognitive–epistemological point of view (George and Velleman), people do not naturally and scientifically reason by means of deductive or inductive processes only. In this regard, scientific reasoning (Lakoff and Nunez) is not a part of our common knowledge reasoning), although we often intuitively compare events, tables etc. Instead, it was remarked that we reason mainly by the association of ideas and sometimes concepts are far from the scientific ones, e.g. heat and temperature, mass, weight and force–weights, the solar system and atomic orbital system in quantum mechanics, the kinetic model of gases and thermodynamics, parallel straight, material points et al. Thus, the current scientific teaching system paradoxically changes the logical basis of reasoning. An hypothetical proposal, of course not the only one possible, could be the introduction within the educational plan of reading passages *ad hoc* centred on mathematics and physics to be analysed in the classroom, main books by Aristotle's mechanics (*mechanical problems*), Euclid (*Elements*), Archimedes (*On equilibrium of planes*), Tartaglia (*Quesiti*), Galilei (*Discorsi*), Torricelli (*Opera*), Lazare Carnot (*Essai*) Lavoiser (*Traité*) Sadi Carnot (*Réflexions*), Faraday (*Experimental Researches*) et al. Reading such passages, together with pre–arranged and effective work shared by several subjects, 1) *the student is placed before a problematic situation and driven to realise the inadequacy of his/her basic knowledge with regard to problem solving.* 2) *When the build up of scientific education begins, in order to overcome such difficulties.* The result will be pedagogy according to which science education (Osborne and Collins; Debru 1997; Id., 1999) essentially means *setting and solving problems* and teaching means re–evaluating the relationship between theory and experience and between history and foundations. They could come together with well–structured and practical interdisciplinary work by means of the history of science. International debate should take into account pedagogical research on foundations for history and learning–teaching science, discovering science teaching and informal learning activities as well. In this way, a student is the *protagonist,* both formally and informally (*hands–on*), of his *learning*. I feel the same about schools training experts, as these also should provide a setting that favours teaching research aimed at the critical re–construction of scientific meanings along with ideas, opinions and proper contents. In the end, this briefly proposed reflection should convey that it is urgent to establish the basis for a debate that ethically appears correct and professionally necessary. Maybe, operating in a different way, we could also contribute to building a school (or university) linked to the new perspectives of science, its image and teaching without limitations on specializations, pushing past disciplinary competences.

4 Final remarks

To sum up, one could think of:

Appealing to students for a scientific culture through the culture of history and philosophy, regardless of the sterile dichotomy between human and scientific disciplines.

Physics in the 20th century changed either the fundamentals of classical physics (and of science as well), or lifestyle (for better and for worse). A reflection based on a program, according to the spirit of research and inter–discipline, and pedagogically–oriented, is always to be regarded as a topic of interest, never obvious.

Inviting a motivated and interested study of physics and mathematics through a wider historical and philosophical knowledge of epistemological criticism.

Trying to re–build the educational link between philosophy and physics–mathematics. For ex., philosophy, from the end of the XIX cent., seems to have no longer found a steady link with physics whose interpretation of a phenomenon is sometimes based on the involvement of an advanced and elaborated mathematics.

Dissemination and sharing of difficult theoretical and experiential works.

Make the students understand that the history of scientific ideas is closely related to history of techniques and of technologies; that is why they are different from one another.

Make the others understand that scientists were once people studying in poor conditions.

Show the real breakthrough of scientific discoveries through the study of the history of fundamentals, not yet influenced by the (modern) pedagogical requirements. For ex: understanding the historical turnover of the principles of classical thermodynamics into the usual teaching of physics.

Let the students experiment discoveries with enthusiastic astonishment through a guided iter reflection on the fundamental stages of progress and scientific thought

Also, a provocative hypothesis: *generally speaking, we should not lose the certainty of a critical thought on science... but if we do not do anything, then nothing changes... but if we do something (a few crucial things), maybe something could be improved.*

> The loss of truth, the constantly increasing complexity of mathematics and science, and the uncertainty about which approach to mathematics is secure have caused most mathematicians [and scientists] to abandon science. With the "plague on all your horses" they have retreated to specialties in areas of mathematics [and physics] where the methods of proof seem to be safe. They also find problems concocted by humans more appealing and manageable than those posed by nature. The crises and conflicts over what sound mathematics is have discouraged the application of mathematical methodology to many areas of our culture such as philosophy, political science, ethics, and aesthetics. The hope of finding objective, infallible laws and standards has faded. The Age of Reason is gone. With the loss of truth, man lost his intellectual center, his frame of reference, the established authority for all thought. The "pride of human reason" suffered a fall which brought down with it the house of truth. The lesson of history is that our firmest convictions are not to be asserted dogmatically; in fact they should be most suspect; they mark not our conquest but our limitations and our bounds.[34]

"Revolution in Science Education[?]: Put Physics First" (Lederman, 44). All of us put a professional teacher first: teachers that teach, research and publish...

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[34]Klein 1980, 7, line 12; 99, line 14. The quotation belongs to the author.

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MATHEMATISATION OF NATURE AND NEW CONCEPTIONS OF CURVES IN THE YEARS 1630

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ABSTRACT

In years 1620-1640, a new approach of the phenomena of nature led to new problems, because the scientists are now more interested to understand phenomena by finding laws than to explain them by their causes. In this worshop we will examine two examples, where the news problems led to new conceptions of curves. Indeed, Greeks geometers studied curves only by geometrical propositions. But with the study of the trajectory of a projectile, Galileo obtained a kinematical conception of curves and with the research of optical curves, Descartes proposed an algebraic conception of curves. In each example, there is also a new conception of a tangent to a point of a curve, which is linked with the phenomena of nature and which permits to obtain methods of invention of tangents.

In 1638, Galileo published his Discourses on two new Sciences, where he proved that the trajectory of a projectile is part of a parabola. For the Greek geometers, the parabola was a static object, defined geometrically by the intersection of a cone with a plane. With Galileo, the parabola becomes a cinematic notion, conceived as the trajectory of a moving point. The Roberval's method for finding the tangent at a point of a curve is based on the kinematical conception of curves. The study of motion in the 17th century become a powerful instrument of invention leading, at the end of the century, to the method of fluxions of Newton

Around 1628, Descartes solved the anaclastic problem, that means the problem to find the form of a lens such that if incident rays of light are parallels then the refracted rays converge to one point. The research of optical curves, like the ovals, led him to a double conception of a curve, as a result of well regulated motions and as an equation, and also to his algebraic method to find circles tangents to a curve.

These historical examples can serve for two purposes: to enrich the mathematical teaching of curves and to build bridges between teachings of mathematics and physics.

Historical texts

Galileo, *Discourses on two new sciences*, 1638.
Roberval, *Observations on the composition of movements and on a way of finding tangents to curves*, 1693.
Newton, *Method of Flusions and infinite series*, 1736.
Descartes, *The Geometry*, 1637.

WHAT CAN WE LEARN FROM A 16th OCCITAN TREATISE OF ARITHMETIC?

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ABSTRACT

We studied the “Cisterna Fulcronica” (Fulconis’ Cistern), published in 1562. Written in the language of Nice by a master in Writing and Arithmetics, it is explicitly intended to the self-training of merchants and craftsmen in commercial arithmetic.
Fulconis mentions accurately his sources: Jacques Grant, Estienne De La Roche, Gemma Frisius, allowing us to look at his theoretical and pedagogical choices: for instance, the choice and place of definitions, examples and applications, the memorizing methods as repetition, striking denominations or visual layout. Moreover we can compare with another reading of Gemma Frisius, namely that of Pierre Forcadel. This study allows teachers to discuss the ways of presenting and passing on a mathematical notion to our students.
By the way, Fulconis opens a window to the life in the County of Nice, whose history is not taught in France.
Some extracts of this book have been used in interdisciplinary work, in order to impart to students both some knowledge of Occitan culture, as well as arithmetical notions, rather basic but not that well known by non-scientific students.

Though discussions were in English, all the source-texts (Fulconis, De la Roche, Frisius, Forcadel) were translated only into modern French.

SOME MATHEMATICAL TOOLS FOR NUMERICAL METHODS FROM 1805 TO 1855

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ABSTRACT

This work covers the first half of the XIX^{th} century, and more specifically the period 1805 to 1855, which concerns our quest to find roots to numerical mathematics. We chose the date of 1805 with Legendre on least squares, and Gauss on the Fast Fourier Transform (FFT), also the closing date of 1855 with Chebychev and his discrete generalized Fourier series. During this period, at least four great mathematicians: Gauss, Cauchy, Jacobi and Dirichlet contributed greatly to the "approximation mathematics", and we shall present some of their motivations. In this present work, what concerns us mostly is what is called the "almost lost and found again". We have selected by chronological order, the 1833 Duhamel principle on convolution, and the 1841 Sarrus article about systems of non-linear equations, with the concept of robustness in numerical mathematics.

1 Introduction

This work covers the first half of the XIX^{th} century, and more specifically the period from 1805 to 1855, which concerns our quest to find roots to numerical mathematics. We selected the date of 1805 and the least squares method for the wave it produced in applied mathematics, and the closing date of 1855 with Chebychev's work on discrete generalized for its modern applications in signal processing. Our objective is to present some mathematical tools which were almost forgotten such as the Duhamel principle with its links with the convolution integral, and the solution of systems of non-linear equations with Sarrus' ideas and the evolution towards robustness in mathematics. Firstly, let us quote Odifreddi (p. 92, 2004) about pure and applied mathematics:

> "Mathematics like the Roman god Janus, has two faces. One is turned inward, towards the human world of ideas and abstraction, while the other looks outwards, at the physical world of objects and material things… The second face constitutes the applied side of mathematics, where the motives are interested, and the aim is to use those same creations for what they can do. "

Householder (p. v, 1970) argued that the adjectives "*pure*" and "*applied*" are meaningful only to describe mathematicians, but not branches of mathematics! The truth is that some mathematicians tried to build mathematical tools for practical applications and their usefulness. They were also motivated by the validity of their approach. This present work concerns a period of fifty years from 1805 to 1855, i.e., from the Napoleonian regime to the Second Empire in France. From this epoch, two men seem to dominate: C.F. Gauss for the fertility of his methods and ideas, and Cauchy for the validity of mathematical tools. If we take Cauchy as an example, he developed the existence theorems for ordinary differential equations, the solution of systems of linear differential equations, the interpolation theory, the spectral theorem for matrices, the convergence properties of the Newton method for finding roots, and the steepest descent for minimization methods, etc. Gauss' contributions on the solution of linear equations were a major breakthrough, and also his Gaussian quadrature for the numerical solution of integrals. These contributions are well presented in books on the history of numerical methods, such as Chabert et & al. *Histoire d'algorithmes* (1993). Let us list by chronological order some main mathematical tools that mathematicians developed for this period from 1805 to 1855:

1805: Legendre: least squares
1805: Gauss: Fast Fourier Transform (FFT)
1809- 1810 : Gauss: least squares
1809-1810: Gauss: gaussian elimination for systems of linear equations
1809 : Gauss: solution of systems of two non-linear equations by *Regula Falsi*
1816: Gauss: gaussian integration
1823: Gauss: iterative method for systems of linear equations
1819,1820, 1830: Horner and Holdred: root computation
1824-1835: Cauchy: on the existence of solutions of ordinary differential equations (ODES's)
Cauchy on systems of ODE
1826: Jacobi: on Gaussian quadrature
1829: Cauchy: convergence of Newton's method for root finding
1829: Lejeune Dirichlet: pointwise convergence on Fourier series
1831: Gauss: on numerical lattices
1833: Duhamel: Duhamel principle for inhomogeneous differential equations
1840: Cauchy: on interpolation
1841: Sarrus: on the solution of systems of non-linear equations
1845; Jacobi: iterative method for systems of linear equations
1846: Jacobi: algebraic eigenvalue problems
1847: Thomson-Dirichlet principle
1847: Cauchy: steepest descent (optimization theory)
1850: Lejeune-Dirichlet: on tessellations
1850: Sylvester: on matrices
1851: Shellbach: numerical solutions of partial differential equations (PDE's)
1855: Chebychev: generalized discrete Fourier series
1855: Cayley: on matrices

This period of time from 1805 to 1855 saw a blossoming of mathematical tools for numerical problems. This genetic approach for the work could be used for second year students in applied mathematics, who have some knowledge on partial differential equations and the method of separation of variables, while the second part of this research could be used for students having some interest on numerical analysis or optimization theory.

2 Duhamel principle and convolutions

Unfortunately, if the convolution integral is a classical modern tool in mathematical physics and signal processing, its long period of genesis is generally ignored by educators.
In its modern definition, a convolution corresponds to the mathematization of a memory problem, smoothing processes, or more precisely, it involves an operator with translational invariance, and the famous principle of causality being that the future cannot influence the present (Sirovich, pp. 80-82, 1988).

During the $XVII^{th}$ century, physical properties of the convolution appeared in the Huygens *Treatise on Light* (1690). Figure 1 is an illustration of Huygens' principles about the propagation and the superposition of spherical waves. He wrote:

> "But we must consider still more particularly the origin of these waves, and the manner in which they spread. And, first, it follows from what has been said on the production of Light, that each little region of a luminous body, such as the Sun, a candle, or a burning coal, generates its own waves of which that region is the centre. Thus in the flame of a candle, having distinguished the points A, B, C, concentric circles described about each of these points represent the waves which come from them"

Figure 1: Huygens' candle and the propagation and superposition of spherical waves

Convolution integrals appeared at the beginning of the XIX^{th} century in the potential theory, the heat conduction equation, and the wave equation, with Cauchy, Fourier and Poisson. They appeared from trigonometric operations as a convenient way to represent analytical results. For example, Poisson (1823, p. 434) wrote his Euler-Fourier series as:

$$fx = \frac{1}{2\ell}\int_{-\ell}^{+\ell} fx'dx' + \frac{1}{\ell}\int_{-\ell}^{+\ell}\sum \cos\frac{n\pi(x-x')}{\ell} fx'dx' \tag{1}$$

Indeed the history of the convolution integral is also linked to integral equations and Abel's work (1826) who presented a convolution integral for the time of descent of a particle starting at a point *P* sliding down a smooth curve.

From these works emerge the 1815 Cauchy contribution *Wave propagation in deep water* (Cauchy 1815-1827; Dahan Dalmedico, 1989) and the Duhamel principles (1833) on radiating heat processes, with complex radiating boundary conditions. These Duhamel principles are occasionally mentioned in books on mathematical physics such as Courant and Hilbert *Methods of Mathematical Physics* (vol. 2, 1953-1962), or Hildebrand *Advanced calculus for Applications* (pp.464-465, 1976). Jean-Marie Constant Duhamel was born in Saint-Malo, France in 1797, and died in Paris in 1872. He became a professor at the École polytechnique in Paris. His third theorem is stated as follows:

> "If a part of the surface of a body is maintained, along with certain points within the interior, at temperatures that vary with time in some way, and the rest of the surface radiates in a medium where all points have some variable temperatures with time, we will obtain in the following way, temperatures of the points of the system, and we will calculate the temperatures of the different points of the body, while supposing that no change takes place in temperatures of the medium, and of all the points of the surface and its interior, of which the law of temperatures is given: we shall consider next the increases produced, at some instant in the temperatures of these last points and of the medium, and we shall calculate the temperatures of various points of the body at the end of time *t*, by giving them zero as initial temperature: we will do the sum of the temperatures of homologous points of these systems, and we will that way obtain temperatures of the same points in the proposed system."

Duhamel introduced into mathematics the Huygens physical concepts about memory, time-delay, and superposition of events. For example, let us assume that $\varphi(t)$ is the variable temperature of the surrounding medium of a one-dimensional heat conducting rod, which is radiating into this medium, at its extremities. Then, $\varphi(t)$ has a complex action on the temperature of the rod. Duhamel assumed that he started from an initial condition $\varphi(0)$ with an action $A(x,t)$. Then, at time θ_1 and the condition $\varphi(\theta_1)$, the action would be delayed as $A(t-\theta_1)$ updated by an additional step $(\varphi(\theta_1)-\varphi(0))$, and so on. It was a discretisation of a complex integration, a technique of quadrature, and the final temperature corresponded to the summation of the action of all rectangular panels. The result would be the classical convolution integral. The summation process and the convolution integral are illustrated in the following equations:

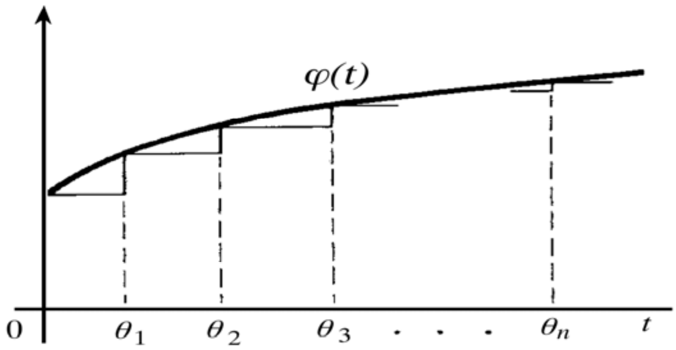

Figure 2: Approximation of the function $\varphi(t)$ by rectangular panels.

$$V(x,t)=\varphi(0)A(x,t)+[\varphi(\theta_1)-\varphi(0)]A(x,t-\theta_1)+[\varphi(\theta_2)-\varphi(\theta_1)]A(x,t-\theta_2)]+\ldots$$
$$+[\varphi(\theta_n)-\varphi(\theta_{n-1})]A(x,t-\theta_n)]$$
$$V(x,t)=\varphi(0)A(x,t)+\sum_{k=0}^{n-1}A(x,t-\theta_{k+1})\left(\frac{\Delta\varphi}{\Delta\theta}\right)_k\Delta\theta_k$$

We obtain the convolution integral by taking the limit version of the above expression:

$$V(x,t)=\varphi(0)A(x,t)+\int_0^t A(x,t-\theta)\varphi'(\theta)d\theta \qquad (2)$$

Duhamel illustrated his technique by applying it to the general solution of the heat conduction equation of a one-dimensional problem with three arbitrary conditions,

$$\frac{\partial V}{\partial t}=a\frac{\partial^2 V}{\partial x^2};\ D=\{(x,t)\,|\,0<x<\ell,t>0\}, \qquad (3)$$

where a is the thermal diffusivity coefficient, ℓ is the length of the rod, and D is the domain of integration. The equation was coupled with radiating boundary conditions at both ends:

$$\begin{aligned} &A\frac{\partial V}{\partial x}(0,t)-h(V-\varphi(t))=0\\ &A\frac{\partial V}{\partial x}(\ell,t)+k(V-\psi(t))=0 \end{aligned} \qquad (4)$$

and the initial condition: $V(x,0)=F(x)$, where $\varphi(x),\psi(x),F(x)$ are arbitrary functions. The radiating boundary conditions corresponded to the Newton law of cooling, and both ends radiated into mediums of thermal conductivities *h* and *k,* while the lateral surface is isolated.
Duhamel proceeded by steps:

a) He first looked for a permanent simplified solution in the limit $t\to+\infty$, with $\varphi(t)=V_1=const\ \ \psi(t)=V_1+\xi=const$.
b) He then solved the time dependant problem for homogenous boundary conditions with $V_1=\xi=0$, and then did a transformation of variables in the case $V_1\neq\xi\neq 0$.
c) Finally, he replaced V_1 and $V_1+\xi$ by the initial values $\varphi(0)$ and $\psi(0)$,
d) At time $t-\theta$, he superposed a solution $V_1 \cong \varphi'(\theta)\,\Delta\theta$, $\xi \cong (\psi'(\theta) - \varphi'(\theta))\,\Delta\theta$

Duhamel ended up with an integral of the type: $\int_0^t e^{-am^2(t-\theta)}\{M\varphi'(\theta)+N\psi'(\theta)\}d\theta$, which has the same behavior as the convolution integral given by Eq. 2. The solution appeared as a time-convolution integral in terms of the derivatives of the boundary conditions.

Convolution integrals also, slowly, appeared in linear non-homogenous ordinary or partial differential equations. An example of this was the Liouville *second memoir* (1837) on differential equations, where the convolution integral appeared in the solution of the differential equation:

$$\begin{aligned} &\frac{d^2U}{dx^2}+\rho^2U=\lambda U\\ &U(x)=A\cos\rho x+B\sin\rho x+\frac{1}{\rho}\int_0^x\lambda(x')U(x')\sin\rho(x-x')dx' \end{aligned} \qquad (5)$$

The famous 1882 Kirchhoff formula for the solution of the three-dimensional wave equation can also be understood as a time-convolution integral (Kline, vol.2, p. 694, 1972).

3 Zeros of a vector equation: towards robustness

The second tool concerns the 1841 Sarrus contribution to the solutions of non-linear vector equations. It is an enlightened anticipation towards the modern concept of robustness in mathematics (Box, 1953), because finding the roots of a vector equation is a very difficult task. A method is said to be robust, if it is reliable and efficient. For example, the classical least squares method is not, because it is too sensitive to outliers. Firstly, we shall review the state of knowledge for finding roots of a single non-linear equation $f(x) = 0$. This is one of the most commonly occurring problems of applied mathematics. At the beginning of the XIX^{th} century, available tools were the Newton-Raphson method, the *regula-falsi* (also known as the method of the false position) and the method of continued fractions. These numerical methods for root finding are iterative methods. The ongoing interest in continued fractions was reflected in Gergonne and Liouville journals, where we found at least 7 articles concerning this topic. In England, important progress was made in computation of a real root of a polynomial equation with the Horner-Holdred rule. In 1818, Fourier (Ostrowski, 1966) emphasized that the Newton-Raphson method was one of the most useful tools in all analysis. This is why it was important to complete it and to overcome its deficiencies, i.e. the divergence problem. Indeed, for the period of time between 1805 to 1855, the 1829 Cauchy contribution about convergence properties of the Newton method dominated. At this epoch, Lagrange book "*Traité de la résolution des equations numériques*" (1798) became a corner stone in this domain. Again, the two volume Legendre book "*Théorie des nombres*" (1830) contained an appendix about numerical root findings and a voluminous chapter on continued fractions. Given this state of knowledge for the roots of a non-linear equation, we examine the problems for the zeros of a vector equation. From Gauss (1809), Fourier (1818), and Sarrus (1841), we were struck by the concern of geometers about the robustness of their methods. This robustness is directly linked to the consequences of the intermediate value theorem for continuous functions, which states that:

> "*between every two values of the unknown quantity, which give results of opposite sign, there must always lie at least one real root of the equation*."

Before Bolzano (1817), proofs of this theorem were based on geometrical propositions. Furthermore, Bolzano utilized a bisection technique on the proof of one of his theorem.

These robustness concepts appeared in the 1809 Gauss article who selected the analogy of the *regula falsi* for two equations with two unknowns. The *regula falsi* utilizes the intermediate value theorem and root bracketing. The concept of the *regula falsi* was to progressively decrease the interval of uncertainty for root findings.

In 1841, Pierre–Frédéric Sarrus published his article in Liouville Journal: *Sur la résolution des équations numériques à une ou plusieurs inconnues et de forme quelconque*. Sarrus was born in 1798 in a small town in France. In 1829, he became professor of mathematics at the University of Strasbourg. He died in 1861. He published regularly in Gergonne's journal, where he had 23 publications from 1820 to 1828.

In his 1841 article, Sarrus proposed three safe methods for root findings. In the first method, he searched for upper and lower bounds for all variables, which include the roots. Then, he proposed a subdivision of the system of bounds for root bracketing. Unfortunately, Sarrus was not explicit enough on his subdivision; probably, it corresponded to a technique of bisection. This technique of division of an interval by a factor 2 was well known as a binary search during the $XVIII^{th}$ century. For example, it was the algorithm for finding a word in a dictionary. We can consider that Sarrus' first method is an ancestor for the modern "cubic" algorithm where roots are included in n-dimensional parallelepiped with sides parallel to the coordinate axes (Hansen, 1980; Galperin, 1988). Sarrus' style was very direct, efficient and modern with almost an algorithmic approach, as we will see in the following sentences:

> "In view of one or more equations of any form:
>
> $L = 0, M = 0, N = 0, ...,$
>
> In one or several unknowns, which number can be different of that of the equations; in view of, besides a system of limits of values of unknowns, find all values of unknowns which can be comprised between the given limits, and satisfy, at the same time the equations $L = 0, M = 0, N = 0, ...$.
>
> *First method*: We will start by looking at inferior limits of values which can be received by the functions $L, M, N, ...$ when we vary $x, y, z, ...$ between the given limits.
>
> But when all calculated inferior limits will be negative, we will proceed to calculate the superior limits of values of the same functions $L, M, N, ...$."

We then have to reduce the intervals of uncertainty for each variable:

> "Accordingly to that, we will subdivide the system of given limits of values $x, y, z, ...$ in several systems of limits more closely, which the entirety will contain the same extent as that of the systems of primitive limits"

Sarrus'second method was also based on interval analysis, but this time, Sarrus transformed a system of non-linear equations into a minimisation problem. In 1847, Cauchy will do the same for his steepest descent algorithm.

> *Second method*. We will take some positive numbers α, β, γ , and doing so, to abridge,
>
> $V = \alpha L^2 + \beta M^2 + \gamma N^2 + ...$
>
> We will then only have to resolve the sole equation $V = 0$. Then we will treat this latter equation by the process of the first method, but with this difference, that it is entirely useless to calculate the upper limits of values of the auxiliary function V. Indeed, according to the compositions of this function, it can never become negative; consequently, his upper limits will always be positive, and that's everything we need to know.

The third method corresponded to a linearization of the equations and a modified generalized Newton method for a system of non-linear equations. Again, Sarrus, being a Fourier follower, was searching upper and lower bounds for all variables. He said to stop calculations, if the system had tendency to diverge.

> *Third method.* It is easy to modify Newton's method, in such a way, that it is never at fault, at least, till the number of unknowns will not surpass that of the equations."

Unfortunately, Sarrus didn't illustrate his methods with examples. Curiously, in 1847, Joseph Liouville, presented a new method for the solution of vector equations, but he hadn't mention Sarrus' contribution!

4 Conclusion

For this work, we could have chosen as well the 1846 Jacobi method on the eigenvalue problem. This method is well covered in any textbook on numerical analysis, but the history of the numerical eigenvalue problem is not. Among other fruitful mathematical tools which were developed during the period of time from 1805 to 1855 are the 1850 Dirichlet work on tessellations, the 1851 Shellbach work on numerical solutions of partial differential equations, and the 1855 Chebychev article on generalized discrete Fourier series.

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SECONDARY SCHOOL STUDENTS' DIFFICULTIES WITH VECTOR CONCEPTS AND THE USE OF GEOMETRICAL & PHYSICAL SITUATIONS

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ABSTRACT

For many years, vectors used to be - and still remain - marginal to the Greek secondary mathematics education. For a long time, young students are left to form their own ideas about vector concepts only through physics courses and everyday life experience, up to grade 11 (16-17 year old), where they are introduced to vector algebra applied to geometry.

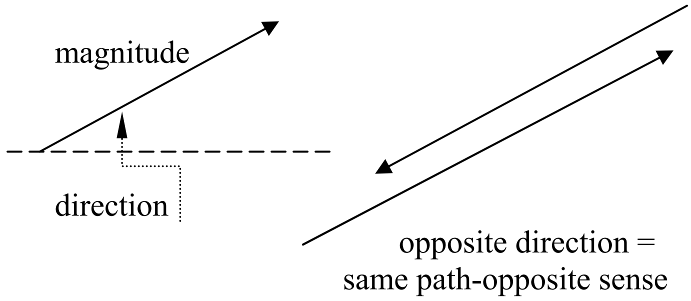
In our previous research with Greek students (9^{th}-12^{th} grades), we identified specific persisting difficulties, concerning certain epistemological aspects of vectorial concepts. Students' difficulties and the historical development of these concepts, led us to a teaching experiment with 8^{th} and 9^{th} graders (14-15 year old), where vector methods & concepts were considered as a new language to be learnt and explored. Inspired by the historical development of vectorial concepts, which was due mainly to problems and situations in physics and geometry, we used privileged situations, based on forces, velocities and displacements, to introduce the concept of vector, vector notation (symbolic representation), geometric representation, comparison between vectors and vector addition. Our experimental approach gave us the opportunity to face, handle and attempt to eliminate a variety of difficulties, related to the multifarious, composite nature of the vector language. The results indicate that our experimental teaching helped students to overcome some of their misconceptions and to proceed to a synthesis of partial concepts in a more coherent and abstract conceptual structure. However, due to space limitations, in this paper we present only some of the activities used in our teaching, focusing on students' misconceptions and difficulties and the role of physical and geometrical situations.

1 Introduction

Vectors in Greek secondary curriculum are marginal in school mathematics teaching. Students form ideas about vector concepts only through physics courses and everyday life experience up to grade 11 (16-17 year old), where they are introduced to vector algebra applied to geometry.

Teaching and understanding difficulties are related either to the epistemological nature of the vector concept because of its multi-dimensional and multi-level character, or/and to the didactical context, since the symbolism and terminology vary in different teaching contexts (geometry, algebra, physics).

For example, in Greek mathematics textbooks, a vector is determined by three components: *magnitude*, *path*, *sense*, whereas, in Greek physics textbooks, a vector quantity is determined by two components: *magnitude* and *direction* (including the concepts of path and sense).



Additionally, for a long time in Greek mathematics courses symbols like $\vec{a}$ or $\overrightarrow{EF}$ have been used, while in physics courses in lower secondary education the symbols used denote only the magnitude of vector quantities and not the vector quantities themselves[1]. This disagreement between mathematics and physics has a negative influence on students understanding of vectorial notation. In a vector's symbol they mainly recognize its "magnitude". For example, they use the notation $\vec{F} = \vec{F}_1 + \vec{F}_2$ for magnitudes of non-collinear forces.

Our previous research with Greek students (9^{th} - 12^{th} grades), has verified specific and persisting difficulties concerning certain epistemological aspects of vector concepts (Demetriadou 1994, 1999, 2002, Demetriadou & Gagatsis 1995, Demetriadou & Tzanakis 2003). Students' difficulties and strong preconceptions concerning vector quantities and operations have also been verified by other researchers, mainly in physics education (Turner 1979, Trowbridge & McDermott 1980, McCloskey 1983, Watts 1983, Aguirre & Rankin 1989, Eckstein & Shemesh 1989, Eisner 1991, Knight 1995).

In our opinion, vectorial notions are too complicated to be introduced in secondary education in their abstract mathematical form. On the other hand, physics is a suitable field to introduce vector concepts and operations in a more intuitive and natural way. Moreover, historically, it was the interplay between physical and purely mathematical situations that led to the emergence of vectorial concepts, operations and methods. In fact, such concepts and methods and the corresponding notation were first established in physics; mathematical practice followed once the efficiency of vectorial methods became clear. This historically undoubted influence of physics is ignored in the Greek secondary education curriculum.

2 The teaching experiment

Based on students' difficulties concerning vector concepts and **implicitly** influenced by the historical development of the subject, we designed and implemented a teaching experiment with 8^{th} and 9^{th} graders (14-15 year old). Vector methods and concepts were treated as a **new language**, which should be learned and its virtues should be explored. Our teaching approach was designed along the following three axes:

1. A historical-genetic approach (Arcavi 1985, Fauvel 1991, Tzanakis 2000) inspired by key issues that were central to the historical development of vector calculus. They are related both to physics and geometry: (a) composition of motions, (b) composition of forces and (c) composition of displacements.
2. Didactical approaches connected to understanding and learning procedures of vector language (Vygotsky 1993, Donaldson 1995, Booth 1981). Vectorial concepts were faced as a language built upon/taking into account pre-existing conceptions and creation of intuition strategies.
3. Pedagogical approaches based on active participation and classroom communication during the learning procedure, for the dynamics of the group/classroom discourse (Piaget 1969, Cobb et al 1992, Radford 2011, Schwarz et al 2009).

The experiment included two phases: (a) a **pilot** teaching of 16 hours on 30 students, 14 year-old, before they had attended any systematic physics course and (b) a **main teaching** experiment of 14 hours with 58 students, 15 year-old, after they had followed part of the

[1]Only in the recently revised version of the curriculum of lower secondary education and the corresponding official textbooks (published in 2007), the vector notation with an overhead arrow is used only for displacements!

physics course. The results of the experimental teaching were compared with those of conventional teaching (following the official curriculum) for an equivalent control group consisting of 53 students.

Due to space limitations, we do not give a detailed account of the whole teaching experiment: In the next section we present only some of the activities used in our teaching experiment, with focus on students' misconceptions and difficulties and the role of physical and geometrical situations. In the last section, we summarize some of the main results that came out of the analysis of our experimental teaching, some of which are based on the comparison with a similar analysis for the corresponding teaching to the control group.

3 Some indicative activities

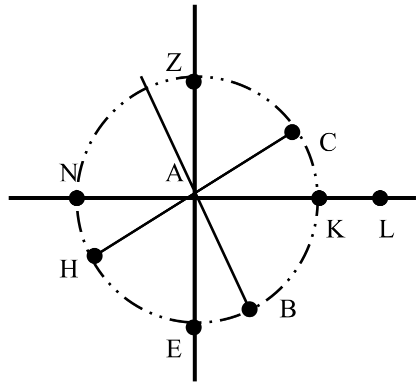
We used didactical activities based on geometrical and physical situations (involving displacements and velocities & forces, respectively), where vector concepts and operations are handled in the context of different conceptual frameworks (Brousseau, 1997). Vectorial methods were treated not as an abstract tool to express, handle and develop logically geometrical and physical concepts, but rather, as a means that clarifies (or even partly determines) their content and meaning, thus becoming crucial for the creation and development of new mathematics (Douady, 1991).

3.1 Introducing the concept of a vector

***1. The introductory activity: An insect's displacement on a plane surface**:*

Peter observes an ant A on his desk, trying to guess where it is going to move[2]. It was easily verified by the students that Peter couldn't do this, since there is an infinite number of possible directions. Given that in 1sec the ant covers 5cm, students were asked to compare possible displacements / trajectories with:

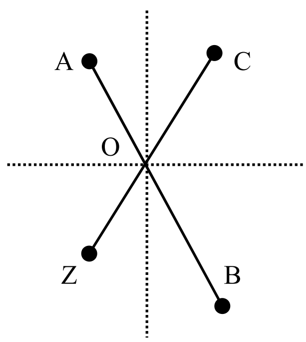
- ✓ Equal magnitude – opposite sense
- ✓ Equal magnitude – different path
- ✓ Same sense-different magnitude



***2. Introductory activity on the terminology: A visitor's displacements**:*

A visitor stands on O. To visit A he is told to move: a) opposite to Z, b) opposite to B, c) in the same direction (or sense) with C.
Is this information exact?

This activity is suitable to distinguish between everyday language and mathematical language. In fact it helps students to make a distinction between path, sense, direction and orientation.



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[2] A related situation is mentioned in a Cyprus mathematics school-book of the 2nd Grade of lower secondary education (Themistocleus & Anastasiadou 1992, p. 216)

3. Displacements between two towns: *Every morning a man travels from town I to town E:*

I ———————————— *E*

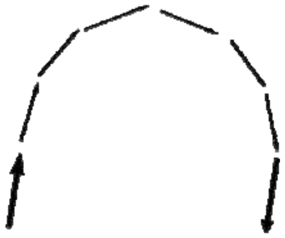
Every evening he travels back to his town I.

I ———————————— *E*

Is there any difference between these two itineraries?

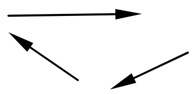
It is a simple and fruitful activity based on collinear vectors, privileged for introducing notation, geometric representation and opposite vectors.

4. Circular displacements: We present step-by-step the vectors of the figure asking: *"Do these arrows have the same direction?"* The student, who answers affirmatively, comes in cognitive conflict with his conception, when in the boundary positions the two vectors become opposite. It is a proposed activity, related to the circular conception of sense verified by our previous research, and contributes to the distinction between direction and orientation.

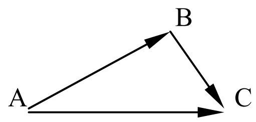


It seems that students' difficulties, due to the confusion between everyday language and mathematical language, lead them to confusion between vector sense (direction) and orientation of a motion (left handed vs. right handed):

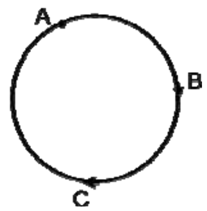
- *Direction of a continuous motion*:
  For some students these three vectors have the same sense.



  The same happened for successive vectors like $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

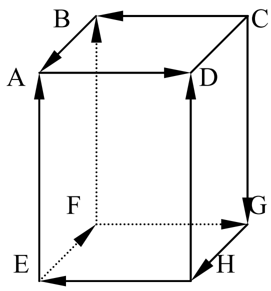


- *Circular motions in physics*
  Some students influenced by the representation of circular motions in physics, suggested these oriented arrows as examples of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ having the same sense.



5. Activities in a geometrical context

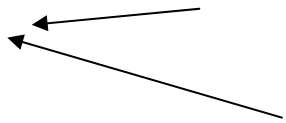
5.a. *Among vectors indicated on the parallelepiped with square bases, distinguish those (i) of equal magnitude, (ii) of the same path, (iii) of the same sense to* $\overrightarrow{AB}$*, (iv) equal to* $\overrightarrow{AD}$*, (v) opposite to* $\overrightarrow{AD}$*[3]*



This is a purely (static) geometric situation, an activity offered for making practice. It is also connected to difficulties due to language, like the following:

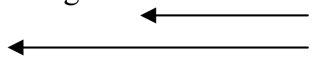
- *Horizontal vs vertical directions means "opposite" directions*: For some students, all vertical vectors were opposite to the horizontal $\overrightarrow{AD}$. Similarly, vectors perpendicular to $\overrightarrow{AD}$, like $\overrightarrow{EF}$, were considered to be opposite to the horizontal $\overrightarrow{AD}$.
- *"Opposite" means "different"*: This is a strongly persisting difficulty, much stronger

[3]This exercise is included in a Greek mathematics textbook for the 3rd grade of high school (9th grade) that was in use for many years (Alibinishis et al 1998, p.247).



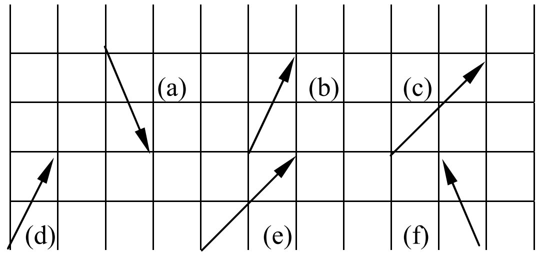
than the circular concept of sense.

- ✓ *Teacher:* Can you compare these vectors?
- ✓ *Student:* No, since these have opposite sense and path.



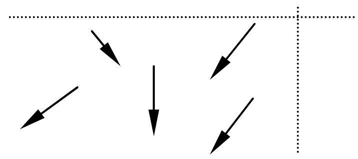
- ✓ *Teacher:* What's the relation between their magnitudes?
- ✓ *Student:* They are opposite.

5.b. *Find vectors which are: i) of the same magnitude ii) of the same path iii) of the same sense iv) equal.*

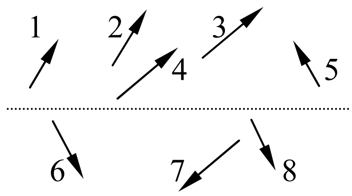


This activity is related to specific difficulties due to confusion between sense and orientation, which is a strong difficulty, still persisting after three months! Some students tended to separate the plane in semi-planes or quadrants, where vectors have the same sense, e.g.:

- Approximately parallel vectors were considered to have the same path, like c and d above.
- Sense & Quadrants: Vectors with the same orientation (SW or left down) were considered to have the same sense.



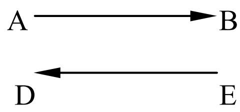
- Sense & Semi-planes: The sense of vectors 1, 2, 3, 4, 5 is upwards. The sense of vectors 6, 7, 8 is downwards.



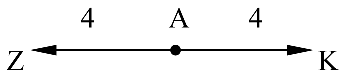
3.2 Introducing vector notation

Students' inventions negotiated in the class, for denoting the displacement between the two cities I and E (§3.1.3) were of two types: at the very beginning they suggested symbols like IE, AB, x or y, strongly related to a line segment. Later on they suggested $\overrightarrow{\mathrm{IE}}$ and I $\longrightarrow$ E. For practical reasons, they soon rejected the last one.

A very interesting invention was: $\overleftarrow{\mathrm{AB}}$ for the vector: A $\longleftarrow$ B, where students attempted to denote the sense of the vector, as well. This is an important result, indicating students' inventiveness. It led to a long discourse in the classroom and was finally rejected by the majority, presumably influenced by the didactical contract (*"Is it legal to use it in exams?"*). Only two students kept it until the end, but they failed to use it correctly in the final exam.



Students met difficulties with the symbol of the opposite vector in general. The confusion becomes obvious in their suggestions for the vector opposite to $\overrightarrow{AB}$: $-\overrightarrow{AB}$, $\overrightarrow{BA}$, $-\overrightarrow{BA}$, $A'B'$, $\overrightarrow{DE}$, $\overrightarrow{ED}$, $-\overrightarrow{ED}$, $\overleftarrow{AB}$.
Moreover, the opposite vector was also related to language problems, since for some students "different" means "opposite". Sometimes they used an idiosyncratic notation of the correct conception of opposite vectors (≠), e.g.:



- ✓ *Student A:* These are opposite.
- ✓ *Teacher:* How should we denote them?
- ✓ *Student B:* $\overrightarrow{AZ} \neq \overrightarrow{AK}$.

The classroom discourse on notation and the use of terms in everyday language raised the issue of the meaning of the arrow in vector's symbol. According to some students, it signifies the vector itself (indicating its vectorial nature), the terminal points, the path or the sense. They got confused on this point, when only one (small) letter was used to denote the vector.

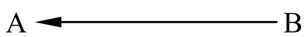
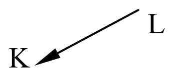
For a small percentage of students, scalar quantities were considered as vectors and vice versa, e.g: *"*$\vec{2}\,^0C - \vec{5}\,^0C = -\vec{3}\,^0C$*"*, *"*$\overrightarrow{10}$ *m"*, *"* $\vec{\upsilon}$ *= 38 m/sec"*, *"+ 2 – 5 = –* $\vec{3}$ *"*.

Students were also asked to suggest symbols for a vector's magnitude in the case of forces. Although the symbol $|\vec{\alpha}|$ was accepted for displacements by analogy to the absolute value of numbers (keeping the number line in mind), there was a confusion in the case of forces, because of the strong context implied by physics. In physics textbooks[4] occasionally vectors are denoted with arrows, or in boldface letters (a single capital letter)! The following symbols were used by the same student: *"*$\vec{F}$*=6N"*, *"*$|\vec{F}|$*=20kg*"*, *"*$\vec{F}$ *= 4cm"*.

3.3 Introducing the geometrical representation of vectors

The following are students' proposals for the displacement between the cities I and E (§3.1.3). This is a privileged activity to combine notation with geometrical representation.

In some cases, it seems that they really saw a text when "reading":

- ✓ from left to right : $\overrightarrow{AB}$ for 
- ✓ or forward to backward : $\overrightarrow{KL}$ for 

3.4. Comparing vectors

Equal vectors are equivalent in magnitude, path and sense. In that sense, "equality" means "equivalence" (relation), not "identity". Students' main difficulties on equality are related

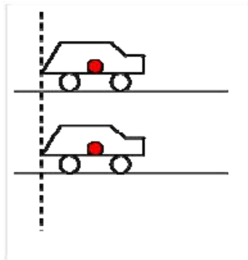
[4]cf. footnote 1.

to the fact that it is conceived as equality of line segments and it is limited to equality of magnitudes.

We first organized a discussion on the conventional use of equality as equivalence, using examples with line segments or numbers (e.g. 3+5=8 or $\frac{2}{3}=\frac{4}{6}$), and concluding for example, that the **equal** vectors $\overrightarrow{BA}$ and $\overrightarrow{GH}$ in (§3.1.5) are **not identical**; they **differ**, since they are in different places; however they are **similar** with respect to **some** of their elements. Then, we used situations realizing vector equivalence. Here vectors with different initial point have **equivalent results** concerning the displacement. Vectors were considered **as operators**, not static objects (the same holds for numbers when teaching multiplicative structures). Physical and geometrical situations are privileged in this respect. The following are some related activities given to our students.

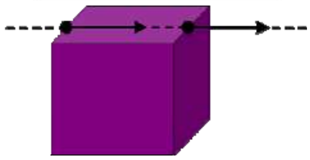
*Example 1***.** Situations with velocities of different initial points.

Two marbles lying in two cars are moving with the same velocity. The observer sees that the two marbles cover the same distance in the same time.



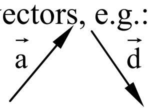
Example 2. Situations with forces (sliding along the same line). This is more difficult to understand, since physics is involved here in a more essential way.

On a wooden compact cube, equal forces are applied, at different points, parallel to the edges.

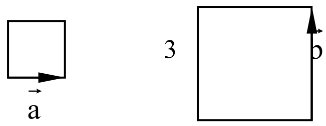


Students' main difficulty when comparing vectors was their misconception that equality of magnitudes is sufficient for equality of vectors, e.g.:

1. Equal magnitudes ≡ Equal vectors; e.g. $\vec{a}=\vec{d}$

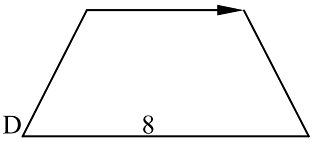


2. Vectors with proportional magnitudes
   - Non-collinear: e.g. $\vec{b}=2\vec{a}$



   - Collinear: Isosceles trapezoid: $\overrightarrow{DC}=2\overrightarrow{AB}$

   Equal magnitudes and different orientation lead to opposite vectors, e.g. $\overrightarrow{AD}=-\overrightarrow{CB}$



3.5 Adding non-collinear vectors

An important didactical comment: **Start teaching vectors in 2-dimension situations!** Our research indicated that 1-dimensional situations are often more confusing than n-dimensional ones because of lack of rich enough geometrical context.[5]

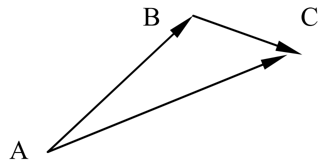
***1. The triangle law*:** It is worth mentioning that vectors were used as **operators** on real objects (bodies). Thus operations between vectors were conceived as the **final effect** on

[5]E.g. think of matrix algebra in 1D (where it is trivial and reveals none of the virtues and subtleties that appear in two or more dimensions); or key concepts in differential geometry, like (intrinsic) curvature, which in one dimension is identically zero, but not so in two or more dimensions.

objects.

1.a Situations with displacements

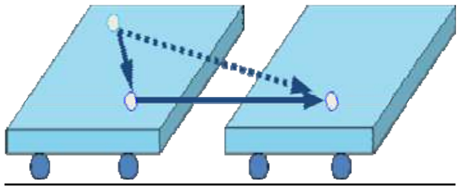
Hercules travels from town A to town C, via B. Indicate his trajectory. Hercules' brother travels directly to C. Compare the final displacement of Hercules to that of his brother.



1.b Situations with velocities

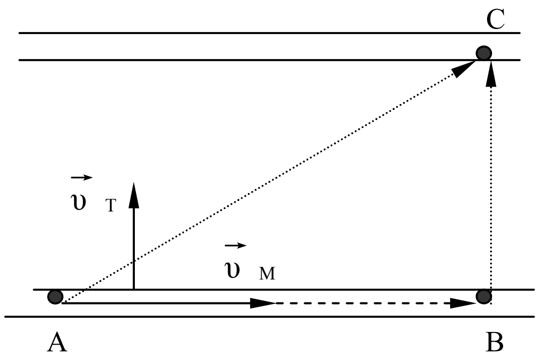
The situations used were thought experiments, based on reconstruction of time, where velocities were treated as successive displacements in a unit time interval. These thought experiments were inspired by Galileo's great achievement, the appreciation of the independence of motions, that was raised to a main epistemological principle of what came to be understood as physics since then[6]. We preferred this presentation, since it is known that many students face difficulties in understanding the concept of velocity in high-school physics. They mostly know only the formula for (constant) speed, and velocity is conceived as a scalar quantity.

***1.** The board problem*: A marble is moving on a board, 1m per sec, while the board is moving 5m per sec to the right. The motion is analyzed in two *successive motions:* the marble moves first for 1sec, then the board moves for 1 sec too (this is really deep physics!).



***2.** The tube problem*:[7] A marble is moving along a tube with 4cm/sec, while the tube is moving upwards by 3cm/sec. In 2sec:

1. If the tube is not moving, marble's displacement is $\overrightarrow{AB}$.
2. If only the tube is moving, marble's displacement is $\overrightarrow{BC}$.
3. In simultaneous motions, marble's displacement is $\overrightarrow{AC}$.



Teaching improved students' understanding of the triangle law. They escaped from using rules of addition depending on the context in each case (see Booth, 1981).

***2. The parallelogram law*:** Simultaneous events with forces were used. However, a preparatory work had been done in the classroom, to introduce the commutativity of vector addition and the parallelism of equivalent vectors.

2a. The commutativity of vector addition follows from the independence of motions/displacements. It leads to the parallelogram rule as law **equivalent** to the triangle law, in the sense that it leads to the **same** result (vectors being conceived as operators). A

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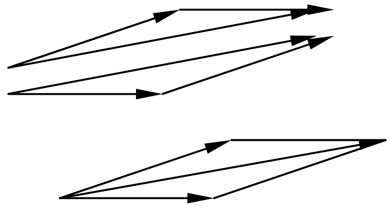
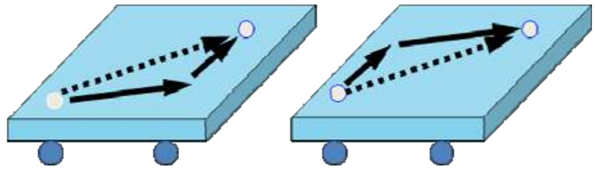
[6]Motion, which was a *property* of bodies in Aristotelian physics, in Galileo's conception of nature became a *state* of bodies, for which bodies were "*indifferent*". This is intimately related to the "principle of independence of motions", which, seen in a modern mathematical context, is nothing less than the vectorial character of velocity.

[7]The idea for this activity is based on a similar one, included in a Cypriot physics schoolbook for the 11th grade (2nd year of upper secondary education; Gavriilidis & Papadopoulos 1992, p. 66-67).

word of caution however: The triangle and parallelogram laws suit better in different situations each! The former is more suitable for velocities and displacements, whereas, the latter is more suitable for forces.

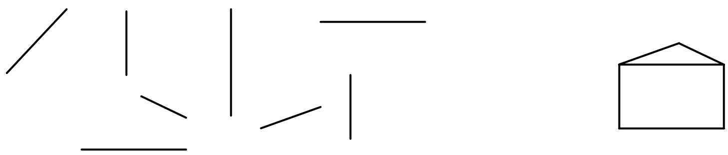




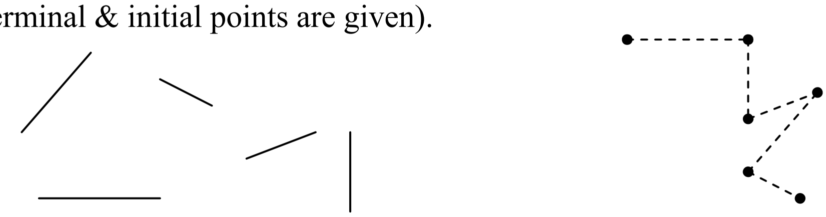
The final displacement of the marble is the same, no matter how we observe/analyze the order of motions. The activity is suitable for the study of the commutativity of addition (triangle law) and as a preparatory one to introduce the parallelogram law.

<u>*2b. Preserving parallelism*:</u> Situations where parallelism should be preserved.

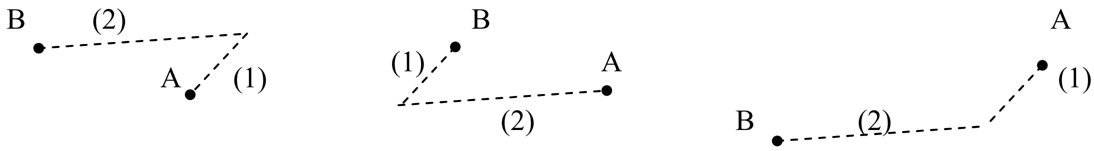
- Line segments moving parallel to form a concrete figure.



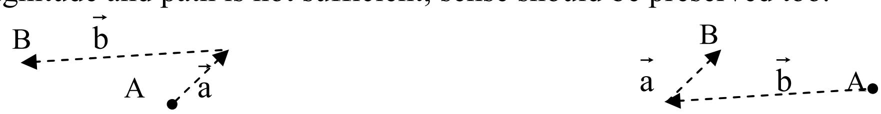
- Situations with line segments where students cannot avoid preserving parallelism (terminal & initial points are given).



- Trajectories between points A & B by moving 2 given segments, while parallelism is preserved.

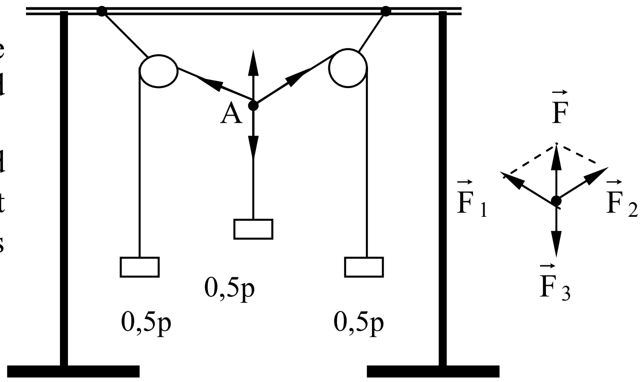


- Trajectories between two given points, by moving two vectors. Preserving only magnitude and path is not sufficient; sense should be preserved too.



After these preparatory activities, a real experimental activity with forces was performed in the classroom:

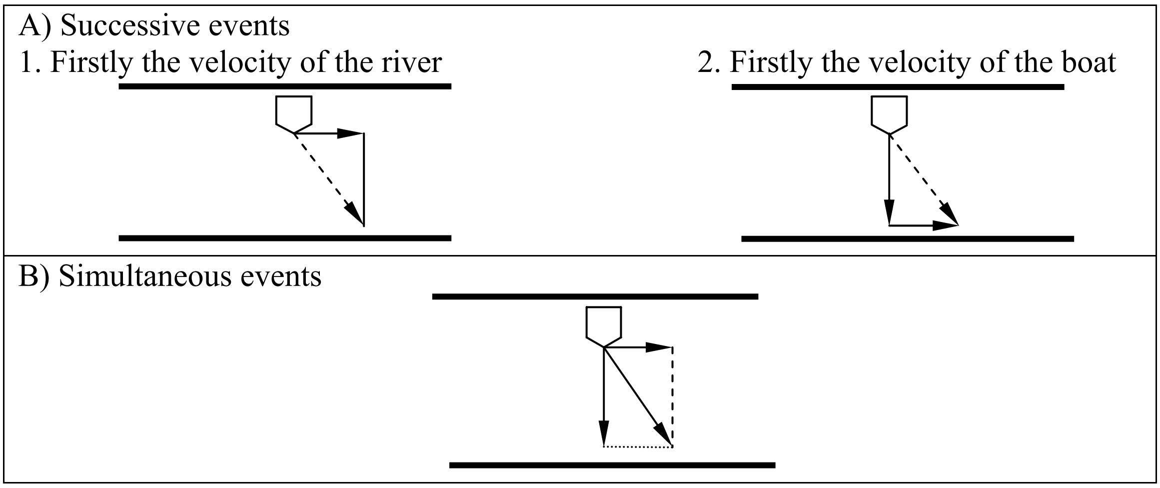
What weight should be hung from A, to ensure equilibrium?
Students invented/discovered the parallelogram law through trial and measurements.



The experiment may also lead some students to a cognitive conflict with their misconceptions. If it is considered that:
$\vec{F}_1+\vec{F}_2=\vec{F}$ and $|\vec{F}_1|+|\vec{F}_2|=|\vec{F}|$
for all directions, then:

$$0,5 + 0,5 = 0,5\ !$$

***3. Equivalence between triangle and parallelogram laws*:** Students' faced a difficulty of didactical origin: The parallelogram law was connected to the «resultant» force in the physics context, while the triangle law was connected to vector «addition» in the geometrical context. We used didactical activities with successive events. Students realized that the result was the same, either for simultaneous, or for successive motions. The privileged concept for such situations is velocity (recalling Galileo's principle of the independence of motions) in problems like those with the board or the tube (§3.5.1). The problem with a boat moving on a river was also used in the classroom, based implicitly on the independence of motion, by "reconstructing" time.

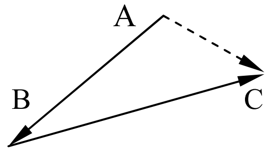


The final conclusion drawn by the students was that the two laws do not replace each other, since they are useful / applicable in different situations: the triangle law fits better to successive vectors, while the parallelogram law to vectors of common initial point. However, their common characteristic is that both lead to the same physical result.

4. Difficulties associated with the triangle and parallelogram laws: We next present some of the difficulties we identified both in connection with the triangle and parallelogram laws. Our research indicated that students managed better the triangle law than the parallelogram one.

(1) *Similarities in form*, *led to intuitive strategies for addition of non-collinear vectors*, which differ from the typical models of the two laws (see Donaldson 1995, Booth 1981). We give some examples:

(1a) *Models similar to the triangle law*: According to this law,

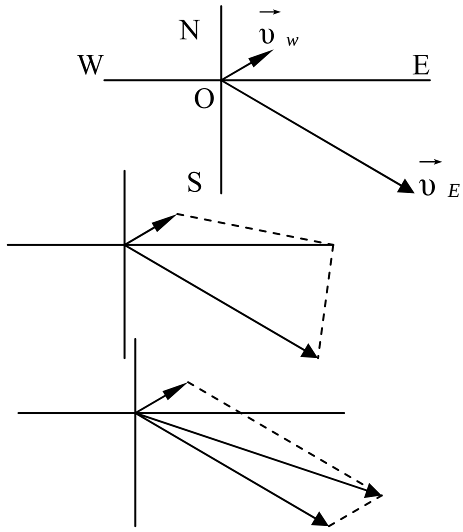


"*the sum of 2 successive vectors has initial point the initial point of the 1st and terminal point the terminal point of the 2nd*": $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

Some students applied the law, even in cases without all the required conditions, e.g.:

- *Successive but not ordered* (according to the law) *vectors lead to zero or opposite vector:* $\overrightarrow{BC}+\overrightarrow{AB}=\overrightarrow{CA}$ *or* $\vec{0}$
- *Vectors with a common initial point lead to a vector with edge points their edge points:* $\overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{BC}$

(1b) *Incomplete parallelograms*: Two vectors with common initial point and a half-line between them, e.g.:

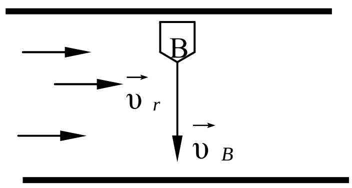


- ✓ *Teacher:"What is a plane's velocity for an observer when NE winds blows 30 miles/h and the engine's velocity is 150 miles /h, in SE direction?"*
- ✓ *John:* E (East!)
- ✓ *Hercules:* By the parallelogram law.

John draws a "parallelogram" with OE as its diagonal. Only after we remind him that the opposite sides should be parallel, he draws the right model.

(2) *Difficulties related to the initial point*

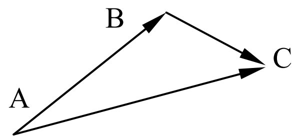
Some students had problems with parallel the displacement of vectors for applying both laws. It was not clear for them how to make the vectors successive or with the same initial point.



(3) *Difficulties related to the magnitude of the sum of two vectors*; *the symbols* "+" & "=": Students used them as if they were identical to those in arithmetic or algebraic operations they were familiar with. This is an epistemological obstacle. The same difficulty is often encountered by students of mathematics, or even mathematicians, when the same symbols are used while treating isomorphic algebraic structures!

A lot of work had been done to make clear the distinction between vector addition and addition of line segments or numbers. For example:

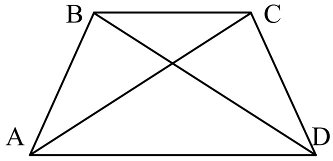
- In the case of the triangle law, for example, students measured: AB= 32, BC=24, AC=45, 32+24 = 56 ≠ 45, and concluded that the symbol "=" means that the two displacements have the same terminal point, the same result. Hence, even though they are equivalent, they are not equal.



- Similarly, for the parallelogram law, when using the experimental setting of §3.5.2, with balance weights 10g, we wrote: $\vec{F}_1+\vec{F}_2=\vec{F}$, 10 + 10 = 10, that is 20=10, and asked for comments:
  - ✓ *Ares:* This is not a simple addition of line segments.
  - ✓ *Alexis:* Vectors are not collinear. Only for collinear vectors of the same sense, the relation $\vec{F}_1+\vec{F}_2=\vec{F}$ and $|\vec{F}_1|+|\vec{F}_2| = |\vec{F}|$ holds.

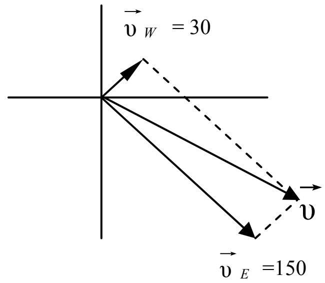
However for some students, addition of non-collinear vectors was treated as addition of scalars, e.g.:

- ✓ *Teacher:* Compare the sums: $\overrightarrow{AB}+\overrightarrow{BD}$, $\overrightarrow{BD}+\overrightarrow{AB}$, $\overrightarrow{AC}+\overrightarrow{CD}$, $\overrightarrow{CD}+\overrightarrow{AC}$.
- ✓ *Alexander:* All are equal, since these are associated to equal triangles.



Also when estimating the resultant's magnitude:

- ✓ *John:* (trying to find the magnitude of the resultant velocity)
  $\vec{\upsilon}_E + \vec{\upsilon}_W$ = 150 miles + 30 miles = 180 miles.



(4) *Difficulties with composite motions*: In the tube problem (§3.5.1), one of the two motions was ignored. Also students expressed misconceptions based on common sense:

- ✓ The tube will swing!
- ✓ The marble cannot go up, because there is the glass of the tube!
- ✓ The marble will roll a little!
- ✓ The marble remains at rest!

In the airplane problem (§3.5.3), they were unable to distinguish between the two systems of reference: "*The motion as motion! No matter who observes, the motion is vertical!*". For some students the passenger sees both movements, while for others the observer at rest sees only the downfall. Here physics may introduce genuine difficulties, which are known historically. As far as the introduction of vector addition is concerned, and following our experimental teaching, we would suggest: (a) to put emphasis on the independence of motions, (b) to analyze composite motion in successive motions (as displacements per unit of time) and (c) to avoid the use of moving frames.

4 Some concluding remarks

Vectorial concepts exhibit many different aspects. Because of this epistemological characteristic and the appearance of vectors in different parts of the secondary education curriculum, students conceive vectors in the following contexts:

Algebraic: As scalars, characterized solely by their magnitude.

Geometric: As linear segments, characterized by their length.

Physical: Physical terminology, symbols and concepts are used to characterize vectors in their abstract form.

Experiential: Vectors are understood via "spontaneously" generated concepts offered by the everyday life social environment.

Our teaching approach attempted to reveal the multifarious nature of vectorial concepts as much as possible, emphasizing their relation to geometrical and physical situations, without ignoring their more abstract algebraic aspects. In this connection, our approach was historically inspired, profiting **indirectly** from the complicated historical development of the basic vectorial concepts and operations, in the sense described in Tzanakis & Arcavi 2000 (§7.3.2, p.210) for an **implicit** integration of historical elements into teaching. More specifically, this development clearly shows that the prototypical (and generic) examples referred to both physical situations (forces and uniform motions, analyzed to displacements per unit time) and geometrical ones (displacements). It took quite a long time to establish these notions following a complicated path, based on both disciplines. One main point of our approach is that this fact cannot be ignored; instead, it permeated our teaching, somehow setting the agenda for the order and the way the various topics were presented, at the same time helping the teacher to get a deeper awareness of the (epistemological) difficulties inherent in the subject (this is close to Jankvist's concept of a *history-based approach*; Jankvist 2009, §6.3).

It was a difficult task to lead students to overcome the partial conception of vectors that they had gained in the four different contexts above and develop a deeper conception in which aspects coming and/or prevailing in each context are integrated into a coherent whole. This was the main positive result of our experimental teaching that was verified by the statistical analysis of correlations between the answers to tests given before, immediately after and three months after the teaching to both the experimental and control group. The experimental group, both in its own development and in comparison with the control group exhibited a coherent understanding of vectorial concepts, in the sense that students succeeded to put together the different aspects of vectorial concepts into coherent conceptual objects. Nevertheless, students' difficulties and misconceptions greatly varied in character, depth and intensity and it has not been possible to overcome all of them. Below is a short summary of those mentioned in section 3.

Students encountered difficulties to understand the *sense* of vectors: Almost parallel vectors were compared with respect to their sense (§3.1.4), a misconception persisting 3 months after teaching. On the other hand the confusion between *sense* and *orientation* (§3.1.4) and *opposite* and *different* vectors (§3.1.5) that are due to the everyday life use of language were almost completely overcome after teaching.

There were lengthy classroom discussions and debates on the most appropriate and convenient vector notation and students exhibited great inventiveness (§3.2). This active involvement of students in classroom activities and discourse helped them overcome to a large extent the difficulties they faced in connection with vector notation. The analysis of the data from the teaching experiment indicates that the experimental group understood better vectorial notations, in the sense that they used them more consistently. Despite this fact, some students failed to distinguish vectorial from scalar quantities (§3.5.4), which is at least partly due to the use of the familiar symbol of numerical/algebraic equality, "=", for vector equality as well (§3.5.4.3). An associated difficulty is the addition of the magnitudes of non-collinear vectors to get their sum, presumably related to the fact that both equality and addition of vectors are denoted by the same symbols used for scalar (numerical) quantities "=" and "+".

Concerning the triangle law, our teaching helped students to escape from using intuitive, context and case-dependent rules. Using such rules was maintained to a moderate level, in contrast to the control group, where this phenomenon remained strong.

Showing the equivalence of the triangle and parallelogram laws of vector addition was not easy, mainly because of a didactically originated difficulty: The triangle law is primarily used in geometry to find the "sum" of two vectors, whereas the parallelogram law is used in physics to find the "resultant" of two vectors. Given the different conceptual framework and the different terminology and notation, some students faced difficulties to understand the equivalence of the two laws, which became possible, however, with the use of appropriate physical examples (§3.5.4).

Finally, vector addition in the context of situations based completely on composite motions and/or moving frames, did not help much our students, especially in the case of non-collinear vectors, mainly because of interference with experiential conceptions about motion of a pre-Galilean nature (a phenomenon already know in the literature) and the difficulties inherent to physical situations involving moving frames/observers (§3.5.4.4).

However, our teaching approach indicates that leaning upon situations from **both** physics and geometry increased students' ability to interconnect different aspects of vectorial notions, hence, to reach a more coherent understanding and view; not an easy task for such multifarious concepts. We do believe that as far as vector operations are concerned, it is advisable to benefit from situations involving displacements, whereas, if velocities are used, it is better to "translate" them into successive displacements.

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HISTORICAL MINI-THEORIES AS AWAY TO REFLECT ABOUT THE MEANING OF PROOF

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ABSTRACT

The paper examines in the first part the historical fact that the Greeks invented not only mathematical proof but also and simultaneously 'theoretical physics'. This simultaneity was not accidental; rather, the two events were connected and influenced each other. The link between them was an idea in the Greek philosophy of science called 'saving the phenomena'. The paper establishes a connection between this idea and the pre-Euclidean meaning of the term 'axiom'. The astronomical problems by which the Greeks were led to the idea of 'saving the phenomena' are well suited to be explored in the teaching of mathematics at school with the intention to enter a discussion about the origins of the axiomatic method and the meaning of proof.

1 The origin of proof

In 1960 the Hungarian historian of mathematics Árpád Szabó proposed a thesis about the origin of the axiomatic method in ancient Greece (Szabó 1960). Szabó's investigations are not only interesting from a historical point of view but they establish also a framework for a didactical conception of proof. In the first section of this paper we present those parts of Szabó's study which seem especially interesting from a didactical point of view. In the second section we relate these ideas with reflections in Greek science about the relation between theory and empirical evidence. This leads to an overall picture of the genesis and meaning of proof whose didactical fruitfulness is shown by a teaching unit which was discussed in the workshop at ESU 6 and which is sketched in the third part of the present paper. For reasons of space we suppress most of the references and documentary evidence. The interested reader may consult (Szabó 1960), (Maté 2006) and (Jahnke 2009).

Euclid divided the foundations of the 'Elements' into three groups of statements: (1) Definitions (greek *Horoi*), (2) Postulates (*Aitemata*) and (3) Axioms or Common Notions (*Axiomata* or *Konai Ennoiai*). Earlier manuscripts contain the terms (1) *Hypotheseis,* (2) *Aitemata* und (3) *Axiomata*.

Szabó investigated the etymology of the terms and showed that *Hypothesis*, *Aitema* und *Axioma* were common terms of pre-Euclidean and pre-Platonic dialectics and still played a considerable role in Plato's dialogues and Aristotle's treatises. Dialectics was the art of exchanging arguments and counter-arguments in a dialogue debating a controversial proposition. The Greeks treated dialectics as part of rhetoric, Plato (427-348 BC) considered it as a method of philosophical debate and of generating new knowledge.

From the times of Plato and Aristotle up to the 19th century mathematicians and philosophers understood axioms as statements which are self-evident and absolutely true. The most important ancient advocates of this view were Aristoteles (384 – 322 BC), especially in his treatise *Analytica Posteriora,* and the Neo-Platonist commentator Proclus (412 – 585 AD) in his influential commentary to book I of Euclid's 'Elements'.

The view of axioms as self-evident truths undergoes a remarkable modification when Szabó's studies are taken into account. At first, let us consider the term *Hypothesis*.

Literally, *hypothesis* is something which is *underlying* and consequently can function as a foundation of something else. Hence, it is an unproven assumption in a dialectic discourse whose validity is assumed in order to derive further statements. In Plato’s works the reader finds a multitude of situations where this term is used. Plato knows *Hypothesis* as a concept of mathematical terminology, but he used it even more frequently as a designation of an assumption in philosophical discourses. E. g. Socrates “asked” his partners to “allow” him to begin with this or this hypothesis. The dialogue can only continue if the partners explicitly agree to this request. Therefore, Plato designated hypotheses frequently as *Homologemata* (the “things which were conceded”).

As a rule a person will introduce in a discourse such hypotheses which he considers as especially strong and which he assumes to be accepted by his partners. At various places of Plato’s dialogues Socrates reflected this use of hypotheses and asked his partners to take a great deal of care over the starting point of a debate.

But it is also possible to propose a hypothesis with the intention of its critical examination. In a philosophical discourse the participants derive consequences from this hypothesis. The consequences might be *desired* or *plausible* and then lead to strengthen or even accepting the hypothesis. Or they might be *not desired* or *implausible* and thus lead to rejecting it. In a debate on ethical questions a participant might propose a certain maxim of behavior. Then the participants jointly investigate consequences of the maxim. In case, it will imply not desired behavior patterns they will reject the maxim, in the opposite case they will judge it of being worth further consideration or even accept it. An extreme case of a not desired consequence is a logical contradiction. In that case mathematicians speak of an indirect proof.

All in all, Szabó found in regard to the term ‘hypothesis’ three different variants of meaning:

(1) Hypothesis = strong, unproven proposition. To convince his partners of a certain statement person *A* proposes a hypothesis which *A* assumes to be accepted by his partners and from which the statement in question can be derived;

(2) Hypothesis = proposition whose validity is assumed for a time in order to critically investigate it. The participants derive consequences and check whether these are desired or not desired. An extreme case of an undesired consequence is a logical contradiction.

(3) Hypothesis = definition. This variant of meaning is especially relevant to mathematics. E.g. somebody proposes definitions of say ‘odd number’ and ‘even number’ and in the course of work a whole bunch of theorems can be deduced which are considered relevant. This leads to accepting the definition as fruitful.

Szabó’s investigations showed that the three terms *Hypothesis*, *Aitema* and *Axioma* had a similar meaning in pre-Platonic and pre-Arisotelean dialectics (with the exception of the meaning ‘definition’). They designated those initial propositions in a dialectical debate whose acceptance is *required* by one of the participants. If the participants agreed on the proposition it was frequently called *Hypothesis*. If, however, acceptance was left undecided the proposition was called *Aitema* or *Axioma* (Szabó 1960, 399). This meaning of *Aitema* was still known to Aristotle.

From the study of the genesis of the term *Axioma* in Greek dialectics we see that, originally, it did not designate a statement which is absolutely true and cannot be doubted but, for a long time, had a connotation in the sense of the modern concept of hypothesis.

Probably, at the times of Euclid philosophers and mathematicians were still aware of this hypothetical connotation and thus used the term *Konai Ennoiai* (= *the ideas common to all human beings*) instead of *Axioma*.

In this way, the concept of an axiom made a career in Greek philosophy and mathematics whose starting point lay in the art of philosophical discourse; later it played a role in both philosophy and mathematics. More importantly, it underwent a concomitant change in its epistemological status. In the early context of dialectics, the term axiom designated a proposition that in the beginning of a debate could be accepted or not. However, axiom's later meaning in mathematics was clearly that of a statement which itself cannot be proved but is absolutely certain and therefore can serve as a fundament of a deductively organized theory. Since Plato and Aristotle this was the dominant view among mathematicians and philosophers to whom mathematics served as the ideal of a certain science for more than two thousand years.

2 Saving the phenomena

We can draw two consequences from Szabó's etymological studies:

(1) The *practice of a rational discourse* served as a model for the organisation of a mathematical theory according to the axiomatic-deductive method. The terms *hypothesis*, *aitema* and *axiom* are crystallized rules of behavior in a dialectical discourse. They say that the participants have to accept and obey certain rules of behaviour which entail their obligation to exhibit their assumptions.

(2) The second consequence refers to the universality of dialectics. Any problem can become the subject of a dialectical discourse, regardless whether it is a problem of ethics, of physics or mathematics. The axiomatic-deductive organisation of a group of propositions is not confined to arithmetic and geometry, but can, in principle, be applied to any field of human knowledge. The Greeks realized this insight at the time of Euclid, and it led to the birth of theoretical physics.

Within a short period of time Euclid and other mathematicians/scientists applied the axiomatic organisation of a theory to a number of areas in natural science. Euclid wrote a deductively organised optics and a music theory, whereas Archimedes provided axiomatic-deductive accounts of statics and hydrostatics. Also in astronomy the concept of hypothesis was systematically used.

In the latter domain, the Greeks discussed, in an exemplary manner, philosophical questions about the relation of theory and empirical evidence with influences even upon Kepler and Galileo. This discussion started at the time of Plato and concerned the paths of the planets. In general, the planets apparently travel across the sky of fixed stars in circular arcs. At certain times, however, they perform an irregular retrograde motion. This caused a severe problem; the Greeks held a deeply rooted conviction that the heavenly bodies perform circular movements with constant velocity. But this could not account for the irregular retrograde movement of the planets. Consequently the Greeks had to adjust the hypotheses about the movements of the planets in order to fit astronomical theory with empirical evidence. They did so by inventing new hypotheses, the so-called eccentric and epicyclic hypotheses. The process of adaptation was reflected under the heading „sozein ta phainomena" ("saving the phenomena") (Mittelstrass 1962). Saving the phenomena designated the process of devising hypotheses in a way that their deductive consequences agree with the observed data. Hence, saving the phenomena agrees exactly with the second variant of meaning Szabó has found for the use of hypotheses in dialectical discourses. A hypothesis is stated and the participants investigate whether its consequences are desired i.e. agree with the data, or not. In the former case it is accepted, in the latter rejected or modified.

Szabó doesn’t hint at the fact that the investigation of a hypothesis in a dialectic discourse has a logical structure similar to the process of accommodating a hypothesis to empirical phenomena (see however Szabó 1974). Nevertheless, this is the case. In ancient times the latter process was coined ‘saving the phenomena’ and was, as far as we know, confined to astronomy. Since Galileo and Huyghens it was considered as the general method of research in the mathematised physical sciences. By the end of the 19th century scientists called the same procedure the ‘hypothetico-deductive method’.

4 Reflecting about the meaning of proof by studying a historical mini-theory

In the teaching of mathematics at the secondary level it is not easy to answer the question of what a proof is. In fact, proof is not a stand-alone concept and cannot be explained without recourse to the notion of theory. But it is widely agreed that it does not make sense to treat axiomatic theories in schools. Nevertheless, it is an urgent requirement to provide also to students of the secondary level an adequate idea of what mathematical proof is and which meaning it has for man‘s understanding of the world.

Which knowledge about proof should students acquire? To say it in a few words, they should understand that a mathematical proof does not establish ‚facts‘ but ‚if-then statements‘. A mathematician does not proof a fact *B* but an *implication* „If *A*, then *B*“. E. G. we don’t prove that all triangles have an angle sum of 180°. Rather, this is a consequence of certain hypotheses, namely the axioms of Euclidean geometry. If we suppose different axioms we arrive at different conclusions. The *certainty* of mathematics does not reside in the facts, but in the *inferences*. Whether a mathematician believes in the facts of his theory, e.g. that there are infinitely many prime numbers or that triangles have an angle sum of 180°, depends on his *evaluation* of the acceptability of the axioms of his theory. In professional mathematics this process of evaluation or assessment of a theory remains mostly implicit. *In teaching however students should be made aware of it*. For example, in arithmetic mathematicians deal with objects which are generated by a uniform process of counting. Hence, there is a high amount of control. But in geometry mathematicians touch the area of physics. If geometrical theorems are applied to physical space they are subject to empirical measurements. The fact that Euclidean geometry is extraordinarily reliable as an empirical theory of “medium-sized objects” is a matter of experience and not a fact of pure thought.

Basically, there are two ways of evaluating the axioms of a theory. Classically axioms are considered as self-evident and absolutely true. Consequently, the theory which is built upon them is true. The other way is to evaluate a theory on the basis of whether or not its consequences agree with what the theory is expected to explain. This corresponds to the above mentioned “hypothetico-decductive method”. In contrast to traditional teaching of mathematics this paper supports the thesis that also the latter way of evaluating the axioms of a theory should be made a theme with students. This seems to be a promising way of explicitly reflecting about what axioms of a theory are and where they come from.

A possible idea for doing this is the development of *mini-theories* which are accessible to learners and sufficiently substantial to discuss meta-issues. The idea of such min-theories is not completely new. It bears resemblance with Freudenthal's (1973) concept of 'local ordering' or with using a finite geometry as a surveyable example of an axiomatic theory. Of course, finite geometries are not feasible in secondary teaching. In regard to Freudenthal's concept of local ordering the proposed idea of a mini-theory is different in two aspects. First, the teaching of a mini-theory would include phases of explicit reflection about the structure of axiomatic theories, the conditionality of mathematical theorems and the evaluation of its truth. Second, the proposal would also take into account 'small theories' from physics like Galileo's law of free fall and its consequences and other mathematised empirical sciences (Jahnke 2007).

In the following section a teaching unit for 9th graders on a historical mini-theory will be described which serves as an example of the conception sketched above. It concerns the so-called 'anomaly of the sun' which was discussed for the first time by the great astronomer and mathematician Hipparchos in the second century BC. Roughly speaking, the term referred to the observation that the half-year of summer is about one week longer than the half-year of winter. This contradicted the basic convictions of Greek astronomers that all heavenly bodies move with constant velocity in circles around the centre of the earth. Therefore Hipparchos invented a new hypothesis to "save the phenomena". Hipparchos' proposal can be found in Ptolemy's 'Almagest' (Toomer 1984). In the teaching unit we will not stick to the details of Ptolemy's exposition and especially will calculate with modern data.

To study a historical mini-theory seems to be especially rewarding in regard to the epistemological situation. The students can well imagine the situation that the ancient astronomers were forced to find out something about the structure and functioning of the universe without any direct access to it. This is reinforced by their knowledge of the fact that Hipparchos' theory does not agree with modern views.

5 A Teaching unit on hypotheses about the path of the sun

The following teaching unit is described by a series of worksheets for pupils of grade 9. The reader should take into account that the worksheets serve the twofold function of providing him with necessary informations and giving him an idea of possible work for pupils. Hence, they are not intended as the final sheets for the pupils.

5.1 Worksheet 1

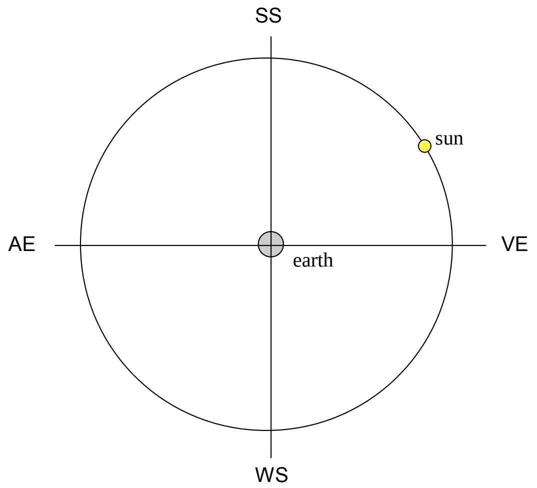
In ancient times Greek astronomers tried to understand the structure of the universe. Of course, they couldn't know how the sky was really built up but they could set up hypotheses and then examine whether the consequences of these assumptions agreed with their astronomical observations. Since the fifth century BC Greek astronomers supposed that the earth is a sphere and located in the center of the universe. Sun, moon and the stars were believed to move around the earth.

In regard to the yearly path of the sun they set up two precise hypotheses:

Hypothesis 1: In the course of a year the sun moves on a circle around the earth exactly one time. This circle is called **ecliptic**.

Hypothesis 2: The sun moves on the ecliptic with constant velocity.

There are four distinguished points on the ecliptic, vernal equinox (VE), summer solstice (SS), autumnal equinox (AE) and winter solstice (WS). When the sun is in one of the equinoxes day and night are equally long, summer solstice is the longest day and winter solstice the shortest. From their measurements the astronomers knew that these four points on the ecliptic are separated by angles of exactly 90°. They separate the four seasons spring, summer, autumn and winter from each other. Hence the path of the sun looked like this:



Tasks:

a) The length of a year is approximately 365,25 days. Derive from hypothesis 1 and 2, how many days spring, summer, autumn and winter comprise. Enter your results in table 1.

b) In table 2 you find the dates of vernal equinox (VE), summer solstice (SS), autumnal equinox (AE) and winter solstice (WS) for the present year 2010/11. Compute the number of days for the different seasons which result from these data. Enter your results in table 1.

c) Compare and comment about what you have found.

Table 1

	Duration of the seasons calculated from hypotheses 1 and 2	**Duration of the seasons calculated from the data for 2010/11**
Spring	91,3 d	92,7 d
Summer	91,3 d	93,7 d
Autumn	91,3 d	89,9 d
Winter	91,3 d	89 d

Table 2

Vernal equinox 2010	March 20, 2010, 8 pm
Summer solstice 2010	June 21, 2010, 2 pm
Autumnal equinox 2010	September, 23, 2010, 6 am
Winter solstice 2010	December 22, 2010, 2 am
Vernal equinox 2011	March, 21 2011, 2 am

5.2 Worksheet 2

We have found that the four seasons are not equally long. The “half year” of summer (= spring + summer) is about a week longer than the “half year” of winter (= autumn + winter). This contradicts hypothesis 1 and/or 2. What can we do? What are your ideas?

The Greek astronomers had an ingenious solution for this problem. They kept the idea of a circular path and the constant velocity of the sun and nevertheless provided for the apparent variation of its velocities. The new hypotheses were:

Hypothesis 1': In the course of a year the sun moves on a circle around the earth exactly one time. The center of this circle is **not** in the center of the earth. This circle is called ecliptic.

Hypothesis 2: The sun moves on the ecliptic with constant velocity.

Hypothesis 1' has been proposed by the Greek astronomer Hipparchos (2nd century BC). Therefore we call it Hipparchos' hypothesis.

Of course, it is still true that, observed from the earth the four seasonal points on the ecliptic are separated by right angles.

Tasks:

Below you find four different cases of relative positions of the earth (E), the vernal equinox (VE) and the center (C) of the path of the sun.

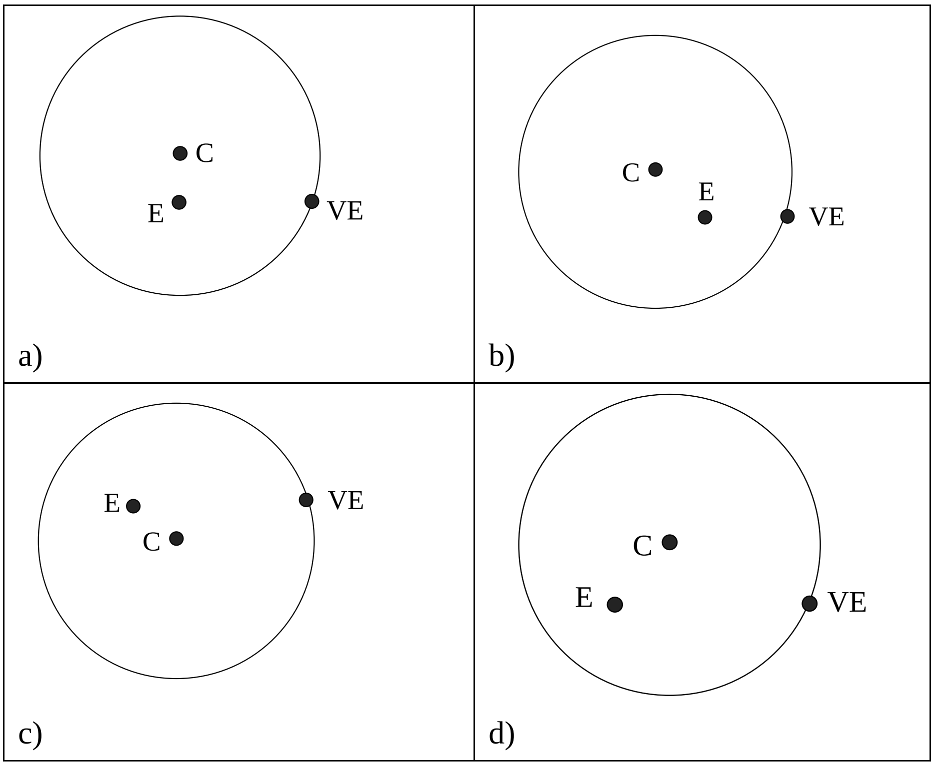
a) Please, add in each case the positions of the missing seasonal points.

b) What can you say about the length of the seasons in each case?

c) Write down your observations concerning the position of the center of the eccentric circle in relation to the lengths of the seasons.

d) The Greek astronomers called the angle between the vernal equinox (VE), the earth (E) and the sun (S) the **"true sun"**. They called the angle between the vernal equinox (VE), the center (C) of the path of the sun and the sun (S) the **"mean sun"**. Please, add in one of the figures the sun (S) at an arbitrary position and draw true and mean sun.

e) Please, speculate why these angles were called "true sun" and "mean sun".



5.3 Worksheet 3

Now we are going to investigate the properties of the system which is created by Hipparchos' hypothesis. The system consists of the earth *E*, the eccentric center *C* and a circle around *C* on which the sun *S* moves with constant velocity. It is useful to connect with *E* two orthogonal axes which represent the observed positions of the distinguished points on the ecliptic. By moving the earth with this cross you can experiment with different relative positions of *C* and *E*.

For your investigation you can use a software tool like Geogebra (or Cabri, DynaGeo) or you take two transparencies. On the one you draw the center *C* with the circle on which the sun moves. On the other you draw the earth *E* with the orthogonal axes connecting the solstices and the equinoxes. Then you can move the transparencies and observe what happens in different relative positions of *C* and *E*.

You may investigate the following questions:

1. What can you say about the <u>distances</u> between earth and sun during a full circle of the sun?

2. What can you say about the varying <u>velocities</u> of the true sun?

3. What are possible lengths of seasons which are consistent with our theory. Or, to put it differently: is it possible to prescribe arbitrary lengths of seasons (of course, their sum has to be 360°) and find a center *C* which fits to these lengths?

You are asked to make as many observations as possible. If you are able to derive these observations from the hypotheses they will become theorems in our theory.

What do you think about the truth of these theorems? Write down your opinion.

5.4 Worksheet 4

Now we will investigate "how good" the hypothesis of Hipparchos is.

a) Please, guess in which quadrant the center C of the eccentric circle will lie if we take the data of table 1.

b) For the construction of the exact position of C we use a software like Geogebra.

First of all, we apply a trick physicists often use. The Greeks didn't know the (mean) distance of the sun from the earth. Therefore we set this distance equal to 1. Then we can represent the distance of C from E by a fraction like for example 1/3. This fraction would say that the distance of *C* from *E* is 1/3 of the distance of the sun from the earth. Already Hipparchos applied this trick and today's astronomers call the distance earth – sun the "astronomical unit".

Since working with real data involves numbers with a lot of digits and clumsy conversions of days in angles and vice versa we simplify the situation by using "easy numbers". That is we suppose that we have already done the conversion of the numbers of days of spring and summer in angles and that we have got 95° for spring and 105° for summer. In sum, the "half year" of summer lasts 200°. If you now consider that the equinoxes VE and AE lie at the endpoints of this arc of 200° you should be able to construct the position of the earth *E* relative to the position of the center *C*.

5.5 Worksheet 5

The same with real data.

Results:

With the center *C* in the origin of a Cartesian system we get the coordinates of the earth E

$$E = (0{,}007801; -0{,}032506).$$

When E is in the origin we get

$$C = (-0{,}007801; 0{,}032506).$$

The distance between E and C is

$$e = \underline{0{,}033429.}$$

This is approximately 1/30. Ptolemy got with the data of his time 1/24.

The angle between the apogee and the vernal equinox is with our modern data 103,475°. Ptolemy got with his data ca. 69°.

When we take 12 exact dates (one per month) and compare modern data for the positions of the sun with data calculated by means of Hipparch's hypothesis we get a mean deviation of 3' and a maximal deviation of 20'.

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ENHANCING STUDENTS' UNDERSTANDING OF VARIANCE

Physical experiments based on a historically inspired model

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ABSTRACT

A didactically oriented study that we have done on the history of statistics[1] points out the importance of the context of the treated situations in the emergence and evolution of basic statistical concepts and methods, like variance, the method of least squares etc. In particular, it points out the importance of the complexity of the involved situations. Although these concepts and methods were initially conceived and developed in the context of the treatment of measurements' errors in situations and problems of astronomy and geodesy, their transfer to the social sciences encountered important conceptual difficulties and followed a complex evolutionary process that lasted for almost a century. Furthermore, history points out that there has always been an intimate relation between statistics and physics, not limited to the fruitful developments related to problems of geodesy and astronomy, during the 18th and 19th centuries; among others, it also concerns the kinetic theory of gases and statistical mechanics. This relation has gradually permitted to understand that basic statistical concepts have a deep physical meaning (like the absolute temperature, the kinetic energy of the molecules of an ideal gas, Brownian motion etc).

In the usual introductory statistics courses, it is very common to use (almost) exclusively examples related to social phenomena (labeled also as "every day life" phenomena), whereas meaningful examples from other domains - like physics and geometry - are absent.

Our analysis of students' behavior points out that: (i) at the introductory level, it is difficult for them to get a coherent meaning of variance only on the basis of situations related to social phenomena; (ii) the restriction to such examples in introductory statistics teaching can activate important epistemological obstacles against students' understanding of variance. On the other hand, the study of the history of statistics points out that one of the most promising ways may be based on the use of adequate physical models. Indeed, looking at the rich reservoir of examples provided by history allowed us to identify such models.

As a first step, we explored the didactical use of two generic such models in introductory teaching of statistics; namely *the moving particles' model* and *the springs' model*, which have important interpretative virtues concerning variance and its properties and are the simplest ones, presupposing only rudiments of elementary physics. Hence, they were adequate for students like ours (prospective elementary school teachers). The results of the use of these two generic models in our previous teaching were very encouraging, where only paper-pencil and blackboard work was involved[2]. However, last year, in collaboration with the *Laboratory for Science Teaching* of the Department of Education of the University of Crete, we designed an experimental setting that constitutes a macroscopic realisation of *the moving particles' model.*

This setting involves small vehicles of different mass, moving in one dimension (on a rail). The dispersion of the vehicles' velocities is realized mechanically (using stretched springs incorporated in the vehicles). Last year we used the experimental setting in an introductory statistics course to students of our department. In this work we analyze some of the benefits on students' understanding of variance that followed from the implementation of this experimental setting: (i) Its use, combined with the generic model and adequate references to the historical modelisation of fundamental physical phenomena offers students the vivid picture of variance as expressing the dispersion energy of the elements of a physical system and that this holds for many fundamental physical phenomena. This greatly facilitates students' initial understanding and acceptance of variance as a basic dispersion measure. (ii) It allows for a clear and simple physical explanation (proof) of

[1] E.g., see our paper in the proceedings of HPM2008 and references there in.
[2] In some cases Excel was also used to treat other, more complex related examples of data.

the basic property of variance $\overline{(x-\bar{x})^2} = \overline{x^2} - \bar{x}^2$. [3] (iii) Its use facilitates students' conception of the following important characteristic: the final distribution of vehicles' (particles') velocities can be realized in different ways: Not only with a unique dispersion from their common initial velocity, but also through a succession of dispersions of the vehicles' velocities. In fact the succession of dispersions in the context of the model is equivalent to a sum of random variables. Thus, students' research work on the successive dispersions that lead to the final distribution of vehicles' (particles') velocities facilitated the subsequent introduction of the concept of the sum of random variables

[3]When only the generic model of moving particles is used, this property has a very interesting and clear interpretation (in fact, it is equivalent to an important physical property), but its justification cannot be established on simple physical grounds, but otherwise, e.g. algebraically; in our previous teaching, students' request to justify this property on physical grounds using this model was not satisfied.

THE BEGINNING OF POTENTIAL THEORY

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ABSTRACT

Potential theory is a central theme of analysis. It originated from attempts to understand the force fields defined by Newton's universal law of gravitation. The subject was developed as a self-contained mathematical discipline in the end of the 18th century and in the beginning of the 19th century with the works of Lagrange, Laplace, Legendre and Poisson. In his Essay on the application of mathematical analysis to the theory of electricity and magnetism of 1828, Green introduces the name of «potential», defines the «Green's function» for the Laplace equation in any finite region, and gives new physical interpretations of potentials.
This history and especially the part played by Green, offers an interesting example for students about the dialectical relations between physics and mathematics.

FROM ARTS TO MATHEMATICS AND BACK

Contextual Learning – Practical Interdisciplinarity

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ABSTRACT

School mathematics is often perceived by learners as an abstract and distant subject. To change this attitude, the abstract nature of the subject is to be situated in real life contexts via interesting links to the attractive, unexpected fields. To eliminate a persisting influence of abstract and instructive way of mathematics teaching/learning complicating the understanding of aforementioned subjects, we attempted to use a contextual teaching/learning to connect new terms and operations to the real world. While contextual approach is supposed to be applied in primary or in higher secondary education, we attempted to connect mathematics to the arts at school in the K-6 class (12-13 years). We interconnected a development of the mathematical notion of “symmetry”, its rules and their use in the real historical artistic production. Three graduated levels were realized – ornaments with periodical repetition of basic elements, decorative plain-initials and garden labyrinths as a control of acquisition and application. In spite of accidental impossibility to present planned historical resources, children arrived to all notion of geometrical symmetries by exploring and discovering their properties and have applied them to realise their artistic goal.

USING HISTORY OF ALGEBRA AS A TOOL FOR MOTIVATING MATHEMATICAL THINKING OF SECONDARY SCHOOL STUDENTS

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ABSTRACT

The aim of my thesis is to analyze the appropriateness of introducing the unknown, solving methods, which affect today's understanding of the way to solve equations and understand this course. My task is comparing the historical progress in symbolization and schematization with that of our modern students in learning and teaching process. Such treatment of the history of algebra arises material useful for primary and secondary schools. Teachers can understand the evolution of the symbol and then address the issue of implementation unknown at primary school.

Primary school pupils enter to the world of abstraction, generalization, understanding letters in terms of numbers in the 7th year of primary school. From my own experience, but also testimonies of students, teachers, is generally the problem with the strategy for editing expressions (whether alone or in solving equations). To solve problems (based on real situations, respectively, solving word problems), leading to the formulation and subsequent solution of equations, students should be able to interpret the problem, use algebraic notation to model the situation of the task. Let us be inspired by the history of algebra as a tool for research.

COURBES DE RACCORDEMENT

dans les chemins de fer au XIXème siècle

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ABSTRACT

S'agissant de la construction des routes et des voies ferrées parcourues par des véhicules à une vitesse relativement élevée, la connexion entre deux lignes droites (ou alignements) ne peut en aucun cas se traduire uniquement par un arc de cercle. En effet la force centrifuge conduirait à l'accident.
L'étude des courbes de raccordement commence à l'époque de la construction des lignes de chemin de fer au XIXème siècle. Différentes solutions ont été utilisées, comme la juxtaposition d'arcs de cercle de rayons décroissants avant d'arriver à la clothoïde. Celle-ci présente un rayon de courbure variant linéairement et utilise les intégrales de FRESNEL, dont l'origine vient de … l'optique. Les mémoires des différentes assemblées et réunions des ingénieurs concernés rendent compte à l'époque de leurs travaux et solutions.

Die Übergangsbogen in Eisenbahnlinien (XIX Jahrhundert)

Was die Absteckung der Eisenbahn und der Straßen betrifft, die mit einer ziemlich hohen Geschwindigkeit durchquert wurde, kann nicht die Verbindung zwischen zwei geraden Linien ein Kreisbogen sein. Die Zentrifugalkraft würde offenbar das Fahrzeug außerhalb des gewünschten Weg bewirken.
Die Studie der Übergangsbogen kommt mit der Konstruktion der Linien von Eisenbahn im neuzehnte Jahrhundert auf. Verschiedene Lösungen werden benutzt, zum Beispiel die Nebeneinanderstellung von Bögen abnehmenden Strahlkreises für die Linie des Brenners in Österreich, bevor man zur Lösung „Klothoïde" gelangt. Diese stellt einen Krümmungsradius vor, der linear variiert, und nimmt die Integrale von FRESNEL in Anspruch, deren Ursprung zu suchen ist in… optisch. Die Aufgaben der verschiedenen Sitzungen der betroffenen Ingenieure geben ihren Arbeiten und Lösungen zurück.

Curves of adjustement on the railway (XIX century)

Concerning the railroad and road drawing, traversed at quite a high speed, the connection between two straight lines cannot be an arc of circle. The centrifugal force would obviously pushthe vehicle out of the wanted trajectory.
The study of the curves of adjustement was born with the construction of the railway lines in the 19th century. Various solutions are used, such as the juxtaposition of arcs of circle with decreasing radius for the line of the Brenner in Austria, before the solution of the Clothoid. This one presents a linearly varying radius of curvature and use the integrals of FRESNEL, whose origin is to be sought in… optical. The reports of the various assemblies and meetings of the engineers concerned return account of their work and solutions.

MATHEMATICS FOR STUDENTS OF DIGITAL ARTS AND GRAPHIC DESIGN: A HISTORICAL APPROACH

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ABSTRACT

In this paper we present a didactic activity developed to motivate our students of no-Engineering university programs to study mathematics. Digital Arts and Graphic Design students have not taken mathematics courses in two years so they do not want to solve exercises of calculus or any other topic. “Animated Mathematics’ History” is an activity where students have to search information and develop an animation of three minutes about a specific mathematics topic. We have only applied once the activity and the students developed animations on the history of π, trigonometry in Egypt and Babylon, Newton–Leibniz invention of calculus, and others. Even we had to negotiate some characteristics of the animations and the way it would be graded, students did a great work over our expectations. Finally, students found unexpected reasons to study mathematics. Some students’ remarks are about the possibility of helping other students to discover mathematics through animations.

Cultures and mathematics

PRACTICAL GEOMETRIES IN ISLAMIC COUNTRIES

The Example of the Division of Plane Figures

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ABSTRACT

The division of plane figures is a geometrical chapter developed in numerous works written in Arabic. In the extension of Greek practices, this chapter is also found in original developments in Islamic countries. The aim of the presentation is to show the diversity from several books of the Muslim Orient and Occident from the 9th century until the 14th century.
This diversity is first based on the multiple origins of the problems. They are linked, among others, to the practices of craftsmen, architects, or jurists. For example, jurists had to decide on the sale or the sharing of fields. To divide a geometrical figure in a certain number of similar figures is an important problem for the decorators who embellished palaces, *madrasas*, and other mosques and mausoleums. Moreover, these problems are illustrated in some writings of eminent geometers.
This diversity also expresses itself by the wealth of procedures of construction and resolution for which the whole mathematical knowledge is included.

1 Introduction

It is unfortunate that the history of scientific activities in Islamic countries is so poorly known to Europeans and this history repeats itself. Today, some European scholars (certainly ideologically tainted) neglect the original developments of science in the Islamic era and even deny their appropriation by Christian Europe from the 12th century.

Even though studies on the advent and development of scientific activities in Islamic countries are still incomplete or deficient, our goal in this paper is to give some general and major features of their history. Then, we discuss the interactions that have been established between scientific practices and several social, cultural, political, and religious aspects.

2 Sciences in Islamic Countries.

2.1 Generalities on the Scientific Development in Islamic Countries.

From Hegira (632), the Islamic countries, that is to say all regions dominated and unified by a single religion – Islam – gradually correspond to a huge Empire. It extends from the Pyrenees to Timbuktu (from North to South), and from Samarkand to Saragoza (from East to West). In this controlled and pacified geographical area, Islamic law is the canon law for the society. All roads, in particular those of trade, are free. In the history of scientific practices in Islamic countries, three major periods can roughly be distinguished[1].

The first one is a period of appropriation of the knowledge of the Ancients (Syrian, Persian, Sanskrit or Greek, of course). Scientific activities benefit from the fluidity of movement of men and books. The local scientific practices (weights and measure, calculation of inheritance, decorative art, astrology, for example) are not neglected and the

[1]For further details, see Djebbar, A., 2001, *L'âge d'or des sciences arabes*, Paris: Seuil.

scholarly knowledge consolidates them and introduces rational approaches. Arabic, the language of the *Qur'ān*, is needed as the language of scientific communication[2]. An important movement of translation grows, from the 8th century until the mid-10th century, to become acquainted with the whole knowledge of the Ancients[3]. From the first conquests of new territories, Islam held a leading role for the sciences.

The second period of the history of scientific practices in Islamic countries runs from the 9th century to the 12th century. It is the so-called "Golden Age of sciences in Islamic countries", corresponding to a period of scientific creation and development. When the knowledge from the Ancients was assimilated, scientists from Islamic countries taught, commented, and surpassed them. Many scientific foyers emerged, both in the East and in the West, such as in Baghdad, Samarqand, Cairo, Cordoba, but also in Kairouan, Nishapur, and Marrakesh. In addition to improving results inherited from the Ancients, innovations are observed, for example, in medicine, astronomy, and mathematics. New disciplines emerged such as trigonometry, algebra, and combinatorics in mathematics. These developments were widely promoted by several factors including the patronage of Princes and foremost that of the Caliph himself, by various social demands or by the non-intervention of religion in scientific practices.

The third period is not a period of decline as it has often been characterized. Original scientific researches continued to occur (in mathematics and astronomy) but they were more isolated in several parts of the Islamic area, both in the West with the Maghreb and Andalus and in the East with actual Iran. This period is characterized, from the late 12th century, by the disappearance of Arabic as the only language of scientific discourse. Indeed, three other languages competed with Arabic: Persian in the East and Hebrew and Latin in the West.

The men of science in Islamic countries, whatever their profile, took advantage of collective management and its contingencies to guide and deepen part of their production. We now discuss a few aspects of the "Islamic culture" (taken in its widest meaning) that are considered as an *impetus* of scientific research in Islamic countries.

2.2 Mathematics and Cultural Aspects in Islamic Era.

To be encouraged and developed, any scientific knowledge needs institutional support and cultural values. Science in Islamic countries is no exception. First of all, institutionally, we mention the strong political will of many successive Caliphs to support scientific research and teaching. Then, culturally speaking, several direct or indirect evidences guarantee the existence of interaction between the scholarly science and the know-how of artists and crafstmen. We shall even see that some of these evidences give the proof of a certain stimulation of scholars by or for craftsmen.

The Umayyadds (661-750) and the first Abbasids (especially between 750 and 850) expressed the same desire by supporting scientific activities. The Caliph al-Ma'mūn (813-833) is probably the more significant. To provide the library *Bayt al-ḥikma* [House of Wisdom] in Baghdad, he interceded with Leo V (813-820), Emperor of Byzantium, to obtain books on philosophy and science. He also supported delegations of scholars in Asia

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[2]Thus, due to an abuse of language, science in Islamic countries is sometimes called « Arabic science ». The term «arabic» must have been understood in the meaning of the language used to write and teach the science. It does not refer to geographic, cultural or religious origins.

[3]Gutas, D., 1998, *Greek Thought, Arabic Culture*, New York: Routledge.

Minor and Cyprus to bring books written in Greek. He organised the measurement of the diameter of Earth. He gave assignments to scientists in order to determine the geographical locations of various events described in the *Qur'ān*. This same Caliph encouraged al-Khwārizmī to "*compose a short book on algebra and muqabala*[4]*"*, namely *Mukhtaṣar fī ḥisāb al-jabr wa l-muqābala* [Compendium on calculating by completion and reduction], which can be regarded as the official birth of Algebra as a new field of knowledge. This kind of patronage was still present at least until the first half of the 15th century. This is confirmed by the astronomer and mathematician al-Kāshī (d. 1429) in his correspondence with his father. Member of the scientific staff of Ulūgh Beg (1394-1449) in Samarqand, one of the most important scientific foyer of the Muslim Orient, he wrote on the construction of a *mihrāb*[5] according to the wishes of the Sultan: "*His Majesty [once] said: ''We would like to make a hole in the wall of a* mihrāb *in such a way that the sun may shine through that hole for a short while at the afternoon [prayer] time both in summer and in winter. That single hole must be round from inside, but from the outside it must be in such a way that sunshine cannot pass through it at times other than the afternoon [prayer time]. This [royal wish] had been [already] expressed before my arrival, and nobody had been able to realize it; [but] when I came [here], I did this also*[6]*.*"

This last evidence allows us to evoke the idea that some of the scientific production (research and teaching) can be considered as replies to individual or collective societal needs. These responses are directed to several practitioners such as architects, craftsmen decorators and, according to al-Khwārizmī himself, "*inheritance, legacies, partition, law-suits, and trade, and in all their dealings with one another, or where the measuring of lands, the digging of canals (...) are concerned*[7]". The duality between the scholar and the practitioner is not widely known but is nevertheless real. It is illustrated by three distinguished scholars of Islamic countries who provide evidence on meetings between mathematicians and craftsmen, which can be considered as a forum to discuss methods for designing ornamental patterns in several materials (wood and tile, for example). The first one, Abū l-Wa'fā' al-Buzajanī (940-998), is a Persian astronomer and mathematician who worked in Baghdad from 959. In his book *Kitāb fīmā yaḥtāju ilayhi as-sanī*^{c} *min al-a*^{c}*māl al-handasiyya* [Book on What is Necessary from Geometric Constructions for the Craftsmen], he specified that he participated in such a meeting in which was discussed the construction of a square from three equal squares[8]. The second one is the famous Persian mathematician and poet: ^{c}Umar al-Khayyām (1048-1131). In an untitled work, he reported a solution of a problem (by using a cubic) settled in a meeting, which could be in Isfahan with mathematicians, surveyors and craftsmen[9]. Al-Khayyām's collaboration with craftsmen does not seem to be limited to this. Indeed, he could be the designer of the

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[4]Rashed, R., 2007, *al-Khwārizmī, le commencement de l'algèbre*, Paris: Blanchard, p. 95.

[5]A *mihrāb* is an alcove in the mosque indicating the direction of prayer.

[6]Bagheri, M.. 1997, "A newly found letter of al-Kāshī on Scientific Life in Samarkand", *Historia Mathematica* **24**, 241-256.

[7]R. Rashed. *al-Khwārizmī...*, p. 95.

[8]Abū l-wafā' al-Būzajānī, 1979, *Kitāb fīmā yaḥtāju ilayhi aṣ-ṣānī*^{c} *min a*^{c}*māl al-handasiyya* [Book on What is Necessary from Geometric Constructions for the Craftsmen], Introduction and critical edition by Al-^{c}Ali, S.A., Baghdad: Imprimerie de Bagdad, p. 145.

[9]Amir-Moez, A., 1963, "A Paper of Omar Khayyam", *Scripta mathematica* **26**, 323-337.

North Dome Chamber of the Friday Mosque of Isfahan[10]. The third and last evidence dates from the 15th century. In a letter to his father, al-Kāshī described its resolution of a problem settled during a meeting between craftsmen, mathematicians and other dignitaries[11]. He also showed an excellent knowledge of architects, designers, and other craftsmen by measuring domes and *muqarnas* in his major *Miftāḥ al-ḥisāb* [Key of Arithmetic][12].

Now, we will illustrate the close relationship between the speculative developments of scholars and real issues of practitioners, whatever their original corporation, with a kind of geometrical problems: the division of plane figures.

3 The problems of the Division of Plane Figures

Our purpose is not to give a historical background of this kind of problem[13]. Nevertheless, it is necessary to detail what we understand by "division of plane figures". It is an old and very diverse mathematical chapter with problems issued by Old Babylonian scribes dating from 1800BCE. Typically, cutting off or dividing a plane figure is sharing it according to several constraints set *a priori*. These constraints are related to geometric properties of the transversal(s) or desired figures. And, they are on magnitudes with several conditions on ratios given on parts obtained after the division. For example, we have to divide a rhombus in two parts according to a given ratio by a line parallel to one of its sides. The problems can also consist in dividing a (or several) given figure(s) in order to acquire an (or many) other(s) respecting conditions of similarity for example. In our present paper, the problems of inscription (or circumscription) and the cutting off of figures necessary to the measurement (as the triangulation, for example) are not taken into account even if they are quite important in their cultural aspects (ornamentation and surveying, for example).

It is not difficult to guess that these kinds of problems could easily be related to professional activities in everyday life. Craft or legal traditions of such sophistication involved a significant amount of technical and mathematical knowledge. Even if this knowledge was above all transmitted from master to apprentice or from a brother to another in a same corporation rather than being written down, it is possible to illustrate it from books dealing with mathematics. First of all, we present briefly the scholarly tradition. Then we will devote our purpose to two main cultural topics: inheritance and ornamentation.

3.1 Division of Plane Figures and Speculative Geometry[14]

The scholarly orientation of the division of plane figures in Islamic science is roughly characterized by the reception and the appropriation of an Euclid's text. First of all,

[10] Özdural, A., 1998, "A Mathematical Sonata for Architecture: Omar Khayyam and the Friday Mosque of Isfahan", *Technologie and Culture* **39**, 699-715.

[11] Kennedy, E.S., 1960, "A Letter of Jamshīd al-Kāshī to His Father", *Orientalia* **29**, 191-213. p. 198

[12] Dold-Samplonius, Y., 1992, "Practical Arabic Mathematics: Measuring the Muqarnas by al-Kāshī", *Centaurus* **35**, 193-242.

[13] Moyon, M., 2008, *La géométrie pratique en Europe en relation avec la tradition arabe, l'exemple du mesurage et du découpage : Contribution à l'étude des mathématiques médiévales*, PhD in Epistemology and History of Sciences supervised by Djebbar, A., University of Lille1.

[14] We only give major features. We have largely studied this subject in our thesis. Moyon, M., 2008, *La géométrie pratique…*

as-Sijzī, one of the most prolific Islamic geometers in the tenth century, wrote an opuscule which explicitly refers to the *Kitāb Uqlīdis fī l-qismat* [Book of Euclid on the divisions][15]. It could be a partial Arabic translation of *On the divisions*, a lost book of Euclid. As-Sijzī introduced thirty-five problems, of which only four were proved. The others were considered as easy by the Persian mathematician[16].

Another author, Muḥammad al-Baghdādī, proposed the same kind of problem in a scholarly book, the *De superficierum divisionibus Liber*, inspired by the previous one. Only the Latin translation, completed in the twelfth century, has survived. The author, probably active between the tenth and the twelfth century in the Muslim Orient, remains unknown.

In these two books, the statements are very general and the given proofs are constructed according to the Euclidean model (hypothetico-deductive). They are based on the *Elements*, in particular on Books I, II, V and VI.

3.2 The Division of Plane Figures as a Source of Decorative Patterns.

In this section, we present a unique example from the *Kitāb fīmā yaḥtāju ilayhi as-sanī^c min al-a^{c}māl al-handasiyya* of Abū l-Wafā‘ that we have already mentioned. Indeed, in the tenth chapter, he gives us valuable evidence comparing, for a same problem, the empirical approach of the craftsman to the speculative one of the geometer[17]. This problem concerns the division of three equal squares in order to compose an other one.

In this context, Abū l-Wafā‘ opposes two methods of the decorator craftsmen based on know-how, intuition, experimentation, and trial and error to the application of a theoretical result established by mathematicians disconnected from design and material reality. *“A group of geometers and craftsmen were wrong in the matter of these squares and their assembling, the geometers because they have little experience in practice, and the craftsmen because they lack knowledge of proofs. The reason is that, since the geometers do not have experience in practice, it is difficult for them to approximate according to the requests of the craftsman what is known to be correct by proofs by means of lines. And, the purpose of the craftsman is what makes the construction easier for him, and correctness is shown by what he perceives through senses and by observation and he does not care about the proofs of the imagined thing and <the correctness of> the lines. (...)The geometer knows the correctness of what he wants by mean of proofs when he is the one who has extracted the notions on which the craftsman and the surveyor work. But, it is difficult for him to apply the proofs to a construction when he has no experience with the work of the craftsman and the surveyor. If the most experienced among these geometers are asked about something in dividing figures or something in multiplying lines, they hesitate and they need a long time to think. Perhaps it is easy for them but perhaps it is difficult for them and they do not succeed in its construction*[18].”

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[15]Hogendijk, J.P., 1993, “The Arabic version of Euclid’s On Divisions”, in *Vestigia Mathematica, Studies in Medieval and Early Modern Mathematics in Honor of H.L.L. Busard,* M. Folkerts & J.P. Hogendijk (eds), Amsterdam-Atlanta: Rodopi, pp. 143-157; p. 149

[16]Hogendijk, J.P., 1993, “The Arabic…”, p. 159.

[17]Abū l-Wafā’ did not forget problems settled and solved in a more scholarly way in the chapter eight for triangles and the ninth one for quadrilaterals. Abū l-Wafā’. *Kitāb fīmā…*, p. 103-127.

[18]Abū l-Wafā’. *Kitāb fīmā…*, p. 144-145.

Abū l-Wafā‘ also acknowledged that some of the methods used by craftsmen were wrong even though they seemed correct in appearance to an observer uneducated in scholarly geometry. He then explained some of the methods known to craftsmen in order to distinguish those that are correct (that is to say demonstrable by geometers) from others. The Persian mathematician also proved why certain methods are inaccurate[19]. Once he studied the methods of craftsmen, he gave his own construction that inspires a new decorative pattern[20].

3.3 Division of Plane Figures and Islam: the Inheritance

Rituals and religious prescriptions in the respect of *al-qur‘ān* and *al-ḥadīth* [words of the Prophet Muḥammad] involve technical problems that encourage scientific researches. We cite, for example, the determination of *qiblā* (the direction of Mecca), the lunar month, and especially the observance of *Ramadan* for thirty days a year, or the calculation of the exact time of the day for all the prayers.

Here, we focus on another aspect of the Islamic law : the *ᶜilm al-farā'id* [science of partitions of the inheritance]. In addition to the questions on arithmetic or algebra[21], this science also raises geometrical problems including the sharing of land between partners or beneficiaries. All of these problems seem to be issues (at least in form) presented to the surveyor when fields are separated. They can be found in books of geometry or in textbooks on reckoning in the chapter dealing with *misāḥa* [measurement]. We present here three examples of sharing of lands between co-owners or eligible parties by creating a way-in for each of them. These three examples have different solutions[22].

The first one is a problem from the last part "On the construction of a way"[23] of the ninth chapter dealing with the divisions of quadrilaterals of the geometric book written by Abū l-Wafā‘ already encountered. The problem is solved with a geometrical construction based on a typical Euclidean style of the *Elements*. No proof is given to justify the exactness of the construction[24].

The second example is extracted from the book of Ibn Ṭāhir al-Baghdādī (d. 1037), an 11th-century mathematician, intitled *Takmila fī l-ḥisāb* [The Completion of Arithmetics][25]. The resolution given by the mathematician of Baghdad is algorithmic. Indeed, he established a general algorithm in order to use it in a special case where it is necessary to share a rectangular field between three brothers with an access for the different parts.

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[19] See appendix 1.

[20] Özdural, A., 2000, "Mathematics and Arts: Connections between Theory and Practice in the Medieval Islamic World", *Historia Mathematica* **27**, 171-201.

[21] Al-Khwārizmī devotes a substantial part of his *Mukhtaṣar* to the resolution of problems on legacies. Rashed, R.. *Al-Khwārizmī…*, p. 232-291. For further details on this topic, see Laabid, E., 2006, *Les techniques mathématiques dans la résolution des problèmes des partages successoraux au Maghreb médiéval : l'exemple du Mukhtaṣar d'al-Ḥūfī (m.588/1192)*, PhD in History of Mathematics supervised by Lamrabet, D. and Djebbar, A., University of Rabat.

[22] See appendix 2.

[23] Abū l-Wafā'. *Kitāb fīmā…*, p. 127-132.

[24] In this section, Abū l-Wafā' exposes five problems (two in a square, two in a triangle, one in a trapezium). Note that the last three constructions are erroneous.

[25] Ibn Ṭāhir al-Baghdādī, 1985, *At-takmila fī l-ḥisāb* [The completion of Arithmetics], Introduction and critical edition by Saïdan, A.S., Koweit: Ma'had al-makhṭūṭāt al-‘arabīyya.

The third and last example can be read in the *Kāfi fī l-ḥisāb* [The Sufficient in Arithmetic] of al-Karajī (d. 1023)[26]. This example is interesting for two main reasons. First of all, the statement of the situation seems dictated by Islamic law. Consequently, this leads to a strange mathematical problem with a sharing between the half, the third and the fourth! This is not an isolated case in the calculations of inheritance[27]. The mathematical solution takes into account this situation and the beneficiaries must accept a smaller part than that given by Islamic law. Secondly, it is the occasion for the Persian mathematician to use the objects and the operation characteristic of Arabic algebra.

4 Conclusion

Division of plane figures is indeed an interesting geometric practice to illustrate the scientific development in Islamic countries. At first, it seems to be based on the local tradition with traditional skills. Actually, however, it is the appropriation of the knowledge of the Ancients, the quest for rationality, and the use of new disciplines that progressively provide answers to these problems in everyday life.

Even if scientists and craftsmen of Latin Europe have different cultural and religious needs as their counterparts in Islamic countries, the problems of division of plane figures are part of the many practices and knowledge that Latin scholars approppriated from the 12th century. We have mentioned the Arabic-Latin translation of the text of Muḥammad al-Baghdādī. We also discussed the *Liber Embadorum* of Platon de Tivoli, the Latin version of the *Hibbur ha-Mesihah we-ha-Tisboret* of the Hebrew scientist Abraham Bar Ḥiyya. These books both dedicate a whole chapter to this topic, the study of which highlights a dual origin: scholarly, but also practical with sharing between beneficiaries. Finally, it is also the case of the fourth distinction of the famous *Practica geometriae* written by Fibonacci in the 13th century that will influence subsequent treaties of geometry.

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[26]Al-Karajī, 1986, *Al-Kafī fī l-ḥisāb* [The Sufficient in Arithmetic], Introduction and critical edition by Chalhoub, S., Alep: Institut d'Histoire des Sciences Arabes.
[27]Laabid, E., 2006, *Les techniques mathématiques* ..., p.41.

APPENDIX 1

Abū'l Wafā': *On the composition and division of squares*[28].

One of the craftsmen placed one of the squares in the center. And he bisected the second by means of the diagonal and he placed the halves on two sides of the <first> square. And he drew two straight lines from the center of the third <square> to two of its angles, not on one diagonal and a line from the center to the midpoint of the opposite side of the triangle which came with the two <previous> lines. Thus the square is divided into two trapezia and a triangle.
Then he placed the triangles below the first square and he placed the two trapezia above it. He combined the two longer sides <of the trapezia> in the middle. Thus he obtained a square as in this figure (fig. 1)

(Abū l-Wafā' said): As for the figure which is constructed, it is in the imagination, and someone who has no experience in the art and in geometry see that it is correct. But if we show it to him, he knows that it is false.
The fact that he imagines that it is correct is explained by the correctness of the angles and the equality of the sides. The angles of the square are correct, each of them is right, and as for the sides, they are equal. Because of this, he imagines that it is correct.
The angles of the <three> triangles – B, G and D – which are the angles of the square, are all right. And the fourth angle is assembled from the two angles which are all <equal to> half a right <angle>, they are the angles of the <two> trapezia.
The sides are straight and equal because each of these sides is assembled from a side of one of the squares and half its diagonals, which are equal. The fact that they are straight with the assembling is clear because the angles gathered at the meeting points of the lines are all equal to two right <angles>.
The three angles which are at point G are equal to two right <angles> because they are one angle of a square and two angles of a triangle which are, each of them, equal to half a right angle. And it is the same for angle T.
Angle I, meanwhile, has two angles, one of them is the angle of the triangle, that is half a right <angle>, and the other is the angle of the trapezium, that is one right <angle> and half <a right angle>. And it is the same for the two angles which are at point K.
As the angles are right and the sides are straight <and equal>, for everybody, it appears that a square has been constructed from three squares. And they do not realize the place where the error is introduced.
This is clarified for us if we know that each side of this square is equal to the side of one of the <first> squares and half of its diagonal. It is not permissible that the side of the square composed of three squares has this magnitude. Indeed, it is greater than that. And that is because if we consider the side of each <first> square <equal to> ten cubits to make it easy for the student, the side of the square composed of three squares is approximately seventeen cubits and one-third <of one cubit>. And the side of this

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[28] Abū l-Wafā'. *Kitāb fīmā…*, p.146-148.

square is seventeen cubits and half of one-seventh [of one cubit], and between them, there is a gap.
What’s more, when we bisected <the first> square ABGD and we placed each half of it next to the side of the square A, the diagonal of the square BG falls on two lines, HY and TK. And, it is not permissible that it falls on them for two reasons. One of them is that the diagonal of square BG is not expressible whereas line HY is expressible and it is as the side of square BG and half of it. The second is that it is less than that because the diagonal of square BG is approximately <equal to> fourteen and one-seventh, and side HY is <equal to> fifteen. Thus, the incorrectness of this division and this assembling clearly appears.

APPENDIX 2.

Abū l-Wafā’[29]
And if someone says: how divide a square ABGD into three equal parts and establish between them a path, whose width MQ is known, between two of the equal parts.
We extend GA towards I.
We construct AI equal to GM.
We extend BA towards E.
We construct <the circle> with center M and radius ME, and the circle intersects line BE (sic BA) at point E.
We join GE.
We cut off line ER from line GE equal to line GM.
We draw from point R line RHL parallel to line BAE.
And from two points M <and> Q, <we draw> two lines MT and QK both parallel to AG.
Thus there are equal surfaces MGHT, KLDQ, ABLH. And this is the figure for it. (fig. 2)

Ibn Ṭāhir al-Baghdādī[30]
And if we want to fit out a path in a land with right angles, or equal sides, or unequal its length or its width and that the land be divided between three people, or four people, or five people or whatever [the number of people], the method for it is to multiply the side on which we want to fit out the width of the path by the number of shares by which the land is divided, for example shares of the sons, of the daughters, of the two parents and of the husband. We subtract from it the width of the path and the rest is the divisor. Then, we multiply the area of the path by the number of heirs, minus the part of one who had charge of the path.
The result of the division is the length of the path. And when we know the length and the width of the path, the rest of the land can be shared between them according to <the rules of> sharing of God the Almighty.
Example of this: a land twenty by thirty that we want to divide between three brothers establishing, between them, a path whose width is two cubits but establishing the path from the thirty. We need to know how must be its length. We multiply thirty by three, and it results ninety. We subtract from it the width of the path and it is two cubits. It remains eighty-eight. This is the divisor that we keep. Then, we multiply the area by two, and this

[29] Abū-l-Wafā', *Kitāb fīmā ...,* p. 128-129.
[30] Ibn Ṭāhir al-Baghdādī. *At-takmila…*, p. 372-373.

is the number of son minus one, it will be one thousand and two hundred. It is necessary to multiply by two because the path is used for the passage of two people. We divide one thousand and two hundred by eighty-eight. The result of the division is the length of the path. And this is the figure for it (fig. 3).

And if the division is between two sons and one daughter, it will be between five parts. If it is between two daughters and one son, it will be between four parts. And everything that reaches to you in this chapter <resolves> it according to this method and it will be its solution. And this is the figure for it. The length of the figure is thirty, its width is twenty, the length of the path is thirteen and seven elevenths of cubits, the width of the path is two cubits, the area of the path is twenty-seven and three elevenths and the area of the whole land is six hundred. If the area of the path is taken off, it remains five hundred and seventy-two and eight eleventh of cubits.

The verification of its <validity consists in> measuring the part of the one who is below and in seeing if it is equal to the part of each of them. If the area of >the part of> the one is below is equal to the part of each of them, we will know that <the result> is correct. If it is different from them, it will be the opposite.

Al-Karajī[31]

If someone says: you have a rectangle whose length is twenty *bāb*[32] and width ten *bāb*. Divide it between three people : one half for one of them, one third for another and one fourth for another in order to there is, in his center, a way whose width is two *bāb* that entries of the three quotas lead by the length, one by the front, another by the right and another by the left so that the quota of the owner of the third is on the front according to this figure (fig. 4).

The computation for that is to call the length of the way thing. You multiply it by the width of the way, and it results two things and this is the measurement of the way. And you consider the remaining eighteen <to divide> into two parts between the owners of half, and of fourth because they take their part from the right of the way, and from his left. One of the two parts is twelve. This is the width of the part of the owner of half.

And the remaining six is the width of the part of the owner of fourth. And the length of each <part> is the length of the way, and this is the thing. It results the measurement of the part of the owner of half : twelve things. And the measurement of the part of the owner of fourth is six things. And from this computation, the measurement of the part of the owner of third has to be eight things, and the measurement of the way is two things, and the whole measurement of this surface is twenty-eight things. And this is equal to two hundred.

And the only thing is equal to seven *bāb* and one-seventh *bāb*. And this is the length of the way. And it remains the width of the part of the owner of third from the whole width of ten *bāb*: two *bāb* and six-seventh *bāb*.

[31]Al-Karajī, *Al-Kafī* …, p. 202-204.

[32]*bāb* is an unit of measurement used in Islamic countries in both West and East.

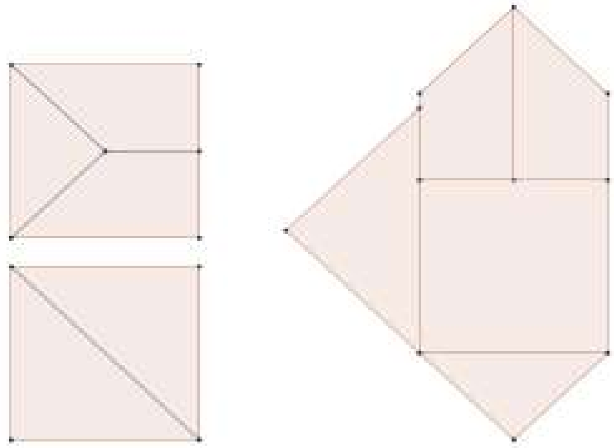

Figure 1 : Abū l-Wafā’

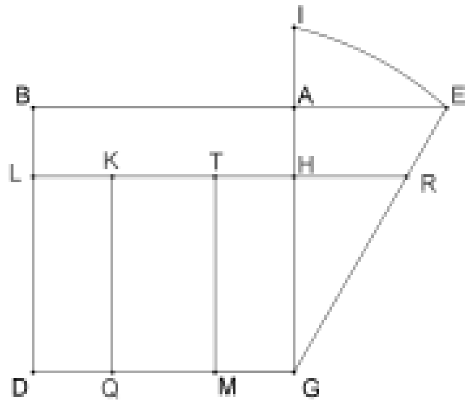

Figure 2 : Abū l-Wafā’

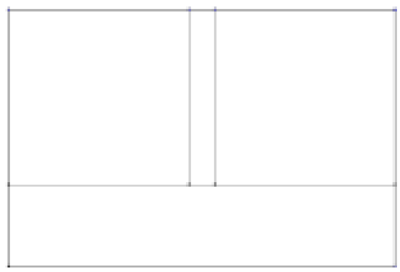

Figure 3 : Ibn Ṭāhir al-Baghdādī

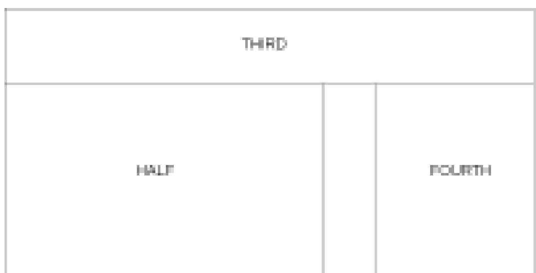

Figure 4 : al-Karajī

THE PERFECT COMPASS: CONICS, MOVEMENT AND MATHEMATICS AROUND THE 10$^{\text{TH}}$ CENTURY

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ABSTRACT

Geometry instruments certainly exist since men are interested in mathematics. These theoretical and practical tools are at the crossroads of the sensible world and mathematical abstractions. In the second half of the 10$^{\text{th}}$ century, the Arabic scholar al-Sijzī wrote a treatise on a new instrument: the perfect compass. At that time, several other mathematicians have studied this tool presumably invented by al-Qūhī. Many works are now available in French and English translations. After an historical presentation of the perfect compass, this article deals with a few passages of al-Sijzī's treatise which show the importance of continuous tracing of curves and provide interesting elements on the role of instruments in the mathematical research process. All these texts can help to understand the importance of motion in geometry which can be easily simulated by geometry softwares and used in a geometry lesson or in teacher training sessions.

1 Introduction and historical context

It is well known that the Arabic medieval period has been marked by the creation of algebra. Between the 9$^{\text{th}}$ and the 13$^{\text{th}}$ centuries, many researches are engaged and the emergence of new theoretical questions is one of the main consequences of the elaboration and development of this new field.
For example, let us consider the equation:

$$x^3 + 2x^2 + x = 4$$

This equation is equivalent to:

$$x(x^2 + 2x + 1) = 4$$

That is to say:

$$x(x+1)^2 = 4$$

And thus (for $x \neq 0$):

$$\frac{4}{x} = (x+1)^2$$

The roots, if they exist, are the intersection points between the hyperbola $y = \frac{4}{x}$ and the parabola $y = (x+1)^2$ (figure 1). In this situation, the existence of the roots is based on the geometrical existence of the intersection points. Both curves are conics and

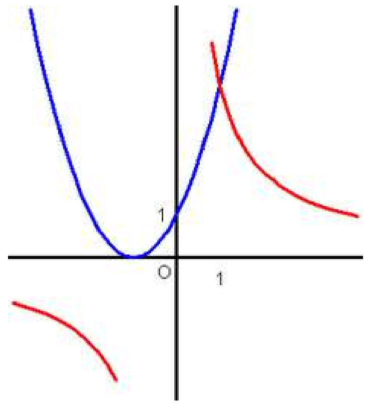

Figure 1: A simple example

these objects are complex enough to create a doubt on the reality of such intersections. Of course, the mathematician can not just say that the figure shows the trueness of the result and this simple example raises a crucial theoretical question. The point-by-point construction of conics has been well known since Antiquity (see the Apollonius' book entitled the *Conics*, for example), and that method is efficient enough for the analysis of the main properties of those curves. Algebraic equations can be solved by intersecting conics curves (ellipsis, parabola, and hyperbola) and the necessary taking into account of these intersections creates new difficulties. Indeed this possibility is based on the continuity of the different curves which is difficult to *prove*. The solution that has therefore been chosen is to associate the curve with a tool that enables a real construction. As the ruler and the compass allow straight lines or circles to be drawn and so justify their continuity, a new tool had to be invented to draw all the conics.

2 A new tool: origin and modelisation

As mentionned by R.Rashed[1], in the second half of the 10^{th} century, a large research movement is engaged by the Arabic scholars on the continuous tracing of curves. In his treatise *On the perfect compass*[2], Abū Sahl al-Qūhī (about 922 - about 1000) presents the results of his own research on this question.

> Abū Sahl Wayjan ibn Rustam al-Qūhī said : This is a treatise on the instrument called the prefect compass, which contains two books. The first one deals with the demonstration that it is possible to draw measurable lines by this compass - that is, straight lines, the circumferences of circles, and the perimeters of conic sections, namely parabolas, hyperbolas, ellipses, and the opposite sections. The second book deals with the science of drawing one of the lines we have just mentionned, according to a known position. If this instrument existed before us among the Ancients and if it was cited and named, but if its names as well as the names of the things associated with it were different from the names we have given them, then we would have an excuse, since this instrument has not come down to us, any more

[1] al-Qūhī et al-Sijzī: sur le compas parfait et le tracé continu des sections coniques, *Arabic Sciences and Philosophy*, vol.13 (2003) pp.9-43

[2] R.Rashed, *Geometry and dioptrics in classical Islam*, al-Furqān, 2005, pp.726-796

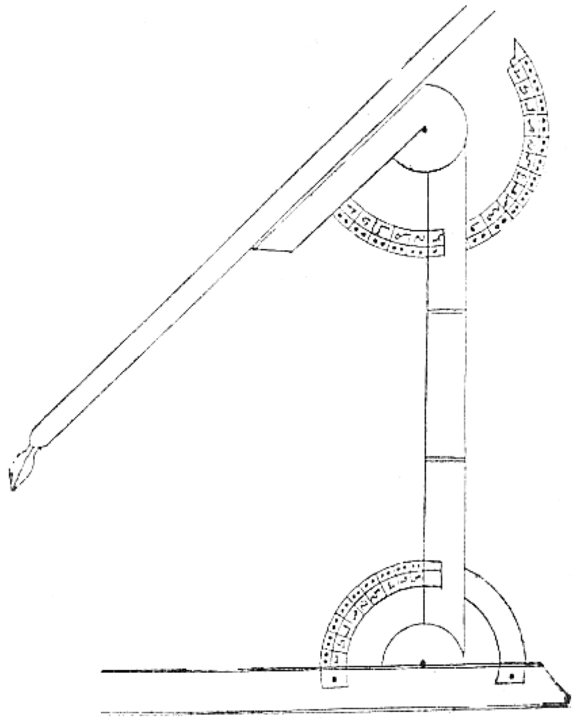

Figure 2: Sketch based on *Kitāb al-Qūhī fī al-birkām al-tāmm*, MS Istanbul, Raghib Pasha 569, fol.235^{v}.

> than has its mention; thus it is possible that this instrument, as well as the demonstration that is draws the lines we have just mentioned, may have existed without its use being the one we have made of it in the second book of this treatise.

In the second part of the short introduction (above quoted *in extenso*), al-Qūhī carefully explains that he has not found any texts on this instrument and that is why he wrote his treatise. The recent historical research seems to confirm that al-Qūhī's book is the first treatise on this tool[3]. Until now, no older descriptions have been found and there is no evidence of an implicite use of it before al-Qūhī. The new tool (figure 2[4]) is a kind of super-compass which can draw circles but also all the conic sections, even the degenerated ones like the straight lines.

> If, at a point of a plane, we raise a straight line that moves in one of the planes perpendicular to this plane, and if, throught another point of this straight line, there passes another straight line which has three motion - one around the straight line raised upon this plane, the second in the plane on which this straight line is situated, and the third on its extension simultaneously on both side - then if an instrument is described in this way, it is called a perfect compass.

The construction, in its principle, is very simple and other texts give lots of technical details that leads to think that some instruments have really been built. Unfortunately,

[3]For a complete analysis of the historical sources that have reached us, see R.Rashed (2003)
[4]Original picture in Rashed (2005) p.860

until now, no ancient perfect compass has been discovered. Nontheless, with this first description, one can construct such a tool. Some informations about modern replications of a perfect compass made for museum expositions, pedagogical or technical experiments, or just for pleasure can be found on the Internet[5]. In the educational context, a computer simulation is possible. A simple geometry software can give a good preview of what the tool can be. The example given in appendix produces the result below (figure 2). The first plane is (Oxy), the second (perpendicular to the first) is (Oxz). (OS) is the main line (the axis), the second one (SM) can move around the axis so that the point X leave a trace on the first plane. Let us call $\alpha = \widehat{SOx}$ and

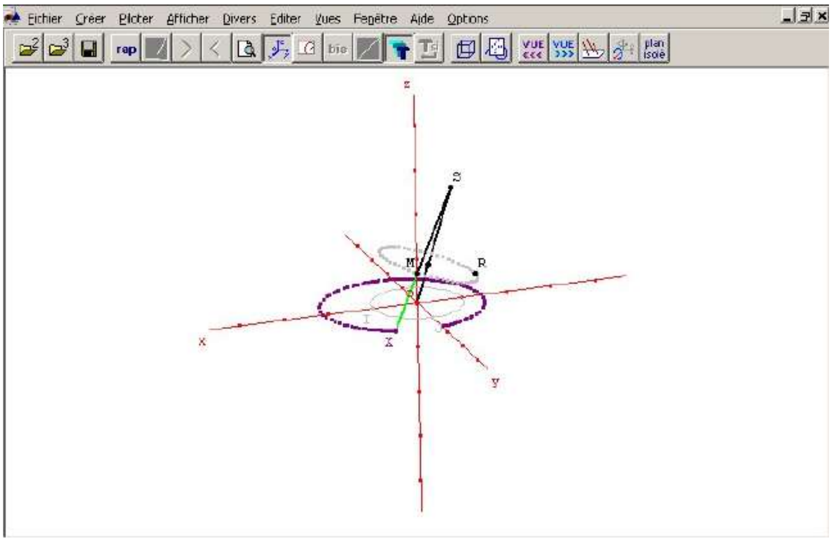

Figure 3: Perfect compass : a simulation with Geospace

$\beta = \widehat{OSM}$. Depending of the position of all the elements of the compass, the point X will trace:

- nothing, if $(OS) \perp (Oxy)$ and $(SM) \perp (OS)$
- a straight line, if $(SM) \perp (OS)$ with (OS) not perpendicular to (Oxy)
- a circle, if $(OS) \perp (Oxy)$ with (SM) not perpendicular to (OS)
- an ellipsis, if $\alpha + \beta < 180^o$
- a parabola, if $\alpha + \beta = 180^o$

[5]Some pictures of perfect compasses are available for instance in the virtual exposition *Theatrum machinarum* on Modena Museum website:
`http://www.museo.unimo.it/theatrum/macchine/017ogg.htm`
or on the pages about the exposition *Beyond the compasses* on the Garden of Archimedes Museum website:
`http://php.math.unifi.it/archimede/archimede_NEW_inglese/curve/guida/paginaindice.php?id=2`
A video of a pseudo-perfect compass *in action* is available on Professor Khosrow Sadeghi's personnal website:
`http://khosrowsadeghi.com/conic_compass.php#demo`

- an hyperbola, if $\alpha + \beta > 180^o$

Directly or not, this first description of the perfect compass is then reused by many other scholars. For instance, al-Bīrūnī (973 - 1048) in his book *Account of the perfect compass, and description of its movements*[6] explains:

> Abū Sahl has said: If, upon a point of a plane, we erect a straight line that moves on one of the planes perpendicular to this first plane, and if through another point on this straight line there passes another straight line, having three movements, of which one is around the straight line erected on this plane, the second is on the plane on which this straight line is situated, and the third is rectilinear in both directions; then if the instrument so described exists, we will call it a perfect compass.

And, in a same way al-Abharī (d. 1264) says in his *Treatise on the compass of conic sections*[7]:

> If, on a given straight line in a given plane, we erect a straight line, and if through the other end of the straight line we have erected there passes another straight line that comes to meet the given straight line in the plane, then we call these straight lines, in this configuration, the compass of the conic sections.

This last quotation gives the opportunity to read one of the other names of the perfect compass. Called by al-Abharī the *compass of the conic sections*, this instrument is sometimes simply named a *conic compass* or *cone compass*. All these treatises contain many mathematical propositions about the way to calculate the good angles corresponding to a given conic section. I do not detail this part but I strongly encourage the reader to have a look at these beautiful texts (see References).

3 Mathematical instruments in the research process

Ahmad ibn Muhammad ibn 'Adb al-Jalīl al-Sijzī (about 945 - about 1020) was born and lived in Iran. Son of mathematician, he worked between 969 and 998 and he wrote exclusively books on geometry. In all, he has written approximately fifty treatises and lots of letters to his contemporaries. Following his predecessors (Banū Mūsā brothers, Ibrāhīm ibn Sinān...) from whom he quoted in a precedent book on the description of the conic sections, al-Sijzī engages himself too in a treatise specifically on the *Construction of the perfect compass which is the compass of the cone*[8]. Like the other scholars, al-Sijzī wants to "construct a compass by means of which he shall draw the three sections mentionned by Apollonius in his book of *Conics*." He first notes that all the conics can be obtained from the right cone (depending on the position of the cutting plane), and afterwards he proposes three possible structures for the perfect compass. The beginning of the study gives technical recommandations. Here is a small quotation:

[6] R.Rashed, *Geometry and dioptrics in classical Islam*, al-Furqān, 2005, pp.816

[7] R.Rashed, *Geometry and dioptrics in classical Islam*, al-Furqān, 2005, pp.828

[8] R.Rashed, *Geometry and dioptrics in classical Islam*, al-Furqān, 2005, pp.798-806; the texts are also available in a French translation in *Œuvre mathématique d'al-Sijzi. Volume 1: Géométrie des coniques et théorie des nombres au Xe siècle*, Trad. R.Rashed, *Les Cahiers du MIDEO*, 3, Peeters, 2004.

> We must now show how to fashion a compass by means of which we may draw these sections. We fashion a shaft; such as AB. We place a tube at its vertex, such as AN, and to its extremity we attach another tube, such as AS. We can accomplish this with the help of a peg, or with anything else, so that the tube AN turns around shaft AB [...]

The instructions should enable the reader to really build such a compass. But for al-Sijzī, the aim of his work on the perfect compass is not only to draw conics. In *Sur la description des sections coniques*[9], the text[10] shows that this compass is also a theoretical tool and a tool for the discovery of new concepts.

> Mais puisque les propriétés de l'hyperbole et de la parabole sont proches des propriétés du cercle et que les propriétés de toutes les autres figures composées de manière régulière à partir de droites et qui ne subissent ni révolution ni rotation sont éloignées des propriétés du cercle, il est donc nécessaire que ces deux figures aient un rapport au cercle et une similitude à celui-ci, comme il en était pour l'ellipse. J'ai toujours réfléchi à l'existence de ce rapport entre elles et le cercle et à leur similitude et cherché à saisir ce rapport ; or la connaissance de ceci ne m'a été possible qu'une fois appris comment faire tourner le compas conique suivant les positions des plans. En effet, cette existence s'ordonne à partir de la rotation du compas conique sur la surface latérale ; la rotation régulière convient au cercle et cette rotation est commune au tracé du cercle sur une surface plane et au tracé de toutes les autres sections coniques ; étant donné que le cercle provient du tracé avec ce même compas si la position du plan est perpendiculaire à son axe, alors que pour les autres sections, leurs formes diffèrent suivant la position du plan par rapport à l'axe du compas. Quant à l'ellipse, sa conception est facile de plusieurs manières, soit à partir d'une section du cylindre soit à partir de la projection des rayons traversant une ouverture circulaire sur un plan de position oblique qui tient lieu aussi d'une section du cylindre ou d'une section du cône. Ce que nous voulions montrer.

Al-Sijzī explains that the link between the circle and the ellipsis is quite obvious. Indeed, the construction of the ellipsis by orthogonal affinity and the formula for the area are both well known. But what are the links between the circle and the parabola or the hyperbola? Now oriented towards the exploration and the solving of new problems, the practical tool becomes an instrument of discovery and as stated by al-Sijzī himself,"I always thought that there was a relationship between these two figures and the circle and their similarities and tried to get it but the knowledge of this has only become possible to me once I had learned how to turn the perfect compass following the positions of the plans". Confronted with the theoretical problem of the continuity of curves, the scientist suggests the use of a new instrument. The experimentation with this instrument creates new theoretical results that create new questions and so on and so forth. In his text, through the comings and goings between theory and practice al-Sijzī clarifies the role of mathematical instruments. They are objects as much as models and this dual status facilitates the theory-experiment passage.

[9]R.Rashed, *Les Cahiers du Mideo*, 3, Louvain-Paris, Éditions Peeters, 2004.
[10]Œuvre mathématique d'al-Sijzî, p.254

4 Conclusion

Halfway between philosophy and science, the acceptance of movement as a valid principle in geometry is one of the important topics[11] during the 10^{th} and 11^{th} centuries. Not only interesting from a mathematical point of view, the perfect compass is also useful in technological areas such as the construction of astrolabes and sundials where conics are essential. At the end of the Middle-Age, this instrument disappears and comes back at the Renaissance as a drawing tool (see figure 4[12] and Raynaud (2007)). The mathematics have changed and such an artefact between theory and practice is

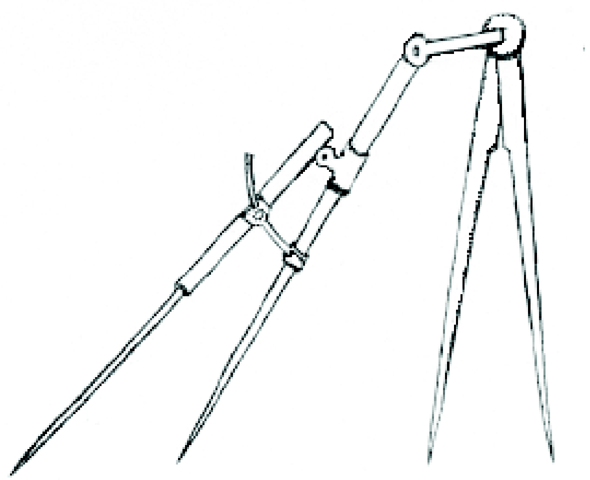

Figure 4: Renaissance : the perfect compass as a drawing tool

now useless. Mathematicians rarely expressed themselves on their relationships to the experiments. However when they did so, they gave us the opportunity to see the complexity of the links between theory and the use of technical instruments. The history of science assures us that: mathematical theories never emerge from nothingness. The scientist describes, builds and explores multiple examples before proposing an analysis or a system. The Arabic developements around the perfect compass is a model of such a process. In education in France and in many countries, the recent official instructions claim the importance of investigation in the learning process. The perfect compass can give a good entry point for an activity that helps the students to understand the way a new mathematical theory is elaborated. In secondary school, the conics are often studied only from a cartesian point of view. For students, and for teachers too[13], the

[11]The movement as a theoretical geometry principle appears a first time in Ibn al-Haytham's tries of a new definition of the geometrical space (see Rashed 2002 or de Vittori 2009)

[12]Sketch. Original pictures can be found in Raynaud (2007), for instance : (a) Venise, B. Naz. Marciana, 5363 (olim Ital. cl. IV 41), fol. 18r, P. Sergescu, "Leonardo da Vinci et les mathématiques", Leonardo da Vinci et lexpérience scientifique (Paris, 1952): 73-88, C. Pedretti, Studi vinciani (Genve, 1957), Idem, Leonardo da Vinci architecte, op. cit., p. 336, Idem, "Leonardo discepolo della sperientia" ,F. Camerota, d., Nel Segno di Masaccio (Firenze, 2001), p.184-185. (b) Vienne, Albertina, Inv. 22448 (olim 164). O. Kurz, "Dürer, Leonardo and the invention of the ellipsograph" , Raccolta Vinciana, 18 (1960): 15-25, sur les problèmes dattribution, cf. infra IV, note 51. (c) Sienne, Biblioteca deglIntronati, ms. L. IV. 10, fol. 92r-98v, G. Arrighi, "Il compasso ovale invention di Michiel Agnelo" , Le Machine, 1 (1968): 103-106; P. L. Rose, "Renaissance Italian methods..." .

[13]Al-Sijzī's texts and the Geospace model have been successfully used during a teacher training workshop (Colloque IREM, Brest 2008).

work on the perfect compass restores the links between solids and curves, practice an theory, real world and mathematics models... Mathematics teaching is always renewing itself and the historical sources give many elements that enrich it and give it sense.

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Some websites:
http://www.museo.unimo.it/theatrum/macchine/con1_04.htm
http://php.math.unifi.it/archimede/archimede_NEW_inglese/curve/guida/paginaindice.php?id=2
http://khosrowsadeghi.com/conic_compass.php#demo

5 Appendix

Geospace[14] figure *(text file[15])*

Figure Géospace
Numéro de version: 1
Uxyz par rapport à la petite dimension de la fenêtre: 0.1
Rotations de Rxyz: verticale: -72 horizontale: 19 frontale: 1
Repère Rxyz affiché
Dessin de o: marque épaisse
I point de coordonnées (1,0,0) dans le repère Rxyz
    Dessin de I: gris
J point de coordonnées (0,1,0) dans le repère Rxyz

[14]This geometry software is available on http://www.aid-creem.org/telechargement.html
[15]Figure available on http://devittori.perso.math.cnrs.fr/sijzi/compas_parfait.g3w

Dessin de J: gris
C cercle de centre o passant par I dans le plan oxy
Dessin de C: gris
S point libre dans le plan ozx
Objet libre S, paramètres: 2.5, 0
Dessin de S: marque épaisse
Segment [So]
Dessin de [So]: trait épais
K point libre sur le segment [So]
Objet libre K, paramètre: 0.5
Dessin de K: marque épaisse, nom non dessiné
P1 plan passant par K et perpendiculaire à la droite (So)
D1 droite d'intersection des plans ozx et P1
Dessin de D1: non dessiné
R point libre sur la droite D1
Objet libre R, paramètre: 0.5
Dessin de R: marque épaisse
C1 cercle d'axe (So) passant par R
Dessin de C1: gris, trait épais, points non liés
M point libre sur le cercle C1
Objet libre M, paramètre: -2.4
Dessin de M: marque épaisse
X point d'intersection de la droite (SM) et du plan oxy
Dessin de X: rose foncé, marque épaisse
Segment [SM]
Dessin de [SM]: trait épais
Segment [MX]
Dessin de [MX]: vert, trait épais
Sélection pour trace: X
Fin de la figure

THE MEDIAEVAL GEOMETRIES

A way to use the history of mathematics in the classroom of mathematics

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ABSTRACT

This workshop was in the continuity of the plenary lecture on the practical geometries in the Islamic countries. We worked in particular on Arabian texts dealing with practices of measurement. For example, we read authors like Abū l-Wafā' (m. 998), al-Karajī (m. 1023) or Ibn Ṭāhir al-Baghdādī (11th century). We tried to present all these excerpts with a pedagogical perspective of the teaching of mathematics. We illustrated notably the use of algorithms in the resolution of problems.
We also attempted to illustrate the transmission of the Arabian mathematical practices toward the Latin world with the presentation of mediaeval texts on practical geometry written in Latin. We studied above all some problems of the Fibonacci's *Practica geometriae* (1220).
The workshop was animated in French but the slides of our presentation and all the translations of (Latin or Arabian) original texts were in English.

INSCRIBED SQUARE IN A RIGHT TRIANGLE

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ABSTRACT

Given a right triangle ABC with AC as the hypotenuse, it is required to inscribe a square in it, that is, to construct a square BHIJ with H on AB, I on AC and J on BC. The question appears as Problem 15 in Chapter 9 of the ancient Chinese mathematical text Jiuzhang Suanshu (Nine Chapters of the Mathematical Art), which is believed to be compiled between 100 B.C. and 100 A.D. Although the result does not appear in Euclid's Elements (c.300 B.C.), it appears as a particular case of Added Proposition 15 in Book VI of Euclidis Elementorum Libri XV compiled by Christopher Clavius in 1574, the first six chapters of which were translated into Chinese by Matteo Ricci and Xu Guang-qi as Jihe Yuanben (Source of Quantity) in 1607. Two years later Xu Guang-qi wrote Gougu Yi (Principle of the Right Triangle) in which he attempted to integrate knowledge about a right triangle contained in Jiuzhang Suanshu and in Elements. In particular, Problem 4 is about an inscribed square in a right triangle. In hindsight a generalization of the result appears as Problem 161 in Chapter 5 of Lilavati composed by the Indian mathematician Bhaskaracharya in the 12th century. Through working together on relevant passages taken from original sources we try to compare the different styles and emphases of doing mathematics in the ancient traditions of the east and of the west, to discuss the benefits obtained and difficulties encountered in trying to integrate both, and to see how we can relate the study to the modern context, especially to the classroom context.

[A worksheet with reading material from the original sources mentioned within the abstract has been prepared (translated into English) by the author and distributed in the workshop. The workshop was primarily for school teachers, but can also be adapted to be used in a class at the secondary level with pupils at the age of 13 to 15.]

THE YOKE

(Ethno)materials for math classes*

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ABSTRACT

We present some contributions to the identification of the existing processes in a mathematical artefact of the agricultural region of Trás-os-Montes and Alto Douro (in the northeast of Portugal) – the yoke.
The main aims of this study were to collect material on traditional jobs endangered in Portugal and analyse it, looking for mathematical contents to adapt its use in the classes of mathematics.
We report the implementation of tasks inspired by the yoke in mathematics' lessons of three classes of the 9th grade (students with 14-15 years old) in a school of Vila Real in the school year of 2007/2008.
This is a work in the field of etnomathematics, focusing on the study of the mathematical knowledge and the know-how of a traditional job.

1 The yoke

A yoke is a wooden beam which is used between a pair of oxen to allow them to pull a load. There are several types used in different cultures and for different types of oxen. In Portugal there are two types of yokes (see figure 1): Yoke board (*Jugo de tábua*) and Yoke beam (*Jugo de trave*).

Figure 1: Yoke board[1] (on the left) and Yoke beam (on the right)

The yokes beam are all over the country, while the yokes board are characteristic of the western region (districts of Aveiro, Porto, Braga and Viana do Castelo). The yokes have peculiarities depending on the region.

> "*Besides the ornamented yokes, there are also simple and modest yokes, who generally make the transition between those zones and higher elevations, except as*

[1] Costa's drawing. All the other photos included in the paper were taken by the authors.

part of the Alto Douro and Trás-os-Montes, where we find the "molhelhas (…) leather and padded to give the horse a look full of surprises." (Mattos, 1942, p. 68)[2]

In Trás-os-Montes e Alto Douro there are three traditional types of yokes, namely the yoke of Vilar de Ossos – Vinhais, the yoke of Agarez – Vila Real and the yoke of S. Salvador de Viveiro – Boticas, illustrated in figure 2.

Figure 2: The three traditional yokes of Trás-os-Montes e Alto Douro

In this region of Portugal the yokes are used with "molhelhas" (see figure 3), a kind of pillows put between the oxen and the yoke.

Figure 3: A yoke with "molhelhas"

The motorization of agriculture made the use of animal's work a method obsolete. This is a strong reason for the disappearance of many traditional occupations related to agriculture. Nevertheless in some small farms of the region the use of tractors and other equipments cannot be used because of the characteristics of the land and not worth the investment (Hopfen, 1981).

This study about yoke makers continue other studies made by some of the authors about other traditional jobs of this region, such as wine coopers (Costa et al, 2008a), tin men (Costa et al, 2008b).

2 The yoke makers

The yoke makers were farmers or carpenters. Nowadays they only make repairs in yokes. This traditional job is endangered. We interviewed three yoke makers of the region of Trás-os-Montes e Alto Douro, namely from Justes, Samardã and Frieira.

[2] Authors' tradution.

2.1. Methodology and collecting of data

In this study we used a qualitative methodology (Bogdan and Biklen, 1994) consisting of data collection done by the authors at the places where yoke makers live or work. This data collection was based on unstructured interviews recorded in audiovisual media, field notes and direct observation. The craftsmen interviewed have been possible within the local contacts provided by acquaintances. The data we present are transcripts of interviews, field notes, photographs as possible while respecting the manner in which they were recorded. For the preparation of the interviews we made a documentary analysis of the literature on this subject available in ethnographic studies (Mattos, 1942), (Oliveira, 1985), (Vasconcelos, 1938).

2.2. The interviews

In this section we present some parts of the interviews made to the three yoke makers.

Marcelino Pereira (see figure 4), 70 years old, was born at Justes. He had several jobs, but he always worked in agriculture. He is yoke maker for more than half a century. He summarized his work as yoke maker by saying: “I was born in it and I have a passion for it!”

Figure 4: Mr. Marcelino Pereira, a yoke maker

José Domingos Costa (see figure 5), 52 years old, was born at Samardã. He worked as yoke maker until get married. Since then he is carpenter and he still makes repairs in yokes. He had times that he went to mountain for two weeks to make yokes. As for his initiation as yoke maker, he said that he learned because of necessity: *“I was working on the field and the yoke broke ... I thought that I had to pay to others to repair it. So, I did it by myself. It was not well as it should be, but then I learned."*

Figure 5: Mr. José Domingos Costa, a yoke maker

Daniel Augusto Morais (see figure 6), 75 years old, was born at Frieira. Mr. Daniel is retired from a life dedicated to carpentry. He still makes handicrafts.

"I formerly did everything… Nowadays, I don't work to anybody. I am doing a piece of the plow, but this is just for decoration."

He learned the art of carpentry with his father. In his words: *"My father was a carpenter and my brother too, but then my brother went to Brazil."*

While still working, Mr. Daniel had to travel frequently to other villages to build farm tools. He used a motorcycle to carryout his tools. *"I bike in the rain and snow everywhere… I took this box full of tools and with this middle mold of a yoke."*

The three yoke maker referred that the know-how based on comparison with a model (another yoke) was the "technique" used by them.

Figure 6: Mr. Daniel Augusto Morais, a yokemaker

3 The teaching experience

In this section, we report the implementation of tasks inspired by the yoke in mathematics' lessons of three classes of the 9th grade (students with 14-15 years old) in a school of Vila Real in the school year 2007/2008. This teaching experience was integrated in the Project Ciência Viva VI – number 771 (Costa et al, 2008).

3.1. Aims, methodology and collecting of data

The aims of the teaching experience were the:

- Identification of mathematical procedures used in traditional jobs of the region;
- Explanation, interpretation and register of the same framed in the mathematical abilities in the different school years;
- Creation, by the students, of educational resources (didactical material) that illustrate this knowledge and enable its divulgation and reproduction;
- Application of experimental methodologies.

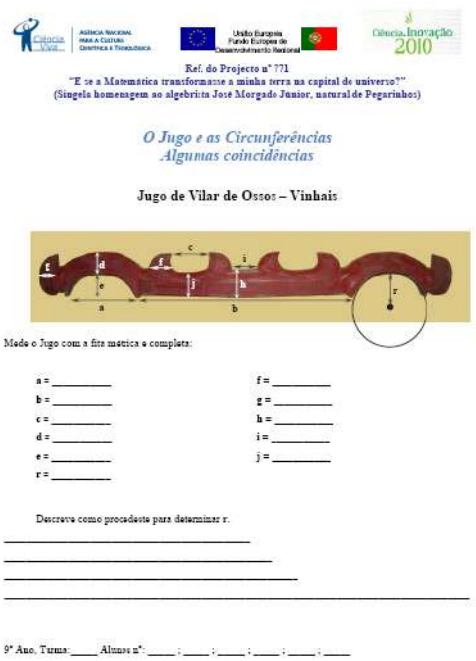
In what concerns the methodology and the collecting of data we used a qualitative methodology (Bogdan and Biklen, 1994). The data was collecting by direct observation, the teacher diary and by the students' productions.

3.2. The teaching experience phases

The teaching experience had two different phases: one of teaching planning and another of classroom implementation. The teacher also had to study similar educational experiences to plan his own, analyzing for instance (Gerdes, 2007), (Gerdes, 2008), (Moreira, 2004).

***Phase 1** – Lessons planning*

First the teacher analyzed the yoke makers' interviews and the mathematical potentialities of the yoke. After this, the teacher prepares tasks involving experimental methods and the yoke. The teacher constructs three worksheets (see figures 7, 8 and 9) for each yoke/class. Finally the teacher collects the three typical yokes of the region of the school (Trás-os-Montes e Alto Douro) and takes the yokes to the classes.



Ref. do Projecto nº 771
"E se a Matemática transformasse a minha terra na capital do universo?"
(Singela homenagem ao algebrista José Morgado Júnior, natural de Pegarinhos)

O Jugo e as Circunferências
Algumas coincidências

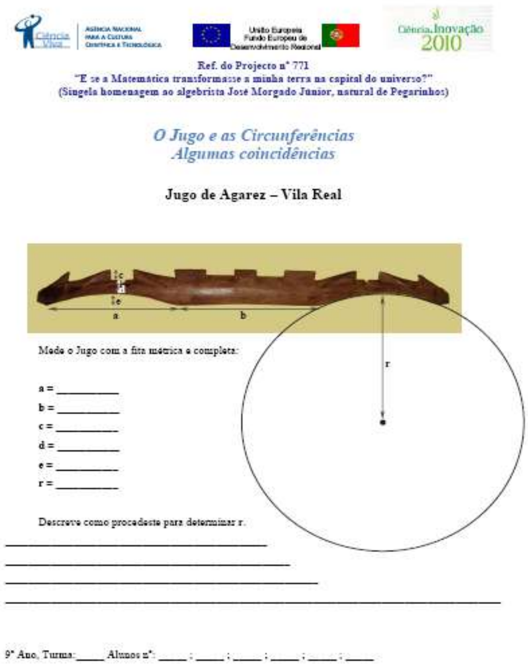
Jugo de Vilar de Ossos – Vinhais

Mede o Jugo com a fita métrica e completa:

a = ________ f = ________
b = ________ g = ________
c = ________ h = ________
d = ________ i = ________
e = ________ j = ________
r = ________

Descreve como procedeste para determinar r.

9º Ano, Turma:____ Alunos nº: ____ ; ____ ; ____ ; ____ ; ____ ; ____



Ref. do Projecto nº 771
"E se a Matemática transformasse a minha terra na capital do universo?"
(Singela homenagem ao algebrista José Morgado Júnior, natural de Pegarinhos)

O Jugo e as Circunferências
Algumas coincidências

Jugo de Agarez – Vila Real

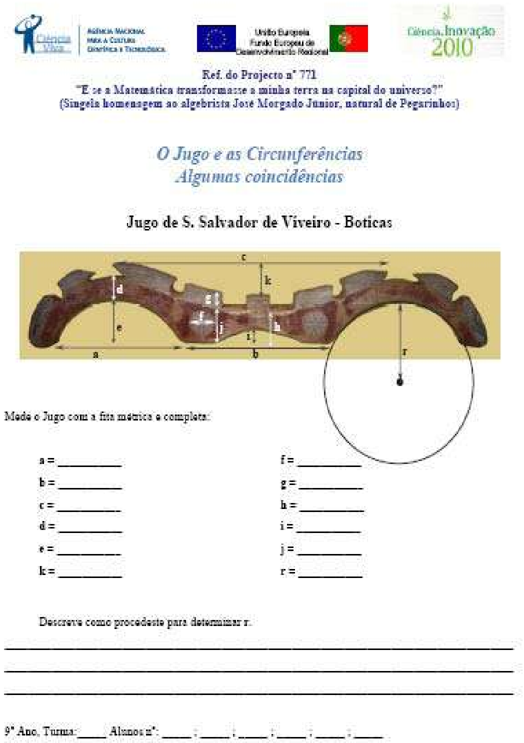
Mede o Jugo com a fita métrica e completa:

a = ________
b = ________
c = ________
d = ________
e = ________
r = ________

Descreve como procedeste para determinar r.

9º Ano, Turma:____ Alunos nº: ____ ; ____ ; ____ ; ____ ; ____ ; ____

Figure 7: Worksheet of Vilar de Ossos' yoke   Figure 8: Worksheet of Agarez' yoke


Ref. do Projecto nº 771
"E se a Matemática transformasse a minha terra na capital do universo?"
(Singela homenagem ao algebrista José Morgado Júnior, natural de Pegarinhos)

O Jugo e as Circunferências
Algumas coincidências

Jugo de S. Salvador de Viveiro - Boticas

Mede o Jugo com a fita métrica e completa:

a = ________ f = ________
b = ________ g = ________
c = ________ h = ________
d = ________ i = ________
e = ________ j = ________
k = ________ r = ________

Descreve como procedeste para determinar r.
____
____
____

9º Ano, Turma:____ Alunos nº: ____ ; ____ ; ____ ; ____ ; ____ ; ____

Figure 9: Worksheet of S. Salvador de Viveiro' yoke

Phase 2 – *Lessons implementation*

In the classroom, in groups, students and teacher explore the mathematics involved in the yoke (see figure 10) and construct final products that will be presented in a public meeting in the University of Trás-os-Montes e Alto Douro.

Figure 10: Students' study of the yoke

The research on teaching practices in each of the three yokes originated surprising results that students have summarized in their notes and final products, which images are reproduced in figures (see figures 11, 12, 13 and 14). These images illustrate the relations found between the three yokes and circumferences.OuvirLer foneticamente

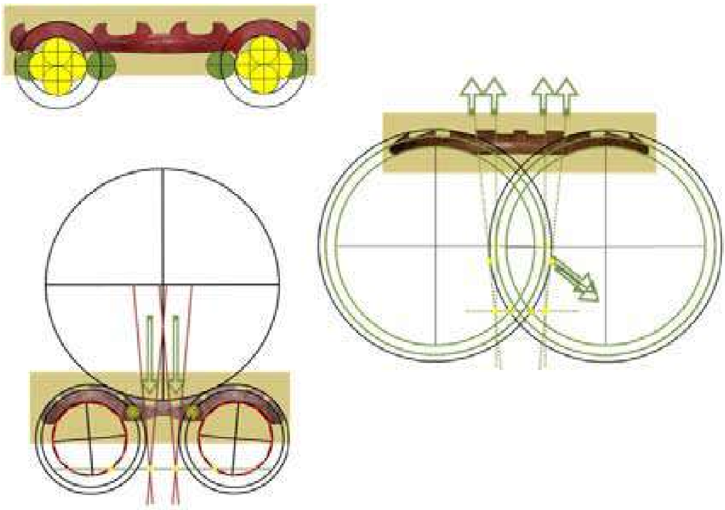


Figure 11: Students' final productions

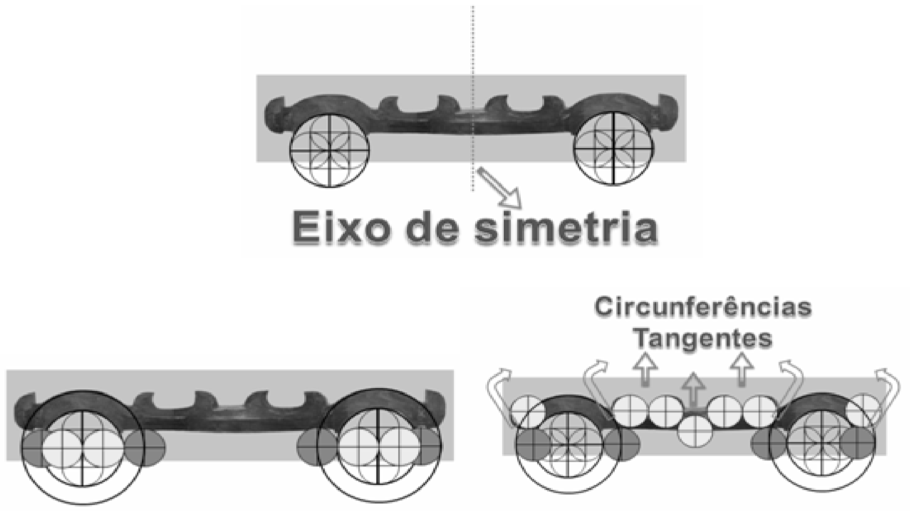


Figure 12: Students' final productions

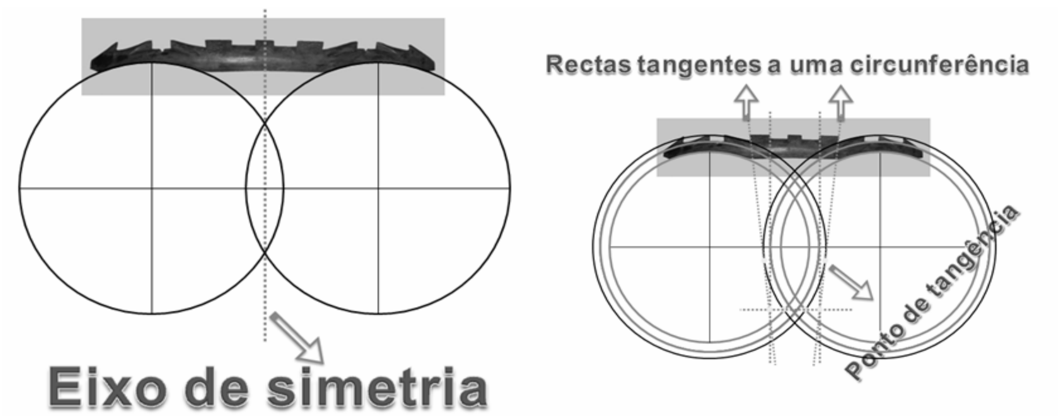


Figure 13: Students' final productions

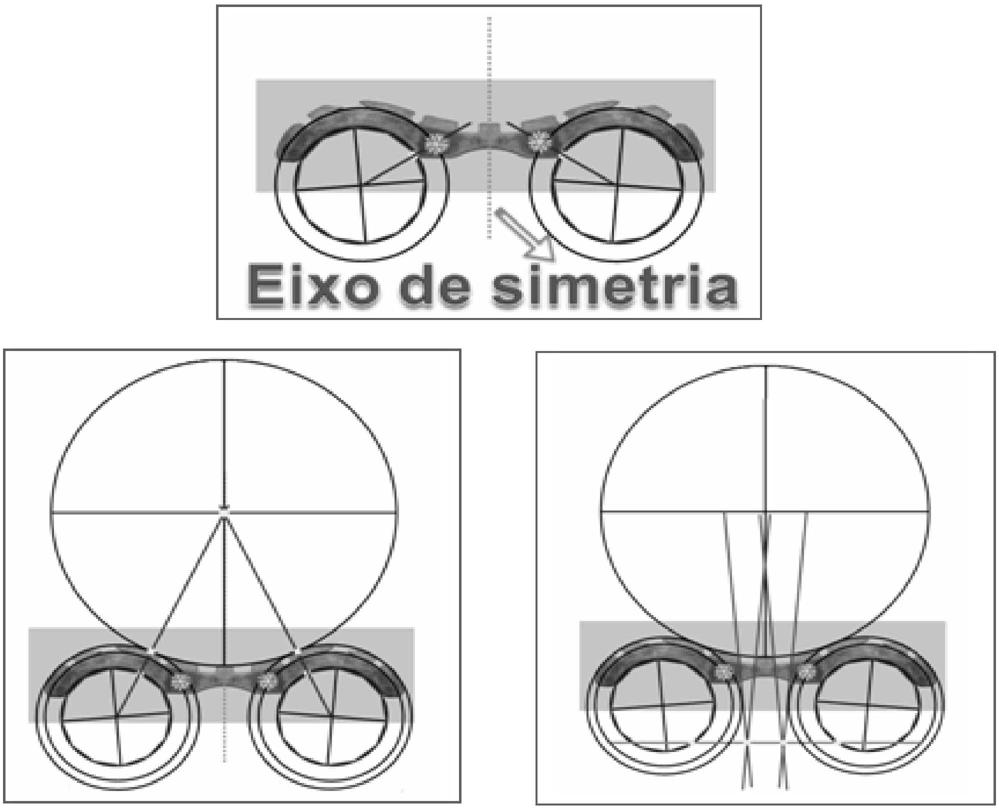


Figure 14: Students’ final productions

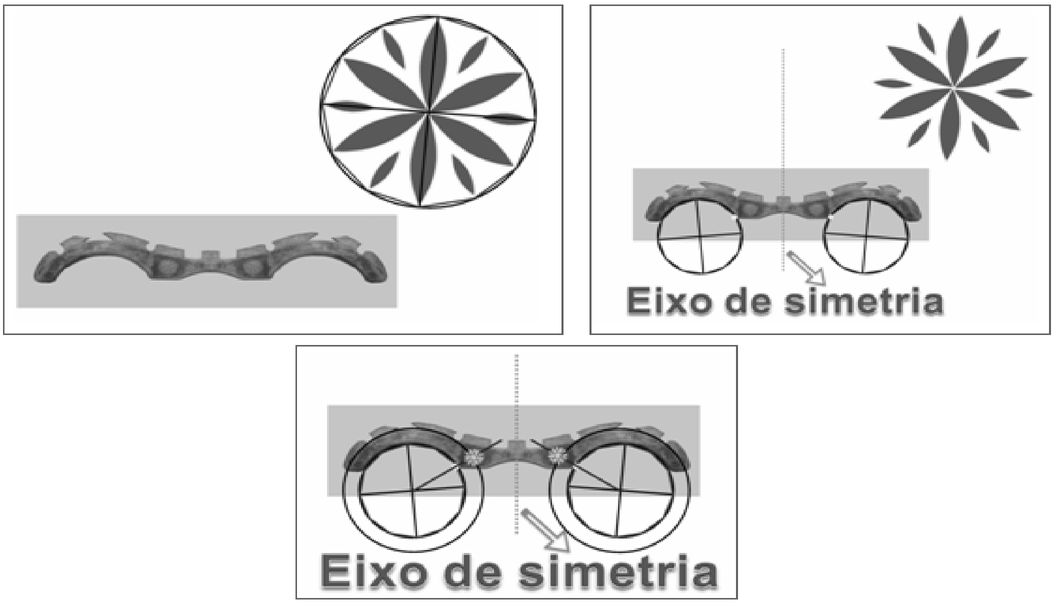


Figure 15: Students’ final productions

This teaching experience allowed the students to study many contents of geometry, namely: angles, regular polygons, trigonometric ratios, circular arcs, circumferences, tangents’ lines of circumferences, symmetry, displacements and rotations.

4 Final remarks

The yoke makers' interviews were gratifying for us, because of the way they welcomed us and made available to talk about their art.

The yoke makers proved to be thankful for the recognition, appreciation and divulgation that we would give to their work.

The study, of the processes used by yoke makers, allows us to identify the existence of mathematical processes which use measurements, symmetry and proportions. This fact is not new, because other studies conclude the same about other artisans (Palhares, 2008).

The use of experimental activities in the classroom motivated the students. They participated more actively than was usual.

The research done proved to be mathematically rich, creative, engaging and stimulating for students.

We consider that it is important that teachers should promote activities that could contribute to motive students and the involving community, and that reinforce the cultural and scientific identity of the students.

With this teaching experience we verified that it was very important for students doing experimental activities. They improved their abilities of measurement and be aware of the importance of the rigor and accuracy of measurements.

The relations between the yoke and circles were not found in the processes used by yoke makers.

The collection of aspects related with traditional jobs contributes for a future memory of the region history.

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THE CONCEPT OF BEAUTY DEFINED BY THE MAKONDE PEOPLE

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ABSTRACT

One of the most impressive occupation among Makonde men is the art of carving. At least since 1902, there exists evidence that Makonde sculptors have been producing all kinds of wood carving-pieces such as chairs, wood masks and walking-sticks. At first, these had been used for trading with the neighboring ethnic groups; in exchange the Makonde people could obtain clay jars and pots (Adams, 1902). Today, Makonde sculptors are mainly dedicated to carve feminine figures, which can be found in all types of handcraft markets and shops. The arrival and settlement of Portuguese colonizers and missionaries at the Makonde Plateau in the 1930s provoked an upsurge in the sales of such sculptures since they had started to order all kinds of human figures, from religious to political "eminences" (Kacimi & Sulger, 2004). This suggests an introduction of the classical European style into the traditional Makonde style. The authors discuss and present how Makonde sculptors define the idealized concept of beauty in their sculptures and how this is related to what has been described as mathematical beauty by the Western culture, i.e., the golden ratio. If the production of ebony-wood sculptures has become, in some sense, more industrialized, to what extent is it posible to consider sculptors to be artists? Some answers will be discussed.

1 Introduction

Mozambique is located in the southeast of Africa, bordering with Swaziland and South Africa from the south, Zimbabwe, Zambia and Malawi from the west, Tanzania from the north and the Indian Ocean from the east. Until the arrival of Vasco da Gama in 1498, the land was inhabited by several Bantu groups who migrated from southwestern Africa. Mozambique was settled and remained a Portuguese colony until its independence on the 25th of June 1975. Today, there are still over 20 ethnic groups living in Mozambique, but they are no longer as isolated as they were before. One of these ethnic groups is called Makonde.

1.1 About the Makonde people

The Makonde Plateau is located between southern Tanzania and the northern part of Mozambique. It is divided by the Rovuma river, also the border between these two countries, and its biggest city in the Mozambican territory is Mueda. According to oral tradition, the Mozambican Makonde people have arrived and settled at this region about three centuries ago, when they were escaping the drought and tribal wars in the south of the Niassa Lake. Although it is not possible to determine whether the migration movements had started even before these dates, it can be affirmed that these movements have lasted until after the colonization period (Fouquer, 1972). The term *makonde* not only describes a geographic region, but also a person who carries this specific culture (Kacimi & Sulger, 2004).

The Makonde people have a cultural concept of nationality and not a racial concept or a concept of blood. Moreover, and as a consequence of it, when a Makonde man marries a woman from another ethnic group, she will certainly become Makonde. Inside, the Makonde's society is matrilinear, i.e., the descendancy is uterine and, in this case, the leader of the family is usually an uncle, a brother or the eldest nephew of the mother.

The main characteristics in the Makonde physiognomy used to be easily recognizable; mostly women, sometimes men as well, were tattooed in their face and some parts of the body, had piercings in their upper lip, and had their teeth sharpened. These characteristics were used to, on the one hand, make evident their ethnic roots and, on the other hand, to represent virtues; in the case of the tattoos they were said to have a symbolic value and the lip-piercing was considered as jewelry and also to have magical virtues (Fouquer, 1972, pp. 11 ff). These practices were banned by the first independent government of Mozambique, in 1975 (Kacimi & Sulger, 2004).

1.2 About the Makonde sculptures

Adams (1902) had mentioned in his book that Makonde men carved different wood pieces such as chairs, wood masks and walking-sticks. These were exchanged for clay jars and pots from other neighboring ethnic groups. Inside their own huts, one could often find some feminine figures. According to one of the legends explaining the origin of the Makonde people, their first mother had originated from a piece wood that had been carved by the first Makonde man (Adams, 1902, p. 41).

The art of carving is an occupation that can only be practiced by Makonde men. Women have too many obligations such as household, taking care of children and pottery, leaving no time for other activities. According to Breutz (1971), sculptures have mainly a religious purpose and these are not the concern of women. As Raum had expressed: "women, who since paleolithic times form one of the most common subjects of art, only rarely create art objects." (Raum, 1966, p. 6)

After the 1930s, the Portuguese colonizers and other missionaries arrived at the Makonde plateau. They immediately showed great interest and fascination for the Makonde wood carvings and began to order different pieces, from religious until political "eminences." The Makonde sculptors, after noticing such interest, decided to carve the new pieces using *pau-preto* (ebony wood, *Diospyros ebenum*) and *pau-rosa* (*Swartzia spp.*) instead of the soft and non long-lasting wood they had used before. This first contact with the Western culture can be considered to be the first introduction of the classical european style into the traditional Makonde style.

With the increase of tourism in Africa, from the 1960s and up to the present days, the styles and types of Makonde sculptures have adapted and diversified in an enormous way. It is possible to find, e.g., nativity sets during the christmas season. The following list describes the most popular types of Makonde sculptures:

1. *Ujamaa.* This concept had been inspired in the visions of Julius Nyerere (Tanzania's first president). It consists of a human tower where not only men and women are climbing one another, but also children. Each person has a determined task or role to fulfill in her or his community that, in this case, is the one consisting of all people in this human tower (Kacimi & Sulger, 2004).

2. *Shetani.* The term *mashatani*, plural of *shetani*, refers to the various spirits of the Makonde cosmogony. These sculptures consist of grotesque human representations and human-animal forms. They have also been considered to be the Modern Makonde Art that appeared after the 1960s (Kacimi & Sulger, 2004; Stout, 1966).

3. *Sculpture in Relief.* This type of sculpture is also called *sculpture in high relief*, since what remains from the bole is its center. Within this style one finds busts, human and animal shapes. The first of these consists of the starting point of an apprentice who wants to become a sculptor or, in their own words, a master (Rohrer, 2010).

4. *Mapiko.* These sculptures are masks consisting of, either the facial side, or they may cover completely the head and some part of the bust. Most probably, the first motivations Makonde men had to carve wood was the making of masks for ritual purpose or for the personification of ancestors for adoration and contemplation (Kammerer-Grothaus, 1991).

After the beginning of the commercialization of Makonde scultpures, some debates regarding the type of occupation of Makonde carvers became an issue. Many anthropologists have disscussed whether it is possible to acknowledge these producers to be artists, or simply artisans. For example, Grohs (1971) had discussed this dilemma and, according to her, there exist two tendencies: the first has considered ebony sculptures as "commercial art," and the second has accounted Makonde art as a highly valued sequel of the art of carving. Both of them have been able to find arguments that validate their statements, making it even more difficult to establish Makonde sculpture as a specific art or artisanry.

In this paper, we would like to discuss and consider a new approach to this debate: "the idealized concept of beauty." Some answers to how the Western culture has defined beauty throughout the history of humankind, and how this definition has affected (or influenced) the concepts and interpretations present in the Makonde tradition will be discussed.

2 Theoretical framework

The various conceptions and approaches proposed for research in ethnomathematics embrace notions, concepts, and methodologies from a number of disciplines within the social sciences. In particular, the following research and its evaluation, as exposed in the following sections, require the implementation of relevant contributions from ethnography and ethnology. Furthermore, the type of analysis used for this research is based on case study, i.e., we proceeded with a comparative case method. Hence, we chose the golden section to be the theoretical proposition and reference point for the analysis of the collected data.

2.1 Ethnography and Ethnology

Ethnographic methods may be considered as one of the fundamentals for any research undertaken in ethnomathematics; it is the science used to describe different cultures and societies. Ethnographers need to be open minded with respect to the cultures they will study. Being open minded allows the ethnographer to uncover relevant information that is outside the aims' scope; it gives freedom to interpret the collected data in diverse forms, even though s/he accounts for an understanding from an insider's perspective (Fetterman, 1989).

This type of research is generally inductive, i.e. although the ethnographer bases her or his investigation on a certain model or theory, it ends up with general conclusions or new theories. In other words, and particularly with regard to ethnography, the subjective reality of an individual is no less valid than the objectively defined one (Fetterman, 1989); this new way of interpreting reality may lead to a reformulation of the model being used.

Ethnology is concerned with the dynamic phenomena of cultural change and assumes only the historical facts that come from archaeological findings, thus the method used for determining such changes "is based on the comparison of static phenomena combined with the study of their distribution." (Boas, 2007) It does not intend to explain the foundations of civilization and society; rather it regards each culture as having an independent cultural history that agrees with its own social developments and adaptations of external influences.

Ethnology uses the results obtained in an ethnographic research to study cultural change. Hence, the respect of the ethnologist towards any ethnic groups is the key to give, in a broad sense, a scientific meaning and acknowledgment to their theories. However, Lévi-Strauss suggested that equality and acceptance between all different cultures was not possible without endangering their differences (cit. Lévi-Strauss in Ritter, 2009).

2.2 The golden section

The *golden section* was firstly defined as *division in extreme and mean ratio* by Euclid in 300 B.C., and later as *proportion having a middle and two ends* in some Arabic books (Herz-Fischler, 1987). From the sixteenth century on, many names had been given to refer to this description, e.g., *divina proportione*, also the title of Paccioli's book, *continuous proportion* used by Kepler in 1597 and *goldene Schnitt* officially introduced by Martin Ohm in 1835. This latter, translated to *golden section*, is the most popular nowadays. According to Schubring, this term had already existed prior to 1835, but only orally, most probably among artisans and engineers (cit. Schubring quoted in Herz-Fischler, 1987).

The golden ratio, apart from describing a geometric concept, has also been used to idealize the proportions of the human body. The Vitruvian Man, by Leonardo da Vinci, was probably obtained from the oldest representation of this ideal of human proportions, made by Marcus Vitruvius Pollio in the Ancient Greece. It consisted of Zeus inscribed in a circle of radius equal to the navel's height and in a square of size length equal to the height. Since those times the relation between the height and the navel's height of a person has been considered beautiful if it satisfies the *golden section*, and this ideal has been able to survive through ages in the history of mankind, at least in the Western Culture (Rohrer, 2010). To the particular interest of this research are the studies developed by Le Corbusier, Albrecht Dürer and Robert Ricketts:

Le Corbusier (Swiss, 1887-1965) had studied the proportion given by the golden section with the goal of achieving an architecture that was simple and functional and yet regarded space and proportions (Atalay, 2004). After World War II he had been offered to design a large-scale residential complex, the *Unité d'habitation*. All the dimensions in these buildings were determined by the *modulor*: a system based on the idealized human proportion, given by the golden ratio and further developed by Le Corbusier himself (Le Corbusier, 1954).

Albrecht Dürer (German, 1471-1528) was a painter and a printmaker who had tried to solve the problem of distinguishing between true and beautiful human proportions with graphics. In 1528, the book "*Hierin sind begriffen vier Bücher von menschlicher Proportion*" containing the study he had developed with at least least thirteen different body types of, each, women, men and children was published. According to Dürer, the concept of human beauty was subject to change in time, but this change should oscillate about a midpoint, the golden section (Dürer, 1528).

Robert Ricketts (American, 1920-2003) was a specialist in aesthetic and orthodontic surgery, who had spent many studies on what he later patented as *golden divider*. In one of them, Ricketts had conducted several measurements about the face on a number of female photographic models. The conclusions he had obtained suggested the possibility of determining some facial features through the golden ratio, e.g., the ratio between the distance from the nose to the forehead and the distance from the chin to the nose (Atalay, 2004).

3 The proportions of Makonde sculptures

For the study of the proportions of Makonde sculptures, a field research between September and October of 2008, had been pursued at the National Museum of Ethnology. The National Museum of Ethnology is located in Nampula, in the north of Mozambique and at about 180 km west from Ilha de Moçambique (UNESCO world heritage since 1991). The museum's administration and the University of Lúrio had decided and openned a working place for many Makonde sculptors in the backyard of the same museum. At least two huts had been built, and the same sculptors can exhibit and sell their own ebony wood carvings.

3.1 Carving Makonde sculptures

The field research was divided into two periods: during the first period, three sculptors had been interviewed and observed while they were carving different ebony wood pieces. Martins Bernardo, Pakholo Laza and Júlio Carlos carved *mulher* (sculpture in relief), *likhomba* (*mapiko*) and *ujamaa*, respectively. The materials used for carving these three sculptures were a handsaw, an ax (called *anchô* in Portuguese), several wood chisels, including some curved, a knife, a file and a batter (called *batedor* in Portuguese).

In all cases, there was no use of a measuring tape: the boles and all different measures and proportions the sculptors had required during the carving of their pieces were measured using their own hands: palms, fingers and thumbs, and phalanges.

The method used for carving the different pieces consisted mostly of beginning in the top of the bole and finishing it in the bottom, i.e., its base. Martins Bernardo started with the head of the woman, Pakholo with the eyes of the mask and Júlio Carlos with the children standing at the most top of the human tower. Carving symmetric body parts, such as ears, eyes, breasts and hair, consisted of using any of the two following methods: from the outer sides to the middle, or carving one side and then reproducing it on the other side.

In the last and shortest period, a complete sample of ca. 150 pictures of different ebony wood sculptures, exhibited in the shop, from several Makonde sculptors were taken with a camera. Many of these images were shot at least three times, in order to guarantee at least one good quality image for further analysis. The camera was always placed as drawing an imaginary parallel line with respect to the sculpture, and the distance between these two was variable.

3.2 Measuring Makonde sculptures

A reduced sample of 16 images of whole body sculptures and 33 pictures consisting only of the facial features was selected and had their background removed using Photoshop®.

The whole body scultures were fitted to the *modulor* drawn with AutoCAD®. This adjustment followed the steps given below:

i. Position the midpoint of the *modulor* in the navel. In most cases, the navel does not appear in the sculpture. The height of the elbow is used to replace the height of the navel since they are equal.

ii. Having the midpoint fixed, it becomes the first reference point for scaling the *modulor* according to the size of the sculpture; the second reference point are the feet.

iii. The *modulor* has been scaled to the size of the sculpture. The complete results of this procedure can be found in Rohrer (2010, cf. Figure 5.19).

The 33 pictures consisting of facial features were measured according to the description of the following steps:

i. Draw the lines marking the beginning of the forehead, the eyes, the nose, the mouth and the chin.

ii. Draw a perpendicular line passing through the middlepoint of the inner canthal distance. The canthus is the corner of each side of the eye, formed by the junction of the upper and lower lids.

iii. Measure the distances from the chin to the forehead (TE), from the chin to the nose (NA) and from the chin to the mouth (BO).

iv. Finally compute the distance $TE - NA$ of each image. All these values created a table that can be found in Rohrer (2010, cf. Table 5.1).

The following section presents the results obtained after comparing and analyzing the different values collected during the field research and those presented in the previous section 2.

3.3 Comparing proportions – t-test

Only three figures, out of 16, matched best to Le Corbusier's *modulor*, although the raised arm did not reach the height of 226 cm in neither of them. From our point of view, they can nonetheless be considered to be very accurate, since we know that

Makonde sculptors have only used their hands and fingers to measure any part of the carving piece. Interestingly, one of these sculptures has a height of about 160 cm in real size; this could suggest that, because of the proximity with the actual height of a person, it would become easier to follow the proportions given by the golden ratio, since it needs no scaling, and thus to carve an "ideally beautiful" human sculpture. But this conclusion is not possible to achieve since the other two figures have the same average size as the other sculptures (they are not higher than 50 cm).

In some cases it was possible to obtain two different results for the same whole body sculpture. This happened because the navel had been carved, but it did not coincide with the height of the elbow. Interestingly, the fitting was usually more accurate if the midpoint had been fixed at the height of the elbow and not that of the navel.

A total of 8 figures showed a similar result: the navel represented the exact midpoint that divides the body into two equal halves. This does not coincide with the midpoint defined in the *modulor* since, in this last case, it halves the total height from the feet until the end of the raised arm. Unfortunately, the sample did not show sufficient results that could present a formal generalization of these analyses.

In the case of the computations made for the facial features and according to the studies pursued by Robert Ricketts, as described in the previous section 2, the following equalities satisfy the idealized concept of beauty as defined by the golden ratio:

$$\frac{TE}{TE-NA}=\frac{TE-NA}{NA}=\frac{NA}{BO}=\varphi \tag{1}$$

The following table shows the results obtained after applying the t-test to the values obtained from the measurements taken in the second part of the study above, i.e., those obtained from the facial features, and by considering the null hypothesis to be the golden ratio φ:

	$\frac{TE}{TE-NA}$	$\frac{TE-NA}{NA}$	$\frac{NA}{BO}$
Mean	1.63824644	1.6157760	1.9140188
Standard Deviation	0.1124368	0.2986213	0.2146197
One-sample t-test	1.032684	−0.04344	7.9224

Table 1: Statistical Results for the t-test computed with respect to the facial features.

One can easily see that the result obtained in the last column (NA/BO) allows a 99% of confidence to claim that sculptors did not use the golden section to compute the distances from chin to nose and from chin to mouth. For the the first two cases, the null hypothesis could not be rejected. The t-test did not give further information on the validity of these results in any of the confidence levels from 99.99% until 90%, but it was possible to compare the means and arrive to some conclusions.

The mean computed for the values of $\frac{TE-NA}{NA}$ differed from the golden ratio φ only after the third decimal place, i.e., $|\frac{TE-NA}{NA}-\varphi|=0,002\ldots$; and for the values obtained in the first column $\frac{TE}{TE-NA}$ the difference is $|\frac{TE}{TE-NA}-\varphi|=0,02\ldots$. Hence, we may conclude that these two results are quite accurate with respect to the proportions given by the golden section and Ricketts' results.

4 Final Conclusions and Remarks

In our opinion, the proportions given by the golden section are not a requirement that Makonde sculptors use to distinguish a sculpture of high quality. But we cannot totally reject them since they appear in some cases, mostly in the facial features. On the other hand, how Makonde carvers manipulate and work the ebony wood shows an enormous ability in determining sizes and proportions, suggesting the existence of other rules of proportions. Unfortunately the sample of this study did not suffice to give a more general conclusion. Furthermore, the same sculptors explain that their knowledge has been based on pure observation, and that is also connected to experience. Many carvers have learned this profession by reproducing the side that has been carved by his master in the other half of the same bole.

It is important to remark that Makonde sculptors tend to represent real persons in their carvings, that is, people are placed in a very real and concrete situation in life. Either a woman is going to the *machamba* to work and her children are accompanying her, or a community is working very solidly united. They have the desire to represent themselves in the sculptures. Following this way of thinking, we could say that the beauty in the Makonde sculptures relies on how far they are able to emulate a real life event or situation.

If we only regard the facial features, the results presented in the previous section 3 show that it is possible to find the proportions related to the golden section: in the ratio between the distances from the short nose until the forehead and from the chin until the short nose, and in the ratio between the distances from the chin until the forehead and from the short nose until the forehead.

These results suggest that up to a certain scale, that of sculpting a face, especially of a woman, Makonde men seem to attempt to encounter the same ideal of beauty that the one defined by the Western culture. This ideal represents the image of their first mother, the woman who had originated from ebony wood. Furthermore, masks constitute the first sculptures created by the Makonde people for the realization of rituals. These masks have the function of representing bad and good spirits during these ceremonies. And the good spirits correspond to the Makonde ancestors, including their first mother. The facial features have thus been idealized since the beginnings of Makonde art.

The question that remains open is whether their initial idealized concept of beauty coincides with that given by the Western culture, i.e., the golden section. The sample does not allow us to answer this since the images correspond to sculptures carved at least after the year 2000. There exists the possiblity that Makonde art has suffered from "cross-cultural transmission" or from "acculturation adjustment."[1]

To conclude, an artist is the person who creates a piece of art, and the person who reproduces this piece is called artisan. In the Western culture, Benjamin (1955) had already suggested an end of the auratic art due to technical reproducibility and commercialization. Furthermore an artist will be considered as such because there exists an "art patron" stimulating, encouraging and promoting her or his style (Raum, 1966).

Hence, instead of regarding Makonde sculptors as being artists or artisans, we suggest that they should be considered to be masters. The term master is used by the same

[1]Man Keung Siu has introduced the term "cross-cultural transmission" in his talk "1607, a year of (some) significance: Translation of the first European text in mathematics - Elements - into Chinese" presented in July 2010 at the 6th European Summer University on the History and Epistemology in Mathematics Education in Vienna.

community of sculptors to always distinguish the best, most creative and experienced carvers. Clearly, not every Makonde sculptor will considered to be a master: only those who achieve to reproduce the so called ideal of beauty within the facial features will be acknowledged as such by their own ethnic group.

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Finally, thank you to those persons who bought the sculptures used during this field research; without them, the Makonde carvers would not have easily accepeted the fact of being constantly observed while they are working.

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1607, A YEAR OF (SOME) SIGNIFICANCE: TRANSLATION OF THE FIRST EUROPEAN TEXT IN MATHEMATICS — *ELEMENTS* — INTO CHINESE

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ABSTRACT

The Italian Jesuit Matteo Ricci and the Chinese scholar-official XU Guang-qi of the Ming Dynasty collaborated to produce a translation of the first six books of *Elements* (more precisely, the fifteen-book-version *Euclidis Elementorum Libri XV* compiled by Christopher Clavius in the latter part of the sixteenth century) in Chinese in 1607, with the title *Ji He Yuan Ben* (Source of Quantity). This paper attempts to look at the historical context that made *Elements* the first European text in mathematics to be translated in China, and how the translated text was received at the time as well as what influence the translated text exerted in various domains in subsequent years, if any, up to the first part of the 20th century. This first European text in mathematics transmitted into China led the way of the first wave of transmission of European science into China, while a second wave and a third wave followed in the Qing Dynasty, but each in a rather different historical context. Besides comparing the styles and emphases of mathematical pursuit in the Eastern and the Western traditions the paper looks at the issue embedded in a wider intellectual and cultural context.

1 Introduction

The title of this paper (which is the text of a talk given at the ESU6 in July of 2010) is inspired by that of a well-received book by the historian Ray Huang [Huang, 1981]. Huang's book *1587*, *A Year of No Significance* was translated into Chinese soon after its publication and was given a more informative but perhaps less pithy title *Wanli Shiwu Nian* (In the Fifteenth Year of the Reign of Emperor Wanli). Huang begins his book with the passage:

> "Really, nothing of great significance happened in 1587, the year of the Pig. […] Let me begin my account with what happened on March 2, 1587, an ordinary working day."

His intention is to give an account of history from a "macrohistory" viewpoint, which he further exemplifies in a subsequent book titled *China: A Macrohistory* [Huang 1988/1997]. The purpose is to give an analysis of events that occurred in a long span in time, viewed from a long distance with a broad perspective. In this respect events, some of which might not reveal its true significance when it initially happened, cumulated in time to produce long-term effects. It is in a similar vein that this author tries to tell the story of the event that occurred in 1607 depicted in the title.

This paper attempts to look at the historical context that made *Elements* the first European text in mathematics to be translated in China, and how the translated text was received at the time as well as what influence the translated text exerted in various domains in subsequent years, if any, up to the first part of the 20th century. This first European text in mathematics transmitted into China led the way of the first wave of transmission of European science into China, while a second wave and a third wave followed in the Qing Dynasty, but each in a rather different historical context. Section 4 of this paper (which is a record of an accompanying three-hour workshop conducted at the ESU6) deals with a comparison of the styles and emphases of mathematical pursuit in the Eastern and the Western traditions.

The readers may query whether it would be more appropriate to give such a talk in 2007, which coincided with the 400th anniversary of the translation of *Elements* into Chinese. Indeed, several symposiums were held on this theme in 2007. In particular, on that occasion this author gave a talk that touches on the influence of *Elements* in Western culture and in China, as well as the pedagogical influence of *Elements*. The text of the 2007 talk (given at the Institute of Mathematics of Academia Sinica in Taipei) was published in a paper in Chinese in that same year [Siu, 2007]. The content and emphasis of that paper differ from those in this paper, but naturally are related to it. We do have a historical reason for giving this talk at the ESU6 held in 2010, for the year marks the 400th anniversary of the passing of Matteo Ricci, one of the two protagonists in this endeavour of enhancing understanding between Europe and China.

2 Translation of *Elements* into Chinese

The story started with the "era of exploration" when Europeans found a way to go to the East via sea route. Various groups took the path for various reasons, among whom were the missionaries. As a byproduct of the evangelical efforts of the missionaries an important page of intellectual and cultural encounter between two great civilizations unfolded in history.

From around 1570 to 1650 the most prominent group of missionaries that came to spread Christian faith in China were the Jesuits sent by the Society of Jesus, which was founded by Ignatius of Loyola in 1540. Of the many Jesuits this paper focuses attention on only one, Matteo Ricci (1552-1610), and of the many contributions of Ricci in the transmission of Western learning into China this paper focuses attention on only one, his collaboration with XU Guang-qi (1562-1633) in translating Euclid's *Elements* into Chinese.

The translation was based on the version of *Elements* compiled by Christopher Clavius (1538-1612) in 1574 (with subsequent editions), a fifteen-book edition titled *Euclidis Elementorum Libri XV*. Ricci learnt mathematics from Clavius at Collegio Romano where he studied from September 1572 to May 1578 before being sent to the East for missionary work.

On August 7, 1582 Ricci arrived in Macau, which was a trading colony in China set up by the Portugese with the consent of the Ming Court in 1557. Macau is the first as well as the last European colony in East Asia, being returned to Chinese sovereignty as a Special Administrative Region of China in 1999. Together with its neighbouring city of Hong Kong, which became a British colony in 1842 and returned to Chinese sovereignty in 1997, the two places played an important role in the history of the rise of modern China in a rather subtle way.

From Macau Ricci proceeded to move into mainland China and finally reach Peking (Beijing) in January of 1601. He became the most prominent Catholic missionary in China. When he passed away on May 11, 1610, he was the first non-Chinese that was granted the right to be buried on Chinese soil, an indication of the high esteem he was held in at the time. (Incidentally, Protestant missionary work also began in Macau with the arrival of Robert Morrison (1782-1834) of the London Missionary Society in 1807.)

Ricci left with us a very interesting and informative account of his life and missionary work in China in the form of a journal that was prepared for publication by a contemporary Jesuit Nicolas Trigault (1577-1628) in 1615. Let us quote a few passages from this journal of Ricci's [Ricci/Gallagher, 1942/1953, p.235, p.476].

"[…] Whoever may think that ethics, physics and mathematics are not important in the work of the Church, is unacquainted with the taste of the Chinese, who are slow to take a salutary spiritual potion, unless it be seasoned with an intellectual flavouring. […] All this, what we have recounted relative to a knowledge of science, served as seed for a future harvest, and also as a foundation for the nascent Church in China. […] but nothing pleased the Chinese as much as the volume on the *Elements* of Euclid. This perhaps was due to the fact that no people esteem mathematics as highly as the Chinese, despite their method of teaching, in which they propose all kinds of propositions but without demonstrations. […] The result of such a system is that anyone is free to exercise his wildest imagination relative to mathematics, without offering a definite proof of anything. In Euclid, on the contrary, they recognized something different, namely, propositions presented in order and so definitely proven that even the most obstinate could not deny them."

Is it really true that the notion of a mathematical proof was completely absent from ancient Chinese mathematics as Ricci remarked? This is a debatable issue [Siu, 1993, pp.345-346]. In Section 4 we will see one example (**Problem 2**) that would have made Ricci think otherwise, had he the opportunity of having access to the commentaries of LIU Hui of the 3rd century.

To Ricci, who studied mathematics under Clavius, the treatise *Elements*, compiled by Euclid (c.325-265 B.C.E.) in the early third century B.C.E., was the basis of any mathematical study. He therefore suggested to his Chinese friend XU Guang-qi that *Elements* should be the first mathematical text to be translated. XU Guang-qi set himself to work very hard on this project. He went to listen to Ricci's exposition of *Elements* every day in the afternoon (since he could not read Latin, while Ricci was well versed in Chinese) and studied laboriously, and at night he wrote out in Chinese everything he had learnt by day. We are told according to an account by Ricci: "When he [XU Guang-qi] began to understand the subtlety and solidity of the book, he took such a liking to it that he could not speak of any other subject with his fellow scholars, and he worked day and night to translate it in a clear, firm and elegant style. […] Thus he succeeded in reaching the end of the first six books which are the most necessary and, whilst studying them, he mingled with them other questions in mathematics." We are further told that "He [XU Guang-qi] would have wished to continue to the end of the Geometry; but the Father [Matteo Ricci] being desirous of devoting his time to more properly religious matters and to rein him in a bit told him to wait until they had seen from experience how the Chinese scholars received these first books, before translating the others." [Bernard, 1935, pp.67-68]

The translated text was published in 1607 and was given the title *Ji He Yuan Ben* (Source of Quantity). In the preface Ricci said:

> "[……] but I said: "No, let us first circulate this in order that those with an interest make themselves familiar with it. If, indeed, it proves of some value, then we can always translate the rest." Thereupon he [XU Guang-qi] said, "Alright. If this book indeed is of use, it does not necessary have to be completed by

> us." Thus, we stopped our translation and published it, […]".

But in his heart XU Guang-qi wanted very much to continue the translation. In a preface to a revised edition of *Ji He Yuan Ben* in 1611 he lamented, "It is hard to know when and by whom this project will be completed." This deep regret of XU Guang-qi was resolved only two and a half centuries later when the Qing mathematician LI Shan-lan (1811-1882) in collaboration with the English missionary Alexander Wylie (1815-1887) translated Book VII to Book XV in 1857 (based on the English translation of *Elements* by Henry Billingsley published in 1570) [Xu, 2005].

XU Guang-qi was a Chinese scholar brought up in the Confucian tradition, upholding the basic tenet of self-improvement and social responsibility, leading to an aspiration for public service and an inclination to pragmatism. He first got to know Catholic missionaries by an incidental encounter with the Jesuit Lazzaro Cattaneo (1560-1640) in the southern province of Guangdong, who probably introduced him to Ricci. XU Guang-qi was baptized (under the Christian name Paul) in 1603. He saw in Western religion and Western science and mathematics an excellent way to cultivate the mind and a supplement to Confucian studies. He also saw in Western science and technology the significant role it would play in improving the well-being of his countrymen. This eagerness on his part to study Western learning was very much welcomed by Ricci as it was in line with the tactics adopted by the Jesuit missionaries in making use of Western science and mathematics to attract and convert the Chinese literati class who usually occupied important positions in the Imperial Court. Matteo Ricci impressed the Chinese intellectuals as an erudite man of learning, thereby commanding their trust and respect [Siu, 1995/1996, p.148].

This is a good point to insert an explanation of the term "*ji he*" in the title of the translated text. This term has become the modern Chinese terminology for geometry. Some people suggests that it is a transliteration of the Western word "geometria (geometry)", Two reason can be raised against this view: (1) The word "geometria" does not appear in the title of Clavius' fifteen-book version of *Elements*. (In fact, nowhere in Euclid's *Elements* does the word "geometria" appear.) (2) Jesuit missionaries in those days were rather cautious about employing anything "un-Chinese" and transliteration was considered to be one such. A reading of the translated definitions in Book V, which is on Eudoxus' theory of proportion, will reveal that "*ji he*" is the technical term for "magnitude". In traditional Chinese mathematical classics the term "*ji he* (how much, how many)" frequently appears to begin a problem. We may conjecture that XU Guang-qi, who was familiar with this term because of his knowledge on traditional Chinese mathematics (the part that he had access to), thought of borrowing it to translate the technical term "magnitude" in *Elements*. By putting the term as a keyword in the title XU Guang-qi probably noticed the significance of the notion of "magnitude" in *Elements*. With the passing of time the original technical meaning of "*ji he*" (as "magnitude") was forgotten. Instead, because *Ji He Yuan Ben* (comprising the first six books of *Elements*) deals with properties of geometric figures such as triangles, squares, parallelograms and circles, the term acquires meaning as the name of the subject, replacing the term *xing xue* (study of figures) employed in the nineteenth century [Siu, 1995/1996, pp.160-161].

3 View of XU Guang-qi about *Elements*

By the Ming Dynasty the mathematical legacy in China was no longer preserved and nurtured in the way it should be. Quite a number of important mathematical classics were either completely lost or left in an incomplete form. As a scholar brought up in the Confucian tradition XU Guang-qi was aware that mathematics had once occupied a significant part of education and statecraft in China and should be restored to its former position of importance. He ascribed the unsatisfactory state of the subject at his time to two factors, which he expressed in 1614 in the preface to another translated European mathematical text (*Epitome Arithmeticae Practicae* compiled by Christopher Clavius in 1583, translated by LI Zhi-zao (1565–1630) also in collaboration with Matteo Ricci):

> "There are two main causes for negligence and dilapidation of mathematics that set in only during several past centuries. Firstly, scholars in pursuit of speculative philosophical studies despise matters of practical concern. Secondly, sorcery encroaches upon mathematics to turn it into a study filled with mysticism."

He saw in the introduction of Western mathematics, which was novel to him, a way to revive the indigenous mathematical tradition. He had a wider vision of mathematics, not just as an intellectual pursuit but as a subject of universal applications as well. In an official memorial submitted to the Emperor in 1629, he said, "Furthermore, if the study of measure and number [mathematics] is understood, then it can be applied to many problems [other than astronomy] as a by-product." Such problems were labelled by him in ten categories: (1) weather forecast, (2) irrigation, (3) musical system, (4) military equipment, (5) accounting, (6) building, (7) machine, (8)topography, (9) medical practice, (10) timepieces.

Despite XU Guang-qi's emphasis on utility of mathematics, he was sufficiently perceptive to notice the essential feature about *Elements*. Commenting on the merits of the book in the preface to *Ji He Yuan Ben*, he said:

> "As one proceeds from things obvious to things subtle, doubt is turned to conviction. Things that seem useless at the beginning are actually very useful, for upon them useful applications are based. It can be truly described as the envelopment of all myriad forms and phenomena, and as the erudite ocean of a hundred school of thought and study."

He stressed this point in another translated text (also in collaboration with Matteo Ricci) in 1608, that of parts of *Geometria practica* compiled by Christopher Clavius in 1606, retitled as *Ce Liang Fa Yi* (Methods and Principles in Surveying):

> "It has already been ten years since Master *Xitai* [Matteo Ricci] translated the methods in surveying. However, only started from 1607 onwards the methods can be related to their principles. Why do we have to wait? It is because at that time the six books of *Ji He Yuan Ben* were just completed so that the principles could be transmitted. As far as the methods are concerned, are they different from that

> of measurement at a distance in *Jiu Zhang* [*Suan Shu*] and *Zhou Bi* [*Suan Jing*]? They are not different. If that is so, why then should they be valued? They are valued for their principles."

In the same year XU Guang-qi published *Ce Liang Yi Tong* (Similarities and Differences in Surveying) in which he tried to explicate traditional Chinese surveying methods by the Western mathematics he had just learnt from *Elements*. In the introduction to the book he said:

> "In the chapter on *gou gu* (study of right-angled triangles) of *Jiu Zhang Suan Shu* there are several problems on surveying using gnomon and the trysquare, the methods of which are more or less similar to those in the recently translated *Ce Liang Fa Yi* (Methods and Principles in Surveying). The *yi* (principles) are completely lacking. Anyone who studies them cannot understand where they are derived from. I have therefore provided new *lun* (proofs) so that examination of the old text becomes as easy as looking at the palm of your hand."

A more explicit explanation can be found in an official memorial he submitted to the Emperor in 1629:

> "[… not knowing that] there are *li* (theory), *yi* (principle), *fa* (method) and *shu* (calculation) in it. Without understanding the theory we cannot derive the method; without grasping the principle we cannot do the calculation. It may require hard work to understand the theory and to grasp the principle, but it takes routine work to derive the method and to do the calculation."

As a scholar brought up in the Confucian tradition XU Guang-qi even saw in *Elements* the derived benefit in moral education. In an essay titled *Ji He Yuan Ben Za Yi* (Various Reflections on *Ji He Yuan Ben*) written in 1607 he said, "Five categories of personality will not learn from this book: those who are impetuous, those who are thoughtless, those who are complacent, those who are envious, and those who are arrogant. Thus to learn from this book one not only strengthens one's intellectual capacity but also builds a moral base."

For an in-depth analysis of the translation of *Ji He Yuan Ben* readers are strongly recommended to consult the book by Peter Engelfriet, which is a revised and expanded version of his 1996 doctoral dissertation at Leiden University [Engelfriet, 1998]. For an analysis of the work of XU Guang-qi in synthesizing Western mathematics and ancient Chinese mathematics in the context of mathematics in the Ming period readers may consult a paper of Peter Engelfriet and this author [Engelfriet & Siu, 2001]. For a general discussion on the contribution of XU Guang-qi in fostering development in science in 17th century China, readers may consult a paper of this author [Siu, 1995/1996].

It may be of interest, if just for the sake of speculation, to raise a few hypothetical questions:

(1) How much would XU Guang-qi have achieved in mathematics if he had concentrated his effort on this one discipline?

(2) What would have happened if he had known about the various commentaries on the controversial Fifth Postulate?
(3) What would have happened if he had mastered Latin just as Ricci had mastered Chinese?
(4) What would have happened if he had the chance and the inclination to actually pay a visit to Europe at the time and to return to China with what he experienced and observed over there?

Nothing of that sort happened in history. Besides cultural obstacle there were at the time adverse social and political factors that did not work in favour of the first dissemination of Western learning in China. "Ironically, the ready acceptance of Western science by this small circle of open-minded scholar-officials, as exemplified by XU Guang-qi, also turned out to be a reason for their failure, for in the eyes of the conservative ministers and the general populace, this small group of converts were over-enthusiastic about the alien culture. They lacked the support of the host culture, so to speak."[Siu, 1995/1996, p.171].

4 An inscribed square in a right-angled triangle

Through working out a series of problems built around one specific question, that of an inscribed square in a right-angled triangle, we will compare the styles and emphases of mathematical pursuit in the Eastern and the Western traditions. In the following problems the labelling in the figure refers to that specified in the corresponding passage, sometimes with an accompanying figure. (This exercise was actually carried out in a three-hour workshop in the ESU6.)

Problem 1: Given a right-angled triangle *ABC* with *AC* as its hypotenuse, how would you inscribe a square in it, i.e., construct a square *BFED* with *D* on *AB*, *E* on *AC*, and *F* on *BC* (Figure 1)?

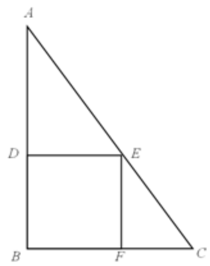

Figure 1

Remarks: There are various ways to solve this problem. One way that is close to the style of Euclid would be to bisect ∠ *ABC* by *BE* (with *E* on *AC*) [justified by I.9], then drop perpendiculars *ED*, *EF* (with *D* on *AB* and *F* on *BC*) [justified by I.12]. It can be proved that *BFED* is the inscribed square we want. (Throughout this section, I.9 means Proposition 9 in Book I of Euclid's *Elements*, etc.)

Another way is to first construct a square *ABB′A′* with *AB* as one side. Join *BA′*, which intersects *AC* at *E*. Drop perpendiculars *ED*, *EF* (with *D* on *AB* and *F* on *BC*). It can be proved that *BFED* is the inscribed square we want. The second way may look just like the first way, but the second way can be generalized readily to construct an inscribed square in an arbitrary triangle *ABC*, which is not necessarily right-angled. To do this, drop a perpendicular *AH* to *BC* (with *H* on *BC*). Construct the square *AHB′A′* with *B′* on *BC* and on the other side of *AH* as *B*. Join *BA′* to intersect *AC* at *E*. Draw *EDG* parallel to *BC* (with

G on *AB* and *D* on *AH*). Drop perpendiculars *EF*, *GI* (with *F*, *I* on *BC*). It can be proved that *IFEG* is the inscribed square we want.

There are yet other ways to construct an inscribed square in an arbitrary triangle *ABC*. For instance, erect a square *WZYX* inside the triangle *ABC* (with *X* on *AB* and *WZ* on and inside *BC*). Join *BY* and produce to intersect *AC* at *E*. Draw *EG* parallel to *BC* (with *G* on *AB*) and drop perpendiculars *GI* and *EF* (with *I*, *F* on *BC*). It can be proved that *IFEG* is an inscribed square in triangle *ABC*. Or one can carry out a similar procedure by starting with a square on BC that lies outside the triangle *ABC*.

It is interesting to note a construction by the English mathematician John Speidell in his book *A geometrical extraction, or , A compendious collection of the chiefest and choisest problems* (1616), which somehow combines the feature of the problem for a right-angled triangle and an arbitrary triangle (Figure 2).

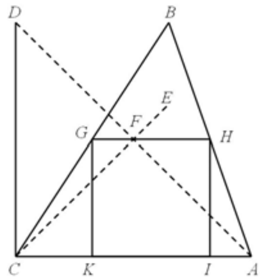

Figure 2

Erect a perpendicular *CD* to the base *CA* with *CD* equal to the height of *B* above *CA*, then bisect $\angle ACD$ (= a right angle) by *CE*. Let *CE* intersect *AD* at *F*. Draw *GFH* (with *G* on *BC* and *H* on *AB*) parallel to *CA*. Drop perpendiculars *GK*, *HI* (with *K*, *I* on *CA*). It can be proved that *KIHG* is the inscribed square we want.

There is a common feature in all of the different methods exhibited above that is characteristic of the style of Greek geometry expounded in Euclid's *Elements*. In Euclid's exposition of geometry a definition (for instance, an inscribed square in a given triangle) does not guarantee existence. Existence is justified by a construction. Each one of these methods actually constructs such an inscribed square in a given triangle. Before one is not even certain whether such an inscribed square exists or not, one would not go ahead to calculate the length of its side.

Now that we know such an inscribed square exists we can ask what the length of its side is. It can be shown from each construction that the side *x* of the inscribed square in a right-angled triangle with sides of length *a*, *b* containing the right angle is given by $x = \dfrac{ab}{a+b}$ (Exercise). More generally, for an arbitrary triangle *ABC* with base *BC* = *b* and altitude *AH* = *h*, the side *x* of the inscribed square *IFEG* (with *I*, *F* on *BC*, *G* on *AB* and *E* on *AC*) is given by $x = \dfrac{hb}{h+b}$ (Exercise).

Problem 2: **Problem 1** appears as Problem 15 of Chapter 9 in *Jiu Zhang Suan Shu* (Nine Chapters on the Mathematical Art) compiled between 100 B.C. and A.D. 100. Study the original text (English translation in **Appendix 1**) and explain the formula by **two** different proofs given in the commentary by LIU Hui in the mid 3rd century (Figure 3).

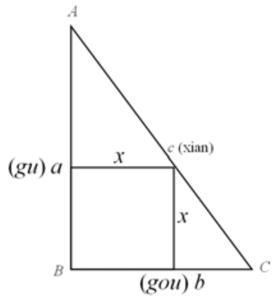

Figure 3

Remarks: The first method is a "visual proof" of the formula $x = \dfrac{ab}{a+b}$ by dissecting and re-assembling coloured pieces (Figure 4). A similar but more interesting computation was devised by LIU Hui for the next problem in the book, of an inscribed circle of a right-angled triangle [Siu, 1993, pp.349-352].

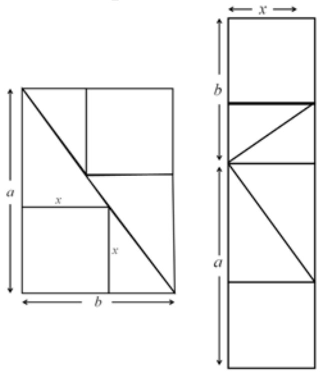

Figure 4

The second method is based on the theory of proportion, making use of the so-called *Jinyou method* (known as the *Rule of Three* in the western world) and the *principle of invariant ratio* (which is basically the same as the content of I.43). Although there was no theory of similar triangles developed in ancient Chinese mathematics, a special case of it in the situation of right-angled triangles was frequently employed with dexterity and proved to be rather adequate for most purposes.

Problem 3: **Problem 1** does not appear in Euclid's *Elements* but appears as a particular case of Added Proposition 15 of Book VI in *Euclidis Elementorum Libri XV* compiled by Christopher Clavius in 1574, which was translated into Chinese in *Ji He Yuan Ben* of 1607. Study the original text (English translation in **Appendix 2**) and compare this explanation with that of LIU Hui's, or that of your own (in **Problem 1**). Is there a different emphasis in these explanations?

Remarks: The construction is effected by dropping the perpendicular AD on BC (with D on BC), and divide AD at E such that $AE : ED = AD : BC$. In the original text this construction was justified by Added Proposition 1 of Book VI of *Ji He Yuan Ben*, which may be wrongly ascribed; the more likely justification seems to be Proposition 10 of Book VI within the same book (Figure 5).

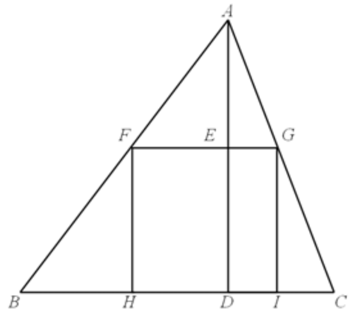

Figure 5

There is a supplemented method that is just a specialization in the case when *ABC* is a right-angled triangle with $\angle ABC$ equal to a right angle (Figure 1). An appended remark at the end says that the side of the inscribed square *BFED* in a right-angled triangle *ABC* must be a mean proportion of *AD*, *FC*, thus affording a motivation of the construction.

Problem 4: In 1609 XU Guang-qi wrote *Gou Gu Yi* (Principle of the Right-angled Triangle) in which he attempted to synthesize knowledge about a right-angled triangle contained in *Jiu Zhang Suan Shu* and in Euclid's *Elements* (or more precisely, the version by Clavius from which he learnt Euclidean geometry). In particular, Problem 4 is about an inscribed square in a right-angled triangle. Study the original text (English translation in **Appendix 3**). In your opinion to what extent did XU Guang-qi succeed in accomplishing the synthesis (Figure 6)?

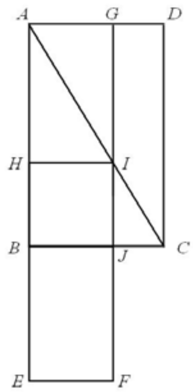

Figure 6

Remarks: Xu Guang-qi started with the formula

$$HB = BJ = JI = IH = \frac{AB \times BC}{AB + BC}$$

and tried to reduce back to Added Proposition 15 in Book VI of *Ji He Yuan Ben*, that is, prove that *H* divided *AB* such that *AH* : *HB* = *AB* : *BC*. He made use of a number of results of reciprocally related figures in Book VI of *Ji He Yuan Ben*. His proof may sound rather round-about and awkward, an indication of an "unnatural" attempt to combine two different styles that may not be as compatible! But we should admire the intention of XU Guang-qi in this effort of what he described as "*hui tong* (to understand and to synthesize)".

Problem 5: Added Proposition 15 of Book VI in *Euclidis Elementorum Libri XV* actually gives the answer to a more general problem, which specializes to the formula for the case of a right-angled triangle. **(a)** Devise a proof by dissection along the line of thinking of LIU Hui. **(b)** In the case of a right-angled triangle the answer to the general problem would give two different ways to "inscribe a square in a right-angled triangle". Compare these two ways.

Remarks: Not to spoil the fun of the reader, a solution to **(a)** will be left as an exercise. The two ways in **(b)** give different answers.

Problem 6: In what way is the result in **Problem 2** a special case of the formula offered by the Indian mathematician Bhaskara (also known as Bhaskara II or Bhaskaracharya) in Problem 161 of Chapter 5 of *Lilavati* (12^{th} century)? Problem 161 is about two vertical poles, the top of each being connected by a string to the bottom of the other. One is asked to compute the height of the intersecting point of the strings from the ground.

Remarks: If the two poles of height a, b are at a distance ℓ apart, then it can be seen that the height x of the intersecting point above ground is given by $x = \dfrac{ab}{a+b}$ (Exercise). In other words, x is the harmonic mean of a and b, independent of ℓ ! (Explain this independence geometrically.) The form of the relationship rings a bell. When ℓ is made equal to a, it becomes apparent that x is nothing but the side of the inscribed square in a right-angled triangle.

A comparison of the methods in **Problem 1** and **Problem 2** will show a general difference in approach between ancient Chinese mathematics and Greek mathematics. Roughly speaking we can borrow the terms "algorithmic mathematics" and "dialectic mathematics" coined by Peter Henrici [Henrici, 1974, p.80] to describe the two approaches. Ideally speaking these two approaches should complement and supplement each other with one containing some part of the other like *yin* and *yang* in Chinese philosophy. Further discussion on these two approaches and cognitive thinking in the West and East revealed in the activity of proof and proving may be found in another two papers of this author on mathematical proofs [Siu, 2009a; Siu, 2012].

5 Influence exerted by *Ji He Yuan Ben* in China

In his essay, *Ji He Yuan Ben Za Yi* of 1607 XU Guang-qi commented:

> "The benefit derived from studying this book is many. It can dispel shallowness of those who learn the theory and improve their concentration. It can supply fixed methods for those who apply to practice and kindle their creative thinking. Therefore everyone in this world should study this book."

But realizing the actual situation he also commented in the same essay:

> "This book has wide applications and is particularly needed at this point in time. […] In the preface Mister Ricci also expressed his wish to promulgate this book so that it can be made known to everybody who will then study it. Few people study it. I surmise everybody will study it a hundred years from now, at which time they will regret that they study it too late. They would wrongly attribute to me the foresight [in introducing this book], but what foresight have I really?"

However, near to a hundred years later, the situation was still far from what he would like to see. In the preface to *Shu Xue Yao* (The Key to Mathematics) written by DU Zhi-geng (second part of 17th century) in 1681, LI Zi-jin (1622-1701) said, "Even those gentlemen in the capital who regard themselves to be erudite scholars keep away from the book [*Elements*], or close it and do not discuss its content at all, or discuss it with incomprehension and perplexity."

The Chinese in the 17th and 18th centuries did not seem to feel the impact of the essential feature of Western mathematics exemplified in Euclid's *Elements* as strongly as XU Guang-qi. Thus, the influence of the newly introduced Western mathematics on mathematical thinking in China was not as extensive and as directly as XU Guang-qi had imagined. The effect was gradual and became apparent only much later. However, the fruit was brought forth elsewhere, not in mathematics but perhaps of an even higher historical importance.

Three leading figures responsible for the so-called "Hundred-day Reform" of 1898 — KANG You-wei (1858-1927), LIANG Qi-chao (1873-1929), TAN Si-tong (1865-1898) — were strongly influenced by their interest in acquiring Western learning. In 1888 KANG You-wei wrote a book titled *Shi Li Gong Fa Quan Shu* (Complete Book on Concrete Principles and Postulates [of Human Relationship]), later incorporated into his masterpiece *Da Tong Shu* (Book of Great Unity) of 1913. It carries a shade of the format of *Elements*, as the title suggests. The book *Ren Xue* (On Moral Philosophy) written by TAN Si-tong and published posthumously in 1899, carries an even stronger shade of the format of *Elements*, reminding one of the book *Ethics* by Baruch Spinoza (1632-1677) of 1675 that began with definitions and postulates. To educate his countrymen in modern thinking TAN Si-tong established in 1897 a private academy known as the Liuyang College of Mathematics in his hometown, stating clearly in a message on the mission of the college that mathematics is the foundation of science, and yet the study starts with mathematics but does not end with it. Apparently, he was regarding mathematics as assuming a higher position than just a technical tool in the growth of a whole-person in liberal education. In his famous book *Qing Dai Xue Shu Gai Lun* (Intellectual Trends of the Qing Period), originally published in *Reform Magazine* in 1920/1921, LIANG Qi-chao remarked (English translation by Immanuel C.Y. Hsü [Liang, 1959]):

> "Since the last phase of the Ming, when Matteo Ricci and others introduced into China what was then known as Western learning (*xi xue*), the methods of scholarly research had changed from without. At first only astronomers and mathematicians credited [the new methods], but later on they were gradually applied to other subjects."

The "Hundred-day Reform" ended in failure despite the initiation and support of Emperor Guangxu (reigned 1875-1908) because of the political situation of the time. TAN Si-tong met with the tragic fate of being arrested and executed in that same year, while KANG You-wei and LIANG Qi-chao had to flee the country and went to Japan. This was one important step in a whole series of events that culminated in the overthrow of Imperial Qing and the establishment of the Chinese Republic in 1911.

Within mathematics itself, *Ji He Yuan Ben* did have some influence, gradual as it was. For a more detailed discussion readers are recommended to consult the book by Peter Engelfriet [Engelfriet, 1998]. By the first part of the 20th century the Chinese began to appreciate the deeper meaning of *Elements*. An illuminating remark came from an eminent historian CHEN Yin-ke (1890-1969) who said in an epilogue to the Manchurian translation of *Ji He Yuan Ben* in 1931 (translated into English by this author):

> "The systematic and logical structure of Euclid's book is of unparalleled preciseness. It is not just a book on number and form but is a realization of the Greek spirit. The translated text in the Manchurian language and the version in *Shu Li Jing Yun* (Collected Basic Principles of Mathematics) are edited to lend emphasis on utility of the subject, not realizing that, by so doing, the original essence has been lost."

6 The three waves of transmission of European science into China

The translation of *Elements* by XU Guang-qi and Matteo Ricci led the way of the first wave of transmission of European science into China, while a second wave and a third wave followed in the Qing Dynasty, but each in a rather different historical context.

The gain of this first wave seemed momentary and passed with the downfall of the Ming Dynasty. "Looking back we can see its long-term influence, but at the time this small window which opened onto an amazing outside world was soon closed again, only to be forced open as a wider door two hundred years later by Western gunboats that inflicted upon the ancient nation a century of exploitation and humiliation, thus generating an urgency to know more about the Western world." [Siu, 1995/1996, pp.170-171].

The second wave came in the wake of the first wave and lasted from the mid 17th century to the mid 18th century. Instead of Chinese scholar-officials the chief promoter was Emperor Kangxi of Qing Dynasty (reigned 1662-1722). Instead of Italian and Portugese Jesuits the western partners were mainly French Jesuits, the so-called "King's Mathematicians" sent by Louis XIV, the "Sun King" of France (reigned 1643-1715), in 1685 [Du & Han, 1992].

This group of Jesuits, led by Jean de Fontaney (1643-1710) reached Peking in 1688. An interesting account of their lives and duties in the Imperial Court was recorded in the journal written by one of the group, Joachim Bouvet (1656-1730) [Bouvet, 1697]. By imperial decree an intensive course of study on Western science and mathematics was organized to take place in the Imperial Palace, with the French Jesuits as tutors, for Emperor Kangxi and some of the princes. The happenings of this second wave form an interesting and intricate story that cannot be discussed in detail in this paper for want of space. It reflects an attitude of learning when the student (Emperor Kangxi) regards himself in a much more superior position than his teachers! A main conclusion is the compilation of a monumental one-hundred-volume treatise *Lu Li Yuan Yuan* (Origins of Mathematical Harmonics and Astronomy) commissioned by Emperor Kangxi, worked on by a large group of Jesuits, Chinese scholars and official astronomers. The project started in 1713 and the treatise was published in 1722/1723, comprising three parts: *Li Xiang Kao Cheng* (Compendium of Observational Computational Astronomy), *Shu Li Jing Yun* (Collected Basic Principles of Mathematics), *Lu Lu Zheng Yi* (Exact Meaning of Pitchpipes). Interested readers will find a more in-depth discussion of this second wave in a paper of Catherine Jami [Jami, 2002].

The third wave came in the last forty years of the 19th century in the form of the so-called "Self-strengthening Movement" after the country suffered from foreign exploitation during the First Opium War (1839-1842) and the Second Opium War (1856-1860). This time the initiators were officials led by Prince Gong (1833-1898) with contribution from Chinese scholars and Protestant missionaries coming from England or America, among whom were LI Shan-lan and Alexander Wylie who completed the translation of *Elements*. In 1862 *Tong Wen Guan* (College of Foreign Languages) was established by decree, at first serving as a school for studying foreign languages to train interpreters but gradually expanded into an institute of learning Western science. The slogan of the day, which was "learn the strong techniques of the "[Western] barbarians" in order to control them", reflected the purpose and mentality during that period. In 1866 a mathematics and astronomy section was added to *Tong Wen Guan*, with LI Shan-lan as its head of department. In 1902 *Tong Wen Guan* became part of Peking Imperial University, which later became what is now Beijing University [Siu, 2009b, pp.203-204]. For a general discussion on the history of the rise of modern China readers may consult some standard texts [Fairbank & Reischauer, 1973; Hsü, 1970/2000].

The theme and mood of the three waves of transmission of European science into China were reflected in the respective slogans prevalent in each period. In the first part of the 17th century the idea was: "In order to surpass we must try to understand and to synthesize." In the first part of the 18th century it became: "Western learning has its root in Chinese Learning." In the latter part of the 19th century the slogan took on a very different tone: "Learn the strong techniques of the '[Western] barbarians' in order to control them."

In a paper on European science in China Catherine Jami says:

> "[…] the cross-cultural transmission of scientific learning cannot be read in a single way, as the transmission of immutable objects between two monolithic cultural entities. Quite the contrary: the stakes in this transmission, and the continuous reshaping of what was transmitted, can be brought to light only by situating the actors within the society in which they lived, by retrieving their motivations, strategies, and rationales within this context." [Jami, 1999, p.430]

In a paper on the life and work of XU Guang-qi this author once suggested:

> "It will be a meaningful task to try to trace the "mental struggle" of China in the long process of learning Western science, from the endeavour of XU Guang-qi, to the resistance best portrayed by the vehement opposition of YANG Guang-xin, to the promulgation of the theory that "Western science had roots in ancient China", to the self-strengthening movement, and finally to the "naturalization" of western science in China. It is a complicated story embedded in a complicated cultural-socio-political context." [Siu, 1995/1996, p.171]

In the words of the historian Immanuel Hsü, this "mental struggle" is "an extremely hard struggle against the weight of pride and disdain for things foreign, and the inveterate belief that the bountiful Middle Kingdom had nothing to learn from the outlandish barbarians and little to gain from their association.” [Hsü, 1970/2000, p.10] Viewed in this light the attempt and foresight of XU Guang-qi stand out all the more unusual, visionary and admirable.

Appendix 1

Now given a right-angled triangle whose *gou* is 5 *bu* and whose *gu* is 12 *bu*. What is the side of an inscribed square? The answer is 3 and 9/17 *bu*.

Method (See Figure 3): Let the sum of the *gou* and the *gu* be the divisor; let the product of the *gou* and the *gu* be the dividend. Divide to obtain the side of the square.

Commentary of Liu Hui: The product of the *gou* and the *gu* is the area of a rectangle comprising crimson triangles, indigo triangles and yellow squares, each in two. Place the two yellow squares at the two ends; place the crimson triangles and indigo triangles, with figures of the same type combined together, in between so that their respective *gu* and *gou* coincide with the side of the yellow square. These pieces form a rectangle. Its width is the side of the yellow square; its length is the sum of the *gou* and *gu*. Hence the sum of the *gou* and the *gu* becomes the divisor. In the figure of the right-angled triangle with its inscribed square, on the two sides of the square there are smaller right-angled triangles, for which the relation between their sides retains the same ratio as that of the original right-angled triangle. The respective sums of the smaller *gou* and *gu* of the right-angled triangle on the *gou* [which is equal to the *gou*] and that of the smaller *gou* and *gu* of the right-angled triangle on the *gu* [which is equal to the *gu*] become the mean proportion. Let the *gu* be the mean proportion and the sum of the *gou* and the *gu* be the other term of the ratio. Apply Rule of Three to obtain the side of the inscribed square with the *gou* being 5 *bu*. Let the *gou* be the mean proportion and the sum of the *gou* and the *gu* be the other term of the ratio. Apply Rule of Three to obtain the side of the inscribed square with the *gu* being 12 *bu*. This [second] method does not follow the method explained at the beginning, but it produces the dividend and the divisor. In the next problem on the inscribed circle of a right-angled triangle when we utilize Rule of Three and Rule of Proportional Distribution, this method again becomes apparent.

Appendix 2

Added Proposition 15 of Book VI: Given a triangle, it is required to produce its inscribed square.

Method (See Figure 5): If *ABC* is an acute-angled triangle and it is required to produce its inscribed square, through *A* construct *AD* [*D* on *BC*] which is perpendicular to *BC*. Divide *AD* at *E* such that *AE* : *ED* = *AD* : *BC* (Book VI, Proposition 1, Added Proposition (?)) . Through *E* construct *FG* [*F* on *AB* and *G* on *AC*] parallel to *BC*. From *F* and *G* respectively construct *FH* [*H* on *BC*] and *GI* [*I* on *BC*] parallel to *ED*. The figure *FHIG* is what is required to produce. If the triangle is right-angled or obtuse-angled, then drop the perpendicular from the right angle or the obtuse angle respectively and proceed as before.

Proof: FEG is parallel to *BC*, so *BD* : *DC* = *FE* : *EG* (Book VI, Proposition 4, Added Proposition). By ratio componendo *BC* : *DC* = *FG*: *EG*. We also have *DC* : *AD* = *EG* : *AE* (Book VI, Proposition 4, Corollary). By ratio ex aequali *BC* : *AD* = *FG* : *AE*. We also have *AD* : *BC* = *AE* : *ED* [and *BC* : *AD* = *FG* : *AE*]. By ratio ex aequali *BC* : *BC* = *FG* : *ED*. Since *BC* and *BC* are equal, we have *FG* and *ED* are equal. *FG* is equal to *HI* (Book I, Proposition 34). *ED*, *FH* and *GI* are all equal, so the four sides *FG*, *GI*, *IH*, *HF* are all equal. *EDH* is a right angle, so *FHD* is a right angle (Book I, Proposition 29). The other angles are also right angles. Hence *FHIG* is a square.

Supplemented method (See Figure 1): If in a right-angled triangle *ABC* it is required to produce its inscribed square with *ABC* as one of its right angle, then divide the perpendicular *AB* at *D* such that *AD* : *DB* = *AB* : *BC* (Book VI, Proposition 10). Through *D* construct *DE* [*E* on *AC*] parallel to *BC*. Through *E* construct *EF* [*F* on *BC*] parallel to *AB*. The figure *DBFE* is what is required to produce.

Proof: *BC* : *AB* = *DE* : *AD* (Book VI, Proposition 4, Corollary) and *AB* : *BC* = *AD* : *DB*. By ratio ex aequali *BC* : *BC* = *DE* : *D*B. Since *BC* and *BC* are equal, we have *DE* and *DB* are equal. Hence *DBFE* is a square.

Appended: In the right-angled triangle *ABC* it is required to produce its inscribed square with *ABC* as one of its right angle. The side of this inscribed square must be a mean proportion of *AD* and *FC*. This is because *AD* : *DE* = *EF* : *FC* (Book VI, Proposition 4, Corollary).

Appendix 3

Method (See Figure 6): *Gu AB* is 36, *gou BC* is 27. It is required to produce its inscribed square. Let the product of *gou* and *gu* be the dividend. Let the sum of *gou* and *gu* be the divisor, which is *AE* equal to 63. Divide and obtain each side of the inscribed square, *HB* and *BJ*, to be 15.428.

Proof: *AB* = 36, *BC* = 27. Let their product 972 be the dividend. This is the [area of the] rectangle *ABCD*. Let the sum 63 be the divisor. This is the straight line *AE*. Divide to obtain the side *EF* to be 15.428. This makes the rectangle *AEFG* equal in area to the rectangle *ABCD* (Book VI, Proposition 16). Let *FG* intersect *BC* at *J* and *AC* at *I*, then the figure *BHIJ* is an inscribed square of the right-angled triangle *ABC*.

Why? *ABCD* and *AEFG* are reciprocally related figures, that is, *AB* : *AE* = *BJ* : *BC* (Book VI, Proposition 15 (14?)). By ratio dividendo *AB* : *BE* = *BJ* : *JC*, that is, *AB* : *BC* = *BJ* : *JC* (because *BC* = *BE*). We also have *AH* : *HI* = *IJ* : *JC* (Book VI, Proposition 4). By ratio alternando *AH* : *IJ* = *HI* : *JC*. But *HB* = *IJ*, *BJ* = *HI*. Therefore *AH* : *HB* = *BJ* : *JC*. Since *AB* : *BC* = *BJ* : *JC* and *AH* : *HB* = *BJ* : *JC*, we have *AB* : *BC* = *AH* : *HB*. Hence *BHIJ* is an inscribed square (Book VI, Added Proposition 15).

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USING ART AND MUSIC FOR SECONDARY SCHOOL MATH EDUCATION

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ABSTRACT

Recent survey has presented that Korean secondary school students and math teachers demanded the authentic applications of mathematics in math curriculum.

Such applications in the math curriculum prevent students from getting exhausted of formulas and logic, and help students enhance interests towards mathematics. Throughout the long history of math, there has been close relationships between math and real life. Some of the connections, such as relationships between sine waves and music will be introduced in this Poster.

INFINITE SERIES BEFORE ALGEBRA AND CALCULUS

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ABSTRACT

In calculus books infinite series usually are studied after derivatives and integrals, mainly in Taylor Series' form. This way we taught students that infinite series are Taylor series' and can be expanded using derivatives only.
Indian mathematicians found many series before calculus was discovered in Europe, many of those series were expanded based in geometry. Some series were used to approximate sine values tables and π with thirteen exact decimals, centuries before European mathematicians.
In this poster we give a brief collection of Indian advances of infinite series expansions without using calculus. Some expansions will be shown in geometrical form.

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Topics in the history of mathematics education

FROM ROME TO ROME: EVENTS, PEOPLE, AND NUMBERS DURING ICMI'S FIRST CENTURY

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ABSTRACT

On the occasion of the ICMI centenary the authors of the present paper have built the website "The first century of the International Commission on Mathematical Instruction (1908-2008). History of ICMI", which contains information and documents about people, events, and bodies intervening in the first century of this Commission.

Working on this enterprise offered an occasion for reflecting on some aspects of ICMI history, in particular, on how the key ideas (internationalization, communication, solidarity) that inspired its founding were realized, as well as on contributions by paramount ICMI members to the evolution of both the Commission's policies and its methodological approaches to problems.

1 Introduction

In 1908, during the fourth International Congress of Mathematicians held in Rome, a commission aimed at studying the problems of mathematical instruction at an international level was founded. Through successive events and changes this commission developed into the body now known as ICMI (International Commission on Mathematical Instruction).[1] A hundred years later, the first centenary of the Commission was celebrated at a Symposium during which about 200 scholars from all around the world gathered in the same sites where ICMI was founded. The celebrations provided a good occasion for revisiting ICMI history through talks centered on particular moments in the history of the Commission. The texts of these talks, reported in the proceedings of the Symposium (Menghini et al., 2008), add further information to the existing works in the field.[2] In the aftermath of this event further publications have appeared: the special issue of the *International Journal for the History of Mathematics Education*, the Regular lecture (Arzarello et al., to appear) delivered at ICME-11, some contributions in the Iceland conference on the history of mathematics education (see Bjarnadóttir, Furinghetti, Schubring, 2009).

The website entitled "The first century of the International Commission on Mathematical Instruction (1908-2008). History of ICMI",[3] designed by the authors of the

[1]The name "International Commission on Mathematical Instruction" and acronym ICMI that we use in this paper exist from the early 1950s. Earlier, the Commission was known by the French acronym CIEM (Commission Internationale de l'Enseignement Mathématique) or, especially under Klein's presidency, by the German acronym IMUK (Internationale Mathematische Unterrichtskommission).

[2]General information on ICMI is in (Bass, Hodgson, 2004). The history of the first 75 years of ICMI is outlined in (Howson, 1984). In his book on the history of the International Mathematical Union (IMU), Lehto (1998) illustrates aspects of ICMI history, with particular reference to its relationship to the community of mathematicians. The early days of ICMI and the conditions for its creation are presented in (Furinghetti, 2003; Schubring, 2008). For general information on ICMI see the ICMI website: http://www.mathunion.org/ICMI/.

[3]See (Furinghetti, Giacardi, 2008).

present paper, is intended as a useful complement to the reconstruction of the history of ICMI. Besides offering a timeline of the events and cameos of the main figures, it presents a large and rich selection of documents, which are often difficult to find.

In this paper we briefly illustrate the six sections of the website and use the materials to review some aspects of the history of ICMI.

2 The website for the history of ICMI

The historical roots of ICMI can be traced to a suggestion by David Eugene Smith (1905) that appeared in the journal *L'Enseignement Mathématique*[4] (hereafter "*EM*"), founded in 1899 by two mathematicians, the French Charles-Ange Laisant and the Swiss Henri Fehr. Smith advocated more international co-operation and the creation of a commission to be appointed during a conference for the study of instructional problems at an international level. As mentioned earlier, this project was realized in 1908 during the fourth International Congress of Mathematicians held in Rome (6-11 April 1908), when ICMI was founded. It is important to note that the Commission was founded and developed inside the community of mathematicians and for a long time the mandates were established each four years during the International Congress of Mathematicians. In what follows we will see that this fact affected the course of ICMI.

In the century of ICMI's existence it is possible to identify five periods, outlined below, that were the result of both external events that influenced the Commission as well as the changing centers of interest and activities of the Commission itself.

1. Foundation and early period up to WW1 (President: Felix Klein)

TABLEAU D'ENSEMBLE DE LA RÉPARTITION PAR PAYS

	Fasc. ou volumes.	Nombre de rapports.	Nombre de pages.
Comité Central	11	19	561
Allemagne	53	53	5571
Rép. Argentine	1	1	24
Australie	1	6	79
Autriche	13	12	776
Belgique	2	5	366
Danemark	1	10	107
Espagne	3	10	165
Etats-Unis	18	18	1499
France	5	45	674
Hollande	1	13	151
Hongrie	9	9	294
Iles Britanniques	32	39	921
Italie	10	11	268
Japon	2	16	788
Roumanie	1	1	16
Russie	7	11	295
Suède	8	8	229
Suisse	9	13	781
	187	310	13565

Figure 1. Publications of the Commission (Fehr, 1920-1921, p. 339).

[4]After the foundation of ICMI, *L'Enseignement Mathématique* became the official organ of the Commission.

During this phase an important international network of national subcommittees was established for the preparation of reports on the state of mathematical instruction as well as on thematic issues. The account by Fehr (1920-1921) on the activity of the Commission from 1908 to 1920 encompasses an impressive list of the publications of the Central Committee and of the national sub-commissions (Argentine, Australia, Austria, Belgium, Denmark, France, Germany, Holland, Hungary, Italy, Japan, Romania, Russia, Spain, Sweden, Switzerland, UK, and USA). During this period nine international inquiries on different aspects of mathematics teaching were launched through questionnaires, followed by the relative reports, with the exception of the last inquiry concerning teacher education, presented only in 1932 because of WW1 and the ensuing political problems.[5]

2. Crisis and dissolution in 1920-21 and ephemeral rebirth in Bologna 1928 between the two World Wars (Presidents: David E. Smith, Jacques Hadamard)

After WW1 scientific associations dissolved and the shocking decision was made to ban the researchers of the Central Powers from most international activities. In Rome, Schubring (2008) provided a vivid picture of the obstacles to cooperation, and the political pressures that resulted in the dissolution of the Commission after WW1, underlining the role of the secretary[6] Fehr. Only during the International Congress of Mathematicians in Bologna (1928) would international collaboration be re-established, reintegrating the countries that had been excluded. However, the reconstituted Commission for mathematics teaching was not capable of producing new ideas and projects, and was limited to carrying out the old agenda, until WW2 forced a second arrest of activities.

3. The rebirth in 1952 as a permanent sub-commission of the IMU (Presidents: Albert Châtelet, Heinrich Behnke, Marshall Stone, André Lichnérowicz)

As reported in the short history in the IMU website (http://www.mathunion.org) "the Constitutive Convention in 1950 in New York created IMU *de facto*. By the Statutes adopted there, IMU came into being in 1951 *de jure*". During the first General Assembly held in Rome in 1952 the Commission became a permanent sub-commission of IMU, while maintaining its original aims, but friction between IMU mathematicians and ICMI very soon made it necessary to better define ICMI's structure (composition, relationship with the IMU, the organization of the national sub-commissions, etc.). Precise Terms of Reference[7] were adopted during the second General Assembly of IMU (The Hague, 31 August-1 September 1954). Behnke was to play an important role in this period. In the 1960s the action of ICMI broadened considerably: thanks to Stone and Lichnérowicz, collaborations both scientific and organizational were established with other associations such as OEEC (Organization for European Economic Cooperation) and UNESCO (United Nations Educational Scientific and Cultural Organization). These led to a greater internationalism and to the organization of numerous thematic congresses in various parts of the world. After WW2 lines of research broadened and new approaches to mathematics

[5]See *Questionnaires and reports* in (Furinghetti, Giacardi, 2008).
[6]The term "secretary general" was used in the first few decades of the ICMI, and became "secretary" after the WW2. In the meeting of April 2002 in Paris new Terms of Reference for ICMI were approved by the Executive Committee of IMU: among the modifications there is a change in the name of the position of "secretary", which is now designated by the term "secretary general", as it was in the past.
[7]The successive Terms of Reference of ICMI from 1954 to 2002 are in (Giacardi, 2009) and on the ICMI website http://www.mathunion.org/icmi/home/

education were carried out in different arenas. In the USA the *University of Illinois Committee on School Mathematics* (UICSM), headed by Max Beberman, was established in 1951. In 1958 the School Mathematics Study Group (SMSG) was created under the directorship of Edward G. Begle[8], a member of the ICMI Executive Committee from 1975 to 1978. In Europe CIEAEM (*Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques*), which had already begun its activities in 1950, was officially founded in 1952. This Commission, whose members included mathematicians, pedagogists, secondary teachers, psychologists, and epistemologists, focused mainly on the importance of working in the field of didactic research while maintaining close contacts with the classroom. As discussed in (Furinghetti et al., 2008), CIEAEM was particularly influential in the evolution of ICMI's approach to educational problems. As a matter of fact, some of the founding members of CIEAEM were important members of ICMI Executive Committee (Evert W. Beth,[9] Hans Freudenthal, Lichnérowicz). One of the most debated themes in this context was the approach proposed by the movement of Modern/New Mathematics. This movement did not have the desired effects in the mathematical instruction of the various countries, but promoted the circulation of ideas at an international level as well as attempts to modernize the teaching of mathematics in different directions, see (Charlot, 1984; Corry, 2007; Walmsley, 2003). Here we cite only a few of these initiatives. The *School Mathematics Project* (1961) and the Nuffield Project for mathematics (1964) were launched in the UK. The book *Mathématique moderne* by Georges and Fréderique Papy appeared in 1963. This ferment led to the rapid increase in the number of mathematics educators and provided a significant impulse to activities that would develop further during the next period (see below).

4. The Renaissance in the late 1960s and consolidation (Presidents: Hans Freudenthal, Michael James Lighthill, Shokichi Iyanaga, Hassler Whitney)

Changes in the needs of mathematics education went hand in hand with changes in society. Freudenthal, president in the years 1967-1970, realized that the earlier trends of ICMI activities were no longer suitable to meet these changing needs. Once again the relationship with the community of mathematicians was strained, and Freudenthal contrived to act independently from them in launching two important initiatives that revitalized the Commission: the founding of the new journal *Educational Studies in Mathematics* (1968), explicitly devoted to mathematics education, and the establishment in 1969 of the tradition of a periodic International Congress on Mathematical Education (ICME). Mathematics education was growing as an autonomous discipline, supported by important international and national initiatives that make evident the ferment of those years. Two new journals soon appeared: in 1969 the German *Zentralblatt für Didaktik der Mathematik* (now *ZDM – The International Journal on Mathematics Education*), and in 1970 the USA *Journal for Research in Mathematics Education*. In 1968 the *Zentrum für Didaktik der Mathematik* (Center for the didactics of mathematics) was founded in Karlsruhe by Hans George Steiner and Heinz Kunle, followed in 1973 by the IDM (Institut für Didaktik der Mathematik) founded in Bielefeld by Steiner with Michael Otte and Heinrich Bauersfeld. In 1969 the first IREM (Institut de Recherche sur l'Enseignement des Mathématiques) was established in Paris. In 1967 the Nordic

[8]See Jeremy Kilpatrick's cameo in (Furinghetti, Giacardi, 2008).
[9]See Giorgio T. Bagni's cameo in (Furinghetti, Giacardi, 2008).

Committee for the Modernisation of School Mathematics (Denmark, Finland, Norway and Sweden) presented a new syllabus inspired by New Math. Among the best-known members of this Committee was Bent Christiansen (Denmark). In the early 1970s the *Collaborative Group for Research in Mathematics Education* was established at the University of Southampton Centre for Mathematics Education: Geoffrey Howson and Bryan Thwaites were among its collaborators. In 1971 Freudenthal himself founded the Institut Ontwikkeling Wiskunde Onderwijs (IOWO, Institute for the Development of Mathematics Teaching). It is remarkable that members of ICMI executive committee are among the supporters of the various initiatives. During this period an important event for the history of mathematics education was the establishing of the first Study Groups affiliated with the ICMI during the third ICME in Karlsruhe (1976): HPM (*The International Study Group on the relations between the History and Pedagogy of Mathematics*) and PME (*International Group for the Psychology of Mathematics Education*).[10] Meanwhile the ICMI continued to organize or support international conferences, in particular in developing countries.

5. Gaining autonomy from IMU and new trends in ICMI action (Presidents: Jean-Pierre Kahane, Miguel de Guzmán, Hyman Bass, Michèle Artigue)

In the most recent decades an important change in the relationship between mathematicians and mathematics educator has taken place. Many activities such as conferences and working sessions were organized, and publications were edited by the Affiliated Study Groups of ICMI.[11] In 1984 the ICMI Studies[12] were launched under the presidency of Kahane, with Geoffrey Howson as a secretary. The former tradition of international inquiries was resurrected with new paradigms: the Studies are launched through a Discussion Document published in *L'Enseignement Mathématique* and in other journals; researchers submit their contributions on the theme of the Study; on the basis of the contributions received the Program Committee delivers the invitations to the ICMI Study meeting; at the end, a book (the ICMI Study volume) is published to disseminate the results. Successive presidents acted to promote further action in favor of developing countries and to strengthen independence from IMU. After the ICM-2006 held in Madrid, important changes were achieved. According to the Terms of Reference of 2007, the Executive Committee of ICMI is elected by the General Assembly of ICMI itself. Thanks to the changes in the Terms of References, which occurred for the first time in 2007, during ICME-11 (2008) the Executive Committee of ICMI for the period 2010-2012 was elected by the General Assembly of ICMI.[13]

The data and ample documentation on the website make it possible to go deeper into the story that we have delineated briefly here, and learn more about the protagonists, events, and official and non-official publications. The website contains the following sections: *Timeline; Portrait Gallery; Documents; The Affiliated Study Groups; The*

[10]For the history of the Study Groups, see the section *The Affiliated Study Groups* in (Furinghetti, Giacardi, 2008).

[11]New groups were created: in 1987 IOWME (*The International Organization of Women and Mathematics Education*), in 1994 WFNMC (The World Federation of National Mathematics Competitions), in 2003 ICTMA (The International Study Group for Mathematical Modelling and Applications).

[12]See the section *The ICMI Studies and Study Volumes: The past Studies - Studies in progress* in (Furinghetti, Giacardi, 2008).

[13]See http://www.mathunion.org/icmi/about-icmi/icmi-as-an-organisation/terms-of-reference/

International Congresses on Mathematical Education; Interviews and Film Clips.

• The *Timeline* is organized on two levels divided in nine periods, with the first level showing the most significant events, and the second adding further details. The aim of this section is to identify the most important moments in the history of ICMI (people, congresses, interactions with other entities, etc.). Each fact is amply documented, with references to the original sources, in particular to *L'Enseignement Mathématique* with links to its website, to the official publications of the Commission, to the *Internationale Mathematische Nachrichten*, to the ICMI *Bulletins*, and to all other documentation that was deemed of interest. Many images, photos and quotations by the protagonists have been inserted. It ends in 1976, when the first Affiliated Study Groups of ICMI were created and activities on mathematics education developed in many different directions. The ICMI *Bulletins*, started in 1972 and available on the website, provide the most important information and news is continually updated on the ICMI website.

• The *Portrait Gallery* contains the complete list of ICMI officers,[14] and cameos (53) of the officers who passed away during the period 1908-2008, of those awarded honorary membership during the International Congress of Mathematicians in Oslo (1936), and of other figures who occupy an important place in the ICMI history, such as Charles-Ange Laisant, one of the founders of the journal *L'Enseignement Mathématique*, the official publishing organ of ICMI since 1908. Precise criteria were used in compiling the biographies so as to respect the nature and aims of the website. The goal was to make evident each person's role within the ICMI, the contributions to research on the problems of teaching, and the publications expressly dedicated to education.

As far as possible the authors of the contributions to this section were chosen from among colleagues from the country of the officer concerned.

• At present, the section dedicated to *Documents* contains: the publications of the Central Committee, with links to digitalized versions in pdf format; the texts of the questionnaires used during inquiries and the relative reports; the list of the ICMI Studies and of the relative volumes; the ICMI *Bulletins*, with links to the digitalized versions; the successive Terms of Reference of ICMI and the list of the documents held in the ICMI Archives.[15]

• The section dedicated to *Affiliated Study Groups* (HPM, ICTMA, IOWME, PME, and WFNMC) presents their histories beginning with their creation, in some cases supplemented by an ample photo gallery.

• The section *The International Congresses on Mathematical Education (ICME)* lists the ten congresses that have taken place up to 2008 and offers general information about each of them, with bibliographical references for the *Proceedings* and their contents, and the *Resolutions of the Congress.*

• The section *Interviews and Film Clips* is dedicated to the testimony of some of the protagonists of the history of the ICMI – Emma Castelnuovo, Trevor Fletcher, Geoffrey

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[14]From 1908 to WW2 ICMI was ruled by a Central Committee consisting of the president, one or more vice-presidents, a secretary-general, co-opted members or members-at-large. When ICMI was reconstructed in 1952 it was ruled by an Executive Committee consisting of the president, two vice-presidents, a secretary (later secretary-general), members-at-large, and ex officio members. In different periods the ex officio members were past presidents of ICMI, presidents of IMU, secretaries of IMU, representatives of IMU in CTS (Committee on the Teaching of Science) /ICSU (International Council of Scientific Unions).

[15]The documents referring to ICMI are in the folders 14 A-G of the IMU files stored at the central Archives of the University of Helsinki. Here we refer to them as "ICMI Archives".

Howson, Maurice Glaymann, Jean-Pierre Kahane, Heinz Kunle, André Revuz, Bryan Thwaites – who describe, through their own experiences and the people they knew, little known aspects of this history.

3 Numbers regarding internationalization, communication, and solidarity in ICMI history

We have seen that the idea of a Commission studying problems of mathematical instruction in different countries germinated in the pages of the journal *L'Enseignement Mathématiqu*e. The vision and mission of this journal – internationalism and the related ideas of communication and solidarity – were inherited by the Commission. To follow how internationalism was realized in the various periods we consider the network of the nations involved in the enterprise during ICMI's first century.

At the beginning the Commission was made up of delegates from countries which had participated in at least two International Congresses of Mathematicians with an average of at least two members. They were: Germany, Austria, Belgium, Denmark, Spain, France, Greece, Holland, Hungary, Italy, Japan,[16] Norway, Portugal, Romania, Russia, Sweden, Switzerland, the British Isles, and USA (19 countries).[17] Each country had either one or three delegates.[18] These countries were joined by a number of "associated countries", whose delegates were permitted to follow the activities of the Commission, without having the right to vote: Argentina, Australia, Brazil, Bulgaria, Canada, Chile, China, the Cape Colonies, Egypt, The Indian Raj, Mexico, Peru, Serbia and Turkey.

In 1952, when ICMI was transformed into a permanent sub-commission of IMU, the new Terms of Reference, adopted by the General Assembly of IMU in 1954 in The Hague, established that ICMI consisted of 10 members-at-large and two national delegates named by each National Adhering Organization of IMU. In 1955, of the 29 countries which were part of IMU[19] only 15 (to which 6 would be added soon) designated their two delegates.[20] The Indian Ram Behari was elected member of the Executive Committee of ICMI: he was the first officer from outside Europe and North America.

ICMI's mission of internationalization, communication and solidarity was strengthened in the following years in synergy with international organizations such as UNESCO and OEEC (now OECD); conferences and other activities were organized outside Europe. In the meeting of ICMI in Paris (14-15 February 1964) ICMI, in agreement with the President of the IMU, decided to acknowledge the actual status of national sub-commission to national Commissions representing countries which were not members of IMU.[21] This decision was immediately put into effect in the case of Luxemburg and successively in that of Senegal, making ICMI even more international.

As of 2010[22] there are 85 member countries of ICMI, 68 of which are also members of

[16]In (*EM*, 1908, 10, Rapport préliminaire) Japan was listed by mistake among the associated members. In (*EM*, 1911, 13, Circulaire n. 4) this country is declared to have the right to full membership.

[17] Here and elsewhere we report the names (translated into English) of the countries as they appear in *L'Enseignement Mathématiqu*e. Changes in names, territories, and status of the countries occurred during the century in question.

[18]See (Giacardi, 2008; *EM*, 1908, 10, 445-458; *EM*, 1909, 11, 193-204).

[19]The member nations of IMU are listed in the Appendix 1 of (Lehto, 1998).

[20]See (Giacardi, 2008; *EM*, II s., 1955, 1, 195-198, *EM* II s., 1955, 1, 202).

[21]See (Giacardi, 2008; *EM*, II s., 1966, 12, 134).

[22]See *ICMI members* in http://www.mathunion.org/icmi/about-icmi/members/.

IMU, and 4 of which are associate members of IMU, a rather small number compared to the 192 member countries of United Nations (see Table 1).

Table 1. List of the 85 member countries of ICMI. (*) indicates the 13 members of ICMI that are not members of IMU; (am) indicates associated members of IMU.

Argentina	Egypt	Republic of Korea	Saudi Arabia
Armenia	Estonia	Kuwait (*)	Senegal (*)
Australia	Finland	Kyrgyzstan (am)	Serbia
Austria	France	Latvia	Singapore
Bangladesh (*)	Georgia	Lithuania	Slovakia
Belgium,	Germany	Luxembourg (*)	Slovenia
Bosnia and Herzegovina	Ghana (*)	Malawi (*)	South Africa
Botswana (*)	Greece	Malaysia (*)	Spain
Brazil	Hong Kong	Mexico	Swaziland (*)
Brunei Darussalam (*)	Hungary	Mozambique (*)	Sweden
Bulgaria	Iceland	Netherlands	Switzerland
Cameroon	India	New Zealand	Thailand (am)
Canada	Indonesia	Nigeria	Tunisia
Chile	Iran	Norway	Turkey
China	Ireland	Pakistan	Ukraine
Colombia	Israel	Peru	United Kingdom
Costa Rica (*)	Italy	Philippines	United States of America
Croatia	Ivory Coast	Poland	Uruguay
Cuba	Japan	Portugal	Venezuela
Czech Republic	Kazakhstan	Romania	Vietnam
Denmark	Kenya (am)	Russia	Zambia (*)
Ecuador (am)			

The ICMI website provides the following information regarding the organization of ICMI:

> Each state, whether an IMU country or not, is invited to appoint a Representative to ICMI, who acts as a liaison between ICMI and the mathematics education community in the country. Moreover every four years, the Representatives elect the ICMI Executive Committee during the ICMI General Assembly. In 16 countries (Australia, Belgium, Chile, Denmark, France, Germany, Japan, Korea, Mexico, New Zealand, Portugal, South Africa, Spain, Sweden, UK, USA) Sub-Commissions of ICMI have been established with two purposes. The first is to provide an organized local forum for dealing with issues of mathematics education and for exchange of information within the country. The second purpose is to offer an interface between the country and the international mathematics education community as represented by ICMI. The Sub-Commission includes among its members the Representative to ICMI, who is often the chairperson.

In recent decades the path towards true internationalization, communication and solidarity was marked by a particular attention to developing countries and with solidarity projects (see Hodgson, 2009 and Jaime Carvalho's cameo of De Guzmán in Furinghetti, Giacardi, 2008). Thus, formally the mission and vision of ICMI has been realized. However, an examination of the data provided on the website reveals a slightly different situation.

From 1908 to 2008 there were 107 ICMI officers[23] coming from 33 countries,[24] 24 of which are European, as can be seen in Table 2. Before WW2 only Europe, USA and Canada had officers. In 1955 an officer from Asia was appointed (the Indian Ram Behari);

[23]Some officers served for more than one mandate.

[24]The country attributed to the officers is that where they were mainly working when serving as ICMI officers.

in 1979 one from South America was appointed (the Brazilian Ubiratan D'Ambrosio); in 1979 one from Australia was appointed (Bernhard H. Neumann). Africa had its first officer in 2003 (Jill Adler). The movement towards internationalism was gradual. Further information concerning this is provided in the list of the presidents and the secretaries in Table 3.

Table 2. Number of officers per country

Argentina (1)	Finland (2)	Netherlands (3)	South Africa (1)
Australia (4)	France (12)	New Zealand (1)	Spain (2)
Austria (1)	Germany (5)	Norway (2)	Sweden (2)
Brazil (2)	Hungary (2)	Philippines (1)	Switzerland (6)
Bulgaria (1)	India (2)	Poland (2)	UK (8)
Canada (4)	Italy (4)	Portugal (1)	USA (14)
China and Hong Kong (3)	Japan (5)	Russia (4)	USSR (4)
Colombia (2)	Mexico (1)	Singapore (1)	Yugoslavia (1)
Denmark (3)			

Table 3. List of the presidents and the secretaries of ICMI

Years	*Presidents*	*Country*	*Secretary*	*Country*
1908-12	Felix Klein	Germany	Henri Fehr	Switzerland
1912-20	Felix Klein	Germany	Henri Fehr	Switzerland
1928-32	David E. Smith	USA	Henri Fehr	Switzerland
1932-36	Jacques Hadamard	France	Henri Fehr	Switzerland
1936-	Jacques Hadamard	France	Henri Fehr	Switzerland
1952-54	Albert Châtelet	France	Heinrich Behnke	Germany
1955-58	Heinrich Behnke	Germany	Julien Desforge	France
1959-62	Marshall H. Stone	USA	Gilbert Walusinski	France
1963-66	André Lichnérowicz	France	André Delessert	Switzerland
1967-70	Hans Freudenthal	Netherlands	André Delessert	Switzerland
1971-74	James Lighthill	UK	Edwin A. Maxwell	UK
1975-78	Shokichi Iyanaga	Japan	Yukiyoshi Kawada	Japan
1979-82	Hassler Whitney	USA	Peter Hilton	USA
1983-86	Jean-Pierre Kahane	France	A. Geoffrey Howson	UK
1987-90	Jean-Pierre Kahane	France	A. Geoffrey Howson	UK
1991-94	Miguel de Guzmán	Spain	Mogens Niss	Denmark
1995-98	Miguel de Guzmán	Spain	Mogens Niss	Denmark
1999-02	Hyman Bass	USA	Bernard R. Hodgson	Canada
2003-06	Hyman Bass	USA	Bernard R. Hodgson	Canada
2007-09	Michèle Artigue	France	Bernard R. Hodgson	Canada

The country which has had the largest number of presidents is France with 5 (one of whom had 2 mandates), followed by the USA with 4 (one of whom had 2 mandates). In only three cases were the country of the President and that of the secretary the same (UK for Lighthill and Maxwell, Japan for Iyanaga and Kawada, USA for Whitney and Hilton).

4 People in ICMI

ICMI activities fall into two categories: political (relationships with mathematicians, with governments, equity issues, policy for developing countries, etc.) and educational/instructional (curricula, inquiries, conferences, ICMI studies, teacher education and recruitment). The activities of ICMI are decided on by the Executive Committee (up to WW2 they were decided on by the Central Committee), but the main imprinting of ICMI's activity is generally due to the president or, in some periods, to the synergy and impulse that derive from the duo of president-secretary. The kind of collaboration carried out by this duo varies according to the personality of the two officers involved, the historical moment, and the actions carried out. There are moments in which the secretary's role was limited to that of a mere executor of the president's resolutions.

Up to and throughout 2006 the officers of ICMI were appointed by mathematicians during their International Congresses. Appointments of members of the Central/Executive Committee, especially the positions of president and secretary, have often been influenced by political issues. First, because the officers were appointed by the mathematicians, it was necessary for them to reach a certain agreement. Second, due to the international character of the Commission, a geographical balance of representation was hoped for (though in fact this was not always achieved). Moreover the twentieth century suffered two large blights during the two world wars and the post-war periods that followed them. It is understandable that a certain caution was exercised in choosing the officers in order to ease the situations.

The reasons underlying the choice of Felix Klein as president and Henri Fehr as secretary at the founding of ICMI in Rome are rather obvious. Felix Klein[25] was one of the most prominent mathematicians of his day, and enjoyed an international reputation; moreover, his commitment to education (reforming of curricula and teacher education) was acknowledged worldwide. His organizational and scientific contributions are illustrated in Schubring's cameo in (Furinghetti, Giacardi, 2008) and in (Schubring, 2003; Schubring, 2008), so here we limit ourselves to mentioning that not only did he manage to create a genuine international network, but he also directed the Commission's efforts to study the two topics that corresponded to the two main issues in his reform agenda: the introduction of the concept of function and elements of differential and integral calculus into the upper years of middle school, and the role of mathematics in higher technical instruction.[26] Henri Fehr[27] was one of the founders of *L'Enseignement Mathématique*, the journal that played an important role in the emergence of international communication in the sector of education, and he strongly supported the idea of international studies on curricula. He was an untiring organizer of the Commission until his death in 1954.

In the years that followed the Commission's founding, the appointment of presidents and secretaries was inspired by various criteria. With the exceptions of David E. Smith and Michèle Artigue, all presidents have been university professors primarily involved in mathematical rather than in educational research. Nevertheless, they also showed genuine interest in mathematics education and addressed this subject in their writings. The cameos featured on our website, which also focus on the educational contributions of these figures, offer significant insights on this point. In what follows we will provide some notes on the activities of the presidents no longer living in order to show their contributions to ICMI in terms of organization as well as their contributions to educational problems. We will also mention some of the secretaries who actively contributed.

As mentioned before, Smith (1928-1932) was not a professional mathematician, but he nevertheless enjoyed excellent contacts with mathematicians, serving from 1902 until 1920 as an associate editor of the *Bulletin of the American Mathematical Society* and from 1916 on as associate editor of *The American Mathematical Monthly*. His publications were

[25]See Gert Schubring's cameo in (Furinghetti, Giacardi, 2008).
[26]See the report on the ICMI congress in Paris (1-4 April 1914) in (*EM*, 1914, 16, 245-356).
[27]See Gert Schubring's cameo in (Furinghetti, Giacardi, 2008).

decisive in shaping mathematics education in the United States. The books *The Teaching of Elementary Mathematics* (1900), *The Teaching of Arithmetic* (1909), and *The Teaching of Geometry* (1911), concerning methodological and didactical aspects of teaching, are directed to the professional formation of teachers. His textbooks in arithmetic, algebra, and geometry and accompanying handbooks, published since 1904, were dominant during the 1910s.[28]

Jacques Hadamard, president from 1932 until WW2, was one of the top mathematicians of his day. His academic life was full of obligations that clashed with ICMI duties (for example, in 1936 he did not report on the work of the Commission because he was in China). He wrote textbooks, essays and articles about mathematics teaching and made an important indirect contribution to mathematics education through his famous book, *An Essay on the Psychology of Invention in the Mathematical Field* (1945, Princeton NJ: Princeton University Press, with many editions), in which he draw attention to the role of psychology in mathematical activity. During his presidency he benefitted from the experience and passionate involvement of two experienced officers, Walther Lietzmann and Fehr.

Albert Châtelet (1952-1954) contributed to the study of topics such as the laboratories of mathematics and the use of concrete materials. He collaborated with Jean Piaget on a well known book on introducing children to arithmetic.[29] As *Directeur de l'Enseignement du second degré* he worked for the modernization of the educational system in France. Moreover he was actively involved in French politics (see Condette, 2009). Probably for this reason during his mandate the Commission's activities were largely overseen by the secretary Behnke.[30]

Marshall Harvey Stone[31] was ICMI president from 1959 to 1962, during the period of the heated discussion about Modern Mathematics, when new curricula were launched in the US and Europe. Stone chaired the milestone meeting in Royaumont (23 November - 4 December 1959). On that occasion he formulated a veritable "program of research in the teaching of mathematics" (study and experimentation), expressing his hopes for the creation of ad hoc institutes for research and the insertion of research projects regarding the teaching of mathematics into universities. He pointed out that teaching must address problems concerning mass education, meet the needs of applications, adjust to society's increasingly urgent demand for the services of scientists, and devise new methodologies. He also stressed the exigency of not widening the gap between school mathematics and university mathematics.[32] During his mandate ICMI co-sponsored important international meetings[33] with the collaboration of OEEC and UNESCO. He was flanked by secretary

[28]See Gert Schubring's cameo in (Furinghetti, Giacardi, 2008).

[29]See Piaget, J., Boscher, B., Châtelet, A., 1949, *Initiation au calcul. Enfants de 4 a 7 ans*, Paris: Bourrelier.

[30]See Gert Schubring's cameo in (Furinghetti, Giacardi, 2008). The ICMI Archives (14 A, 1952-1954) show that Châtelet did not play a very significant role during his mandate, and this is confirmed by the documentation held in the French archives examined by J.-F. Condette, who we thank for this information.

[31]See Jeremy Kilpatrick's cameo in (Furinghetti, Giacardi, 2008).

[32]See *New Thinking in School Mathematics*, OEEC, 1961, pp. 28-29.

[33]We mention for example: *Symposium on The teaching of Geometry in Secondary School* (30 May - 2 June 1960) in Aarhus (Denmark); *Symposium on The Co-ordination of the Teaching of Mathematics and Physics* (19-24 September 1960) in Belgrade (Yugoslavia); *Seminar on the teaching of analysis and relative manuals* (26-29 June 1961) in Lausanne; Seminar on A discussion of the Aarhus and Dubrovnik reports on the teaching of geometry at the secondary level (4-8 October 1961) in Bologna (Italy); *Inter American*

Gilbert Walusinski,[34] president of the *Association des Professeurs de Mathématiques de l'Enseignement Public* (APMEP) from 1955 to 1958, and very involved in curriculum innovation and in teacher training in France.

Other presidents were passionate university teachers and lent their enthusiasm and experience in support of educational projects for school and teacher associations. This is the case of the British Michael James Lighthill[35] (1971-1974), applied mathematician, who was involved in an advisory capacity in the creation of the School Mathematics Project (SMP) in 1961, aimed at secondary school students 11 years and older. In 1970 he was elected president of the British Mathematical Association. Lighthill's contributions to mathematics education were mainly made in the 1960s and 1970s. He served on several important committees in England. In addition to being president of ICMI at the time of the second International Congress on Mathematics Education (Exeter, England, 1972), he was also chairman of its Organizing Committee. During his term of office a new policy of holding Regional Symposia "to facilitate wider discussion of mathematical education outside those areas of Europe and America where international meetings on the subject have mainly been held hitherto"[36] was adopted, and numerous symposia were held with the co-sponsorship of the ICMI[37] One example is the symposium held jointly with UNESCO, in Nairobi, Kenya (1-11 September 1974) on "Interactions between linguistics and mathematical education", which was of particularly interest for African countries. He also had the opportunity to collaborate with a secretary who was an exceptional teacher and a fine author of works aimed at popularizing mathematics, the British Edwin Maxwell.[38] Maxwell was an active supporter of the SMP, for which he wrote the book *Geometry by transformations* (1975); he too was president of the Mathematical Association (1960) and was editor of the *Mathematical Gazette* (1963-1971). His book *Fallacies in mathematics* (1959) is quite well known.

The Japanese Shokichi Iyanaga,[39] president in 1975-1978, wrote many mathematical textbooks in Japanese for primary and secondary schools, even before his presidency. During those same years the ICMI secretary was the Japanese Yukiyoshi Kawada,[40] who initiated the Southeast Asia Conference on Mathematical Education (SEACME) series in 1978 with the inaugural conference in Manila. This conference was very important for the involvement of the Eastern countries in the international movement of math education (see Lim-Teo, Suat Khoh, 2008).

Hassler Whitney[41] (1979-1982) had developed an interest in mathematics education, which occupied the last two decades of his life, even before his official retirement in 1977. His main interest was primary education. As Kilpatrick underlines in his cameo, Whitney was opposed to formal instruction in arithmetic in the early grades, and he criticized the habit of mathematics teachers of focusing on passing tests rather than what he called "meaningful goals." He was particularly disturbed by national reports that called for more

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Conference on Mathematical Education. (4-9 December 1961) in Bogotá (Colombia). See (Giacardi, 2008, *Timeline* 1960-1966).

[34]See Eric Barbazo's cameo in (Furinghetti, Giacardi, 2008).

[35]See Adrian Rice's cameo in (Furinghetti, Giacardi, 2008).

[36]*EM*, 1975, s. 2, 21, 330.

[37]*EM*, 1973, s. 2, 19, 171, and (Giacardi, 2008, *Timeline* 1972-1976).

[38]See Adrian Rice's cameo in (Furinghetti, Giacardi, 2008).

[39]See Shigeru Iitaka's cameo in (Furinghetti, Giacardi, 2008).

[40]See Shigeru Iitaka's cameo in (Furinghetti, Giacardi, 2008).

[41]See Jeremy Kilpatrick's cameo in (Furinghetti, Giacardi, 2008).

mathematics to be taught earlier in school:

> The most pressing need I see is for us to face fully the consequences of interventions we make, and hold up on those with bad results. I speak, of course, of mandating more work in mathematics for failing students, raising standards for these without helping them toward meeting the standards, and starting mathematics teaching at an earlier age. It is unthinkable to market drugs without a thorough study of all effects; in education I see no parallel concern, though there should be (Whitney, 1985, p. 233).

The commitment to education of Miguel de Guzmán[42] (1991-1998), who presided over ICMI for two mandates, began very early and permeated his work. Convinced that "teaching in any form is very attractive" but also that the "nature of the mathematical task makes it capable of stimulating important ethical aspects", he worked to involve other mathematicians in mathematics education problems, while he himself contributed to mathematics education writing several "popular" books on mathematics. According to de Guzmán, mathematics teaching should pay particular attention to problem solving, with an emphasis on the thought processes, to the exploration of applications, games, etc., to the impact of calculators and computers, and to the history of mathematics. One of his major contributions to ICMI was the Solidarity Program, aimed at supporting the improvement of mathematics education in developing countries.

It is worth noting that some of the presidents and secretaries we have mentioned were involved in the activity of national associations of teachers (notably Behnke, Lichnérowicz, Lighthill, Maxwell and Walusinski). Some were active in the discussion for curriculum innovation in their countries (notably Behnke, Lichnérowicz, Maxwell, Smith and Stone). Some presidents and secretaries brought into ICMI the ideas and ferments developed in other environments: this is true, in particular, of Freudenthal and Lichnérowicz, who were among the founders of CIEAEM, the commission that contributed to the creation of new approaches to problems of mathematics education (see Furinghetti et al., 2008).

Particularly worthy of note are two presidents whose activities in ICMI played a significant role in changing the status and policy of the Commission: Behnke (1955-1958) and Freudenthal (1967-1970). Both relegated their ICMI secretaries to a marginal role. Under Behnke the secretary was the French Julien Desforges,[43] who was quite involved in mathematics education, first as president of APMEP in 1931-1932 and 1934-1937, and then as a member of *Conseil supérieur de l'instruction publique*, to which he was elected in 1936. Under Freudenthal the secretary was the Swiss André Delessert. As the president of IMU Henri Cartan complained in a letter to Lighthill dated August 20, 1970 (IA, 14B 1967-1974), he was reduced to a "mailbox".

Behnke and Freudenthal were active during crucial moments in the history of ICMI: the former during the reconstruction that followed WW2, the latter during the period of major social changes that also affected the school and academic worlds. The key issues in their work were the relationship with the community of mathematicians and the search for

[42] See Jaime Carvalho e Silva's cameo in (Furinghetti, Giacardi, 2008).
[43] See Eric Barbazo's cameo in (Furinghetti, Giacardi, 2008).

autonomy from it.[44] Behnke was the first secretary of the renewed Commission after WW2, then president of the ICMI from 1955 to 1958, vice-president from 1959 to 1962, and a member of the Executive Committee from 1963 to 1970. The beginnings of the new Commission were not easy, and relations with the IMU were characterized by constant friction[45] derived from the lack of precise Terms of Reference for governing the activities of the Commission. As ICMI secretary under Chatêlet, Behnke was very active and succeeded, in spite of several difficulties, in organizing the intervention of ICMI at the International Congress of Mathematicians in Amsterdam in 1954. The General Assembly of the IMU in The Hague (31 August - 1 September 1954) introduced the Terms of Reference for governing the activities of the Commission and the relationships with IMU: according to these, ICMI had a relatively free hand in its internal organization, but IMU retained control over important points: the President and the ten members-at-large of ICMI would be elected by the General Assembly of IMU based on nominations by the Union's president. Moreover, the national delegates would be named by each National Adhering Organization of IMU. The Executive Committee of ICMI was renewed, with Behnke nominated president. As his correspondence kept in the ICMI Archives shows, he was completely aware of the problems he had to face in revitalizing the Commission: the difficulty of finding mathematicians active in research who were interested in teaching; the difficulty of being recognized in the world of mathematical research, and thus how important it was that the work of the Commission be visible at the international congresses; the difficulty of obtaining funding; the need to improve relations with IMU, and finally, the relevance of the collaboration of mathematics teachers at all levels.[46]

Freudenthal (1967-1970) continued this work in an even more radical way. He sought and obtained funding beyond that provided by the IMU, carried on the collaboration with UNESCO, already well established by his predecessors, and undertook the two initiatives that were mentioned in section 2: in May 1968 the first issue of the new journal *Educational Studies in Mathematics* appeared, and in August 1969 the First International Congress on Mathematical Education was held in Lyon, France. The IMU was faced with decisions already made. Freudenthal grasped the spirit of the times and the new needs of society, and suggested new topics for discussion that included motivation, comparative evaluation of the contents of mathematics courses, criteria of success, research methodology, evaluation of the results of research in mathematics education, international cooperation, the permanent training of teachers, and the place of "the theory of mathematical education" in universities or research institutes.[47] He also fostered the recognition of mathematics education as a separate academic discipline.

The change promoted by Freudenthal was radical, and after his mandate things were no longer the same: on the one hand, mathematics education was becoming an autonomous discipline with its own congresses and journals; on the other, ICMI was performing a

[44]See (Arzarello et al, to appear; Furinghetti, 2008a). The action of these two presidents in relation with the community of mathematicians is analysed with reference to the documents kept in ICMI archives in (Furinghetti, Giacardi, 2010).

[45]See for example Stone to Châtelet, Chicago, November 3, 1952, in *IA*, 14A, 1952-1954; Hodge to Stone, May 31,1954, *IMU Archives*, quoted in (Lehto, 1998, p. 111); Stone to Châtelet, Chicago, July 29, 1954, and Behnke to Stone, Oberwolfach, August 11, 1954, in *IA*, 14A, 1952-1954; see (Giacardi, 2008, *Timeline*, 1937-1954).

[46]*Report of the president of the International Commission of Mathematical Instruction to the president of the International Mathematical Union*, April 20, 1955, in *IA*, 14A, 1955-1957.

[47]See *EM*, 1967, s. 2, 13, 245-246, and (Giacardi 2008, *Timeline* 1967-1971).

more institutional role, building on what he had begun and extending his influence beyond Europe and the US. Papers specifically dealing with education were appearing in the new journals founded from 1968 on. The research in mathematics education was strengthened by the creation in 1976 of the first affiliated Study Groups, those of History and Pedagogy of Mathematics (HPM) and Psychology of Mathematics Education (PME). Their founding was fostered by the opportunities to compare and contrast different points of view provided by ICMEs. ICME conferences made it possible to realize the concept of internationalization in new, more efficient ways.

Let us conclude by briefly mentioning the role of women in ICMI. The first woman included in the Executive Committee was the Canadian Anna Sierpinska, appointed a member-at-large in 1994. Up to 2008, of the 107 officers, only nine are women. At ICM-2006 the French Michèle Artigue became the first woman appointed as president of ICMI. In the trend to include women ICMI was more advanced than its parent body IMU. It was only during the ICM-2002 in Beijing, China, that a woman was first appointed as a members of IMU Executive Committee: the Norwegian Ragni Piene (the daughter of the ICMI officer Kay Waldemar Kielland Piene) (see Li, 2002).

However, even though they were not part of the Central/Executive Committee, several other women also contributed to the development of ICMI and to the discipline of mathematics education in different ways. This is not surprising because since the beginning of the twentieth century women joined the profession of teaching, and some of them delved quite deeply into various aspects of mathematics education (see Furinghetti, 2008b). As early as 1928, a secondary teacher, Maria Giovanna Sittignani, presented a paper specifically addressing didactics in the section devoted to mathematics teaching at the 1928 International Congress of Mathematicians held in Bologna. It is also worth noting that Freudenthal included in the editorial board of *Educational Studies in Mathematics* three women who played remarkable roles in the development of mathematics education after WW2: Emma Castelnuovo, Anna Zofia Krigowska and Lucienne Félix. Papers by Castelnuovo, Krygowska, Galina G. Maslova, and Frédérique Papy are included in the proceedings of the first ICME. The number of women participating as researchers or teachers in ICMI activities continued to grow with the passing of time.

We would also like to mention some of the women who worked behind the scenes, but who nevertheless played an important role. A first interesting female presence is that of Tatiana Ehrenfest-Afanassjewa, a Ukrainian mathematician who, after moving to Leiden, became a leading figure in the development of mathematics education in the Netherlands. As a chair of a discussion group she invited Freudenthal, who had not yet published didactical papers, to deliver a talk as a mathematician. Freudenthal later became a regular participant of the group, and in 1950 he became its chairman. As Smid (2009, p. 218) notes “There can be no doubt that the monthly meetings of this group helped him to form, shape and develop his, at that time still vague, ideas on math teaching” (see also la Bastide-van Gemert, 2006, pp. 126-138).

It must be acknowledged that the large amount of business conducted by Behnke and Freudenthal during their presidencies was made possible by the exceptional involvement of their personal secretaries. Behnke’s secretary, Renate Wohlert, was fluent in several languages; she not only translated his letters into English and French, but sometimes responded in his stead and accompanied him on his trips abroad.[48] Freudenthal's secretary, D. Breughel-Vollgraff, assisted him with his work on a daily basis, thus making it possible for him to dedicate himself completely to scientific research and related activities. During his presidency she also fulfilled a large part of the duties of the secretary of ICMI,[49] and Freudenthal acknowledged this fact on many occasions. In the introduction to his book *Didactical phenomenology of mathematical structures* (1983, Dordrecht / Boston / Lancaster: D. Reidel), we read: "Let me add that my secretary and collaborator for almost 25 years, Mrs. Breughel read and wrote the last line of the illegible Dutch manuscript of this book the day before she retired" (p. ix). The following passage in a letter to Howson epitomizes Freudenthal's appreciation of this woman:

> If you wonder how anybody could travel, lecture, edit, publish so much at a time, my explanation is that for 25 years I had a secretary, Mrs. Breughel, who was unsurpassable. If I die early enough to get an obituary, her name should not be forgotten. [50]

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[48]See, for example, Behnke (signed by R. Wolhert) to E. Bompiani, 15 August 1955; R. Wohlert to E. Bompiani, 24 September 1955, in IA 1955-1957; see also all the letters with the initials Be/Wo in the list in (Giacardi, 2009, *Documents, ICMI Archives*).
[49]See A. Delessert to O. Frostmann, 22 March 1959, IA 1967-1980.
[50]Freudenthal to Howson, 19 July 1983; RANH, Hans Freudenthal Papers, inv. nr. 38, in (Bastide-van Gemert, 2006, p. 63), We thank Jan van Maanen for this information.

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17th and 18th century European arithmetic in an 18th century Icelandic manuscript

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ABSTRACT

Icelandic arithmetic books from the 18th century, printed and in manuscripts, adhered to the European structure of practical arithmetic textbooks, formed in the late Middle Ages: The number concept, numeration, the four operations in whole numbers and fractions, monetary and measuring units, extraction of roots, ratio, progressions and proportions. A manuscript textbook, *Arithmetica – That is reckoning art*, dated in 1721, deviates from the general model in that it does not treat monetary and measuring units but includes theoretical sections on the number concept and common notions on arithmetic. No specific model for the manuscript has been spotted, while similarities to works by authors as Ramus, Stevin, Suevus, Meichsner and Euler were found.

1 Introduction

According to Swetz (1992), basic types of texts on European arithmetics up to the fifteenth century were *theoretical tracts*, transmitting neo-Pythagorean speculations of Nicomachus of Gerasa (ca. 100); *computi*, ancestors of almanacs; *abacus arithmetic* for Roman numerals and *algorisms*, evolving from descendant translations of works of Al-Kwarizmi (ca. 825), such as *Carmen de Algorismo* (c. 1200) by Alexander Villa Dei. *Abacus arithmetic* and *algorisms* that discussed problems related to trade and commerce were called *Practica*, reflecting their practical use. Their influence prevailed in the teaching of arithmetic until the beginning of the 20th century.

Commercial arithmetics were not intended to be *theoretical tracts* devoted to philosophical speculations on the nature of number, rather; they were handbooks on readily usable mathematics. Generally, their content was numeration, monetary and measuring units, the four operations in whole numbers and fractions, i.e. addition, subtraction, multiplication, division, and often extraction of roots, progressions and proportions in the form of *Regula Trium*, the ‘Rule of Three’.

There were two cathedral schools in Iceland until 1800. Their first regulations, issued in 1743, prescribed knowledge in the four arithmetic operations in whole numbers and fractions (Jónsson, 1893). The first substantial printed arithmetic textbooks in Icelandic were published in the 1780s, while textbooks in manuscripts, written in the vernacular, were dispersed from person to person in early modern times. These books generally belonged to the *Practica*.

The existence of the manuscripts shows that Icelanders made efforts to adapt European education and literature to their language as had been customary since the Middle Ages. Before the Lutheran Reformation, in 1550, translations had been made from Latin. Later, texts from the Northern-European protestant countries, written in e.g. German or Danish, seem to have been preferred for translation.

The following manuscripts of arithmetic textbooks from the seventeenth century onwards are preserved in the National and University Library of Iceland – manuscript department:

- *Gandreið*, ÍB. 35 fol. (1660, copy 1770-80).
- *Arithmetica Islandica,* Lbs. 1694, 4to (1716/1733?)
- *Arithmetica – Það er reikningslist* ÍB 217, 4to (1721, copy c. 1750)
- *Limen Arithmeticum* Lbs. 1318, 8vo (1736, copy 1777), by Stefán Einarsson.
- *Stutt undirvísan um Arithmeticam Vulgarem eða almenniliga reikningslist.* Lbs. 409, 8vo (copy 1770-1780).

The question arises if these textbooks were an Icelandic creation or adapted translations, and in the latter case: which European textbooks were they modelled after? In this paper we shall mainly concentrate on *Arithmetica – Það er reikningslist / That is Reckoning Art*, and only mention *Arithmetica Islandica* briefly.

Likely models for *Arithmetica – That is Reckoning Art* were written in German language, while other influences have been spotted. *Arithmetica* even contains the same examples as Euler's (1738) *Einleitung zur Rechenkunst*. It would be natural to think that Euler's textbook was the model, if the estimated date of the original for the manuscript was not 1721, so they seem to have an ancestor in common. Other possible models will be explored.

2 Arithmetic Textbooks in Iceland in Early Modern Times

Several well-known European textbooks have been recorded in inventories of churches and estates at deaths:

- Gemma Frisius (Antwerpen, 1540). *Arithmeticae practicae methodus facilis*. (Latin).
- Ramus, Petrus (Basel, 1569). *Arithmeticae libri Duo*. (Latin).
- Frommius, Geo (Copenhagen, 1649). *Arithmetica Danica*. (Latin).
- Matthisen, Søren (Copenhagen, 1680). *Compendium Arithmeticum eller Vejviser* (Danish).

Other books are mentioned in forewords to printed books or may have exerted influence on Icelandic textbooks:

- Stevin, Simon (Leiden, 1585). *L'Arithmetique* (French).
- Suevus, Sigismund (Breslau, 1593). *Arithmetica Historica. Die löbliche Rechenkunst* (German).
- Meichsner, G. (Rothenburg ob der Tauber, 1625a). *Arithmetica Historica. Das ist: Rechenkunst* (German).
- Cocker, Edward (London, 1679). *Cocker's Arithmetick* (English).
- Hatton, Edward (London, 1695). *Tradesman's Treasury* (English).
- Hatton, Edward (London, 1721). *An Intire System of Arithmetic* (English).
- Clausberg, Christlieb (Leipzig, 1732, 1748, 1762). *Der Demonstrativen Rechenkunst oder Wissenschaft gründlich und kurz zu rechnen* (German).
- Cramer, Christian (Kaupmannahöfn, 1735). *Arithmetica Tyronica* (Danish).
- Euler, Leonhard (St. Pétursborg, 1738). *Einleitung zur Rechenkunst zum Gebrauch des Gymnasii* (German, translated into many languages).

3 Arithmetica – That is Reckoning Art

Arithmetica – That is Reckoning Art, estimated to be composed in 1721 by an anonymous author, is so devoid of any reference to Icelandic society that it must be a translation of or be modelled on some foreign source. The date of its extant manuscript has not been examined scholarly but is estimated to be 1750. However, the texts of *Arithmetica* and its possible sources contain examples that reveal their dates of origin, a hypothesis supported by Jan van Maanen (1993), who quotes many examples of the ‘present time’ in Dutch manuscripts and textbooks. None of his examples are identical to ours, while our examples are found in other texts.

Arithmetica – That is Reckoning Art contains chapters on the number concept and the unit, *Common Notions* on arithmetic, numeration, the four arithmetic operations in whole numbers and fractions, arithmetic and geometric progressions, extracting a square root and proportions in the form of the *Regula Trium*.

Our *Arithmetica* has thus some of the characteristics of the *Practica*, while a section on monetary and measuring units is missing. Instead, Neo-Pythagorean speculations on the number concept and the unit, usually confined to more theoretical books, are included, as are the *Common Notions* on arithmetic. *Arithmetica* seems to have drawn upon various sources.

Two of our possible models are of religious origins; *Arithmetica Historica*. *Die löbliche Rechenkunst*, by Suevus (1593) in Breslaw, and *Arithmetica Historica*. *Das ist Rechenkunst*, by Meichsner (1625a) in Rothenburg ob der Tauber, both Lutheran protestant towns at that time. Their subtitles, *Rechenkunst*, and the Icelandic subtitle, *Reikningslist*, are identical. These books may have been known to Icelanders through strong Lutheran impact from Germany through Denmark, while no evidence exists that they were possessed in Iceland. A number of their examples are quoted in our *Arithmetica*, but their expository text is different, and they do not, for example, treat fractions, which are thoroughly treated in our *Arithmetica*.

In the following, relations between foreign arithmetic textbooks and our *Arithmetica* manuscript will be explored; ideas on the number concept and *Common Notions*, and thereafter examples of numeration and the various arithmetic operations.

3.1 The Number Concept

Discussion of the number concepts was an integral part of the *theoretical tracts*, reflecting classical Greek mathematical concerns. In this respect *Arithmetica* belongs to that category. Icelanders seem to have been acquainted with *Algorismus* through the centuries, a medieval translation of Alexander de Villa Dei’s *Carmen de Algorismo*, which belonged to the *algorisms*. The translator has added this phrase: ‘One is neither [even nor odd number] as it is not a number but the origin of all numbers.’ (Jónsson, 1892–96: p. 418). Neo-Pythagorean speculations on that one is not a number remained in many Icelandic and North-European textbooks up to the 19^{th} century (Bjarnadóttir, 2007).

The author of *Arithmetica* knew this idea, but also different ones, possibly originating from Ramus’s *Arithmeticae libri Duo* (1569), which was in the possession of Bishop B. Sveinsson (1605–75) (Helgason, 1948). The ancient understanding was further questioned by Stevin (1585) and echoed by Cocker (1677, 1715) and our *Arithmetica*, see examples, listed in Table 1.

Author and title	Original text	Translation
Euclid, 300 BC. *The Elements*,		A **unit** is that by virtue of which each of the things that exist is called one.

edited by Heath, 1956, 2, p. 277.		A **number** is a multitude composed of units.
Muhamed ibn-Musa al-Kwarizmi, 1992. *Kitab al-jam´val tafriq bi hisab al-Hind* (~800), Latin translation: *Dixit Algorizmi* (~1100), p. 1	Et iam patefeci in libro algebr et almucabalah, id est restaurationis et oppositionis, quod uniuersus numerus sit compositus et quod uniuersus numerus componatur super unum. Unum ergo inuenitur in uniuerso numero. Et hoc est quod in alio libro arithmetice dicitur quia **unum est radix uniuersi numeri et est extra numerum :**	And I have already explained in the book on algebra and almucabalah, that is on restoring and comparing, that every number is composite and every number is composed of the unit. The unit is therefore to be found in every number. And this is what is said in another book on arithmetic that **the unit is the origin of all numbers and is outside numbers** :
Jónsson, 1892–96. A*lgorismus*, AM 544, 4to. (1306–08), p. 418.	Jafnir fingur eru fjórir; 2, 4, 6, 8 en ójafnir aðrir fjórir; 3, 5, 7, 9**. En einn er hvorki því að hann er eigi tala heldur upphaf allrar tölu**.	Even digits are four; 2, 4, 6, 8, and uneven another four; 3, 5, 7, 9. **But one is neither as it is not a number but the origin of all numbers**.
Ramus, 1569. *Arithmeticae libri Duo*, p. 1.	Numerus est, secundum quem unum quodque numeratur : ut secundum unitatem unum, secundum binarium duo, secundum ternarium tria; & sic deinceps omnes numeri: Itaque **numerus est unitatis aut multitudinis**: potesque esse minimus, ut unitas: ...	A number is that which each is counted by one, thus a unit by one, a couple by two, a triple by three, and thus in a sequence all numbers. Therefore **a number has the properties of a unit and a multitude**: it can be as small as a unity: ...
Stevin, 1585. *L'Arithmetique*, edited by Struik, 1958. pp. 494–504.	La partie est de mesine matiere qu'est son entier, Vnité est partie de multitude d'vnitez. Ergo l'vnité est de mesine matiere qu'est la multitude d'vnitez. Mais la matiere de multitude d'vnitez est nombre, Doncques la matiere d'vnité est nombre. Et qui le nie, faiĉt comme celui, qui nie qu'vne piece de pain foit du pain. Nous pourrions aussi dire ainci: Si du nombre donné l'on soubtraiĉt nul nombre, le nombre donné demeure. Soit trois le nombre donné, et de mesme soubtrahons vn, qui n'est point nombre comme tu veux. Doncques le nombre donné demeure, c'est à dire qu'il y restera encore trois, ce qui est absurd.	The part is of the same matter as its whole, unity is part of a multitude of unities, hence unity is of the same matter as the multitude of unities, But the matter of a multitude of unities is number. Hence the matter of unity is number, Who denies this behaves like one who denies that a piece of bread is bread. We can also say: If we subtract no number from a given number, then the given number remains. If three is the given number, and if from this we subtract one, which – as you claim – is no number, then the given number remains, that is three remains, which is absurd.
Cocker's ARITHMETICK (1715, first published 1677)		Unit is *Number*; for the part is of the same Matter that is his whole, the Unit is part of the Multitude of Units, therefore the Unit is of the same Matter that is the Multitude of Units; but the matter of the multitude of Units is *Number*, therefore the Matter of Unit is *Number*; for else if from a *Number* given no *Number* be subtracted, the *Number* giveth remaineth; let three be the *Number* given ; from which *Number* subtract or take away one (which as some conceive, is no number) therefore the *Number* given remaineth, that is to say, there remaineth three which is absurd.
Arithmetica –	Margir af þeim Lærdu Mathematicis	Many of the learned mathematicians

Það er reikningslist / That is Reckoning Art. ÍB 217, 4to. 1721/1750. pp. 1–2.	hafa viliad halda að 1. / unitas / væri ej Tala /numerus/ helldur væri Talann Fiølldi af 1. /:ex unitatibus / til samans lagdur vid Evcl: Elementa Lib: 7. Def. 2., hvad um 1 kyni ej seigiast, einasta álited unitatem, sem upphaf og undirrót til allrar Tølu. Adrir þar í mót meina að 1 eigi ad kallast Tal, þvj hann gieti i sier innibundid marga Parta af þessum hann sie samsettur. Þar ad auki ef 1 væri ej Tal skylldi annad Tal / til dæmis 5/ vera eins margt þó 1 væri þar af tekinn, þar þó allir skilja, ad ej þann aftur utan 4.	have maintained that 1 /*unitas*/ was not a number /*numerus*/, rather the number was a multitude of 1 */:ex unitatibus*/ added together, Evcl: Elementa Lib. 7. Def. 2., what could not be said about 1, only supposed to be *unitatem*, as the origin and basis of all number. Others, on the other hand, maintain that 1 is to be counted as a number, as it can contain many parts of which it is composed. In addition, if 1 was not a number, another number / for example 5/ should be as many, even though 1 was removed therefrom, there though everyone understands that not that one again, but 4.
Christlieb von Clausberg, C.v. 1732. *Der Demonstrativen Rechnenkunst*, pp. 14–15.	... alle diese Dinge, bey denen man eben solche Eigenschaften findet, machen gleichfalls eine **Eins** aus, und diese Einheiten zusammen genommen, geben eine **Zahl** ... Hieraus ist klar, dass die Unität oder **Eins** vor sich selbst nur der Nadir oder Wurzel der Zahlen ist, nemlich eine angenommene Grösse, wornach die Zahlen erwachsen und betrachtet werden ... indem sich keine **Zahl** benennen läst, wo nicht eine gewisse Grösse für eine **Eins** angenommen wird.	... all those things, with which one finds such characteristics, are similarly a **one**, and these units taken together, constitute a **number** ... From this it is clear that the unit or **one** for itself is only the nadir or the root of the numbers, that is an accepted magnitude from which the numbers grow and are observed … ; as no **number** may be mentioned, without a certain quantity for a **one** being stated.

Table 1. Elaborations on the number concept in a choice of textbooks.

Clearly, the author of *Arithmetica* is aware of Stevin’s reasoning that one, the unit, is indeed a number.

3.2. Common Notions

The *Common Notions* in Euclid’ *Elements*, referring to magnitudes, were found in theoretical books on arithmetic, while they were seldom seen in the *Practicae*. *Arithmetica* contains thirteen *Common Notions*: ‘Incontestable procedures, that is, so obvious rules that each one’s understanding must acknowledge them (p. 7).’
Three of these axioms match Euclid’s *Common Notions* as presented in Heath’s 1956 publication; while other three match Mersenne’s 1644 publication of the *Elements*, Book Seven, see Table 2, below. Both lists of thirteen *Common Notions* refer to measuring, *metitur*, of *metior* : measure, in, Mersenne’s in particular.

The *Common Notions* as listed in Heath’s edition:

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part. (Euclid, 300 BC, vol. 1, p. 155)

Mersenne: *Universe Geometiae synopsis*. Paris, 1644, pp. 22–23.	*Arithmetica – That is Reckoning Art*. 1721/1750, p. 11.

1. *Quicumque eiusdem, vel æqualium æquemultiplices fuerint & ipsi inter se sunt æquales.*	1. The whole is more than its part (Heath, 5).
2. *Quorum idem numerus æquemultiplex fuerit, vel quorum æquemultiplices fuerint æquales, & ipsi inter se æquales sunt.*	2. The whole equals all its parts.
3. *Quicumque eiusdem numeri, vel æqualium eadem pars, vel eædem partes fuerint, & ipsi inter se sunt æquales.*	3. That which parts all are equal, are equals (Mersenne, 3).
4. *Quorum idem, vel æquales numeri eadem pars, vel eædem partes fuerint, & ipsi inter se sunt æquales.*	4. If equals be added to equals, [they] become between themselves equal (Heath, 2).
5. *Omnis numeri pars est unitas ab eo denominata, binarij enim numeri unitas pars est ab ipso binario denominata, quæ dimidia dicitur, ternarij vero unitas est pars, quæ à ternario denominata tertia dicitur : quaternarij quarta, & ita in alijs.*	5. If equals be subtracted from equals, the remainders are equal (Heath, 3).
6. *Unitas omnem numerum metitur per unitates quæ in ipso sunt.*	6. If equals be multiplied by equals, the wholes are equal (Mersenne, 1).
7. *Omnis numerus seipsum metitur.*	7. Of equals, their halves, thirds, fourths etc. are equal.
8. *Si numerus metiatur numerum, & ille, per quem metitur, eundem metietur per eas, que sunt in metiente, unitates.*	8. The squares of equals are equal.
9. *Quicumque numerus alium metitur, multiplicans eum, vel multiplicatus ab eo, per quem metitur, illum ipsum producit.*	9. If square numbers and cubic numbers are respectively equal, then also their roots are equal.
10. *Si numerus numerum alium multiplicans, aliquem produxerit, multiplicans quidem productum metitur per unitates quæ sunt in multiplicato : multiplicatus verò metitur eumdem per unitates quæ sunt in multiplicante.*	10. Thus also their halves and doubles are equal.
11. *Quicumque numerus metitur duos, vel plures, metietur quoque eum, qui ex illis componitur.*	11. No one can for himself measure all numbers.
12. *Quicumque numerus metitur aliquem, metietur quoque eum, quem ille ipse metitur.*	12. All numbers have their measure of units (Mersenne, 7).
13. *Quicumque numerus metitur totum & ablatum, etiam reliquum metietur.*	13. The unit neither multiplies nor divides.

Table 2: Comparison of *Common Notions*, listed in Mercenne's edition of Euclid's *Elements*, Book Seven, to those in *Arithmetica – That is Reckoning Art*.

Mersenne's edition also contains ten *Axiomata, Communes Notiones decem*, in Book One (p. 3), some of which correspond to the *Common Notions* in Heath's edition and others to no. 2 and 10 in *Arithmetica*.

3.3 Numeration

Arithmetic textbooks explain how to write numbers by place value notation, often demonstrated by large numbers. All the five examples on numeration in *Arithmetica* have corresponding examples in the German textbooks.

Arithmetica – That is Reckoning Art. (1721/1750)	Suevus (1593). *Arithmetica Historica. Die löbliche Rechenkunst.*	Meichsner G. (1625a). *Arithmetica Historica. Das ist: Rechenkunst*	Euler. L. (1738). *Einleitung zur Rechenkunst*
The number of years from the creation of the world until Christ was born: 3,970 years (p. 4).	3,970 years (p. 4)	3,970 years (p. 3)	
The cost of the building of King	13,695,380,050	13,695,380,050	13,695,380,050

Salomon's temple: 13,695,380,050 Coronatos Crowns (p. 5)	Crowns (p. 4)	Crowns (p. 2)	Crowns (p. 22)
The yearly cost of the government of the Emperor Augustinus: 1,200,000 Coronatos Crowns (p. 6)	12,000,000 Crowns (p. 5)	12,000,000 Crowns (p. 2-3)	1,200,000 Crowns (p. 22)
The fortune of Sardanapalus, the King of Assyria: 145,000,000,000 Guilders (p. 6)	154,000,000,000 Crowns (p. 6–7)	154,000,000,000 Crowns (p. 3–4)	145,000,000,000 Guilders (p. 22)
The number of grains of sand to fill the world: computed by Archimedes as 10^{63}, the unit with 63 zeros (p. 6)	$8 \cdot 10^{37}$ (p. 7)		10^{51} (p. 22)

Table 3. Examples on numeration in four textbooks.

Two of the above examples match Euler's text better than the two *Arithmeticas Historica*; Sardanopalus's fortune of 145 billion vs. 154 billion Guilders, and 1.2 million vs. 12 million Crowns of Emperor Augustus's yearly cost. This supports a hypothesis of a missing ancestor to both Euler's and the anonymous Icelandic text, as it is highly unlikely that one of them is copied from the other.

3.4. Addition and subtraction

The *Arithmetica* explains addition of multi-digit numbers in a well known fashion, still practiced, beginning from the right side, adding up the units, and proceeding to the left. All its examples, but one, listed in Table 4, are contained in the foreign sources by Suevus, Meichsner and Euler, which have more examples.

Arithmetica – That is Reckoning Art. (1721/1750)	Suevus (1593). *Arithmetica Historica.*	Meichsner (1625a). *Arithmetica Historica.*	Euler (1738). *Einleitung zur Rechenkunst.*
The age of Methusalem, who according to Holy Scripture was 187 years old when he begat Lamech, whereafter he lived for 782 years, to the age of 969 (p. 10).	969 years (p. 22–23)	969 years (p. 5)	969 years (p. 32)
Example revealing the 'current' year: 'I want to know how many years there are since the poet Homerus lived. Aulus Gellius writes that he lived 160 years before Rome was built, but the city of Rome was built 752 years before the birth of Christ, and the number of years since Christ was born until now is **1721** years.' (p. 10)	Number of years since the creation of Earth, 3,970 + **1,590** (current year) (p. 47)	Number of years since the creation of Earth, 3,970 + **1,625** (current year) (p. 4)	The number of years since Homerus lived, 160 plus 752 plus the current year, **1737** (p. 33)
The number of the Greeks, 880,000, and Trojans, 686,000, deceased in the Trojan War was in total 1,566,000. (p. 11)	1,566,000 men (p. 45–46)		1,566,000 men (p. 32)
Four men owe me 6,952, 8,346, 6,259 and 5,490 each, a total of 27,047 monetary units.			27,047 Rubles (p. 32)

Table 4. Comparison of addition examples in four textbooks.

The examples of the age of Methusalem and of four men owing 27,047 units with the same four amounts, in that case rix-dollars, are also found in *Arithmetica Islandica*, dated on its front page in 1716. Several examples count the present year as 1733, possibly a date of its extant copy (pp. 21, 36, 64, 66).

The example of the poet Homerus, revealing the date of all the four books, is also found in Ramus's (1569) *Arithmeticae libri Duo*, mentioned earlier. 'Ut si quaeratur quampridem vixerit Homerus, & respondeatur e Gellio, 160 annis ante conditam Romam, quæ condita sit ante natum Christum annis 752. Christum vero natum anno abhinc 1567. addantur hi tres numeri : Summa inductionis indicans Homerum annos abhinc 2479 floruisse, erit hoc modo' (p. 3). This example testifies that Ramus's book was written in 1567.

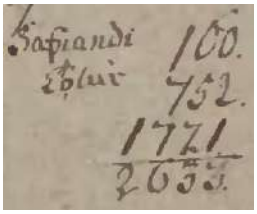

Figure 1: The example in *Arithmetica – That is Reckoning Art* on the number of years since Homerus lived.

Only three subtraction examples are found in our *Arithmetica* and none of them is historical. A demonstration follows on how subtraction and addition can be used for testing each other.

3.5. Multiplication

Arithmetica – That is Reckoning Art has several multiplication problems, similar or identical to other sources, such as on the circumference of the earth, see Table 5 below. *Arithmetica* gives an example on the average number of hours in a year, see Figure 2, as does Suevus, and Meichsner too in another book from the same year, *Arithmetica Poetica*. Euler counts the number of hours in a regular year.

Arithmetica – That is Reckoning Art (1721/1750).	Suevus (1593). *Arithmetica Historica.*	Meichsner (1625a). *Arithmetica Historica.*	Euler (1738). *Einleitung zur Rechenkunst.*
Circumference of the earth,360°·15 = 5,400 miles (p. 23).	360°·15 = 5,400 miles (p. 128).	360°·15 = 5,400 miles (p. 14–15).	360°·105 = 37,800 Werste (p. 69).
Number of hours in a year, 365·24 + 6 = 8,766 hours (p. 22).	(52·7+1)·360 +6 = 8766 hrs. (p. 127).	52·7·360 = 8,736 hrs. in 52 weeks in *Arithmetica Poetica* (1625b), (p. 17).	365·24 = 8760 hours (p. 69).
Size of a military group, 264·100 = 26,400 soldiers (p. 27).			156·97 = 15,132 soldiers (p. 69).
Fortune in King David's grave, 3000·600 = 180,000 Crowns (p. 28).	3000·600 = 180,000 Crowns (p. 171–172).		

Table 5. Comparison of multiplication examples.

The number of soldiers in the military group exceeds the number of adult men in Iceland in the early 18^{th} century, where no army existed.

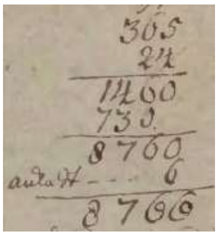

Figure 2. Computing hours in a year in *Arithmetica – That is Reckoning Art*.

3.6. Division

Only two division problems, both concerning calendar computations, are found in other books. Dates of the books are again revealed, as shown in Table 6.

Arithmetica – That is Reckoning Art (1721/1750)	Suevus (1593). *Arithmetica Historica.*	Meichsner (1625a). *Arithmetica Historica.*	Euler (1738). *Einleitung zur Rechenkunst.*
Is the coming year a leap year? Coming year, **1721**:4 = 430, remainder 1 (p. 35)	**1591**:4 = 397, remainder 3 (p. 175)	Check on which of the years 1620–1624 were leap-years (p. 25)	
Golden number of a year (**1622**+1):19 = 85, remainder 8 (p. 38)	(**1591**+1):19 = 83, remainder 15 (p. 176)		

Table 6. Comparison of division examples.

The date 1622 in *Arithmetica* could point to copying a book dated that year. Problems of this kind are not found in Euler’s *Rechenkunst*.

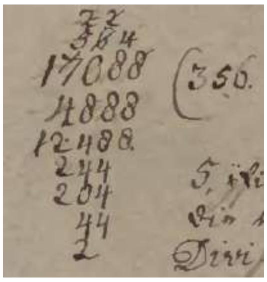

Figure 3. Dividing 17,088 by 48 in *Arithmetica – That is Reckoning Art*. The divisor is written below the dividend and disappears in a mess of different products.

3.7. Square Root and the Theorem of Pythagoras

The first example on extracting a square root in *Arithmetica – That is Arithmetic Art* concerns a general arranging in a square a group of 54,756 soldiers, exceeding the population of Iceland.

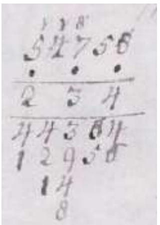

Figure 4. Extracting square root.

In continuation extracting square root is applied to the Pythagorean Theorem, a rule not found in other textbooks inspected. No tower of the kind drawn in the manuscript existed in Iceland, while stories of kings, queens, knights and princesses in castles were a favoured branch of literature.

Root extraction is not contained in the books by Suevus, Meichsner and Euler, while e.g. Gemma Frisius's (1567) *Arithmeticae practicae methodus facilis* contains extractions of square and cube roots.

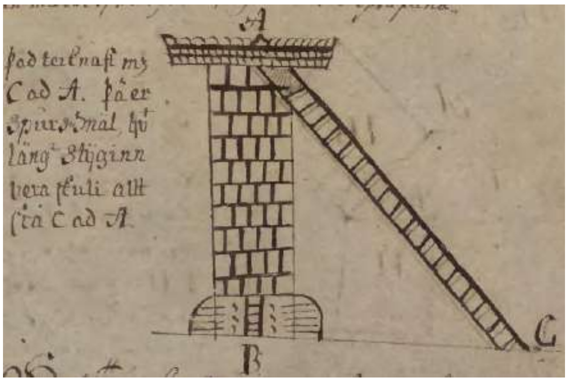

Figure 5. Demonstration of the Pythagorean Theorem:
The height of the tower AB is 30 feet and the width of the dike BC is 28 feet.
$30^2 + 28^2 = 1684$
$1684 - 41^2 = 3$
The length of the ladder AC is 41 and 3/(2·41 + 1), that is 41 3/83 feet.

3.8. The remaining text

Arithmetica – That is Reckoning Art continues with a thorough treatment of fractions and their four operations, and a large section on the *Regula Trium*, applied to proportional problems for centuries (Tropfke, 1980, pp. 359–378). The manuscript ceases abruptly late in that section, which reminds the reader that it is a copy and not an original.

The topic of fractions is not contained in the textbooks by Suevus and Meichsner, and *Regula Trium* does not exist in Euler's book. Examples using the rule, matching those in our *Arithmetica*, have not been found in Suevus's and Meichsner's books.

4. Summary and conclusions

The quotations and comparisons above witness that the European tradition of arithmetic textbooks was practiced in Iceland. The Icelandic *Arithmetica – That is Reckoning Art* has clear connections to the books *Arithmetica Historica, Die löbliche Rechenkunst* by Suevus (1593) and *Arithmetica Historica. Das ist: Rechenkunst* by Meichsner (1625a), both originating in Lutheran protestant towns.

There are even more examples common to Euler's *Einleitung zur Rechenkunst* than the other two books. Euler's book contains, however, neither root extractions nor the *Regula Trium*, and its second part is devoted to 'benannten Zahlen', monetary and measuring units, which *Arithmetica* does not touch upon. Moreover, Euler's book was published in 1738, while the Icelandic *Arithmetica* probably origins in 1721, when Euler was 14 years old.

Probably there existed more books of similar origin. More work is needed to trace the origin of *Arithmetica – That is Reckoning Art*.

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THE PLACE OF GEOMETRICAL CONSTRUCTION PROBLEMS IN FRENCH 19th CENTURY MATHEMATICS TEXTBOOKS

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ABSTRACT

The place dedicated to construction problems in 19th century French textbooks shows a striking disparity. In fact, it turns out that the destination, and the notion itself, of the geometrical construction problem is radically different from one textbook to another. This article surveys a certain number of characteristic examples of such textbooks. For each one of them, the analysis ends up unveiling some of the author's fundamental conceptions about geometry and geometry teaching.

1 Introduction

At the dawn of 19th century, in 1794, Legendre writes his *Éléments de Géométrie* as a comeback to Euclid's rigour and exactness. Though, this strong reference to the Ancients is contradictory on the following point : the place given to construction problems is radically different from what occurs in Euclid's text. They are no longer part of the rational knowledge of the figures that constitutes geometrical truth, and this will remain true in most 19th century textbooks. Then, what place do geometrical constructions hold in those textbooks? For what reasons and with what intentions do authors integrate construction problems? These questions are analysed through seven different, and at the same time characteristic, textbooks of the 19th century. First of them, the *Elements* from Legendre, as we just mentioned, focus the geometrical truth on the discourse on the figures, and put aside, as far as possible, the construction problems. On the contrary, Lacroix brings them back into the discourse, for innovative pedagogical reasons. A glance to the primary degree reveals, in the 1830's and for the sake of industry, the introduction of geometry classes consisting in reproducing geometrical figures. This leads to the edition of specific textbooks in which construction problems are torn between the pursuit of material precision and the initiation to geometrical reasoning. On the other end of the scholarship, the entrance exam to the Ecole Polytechnique includes questions about construction problems, as can be seen in a published collection of given subjects. Actually, geometrical construction problems have taken an independent place in many textbooks since the curriculum is officially scattered in a list of items to be studied. In the work of Briot, they are dedicated a specific chapter. Anyhow, the crucial question of the methods available to solve construction problems is approached later in the century, and two textbooks focusing on this question will be presented.
As can be seen, the textbooks chosen correspond to different contexts and levels, nevertheless they all deal with school geometry teaching. And our analysis will, for each one of them, tell much about the underlying conception of geometry and of geometry teaching.
All the translations of historical texts given here are mine. In case of any doubt, please refer to the original texts.

2 The reference work : Euclid *Elements*

Euclid's *Elements* keep being a very strong reference in elementary geometry along the 19th century. In particular, it is evident that all the authors we will read know this book perfectly well. It is based on a logico deductive structure, whereby each new proposition is deduced from prior knowledge. So that all the propositions are deduced from a couple of basic assumptions, namely axioms and postulates. The propositions are of two types : the *theorems* are propositions that establish properties of given figures, they constitute the discourse on the figures, while the *problems* ask something to be done, that is, a construction fulfilling given conditions. Consequently, in Euclid's work, the constructions are a type of proposition, as are theorems. All the constructions are deduced from the three basic assumptions :

POSTULATES.

Let it be granted:

1. **That a straight line may be drawn from any one point to any other point.**
2. **That a terminated straight line may be produced to any length in a straight line.**
3. **That a circle may be described from any centre, at any distance from that centre.**

All along Euclid *Elements*, theorems and problems are interlinked, in the sense that a theorem about a figure shall only be expressed if the figure was first constructed, and conversely, a theorem shall be proven with the aid of constructions justified before. Geometry is then a rational body of propositions that tell on the one hand how to construct geometrical figures (problems) and on the other hand the properties of these figures (theorems).
Now, in this context, let's see how 19th century textbooks deal with construction problems.

3 Legendre

Legendre writes elements of geometry in 1794, after a long experience of teaching. His intention is to write elements that would be as perfect as possible in rigour and clearness. In this, he refers himself to the method and rigour of the Ancients, particularly Euclid. Though, there are no problems in his elements until the end of book II, which is a complete break with Euclid's text. According to Legendre, geometry is a discourse on the figures constituted only by theorems, and problems of construction are whether corollaries of these theorems, or means to reach their demonstrations.

We give here, to help the general understanding of the paper, a list of problems given in Legendre's *Elements*, which are also problems present in Euclid's *Elements*.

Problems related to the two first books :

- To divide the given line AB in two equal parts
- Through a point A, given on the line BC, draw a perpendicular to this line
- From a point A, given outside the line BD, draw a perpendicular to this line
- At the point A of the line AB, make an angle equal to the given angle K
- To divide a given angle or arc in two equal parts
- Through a given point A, to draw a parallel to the given line BC
- Being given two sides B and C of a triangle and the angle A they include, to draw the triangle
- Through a given point to draw a tangent to a given circle
- To inscribe a circle in a given triangle ABC

Problems appended to book III :

- To divide a given strait line in as many equal parts as required, or in parts proportional to given lines
- To find a fourth proportional to three given lines A, B, C
- To find a proportional mean between two given lines A and B
- To make a square equivalent to a given parallelogram or triangle

4 Lacroix

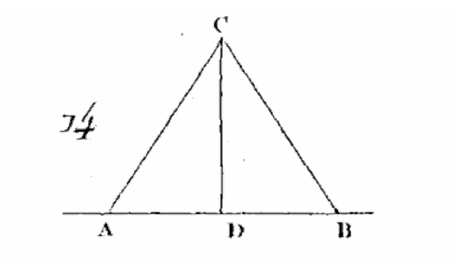
Lacroix has, at the turn of the 19th century, a strong experience of teaching and is very much involved in post Revolution mathematics education. He writes in 1805 an essay to expose his views on education, in which he takes up an explicit position on the place to give to constructions in mathematics textbooks. As for Legendre, problem is here synonymous of geometrical construction. Lacroix writes (1838, p. 298.) :

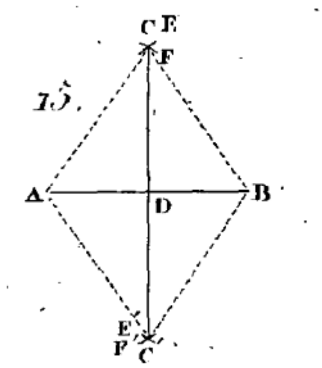
> The permanent custom of proposing problems to the pupils made me conscious of the inconvenient there would be in presenting a whole section of theorems, and report later, the problems that are their continuation. This arrangement, for the less singular, not to say more, that brings out the problem when the theorem on which it is based, and which it would have enlightened or confirmed, deprives the reader from the means to construct his figures with some care ; and although I know as well as anybody, that it is on the rigour of the reasoning, rather than on the exactness of figures

that is based the geometrical truth I nonetheless believe that the practice of the drawing is not less necessary in geometry, than that of the calculus in arithmetic, for the multiple usages of the former science depend on the construction of the figures, as those of the latter, on the practice of the rules [...] Guided by these motives and by experience, I placed the problems as soon as they resulted from the theorems, or were necessary for the construction of the figures ; and I always observed that this order was the most convenient for all minds.

As Legendre, Lacroix states that the discourse is the place where stands geometrical truth. Nevertheless, he attributes to constructions a main role as being the proper activity of those who employ geometry. Besides, his experience of teaching provides him with the intuition that problems enlighten and confirm the theorems. Consequently, he opposes explicitly to Legendre's exposition of theorems one after the other, and asserts that the problems must be placed in the textbooks along with the theorems on which they are based, or that they enlighten or confirm. This idea is not much developed by Lacroix, but it is very interesting from the pedagogical point of view.
Lacroix had actually written, in 1799, his own *Éléments de géométrie* (1808). An example will help us interpret what he means when saying that a problem "results from a theorem" and is "necessary for the construction of figures". There is, first, the theorem (Lacroix 1808, p. 15):

Theorem. Lines AC and CB, fig. 14, that go from a given point C of line CD, perpendicular on AB, and that divert equally from the foot of this perpendicular, i.e. from point D, where it meets line AB, are equal [...]
Corollary. [Line CD] has all its points equally distant from points A and B;





The theorem leads to the property that the perpendicular raised on the middle of a line AB has all its points equally distant from A and B. This property brings out the procedure that permits to draw a perpendicular on any given line.

Problem. Draw on line AB, fig 15, a perpendicular that divides it in two equal parts.
Solution. From points A and B, taken successively as centres, and with a compass opening greater than the half of AB, will be described two arcs, CE and CF, that cross on a point C. The same thing will be done below AB; and joining points C and C', the line CC' will be the requested perpendicular.

The problem results from the theorem, and is necessary for the construction of any figure involving the draw of a perpendicular.
For Lacroix, geometrical truth is embedded in the discourse, but geometry's destination is the construction of the figures. Now the place of the problems is dictated by the pedagogical intuition that problem solving enlightens and confirms the theory. The following example shows that, in Lacroix's book, constructions are propositions that assume a pedagogical role. For this problem, Lacroix gives three solutions. This can be explained as each solution enlightens a different theorem. On the left side below stand the theorems, and and the right side the different solutions of the problem, each corresponding to the use of a different theorem (Lacroix, 1808, p. 38):

64. *Theorem.* When two triangles ABC and EDF have their angles equal one to another, their homologous edges are proportional, and they are consequently similar.	68. Problem. To draw on a given line EF, a triangle similar to triangle ABC.
65. *Corollary.* It follows from former proposition, that two triangles are similar, when they have only two equal angles one to another, for the third angle of the first one is necessarily equal to the third angle of the other one ;	*Solution.* It can be drawn a triangle similar to an other one, starting from the various characteristics by which the similarity of these figures may be established. If it is then requested to form on line EF, a triangle that be similar to triangle ABC, it could be achieved, 1°. drawing through points E and F, lines that make with EF angles E and F, respectively equal to angles B and C (65);
66. *Theorem.* Two triangles are similar when they have an angle equal one to another, situated between proportional edges.	2°. Drawing on point E, on line EF, an angle equal to angle B, and bringing on edge DE of this angle a distance DE fourth proportional to the three lines BC, EF, AB ; this way, the two triangles will still be similar, having, one to another, an equal angle placed between proportional edges (66);
67. *Theorem.* Two triangles that have proportional edges, one to another, are similar.	3° At last, searching for a fourth proportional to the three lines BC, EF, AB, an other to the three lines BC, EF AC, and drawing on the two resulting lines and on EF, a triangle DEF ; the triangles DEF and ABC will be similar, having their edges proportional (67).

The three solutions given by Lacroix to problem 68 refer explicitly to the three theorems 65, 66 and 67. This shows that the problem is placed here not only because it results from the theorem, but also to give sense to each of the three theorems by showing how they may be useful in the application of geometrical knowledge.

5 The *dessin linéaire*

The *dessin linéaire* consists in drawing geometrical figures, first without instruments, and then with the instruments. This latter part concerns directly our subject as it is nothing else than geometrical constructions. The *dessin linéaire* is taught from the years 1820 in primary schools in France for the needs of industry and craftsmen. It appears in the official curriculum from the 1830's up to 1850. During this period, many textbooks are printed on that subject. It is interesting to note that many of them mix two different objectives, that are :
- to reach good precision in the drawing, from the concrete point of view, which is the primitive objective of this teaching. The geometry involved here consists in a set of rules saying how to use the instruments to obtain certain figures.
- to prepare to the geometrical reasoning. This objective is explicitly defended on the argument that to draw and then to construct the figures is a fruitful preliminary to the elements. This point is actually contested when the *dessin linéaire* is officially introduced in the primary school programs, and some basic elements of geometry are eventually placed ahead in the curriculum.

It should be noticed that these two conceptions are reverse one from another : the first one goes from the rational discourse to the constructions, and the other one from the constructions to the rational discourse. This questions us as mathematics teachers : should we go from practice to theory or from theory to its applications? What is of particular (historical) interest here is that many authors, like Lacroix in the preceding chapter, have an empirical approach : they *observe* that going from practice to theory seems fruitful.

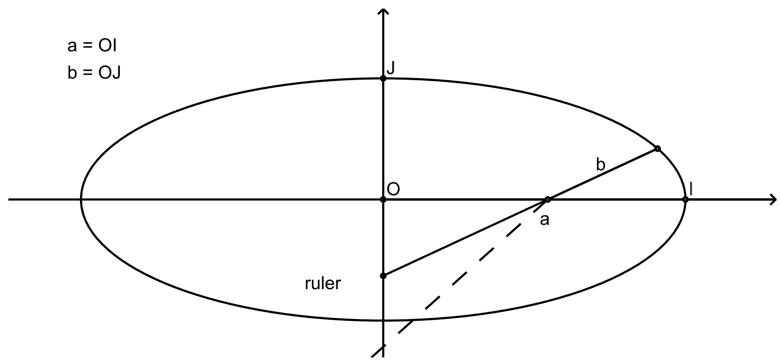
Here is, as an illustration, an extract of the introduction to the *Cours méthodique de dessin linéaire* from the schoolteacher Lamotte (1832):

> 1. The linear drawing is, in a large sense, the art of representing the different bodies with the help of mere lines. That kind of drawing is based on the principles of geometry, and is mainly aimed at representing industrial arts productions, machines, etc., etc. [...] The experience showed that pupils whose eye and hand were exercised with linear drawing were improving fast in the drawing of the figure, and obtained a marked advantage on their mates in mathematical constructions. All industrial professions need the linear drawing, the workers to do their job, the workshop chiefs to prepare it. Which is the man, even in the highest social position, who does not need in a thousand occasions to pass on clearly his thought to an architect, a builder, a cabinet-maker, or any other worker, with a figure, with a quick draw? Long explanations, unintelligible for the worker, are replaced so much advantageously by a linear drawing ! But in this case the linear stroke does not need to be executed with mathematical precision, it might be an indication, an approximation. The accuracy, that is not essential here, must be supplied with rapidity. If on the contrary a master prepares his workers' task, measures must be perfectly exact. It is only with the instruments of geometry that we may reach a convenient degree of precision. We will consequently distinguish two kinds of linear drawing, the

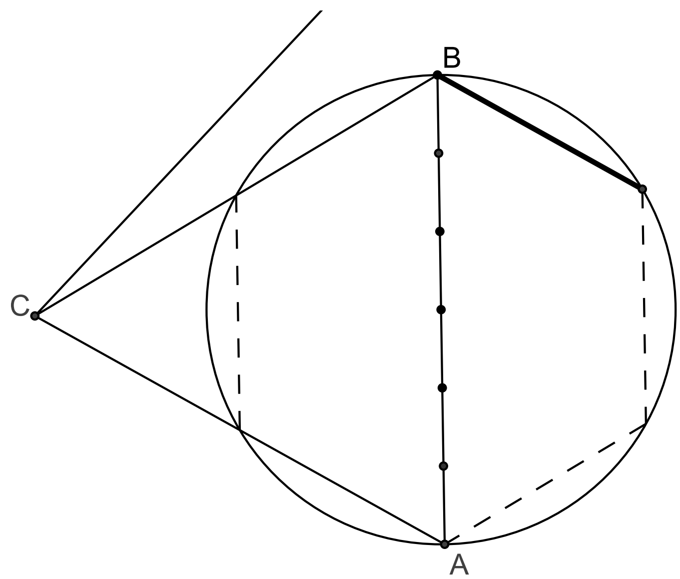
on-sight drawing or without instruments, and the geometrical drawing or with instruments. These two kinds of drawing are equally utile, depending on the circumstances. They shall be studied with same care.

To illustrate the paradox mentioned before between the two objectives aimed at by such textbooks, namely the precision of the drawing and the preparation to the geometrical reasoning, let's detail two items of the textbook.

- Lamotte proposes the following method to check an ellipse drawn by a pupil : Mark, from one end, on the ruler, the lengths a and b (parameters of the ellipse) and make the corresponding points coincide with each of the axes; the end of the ruler must then coincide with the ellipse.



- Later comes a general construction to divide a circumference in as many divisions as requested.



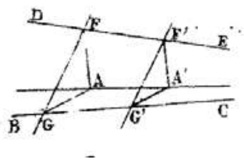
"From ends A and B of diameter AB, and with a compass opening equal to AB, describe two arcs that cross in C. ACB is an equilateral triangle. Divide the diameter in as many parts as you want divisions, and draw from point C a line that will pass through the second division of the diameter, and will reach the circumference, the chord of the intercepted arc is the side of the requested polygon"

Now if these two constructions give satisfaction from the practical point of view in many cases, one of the two is, in general, wrong from the deductive point of view. The reader probably knows which one?

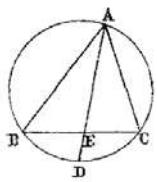
6 Construction problems as questions to be solved

Until the years 1870, the admission exam at the École Polytechnique is mainly an oral examination about mathematics. If analytic geometry, algebra and trigonometry are mostly represented, some subjects deal with constructions as we can see in the examples below. They are taken from a collection of questions given at the exam edited by Lonchampt, supervisor of the École Polytechnique (Lonchampt, 1865, p. 6):

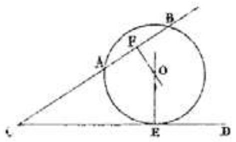
8. — Par un point A mener une droite qui passe par le point de concours de 2 droites BC, DE que l'on ne peut prolonger.



9. — Construire un triangle ABC, connaissant la base BC, l'angle opposé A et le rapport $\frac{m}{n}$ des deux autres côtés AB et AC.



10. — Décrire une circonférence qui passe par 2 points donnés A et B et qui soit tangente à une droite donnée CD.



Geometry is, in this context, the ability to work on any new question and find its solution or demonstration. That conception is challenging for the teacher : how to prepare somebody to work on any question ? Two answers can be found in 19th century textbooks. The first one consists in intensive training. Many collections of problems are edited in this intention. The second one consists in rationalizing the problem solving activity, i.e. to identify and describe general methods to solve whole classes of problems. We will come back on this point more specifically about construction problems in chapter 8.

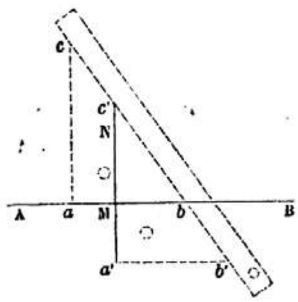
7 Construction problems as part of the curriculum

French Revolution settles a public instruction minister, in charge of defining the official program to be studied in schools. At first, the program simply refers to textbooks, in fact those of Lacroix (1808) and Legendre (1794) for elementary geometry. But from the 1830's on, the program is wholly detailed in a list of various items. From then on, textbooks are, conversely, written in accordance to the program. As a consequence, the knowledge is scattered in a succession of items, and is no more part of a coherent body. In particular, the axiomatico deductive structure vanishes. Concerning the construction

problems, they are departed from the rest and constitute somehow an independent part of elementary geometry. Besides, the paradigm of constructions is clearly enough that of practical applications.
Briot, teacher at the lycée Saint Louis in Paris, writes as an introduction to the chapter about geometrical constructions in his *Éléments de Géométrie* (Briot, 1865, p. 61.):

> Up to now [in the textbook], the figures that were used in the demonstration of theorems, were drawn with the hand only, with little precision ; to represent roughly the figure, to help the mind follow the reasoning. But, in practice, when it is asked to determine with precision, whether the position of a point, whether an unknown magnitude, the graphic constructions must be done on the paper with great exactitude.

The text says clearly that geometrical truth belongs to the discourse, and that the constructions are only necessary to obtain precision in the practical applications.
Moreover, the construction procedures are freed from the restrictive frame of the Euclidean postulates (see p. 2) which request that any construction be drawn out of straight lines and circles. This allows new procedures and new instruments.
As example, here is the problem to rise a perpendicular to a given line AB on a given point Briot proposes after a classical method, another one that consists in sliding a set square along a ruler.

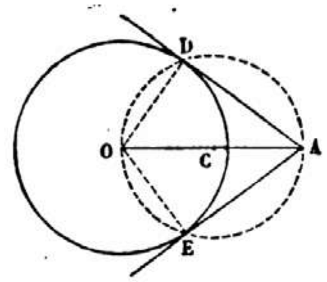


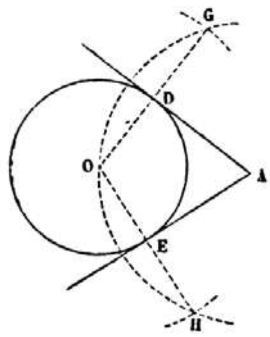
This is, in comparison to what Lacroix and Legendre wrote, a new instrument, as well as a new procedure as it involves movement.
Another example shows that what is searched for is the simplicity of the material realisation of the figure. Briot gives two different constructions to draw a tangent line to a given circle of centre O through a given point A.

The first one has two steps :
- draw the midpoint C of line OA
- draw the circle of centre C through O

The second one has four steps :
- draw the circle of centre A through O
- give the compass an opening twice the radius of the given circle
- draw the circle of centre O with the obtained opening
- connect O with the intersection points of the two drawn circles

Then Briot says the second construction is simpler, as it avoids the construction of the midpoint of line OA. The idea of simplicity involved here is the simplicity of the practical realisation of the figure, not the simplicity of the rational deduction of the solution to the problem.





8 Construction problems and methods

The question of the method has long been salient in solving construction problems of elementary geometry. In his essay of 1805, Lacroix writes (Lacroix, 1838, p. 299):

> Besides, the choice of geometry problems is more embarrassing than that of analysis problems, because the latter depend only on a small number of rather general methods, having together evident connexions, although the former require various constructions difficult to devise.

Indeed, the first half of 19th century sees a revival of synthetic geometry, in opposition with the analytic geometry of coordinates. In particular, new methods are developed. Consequently, some authors undertake to fulfil the long time noticed lack of general methods in synthetic geometry, notably in solving construction problems. The book of Julius Petersen (1880), the translation of which was successful in France, is of particular interest here. The preface says :

> Although modern mathematicians have not ceased to be interested in this branch of science [construction problems], the means to treat rationally this class of problems have developed relatively less rapidly [...] This situation has been the cause that many people have considered geometrical construction problems as kinds of enigmas whose solution could barely be attempted only by few minds gifted with very special faculties. It resulted that these questions have hardly penetrated schools where though they should naturally have been cultivated ; for there exist no problems that sharpen as well the observation and combination faculty and give the mind clearness and logic ; there are none that present as much attraction for the pupils. This book aims at teaching them how one should tackle a construction problem.

The practice of construction problems, writes Petersen, sharpens the mind, giving it clearness and logic. For him, construction problems have a main role to play in mathematics education. The book then exposes a range of general methods in solving construction problems, and tries to identify the class of problems solvable by each method. In this it is rather an innovative book. For example, Petersen (1880, p.71) develops a quick theory on rotation (we would say similarity) and immediately follows

> 4- By the mean of propositions we just developed, we are able to solve the following general problem : To place a triangle, similar to a given triangle, so that one of its vertices be on a given point and the two others be on two given curves.

Then many examples of applications of this general method are given. They begin with [1] :

> 349. To place an equilateral triangle, so that its vertices be on three given parallels.
> One of the vertices might be put on any point of one of the lines ; let it be the centre of rotation ; $\tau = 60°$, $f = 1$.
> 350. To place an equilateral triangle so that its vertices be on three given concentric circles.

Another example is the textbook from frère Gabriel Marie, member of a congregation devoted to scholar education that edited a complete collection of mathematics textbooks in the second half of 19th century. The textbook mentioned here is a collection of a large number of questions to be solved. It begins with an important part exposing various methods, that are referred to along the questions treated in the body of the text. One of the methods presented is the similarity [2]:

> To solve a problem with the assistance of similarity, one constructs a figure similar to the given one, and one compares a single dimension to its given homologue. This is to be done especially when the problem requested depends on a single given line.

The method is illustrated with the following problem :

> 207. Problem. To construct a square, knowing the sum or the difference of its diagonal and its edge.

This problem may be solved by the mean of a synthetic, rather elaborated, construction, as it is done in another place in the textbook. But the solution proposed here follows from the observation that the problem depends on a single given line. It suffices to draw any square, to construct the sum, or difference, of its diagonal and its edge, and finally to enlarge or reduce the whole figure to make the obtained line coincide with the given.

These two textbooks highlight the importance of the method. They say that the ability to work on any new question in geometry rests on the knowledge of various methods together with the skill to use them adequately.

9 Conclusion

Throughout the various textbooks surveyed here, it appears that depending on the conception one has of geometry, and on what are the pedagogical objectives, the status and the role attributed to the geometrical construction problems can be totally different. From an epistemological point of view, Legendre considers these construction problems as corollaries, or lemmas of the discourse on the figures. Lacroix, on the contrary, claims they are the final outcome of geometry. Other authors consider them whether as precise drawings, or theoretical questions to be tackled.

[1] The problem 350 was given in the *concours général des collèges de Paris* in 1814 and the students renounced, while the problem given in replacement does not seem simpler to us.

[2] FGM, 1896, p. 100.

From a pedagogical point of view, construction problems are not only mere items of the curriculum, since various authors consider them as a pedagogical tool. Lacroix says they enlighten the theorems, many primary degree teachers remark that drawing geometrical figures improves the geometrical reasoning, FGM and Petersen employ them to illustrate general methods, and the latter even claims that the practice of these problems develops mathematical faculties more than any other type of problems.
Thus, even deprived from their deductive necessity in the discourse, geometrical construction problems keep occupying a determinant place in main textbooks. Certainly because, as writes R. Bkouche, they "guaranty the coherence between the theoretical discourse and the matter-of-fact situation"[3].

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[3]Bkouche, Rudolf, *La géométrie entre mathématiques et sciences physiques*, http://michel.delord.free.fr/rb/indexrb.html, p. 14.

LES DÉFINITIONS LES PLUS RIGOUREUSES SONT-ELLES PLUS FACILES À COMPRENDRE ?

Charles Méray et la proposition d'une définition « naturelle » des nombres irrationnels

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ABSTRACT

Nous analysons la définition des nombres irrationnels proposée par Charles Méray en 1869 en soulignant ses motivations d'ordre épistémologique et didactique. Nous prétendons contribuer ainsi à la discussion sur le rôle de la question des fondements dans l'enseignement de l'analyse.

1 Introduction[1]

À l'heure actuelle il est assez connu dans l'historiographie que le mathématicien français Charles Méray a été le premier à proposer une définition des nombres irrationnels dans le style qui serait consacré par Cantor peu après. Dans son *Histoire de l'Analyse*, Pierre Dugac a introduit un chapitre sur Méray ayant pour titre *Premier exposé publié de la théorie des nombres irrationnels*. Il fait référence au mémoire « Remarques sur la nature des quantités définies par la condition de servir des limites à des variables données », publié en 1869 dans la *Revue des Sociétés Savantes*.

Le titre de ce mémoire indique que Méray veut traiter la « nature » de certaines quantités, à savoir les incommensurables. En fait, l'auteur soutient à plusieurs reprises que la façon comme il définit les irrationnels est la plus « naturelle », ou la plus « conforme à la nature des choses », et pour cette raison, serait la plus facile à comprendre.

Nous exposerons d'une manière assez résumée la définition de Méray, à fin de rechercher, par la suite, dans quel sens peut-on dire qu'il s'agit d'une définition « naturelle ». Il n'est pas dans notre but de discuter de la valeur mathématique de la définition de Méray, dont la rigueur a été mise en relief par Dugac et d'autres avant lui. Notre intérêt ici est plutôt de comprendre pourquoi cette définition est considérée comme étant naturelle et, par conséquence, la plus adaptée pour l'enseignement.

Nous verrons à partir des échanges de lettres pendant les années 1880 que même les définitions fournies par Weierstrass peu après Méray n'ont pas êtes unanimement considérées comme les plus faciles à être comprises par les étudiants. Malgré son indéniable valeur mathématique et la reconnaissance d'un étalon de rigueur plus convainquant posé par le mouvement de formalisation de l'analyse, on envisage

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[1] Ce texte n'est qu'une partie du travail presenté dans le workshop de ESU6, qui traite plus généralement des rapports entre la recherche et l'enseignement des mathématiques en France entre les années 1860 et 1890. Ce projet a été realisé avec Gerard Grimberg et João Bosco Pitombeira. J'aimerais les remercier, notamment Gerard qui a donné l'idée de travailler sur ce thème. Je remercie aussi les participants du workshop, pour leurs observations précieuses, et à Gert Schubring, qui m'a suggeré des références importantes.

l’hypothèse qu’il ne serait peut-être pas tellement profitable d’introduire les nouvelles notions tout de suite aux étudiants. Le plus approprié serait de commencer par des définitions plus intuitives pour n’aboutir qu’à la fin à une présentation rigoureuse de l’analyse (selon les nouveaux critères).

Il est particulièrement remarquable la façon comme des mots relatifs au caractère « naturel » des nouvelles définitions étaient employés pour soutenir la position contraire à celle citée ci-dessus, dite « intuitive ». L’appel à la nature et à l’intuition servait, respectivement, à la défense de deux positions contradictoires sur l’enseignement de l’analyse : introduire dès le début cette matière de façon considérée comme la plus rigoureuse à l’époque ou maintenir la façon ancienne et garder des précisions supplémentaires pour la fin du cours.

Nous n’arriverons pas jusqu’à discuter les nombreuses études sur l’enseignement de l’analyse jusqu’à nos jours, mais l’actualité de la discussion traitée ici se retrouve dans quelques-uns de ces travaux. Pour n’en donner qu’un exemple, nous ne sommes pas loin de la dualité entre les positions citées ci-dessus lorsqu’on questionne si un cours d’analyse doit forcement commencer par l’introduction de la notion de limite, où s’il est mieux d’exposer cette notion après des expériences intuitives – par exemple, graphiques – d’approximation des courbes par de droites[2].

Même en nous éloignant de la discussion sur l’enseignement de l’analyse, des questions plus générales se posent : Est-ce qu’exposer les mathématiques de la façon la plus rigoureuse est la meilleure stratégie pour obtenir une compréhension plus profonde ? Est-ce que la clarté suit immédiatement de la rigueur et de la concision d’un raisonnement ?

Nous espérons contribuer à ce débat en revenant à la discussion au sujet de l’enseignement de l’analyse au moment même où cette matière souffrait une restructuration.

2 La définition des nombres incommensurables

Au début du mémoire publié par Méray en 1869 (Méray, 1869), il énonce deux principes de la théorie des nombres incommensurables :

1) Une quantité variable v qui prend successivement les valeurs $v_1, v_2, ..., v_n, ...$ tend vers une certaine limite si les termes vont toujours soit en augmentant, soit en diminuant, pourvu qu’ils restent dans le premier cas inférieurs, dans le second supérieurs à une quantité fixe quelconque.
2) La variable v définie comme ci-dessus jouit encore de la même propriété si la différence $v_{n+p} - v_n$ tend vers zéro quand n augmente indéfiniment, quelque relation qu’on puisse établir entre n e p.

Ces deux propositions ont été regardées jusqu’au moment en question comme étant des axiomes, mais Méray veut montrer qu’on peut ramener la seconde à la première, dont selon lui « le caractère d’évidence est plus prononcé ».

Malgré ce caractère d’évidence, le premier principe pose problème, puisqu'il affirme que la limite existe, sans fournir une méthode pour découvrir sa valeur. Cette difficulté est

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[2] Cette discussion a été centrale dans la recherche inaugurée par David Tall dans les années 1980, connue de nos jours comme *Advanced Mathematical Thinking*. Pour un aperçu récent voir (Tall, 2010).

due à la façon de concevoir les quantités incommensurables. Si la limite est un nombre (c'est-à-dire, un nombre rationnel), on sait que la limite existe parce qu'on calcule sa valeur. Sinon, il paraît contradictoire de « vouloir rattacher l'existence analytique d'un objet à une hypothèse qui ne l'assujettit à correspondre à aucun nombre » (Méray, 1869, p.283, italiques de lui). Il a fallu donc ressaisir la véritable nature des quantités incommensurables, et Méray passe ensuite à la définition de ces quantités conçues comme des entités fictives.

Il réserve la dénomination de « nombre » ou « quantité » aux entiers et aux fractions. Une quantité v qui reçoit successivement diverses valeurs v_n en nombre illimité est appelée « variable progressive ». Si n croissant à l'infini, il existe un nombre V tel que, à partir d'une valeur convenable de n, $V - v_n$ reste inférieure à une quantité quelconque aussi petite qu'on puisse la supposer, on dit que v a pour limite V.

Si ce nombre n'existe pas, il n'est plus permis d'affirmer que v a une limite. Voici la difficulté posée par la non considération des quantités incommensurables comme étant des nombres.

Mais dans le cas où v a pour limite V, la différence $v_{n+p} - v_n$ converge vers zéro, ce qui est aussi vrai du cas où v ne tends pas vers un nombre (admis comme tel, soit un rationnel). Dans cette situation, Méray observe que la nature de v offre une « ressemblance extraordinaire avec des variables réellement douées de limites » (idem, p.284). On peut dire alors que la variable progressive v est « convergente », qu'elle ait ou non une limite « numériquement assignable ».

On devine ainsi le raisonnement de Méray : si la « variable progressive » (pour nous, une « suite ») est convergente, même si la limite vers laquelle elle converge n'est pas un nombre à proprement parler, nous pouvons le définir comme en étant un. On procède ainsi à une extension du concept de nombre. Des quantités qui ne sont pas « numériquement assignables » sont considérées comme étant des nombres afin de donner consistance au fait que toute suite convergente doit avoir une limite, même si cette limite est fictive.

En effet, Méray affirme dans la continuation du mémoire qu'il est une convention utile d'exprimer la convergence de la variable progressive en disant qu'elle a « une limite (fictive) ». Pour pouvoir définir des quantités incommensurables par le moyen de ces limites, il faut d'abord savoir quand on peut affirmer que deux variables progressives sont équivalentes. En aboutissant à cette définition, Méray peut affirmer qu'à toute quantité dite incommensurable correspond une infinité de variables progressives (commensurables) convergentes qui sont équivalentes.

Il définit ainsi des variables progressives équivalentes :

Si m et n augmentant tous deux à l'infini, la différence $u_m - v_n$ de deux variables progressives convergentes tend vers zéro pour une certaine dépendance mutuelle entre ces indices, on prouve aisément qu'elle reste infiniment petite pour toute autre loi. On dit alors que les variables progressives u et v sont équivalentes.

Et leurs limites (vraie ou fictive) :

Supposons que u et v aient des limites U et V (nous ajoutons, des nombres rationnels). Si u et v sont équivalentes, U et V seront égales. Supposons au contraire que u et v n'aient point de limites (numériques), Méray affirme que « il y aura avantage à dire toujours (au figuré) qu'elles ont des limites égales ». Ces limites seront nommées, par opposition aux limites numériques, des « limites fictives » des variables progressives.

Nous savons que ce qu'on nomme aujourd'hui une « suite de Cauchy » peut ou non avoir une limite rationnelle. Méray définit donc un irrationnel comme la limite d'une suite de Cauchy lorsque cette limite n'est pas rationnelle. La nécessité de cette définition s'explique par la volonté de garder l'uniformité de la définition d'une suite convergente comme étant celle qui tend vers une limite. Il en était déjà ainsi lorsqu'on avait une limite numérique; un nouveau type de nombre, fictif, doit être proposé à fin d'obtenir une définition semblable pour tous les cas.

L'objectif de Méray devient plus clair dans son traité de 1872, *Nouveau Précis d'Analyse Infinitésimale*. Les variables progressives seront rebaptisées « variantes », et il énonce une propriété fondamentale sur les variantes ayant des limites numériquement assignables (rationnels pour nous) : la somme, le produit (ou produit de puissances) d'un nombre déterminé quelconque de variantes pourvues de limites et de quantités invariables, ont pour limites les résultats qu'on obtient en substituant, dans les mêmes calculs, les limites aux variantes (Méray, 1872, p.2).

Il passe ensuite à la définition des quantités incommensurables comme des limites des variantes convergentes, de la même manière que dans le mémoire de 1869 (seulement en rendant l'exposé plus clair et détaillée). La définition des variantes équivalentes joue ici un rôle tout aussi fondamental. Les quantités incommensurables permettent d'étendre la proposition citée ci-dessus aux cas des variantes qui ne tendent pas vers des limites numériquement assignables : la somme, le produit (ou produit de puissances) d'un nombre déterminé quelconque de variantes convergentes et de quantités invariables, est une variante convergente qui reste équivalente à elle-même quand on substitue aux variantes proposées d'autres qui leur soient respectivement équivalentes (Méray, 1872, p.3). Il ajoute ensuite que cette proposition exprime la même chose que le théorème précédant lorsque quelques variantes ne tendent pas vers des limites numériquement assignables.

> « Toutefois, précise-t-il, on peut convenir de dire au figuré qu'une variante tend vers une limite fictive *incommensurable*, quand elle est convergente et n'a point de limite numériquement assignable » (Méray, 1872, p.3, italique de lui).

Les limites incommensurables de deux variantes convergentes sont égales quand celles-ci sont équivalentes et cela permet d'étendre à toutes les variantes les propriétés sur la somme et le produit des variantes ayant des limites numériquement assignables. Méray explicite que cette généralisation n'est possible qu'à l'aide d'une « convention » :

> « Telle est pour nous la nature des nombres incommensurables ; ce sont des fictions permettant d'énoncer d'une manière uniforme et plus pittoresque toutes les propositions relatives aux variantes convergentes » (Méray, 1872, p. 4).

3 Une définition conforme à la nature des choses

Dès la préface de ses Leçons, Méray affirmait que sa motivation était de proposer des définitions plus naturelles. En se référant à la définition qu'on vient d'exposer, il ajoute que « si singulière qu'elle puisse paraître, comparée aux traditions classiques », il la croit « plus conforme à la nature des choses que des exemples physiques dont il faut illustrer les autres».

Il témoigne du sentiment commun à son époque qu'il faut aller au-delà des arguments physiques afin de justifier des définitions mathématiques. Mais dans quel sens peut-on dire que la définition qu'on vient de voir est plus conforme à la nature des choses ? Dit autrement, comment expliquer aux étudiants par rapport à quelle « nature des choses » la définition exposée ci-dessus peut être considérée comme la plus adaptée ?

Selon l'avis de Méray le problème est d'amener les propositions sur des nombres incommensurables à exprimer des relations existantes entre des nombres au sens propre (c'est-à-dire, des nombres rationnels). La question pour lui est donc celle d'énoncer pour des nombres incommensurables, des nombres au sens figuré, des propriétés connues des nombres au sens propre.

La définition d'un nombre incommensurable proposée par Méray est considérée comme la plus naturelle parce qu'elle garantit le plus d'homogénéité, le plus d'uniformité dans les énoncés mathématiques. La « nature de choses » dont il est question reflète une conception des mathématiques selon laquelle les objets doivent être les plus uniformes et les propositions doivent être les plus générales possibles. Cela implique une notion particilière de nombre. D'ailleurs une conception qui n'est absolument pas facile à comprendre comme étant la plus naturelle.

Ayant considéré l'exposition des nombres incommensurables fournie dans les écrits précédents trop mélangée aux principes de l'analyse, Méray a publié en 1887 un autre article afin d'éclaircir le sens qu'il faut attribuer à la notion de « nombre incommensurable ». Le titre de cet article, paru dans les Annales Scientifiques de l'École Normale Supérieure, exprime bien son but « Sur le sens qu'il convient d'attacher à l'expression nombre incommensurable et sur le critérium de l'existence d'une limite pour une quantité variable de nature donnée ». Il a proposé la même définition déjà donnée dans les Nouveaux Précis ayant pour but d'éclaircir un point clef de la théorie des incommensurables:

> « En examinant attentivement les divers points de cette théorie, on constatera qu'il n'existe aucune propriété des nombres incommensurables qui ne soit la traduction, dans un autre langage, de quelque propriété intéressant exclusivement des nombres proprement dits variables.
>
> En dehors de cette conception, rien ne semble satisfaire pleinement l'esprit ».

Il s'agit surtout, à son avis, d'étendre le concept de nombre, et des opérations sur des nombres, aux incommensurables. Il faut uniformiser les opérations de l'algèbre et les faire valoir pour tout type de nombre qui peut servir de limite à une suite convergente de rationnels.

Quelques années après, en 1894, Méray a publié un deuxième traité d'analyse *Leçons nouvelles de l'analyse infinitésimale et ses applications géométriques*. Dans la préface il affirme que la conception des quantités incommensurables comme des fictions utiles appartient entièrement au domaine de l'analyse pure. Si l'on définit le nombre de la manière usuelle, on ne peut pas affirmer que toute suite convergente ait une limite. Il faut donc introduire la notion de limite idéale (ou fictive) pour aboutir à un langage uniforme pour l'analyse. Méray exprime la convergence d'une variante quelconque en disant qu'elle tend vers une limite effective ou idéale (suivant le cas) et procède à une extension de toutes les opérations de l'analyse à tous types de nombre :

> « Nous n'avons donc plus aucune raison pour maintenir la restriction faite jusqu'ici tacitement par nous sur le sens du mot *quantité* ; désormais nous l'appliquerons aussi bien aux limites idéales de variantes convergentes qu'aux quantités positives et

négatives ayant exclusivement des nombre entiers ou fractionnaires pour valeurs absolues » (Méray, 1894, p. 36, italique de lui).

On pourrait résumer ainsi les principaux aspects concernant la nouvelle définition :

1) Il est nécessaire de définir un nouveau type de nombre, dit autrement, de considérer comme étant un nombre un nouveau type d'objet (de nature fictive), afin d'uniformiser la notion de convergence. Toute suite convergente doit avoir une limite.
2) Le nombre est ainsi défini, par le moyen d'une relation d'équivalence, comme étant la limite elle-même (la suite convergente *est* le nombre).
3) Cela permet d'étendre les opérations ordinaires aux nouveaux nombres et d'avoir une définition uniforme des limites comme des nombres

Aucun de ces aspects ne peut être facilement admis comme plus naturel, ni plus conforme à la nature des choses, sans que cette nature soit explicitement choisie, sans qu'on soit convaincu que *uniformité* et *généralité* sont des traits obligatoires d'une mathématique qui se veut rigoureuse.

4 Les réactions à la définition de Méray

En 1873, H. Laurent a publié une analyse des *Nouveaux Précis* dans le *Bulletin des Sciences Mathématiques*. Il voit dans le titre une indication de l'intention de Méray d'exposer les bases du Calcul infinitésimal. Mais à son avis il est impossible d'admettre que l'auteur ait atteint ce but, vu que « les méthodes employées dans cet Ouvrage sont tellement subtiles, tellement délicates, qu'elles ne sauraient être bien comprises que par des personnes déjà familiarisées avec les spéculations de la haute Analyse (…) et il faut ne pas rompre brusquement avec des habitudes consacrées par une longue expérience » (Laurent, 1873, p. 25).

Méray a répondu dans la préface du nouveau traité de 1894. Il affirme que ces appréhensions sur la valeur didactique de ses méthodes ont été contredites par le résultat de sa propre expérience. Méray témoigne avoir eu, depuis 1872, une bonne réception de la part de ses étudiants. Au départ, dit-il, il a été un peu gênant de rompre avec les habitudes consacrées, mais il ajoute « ce n'est pas ma faute si elles sont mauvaises au point de rendre une rupture nécessaire » (Méray, 1894, p. xxxii).

Le rôle de Méray dans la communauté mathématique française des années 1870-1890 est mineur. Le souci de fonder autrement le Calcul infinitésimal, qui animait l'école de Weierstrass en Allemagne depuis 1860[3], n'a été ressenti vraiment en France qu'à partir des années 1885. Dans l'ouvrage de Tannery *Introduction à la Théorie des fonctions d'une variable*, publié en 1886, on trouve des références aux travaux des allemands sur les fondements, mais ce n'est qu'avec la publication du deuxième mémoire de Jordan en 1893 que l'analyse sera exposée d'une façon plus conforme à la nouvelle conception de rigueur proposée par Weierstrass et son école, comme Gispert le montre dans (Gispert, 1982)[4]. La grande majorité des traités d'analyse publiés en France entre 1870 et 1893 se basait sur les

[3] Sur la conception de Weierstrass concernant le fondement de l'analyse comme étant basé sur la notion de nombre, voir (Ullrich, 1989).
[4]On trouve aussi une analyse de la réception des travaux allemands en France dans (Schubring, 2005, p.603).

principes traditionnels, hérités de Cauchy. L'étude de la convergence des séries et des principes concernant l'existence des limites des suites suivait les pas de Cauchy, sans besoin de définir les nombres irrationnels.

En examinant les lettres échangées entre Darboux et Houel, de 1872 à 1882 (Gispert, 1982), on voit que dès 1873 le premier insiste sur la nécessité de rendre les démonstrations de l'analyse plus rigoureuses, par le moyen des critères solides qu'on trouve depuis longtemps en géométrie euclidienne. Mais Darboux sera lui-même déçu par la faible réception de son appel dans la communauté mathématique française (Gispert, 1982 ; 1983).

On repère très peu de références à Méray dans des textes publiés à son époque. Même Darboux ne le cite pas, peut-être parce que la question de la définition des irrationnels ne semble pas l'inquiéter autant que les fonctions continues.

Du côté de l'enseignement il ne semble pas que cela soit différent. Aucun mémoire ne fait référence à la définition des incommensurables proposée par Méray dans les *Nouvelles Annales de Mathématiques*, journal qui servait à la préparation des étudiants au concours d'entrée à l'École Polytechnique et à l'École Normale et dont les articles ne se souciaient pas des questions de fondements.

Des rares exemples de mathématiciens à avoir cité Méray sont Riquier, Tannery et Poincaré. Comme on peut voir dans (Dugac, 2003, p.149), ce dernier mentionne Méray seulement pour faire justice au caractère novateur de sa définition des irrationnels et tient à souligner la superiorité de Weierstrass.

Charles Riquier a été élève de Méray et s'est occupé des différentes extensions de la notion de nombre à la manière de son professeur. Il a écrit un article sur le sujet avec Méray (Méray et Riquier, 1889).

Dans le premier tome des *Éléments des fonctions elliptiques*, Tannery et Molk mentionnent la priorité de Méray au sujet de la définition des irrationnels, mais leur inspiration est weierstrassienne. A. Pringsheim et J. Molk observent, dans *l'Encyclopédie des Sciences Mathématiques* (Molk, 1904, p.147), que quelques mathématiciens cherchaient, indépendamment les uns des autres, à édifier une théorie nouvelle entièrement dégagée de toute considération impliquant des grandeurs concrètes et étaient ainsi amenés à approfondir la notion de nombre irrationnel ; Méray est le premier qui a élaboré une telle théorie.

Pour plus de détails au sujet du contexte des travaux de Méray en France, voir (Dugac, 1970). Notre but dans cet article n'est pas d'analyser l'histoire de l'analyse en France à la fin du XIXème siècle[5], mais seulement d'explorer, par le moyen d'un exemple historique, la question du rapport entre la rigueur d'une nouvelle définition et la manière de l'intégrer dans l'enseignement.

5 La rigueur et l'enseignement

Des notions et des outils forgés dans la recherche des nouveaux fondements pour l'analyse ont été utiles et nécessaires aux progrès cette discipline[6]. Nous savons que ce mouvement est à l'origine des notions clefs en mathématiques aujourd'hui. Mais qu'elles ont été ses

[5] Des reférences sur l'histoire des traités à cette époque sont (Gispert 1982;1983) et (Zerner, 1994).
[6] Comme montre (Gispert, 1982, p.36).

motivations ? Est-ce que ce mouvement s’explique seulement par la volonté gratuite et formelle de fournir des démonstrations correctes des théorèmes sur les principes de l’analyse ? Bien sûr que non.

Au delà de l'utilité des concepts pour l'avancement de la recherche mathématique, il y a aussi le besoin de mieux enseigner des notions complexes qui constituent les principes de l'analyse. Méray tenait à justifier par des raisons didactiques la portée de ses nouvelles définitions. Néanmoins, il ne va pas de soi que les principes exposés de la manière la plus rigoureuse (selon ses critères) soient effectivement plus faciles à comprendre, et nous donnerons des exemples des mathématiciens de l'époque qui pensaient autrement.

Vers 1880, Mittag-Leffler devient un des principaux divulgateurs des travaux de l’école de Weierstrass en France. Picard et Molk sont, à cette époque, les seuls mathématiciens français à avoir lu les notes des cours de Weierstrass, envoyées par Mittag-Leffler (Dugac, 2003). Certains mathématiciens, pourtant, tout en admettant la nécessité des nouveaux fondements pour l’analyse, ne jugeaient pas que les résultats obtenus étaient indispensables à une bonne exposition des principes du Calcul. Le produit de la nouvelle recherche mathématique n’était pas considéré comme pouvant profiter à un meilleur enseignement.

En 1881, suite à l’annonce faite par Mittag-Leffler qu’il allait enseigner le nombres irrationnels à la façon de Weierstrass, Charles Hermite a écrit:

> « Je crois, mon cher Ami, qu'il ne serait point sans péril d'exposer à des commençants ces mathématiques nouvelles, si incontestablement meilleures et plus rigoureuses que les anciennes. Mon sentiment est qu'il faut d'abord *préparer* à ces nouvelles théories, et suivre l'ancienne route, en montrant soit des erreurs, soit des insuffisances des démonstrations restées longtemps inaperçues, et annonçant que d'autres méthodes les feront disparaître. Et la raison est que quelque chose du développement historique de la science doit se trouver dans l'enseignement. Je m’explique. C’est un fait d’expérience absolument certain que l’erreur a été bien souvent plus utile que des vérités parfaites pour la marche de l’esprit et le progrès de la science.(…)
>
> J’en tire, peut-être en me trompant, la conclusion que l'appareil si complexe de la rigueur moderne, et le caractère abstrait qu'elle revêt, peut n'être pas absolument profitable pour les commençants, ou du moins qu'il est utile de reléguer à la fin, en la réservant pour le couronnement de l'édifice, cette rigueur, qui n'est point toujours *suffisamment instructive* » (Dugac, 2003, p.199, italiques de Hermite).

En octobre 1881, Mittag-Leffler a répondu :

> « C’est vrai que les erreurs ont profité à la science, mais alors on a été naïf et on croyait à l’erreur. Mais comment voulez-vous enseigner une erreur quand vous savez que c’est une erreur? (…) Je ne crois pas non plus qu'il soit juste de regarder le système de Monsieur Weierstrass comme compliqué. C'est au contraire simple et naturel, en même temps que rigoureux, mais c’est vrai qu’il faut beaucoup de temps pour le développer » (Dugac, 2003, p.199).

L’adjectif « naturel » intervient ici à nouveau pour qualifier un raisonnement rigoureux. Le sens n’est pas loin de celui employé par Méray.

En 1892, Méray a publié un mémoire dans lequel son point sur l'enseignement de vue se clarifie, en conformité avec des principes mathématiques alors déjà connus dans le milieux français. Dans ses *Considérations sur l'enseignement des mathématiques*, il expose dans quel sens un raisonnement mathématique doit être considéré « artificiel » :

> « Les principes de chaque démonstration doivent être puisés dans la théorie à laquelle se rattache la proposition correspondante et non en dehors, encore moins dans des théories subséquentes. Il faut épuiser les conséquences qui doivent être tirées d'un principe avant d'en introduire un nouveau. Autrement les théories s'enchevêtrent et perdent à la fois leur clarté et leur indépendance (...)
> Ce qu'on nomme des *artifices* sont des infractions à cette règle; ils peuvent séduire par leur facilité et leur imprévu; mais ils n'ont point de véritable portée, et leur emploi habituel ne produit que des théories décousues, ne donne que l'apparence du savoir » (Méray, 1892, pp.15-16, italique de lui).

C'est pour cette raison qu'il faut enseigner de la façon la plus naturelle : « Les raisonnements bien construits finissent toujours, malgré des complications, par pénétrer dans l'esprit des élèves qui les reproduisent facilement ensuite ».

Naturel est donc l'attribut de ce qu'on peut déduire par démonstration d'une suite d'axiomes, ou des principes, qu'on énonce explicitement au commencement d'une théorie. Par conséquent, dire qu'une définition est naturel revient à la déduire des principes d'une théorie, sans avoir recours à aucune connaissance venue de l'extérieur.

Mais pour quoi les définitions les plus naturelles, dans ce sens, sont les plus adaptées à être enseignées ? On comprend du point de vue de Méray qu'enseigner l'analyse d'une façon naturelle consiste à énoncer d'abord les principes et ensuite les théorèmes qu'on peut en déduire. Ce type de raisonnement, selon lui, finit par pénétrer dans l'esprit des élèves justement parce qu'il est bien construit, c'est-à-dire, il nous permet d'oublier les chemins entrepris dans sa construction.

Comment expose-t-on la définition des nombres réels aujourd'hui ? En fait, dans la plupart des cas l'on définit les corps des réels comme étant un corps complet bien ordonné, et l'on se passe de construire les réels. Mais si l'on veut enseigner comment construire les réels, on peut commencer par la définition d'un réel à la manière de Cantor (qui est essentiellement la même que celle de Méray) ou de Dedekind.

Notre but ici a été d'expliciter que la définition d'un réel comme classe d'équivalence des suites de Cauchy n'est pas si naturelle qu'on le croit, même si elle est très conforme à la nature – mathématique – des choses. Il n'est jamais assez explicité par l'enseignement que cette « nature des choses » implique un choix qui a été déterminé historiquement, un choix sur le type de mathématique qu'on a convenu être la meilleure. Tel pourrait être, à notre avis, l'importance de l'histoire dans l'enseignement de la construction des réels : rendre ce choix explicite; ce qu'implique montrer le rôle prépondérant des critères d'uniformité et de généralité.

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THE ROLE OF THE EUROPEAN UNIVERSITIES IN THE FORMATION OF THE BRAZILIAN UNIVERSITIES:

The case of the university of São Paulo

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ABSTRACT

This preliminary survey aims at studying “principles” heading "scientific knowledge" in Brazil from the early 1930’s in the context of “globalization” of knowledge. In order to carry out such a study, this article focus on the ideology of the university as a “need” (Baudrillard, 1972) for a country, particularly in Brazil. In fact, until 20th century Brazil survived in or resisted to adopt this European ideal: the university.

University became a global ideal since it got space everywhere. In fact, the ideology of “being a nation implies in adopting the university system” acted to extend to another countries such kind of organization, so the ideology is “to become global”. No one will deny that the University of São Paulo – USP -- was a crucial initiative in the consolidation of science in Brazil. As a result Brazil widened its borders.

The general culture taught at the USP was originally conceived to be introduced by the European celebrities inasmuch as the it was believed that the university sprit would be established and recognised if transmitted by such scholars. To some extend such a belief succeeded since the globalization of scientific knowledge in Brazil would also occur under the influence of these European celebrities.

A SURVEY ON THE PORTUGUESE MATHEMATICIAN JOSÉ MORGADO JÚNIOR

Life and work

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ABSTRACT

José Morgado Júnior (1921-2003) was a great Professor, a distinguished algebraist and a very special human being. His influence was more evident in his own country (Portugal) and in Brazil, where he was exiled for a long time. His work is known in some other countries. With this study we would like to contribute to an enhanced knowledge of José Morgado Júnior and of his work

We present some phases of José Morgado Júnior's life and work, with more emphasis in his first book (1956) on lattices and in his first paper (1983; 1984) on number theory involving the Fibonacci sequence. This book of 1956 was written during the time that he was prisoner, without any contact outside. The referred paper contains a very nice generalization of a result of Hoggat and Bergum on Fibonacci numbers.

BERNT MICHAEL HOLMBOE'S TEXTBOOKS

and the development of mathematical analysis in the 19th century

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ABSTRACT

Bernt Michael Holmboe (1795–1850) was a teacher at Christiania Kathedralskole in Norway, from 1818 till 1826. After that he was lecturer at the University of Christiania until 1834, when he was appointed professor in pure mathematics, a position he held until his death in 1850. Holmboe wrote textbooks in arithmetic, geometry, stereometry and trigonometry for the learned schools in Norway, and one textbook in higher mathematics, and some of them came in several editions. This paper will focus on the first three editions of the textbook in arithmetic, and the way he dealt with the rigour in mathematical analysis.

In the first half of the 19th century there was a growing demand for rigour in the foundations and methods of mathematical analysis, and this led to a thorough reconceptualisation of the foundations of analysis. Niels Henrik Abel (1802–1829) complained in a letter to Christopher Hansteen (1784–1873) in 1826 that mathematical analysis totally lacked any plan and system, and that very few theorems in the higher analysis had been proved with convincing rigour.

The teaching of mathematics was an important motivation for this rigourisation, and I will try to demonstrate that the textbooks of Holmboe reflected the contemporary development in mathematical analysis.

1 Introduction

The 19th century was an important century in the development of the modern number concept. After the introduction of new methods like analytic geometry, differential and integral calculus, and algebraic transformation of equations in the 17th century, the 18th century gave the world influential results by the use of these methods. Calculations with *negative numbers* became unproblematic in this century, and we also saw the start of definitions of what later developed into what we now know as irrational numbers. What characterized the 19th century was the demand for rigour with the basis of the methods and fundamental concepts. From the beginning of the century the concept of number was thoroughly reinvestigated. The demand for purity in methodology did *not* allow the use of geometrical or purely intuitive arguments as basis of the number concepts, and the development led finally to the definition of the *real numbers* in 1872 by Richard Dedekind (1831–1916) and other equivalent definitions by mathematicians like Cantor, Weierstrass, Meré etc. There were two mathematicians who made important contributions in the definition of *irrational numbers* in the beginning of the 19th century, namely Bernard Bolzano (1781–1848) and Martin Ohm (1792–1872). (Gericke 1996)

The kingdom Denmark/Norway introduced a new school reform around the year 1800 which in many ways strengthened the position of the discipline of mathematics, and from now on may we talk about proper teaching in mathematics in the higher education (Piene 1937). There was much work done in the last decades of the 18th century to reintroduce mathematics as a subject in school.

The motivation and background for my research is to make an analysis of the development of mathematics education – and the didactical debate – in the first half of the 19th century in Norway, in view of the development of mathematical analysis, and to throw some light on Bernt Michael Holmboe's significance.

This paper is based on an oral presentation given at the ESU–6 conference in Vienna. It is part of an ongoing PhD project about Bernt Michael Holmboe and his textbooks in mathematics, and it is work in progress.

1.1 Bernt Michael Holmboe

Bernt Michael Holmboe was born on the 23rd of March 1795 in Vang in Valdres, centrally situated in Southern Norway, and he died on the 28th of March 1850, at the age of 55 years and 5 days. Holmboe was a teacher at *Christiania Kathedralskole* from 1818 till 1826, and after that he was lecturer at the *University of Christiania* until 1834, when he was appointed professor in mathematics, a position he had until his untimely death. Holmboe's home burnt down shortly after his death, and some of his letters and works were lost, in addition to some of Abel's letters and works. (Bjerknes 1925: 56,79)

Among Holmboe's students we find great mathematicians like Niels Henrik Abel, Ole Jacob Broch and Carl Anton Bjerknes. Holmboe proclaimed that in no other subject did novices complain more than in mathematics (Piene 1937), and his aim was to make the students familiar with mathematical signs *before* a more methodical study. He further stated that unless pupils engaged in «uninterrupted practice»[1] even persons with several years of education would find that mathematics is «something of a mind consuming and boring matter».[2] The lecture notes of Carl Anton Bjerknes shows, however, that Holmboe's teaching was characterized by *pre-abelian* times, in spite of his knowledge of Abel and his works (Bjerknes 1925).

Holmboe wrote in a letter to the then 24 years old Carl Anton Bjerknes (Holmboe 1849) that he is inspired by the great French mathematician Joseph-Louis Lagrange (1736–1813). Bjerknes had asked Holmboe to advise him about studies in mathematics, and Holmboe wrote «*The best I have to state in this respect is to inform you about some notes from Lagrange and some rules and remarks by him regarding the study of mathematics, which I found in Lindemanns and Bohnebergers* Zeitschrift für Astronomie *about 30 years ago ... Those who really want, should read Euler, because in his works all is clear, well said, well calculated, because there is an abundance of good examples, and because one should always study the sources*».[3]

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[1] «idelig øvelse»

[2] «noget åndsfortærende og kjedsommelig tøi»

[3] «Som det bedste jeg her i denne Henseende veed at anføre, vil jeg meddelle nogle Notiser om Lagrange og en Deel Regler og Bemærkninger av ham angaaende Mathematikens Studium, hvilke jeg for omtrent 30 Aar siden fandt i Lindmann's (sic!) og Bohnebergers *Zeitschrift für Astronomie* ... De som virkelig vil, bør lese Euler, fordi i hans skrifter alt er klart, godt sagt, godt regnet, fordi de vrimler av gode eksempler og fordi man altid bør studere kildene.»

1.2 Holmboe's textbooks

These are the textbooks that Holmboe wrote, and I will in this paper focus on the three first editions of the Arithmetic (Holmboe 1825, 1844, 1850), the three editions that came out in Holmboe's lifetime.

Title	Edition	Edited by	Publisher
Lærebog i Mathematiken. Første Deel, Inneholdende Indledning til Mathematiken samt Begyndelsesgrundene til Arithmetiken	1825		Jacob Lehmann
	1844		J. Lehmanns Enke
	1850		J. Chr. Abelsted
	1855		R. Hviids Enke
	1860		R. Hviids Enke
Lærebog i Mathematiken. Anden Deel, Inneholdende Begyndelsesgrundene til Geometrien	1827		Jacob Lehmann
	1833		Jacob C. Abelsted
	1851	Jens Odén	R. Hviids Enke
	1857	Jens Odén	J. W. Cappelen
Stereometrie	1833		C. L. Rosbaum
	1859	C. A. Bjerknes	J. Chr. Abelsted
Plan og sphærisk Trigonometrie	1834		C. L. Rosbaum
Lærebog i den høiere Mathematik. Første Deel	1849		Chr. Grøndahl

2 The development of mathematical analysis

I will now take a closer look at three important participants in the development of mathematical analysis.

2.1 Bernard Bolzano

An intuitive and geometrical interpretation of real numbers did not satisfy the 19th century mathematicians' demand for purity in methodology, and the need for a new understanding of real numbers did arise in connection with the proof of the *intermediate value theorem*. This theorem was proven by Bolzano (1817), and both continuity of functions and convergence of infinite series are defined and used correctly in this paper, as it is understood in a modern sense. Bolzano's comprehensive paper was later translated into English (Russ 1980, 2004), and the latest of these two translations forms the basis of the following short description.

According to Russ (1980: 157), Bolzano's paper includes «*the criterion for the (pointwise) convergence of an infinite series, although the proof of its sufficiency, prior to any definition or construction of the real numbers, is inevitably inadequate*». This criterion is, however, *not* concerning the definition of convergence, but the Bolzano-Weierstrass theorem, which states that every bounded sequence has a convergent subsequence.

Bolzano's only strict requirement is «*that examples never be put forward instead* *of* proofs *and that the essence* [Wesenheit] *of a deduction never be based on the merely figurative use of phrases or on associated ideas, so that the deduction itself becomes void as soon as these are changed*»[4] (Russ 2004: 256).

The comprehension of the concepts of real numbers is clear when Bolzano emphasizes that «*between any two nearby values of an* independent variable*, such as the root x of a function, there are always infinitely many* intermediate values.» He continues that for any continuous function, there is no *last* x which makes it negative, and there is no *first* x which makes it positive (Russ 2004: 258). Bolzano states in a theorem (Russ 2004: 266–67) that if in a series[5] of quantities, the difference between the nth term and every later term is smaller than any given quantity if n has been taken large enough, then there is always a constant quantity, and only one, which the terms of the series approach, and to which it can come as near as desired if the series is continued far enough. In a following note (Russ 2004: 268), he explains that if one is trying to determine the value of such a quantity as described above, namely by using one of the terms of which the given series is composed, then the value cannot be determined entirely accurately unless all terms after a certain term are equal to one another. One cannot, however, conclude that the value of a quantity is irrational if it cannot be determined accurately by the terms of a certain series.

It is worthwhile to emphasize the fact that Bolzano's notion of «irrational» is *not* the classical definition by the Greeks. Russ (2004: 2) writes that «*In the 1830s, he* [Bolzano] *began an elaborate and original construction of a form of real numbers – his so-called 'measurable numbers'. ... He formulated and proved (1817) the greatest lower bound property of real numbers which is equivalent to what was to be called the Bolzano-Weierstrass theorem. He later gave a superior proof with the aid of his measurable numbers.*»

Bolzano introduces *measurable numbers* in his *Pure Theory of Numbers*[6] (Russ 2004: 347–49, 360–61). According to the definition, S is *measurable* when

$$\forall q \in \mathbb{N} \quad \exists p \in \mathbb{Z} \quad \text{such that} \quad S \;=\; \frac{p}{q} + p_1 \;=\; \frac{p+1}{q} - p_2 \quad \text{where} \quad p_1 \geq 0, \quad p_2 > 0$$

In other words

$$\frac{p}{q} \;\leq\; S \;<\; \frac{p+1}{q}$$

Bolzano explains that p_1 and p_2 denotes a pair of strictly positive number expressions, the former possibly denoting zero (Russ 2004: 361).[7]

[4]Bolzano also criticised Gauss's original proof of the fundamental theorem of algebra of 1799, because Gauss here used geometrical considerations to prove an algebraic theorem (Otte 2009: 53). Bolzano did not doubt the validity of the theorem, but he criticised the «impurity» of the method.

[5]Russ uses the word «*series*» in this English translation of Bolzano's theorem, where the text might indicate that «*sequence*» would be correct. An equivalent use of «*series [Reihe]*» is found in Bolzano's *Paradoxien des Unendlichen*, (Russ 2004: 602).

[6]«Reine Zahlenlehre». Was not published until late 20th century (Russ 2004: 681). Bolzano wrote in a letter, dated 5th of April, 1835, to one of his former students, that he had «*one book near completion with the title* Pure Theory of Numbers *consisting of two volumes: an* Introduction to Mathematics*, the first concepts of the general theory of quantity, and then the* Theory of Numbers *itself*» (Russ 2004: 347).

[7]Bolzano denoted them p^1 and p^2, but the superscripts only distinguished, they where not powers (Russ 2004: 349). I have chosen to use subscripts to avoid confusion.

The measurable number may be used to measure, or determine by approximation, the magnitude or quantity. Bolzano called the fraction $\frac{p}{q}$ the *measuring fraction*, and the fraction $\frac{p+1}{q}$ the *next greater fraction*. p_1 is called the *completion* of the measuring fraction since $S = \frac{p}{q} + p_1$. Every *rational number* is a measurable number where $p_1 = 0$, and indeed a complete measure.

Abel is the only known reference that Bolzano was known already in the 1820s, as he mentions Bolzano in his Paris notes (Schubring 1993: 45). Schubring (1993: 50) writes that during the four months Niels Henrik Abel stayed in Berlin in 1825, he was in close contact with August Leopold Crelle and his mathematical circle, where he was engaged in intensive conversations on all mathematical issues. Crelle had Bolzano's three booklets in his personal library, and Abel's reading of Bolzano was part of this process of communication.

2.2 Augustin Louis Cauchy

The major work of AUGUSTIN-LOUIS CAUCHY (1789–1857), *Cours d'analyse* from 1821 (Bradley and Sandifer 2009), was designed for students at *École Polytechnique*, and Cauchy was concerned with developing the basic theorems of the calculus as rigorously as possible.

Cauchy makes a clear distinction between *number* and *quantity* (Bradley and Sandifer 2009: 5–6,267–68), where *numbers* arise from the absolute measure of magnitudes, and *quantities* are real positive or negative quantities, that is to say numbers preceded by the signs «+» or «−». Quantities are intended to express *increase* or *decrease*,[8] and to indicate this intention, we represent the sizes that ought to serve as increase or decrease as numbers preceded by a sign. The *numerical value*, or absolute value, of a quantity is the number that forms the basis of the quantity, and we may perform arithmetical operations on a quantity. Absolute numbers are considered to be only the *proper* numbers. Negative numbers are quantities, and the difference between a number and a quantity is the difference that is between an absolute number, and a number with an additional quality, for instance a sign. A positive number may therefore be *both* a number and a quantity (Schubring 2005: 446).

Cauchy says that a variable is called *variable* if it is able to take on successively many different values, as opposed to a *constant* when it takes on a fixed and determined value. A variable quantity becomes *infinitely small* when its numerical value decreases indefinitely so that it converges towards the limit zero. Corresponding, a variable quantity becomes *infinitely large* when its numerical value increases indefinitely so that it converges towards the limit ∞ (Bradley and Sandifer 2009: 6,21-22).

He had to leave France for political reasons, and he met Bolzano in Prague (Schubring 2005: 429). Some historians (*see* Grabiner 2005: 11) assert that Cauchy was very influenced by Bolzano, and especially his proof of the *intermediate value theorem* (Bolzano 1817), when he wrote *Cours d'analyse*. His works made, however, a new standard for the demands of rigour in connection with concepts like *limit* and *convergence*. Cauchy neither started nor completed the rigourization of analysis, but he was, according to Grabiner (2005: 166), more than any other mathematician responsible for the first great revolution in mathematical rigourization since the Greeks in the Antiquity.

[8] *accroissements* or *diminuitions* (Schubring 2005: 446)

2.3 Martin Ohm

Martin Ohm was autodidact in mathematics, and a teacher, which may have lead him to the idea that mathematics needed to be completely revised, in order to reach a broader public and to achieve a higher logical clarity. He made an attempt to make a completely consistent foundation for mathematics (Ohm 1832) – to found a concept of numbers free of contradictions, and to provide an appropriate conception of negative numbers for it. Prior to that, he wrote an elementary book on number theory (Ohm 1816), and a book in pure elementary mathematics in two volumes.[9] The separation of number and quantity was the core of Ohms number concept, and there are no other numbers than «absolute integer numbers». Zero, negative numbers, rational and irrational numbers are extensions. (Schubring 2005: 521pp)

Ohm defines the arithmetical operations *addition* and *multiplication* traditionally in the first volume of pure elementary mathematics (Gericke 1996: 136). One adds two number by imagining a number that has as many units as the two numbers together, and one multiplies a number a by a number m by adding a, m times to itself. The quotient $\frac{a}{b}$ is defined as the number that gives a when it is multiplied by b.

3 Textbooks and irrational numbers

In the following sections, I will show some of my findings concerning the development of Holmboe's definitions of irrational numbers, and compare that with definitions found in the other textbooks in arithmetics and algebra in the Danish language, that may have been used or known in the learned schools in Norway in the first half of the 19th century. The focus will be on their way of handling the rigour in mathematical analysis.

The mathematical notation in the English translations are in some cases slightly modernized by me, but the original quotation in the footnotes are unaltered.

3.1 Bernt Michael Holmboe

In the first edition of Holmboe's textbook in arithmetics (Holmboe 1825), irrational numbers are defined as

> *Any number, that can be expressed neither as a whole number nor as a fraction, whose numerator and denominator are whole and finite numbers, is called an* irrational *number.*[10]

This definition tells us what an irrational number *isn't*, it doesn't tell us what it *is*. Opposed to irrational numbers, all whole numbers and fractions, whose numerator and denominator are whole and finite numbers, are called *rational* numbers. The definition has some corollaries, or supplements, and the following corollary explains intuitively that

> *One can always find a rational number, whose value approaches the value of a given irrational root, so that the difference between them is less than any given unit fraction.*[11]

[9] *Die reine Elementar-Mathematik*, 1825 and 1826

[10] «Ethvert Tal, der hverken kan udtrykkes som et heelt Tal eller som en Brøk, hvis Tæller og Nævner ere hele og endelige Tal, kaldes er *irrationalt* Tal.» (Holmboe 1825: 134)

[11] «Man kan altid finde er rationalt tal, hvis Værdie nærmer sig Værdien af en given irrational Rod saameget, at Forskjellen mellem begge er mindre end en given Brøkeenhed.» (Holmboe 1825: 136)

The definition of irrational numbers in Holmboe (1844), and subsequent editions, may indicate an influence by Bolzano and his definition of measurable numbers. It is also noteworthy that Holmboe now specifies *magnitude*, and not number.

> *Any magnitude, that can be expressed neither as a whole number nor as a fraction, whose numerator and denominator are whole and finite numbers, but whose value always falls between two fractions* $\frac{t}{n}$ *and* $\frac{t+1}{n}$*, where* t *and* n *are whole numbers, and where one can make* n *larger than any given number, is called an* irrational *number.*[12]

This following corollary from Holmboe (1844) shows how two magnitudes between the same two limits must be the same magnitude. In Holmboe (1850), the specification of *irrational* is taken out – the statement is valid for *real* magnitudes, rational and irrational.

> *If two irrational, positive magnitudes* P *and* Q*, both independent of* n *and between boundaries of the form* r *and* $r + \frac{a}{n}$*, are in such a way that* $r < P < r + \frac{a}{n}$ *and* $r < Q < r + \frac{a}{n}$*, where one can make* n *larger than any given number and* a *is finite. Then is* $P = Q$*.*[13]

Finally, this definition from Holmboe (1850) shows that if x and y are two irrational magnitudes, squeezed between two pairs of rational limits, then the sum $x+y$ is squeezed between the sums of the lower and of the upper limits, and the difference of these to sums of limits also disappears when n grows infinitely, which makes the sum $x + y$ unique.

> *If one or both of two magnitudes,* x *and* y*, are irrational, and where* $\frac{t}{n} \leq x < \frac{t+1}{n}$ *and* $\frac{p}{n} \leq y < \frac{p+1}{n}$*, and one can make* n *larger than any given number. The sum* $x + y$ *is then to be understood as the common boundary for the sums* $\frac{t}{n} + \frac{p}{n}$ *and* $\frac{t+1}{n} + \frac{p+1}{n}$*, whose difference is* $\frac{2}{n}$*, which disappears when* n *grows infinitely.*[14]

3.2 Hans Christian Linderup

The Danish teacher of mathematics, HANS CHRISTIAN LINDERUP (1763–1809) published a basic textbook (Linderup 1807), where he proves that a power of an irreducible fraction never can be a whole number. As a corollary to this he claims that *when the root of a whole number is not a whole number, it is also not a fraction, whose numerator and denominator are finite numbers, but as any quantity must be expressed by whole numbers or fractions, must this root necessarily be a fraction; it is consequently a fraction, whose*

[12] «Enhver Størrelse, der hverken kan udtrykkes som et heelt Tal eller som en Brøk, hvis Tæller og Nævner ere hele og endelige Tal, men hvis Værdie altid falder mellem to Brøker $\frac{t}{n}$ og $\frac{t+1}{n}$, hvor t og n ere hele Tal, og hvor man kan gjøre n større end ethvert givet Tal, kaldes er *irrationalt* Tal. (Holmboe 1844: 128)»

[13] «Falde 2 irrationale positive af n uafhengige Størrelser P og Q mellem Grændser af Formen r og $r + \frac{a}{n}$, saaledes at $P > r$, $P < r + \frac{a}{n}$, $Q > r$, $Q < r + \frac{a}{n}$, hvor stor man gjør n, naar a er en endelig Størrelse: saa er $P = Q$;» (Holmboe 1844: 128–29)

[14] «Er af Størrelserne x og y den ene eller begge irrationale, $x =$ eller $> \frac{t}{n}$ og $x < \frac{t+1}{n}$, $y =$ eller $> \frac{p}{n}$ og $y < \frac{p+1}{n}$, hvor stor man gjør n, saa forstaaes ved Summen $x + y$ den fælles Grændse for Summerne $\frac{t}{n} + \frac{p}{n}$ og $\frac{t+1}{n} + \frac{p+1}{n}$, hvilke Summers Forskjel er $\frac{2}{n}$, der forsvinder med $\frac{1}{n}$, det er, naar n voxer i det Uendelige.» (Holmboe 1850: 115–16)

numerator and denominator are infinitely large, and therefore may never be expressed exact. Such root magnitudes are called irrational numbers.[15] The term *infinitely large* must here imply that the fraction is a limit, reached by stepwise expanding the numerator and denominator. This mode of expression is intuitive, and does not have an evident mathematical definition.

Linderup is here talking about *any* root, and he specifies this in the next corollary, by defining that if the square root of a number is a whole number, then the original number is a *perfect square number*,[16] if the cube root of a number is a whole number, then the original number is a *perfect cube number*,[17] and likewise for any power. If the root on the other hand is *not* a whole number, then the number is called an *imperfect square and cube number*,[18] or in general, an *imperfect power*.[19] He concludes that *all* roots of imperfect powers are irrational numbers.

3.3 Ole Jacob Broch

Holmboe's former student, and successor as professor in mathematics at the university, Ole Jacob Broch (1818–1889), published in 1839 a collection of exercises and examples (Broch 1839) to Holmboe's textbook. He also published in 1860 a textbook in arithmetic and algebra which succeeded Holmboe's textbook in the learned schools. In this book he defines in the introduction that *mathematics* is the science about *magnitudes* and their connections. The study of magnitudes separate from any matter is called *pure mathematics*, and the study of magnitudes belonging to material objects is called *applied mathematics*.[20] He continues to say that if we look at several magnitudes of the same kind, and we focus first on one specific magnitude, and then on the whole collection of magnitudes, then we have the conception of *one* magnitude and of *several* magnitudes. The word *unit* is used to characterize any of these homogeneous magnitudes, and the word *number* is used to characterize a collection of units as well as the unit itself.[21]

He shows in the section about powers and roots that when the root value of a natural number is *not* a natural number, it can *not* be a fraction either, where both numerator and denominator are whole and finite numbers (Broch 1860: 184). He continues that *one may always approximate its value by a fraction, such that the difference between the root value and the fraction is smaller than any given finite positive magnitude, however*

[15] «Naar altsaa Roden til et heelt Tal ikke er et helt Tal, er det heller ingen Brøk, hvis Tæller og Nævner ere endelige Tal; men da enhver Mængde maa kunde udtrykkes ved hele Tal og Brøk, maa denne Rod nødvendig være en Brøk; den er altsaa en Brøk, hvis Tæller og Nævner ere uendelig store, og følgelig aldrig nøiagtig kan udtrykkes. Saadanne Rod-Størrelser kaldes irrationale tal.» (Linderup 1807: 99).

[16] fuldkomment Quadrat-Tal

[17] fuldkomment Cubik-Tal

[18] ufuldkomment Quadrat- og Cubic-Tal

[19] ufuldkommen Potens

[20] «*Mathematik* er Læren om *Størrelser* og deres Forbindelser. Forsaavidt den betragter Størrelserne afsondrede fra enhver Materie, kaldes den *reen Mathematik*; betragter den derimod Størrelserne som henhørende til materielle Gjenstande, kaldes den *anvendt Mathematik*.» (Broch 1860: 1)

[21] «Naar vi betragte flere Størrelser af samme Art, og vi henvende vor Opmærksomhed først paa een af disse Størrelser i Særdeleshed, dernæst paa den hele Samling af Størrelser, saa have vi Begrebet om *een* Størrelse og om *flere* Størrelser. Man benytter ordet *Enhed* for at betegne en hvilkensomhelst af disse eensartede Størrelser, og man betegner ved ordet *Tal* saavel den hele samling ef Enheder som ogsaa Enheden selv.» (Broch 1860: 1)

small it is chosen.[22]

$$p \ < \ \sqrt[n]{a} \ \ < \ p + \frac{1}{x}$$

p is here simply called a *number*,[23] but it is obvious that Broch means a rational number, and $\frac{1}{x}$ is a *unit fraction*.[24] An *irrational magnitude* is then defined as a magnitude that cannot be expressed by a finite number of digits, either as a whole number or as a fraction, whose numerator and denominator are whole and finite numbers, but whose value always may be given within two limits, specified by a finite number of digits, and whose difference may be made smaller than any positive magnitude, or may approach zero as much as one wants. If the limits that the irrational magnitude is bounded by are positive, then the magnitude is an *irrational number* (Broch 1860: 186–87).[25] Broch's definition is general, since the irrational magnitude does not need to be a root. The irrational magnitude may also be a magnitude that cannot be expressed by a root with a finite number of digits under the radical sign.

Unlike a rational number, an irrational may never be transformed into a periodic decimal fraction. He also shows that two irrational magnitudes are equal if they are bounded by the same two limits, whose difference may be made smaller than any given positive magnitude, however small it is chosen. In other words

$$p \ < \ a \ < \ p + \frac{1}{x} \quad \wedge \quad p \ < \ b \ < \ p + \frac{1}{x} \quad \Longrightarrow \quad a \ = \ b$$

Furthermore, by adding or multiplying these limits of two irrational numbers respectively, the difference between the sum or product may be made smaller than any given positive magnitude, however small it is chosen. This extends the concept of addition and multiplication to include irrational numbers. All theorems regarding addition and multiplication are only proven for rational numbers, and must therefore also be valid for the rational limits of irrational numbers. As the difference of these limits may be made smaller than any finite positive magnitude, and these limits determine the irrational numbers, then must consequently all these theorems also be valid for irrational numbers.

The doctrines about subtraction and division are based on the doctrines about addition and multiplication, and consequently must all theorems valid for rational numbers also be valid for irrational numbers. Likewise must the doctrines about powers be valid if the root is irrational. (Broch 1860: 189–90)

[22] «Men man kan isaafald dog stedse tilnærmelsesviis udtrykke dens Værdi ved en saadan Brøk, saaledes at dennes forskjel fra Rodstørrelsen bliver mindre end enhver given endelig positiv Størrelse, hvor liden denne end er valgt.» (Broch 1860: 185)
Broch's original symbolic notation is «$\sqrt[n]{a} \ \ > \ p$ og (*and*) $< \ p + \frac{1}{x}$»

[23] Tal

[24] Brøkenhed

[25] «En Størrelse, som ikke kan udtrykkes med et endelig Antal Ziffre, hverken som et heelt Tal eller som en Brøk, hvis Tæller og Nævner ere hele og endelige Tal, men for hvis Værdi der altid kan angives to med et endeligt Antal Ziffre udtrykte Grændser, hvis Differents kan gjøres mindre end enhver given endelig positiv Størrelse, eller bringes til at nærme sig Nul saameget man vil, kaldes en *irrational Størrelse*, eller, hvis dens Grændser ere positive, et *irrationalt Tal.*»

4 A brief summary

The following is an overview of the significant events mentioned in this paper.

1816–17: Bernhard Bolzano publishes *binomische Lehrsatz*, and proof of the *intermediate value theorem*
1816: Martin Ohm publishes elementary number theory
1821: Cauchy publishes «*Cours d'analyse*»
1822–: Martin Ohm works on «*Complete system ...* »
1825: First edition of Arithmetiken
1825–26: Abel is in Berlin where he meets Crelle and Ohm, and in Paris where he meets Cauchy.
1830ies: Bolzano writes about *measurable* numbers (unpubl.)
1844: Second edition of Arithmetiken
1850: Third edition of Arithmetiken

Abel, who was Holmboe's student at Christiania Kathedralskole and who remained Holmboe's confident friend the rest of his life, is the only known reference that Bolzano was known already in the 1820ies. He mentions Bolzano in his Paris notes. August Leopold Crelle *had* the three books that Bolzano published in 1816–17. The way Holmboe writes about irrational numbers in the editions from 1844 and later is very similar to Bolzano's description of measurable numbers, and not similar to other contemporary textbook authors.

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TEACHING LINEAR ALGEBRA IN THE PORTUGUESE UNIVERSITIES

The case of the University of Coimbra

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ABSTRACT

We study the evolution of the teaching of Linear Algebra in Mathematics, Sciences and Engineering courses in Portuguese universities.
Initially elementary Linear Algebra topics, such as: matrices, determinants and linear systems, were taught in other disciplines of Mathematics.
With a great reform of university studies in the early 1970's, Linear Algebra emerged as a separate discipline for Mathematics and Engineering courses in Portuguese universities.
In this work we specify the case of the University of Coimbra.

AN ANALYSIS OF THE UNIVERSITY MATHEMATICAL EDUCATION IN PORTUGAL IN THE LATE EIGHTEENTH CENTURY

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ABSTRACT

The Portuguese University was founded in Lisbon in the late 13th century (by Papal Bull of Nicholas IV in August 9, 1290) and was composed by the Faculties of Arts, Medicine and Laws. This Portuguese institution, whose localization alternated between Lisbon and Coimbra, where it was established in 1537, held the monopoly of superior education in Portugal during about five centuries.

The teaching of Mathematics at the University had gone through long periods without teachers and its importance in the University curricula was secondary. This situation lasted until the reform of the University of the Marquis de Pombal, enacted by the Statutes of the University of Coimbra (in September 29, 1772). Those Statutes were made to place the Portuguese University as one of the best of the European enlightened nations and, according to Castro Freire (1872), so it happened. We do not agree that it had been exactly like that, but it is also true that this reform has fundamentally altered the structure and modernity of the Portuguese University.

Regarding Mathematics, the Statutes stressed the importance of this science in order to take a prominent place on the University and assigned a significant importance to the mathematical studies. This importance given to Mathematics by the Statutes led to the creation of the Faculty of Mathematics that presents a mathematical curriculum that we believe to be in tune with the ones in enlightened Europe.

We intend to present some of the most innovative aspects introduced in the teaching of Mathematics in Portugal at the time of the 1772 reform. We will give some special attention to the discipline of the second year of the Mathematical Course, Algebra, once the Statutes themselves value their teaching.

1 Brief description of the Portuguese educational system in the 18th century

The late 18th century was a period in which Portugal produced several reforms in the studies, both at university and non-university studies. The aim of these reforms was to improve the quality of our educational system and to place our University at the same level of the European Enlightenment Universities. But, why such reforms were needed? Was it crucial for the survival of the Portuguese educational system or was it just a need to change the ruling class, the Jesuits?

In fact, we can not talk about the Portuguese education until the 18th century without talking about the Society of Jesus because they held almost complete control of the Portuguese educational system between 1540 and 1759.

The religious order Society of Jesus was founded by Saint Ignatius of Loyola (1491-1556) in August 15, 1534 and it was confirmed by Pope Paulus III (b. 1468, 1534 − 1549) in September 27, 1540. In that same year the first two Jesuits arrive to Portugal, two of its founders: Simão Rodrigues (1510-1579) and Saint Francisco Xavier (1506-1552). While

Francisco Xavier leaves to evangelize the East, Simão Rodrigues stays in Portugal, to start the implementation of the Jesuit education in our country.

The first house that the Jesuits owned in Portugal (and in the world) was the monastery of St. Antão, founded in Lisbon on January 5, 1542. Their first school, for the exclusive use of Jesuit members, was the Jesus' College in Coimbra, founded in July 2, 1542. In 1551 is established the Espírito Santo College in Évora, where classes began public in August 28, 1553. Earlier, in February 1553, the monastery of St. Antão becomes the first Jesuit public school in Portugal and, we believe, the first one in the world. From its inception, the College of St. Antão had a large influx of students and, as we shall see, it was here that Mathematics gains more importance in Portugal until the first half of the 18^{th} century, and not in the University.

In Coimbra, where the University was settled since 1537, the Society of Jesus gains control of the College of Arts. At that time, this College had the monopoly of the public education in Coimbra and was under the administration of the University of Coimbra. The interference of the Jesuits in the College of Arts, dependent of the University, causes several problems between both institutions. The disputes between these two institutions were reported in *Compêndio Histórico*; a compendium that gives a very anti-Jesuit point of view of their influence in the University.

The expansion of Jesuit education in Portugal continues: the Professed House of S. Roque in Lisbon in 1553, the colleges of Braga and Porto in 1560, Bragança in 1562, Funchal and Angra in 1570, Ponta Delgada in 1591, Faro in 1599, Portalegre in 1605, Santarém in 1621, Elvas in 1644, Faial in 1652, Setúbal in 1655, and so one. On this list are missing, for instance, the colleges that the Jesuits ran in the Portuguese domains. We do not have the data of their growth overseas. We can imagine that it was not as fast and extended as it was in Continental Portugal (especially in African territories), but certainly there were several Jesuit colleges in those territories to add to those listed above.

Consequently, from the arrival of the first Jesuits in 1540 until they were expelled in 1759 they had progressively increasing their dominance and control of the Portuguese educational system. What was the reason for such rapid growth in our educational system? We can explain it as a natural response to the necessity for training missionaries to go to the vast territories that the Roman Jesuit curia assigned to Portugal, like some parts of China, India or Japan.

One exception to the Jesuit domain in the Portuguese educational system was the University of Coimbra. Although it has never been subdued to the Society of Jesus, its influence in the University was felt and reported in the *Compêndio Histórico*.

The Jesuit educational characteristics were, at that time, a severe code and strict rules and also a very conservative attitude to education. For example, scientific investigation was not highly valued. Although some Jesuit mathematicians are known for developing mathematics, their main objective was not the teaching but the preaching of the Catholic religion.

As we know, the Jesuit studies were regulated by the *Ratio Studiorum* that, regarding Mathematics, was not very detailed. There were only three items concerning Mathematics. The first one describes the time of the classes, to whom it was intended, and its program: Euclid's Elements and some Geography or Astronomy. The second one state that the students should, once a month, resolve some celebrated mathematical problem and the third item was related to the revisions that should be made once a month.

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REGVLAE
PROFESSORIS
MATHEMATICÆ.

*Qui auctores quo tẽpore quibus explicandi. *p. 4. c. 12. C.*

1 PHYSICAE auditoribus explicet in schola tribus circiter horæ quadrantibus Euclidis elementa: in quibus postquã per duos menses aliquantisper versati fuerint, aliquid Geographiæ, vel Sphæræ, vel eorum, quæ libenter audiri solent, adiungat: idque cum Euclide, vel eodem die, vel alternis diebus.

Problema 2 Singulis, aut alternis saltem mensibus ab aliquo auditorum magno Philosophorum, Theologorumque conuentu illustre problema mathematicum enodandum curet; posteaque, si videbitur, argumentandum.

Repetitio 3 Semel in mense, idque fere die sabbathi, prælectionis loco præcipua quæque per eum mensem explicata publicè repetantur.

RE-

Ratio Studiorum, 1603, p.76

The Mathematical program in the Ratio Studiorum was not very ambitious. So, in the Portuguese Jesuit schools the teaching of Mathematics did not go beyond elementary Mathematic. Meanwhile, responding to a request of the Portuguese king, it was founded in Lisbon the Aula da Esfera. This was a public Mathematics class that took place in the College of Santo Antão and it worked continuously from around 1590 until 1759. Since this Aula was a result of a request by the Portuguese Corte, his curriculum wasn't the Ratio Studiorum curriculum but a mixture of it and other subjects, such as Astronomy, Nautical Science or Military Architecture, more advanced and recent concepts than those that would be taught in other Portuguese Jesuit schools.

Due to the lack of Portuguese teachers in the early years, most of them were foreigners and some of them well known in European science like Cristoforo Borri. These Jesuit scientists came to Portugal with the intention to go in mission to the East but most of them were kept in Portugal to teach Mathematics in *Aula da Esfera*. The best mathematicians were selected to teach there, so this class became the center of the exchange of mathematical knowledge in Portugal, instead of the University of Coimbra as expected. For these reasons, the *Aula da Esfera* was the incoming center to the science discovered in Europe and it was from it that the Portuguese science was know; essentially Astronomical records and Portuguese Nautical science. To testify this opinion, we would like to mention that all Planetary Systems (the Ptolemaic, the Copernican and the Tycho Brahe) were studied and discussed in the *Aula da Esfera*. The reasons for non-acceptance of the Copernican system were not Astronomical or Physical, but Biblical grounds.

If we consider the hierarchy that the Jesuits assigned to their disciplines[1] and the fact that the Mathematics in *Aula da Esfera* was taught outside the curriculum of Philosophy, this Aula was considered subordinate to that of Philosophy and therefore it was considered of less importance and prestige for Jesuit teachers and leaders. This was the main reason why the profession of mathematics was, according to Baldini (2004), considered an unpleasant burden by the foreign missionaries and a transition job, to a more important one, by the Portuguese teachers. To this unimportance of Mathematics Leitão (2007) adds the small interest in the scientific disciplines that had always characterized the Portuguese culture and education. As a consequence, this led to a low quality of Jesuit education, comparatively to Central Europe. Furthermore, the areas of development of mathematics in Portugal fluctuate according to national contexts; it is often that the guidelines of the Portuguese leaders overrode the Jesuit leaders in Rome so that, in the words of Baldini (2004), «the Society's curia in Rome was not responsible for the situation in Portugal, and at one point, it acted decisively against it». Baldini refers to the *Ordenatio*[2] that the Leader of the Society of Jesus, Father Tirso Gonzalez, sends to Portugal in 1692. In this *Ordenatio*, Tirso Gonzales requires several improvements in the teaching of Mathematics in Jesuit schools and point out a series of orders to be fulfilled by the Portuguese leaders.

To these arguments we may add the Portuguese Inquisition, which in this period had a strong power repressing and censoring books making it difficult and very dangerous the entrance of modern books in Portugal, to the reasons that hindered the implementation and the development of the science in Portugal.

Nevertheless, in the beginning of the 18th century, in the reign of King João V (helped by the gold of Brazil) the European scientific culture started entering in our country. For example, some scientific shows, like astronomical observations, were promoted, and several Academies were founded all over the country. For these reasons we can say that this king was a fan of cultural magnificence but not particularly of the scientific results. Even so, it is in the beginning of the 18th century that the Portuguese culture and science began to develop.

In 1758 there was an attempted murder to the king and, a year latter, the Jesuits were considered co-responsible and were expelled from Portugal. Once they were in control of the Portuguese educational system, it is easy to imagine the damage that their expulsion causes to our educational system. In these early years it will goes through serious

1 Theology was the most important and prestigious discipline, followed by the discipline of Philosophy, then Rhetoric and lastly Grammar.

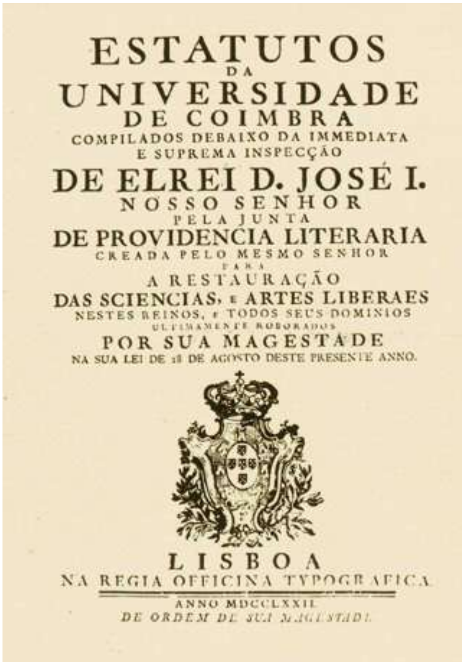
2 Ordenatio ad suscitandum fovendumque in Provincia Lusitaniae Studium Mathematicae. This document was transcribed from the original Latin by Ugo Baldini and Henrique Leitão on pages 648-664 of "appendix A: Documents and Letters" in Baldini (2004). In the same book we can find a translation into Portuguese, on pages 704 to 723, developed by Henrique Leitão and Bernardo Mota, from a partial translation by Vitor Manuel Leal Frost, published by Monteiro, A., 1998, *Inácio Monteiro e o Ensino da Matemática em Portugal no século XVIII*, Coimbra: Departamento de Matemática, pp. 186-192.

difficulties, particularly a severe lack of teachers.

In the same year that the Jesuits were expelled from Portugal the non-University studies were reformed. The aim of this reform was to widespread all over the country the elementary education, which should be financed and managed by the State. But, as we just say, because the education in Portugal was almost completely dependent on the Jesuit teachers, their expulsion led to a strong lack of teachers; both in non-University studies and in University studies. Therefore, the non-Universities studies had to be reformed again by Queen Maria I, only 20 years after their first reform.

2 The Faculty of Mathematics

The University of Coimbra was reformed in 1772; thirteen years after the Jesuits were expelled from Portugal.



ESTATUTOS
DA
UNIVERSIDADE
DE COIMBRA
COMPILADOS DEBAIXO DA IMMEDIATA
E SUPREMA INSPECÇÃO
DE ELREI D. JOSÉ I.
NOSSO SENHOR
PELA JUNTA
DE PROVIDENCIA LITERARIA
CREADA PELO MESMO SENHOR
PARA
A RESTAURAÇÃO
DAS SCIENCIAS, E ARTES LIBERAES
NESTES REINOS, E TODOS SEUS DOMINIOS
ULTIMAMENTE ROBORADOS
POR SUA MAGESTADE
NA SUA LEI DE 28 DE AGOSTO DESTE PRESENTE ANNO.
LISBOA
NA REGIA OFFICINA TYPOGRAFICA.
ANNO MDCCLXXII.
DE ORDEM DE SUA MAGESTADE.

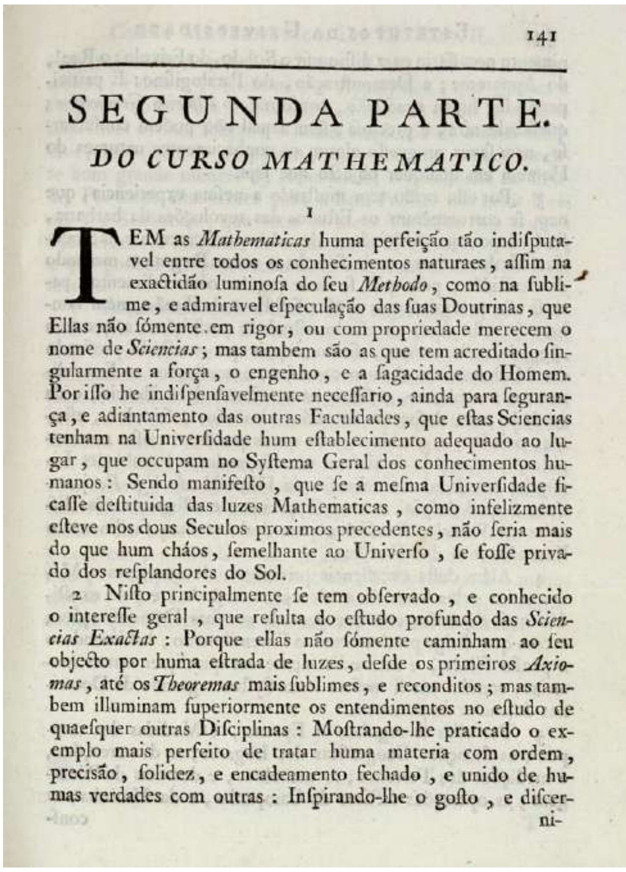
Statutes of the University of Coimbra, 1772

This reform intended to place our University in the top group of the European Enlightenment Universities.

Regarding Mathematics, before 1772, we should point out that for almost two hundred years no one taught Mathematics more than a few years and for more than sixty years that no one teaches Mathematics at all. At that time there were only two disciplines of Mathematics, namely Geometry and Astronomy which were included in the curriculum of Medicine. Since the majority of the University students were from the Canon and Laws, the number of students assigned to Mathematics was small.

The reform of the University of 1772 creates two new Faculties: the Faculty of [Natural] Philosophy (or, as we would call it today, Faculty of Sciences), and the Faculty of Mathematics - which was the first one in the World. This illustrate the importance that Mathematics had in this reform to become an autonomous Faculty in the University. Moreover it provides the infrastructures and resources for the improvement of these two newest Faculties like an Astronomic Observatory, a Chemical Laboratory, a Botanical Garden, a Museum of Natural History and the University Press. The University Press was not for the exclusive use of the newest Faculties; it represents an important new tool available to the whole University.

The importance given to Mathematics by the promoters of this reform is well expressed in the introduction of the Second Part of the Statutes of the University of Coimbra, which concerned to the Course of Mathematics.



141

SEGUNDA PARTE.

DO CURSO MATHEMATICO.

I

TEM as *Mathematicas* huma perfeição tão indiſputavel entre todos os conhecimentos naturaes, aſſim na exactidão luminoſa do ſeu *Methodo*, como na ſublime, e admiravel eſpeculação das ſuas Doutrinas, que Ellas não ſómente em rigor, ou com propriedade merecem o nome de *Sciencias*; mas tambem são as que tem acreditado ſingularmente a força, o engenho, e a ſagacidade do Homem. Por iſſo he indiſpenſavelmente neceſſario, ainda para ſegurança, e adiantamento das outras Faculdades, que eſtas Sciencias tenham na Univerſidade hum eſtablecimento adequado ao lugar, que occupam no Syſtema Geral dos conhecimentos humanos: Sendo manifeſto, que ſe a meſma Univerſidade ficaſſe deſtituida das luzes Mathematicas, como infelizmente eſteve nos dous Seculos proximos precedentes, não ſeria mais do que hum cháos, ſemelhante ao Univerſo, ſe foſſe privado dos reſplandores do Sol.

2 Niſto principalmente ſe tem obſervado, e conhecido o intereſſe geral, que reſulta do eſtudo profundo das *Sciencias Exactas*: Porque ellas não ſómente caminham ao ſeu objecto por huma eſtrada de luzes, deſde os primeiros *Axiomas*, até os *Theoremas* mais ſublimes, e reconditos; mas tambem illuminam ſuperiormente os entendimentos no eſtudo de quaeſquer outras Diſciplinas: Moſtrando-lhe praticado o exemplo mais perfeito de tratar huma materia com ordem, preciſão, ſolidez, e encadeamento fechado, e unido de humas verdades com outras: Inſpirando-lhe o goſto, e diſcerni-

Statutes of the University of Coimbra, 1772, book III, p.141

“Having Mathematics a perfection so unquestionable among all natural knowledge, both in luminous accuracy of his Method, as in the sublime and admirable speculation of its Doctrines, that They, not only for the accuracy or property, deserve the name of Sciences; but also are those who have singularly believed the strength, the creativity and the shrewdness of Man. It is therefore indispensably necessary that, even for security and progress of other Faculties, these Sciences have in the University a suitable establishment for their place in the General System of human knowledge. Being clear that if the University stay devoid of the lights of Mathematics, as unfortunately it stayed in the previous two centuries, there would be no more than a chaos, like the universe, if it were deprived of the splendors of the Sun.”

A very curious requirement of the Statutes was that all University’s students (including Theology’ students!) had to assist to some Mathematics disciplines. This implied an increase in the number of students attending mathematics and consequently its importance in the University.

As stated before, the Second Part of the Statutes concerned the Course of Mathematics, which was established at the Faculty of Mathematics. This was a four years course. During the first year students had to attend to a class of Geometry and one of Natural History, in the Faculty of Philosophy. On the second year they had to attend two classes, one of Algebra and other of Experimental Physics, in the Faculty of Philosophy. The third year is devoted to Mechanics and the fourth to Astronomy.

This Course was clearly a lever for the Faculty of Mathematics promptly begin to graduate professionals in Mathematics.

The Statutes were very precise on writing the program for these disciplines. In Geometry it had to be taught the "*Elements of Arithmetic, and of Geometry, and the Plane Trigonometry; with applications to Geodesy, Stereometry, &c*" [Estatutos, page 166; a more detailed description of the program of this discipline is found in the pages 169-175]. In the discipline of the second year, Algebra, the professor had to teach "*Elements of Literal Calculus; or Elementary Algebra, the Principles of Direct Infinitesimal Calculus, and Inverse; with their application with Sublime Geometry, and Transcendent*" [Estatutos, page 166; with further description on pages 175-182]. In Mechanics it should be taught the "*General Science of movement, with its application to all branches of Kinetics, that constitute the Body of Physical-Mathematical Sciences; as Mechanic, Static, Dynamic, Hydraulic, Hydrostatic, Optic, Dioptrics, &c*" [Estatutos, page 166, also on pages 182-188]. Finally, in Astronomy it should be taught the "*Theory of Planetary motion, both Physical as Geometrical, with the Practice of Calculus, and Astronomical Observations; and with other Sciences, that depend on Astronomy*" [Estatutos, pages 166-167, more detailed on pages 189-195].

The Statutes also predicted the kind of exercises that students had to do: daily, weekly and monthly exercises, written exercises and small dissertations. [Estatutos, pages 197-205].

The acceptance age for the Course of Mathematics was 15 years old and the Mathematical prerequisites were just the knowledge of the four fundamental rules of arithmetic: add, subtract, multiply and divide numbers [Estatutos, pages 150-152 and 154-157].

In order to attract students to this Course, various privileges were given to those who finished their studies in Mathematics, like job and military privileges [Estatutos, pages 149-150].

Regarding the second year discipline, Algebra, the statutes describe it as "the first Science in Mathematics" [Estatutos, page 162], which gives an idea about the importance of Algebra in the Course of Mathematics.

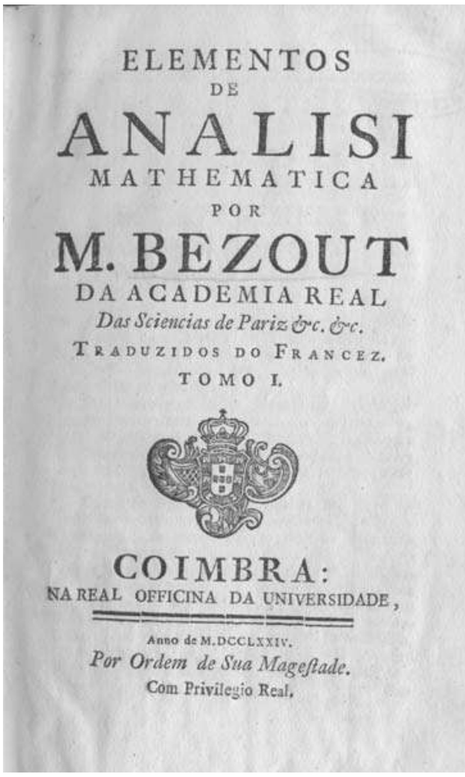
The program of this discipline was:

Elementary Algebra
- Preliminary notions
- Algebraic operations
- Equations
- Applications
- Conic sections

Infinitesimal Algebra
- Differential Calculus
- Differentiation rules
- General theory of curves
- Integral calculus
- Applications

The bibliography was not fixed in the Statutes; it would be chosen by resolution of the Congregation of Mathematics. This means that there is no indication to follow a particular book and it was implicit an update of the bibliographical references. The *Elementos de Analisi Mathematica por Bezout*, translated into Portuguese from his *Cours de*

Mathematiques, was chosen for the discipline of the second year, and it was used as a textbook until 1836, with five editions.


ELEMENTOS
DE
ANALISI
MATHEMATICA
POR
M. BEZOUT
DA ACADEMIA REAL
Das Sciencias de Pariz &c. &c.
TRADUZIDOS DO FRANCEZ.
TOMO I.

COIMBRA:
NA REAL OFFICINA DA UNIVERSIDADE,
Anno de M.DCCLXXIV.
Por Ordem de Sua Mageſtade.
Com Privilegio Real.

Elementos de Analisi Mathematica por Bezout, 1774, 1st edition.

The *Professors of Algebra* between 1772 and 1836 are presented in the following list. In the right column we point out the number of *Substitute Professors*, that is, those who substitute for a certain period of time the responsible Professors that were called to Lisbon by king's request.

Year	Professor	Nº Subst.
1772/1773	--	----------
1773/1774	Miguel Franzini	
1774/1775		
1775/1776		
1776/1777		
1777/1778		1
1778/1779		1
1779/1780	Vitúrio Lopes Rocha	
1780/1781		
1781/1782		
1782/1783		
1783/1784	Manuel José Pereira da Silva	
1784/1785		
1785/1786		
1786/1787		1
1787/1788		1
1788/1789		1

1789/1790	Manuel José Pereira da Silva	
1790/1791		1
1791/1792		
1792/1793		
1793/1794		1
1794/1795		1
1795/1796	José Joaquim de Faria.	1
1796/1797		3
1797/1798		2
1798/1799		2
1799/1800	António José de Araújo Santa Bárbara	
1800/1801	---	-----------
1801/1802	Tristão Alves da Costa Silveira	
1802/1803	José Joaquim Rivara	
1803/1804		
1804/1805		
1805/1806	Tristão Alvares da Costa Silveira	1
1806/1807		1
1807/1808		
1808/1809		
1809/1810	José Joaquim Rivara	
1810/1811	*THE UNIVERSITY WAS CLOSED*	-----------
1811/1812	José Joaquim Rivara	
1812/1813		
1813/1814		
1814/1815		
1815/1816		
1816/1817		1
1817/1818		
1818/1819		
1819/1820		
1820/1821		
1821/1822		1
1822/1823		1
1823/1824		1
1824/1825		2
1825/1826	Joaquim Lebre de Sousa e Vasconcelos	2
1826/1827		2
1827/1828		2
1828/1829	Manuel Pedro de Melo	2
1829/1830	Joaquim Lebre de Sousa e Vasconcelos	1
1830/1831		1
1831/1832		1
1832/1833		1
1833/1834		1

1834/1835	Guilherme António Dias Teixeira	1

The scientific production of this academic community was relatively scarce, as can be seen in the Curriculum Vitae of the professors. José Joaquim de Faria improved of the 2nd edition of the *Elements of Analysis por M. Bezout*, (1793-94); Manuel Pedro de Melo published, in 1815, at the Royal Academy of Sciences of Lisbon (recently founded in 1779) "Memória sobre os binomiais", *História e Memórias da Academia Real das Sciencias*, **IV**, pp. 41-51; and José Joaquim Rivara published, in 1818, *Resolução analytica dos problemas geometricos, e indagação da verdadeira origem das quantidades negativas*, Coimbra: Imprensa da Universidade.

The ambition to create the Faculty of Mathematics and its Mathematics Course was great. However, there were some obstacles to the progress of this Faculty, and of the Portuguese educational system and science in general. Besides the lack of teachers and students (for instance, there were only six students in the first five years of the Course of Mathematics) we must consider the power of the Portuguese Inquisition and the State influence in the Portuguese educational system. The social and political environment were also not propitious; in this period Portugal has fought several wars: the war against Spain, in 1801; the three French invasions, in 1807, 1809, 1810, and consequently the transfer of the Portuguese court to Brazil, between 1807 and 1821; and the Portuguese civil war, between 1828 and 1834.

Nevertheless, the first step to improve scientific knowledge in Portugal was taken and there was no turning back.

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THE TRAINING OF MATHEMATICS TEACHERS IN ITALY (1875-1923)

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ABSTRACT

In order to respond to the need to train future teachers and thus of guaranteeing a higher level of secondary schools, in 1875 the Minister of Education R. Bonghi instituted the so-called Scuole di Magistero, which performed this function, through successive modifications, until their suppression in 1920. In 1921 the Minister of Education O.M. Corbino established a combined degree in mathematics and physics, intended to prepare graduates to teach scientific subjects in secondary schools; in the following year a course in "complementary mathematics", regarding the advanced areas of mathematics that were more closely related to elementary mathematics, was established for the combined degree program, flanked by didactic and methodological practice.

The history of the Scuole di Magistero was especially troubled, as shown by the great number of decrees that concerned them, and in many cases they were completely inadequate for reliably addressing the problem of teacher training. There were many reasons for this: above all, the professors who taught there, were the same ones who taught institutional courses, and as these professors had, with rare exceptions, no experience in secondary teaching, they were unprepared to addresses questions about pedagogy and method. Furthermore, supporting structures (libraries, laboratories, etc.) and teaching materials were practically nonexistent, the number of assigned course hours was inadequate, and there was scant funding. In addition to these factors, secondary school teachers were relegated to a minor role with respect to university teachers, a fact that had inevitable repercussions for the significance that was given to their training.

In my talk I will present a brief institutional history of the Italian Scuole di Magistero from 1875 to 1923, discussing the most significant legislative measures regarding them, the contribution of the Mathesis Assocation starting with its first national congress in 1898 in Torino up to the Naples congress of 1921, and the debates between the mathematicians (A. Padoa, G. Loria, S. Pincherle, G. Castelnuovo, G. Fano, etc.). Then I will concentrate on the contributions of the Italian geometers — such as Corrado Segre, Guido Castelnuovo, Federigo Enriques — to teacher training: the lessons given at the Scuole di Magistero, the different publishing initiatives (articles, textbooks, series of books for teachers), proposals of an institutional nature, etc. I will also discuss the methodological assumptions that guided them, highlighting the influence exerted by Felix Klein's ideas and initiatives. I will conclude by mentioning some of the debates about teacher training that followed the Reform enacted in 1923 by the Neo-Idealist philosopher Giovanni Gentile, who equated "knowledge" with "knowing how to teach", and who maintained that teacher training consisted solely in "genuine, profound and authentic scientific preparation".

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ALOIS STRNAD—LEADING PERSONAGE OF THE I. AND R. MATHEMATICAL OLYMPIAD

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ABSTRACT

This paper deals with Alois Strnad, who was a teacher at Czech secondary schools. His curriculum vitae as well as his scientific and pedagogical publications are mentioned in this contribution. Strnad played a leading role in the running of the mathematical competition organized by the Union of Czech Mathematicians and Physicists. Strnad published more than 500 tasks, which was about one third of all the tasks published between 1872 and 1918. Majority of these tasks were on algebra and geometry and they covered all subject matters taught at secondary schools.

1 Introduction

At the 5th ESU, which took place in 2007 in Prague, I talked about the writing competition that was organized by the Union of Czech Mathematicians and Physicists and published in the *Journal for the Cultivation of Mathematics and Physics* (further in this paper abbreviated as *Journal*). I call this activity *Imperial and Royal Mathematical Olympiad*. Although this is not the official title, I will use it in this article. For the sake simplicity, I will often use it in its short form, *Olympiad*. It was Alois Strnad who, in my opinion, should be given the greatest credit for helping the *Olympiad* survive its critical times after Studnička, the editor-in-chief of the *Journal*, had given up the editorship. After that, there was nobody who could publish enough tasks every year. It was precisely Strnad who took up this job and started to publish dozens of tasks each year till his death. This paper brings some information about Strnad's life and his scientific and pedagogical activities.

2 Curriculum vitae

Alois Strnad was born on the 1st of October 1852 in the Prague district Malá Strana (Little Quarter). He spent his youth in that marvelous part of Prague and here he also studied at Czech reálka[1] in Panská street. After passing the school-leaving examination in 1870 he started to study at Czech Provincial Polytechnical Institute (technical university) in Prague. His study results were always excellent and Strnad was several times awarded prize for them. In winter semester he was appointed assistant to Prof. Tilšer[2] at that school and he taught descriptive geometry. Since 1875, he also gave lectures instead of Prof. Tilšer.

Strnad, however, decided to leave the university career, passed the teacher's exam

[1]Reálka (in German Realschule) was a kind of a secondary school which focused on technical subjects rather than humanities.

[2]František Tilšer (1825–1913), Professor at Czech Provincial Polytechnical Institute.

and in 1876 was appointed professor[3] at reálka in Hradec Králové in eastern Bohemia. At this school he taught for fifteen years, till 1891, when he returned to Prague. For the following five years, he taught at the Czech reálka in Ječná street. His pedagogical career culminated in 1896, when he was appointed director at the reálka in the marvellous old town of Kutná Hora.[4] Strnad led this school for fifteen years, although he had to take leaves for health holiday, because his health got worse. He died on 26th May 1911.

Strnad was a very modest man, he did not struggle for worldly fortune. We can say teaching was the sense of his life. On the other hand he was rather disappointed that he could not devote more time to scientific work owing to his school activities. Despite this, Strnad published about 30 papers, mostly in Czech journals. He is also the author of textbooks and exercise-books for the *Olympiad*. Owing to his contributions to mathematics and its teaching, he was appointed corresponding member of the Czech Academy and honorable member of the Union of Czech Mathematicians and Physicists.

3 Strnad and the Olympiad

In the *Journal for Cultivation of Mathematics and Physics*, volume 13, issue 2, we can find this task: *A cone with radius 15 cm and altitude 30 cm is given. At what height must we cut the cone by a plane parallel with the base so that we can inscribe a sphere into the truncated cone? What is the volume of the truncated cone?* The author of this task was Alois Strnad and except that exercise he published another three tasks in that volume. Maybe nobody supposed that the name of Strnad would be appear at the pages of the Journal every year till 1908, when Strnad published his last task. And nobody anticipated that the signature of Strnad would appear more frequently than the names of all the other authors in the years 1872–1918 together. The total number of exercises published as a part of the Olympiad reached 1351. Strnad published 508 exercises, which means Strnad is the author of about 38 per cent of the total number of published tasks. None of the other authors contributed with an at least comparable number of tasks. I did not considered it necessary to make scale of all authors, so I do not know who would occupy the second position on the scale. However, the man holding the second position, whoever he was, only contributed with several dozens of exercises to the Olympiad. In addition, Strnad started his activity in the Olympiad at the time when organizers were short of tasks suitable for publication. It was just Strnad who every year published dozens of tasks that covered all branches of mathematics taught at grammar schools. We can find there tasks on constructive and analytic geometry, stereometric problems, many kinds of equations, number theory and so on. In some exercises, Strnad asked for a proof of some theorem, especially in geometry. According to my experience, it is really good to occasionally choose some old (not only Strnad's) problems and assign them to students. Contemporary students are usually surprised how excellent their ancestors' knowledge of mathematics was. It is a pity that foreign readers are limited by the fact that the exercises are written in Czech.

It is very difficult to choose a representative selection of Strnad's exercises; what follows is my subjective selection.

[3]Every teacher at secondary school used the title professor, so it is necessary to differentiate between the secondary school professors and professors at universities.

[4]In the middle ages, this town was second biggest in Bohemia and was important because of its silver mines and minting of coins named *Prague grossus*.

1. If n is an integer, then $60|n(n^4+35n^2+24)$. Prove it. (Vol. 25, exercise 1)
2. How many digits does number 777^{777} contain? (Vol. 25, exercise 2)
3. Three mass particles (points) are moving on three given lines in the positive direction. What is the time in which they get to the points A_1, B_1, C_1 forming a triangle of minimal (maximal) area? (Vol. 29, exercise 50).
4. Assess the sum of the series, whose general member is $a_n = (-1)^{n-1}\frac{a^{\frac{m+1}{2}}}{b^{\frac{m-1}{2}}}$, $m = (2n-1)(-1)^{n-1}$. (Vol. 33, exercise 3)
5. Point P and an ellipse are given. Construct a line that crosses the ellipse in two points A, B. To these points construct two associated diameters of the ellipse. (Vol. 36, exercise 37. This is the last exercise which Strnad published in the Olympiad.)

4 Strnad as an author of textbooks

Strnad also wrote three textbooks. Nobody is surprised that Strnad wrote, together with F. Hromádko,[5] *Collection of exercises on algebra.* Strnad was the leading person in publishing this exercise-book and he still improved this book with new exercises. During his life this book was published in seven editions. This work was the first of its kind published in Czech and it was really needed. Strnad did not manage to rework this book according to the new school programme, because when the new programme had been devised, he was already seriously ill. For that reason he passed the copyright to the Union of Czech Mathematicians and Physicists.

Strnad wrote two exercise-books on geometry, namely *Geometry for higher reálka* and *Geometry for higher gymnasium.*[6] Although the titles of these exercises-books are similar, we must consider them as two different works because the school programmes were different at those two kinds of schools. The textbooks covered all branches of geometry taught at secondary schools. It deals with such terms as line, angle, triangle, circle, tetragon, polygon and their equivalency and similarity in planimetry. A section is devoted to stereometry and to reckoning surfaces and volumes of various solids. Trigonometry is both in the plane and on the sphere[7], exercises are mostly intended for practice. Analytic geometry is the last part of these textbooks, and apart from the parts devoted to the line and the plane, we can also find a treatise about conics here.

The textbook on geometry for gymnasium was also translated into Bulgarian. Explaining this fact is really simply. In the second half of 19th century some Czech teachers helped to build secondary educational system in other Slavonic countries. They partly wrote new textbooks and partly translated textbooks used in the Czech lands. For that reason even Strnad's textbook was translated into Bulgarian by professor Šourek[8].

Strnad's textbooks are written in 'Euclid style', i.e. definition, theorem, proof. Strnad proved every theorem. The reader does not find here any example for motivation,

[5]František Hromádko (1831–1911), teacher at Czech grammar schools.

[6]Gymnasium was a kind of secondary school, it gave students universal education.

[7]This part is not taught at Czech grammar schools nowadays.

[8]Antonín Václav Šourek (1857–1926), teacher at various Bulgarian secondary schools and later Professor of mathematics at University in Sofia. Further details in (6).

commentary, nor explanatory text, except historical notes, which are used quite often. Strnad added also examples, with every example solved. Strnad also planned to write a collection of exercises, but owing to his duties as a director and to his illness, he did not manage to finish it. While Strnad's exercises are used by teachers even nowadays, I am not sure contemporary studends would like to learn geometry from Strnad's textbooks.

5 Strnad's scientific papers

Strnad published 32 papers, and a majority of them (19) was published in the *Journal*. Further papers were published in annual bulletins of secondary schools where he taught and in others journals. One paper was published in *Archiv für mathematik* (Grunert-Hope) and one in *Rad jugoslavenske akademiji*. Majority of his papers are devoted to various problems on geometry (triangle, Simson straight line, surface of the cone, construction of a regular seventeen-angle). Other papers deal with various problems in arithmetics and analysis. Strnad was not a world-wide known scientist as some of his contemporaries, but his papers were important especially for students, because they enabled them to learn things that were not taught at schools. In the first published paper (3) Strnad proved the theorem of Moret-Blanc which was published a year ago in a different way. For patriotic reason we demonstrate Strnad's style on another paper.

Lerch in (7) presented the following two formulas:

$$\sum_{\varrho=0}^{[\frac{n}{2}]} \psi(n-\varrho,\varrho) = n \tag{1}$$

$$\sum_{\varrho=0}^{n} \psi(n+\varrho,\varrho) = 2n, \tag{2}$$

where $\psi(\alpha,\beta)$ is a number of divisors of α greater than β.

Lerch developed these formulas from the equation

$$\frac{x}{(1-x^2)} = \sum_{\nu=1}^{\infty} \frac{x^\nu}{(1-x^\nu)(1-x^{\nu+1})}. \tag{3}$$

From the formula for geometric series, it follows that

$$\frac{x^\nu}{1-x^\nu} = \sum_{\mu=1}^{\infty} x^{\mu\nu}$$

and

$$\frac{1}{1-x^{\nu+1}} = \sum_{\varrho=0}^{\infty} x^{\varrho(\nu+1)}.$$

The equation (3) changes into

$$\sum_{n=1}^{\infty} x^n = \sum_{\mu,\nu\varrho} x^{\mu\nu+\varrho(\nu+1)},$$

where n is a number of the solutions of the equation

$$\mu\nu + \varrho(\nu + 1) = n.$$

Strnad denotes by $\Psi(\alpha, \beta)$ all divisors of α which are greater than β and by $\Psi_{\varrho=0}^{n-1}(n-\varrho, \varrho)$ all divisors of the sequence

$$\Psi(n, 0), \quad \Psi(n-1, 1), \ldots \Psi(1, n-1)$$

He proved that all the members of this sequence form an arithmetic series $1, 2, \ldots, n$ which means Lerch's formula (1) is correct. The equation $n - \varrho = mp$ is solved in integers for every $p > \varrho \geq 0$. It follows from the formula

$$\frac{n}{p} \geq m > \frac{n}{p} - 1.$$

Strnad added also the formula

$$\sum_{\varrho=0}^{n} \Psi(n-\varrho, \varrho) = \frac{n(2n+1)}{2}. \tag{4}$$

In a similar way he proved Lerch's formula (2) and added also the formula

$$\sum_{\varrho=0}^{n} \Psi(n+\varrho, \varrho) = n(2n+1) \tag{5}$$

Lerch's proof of this formula is much longer, so we refer the reader to Lerch's original paper (7). Lerch returned to this problem some years later in paper (8), where he presented additional formulas and quoted Strnad's paper.

Except for normal papers, Strnad published also many so-called Little notes in the *Journal*, in which he made readers familiar with various papers published in foreign journals. Between years 1886–1892 when articles of such kind were published in the *Journal*, Strnad published twenty-one contributions in these columns. In the first Little note Strnad informed the readers that Landy proved that Fermat number F_{26} is not a prime, but a composite. There is a number of such descriptions in one article. Although he was not the only author of such articles, he was again the most hardworking in this activity.[9] This activity proves that Strnad had great knowledge of facts in mathematics and that he was able to reproduce it in an intelligible form. On the other hand, he published several notices about Czech mathematical papers in *Revue trimestrielle des publications mathématiques* and in *Répertoire bibliographique des sciences mathématiques*.

At the end of this section I would like to mention a lovely paper (5), which was published in 1889, on the occasion of the 100th anniversary of The Great French Revolution. Strnad mentioned French mathematicians who played an important role during the revolution. Readers came to know fates of excellent scientists who, apart from their scientific activities, were also leaders of the French revolution. Many lines are devoted to the Organizer of Victory Lazare Carnot, the chief of the parliament Bailly, the favorite of all governments Laplace, a member of Egypt expedition Gaspard Monge, and

[9] Well known scientists as Matyáš Lerch (1860–1922) and Augustin Seydler (1849–1891) also contributed to this column.

so on. Strnad wrote primarily about their life, their scientific results are described only in words and the reader cannot find any formulas in this paper. Other scientists, who did not play so important a role in politics (Legendre, Fourier etc.), are mentioned only shortly. Contrary to Strnads text-books, this paper is written in a nice and eloquent Czech language. It resembles a work of fiction rather than a scientific work. Strnad uses many quotations in French without translation, too, and thus we can presume that the knowledge of French was on a high level at that times, at least in Hradec Králové and its surroundings. This paper is one of the first publications on history of mathematics in the Czech lands.

6 Conclusion

We cannot say Strnad was a world-famous scientists, but we must appreciate him in consequence to the situation in the Czech lands in the second half of 19th century. From this point of view we can appreciate Strnad as a significant personage of Czech scientific and scholastic community. He was an author of textbooks, his papers allowed students to get acquainted with new discoveries in mathematics, and his activity in the Olympiad was essential for continuing this matter. From 1884 till 1904 he was the editor for geometry in the *Journal*. Strnad contributed to *Otto's Encyclopedia*[10], he is the author of about 70 entries mostly on geometry and of the curriculum vitae of Professor Tilscher. We cannot overlook his educational activities. Strnad has his place in the history of Czech mathematics and his heritage is still alive, especially in the educational area. We should not forget this personality, although one hundred years elapsed since his death.

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[10]This excellent and, in the author's opinion, up to the present days unsurpassed, encyclopedia in Czech countries, has 28 volumes. It has been named after its publisher.

SHAPING A MODERN MATHEMATICS PEDAGOGICAL CONTENT KNOWLEDGE

The case of *Telescola* in Portugal in the middle 1960s

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ABSTRACT

Modern mathematics reform spread through many countries during the 1960s and 1970s producing changes in the representations of learning, of mathematical content, of the social roles of mathematics, and in classroom teaching practices. This paper focuses on changes in pedagogical content knowledge to appropriate the new ideas. It is a part of a larger comparative study on modern mathematics in Brazil and Portugal.

From 1965, a network of schools for 10 and 11 years-old supported by televised lessons was gradually put in place by the Portuguese Ministry of Education, in an effort to enlarge schooling after primary school, as demanded by economic development. Students that attended the Postos (the name of those schools) and finished the two-year course could enrol in the 7th grade of secondary schools. By 1968, this system that became known as Telescola covered the entire country, especially in remote areas.

Modern mathematics was gradually incorporated into these televised classes providing an experimental field for their later generalization to the entire population of 5th and 6th graders. Mathematics classes in Telescola were actually the first experience in the dissemination of the new ideas through an entire school sub-system in Portugal.

This study encompasses three dimensions aiming at understanding changes in pedagogical content knowledge (PCK) at this level:

1) a longitudinal study from 1965 to 1970 of all the broadcasted mathematical lessons;
2) interviews of the teacher which prepared and enacted them on the screen, António Augusto Lopes;
3) documental analyses of legal papers, articles, gray literature, among other materials.

Qualitative procedures, especially content analysis, were performed.

The gradual shaping (bricolage) of PCK was observed, as the new ideas were put to practice, either in front of the TV cameras, in small-scale experiments in actual schools, or in teacher education settings. The study also traces the influences of other educators and institutions in the ways in which PCK was developed.

THE CONCEPT OF TANGENT LINE

Historical and didactical aspects in Portugal (18^{th} century)

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ABSTRACT

The concept of a tangent line to a curve is, from the beginning, present in the history of mathematics. The study of the historical evolution of this concept is a source of knowledge and enlightenment that can lead to a better understanding of the concept.

By 1772 Marquês de Pombal reorganized the University of Coimbra, establishing in Portugal the first Faculty of Mathematics in the world. The Statutes that ruled that Faculty stated, in particular, that the concept of a tangent line to a curve was to be taught in the first and second years of the Mathematics degree. The books for these years included the Portuguese translation of Euclid's *Elements* and translations, made by José Monteiro da Rocha, of the work of Etienne Bezout. By then, José Anastácio da Cunha wrote a manuscript "*Principios de Geometria tirados dos de Euclides*" where he strongly criticizes the Euclidean definition of a tangent line, presenting an alternative definition in his treat "*Principios Mathematicos*".

In this article we approached the teaching of the concept of a tangent line in the Faculty of Mathematics at University of Coimbra and we focus on the alternative solution presented by the Portuguese mathematician José Anastácio da Cunha.

Keywords: Concept of tangent line; 18^{th} century; University of Coimbra.

1 The concept of a tangent line in Portugal in the 18^{th} century

1.1 Context

The Faculty of Mathematics at University of Coimbra is the first Faculty of Mathematics in the world, being created in 1772, in the reign of D. José and under the influence of Marquês de Pombal, the Minister of the Kingdom, when this University was reformed. With this reform D. José was aiming to disseminate the knowledge in Liberal Arts and Sciences, creating five Faculties (Theology, Law, Medicine, Mathematics and Philosophy). The Portuguese mathematician José Monteiro da Rocha (1734 – 1819) was the responsible for creating the Statutes for the Faculty of Mathematics, in which it was stated that Mathematics is vital to strengthen both the spirit and the wisdom of mankind; as a consequence the study of Mathematics was made compulsory to mathematicians as well as to every student from the other faculties[1] (see [10], [14], [15]).

The degree in Mathematics was, according to the Statutes, composed by four courses: Geometry, Algebra, Phoronomy and Astronomy, each of them in every of the four years[2].

The textbooks used by that time were portuguese translations of the treats published in other languages, and the faculty teachers were strongly encouraged by the Statutes to write their own treats in Portuguese, so that they were used in classes.

[1] Do Curso Mathematico, 1772, p. 141 – 142.

[2] The distribution of the courses, subjects, teachers and textbooks through the years is presented in table 1 of the Appendix.

The concept of a tangent line, in particular, should be taught in the 1^{st} year, in Geometry, as presented in Euclid's *Elements*[3], as well as in the 2^{nd} year of the degree, in Algebra classes, using Bezout's *Elementos de Analisi Mathematica*, a translation into Portuguese made by Monteiro da Rocha from Bezout's *Cours de mathématiques à l'usage des gardes du pavillon et de la marine* that was published in Paris between 1764 and 1769[4]. José Anastácio da Cunha (1744 – 1787), the Geometry professor[5], also studied the concept of tangent line to a curve in a manuscript entitled *Principios de Geometria tirados dos de Euclides*[6] where he criticized the Euclidean definition. In his master work *Principios Mathematicos para instrucção dos alumnos do collegio de São Lucas, da Real casa Pia do Castello de São Jorge* (see [7]), Anastácio da Cunha offered an alternative definition of the concept.

We, therefore, faced a dichotomy: one the one hand the definitions for a tangent line presented in the textbooks chosen to be used in the newly created Faculty of Mathematics and, on the other hand, the alternative definitions as stated by the Geometry professor José Anastácio da Cunha.

1.2 The concept of a tangent line in the Geometry course

The definition of a tangent line, according to *Elementos de Euclides dos seis primeiros livros* (…) is, as expected, the usual "touch without cut", namely:

> *Elementos III, Def. 2:* A straight line is said it touches a circle or that it is a tangent to a circle when it is at the same level as the circle and it meets the circle without cutting it[7].

As a complement, the propositions involving our concept are the 16 to the 19, in Book III. One may notice however that proposition 16, a theorem, is in the Portuguese edition following the Robert Simson edition which is, in its last part, different of Heiberg's:

> *Elementos* III, 16 (*Theorem*): The straight line that from an extremity of the diameter of a circle raises perpendicularly above the same diameter, will fall out of the circle. And between this straight line and the circle it cannot exist any other straight line; this is the same as saying that the circle will pass between the perpendicular and the diameter, and the straight line that with the diameter makes an acute angle as big as it may be; or also, that same circumference will pass between that perpendicular and another straight line, that does the same perpendicular of any other angle, as small as it may be[8].

[3] The Portuguese version of Euclid's *Elements* is a translation by João Ângelo Brunelli, dated from 1756, from the Robert Simson's edition of the geometrical books (I to VI and XI and XII).

[4] The first edition of this translation is dated from as soon as 1774 and further editions got a slightly different title, namely *Elementos de Análise*. All translations included in the notes and methods added by Monteiro da Rocha to the original text.

[5] We know that by 1776 José Anastácio da Cunha presented a geometry textbook to the Congregação da Faculdade de Matemática (a governing organ). This Congregação was supposed to have accepted or rejected every textbook presented to them by the professors, but unfortunately we know nothing of the solution given to Anastácio da Cunha's proposal.

[6] This specific manuscript is part of a larger set found recently in Portugal (see [6] and [8]).

[7] This translation, as well as all the others in this article, and the underlines is ours. In this particular case, in the original we can read: *Huma linha recta se diz, que toca hum círculo, ou que he tangente de um círculo, quando estando no mesmo plano do círculo encontra a circunferência sem a cortar* (Euclides, 1768, p. 81).

[8] *A Recta, que de huma extremidade do diâmetro de um círculo se levantar perpendicularmente sobre o mesmo diâmetro, cahirá toda fora do círculo; e entre esta recta, e a circunferência não se poderá tirar outra linha recta alguma; que he o mesmo que dizer, que a circunferência do círculo passará entre a perpendicular ao diâmetro, e a recta, que com o diâmetro fizer hum ângulo agudo por grande que seja; ou também que a mesma circunferência passará entre a dita perpendicular e outra recta, que fizer com a mesma perpendicular hum ângulo qualquer, por pequeno, que seja* (Euclides, 1768, p. 101).

We may conjecture that the author was, with this formulation, avoiding the contingence (cornicular) angle, present in the Heiberg's edition, which was a polemic subject for many years and therefore avoiding a potential problem to students (see [12],[13]).

1.3 The concept of a tangent line in Algebra course

Elementos de Analisi Mathematica, Bezout's work translated to Portuguese for the purpose of becoming a textbook in Coimbra, was divided in two parts "*Elementos de Álgebra*" and "*Elementos de Cálculo*".

In this work we cannot find a definition for tangent line, but the definition of circle is presented in Bezout's *Elementos de Geometria* (see [5]), namely:

> *Elementos de Geometria*, 46: It is called a tangent line that straight line that does not more than to lean to the circle[9].

In *Elementos de Analisi Mathematica* the concept of a tangent line is not defined but it is studied in its relationship with the conic sections. Since no definition is presented, we assumed that a generalization of the definition presented in *Elementos de Geometria*, which is similar to the definition presented in Euclides' *Elementos*, was considered.

The study of the concept of tangent line started with the presentation of a method for drawing and for determining the length of the subtangent. Here we may find that the tangent line was considered as the line that touched the curve in a single point (proposition 298):

> *Elementos de Álgebra*, 298: To draw a tangent line to any point M from the ellipse, produce the vector radius fM to G, such as $MG = MF$; and taking GF, produce by M the perpendicular MT to it by M, which is the tangent, this is, will meet the curve only at the point M.

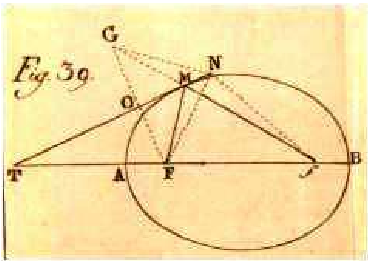

Fig. 1 – Construction of the tangent line from an ellipse.

> ***Proof:*** Suppose MT is not the tangent line, this means that it meets the curve in another point N.
> By the property of the ellipse $FN + Nf = FM + Mf$, or (by Euclid I, 4 e 5) $NG + Nf = Gf$ which is absurd (by Euclid I, 20).
> Therefore, the point N does not belong to the ellipse.■

In addition to this direct method to determine the tangent line to the ellipse the author presents a second method to draw the tangent line, determining the length of the subtangent. Similar methods are presented for the parabola and the hyperbole.

The concept of a tangent line is approached again when, in the second part of *Elementos de Analysi Mathematica,* the differential calculus is presented. In this part, it is stated a general method to draw the tangent line to a curve. This new method, presented in proposition 30, stands from the concept of

[9] *Chama-se tangente aquela que não faz mais do que encostar-se à circunferência* (Bezout, 1817, p. 25).

difference[10] (or “differential” or “fluxion”) and uses the similarity between the infinitesimal triangle $[Mrm]$ and the finite one $[TPM]$ to prove that the subtangent PT is given by $y\frac{dx}{dy}$ where dx and dy are the differences between the abscissa and ordinate, respectively.

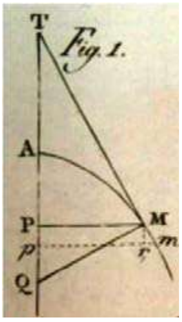

Fig. 2 – Construction of the tangent line using the infinitesimal triangle.

Several examples are presented, such as the conic sections, all genders of parabolas and a logarithmic curve.

The tangent line is not only determined by points on the curve but also by points outside the curve, being presented a general method to draw it.

Elementos de Cálculo, 32: To draw a tangent line to any curve from a point outside the curve[11].

Proof: Consider the curve AM and the point D outside the curve for which it is required to draw the tangent line.

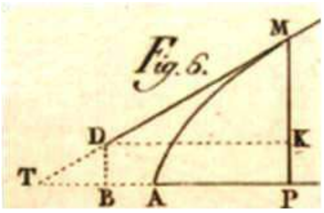

Fig. 3 – Construction of the tangent line to a curve from na outside point.

Considering the abscise $AB = g$, the ordinate $BD = h$ and if M , the contact point, has as coordinates $AP = x$ and $PM = y$ then, because the triangles $[TDB]$ and $[TPM]$ are similar, comes that:

$$\frac{TB}{BD} = \frac{TP}{PM} \Leftrightarrow (y-h)\frac{dx}{dy} = g + x \blacksquare$$

The following propositions studied the properties of the tangent line in curves depending on other curves, stating different methods for the determination of the tangent line for all types of curves, presenting as examples different types of Cycloids, Archimedes’ Spiral and Nicomedes’ Conchoid.

From proposition *42* to *46* the concept of a tangent line is used to state a method of *Maximis & Minimis*[12] where vertical tangent lines are accepted. Multiple points are also studied, being presented examples for the existence of more than one tangent line at a point. The study of tangent lines is

[10] Differential is the difference between two consecutive moments of a varying quantity (Bezout, 1774, vol. 2, p. 10).

[11] *Tirar uma tangente a qualquer curva de um ponto dado fora dela* (Bezout, 1774, vol. 2, p. 34).

[12] Bezout, 1774, vol. 2, p. 46 – 49.

finally concluded with the analysis of inflection points where one finds an example of the tangent line cutting the curve.

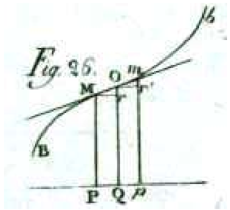

Fig. 4 - Tangent line to a curve at inflections points.

The concept of a tangent line is presented in a contradictory way in Algebra where, in the first part of the textbook, it is assumed as the “touch without cut” principle of a tangent to a curve , or that the tangent cuts the curve at a single point; and, in the second part, there are examples where the tangent line cuts, effectively, the curve at the tangency point and curves that it has more than one point in common with the tangent line, without any reference to a different definition for the concept of tangent line.

1.4 The concept of a tangent line in Anastácio da Cunha’s *Principios de Geometria tirados dos de Euclides*

In the manuscript *Principios de Geometria tirados dos de Euclides*, it was thought to have been composed by Anastácio da Cunha in 1778. There one may find profound reflections about geometry, its teachings, its peculiarities and many historical details concerning, in particular, the importance of definitions.

In this work the author criticizes the lack of generality in definitions, using as an example the definition of the tangent line, referring the traditional definition, “touching without cutting”, according to which several curves would not have a tangent line.

> The lack of generality in definitions has also been the cause of remarkable embarrassments and errors, and still remain to prove several relevant theories. For example: all that followed Euclides have put the contact of two lines in meeting without cutting each other, and did not notice the several lines (and common lines), which due to that do not admit tangents. In fact, the well known algebraic and fluxionary theorems solves this problem; but to do so it is necessary first to reject the imperfect definition, that contradicted them[13].

Anastácio da Cunha comments, as well, the proposition 16 from Book III of Euclid’s *Elements* (where, in Heiberg’s edition, the angle of the semicircle is said to be greater than any acute angle, and the remaining angle less that any rectilinear angle), referring that the angle between the circle and the tangent line should not exist because it goes against the definition 8 of Book I[14] of the same treat.

In this manuscript it is not presented the solutions to the detected problems.

[13] *Falta de generalidade nas definições também tem sido causa de notáveis embaraços e erros, e de estarem ainda por demonstrar varias relevantes theoricas. Por exemplo: todos, seguindo Euclides, põem o contacto de duas linhas em concorrerem sem se cortarem mutuamente, e não reparam na affinidade de linhas (e linhas vulgares), que por esse modo não admitem tangentes. Na praxe os sabidos theoremas algebraicos e fluxionarios emmendam este defeito; mas para se adoptarem he necessário primeiramente rejeitar a imperfeita definição, que os contradiz* (Cunha, 1778, p. 16).

[14] *Elements* I, Definition 8: A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line (Heath,1956, p. 176).

1.5 The concept of a tangent line in Anastácio da Cunha's *Principios Mathematicos*

The exact title of Anastácio da Cunha's main work is ***Principios Mathematicos*** *para instrucção dos alumnos do collegio de São Lucas, da Real casa Pia do Castello de São Jorge*, which means that we are dealing with a textbook expressly written to be used for teaching mathematics to the pupils of Collegio de São Lucas in Lisbon, where there were, at the time, 2 different classes of teaching (an elementary one and a so-called scientific, aiming to prepar pupils to enter the university of Coimbra or abroad). This treat, with 300 pages covering a vast range of mathematics fields, came to be finally published, in 1790, two years after Anastácio da Cunha's death.

The first time the concept of a tangent line appears in *Principios Mathematicos* is in the second book, where the third definition is Anastácio da Cunha's alternative definition to our concept. Anastácio da Cunha presents a general definition for tangency that he applies in his work to tangent lines to curves and to tangent curves, namely:

> *Principios Mathematicos* 2, Definition III: If the sides of an angle meet at the vertex such as it is not possible to draw two straight lines between them it is said that the sides are tangent to each other[15].

This definition settled in the new definition of angle has already been presented in the first book.

> *Principios Mathematicos* 1, Definition VII: Angle is the figure that two competing lines form at a point[16].

The concept of a tangent line is used again in proposition *VI* of the second book[17] where, in the second corollary, Anastácio da Cunha states that the tangent line to a circle is perpendicular to the diameter through the contact point (the proof, left as an exercise according to the authors' methodology for the whole book, uses the fact already proved that if the angles *LIN* and *LMI* are equal then *IN* is a tangent line to the circle).

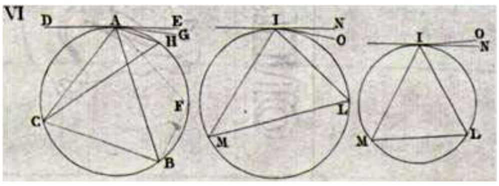

Fig. 5 - – Proof of the perpendicularity between the tangent line and the diameter through the contact point.

The method to draw a tangent line to a circle from a point outside the circle is given in the third proposition[18] of Book 7, being the method presented in Euclid's *Elements*.

Anastácio da Cunha has criticized the 16th proposition of Euclid's *Elements*, namely the problems with the curvilinear angle. In his opinion, and since the only definition of angle presented in the Euclides' *Elementos* was definition 8 of Book I, the concept of curvilinear angle could not exist. To solve the problem with this proposition, Anastácio da Cunha defines curvilinear angle using the concept of tangency already presented.

[15] *Se os lados de hum ângulo concorrerem no vértice, de sorte que deste se não possão tirar duas rectas entre elles, dir-se-ha , que cada hum dos lados he tangente ao outro, ou, que o toca no dito vértice* (Cunha, 1790-1987, p. 13).

[16] *Ângulo é a figura, que duas linhas formam concorrendo em hum ponto. Este ponto chama-se Vértice; e as linhas lados* (Cunha, 1790-1987, p. 1).

[17] Cunha, 1790-1987, p. 15 – 16.

[18] Cunha, 1790-1987, p. 86.

Principios Mathematicos 15, Definition VIII: Quantity of a curvilinear angle is the angle that straight lines do when touching the sides[19].

In book 17 Anastácio da Cunha returns to the study of the tangent line, presenting, in the first proposition[20], a method to draw a tangent line to a conic. In this case we can see the author using the *fluxionary calculus* together with the fact that the subtangent is given by $y\frac{dx}{dy}$. This method is also used to determine the tangent line to Huygens Cycloid[21] (*Principios Mathematicos 17, Proposition* IV) and to draw tangents to an algebraic curve at multiple points[22] (*Principios Mathematicos 17, Proposition* V).

In this treat, Anastácio da Cunha presents a general definition of tangency that has as objective avoiding the lack of generality of the Euclidean definition for tangent line.

2 Final Remarks

The concept of a tangent line was considered, for many centuries, as a "touch without cut" line. This definition, probably very much in use within our teaching/learning strategies of the present times, was also the one taught to students at the Faculty of Mathematics in the University of Coimbra by the end of the eighteenth century. The differential calculus, which was also studied at the University, fomented the appearance of tangent lines that may, in fact, cut the curves[23] going against the usual definition. This contradiction was not referred to in the textbooks used in the Faculty but the professor and Portuguese mathematician José Anastácio da Cunha confronted the "conflict" in his master treat, presenting an alternative definition to our concept, unifying, in addition, Geometry and Differential Calculus. The solution presented by Anastácio da Cunha is very much tuned with the standards of rigor that he got us used to, that is, we are faced with a definition that is, in our opinion, by all means remarkable: it is a skillful definition for the tangent line that being geometrical is also consistent with the use of the differential calculus.

Acknowledging the evolution of our concept of a tangent line in Portugal enables us to better understand the teaching of the concept as we perform it nowadays and to prepare for future research using new methods, new materials, new examples all "imported" from old/renewed mathematics.

[19] *Quantidade de um ângulo curvilíneo he o ângulo que fazem as rectas que no vértice tocam os lados* (Cunha, 1790-1987, p. 229).

[20] Cunha, 1790-1987, p. 236.

[21] Cunha, 1790-1987, p. 239.

[22] Cunha, 1790-1987, p. 240.

[23] We are, in the case of dealing with tangents within the field of Differential Calculus reporting, as well, to a situation very much common to nowadays classes.

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APPENDIX

Table 1 – The Mathematics Degree in Faculty of Mathematics at University of Coimbra

Year	Course	Subject	Teacher	Textbook
1st	Geometry	Elements of arithmetic, geometry and trigonometry	José Anastácio da Cunha	*Elementos*, Euclides *Elementos de Arithmetica* and *Elementos de trigonometria plana*, Bezout
2nd	Algebra	Literal calculus, differential and integral calculus	Miguel Franzini	*Elementos de Analisi Mathematica*, Bezout
3rd	Phoronomy	Movement science applied to all branches	José Monteiro da Rocha	*Tratado de Mecânica*, Marie *Tratado de Hidrodinamica*, Bossut *Optica*, La Caille
4th	Astronomy	Movement of the stars, the practice of calculation and the observations	Miguel Ciera	*Astronomia*, Lalande

MATHEMATICS TEXTBOOKS FOR SCHOOLS (1898-1939)

The cultural proposal of the School of G. Peano

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ABSTRACT

Between 1890 and 1940, the School of Peano formulated many proposals for the renewal of the teaching of mathematics, showing a sensitivity regarding the pedagogical tasks and the syllabuses.
An examination of the textbooks by G. Peano, C. Burali-Forti, S. Catania, M. Nassò and A. Padoa can make explicit their epistemological assumptions and concrete solutions. The reflexions on the possibility of presenting maths in classroom as an hypothetical-deductive abstract system, the use of the logical symbolism, the observations about the educational value of the history, and the necessity of introducing concrete applications and games characterize these manuals. Of great interest is also the writing of books for the youngest pupils that show attention for the psychological aspects of the teaching at the elementary and pre-school levels, which at the time was still lacking in Italy. In 1939, for example, R. Bettazzi stresses the parents' contribution in guiding the child to the discovery of the mathematical world, the usefulness of collaboration between teachers, educationalists and families and the value of domestic help, aimed at promoting in the child the geometric sense, without creating pseudo-intuitions and automatisms.
The School of Peano wanted to act not only on the side of the theoretical reflection, but also on that of the circulation of information on educational topics. Apropos this, a suggestive experiment was that of the journal Schola et Vita (1926-39), which was founded in Milan by N. Mastropaolo, with the collaboration of Peano. This was an international periodical which gave hospitality to reports of didactic conferences, news about the scholastic systems in the world, reviews and articles on general, comparative, social and even special pedagogy, with works on the education and recovery of abnormal subjects, all the more important if we consider the context in which they appeared.
The reflection launched by the School of Peano presents elements of modernity in its perception of the teaching and learning of mathematics as global problems, in its desire to face new challenges, regarding the structure and the language of mathematics, and in its synergy between research of a scientific and of a didactic nature. By the way, some of the solutions adopted in textbooks show insufficient attention to the cognitive obstacles. This is explicitly the case of the introduction of logic: even those authors who had a high scientific culture into this sector ran the risk of reducing it to a mechanical activity, suited only to exceptionally gifted or mediocre students.
The advisability of making rigorous the introduction of the concept of number and of the geometric figures at pre-school level; the stress placed on the teaching of algebra, considered equally formative as geometry; the strategies to facilitate the integrated acquisition of the numerical, graphic and symbolic aspects; the narrative and ludic approach are aspects underlined by Peano's collaborators that are fully recognized and still appreciated today.

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SKJOLDAGER , Sif – Denmark

SMESTAD, Bjorn – Norway

SMID, Harm Jan – Netherlands

THOMAIDIS, Yannis – Greece

TOURNES, Dominique – France

TSAMADIA, Maria – Greece

TSCHUGG, Ingrid – Austria

TSIGRIS , Miltiadis – Greece

TZANAKIS, Costas – Greece

VAN HOORDN, Martinus – Netherlands

VAN MAANEN, Jan – Netherlands

VERDUGO ROHRER, Andrea – Germany

VICENTINI, Caterina – Italy

WEEKS, Chris – United Kingdom

WEISS-PIDSTRYGACH, Ysette – Germany

WIELAND, Reiner – Austria

WIERNIK, Rose – Israel

WITTEK, Irmtraud – Austria

ZEIMETZ, Antonia – Germany