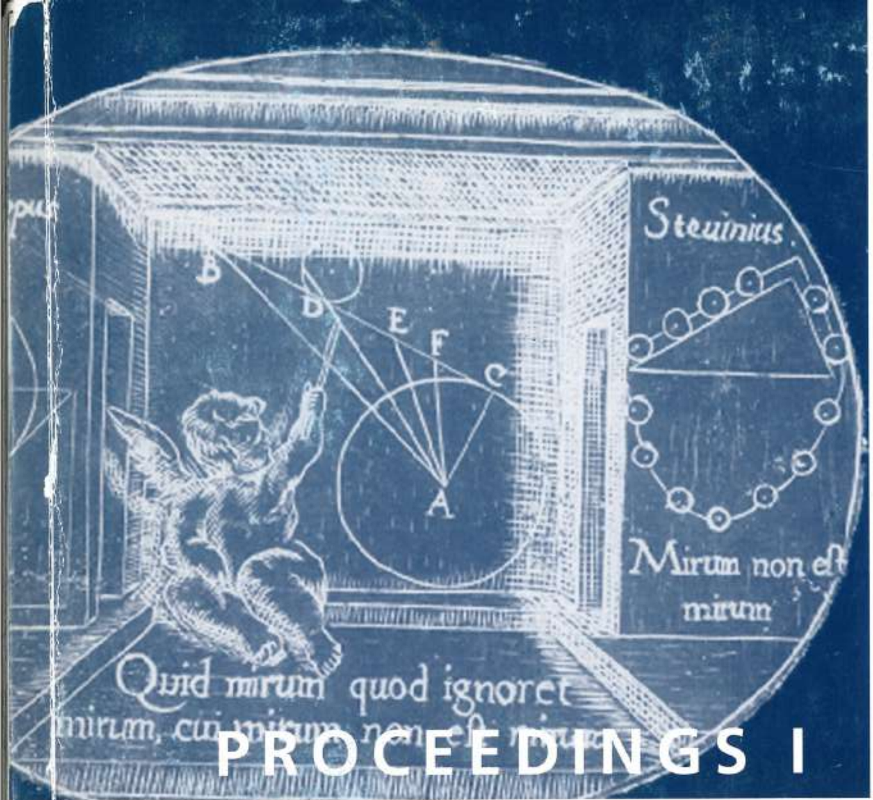

PROCEEDINGS I

Troisième université d'été européenne
Derde europese zomeruniversiteit
Third european summer university

Histoire et épistémologie dans l'éducation mathématique
«De la maternelle à l'université»

Geschiedenis en Epistemologie in de Wiskundedidactiek
«Van de kleuterschool tot de universiteit»

History and Epistemology in mathematical Education
«From play school to university»

UCL Université catholique de Louvain
KULeuven Katholieke Universiteit Leuven
LOUVAIN-LA-NEUVE (15/07/1999 - 18/07/1999)
LEUVEN (18/07/1999 - 21/07/1999)

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1999

Third european summer university
Troisième université d'été européenne
Derde europese zomeruniversiteit

PROCEEDINGS
Volume I

Editeur : PATRICIA RADELET-DE GRAVE
avec la collaboration de C. BRICHARD

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History and epistemology in the education of mathematics
"From infant school to university"

This summer university had as a goal to supply extra formation to teachers of mathematics. This formation refers to contents as well as methods of education of mathematics. This summer university enriched all participants, because teachers (from infant school to university) as well as research-workers (in didactics, history or epistemology of mathematics) could exchange views about the importance of history and epistemology. Furthermore they could be complementary to one another in the working-out of lessons of mathematics and education-projects. This summer university directed also towards those who are involved in the teacher training of the different levels.

The history throws light upon different facets of mathematical notions and theories, their turbulent course of life and the change they have endured. By this one understands in a better way which obstacles one has to face during the acquirement of mathematical notions. The epistemology draws up an inventory of the nature of obstacles that have to be overcome. By epistemological thresholds is meant : the enormous distance between a notion of daily life and the corresponding theoretical concept. These two disciplines -history and epistemology- show that the argumentation (of an exposition), the thinking about the exactness (precision), the way of working, the link with other branches (scientific subjects), the prevailing paradigms, evolve in time.

The necessity of particular intermediate steps during the learning of one or another part of mathematics can become more clear by the study of history and epistemology. In this way history and epistemology are inseparably connected with mathematics, in particular, with "the act of mathematics". If the education breaks the links between history and mathematics, the result is poverty, scantiness.

An historical study of the origin and evolution of notions and theories, forms an important source of inspiration for teachers and developers of curricula. On the one side they can draw upon meaningful examples and contexts to use in the class-room. On the other hand it is interesting for them to know the evolution of the theories that they are teaching. The history should hold an important place in the initial and continued training of teachers so that future teachers discover for a second time and in another way the subject-matter, they are familiar with in principle. The history leads to a reflection on mathematical contents and urges on "doing mathematics". All this means "doing epistemology".

There also exists an epistemological study of the origin and maturing of a notion or theory and this without referring to history. One can make a study of several phenomena which can be explained by the study of mathematics or lead to mathematics; and this with special attention to the steps of thinking within mathematics, the intermediate steps which cannot be omitted with impunity and the factors which prevent the maturing of ideas. In this context we refer to the work of Mach[1], who has analised in which way the sensorial perceptions produce geometrical thinking (for example connected with symmetries), Poincaré, Hadamard, Polya[2], Gonseth, Wertheimer[3], in his study of the psychology of the form, Freudenthal[4].

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[1]For example in Ernst Mach, *Die analyse der Empfindungen und das Verhältnis des Physischen zum Psychischen*, first edition, Iena 1886, Or sixth edition, (far more elaborated) 1911; Or also *L'analyse des sensations, Le rapport du physique au psychique*, translated by F. Eggers and J.M. Monnoyer, Editions Jacqueline Chambon, Nimes 1996.

[2]For example in G. Polya, *Induction and Analogy in mathematics*, Princeton Un. Press 1959.

[3]For example in M. Wertheimer, *Productive thinking*, Harpers and Brothers, New York 1945.

[4]For example in H. Freundenthal, *Didactical phenomenology of mathematical structures*, D. Reidel, Dordrecht 1983.

Histoire et Épistémologie dans l'Éducation mathématique
"De la maternelle à l'université"

L'université d'été avait pour objectif de donner une formation aux enseignants de mathématiques, du point de vue des contenus et des méthodes. Elle proposait de tirer parti de la richesse due à la participation conjointe d'enseignants (de la maternelle à l'université) et de chercheurs (que ce soit en didactique, en histoire ou en épistémologie des mathématiques) pour discuter l'intérêt de l'histoire et de l'épistémologie, et pour exprimer leur complémentarité, dans l'élaboration et le vécu de projets d'enseignement. Elle s'adressait aussi aux personnes chargées de formation initiale ou continuée.

L'histoire permet de voir les multiples facettes des concepts et des théories, de leurs avatars, de leurs mutations. Elle peut sinon expliquer, du moins faire comprendre les obstacles rencontrés dans la maîtrise de ces concepts. L'épistémologie inventorie la nature et la raison d'être des seuils (on désigne par le terme *seuil* la distance qui sépare bien souvent une notion familière du concept ou de la théorie qui lui correspond en mathématiques). Les deux disciplines montrent l'ancrage des objets mathématiques dans des problématiques déterminées, l'évolution de la rigueur, des idéologies, des formes du discours, des méthodes, des liens avec d'autres disciplines; elles posent le problème des modèles généraux, c'est-à-dire des paradigmes. Elles peuvent aussi mettre en évidence des passages obligés, incontournables, dans l'apprentissage de telle ou telle théorie mathématique. En ce sens alors, histoire et épistémologie ne sont pas dissociables de la mathématique, et en particulier de la mathématique en train de se faire. Lorsque l'enseignement casse les liens de l'histoire avec la mathématique, il y a étiolement.

L'étude historique de la production et de l'évolution des concepts et des théories est une ressource importante pour les enseignants et pour ceux qui sont amenés à concevoir des enseignements. D'une part, ils peuvent trouver tels quels (ou presque...) des exemples significatifs, des situations à exploiter en classe. D'autre part, il est intéressant pour eux de connaître les évolutions marquantes des théories à enseigner : c'est en quelque sorte leur culture qui est ainsi enrichie. L'histoire a une place importante à prendre dans la formation initiale et dans la formation continuée des maîtres : elle entraîne ces derniers à découvrir autrement une matière qu'en principe ils connaissent bien, et à se poser de nouvelles questions à son propos; elle amène une réflexion sur les matières mathématiques, une occasion de faire des mathématiques. Poser ces questions, c'est faire de l'épistémologie.

L'étude épistémologique de la maturation d'un concept ou d'une théorie mathématique peut ne pas faire référence à l'histoire. On peut étudier des phénomènes de tous ordres qui se résolvent au moyen de mathématiques, qui provoquent, enclenchent une pensée mathématisante, en s'intéressant particulièrement à l'évolution de cette pensée, à ses passages obligés, à ce qui vient entraver cette maturation. Illustrons ce propos par quelques noms : Mach[5] lorsqu'il analyse comment les sensations engagent la pensée géométrique (en particulier, en ce qui concerne les symétries), Poincaré, Hadamard, Polya[6], Gonseth, Wertheimer[7] dans son Étude de la psychologie de la forme, Freudenthal[8],...

[5]Par exemple dans Ernst Mach, *Die analyse der Empfindungen und das Verhältnis des Physischen zum Psychischen*, première édition, Iena 1886, ou sixième édition, (beaucoup plus volumineuse) 1911; ou encore *L'analyse des sensations, Le rapport du physique au psychique*, traduit par F. Eggers et J.M. Monnoyer, Editions Jacqueline Chambon, Nimes 1996.

[6]Par exemple dans G. Polya, *Induction and Analogy in mathematics*, Princeton Un. Press 1959.

[7]Par exemple dans M. Wertheimer, *Productive thinking*, Harpers and Brothers, New York 1945.

[8]Par exemple dans H. Freundenthal, *Didactical phenomenology of mathematical structures*, D. Reidel, Dordrecht 1983.

Geschiedenis en epistemologie in het wiskundeonderwijs
"van kleuterschool tot universiteit"

Deze zomeruniversiteit had tot doel een bijkomende vorming te geven aan wiskundeleerkrachten. Deze vorming heeft zowel betrekking op inhoudelijke als op methodische aspecten van het wiskundeonderwijs. De zomeruniversiteit was verrijkend voor alle deelnemers omdat zowel leerkrachten (van de kleuterschool tot universiteit) als onderzoekers (in didactiek, geschiedenis of epistemologie van de wiskunde) met elkaar van gedachten kunnen wisselen over het belang van de geschiedenis en de epistemologie en elkaar kunnen aanvullen bij het uitwerken van wiskundelessen en onderwijsprojecten. De zomeruniversiteit heeft zich ook tot diegenen die betrokken zijn bij de opleiding van leerkrachten van de verschillende niveaus gericht.

De geschiedenis brengt verschillende facetten aan het licht van wiskundige begrippen en theoriën, hun woelige levensloop en de wijzigingen die ze hebben ondergaan. Hierdoor begrijpt men beter welke hindernissen in de weg liggen bij de verwerving van wiskundige begrippen. De epistemologie inventariseert de aard en het waarom van drempels die dienen overschreden te worden. Onder een (epistemologische) drempel wordt de enorme afstand verstaan die een begrip uit het dagelijks leven scheidt van het overeenkomstige theoretische concept. Deze twee disciplines tonen hoe de wiskundige objecten geworteld zijn in probleemsituaties en hoe de vorm van een betoog, het denken over exactheid, de manier van werken, de band met andere vakgebieden en het heersende paradigma alle in de tijd evolueren. De noodzaak van bepaalde tussenstappen bij het leren van een of ander stuk wiskunde kan door de studie van geschiedenis en epistemologie ook duidelijker worden. In deze zin zijn geschiedenis en epistemologie onafscheidelijk verbonden met wiskunde, meer in het bijzonder het "wiskunde doen". Als het onderwijs de banden van de geschiedenis met de wiskunde verbreekt, betekent dit een verschraling. Een historische studie van het ontstaan en de evolutie van begrippen en theorieän vormt een belangrijke inspiratiebron voor leerkrachten en curriculumontwikkelaars. Enerzijds kunnen ze er betekenisvolle voorbeelden en contexten uit putten om in de klas te gebruiken. Anderzijds is het voor hen interessant om de evolutie te kennen van de theoriën die ze onderwijzen. De geschiedenis zou een belangrijke plaats moeten innemen in de initiâle en voortgezette lerarenopleiding, zodat toekomstige leerkrachten de (in principe vertrouwde) leerstof op een andere manier herontdekken en er zich nieuwe vragen over stellen. De geschiedenis leidt tot een reflectie over wiskundige inhouden en zet aan om wiskunde te bedrijven. Dit alles is aan epistomologie doen.

Er bestaat ook een epistomologische studie van het ontstaan en de rijping van een begrip of een theorie zonder te verwijzen naar de geschiedenis. Men kan allerlei verschijnselen bestuderen die met wiskunde op te lossen zijn of tot mathematisering leiden, met bijzondere aandacht voor de denkstappen bij die mathematisering, de tussenstappen die je niet ongestraft kunt weglaten, de factoren die de rijping van ideeën in de weg kunnen staan. We denken hierbij aan het werk van Mach[9], die geanalyseerd heeft hoe de zintuiglijke waarnemingen het meetkundig denken voortbrengen (bijvoorbeeld in verband met symmetriën), Poincaré, Hadamard, Polya[10], Gonseth, Wertheimer[11], in zijn studie van de psychologie van de vorm, Freudenthal[12].

[9]Bijvoorbeeld in Ernst Mach, *Die analyse der Empfindungen und das Verhältnis des Physischen zum Psychischen*, eerste editie, Iena 1886, of de zesde editie, (heel wat uitgebreider) 1911; of nog *L'analyse des sensations, Le rapport du physique au psychique*, vertaald door F. Eggers en J.M. Monnoyer, Editions Jacqueline Chambon, Nimes 1996.

[10]Bijvoorbeeld in G. Polya, *Induction and Analogy in mathematics*, Princeton Un. Press 1959.

[11]Bijvoorbeeld in M. Wertheimer, *Productive thinking* Harpers and Brothers, New York 1945.

[12]Bijvoorbeeld in H. Freundenthal, *Didactical phenomenology of mathematical structures*, D. Reidel, Dordrecht 1983.

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Figures et lettres mathématiques : nécessité visuelle et nécessité discursive

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Abstract

Un texte mathématique est un texte qui se lit, où se tient un discours, mais le texte mathématique est aussi un texte qui se regarde, où des traces sur le papier demandent et font compréhension, qu'il s'agissent de figures géométriques, de lettres ou de symboles. Nous nous proposons d'examiner, à partir de textes historiques, quelques aspects visuels du texte mathématique.

Dans une première partie, nous comparons nécessité visuelle et nécessité discursive à partir de démonstrations de théorèmes de géométrie élémentaire, celui sur les trois angles d'un triangle et celui dit de Pythagore[1]. Dans une seconde partie, nous nous intéressons à distinguer l'usage des lettres, selon que la stabilité visuelle de leurs traces sert uniquement à la représentation de choses, ou selon que l'agencement des traces permet aussi, de plus, d'exprimer visuellement une composition des choses représentées. Nous utiliserons les deux termes de *littération* et de *littéralisation*, pour marquer cet ajout dans le fonctionnement des lettres d'un texte mathématique. Ceci est examiné dans des textes d'Euclide, de Peletier du Mans, d'Arnauld et de Bellavitis, où nombres et grandeurs sont représentés et composés par des traces géométriques et littérales. Dans une troisième partie, la proposition VI du livre II des *Éléments* d'Euclide, telle qu'elle se lit et telle qu'elle se voit chez Euclide, Hérigone, Arnauld et Bellavitis, nous fournit un exemple très simple de la pulsation entre le discursif et le visuel[2] quand il s'agit de voir des figures comme des mots faits de droites et de voir les formules comme des figures faites de lettres.

[1]Nous reprenons ici brièvement une partie de l'article "La démonstration : pulsation entre le visuel et le discursif", in *Produire et lire des textes de démonstration*, IRMAR, Université de Rennes I, pp.39-67.
[2]voir GUITART, *La pulsation mathématique*, L'Harmattan, Paris, 1999, n°157, pp.160-167.

1 Nécessité visuelle et nécessité discursive

La nécessité d'une proposition comme celle qui concerne les trois angles d'un triangle peut être d'ordre visuelle. Voir de multiples triangles aux formes variées ne permet pas d'énoncer quelque chose, mais voir se composer trois copies d'un même triangle permet d'affirmer que les trois angles d'un triangle se juxtaposent pour former un angle plat ou deux angles droits (fig.1). La composition du dessin n'est pas fortuite, elle dépend d'une intelligibilité, mais c'est la stabilité de la composition, valable pour n'importe quel triangle, qui fait nécessité. La vision dépend aussi d'une intelligibilité, il faut voir en même temps les trois angles comme ceux d'un même triangle, et comme ceux formant un angle plat. C'est un regard théorique, mathématique, qui est porté sur le dessin. On voit un dessin, mais on regarde une figure. Le seul discours qui suffirait donc serait : "Regardez".

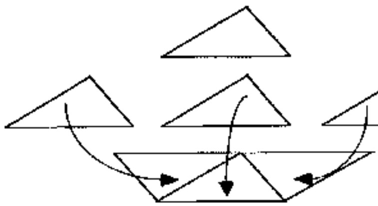


FIGURE 1

La proposition XXXII du Livre I des *Éléments* d'Euclide énonce que

> dans tout triangle, un des côtés étant prolongé, l'angle extérieur est égal aux deux angles intérieurs et opposés, et les trois angles intérieurs du triangle sont égaux à deux droit[1].

Puis, ce qui est rituel dans l'ouvrage euclidien, cet énoncé impersonnel est repris en disant une figure. Dire une figure (Figure 2) demande une littération des points, des droites[2] et des triangles, c'est-à-dire une lettre pour dire un point, deux pour une droite, trois pour un triangle. Le texte se poursuit par :

> Soit le triangle ABC, et qu'un des côtés BC, soit prolongé au-delà jusqu'en D. Je dis que l'angle extérieur, celui sous ACD, est égal aux deux angles intérieurs et opposés, ceux sous CAB, ABC, et que les trois angles intérieurs du triangle, ceux sous ABC, BCA, CAB, sont égaux à deux droits.

Dire une figure permet d'affirmer, de manière personnelle, la proposition : *je dis que*.

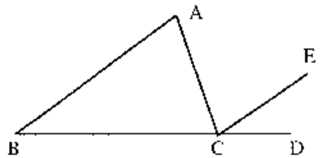


FIGURE 2

[1]EUCLIDE, *Les Éléments*, trad. Vitrac, vol.I, PUF, Paris, pp. 255-56.

[2]comme Euclide, nous appelons droite une droite finie (ce que l'on appelle aujourd'hui segment).

La démonstration qui suit est l'expression d'une nécessité discursive, selon deux modes.

D'une part, le discours exprime une nécessité déductive : cette proposition prenant place dans un système axiomatico-déductif, elle doit être déduite des axiomes et des propositions précédentes. Ces propositions sont réactivées grâce aux lettres de la figure, et le discours fait usage des mots "en effet", "puisque", "donc", etc. Euclide écrit :

> En effet, que par le point C, soit menée CE parallèle à la droite AB (prop. 31). Puisque AB est parallèle à CE et que AC tombe sur elles, les angles alternes, ceux sous BAC, ACE sont égaux entre eux. Ensuite, puisque AB est parallèle à CE et que la droite BD tombe sur elles, l'angle extérieur, celui sous ECD, est égal à celui sous ABC, intérieur et opposé (prop. 29).

D'autre part, le discours indique quels sont les angles qu'il faut regarder pour les comparer et les juxtaposer. Euclide écrit :

> Et il a aussi été démontré que celui sous ACE est égal à celui sous BAC. L'angle tout entier sous ACD est donc égal aux deux angles intérieurs et opposés, ceux sous BAC, ABC (n.c.2). Que soit ajouté de part et d'autre celui sous ACB. Ceux sous ACD, ACB sont donc égaux aux trois sous ABC, BCA, CAB. Mais ceux sous ACD, ACB sont égaux à deux droits (prop. 13); donc ceux sous ABC, BCA, CAB sont aussi égaux à deux droits (n.c.1).

Ainsi, ce discours correspond à la nécessité visuelle de la première figure (Figure 1), mais l'abréviation de la figure de la proposition (Figure 2), le fait qu'elle ne soit qu'une partie de la précédente, réclame un discours plus long que "regardez", avec des "aussi", "mais" et "donc". La démonstration est ici une "lecture raisonnée du dessin"[3].

Le discours d'Euclide est un discours raisonné sur des figures, où les figures participent elles-mêmes du raisonnement. En effet, les figures des propositions doivent nécessairement être construites à la règle et au compas, et inversement, la possibilité de telles constructions découle d'une nécessité discursive et axiomatique. Ainsi, les trois premières demandes du Livre I permettent la construction de droites et de cercles à partir de points :

1. Qu'il soit demandé de mener une ligne droite de tout point à tout point.
2. Et de prolonger continûment en ligne droite une ligne droite limitée.
3. Et de décrire un cercle à partir de tout centre et au moyen de tout intervalle.

Les figures sont donc obtenues par composition de traces géométriques élémentaires que sont la droite et le cercle. Au long du Livre I, les propositions relatives aux constructions, par exemple plus haut celles de droites parallèles, s'entremêlent à des propositions qui, à la fois, sont nécessaires à leur déduction, et rendent nécessaires leurs possibilités.

Les autres demandes peuvent être interprétées, de ce point de vue, comme les prémisses à un discours axiomatique sur des figures. Par exemple, la demande 5, *si une droite tombant sur deux droites fait les angles intérieurs et du même côté plus petits que deux droits, les deux droites indéfiniment prolongées, se rencontrent du côté où sont les angles plus petits que deux droits*, permet de dire l'existence d'un point de rencontre éventuellement inaccessible sur la feuille à partir de traces locales[4]. Les notions communes concernent alors la possibilité de dire l'égalité des choses, les figures, à partir de l'ajustement ou de la juxtaposition des traces, les dessins :

1. Les choses égales à une même chose sont aussi égales entre elles.
2. Et si, à des choses égales, des choses égales sont ajoutées, les touts sont égaux.

[3]BKOUCHE, De la démonstration en géométrie, in *Le dessin géométrique, de la main à l'ordinateur*, IREM de Lille, 1996, pp. 197-98.

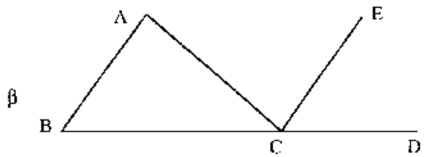
[4]GUITART, *op.cit.*, n°70, pp. 212-214.

3. Et si, à partir de choses égales, des choses égales sont retranchées, les touts sont égaux.
4. Et si, à des choses inégales, des choses égales sont ajoutées, les touts sont inégaux.
5. Et les doubles du même sont égaux.
6. Et les moitiés du même sont égales entre elles.
7. Et les choses qui s'ajustent les unes et les autres sont égales entre elles.
8. Et le tout est plus grand que la partie.

La dernière notion commune, *deux droites ne contiennent pas une aire*, permet de dire une droite par les deux lettres qui désignent les points de ses extrémités, puisque AB désigne sans ambiguïté l'unique droite joignant A à B. Dans le texte euclidien, il peut aussi être mentionné un rectangle AB ou un carré AB, et ces deux figures ont alors A et B comme sommets opposés. Ainsi, la désignation ne peut se passer de ce qu'elle désigne, de la trace, du dessin.

L'ouvrage d'Euclide est traduit et commenté dans de nombreux ouvrages des $16^{\text{ème}}$ et $17^{\text{ème}}$ siècles. Le *Cours mathématique* d'Hérigone de 1634 est l'une de ces reprises des *Éléments*, elle est intéressante du point de vue de l'usage des lettres dans les figures et dans le discours mathématique. Dans les *Éléments*, l'usage des lettres ne vise pas à abréger le discours, les lettres n'ont pas un rôle opératoire sur le discours mais pour le discours. Il n'en est pas de même dans le *Cours mathématique*, qui est voulu comme un "cours mathématique démontré d'une nouvelle, brève, et claire méthode, par notes réelles & universelles, qui peuvent être entendues facilement sans l'usage d'aucune langue". La langue dont Hérigone ne fait point usage est celle de l'articulation grammaticale de vocables, car les mots de la langue vont être remplacés par des lettres ou des symboles, et la disposition en colonne va suppléer aux conjonctions de subordination.

La démonstration de la proposition XXXII, *De tout triangle, l'un des côtés étant prolongé, l'angle externe est égal aux deux internes & opposés : & les trois angles internes de tout triangle, sont égaux à deux droits*, est la suivante[5] :



	Hypoth	$\alpha.29.1$	$< ecd\,2/2 < b.$
	abc est Δ.	1.concl.	$< acd\,2/2 < a+ < b.$
	bcd est $----$.	$2.a.1$	
	Requ.π.demonstr.	β	$< a+ < b2/2 < acd$
	$< acd\,2/2 < a+ < b.$		$< acb$ commun.add.
	Prepacr.	$2.a.1$	$< a+ < b+ < acb\,2/2 < acd+ < acb.$
31.1	$ce = ba.$	13.1	$< acd+ < acb\,2/22_x.$
	Demonstr.	2.concl.	$< a+ < b+ < acb\,2/22_x$
$\alpha.29.1$	$< eca\,2/2 < a.$	$1.a.1$	

[5] HÉRIGONE, *Cours mathématique*, Le Gras, Paris, 1634, pp. 37-38.

La démonstration du théorème de Pythagore dans les *Éléments* d'Euclide est un autre exemple pour comparer nécessité visuelle et nécessité discursive. Il faut montrer que le carré $BCDE$ est égal (en aire) à la somme des carrés $ABFG$ et $ACKH$ (Figure 3).

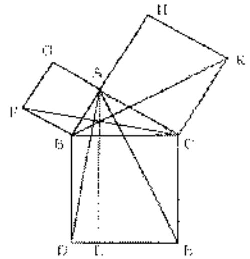


FIGURE 3

Cette proposition est l'avant-dernière du livre I, et la démonstration est donc déduite d'un bon nombre de propositions qui précèdent, en particulier, des propositions qui énoncent que deux parallélogrammes de même base et situés entre les mêmes parallèles sont égaux (en aire), et qu'un triangle est égal (en aire) à un parallélogramme de même base et situé entre les mêmes parallèles. Elle est relativement longue[6], puisqu'il faut démontrer que la figure construite est valide, par exemple que AC est aligné avec le côté AG du carré construit sur AB.

Schopenhauer, dans *Le monde comme volonté et comme représentation*, qualifie cette démonstration d'étrange et d'absurde. Il écrit :

> Elle se donne une peine infinie pour détruire l'évidence, qui lui est propre, et qui d'ailleurs est plus à sa portée, pour lui substituer une évidence logique. [. . .] A nos yeux, la méthode d'Euclide n'est qu'une brillante absurdité. [. . .] La démonstration boiteuse et même captieuse d'Euclide nous abandonne au pourquoi, tandis que la simple figure [Figure 4] [. . .] nous fait entrer du premier coup, et bien plus profondément que la démonstration, au cœur même de la question ; elle nous amène à une plus intime conviction de la nécessité de cette proposition et de sa liaison avec l'essence même du rectangle[7].

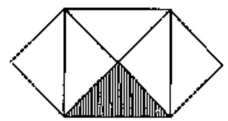


FIGURE 4

En effet, en copiant huit fois un triangle rectangle isocèle, et en faisant deux assemblages différents, on peut donner à un cas particulier du "théorème de Pythagore" un caractère de nécessité visuelle (Figure 5).

[6] EUCLIDE, *op.cit.*, pp. 282-84.
[7] SCHOPENHAUER, *Le monde comme volonté et comme représentation*, trad. Burdeau, PUF, Paris, 1966, pp. 106-110.

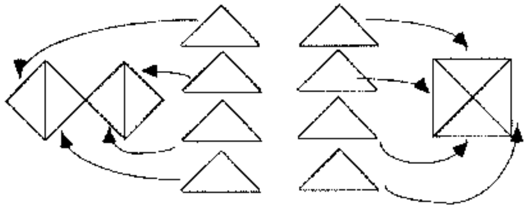


FIGURE 5

La figure de Schopenhauer peut être rapprochée de celle du dialogue du *Ménon* de Platon[8]. Ceci afin de remarquer que, dans le texte de Platon, le discours sur la figure vient suppléer à une impossibilité de dire un nombre. En effet, la première question que pose Socrate à l'esclave est de dire le côté d'un carré dont l'aire soit double de celle d'un carré d'aire deux. Cette question est remplacée, dans la deuxième partie du dialogue, par celle de montrer un carré dont l'aire soit double de celle d'un carré donné. Chaque étape de la construction indiquée par Socrate peut-être vue comme celle d'une composition de traces géométriques (Figure 6).

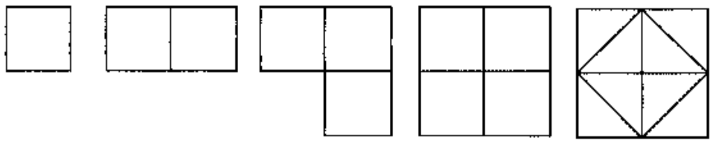


FIGURE 6

Une généralisation de la figure de Schopenhauer, produisant une nécessité visuelle du théorème de Pythagore, peut être celle de la démonstration du 3ème siècle de Liu-Hui (Figure 7) ou celle des *Éléments de géométrie* de 1753 de Clairaut (Figure 8).

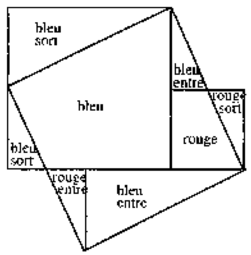


FIGURE 7

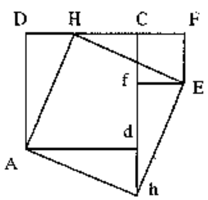


FIGURE 8

Pour Clairaut, le théorème de Pythagore est la conséquence de la construction d'un carré égal à la somme (géométrique) de deux carrés, construction qui généralise la construction d'un carré double d'un autre, c'est-à-dire la construction de Socrate. Il écrit :

[8]PLATON, *Œuvres complètes*, vol. I, trad. Robin, Gallimard, Paris, 1950, pp. 530-535.

> Supposons présentement qu'on veuille faire un carré égal à la somme des deux carrés inégaux $ABDd$, $CFEf$, ou, ce qui revient au même qu'on se propose de changer la figure $ADFEfd$ en un carré. En suivant l'esprit de la méthode précédente [Faire un carré double d'un autre], on cherchera s'il n'est point possible, de trouver dans la ligne DF, quelque point H, tel 1° Que tirant les lignes AH et HE, et faisant tourner les triangles ADH, EFH, autour des points A et E, jusqu'à ce qu'ils aient les positions Adh, Efh, ces triangles se joignent en h; 2° Que les quatre côtés AH, HE, Eh, hA, soient égaux et perpendiculaires les uns aux autres.
>
> Or ce point H se trouvera en faisant DH égal au côté CF ou EF. Car de l'égalité supposée entre DH et CF, il suit premièrement que si l'on fait tourner ADH autour de son angle A, en sorte qu'on lui donne la position Adh, le point H arrivé en h sera distant du point C d'un intervalle égal à DF.
>
> De la même égalité supposée entre DH et CF, il suit encore que HF égalera DC, et qu'ainsi le triangle EFH tournant autour de E pour prendre la position Efh, le point H arrivera au même point h, distant de C d'un intervalle égal à DF.
>
> Donc la figure $ADFEfd$ sera changée en une figure à quatre côtés $AHEh$. Il ne s'agit donc plus que de voir si ces quatres côtés sont égaux et perpendiculaires les uns aux autres.

Ce discours est celui de la visualisation de figures en mouvement (Figure 9).

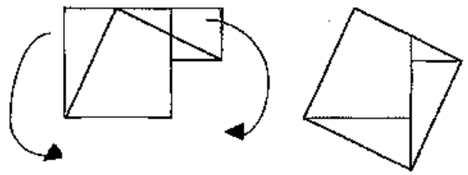


FIGURE 9

2 Nombre et grandeur : traces géométriques et traces littérales

Dans les mathématiques d'Euclide, nombre et grandeur sont des choses de genres différents : le nombre est du domaine du discret, de l'arithmétique, tandis que la grandeur est du domaine du continu, du géométrique. Mêler nombres et grandeurs serait faire un "mélange de genres", et leur traitement spécifique fait donc l'objet de livres différents, les livres VII, VIII et IX pour les nombres, les livres VI et X pour les grandeurs. Cependant, les nombres sont représentés par des droites. Ainsi, après l'énoncé de la proposition 1 du livre VII, "Deux nombres inégaux étant proposés et le plus petit étant retranché du plus grand de façon réitérée et en alternance, si le reste ne mesure jamais [le reste] précédent jusqu'à ce qu'il reste une unité, les nombres initiaux seront premiers entre eux",

Euclide écrit que

> En effet, que deux nombres AB, CD, le plus petit étant retranché du plus grand de façon réitérée et en alternance, le reste ne mesure jamais le [reste] précédent jusqu'à ce qu'il reste une unité. Je dis que AB, CD sont premiers entre eux, c'est à dire qu'une seule unité mesure AB, CD.[9]

Les nombres sont donc visualisés géométriquement par des figures géométriques, qui, elles-mêmes, sont désignées par des lettres (Figure 10).

[9]EUCLIDE, *Les Éléments*, trad. Vitrac, vol.II, PUF, Paris, p. 290.

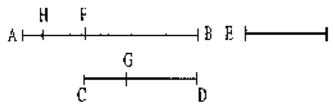


FIGURE 10

Cette littération permet de voir et de dire les différences successives de nombres sans les particulariser par des valeurs, la décomposition des traces géométriques donne à voir et à dire celle des nombres :

> Car si AB, CD ne sont pas premiers entre eux, un certain nombre les mesurera. Qu'il les mesure et que ce soit E. Et, d'une part que CD mesurant BF, il reste FA, plus petit que lui, et d'autre part que, AF mesurant DG, il reste GC, plus petit que lui, et que GC mesurant FH, il reste une unité HA.

C'est ainsi que la visualisation et la littération de la proposition 2 du Livre X sur les grandeurs, trouvent une similarité avec celles de la proposition précédente. Dans celle-ci, il s'agissait de dire une condition de primalité pour les nombres, dans l'autre la condition concerne les grandeurs :

> Si, de deux grandeurs inégales la plus petite étant retranchée de la plus grande de façon réitérée et en alternance, le dernier reste ne mesure jamais le [reste] précédent, les grandeurs seront incommensurables.[10]

Euclide écrit :

> En effet, AB, CD étant deux grandeurs inégales et AB la plus petite, la plus petite étant retranchée de la plus grande de façon réitérée et en alternance, que le reste ne mesure jamais le reste précédent. Je dis que les grandeurs sont incommensurables.

La figure littéralisée et dite est semblable à la précédente (Figure 11).

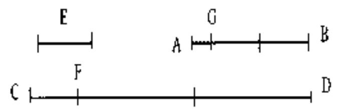


FIGURE 11

Dans l'algèbre symbolique du 16ème siècle, les irrationnels sont dé-géométrisés. D'une part, ils interviennent comme racines d'équations, et non pas dans un rapport géométrique. D'autre part, ils vont être vus et dits avec les symboles de l'algèbre, et non à partir d'une figure géométrique. Ce qui, du coup, met en évidence le statut symbolique des figures géométriques et les figures numériques. C'est ainsi que, Jacques Peletier du Mans défend l'usage des nouveaux signes de l'algèbre en écrivant :

[10]EUCLIDE. *Les Éléments*, trad. Vitrac, vol.III, PUF, Paris, p. 94.

> Et qui blamera mon Livre, pour contenir nouveaux signes ou caractères : qu'il pense, qu'à nouvel art, nouveaux commencements et nouvelle matière. Qu'il pense encore que toute l'Arithmétique ne se saurait passer de figures élémentaires : lesquelles, combien qu'elles semblent plus servir à l'œil sensitif qu'au spirituel (comme sont 1, 2, 3, 4, etc.) toutefois sans elles ne se sauraient faire aucune opérations arithmétiques sinon en l'air. La Géométrie même a ses lignes, et encore ses superficies & corps materiels : pour montrer que les sens extérieurs sont messagers sujets & moyenneurs de ceux de dedans[11].

Les signes de l'algèbre sont donc des traces assimilées à celles des nombres et des figures.

Les symboles pour les nombres radicaux s'écrivent en suivant la progression arithmétique :

prog. arithmétique :	0,	1,	2,	3,	4,	5,	...16
nombres radicaux :	1,	$\mathcal{R}$,	$\mathcal{C}$,	$\mathcal{T}$,	$\mathcal{CC}$,	$\mathcal{B}$,	...$\mathcal{CCCC}$.

Peletier remarque que les nombres radicaux sont ainsi plus aisés à voir qu'à dire :

> Au second rang, sont les caractères des nombres radicaux qui appartiennent à l'Algèbre, portant leur dénomination. Savoir est, $\mathcal{R}$, Racine, $\mathcal{C}$, Cense, $\mathcal{T}$, Cube, $\mathcal{CC}$, Censicense, etc. Le dernier terme est Censicensicensicensique : comme vous voyez par le signe $\mathcal{CCCC}$. Et encore que le mot semble être rude, il suffit qu'il soit signifiant. Car c'est beaucoup d'avoir donné nom à choses si inusitées et si peu pratiquées[12].

Les trois sortes de "nombres appartenant à l'algèbre" sont caractérisées par la manière dont sont composés littéralement les différents symboles, ce qui permet aussi de les "prononcer". Les premiers nombres de l'algèbre sont les "nombres cossiques", ils ont un signe postposé, *comme 3 $\mathcal{R}$, 6 $\mathcal{C}$, 25 $\mathcal{T}$ qui se prononcent trois racines, six censes, 25 cubes.* Les seconds sont les "nombres irrationaux", ils ont un signe préposé, *comme $\sqrt{\mathcal{C}}$ 20 qui se prononce Racine censique de 20.* Les troisièmes ont un signe préposé et l'autre postposé, *comme $\sqrt{\mathcal{C}}$ 8 $\mathcal{T}$ qui se prononce Racine censique de huit Cubes.* Les algorithmes des opérations sur les nombres de l'algèbre sont ensuite donnés à voir et à dire à l'aide de ces symboles, et de nouveaux signes, comme $p.$ et $m.$ pour l'addition et la soustraction.

Peletier parle des "nombres de l'algèbre" pour des choses qui ne sont pas des nombres entiers, tout comme il considère que les irrationaux sont des nombres. Il écrit :

> Non sans propos se fait un doute sur les nombres irrationaux, s'ils sont nombres ou non. Car d'une part il est certain qu'ils sont quelque chose : vu que par leur aide, on parvient, non seulement à la preuve, mais aussi à la précision de nombreux théorèmes, dont les nombres rationaux ne font qu'approcher [...]. Davantage ils ont leur algorithme, leur ordre et règles infaillibles, tout ainsi que les rationaux comme nous avons à voir.

Ainsi, l'assimilation des irrationaux à des nombres repose sur celle des algorithmes des opérations sur les uns et les autres, qui repose à son tour sur une similitude visuelle et discursive.

La "grande règle générale de l'algèbre" explique comment se construit le discours de l'algèbre :

[11]Poème de Jacques Peletier sur le premier livre de son Algèbre, in PELETIER, *L'Algèbre*, Jean de Tournes, Cologny, 1609.

[12]PELETIER, *op.cit.*, pp. 8-9.

Au lieu du nombre inconnu que vous cherchez, mettez 1 $\mathcal{R}$. Avec laquelle faites votre discours selon la formalité de la question proposée : tant qu'ayez trouvé une équation convenable, et icelle réduite, si besoin est.

L'algèbre de Peletier fait usage des traces géométriques seulement lorsqu'il s'agit de résoudre un problème du second degré : "il y a deux nombres, desquels les quarrés ajoutés, font 205 : & les deux nombres multipliés l'un par l'autre font 78". Il commente :

C'est comme s'il se proposait, il y a une ligne divisée en deux parties inégales : le quarré de laquelle est fait de deux quarrés particuliers avec leurs deux suppléments [...] : les deux quarrés joints ensemble faisant 205, et l'un de suppléments 78. Quelles sont les parties de la ligne ? Je mets ceci au long à fin d'apprendre au lecteur à approprier les questions Arithmétiques aux Géométriques: lesquelles se rapportent les unes aux autres quasi partout.[...] Soit donc la ligne AB, divisée au point C. Et mettons pour la portion AC, $1\mathcal{R}$. Dont le quarré est $1\,\mathcal{C}$. Partant l'autre quarré sera 205 $m.1\mathcal{C}$. Duquel la racine est $\sqrt{\mathcal{C}}$ 205 $m.1\mathcal{C}$[13].

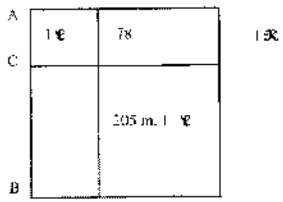


FIGURE 12

Par conséquent, le nombre inconnu est vu et dit, à la fois, à l'aide d'une lettre, par 1 ℝ, et d'une droite, par AC. Du coup, la droite AC n'est plus littéralisée par deux lettres, A et C, mais par une seule, $\mathcal{R}$. De la même façon, une seule lettre, $\mathcal{C}$, représente un carré. Les lettres se mêlent dans le discours de l'algèbre à des symboles de nombres et d'opérations. Alors que la géométrie grecque séparait nombres et grandeurs, l'algèbre les mélange, car, comme le dit Peletier, il est bon d'apprendre à approprier les questions arithmétiques aux géométriques, à associer les signes de l'algèbre à ceux de la géométrie.

Près d'un siècle plus tard, dans ses *Nouveaux éléments de géométrie* de 1667, Arnauld littéralise la géométrie. Mais, il ne veut rien voir derrière la lettre, que la lettre elle même. Arnauld écrit au début de son ouvrage à titre de "supposition" :

Je suppose enfin qu'on s'accoutume à concevoir généralement les choses en les marquant par des lettres sans se mettre en peine de ce qu'elles signifient, puisqu'on s'en sert que pour conclure que b est b, que c est c, ou ce qui est pris pour la même chose en matière de grandeur, sur tout en général, que b est égal à b, et c à c, ou que b multiplié par c est égal à b multiplié par c [...]. L'une des plus grandes utilités de ce traité, est d'accoutumer l'esprit à concevoir les choses de manière spirituelle sans l'aide d'aucune image sensible, ce qui sert beaucoup à nous rendre capables de la connaissance de Dieu & de notre âme[14].

[13] PELETIER, *op.cit.*, pp. 194-196.

[14] ARNAULD, *Nouveaux éléments de géométrie*, Savreux, Paris, 1667, p. 4.

Pour lui, l'étude de la géométrie permet de parvenir à la piété, parce qu'elle éloigne l'esprit de choses sensibles et sensuelles.

Cependant, dans le choix des mots "grandeur linéaire" et "grandeur plan" d'Arnauld, il est clair que les lettres se réfèrent à du géométrique. Ainsi, une seule lettre renvoie à la ligne tandis que la concaténation de deux lettres renvoient au rectangle, et donc au plan. La composition des lettres a donc une signification géométrique, on parlera ici d'une "littéralisation" de la géométrie. Arnauld écrit :

Multiplier ou Multiplication s'exprime ainsi b en c, et se marque ainsi $b \times c$, ou plus brièvement bc. [...] Où il faut remarquer qu'une grandeur marquée par un seul caractère comme b, ou c, s'appelle grandeur linéaire [...]. Que s'il n'y a eu que deux grandeurs linéaires qui aient été multipliées l'une par l'autre, ce produit s'appelle grandeur plane ou plan. [...] Lorsque les deux grandeurs ont chacune deux termes, parce que deux fois deux font 4, il faudra faire 4 multiplications partielles pour avoir le produit total

b + c
En produit $pb + pc + qb - qc$[15].
p + q

La figure du rectangle est assimilée à la multiplication de deux grandeurs droites figurée par la concaténation de deux lettres. Mais les lettres ou les concaténations de lettres peuvent aussi représenter des nombres, des plans, des solides. La possibilité de concaténer indifféremment toutes ses lettres signifie-t-il la liberté de multiplier indifféremment nombres et grandeurs géométriques de toutes dimensions ? c'est-à-dire des grandeurs hétérogènes ?

Des grandeurs de même genre s'appellent homogènes, comme deux nombres, deux lignes, deux surfaces. De divers genres hétérogènes, comme un nombre et une ligne; une surface et un solide.

Arnauld répond affirmativement :

On croit ordinairement que les grandeurs de divers genres qu'on appelle hétérogènes ne se peuvent pas multiplier. Cela ne me paraît pas vrai, ou a besoin d'explication. [...] Ce qui ne peut se multiplier par la nature se peut multiplier par une fiction d'esprit par laquelle la vérité se découvre aussi certainement que par les multiplications réelles. [...] On multiplie aussi par la même fiction d'esprit des surfaces par des surfaces, quoique cela donne pour produit une étendue de quatre dimensions qui ne peut être dans la nature.[16]

Avant lui, Descartes a montré dans *La géométrie* de 1637, comment, à condition d'introduire une droite unité, la multiplication de deux droites est une droite. Mais pour Arnauld, la vérité des "produits imaginaires" de surfaces ne saurait dépendre de la linéarisation cartésienne, qui leur est visiblement étrangère.

Tout comme on somme et on multiplie des nombres, et sans changer les formules littérales exprimant les algorithmes opérationnels, dans la méthode analytique, on somme et on multiplie des droites ayant même direction. Dans son *Exposition de la méthode des équipollences* de 1854, Bellavitis se propose de faire de même pour des droites n'ayant pas la même direction. Il introduit pour cela une méthode et un nouveau "graphisme" :

[15] ARNAULD, *op.cit.*, p. 6-11.

[16] ARNAULD, *op.cit.*, p. 38.

Cette méthode donne satisfaction au désir exprimé par Carnot, de trouver un algorithme qui représente à la fois la grandeur et la position de diverses parties d'une figure; il en résulte directement des solutions graphiques, élégantes et simples, des problèmes de géométrie[17].

Le calcul des équipollences a des règles qui s'exprime graphiquement. La règle I indique que, quels que soient les trois points A, B, C on a toujours

$$AB + BC \,\underline{\Omega}\, AC.$$

Cette règle s'exprime visuellement par la disparition de la lettre B. Bellavitis la commente en montrant qu'elle peut servir à des démonstrations visuelles, où il ne s'agit pas de regarder des figures mais d'avoir l'œil sur des lettres et des formules. Il écrit :

Cette règle est d'un continuel usage pour substituer une droite à d'autres [...]. Nous y joindrons une règle pour distinguer d'un coup d'œil qu'elles sont les équipollences identiques, de manière qu'on puisse s'assurer de leur exactitude sans aucun effort d'esprit, et sans regarder le moins du monde la figure [...]. Ainsi, par exemple,

$$AB + BC \,\underline{\Omega}\, AD - CD$$

est une équipollence identique, puisque

$$AZ - BZ + BZ - CZ \,\underline{\Omega}\, AZ - DZ - (CZ - DZ).$$[18]

D'autres règles concernent l'addition et l'inclinaison des droites :

Règle II : Si $mIL \,\underline{\Omega}\, nMN$ et $inc.IL \neq inc.MN$
alors $m = n = 0$.

Règle III : si $LM + MN + NL \,\underline{\Omega}\, 0$ et $inc.LM = inc.MN$
alors $inc.NL = inc.MN \pm 180°$
et $gr.NL = gr.LM + gr.MN$.

Règle IV : si $LM + MN + NL \,\underline{\Omega}\, 0$
et $inc.LM + inc.LN = 2inc.MN$
alors $gr.LM = gr.NL$.

Ceci indique que la géométrie vectorielle, comme la géométrie analytique, s'appuie sur la vision des lettres et des signes autant ou plus que sur celle des figures géométriques.

3 Visuel et discursif : la proposition VI du livre II d'Euclide

La proposition énonce que

si une ligne droite est coupée en deux parties égales, et si on lui ajoute directement une droite, le rectangle compris sous la droite entière avec la droite ajoutée, et sous la droite ajoutée, avec le quarré de la moitié de la droite entière, est égal au quarré décrit avec la droite composée de la moitié de la droite entière et de la droite ajoutée, comme avec une seule droite.

Puis Euclide tient un discours sur la figure lettrée (Figure 13) :

Qu'une ligne droite AB soit coupée en deux parties égales au point C; qu'on lui ajoute directement une autre droite BD; je dis que le rectangle compris sous AD, DB, avec le quarré de CB est égal au carré de CD.[19]

[17]BELLAVITIS, *Exposition de la méthode des équipollences*, trad. Laisant, Gauthier-Villars, Paris, 1874, p. 2.
[18]BELLAVITIS, *op.cit.*, p. 10.
[19]EUCLIDE, *Les Éléments*, trad. Vitrac, vol. I, PUF, Paris, p. 335.

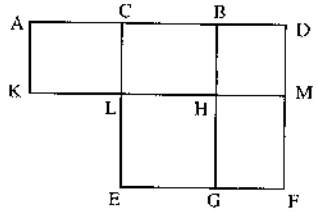


FIGURE 13

La nécessité visuelle du résultat peut tenir en un simple mouvement de la figure initiale (Figure 14). Tandis que la démonstration d'Euclide est assez longue, elle exprime une nécessité discursive en s'appuyant sur deux propositions du Livre I : la proposition XXXVI sur l'égalité des (aires de) parallélogrammes de même base et de même hauteur, qui permet d'affirmer l'égalité des (aires des) rectangles AL et CH, et la proposition XVIII qui permet d'affirmer l'égalité des (aires des) rectangles CH et HF. Le discours raisonné sur la figure indique les parties qu'il faut comparer et juxtaposer :

Que LG - qui est égal au carré sur BC - soit ajouté de part et d'autre. Le rectangle contenu par AD, DB, avec le carré sur CB est donc égal au gnomon NOP avec LG. Mais le gnomon NOP et sont le carré $CEFD$ tout entier - qui est celui décrit sur CD. Donc le rectangle [...].

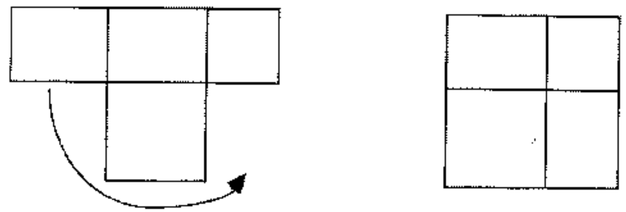
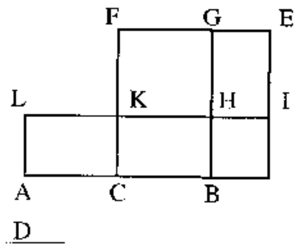


FIGURE 14

La démonstration correspondante d'Hérigone abrège le discours[20]. La disposition en colonnes évite l'utilisation des "donc", "car", "puisque", etc., et la numérotation des propositions précédentes permet d'indiquer quelle est la proposition appliquée pour affirmer une nouvelle conséquence logique. De nouveaux symboles sont introduits à la place des mots "carré", "rectangle", "parallèle". La propriété de la transitivité de l'égalité est ainsi vue sur des écritures symboliques. De plus, l'indication de la juxtaposition des aires est marquée par le signe arithmétique +. Tout ceci permet de voir cette juxtaposition comme une somme.



[20]HÉRIGONE, *op. cit.*, pp. 77-78.

	Hypoth.		*Demonstr.*
	ac 2/2 cb,	const.	ce est □ . cd,
	abd est ⋯.	1.c.4.2.	kg & bi snt □
	Requ. π. demonstr.	43.1	□ he 2/2 □ ch,
	□. adb+□.cb 2/2 .cd,	α.36.1	□ak 2/2 ⊔ ch,
	Prepaer.	1.a.1	□he 2/2 □ ak,
46.1	ce est □.cd,		□ci *commun.add.*
1.p.1	fd *est diamet.*	2.a.1	*gnom*.kdg 2/2 □ ai,
31.1	bg = cf ⋃ de,		□ kg *commun. add.*
31.1	al = cf,	1.a.1	*gnom*.kdg+□kg 2/2 ai □.adb+kg□.ch
31.1	lhi=ad,	1.a.1	*gnom*.kdg+□kg 2/2 ce□.cb,
		concl. 1.a.g.	□.adb+□.cb 2/2 □.cd.

L'abréviation symbolique d'Hérigone n'induit pas une nécessité autre que celle du discours euclidien. En revanche, la littéralisation de la géométrie d'Arnauld procure un nouveau type de nécessité, une nécessité que l'on peut qualifier de "littérale". Le livre XIV des *Nouveaux éléments de géométrie* est consacré "aux figures planes selon leurs aires, c'est-à-dire selon la grandeur des surfaces qu'elles contiennent" et commence avec le rectangle. Dans la partie intitulée "De la puissance d'une ligne comparée avec la puissance de ses parties", Arnauld énonce comme troisième axiome :

> C'est la même chose de multiplier le tout par le tout, & de multiplier le tout par chacune de ses parties, ou de multiplier chaque partie par toutes les parties.

Puis il écrit comme "avertissement" :

> Ainsi le plus grand mystère pour ne point se brouiller est de nommer chaque ligne autant que l'on peut par un seul caractère, afin que deux caractères joints ensemble puissent marquer une multiplication, c'est-à-dire un rectangle, & de marquer par un même caractère les lignes égales[21].

Dans la littération des figures par Euclide, une ligne droite (finie) est désignée par deux lettres qui dénomment deux points. Tandis que dans la littéralisation des figures par Arnauld, une ligne droite est désignée par une seule lettre. La composition ou la concaténation de deux lettres a un sens opératoire et géométrique, celui de fabriquer à partir de deux droites une "chose" hétérogène et d'une complexité supérieure, un rectangle.

Les formulations lettrées des propriétés de la somme et de la multiplication correspondent à des propriétés géométriques. Ainsi, Arnauld donne comme exemple du troisième axiome :

> la ligne b soit divisée en trois portions inégales que j'appellerai $c.d.f.$ il est visible que c'est la même chose de multiplier b par b, ce qui donne bb, que de multiplier b par toutes ces parties; c'est-à-dire par c, par d; & par f, ce qui donne $bc.bd.bf$: & par conséquent
>
> $$bb = bc + bd + bf.$$

[21] ARNAULD, *op. cit.*, p. 285.

Le discours est accompagné d'une figure (Figure 15), et la question est de savoir à quelle vision se réfère Arnauld quand il écrit "qu'il est visible que" : celle de la figure ou celle de la formule ? S'il s'agit de s'appuyer sur la vision de la figure, n'est-ce pas contraire à la "supposition" du premier livre qui demande de concevoir les choses sans l'aide d'images sensibles ? En tous les cas, à partir de là, la vision de la formule algébrique, où la lettre b se distribue sur ses parties sera suffisante pour valider des propositions géométriques.

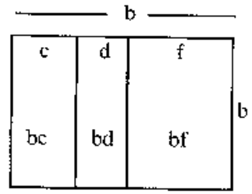


FIGURE 15

En effet, Arnauld littéralise tout le livre II des *Éléments* d'Euclide, en remarquant que :

> ainsi presque toutes les propositions du second livre d'Euclide ne sont que des corollaires de cet axiome & de cet avertissement[22].

La démonstration de la proposition VI tient seulement en la formule

$$b(b+2c) + cc = (b+c)^2.$$

Avec le "graphisme" de Bellavitis, la vision de cette formule se complexifie, car elle va être transformée en une équipollence. Une telle transformation fait l'objet d'un "théorème général" qui produit de nouveaux théorèmes. Bellavitis énonce :

> Théorème général : toute propriété des points d'une droite donne un théorème relatif aux points d'un plan, par le seul changement des équations en équipollences[23].

Suivons le premier exemple qu'il donne :

> Prenons pour exemple la formule algébrique
>
> $$b(b+2c) + c^2 = (b+c)^2 \qquad (1)$$
>
> qui conduit à la propriété VI du second livre d'Euclide : si une droite BD est divisée en C également, c'est-à-dire si $BC = CD$, et que A soit un point quelconque dans son prolongement, on aura
>
> $$AB.AD + (BC)^2 = (AC)^2. \qquad (2)$$

Ainsi, Bellavitis part de la formule algébrique (1), dont il commence par donner une vision géométrique (Figure 16), la formule correspondant à une littéralisation de la figure. Puis il lettre la figure et il obtient ainsi l'équation (2), qui correspond à la littération de la figure (Figure 17).

[22] ARNAULD, *op. cit.*, p. 287.

[23] BELLAVITIS, *op. cit.*, p. 21.

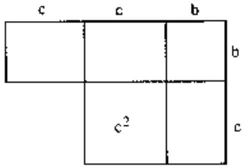


FIGURE 16

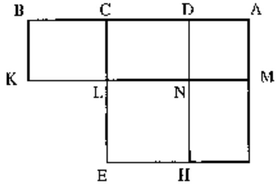


FIGURE 17

L'équation (2) concerne trois points A, B, et D d'une droite (Figure 18), donc d'après le théorème général, elle reste valable si les trois points sont des points du plan (Figure 19), c'est-à-dire encore, que l'équation (2) peut être transformée en une équipollence. Ceci est facilement démontré en utilisant la règle I, que nous avons commenté plus haut. Bellavitis écrit :

> Pour vérifier cette équation [. . .], observons si l'équation
>
> $$(AZ - BZ)(AZ - DZ) + (BZ - CZ)^2 = (AZ - CZ)^2$$
>
> devient identique dans l'hypothèse $BC = CD$, c'est-à-dire
>
> $$BZ - CZ = CZ - DZ.$$
>
> Après avoir fait cette vérification, nous avons le théorème suivant : Si un côté BD du triangle ABD est divisée par le milieu en C, l'équipollence suivante aura toujours lieu
>
> $$AB.AD + (BC)^2 \,\Omega\, (AC)^2. \qquad (3)$$

En effet, le calcul est le même, que les points A, B et D soient alignés ou non, donc l'équipollence est vraie lorsque ABD est un triangle. Le passage de l'équation à l'équipollence correspond à une transformation de la figure géométrique initiale (Figure 18 et Figure 19), nous avons maintenant un théorème sur la figure du triangle.

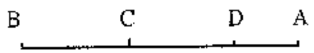


FIGURE 18

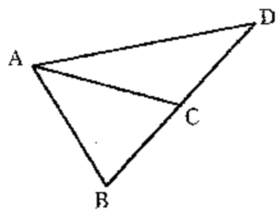


FIGURE 19

Comme le remarque Bellavitis, ce nouveau théorème, sous sa forme générale, présente peu d'utilité. Cependant, il montre qu'il admet deux corollaires importants. Le premier est le théorème dit de Pythagore, le second est l'inscription de l'angle droit dans un demi-cercle. Examinons, la déduction du théorème de Pythagore. Bellavitis écrit :

> Corollaire I. Si $2inc.CB = 2inc.AC \pm 180°$, les deux premiers termes de l'équipollence trinôme auront la même inclinaison ; d'où, par la règle III,
>
> $$gr.(AB.BD) = gr.(AC.CA) = gr.(CB)^2$$
>
> et, en outre,
>
> $$inc.BA + inc.AD - 2\,inc.CB = 2\,inc.DB.$$
>
> Cette dernière relation, appliquée à l'équipollence
>
> $$BA + DB + AD \,\Omega\, 0,$$
>
> donne, d'après la règle IV,
>
> $$gr.AB = gr.AD.$$
>
> La condition $inc.CB$ - $inc.AC = \pm 90°$ signifie que l'angle ACB est droit, et les deux équations relatives aux grandeurs donnent le théorème de Pythagore
>
> $$(AB)^2 = (AC)^2 + (CB)^2.$$[24]

La déduction est purement calculatoire, elle repose sur des transformations de formules par application des règles III et IV. D'une certaine façon, le calcul de Bellavitis parfait celui d'Arnauld, puisqu'il permet de se passer de l'image de la figure. Ainsi, pour le théorème de Pythagore, il n'y a même plus de figures carrées, mais des puissances de grandeurs. Le "théorème général" du graphisme de Bellavitis demande une multivision d'une formule lettrée, une formule algébrique devenant une équation géométrique, une équation devenant équipollence par le seul jeu des significations accordées aux symboles. Mais notons que cette multivision correspond à des littéralités différentes d'une figure géométrique, selon que les lettres désignent des points ou des droites (finies). Notons aussi que la multivision des lettres correspond à une multivision des figures, car un triangle peut être vu comme trois points, trois droites ou une aire délimitée par trois droites. Le mathématicien tient un discours sur des figures et des lettres, mais ce discours ne dit pas toujours la multivocité de son regard.

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[24] BELLAVITIS, *op. cit.*, pp. 22-23.

1. [Aristippus image]

Caption : Euclid's *Elements* (ed. David Gregory), Oxford 1703, frontispiece

Can mathematics education learn from its history ?

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Abstract

In 1570, at the University of Oxford, a young lecturer called Henry Savile opened his lecture course on mathematical astronomy by telling the story of Aristippus the Socratic philosopher. According to the Roman writer Vitruvius, Aristippus was shipwrecked on the coast of Rhodes, and struggling with his comrades out of the storm-tossed waves came to a beach on which there were geometric drawings in the sand. "Be of good cheer" he said to his comrades, "for I see the marks of human-kind." Humanity is defined and characterised by its being a mathematizing species; wherever there is mathematics there is civilisation. The image from antiquity became a symbol of the University of Oxford's mathematical aspirations, forming a background to common beliefs about the significance and role of mathematics in the ensuing centuries.

This use of the past as a guide and inspiration, testifying to mathematics as a symbol and characterization of humanity, can be seen as forming a thesis which we can hold up in opposition to a second image, also produced in Oxford, three centuries later.

In a book called *Picture logic: an attempt to popularise the science of reasoning* (1874), Alfred Swinburne drew a picture showing all the living world passing into a monster logical sausage-machine, emerging in uniform parcels under the watchful gaze of Professor Logic, who cheerily reflected:

> "It's with our machine here as it is with the ordinary sausage machine, never mind what the material is-so long as the shape is right. Large or small, young or old, flesh or stone, you must all pass through, for Professor Logic isn't particular about the matter, all he concerns himself with is the form." [p. 37]

This symbolic image can be taken to stand for the numerous interventions down the ages suggesting that mathematics is involved in something inhuman, mechanical; that mathematical modeling misses the point and unweaves the rainbow; that mathematics teaching is an anti-humanistic shuttering, a closing down, of human aspirations. And as we will see, the critique made of mathematics and mathematics education as being mechanistic can be extended to the educational policies within which the education is experienced.

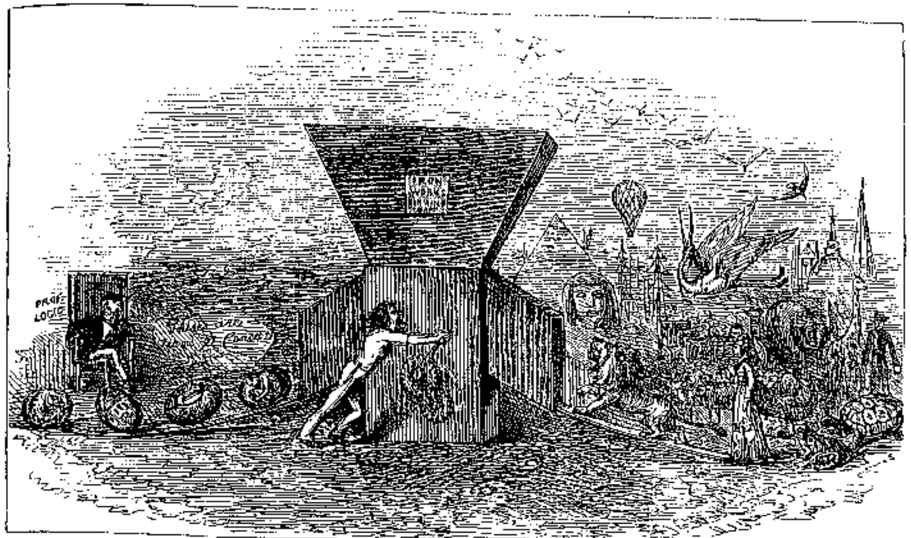
Somewhere in the historical dialectic between the conceptions underlying these two images there may lie a resolution which with strength and energy can be used in constructing the best possible framework for mathematics learners of the future. I'll come back to these symbols later and draw more out of them for this purpose, after making some observations on internationalism in our community.

1 Reflections on internationalism: states, people and the New World Order

I'm drawing today particularly on the British experience because that's what I know best, but I would like to think that any ideas and insights we may be able to glean from that situation are transferable elsewhere and make sense in other national contexts too. One of the few reassuring aspects of the "New World Order" is the sense that, through meetings such as the European Summer University and the growth of international travel, people in different countries are able to develop a new sense of kinship and shared experiences in one world despite the selfish self-importance of politicians. We can learn from each others' experiences of the modern state how to improve individual lives as much as possible even while politicians try to impose and magnify divisions.

After all, one of the most moving and important insights into the Great War, which men were fighting not very far from here 85 years ago, came from that incident at Christmas 1914, when the opposing troops came out of their trenches to exchange cards and cigarettes and food rations, and sang together *Silent Night-Stille Nacht, heilige Nacht*-until the officers on both sides discovered what was going on and put a stop to it immediately, pointing out that if this kind of thing spread there would be no point in having a war, and that would not be at all good for armaments manufacturers.

One thing we can learn from international gatherings, such as the European Summer University, is how we foot-soldiers in the educational trenches can learn to control our officer class, the politicians, better. This involves learning from those in other countries how they have achieved certain improvements in educational provision: we in Britain want to find out from you how *the Danes* have managed to get the importance of history of mathematics recognized and written in to the National Curriculum; how *the French* have managed to build up and sustain the splendid IREM structure which enables teachers throughout France to participate in research activities promoting the better utilization of history of mathematics in the classroom; how *the Dutch* have got to the point of talking about the possibility of teachers being seconded to study for PhDs with funding cover supplied to their schools; how *the Italians* have managed to sustain the importance of geometry and its history in their schools when it has nearly vanished in some other countries; how *the Portuguese* managed to support the sending of some 250 of their mathematics teachers to the previous European Summer University, the wonderful History and Pedagogy meeting in Braga three years ago; how *the Swiss* manage to trust their teachers enough to keep them free from the layers of constraints and bureaucracy that keep teachers elsewhere in a state of stress with no pedagogic advantage; and above all, after four happy days in Louvain-la-Neuve, we want to learn how *the Belgians* have persuaded their town planners to design a university city free from cars and on a human scale, embodying so impressively the humanistic structural values of the Middle Ages and High Renaissance. In short, each country has aspects to its educational provision, and its social-political context, from which other countries can learn-both positively and negatively: what to cherish, what to avoid.



THE LOGICAL SAUSAGE MACHINE.

2. [Logical sausage machine]
Caption : Alfred Swinburne, *Picture logic*, 1874, p.35

2 History and mathematics education over 400 years

What have we in the UK to offer you? There may be an object lesson here, in contrasting two apparently contradictory aspects of British history. On the one hand, we have one of the longest traditions of taking the history of mathematics seriously, both intrinsically and in connection with education. On the other hand, the UK doesn't seem to have benefited from all this history in terms of what is actually happening in the majority of our schools at the moment: there is no mention of history in the National Curriculum for mathematics, and pressures on teachers make it hard for them to develop historical interests. So there is some dissonance here which needs to be understood.

From the very first printed mathematics textbook in English, history of mathematics has been invoked as an aid to the pedagogic strategy-in a variety of ways, just like now. Robert Record in his algebra text *The whetstone of wit* (1557), for example, tried to create interest in the activity of taking a cube root, by including a memorable anecdote about the Greek heritage of the problem. In Record's account (derived from Eratosthenes), the Delian problem, that of doubling the cube, was proposed by the gods as a means of deliverance from a plague. But the first mathematics teacher in England, and maybe anywhere, to take the history of mathematics seriously as an integral part of mathematics research, teaching and exposition was John Wallis at the University of Oxford in the seventeenth century. He was working within an ideology already tilting that way, set in train by Sir Henry Savile when he founded two chairs in mathematical sciences, in geometry and astronomy, together with detailed instructions for how the subject was to be taught. The core of the teaching of both subjects, as laid down by Savile, was *explanation* *of ancient texts*. Thus the professor of geometry was to give public expositions of the whole of Euclid's *Elements*, Apollonius' *Conics*, and all the works of Archimedes; in other words, the method for mathematics teaching was historically based, and primary source based. (This may be to do with the Reformation: after all, that is just what protestants do with the Bible). At all events, Henry Savile is a prime example of a great teacher who developed a style of teaching mathematical sciences based on the use of history as intrinsic to the subject and vital for communicating the traditions of the past to the next generation.

John Wallis was the third Savilian professor, as they were called, of geometry, and took up and developed this pedagogic challenge by writing a great *Treatise of algebra* (1685). This is at once the first major algebra text in English of this size and the first history of algebra, or indeed of any mathematical subject, in English. For Wallis, algebra *is* its history: to learn algebra is become familiar with the progress of the subject up to now, and to do research in algebra is to stand at the frontiers, with the knowledge of the past around you, and move forwards, while dipping in to the past, rediscovering and reinventing the sometimes confusing or hidden messages from long ago.

In this way, there was already by 1700 influential interest in England in the mathematical past, and examples of its use in a variety of ways in teaching the subject. But of course, these studies were at university level and we are talking about a very small number of people. There was no national school system, and such schools as there were taught mathematical skills in practical contexts such as navigation, book-keeping, and so on.

Let the centuries roll on. The nineteenth century was that in which the framework of today's national education system was laid down. In the aftermath of our 'Industrial Revolution', various components began to emerge and take shape, for a range of reasons, including liberal philanthropic responses to the human misery of industrialization as well as its need for an educated workforce. These components included such things as development of examinations; training of teachers; compulsory education, first primary then secondary, in Acts of 1870 and 1902; development of scholarships and State funding mechanisms; and eventually, only recently, a National Curriculum.

Surprisingly, perhaps, the nineteenth century was one in which mathematics education was more historically imbued than at any other time. This was for two main reasons: one was the extraordinary place which was occupied by one 2000-year-old textbook, Euclid's *Elements*. The other, which we may think was not altogether compatible, was a growing educational ideology, that mathematical ideas in the individual develop along the same lines as their historical development-from which it would seem to follow that mathematics teachers, or at least curriculum designers, needed to know the history of mathematics to order materials for presenting to pupils.

First, Euclidean geometry. Other countries taught Euclid, of course, but not with quite such single-mindedness, nor with quite the English obsession with the text of Euclid's *Elements* itself. The French were relaxed about teaching geometry from Clairaut or from Legendre, but in England only Euclid would do. This was widely commented upon. The great British algebraist James Joseph Sylvester recalled in 1869 his mathematical education in the 1820s in these terms:

> The early study of geometry made me a hater of geometry.... I know there are some who rank Euclid as second in sacredness to the Bible alone, and as one of the advanced outposts of the British Constitution.

To account for the dominant and sacred status of Euclid's *Elements* it is sufficient to notice that

it served other functions too. In particular, the other skills it was thought to teach were just what were needed for the intellectual part of governing an Empire (Sylvester's remark about an advanced outpost of the British Constitution was shrewdly positioned.) The intellectual case for Euclid's *Elements* was well stated by William Whewell, the Master of Trinity College Cambridge:

> There is no study by which the Reason can be so exactly and rigorously exercised. In learning Geometry the student is rendered familiar with the most perfect examples of strict inference; he is compelled habitually to fix his attention on those conditions on which the cogency of the demonstration depends. He is accustomed to a chain of deduction in which each link hangs from the preceding, yet without any insecurity in the whole; to an ascent, beginning from the solid ground, in which each step, as soon as it is made, is a foundation for the further ascent, no less solid than the first self-evident truths. Hence he learns continuity of attention, coherency of thought, and confidence on the power of human Reason to arrive at the Truth.

Of course, in the hands of unimaginative or badly trained teachers all this was for nothing, and it is not surprising that the controversies around Euclid grew louder as the century wore on. Indeed the UK's main organization for mathematics teachers, the Mathematical Association, was founded out of a society set up to replace Euclid's own text by other geometry textbooks. One of the leading campaigners on this issue at the turn of the century, the engineer John Perry, wrote

> Like all the men who arrogate to themselves the right to preach on this subject, I was in my youth a keen geometrician, loving Euclid and abstract reasoning. But I have taught mathematics to the average boy at a public school, and this has enabled me to get a new view. I have seen faces bright outside my room covered as with a thin film of dullness as they entered; I have seen men [...] lose in half an hour (as men did in the first day of slavery in old times) half their feeling of manhood; and I have known that, as an orthodox teacher of mathematics, I was really doing my best to destroy young souls.

As the result of feelings and rhetoric such as this, the primacy of Euclid was successfully challenged and overthrown, followed barely half a century later by the virtual disappearance of geometry as a school subject, though that's another story. The main point is that, for a long period of time in England, learning and reproducing the text of Euclid was more important than developing geometrical understanding, which is the kind of thing to give history (and mathematics) a bad name.

Another strong educational ideology of that century was the view that a young person's mathematical development mirrors the history of mathematics itself. The didactic implication is that mathematical topics should be taught in an order, and perhaps in a way, which is supposed to reflect the historical evolution of mathematical discovery. On the whole this idea was, and is, a force for good as it focuses concern on the development of the individual, thus happily fitting in with other notions of child-centred learning which were becoming consolidated at this time. Throughout its long history, the idea has provided further encouragement for the history of mathematics in an educational context. Indeed, this precise argument and connection was made by the great French mathematician Henri Poincaré, at the turn of the century, who was very interested in mathematics education and wrote:

> One cannot reflect on the best method of imbuing virgin brains with new notions without, at the same time, reflecting on the manner in which these notions have been acquired by our ancestors, and consequently on their true origin-that is, in reality, on their true nature. . . .

> Zoologists declare that the embryonic development of an animal repeats in a very short period of time the whole history of its ancestors of the geological ages. It seems to be the same with the development of minds. The educator must make the child pass through all that his fathers have passed through, more rapidly, but without missing a stage. On this account, the history of any science must be our first guide.

Even at the time, the idea did not go unchallenged. John Perry, who we saw earlier destroying young souls through teaching Euclidean geometry, was equally eloquent in his criticism of a mindless application of the growing recapitulationist orthodoxy:

> Because the embryo passes through all the stages of development of its ancestors, a boy of the nineteenth century must be taught according to all the systems ever in use and in the same order of time. [...] Think of compelling all emigrants to pass to America through Cuba, because Cuba was discovered first. Think of making boys learn Latin and Greek before they can write English, because Latin and Greek were the only languages in which there was a literature known to Englishmen 450 years ago. [...] It is not merely in arithmetic and geometry, but in the higher parts of mathematics that this waste goes on. Newton employed geometrical conics in his astronomical studies, and mechanics was developed; and therefore it is that every young engineer must study mechanics through astronomy, and he dare not think of the differential calculus until he has finished geometrical conics.

Of course there is still a great deal to be said for a more sophisticated version of recapitulationism, and indeed in our ICMI Study on *The Role of the History of Mathematics in the Teaching and Learning of Mathematics* there is a whole chapter on the historical formation of mathematical thinking. So any problem isn't the model as such, but the crassly unintelligent application of it which has been too prevalent in the past. Whatever its failings, the recapitulationist postulate did have the effect of forcing serious consideration of history of mathematics in education.

Through these and in other ways the history of mathematics has been a regular theme of English discourse about mathematics education during this century, and indeed for several centuries before that, one way or another. By the 1990s, there has developed a lot of serious interest among a relatively small group of people, and more general interest in, or lip-service to, the history of mathematics among a much wider group of people; but no hint of history in the National Curriculum, though of course no positive bar either, on using history as a resource for better delivery of the mathematical objectives and teaching outcomes of the Curriculum. We do know, from surveys prepared for our ICMI Study, that many countries make a stronger encouragement to teachers to use history-of which the Dutch-speaking part of Belgium is a good example-or even require it in some places, such as Denmark and Norway. So to some extent the problem is a British one. But in the structures of what is happening perhaps there are wider lessons too.

3 Analysis of the problem

To put this into perspective, and give us a framework for analysis, I want to use the recent work of the young Hong Kong researcher Lit Chi Kai. For his MPhil thesis at the Chinese University of Hong Kong, Chi Kai has investigated aspects of introducing history of mathematics in classrooms, including teacher reactions. He garnered the views of 360 mathematics teachers in 41 Hong Kong secondary schools, and found that although most think highly and positively towards history of mathematics, most, too, do not make use of it in their teaching. He then went on to ask them why not, why there is a dissonance between their beliefs and their actions, and he found four reasons in particular

- lack of knowledge
- lack of supporting materials
- students and parents too concerned about examination scores
- curriculum too packed.

This overall perspective, the four main reasons given by Hong Kong teachers for their hesitation in making use of historical materials, ties in with our experience in the UK, in the sense that it is very difficult to find anyone-any teacher, any member of the mathematical community-who will not say that they think history of mathematics is a good thing. And yet the use of history remains a minority pursuit for those relatively few pupils lucky enough to have teachers with the confidence or enthusiasm to over-ride these difficulties.

So we need to address these reasons. They fall visibly into two parts. The first two, the lack of knowledge and the lack of supporting materials, is being rectified actively by teacher trainers in collaboration with organizations such as the Summer University, the HPM committee of ICMI, and national bodies in many countries. So in time these need not be major stumbling blocks.

The second two, pressures of examinations and curriculum, are to a great extent outside direct control of teachers and take us to the effects of what are primarily political decisions. This is true even though the proponents of using history might want to claim that deeper understanding would benefit students' examination scores and would make a crowded curriculum easier, not harder, to assimilate. But there remains a problem of perception.

I should observe too that the first two reasons are also the result of political judgements. The increased difficulty, in the UK, in supporting teachers with issues such as historical knowledge and resources arises directly out of political pressures for local management of school budgets. Ten or so years ago the Thatcher government forced local education authorities to devolve budgets down to schools, which has had the entirely predictable consequence that there is less money for things which only larger budgets have the flexibility and vision to accommodate. Teachers who used to be able to apply for central funding, for such things as attending meetings to be informed and trained in using history, now must persuade their headmaster or school manager to divert funding away from repairing the classroom roof. That is just one instance of the rippling consequences of decisions taken for other reasons.

This is not to say that politicians are unaware of the importance of history. In 1999 the UK's Education Secretary, David Blunkett, made an apparent exception to the lack of history of mathematics in the National Curriculum when he advised schools that the wartime exploits of the Bletchley Park codebreakers should be on the history curriculum for all pupils over the age of eight. It was at Bletchley Park, a Victorian mansion in Buckinghamshire, that from 1939 to 1945 a team of mathematicians and other codebreakers, among them Alan Turing, worked with great success to decode German Enigma signals and others. Among the decoding apparatus constructed by workers at Bletchley Park was the world's first large electronic programmable computer, called Colossus. The Education Secretary was impressed with the way in which studying this history could motivate young people. "This is a good opportunity", he said, "to turn them on to something that brings history alive and which makes them think positively about how they might contribute in their own lives." [Carvel 1999, p. 4.] It should be noted, though, that this use of history is for its morale-raising value, not for any contribution it might make to mathematics education.

Overall, we can see that the state of mathematics education, and the role history is allowed to play in that, are deeply political issues. To understand where we have got to and how, and what to do, involves looking again at the political and social influences and constraints on educational policy and practice. I now pull out again the two introductory images, and explore the idea that mathematics education has developed within a tension between humanization and mechanization over the past centuries. I will just give a few pointers to ways in which these show themselves.

4 Mechanization versus humanization

The specific image of the logical sausage machine is interesting because it captured the spirit of the age, and we find Henri Poincaré, once again, using similar imagery in his analysis of current trends within mathematics. Poincaré was rather suspicious of the formalist approach to mathematics, of the school of David Hilbert, and satirized it in these terms:

> Thus it will be readily understood that, in order to demonstrate a theorem, it is not necessary or even useful to know what it means. We might replace geometry by the reasoning piano imagined by Stanley Jevons; or, if we prefer, we might imagine a machine where we put in axioms at one end and take out theorems at the other, like that legendary machine in Chicago where pigs go in alive and come out transformed into hams and sausages. ['Mathematics and logic', in *Science and method*, p. 147].

This kind of critique has been an important thread in mathematics education down the ages. Whether you need to understand what you are doing, in demonstrating some technical skill, is a perennial issue that keeps returning. Its *locus classicus* is perhaps the dispute between William Oughtred and Richard Delamain in the 1630s over whether in order to use a slide rule you needed to be taught the principles of logarithms. Ever since then question has been a recurrent one of how to strike the right balance, in education, between deep understanding and technical facility.

The perception that mathematics is somehow damagingly close to the mechanistic aspects of the age has long been held in some quarters. A famous early proponent of this cultural criticism was the English poet William Blake:

> I turn my eyes to the Schools & Universities of Europe
> And there behold the Loom of Locke, whose Woof rages dire,
> Washed by the Water-wheels of Newton: black the cloth
> In heavy wreathes folds over every nation: cruel Works
> Of many Wheels I view, wheel without wheel, with cogs tyrannic
> Moving by compulsion each other, not as those in Eden, which,
> Wheel within Wheel, in freedom revolve in harmony and peace.

One of the most influential treatments was that of the English novelist Charles Dickens, in his novel *Hard Times* (1854). Sometimes considered Dickens' greatest novel, *Hard Times* is *about* precisely the dialectic of my opening images, the rival claims of humanity and mechanization in education. Early in the book, the schoolmaster Mr M'Choakumchild (who would be *Master Kindkeler* or *M. Etouffélève* in Belgium) is introduced as both a victim and a symptom of the mechanization of education:

> He and some one hundred and forty other schoolmasters had been lately turned at the same time, in the same factory, on the same principles, like so many pianoforte legs. . . . Orthography, etymology, syntax, and prosody, biography, astronomy, geography, and general cosmography, the sciences of compound proportion, algebra, landsurveying and levelling . . . were all at the ends of his ten chilled fingers. . . . Ah, rather overdone, M'Choakumchild. If he had only learnt a little less, how infinitely better he might have taught much more! [pp. 52-53].

This message has been well absorbed in the University of Leuven, it is gratifying to notice, whose publicity brochure contains an explicit disavowal of mechanistic *kindkeler* principles: speaking of Leuven's commitment to the broad ethical, cultural and social context of education, the brochure goes on to say that "Rather than passing on mere factual knowledge, [KUL] promotes the skills of identifying, formulating and solving problems. It creates the necessary conditions for a stimulating educational experience."

Dickens' criticism is just one part of a long tradition of socio-cultural awareness from long before him, until now-Charlie Chaplin was making a similar point in his great film of the 1930s, *Modern Times*. The same mechanistic ideology criticized by Dickens in education, and Chaplin in industrial management, fuels the limited imaginations of our present politicians. The havoc being created in mathematics education today, and education generally, through managerialism and market solutions-the pressure towards examining and assessing everything so that what is unassessable drops out of the aims of education-results from just the same attitudes of unimaginative self-righteousness that Dickens noted.

We may ask what the forces of humanization have been doing, whether, apart from criticizing the mechanistic trends, there is any countervailing force with a positive constructive ideology. There is, of course, the whole movement in which we are involved, towards the incorporation of history as a humanizing influence, restoring and sustaining for young people and their teachers some of the important human values in education. But we have other examples from history too and I mention just one, the case of Mary Boole (1832-1916). She is not very well known now, and indeed was never very influential outside a small circle, but English mathematics educators have in recent years increasingly found inspiration in her work, particularly at primary and middle school levels. Mary Boole was the widow of George Boole, the great English mathematician after whom Boolean algebra is named. He died young, in the 1860s, and she lived for another 50 years and contributed a number of books towards developing the mathematical imagination of the child. Her books include such titles as *Logic taught by love*, and *Philosophy and fun of algebra*. These give enough idea, for now, of people who have been striving to promote other than mechanistic values and whose influence within educational circles is a good one.

5 Conclusions

The main direction of these remarks is that there are grounds for guarded optimism. My reading of history is basically a cyclic one: those who forget history are doomed to repeat it. Few are more vulnerable to this inexorable law than the politicians who pride themselves on starting afresh, discarding the past and obliterating it from their memories.

In 1861, for example, nearly 140 years ago, a Royal Commission in the UK reported that the best and cheapest way to extend elementary education was to fund teachers according to the examination successes of their pupils, so-called 'payment by results'. Because this system looked to be very cheap, the government of the day enthusiastically adopted it. In vain did wiser voices urge that it would lead to terrible consequences for the education system: it would lead to a reduced curriculum, where only the minimum that was examinable would be taught; it would encourage the neglect of moral and other dimensions of a rounded education; the status of teachers would be lowered; grants to schools would be reduced; so would local interest in schools;. . . and on it went. All this happened just as foreseen, and it took 30 years to get the system of 'payment by results' stopped and to repair the damage. Those who notice tendencies in later governments towards payment by results have a responsibility to point out the historical precedents, and hope that a government is not so set on abolishing the very idea of history that it cannot heed the omens.

In his great novel on mechanization in education, *Hard Times*, Charles Dickens foresaw, it is interesting to observe, precisely the consequences of Reagano-Thatcherite policies of the 1980s:

> It was a fundamental principle of the Gradgrind philosophy, that everything was to be paid for. Nobody was ever on any account to give anybody anything, or render anybody help without purchase. Gratitude was to be abolished, and the virtues springing from it were not to be. Every inch of the existence of mankind, from birth to death, was to be a bargain across a counter. And if we didn't get to Heaven that way, it was not a politico-economical place, and we had no business there. *Hard Times* (1854), p.304.

It is the application of such principles to education that has led to some deplorable consequences in current practice, but the lesson of history is still that these things will pass. Mathematics education can indeed learn from its history, but most effectively if those responsible for educational policies will listen.

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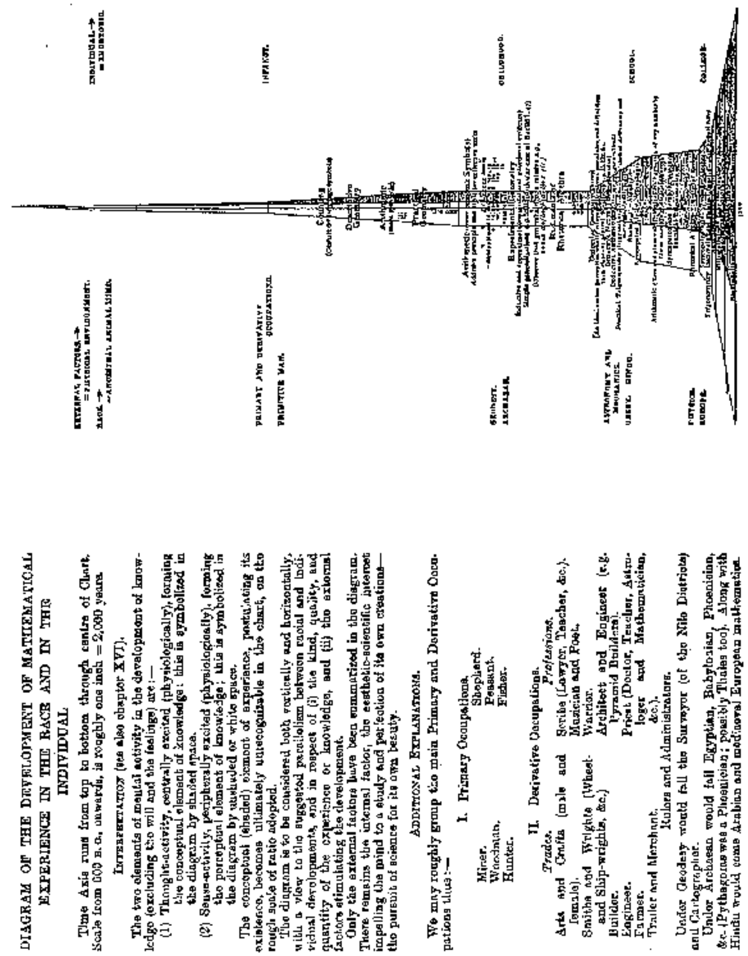

3. [Branford 'Diagram of the development of mathematical experience in the race and in the individual']

Caption Benchara Branford, *A study of mathematical education*, 1908, frontispiece.

Simon Stevin (1548-1620)

Arithmetic in the Low Countries up to 1600 :
trade, tradition, terminology

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Abstract

During the Middle Ages merchants went along the houses with their basket with goods. They tried to barter and their job was very different from the occupations of their later colleagues. In the sixteenth century merchants stayed in their offices and sent their travelling salesmen out. Business travels became longer, merchants went overseas to different countries, they had to pay salaries, custom rights, costs of transport, assurances of goods, etc. They needed to change money in many different ways, because each city had its own money system. They visited exchange banks where moneychangers took care of their affairs. Bankers and bookkeepers were needed. Many merchants earned a lot of money and they needed carpenters, bricklayers, gold- and silversmiths and others to spend this money by building houses and filling these with luxury goods.

The job of the merchant and his calculations became more and more complicated and he felt the need of a new arithmetical method, a written arithmetic that could help him to write down his big numbers and his extensive calculations. During the fifteenth and sixteenth century the new scriptural arithmetic with Hindu-Arabic numbers came into use in the Netherlands. Many arithmetic books were written in the vernacular, but in spite of that it lasted several centuries before the new arithmetic had superseded the old one completely.

What are the contents of the Dutch arithmetic books of the fifteenth and sixteenth centuries? Why does it take such a long time before the new arithmetic was the one and only method? At which schools was the new arithmetic taught? These are the main problems dealt with in this article.

1 Schools you could count on

During the fifteenth and sixteenth centuries most children visited primary school during a few years and then left school because they had to help their parents and work for a living. Some children went to Latin school. There all education was in Latin. The pupils were taught the subjects of the trivium: grammatica, rhetorica, dialectica. They also learned Latin prayers and church-singing because they had all kinds of tasks in the Latin service in church. Arithmetic was not a subject on the timetable of this school.

The sons of the merchants, who were predestined to follow in the steps of their fathers, were not interested in Latin and church-singing. They needed a school where they could learn practical and applied arithmetic. Such a school did not exist, but merchants, bankers and other financial and administrative practitioners started their own private schools where their sons could learn arithmetic, book-keeping and French. French was the most important business language in that time. These schools were called 'French schools' although the other subjects were taught in vernacular. It is clear that these schools were good nurseries for future merchants, bankers and money changers.

The arithmetic books used at the French schools were written in vernacular and in the prologue the authors mention the kind of pupils they are aiming at: merchants, bankers, bookkeepers, money changers, but also technical practitioners like carpenters, smiths, mintmasters, etc. The books contain many financial and technical arithmetical problems that had to be solved by these people in their daily practice. The arithmetic with the Hindu-Arabic numbers is very suitable to solve these problems and the authors give long explanations about this new arithmetic. The new arithmetic is the most important subject in the arithmetic books, but besides that many authors also include a chapter on the old arithmetical method, the traditional calculating with coins. Using this old method you don't need to write and that was a big advantage, even in the sixteenth century, when the ability of writing was still quite rare. In primary school the children first learned to read and then to write. Most children left school before they got writing lessons. Around 1600 about 60% of all men and 40% of all women were able to sign their marriage certificate. That means that during the sixteenth century many people could not write. If you can't write but yet want to do arithmetic, you can choose the traditional way of calculating, arithmetic with coins, for which you only need a few lines and some coins.

2 The old and the new arithmetic

In the sixteenth century the counterboard has horizontal lines. You can place coins on or between the lines and in this way you can express numbers and calculate. This board is a variation of the traditional abacus of the old Greeks and Romans. They used a board with vertical columns. During the Middle Ages this abacus underwent some changes and was turned a quarter.

At the end of the Middle Ages some people even use another variation of the abacus. They calculate with coins without lines. In figure 1 you see a picture from the fifteenth century French arithmetic book 'Le livre de getz'. Three merchants calculate with coins without using lines. This method is also found in a Dutch arithmetic book by Christianus van Varenbraken from 1532.

In using this method you start by placing coins on a vertical line. These are the so called 'liggers' in Dutch, or 'layers' in English. The first layer indicates the ones. The second layer

indicates the tens, the next one the hundreds, etc. The fields between the layers also have a value, increasing from 5 to 50, 500, etc. Figure 1 shows what the number 3767 looks like in coins.

If you can express numbers with coins you can make calculations with coins and it is indeed quite easy to add up and subtract. Multiplying and dividing is a bit more complicated, but for people who can't write calculating with coins is a good alternative.

During a very long time both arithmetical methods, the old and the new one, stayed in use. In figure 2 you see them together being practised at the same table. On the left you see the modern scriptural method with the Hindu-Arabic numbers and in the middle you see the traditional way of calculating with coins. This picture is from the title page of the arithmetic book by Adam Ries from 1533. Ries explains that the traditional calculating with coins

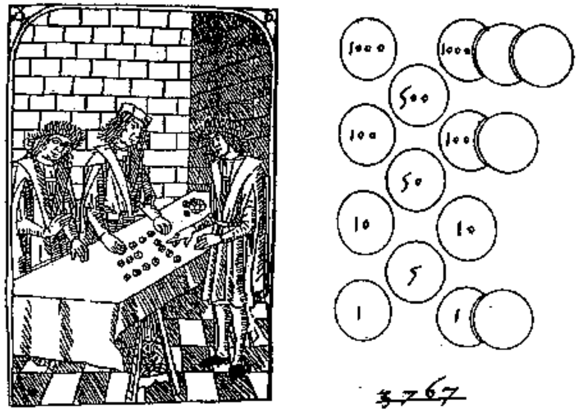

Figure 1: On the left: Three merchants making a calculation with coins without lines, from the French 'Livre de getz', 15th century. On the right: The number 3767 expressed in coins, from the Dutch arithmetic book by Christianus van Varenbraken, 1532.

Figure 2: The old and the new arithmetic at the same table. On the left the modern scriptural method with Hindu-Arabic numbers, in the middle the traditional calculating with coins. Title page from the arithmetic book by Adam Ries, 1533.

is a good preparation for the new method with the pen. He describes both methods. Other authors, also Dutch ones, do the same. It seems that many people in those days could work with both methods. The mathematician Petrus Ramus used the new arithmetical method in his 'Arithmeticae libri tres', but in private, he said, he prefered the traditional way with coins. It sounds more or less like the way we use our pocket calculator nowadays. Sometimes we cipher on paper and sometimes we choose a calculator. It depends upon the problem and the circumstances.

Finally the modern way of calculating with the pen was prefered to the old manner. But this happened only after a very long period. Even in 1698 coins were struck in the Southern Netherlands. And in 1707 Leonhard Sturm explained in his 'Kurtzer Begriff der Gesammten Mathesis' how to calculate with coins.

Why did it take such a long time before the new method was accepted everywhere? It is clear that it has many advantages. For instance you can easily check your calculation afterwards. In arithmetic with coins, the numbers you start with disappear from your counterboard during your calculation. Of course you can check by using the check of nines, but it is impossible to read again the process afterwards. In the new arithmetic you can.

The new method has more advantages. In using Hindu-Arabic numbers it is easy to write big numbers and to extract roots or calculate with fractions. And finally, you only need one instrument for making your calculation and noting the result. So why the delay?

First because many people could not write, secondly the use of zero in the new method. If you are calculating with coins the zero is not necessary. Zero means an empty space on your countingboard and that is no problem at all: nothing means nothing. In the Roman numbers it is the same. You don't need a zero.

But in the new system you do. You have to write a sign, although you mean nothing. And at the same time this magical sign can change the value of a number when it is added to it. 4 doesn't mean the same as 40! People found this difficult, so that authors gave long explanations about the function of the zero.

Some people were even opposed to the new number system. In Florence the guild of money changers forbade their members to use the new numbers in their cashbooks for fear of fraud. This hap- pened as late as 1299. In 1202 Leonardo of Pisa had explained the new method in his 'Liber abaci', when it was completely unknown in the Netherlands.

The oldest Dutch arithmetic book in which the new arithmetic is described, was written in 1445. There are two other Dutch arithmetic manuscripts from the fifteenth century, but most of the Dutch arithmetic books were written and printed in the sixteenth century. At present we know 36 arithmetic books from these two centuries: 12 manuscripts and 24 printed editions. Eleven of the 36 arithmetic books are either a reprint or a copy of an earlier work. On first view the books do not look like arithmetic books. Dutch authors didn't use arithmetical symbols, but described their calculations in words and complete sentences. The German Johann Widman already used arithmetical symbols in 1489 but he was hardly followed by his Dutch colleagues.

3 What could you learn from a sixteenth-century arithmetic teacher?

The most important aim of the authors of the Dutch arithmetic books was to teach their pupils the new scriptural arithmetic with Hindu-Arabic numbers in order to apply this to all kinds of

technical and financial arithmetical problems.

First the pupils had to learn how to read and write the new Hindu-Arabic numbers. The authors wanted to show that the new number system is very suitable for writing down big numbers systematically. Christianus van Varenbraken gives an example of a number consisting of 18 figures. Try to imagine what this number looks like in Roman numbers and you will realize the advantages of the new system.

If you know how to read and write the new numbers you will have to learn how to do calculations with the pen. All Dutch arithmetic books start explaining how to add, subtract, multiply and divide. Sometimes the authors even treat halving and doubling. The arithmetical operations look rather modern. We use the same methods nowadays, with a few differences when it comes to division.

The different money systems of the cities could make calculating rather complicated. In figure 3 you see a subtraction with two amounts of money: 298 pounds, 19 shillings, 10 pennies and 16 mites are subtracted from 334 pounds, 13 shillings, 9 pennies and 13 mites. You have to know the Ghent system, in which: 1 pound equals 20 shillings, 1 shilling equals 12 pennies and 1 penny equals 24 mites. It is clear that this made calculations very difficult and there were many mistakes, as you see in the final result of this example: 11 pennies ought to be 10 pennies.

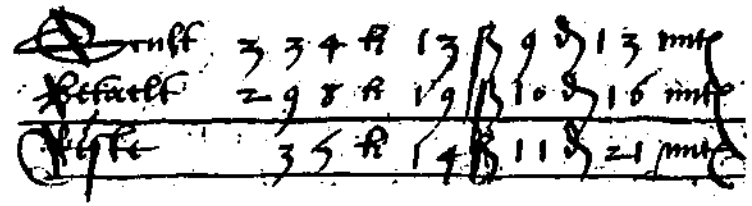

Figure 3: Subtraction of two amounts of money from the arithmetic book by Christianus van Varenbraken, 1532.

When the authors have explained the arithmetical operations they next discuss rules to solve the applied and practical arithmetical problems. The rule of three is the most important one. It is used to find the fourth number in proportion to three given numbers. Because of its importance some authors introduce this rule in a beautifully decorated frame, Figure 4. This picture is from the arithmetic book of Peter van Halle, 1568. The text says: *The rule of three, how you can find the fourth number out of three numbers.*

Figure 4: Introduction of the rule of three in the arithmetic manuscript by Peter van Halle, 1568.

With the rule of three you can solve all kinds of problems in the life of merchants and financial, administrative and technical practitioners : problems about buying, selling or exchanging of goods, about partnership, changing money, calculating of interest, about insurance, profits, losses, making alloys, etc. Look for instance at the problem on selling cloth in figure 5. This problem is from an anonymous arithmetic book from 1578.

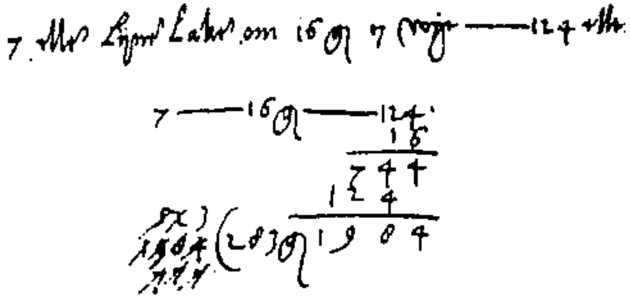

Figure 5: Linen sale from an anonymous arithmetic book, 1578, fol. 32r.

If 7 ells of linen cost 16 pennies, how much do 124 ells cost?
Place the three given numbers on a line. Multiply the last two numbers and divide the product by the first one: $(16 \times 124) : 7 = 283$ pennies. In figure 5 you see that 3 is not ticked off. That is the remainder.

The following problem is a bit more difficult. You can read it in figure 6. It is from the arithmetic book by Peter van Halle, 1568: *If 10 mowers can mow 15 hectares of land in 7 days, how many days do 16 mowers need to mow 20 hectares of land?*
In this problem you have to use the rule of three twice: once in the normal order and once in the inverse direction. The final outcome is 5 5/6 days.

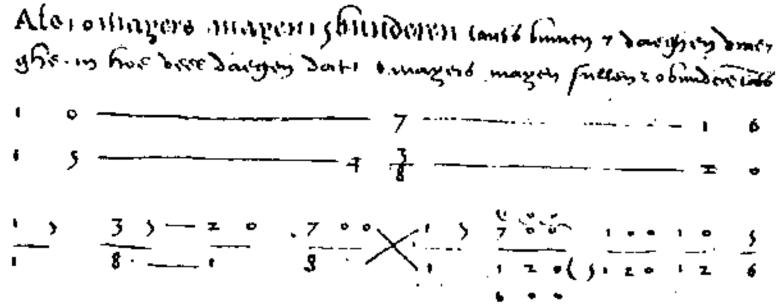

Figure 6: The problem of the ten mowers from the arithmetic book by Peter van Halle, 1568, fol. 95r.

Money changers had to solve problems like the one in figure 7: *A merchant from Florence went to the exchange-bank in London in order to change 120_ ducats at 42.. pennies each into angelots at 66_ pennies each. The question is: How many angelots will he get in London?*
The calculation here is: (120_ × 42_) : 66 _ = 76 angeloten and the remainder is 594.

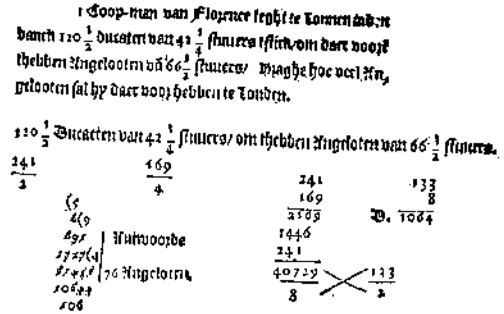

Figure 7: The problem of changing money from the arithmetic book by Adriaen van der Gucht, 1569, fol. 96r.

4 Words used for arithmetic

Mostly the teachers at the French schools were the authors of their own arithmetic books. They did their very best to make the arithmetic as clear as possible to their pupils. They gave long explanations and very many problems to prepare their pupils for their future jobs.

Alas, especially in the early arithmetic books, there is one aspect that made their explanations a bit obscure. Although the authors wrote their books in the vernacular, they used many traditional Latin arithmetical terms and that was difficult for pupils who did not understand Latin. But an arithmetical vocabulary in Dutch did not exist.

In the sixteenth century the authors tried to make the Latin terms clear in different ways:

1. they gave the Latin words Dutch endings;

For instance: addito — *adderen*
summa — *summe*
unitas — *uniteit*
quadratus — *quadraet*

2. they tried to describe the Latin word and added Dutch expressions to it;
For instance: *in sich selve multipliceren*
(multiply in himself = to square) *getal dat gi begeert te delen*
(number that you want to divide = division number) *gebroken getal*
(broken number = fraction)

3. they tried to find Dutch translations of the Latin words. Most of the time they used existing Dutch words:
For instance: differentia — *verschil*
radix — *wortel*
substractio — *aftreckinge*

4. sometimes the authors created completely new Dutch words:
For instance: arithmetica — *rekenconste*
additio — *optellen*

and soon.

Some of these inventions became part of the Dutch arithmetical vocabulary.

If a sixteenth century arithmetic teacher wanted to write his own book he started by using books of his predecessors. We would call this plagiarism, but in those days it was quite normal to compose a new book with material from existing books. Of the 36 transmitted Dutch arithmetic books at least 31 are based on one or more sources. There is a dense network of relations between the arithmetic books. Take for instance the book of Gielis van den Hoecke from 1537. He may have used the works of Euclid, Ptolemy and Regiomontanus. He translated and adapted fragments from Grammateus, de la Roche and Rudolf. In the same way Van den Hoecke used the works of his predecessors, his work was used by later authors. Some colleagues copied his whole book, others used fragments or made adaptations. Some books seems to have some relation with the book of van den Hoecke although it is difficult to prove that. Probably some linking books between those have been lost in the course of time. The book of van den Hoecke is just one example. Arithmetic books of this time heavily depend upon earlier books.

If an author copied pieces from the works of his predecessors he also copied their terms and if he feared that the copied terms were not clear enough for his pupils, he added some extra terms or descriptions. The consequences of this practice are that the amount of arithmetical terms in the books is growing during the age. Authors made no choices between terms and used all synonyms they knew.

Sometimes several synonyms appear in the same sentence. Gielis van den Hoecke starts his chapter on addition with the title: "Additie, vergaerderijnghe, sommerijnghe". He uses three different words to announce 'to add'. Van der Gucht has used the work of Van den Hoecke, he starts addition with the title: "Additie, vergaerderijnghe, sommerijnghe ofte toedoeninghe". He even added a fourth term.

Van der Gucht used many different sources in composing his book. The consequence is that he

has a striking number of arithmetical synonyms. For instance: he uses 13 different words for 'To add': *adderen, brijnghen, inrekenen, rekenen, sommeren, toebrijnghen, toedoen, verzaemen, tegader adderen, tegader doen, in een somme brijnghen, tegader tellen, tsamen tellen.*

In our eyes the big amount of synonyms in the sixteenth century arithmetic books is remarkable and it seems confusing, but in these days there was no standardisation of terms. Finally, a long time after the sixteenth century, the biggest part of these terms disappeared, but it is striking that among the sixteenth-century arithmetical terms that many of them are still in use nowadays in the Dutch arithmetic.

An important author was Simon Stevin. He was born in Brugge, probably in 1548. He moved to the North in 1581 and became a student in Leyden. He published many works on mathematics, astronomy, physics, law. He was the teacher of Prince Maurits and published his instructions in the 'Wisconstige gedachtenissen'. Stevin was convinced that of all languages Dutch was by far the best for scientific purposes. In his works he introduced many new Dutch words. Some became the only correct terms in the Dutch scientific language. Words like: *wiskunde* for 'mathematics', *evenwijdig* for 'parallel' and *evenredigheid* for 'equality of two ratios', etc. In 1585 Stevin published his "De Thiende". He described a new method to write decimal fractions and how to use them in fundamental arithmetical operations. None of the Dutch terms he used here are his own invention; he made a selection from the existing 16th century terms. We may conclude that Simon Stevin invented many new Dutch scientific terms, but for our Dutch arithmetical terms we must thank his predecessors.

5 The combination of business with pleasure

Let us return now to the contents of the arithmetic books.
Among the many serious practical arithmetical problems for future merchants, bankers, money changers, etc. you find problems that are not realistic at all. These are mostly funny story-problems, probably to amuse the pupils and make the program less serious.

Christianus van Varenbraken (1532) starts this kind of problems always with the title: *Questie uut genouchten.* which means: Problem for pleasure. Martin van den Dijcke (1591) has collected all his recreational problems in a special chapter at the end of his book. He introduces his collection as follows: *Here you will find many different beautiful problems to sharpen and enjoy your mind.* It is clear that these problems have a recreational function.

The following example is taken from this collection:
Hanneken Vinc and Lysken Rinck are getting married. The bridegroom and his family are waiting for the bride. The bride is walking towards her bridegroom, but suddenly she stops because her family has told her that she is breaking the rules. The horse has to go to the hay and not the reverse. There is a fight between the two families but finally they reach a compromise. The bride goes 7/8 part of the distance between her and her bridegroom backwards end then again 9 2/3 steps forward. The bridegroom goes forward 1/5 part of the distance. Finally the distance between the couple is only 3/5 part of one step. So they can kiss each other and the audience laughs loudly. The question is: How many steps were there between bride and groom when the bride stopped walking?

Probably this problem was nice entertainment in the sixteenth century classroom, but it could play the same part in a modern mathematics lesson. Funny problems are timeless.

In the following example from the same source we see that rivalry between cities is of all times.

Once in an inn two men, one from Louvain and one from Mons, were boasting. The man from Louvain said: 'Compared with Louvain, Mons is just a pigeon house.' The man from Mons answered that the length of the Mons citywalls was 5/8 of a mile. But then it seemed that the Louvain walls were 1 4/11 miles long. The conversation ended with a bet: 'I believe it is possible to place Mons more than 4 times inside the Louvain walls. If not, I will pay for the drinks.' Did the man from Louvain win?

These types of problems were not only solved in the classroom but also probably in pubs or at parties. It is like a modern quiz: a way to challenge each other.

We do not know how old Van den Dijcke's problems are. He copied them from Godevaert Gomparst, a sixteenth-century arithmetician from Antwerp and probably they have not been used in books of an earlier date. But some of these arithmetical 'riddles' are extremely old and can be found in books from different times and cultures, like the 'world famous' problem of the ring.

In a group of people there is a person who has a ring on one of his fingers, around a certain phalanx of that finger. The leader of the game doesn't know where the ring is, but he asks the group to do some calculations and finally when he sees their results he knows where the ring is. The calculations are:
Double the number of the person that has the ring. (Each person has a number)
Add 5 to this.
Multiply the result with 5.
Add the number of the finger
Place the number of the phalanx behind the result and subtract 250.

If for instance the third person has the ring on his second finger around his first phalanx, the final result of the calculations will be: 321!

I believe that this problem can also be a big success in our modern classrooms. You can add extra questions. For instance you can ask your pupils what will happen if the ring is on the tenth finger. You can ask them why the trick works and if it will work with all numbers. And perhaps you can even challenge them to create their own hidden ring trick. In doing this you can turn this exciting historical game into a rich and valuable math problem.

I found this problem in the arithmetic book by Peter van Halle (1568) and it also appeared in the arithmetic book by Adriaen van der Gucht (1569). But the problem is much older. It appeared already in the problem collection of Beda in the 8th century and also in the works of Abu mansur (eleventh century), Fibonacci (thirteenth century), Chuquet and Calandri (fifteenth century) and in many sixteenth century books: De la Roche, Ghaligai, Rudolff, Apian, Tartaglia and Trenchant. And last but not least, the Dutch arithmetician Willem van Assendelft described the ring problem in his book of 1621. It is clear that long after the sixteenth century these problems were as popular as they turned out to be at a summer university in Louvain in 1999. Which shows us that good problems will keep their charm forever.

6 Conclusions

In the title of this article there are three T-words: Trade, Tradition and Terminology. During the fifteenth and especially the sixteenth century the increasing trade caused an increasing need for a written arithmetic method. Merchants and practitioners of financial, administrative and

technical professions learned to apply this new method to practical arithmetical problems.

During a long time both the new arithmetical method and the traditional method of calculating with coins stayed in use and many new books contained old and well-known problems, that used to be solved for fun.

Many of the modern Dutch arithmetical terms have been invented by sixteenth century authors of arithmetic books.

The sixteenth century was a time of tradition and ambition, but people also liked partying. During these parties people may have been challenging each other by trying to solve arithmetical problems. Probably these occasions were a kind of Summer university-banqets-avant-la-lettre. The most important lesson we can learn from our sixteenth century colleagues is: Always try to combine business with pleasure!

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La géométrie et la nature des choses

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Abstract

Les perceptions portent sur des choses particulières, individuelles. Mais les symétries et les parentés de symétries induisent la formation d'objets mentaux géométriques, qui sont des classes d'objets parcourus en imagination et aussi cernées par le langage. Les déterminismes géométriques reconnus dans l'action (par exemple le dessin aux instruments) fournissent des inférences, dont les premières sont évidentes, relèvent de la "lumière naturelle". Et celles-ci ensuite se combinent en inférences non évidentes, en preuves discursives. Nous essayons ici de comprendre ce cheminement, en nous appuyant sur quelques auteurs du XX^{ème} siècle, en particulier E. Mach, qui ont contribué à en éclairer certaines étapes.

[1]Cet article a fait l'objet d'un exposé à l'Université d'Été européenne d'histoire et épistémologie des mathématiques de Louvain-la-Neuve en 1999, dont les Actes sont en cours de poublication. Il est reproduit ici avec l'aimable autorisation des organisateurs de cette Université.

Si peu que nous sachions encore du monde environnant historique des premiers géomètres, il est toutefois certain [...] que c'était un monde de "choses" (parmi lesquelles les hommes eux-mêmes en tant que sujets de ce monde).

EDMUND HUSSERL

Un grand avantage de la géométrie, c'est précisément que les sens peuvent y venir au secours de l'intelligence, et aident à deviner la route à suivre.

HENRI POINCARÉ

1 Introduction : la lumière naturelle

PASCAL écrivait en 1658 : "[L'ordre de la géométrie] ne suppose que des choses claires et constantes par la lumière naturelle, et c'est pourquoi il est parfaitement véritable, la nature le soutenant au défaut du discours."

Cette citation appelle plusieurs commentaires. Le premier est que l'adjectif *constant* n'a pas ici son sens habituel d'*invariable*. Dans le dictionnaire de FURETIÈRE [1694], le premier sens de *constant* est "ce qui est certain de toute certitude", et il en donne pour exemple : "Il est constant que deux et deux font quatre." Ainsi la géométrie dit la vérité, elle est selon les termes de PASCAL "parfaitement véritable".

En l'occurence il s'agit d'abord de la vérité des premières assertions géométriques, celles que l'on suppose et ne démontre pas. En effet, on ne peut tout démontrer, donc il faut bien supposer des propriétés "sans discours", ce qui veut dire sans preuve. Ces propriétés sont garanties par "la nature", ou plutôt elles sont reconnues sans prêter au doute par "la lumière naturelle". Dans *lumière naturelle,* il y a *naturelle* qui renvoie à la nature, siège des propriétés, et *lumière* qui renvoie à l'esprit qui comprend, qui souscrit à l'évidence.

Passons maintenant du XVII$^{\text{ème}}$ siècle à notre époque, et demandons-nous si la lumière naturelle au sens de PASCAL existe encore aujourd'hui. Est-ce qu'un être humain de nos jours conviendra comme PASCAL de l'existence de plans, de droites et de points, de leurs intersections, de l'existence des parallèles et des perpendiculaires, etc. ? Oui sans doute, à condition qu'il n'ait entendu parler ni des géométries non euclidiennes, ni de RIEMANN, ni de la relativité. Il doit s'agir d'un être humain naïf, quelqu'un qui, armé de son seul bon sens, commencerait à s'intéresser aux phénomènes géométriques et les soumettrait à l'expérience et au raisonnement.

Il existe encore aujourd'hui, comme au XVII$^{\text{ème}}$ siècle, des évidences communes. La différence – considérable il est vrai –, est qu'elles n'ont plus la même valeur de vérité. Nous savons dorénavant que la nature ne se révèle pas dans sa vérité profonde au regard naïf. La lumière naturelle n'est plus ce qu'elle était. Mais les évidences communes engendrent encore une géométrie naturelle à l'homme. Elles ne sont pas vraies au sens de la physique, mais elles le paraissent. Elles sont le fondement de la géométrie approximative adaptée à l'échelle humaine.

Si on accepte qu'un savoir ne peut commencer à se construire que dans l'univers intelligible de celui qui apprend, alors la géométrie, quelles que soient les révisions radicales par lesquelles elle devra passer pour rejoindre ce qu'on appelle aujourd'hui les géométries, commence par les évidences communes, par la lumière naturelle.

L'objet de ce travail est de remonter, dans la mesure du possible, aux origines des évidences communes de la géométrie. PASCAL constatait ces évidences. Nous essayerons de les expliquer en partant de la notion d'objet invariable chez MERLEAU-PONTY [1947] et des observations de

MACH [1997] sur la perception des objets symétriques. Notre espoir est que ces explications aident à comprendre et organiser les cheminements des élèves dans les débuts de la géométrie.

La substance de cet article est tirée d'une étude réalisée par le CREM[1]. Pour ne pas allonger l'exposé, nous avons décidé – non sans regret – de n'évoquer ici que des objets plans. Nous renvoyons à CREM [1999] les lecteurs qui voudraient approfondir la question, entre autres en prenant en compte les objets à trois dimensions.

2 Percevoir

2.1 La constance de la grandeur et de la forme

Considérons un objet plan (par exemple une forme découpée dans du carton), d'une taille telle que le regard puisse l'embrasser lorsqu'il se trouve devant l'observateur, en position frontale, à distance de toucher. Nous avons l'impression que lorsque l'objet est dans cette position, nous le voyons véritablement tel qu'il est. *Nous privilégions cette position de perception commode,* par rapport à toutes celles correspondant à des distances et des orientations différentes. Toutefois, notre connaissance de l'objet englobe les souvenirs de toutes ces positions variées.

Que faisons-nous pour bien voir un objet de ce type rencontré par hasard ? Nous tournons notre regard vers lui, nous l'éloignons ou le rapprochons de nos yeux pour le voir sous un angle approprié, nous ajustons la courbure de nos cristallins pour le voir nettement, nos ajustons l'ouverture de nos pupilles pour qu'il ne nous paraisse ni trop sombre, ni trop brillant, et nous l'amenons en position frontale[2].

Aucune de ces manœuvres n'est inspirée par des *considérations scientifiques* relatives à la distance, l'orientation, l'éclairement, etc. Selon MERLEAU-PONTY, elles se situent à *un niveau prélogique*, et la position privilégiée est perçue comme un point de maturité de la perception, comme la position qui équilibre les influences de l'objet sur l'observateur. R. ARNHEIM [1976] parle en l'occurence de *l'intelligence visuelle.* Toutes les manœuvres décrites ci-dessus s'enchaînent en tout cas dans un ordre approprié au résultat escompté et sont guidées par une intention de connaître.

On acceptera sans doute volontiers que la grandeur et la forme objectives d'un objet du type considéré soient celles que nous lui attribuons dans la position privilégiée. Mais alors comment reconnaissons-nous que l'objet *conserve sa grandeur et sa forme*, bien que son apparence change selon les positions qu'il occupe par rapport à nous ? En fait, l'ensemble de ses apparences liées aux conditions de sa présentation forme un système structuré et, selon MERLEAU-PONTY, la constance de cette structure – nous pouvons toujours retrouver une apparence particulière quelconque – s'identifie à ce que nous appelons la constance de la grandeur et de la forme.

Qui plus est, les changements de position de l'objet par rapport à nous et les changements correspondants de la perception sont continus, ce qui contribue à assurer l'identité de l'objet perçu. Comme le dit R. ARNHEIM : "Dans la perception, les divers aspects d'un objet, loin de constituer une "déroutante variété", s'enchaînent en séquences continues. Ce sont plus qu'une multitude de cas éparpillés au hasard, des transformations progressives." Et il conclut : "l'identité n'a donc pas à être extrapolée au hasard."

[1] Voir CREM, [1999]

[2] D'autres manœuvres s'ajoutent à cela, qui dépendent de la forme particulière de l'objet. Par exemple, s'il a la forme d'une lettre, nous le tournons dans le plan frontal pour que la lettre apparaisse dans la position normale de lecture. Sur certains de ces phénomènes qui dépendent de la forme de l'objet, voir section 3.

2.2 Reconnaître les congruences

Continuons à nous intéresser à des objets plans indéformables et de dimensions modérées. Jusqu'à présent, nous avons cherché à voir comment nous connaissons et arrivons à connaître un tel objet. Mais nous n'avons envisagé qu'un objet quelconque. Autrement dit, nous n'avons pas pris en compte le rôle joué par les diverses formes possibles de l'objet.

Or notre environnement – minéral, végétal, animal, humain, fabriqué – est peuplé d'objets présentant des régularités, des symétries. Nous avons tendance à l'oublier à force de vivre au milieu d'eux. Or ce sont seulement ces objets qui ont permis la constitution de la géométrie. Celle-ci n'aurait pas eu de matériau dans un univers complètement désordonné.

Voyons maintenant comment nous arrivons à reconnaître les symétries les plus simples, à savoir certaines congruences[3]. Dans l'immédiat, nous examinerons deux objets congruents, mais nos conclusions s'appliqueront à deux parties congruentes d'un même objet.

FIGURES 1, 2 et 3

Soient par exemple les deux taches noires de la figure 1. Elles sont congruentes, mais on ne s'en aperçoit pas au premier regard. Par contre, si on les découvre en position symétrique comme sur la figure 2, ou translatées l'une de l'autre comme sur la figure 3, on voit d'emblée qu'elles sont congruentes. Sur la figure 2, l'axe de symétrie est "vertical", et de ce fait il est dans le plan de symétrie de l'observateur. Sur la figure 3, la direction de la translation est horizontale. Sur les deux figures, le regard se déplace horizontalement chaque fois qu'il passe d'une partie d'une des deux taches à la partie homologue de l'autre (voir figures 5 et 6). Par contre, les segments joignant des points homologues sur la figure 1 paraissent disposés de manière anarchique (voir figure 4).

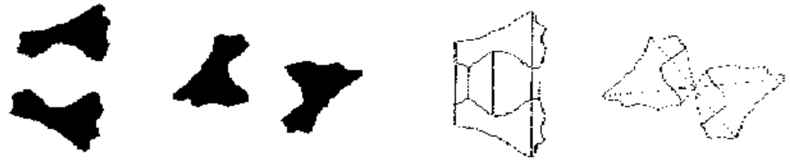
FIGURES 4, 5 et 6

La congruence de deux taches se perçoit encore aisément – un peu moins toutefois que dans le cas des figures 2 et 3 – lorsqu'elles sont symétriques avec un axe horizontal (figure 7), ou lorsqu'elles sont en situation de symétrie centrale (figure 8). Dans le premier de ces deux cas, le regard se meut verticalement[4] pour passer d'un point à son homologue (figure 9), et dans le second, il passe par le centre de symétrie[5] (figure 10).

[3]Nous préférons le terme *congruent* à *isométrique*, parce que ce dernier renvoie à des mesures, qui sont hors de notre propos. Deux objets plans sont *congruents* s'ils sont superposables, fût-ce après retournement de l'un deux.

[4]Il semble établi (cf. *CREM* [1999]) que l'être humain compare plus facilement deux objets lorsqu'ils sont situés sur une horizontale que lorsque le regard doit descendre ou monter pour passer de l'un à l'autre.

[5]Voir *CREM* [1999] pour une analyse plus détaillée des perceptions de congruences.



FIGURES 7, 8, 9, 10

En conclusion, il y a des situations où la congruence se reconnaît aisément, ou sans trop de peine. Il en existe d'autres où elle ne se reconnaît pas : le regard est impuissant. D'où la question : que peut-on faire pour établir la congruence de deux objets lorsqu'on ne la perçoit pas directement ? On peut d'abord recourir à des *opérations mécaniques*, comme de les amener dans une position où on perçoit leur congruence, ou encore de les superposer. Mais on peut aussi, si on en dispose, recourir à un *critère intellectuel*. C'est ce qu'illustre l'exemple suivant. Soient deux triangles disposés de telle façon qu'on ne peut pas les comparer à vue. Si on sait que leurs côtés sont égaux deux à deux, alors on sait aussi que les deux triangles sont congruents. La figure 11 illustre une situation de ce type.

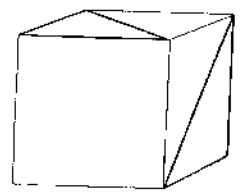


FIGURE 11

Faisons le point. Nous observons dès à présent, dans le cadre de notre analyse, un registre élémentaire de la pensée géométrique. Nous nous sommes en effet posé la question suivante : quand peut-on dire que deux objets ou deux parties d'un objet sont "les mêmes" du point de vue de la grandeur et de la forme ? On peut croire que c'est là l'un des premiers problèmes d'une géométrie naturelle. Les critères perceptifs et mécaniques, qui permettent parfois d'y répondre, relèvent de la vie et de l'intelligence pratiques. On peut dire que la géométrie intervient dès que des critères intellectuels permettent d'*inférer* la congruence.

Vue sous cet angle, la géométrie apparaît comme un système d'instruments intellectuels qui pallie les insuffisances de la perception et permet de les dépasser. Elle étend notre connaissance des objets au delà des limites, à vrai dire étroites, de notre perception[6]. Elle est comme un instrument d'optique qui donnerait plus de puissance à notre vue. En ce sens on peut dire qu'elle nous aide à *saisir l'espace*. Comme dit H. FREUDENTHAL [1973], "la géométrie au niveau de base, c'est saisir l'espace". En éclairant les situations spatiales, elle nous met à l'aise dans l'espace.

Toutefois, si la géométrie commence avec les opérations intellectuelles, il faut souligner avec force qu'elle se constitue sur le terrain des expériences sensorielles et mécaniques et serait impossible sans ces dernières. MACH insiste beaucoup sur cet ancrage de la géométrie dans la

[6]Nous percevons mal, ou nous ne percevons pas du tout, non seulement les objets trop proches ou trop éloignés, ou mal orientés dans le champ de notre regard, mais encore ceux qui sont trop petits ou trop grands. Il est par exemple impossible percevoir la forme et les dimensions d'une parcelle de terrain, à moins de la survoler. C'est bien pour cela qu'on en dresse un plan à échelle humaine.

réalité sensible et l'action. Il écrit : "Ce sont [...] les sensations d'espace [...] qui servent de point de départ et de fondement à toute géométrie." Et encore : "Les propriétés physiologiques ont probablement donné la première impulsion à la recherche en géométrie." Cette affirmation nous renvoie sans doute à une préhistoire mystérieuse où les êtres humains, reconnaissant les configurations symétriques les plus simples, ont commencé à explorer plus avant l'espace à la force de leur esprit.

MACH affirme enfin que la géométrie, née sur le terrain des perceptions et du fait de leurs limitations, ne perd pas le contact avec elles. "Une géométrie scientifique, écrit-il, est impensable hors de la coopération de l'intuition sensible et de l'entendement."

Ces affirmations ont des conséquences importantes, entre autres pour les enseignements maternel et primaire. C'est sur le terrain des perceptions et de l'action que *se prépare* l'apprentissage de la géométrie.

2.3 Symétries perçues et esthétique

Il nous reste à expliquer un dernier aspect des perceptions, avant de passer de celles-ci aux concepts : il s'agit de leur valeur esthétique. Commençons par quelques observations. Bien que le motif de base de la figure 1 – une tache quelconque – soit irrégulier, on éprouve une sensation d'équilibre et d'agrément lorsqu'on en découvre deux en position symétrique comme sur la figure 2. Une sensation analogue émane de la figure 3, et davantage encore lorsqu'on la reproduit plusieurs fois pour former une frise (figure 12).

FIGURE 12

A contrario, cette sensation agréable s'évanouit lorsque les deux objets congruents sont orientés de façon quelconque : ils donnent alors au contraire une sensation de déséquilibre, de désordre.

Cette sensation esthétique est liée à la présence des droites virtuelles qui joignent chaque point à son homologue, au parallélisme de ces droites et à leur horizontalité. D'autre part, comme le note MACH, le droite provoque par elle même une sensation d'équilibre, particulièrement vive lorsque son orientation s'accorde aux éléments de symétrie de l'appareil perceptif de l'observateur. Il écrit : "La *ligne droite*[7] en tous ses éléments, conserve la même direction, et excite partout la même sensation d'espace. D'où son avantage esthétique évident. Par ailleurs les lignes droites qui se trouvent dans le plan médian[8] ou qui lui sont perpendiculaires bénéficient d'une situation intéressante, en ce qu'elles occupent une situation de symétrie et se comportent de la même manière par rapport aux deux moitiés de l'appareil visuel. On ressent toute autre position des lignes droites comme une distortion par rapport à la symétrie et comme "allant de travers."

S'il est vrai, comme nous l'avons vu, que la reconnaissance perceptive de certaines symétries simples soit au seuil d'une emprise mentale sur l'ordre des choses, qu'elle soit la porte d'entrée

[7]Dans cette citation, c'est MACH qui souligne.

[8]MACH entend par là le plan de symétrie du corps humain.

et la condition d'intelligibilité géométrique du monde, cela nous paraît d'une grande importance, entre autres sur le plan pédagogique, que les symétries perçues facilement soient aussi porteuses d'un pouvoir esthétique élémentaire[9].

3 Concevoir

3.1 Les deux faces des concepts

Lorsqu'on reconnaît une congruence visuellement ou mécaniquement, il s'agit toujours de deux objets congruents particuliers. Par contre les critères intellectuels, c'est-à-dire géométriques, sont eux applicables à des catégories d'objets, à des concepts. Nous en arrivons maintenant à l'étude de ceux-ci. Mais comme nous nous occupons des origines de la géométrie, nous envisagerons plutôt les *objets mentaux*, au sens de FREUDENTHAL [1983], que les concepts inscrits dans une théorie mathématique de grande ampleur. Les objets mentaux sont des concepts encore assez proches de ceux de la pensée commune et toutefois assez élaborés pour être déjà des instruments efficaces d'organisation d'un champ de phénomènes[10].

Un objet mental renvoie à un ensemble de choses possédant des caractères communs[11]. Selon la distinction classique, un tel ensemble peut en principe être saisi de deux façons : soit *en extension*, c'est-à-dire par passage en revue de tous ses éléments, soit *en compréhension*, c'est-à-dire par l'ensemble des propriétés que possèdent ses éléments et qu'ils sont seuls à posséder.

Chaque objet mental géométrique est inévitablement saisi des deux façons. La saisie en extension est première, l'esprit se représentant les objets un à un. Mais elle devient difficile lorsque le nombre d'objets va croissant, et impossible lorsque ce nombre est infini. La saisie en compréhension est seconde, elle s'exprime dans une définition, elle renvoie au langage. Les objets saisis en extension sont les référents de la définition, ils constituent son *domaine de sens*. Ils sont les sources de l'intuition, tandis que la définition se situe du côté symbolique et formel de la pensée.

Ceci dit, il devient assez clair que le savoir géométrique aura tendance à se constituer d'abord là où l'accès au sens – la vue des concepts en extension – est le plus facile. Essayons donc, en nous bornant dans un premier temps aux objets mentaux qui renvoient à des figures, de discerner précisément les types de figures dont les variantes possibles peuvent être imaginées sans trop de difficulté, celles par conséquent qui opposent peu d'obstacles à un parcours en extension.

3.2 Formes libres à symétrie simple

La tache arbitraire de MACH (voir section 2.2), prise isolément, ne renvoie à aucune catégorie particulière d'objets géométriques. Par contre, l'univers quotidien offre à notre perception

[9]Il semble bien que les symétries aisément perçues, non seulement soient à l'origine de la pensée géométrique, mais encore soient le matériau de base des arts plastiques et de la musique. Comme l'écrit PICASSO (cité par R. ARNHEIM [1976]) en ce qui concerne son art : "La peinture est poésie ; elle s'écrit toujours en vers avec des rimes plastiques, jamais en prose. Les rimes plastiques sont des formes qui riment entre elles ou qui créent des assonances avec d'autres formes ou l'espace qui les entoure." Cette comparaison de la géométrie et des arts du point de vue de leur matériau de base est approfondie et illustrée dans *CREM* [1999]

[10]Sur la notion d'*objet mental* et celle de *phénomène*, cf. H. FREUDENTHAL [1983] ou *CREM* [1999]

[11]Nous n'envisageons pas ici les concepts primitifs dont parle par exemple L. VYGOTSKI [1997]. Observés surtout chez les petits enfants, ils regroupent des objets ayant entre eux des liens parfois fortuits, et qui en tout cas ne se réduisent pas à un ensemble de caractères communs.

une foule d'objets ou d'assemblages d'objets de formes libres – des formes non géométriques au sens ordinaire de ce terme – et qui pourtant manifestent une de ces symétries simples dont parle MACH : symétrie orthogonale, translation, symétrie centrale. Ces formes symétriques libres sont tellement répandues qu'il est à peine utile d'en donner des exemples. Mentionnons, pour les symétries orthogonales, les papillons, de nombreuses feuilles, les photos du visage humain de face[12], ... ; pour les translations, les frises, les papiers peints, des mots écrits tels que *flonflon*, ... ; pour les symétries centrales, d'assez nombreux motifs ornementaux.

À propos des objets de ce type, renversons maintenant le point de vue de MACH. Celui-ci disait : si deux formes sont transformées l'une de l'autre par une de ces symétries simples, et si en outre elles se présentent à l'observateur en position privilégiée, alors celui-ci reconnaît sans peine leur congruence. Or ce qui se passe aussi, c'est que dans les situations décrites, non seulement la congruence des figures est reconnue, mais encore *le type de symétrie dont elles sont le siège est identifié lui aussi.*

Un objet présentant une symétrie orthogonale est reconnu, en position privilégiée, moins par l'évocation du retournement (ou du mouvement de pliage) qui envoie un côté de l'axe sur l'autre et réciproquement, que par la correspondance immédiatement perçue entre ses parties gauche et droite. Les objets symétriques forment une catégorie, un objet mental, correspondant à ce type de perception.

Une translation est reconnue comme telle, c'est-à-dire comme correspondant à un glissement d'une forme vers l'autre sans changement de direction. Les objets ou figures se présentant sous forme de parties translatées les unes des autres forment une catégorie, un objet mental, le lien entre eux étant précisément ces glissements sans changement de direction, un type de mouvement qui se retrouve dans tous.

La parenté des objets présentant une symétrie centrale est sans doute reconnue par l'évocation mentale, pour chacun d'eux, du demi-tour qui applique l'objet sur lui-même.

Insistons, car c'est important, sur le fait que les reconnaissances dont nous parlons ne portent que sur les symétries simples, celles identifiées par MACH[13], et qu'elles exigent en outre la présentation en position privilégiée. Les lois mathématiques plus compliquées de correspondance entre formes ne sont pas *perçues*, et ne sauraient être considérées au stade naissant de la géométrie.

3.3 Formes simples à symétrie modérée

Bornons-nous aux formes polygonales[14], et commençons par les rectangles. Si on partitionne l'ensemble des rectangles en classes de similitude, ce qui revient à considérer que deux rectangles sont *les mêmes* s'ils ont même proportion, alors un rectangle est déterminé par un seul paramètre, à savoir le rapport de grandeur de ses côtés. Ainsi, la classe des rectangles considérés *à similitude près* est *une famille à un paramètre*. Cette propriété est importante : elle se traduit par le fait que l'on peut représenter l'ensemble des rectangles par une vue ordonnée de certains d'entre eux, en les disposant en une seule rangée. C'est ce que montre la figure 13, qui se parcourt aisément. Une manière naturelle d'appréhender l'ensemble des rectangles consiste à partir de l'un d'entre eux et, dans un premier temps, à faire tendre sa base mentalement de manière continue vers zéro, puis dans un deuxième temps à faire tendre sa base vers l'infini. On s'efforce en ce faisant de parcourir l'ensemble des rectangles en extension.

[12]Nous nous bornons ici à des objets faciles à décrire en deux dimensions.

[13]Celles-ci s'étendent jusqu'à certaines similitudes, que nous n'avons pas mentionnées pour ne pas allonger l'exposé (cf. *CREM* [1999]).

[14]Pour un examen plus complet de la question, voir *CREM* [1999]

FIGURE 13

Par ailleurs, les propriétés du rectangle sont nombreuses. Seules les plus prégnantes, celles qui s'imposent le plus facilement à l'observateur, vont jouer lorsqu'on s'efforce de le saisir en compréhension. Mais encore chaque observateur a-t-il sans doute sa manière de connaître un rectangle. Par exemple, le rectangle en position privilégiée peut être saisi par le fait qu'il a deux côtés verticaux et deux horizontaux. Ou encore parce qu'il a deux paires de côtés égaux et quatre angles droits.

Venons-en maintenant aux triangles isocèles. Ils sont caractérisés, à similitude près, par le rapport de leur base à leur hauteur. Ils forment donc aussi *une famille à un paramètre*. La figure 14, aisée à parcourir d'un bout à l'autre, donne une idée raisonnable de tous les triangles isocèles possibles : il s'agit d'un essai de parcours en extension. Une autre idée de ces triangles s'obtient à partir de l'un d'eux, dont on maintient la base constante, en faisant tendre sa hauteur de manière continue successivement vers zéro et vers l'infini.

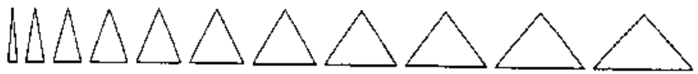


FIGURE 14

Par ailleurs on saisit les triangles isocèles en *compréhension*, en évoquant l'une ou l'autre de leurs propriétés simples : par exemple, en position privilégiée, l'égale inclinaison de deux côtés sur la base horizontale et la propriété de symétrie orthogonale.

Considérée de même à similitude près, la classe des parallélogrammes est *une famille à deux paramètres*. Pour déterminer un parallélogramme, on doit se donner le rapport de deux côtés adjacents, et l'angle qu'ils font entre eux. Du fait qu'il dépend de deux paramètres, le parallélogramme est une figure plus compliquée que le rectangle ou le triangle isocèle. La figure 15 se présente sous forme d'un tableau à double entrée. Ce tableau, évidemment moins aisé à parcourir que la rangée des rectangles ou celle des triangles isocèles, n'oppose néanmoins pas trop de difficulté à l'imagination : on accède sans trop de peine en extension à la catégorie des parallélogrammes.

Pour saisir les parallélogrammes en compréhension, on considérera par exemple, en position privilégiée, l'horizontalité et la congruence de deux côtés, l'égale inclinaison et la congruence des deux autres.

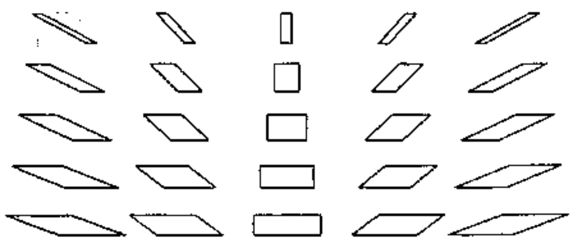


FIGURE 15

Il est raisonnable de penser que la première géométrie raisonnée va se constituer à partir de figures simples telles que le rectangle ou le triangle isocèle, ou relativement simples comme le parallélogramme. Ces trois exemples suffisent à illustrer ce que nous entendons par figure simple : une *figure simple* est une figure aisément accessible en extension et en compréhension, donc une figure dont les variétés se parcourent sans peine en imagination et dont certaines propriétés caractéristiques sont saisies facilement.

Il existe, outre les rectangles, les triangles isocèles et les parallélogrammes, d'autres exemples de figures simples à symétrie modérée. Par exemple, les figures formées de deux droites sécantes (un paramètre), ou de deux parallèles et une sécante (un paramètre), les triangles (deux paramètres)[15]. Pour montrer par contraste ce qu'est une figure simple, montrons deux sortes de figures compliquées.

D'abord on ne reconnaît pas à vue un heptagone (ou un octogone), même s'il est régulier. Il faut pour l'identifier une intervention de l'intelligence, que ce soit par comptage du nombre des côtés ou par analyse détaillée des symétries. Il en va de même pour toutes les classes de polygones réguliers comportant beaucoup de côtés.

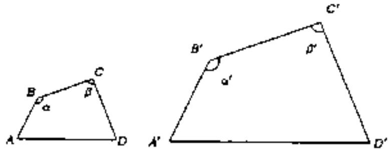


FIGURE 16

Considérons maintenant la classe des quadrilatères. Montrons qu'elle est *une famille à quatre paramètres*. Soit par exemple, à la figure 16, un quadrilatère quelconque $ABCD$, et soit à construire un quadrilatère semblable à celui-là. On pourra se donner un côté $[A'B']$ quelconque pour correspondre à $[AB]$. Ensuite on reproduit l'angle α. On construit $[B'C']$ avec un rapport à $[A'B']$ égal au rapport de $[BC]$ à $[AB]$. On reproduit ensuite l'angle β, puis on construit le côté $[D'C']$ avec un rapport à $[B'C']$ égal au rapport de $[DC]$ à $[BC]$. Comme nous avons dû nous donner quatre mesures (deux angles et deux rapports de côtés) pour construire la figure, la famille a bien quatre paramètres.

À cause du nombre de paramètres (on pourrait dire aussi de *degrés de liberté*), des phénomènes nouveaux apparaissent, que l'on n'avait pas rencontrés dans le rectangle, le triangle isocèle ou le parallélogramme, à savoir la non-convexité et les points doubles (lorsque deux côtés se croisent).

La variété des quadrilatères, dont la figure 17 ne donne qu'une faible idée, défie l'imagination. On ne peut pas représenter cette famille par une figure analogue à celle que nous avons proposée pour les parallélogrammes : il faudrait ici construire un réseau de figures à quatre dimensions.

[15]Pour être complet, il faudrait encore citer les figures à zéro paramètres, c'est-à-dire celles qui sont toutes semblables entre elles : c'est le cas des carrés, des cercles, des paires de parallèles, etc. (cf. *CREM* [1999])

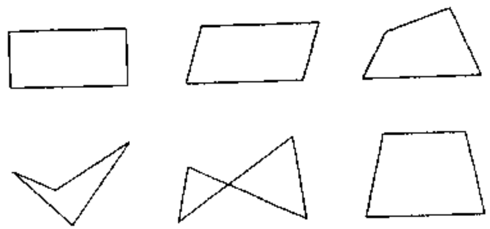


FIGURE 17

Ainsi la famille des quadrilatères, aisée à saisir en compréhension (quatre côtés rectilignes enchaînés) est extrêmement difficile, sinon impossible, à saisir en extension. Seule la géométrie raisonnée permettra d'acquérir des connaissances générales sur cette famille.

3.4 Le sens étroit et le sens large

Nous avons vu à la section précédente que les figures simples à symétrie modérée, celles qui sont au départ du savoir géométrique, sont saisies en compréhension par quelques propriétés caractéristiques. Toutefois, dès que nous réfléchissons à une telle figure, nous pouvons faire état de toute l'expérience que nous en avons et d'une foule d'autres propriétés. Celles-ci s'expriment en termes de perceptions, de relations aux directions privilégiées du monde physique, aux symétries de notre corps, et à toutes sortes d'expériences que nous avons de la figure en question.

Prenons l'exemple du rectangle. Nous savons qu'il possède quatre angles droits, des côtés opposés parallèles et égaux, que si deux de ses côtés opposés sont verticaux, les deux autres sont horizontaux. Mais nous nous connaissons bien d'autres propriétés du rectangle, acquises lors de constructions, dessins, pliages divers. Par exemple :

chaque médiane d'un rectangle le divise en deux rectangles congruents (figure 18(a) et (b));

les deux médianes d'un rectangle divisent celui-ci en quatre rectangles congruents (figure 18(c));

les médianes d'un rectangle sont perpendiculaires et se coupent en leur milieu;

le rectangle possède deux axes de symétrie orthogonale ; ces deux axes sont orthogonaux entre eux.

Les diagonales font apparaître de nouvelles propriétés :

chaque diagonale d'un rectangle décompose celui-ci en deux triangles rectangles congruents (figure 18(d) et (e)) ; l'un quelconque des deux est superposable à l'autre par un mouvement d'un demi-tour autour du milieu de la diagonale en question.

Les deux diagonales décomposent le rectangle en deux couples de triangles isocèles congruents (figure 18(f)) ; dans chaque couple, les deux triangles sont symétriques orthogonaux l'un de l'autre.

Si on trace les médianes et les diagonales d'un rectangle, on obtient une nouvelle décomposition intéressante :

les médianes et les diagonales d'un rectangle décomposent celui-ci en huit triangles rectangles congruents (figure 18(g)).

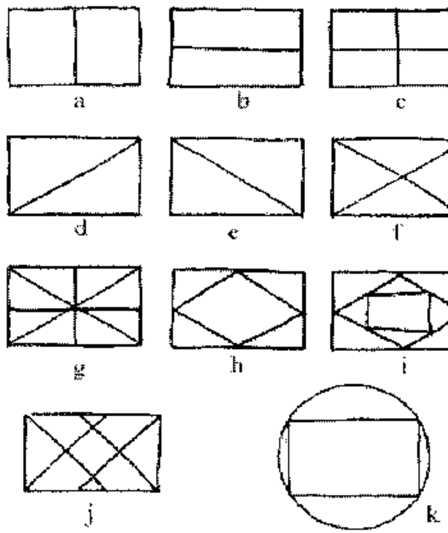


FIGURE 18

Comme nous l'avons dit, toutes ces propriétés contribuent à la connaissance familière du rectangle, plus ou moins riche d'une personne à l'autre. Les structurations plus complexes du rectangle n'en font habituellement pas partie. Par exemple, la plupart des gens ne pensent pas spontanément au losange que l'on obtient en joignant les milieux des côtés d'un rectangle (figure 18(h)), et beaucoup moins encore au carré que forment les bissectrices d'un rectangle (figure 18(j)). L'essentiel pour nous maintenant et pour ce qui va suivre, est que le rectangle possède pour chaque personne un visage familier fait d'un certain nombre de traits fondamentaux, un paquet de propriétés. Il ne se réduit pas à la définition qu'en donnent les dictionnaires : un quadrilatère à quatre angles droits. Cette définition cerne son *sens étroit*. Mais il possède un *sens large*, soutien de la pensée créative.

Notons enfin que le rectangle saisi en compréhension ne se ramène pas à un ensemble de propriétés juxtaposées, immobiles dans l'esprit, vouées à la seule contemplation. En effet, ces propriétés, acquises dans le registre perceptivo-moteur, manifestent d'emblée des liens de causalité, d'implication. On sait par exemple que si on a dessiné une médiane d'un rectangle, lorsqu'on dessinera l'autre, on la trouvera perpendiculaire à la première. Par exemple encore, *si* on coupe un rectangle suivant une diagonale, *alors* on obtient deux triangles et on peut superposer l'un à l'autre par un demi-tour autour du milieu de la diagonale. Ou encore, *si* on joint les milieux des côtés successifs d'un rectangle, *alors* on obtient un losange.

Ainsi la constitution des objets mentaux en compréhension fournit immédiatement des amorces d'une pensée déductive. Regardons maintenant ce phénomène de plus près, en tâchant d'expliquer comment naissent les premières inférences.

4 Inférer

Pour nous éclairer, commençons par quatre exemples d'implications, en regardant pour chacune ce qui en fonde l'évidence. Les trois premières sont des conditions suffisantes pour qu'un

quadrilatère soit un rectangle. La dernière est une condition suffisante pour qu'on puisse assembler trois segments en forme de triangle.

4.1 Quatre exemples

DEUX CÔTÉS CONGRUENTS PERPENDICULAIRES À UN TROISIÈME

Dans un plan frontal (un tableau noir par exemple), on fait le dessin suivant (figure 19) :

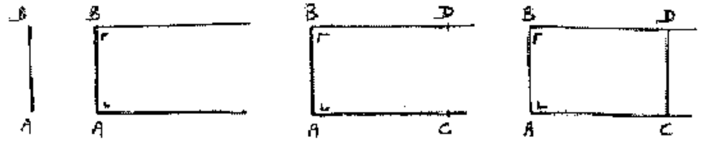


FIGURE 19

on trace un segment vertical $[AB]$;

à partir de A et de B, et du même côté de $[AB]$, on dessine deux demi-droites horizontales ;

on porte une même distance sur chacune d'elles, ce qui détermine deux points C et D ;

on joint C et D ; le quadrilatère $ABCD$ est un rectangle.

Cette construction faite ou refaite – en réalité de nombreuses fois refaite en pensée – suffit à m'assurer de la propriété suivante :

Si un quadrilatère possède deux côtés congruents perpendiculaires à un même troisième et situés du même côté, il est rectangle.

DIAGONALES CONGRUENTES SE COUPANT EN LEUR MILIEU

On articule en leur milieu deux tiges d'égale longueur. On dispose l'objet ainsi obtenu dans un plan frontal, de sorte que les deux tiges aient une pente égale, en deux sens opposés (figure 20(a)). Les extrémités A et B sont alors sur une horizontale, de même que les extrémités D et C. De même A et D sont sur une verticale, ainsi que B et C. Si on dessine le quadrilatère $ABCD$, on trouve un rectangle (figure 20).

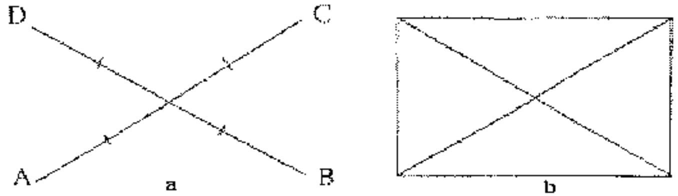


Figure 20

D'où la propriété suivante :

Si les diagonales d'un quadrilatère sont congruentes et se coupent en leur milieu, le quadrilatère est un rectangle.

DÉFORMER UN PARALLÉLOGRAMME

Voyons maintenant comment un mouvement continu peut aussi suggérer une détermination de la forme rectangulaire. On réalise un parallélogramme avec des tiges articulées, et on le dispose devant soi de manière que deux de ses côtés soient horizontaux. Les deux autres ont un certaine inclinaison par rapport à la verticale. Mais ils sont parallèles, et qui plus est, ils le demeurent lorsqu'on déforme continûment l'objet. On redresse un de ces deux côtés en le faisant tendre vers la verticale. L'autre arrive à la verticale en même temps, et le quadrilatère obtenu ainsi est bien un rectangle. On arrive ainsi à la propriété suivante :

Si un parallélogramme possède un angle droit, il est rectangle.

L'INÉGALITÉ TRIANGULAIRE

Considérons trois segments (ou trois tiges) dont l'un est plus grand que la somme des deux autres. Distinguons trois cas.

Soit les deux plus petits mis bout à bout forment un segment plus petit que le troisième. Dans ce cas, on n'arrive pas à les assembler en triangle (figure 21(a)).

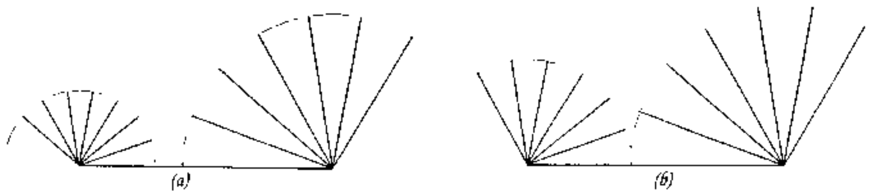


FIGURE 21

Soit les deux plus petits mis bout à bout forment un segment égal au troisième. Alors on n'arrive pas non plus à les assembler en triangle. En effet, lorsqu'on essaie, les deux plus petits ne se touchent que lorsqu'ils viennent s'aligner sur le grand (figure 21(b)).

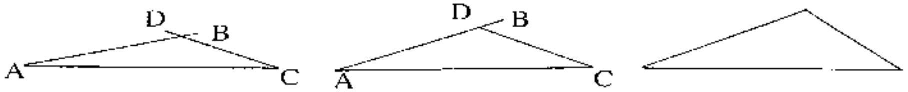


FIGURE 22

Soit les deux plus petits mis bout à bout forment un segment plus grand que le troisième. Alors on peut les assembler en triangle. En effet, on peut disposer les deux plus petits (appelons-les $[AB]$ et $[CD]$) de sorte qu'ils se coupent. Tournons ensuite le segment $[AB]$ jusqu'à ce qu'il passe par D. Faisons ensuite glisser D sur $[AB]$ jusqu'à ce que D vienne coïncider avec B. Nous avons obtenu un triangle (figure 22).

Ces manipulations conduisent à la proposition suivante :

Si trois segments (inégaux) sont tels que le plus long est plus petit que la somme des deux autres, alors on peut les disposer en triangle. Sinon ce n'est pas possible.

4.2 Des conditions déterminantes évidentes

Après avoir parcouru ces quatre exemples d'inférence élémentaire, voyons maintenant ce qu'elles ont en commun. Dans chaque cas, on a sous les yeux – ou comme image mentale –, un objet, une situation : un rectangle que l'on cherche à dessiner, deux tiges que l'on assemble, un parallélogramme articulé, trois tiges que l'on cherche à assembler en triangle. Il s'agit de situations simples, d'objets que l'on reconnaît sans analyse, dont on connaît par ailleurs diverses propriétés et que l'on sait manipuler. Dans chaque cas, on réalise réellement ou en pensée une expérience, une action, un dessin et on conclut de la même façon : quand on fait telle ou telle chose, on obtient tel ou tel résultat. En d'autres termes, telles ou telles propriétés que l'on mobilise entraînent que l'on obtient (ou que l'on ne peut obtenir) la figure souhaitée. On reconnaît celle-ci sans analyse, ou au moins de manière implicite, sans recourir à une définition. Par exemple, on n'a pas besoin de *formuler* que si deux côtés sont horizontaux et les deux autres verticaux, on a bien un rectangle. Ou encore on reconnaît un triangle à vue.

Nous appelons *conditions déterminantes* d'un type de figure ou de configuration géométrique, des conditions suffisantes obtenues ainsi par expérience réelle ou mentale.

Les conditions déterminantes ne portent bien entendu pas sur une seule figure, une seule situation. Lorsqu'il était ci-dessus question du rectangle, il s'agissait de tous les rectangles, et dans le quatrième expérience, il s'agissait de tous les triplets de tiges. En d'autres termes, l'inférence porte dans chaque cas sur une infinité de figures, même si l'objet sur lequel on expérimente en réalité ou en pensée est un objet singulier. Il représente tous les autres objets de son espèce. Il est *paradigmatique*.

A. ARNAULD [1683] observait déjà au XVII$^{\text{ème}}$ siècle que les ensembles de figures dont traite la géométrie sont infinis : "On peut dire que toutes les propositions géométriques sont de même infinies en étendüe ; parce que l'on n'y conclut pas ce qu'on démontre d'une seule ligne, d'un seul angle, d'un seul cercle, d'un seul triangle, mais de toutes les lignes, de tous les angles, de tous les cercles, de tous les triangles ; et qu'ainsy l'esprit les renferme et les comprend tous en quelque sorte quelques infinis qu'ils soient."

Mais qu'est-ce qui permet, dans le cas des expériences que nous avons décrites, d'étendre la conclusion d'une figure à toutes celles de la même espèce ? Deux circonstances rendent cette extension possible et naturelle : le première est que nous faisons l'expérience en position privilégiée, c'est-à-dire que nous nous mettons dans les conditions les meilleures pour voir ce qui se passe ; la seconde est que les figures ou situations sur lesquelles nous expérimentons sont simples. Elles sont définies, à similitude près, par un ou deux paramètres. L'imagination glisse facilement d'une figure à une autre de la même famille, et ceci d'autant plus que les paramètres définissant la famille peuvent varier continûment. L'imagination entame en douceur le parcours de tous les cas possibles.

Notons cependant que tout le monde ne partage pas les mêmes évidences.

Il est sans doute vrai que certains phénomènes très simples et très symétriques s'imposent comme évidents. Ce pourrait être le cas par exemple pour l'égale inclinaison des deux montants d'une échelle à montants égaux dressée sur un sol horizontal. Mais il existe des phénomènes évidents pour certains esprits, et non pour tous.

Ainsi les évidences ne sont pas nécessairement des données immédiates. Elles s'appuient fréquemment sur un vécu, sur une expérience. Il semble clair que certains phénomènes apparaissant comme non évidents à certaines personnes, deviennent évidents pour elles à la suite d'observations, de constructions, de manipulations appropriées. C'est ainsi que peut être installé, dans un groupe d'élèves hétérogène, un consensus minimal pour servir de base aux premiers éléments d'une géométrie raisonnée.

4.3 Des inférences inductives

Il semble assez clair que les conditions déterminantes trouvent leur fondement dans la causalité physique. Comme nous venons de le voir, le schéma est constant : on construit, on dessine, on dispose des objets de telle ou telle façon, puis on constate tel résultat. Et ensuite la propriété observée est étendue à l'ensemble des figures du même type. Il s'agit donc au départ d'inférences inductives.

Mais qu'est-ce que cela veut dire au juste ? "La méthode des sciences physiques, écrit HENRI POINCARÉ [1943], repose sur l'induction qui nous fait attendre la répétition d'un phénomène quand se reproduisent les circonstances où il avait une première fois pris naissance." Cette définition énonce le principe même de l'induction. Mais il faut la compléter pour en discerner la portée

Tout d'abord les circonstances de lieu d'un phénomène peuvent varier, en particulier sa situation par rapport à l'observateur : nous reviendrons sur ce point à la section 4.4. Mais, et ceci est important, la forme de l'objet en cause peut aussi varier.

Reprenons l'un de nos exemples. Nous observons une première fois qu'en dessinant deux segments congruents qui se croisent en leur milieu, nous obtenons les diagonales d'un rectangle. Quel serait l'intérêt de cette observation si nous n'en tirions que ceci : chaque fois que nous dessinons deux segments de même longueur que ceux de la première expérience et que nous les faisons se croiser en leur milieu, de sorte en outre qu'ils fassent entre eux le même angle que dans la première expérience, nous obtenons un rectangle (congruent à celui obtenu la première fois). Ce n'est pas cela qui nous intéresse. L'inférence utile et productive est ici celle qui nous dit ceci : quelle que soit la longueur commune des deux segments, et quelle que soit l'angle qu'ils font entre eux, nous obtenons bien un rectangle. Ainsi, nous maintenons certaines circonstances de l'expérience, mais nous en libérons d'autres, ce qui nous permet d'étendre la propriété observée à la catégorie tout entière des rectangles, de nous dire que les choses se passeront *forcément toujours* comme cela. Seul un tel type d'induction nous permettra d'enclencher des raisonnements géométriques intéressants. On qualifie de *paradigmatiques* les expériences de cette sorte, qui sont représentatives d'une infinité d'expériences analogues.

Montrons *a contrario* une expérience non paradigmatique, c'est-à-dire dont on ne voit pas avec évidence qu'elle donnera le même résultat dans tous les cas de figure possibles. Il n'est pas d'emblée évident que la somme des angles d'un triangle soit égale à deux droits. Constater cette propriété sur quelques triangles particuliers, par exemple en mesurant les angles et faisant la somme des mesures, n'aide en rien à se convaincre que les choses se passeront toujours de la même façon. Il faut aménager l'expérience de sorte qu'on voit l'*inéluctabilité* du résultat, sa *nécessité*. Il faut créer les conditions qui rendent la généralisation – autrement dit l'induction –, évidente.

4.4 Extension aux situations non privilégiées

On peut se demander à quoi servent les conditions déterminantes. Par exemple, à quoi bon considérer des conditions qui déterminent la forme rectangulaire, puisque nous reconnaissons habituellement les rectangles sans analyse ? Il est vrai que nous les reconnaissons sans analyse, mais à condition qu'ils soient en position privilégiée. Qui plus est, nous l'avons dit, c'est aussi et forcément en position privilégiée que nous reconnaissons l'évidence des conditions déterminantes. *Mais ces conditions sont exportables en position non privilégiée.* En d'autres termes, si nous savons par une raison quelconque – ou un raisonnement – qu'un quadrilatère situé n'importe comment répond à des conditions déterminantes de la forme rectangulaire, nous

en concluons – sans aucun besoin de nous en convaincre par la vue – que ce quadrilatère est un rectangle. Ceci montre par quel mécanisme les conditions déterminantes augmentent notre capacité de saisir l'espace.

Un exemple concret suffira à illustrer cela. Supposons que nous doutions de la forme rectangulaire d'une vaste place publique bordée de maisons. Impossible d'amener cette place en position privilégiée : elle est trop grande et trop lourde ! Mais nous pouvons vérifier par des opérations de visée que ses bords sont rectilignes, et par des mesures exécutées localement, qu'elle a deux côtés égaux et perpendiculaires à un troisième. Nous saurons alors que la place est rectangulaire, sans pourtant avoir jamais pu la saisir d'un coup d'œil comme un rectangle.

Pour pouvoir exporter dans toutes les situations possibles une inférence constatée en situation privilégiée, il faut être assuré que les objets que nous éloignons de nous, que nous orientons arbitrairement, que nous ne percevons que partiellement, demeurent inchangés pendant qu'on les transporte ou qu'on les soumet à des observations particulières. Nos inférences seraient inopérantes dans un univers de terre glaise humide ou de caoutchouc. Il nous arrive d'ailleurs de nous laisser surprendre par des variations inattendues d'un objet. Autrement dit, notre géométrie débutante s'applique à des objets *qui se conservent*, à des objets, comme dit Freudenthal [1983], qui ne subissent que des *gentle transformations*, des transformations douces[16].

4.5 Du réel à l'idéal

Nous avons parlé jusqu'ici des objets et des figures géométriques comme de choses réelles obtenues en dessinant, en articulant des tiges, en découpant et pliant des papiers, etc. Or aucune de ces choses ne s'identifie à une figure idéale. Par exemple nous ne pouvons jamais dire si un rectangle réel est un rectangle, absolument parlant. Il y a des cas où nous sommes incapables de discerner si deux segments bout à bout sont à eux deux plus longs, égaux ou plus courts qu'un segment donné. Rappelons qu'il y a deux raisons à cela. La première est physiologique : nos sens ne nous communiquent que des images approximatives. La seconde est physique : les objets réels se résolvent en atomes et particules qui leur enlèvent, dans le monde microscopique, toute possibilité de correspondre exactement aux objets idéalisés de la géométrie.

Il est par conséquent exclu que l'évidence de la première condition déterminante ci-dessus (cf. 4.1) se ramène à ceci : si je suis sûr, *absolument*, qu'une figure est un quadrilatère, et si j'ai vérifié, *sans marge d'erreur*, qu'elle a deux côtés égaux et perpendiculaires à un troisième, alors je sais *sans erreur possible* que le quatrième angle est droit. L'évidence n'est pas de cet ordre-là.

L'évidence n'ayant pas un tel fondement absolu, elle s'appuie sur une *expérience* entourée d'une marge d'indétermination. À propos du rectangle par exemple, je pourrais dire ceci, ou quelque chose d'approchant : chaque fois que j'ai construit un quadrilatère avec trois angles dessinés soigneusement à l'équerre, j'ai pu vérifier que le quatrième angle correspondait, aussi précisément que je pouvais le voir, à l'angle de l'équerre. Dans tous les cas où, peut-être parce que j'étais fatigué ou distrait, j'ai obtenu un "rectangle" un peu bancal, j'ai recommencé la construction et j'ai obtenu un rectangle que j'estimais satisfaisant. Bien sûr je n'ai jamais, et pour cause, construit un rectangle d'un kilomètre de long et d'un millimètre de large. Qui plus est, j'ai même rarement pensé à un tel rectangle. Pourtant j'arrive un peu à l'imaginer.

[16]Cette affirmation provoque un paradoxe. En effet, comment pourrions-nous nous convaincre que les objets qui nous occupent se conservent, ou en d'autres termes ne subissent que des transformations douces, sans leur appliquer des mesures relevant de la géométrie que nous sommes précisément en train de construire ? Ce paradoxe se résout pratiquement au niveau du bon sens : nous verrons bien à l'usage quand notre géométrie ne fonctionnera plus. On le résout au niveau théorique en *postulant* l'existence des transformations douces.

Ainsi mon sentiment d'évidence est tempéré par ma conscience des imprécisions des objets réels et de mes sens, et par mon impuissance à saisir la famille infinie des quadrilatères, et celle des rectangles. Mon évidence au départ est d'ordre pratique et s'étend à des catégories d'objets à mon échelle[17].

Mais bien que nos premières implications ne soient jamais vérifiables qu'approximativement, elles s'énoncent avec des mots (des concepts) qui ne sont guère connotés par l'indétermination des choses. Quand nous pensons à un rectangle, un angle droit, une égalité de segments, nous ne nous embarrassons pas spontanément du fait qu'il n'existe ni rectangle, ni angle droit, ni égalité absolue. Les concepts sont univoques par nature. Et puisque nous savons d'expérience que les propriétés que nous évoquons, quoique vérifiées imparfaitement par les objets réels, sont néanmoins vérifiées par eux avec une marge d'erreur d'autant plus petite qu'ils ont été construits ou dessinés avec plus de précision, nous appliquons spontanément ces propriétés à des objets infiniment précis.

Les choses et les phénomènes s'installent donc dans la pensée avec une netteté qu'ils n'ont pas dans la réalité. Ce qui prend la forme d'une idée devient par là même idéal. Il en résulte que les ensembles infinis d'objets auxquels renvoient les mots sont moins réels qu'imaginaires.

L'évolution de la physique au XX^{ème} siècle nous a fait perdre par ailleurs l'illusion que cet idéal pourrait aussi être vu dans la nature "si nous disposions d'instruments infiniment précis". En ce sens la géométrie est une construction de l'esprit[18].

4.6 Vers une théorie

Nous sommes passés de quelques expériences paradigmatiques à des évidences portant chaque fois sur une famille infinie de figures en situation privilégiée. Nous avons ensuite étendu les conditions déterminantes ainsi obtenues à toutes les figures en situation quelconque. Mais la vocation des conditions déterminantes est de servir de points de départ pour une géométrie qui prouve des propositions non évidentes, des propriétés de figures compliquées.

Les conditions déterminantes, basées sur des expériences, ont la forme d'implications entre propriétés, et donc elles se situent au niveau de la saisie des figures en compréhension. Toutefois, leur évidence est telle qu'on les voit assez clairement fonctionner en extension. On étend sans peine ces expériences en pensée à l'ensemble des cas de figure. Tel ne sera évidemment plus le cas pour les propriétés des figures compliquées, celles où l'imagination en est réduite à des parcours très partiels des cas de figure. La part de l'intuition se réduit ainsi par nécessité. Bien entendu, les intuitions appliquées à certains cas de figure demeurent essentielles sur le plan heuristique. Mais les cas de figure accessibles ne sont plus représentatifs de tous les cas possibles. Ainsi les intuitions se font hasardeuses et la pensée déductive s'impose comme unique alternative.

Montrons sur un exemple comment une propriété déterminante peut être appliquée à la preuve d'une propriété a priori non évidente. Quelques expériences montrent que, de quelques points d'un cercle, on voit un diamètre de celui-ci sous un angle droit. Mais deux questions se posent : est-ce qu'il en est toujours ainsi ? et si oui, à quoi est due cette propriété remarquable ? On dessine alors dans un cercle un angle inscrit interceptant un diamètre (figure 23). On ajoute à la figure le diamètre issu du sommet de l'angle. Les deux diamètres sont égaux et se coupent en leur milieu. Leurs extrémités sont donc les sommets d'un rectangle. L'angle inscrit de départ est donc bien un angle droit.

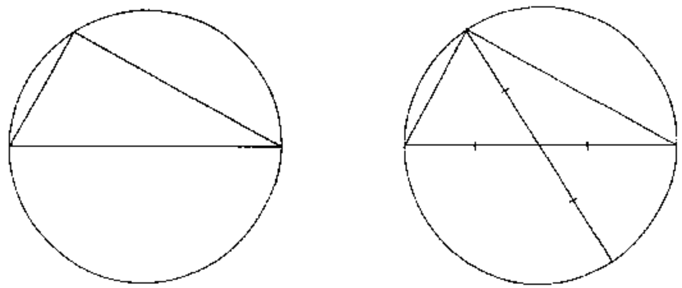


FIGURE 23

Remarquons qu'une telle preuve consiste essentiellement à amener la propriété en cause à l'évidence. Elle ne consiste pas *d'abord* à montrer que la propriété découle d'autres propriétés connues. Dans les débuts de la géométrie, on s'intéresse à des phénomènes, à des propriétés. Il faudra toute une évolution de la pensée pour qu'on s'intéresse à la cohérence d'une théorie, et que corrélativement la portée, la visée – sinon la nature – des preuves change, pour que l'attention se déplace de l'évidence du phénomène vers l'évidence des implications[19].

4.7 L'univers de la géométrie commençante

Nous avons structuré cet exposé en trois parties principales : percevoir, concevoir, inférer. Cette division marque les étapes d'une analyse, plutôt que des moments clairement discernables et successifs de la pensée géométrique en formation. Il est rare en effet que nous percevions un objet sans le rattacher à l'une ou l'autre catégorie. Par exemple un triangle particulier est sans peine rattaché – fût-ce implicitement –, à la forme triangulaire en général. Qui plus est, les formes géométriques simples ne sont pas seulement des objets de contemplation. Comme nous l'avons vu, la connaisssance que nous en avons comporte des expériences de déplacements, de constructions, de déformations, et plus généralement de manipulations génératrices d'inférences spontanées : quand je fais ceci, j'obtiens tel résultat. En cela consiste l'expérience des choses, sur laquelle se bâtit, entre autres, une première connaissance géométrique. Ainsi,

[17] De telles indéterminations laissent la porte ouverte à d'autres géométries. Si l'espace est doté d'une courbure imperceptible à mon échelle, il se pourrait bien que mon évidence sur les quadrilatères possédant deux côtés égaux perpendiculaires à un troisième soit mise en défaut. Mais ces phénomènes étranges se passeraient sous le seuil de mes perceptions claires.

[18] Le phénomène constaté de manière imprécise dans la réalité se mue en évidence précise dans l'esprit. C'est ce dont témoigne d'ALEMBERT lorsqu'il écrit, évoquant la congruence de deux figures : "Ce dernier principe n'est point, comme l'ont prétendu plusieurs Géomètres, une méthode de démontrer peu exacte et purement mécanique. La superposition, telle que les Mathématiciens la conçoivent, ne consiste pas à appliquer grossièrement une figure sur une autre, pour juger par les yeux de leur égalité ou de leur différence, comme l'on applique une aune sur une pièce de toile pour la mesurer ; elle consiste à imaginer une figure transportée sur une autre, et à conclure de l'égalité supposée de certaine parties des deux figures, la coïncidence du reste : d'où résulte l'égalité et la similitude parfaite des figures. Cette manière de démontrer a donc l'avantage non seulement de rendre les vérités palpables, mais d'être encore la plus rigoureuse et la plus simple qu'il est possible, en un mot de satisfaire l'esprit en parlant aux yeux." (Cité par *R. Bkouche*).

Le mode de preuve évoqué par d'ALEMBERT n'a plus cours dans les mathématiques d'aujourd'hui. On peut penser toutefois qu'il demeure un palier incontournable dans l'apprentissage de la preuve en géométrie.

[19] Cf. N. ROUCHE [1990].

les trois plans que discerne l'analyse, – percevoir, concevoir et inférer –, s'intègrent dans un mode spontané d'activité qui est à la fois manuel, perceptif et intellectuel.

L'univers de cette activité, qui est aussi celui des premières acquisitions géométriques, n'est pas constitué d'un espace et de parties immobiles de cet espace, dont on étudierait les propriétés. Il est peuplé d'objets et de figures déplaçables soumises à des mouvements continus et comportant des symétries. Celles-ci sont mises en relation avec les symétries du corps humain et les directions physiques privilégiées – la verticale et l'horizontale. Comme nous l'avons annoncé, nous n'avons considéré dans cette étude que des objets plans. Observons, dans ce cadre restreint, que la géométrie plane n'est pas d'abord la géométrie *du plan*, mais bien la géométrie *des objets plans*. Il y a un long chemin à parcourir pour passer de la géométrie commençante à la géométrie constituée. Nous nous contentons ici d'évoquer ce parcours, en soulignant que ces étapes dans l'enseignement méritent d'être soigneusement motivées.

5 Mais aussi chercher

Examinons pour terminer un quatrième registre de la pensée géométrique à ses débuts. Il est évidemment déjà intéressant et utile de posséder quelques vérités géométriques évidentes. Mais pour avancer, il faut arriver à en trouver et prouver de nouvelles. On trouve en se posant des questions, en expérimentant, en tâtonnant. On prouve en construisant des preuves. Et pour prouver, il faut des arguments. L'exemple ci-dessus de l'angle inscrit interceptant un diamètre montre que l'argument clé d'une preuve – en l'occurence "deux segments égaux se coupant en leur milieu déterminent un rectangle" – n'est pas révélé d'office à qui contemple passivement la situation. Il faut puiser dans ses souvenirs, faire fonctionner son imagination, mobiliser sa pensée dans un univers bien peuplé de choses et de propriétés diverses, avec entre elles le plus possible de relations significatives. Il ne suffit pas d'avoir la tête bien faite, il faut aussi qu'elle soit bien pleine, et encore pas de n'importe quoi.

On retrouve ici le sens large, mentionné à la section 3.4. La pensée en recherche s'appuie sur le sens large, sur toutes les choses que l'on est capable d'associer à chaque figure, à chaque phénomène, à chaque mot, à chaque symbole. Une condition déterminante est une condition suffisante pour obtenir telle ou telle configuration. Mais elle apporte avec elle et rend disponible pour la recherche toutes les propriétés connues de la figure ou de la configuration.

Notons enfin l'importance, du point de vue heuristique, de ce que nous avons appelé les figures simples à symétrie modérée, celles qui forment des familles à un ou deux paramètres. On comprend leur rôle particulier dans la recherche des preuves en géométrie élémentaire. En effet, d'une part – nous l'avons abondamment expliqué – leur simplicité fait que l'on parcourt de telles familles en extension sans trop de difficulté. Mais d'autres part, comme elles sont plus variées que les figures à zéro degrés de liberté, on les retrouve plus souvent comme sous-figures dans des figures compliquées qui posent problème. Elles jouent donc plus souvent le rôle de *figures clés*, génératrices de clarté. Une figure compliquée est comme une forteresse qui décourage les attaques. Une manière efficace d'en percer les secrets consiste à y chercher, et bien souvent à y introduire (comme un cheval de Troie), une figure clé qui l'éclaire.

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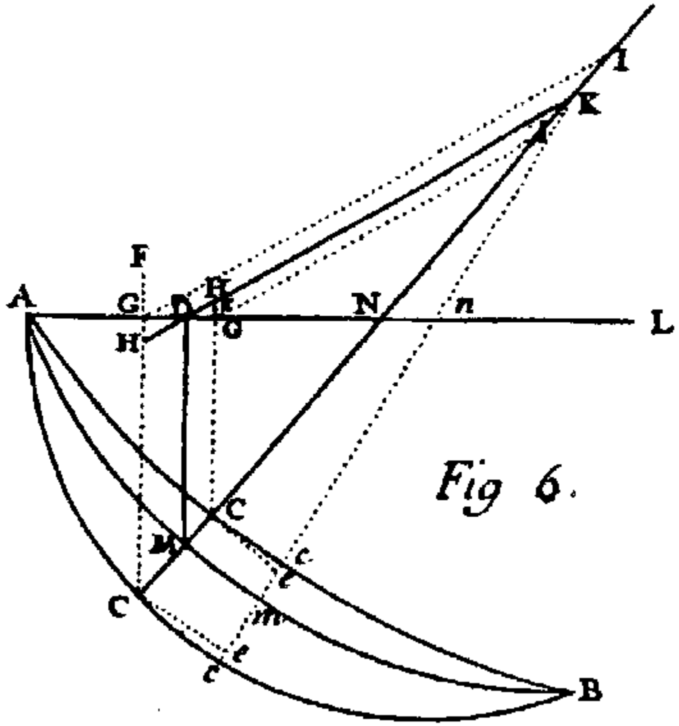

Fig 6.

"Mathematical Physics" and "Physical Mathematics"[1] : A historical approach to didactical aspects of their relation

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Abstract

A. The issue of the relevance of the history of mathematics to mathematics education is addressed and it is suggested that there are three possible ways to integrate historical aspects in the presentation of mathematics:

- By providing direct historical information, the emphasis being on learning about history.
- By following a teaching approach inspired by history, the emphasis being on learning mathematical topics.
- By presenting social and cultural aspects of Mathematics in a historical perspective, the emphasis being on mathematical awareness.

These possibilities are not restricted to mathematics only, but can be realized in the presentation of physics as well.

B. On the other hand, the historically continuous, close relation between mathematics and physics suggests that:

- mathematics and physics, as general attitudes towards the description and understanding of empirically and mentally conceived objects, are so closely interwoven, that any distinction between them, is related more to the point of view adopted while studying particular aspects of an object, than to the object itself. A historically inspired approach, though not necessary, is well suited to illustrate this point.
- The interwining referred to above, is expressed by both, the use of mathematical methods in physics (mathematical physics), and the use of physical concepts, thinking and arguments in mathematics ("Physical Mathematics").

According to the above points (supported by many historically important examples), it is legitimate to consider mathematics and physics as different, but complementary, views of the world. This can be fruitful in teaching and understanding both disciplines.

C. The issues **A** and **B** raised above, are illustrated in some details by means of an example at the high-school level, namely, geometrical optics and differential calculus. This example admits considerable generalization, hence it is virtually important at the university level as well.

[1] The term "Mathematical Physics" is familiar, but the term "Physical Mathematics" is not; it has been taken from POLYA 1954, ch.IX.

1 Introduction

This paper is divided into two parts, both of which are related to two out of the twelve questions raised in the "Discussion Document" (DD) (FAUVEL et al. 1997), which motivated the writing of an ICMI Study Volume on *The role of the history of mathematics in the teaching and learning of mathematics* (FAUVEL et al., 2000). The first and smaller part[2] is related to question n°8 : "*What are the relations between the ... roles we attribute to history and the ways of introducing or using it in education?*". Its analysis suggests that "The [answer] ... involves a **listing of ways of introducing or incorporating a historical dimension**" (FAUVEL et al. 1997, p. 257, our emphasis). The first point to be made here is that it will become clear that this listing could be the **same** in both mathematics and physics.

The second and longer part is related to question n°5 of the DD : "Should different parts of the curriculum involve history of Mathematics [HM] in a different way?" Its analysis suggests that "Bearing in mind that history extends into the future, ... [this] could lead to suggestions for **new topics to be taught**" (FAUVEL et al. 1997, p. 256, our emphasis). The second point to be made here is that indeed, **history strongly suggests** the existence of a **close relation** between mathematics and physics, which should not be ignored in teaching and learning either of these disciplines.

Therefore, this paper is organized as follows: In Section 2 we provide a list of the possible reasons for introducing a historical dimension in mathematics education (ME), that have been or could have been put forward. This list clearly suggests, on one hand the possible general ways of introducing a historical dimension and on the other hand that they are equally valid in physics education (PE) as well. This is not accidental, but in our opinion it is related to two epistemological and historical theses concerning the relation between mathematics and physics. Their formulation and clarification is the subject of section 3. In sections 4 and 5 we present an outline of some historically important examples that illustrate these theses, at the same time providing evidence for their correctness. Finally, in section 6, the possibility to implement in the teaching process the historically and epistemologically suggested close relation between mathematics and physics described in sections 3 to 5, is illustrated by means of an example.

2 Arguments for integrating history of mathematics in mathematics education[3]

Integrating HM in ME may support, enrich and improve:

1. *The learning of mathematics* by (a) contrasting the historical development of mathematical knowledge vs. its final form presented as a deductive structure; (b) using history as a resource of relevant questions, problems and ideas that may motivate, interest and engage the learner; using history as a bridge between different mathematical domains or between mathematics and other disciplines.

2. *The development of views on the nature of mathematics and mathematical activity.* This concerns the appreciation of the fact that both the **form** of mathematics (notation, terminology, computational methods) and its **content** have an evolutionary nature that underlines the relative —with respect to time—character of fundamental metaconcepts like rigor, proof, evidence, error etc.

3. *The didactical background of teachers and their pedagogical repertoire*, by helping them (a) to identify the **motivations** for the introduction of (new) mathematical knowledge, (b) to become aware of the **difficulties** that appeared in the past and **may** reappear in the classroom, (c) to get involved into the **creative process** of "doing mathematics", e.g. in the context of historically inspired projects, (d) to enrich **their didactical repertoire** of questions, problems, teaching sequences etc, (e) to become more sensitive and tolerant towards **nonconventional ways of doing mathematics**.

4. *The affective predisposition towards mathematics*, by helping both students and teachers (a) to see mathematics as an **evolving human endeavour** requiring intellectual effort, (b) to appreciate the creative nature of failure, mistakes, misunderstanding etc.

5. *The appreciation of mathematics as a cultural-human endeavour* by letting both students and teachers appreciate (a) the fact that mathematics evolves under the influence of **both social and cultural factors** and by **intrinsic ones** like aesthetics, curiosity, challenge, recreational purposes etc and (b) mathematics as part of the cultural heritage of particular civilizations and societies, and the role it played in this context.

A closer inspection and analysis of the above arguments suggests that there are three **different but complementary general directions** and **emphases** for introducing the historical dimension in ME:

(a) To **learn history** by providing **direct historical information**.

(b) To **learn mathematical topics** by following a **teaching approach inspired by history**.

(c) To develop what may be called **mathematical awareness** (i) by learning **about mathematics** and (ii) by highlighting **social and cultural aspects of mathematics in a historical perspective**.

These general ways for introducing the historical dimension in ME can be implemented in practice in a variety of ways, from employing original sources, worksheets, research projects etc, to using theater plays, movies, the Internet etc (e.g. see section 6 for an outline of such an implementation of (b)). However, we are not going to describe them here. Instead, we would like to point out that both the arguments presented in this section and (a)-(c) above, can equally well be valid in PE. This is not accidental but in our opinion it is related to two epistemological / historical theses, which form the subject of the next section.

3 The relation between mathematics and physics

The following two theses form the central core of the present paper and partly explain why the historical dimension plays a similar role and has a similar nature in ME and in PE.

Thesis A: Mathematics and physics have always been closely interwoven, in the sense of a "two-ways process":

[2]It summarizes part of the work done by the group, which was responsible for writing chapter 4.1 of the above mentioned ICMI Study Volume on *An analytical survey of the possible ways of integrating History of Mathematics in the classroom*; see acknowledgements here.

[3]This is a summary of the detailed analysis provided in Ch. 4.1, Sections 2 and 3 in FAUVEL et al. (to appear); see footnote 2 and acknowledgements here.

- Mathematical methods **are used** in physics
- Physical concepts, arguments and modes of thinking **are used** in mathematics (for this fact seen in a somewhat different perspective see TZANAKIS 1996, TZANAKIS 2000).

Apparently, this thesis seems more easily acceptable than thesis **B** below. Nevertheless, the term "use" will be clarified and deepened after stating thesis **B**, in a way that makes thesis **A** to appear less naive and readily acceptable.

Thesis **B**: Any distinction between mathematics and physics, seen as general attitudes towards the description and understanding of an object[4], is related **more** to the point of view adopted while studying particular aspects of this object, than to the object itself.

In sections 4 and 5 we will provide some evidence for theses **A** and **B**, by commenting on some historical examples. However, if these theses are accepted, then the following conclusions can be drawn:

- Any treatment of the HM independent of the history of physics (HP) is necessarily incomplete (and vice versa).
- By accepting the importance of the historical dimension in education (for the reasons given in section 2), the relation between mathematics and physics should not be ignored in teaching these disciplines.

Before illustrating theses **A** and **B** by means of examples, we will elaborate more on thesis A. Conventionally thesis **A** is interpreted as follows:

(1) *From mathematics to physics*: Mathematics is simply the **language** of physics.

(2) *From physics to mathematics*: (i) Physics is an **exterior to mathematics, huge reservoir of problems** to be solved mathematically; (ii) Physics is simply a **domain of application** of already **existing** mathematical **tools**.

Though both (1) and (2) are true, they do not exhaust the multifarious interconnection of the two disciplines and they need refinement in the following sense:

For (1): Mathematics is not only the "language"of physics (i.e. the tool for expressing, handling and developing logically physical concepts and theories), but also, it often **determines** to a large extent the **content** and **meaning** of physical concepts and theories themselves. It is in this broader sense that the term "**mathematical physics**" is used in this paper.

For (2): Physics provides, not only problems "ready-to-be-solved"mathematically, but also **ideas, methods** and **concepts** that are crucial for the **creation** and **development** of **new** mathematical concepts, methods, theories, or even whole mathematical domains. It is in this broader sense that the term "**physical mathematics**" is used in this paper.

In the next two sections we provide some evidence for the above by means of two groups of examples, one illustrating mathematical physics and the other one physical mathematics. At the same time, some of them provide evidence for thesis **B** as well. However, it should be

[4]By this term we mean not only concrete, empirically conceived objects, but also mental objects like concepts, questions, problems etc.

emphasized that lack of space makes our presentation sketchy, hence incomplete, and more details can be found in the litterature.

4 Examples : Mathematical Physics

The first two examples illustrate the way in which some strictly mathematical development can lead to the introduction and the specification of meaning of an important physical concept.

1. *The concept of antimatter*: After the invention of quantum mechanics, Dirac tried to develop a relativistic theory of the electron. Based partly on mathematical criteria of symmetry, he arrived in 1928 at the relativistic equation now bearing his name, by an essentially mathematical approach. However, the trouble with this equation, was that it admits solutions with negative energy for the electron, a physically unacceptable result. Instead of rejecting his equation on the basis of this physically absurd result, Dirac proposed in 1931 that these negative energy solutions should be retained as describing, not electrons, but "antielectrons" (or positrons as they are now called), a different kind of particles with energy and charge opposite to those of ordinary electrons, with which they are mutually annihilated when they interact. Originally, Dirac believed that these negative energy solutions would correspond to protons, but finally he changed his mind on the basis of objections mainly of a mathematical nature that had been raised by Weyl (KRAGH 1990 pp. 102-103). In this way the bold new concept of antimatter was introduced into physics by interpreting the result of a strictly mathematical deduction. Apparently this could not have been done otherwise if one wanted to avoid the rejection of Dirac's theory (for a detailed historical account see SCHWEBER 1994 section 1.6, KRAGH 1990 pp. 57-59, 87-103).

2. *The wave nature of matter*: In 1900, Planck introduced his quantum hypothesis for the energy E of light as a function of its frequency ν, $E = h\nu$ (h being Planck's constant)[5]. In 1908, three years after the original formulation of special relativity (SR), it became clear through Minkowski's work that the momentum p and the energy E of a particle are **different coordinates** of the **same** four dimensional (4D) vector in space time (p, E) and the same is true respectively for the wave number k and the frequency ν of any wave (k, ν) (PAULI 1981 sections 29, 37, PAIS 1982 section III.7(c), PIERSEAUX 1999, section 2.IV.4-1)[6]. In 1924, de Broglie observed that given the analogy between geometrical optics and classical mechanics (see example 5 below), if one wants to accept both Planck's relation and the above consequences of SR, then one is **mathematically** led to the relation $p = hk$ which clearly suggests that not only the light, but also any kind of particles has (is associated to) a wave nature (wave phenomenon) (DE BROGLIE 1925, ch.II section V). This idea, on the one hand stimulated Schrödinger's effort towards the formulation of wave mechanics and on the other hand was confirmed experimentally a few years later (SCHRÖDINGER 1982 p. 20, JAMMER 1965 pp. 257-258, KRAGH 1982 pp. 155-157).

The next two examples illustrate how the introduction of a new mathematical concept may accelerate the development of a physical theory, or, conversely its absence may prevent its development. Both examples provide evidence for thesis **B** as well.

[5]Strictly speaking, originally Planck's relation concerned the exchange of energy between matter and light. It was Einstein who, in 1905, extended this relation to the light itself

[6]This is already implicity contained in Einstein's original paper (reprinted in SOMMERFELD 1952 paper III, p. 56) and can be inferred as long as the geometric formulation of SR based on the concept of spacetime is used. This is clear in de Broglie's work (DE BROGLIE 1925 ch.II section IV).

3. *The concept of spacetime and the theory of relativity*: It is known that SR is based on the so-called "Lorentz coordinate transformations" (LT) between inertial coordinate systems moving relative to each other at constant velocity. Originally they have been derived in a **physically oriented way** (*thesis* **B**) by Lorentz (1904) as those transformations leaving invariant Maxwell's electrodynamic equations (SOMMERFELD 1952, paper II). Einstein's 1905 derivation was also of this nature, but it was based on an epistemological analysis of the intuitive concept of simultaneity (SOMMERFELD 1952, paper III). However, in 1908, Minkowski followed a more **mathematically oriented approach** (*thesis* **B**). He introduced the crucial **geometric** concept of spacetime as a 4D manifold with a particular type of (pseudo)distance for any two of its points. He then derived the LT as those transformations that leave invariant this (pseudo)distance[7] (SOMMERFELD 1952, paper V; for a didactically appropriate reconstruction with more historical comments and references to the original litterature, see TZANAKIS 1999). This was a crucial step which accelerated the development of SR, by unfolding the geometrical ideas hidden in Einstein's original paper. Nevertheless, one could imagine the development of SR **without** the concept of spacetime. However, and this is the second point to be made here, it is completely impossible to imagine the development of the general theory of relativity (GR) without this concept. The reason is simple: without it, riemannian geometry and tensor calculus could not have been used as the absolutely indispensable tool for the formulation of Einstein's physical ideas about gravitation, which led to GR. This last point is even more clearly illustrated by the next example.

4. *The concept of a singularity in spacetime*: We all have heard about the existence of a singularity in the original "big-bang" of the universe, or inside blackholes. Originally, a singularity was conceived in a rather **intuitive physical way** (*thesis* **B**), as a point (or region) of spacetime in which (some) geometrical and physical quantities become infinite (see e.g. WHEELER 1964 p. 317, HARRISON et al. 1965, ch.11, p. 141). This idea does not permit to understand whether the already (theoretically) known existence of singularities in particular cases, is an accidental fact, or is an intrinsic feature of GR (HAWKING et al 1973 pp. 261-262, JOSHI 1993 pp. 157-163).

On the other hand, in the 1960's a different **mathematical approach** (*thesis* **B**) was initiated by Penrose and developped by him, Hawking and Geroch. It was based on the **quite familiar idea of singularity in riemannian geometry** as a limit point of a curve that does not belong to the manifold (see e.g. CLARKE 1993 section 1.2. for a rigorous definition)[8]. This simple, but radically different idea was the necessary crucial step for the formulation and proof of the famous "singularity theorems" in GR by Penrose, Hawking and Geroch (1965-1970), which showed that the existence of singularities is essentially an intrinsic characteristic of GR (HAWKING et al. 1973 ch.8). At the same time, we have here a fruitful **feedback** to Mathematics, namely the development of (pseudo)riemannian geometry as an independent mathematical discipline with prototype example the geometry of spacetime, which remains however, always closely connected to its physical applications (good examples of such monographs are those by O'NEILL 1983 and BEEM et al. 1981).

[7]The prefix "pseudo" stems from the fact that this spacetime distance may be positive, zero or negative for noncoinciding points.

[8]E.g. a spherical surface in which a point has been removed has a singularity in this sense; that is, it has curves which are **incomplete** in the way described above. In riemannian geometry, the opposite concept of a **complete** curve (i.e. loosely speaking, a curve containing its limits points) was quite familiar (see e.g. the classical treatises by HICKS 1971 and HELGASON 1962 and original references therein).

5. *The invention of wave mechanics*: Schrödinger in 1926 laid the foundations of his wave mechanics, based on the analogy between classical mechanics and geometrical optics (SCHRÖDINGER 1982, paper II and lecture I). This analogy was known long before. In the period 1833-1835, based on this analogy, Hamilton developped a unified mathematical approach to the description of these two theories, in which they appear as different but isomorphic structures. Schrödinger's crucial argument was based on the remark that we know that geometrical optics is only an approximation to the exact wave optics. Therefore, we may look for a new (wave) mechanics, such that classical mechanics is an approximation to it, in such a way that the above mentioned isomorphism is preserved. This was sufficient for arriving mathematically at the formulation the partial differential equation (PDE) now bearing his name and which is the cornerstone of wave mechanics (for a reconstruction of Schrödinger's approach stressing the role of analogy as a pattern of discovery, see TZANAKIS 1998, see also TZANAKIS et al. 1988 and references to the original litterature therein).

This example also illustrates thesis **B** at the teaching level: In a **mathematically** oriented treatment, Hamilton's approach may be considered as a method for solving a first order PDE, the so-called Jacobi method, or its equivalent, the solution of such an equation by using its system of characteristic ordinary differential equations. In a **physically** oriented treatment it can be seen as a starting point for developing analytical mechanics and more precisely, the Hamilton-Jacobi theory of solving a mechanical problem, or its equivalent, solving it with the aid of the corresponding system of Hamilton's canonical equations[9] (TZANAKIS, 2000, section 3.3).

5 Examples : Physical Mathematics

The first two examples illustrate the fact that a **physical** concept or idea may act as a (partial) motivation for the emergence of important **mathematical** concepts.

1. *Velocity and the derivative concept*: In its modern form, the velocity concept was the product of a long and complicated evolution over almost three centuries and it was partially formulated by Galileo, who discussed only uniformly accelerated motion (BOYER 1959 pp. 72-73, 82-83, 113-114, DUGAS 1988 pp. 57, 59-61, 66-67, WHITROW 1980 pp. 181, 183-184). It is known that this fact influenced the emergence of the concept of instantaneous velocity (BOYER 1959 pp. 177, 180), which in turn acted as a basic motivation for the formulation of the derivative concept (HALL 1983 pp. 288-289). This fact can be illustrated by a short extract from Newton's "Principia". After Newton introduces his conception of the derivative as "an ultimate ratio of evanescent quantities", he tries to refute possible objections to it by writing:

> Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But **by the same argument** it may be alleged that a body arriving at a certain place, and there stopping, has no ultimate velocity; because the velocity, before the body comes to a place, is not the ultimate velocity; when it has arrived, is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives... **And in like manner**, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish. (NEWTON 1934, p. 39, our emphasis)

It is clear that Newton tried to legitimize the concept of the derivative on the basis of the physical

[9]Actually, the latter is the system of characteristic ordinary differential equations for the Hamilton-Jacobi equation which is the basis of the Hamilton-Jacobi theory.

concept of instantaneous velocity which he considered intuitively more clear (and apparently he never defined it exactly; BOYER 1959 pp. 193-194), which thus appears as the prototype example of a derivative.

2. *Dirac's δ - function and generalized functions*: A similar, more recent example is provided by Dirac's δ-function in quantum mechanics introduced in 1927 (KRAGH 1990 pp. 40-41). In his famous book on the foundations of quantum mechanics, first published in 1930 (DIRAC 1958), he introduces the δ-function

$$\delta(x) = 0 \text{ for } x \neq 0, \int \delta(x)dx = 1.$$

It is easily seen that strictly speaking, this object has no mathematical meaning; actually only an approximate physical picture of it as a highly peaked graph can be given. Dirac was well aware of this fact when he wrote that

> ... $\delta(x)$ **is not a function** of x according to the usual mathematical definition of a function, which requires a function to have a definite value for each point in its domain, but **is something more general**, which may be called an 'improper function'... Thus $\delta(x)$ is not a quantity which can be generally used ... as an ordinary function, but **its use must be confined** to certain simple types of expression for which it is obvious that **no inconsistency can arise**... although an improper function does not itself have a well-defined value, **when it occurs as a factor in an integrand the integral has a well-defined value.** (DIRAC 1958, p. 59, our emphasis)

This physical picture, the operational effectiveness of $\delta(x)$ in quantum mechanical calculations and its mathematically self-contradictory nature, acted as a partial motivation for the emergence of the concept of generalized function introduced originally by Sobolev in 1936 and more systematically by L. Schwarz from 1945 onwards, as a functional (i.e. a function) acting on an appropriate space of functions (KRAGH 1990 p. 41, BOYER 1968 p. 671, DIEUDONNÉ 1981 pp. 225-226). Actually, the last sentence of the above quotation implicitly expresses the idea of the δ-function as a functional.

Finally, this example provides evidence for thesis **B**: In the above quotation, Dirac adopts a **physical** attitude by admitting that although this concept is mathematically unacceptable, still he uses it since it is operationally effective[10]. On the other hand, a **mathematical** approach to it, was to accept that such a concept may be useful in applications and possibly in pure Mathematics, therefore one should try to make it a logically consistent concept.

3. *Brownian motion and stochastic differential equations*: In a similar way, the study of Brownian motion was the main motivation for the development of a whole mathematical domain, namely the theory of stochastic differential equations.

Brownian motion is the irregular motion done by a heavy particle suspended in a fluid, due to its random collisions with the (much lighter) molecules of the fluid. Though it was first observed by Brown in 1828, it was not until 1908 that the first **mechanical** model was proposed by Langevin (LANGEVIN 1908), who wrote for the velocity v of the Brownian particle as a function of time t, the equation

$$\frac{dv}{dt} = -\beta v + F(t), \beta = \text{constant}$$

[10]For more details on this attitude of Dirac towards Mathematics, see KRAGH 1990 pp. 280-281.

The novel feature here is the nature of the force $F(t)$ due to the collisions of the particle with the molecules of the fluid. Since the motions of the latter are random, only the **average** properties of F could be postulated upon physical considerations. This was an important equation since it was the first **mechanical** model which allowed for the experimental verification of the molecular structure of matter[11]. On the other hand, Wiener (WIENER 1923) proved that, although the velocity $v(t)$ of the Brownian particle seen as a stochastic process (due to the random nature of $F(t)$), is continuous (and of unbounded variation) with probability one, it is **nowhere** differentiable with probability one. Hence, Langevin's equation has no meaning as an ordinary differential equation. This contradiction, together with the average properties of $F(t)$, postulated on physical grounds, were the main input for the emergence of the concept of a **stochastic integral** introduced by Itô in 1951 (see ARNOLD 1974, Introduction and references therein). This is the cornerstone for the development of the theory of stochastic integration and of stochastic differential equations. In this context, Langevin's equation is no longer an unacceptable object, but acquires a meaning as an equation of this kind.

4. *The development of vector analysis in the 19th century*: Often an intuitive physical method for tackling some problems quantitatively, may lead to the development of new mathematical methods and theories. This is the case of Bernoulli's "brachistochrone problem" as a main motivation for the development of the calculus of variations. This example will be discussed from a somewhat different perspective in the next section. Below, we will consider vector analysis as another such example, in which the new mathematical concepts and methods were developped in their final form mainly by physicists. Here we have a very complicated interaction between mathematics and physics, hence the discussion that follows is necessarily sketchy, mainly confined on Maxwell's contribution.

By the mid 19th century there were important physical investigations containing deep mathematical insights that form part of the foundations of modern vector analysis; Green's essay on "The applications of Mathematical Analysis to the theories of electricity and magnetism" (1828) containing his and Gauss' theorems. W. Thompson's early work on analogies between electric phenomena with heat conduction and elasticity (1846-47; WHITTAKER 1951, pp. 241-242) and Stokes' Smith Prize Essay of 1854 in which the theorem bearing now his name is contained (for more details on the history of these theorems, see CROWE 1985, note 29 pp. 146-147). The general significance of these theorems, as well as the importance of vector methods, were explicitly acknowledged by Maxwell in his classical "Treatise on Electricity and Magnetism" published in 1873 (MAXWELL 1954, vol. I, sections 16, 21, 24, 95b, see also below). Actually, Maxwell was well aware of this fact when in 1871 he wrote in a more general context:

> ... when the student has become acquainted with several different sciences [i.e. domains or theories in physics], he finds that the **mathematical processes and trains of reasoning on one science resemble those in another** so much that his **knowledge of the one science may be made a most useful help in the study of the other.**
> When he examines the reason of this, he finds that in the two sciences he has been dealing with systems of quantities, in which the **mathematical form of the relations** between the quantities **are**

[11]Einstein's paper of 1905 (EINSTEIN 1956, paper I) was the first theoretical work on which conclusive experiments on the existence of molecules could be based, like those of Perrin in 1908 (PERRIN 1991, ch.IV). However, Einstein's theory was not a genuine mechanical model. Such a model was provided by Langevin (see the equation above) who rederived Einstein's basic result. His model was further elaborated by others and especially by Ornstein and Uhlenbeck (UHLENBECK et al. 1930) and played an important role in the development of the theory of stochastic processes and of stochastic differential equations (for a brief historical survey see BLANCHARD et al. 1987 section I.1a; cf. NELSON 1967 sections 3, 4, 9).

the same in both systems, though the physical nature of the quantities may be utterly different. (quoted in CROWE 1985, p. 130, our emphasis)

In this quotation it is evident that Maxwell stresses the importance for both mathematics and physics of the determination of isomorphic mathematical structures, helpful for developing mathematical methods on the basis of which problems in different domains can be tackled in the same way. He becomes more explicit in 1872 when he stressed the importance in this context of the "calculus of quaternions" developped mainly by Hamilton and Tait:

> A most important distinction was drawn by Hamilton when he divided the quantities with which he had to do into Scalar quantities... and Vectors... The invention of the calculus of Quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance, with the invention of triple coordinates by Descartes. **The ideas** of this calculus, as **distinguished from its operations and symbols**, are fitted to be of the greatest use in all parts of science. (quoted in CROWE 1985, p. 131, our emphasis)

These qualitative remarks were transformed into exact mathematics in his "Treatise", in which on the basis of the calculus of quaternions he stresses the importance of the basic operators *grad, div, curl* of modern vector analysis (originally introduced by Tait; MAXWELL 1954, section 25) and revealed the **general significance** of its basic theorems, already known in special cases (theorems of Stokes, Gauss and Green). In fact, according to Maxwell "... the doctrine of Vectors... is a method of **thinking** and not a method for saving thought..." (quoted in CROWE 1985, p. 133, our emphasis). Thus in his view, "... by means of the vectorial approach, the physicist attains to a direct mathematical representation of physical entities and is thus aided in seeing the physics involved into the mathematics" (CROWE 1985, p. 134). In fact, in the preface to his "Treatise", he expresses very clearly the role of physical insight for appreciating the significance of mathematical results:

> I also found that several of the most fertile methods of research discovered by mathematicians could be expressed **much better** in terms of ideas derived from Faraday than in their original form... Hence many mathematical discoveries of Laplace, Poisson, Green and Gauss **find their proper place** in this treatise and **their appropriate expressions in terms of conceptions mainly derived from Faraday**. (MAXWELL 1954, pp. ix-x, our emphasis)

It is through such deep insights into the mathematical structure of physical theories, together with an outline of vector methods and concepts contained in his "Treatise" that modern vector analysis emerged and was established in the hands of physicists like Gibbs (1881-1884) and Heaviside (from 1883 onwards) (CROWE 1985 pp. 138-139).

5. *Quantum mechanics and functional analysis*: As the last example we mention the role of quantum mechanics (QM) in stimulating the development of functional analysis. The subject is vast and here we only mention the role of physics in the emergence of the abstract concept of a Hilbert space.

In 1925, originally Heisenberg and later Heisenberg, Born and Jordan developped matrix mechanics as a new theory of atomic phenomena, based on matrix algebra, a subject unfamiliar to physicists at that time (VAN DER WAERDEN 1967, papers 12, 13, 15). In 1926 Schrödinger founded wave mechanics (cf. section 4.5) based on the familiar theory of PDE. The main mathematical problem of the two theories was respectively, the diagonalization of a certain (often infinite dimensional) matrix, the hamiltonian matrix and the solution of Schrödinger's equation (see e.g. HEISENBERG 1949, Appendix). The strange thing was that these two **conceptually**

totally **different** theories, gave **identical** results, compatible with experiments. Hence the question of finding their relation naturally arose. It was tackled by both Schrödinger (1926) and von Neumann (1927-1932) in a different way that provides some support for thesis **B**:

Schrödinger provided a **formal** proof that, by choosing a basis for the wave functions he was using, solving his PDE becomes a matrix eigenvalue problem identical to that of matrix mechanics and vice versa (SCHRÖDINGER 1982, paper 4).

Von Neumann's approach was of a rather different character (VON NEUMANN 1947, ch.I, particularly section 4 and p. 19). He tried to identify the basic properties of the objects with which the two theories were dealing of and in this way he was led to define axiomatically what became known as a separable Hilbert space (VON NEUMANN 1947, ch.II, STONE 1932, p. 2)[12]. Then he proved that all such spaces are isomorphic, thus giving a definite answer to the question above: the two conceptually different theories were just different representations of the same abstract mathematical structur that forms the mathematical substratum of the formalism of QM (VON NEUMANN 1947, ch.II, theorem 9 and pp. 41-42). For an outline of a possible didactical sequence, see TZANAKIS 2000, section 3.4).

6 Implementing the relation between mathematics and physics in teaching : an example

The examples presented in sections 4 and 5 provide enough evidence for the deep interplay between mathematics and physics. Therefore, in this section we will describe how their close relation could be implemented in practice, by analyzing an example on the basis of an approach inspired by history (see (b) in section 2). To this end, the following general scheme will be employed (TZANAKIS, 2000, section 1, TZANAKIS 1996 section 1 and in more detail, FAUVEL et al., 2000, ch.7, section 3.2 - cf. acknowledgements here):

(a) The teacher has a **basic** knowledge of the historical evolution of the subject.

(b) On the basis of this knowledge he identifies the **crucial steps** of this evolution (key ideas, questions and problems that stimulated it, difficulties and errors that have been faced and possibly overcome etc).

(c) These crucial steps, are **reconstructed**, probably using modern terminology and notation, so that they become didactically appropriate.

(d) To keep the presentation to a reasonable size, many **details** in (c) can be given as sequences of historically motivated exercises of an increasing level of difficulty, such that each one presupposes (some of) the preceding ones.

As an example we outline below a **possible** teaching sequence for Bernoulli's "brachistochrone problem" mentioned in section 5, as a subject for making practice in differential calculus at the high school level or early undergraduate level (TZANAKIS et al. 2000; for a similar approach to this subject see CHABERT 1993). Following the above mentioned general scheme, we have:
The basic historical steps (for (a) and (b))

[12] The term "Hilbert space" was used earlier to denote only the space ℓ^2 of complex sequences of numbers having a finite sum of the squares of their norms (DIEUDONNÉ 1981, p. 172, VON NEUMANN 1949, p. 23).

(i) Hero's proof of the law of reflection on the basis of the assumption that light moves between two points by following the shortest path. As a geometrical extremum problem the proof is elementary and well known, see figure 1 (THOMAS 1941, p. 496-499).

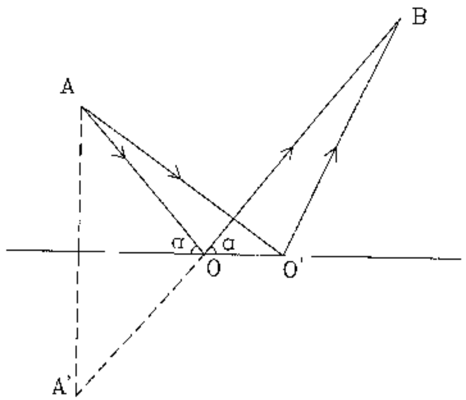

FIGURE 1

(ii) Fermat's derivation of the law of refraction (1662), already formulated empirically by Harriot (1601), Snel (1621) and Descartes (1637) (HALL 1983 p. 197), on the basis of his principle of Least Time "... Nature always acts in the shortest ways" which he interpreted in the present context as follows (see figure 2): Light goes from a point A in which it has speed v_1 to a point B in which it has speed v_2 by following the shortest path (DUGAS 1988, p. 254; here $v_1 > v_2$).

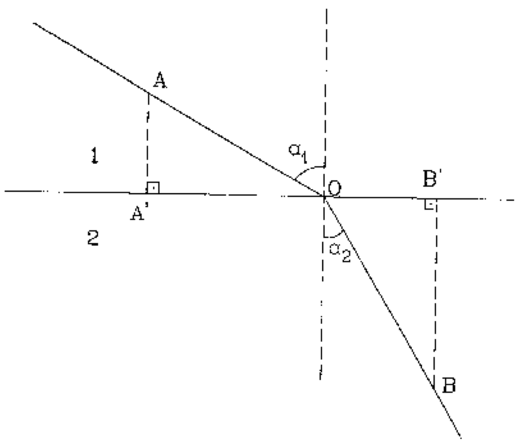

FIGURE 2

In this way he specified point O by deriving the relation (DUGAS 1988, part III, section V.1).

$$A'O/B'O = v_1/v_2 = \sin a_1/\sin a_2 \text{ if } AO = BO \tag{1}$$

(iii) In 1696, Johann Bernoulli formulated the "brachistochrone problem": "To find the trajectory of a point, which starting from a given point A and moving on a vertical plane under its weight only, arrives at a given point B in the least time" (HAIRER et al. 1996, pp. 136-137). His solution was given in **analogy** to Fermat's approach in (ii), by dividing the vertical distance between A and B into thin horizontal layers in which the velocity of the particle could be considered approximately constant. In this way an equation similar to (1) is valid in each layer. By passing to the limit of a vanishing width of the layers, he derived and solved an equation for the unknown curve which turns out to be the cycloid.

A teaching sequence (for (c) and (d))

(i) One may introduce coordinates in figure 2 and express AOB **analytically** as a function of one of them, e.g. of $OA'(AO \neq BO$ in general). Requiring the time to be a minimum, leads to the vanishing of the derivative of this function, which gives the second of equations (1), i.e. the law of refraction in its usual form $\sin a_1/v_1 = \sin a_2/v_2$.

(ii) One may give as an exercise (possibly in several steps) a derivation along these lines of the much simpler law of reflection.

(iii) The idea in (i) may be used to **reconstruct** Bernoulli's solution of the brachistochrone in the form of a differential equation for the unknown curve $y(x)$. Equation (1) and the law of the conservation of the energy of the particle, give (SIMMONS 1974, section 1.6)

$$y(1 + y'^2) = 2c = \text{constant} \tag{2}$$

(iv) That the cycloid, given in parametric form as

$$x = c(\theta - \sin\theta),\ y = c(1 - \cos\theta)$$

satisfies this equation is a simple exercise.

(v) A somewhat more advanced related subject is to use Newton's dynamic law and the differentiation rules (especially the chain rule), to derive the equation of motion of a point particle which is constraint to move along a cycloid under its own weight only. The result is (g = the acceleration of gravity)

$$\frac{d^2}{dt^2}(\sin\frac{\theta}{2}) = -\frac{g}{4c}\sin\frac{\theta}{2} \tag{3}$$

This is the equation of motion of a **simple** pendulum ($\sin\theta/2$ denoting the amplitude of its oscillation; $\sin\theta/2$ is proportional to the arc length of the cycloid). Eq(3) is an important result since, on the one hand it describes a strictly isochronous oscillation (i.e, its period is independent of its amplitude) and on the other hand it is only an **approximation** for the **simple** pendulum, but an **exact** result for cycloidal motion. This was the basis of Huygens' construction (1673) of the first isochronous clock based on the motion along a cycloid (cycloidal pendulum; SOMMERFELD 1964 section 17).

(vi) At the university level, one may use (iii) above to express the time as an **integral** involving y and y'. In analogy with the differential of a function in the Calculus, one may introduce the concept of the variation of an integral (functional), which should vanish for an extremal curve. In this way equation (2) is obtained again and the problem is solved. Thus, one may appreciate clearly the generality of this approach, which supersedes the particular problem of the brachistochrone and indicates the path for a systematic introduction to the calculus of variations (see e.g. CHABERT 1993).

In this paper an effort was made to support the claim that there is a continuous in time, fruitful deep interplay between mathematics and physics, which should be conceived as different, but complementary views of the same world (mentally or empirically conceived). Therefore, this interplay and complementary character should not be neglected in teaching and learning these disciplines; on the contrary, both ME and PE can profit from it, possibly taking into account aspects of the historical evolution of this interplay.

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Exposés
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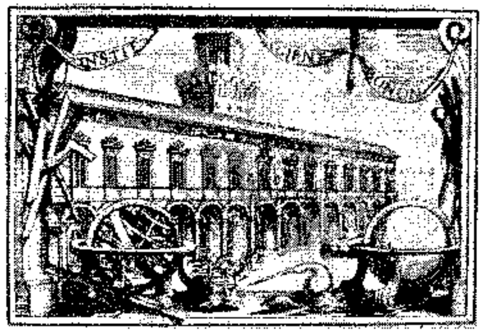

La formation quantitative des économistes à la lumière de l'évolution des rapports entre les mathématiques et l'économie

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Abstract

Il est indéniable que les sciences économiques exploitent de plus en plus l'outil mathématique. En conséquence, à l'université, les futurs économistes doivent suivre des programmes toujours plus lourds en mathématiques.

Nous nous efforcerons de décrire, de comprendre et de justifier cet accroissement progressif des mathématiques dans la formation des économistes en analysant quelque peu l'évolution temporelle des rapports entre ces deux disciplines.

Une attention toute particulière sera portée sur l'enseignement de la finance.

Billet de 20 francs belges avec le portrait de Rubens.

1 L'importance des mathématiques en économie

1. 1 Bref historique

Alors que les mathématiques interviennent depuis longtemps, de façon très naturelle et indiscutable dans les sciences dites "dures", à savoir la physique, la chimie, la biologie, l'art de l'ingénieur, ..., elles sont apparues très tardivement, de manière moins bien établie dans les sciences humaines, spécialement en économie. Cela résulte de la plus grande complexité des situations rencontrées : il est, par exemple, plus difficile d'étudier le comportement d'un groupe d'individus aux intérêts souvent divergents plutôt que le mouvement d'un corps soumis à des forces bien connues. Les multiples paramètres et facteurs d'influence intervenant dans les problèmes économiques n'ont pu être traités efficacement que grâce au développement du calcul matriciel et à l'avènement des ordinateurs qui permirent de traiter de nombreuses informations et d'effectuer des calculs sur un nombre considérable de grandeurs. Certains auteurs avancent, pour expliquer ce retard de l'intervention des mathématiques en économie, une seconde raison d'ordre psychologique : *Les économistes ont longtemps souffert d'un véritable complexe d'infériorité. A voir les splendides succès qu'obtenaient, dans le bâtiment voisin, leurs collègues physiciens, qu'il s'agît de théorie planétaire ou d'électromagnétisme, ils se sentaient jaloux et se disaient qu'il faudrait attendre leur Newton* [DAVIS & HERSH, p. 83].

Les premiers théoriciens de l'économie politique moderne, avec à leur tête l'anglais D. Ricardo (1772-1823), n'employaient aucune technique scientifique; ils faisaient, selon J. R. Hicks (prix Nobel d'Economie en 1972), des *mathématiques dans la coulisse*, c'est-à-dire sans le savoir.

Le philosophe et mathématicien français A. Cournot (1801-1877) apparaît aujourd'hui comme le véritable fondateur de l'économie mathématique : il est notamment l'auteur d'une *théorie mathématique des richesses* (1838). Ses mérites furent toutefois reconnus de façon posthume.

L'italien Pareto (1848-1923) mit en évidence l'importance de la notion de fonction en économie en déclarant : *les économistes littéraires perdent leur temps à chercher des relations de cause à effet là où il n'y a que des relations fonctionnelles réversibles entre les données qui se conditionnent mutuellement.* Pratiquement au même moment naissait, en 1871, la théorie du "marginalisme" développée séparément par le français Walras (1834-1910) et l'austro-anglais Jevons (1835-1882) : on y mettait en évidence l'importance de la dernière unité dans l'influence de chaque variable, ce qui ouvrait la porte à l'utilisation du calcul infinitésimal en économie.

Il est intéressant de constater que les grands économistes de la fin du 19ème siècle et du début de ce siècle ont, très souvent, *utilisé des formes de mathématiques pour trouver le résultat (algèbre, graphiques ou exemples numériques), soit qu'on le sache (Cournot, Marx (1818-1833), ...), soit qu'on le devine (Keynes (1883-1946), Walras, ...), mais ils ont publié leur résultat "en littérature" pour ne pas rebuter des lecteurs. Dans tous ces cas aussi, avancer plus loin dans la théorie exige la mathématisation* [KOLM 1986].

Au cours du vingtième siècle, les économistes se rendent compte de la puissance de la démarche scientifique et vont même jusqu'à créer des théories mathématiques nouvelles pour résoudre les problèmes spécifiques.

Les méthodes d'optimisation de Kuhn-Tucker et les théories modernes de programmation non linéaire ont permis de résoudre des problèmes fondamentaux que se pose l'économiste, comme la recherche de l'achat optimal que peut réaliser un consommateur avec un budget déterminé.

L'économétrie, apparue vers 1930 comme discipline autonome, recourt à la fois à la théorie économique, à la formulation mathématique et à l'analyse statistique. En 1969, le premier prix

Nobel d'Economie fut décerné à Frisch et Tinbergen qui peuvent être considérés comme les fondateurs de l'économétrie.

Par la suite, cette suprême distinction fut fréquemment attribuée à des mathématiciens reconvertis à l'économie et à des économistes très quantitatifs :

- Samuelson (1970), célèbre pour ses travaux fondamentaux en économie et l'introduction du *calculus* et du calcul de probabilité dans la résolution de problèmes économiques.
- Arrow (1972), connu pour son fameux théorème d'impossibilité dans la théorie du choix collectif ainsi que pour ses contributions dans les théories des formes quadratiques sous contraintes et des fonctions quasi-concaves,
- Leontief (1973), créateur de l'analyse input-output,
- Kantorovitch (1975), pionnier de la programmation linéaire,
- Klein (1980), auteur de contributions importantes en économétrie,
- Debreu (1983), mathématicien de formation et auteur d'une célèbre *théorie de la valeur : analyse axiomatique de l'équilibre économique* (1966),
- Allais (1988), ingénieur de formation qui plaide pour un emploi judicieux de l'outil mathématique en économie,
- Markovitz, Sharpe et Miller (1990) pour leurs travaux novateurs sur la théorie économique financière et le financement des entreprises,
- Nash, Harsanyi et Selten (1994) pour leur contribution fondamentale à la théorie des jeux non-coopératifs,
- Merton, Scholes et Black (1997), célèbres pour leurs modélisations stochastiques en finance,
- Sen (1998), réputé en théorie du choix social ayant pour objet l'analyse des relations entre les préférences individuelles et les décisions collectives.

1.2 Mathématiques exploitées par les économistes au cours du temps

Très schématiquement, on pourrait dire que, jusqu'à la fin du siècle dernier, l'économie était essentiellement "littéraire".

Lors de la première moitié du vingtième siècle, l'économiste a recouru à la méthode graphique.

Le troisième quart de ce siècle a vu les théories mathématiques classiques, telles l'analyse et la statistique, être de plus en plus exploitées par les économistes.

Cette fin de siècle peut être caractérisée par l'accroissement considérable de l'utilisation des mathématiques en économie. De nombreuses théories, parfois sophistiquées, dont certaines ont été créées *ex nihilo* pour résoudre des problèmes rencontrés dans l'univers économique, se retrouvent dans les publications spécialisées en économie. Signalons, en guise d'exemples, la théorie moderne de la programmation mathématique, la théorie des modèles non linéaires pouvant déboucher sur le chaos déterministe, les modèles stochastiques faisant notamment appel au mouvement brownien standard, à l'intégrale d'Ito ou encore à la notion de martingale, ...

Depuis plusieurs décennies, il faut bien se convaincre que *le choix véritable n'est pas entre l'emploi ou le non-emploi de l'outil mathématique, mais entre une utilisation consciente et rationnelle et une utilisation inconsciente et désordonnée de cet outil* [ALLAIS 1954]. D'ailleurs, pour un économiste contemporain, *il est devenu difficile de faire publier par les grandes revues internationales un article se voulant avant tout théorique (même s'il comporte aussi une partie empirique) mais utilisant exclusivement des raisonnements littéraires, des mathématiques élémentaires et des figures géométriques* [SALMON 1995].

1.3 Le cas particulier de la finance

Bien que la théorie financière fasse partie intégrante de l'économie, ses rapports avec les mathématiques sont importants et assez particuliers. En effet, ils datent de près de 10000 ans : dès que l'homme s'est sédentarisé, il a senti le besoin de faire des mathématiques pour ses échanges commerciaux; ainsi, plusieurs siècles avant notre ère, les Phéniciens inventèrent le système de numérotation pour s'en servir dans leur commerce. Toutefois, l'arithmétique commerciale a été développée au 15ème siècle par les négotiants italiens qui pouvaient exploiter l'algèbre élémentaire classique naissante, notamment grâce à la découverte des nombres négatifs par N. Chuquet (1484).

L'invention des logarithmes par J. Napier (1550-1617) a permis la naissance de la théorie de l'intérêt composé; il a néanmoins fallu attendre l'emploi courant des ordinateurs pour que le législateur belge propose, en 1992, une méthode d'analyse numérique permettant le calcul du taux réel, appelé le TAEG (taux annuel effectif global), d'un achat à tempérament. Jusque vers les années 1970, *la finance proprement dite -techniques boursières et d'actuariat- ne nécessitait guère plus que la table de logarithmes* (BOULEAU, p. 350]. Les praticiens font appel à l'intérêt simple pour des contrats à court terme, et à l'intérêt composé, ou mieux encore à l'intérêt mixte, pour du plus long terme; ils n'exploitent que rarement l'intérêt instantané qui s'avère pourtant être le plus logique et réaliste : cette théorie de la capitalisation continue fait appel à l'analyse infinitésimale classique et, de ce fait, aurait pu être construite et appliquée depuis longtemps, grâce aux travaux, notamment, de Newton (1642-1727), Leibniz (1646-1716), etc ...

Malgré les travaux précurseurs de L. Bachelier (1905) dont les mérites ne furent reconnus qu'à titre posthume, il fallut attendre la deuxième moitié de ce siècle pour voir un changement essentiel dans l'abord des phénomènes financiers grâce à l'intervention du calcul des probabilités. Le point de départ de ce courant fut les travaux de Markowitz et Tobin (1958) sur l'évaluation d'un capital par la valeur espérée (*expected value*) au lieu de la valeur (*value*) qui était seule considérée auparavant. L'usage intensif du calcul stochastique en finance aurait été impossible sans les travaux purement mathématiques de Lebesgue (1875-1941), célèbre pour l'intégrale qui porte son nom et le développement de la théorie de la mesure, de Kolmogorv (1903-1987), considéré comme étant le "père" de la théorie moderne des probabilités, de Wiener (1894-1964), Lévy (1886-1973) et Ito notamment, qui on profité des résultats théoriques de leurs prédécesseurs pour mettre au point un calcul différentiel et intégral stochastique parfaitement adapté pour étudier les phénomènes de la bourse. Ainsi, *les concepts mathématiques qui ont renouvelé la finance et qui sont utilisés aujourd'hui quotidiennement à Chicago, Paris, Londres ou Singapour ont été élaborés au sein des mathématiques pures, c'est-à-dire de questionnements internes aux mathématiques* [BOULEAU, p. 35].

2 Les mathématiques dans la formation des économistes

2.1 L'enseignement de l'économie

Jusqu'au milieu de ce siècle, des éléments d'économie étaient enseignés au sein d'autres disciplines telles que l'histoire et la géographie dans l'enseignement secondaire, ou encore le droit à l'université. Depuis quelques décennies, l'économie s'est considérablement développée au point de pouvoir proposer des enseignements spécifiques et est devenue une matière d'enseignement à part entière.

Dans l'enseignement secondaire, l'économie est actuellement une matière optionnelle : elle n'est dispensée que dans certaines sections, dites les sciences économiques, qui, malheureusement, sont souvent choisies négativement par des élèves faibles qui sont incapables de réussir dans les sections réputées plus fortes et cherchent fréquemment à fuir des programmes forts, ou même moyens en mathématiques.

A l'Université de Liège, l'enseignement de l'économie vient de fêter ses 100 ans : au départ, il était dispensé essentiellement dans la Faculté de Droit et dans une Ecole de Commerce. Actuellement, il est organisé principalement par la Faculté d'Economie, de Gestion et des Sciences Sociales, qui vient seulement de célébrer son dixième anniversaire et où sont délivrés des diplômes de licence en Sciences économiques, en Sciences sociales, en Sciences de gestion et en ingénieur de gestion, ainsi que des diplômes complémentaires et de troisième cycle qui attirent de plus en plus de diplômés universitaires de deuxième cycle à la recherche d'une formation additionnelle en gestion; mais il existe également des cours d'économie dans toutes les autres facultés, ce qui est assez nouveau. La situation est sensiblement la même dans toutes les autres Universités belges.

La distinction ancienne entre la micro- et la macro-économie tend à disparaître, mais celle entre l'économie dite pure et l'économie appliquée, c'est-à-dire la gestion, s'accentue de plus en plus. Il convient de remarquer une différence assez sensible dans le recrutement de ces deux types d'études en Belgique. D'une part, la gestion attire généralement des étudiants issus d'humanités générales, avec une formation mathématique poussée, surtout chez les ingénieurs de gestion. D'autre part, les sciences économiques recrutent principalement des élèves ayant choisi la section économique dans le secondaire, et ont de ce fait une formation mathématique qui est souvent assez faible, ce qui est paradoxal, puisque les économistes devraient utiliser plus de mathématiques abstraites que les gestionnaires.

2.2 Mathématiques enseignées aux futurs économistes

Il existe un parallélisme frappant entre l'évolution historique de l'intervention des mathématiques en économie et les matières mathématiques actuellement enseignées aux économistes dans les différents stades de leur formation.

Dans l'enseignement secondaire, l'économie est présentée de manière fort littéraire, principalement à l'aide de graphiques. Les directives pédagogiques figurant dans les programmes ambitionnent de formaliser mathématiquement davantage les raisonnements, ce qui n'est guère possible au vu de la formation préalable et de la qualité des élèves.

Le premier cycle universitaire a pour mission de doter les étudiants d'un bagage de base, notamment pour tous les outils qu'ils seront amenés à exploiter ultérieurement. C'est ainsi qu'y sont enseignées les mathématiques générales connues depuis longtemps des mathématiciens, à savoir l'analyse des fonctions à une ou plusieurs variables, l'algèbre linéaire, la statistique et le calcul des probabilités.

Le deuxième cycle est consacré à des matières plus spécifiques et à l'exploitation des outils de base dans des problèmes réels rencontrés dans le monde des affaires. Y sont organisés des cours de mathématiques appliquées développées surtout depuis la fin de la seconde guerre mondiale, comme l'économétrie, la recherche opérationnelle et la programmation mathématique.

Les troisièmes cycles, qui se développent de plus en plus dans les différentes Universités, présentent des théories plus pointues et plus récentes. D'un point de vue mathématique, une attention particulière est, par exemple, portée sur les modèles non linéaires, les processus stochastiques, et même parfois, l'analyse non standard, la statistique robuste, la logique floue, ...

2.3 Le cas particulier de la finance

Il est intéressant de constater que la mathématique financière est enseignée à tous les niveaux de l'apprentissage scolaire, d'une manière assez rationnelle.

L'arithmétique commerciale, comprenant notamment la théorie des nombres entiers et des fractions ainsi que l'utilisation de la règle de trois, est donnée à l'école primaire, car elle fait partie de ce que tout citoyen doit connaître.

Les théories de l'intérêt et de l'escompte simples, ainsi que celles des intérêts composés et des annuités étaient enseignées, il y a quelques années encore, respectivement dans l'enseignement secondaire inférieur et dans le cycle supérieur dans toute section d'humanités; de nos jours, ces notions ne sont plus guère données que sporadiquement comme illustrations de notions mathématiques, sauf dans les sections commerciales et de sciences économiques. Il est à noter que la théorie de la capitalisation mixte, qui, comme son nom l'indique, exploite les deux théories précédentes, est peu souvent dispensée dans l'enseignement général bien qu'elle soit accessible aux élèves et d'usage fréquent dans la pratique.

L'intervention du *calculus* dans les problèmes financiers est essentiellement exposée dans le premier cycle universitaire en sciences économiques ou de gestion : on y voit notamment la notion de taux instantané, des problèmes d'optimisation, le calcul du TAEG, la modélisation de la valeur prise par un capital au moyen d'équations récurrentes (pour le cas discret) ou différentielles (pour le cas continu).

Les techniques économétriques classiques sont vues dans les épreuves de licences organisées par les Facultés d'Economie.

Les processus stochastiques, notamment l'exploitation du mouvement brownien standard dans la modélisation des actifs financiers, sont développés en fin de deuxième cycle universitaire ou, plus souvent, dans les troisièmes cycles spécialisés.

L'enseignement contemporain de la finance au plus haut niveau se fait non seulement dans les Universités, mais aussi dans des Institutions privées et même dans les salles de marchés. Les deux points de vue sont pourtant assez différents. Alors que, dans les Universités, la finance est enseignée aux futurs économistes et gestionnaires de manière collective et démocratique, sur base de connaissances universelles et publiées dans des revues spécialisées accessibles à tous, la formation privée est fréquemment donnée à des ingénieurs ou mathématiciens, au prix d'un enseignement individuel qui se veut directement rentable, sur base de la pratique et de connaissances furtives qu'il n'est pas intéressant de divulguer : par exemple, un spéculateur qui a découvert un arbitrage intéressant n'a pas intérêt à le rendre public et, tout au plus, se contentera de le communiquer à un de ses collaborateurs; dans ce cas, *les connaissances se situent clairement dans une perspective pré-professionnelle et sont d'autant plus onéreuses qu'elles sont susceptibles de déboucher sur de plus grosses rémunérations* [BOULEAU, p. 158].

Cette tendance indique ce que sera peut-être l'avenir. La finance concerne de plus en plus de mathématiciens et il est vraisemblable que des cours très spécialisés seront organisés lors des

licences en sciences mathématiques ou dans des épreuves de troisième cycle spécifiques. Par ailleurs, il est probable que l'enseignement de gestion des portefeuilles se fera bientôt "sur le terrain" grâce à des postes de travail informatiques directement reliés aux données des marchés, comme cela commence à se faire dans des Universités américaines.

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Historical and didactical phenomenology of the average values

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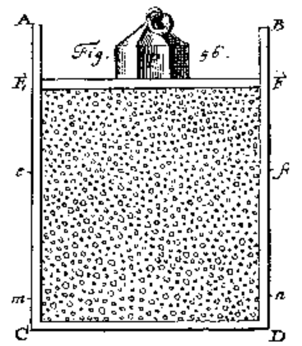
Abstract

This study was carried out as a preparation to the development of instruction material for statistics. The history of statistics was studied with special attention to the development of the average values: the arithmetic, geometric, harmonic mean; median, mode and midrange. Also sampling and distribution are discussed. After an introduction on phenomenology, this article firstly discusses a so-called historical and then a didactical phenomenology of the average values.

The average values form a large family of notions that in early times were not yet strictly separated. There are many parallels between history and the development of students' conceptions. It appears to be important that students discover many qualitative aspects of the average values before they learn how to calculate the arithmetic mean and the median. From history, it is concluded that estimation, fair distribution and simple decision theory can be fruitful starting points for a statistical instruction sequence.



Modèle de gaz ayant permis le traitement statistique de la théorie cinétique des gaz, tiré de l'*Hydrodynamica* de Daniel Bernoulli.

1 Introduction

As a preparation to the development of statistical instruction material for 12-year-old students, I studied the early history of statistics. The reason for this is that I assumed that there are parallels between history and education that give inspiration for hypothetical learning trajectories. These learning trajectories are developed with the aim that students somehow reinvent the mathematical concepts (FREUDENTHAL 1973 and 1991). The Dutch mathematician and historian DIJKSTERHUIS (1990) even believed that students recapitulate history at a higher speed. Problems that students encounter when learning mathematics resemble the problems that former generations of mathematicians dealt with. With this in mind, I hoped to find the historical contexts that led to the development of statistical notions, and to detect the conceptual obstacles that mathematicians and users of mathematics encountered. The historical insights are discussed in relation to the results of four exploratory field tests.

1.1 Phenomenology

In FREUDENTHAL's *Didactical Phenomenology of Mathematical Structures* (1983) we find a method of studying the relations between mathematics, history and education. Freudenthal makes a distinction between *phainomena*, phenomena that we want to understand or structure, and *nooumena*, the entities of thought with which we organize these phainomena. Mathematical concepts are examples of such *nooumena* with which we organize our experiential world. This view explains his special way of defining phenomenology:

> Phenomenology of a mathematical concept, a mathematical structure, or a mathematical idea means, in my terminology, describing this *nooumenon* in its relation to the *phainomena* of which it is the means of organising, indicating which phenomena it is created to organise, and to which it can be extended, how it acts upon these phenomena as a means of organising, and with what power over these phenomena it endows us. If in this relation of *nooumenon* and *phainomenon* I stress the didactical element, that is, if I pay attention to how the relation is acquired in a learning-teaching process, I speak of *didactical* phenomenology of this *nooumenon*. (...) if "is ... in a learning-teaching process" is replaced by "was ... in history", it is *historical* phenomenology. (1983, pp. 28-29).

In the context of my article, I use the following simplified definitions:

- *Phenomenology* of a mathematical concept is the study of the relation between that concept and the phenomena it organizes.
- *Historical* phenomenology is the study of the historical contexts in which certain mathematical concepts arose in order to understand why these arose.
- *Didactical* phenomenology is the study of the relation between the mathematical concepts and the phenomena in which they arise with respect to the process of teaching and learning. See also (GRAVEMEIJER 1994, p. 95).

This suggests studying statistics from two perspectives: a historical and a didactical perspective. In the following I will start with a section on the historical phenomenology of the average values. The examples are ordered chronologically. After that, I will discuss basically the same phenomena from a didactical point of view in the same order.

Philosophically seen, there is a difficulty when we make the distinction between *phainomenon* and *nooumenon*: it is not possible to see the phenomenon separated from concepts, since a concept determines and influences the phenomenon. For an educational purpose, this is not a major problem, as long as we keep in mind that students need not see the same phenomena as we do, with our understanding of certain concepts. Studying history can help us to see certain phenomena through the eyes of people who did not have the same concepts. This may help us to understand the learning process of students.

1.2 Historical phenomenology of the average values

The historical study of the average values is difficult for three reasons. First of all, history is often written with respect to men or books, not with respect to concepts. Second, most historical studies start around 1660, when statistics and probability were born according to Kendall (PEARSON & KENDALL 1970, p. 45) and HACKING (1975). Mathematical statistics is even younger, namely from the late 19th century. For my purpose it was necessary to go back to the very start of statistical inference, because my target group is young children. So where I write 'statistical' some readers might prefer to read 'pre-statistical'. Third, statistics is a subject (unlike probability) that was mainly born outside mathematics and was often not seen as part of mathematics. Its birth and growth was due to astronomy, demography, economics, medicine, genetics, biometry, anthropology, the social sciences and many other areas. This might explain why many 'histories of mathematics' pay very little attention to statistics.

1.2.1 Estimation

The oldest example of using an implicit kind of average I have found concerns the estimation of the number of leaves and fruit on two great branches of a spreading tree in ancient India (HACKING 1975, p. 7). How did Rtuparna, the protagonist of the story, do this? He estimated on the basis of one single twig, which he multiplied by the estimated number of twigs on the branches.

I assume he chose a typical or an average twig. I see this as an implicit use of (an intuitive predecessor of) the arithmetic mean, since one number represents all others and this number is somehow in the middle of the others; it is less than the greater, and greater than the smaller numbers; what is too much on the one hand is too little on the other. This use of an average has therefore to do with compensation and balance.

Another example of estimation is a passage written by the Greek historian Herodotus (485-420 BC) on the Egyptians (RUBIN 1968, p. 31):

> They declare that three hundred and forty-one generations separate the first king of Egypt from the last mentioned [Hephaestus] – and that there was a king and a high priest corresponding to each generation. Now reckon three generations as a hundred years, three hundred generations make ten thousand years, and the remaining forty-one generations make 1,340 years more; thus one gets a total of 11,340 years...

The important point in this quotation is the assumption that three generations is a hundred years. This assumption was made to estimate the total amount of years between the first Egyptian King and Hephaestus.

In daily life we often use this kind of average when estimating things. We can think of estimating the number of people coming to a party, the number of bottles we need for that, and the price of our shopping et cetera.

1.2.2 Thucydides: majority, average and midrange

Other very old examples of statistical or pre-statistical reasoning can be found in the work of one of the first scientific historians, Thucydides (460-400 before Christ). The following quotations are from his *History of the Peloponnesian War*. The reader is invited to decide how he or she would translate these two episodes into modern statistical terms (RUBIN 1971, p. 53):

> The problem was for the Athenians ... to force their way over the enemy's wall. Their method was as follows: they constructed ladders to reach the top of the enemy's wall, and they did this by calculating the height of the wall from the number of layers of bricks at a point which was facing in their direction and had not been plastered. The layers were counted by a lot of people at the same time, and though some were likely to get the figure wrong, the majority would get it right, especially as they counted the layers frequently and were not so far away from the wall that they could not see it well enough for their purpose. Thus, guessing what the thickness of a single brick was, they calculated how long their ladders would have to be...

> Homer gives the number of ships as 1,200 and says that the crew of each Boetian ship numbered 120, and the crews of Philoctetes were fifty men for each ship. By this, I imagine, he means to express the maximum and minimum of the various ships' companies ... If, therefore, we reckon the number by taking an average of the biggest and smallest ships...

In the first example, we could see an implicit use of the mode, here indicated by 'the majority'. Note that 'the majority' probably means 'the most frequent value' and not 'more than half'. So in this situation, the Greeks assumed that the most frequent number would be the correct one. In order to find the total height of this number of bricks, they needed another estimation, supposedly of the expected or the average height of a brick. Note that in the first example and the Indian one, it is the total number that counts; they are not primarily interested in the average. The implicit average value is just used to find the height of the wall or the number of leaves.

In the second example, we again see an estimation with the help of an average value. It seems that Thucydides interprets the given numbers as the extreme values, so that the total amount of men on the ships can be estimated. He suggests that this be done by taking the average of these two extremes. In fact this is called the midrange: the midrange is defined as the arithmetic mean of the two extremes. RUBIN[1] (1971, p. 53) writes about this:

> This technique of averaging the extreme values of the range to obtain the arithmetic mean or mid-range can be justified if certain assumptions are defensible, i.e., that the underlying distribution is at least approximately symmetrical or rectangular.

Resuming, we encountered certain phenomena, or problems, that were organized by certain intuitive variants or predecessors of contemporary concepts. When estimating the number of years between two important persons, Herodotus used a number that we now would call the *average*. And Thucydides used a method that we would call *taking the midrange*.

[1]In the American Statistician, RUBIN discussed Adam Smith (1959), Malthus (1960), Karl Marx (1968), Herodotus (1968), Thucydides (1971), Darwin (1972), a medieval household book (1972), and Shakespeare (1973) from a statistical point of view.

1.2.3 Majority, voting and democracy

A very implicit use of statistics is in voting: the majority (the most frequent or common value) is seen as representative of the population. The majority rules. This is the basis of democracy. Also in the Bible and Talmud there are rules like 'follow the majority'. In the Talmud we can read for instance:

> In the entire Law we adopt the rule that a majority [or the larger portion] is equivalent to the whole. (RABINOVITCH 1973, p. 38).

From a modern point of view we might think of the *mode*[2].

1.2.4 The Greek definition of the arithmetic mean

In Pythagoras' time, three kinds of means were known in Greece: the arithmetic, geometric and harmonic mean (HEATH 1921, p. 85; Iamblichus 1939). The theory of these mean values was developed in his school with reference to music theory and arithmetic. Consider for example the musical proportions 6:8:9:12. 8 is the harmonic mean between 6 and 12, and 9 is the arithmetic mean. The proportions 6:8 = 9:12 (a fourth as a musical interval), 6:9 = 8:12 (a fifth), 6:12 (an octave) all form consonant intervals; 8:9 is a second.

Greek mathematics had a different form and aim than modern mathematics. Greek mathematics, even number theory, was highly geometrical and visual. Numbers were represented by lines. This difference between Greek and modern mathematics can be illustrated with the different definitions of the arithmetic mean. The Greek definition was: the middle number b is called the arithmetic mean if and only if $a - b = b - c$. Note that this definition differs from the modern one, $(a+c)/2$, and that it only refers to two values. The Greek version concentrates on the intermediacy, but is difficult to generalize, whereas the modern version highlights the calculation, and is easy to generalize. With education in mind, it is important to note that the Greek definition shows other qualitative aspects than the modern quantitative one. From the Greek definition we can immediately see that the mean is halfway between the two other values. In the didactical section I will come back to this point.

1.2.5. Aristotle's Doctrine of the Mean

Aristotle defines in his Nicomachean Ethics a more philosophical form of the mean, namely the *mean relative to us*. With this notion he explains what virtue is. About the difference between the arithmetic mean and *the mean relative to us* he writes:

> By the mean of a thing I denote a point equally distant from either extreme, which is one and the same for everybody; by the mean relative to us, that amount which is neither too much nor too little, and this is not one and the same for everybody. For example, let 10 be many and 2 few; then one takes the mean with respect to the thing if one takes 6; since 10-6 = 6-2, and this is the mean according to arithmetical proportion [progression]. But we cannot arrive by this method at the mean relative to us. Suppose that 10 lb. of food is a large ration for anybody and 2 lb. a small one: it does not follow that a trainer will prescribe 6 lb., for perhaps even this will be a large portion, or a small one, for the particular athlete who is to receive it; it is a small portion for Milo, but a large one for a man just beginning to go in for athletics.

[2]We find much more about these matters in RABINOVITCH (1973: e.g. section 3.1), a very interesting book on statistical inference in ancient and medieval Jewish literature.

Later in the section he writes about virtue:

> Virtue, therefore, is a mean state in the sense that it is able to hit the mean. (N.E. book II, chapter vi).

From this we may conclude that Aristotle generalized the notion of the mathematical means to situations in daily life. For him, the mean relative to us was an ethical ideal. Another interesting point is that the mean has to do with balance and intuitively means 'not too much and not too little'. This 'definition' is one that students used in all three field tests in which they estimated the number of elephants in a picture. When these students explained their strategies, they defined 'an average box' in a grid as a box in which were 'not too many and not too little' elephants.

1.2.6 Decision theory and Jewish Law

To understand the oldest Jewish examples it is helpful to have some insight into rabbinical Law. First, the Jewish Law is considered a rational pursuit. Although rabbis accepted Divine guidance they insisted on rational methods in coming to decisions. Statistics in this context was solely a decision theory (RABINOVITCH 1973, p. 140). Second, Jewish Law deals mainly with social, ethical and ritual duties. It is not primarily concerned with quarrels and punishing wrongdoing. Rabbis had to decide for example whether food was kosher and how inheritances had to be divided. Gambling was not mentioned since it was considered pagan. I will give one example on kosher food: if 9 out of 10 shops in a city sell kosher meat, and you find a piece of meat in that city, you may consider it kosher (*op. cit.*, p. 45).

A very interesting example concerning multiplicative reasoning and sampling (RABINOVITCH 1973, p. 86) in the Mishnah is found in the Taanit 21a. It concerns whether or not an epidemic has taken place:

> A town bringing forth five hundred foot-soldiers like Kfar Amiqo, and three died there in three consecutive days - it is a plague... A town bringing forth one thousand five hundred foot-soldiers like Kfar Akko, and nine died there in three consecutive days - it is a plague; in one day or in four days - it is not a plague.

In this example, three points are interesting. First, we see that rabbis reasoned proportionally to the total population. Second, a kind of sampling is used: the amount of foot soldiers is used as an indicator of the total population, assuming that foot soldiers form a constant percentage of the population. Third, the rabbis seemed to know normal (or average) death rates: if nine died in one day then it need not be a plague, and also in the case of four days there can be another reason for the nine deaths. So they took into account how the deaths were distributed over the days.

1.2.7 Average as a kind of insurance: fairness and redistribution

In *The World of Mathematics*, MORONEY (1956, p. 1464) writes the following on the history of the average:

> In former times, when hazards of sea voyages were much more serious than they are today, when ships buffeted by storms threw a portion of their cargo overboard, it was recognized that those whose goods were sacrificed had a claim in equity to indemnification at the expense of those whose goods were safely delivered. The value of the lost goods was paid for by agreement between all those whose merchandise had been in the same ship. This sea damage to cargo in transit was known as 'havaria' and the word came naturally to be applied to the compensation money, which each individual was called to pay. From this Latin word derives our modern average. Thus the idea of an average has its roots in primitive insurance.

This view is confirmed by several (etymological) dictionaries (SKEAT 1882) and by WALKER (1931). The Dutch reader may think of 'averij' and the French reader of 'avarie', which is the damage to a ship or to its load after a storm. As a legal term it still refers to the costs of compensating the damage to the load. Here we see that the average has to do with fair redistribution.

1.2.8 Navigation and astronomy: from the midrange to the general form of the arithmetic mean

The *midrange*, the arithmetic mean between the extremes, turns out to be a predecessor to the arithmetic mean (9th - 11th century; Eisenhart 1974, p. 31). Not until the 16th century was it recognized that the arithmetic mean can be extended to n cases: $(a_1 + a_2 - ... + a_n)/n$. When sailors in these days had to determine their position on earth, they often used the midrange. Especially when a ship rocks and the compass needle varies, they had to make many observations or look for a while at the compass and find the middle value. Nowadays we know that many observations and errors follow the normal distribution. So the midrange probably was a sensible value to take.

The method of taking the mean for *reducing observation errors* was mainly developed in astronomy (PLACKETT 1958). In astronomy, we want to know a real value, but we can't measure this value directly. We assume that sum of the errors add up to a relatively small number when compared to the total of all measured values.

It is important to note that until then, and long after that, the notion of the mean was qualitative, rather than quantitative. It gradually became a common method to use the arithmetic mean to reduce errors (PLACKETT 1958, EISENHART 1974). It was used to measure, for example, the diameter of the moon, but also used when weighing gold and silver coins.

I would call this insight into the change of definition and the possibility of generalization of the modern arithmetic definition a typical result of historical phenomenology.

1.2.9 Representative value

It took a long time before the mean was used as a representative or substitute. In the examples up till now, the mean has always been used to find some real value: the number of leaves on branches, how many years passed, what the height of the wall was, how many men there were on the ships, and so forth. The only exception seems to be Aristotle's doctrine of the mean, which is a more philosophical variant of the mean.

The Belgian statistician Quetelet (1796-1874) was one of the firsts to use the mean as the representative value for an aspect of a population. Quetelet, as a student of Laplace, was inspired by the method of physics and called his subject 'social physics'. I assume that he could also have been inspired by Aristotle's doctrine of the mean.

In Quetelet's eyes, the average man (*l'homme moyen*) was the ideal man[3]. Cournot (1801-1879) challenged this view; he pointed out that the mean taken for each side from a great number of right triangles could in no way represent the type of a right triangle, since it would almost certainly not be a right triangle at all (PORTER 1986, p. 172). Also Galton and Darwin disagreed with Quetelet, since they were interested in the deviations from the mean. They both were interested in exceptions and they thought that average people were mediocre. Galton was interested in inheritance of genius; in his book *Hereditary Genius* he introduced regression to the mean. By this he meant that the intellectual ability of children of geniuses, but also of 'stupid' people, were generally closer to the mean. For Darwin, deviations from the mean were important since they are a condition for evolution.

Note that the transition from the real value in astronomy to the ideal value of Quetelet, which is a mathematical construct in the social sciences, was an important conceptual change, which is another result of historical phenomenology. The didactical implications will be discussed in section 2.13 and 3.9.

1.2.10 Sampling

We have already seen examples in which sampling was involved: for instance the estimation of the number of leaves and fruit on two great branches (2.1) and the question if there was a plague (2.6). Many other examples of sampling can be found in the Bible and Torah (RABINOVITCH 1973).

Here we only discuss a nice example of simple secular sampling and quality control: the trial of the Pyx in England (STIGLER 1977). At the Royal Mint of Great Britain gold and silver coins were made. Starting from the 12th century, every day one of these coins was put in the Pyx, a box in Westminster Abbey. After a few months or years, the Pyx was opened and the coins were investigated on weight and pureness. If they turned out to be good, this fact was celebrated with a banquet. Otherwise the coinmakers were punished. This is the first example of quality control I have found. For a statistician it is interesting to see that the tolerance interval did not depend on the number of coins; it was a constant percentage of the weight.

Until 1900, samples of the population were considered dishonest and imprecise. Everybody had to contribute to an investigation. When the Norwegian Kiaer presented the representative sample in 1895, he met a lot of resistance. The method was not accepted until 1903, and even in the following years this method of taking representative samples by stratifying was not very successful. The Pole Neyman proposed the random sample in 1934. After that, sampling became increasingly accepted (BETHLEHEM & DE REE 1999).

1.2.11 Median and mode: a few dates

Cournot seems to be the first who used the term *médiane* for the middle value in 1843 (Monjardet in: FELDMAN et al. 1991; PORTER 1986). The English term *median* was coined by Francis Galton (1822-1911) in 1883, who preferred the middle value because it was more robust than the mean, and easier to determine. He also preferred to use the quartiles instead of the standard deviation. Before he coined the median, Galton used the word 'middlemost value' (in

[3]This is a simplified view; in fact Quetelet's view was much more subtle.

1869) or 'medium' (1980). In a lecture of 1874 he gave the following description but not the current name:

> The object then found to occupy the middle position of the series must possess the quality in such a degree that the number of objects in the series that have more of it is equal to that of those that have less of it. (WALKER 1931, p. 87)

Independently from Galton, Fechner (1801-1887) defined the median but called it *Centralwerth* or C in 1874 (WALKER 1931, pp. 86, 88, 184). WALKER (1931, p. 84):

> In contrast to the practical interest in anthropology which impelled Galton to use the median and related measures, the incentive which Fechner led to discover the median seems to have been a theoretical interest in generalizing the measures of central tendency.

Why was the median called the median? We can find hints for that question in geometry and in the works of Laplace on (continuous) probability density functions: the median is the value such that the left and right parts have equal area. The statistical median of a row of numbers is also in the middle: there are as many numbers left as there are right from the median. If we think of geometry we can imagine another reason why it is called the median: in a triangle, the median is the line that goes to the *middle* of the opposite side and the two parts left and right from the median have equal area. The mode was coined by Karl Pearson in 1894. Fechner spoke of *der dichteste Werth* (D) in 1878. The difficulty with these historical dates is that the notions were used long before in an informal sense, but without being given a name. See for instance the sections on Thucydides and democracy, and the quotation of Galton in this section.

1.2.12 Box plot and stem-and-leaf diagrams

John TUKEY is a famous name connected to exploratory data analysis. He is the inventor and propagator of the box-and-whiskers plot and stem-and-leaf diagram, which became well known in the 1970s (TUKEY 1977). Box plots are suitable for comparing uni-modal distributions. Stem-and-leaf-diagrams can be useful when we have to put numbers into order by hand and if we want to see how they are distributed.

1.2.13 Cultural knowledge

The box plots and stem-and-leaf diagrams are examples of mathematical objects that came late in history, but can come early in education. So we can not simply follow the path of history. This would also be a waste of time, since a lot of mathematical knowledge nowadays is also cultural knowledge. Many children already know in some intuitive sense what *average* and *random* mean and what a survey is, just because they meet these words in their environment. From history, we might conclude that random sampling is a difficult concept because it was only accepted in the 20th century, but this does not imply that students nowadays have the same problems with that notion as people around 1900. Therefore, when making instruction material with history in mind, we should not forget that we could benefit from the students' cultural knowledge.

1.3 Didactical phenomenology of the average values

The phenomena and concepts that were discussed in the historical context will now be discussed in relation to the process of teaching and learning. The order will be basically the same. The protocols come from an exploratory field test with 26 students of age 12 in April 1999. I tested my instruction material in three classes (first year in secondary school) in the period from September to November 1999.

1.3.1 Estimation

The earliest historical examples had to do with estimation. This gives rise to the following question: is it a good idea to start the teaching of the mean with estimations? More than the calculation of the mean can estimation direct the attention to more qualitative aspects of the average: representative, typical, normal, somewhere in the middle, balance, compensation. Generally speaking, estimation does not direct the attention to the average but to the total. I chose to start my instruction material with a picture of a herd of elephants and ask: how many elephants are there in the picture? The 12-year-old children used four main strategies with many variants:

1. Make groups, guess how many are in each group and add, or
2. make a group with a fixed number and estimate how many groups fit into the whole, or
3. make a grid, choose an 'average box' and multiply by the number of boxes in the grid, or
4. count the number of elephants in the length and width and multiply these (Mr. Bean's method of counting sheep).

Strategy 3, which was used most often, is indeed based on an intuitive idea of the average. Discussing why this strategy works students tend to define an 'average box' as one that does not have too many and not too little in it. Compare this with Aristotle's characterization of the mean (2.5).

1.3.2 Translation of terms: majority

Sometimes, the majority is right; see Thucydides on counting the number of bricks (2.2). But in the case of estimating the elephants nobody was right: all students, except the few who used strategy 4, estimated too low. Does the majority have to do with the mode? Indeed, I would anachronistically translate 'the majority' into the mode or modal class. The mode is not a very robust[4] or useful value in statistics. Therefore, I chose to concentrate in my instruction material on the mean and median.

As we saw from the quotations of Thucydides (2.2), it is rather difficult to make implicit aspects of average values explicit. Did Thucydides really think of the midrange in the second quotation? In my field test I encountered a similar difficulty of translating the arguments of students into statistical terms. Often, it was hard to detect the underlying principles of the students' arguments. From only 26 students I got a rich variety of answers to the question what the average is:

[4]The mode is not robust since it is sensitive to outliers. Neither is the midrange. The median though is very robust, even more than the arithmetic mean.

Jennifer:	Yes, the half. The whole, and in between the half, that is the mean.
Charissa:	Everything together.
Bart:	You look between the highest and lowest.
Centina:	The most.
Claire:	What you think it is roughly.
Lisa:	The mean is about a bit in balance.
Kerster:	In between.
Frank:	The midpoint.

Many students said: add and divide by the number.

Keeping the same order, we could translate these answers into: one part of the algorithm (divide by 2), the other part of the algorithm (add everything), midrange, mode, estimation, balance, intermediacy, median or center of gravity and the complete algorithm.

If we turn to philosophy of language, we could see the average values as a large family of related concepts of which the arithmetic mean is just one member (WITTGENSTEIN 1984). Early in history, but also in the development of a child, we see no clear distinctions between all aspects of these concepts. When people organize their world and solve problems, they are urged to become clearer and define more precisely. For the instruction material, we must therefore choose contexts that really ask for clear distinctions. For example, a skewed distribution can show the limitation of the midrange and ask for another measure of central tendency.

1.3.3 The Greek definition: intermediacy

What I conclude from the transition from the visual 'intermediacy' definition of the Greeks (b is the mean of a and c if and only if $a - b = b - c$) to the general form (16th century) is that we should start with a very visual and qualitative notion of average values before teaching the calculation of the arithmetic mean. The students should at least know by experience that the mean is between the extremes. I have noticed that many students forget this aspect if they calculate the mean and sometimes get an answer out of the range. An illustration of this:

Arthur:	How would you estimate the average annual temperature in the Netherlands from this graph or table?
Jennifer:	Add everything.
A:	And then?
Lisa:	Divide by 2.
A:	Why divide by 2?
L:	Because that is the average.

They got 55 degrees Celsius and did not realize that this was very hot.

If students have developed some visual intuition, this might happen less.

The first computer minitool[5] I use for instruction in exploratory data analysis (www.fi.uu.nl/arthur Minitool 1) is designed so that magnitudes are represented by bars. It

[5]The minitools were designed for a teaching experiment of Koeno Gravemeijer, Paul Cobb and others at the Vanderbilt University in Nashville, USA.

turned out that this representation created an opportunity for the students to find the mean by cutting off the longer ones and give these pieces to the shorter ones. When asked to estimate the annual year temperature in the Netherlands from a bar graph, many students spontaneously came up with a compensation strategy. Some said: 'I give a bit of July to January, from August to February and so on'. This visual way of dealing with the mean is also an example of fair redistribution.

1.3.4 Redistribution and fairness

In history, statistical reasoning often had to do with fair division of inheritance or possessions (2.6). Fairness is very important issue to children, so we can profit from that. It can easily be combined with other aspects of the mean, such as compensation and balance. Note that distribution in the sense of division (giving everybody the same amount) is different from distribution in the technical sense (normal distribution, for example).

This is done for example in *Mathematics in Context*, instruction material for American middle schools. Students are asked to rearrange mice in nests in order to answer the question: how many mice does a mother mouse roughly get? If we want to avoid half mice, we could take biscuits instead. The question will then be: how many biscuits does every student get if they are fairly distributed? (See also: MOKROS & RUSSELL 1995; STRAUSS & BICHLER 1988).

1.3.5 Three components

If we consider calculations with the mean, we see three components: the number n, the sum or total Σ and the mean μ. These components can have different roles. When making and analyzing exercises it is useful to categorize the possibilities:

1. Estimation often has to do with finding the total: $n^*\mu = \Sigma$. The used average value mostly stays implicit since the focus is on the total. In this way students get feeling with many aspects of the mean without using it explicitly.

2. Fair redistribution (how much does everyone get?) has to do with finding the mean: $\Sigma/n = \mu$. Sometimes, for example if we want to compare fairly, we need to compensate for the number. By *mean as a measure* is meant the use of the mean as a way of compensating for the number n, e.g. if we use parts per million, a percentage, gross national product per head, et cetera. In other words: the mean is used to make fair comparison possible. In history we see this when a criterion is given whether an epidemic has taken place (2.6).

3. I also asked how many 12-year-old students could go into the basket of a hot air balloon. They got the allowed weight, they estimated the mean weight of students and calculated the number n : $\Sigma/\mu = n$. The same exercise could be done with an elevator instead of a balloon. This exercise also implicitly asks for an average, namely the estimated weight of these students.

1.3.6 Reduction of error

In the counting example of Thucydides the mode is the best value. In astronomy, it often is the mean that is the best value, namely as an approximation of the assumed real value. We only

have data that are approximations of this value, so the mean substitutes the real value. This use of the mean, I fear, is too difficult for students that don't yet have learnt physics at school. We could, however, look for other contexts in which students have some intuition about reducing errors.

1.3.7 From the midrange to the general form of the arithmetic mean

In history, the midrange turned out to be a predecessor of the mean (see 2.8). This suggests allowing the midrange as an average value, for instance in estimation strategies. When estimating the number of elephants in a picture, a few students indeed used the midrange. They counted the box with the least and with the most elephants, calculated the mean of these two, and multiplied by the number of boxes. Of course, the students should experience that the midrange is only useful if the distribution is symmetric, so the following lesson I showed both a symmetric and a skewed distribution to make clear when this method is useful and when it goes wrong. Another disadvantage of the midrange that students should encounter is that it is very sensible to outliers, as can be shown easily.

1.3.8 Balance and midpoint

A very nice way of presenting the mean is by means of a balance. HARDIMAN, WELL & POLLATSEK (1984) have written about the usefulness of the balance model in understanding the mean. For younger students, I fear though, this method is less suitable since they need more knowledge of physics to understand the balance model. Therefore, I preferred to start in my field tests with estimation and redistribution.

1.3.9 Representative value

The aspect of representativeness comes very late in history. It is rather a huge step from the real value, as in astronomy, to an ideal or representative value, as in the social sciences. Quetelet, the man of the *l'homme moyen*, is an example of the latter use. We can't simply say that this latter use is more difficult than the former one. There are several layers of understanding the mean as a representative value. Students have no problem in seeing an average Dutchman as a normal Dutchman, but have more difficulties with artificial constructs like the average size of a family, which is a decimal number.

1.3.10 Sampling

There are several layers in the understanding of sampling. The historical examples can help to define these layers. For example, the Trial of the Pyx is a very elementary form of sampling, whereas random sampling is very abstract. It is clear that sampling should be learned, because students can only understand certain data if they know how these data were created.

Note that sampling in the form of surveys is a common practice in society, so many children intuitively know what random means and what a sample is. Therefore, didactics need not be parallel to history (2.13).

1.3.11 Median and mode

In most statistics courses distributions are described in relation to the mean and standard deviation. For younger students, I prefer the median and quartiles, because they don't know the square root yet. Later on, of course, they should learn the mean and standard deviation, since these are used more often and have great theoretical advantages.

The students should also meet situations where using the median or mode makes more sense (e.g. salaries).

1.4 Conclusions

Historical phenomenology can be helpful when designing instruction material. The obstacles in history tell us where the difficult conceptual transitions are and what might facilitate their teaching and learning. Of course, it is not necessary that students learn all the detours that history made. A shorter route is possible because of the cultural knowledge of the students and the experience of teachers with education.

Still, the parallels between history and education are sometimes remarkable. For example, I have detected many aspects of the mean with only one field test with 26 students. Their answers to the question 'what is the average?' already yielded many associations with: parts of the algorithm (adding or dividing), median, mode, balance, midrange, representativeness, compensation, midpoint and intermediacy. All these aspects and different average values can be seen as a large family of concepts that initially have no precise borders or definitions (WITTGENSTEIN 1984). This implies that we should start very broadly and not stick to the arithmetic mean as the only possible average value. In the 19th century, statistics seemed to be the science of arithmetic means (FELDMAN et al. 1991), but many people such as Galton and Darwin have opposed that tendency, and preferred other values such as the median. About the midrange: this concept is useful for the educator and teacher to describe what students do. We could allow students to use it as long as they learn the limitations as well.

When students see a lot of different situations and aspects of the average values, they develop a good basis and motivation to make distinctions between them. Not until then will they appreciate and understand the precise mathematical definitions of the statistical concepts. My most important point is that students should first discover many qualitative aspects of the mean before they learn the general form or the algorithm. I think that estimation and redistribution can be good starting points, especially when the visual aspects are stressed.

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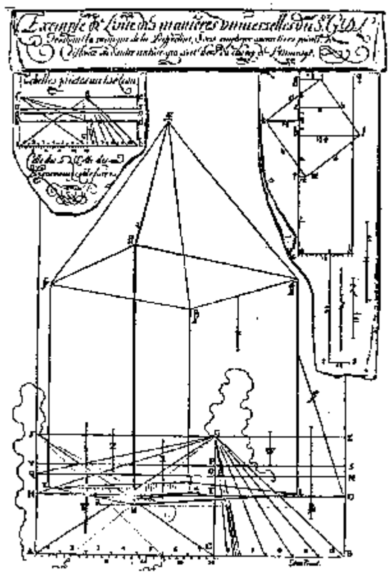

Planche du traité de perspective de Desargues de 1636.

De Brunelleschi à Desargues ou des problèmes liés à la représentation plane d'objets de l'espace . . .

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Abstract

À partir de textes originaux – Leon Battista Alberti, Piero della Francesca, Leonardo da VInci, Albrecht Dürer, Antonio Manetti, Giorgio Vasari, Simon Stevin, . . . – nous tentons de mettre en évidence les difficultés qui surgissent lorsque l'on essaie de représenter dans un plan des objets de l'espace. C'est un problème auquel nos jeunes élèves sont confrontés dans leur cours de géométrie de l'espace. Nous tentons de montrer que ce problème mathématique qui va déboucher sur une nouvelle géométrie -la géométrie prospective- trouve en fait son origine dans un idéal profondément humain de recherche d'esthétique en peinture.

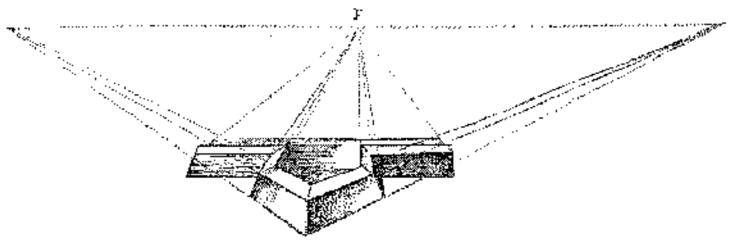

Illustration tirée du *De sterkte Bouwing* de Stevin

1 Introduction

De nombreux élèves du secondaire affichent une certaine réticence quand ce n'est pas un réel dégoût envers le cours de mathématiques. La raison en est très souvent qu'ils ne voient pas bien à quoi les mathématiques peuvent servir. Or il est légitime qu'ils essaient de trouver un sens à ce que nous exigeons d'eux dans le cadre de ce cours. Donner du sens à ce qu'on fait constitue une source importante de motivation.

S'il est relativement aisé de rendre attrayante une revue scientifique traitant même très sérieusement de physique, chimie ou biologie, ... grâce à de belles photographies ou maquettes aux couleurs chatoyantes, le niveau d'abstraction des objets manipulés en mathématiques ne permet guère de recourir à cette forme de vulgarisation. L'aspect esthétique d'un tableau de nombres ou d'une équation est — il faut bien l'admettre — a priori assez limité.

Très souvent, il nous est même impossible de donner une bonne réponse à la question 'à quoi cela sert-il ?', tout simplement parce que les élèves ne disposent pas des connaissances nécessaires à la bonne compréhension de cette réponse.

L'approche historique d'un concept est en général plus efficace et permet de situer le cours de mathématiques dans l'histoire de l'humanité. La nécessité d'évaluer les dimensions d'un terrain, de construire un bâtiment qui sera 'droit', de partager équitablement un avoir, de représenter le monde réel, ... est source de découvertes et d'activités mathématiques auxquelles il est possible de sensibiliser les élèves.

Nous proposons ici une anthologie de quatre textes que nous allons commenter et qui représentent diverses étapes dans la connaissance des règles de la perspective. L'un ou l'autre de ces textes pourra être exploité en classe pour introduire le cours de géométrie de l'espace qui, chez nous en Belgique, fait partie du programme de quatrième année du secondaire.

Pour diverses raisons que nous ne pourrons pas analyser ici, les artistes de la Renaissance n'auront pas les mêmes buts en peinture que leurs prédécesseurs. Au moyen âge, la glorification de Dieu, l'illustration de scènes bibliques sont pratiquement les seuls thèmes abordés dans nos régions. Les arrière-plans dorés suggèrent que les personnages et objets peints coexistent dans des lieux célestes. La réalité a peu d'importance ; c'est plutôt le côté symbolique de la scène qui compte. Les peintres produisent des tableaux plats, comme par exemple la représentation du martyre de Saint Quirin (Figure 1). Les artistes de la Renaissance manifestent eux un intérêt pour la représentation fidèle de la nature et sont dès lors confrontés aux problèmes posés par la représentation de la réalité du monde à trois dimensions sur une toile à deux dimensions. Il y a là une difficulté réelle à laquelle n'échappent pas les élèves. Certains ont beaucoup de mal à interpréter une représentation plane de l'espace, ce qui est sans doute dû au fait que le passage de trois à deux dimensions ne se fait pas sans perte d'information.

Les règles qui seront mises au point à partir du *Quattrocento* sont basées sur des théorèmes mathématiques.

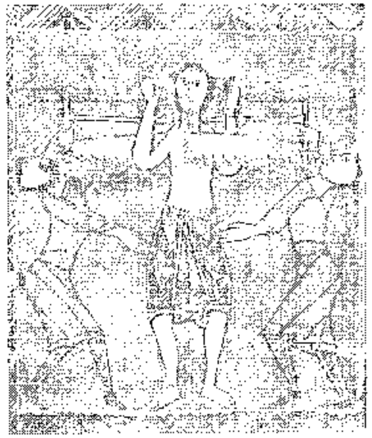

FIGURE 1 : *Le martyre de Saint Quirin, XII^ème^ siècle*

S'il est relativement simple d'exhiber des différences fondamentales entre la peinture de la figure 1 et les deux peintures des figures 2 et 3, il est par contre plus difficile de décider laquelle de ces deux dernières peintures est la plus proche de l'ombre à la lampe d'un damier transparent (4) ou d'une photographie (Figure 5).

FIGURE 2 : *L'annonciation de A. Lorenzetti, 1344*

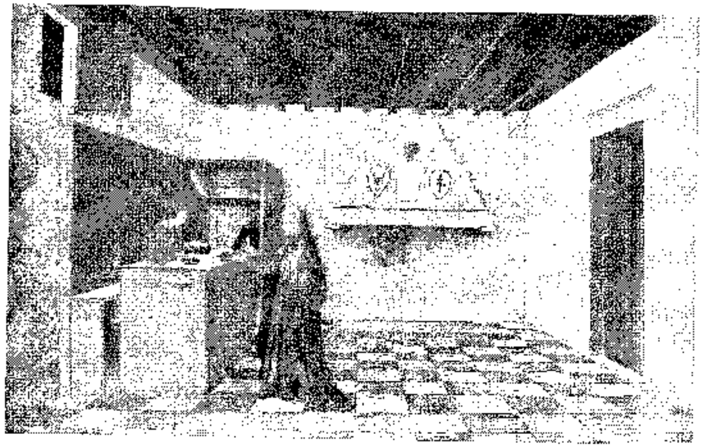

FIGURE 3 : *Le miracle de l'hostie profanée de Uccello, 1469*

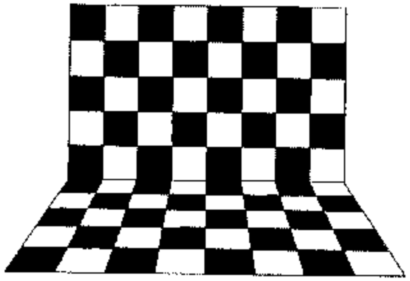

FIGURE 4 : *Ombre à la lampe d'un damier*

FIGURE 5 : *Photographie de carrelage*

Comment se forme une ombre à la lampe ? Que se passe-t-il dans un appareil photographique ? Comment les peintres italiens du *Quattrocento* ont-ils établi les règles de la perspective centrale ? Ces questions sont provoquées par la comparaison des différents documents ci-dessus. Elles ont été abordées sous forme de situations-problèmes dans une brochure élaborée durant cette année par l'équipe du CREM (voir bibliographie). Notons que ces différentes approches permettent également d'introduire la géométrie projective.

Dans le cadre de cet exposé, nous ne parlerons que du dernier point qui concerne quelques étapes historiques de la découverte des règles de la perspective centrale.

2 Filippo Brunelleschi (1377 – 1446)

Selon Giorgio Vasari (1511 – 1574), son intérêt pour les mathématiques l'a conduit à étudier la perspective. Il avait lu Euclide, Hipparche, et appris les mathématiques avec le mathématicien florentin Paolo del Pozzo Toscanelli (1397 – 1482). Pour lui, la peinture était prétexte à pratiquer la géométrie.

Les panneaux que Brunelleschi réalisa pour montrer la validité de la perspective centrale sont perdus. Voici la description qu'en donne Manetti. La scène se passe vers 1415.

> ... La première chose qui révéla cette science de la perspective fut un panneau d'environ une demi-brasse carrée où il peignit l'extérieur de l'église *San Giovanni* de *Firenze*, autant qu'on en voit en la regardant du dehors, comme si pour la représenter il s'était enfoncé de trois brasses environ dans la porte centrale de *Santa Maria del Fiore*. Elle était peinte avec tant de soin et d'artifice, et tant de précision dans les couleurs des marbres blancs et noirs, qu'aucun miniaturiste n'aurait fait mieux. Filippo avait aussi représenté cette partie de la place que voit l'œil du spectateur, c'est-à-dire le côté d'en face de la *Misericordia* jusqu'à la voûte et au *Canto de Pecori*, le côté de la colonne du miracle de *Santo Zanobi* jusqu'au *Canto alla Paglia*, et tout ce qu'on voit au loin ; pour ce qu'on voyait du ciel, c'est-à-dire là où les murailles représentées se détachaient dans l'atmosphère, il était d'argent bruni, afin que l'air et le ciel réel s'y réfléchissent, et de même les nuages entraînés par le vent quand il souffle. Comme le peintre doit supposer un seul point pour voir sa peinture, tant en hauteur qu'en largeur et de biais comme de loin, afin qu'on ne pût se tromper en la regardant, puisque tout changement de lieu entraîne une vision différente, il avait fait dans le panneau supportant cette peinture un trou au point exact de l'église *San Giovanni* où frappait le regard de qui se trouvait à l'intérieur de la porte centrale de *Santa Maria del Fiore*, endroit où il se serait placé s'il l'avait peint sur le motif. Ce trou était petit comme une lentille sur le côté de la peinture, s'élargissant en pyramide, comme un chapeau de paille de femme, du côté du revers, jusqu'à atteindre la circonférence d'un ducat ou un peu plus. Il voulait que celui qui regardait appliquât l'œil au revers, là où le trou était large, qu'une main fût placée près de l'œil et que l'autre tînt, face à la peinture, un miroir plan où celle-ci vînt se réfléchir ; la distance entre le miroir et la seconde main était proportionnellement, en brasses minuscules, pour ainsi dire, la même qu'en brasses réelles entre l'endroit où il supposait s'être mis pour peindre et l'église *San Giovanni* ; si bien qu'en le regardant, grâce aux autres éléments dont on a parlé, l'argent bruni, la place, etc., et de ce point, on croyait voir la réalité même. Ayant eu ce dispositif en main et l'ayant vu plusieurs fois jadis, je peux en porter témoignage.

La Vita di Ser Filippo Brunelleschi, Antonio di Tuccio MANETTI. Republié. Supplément aux *Cahiers de la recherche architecturale*, n°3, C.E.R.A., Paris, 1979.

Manetti insiste sur l'unique point d'où l'œil doit regarder à la bonne distance la scène réelle pour voir exactement ce que le peintre a vu ; ce problème est le même en ce qui concerne l'objectif d'un appareil photographique. Il faut se placer à la bonne distance face à un point précis qui, sur la peinture, porte le nom de point de fuite principal ou point central. C'est précisément en ce point que Brunelleschi avait percé son panneau.

3 Leone Battista Alberti (1404 – 1472)

Descendant d'une illustre et antique famille florentine exilée et dispersée dans toute l'Italie, il est né à Gênes. Il a grandi dans une atmosphère humaniste et est devenu une personnalité très cultivée, que ce soit en sciences, lettres ou arts. Il a beaucoup voyagé et a eu énormément de contacts tant avec les gens du Nord que du Sud. En 1428, les Alberti obtiendront la permission de rentrer à Florence où Leone Battista peut rencontrer Brunelleschi, Donatello, ... En 1435, il écrit son traité *De Pictura* en latin. Il écrira également une *Descriptio urbis Romae*, un autre petit traité *De Statua* et son grand traité d'architecture *De Re aedificatoria*.

Alberti est convaincu de ce que les mathématiques sont essentielles pour traiter les formes de manière à ce qu'elles représentent correctement la réalité. Toute bonne peinture implique une connaissance approfondie de la perspective. Toutefois, au début du *De Pictura*, il précise bien que 'en tout cet exposé je ne parle pas de ces choses en mathématicien mais bien en peintre'. Voici un extrait où il donne sa *costruzióne legittima*.

> [...] Je trace d'abord sur la surface à peindre un quadrilatère de la grandeur que je veux, fait d'angles droits, et qui est pour moi une fenêtre ouverte par laquelle on puisse regarder l'histoire[1], et là je détermine la taille que je veux donner aux hommes dans ma peinture. Je divise la hauteur de cet homme en trois parties et ces parties sont pour moi proportionnelles à cette mesure qu'on nomme vulgairement bras[2]. Car, comme on le voit par la symétrie des membres de l'homme, la longueur la plus commune du corps d'un homme est de trois bras. À l'aide de cette mesure, je divise la ligne de base du rectangle que j'ai tracé en autant de parties qu'elle peut en contenir, et cette ligne de base du rectangle est pour moi proportionnelle à la quantité transversale la plus proche sur le sol et qui lui est parallèle. Je place ensuite un seul point, en un lieu où il soit visible à l'intérieur du rectangle. Comme ce point occupe pour moi le lieu même vers lequel se dirige le rayon central, je l'appelle point central. Ce point est convenablement situé s'il ne se trouve pas, par rapport à la ligne de base, plus haut que l'homme que l'on veut peindre. De cette façon, ceux qui regardent et les objets peints sembleront se trouver sur un sol plat. Une fois ce point central placé, je tire des lignes droites de ce point à chacune des divisions de la ligne de base, et ces lignes me montrent comment les quantités transversales successives changent d'aspect presque jusqu'à une distance infinie.[3]

[1] Selon Schefer, il faut entendre par là un agencement de parties (corps, personnages, choses) doté de sens.

[2] Mesure florentine valant environ 0.58 mètre.

[3] [...] Principio in superficie pingenda quam amplum libeat quadrangulum rectorum angulorum inscribo, quod quidem mihi pro aperta finestra est ex qua historia contueatur, illicque quam magnos velim esse in pictura homines determino. Huiusque ipsius hominis longitudinem in tres partes divido, quae quidem mihi partes sunt proportionales cum ea mensura quam vulgus brachium nuncupat. Nam ea trium brachiorum, ut ex symmetria membrorum hominis patet, admodum commuois humani corporis longitudo est. Ista ergo mensura iacentem infimam descripti quadranguli lineam in quod illa istiusmodi recipiat partes divido, ac mihi quidem haec ipsa iacens quadranguli linea ex proximiori transversae et aequaedistanti in pavimento visae quantitati proportionalis. Post haec unicum punctum quo sit visum loco intra quadrangulum constituo, qui mihi punctus cum locum occupet ipsum ad quem radius centricus applicetur., idcirco centricus punctus dicatur. Condecens huius centrici puncti positio est non altius a iacenti linea quam sit illius pingendi hominis longitudo, nam hoc pacto aequali in solo et spectantes et pictae res adesse videntur. Posito puncto centrico, protraho lineas rectas a puncto ipso centrico ad singulas lineae iacentis divisiones, quae quidem mihi lineae demonstrant quemadmodum paene usque ad infinitam distantiam quantitates transversae successivae sub aspectu alterentur.

> Pour ce faire, certains traceraient à travers le rectangle une ligne parallèle à la ligne de base et diviseraient en trois parties l'intervalle qui se trouve entre les deux lignes. Puis, à cette seconde ligne parallèle à la ligne de base, ils ajouteraient une autre ligne parallèle, placée de telle façon que l'intervalle divisé en trois parties qui sépare la ligne de base de la seconde ligne soit plus grand d'une partie que celui qui sépare la seconde ligne de cette troisième ; et ils ajouteraient ainsi d'autres lignes pour que l'intervalle qui suit un autre intervalle entre les lignes soit toujours, pour employer le terme des mathématiciens, *superbipartiens*. Ceux qui feraient ainsi, même s'ils affirmaient suivre la meilleure voie en peinture, je déclare qu'ils se trompent beaucoup car, ayant posé au hasard la première ligne parallèle, quand bien même les autres lignes parallèles se suivraient selon un même rapport de diminution, le fait est qu'ils n'ont pas le moyen d'obtenir un lieu précis pour la pointe [de la pyramide] qui permet de bien voir.[4]
>
> [...]
>
> J'ai d'ailleurs trouvé cette excellente méthode : dans tous les cas je poursuis cette même division entre le point central et la ligne de base en tirant des droites de ce point jusqu'à chacune des divisions de la ligne de base. Mais pour la succession des quantités transversales, je procède de cette manière-ci. Je prends une petite surface sur laquelle je trace une seule ligne droite. Je la divise en autant de parties que la ligne de base du rectangle est divisée. Je pose ensuite un point unique au-dessus de cette ligne, à la verticale d'une de ses extrémités, aussi élevé que l'est dans le rectangle le point central au-dessus de la ligne de base. De ce point, je trace des droites jusqu'à chacune des divisions de la ligne. Je fixe alors la distance que je désire avoir entre l'œil de celui qui regarde et la peinture, puis, ayant fixé l'emplacement de la section, au moyen de ce que les mathématiciens appellent une ligne perpendiculaire, je produis l'intersection de toutes les lignes qu'elle rencontre.[5]
>
> Une ligne perpendiculaire est celle qui, divisant une autre ligne droite, possède partout autour d'elle des angles droits. Ainsi cette ligne perpendiculaire me donnera par ses points d'intersection les

[4] Hic essent nonnulli qui unam ab divisa aequedistantem lineam intra quadrangulum ducerent, spatiumque, qod interutrasque lineas adsit, in tres partes dividerent. Tum huic secundae aequedistanti lineae aliam item aequedistantem hac lege adderent, ut spatium, quod inter primam divisam et secundam aequedistantem lineam est, in tres partes divisum una parte sui excedat spatium id quod sit inter secundam et tertiam lineam, ac deinceps reliquas lineas adderent ut semper sequens inter lineas esset spatium ad antecedens, ut verbo mathematicorum loquar, superbipartiens. Itaque sic illi quidem facerent, quos etsi optimam quandam pingendi viam sequi affirment, eosdem tamen non parum errare censeo, quod cum casu primam aequedistantem lineam posuerint, tametsi caeterae aequedistantes lineae ratione et modo subsequantur, non tamen habent quo sit certus cuspidis ad bene spectandum locus. [...]

[5] Haec cum ita sint, ipse idcirco optimum hunc adinveni modum. In caeteris omnibus eandem illam et centrici puncti et lineae iacentis divisionem et a puncto linearum ductionem ad singulas iacentis lineae divisiones prosequor. Sed in successivis quantitatibus transversis hunc modum servo. Habeo aerolam in qua describo lineam unam rectam. Hanc divido per eas partes in quas iacens linea quadranguli divisa est. Dehinc pono sursum ab hac linea punctum unicum ad alterum lineae caput perpendicularem tam alte quam est in quadrangulo centricus punctus a iacente divisa quadranguli linea distans, ab hocque puncto ad singulas huius ipsius lineae divisiones singulas lineas duco. Tum quantam velim distantiam esse inter spectantis oculum et picturam statuo, atque atque illic statuto intercisionis loco, perpendiculari, ut aiunt mathematici, linea intercisionem omnium linearum, quas ea invenerit, efficio. Perpendicularis quidem linea est ea quae aliam rectam lineam dividens angulos utrinque circa se rectos habeat. Igitur haec mihi perpendicularis linea suis percisionibus terminos dabit omnis distantiae quae inter transversas aequedistantes pavimenti lineas esse debeat. Quo pacto omnes pavimenti parallelos descriptos habeo, est enim parallelus spatium quod intersit inter duas aequedistantes lineas de quibus supra nonnihil tetigimus. Qui quidem quam recte descripti sint inditio erit., si una eademque recta continuata linea in picto pavimento coadiunctorum quadrangulorum diameter sit. Est quidem apud mathematicos diameter quadranguli recta quaedam linea ab angulo ad sibi oppositum angulum ducta, quae in duas partes quadrangulum dividat ita ut ex quadrangulo duos triangulos efficiat.

limites de chaque écartement qui doit se trouver entre les lignes transversales parallèles du dallage. Je peux de cette façon tracer toutes les rangées transversales de carreaux du dallage. On appelle parallèle l'intervalle séparant deux des lignes parallèles dont nous avons parlé plus haut. J'aurai la preuve que celles-ci ont été correctement tracées si une même ligne droite prolongée sur le dallage peint sert de diamètre aux rectangles juxtaposés. Pour les mathématiciens, le diamètre d'un rectangle est la ligne droite, tirée d'angle à angle opposé, qui divise le rectangle en deux parties de façon à faire deux triangles.

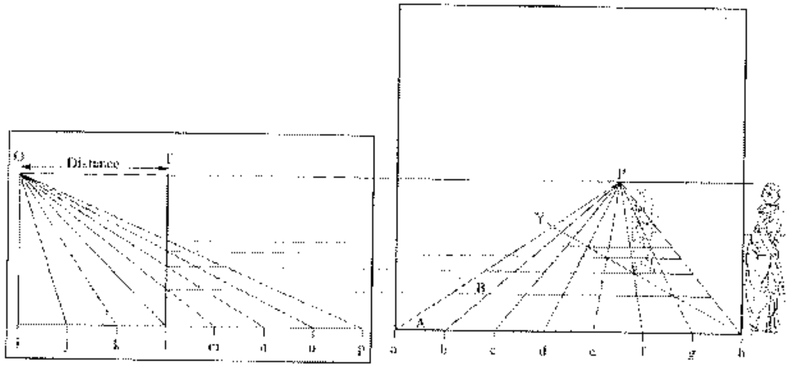

FIGURE 6 : *La costruzióne legittima*

La dernière partie de l'extrait du texte d'Alberti suggère maintenant d'aller vérifier les diagonales des carrelages des figures 2, 3, 4 et 5.

Remarquons que, pour expliquer sa *costruzióne legittima*, Alberti donne une représentation de la situation au moyen de deux projections orthogonales (plan et profil). Et ce sont ces deux projections qui permettent la mise en perspective d'une figure. Il n'aurait certainement pas pu donner une représentation de la situation en perspective cavalière car celle-ci n'est entrée dans la pratique qu'à la fin du seizième siècle en Europe. Elle convenait parfaitement pour réaliser des plans d'architecture classique pour laquelle elle faisait mieux percevoir les alignements, les symétries, le parallélisme, ... La perspective cavalière donne une vue plus globale des objets car en fait, le point de vue de l'observateur est rejeté à l'infini. Elle aura également beaucoup d'applications dans l'art de la guerre, car il est plus facile d'y mesurer les objets, les distances. P. COMAR (voir bibliographie) signale que, dans ses *Cours de mathématique nécessaires à un homme de guerre* (1693), Ozanam écrit :

> Pour représenter les fortifications, on se sert d'une perspective [...] qu'on appelle *Perspective Cavalière* et *Perspective Militaire*, qui suppose l'œil infiniment éloigné du Tableau, [...] et quoiqu'elle soit naturellement impossible, la force de la vue ne pouvant se porter à une distance infinie, elle ne laisse pas néanmoins de faire bon effet.

Signalons encore que depuis l'Antiquité et notamment en Extrême-Orient (peintures japonaises, par exemple), on trouve des utilisations de cette perspective, mais pas comme moyen global de représentation. Ainsi, une partie du 'tableau' qui fait penser à une perspective cavalière peut avoisiner une projection orthogonale, par exemple. C'est surtout une manière empirique de suggérer le relief.

Alberti fait allusion à de fausses perspectives, notamment celle qui utilise la règle des deux tiers. Il s'agit là d'utilisations empiriques de la perspective. Dans son ouvrage sur *Piero della Francesca* (voir bibliographie), J.-P. LE GOFF (note 21 de la page 48) explique comment on peut renverser une certaine problématique de la perspective. À partir du moment où on prend conscience de ce que des objets de même grandeur paraissent de plus en plus petits lorsqu'on les éloigne, si l'on tente de construire un carrelage en en tenant compte, le théorème de Thalès nous offre un point de fuite 'en cadeau'. Et ainsi, dans certains tableaux antérieurs à la découverte des règles de la perspective, on trouve par exemple plusieurs points de fuite en *arête de poisson*, c'est-à-dire disposés sur une même verticale.

Quant à la pyramide dont il est question dans l'extrait, il s'agit de la pyramide visuelle, notion connue depuis fort longtemps en optique, reprise par les artistes de la Renaissance pour expliquer la perspective. Voici par exemple, un extrait des Carnets de notes de Leonardo da Vinci (1452 – 1519) :

> La perspective n'est rien d'autre que la vision d'un objet derrière un verre lisse et transparent, à la surface duquel pourront être marquées toutes les choses qui se trouvent derrière le verre ; ces choses approchent le point de l'œil sous forme de diverses pyramides que le verre coupe.

C'est peut-être cette conception de la perspective qui fait qu'Alberti, dans son explication, trace d'abord tous les rayons joignant l'œil aux différents objets à représenter avant de fixer la distance – en coupant la pyramide visuelle – qu'il désire avoir entre l'œil de celui qui regarde et la peinture.

4 Piero della Francesca (1410-20 – 1492)

Il naît dans le petit village de Borgo San Sepolcro. Son père était cordonnier. Vasari note chez lui un intérêt précoce pour les mathématiques. On ne sait que peu de choses de ses premiers maîtres. Assez vite, il travaille et voyage avec Domenico Veneziano mais en 1439, il est toujours un peintre de renommée plutôt locale. Par la suite on le verra à Urbino, Rimini, ...

Son traité *De prospectiva pingendi*, écrit en italien mais avec un titre en latin, date des années 1470 - 80. Il innove en ce sens qu'il traite la perspective en termes de géométrie et la détache du domaine de l'optique. C'est de la géométrie pratique mise au service de la résolution d'un problème technique. Il écrit pour les peintres mais il donne toutes les démonstrations dans le style des *Éléments* d'Euclide. On lui doit encore un *Trattato dell'abaco* (arithmétique, algèbre, stéréométrie) et un *Libellus de quinque corporibus regularibus* dont son élève, Luca Pacioli s'inspira lorsqu'il écrivit son traité de la *Divine proportion*.

Voici un texte extrait de son traité *De la perspective en peinture* qui montre comment réduire en carré un plan dégradé.

> Comme dans la proposition précédente, soit DC une ligne partagée au point B, menons BF qui lui est perpendiculaire et une autre ligne perpendiculaire au-dessus de D jusqu'en A, placé en son lieu[6] ; tirons une ligne perpendiculaire au-dessus de C, de longueur égale à BC, soit GC, et du point G, menons une parallèle à BC, soit GF ; je dis qu'il s'ensuit un carré de côtés égaux BC, CG, GF et FB. Maintenant, je tire du point A les lignes AC et AG, qui coupent BF en deux points : AC coupe BF au point E, et AG coupe BF au point H. Je dis que E apparaît au point A plus élevé que B, parce que A se tient au-dessus de B, et que H apparaît plus bas que F, parce que A est plus bas que F, comme cela est démontré dans les dixième et onzième propositions du *De*

[6]Ce lieu est l'endroit où se trouve l'œil du peintre.

Aspectuum Diversitate d'Euclide[7]. Je dis que BE apparaît égal à BC dans le lieu déterminé, et que EH apparaît égal à CG dans le même lieu déterminé, et que HF apparaît égal à FG. Tirons AF et AB : nous aurons trois triangles, chacun avec deux bases [opposées à A] : le triangle ABC avec les deux bases BC et BE, le triangle ACG avec les deux bases CG et HE, et le triangle AGF avec les deux bases FG et FH ; d'où par la seconde proposition de ce livre, la base BE apparaît égale à la base BC, parce qu'elles sont sous un même angle A, et la base EH apparaît égale à CG, du fait qu'elles sont sous un même angle, et la base HF apparaît égale à FG, parce qu'elles sont contenues dans un même angle ; la proportion qui est de AE à AC, est celle qui est de DB à DC, la même qui est de EH à CG, mais aussi de AE à AC, [. . .]

Donc je dirai que EH-CG[8] est le plan BE réduit à un carré. Maintenant, menons du point A une ligne parallèle à BC, prolongée sans fin, puis divise la ligne BC en deux parties égales au point I, et tire au-dessus de I la perpendiculaire ; à l'endroit où elle coupe la ligne qui part du point A parallèlement à BC, se trouve le point A' ; puis tire à partir de E une parallèle à BC qui coupe CG au point K, mène à partir de A' une ligne vers B, qui coupe EK au point D', puis tire à partir de A' une ligne vers C qui coupera EK au point E' ; je dis avoir mis au carré le plan dégradé, à savoir $BCD'E'$. La preuve : voyons si $D'E'$ est égale à EH qui apparaît égale à la quantité CG, comme cela a été prouvé ci-dessus. Je dis qu'elle est égale ou semblable, car la proportion qui est de $A'D'$ à $A'B$, est de $A'E'$ à $A'C$, est la même que celle de $D'E'$ à BC, qui est aussi de EH à CG. Étant proportionnelles, elles sont égales ou semblables, mais en fait égales, car nous avons admis que BC de l'une est égale à BC de l'autre[9], ce qui éclaire l'énoncé. Mais si tu me disais : pourquoi mets-tu l'œil au milieu ? Parce que cela me paraît convenir mieux pour voir les opérations ; néanmoins, chacun peut le mettre là où il lui plaît, [. . .]

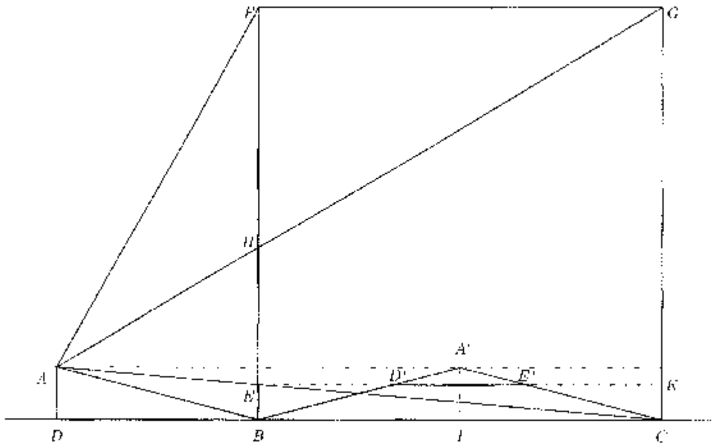

FIGURE 7 : *Réduction en carré d'un plan dégradé*

[7]Euclide, *L'Optique*.

[8]Il s'agit du rapport de réduction.

[9]C'est-à-dire $BC = CG$.

5 Albrecht Dürer (1471 – 1528)

Il est né à Nuremberg qui, à l'époque, était une importante ville marchande à l'artisanat très développé. Son père était orfèvre et ce fut donc sa première formation. C'est dès l'âge de quinze ans qu'il put s'initier à la peinture dans l'atelier du peintre Wolgemut. La présence dans les murs de la ville de Johannes Müller, plus connu sous le nom de Regiomontanus, ne sera pas étrangère à la formation humaniste d'Albrecht Dürer. Il voyagera en Italie et à son retour, s'intéressera de plus en plus aux fondements de l'art pictural. Sa connaissance du latin était sans doute faible et c'est grâce à ses amis humanistes qu'il pourra accéder à la culture classique. L'un des ouvrages les plus connus de Dürer est l'*Underweysung der Messung* de 1525, titre traduit par *Instruction sur l'art de mesurer* ou plus simplement *La Géométrie*. Le livre IV est notamment consacré à la perspective.

Dürer était conscient des difficultés rencontrées par le peintre lorsqu'il essayait de mettre 'le monde à plat'. Il a imaginé différents dispositifs, encombrants et peu maniables, qui ont cependant le mérite de mettre en évidence les grands principes de la perspective centrale. Voici un extrait du livre IV qui décrit un matériel de ce type.

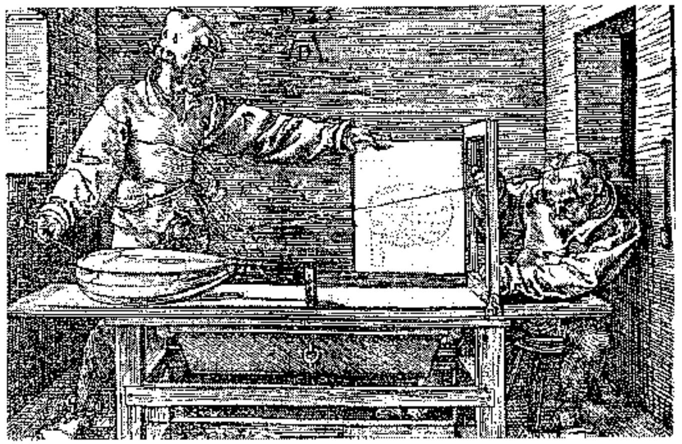

FIGURE 8 : *L'un des dispositifs de Dürer*

À l'aide de trois fils tu peux projeter dans un plan et dessiner sur un tableau tout objet que tu peux atteindre par ces fils. Procède comme suit.

Si tu te trouves dans une salle, enfonce une aiguille au chas très vaste, spécialement conçue à cet effet, dans un mur. Elle jouera le rôle de l'œil. Fais-y passer un fil solide et suspends un poids en plomb à une de ses extrémités. Dispose ensuite une table ou un tableau à une distance arbitraire du chas de l'aiguille où se trouve le fil. Places-y un cadre vertical, transversal par rapport au chas de l'aiguille, plus haut ou plus bas, et du côté voulu. Dans ce cadre se trouvera un portillon qui s'ouvre et se ferme. Ce portillon sera le tableau sur lequel tu peindras. Fixe ensuite par des clous deux fils qui ont des longueurs respectivement égales à la longueur et à la largeur du cadre, l'un au milieu de la traverse supérieure, l'autre au milieu d'un montant, puis laisse-les pendre librement. Prépare ensuite une longue pointe en fer avec un chas dans son extrémité pointue, par lequel tu feras passer le fil long passant déjà par l'aiguille fixée au mur. Amène la pointe avec le fil long dans le cadre et au-delà, et remets-la à quelqu'un d'autre, alors que toi tu t'occuperas des deux autres fils fixés au cadre. Utilise ce dispositif comme suit. Pose le luth ou tout autre objet qui te plaît à la distance voulue du cadre et veille à ce qu'il ne bouge point pendant le temps dont tu en as besoin. Demande à ton compagnon de maintenir tendu le fil passant par l'aiguille et de l'amener sur les principaux points du luth. Dès qu'il s'arrête sur un point et tend le fil, amène les deux fils fixés au cadre, tendus, à se croiser avec le fil long. Attache leurs extrémités avec de la cire au cadre ; ordonne à ton compagnon de ne plus tendre le fil long. Ferme alors le portillon et reporte sur le tableau le point où les deux points se croisent. Rouvre le portillon et procède de même pour un autre point et ainsi de suite jusqu'à ce que le luth apparaisse en pointillé sur le tableau. Joins par des lignes tous ces points que tu as obtenus à partir du luth sur le tableau, et tu verras ce qui adviendra. [. . .]

6 Girard Desargues (1591 – 1661)

Né à Lyon, on connaît peu de choses sur son enfance. C'est à plus de trente-cinq ans qu'on retrouve sa trace à Paris où le graveur Abraham Bosse affirme que dès le début de l'année 1630, il avait déjà obtenu le privilège royal pour divers écrits qu'il avait l'intention de publier. En tant qu'ingénieur, Desargues était semble-t-il fort apprécié de Richelieu. Il eut des contacts avec le Père Mersenne, Descartes, Roberval, Pascal et bien d'autres. Parmi ses œuvres, citons le *Brouillon project d'une atteinte aux evenemens des rencontres du Cone avec un Plan* (1639), qui traite des coniques et trois paragraphes d'esprit géométrique qui terminent le *Traité de perspective*, publié par son élève Abraham Bosse en 1648. Le fonds et la forme de ces paragraphes montrent, selon René Taton, qu'ils doivent être attribués à Desargues lui-même. Le titre que Bosse a donné à son traité est d'ailleurs *Maniere universelle de M. Desargues, pour pratiquer la Perspective, . . .* La première proposition qu'on y trouve est le célèbre 'théorème de Desargues'. Ce théorème, vrai aussi bien dans l'espace que dans le plan, permet de ramener à deux dimensions toute configuration spatiale, ce qui est l'esprit même de la perspective. Cet aboutissement sera à l'origine de la géométrie projective.

Signalons pour terminer que dès 1636, Desargues publiait à Paris, un ouvrage intitulé *Exemple de l'une des manieres universelles du S.G.D.L. touchant la pratique de la perspective sans emploier aucun tiers point, de distance ny d'autre nature, qui soit hors du champ de l'ouvrage*. Cette étude fut en partie plagiée par Du Breuil, révérend père jésuite. Desargues riposte en placardant sur les murs de Paris des affiches dénonçant cet emprunt ; mais cela ne lui profite pas et, découragé, il abandonne. C'est sans doute pour cela que c'est son disciple Bosse qui publiera, comme nous l'avons dit, le fameux théorème. Bosse sera alors admis comme membre honoraire à l'Académie royale pour y enseigner la perspective. Il prône un peu trop que l'art repose tout entier sur les règles de la géométrie et finit par se quereller avec Le Brun, futur premier peintre du Roi et directeur de l'Académie. L'affaire s'envenime et l'Académie prononce l'exclusion de Bosse en 1661. Il fonde alors sa propre école. Cette impudence a pour conséquence un arrêt du Conseil signé par le roi, qui donne à l'Académie le monopole de l'enseignement artistique.

Si le théorème de Desargues marque le point d'aboutissement de toutes les théories sur la perspective et annonce la géométrie projective, il condamne malheureusement cette perspective comme instrument de l'art. L'artiste ne peut se contenter de choses arrivées à terme et donc figées ; il a envie de se tourner vers d'autres voies. Mais si la perspective s'apprête à connaître quelques siècles d''oubli', elle n'est cependant pas définitivement morte. Un peintre comme Delvaux, par exemple, a su l'employer à merveille au vingtième siècle.

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Les nombres complexes, les vecteurs et les quaternions

(L'introduction des quaternions au Portugal par Augusto d'Arzilia Fonseca-1884)

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Abstract

À la fin du XVIIIème siècle, les nombres complexes n'ont pas de statut mathématique.

Avec les travaux de Wessel (1745-1818), Argand (1768-1822), Gauss (1777-1855), les quaternions de Hamilton (1805-1865) et l'algèbre de Grassmann (1809-1877), on voit la représentation géométrique et la multiplication des quantités imaginaires.

En utilisant la multiplication de triples, Hamilton arrive aux quaternions et aux algèbres non-communicatives. Au XIXème siècle on voit apparaître les vecteurs comme une partie de l'ensemble des quaternions. Cependant, Hamilton s'interroge sur la nature de Sab et Vab, au produit des quaternions ab.

Présenter le quaternion, choisir son meilleur mode de représentation, montrer son efficacité et ouvrir des horizons, a été le but de ceux qui ont défendu l'utilisation des quaternions, comme on voit au travail de Arzila Fonseca (Coimbra 1884, Portugal).

1 Les Nombres complexes

Au début du XVI$^{\text{ème}}$ siècle on voit beaucoup de travail sur la résolution des équations algébriques. À ce sujet, nous rappelons les travaux des algébristes italiens (Bombelli, Cardan, Tartaglia), de Nicolas Chuquet en France, de Robert Recorde en Angleterre et de Pedro Nunes au Portugal.

Dans quelques livres de cette époque-là nous pouvons voir beaucoup de problèmes à résoudre, comme par exemple le suivant [STEWART & TALL 1977] :

> *La somme de deux nombres, quels qu'ils soient est 10 et leur produit est 40, quels sont ces nombres ?*

Nous allons transformer le problème en équation et nous avons $x + y = 10$ et $x \cdot y = 40$ ou encore $y = 10 - x$ et $x(10 - x) = 40$. Cela veut dire que $x = 5 \pm \sqrt{-15}$.

Pour les mathématiciens du XVI$^{\text{ème}}$ siècle, $\sqrt{-15}$ était une chose étrange. En supposant que le carré d'un nombre réel est positif, le carré de $\sqrt{-15}$ est (-15), qui est un nombre négatif. D'autre part nous vérifions que l'addition des deux racines trouvées est égale à 10 et leur produit, égal à 40. Cela veut dire que nous pouvons traiter $\sqrt{-15}$ d'une manière *imaginaire* ?

Ces *imaginaires* (c'est Descartes qui a donné ce nom aux nombres dont le carré est négatif, comme nous pouvons voir dans le livre *La Géométrie* [De,1954]), sont entrés dans les opérations usuelles. Cela permet de trouver les solutions des équations et d'arriver à quelques résultats importants.

En 1774, Euler introduit des symboles et il écrit ces nombres $a + b\sqrt{-1}$ (a et b sont des nombres réels).

Les imaginaires prennent du sens et leur représentation géométrique impose de croire à leur existence.

En 1673, John Wallis essaye de représenter géométriquement les solutions imaginaires de l'équation du deuxième degré. [STILLWELL 1989]

$$\boxed{x^2 - 2bx + c^2 = 0, \qquad b, c \geq 0}$$

Cependant il n'a pas réussi car il a traité les racines imaginaires de l'équation comme si elles étaient réelles. Il faut remarquer qu'au XVII$^{\text{ème}}$ siècle les nombres négatifs attendent encore un statut mathématique propre jusqu'au moment (1770) où Euler définit $\sqrt{-a}$ et explicite le produit $\sqrt{-2} \times \sqrt{-3} = \sqrt{6}$.

En 1811, Gauss appelle les imaginaires *nombres complexes* et parle déjà du plan complexe. Il publie quelques résultats sur ce sujet dans *Theoria Residuorum Biquadraticorum* en 1831.

D'autres mathématiciens ont travaillé sur la représentation des *nombres complexes*. En 1797, Wessel (1745-1818) publie l'*Essai sur la représentation analytique de directions*, qui est resté inconnu pendant 100 ans.

L'idée fondamentale qui va permettre d'accepter les imaginaires est d'associer la perpendicularité au signe $\sqrt{-1}$ en relation avec l'unité positive $(+1)$. Ce qui est présent dans les travaux de Wessel, comme nous allons le décrire :

$(+1)$ est le segment unité positif et $(+\epsilon)$ est une autre unité perpendiculaire à $(+1)$. La direction angulaire de $(+1)$ est 0°, de (-1) est 180°, de $(+\epsilon)$ est 90° et de $(-\epsilon)$ est (-90°) ou encore 270°.

En utilisant la règle du produit de deux segments sur le même plan, Wessel est arrivé aux égalités suivantes :

$$\begin{array}{|lll|}
\hline
(+1)\cdot(+1) = (+1) & (+1)\cdot(-1) = (-1) & (-1)(-1) = (+1) \\
 & (+1)\cdot(-\epsilon) = (+\epsilon) & \\
(+1)\cdot(-\epsilon) = (-\epsilon) & (-1)\cdot(-\epsilon) = (-\epsilon) & (-1)\cdot(-\epsilon) = (+\epsilon) \\
 & (-\epsilon)\cdot(-\epsilon) = (-1). & \\
\hline
\end{array}$$

En 1806, Argand (1768-1822) publie *Essai sur une manière de représenter les quantités imaginaires*, où il définit géométriquement les *imaginaires*, leur addition et leur produit. L'acceptation de ces entités dans le champ algébrique revient à considérer i comme la moyenne proportionnelle géométrique entre $(+1)$ et (-1) et nous avons $i^2 = -1$.

2 Hamilton et l'invention des quaternions

Hamilton est né à Dublin en 1805 et il est mort, aussi dans cette ville en 1865. À l'âge de 3 ans il va habiter avec son oncle, qui reconnaît rapidement sa précocité, surtout dans l'étude des langues.

En 1823, Hamilton entre au Trinity College à Dublin et en 1827, il obtient le poste d'astronome royal à l'observatoire de Dunsink et de professeur d'astronomie au Trinity College.

De 1830 à 1840, Hamilton écrit une théorie des *nombres complexes* en utilisant les couples (a, b), a et b étant des nombres réels.

On considère que les résultats déjà connus sur les *nombres complexes*, leur représentation géométrique et la théorie de couples (1837), ont joué un rôle très important dans l'invention des quaternions. La Théorie des Couples a convaincu Hamilton de la légitimité des *nombres complexes* et, en même temps a donné le moyen d'aller au-delà de la dimension 2. Elle a préparé la découverte et la certitude qu'il fallait passer de la dimension 2 à la dimension 4. Cependant la démarche d'Hamilton pour arriver aux quaternions a été longue et a occupé vingt ans de sa vie.

Par analogie avec les nombres complexes, Hamilton écrit les triples $a + bi + cj$, où les vecteurs $1, i, j$ sont perpendiculaires deux à deux et de longueur un. Pour le produit des triples il a considéré qu'il était possible de multiplier terme à terme et que la longueur du vecteur produit était égale au produit des longueurs des vecteurs facteurs (1)

Lors du produit $(a + bi + cj)(x + yi + zj)$ (2), on voit apparaître les expressions i^2, j^2 que Hamilton égale à (-1), par analogie avec $i^2 = -1$ dans la *théorie des couples*. Le produit ij fait surgir beaucoup de doutes, ce qui a amené Hamilton à faire quelques essais que nous allons décrire [HAMILTON 1844] :

1- Si on a $i^2 = j^2 = -1$, alors $(ij)^2 = 1$, mais dans ce cas $ij = 1$ ou $ij = -1$ et au produit (2), la propriété (1) n'est pas vérifiée.

2- Si on a le cas particulier du produit $(a + ib + cj)^2 = a^2 - b^2 - c^2 + 2iab + 2jac + 2ijbc$, en faisant l'addition des carrés des coefficients de $1, i, j$, à droite on a $(a^2 - b^2 - c^2)^2 + (2ab)^2 + (2ac)^2 = (a^2 + b^2 + c^2)^2$. La règle du produit est vérifiée si $ij = 0$, ce qui n'est pas en accord avec la propriété (1)

3- Hamilton est sûr qu'il ne peut pas faire $ij = 0$ et il écrit $ij = -ji$ ou $ij = k.\, ji = -k$. Il reste encore des doutes sur la valeur de k.

4- Hamilton multiplie les triples $(a + ib + jc)$ et $(x - ib + jc)$ et il obtient :

$$ax - b^2 - c^2 + i(a + x)b + j(a + x)c + k(bc - bc).$$

Si on considère $ij = -ji$, le coefficient k a disparu. Les expressions $ax - b^2 - c^2, (a + x)b, (a + x)c$ sont les vraies coordonnées du produit et Hamilton a la confirmation de ce résultat par voie géométrique.

Avec l'égalité $ij = -ji$, Hamilton généralise le produit des triples

$$(a + ib + jc)(x + iy + jz) = (ax - by - cz) + i(ay + bx) + j(az + cx) + k(bz - cy)$$

et il s'interroge sur la véracité de l'égalité

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax - by - cz)^2 + (ay - bx)^2 + (az + cx)^2.$$

Le membre de gauche est plus grand que celui de droite en $(bz - cy)^2$, qui est en même temps le carré du coefficient en k, quand on condidère $ij = k$ et $ji = -k$. Hamilton n'a plus de doutes sur l'existence d'une autre unité imaginaire, qu'il appelle k.

Le 16 octobre 1843 est le jour de la découverte des quaternions. En partant du produit $ij = -ji = k$, Hamilton travaille sur une base avec quatre vecteurs indépendants, $1, i, j$ et k, jusqu'à arriver aux résultats suivants (en supposant $i^2 = j^2 = k^2 = -1$)

$$\begin{aligned} ij &= -ji = k \\ jk &= -kj = i \\ ki &= -ik = j \end{aligned}$$

qui vérifient la propriété (1), pour le produit des quaternions, quels qu'ils soient.

En s'appuyant sur l'ensemble des *nombres complexes*, Hamilton a essayé le produit de triples. Ce travail, par hasard, l'a amené à une troisième unité imaginaire, k et à l'algèbre des quaternions.

Le scepticisme sur les nombres de carré négatif a été dépassé et en même temps, se créent des abîmes de doutes sur la nouvelle théorie des quaternions due à Hamilton qui n'obéit pas aux lois de l'arithmétique usuelle; on a perdu la propriété de commutativité de la multiplication.

Hamilton écrit tout quaternion sous la forme $q = w + ix + jy + kz, (w, x, y, z \in \mathbb{R})$ et il introduit, pour la première fois les mots *scalar* et *vector*. Sur l'ensemble des quaternions, ils déterminent deux entités différentes.

Nous considérons $q = w + ix + jy + kz$, w est le *scalar* et $(ix + jy + kz)$ est la partie vectorielle et Hamilton a écrit

$Q =$ Scal. $Q +$ Vect. Q ou $Q = S.Q + V.Q$ ou encore $Q = SQ + VQ$, pour le quaternion Q.

Considérons le cas particulier du produit $\alpha\alpha'$ des quaternions $\alpha = ix + jy + kz$ et $\alpha' = ix' + jy' + kz'$:

$$\alpha\alpha' = (-xx' - yy' - zz') + i(yz' - zy') + j(zx' - xz') + k(xy' - x'y)$$

$(-S'\alpha\alpha)$ est le produit scalaire et $V\alpha\alpha'$ est le produit vectoriel au sens moderne.

Hamilton a vu que les quaternions ne ressemblaient pas aux nombres complexes et surtout il a eu des doutes sur la représentation géométrique de la partie réelle. À ce propos il écrit [CROWE 1993] :

> Regarded from a geometrical point of view, this algebraically imaginary part of a quaternion has thus so natural and simple a signification or representation in space, that the difficulty is transferred to the algebraically real part; and we are tempted to ask what this last can denote in geometry, or what in space might have suggested it.

En 1989, Simon L. Altmann, publie un article sous le titre *Hamilton, Rodrigues, and the Quaternion Scandal* (What went wrong with on of the major mathematical discoveries of the nineteenth century), dans le *Mathematical Magazine* [ALTMANN 1989], ce qui nous fait croire que l'intérêt pour la théorie des quaternions reste présent à notre époque.

Dans cet article, Altmann met côte à côte Hamilton, considéré l'inventeur des quaternions et Olinde Rodrigues, directeur de la *Caisse Hypothécaire* de la Rue Neuve-Saint –Augustin à Paris. L'auteur dit qu'en 1840 Olinde Rodrigues a présenté quelques résultats sur les rotations dans l'espace, en donnant un sens géométrique aux entités créées par Hamilton trois ans plus tard.

Depuis qu'Hamilton est mort en 1865, la théorie des quaternions a eu quelques disciples. On a resenti l'importance des produits scalaire et vectoriel, grâce au produit des quaternions. À ce sujet nous rappelons les travaux de Hermann Grassmann (1809-1877) en Allemagne et de Heaviside (1850- 1925) en Angleterre.

En 1844 Grassmann a publié la *Théorie de l'Extension*, en présentant deux produits, le vectoriel (*outer product*) et le scalaire (*inner product*). En considérant la notation moderne et la base $e_1, e_2, \cdots, e_n$, le produit vectoriel obéit aux propriétés suivantes $(x, y, z \in \mathbb{N})$

$$\begin{aligned} e_x e_y &= -e_y e_x \\ e_x e_x &= e_y e_y = 0 \\ e_x(e_y + e_z) &= e_x e_y + e_x e_z. \end{aligned}$$

Dans le langage de Grassmann le résultat du produit vectoriel est une entité de deuxième ordre, une surface orientée ou un *bivecteur*. Dans son algèbre nous avons les 0-vecteurs, les 1-vecteurs, les bivecteurs, etc.

David Hestenes [HESTENES 1986] considère que Grassmann a développé le produit vectoriel, à partir d'un travail déjà réalisé par son père Gunther Grassmann, sur le sujet, pour l'enseignement de base.

Dans l'algèbre de Grassmann, à un bivecteur B correspond un parallélogramme unique, déterminé par le produit des vecteurs a et b sur leurs côtés tel que $a \wedge b = B$ et $a \wedge b = -b \wedge a$.

Plus tard Grassmann a additionné les produits $a \cdot b$ (le produit scalaire) et $a \wedge b$ en créant le produit mixte, $ab = a \cdot b + a \wedge b$, qui est à la base des algèbres de Clifford [CNOPS & MALONEK 1995].

Le travail de Grassmann a été considéré comme très difficile à comprendre par les mathématiciens de son époque.

En 1886, Gibbs (1839-1903) a présenté les idées de Hamilton et de Grassmann, côte à côte, sans réussir à établir un lien entre eux et il a écrit à ce propos :

We begin by studying multiple algebras ; we end, I think, by studing MULTIPLE ALGEBRA [CROWE 1993].

Le travail de Hamilton l'a amené à un système de nombres pour représenter les rotations dans l'espace, où il a essayé d'expliquer la géométrie en langage algébrique. Hamilton et Grassmann travaillent en même temps mais de façon indépendante.

En 1878, William Clifford (1845-1879) a publié un article sous le titre *Applications of Grassmann Extensive Algebra*, où il dit que Hamilton et Grassmann sont restés très proches du même sujet, en utilisant différents points de vue et Clifford écrit [CLIFFORD 1878] :

> Now there are two sides to the notion of a product. When we say $2 \times 3 = 6$, we may regard the product 6 as a number derived from the numbers 2 and 3 by a process in which they play similar parts; or we may regard it as derived from the number 3 by the operation of doubling. In the former view 2 and 3 are both numbers; in the latter view 3 is a number, but 2 is an operation, and the two factors play very distinct parts. The Ausdehnungslehre is founded on the first view: the theory of quaternions on the second.

Depuis qu'on a connaît le travail de Grassmann, les mathématiciens se sont divisé entre ceux qui ont défendu les quaternions et ceux qui sont contre cette théorie. Crowe [CROWE 1993] présente les publications, dans différents pays, des travaux sur les quaternions et sur l'algèbre de Grassmann. Au Portugal nous pouvons voir la publication de deux livres sur la théorie des quaternions.

En 1884, Augusto d'Arzilla Fonseca, publie le livre (3) *Principios Elementares do Cálculo de Quaterniões* [FONSECA 1884], suivi par un autre livre, *Applicação dos Quaterniões á Mecânica*, publié en 1885.

En prenant en compte la liste de Alexander Macfarlane [MACFARLANE 1904], nous pouvons considérer que Arzilla Fonseca a été le premier à travailler sur la Théorie des Quaternions au Portugal.

On considère que les mathématiciens portugais connaissent la production en mathématique en Europe. Quelques uns ont fait leur formation à l'étranger et ils ont établi des contacts avec leurs amis d'Europe, c'est l'exemple de Henrique Manuel de Figueiredo (1861-1922) qui a été le premier à travailler et à divulguer les idées de Riemann au Portugal [GRAY & ORTIZ 1999].

En 1877, Francisco Gomes Teixeira a créé le *Jornal de Sciencias Matematicas e Astronomicas*, ce qui a permis la divulgation de travaux mathématiques tant portugais qu'étrangers [SILVA].

Augusto d'Arzilla Fonseca[1] est contemporain de Francisco Gomes Teixeira et de Henrique Manuel de Figueiredo et il a participé au climat dynamique crée à la *Faculdade de Matemática da Universidade de Coimbra.*

Pour obtenir le poste de professeur à Coimbra, Arzilla Fonseca a présenté sa thèse, sous le titre (3). Ce travail se divise en six parties, sous les thèmes suivants :

I – *Propriedades das operações*
II – *Vectores e sua composição*
III – *Produto e quociente de vectores. Quaterniões*
IV – *Interpretação e transformação de expressões*

[1] Augusto d'Arzilla Fonseca est né au Funchal (Madeira) en 21 octobre 1853 et il est mort à Porto le 17 février 1912. Il a été licencié en philosophie (7/07/1883) et en mathématique (3/3/1834) à l'université de Coimbra. Le 27 juillet 1884 il est devenu docteur en mathématique. Parallèlement à la vie académique il a suivi la carrière militaire. Plus tard dans sa vie il a eu de problèmes de santé et il est resté presque aveugle.

V – *Equações do 1º grau. Biquaterniões*
VI – *Diferenciação de quaterniões*

Arzilla Fonseca commence son travail par une référence à l'histoire de la découverte des quaternions, par Hamilton. Il nous parle aussi, d'une arrivée très positive du calcul des quaternions en Angleterre, en Allemagne et en Amérique, et sur les avantages de ses applications.

Nous pouvons lire une référence à Grassmann et à Bellavitis, et surtout l'auteur présente plusieurs remarques sur Hamilton.

Fonseca a défini opération, vecteur, addition et soustraction de vecteurs dans l'espace, produit et division de vecteurs, où il présente un quaternion comme le quotient de deux vecteurs. Il travaille sur les rotations de l'espace en utilisant le nom versor. Il utilise des figures sur la sphère (de rayon 1) pour arriver au produit des unités imaginaires i, j, k :

$$jk = i, ki = j, ij = k, kj = -i, ik = -j, ji = -k.$$

Fonseca a définit le produit de deux vecteurs perpendiculaires (c'est le produit vectoriel moderne). Nous pouvons voir la décomposition d'un quaternion en scalaire et en vecteur, en utilisant les symboles S et V, comme l'a présenté plus tôt Hamilton. L'auteur a fait quelques applications à la trigonométrie, jusqu'à présenter la différentiation des quaternions et l'operateur *nabla*. Pour finir son travail, Fonseca établit une relation entre le calcul des quaternions et la physique.

Nous savons que les exigences de la science du XIX^{ème} siècle n'ont pas fait triompher le système des quaternions, qui a connu une époque de décadence en opposition au système vectoriel de Gibbs-Heaviside qui a réussi.

Au XX^{ème} siècle, R. Fueter, professeur de l'université à Zurich, a récupéré les opportunités perdues de la théorie de quaternions en travaillant sur le sujet. En 1924, il a été invité à participer au *Clube dos Lentes* à Coimbra et plus tard, en 1932, il a présenté un travail, à l'université à Coimbra sur *Quelques résultats de l'algèbre moderne* [FUETER 1932] ce qui nous fait croire que les mathématiciens portugais se sont intéressés au développement de l'algèbre moderne.

Une question se pose :

Quel place occupe aujourd'hui la théorie des quaternions créée par Hamilton ?

Pour répondre à cette question nous citons Jack B. Kuipers [KUIPERS 1999] :

> Our intent in these pages is to explore the use of Hamilton's quaternions in studying certain transformations in ordinary space of three dimensions. It must be said that it was not long after the publication of Hamilton's results that Josiah Willard Gibbs and others began to work out the details of what we know today as the algebra of vector spaces, and Hamilton work seemed quickly to be eclipsed. Recently, however, interest in the use of quaternions has revived, and we want to consider ways in which quaternion algebra may still be more effective than the use of ordinary vector algebra.

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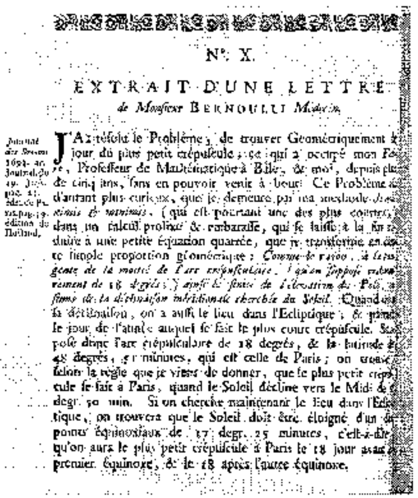

N°. X.

EXTRAIT D'UNE LETTRE

de Monsieur BERNOULLI *Médecin.*

J'AI résolu le Problême, de trouver Géométriquement le jour du plus petit crépuscule ; ce qui a occupé mon *Frere*, Professeur de Mathématique à Bâle, & moi, depuis plus de cinq ans, sans en pouvoir venir à bout. Ce Problême est d'autant plus curieux, que je demeure par ma methode *de maximis & minimis*, (qui est pourtant une des plus courtes) dans un calcul prolixe & embarassé, qui se laisse à la fin réduire à une petite équation quarrée, que je transforme en cette simple proportion géométrique : *Comme le rayon, à la tangente de la moitié de l'arc crépusculaire, (qu'on suppose ordinairement de 18 degrés) ainsi le sinus de l'élevation du Pole, au sinus de la déclinaison méridionale cherchée du Soleil.* Quand on sait la déclinaison, on a aussi le lieu dans l'Ecliptique, & partant le jour de l'année auquel se fait le plus court crépuscule. Supposé donc l'arc crépusculaire de 18 degrés, & la latitude de 48 degrés, 51 minutes, qui est celle de Paris ; on trouve selon la règle que je viens de donner, que le plus petit crépuscule se fait à Paris, quand le Soleil décline vers le Midi de 6 degr. 50 min. Si on cherche maintenant le lieu dans l'Ecliptique, on trouvera que le Soleil doit être éloigné d'un des points équinoxiaux de 17 degr. 25 minutes, c'est-à-dire, qu'on aura le plus petit crépuscule à Paris le 18 jour avant le premier équinoxe, & le 18 après l'autre équinoxe.

Lettres de Johann Bernoulli au Journal des Savans.

Le problème du crépuscule minimum, d'après PEDRO NUNES, dans son ouvrage "De Crepusculis"

DA SILVA VILAR Carlos Alberto
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Abstract

PEDRO NUNES est un mathématicien célèbre portugais du XVI^{ème} siècle. La géométrie, la cosmographie, l'algèbre, l'astronomie et tout ce qui concerne la navigation sont les sujets de plusieurs ouvrages qu'il écrivit, au cours de sa vie. Le "De Crepusculis" n'en est peut-être pas le principal, mais il est certainement l'un des plus beaux et plus connus. C'est dans cet ouvrage que Pedro Nunes résolut le problème du crépuscule minimum et sa duration par une voie géométrique, ce que les frères Jean et Jacob Bernoulli firent aussi, une centaine d'années après, par l'application de la théorie des extrema d'une fonction de plusieurs variables. On fera, ici, au cours de ce travail, un résumé de ce problème.

Johann Bernoulli, *Lectiones de calculo differentialium*. Problema XX, *Invenire brevissimum Crepusculum*, Manuscrit de l'Universitätsbibliothek Basel, L I a 6.

PEDRO NUNES, mathématicien portugais du XVI^{ème} siècle, après avoir fait ses études à l'Université de Coimbra, au Portugal, et plus tard, à Salamanca, en Espagne, fut Professeur à l'Université de Coimbra, et le grand cosmographe du Royaume, pendant de nombreuses années. L'astronomie, la géographie, la géométrie, l'algèbre et tout ce qui concerne la navigation maritime ont été les sujets de ses recherches et de ses nombreux écrits.

D'après Rodolfo Guimarães, dans son essai *Les Mathématiques au Portugal*, beaucoup de mathématiciens se sont intéressés à ses écrits : Ticho-Brahe, Possevino, Stevin, Vossius, Christopher Clavius, Dechasles Kastner, Saverien. Montucla, Dutens, de Zach, Bailly, Lalanne, Poggendorff, Delambre. Paul Serret, Maximilien Marie, S. Gunther, M. Moritz Cantor, etc. Sont donc tout à fait légitimes les paroles de Rodolfo Guimarães, encore lui, au début de son étude biographique *Sur la vie et l'œuvre de* PEDRO NUNES : "Parmi les portugais du XVIème siècle, qui se sont rendus illustres dans les sciences mathématiques, la première place revient, sans conteste, au Dr. PEDRO NUNES", ou encore celles-ci de Garção Stokler, dans son *Essai historique sur l'origine et les progrès des mathématiques au Portugal* : "Ce géomètre, le plus grand que les Espagnes ont produit et, incontestablement, l'un des plus grands de l'Europe du XVIème, . . ."[1]

Son ouvrage *De Crepusculis* est peut-être le plus beau, parmi tous ceux qu'il a écrits et, certainement, celui qui lui a attiré le plus de louanges et de prestige, dans le monde scientifique de son époque. C'est, d'après Garção Stokler, dans son *Essai historique* · · · , un ouvrage "digne, certainement, de mémoire éternelle" ou encore, "celui qui a fait beaucoup plus d'honneur à la sagacité de son esprit" et, d'après la Commission de l'Académie des Sciences de Lisbonne, chargée de sa dernière édition, celle de 1943, "Le *De Crepusculis* est un chef-d'œuvre. Son originalité, l'écho qu'il a eu, l'influence qu'il a exercée, attestent son importance. Le *De Crepusculis*, à lui tout seul, suffit à justifier la reconnaissance qui a été faite à PEDRO NUNES, en astronomie en donnant son nom à l'un des cratères de la Lune".

PEDRO NUNES était un défenseur de la théorie ptolémaique[2], d'après laquelle le soleil se meut, sur l'écliptique[3], autour de la terre, qui est le centre de l'univers. C'est à l'intérieur de cette théorie qu'il développe tous ses raisonnements relatifs aux crépuscules, qui arrivent, chaque jour et partout, c'est-à-dire, une période de lueur, qui précède le lever du soleil -c'est le crépuscule matinal- et une autre de lumière incertaine, qui succède immédiatement au coucher du soleil - c'est le crépuscule vespéral. Dans cet ouvrage, il ne s'occupe pas seulement du sujet central "les crépuscules". On peut même dire que *De Crepusculis* est presque un traité d'astronomie sphérique. Tous les problèmes y sont étudiés géométriquement, depuis la définition du crépuscule, la variation de son amplitude, chaque jour, d'endroit en endroit, et chaque jour et à chaque endroit, tout au long de l'année, jusqu'au problème couronnant toute la théorie exposée - celui du plus petit crépuscule et sa durée.

C'était un problème conséquent, en ce temps-là, celui des crépuscules. En effet, le problème du crépuscule de durée minimum a été étudié, une centaine d'années après, par les frères Jean et Jacob Bernoulli, qui ont utilisé la théorie des extrêmes d'une fonction et, après beaucoup d'efforts, comme d'ailleurs Jean Bernoulli, l'avoue lui-même, sont arrivés à la solution, d'une

[1] *Ensaio histórico, sobre a origem e progressos das matemáticas, em Portugal (Paris, 1819)* : "Este geómetra, o maior que as Hespanhas têm produzido e, incontestavelmente, um dos maiores, que, no século XVI, floresceram na Europa,. . ."

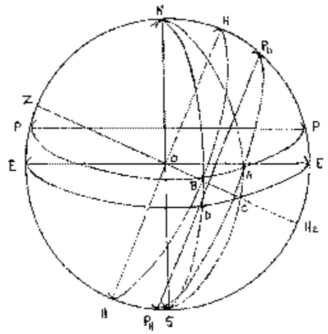
[2] L'ouvrage *Des revolutionibus orbium coelestium libri sex*, où Copernic a développé sa théorie d'après laquelle c'est la terre que se meut sur l'écliptique autour du soleil, a été publié en 1543, c'est-à-dire l'année suivante de celle de la première publication du *De Crepusculis* de PEDRO NUNES.

[3] Grand cercle d'intersection du plan de l'orbite du soleil, dans son mouvement autour de la terre (théorie ptolémaique ou géocentrique), ou du plan de l'orbite de la terre, dans son mouvement autour du soleil (théorie copernicienne ou héliocentrique), avec la sphère céleste.

façon peut-être plus élégante. Voilà un "*Extrait d'une lettre*[4] *de M. Bernouilli, Medecin : J'ai résolu le Problème, de trouver géométriquement le jour du plus petit crépuscule; ce qui a occupé mon Frère, Professeur de Mathématique à Bâle & moi depuis plus de cinq ans, sans en pouvoir venir à bout. Ce problème est d'autant plus curieux que je demeure par ma méthode de maximis & minimis, (qui est pourtant une des plus courtes), dans un calcul prolixe & embarassé qui se laisse, à la fin réduire à une petite équation quarrée, que je transforme à cette simple proportion géométrique :* **Comme le rayon à la tangente de la moitié de l'arc crépusculaire,** *(qu'on suppose ordinairement de 18 degrés),* **ainsi le sinus de l'élévation du Pole, au sinus de la déclinaison**[5] **méridionale cherchée du soleil.** *Quand on a sa déclinaison, on a aussi le lieu dans l'Ecliptique; & partant le jour de l'année auquel se fait le plus court crépuscule. . .*"

En ce qui concerne les crépuscules, voilà des points spécifiques minutieusement développés par PEDRO NUNES, dans cet ouvrage, avant d'arriver au problème du plus petit crépuscule.

1. Le crépuscule matinal en un endroit commence quand le soleil, pendant son mouvement diurne et apparent autour de la terre, sur le parallèle[6] de la position qu'il occupe en ce jour-là dans l'écliptique, se trouve 18 degrés au-dessous de l'horizon[7] de l'endroit[8] et se termine quand le soleil le dépasse, en se levant au-dessus du même. Le crépuscule vespéral commence quand le soleil, à la fin du jour, encore pendant son mouvement diurne sur ce parallèle-là autour de la terre, se couche, en dépassant l'horizon, jusqu'au moment où il se trouve 18 degrés au-dessous du même. Leur durée est la même chaque jour, en un même endroit. C'est pourquoi PEDRO NUNES parle des crépuscules, sans aucune distinction.



N-S axe Nord-Sud;

EE l'équateur,

PP parallèle décrit par le soleil;

HH horizon de l'endroit;

PHPH parallèle à l'horizon, 18º au-dessous;

Z-Nz axe Zénith-Nadir;

AB arc crépusculaire, sur le parallèle PP;

CD arc crépusculaire, sur l'équateur.

[4] Lettre, qui a été publiée dans le *Journal des Sçavans pour l'année M.CD.XCIII (Paris, 1729)*, pg.25, et reproduite dans le t.I. (Lausanne et Génève, 1742), pg.64 de son ouvrage *Opera omnia tam antea sparsim edita quam hactenus inedita*.

[5] Arc de méridien céleste compris entre un astre et l'équateur céleste.

[6] Petit cercle de la sphère céleste, parallèle au plan de l'équateur.

[7] Grand cercle théorique divisant la sphère céleste en deux parties égales, l'une visible, l'autre invisible.

[8] On dit aussi que la dépression du soleil est, alors, de 18 degrés. En ce qui concerne cette valeur de la dépression du soleil, aux limites des crépuscules, il faut noter que ce n'est pas exactement comme ça, mais c'était la théorie, en ce temps-là, la plus vraisemblable. En effet, on a toujours atribué, dès l'antiquité, une valeur constante à cette dépression, quoique pas la même, parmi les astronomes : 18°, d'après Ptolémée; 17° 30', d'après Estrabon; 19°, d'après Allacen; etc. D'après la tradition astronomique et l'opinion généralisée, à l'époque de PEDRO NUNES, c'était 18°. Mais PEDRO NUNES ne l'a pas admis, sauf théoriquement et comme point de départ pour ses raisonements. Il a montré, dans la *Proposition I* de la deuxième partie de son ouvrage *De Crepusculis*, que cette valeur-là n'est pas du tout une valeur constante et a annoncé une méthode pour sa détermination, chaque jour et à chaque endroit, ce qu'il fit, plus loin, vers la fin de l'ouvrage, dans la *Proposition* XVI.

2. Les crépuscules, qui arrivent, à chaque endroit, quand le soleil, pendant son mouvement autour de la terre, sur l'écliptique -le mouvement ascendant et le mouvement descendant- se trouve à même distance d'un même point solsticial, soit celui de l'hiver, soit celui de l'été - ont même durée. Donc, seulement la durée du crépuscule, qui arrive dans un endroit, le jour où le soleil dépasse un point solsticial, ne se répète pas.

3. À des positions du soleil, sur l'écliptique, également éloignées de l'un ou de l'autre des points equinoxiaux -soit celui du printemps, ou le point vernal, soit celui de l'automne, ou le point automnal- correspondent, à chaque endroit, des crépuscules de différentes durées :

dans les endroits de l'hémisphère septentrional, sont plus longs les crépuscules correspondant à la position septentrionale du soleil, sur l'écliptique;

dans les endroits de l'hémisphère méridional, sont plus longs les crépuscules correspondants à la position méridionale du soleil, sur l'écliptique.

4. Quand le soleil, pendant son mouvement autour de la terre, visite les signes septentrionaux de l'écliptique, soit en ascendant soit en descendant, la durée des crépuscules dans un endroit, augmente à mesure que le soleil s'approche, sur l'écliptique, du principe de Cancer. On va voir, plus tard, comme d'ailleurs le fait aussi PEDRO NUNES, que cette croissance de la durée des crépuscules en quelqu'endroit septentrional, se vérifie, à partir d'une valeur minimum, qui arrive quand le soleil, pendant son mouvement ascensional, dépasse un certain point de l'écliptique, encore avant l'équinoxe du printemps.

5. Sur l'équateur, la durée des crépuscules est différente de jour en jour. Elle grandit à mesure que le soleil s'approche des points solsticiaux. Toutefois, la durée des crépuscules est la même en des positions du soleil, sur l'écliptique, également éloignées des points équinoxiaux. Donc, on peut dire que, sur l'équateur, les crépuscules ont durée minimum les jours où le soleil dépasse les points équinoxiaux, et maximum, les jours où le soleil se trouve aux points solsticiaux.

6. Quelle que soit la position du soleil sur l'écliptique, la durée des crépuscules, qui arrivent chaque jour, en deux endroits septentrionaux quelconques, est toujours plus grande dans l'endroit plus boréal. De même, en deux endroits méridionaux quelconques, la durée des crépuscules qui y arrivent chaque jour, est plus grande dans l'endroit plus austral.

7. PEDRO NUNES avant d'étudier le problème du plus petit crépuscule et il ne faut pas oublier qu'il y a dans son ouvrage beaucoup d'autres problèmes, qui ne concernent pas directement les crépuscules -s'occupe des méthodes pour la détermination de la durée des jours et des nuits et aussi des crépuscules, soit dans un endroit quelconque, hors de la ligne équatoriale[9]- quelle que soit la position du soleil sur l'écliptique, et, spécialement, dans le cas où le soleil se trouve sur l'équateur - soit sur la ligne équatoriale[10].

C'est vers la fin de l'ouvrage, dans la *Proposition* XVII, certainement la plus fameuse, que PEDRO NUNES fait une exposition minutieuse de l'évolution de la durée des crépuscules, dans un endroit septentrional, pendant le "voyage" du soleil, le long de l'écliptique, dès le point

[9]C'est-à-dire, dans un horizon oblique quelconque.

[10]C'est-à-dire, dans un horizon droit quelconque.

solsticial de l'hiver[11], jusqu'au point solsticial de l'été[12].

En ce qui concerne les jours -comme d'ailleurs tout le mond le sait- leur durée augmente, de plus en plus, dans un endroit quelconque de l'hémisphère nord, et ils ont même durée que les nuits, quel que soit l'endroit, boréal ou austral, précisément, quand le soleil dépasse l'équateur.

En ce qui concerne les crépuscules, ils y décroissent jusqu'au jour où ils ont une durée minimum, encore avant l'équinoxe du printemps. Après ce jour-là, leur durée augmente continuellement, jusqu'au jour du solstice d'été. En effet, on peut dire et montrer, comme l'a fait PEDRO NUNES, que les crépuscules y décroissent rapidement jusqu'au jour où le soleil dépasse un certain point de l'écliptique tel que le sinus de la latitude de l'endroit soit égal à la raison entre le sinus de la dépression du soleil[13], au début du crépuscule matinal ou à la fin du crépuscule vespéral, et deux fois le sinus de la déclinaison de ce point-là de l'écliptique:

qu'ils y décroissent encore, mais plus lentement, jusqu'au jour, où ils ont même durée que ceux qui arrivent, quand le soleil dépasse l'équateur;

que cette décroissance continue encore, quoique beaucoup plus lentement, après ce jour-là, ce qui implique qu'il y ait une valeur minimum, quand le soleil dépasse un certain point sur l'écliptique encore avant le point équinoxial;

et que c'est après ce passage, que commence la croissance de la durée des crépuscules, jusqu'au solstice d'été.

Après une exposition théorique du problème, sûre et exacte, quoique très longue, sans doute, et même complexe -ce qui n'enlève pas du tout la valeur et le mérite de son raisonnement, en ce temps-là- il est arrivé géométriquement et dans un contexte d'astronomie sphérique aux calculs, soit des jours (deux), où arrivent à chaque endroit, les crépuscules de durée minimum, soit même et aussi de sa durée. PEDRO NUNES a même résolu le problème pour l'horizon de Lisbonne. Voilà les résultats, qu'il a trouvés[14].

1 Les jours de l'année où les crépuscules, à Lisbonne, ont même durée que ceux qui ont lieu, quand le soleil dépasse la ligne équatoriale

1.1 Arc crépusculaire, en ces jours-là

$$sr(ArcCrp) = sr(DpS)st()/sr(HE)$$

[11]Le Tropique du Capricorne.

[12]Le Tropique du Cancer.

[13]Arc du méridien céleste d'un endroit, au-dessous de l'horizon, compris entre le soleil et l'horizon.

[14]PEDRO NUNES, n'utilise que deux fonctions trigonométriques, tout le long du *De Crepusculis* : le *sinus rectus* et le *sinus versus*.
Étant AB un arc d'un cercle, centré en O et dont le rayon soit OA, on dit que le *sinus total*, $st()$, quelque soit l'arc AB, c'est la mesure du rayon OA;

- *sinus rectus* de l'arc AB, $sr(AB)$, c'est la mesure du coté CB du triangle OCB (C, un point en OA), rectangle en C;

- *sinus versus* de l'arc AB, $sv(AB)$, c'est la mesure de $OA - OC$.

On a, alors, $\sin(AB) = sr(AB)/st()$ et $sv(AB) = OAsr(90^\circ - AB))$
Dans son ouvrage *De Crepusculis*, PEDRO NUNES a utilisé des tables de sinus, où le rayon de la sphère céleste était supposé d'être divisé en 100000 parties égales, c'est-à-dire, où $st() = 100000$.

$ArcCrp$	=	Arc Crépusculaire.
DpS	=	Dépression du Soleil, au début du crépuscule matinal et à la fin du crépuscule vespéral. À Lisbonne, $DpS = 16°2'$, valeur auparavant calculée[15].
HP	=	Hauteur du Pôle, (latitude de l'endroit).
HE	=	Hauteur de l'Équateur.

$$DpS = 16°2'$$
$$HP = 38°40'$$
$$HE = 51°20'$$

$sr(DpS) = 27626$
$sr(HE) = 78079$
$st() = 100000$

$$sr(ArcCrp) = 27626 \times 100000 \div 78079 = 35382$$
$$ArcCrp = 20°43'20''$$

auquel correspond la durée 1h 22m 53.3s

1.2 Déclinaison des points P de l'écliptique, où se trouve le soleil, quand les crépuscules, à Lisbonne, ont même durée que les crépuscules équinoxiaux :

$$sr(DclP) = sr(AO)sr(HE)/st()$$

$DclP$	=	Déclinaison des points P de l'écliptique...;
AO	=	Amplitude Ortive[16] du point P de l'écliptique, où se trouve le soleil, pendant son mouvement ascendant, quand les crépuscules, à Lisbonne, ont même durée que les crépuscules équinoxiaux.
AO	=	13°12', valeur auparavant calculée.
HE	=	Hauteur de l'Équateur, ($HE = 51°20'$).

$$sr(AO) = 22551$$
$$sr(HE) = 78079$$
$$st() = 100000$$

$$sr(DclP) = 22551 \times 78079 \div 100000 = 17607.5$$
Déclinaison du point $P = 10°8'30''$.

La déclinaison du point P correspondant, quand le soleil descend l'écliptique, est évidemment la même.

1.3 Identification de ces point-là

$$sr(DistP) = sr(DclP)st()/sr(DME)$$

$DistP$ = Arc de l'écliptique, entre la position P du soleil et le point equinoxial plus proche – le point vernal, pendant le mouvement ascendant, ou le point automnal, pendant le mouvement descendant.
DME = Déclinaison maximum de l'écliptique, ($DME = 23°28'$).
$DclP$ = Déclinaison des points P de l'écliptique,...

$$sr(DclP) = 17607.5$$
$$sr(DME) = 39874$$
$$st() = 100000$$

$$sr(DistP) = 17607.5 \times 100000 \div 39874 = 44158$$
$$DistP = 3°48' \text{ du signe des } \textit{Poissons}$$
$$26°12' \text{ du signe de la } \textit{Balance}$$

On peut donc affirmer, comme l'a fait Pedro PEDRO NUNES, que les jours de l'année *de notre temps*, où les crépuscules à Lisbonne ont même durée que les crépuscules équinoxiaux, sont le 12 Février et le 10 Octobre, quand le soleil se trouve déjà, respectivement, dans les signes des *Poissons* et de la *Balance* du Zodiaque[17].

2 Durée minimum des crépuscules à Lisbonne

$$sr(CrM/2) = st()sr(DpS/2)/sr(HE)$$

CrM = Arc crépusculaire minimum.
DpS = Dépression du Soleil, au début du crépuscule matutinal et à la fin du crépuscule vespéral, (à Lisbonne, $DpS = 16°2'$)[18].
HE = Hauteur de l'Équateur, ($HE = 51°20'$).

$$st() = 100000$$
$$sr(DpS/2) = 13946$$
$$sr(HE) = 78079$$

$$sr(CrM/2) = 100000 \times 13946 \div 78079 = 17865$$
$$CrM = 20°34'40''$$

auquel correspond la durée 1h 22m 18.7s

2.1 Déclinaison des points de l'écliptique, où se trouve le soleil, quand les crépuscules à Lisbonne ont durée minimum

$$sr(\text{Déclinaison}) = sr(AO)sr(HE)/st()$$

[17] Il n'y a ici aucune erreur, au contraire de ce qu'il semble. En effet, d'après le Calendrier, *au temps de* PEDRO NUNES, c'est-à-dire, le Calendrier Julien, les équinoxes du printemps et de l'automne avaient lieu le 11 Mars et le 12 Septembre, respectivement. Par conséquent, le 12 Février était, en ce temps-là, le vingt-huitième jour, avant le printemps, et le 10 Octobre, le vingt-huitième jour, après l'automne.
[18] V. note 15.

AO = Amplitude Ortive du point P de l'écliptique, où se trouve le soleil, pendant son mouvement ascendant, quand les crépuscules, à Lisbonne, ont durée minimum. $AO = 6°28'$, valeur auparavant calculée.

HE = Hauteur de l'Équateur, $(HE = 51°20')$.

$$\begin{aligned} sr(AO) &= 11262 \\ sr(HE) &= 78079 \\ st() &= 100000 \end{aligned}$$

$$sr(\text{Déclinaison}) = 11262 \times 78079 \div 100000 = 8793$$

$$\text{Déclinaison (austral) du point } P = 5°2'40''.$$

La déclinaison du point P correspondant, quand le soleil descend l'écliptique est évidemment la même.

2.2 Identification de ces points-là

$$sr(DistP) = sr(DclP)st()/sr(DME)$$

$DistP$ = Arc de l'écliptique, entre la position P du soleil et le point vernal ou le point automnal – le plus proche.

DME = Déclinaison maximum de l'écliptique, $(DME = 23°28')$.

$DclP$ = Déclinaison des points P de l'écliptique....

$$\begin{aligned} sr(DclP) &= 8793 \\ st() &= 100000 \\ sr(DME) &= 39874 \end{aligned}$$

$$\begin{aligned} sr(DistP) &= 8793 \times 100000 \div 39874 = 22052 \\ DistP &= 17°16' \text{ du signe des } \textit{Poissons} \\ & 12°44' \text{ du signe de la } \textit{Balance} \end{aligned}$$

ce qui, à l'époque de PEDRO NUNES, avait lieu, le 25 Février et le 26 Septembre, respectivement.

On peut donc et finalement affirmer, comme l'a fait PEDRO NUNES que les jours de l'année *de notre temps*, où les crépuscules, à Lisbonne, ont durée minimum, sont le 25 Février et le 26 Septembre[19], quand le soleil se trouve déjà, respectivement, dans les signes des *Poissons* et de la *Balance* du Zodiaque.

D'après Heinrich Dorrie, dans son ouvrage *100 Great Problems of Elementary Mathematics*, PEDRO NUNES a essayé de résoudre le problème du crépuscule minimum, mais n'y a pas réussi; néanmoins, ce problème a été résolu, beaucoup plus tard, par Jacob Bernoulli et D'Alembert, qui sont arrivés d'une façon pas simple à la solution cherchée en utilisant le calcul différentiel. Mais il s'est trompé, en ce qui concerne les efforts de PEDRO NUNES. Tous ces mathématiciens

[19] C'est-à-dire, *au temps* de PEDRO NUNES, le quatorzième jour, avant le printemps, et le quatorzième jour, après l'automne. (V. note 17).

Jacob Bernoulli, D'Alembert et, peut- être, d'autres ont trouvé, par des voies différentes de celles de PEDRO NUNES et après lui, les mêmes résultats, soit pour la durée du crépuscule minimum, soit pour la déclinaison du soleil, le jour où le crépuscule le plus petit arrive en chaque endroit. Dans l'ouvrage ci-dessus, par exemple, on peut voir[20] la détermination d'une formule pour la durée du crépuscule minimum (CrM), dans un endroit ayant latitude ϕ, en supposant, toujours et partout, $h = 18°$, la dépression du soleil, au début du crépuscule matinal et à la fin du crépuscule vespéral, c'est-à-dire,

$$\sin(CrM/2) = \sin(h/2)/\cos(\phi)$$

celle-ci tout à fait analogue à celle que PEDRO NUNES avait déjà trouvée, par des voies différentes, c'est-à-dire,

$$sr(ArcCrp/2) = sr(DpS/2)st()/sr(HE)^{21}$$

et, encore, la détermination d'une autre, celle qui donne la déclinaison d du soleil, ce jour-là,

$$\sin(d) = \sin(f) \text{ tg } (h/2)^{22}$$

celle-ci, différente, sans doute, de celle que PEDRO NUNES a trouvée, mais encore, tout à fait équivalente, comme l'on peut voir, par exemple, quand on cherche la déclinaison du soleil, le jour du crépuscule minimum, à Lisbonne[23]. En effet, on a

$$\begin{aligned} \sin(d) &= \sin(33°40')\text{tg }(8°1') \\ &= 0.624793547 \times 0.140836294 \\ &= 0.08799361 \end{aligned}$$

et la déclinaison du soleil correspondante, c'est-à-dire, $d \approx -5°2'40''$.

Mais, beaucoup plus important, pour mettre fin à cette question, c'est le témoignage de Jacob Bernoulli lui-même[24] :

> [*] Par contre, si je ne me trompe pas, ce problème a été déjà résolu par PEDRO NUNES, l'année 1542, dans son ouvrage *De Crepusculis*. Certes, je n'ai pas pu trouver ce livre-là, mais vers la fin des

[20] Solution qui se trouve dans *Lehrbuch der Spharischen Astronomie* de Brunnow.

[21] $ArcCrp$ = Arc Crépusculaire; DpS = Dépression du Soleil, au début du crépuscule matinal et à la fin du crépuscule vespéral; $(DpS = 18°)$; et HE = Hauteur de l'Équateur, soit $90° HP$, (HP = Hauteur du Pôle ou latitude de l'endroit). D'après la note 14, les formules $sr(ArcCrp/2)/st() = (sr(DpS/2)/st())/(sr(HE)/st())$ et $\sin(CrM/2) = \sin(h/2)/\cos(f)$ sont équivalentes.

[22] C'est, donc, négative la déclinaison du soleil, le jour du crépuscule minimum, dans un endroit septentrional, comme d'ailleurs l'avait déjà montré PEDRO NUNES.

[23] Puisque on veut comparer des résultats, on va mettre $h = 16°2'$. (V. note 15).

[24] "SOLUTIO PROBLEMATIS DE MINIMO CREPUSCULO", Per Jacob Bernoullium *Communicata in litteris, Basileae, die 20 Julii, 1692, datis* :
"Imo, nisi fallor, jam anno 1542 id Problema fuit a P. NONNIO legitime solutum, in Tractatu *De Crepusculis*. Librum quidem reperire non potui, sed exstat ad calcem *Commentarii in Sphaeram* J. DE SACROBOSCO per *Chr.* CLAVIUM . . . *Digressio de Crepusculis*, cujus Auctor in Prooemio prosequitur, se NONNII librum in compendium duntaxat redegisse. Huius autem Digressionis Prop.XXII docet reperire Punctum Eclipticae in quo Sol brevissimum efficit Crepusculum, eiusque Crepusculi magnitudinem definire. Etsi vero non incidit in Analogiam tam simplicem, quam ea est quae hic proponitur, legitimam tamen solutionem esse negari nequit, quaeque facile ad istam reducitur".

Commentarii in Sphaeram J. DE SACROBOSCO per Chr. CLAVIUM···, on trouve une *Digressio de Crepusculis*, où l'auteur affirme, dans le préambule, qu'il a seulement rassemblé dans un résumé le livre de NUNES. La Prop. XXII de cette Digression nous apprend à trouver un point de l'écliptique, où le soleil produit le crépuscule minimum et à déterminer la durée de ce crépuscule. Quoique, en vérité, elle ne soit pas une relation si simple que celle que l'on propose ici, on ne peut pas, toutefois, nier que cette solution-là est légitime, laquelle facilment aussi se réduit à celle-ci.

Et voilà la relation trouvée par Jacob Bernoulli, pour la détermination de la déclinaison du soleil, le jour du crépuscule minimum, dans un endroit[25] : *"Comme le sinus total à la tangente de 9°, ainsi le sinus de l'élévation du pôle au sinus de la déclinaison cherchée"*, c'est-à-dire,

$$1/\mathrm{tg}(9°) = \sin(\phi)/\sin(d).$$

Donc, le jour où le crépuscule de durée minimum arrive, dans un endroit septentrional ayant latitude ϕ, c'est celui où la déclinaison *austral* du soleil, sur l'écliptique, a la valeur d, telle que

$$\sin(d) = \sin(\phi)\mathrm{tg}\ (h/2).$$

Finalement, on peut dire que le problème du crépuscule minimum a été résolu par PEDRO NUNES, qui a établi géométriquement (il ne connaissait pas encore le calcul différentiel) des formules pour la détermination, soit de la durée du crépuscule minimum, dans un endroit septentrional, soit des jours[26], où il a lieu. Il a aussi été résolu, une centaine d'années après, mais partiellement, par JACOB BERNOULLI celui-ci ayant d'autres moyens mathématiques, c'est-à-dire, le calcul différentiel. Lui aussi a trouvé, une formule pour la détermination des jours[27] où le crépuscule, dans un endroit septentrional, a durée minimum.

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[25] Ut Sinus Totus ad Tangent. 9 grad. Sic Sinus Elevationis Poli ad Sinum quaesitae Declinationis *Australis*, quam Sol tempore minimi Crepusculi obtinet.

[26] La déclinaison *austral* du soleil, sur l'écliptique, le jour où le crépuscule, dans un endroit septentrional, a durée minimum.

[27] V. note 26.

Quelles mathématiques pour former des enseignants

Illustration d'une expérience de définition de contenus adéquats, à forte coloration épistémologique et historique, sur le thème "La géométrie : une description de la réalité ?"

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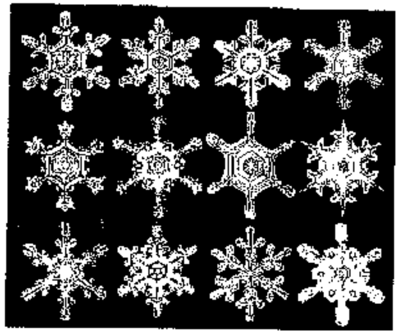
Abstract

Doit-on former des matématiciens qui feront de l'enseignement ou des enseignants qui enseigneront les mathématiques ? Une formation académique en mathématiques donne-t-elle de cette discipline la vue large et contextualisée qui devrait être celle d'un enseignant ou mène-t-elle à une forte spécialisation et à une vision formaliste et trop axée sur la rigueur et la structuration axiomatique ?

Ces questions ont amené, notamment en Suisse, à ce que soit discutée l'idée de formation en mathématiques orientée vers l'enseignement. La définition des buts, des contenus et des processus à mettre en œuvre dans un tel cadre demande un travail approfondi de réflexion ainsi que la réalisation et l'évaluation de nombreux essais.

Cette conférence donnera une présentation d'un tel essai, centré sur une approche épistémologique et historique de la géométrie et de son enseignement, essai qui a été réalisé l'automne dernier à l'Université de Berne, en Suisse, dans le cadre d'une formation post-diplôme destinée à habiliter un groupe de personnes à enseigner la didactique des mathématiques dans les Hautes écoles pédagogiques qui se mettent en place dans ce pays.



1 Le problème général

Doit-on former des mathématiciens qui feront de l'enseignement ou des enseignants qui enseigneront les mathématiques ? Une formation académique en mathématiques donne-t-elle de cette discipline la vue large et contextualisée qui devrait être celle d'un enseignant ou mène-t-elle à une trop forte spécialisation, à une vision formaliste et trop axée sur la rigueur et sur la structuration axiomatique ?

Quelques réflexions préalables permettent d'éclairer ces questions, sans prétendre leur donner une réponse définitive. Partons peut-être du "haut niveau" de savoir qui est généralement attendu pour enseigner les mathématiques et remarquons avec DEVELAY (1994, p. 84) :

> Qu'est-ce qu'un haut niveau : un savoir pointu comme celui de l'agrégation dans quelques domaines triés et sans grande relation avec les programmes du second degré ou ne couvrant au mieux qu'une faible partie d'entre eux ? Nous pensons pour notre part, qu'un savoir de haut niveau est celui qui permet un recul distancié vis-à-vis de la structure de la discipline, un savoir des contenus et de leur épistémologie.

Nous partageons ici l'avis de WITTMANN (1989, p. 294 et 299) selon lequel la formation universitaire ne répondrait pas réellement à cette attente :

> In general, the mathematical training of teachers is not systematically related to educational aspects. Very often we find a formal study of mathematics ignoring the requirements of school [...]. Mathematics must not be seen within the narrow boundaries of a specialised discipline which is represented exclusively by the departments of pure mathematics at the universities; rather it should be seen in the full spectrum of its relationships to science, to technology, to the humanities, and to human life.

L'enseignement universitaire est souvent très formel. Il privilégie des connaissances pointues, une démarche d'enseignement essentiellement transmissive et une présentation des connaissances sous la forme de "théories achevées". La concrétisation par des exemples "pratiques", par des interprétations ou des représentations intuitives est rare.

Cette forme d'enseignement peut s'avérer efficace et permettre une avance rapide pour les quelques étudiants qui deviendront éventuellement des chercheurs en mathématiques, les mieux adaptés et les plus doués pour une telle approche. Cependant, nombreux sont ceux qui se réfugient dans un apprentissage par cœur, pour les examens, seul moyen pour eux de "réussir". La représentation de ce que sont les mathématiques, de la façon dont elles se construisent, se structurent, se présentent ou s'enseignent en sort fréquemment biaisée. Cela se révèle notamment lors des séminaires, dans les examens ou dans les cours de didactique, lorsque les étudiants doivent "se dévoiler".

Sans nécessairement vouloir la condamner, nous pensons que cette forme d'enseignement n'est sûrement pas idéale si l'on vise à former des enseignants. Selon nous, un réel effort doit être entrepris pour infléchir la formation des futurs enseignants vers des contenus moins techniques, en cherchant à leur donner une vision large de la structure des mathématiques, une vision réaliste de la façon dont on les "fait", une vision de leur sens, de leur rôle et de leur histoire[1].

[1]Pour rétablir l'équilibre, il convient de remarquer que, de leur côté, les sciences de l'éducation devraient se pencher plus qu'elles ne le font aujourd'hui sur les particularités des disciplines à enseigner. Ce qu'écrit Shulman (1986, p.7) au sujet de ce qu'il appelle "the missing paradigm" mérite bien une réflexion ! WITTMANN (1992, p. 58) montre selon nous très bien comment peut être réalisée une intégration des aspects liés aux mathématiques et de ceux liés aux disciplines qui contribuent à leur apprentissage ou à leur enseignement.

Ces réflexions ont amené à ce que soit discutée l'idée de **formation en mathématiques orientée vers l'enseignement**. La définition des buts, des contenus et des processus à mettre en œuvre dans un tel cadre demande un travail approfondi de réflexion ainsi que la réalisation et l'évaluation de nombreux essais. Ce texte présente un tel essai, centré sur une approche épistémologique et historique de la géométrie et de son enseignement.

2 Le cadre de l'expérience

Nous l'avons réalisée, mon collègue Th. Rychener et moi, à l'Université de Berne, en 1998, dans le cadre d'une formation post-diplôme destinée à habiliter un groupe de personnes à enseigner la didactique des mathématiques dans les Hautes Écoles Pédagogiques qui se mettent en place en Suisse. Le cours concernait douze personnes enseignant la didactique des mathématiques dans la formation des enseignants du primaire et du secondaire obligatoire depuis plusieurs années, mais elles-mêmes insuffisamment formées pour satisfaire aux exigences qui seront posées pour œuvrer dans les nouvelles structures de formation. La majorité des participants étaient des enseignants du degré secondaire I (élèves de 12-15 ans) réputés avoir "fait leurs preuves". La plupart disposaient d'une formation de premier cycle en mathématiques (comparable au niveau bac+2). Quelques-uns n'avaient jamais étudié les mathématiques à l'université mais étaient porteurs d'un diplôme de psychologie ou de pédagogie.

La formation qu'ils suivaient comportait des compléments académiques en mathématiques, des cours de psychopédagogie et de didactique générale, des cours de didactique des mathématiques ainsi que des modules de formation en mathématiques orientée vers l'enseignement (le terme allemand exact est "unterrichtsbezogene Mathematik"). C'est d'un tel module, organisé sur une semaine intensive de cours, de travaux de groupe et de séminaires, qu'il est question ici.

Le niveau relativement peu élevé des connaissances des participants en mathématiques a évidemment constitué une difficulté particulière. Il nous a contraints à nous en tenir à l'essentiel et nous a évité de nous laisser entraîner dans des considérations formelles et techniques pourtant bien tentantes pour des mathématiciens...
Nous avions carte blanche pour la conception de ce cours si ce n'est qu'il devait porter sur la géométrie et répondre aux attentes d'un "cours de mathématiques orienté vers l'enseignement"[2].

3 Les objectifs poursuivis

Nous nous étions fixés des objectifs de trois types :

Connaissances "dans" les mathématiques :

- Approfondir les connaissances des participants en géométrie euclidienne.
- Travailler dans des théories géométriques diverses, ou dans différentes approches de la géométrie, pour les comprendre.

La géométrie s'étudie d'habitude par une approche élémentaire construite sur le modèle euclidien dans les premières années de scolarité obligatoire. Par la suite, elle se traite sous forme analytique et vectorielle puis, à l'université, sous forme d'algèbre linéaire, de géométrie affine, convexe, projective ou différentielle, par une approche des variétés riemanniennes ou de la topologie. Il est rare qu'un retour sur les fondements euclidiens apparaisse dans les cursus.

[2]WITTMANN (1989, p. 299-300) donne une description qui s'appliquerait très bien à ce concept, mais qui ne nous était malheureusement pas encore connue à l'époque.

Cette trajectoire de formation fait que les futurs enseignants considèrent fréquemment la théorie euclidienne comme peu profonde, élémentaire et naïve. Ils sont en outre souvent très mal à l'aise en géométrie synthétique et recourent volontiers à des formes calculées de géométrie, même lorsque les arguments synthétiques sont plus explicites. Une réflexion au sujet des liens entre les différentes approches joue un rôle important dans la préparation à l'enseignement. Dans un cours classique de mathématiques, nous nous en serions tenus à ces premiers points.

Connaissances "sur" les mathématiques :

- Étudier la genèse des théories géométriques.
- Étudier le va-et-vient entre perception de la réalité, élaboration de schémas, de théories et illustration par des modèles, sur l'exemple de la géométrie.
- Comprendre la portée et le sens de la démarche axiomatique.
- Réfléchir sur les rapports entre le vrai, la réalité, la perception et la géométrie.

Généralement, les étudiants savent peu de choses de l'histoire des mathématiques et des idées mathématiques. Ils savent que la géométrie euclidienne relève d'une construction axiomatique et la conçoivent fréquemment comme un "système axiomatique matériel" (TRUDEAU 1987). Ils y attachent quelques noms mythiques, en connaissent divers axiomes ainsi que des "définitions" pour les notions de point, de ligne et de surface, qu'ils citent avec une pointe d'amusement et de gêne parce qu'ils en craignent confusément les limites. L'idée que la géométrie naïve qu'ils ont à l'esprit n'est que leur représentation mentale d'une géométrie formelle et que ces "définitions", bien que nécessaires pour cette représentation, n'ont pas de raison d'être dans une telle géométrie leur est habituellement étrangère.

Ils décrivent volontiers l'axiome des parallèles, qu'ils savent discutable, mais sans vraiment pouvoir imaginer comment ou pourquoi. Justement parce que leur conception de cette géométrie est celle d'une idéalisation de la réalité et de la vérité[3] et que, dans la pensée commune, le lien entre **la** géométrie et **la** réalité est à la fois extrêmement fort et culturellement bien ancré !

Ils lient l'approche axiomatique aux structures algébriques, ressenties comme formelles, plutôt qu'à la géométrie, ressentie comme bien réelle. Ils sont fréquemment désarçonnés lorsqu'on leur demande si un axiome se prouve ou se justifie et qu'on leur rappelle dans la foulée qu'ils ont systématiquement dû commencer par les "vérifier", dans chaque exemple traité en cours d'algèbre. Pour la plupart, le rapport entre système axiomatique formel et modèle le réalisant concrètement n'est pas clair, comme ne l'est pas celui entre réalité et schéma.

Un cours d'histoire des mathématiques serait entré dans les évènements et les différentes sources existantes de manière plus détaillée[4]. Un cours d'épistémologie aurait approché les questions évoquées ci-dessus plus systématiquement. Il les aurait traitées plus en profondeur, avec un accent plus marqué vers la philosophie. Mais de tels cours n'auraient probablement pas

[3]On ne discute habituellement pas la vérité dans les cours de mathématiques. Elle est supposée être absolue et basée sur la logique, bien qu'elle soit en fait souvent issue de la conviction par "le bon sens", fondée sur la perception "claire" de la réalité ou construite sur l'intuition (GONSETH 1936, p. 327).

[4]Nous pensons avec DEVELAY (1994, p. 85-86) que ce qui importe dans la formation des enseignants ce n'est bien sûr pas "*une histoire descriptive, biographique qui expliquerait par exemple, que la notion de logarithme népérien est due à un baron écossais du nom de John Neper qui vécut au XVI^{ème} siècle* [. . .]. *Mais à une histoire des idées qui évoquerait à quel problème s'est confronté Neper* [. . .], *quels étaient les concepts mathématiques de l'époque, en quoi ils se montraient insuffisants* [. . .]."

abordés les liens de ces questions avec la géométrie enseignée aujourd'hui à l'école, pour en donner une vue large et contextualisée.

Objectifs de type didactique

- Suivre les étapes de l'histoire pour développer le savoir des participants en géométrie et en histoire de la géométrie (démarche génétique).
- Comparer la genèse du savoir géométrique, l'apprentissage de la géométrie et l'évolution de son enseignement.
- Expérimenter une approche socio-constructiviste au niveau tertiaire.
- Comprendre ce qu'est une révolution copernicienne, en la vivant dans le sens de son déroulement.

La démarche génétique se justifie particulièrement dans un cas où l'on veut simultanément, comme ici, étudier la constitution du savoir, faire acquérir ce savoir et réfléchir au processus de son acquisition par des élèves. Couplée à une réflexion sur quelques courants didactiques actuels ainsi qu'à une analyse de manuels, elle donne une grande cohérence à l'ensemble et permet de dresser de nombreux parallèles entre ce qui s'est passé dans l'histoire, ce qui est en train de se passer chez les étudiants et ce qui se passera probablement chez des élèves.

Il est par exemple intéressant de comparer, d'une part, ce qu'on imagine avoir été le processus de constitution du savoir géométrique avant qu'il ne soit formalisé par Euclide (PONT 1986, pp. 62-65 et 75-99), d'autre part, les processus décrits par les théories de l'apprentissage et, finalement, certains processus préconisés aujourd'hui dans l'enseignement de la géométrie : passer par des mathématiques en construction, des mathématiques *informelles* (WITTMANN, 1987); privilégier une approche *réaliste*, favoriser l'élaboration *d'objets mentaux* par le biais d'activités concrètes faisant sens pour l'apprenant, constituer et renforcer graduellement des *îlots déductifs* (CREM 1995, pp. 33, 34 et 135).

Il est de même intéressant d'opposer cela à la démarche axiomatique systématique qui transparaît encore dans de nombreux manuels de géométrie, alors que c'est beaucoup plus rare pour d'autres domaines des mathématiques, et de mesurer ainsi l'influence considérable des travaux d'Euclide. L'utilisation, comme manuel d'enseignement, d'un traité scientifique établissant l'état du savoir à une certaine époque exerce encore ses effets aujourd'hui.

Il vaut également la peine de constater comment, après plus de deux mille ans d'efforts ayant mené aux travaux de Hilbert, si parfaits qu'on a pu penser clos le chapitre de la géométrie, quelques années ont suffi pour retourner la construction et amener à des présentations de la géométrie basées sur les nombres : les nombres s'expliquaient par la géométrie (SESIANO 1999), la géométrie se justifie aujourd'hui par les nombres.

Finalement, la démarche génétique s'impose dans un cas où elle permet de faire vivre une révolution copernicienne à celui qui la suit. Se libérer d'un enfermement dans une vision strictement euclidienne par l'ouverture de perspectives de géométrie hyperbolique, se forger grâce à une théorie mathématique une clé avec laquelle nous pouvons ouvrir notre regard sur une autre perception de la réalité, ressentir le trouble qui résulte du sentiment que la réalité s'en trouve comme modifiée, est une expérience fascinante. Nous partageons sur ce point l'avis de TRUDEAU (1987, viii-ix) :

> This [. . .] will provide [. . .] a rare opportunity to actually experience the intellectual and intuitive disorientation scientific revolutions cause. In fact the opportunity may be unique. If you are an

average educated person it would probably be difficult for you, reading an account of one of the other scientific revolutions I mentioned, to feel the confusion (and excitement !) that originally surrounded the event, because you already believe the once-revolutionary theory to be substantially correct. You have brought up to believe the earth moves around the sun and is held to its path by gravity [...]. With regard to geometry, however, you are almost certainly a committed Euclidean, and consider the possibility of a logical, "truthful" geometry contradicting Euclid's to be absurd. You are alike a 16th-century astronomer hearing of Copernicanism for the first time.

4 La démarche poursuivie

Pour atteindre ces objectifs, nous avons suivi une démarche *d'apprentissage en spirale* (CREM 1995, p. 37) construite autour de la question récurrente des rapports entre vérité, géométrie et réalité, prise comme fil conducteur du cours. La description qui en est donnée ci-dessous livre bien sûr un aperçu très incomplet, en particulier parce qu'elle se centre sur les contenus et les intentions et fait peu apparaître les modes de travail utilisés.
Nous avons suivi le découpage suivant, partiellement imposé par la chronologie des évènements :

4.1 Une entrée dans la géométrie

On renvoie très naturellement celui à qui l'on pose la question : "Décrivez les objets de la géométrie (point, ligne droite, ...)" à une géométrie considérée comme un ensemble de vérités tirées de la réalité, à une géométrie obtenue par idéalisation de faits observés. Il est alors facile de faire naître une série de conflits entre intuition, logique et savoir. Mais c'est bien cette démarche qui était suivie jusque dans les années soixante dans les manuels utilisés en Suisse, manuels par ailleurs construits sur le modèle des *Éléments*, sans cesse présents en filigrane. Ils débutaient invariablement par une description des termes primitifs et une justification de leurs propriétés "sûres" (GONSETH & MARTI 1933, DELESSERT 1960). Leur analyse et leur mise en opposition avec des textes plus récents permettent de mener des discussions avec les étudiants sur les *objets mentaux*, sur leur caractère indispensable lorsqu'il s'agit de penser et de faire des mathématiques. Elle mène à parler de la nécessité, à l'école, de construire, de décrire ou de faire décrire les images idéalisées qu'on se fait des objets de la géométrie, à parler finalement aussi d'une première géométrie, structurée localement en *îlots déductifs*. On bascule ainsi naturellement vers la didactique et les points évoqués dans la discussion des objectifs s'y rapportant. Cette démarche a aussi pour effet, d'ailleurs recherché, de replonger les étudiants dans leur conception pré-universitaire de la géométrie et de recréer l'obstacle épistémologique (BACHELARD 1938, p. 18) qui les empêche de penser à une autre géométrie, par exemple hyperbolique. Les questions posées ensuite au sujet des résultats connus et du foisonnement de leurs liens logiques, les uns aux autres, ramènent les étudiants à un stade pré-euclidien. Désécurisés par une constellation si diffuse, ils conçoivent le besoin de structurer et d'ordonner les connaissances. Le moment est venu de faire le pas et d'entrer dans les *Éléments*.

4.2 Au-delà d'une approche descriptive, première abstraction : une forme mixte

Se plonger dans le texte d'Euclide est proprement fascinant. Tout est à interpréter et à discuter. Que veulent dire les définitions ? S'agit-il vraiment toujours de définitions ? Quelle est la portée des différentes demandes (postulats) ? Pourquoi distinguer entre demandes et notions

communes (axiomes) ? La rencontre avec la culture grecque nous contraint à questionner notre point de vue. Le mode d'expression inhabituel utilisé dans ce texte ouvre la porte aux interrogations sur la pensée de l'auteur et appelle le débat. Les questions ne trouvent pas toutes des réponses mais des sous-entendus sont évoqués, des conjectures sont émises sur des implicites possibles, qu'il s'agira de mettre à l'épreuve à la lecture des constructions, des théorèmes et des justifications qui suivent. L'importance de la langue et du débat d'idées sur les mathématiques apparaît clairement aux étudiants qui goûtent par ailleurs le plaisir d'avoir une marge de manœuvre : grâce au décalage culturel, les mathématiques sont "discutables". Ce qui est paradoxal si l'on songe au but poursuivi par Euclide !

L'analyse des premiers résultats proposés permet de revenir sur les hypothèses émises, de comprendre certains sous-entendus, d'observer la structure impressionnante de la construction mais aussi de déceler un appel somme tout fréquent à l'évidence géométrique ainsi que quelques manques de rigueur, du point de vue des exigences formelles d'aujourd'hui. Ceci constitue une excellente base pour discuter de la relativité de la rigueur et de la précision, mais aussi pour confronter preuve et conviction, dans un contexte a priori plus "élémentaire" que celui des exemples de LAKATOS (1984).

Cette analyse permet de traiter du concept de système axiomatique matériel et de s'interroger à ce propos sur le statut des manuels dont nous disposons ainsi que sur celui des *Éléments* (TRUDEAU 1987, pp. 250-251). Nous avons convenu à ce stade de parler pour Euclide de système mixte : matériel au premier abord, parce que les éléments fondamentaux semblent définis, mais finalement formel par le fait que ces définitions ne sont pas vraiment descriptives et qu'il n'est jamais réellement fait appel à elles: formel aussi par l'absence de toute tentative de justifier les postulats; formel encore dans la volonté de rigueur, mais matériel dans l'appel fréquent à l'évidence géométrique; matériel finalement en ce sens que cette géométrie se veut une idéalisation des faits observés, de la réalité.

D'autre part, une certaine familiarité et de nombreuses similarités avec les manuels se font jour et imposent une conclusion : l'influence des *Éléments* a été telle qu'ils ont très longtemps été simplement transposés dans l'enseignement, quand ils n'étaient pas même directement utilisés tels quels (PONT 1986, pp. 102-105) ! Ce qui constitue là encore une source de discussion didactique.

4.3 Le postulat des parallèles

Avant de lancer une analyse des résultats qui concernent les parallèles dans les *Éléments*, nous avons proposé la lecture et l'interprétation de la définition que donne LOBATSCHEFSKIJ du terme *parallèle*[5]. La prégnance des images euclidiennes est frappante. Elles ramènent les étudiants à estimer, après un certain étonnement, qu'il s'agit là d'une définition inutilement compliquée pour une notion somme tout usuelle et claire. Les potentialités de cette définition ne sont pas reconnues.

Les étudiants savent cependant bien qu'il y a eu polémique au sujet des parallèles. Ils sont d'ailleurs étonnés de ne pas trouver explicitement le postulat qu'ils attendent, qui affirmerait selon les uns l'existence, selon les autres l'unicité des parallèles (axiome de Playfair). La recherche et l'analyse des énoncés ayant trait à cette notion permettent de clarifier le sens du cinquième postulat et de comprendre ce qui le lie à l'axiome de Playfair et au résultat sur la

[5] Toutes les droites tracées par un même point dans un plan peuvent se distribuer, par rapport à une droite donnée dans ce plan, en deux classes, savoir : en droites *qui coupent* la droite donnée, et en droites *qui ne la coupent pas*. La droite qui forme la *limite* commune de ces deux classes est dite *parallèle* à la droite donnée.

somme des angles intérieurs d'un triangle.

4.4 Une échappatoire pour sortir de la polémique sur l'axiome des parallèles : le point de vue axiomatique. Ou : Trancher le nœud gordien par le renoncement à la réalité

La richesse des implications du postulat des parallèles apparaît lorsqu'on étudie les formulations proposées dans l'histoire des mathématiques pour le "prouver" ou le remplacer. Diverses activités portant sur une liste de formulations équivalentes ainsi que sur leurs liens logiques réciproques permettent aux étudiants de s'en imprégner et de sentir les conséquences profondes du postulat sur des résultats dont ils ne mettraient pas en doute la vérité, dans la réalité. C'est notamment le cas de celui portant sur la somme des angles intérieurs d'un triangle, qui semble bien être pour eux un des résultats les moins discutables de la géométrie.

La résistance de cet axiome aux tentatives de montrer qu'on peut s'en passer et l'histoire des échecs successifs de ces tentatives amènent graduellement une prise de liberté. Dans ce jeu, on accepte peu à peu de faire abstraction du lien de la géométrie avec la réalité et la vérité, l'essentiel passant dans la clarification "à tout prix" des questions concernant sa nécessité et sa portée. Ce processus historique correspond aussi à celui suivi par les étudiants dans ce cours. À ce stade, la lecture des *Éléments*, à la recherche de ce qui touche au parallélisme, et l'analyse des points concernés donnent l'occasion de présenter des explications sur la notion de système axiomatique, sur celle d'équivalence entre axiomes et sur la possibilité de réorganiser une théorie. Mais si la disposition et la volonté sont bien là, une deuxième lecture du texte de Lobatchevski montre que la conception usuelle du mot *droite* fait encore barrage à une interprétation qui serait nouvelle.

4.5 La méthode axiomatique. Mot d'ordre : Renoncer à toute description ou justification des concepts et des énoncés fondamentaux

Même si les étudiants sont désormais d'accord de jouer le jeu de l'axiomatique, suivre la démarche de Hilbert ne leur apparaît pas très naturel. Renoncer, comme il le fait, à définir et à décrire les objets de la géométrie, ravalés au niveau de "choses" sans signification propre, suscite des résistances. Cependant, une fois cette situation acceptée, la lecture des *Fondements* de Hilbert se passe sans difficultés apparentes. Les étudiants ont le sentiment de suivre. Les premiers éléments lus concernent uniquement des "évidences", si bien que la concentration n'est pas vraiment élevée et qu'on est facilement convaincu d'avoir compris. Mais on lit plutôt les figures que ce qui est réellement écrit. Un test effectué après une lecture et une réflexion individuelle portant sur les axiomes d'ordre a bien révélé que les étudiants avaient plutôt projeté les images de ce qu'ils pensaient que diraient les axiomes plutôt que d'interpréter ce qu'ils disent exactement. Au fond, le texte de Hilbert est très subtil. Tout compte, tout a une importance. L'analyse d'un des premiers théorèmes prouvés par l'auteur[6] s'est d'ailleurs avérée extrêmement ardue, les arguments verbaux et le rappel précis des axiomes étant toujours court-circuités par l'évidence liée à la figure.

Dès le moment où le principe d'un système formel et d'une démarcation du réel est admis, il est envisageable de postuler des axiomes qui seraient, sinon, contraires au "bon sens". La lecture du texte de Hilbert portant sur la géométrie hyperbolique peut être entreprise. Mais la surprise est alors de taille. Même si le style reste exactement le même, le contraste est total. Le sentiment

[6] Théorème 3. Deux points A et C étant donnés, il existe sur la droite AC au moins un point D situé entre A et C.

de stupeur et d'incompréhension remplace le confort de l'évidence et de la banalité apparente qui avait régné à la lecture de la partie euclidienne du texte. Ceci agit comme révélateur de la superficialité avec laquelle les parties précédentes étaient lues. De même le rôle essentiel qu'avaient joué les figures devient patent. Ici, elles contredisent l'intuition au lieu de la refléter fidèlement. Les droites sont courbes. Les termes, supposés ne désigner que des choses, se révèlent avoir été d'importants supports de lecture et d'interprétation. Désormais ils interfèrent avec la pensée et créent une situation de paradoxe permanente. Plus que l'absence *d'images mentales* c'est l'appel naturel à des images mentales inadaptées qui nuit à la compréhension.

On se rend compte qu'on comprenait les textes euclidiens de Hilbert, malgré la relative rareté des figures et le fait que le texte ne s'y réfère pas explicitement, parce qu'on a en tête suffisamment de représentations issues de l'expérience. Par contre, on ne peut pas lire les textes hyperboliques de Hilbert. On ne dispose justement pas de ces outils, nécessaires pour penser ! On perçoit alors bien l'importance de pouvoir s'appuyer, pour comprendre, sur des représentations, même partielles et mal abouties, sur des images mentales, généralement construites à partir du concret. On pourrait rappeler à ce propos le point de vue de GONSETH (1936, p. 338) pour qui *"la permanence du concret est une condition d'existence de l'abstrait"*. Sans ce concret, sans ces images mentales, les textes mathématiques, bien que corrects de l'avis des spécialistes, ne font tout simplement aucun sens. Ils ne deviennent ni faux, ni dénués de sens. Mais ils se vident de tout sens, glissent dans le "sans sens" (BARUK 1985, chapitre 8). Sans clés d'interprétation, tous les énoncés se ressemblent et se confondent : ceux qui sont faux, ceux qui sont absurdes, ceux qui sont corrects. Seules les personnes qui disposent des connaissances nécessaires ou de représentations préconceptuelles qui guident leur intuition peuvent faire le tri.

Pour des enseignants, vivre une telle situation et en débattre constitue à coup sûr une expérience importante. Placés, face au savoir officiel, dans une situation qui sera peut-être parfois celle de leurs futurs élèves face au savoir du maître, ils vivent l'inconfort de cette situation. Une expérience sans doute a fortiori utile pour un maître de didactique.

4.6 Une approche axiomatique de la géométrie hyperbolique, illustration par le modèle de Poincaré

L'illustration par un modèle, en l'occurrence celui de Poincaré pour le disque, permet d'entrer dans cette géométrie. La traduction et la vérification des axiomes donnent l'occasion de revenir sur différents énoncés de géométrie euclidienne concernant notamment les propriétés des inversions par rapport aux cercles. Celles-ci jouent dans ce modèle de plan hyperbolique le rôle des symétries orthogonales de la géométrie euclidienne. On peut illustrer avec elles la possibilité de "conjuguer les situations" pour passer de situations générales à d'autres, particulières mais favorables, et résoudre ainsi divers problèmes assez simplement. Un passage par la géométrie analytique ou par les nombres complexes montre parfois l'efficacité de ces approches et l'interconnexion de diverses facettes des mathématiques. Ces activités sont l'occasion de remplir certains des objectifs fixés en ce qui concerne les connaissances "dans" les mathématiques : approfondissement des connaissances en géométrie classique, approche de problèmes non élémentaires par des méthodes variées et mise en évidence de principes mathématiques généraux.

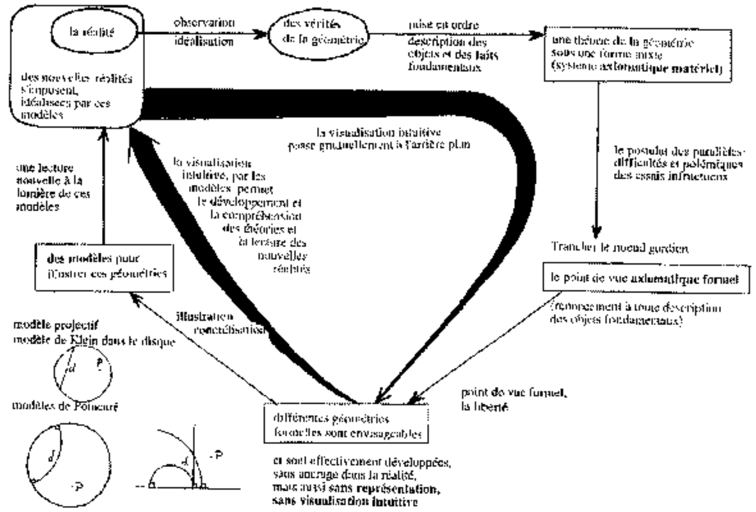
Il est frappant d'observer comment, après quelques réflexions menées dans ce modèle au sujet des premiers axiomes, la situation se débloque. Le texte de Hilbert devient lisible, car désormais interprétable. L'axiome concernant le parallélisme, les définitions qui le suivent au sujet des extrémités des droites, le texte de Lobatchevski, la possibilité pour la somme des angles intérieurs d'un triangle d'être inférieure à 180, tous ces aspects qui avaient offert tant de résistance auparavant s'ouvrent littéralement à la compréhension intuitive ! Le modèle a livré les représentations mentales qui permettent de "penser hyperbolique". La discussion de ce phénomène, vécu par tous les participants, devrait selon nous avoir eu un impact didactique profond. Parler du besoin, pour penser, de disposer d'images mentales est une chose. Avoir l'occasion de ressentir soi-même le déblocage qu'elles peuvent permettre est incomparablement plus marquant.

Ceci dit, qu'une telle géométrie puisse exister ne constitue encore ni la surprise ni la révolution copernicienne attendue. A ce stade, la géométrie hyperbolique est prise comme une vue de l'esprit, un pur produit de l'intellect sans réel rapport à la réalité, même si l'on dispose désormais pour elle d'une concrétisation par un modèle. L'évocation dans un jeu de l'esprit d'une plongée dans le centre de notre modèle de plan hyperbolique et d'un brusque changement d'échelle fait surgir le sentiment trouble que cette géométrie est peut-être bien aussi réelle que nous le pensions de la géométrie euclidienne ! La révolution promise se situe bien là : cette géométrie, dont a priori l'approche a été rendue possible en évacuant le lien à la réalité, se charge d'elle-même de réalité. On ne peut plus choisir vraiment entre elle et celle qu'on pensait initialement être en fait la seule vraie possible. TRUDEAU (1987) décrit très bien une démarche menant son lecteur à un tel basculement et l'on ne peut que répéter avec lui :

> This [...] will provide [...] a rare opportunity to actually experience the intellectual and intuitive disorientation scientific revolutions cause.

4.7 Conclusion : Vérité, géométrie et réalité

Le schéma ci-dessous, établi dans une discussion avec les participants à la fin du cours, donne un aperçu de la démarche suivie et du jeu entre abstraction et représentations intuitives, entre schémas, théories et modèles concrets. Il permet de revenir une dernière fois sur la question du rapport entre géométrie et réalité et de constater l'évolution des points de vue. Il peut d'ailleurs se lire aussi bien comme représentation du processus historique de constitution du savoir géométrique, comme vue d'ensemble récapitulant le déroulement du cours que comme illustration du processus d'apprentissage de ses participants. Ses premiers éléments fournissent l'occasion d'un retour sur quelques éléments didactiques discutés en cours de route : évolution de l'enseignement de la géométrie, évolution des manuels utilisés et processus d'apprentissage des élèves.



Géométrie et réalité, schéma d'une (r)évolution. (A. DALLA PIAZZA & *Th.* RYCHENER, *Berne, 1998*)

5 Résultats et conclusions provisoires

La démarche que nous avons suivie a rencontré un accueil enthousiaste des participants. Nous pensons avoir atteint de manière satisfaisante la plupart de nos objectifs et estimons pouvoir relever à son sujet les points positifs suivants :

- Elle permet de compléter, mais surtout de relier entre elles des connaissances éventuellement fragmentaires.
- Elle permet de relier les processus individuels d'apprentissage aux processus historiques d'élaboration du savoir.
- Elle illustre des correspondances entre difficultés observées chez les élèves et difficultés conceptuelles rencontrées historiquement.
- Elle permet de comprendre des faits **sur** les mathématiques malgré d'éventuelles difficultés ou lacunes de connaissances **dans** les mathématiques.
- Elle donne du recul et une vue d'ensemble sur un domaine des mathématiques.
- Elle illustre des rapports entre mathématiques, courants de pensée et culture, au sens large.
- Elle montre "d'où nous venons", ce que sont les sources de notre savoir ou de notre façon de voir et permet de mesurer le caractère extraordinaire des performances de nos prédécesseurs.

En outre, le rôle et le sens de l'approche axiomatique nous paraissent avoir été clairement perçus, de même que les rapports entre systèmes axiomatiques et modèles. Le fait qu'une théorie a priori abstraite amène à lire des aspects de la réalité auxquels le "bon sens" et l'empirisme n'auraient pas donné accès, qu'elle crée une distance qui permet justement de dépasser l'enfermement dans le bon sens, semble avoir été clairement ressenti. L'universalité de résultats considérés comme absolus a été remise en question, notamment celui concernant la somme des angles dans un triangle, ouvrant un regard désormais plus critique. La démarche génétique a rencontré un grand succès. Grâce à elle, histoire et apprentissage individuel des mathématiques ont été mis en correspondance. Le regard des participants sur les mathématiques s'est véritablement modifié. Ils devraient avoir enrichi leur compréhension des rapports entre mathématiques et réalité.

On doit toutefois tempérer ce tableau très optimiste en relevant que parmi les participants, jusqu'à présent, seul le tiers a finalement réalisé les activités exigées pour la validation du module ! La très grande disparité de leurs connaissances mathématiques, qui avait compliqué la préparation et le déroulement du cours, a finalement rattrapé les moins bien préparés d'entre eux. Le cours n'a pas suffi pour combler les lacunes existantes. Il les a parfois même révélées. Certains doivent encore fournir un travail considérable pour parvenir à remplir les conditions imposées. Ils s'y sont pour la plupart attelés avec un fort engagement, motivés aussi, rappelons-le, par des raisons externes à leur intérêt pour les mathématiques, liées à la nécessité d'assurer leur avenir professionnel.

Le seul objectif insuffisamment atteint concerne l'approfondissement des connaissances en géométrie. Le travail de vérification du modèle de Poincaré par l'étude de constructions faisant intervenir les inversions n'a pas assez pu être développé. Le temps à disposition était trop court. Ou le projet trop ambitieux. Trois blocs de deux jours, séparés par des phases de travail individuel à domicile auraient peut-être fourni de meilleures conditions pour le réaliser. Bien sûr, un cours classique de mathématiques aurait certainement pu amener plus de connaissances, ou des connaissances plus pointues. Par opposition, un cours comme celui que nous avons décrit et réalisé amène des connaissances plus profondes **sur** les mathématiques, en privilégiant une vue contextualisée et large du domaine. Selon nous, cet aspect de la formation en mathématiques des enseignants est celui dans lequel on relève actuellement les plus grandes faiblesses, du moins en Suisse.

Par cet essai, nous entendions clarifier le concept de "formation en mathématiques orientée vers l'enseignement" et ouvrir la voie pour les développements en cours dans notre région. Nous ne savons pas dans quelle mesure nous y avons réussi ni si nous avons suffisamment caractérisé les critères propres à ce type de formation[7]. Néanmoins, cette expérience nous semble révéler la nécessité de travailler simultanément sur trois plans différents si l'on entend répondre aux objectifs visés par une telle formation :

- Le plan des connaissances "dans" les mathématiques, au travers de thèmes liés aux contenus des plans d'études scolaires.

- Le plan des connaissances "sur" les mathématiques : aspects historiques, réflexion sur le fonctionnement des mathématiques, sur leur construction, sur leur sens, sur leur rôle par rapport à d'autres secteurs du savoir.

- Le plan des processus d'enseignement/d'apprentissage, au travers d'une approche dans laquelle le savoir étudié est mis en relation avec l'apprentissage des élèves, au niveau des contenus et des modes de travail.

[7]Voir à ce sujet WITTMANN 1989.

Nous ne prétendons pas que ce type de cours doive se substituer à l'étude des mathématiques[8] ou de la didactique. Nous pensons qu'ils ont à remplir un rôle de pont entre ces deux aspects et qu'ils peuvent orienter la formation en mathématiques vers une formation à l'enseignement des mathématiques.

Ce cours n'était pas notre premier essai. Nous en avions déjà dispensé un autre dans le même cadre. La place de l'histoire y était incomparablement plus faible. On y partait des représentations des étudiants au sujet des nombres. Le cours était construit sur un canevas similaire à celui d'une situation-problème, contraignant les étudiants à expliciter diverses facettes de leurs conceptions des nombres, à établir un débat au sujet des contradictions entre ces facettes, contradictions présentes parfois chez une même personne mais plus généralement entre les différents participants, puis à clarifier (à institutionnaliser) les points de vue. On visait à montrer la signification des théories qu'ils avaient étudiées préalablement dans des cours classiques mais qu'ils ne reliaient pas toujours spontanément aux questions inhabituelles qui étaient posées. Le débat était illustré par des controverses similaires ayant traversé l'histoire et qui avaient débouché sur l'élaboration, justement, des théories étudiées dans leurs cours académiques antérieurs. Là, c'était plus la méthode de travail que les aspects historiques ou épistémologiques qui avaient donné au cours sa teinte et son orientation particulière.

Ces cours ne devaient a priori être pour nous que l'occasion ponctuelle de sortir du cadre de nos activités habituelles. Cependant, le contexte de réformes dans lequel nous nous trouvons engagés nous a donné l'envie de lancer une démarche plus large, en vue de définir, pour la formation des enseignants, des contenus et des approches des mathématiques qui tiendraient compte d'une orientation vers la didactique. Nous ne nous sommes pas guidés pour cela par des considérations et une démarche scientifiques. Nous avons procédé de façon pragmatique et intuitive, sur la base de notre expérience de mathématiciens et d'enseignants[9]. Mais nous entendons prolonger ces premiers pas et suivre le plan de travail suivant, comportant trois phases :

- Faire des essais. Ces cours étaient les premiers. Un prolongement suivra pendant l'année 1999/2000 dans le cadre de la formation continue des enseignants.

- Sur la base de ces expériences, des conclusions provisoires, des conjectures et des hypothèses qui en seront retirées, passer à une théorisation de la démarche et définir des thèmes de recherche et d'expérimentation, avec cette fois une méthodologie précise.

- Construire des modules de formation qui viendraient à l'avenir s'insérer dans la formation des enseignants.

La réalisation de la dernière étape dépendra des choix qui seront faits dans notre région et qui nous échappent pour une grande part. Selon ce qu'il adviendra, le projet en restera peut-être aux deux premières phases. Ce congrès aura été l'occasion d'établir des contacts et d'apprendre. Il est un pas dans la phase de théorisation et vers l'établissement de contacts avec des collègues prêts à collaborer avec nous à un tel projet.

[8]Nous avons bien perçu les limites imposées par le niveau insuffisant des connaissances en mathématiques de certains participants. En soi, ce constat est très préoccupant et nous ne pouvons qu'espérer que les nouvelles institutions de formation des enseignants sauront poser des exigences plus élevées au moment du recrutement de leur personnel.

[9]Nous avons tous deux une expérience de l'enseignement des mathématiques au degré secondaire II (élèves de 15-19 ans) et au niveau des premiers cycles universitaires (étudiants de 18-21 ans) ainsi que, pour la didactique des mathématiques, dans la formation des enseignants du secondaire II (étudiants de 23-25ans).

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Some reflections on the illusion of linearity

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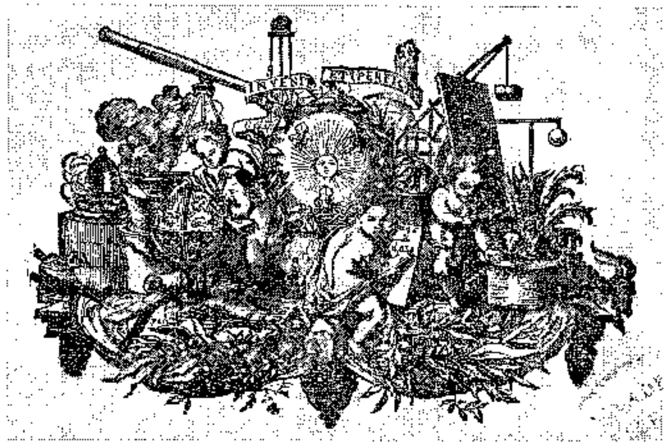
Abstract

Linear (proportional) functions are undoubtedly one of the most common models for representing and solving both pure and applied problems in mathematics education. But according to several authors, different aspects of the current culture and practice of school mathematics develop in students a tendency to use these linear models also in situations in which they are not applicable. In this paper, we first present some historical examples of and comments on this "illusion of linearity". Second, we briefly discuss the results of five recent empirical studies about the occurrence of this phenomenon in 12-16-year old students working on problems about the relation between the linear measurements, the area and/or the volume of similar geometrical figures, as well as about the effect of several task variables on this improper use of linearity. Finally, we analyse the connection between this linear illusion and other intuitive rules and erroneous ways of thinking in mathematics education.



1 Introduction

Linear (or directly) proportional relationships are a major topic in elementary mathematics education. The attention given to linear relationships is mainly due to the fact that they are the underlying basic model for a lot of problems in pure and applied mathematics. Unfortunately, students' growing familiarity and experience with linear models may have a serious drawback: it may lead to the misbelief that these models have a "universal" applicability and to a tendency to deal with each numerical relation as though it were linear (FREUDENTHAL 1983). In the literature, this phenomenon is referred to as the "illusion of linearity", "linear misconception", "linear obstacle" or "linear trap" (see, e.g., BERTÉ 1993; FREUDENTHAL 1973, 1983; ROUCHE 1989). Although these terms may have subtly different meanings (for instance, the term "linear illusion" carries a connotation of visual perception which seems absent in the other terms), we will not differentiate between them and use them intermixedly in the rest of this article. The illusion of linearity in students' reasoning has been frequently described and illustrated with respect to different domains of mathematics in this literature, but it has elicited little systematic empirical research. This is quite remarkable taking into account the large amount of research on proportional reasoning in the last 10-15 years (for reviews of this research see, e.g., BEHR, HAREL, POST & LESH 1992; TOURNIAIRE & PULOS 1985), including research on students' misconceptions about and primitive strategies for solving proportional tasks.

2 Examples of the illusion of linearity

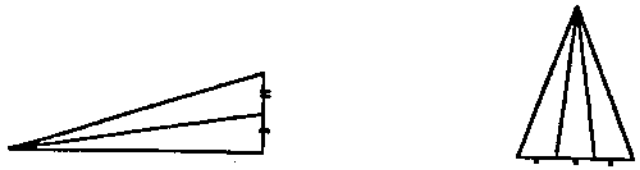
A first series of examples relates to elementary school pupils' modelling of word problems in school arithmetic.

> *"It takes 15 minutes to dry 1 shirt outside on a clothesline. How long will it take to dry 3 shirts outside?"*
>
> *"It costs 2 EUR to send a parcel weighing 500 g. How much will it cost to send a parcel weighing 1500 g?"*
>
> *"John's running record for 100 m is 12 seconds. How long will it take him to run 1 km?"*

Pupils who fall into the trap of proportional reasoning on the first example, obviously neglect the realistic consideration that the drying time doesn't depend on the number of clothes on the line. Rather than using their real-world knowledge about the situation described in the problem - namely: drying clothes - they simply play the "game of school word problems" (VERSCHAFFEL, GREER & DE CORTE, in press), in which the players are assumed not to attend too much to the realities of the situation described in the problem statement and to identify the arithmetic operation(s) with the given numbers that yields the correct answer. In the example of the mail costs, it is less sure that pupils giving the incorrect, directly proportional response neglect to apply their knowledge of the context or meaning of the problem. Probably, they are just ignorant of the specific mathematical model linking mailcost to weight, a "staircase" instead of a linear function (VERSCHAFFEL et al., in press). Apart from that, even if they knew the adequate mathematical model, this knowledge wouldn't enable them to answer the problem correctly; therefore, they have to know the appropriate costs for sending parcels or at least have the possibility to look them up. In a way, the problem about the running time is even more delicate because, strictly speaking, it cannot be solved correctly based on the information given, on the one hand, and the solver's real-world and mathematical knowledge, on the other hand.

An adequate mathematical model for this problem situation, enabling to determine precisely the running time, and thus taking into account the time the runner needs to accelerate at the beginning as well as his tiredness latterly, isn't obvious at all. A student who realizes that the distance and the running time in this "problem" are not proportional, is -in some sense- placed in a dilemma: either acknowledging that the realities of the problem context prohibit that it is solved by means of proportional reasoning and therefore refusing to give a (precise) answer (which is a violation of one of the basic rules of the "game of word problems", namely that word problems always have to be solved by means of a numerical answer), or playing the game by doing as if one had not detected that mathematical modelling complexity and responding with the outcome of a routine proportional reasoning process.

A mathematical subject-matter in which many people (frequently) fall in the proportional trap is *geometry*. For instance, it appears that some pupils regularly apply an incorrect linear reasoning in problems involving the relationships between angles and sides of plane figures. Figure 1, borrowed from ROUCHE (1992a), suggests incorrect methods for bisecting (drawing left) and trisecting (drawing right) an angle.



The first construction assumes a linear relation between an acute angle and its opposite side in a right-angled triangle. The second construction, which is correct for the bisection but not for the trisection of an angle, assumes a linear relation between the angle at the top and the base of an isosceles triangle (or between an angle in a circle and its corresponding chord).

Some other interesting geometrical examples of the linearity illusion can be found in the doctoral dissertation of DE BLOCK-DOCQ (1992). In the margin of her "epistemological comparative analysis of two teaching methods for plane geometry with pupils of the age of twelve", she mentions some typical erroneous reasoning processes, based on an inappropriate application of direct (or inverse) proportionality between non-proportional quantities.

> *"The angle of a regular dodecagon can be obtained by dividing the angle of a regular hexagon by six and multiplying this result by twelve."*
>
> *"To construct an equilateral triangle inscribed in a circle, one has to pace the diameter on the circumference; an inscribed regular dodecagon can be constructed by pacing the half of the radius on the circumference."*
>
> *"If one can split a heptagon into five triangles (by joining one corner to all the other corners), one can split up a 14-sided polygon into ten triangles."*

The field of probabilistic thinking provides also a lot of nice examples of improper applications of proportionality. Confronted with the problem:

"In a game of chance the probability of success is one tenth. What is the probability of at least one success if one plays the game of chance three times?"

many pupils will trap into proportional reasoning.

Very famous in this domain are the two historical problems Chevalier de Méré posed to his friend Pascal (see, e.g., FREUDENTHAL 1973). De Méré knew (by experience?) the advantage of betting on the event "at least one six in 4 rolls of one fair die" and he deduced that it must be equally advantageous to bet on "at least one double-six in 24 rolls of two fair dice". One specific outcome among 6 possibilities in 4 trials must occur equally frequent (and thus justify the same stake) as one specific outcome among 36 possibilities in 24 trials, because 6/4 = 36/24. Later on, because he experienced that, notwithstanding his reasoning, bets on the latter event didn't yield the hoped-for financial gain, he consulted his friend Pascal. Pascal's (correct) answer was as follows: the probability of no six in one roll of a fair die is 5/6, thus the probability of no six in 4 trials is $(5/6)^4$ and thus the probability of at least one six is $1 - (5/6)^4 = 0.5177$, a bit more than a half. The probability of no double-six in 24 rolls of two fair dice is $(35/36)^{24}$ and, consequently, the probability of at least one double-six is $1 - (35/36)^{24} = 0.4914$, a bit less than a half.

The second problem de Méré propounded to Pascal is the so-called "problème des partis". Two persons, let's say A and B, stake the same amount in a game of chance consisting of at most 9 sets, each with the same chance of profit for A and B. Due to circumstances without their consent, they must break the game after 7 sets. At that moment, A has won 4 sets and B 3. How can the stake be honestly shared out? Chevalier de Méré proposes a proportional reasoning based on the three numbers given in the problem (3, 4 and 5), but hesitates between a proportion of 4 over 3 and (5 - 3) over (5 - 4). What is correct? None of both, Pascal judges! Suppose both players would play two more sets, than there would be four possibilities:

A wins, A wins
A wins, B wins
B wins, A wins
B wins, B wins.

In three of these four cases, A would receive the whole stake while B would receive it in only one case. Thus, A has three chances versus the one chance of B. In consequence, the stake should be shared out in a proportion of 3 to 1.

Finally, also in the fields of algebra and calculus one can find several examples that can be qualified as linearity illusions. In this domain, what pupils actually misuse in most cases is not the linear model itself, but rather its properties, especially the preservation of the addition and the multiplication by scalars. Every high school teacher knows examples of students applying "properties" like: "the square root of a sum is the sum of the square roots", "the logarithm of a multiple is the multiple of the logarithm",... BERTÉ (1992) takes up the question how this "linear obstacle" can be removed to open pupils' mind for the acquisition of new mathematical models and their proper range of application. The improper use of linearity in the solution of extremum problems has been described in DE BOCK (1992).

In addition to the numerous examples of pupils' misuse of the linear model in a wide range of situations, the mathematics education literature also contains some reflections and comments

on this phenomenon. Some authors suggest that maybe the simplicity and self-evidence of the linear model are at the root of the illusion of linearity.

> Linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as though it were linear. (FREUDENTHAL 1983, p. 267)

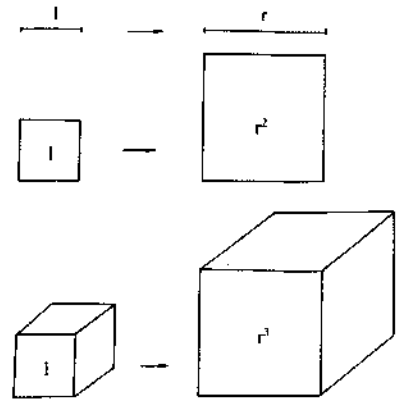
> C'est l'idée de proportionnalité qui vient d'abord à l'esprit, parce qu'il n'y a sans doute pas de fonctions plus simples que les linéaires. (ROUCHE 1989, p. 17).

The faulty solution strategies by de Méré based on a straightforward application of linearity mentioned above, elicit the following (sharp) comment by FREUDENTHAL (1973, p. 585) with respect to mathematical instruction:

> He [de Méré] applied the mathematics he knew, the kind of mathematics which in my childhood was called the rule of three... Maybe he would have performed better if he had never learned mathematics at all! Then there would have been some chance that he would have applied not the mathematics he had learned but the mathematics that he would have to create himself.

3 The effect of a linear enlargement (reduction) on area and volume

Our own research focuses on applied mathematical problems about the relation between the linear measurements, the area and/or the volume of similar geometrical figures. The principle governing that kind of application problems is well-known: an enlargement or reduction by a factor r, multiplies lengths by factor r, areas by factor r^2 and volumes by factor r^3 (Figure 2).



A crucial aspect of understanding this principle is the insight that these factors depend only on the magnitudes involved (length, area, and/or volume), and not on the particularities of the figures (whether these figures are squares, circles, etc.). According to FREUDENTHAL (1983, p. 401), this principle is mathematically so fundamental that it must come first, both from a phenomenological and didactical point of view.

This principle deserves, as far as the moment of constitution and the stress are concerned, priority above algorithmic computations and applications of formulae because it deepens the insight and the rich context in the naive, scientific, and social reality where it operates.

The improper use of a linear proportional model in enlarging and reduction operations is a classical mistake - probably one of the oldest in the history of mathematical thought. The most often quoted example can be found in Plato's dialogue Meno (see, e.g., BERTÉ 1993, ROUCHE 1992b) in which a slave, when asked by Socrates, Plato's master, to draw a square having two times the area of a given square, firstly proposes to double the side of the square. So, the slave spontaneously applies the idea of linear proportionality (between length and area) and changes his mind only when Socrates helps him in diagnosing and correcting the error in his reasoning confronting him with a drawing.

Since then many other authors have argued that getting insight in the above-mentioned relationships between lengths, areas and volumes of similar figures usually is a slow and difficult process. In the American Standards, for instance, it is stated:

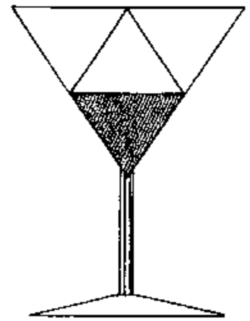
> ... most students in grades 5-8 incorrectly believe that if the sides of a figure are doubled to produce a similar figure, the area and volume also will be doubled. (NCTM, 1989, pp. 114-115).

But scholars of the Freudenthal Institute, who have explored the influence of linear enlargement on area and volume in realistic contexts (like "Gulliver" in TREFFERS 1987, or "With the giant's regards" in STREEFLAND 1984), have claimed and provided some evidence that -at least in the context of realistic mathematics education- this misbelief can rather easily be overcome even in the primary school. However, this assertion is -as far as we know- not supported by systematic empirical research data.

Remarkable and contrary to some other domains in which pupils (frequently) fall in the linear trap, is the ascertainment that real-word or commonsense knowledge doesn't always suffice to grasp correctly the influence of a linear scaling on length, area and volume. In this respect, FEYS (1995, p. 123) describes an experience with his pre-service teachers as follows:

> We ask pre-service teachers what will happen when they lay out two A4 pages side by side on a copier in order to reduce them on one A4 page. Regularly, they answer that the text will be no longer readable because the height and width of the characters and of the drawings should be halved.

Also typical for this phenomenon is the fact that pupils are often surprised that by an enlargement, the area and certainly the volume is enlarged thus much; contrarily, by a reduction, they are often surprised that the area and certainly the volume is reduced thus much (a giant being ten times as tall as an adult man of 70 kg, weighs 70 ton; a goblin ten times smaller than this adult, only weighs 70 g!). A well-known example of pupils' misjudgement of the very strong effect of a reduction on volume is the conic glass that is filled half (of the height) of a full glass: pupils most often realise that its volume is less than the half of a full glass, but when asking them to estimate more precisely which part it is, their answers are mostly more than one eighth. (Pupils' tendency to overestimate this volume probably relates to visual perception: in front view, one can see in fact a triangle whose area is reduced to one fourth, see Figure 3.)



Finally, it is reality itself that, in a certain sense, puts the pupils on the wrong track. The mathematical idea of a "linear enlargement" doesn't always fall together with the physical and biological reality of scaling. Old trees are more plump than younger specimens; tigers have relatively thicker paws than cats. These examples of enlargements "taken from nature" are not similar enlargements and, consequently, the relationships between linear measurements, area and volume, as described above, are not applicable in these situations. The reasons why are not mathematical, but have a physical or biological origin. Let us explain, for instance, why higher trees (must) have relatively thicker trunks than smaller species. Suppose the trunk of a tree being twice as high, should be twice as thick, then, the higher tree's volume should be increased by a factor 8! In fact, the bearing-power of a trunk (pillar, paw,...) is directly proportional to the cross-section of the trunk, that, in the case of a linear enlargement by a factor 2 (in all dimensions), only would quadruple. In order to bear a tree being 8 times as heavy, the diameter of the trunk should increase by a factor $\sqrt{8}$ (which is nearly a triplication!). Also in this respect, we notice that babies are not "linearly reduced" adults: their head and bones have a relatively bigger portion in their weight which makes them relatively heavier than adults.

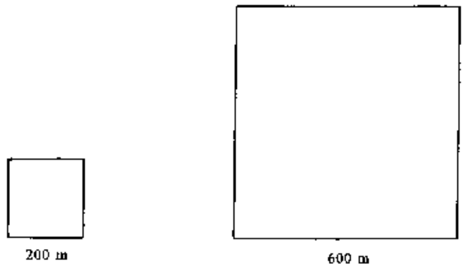
In the next part of this article we report on five closely related ascertaining studies on the illusion of linearity with respect to problems involving length, area and volume of similar figures presented in a school context.

4 Five ascertaining studies on the illusion of linearity

We first summarize the objectives and major results of these five empirical studies (DE BOCK, VERSCHAFFEL & JANSSENS 1998a, 1998b; DE BOCK, VERSCHAFFEL, JANSSENS & ROMMELAERE 1999; DE BOCK, VERSCHAFFEL & CLAES 1999). Study 1 investigates the illusion of linearity in 12-13-years old pupils working on word problems involving length and area of similar plane figures of different kinds of shapes. To answer the question whether self-made or ready-made drawings are helpful in breaking pupils' tendency to overgeneralize the applicability of linear reasoning for that kind of problems, testing took place under various conditions. Study 2 is basically a replication of the first one with 15-16-year old pupils. Both Study 1 and 2 convincingly demonstrated the strength of the illusion of linearity in pupils solving (non-linear) scaling problems presented in a school context. With a view to arrive at a better understanding of the results observed in these studies, three follow-up studies were executed focusing on the influence of different aspects of the testing context on pupils' solutions. Study 3 was set up to examine the resistance to change of the illusion of linearity by providing pupils adequate metacognitive and adequate visual support. Subsequently, Study 4 investigates to what extent the tendency towards improper proportional reasoning is caused by particularities of the problem formulation, more specifically by the missing-value type of problem that pupils learned to associate with proportional reasoning throughout their school career. At last, Study 5 explores yet another possible explanation for pupils' misuse of the linear model, namely the inauthentic or unrealistic nature of the problem context.

4.1 Study 1

Hundred-and-twenty 12-13-year old pupils, divided in three equal groups, participated in this study. The experiment consisted of two phases. During the first phase all pupils were administered the same paper-and-pencil test consisting of 12 experimental items and several buffer items. No hints or special instructions were given. All 12 experimental items involved similar plane figures, and belonged to either one of three categories: 4 items about squares, 4 about circles, and 4 about irregular figures. Within each category of figures, there were 2 proportional items (e.g. "Farmer Gus needs approximately 4 days to dig a ditch around his square pasture with a side of 100 m. How many days would he need to dig a ditch around a square pasture with a side of 300 m?") and two non-proportional items (e.g. "Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m?"). Two weeks after the first test the three groups of pupils were confronted with a parallel version of the first test. The problems in this second test were the same for all three groups, but the instructions were different. In Group I, which functioned as a control group, the testing conditions were exactly the same as during the first test. The students of Group II were explicitly instructed to make a drawing or a sketch of the problem situation before computing their answer. In Group III, finally, every problem was accompanied by a correct drawing (like the one given in Figure 4). The influence of the task variables on pupils' performance was determined by means of an analysis of variance and a posteriori Tukey tests.



The results confirmed the hypothesis that the predominance of the linear model would be a serious obstacle for the vast majority of the pupils. Indeed, the analysis of variance revealed an extremely strong main effect of the task variable "proportionality": while the proportional items elicited 92% of correct responses, only 2% of the non-proportional items was answered correctly. Second, we unexpectedly did not find any beneficial effect of the self-made or given drawings, nor in general, nor for the non-proportional items in particular. Third, the type of figure had a significant effect. Percentages of correct responses were also in the expected direction (problems about squares > circles > irregular figures).

For more than one reason, we found it appropriate to set up a follow-up study with an older target group. First, the small number of correct responses on non-proportional items made us wonder how strong the predominance of the linear model would be for pupils who were older and -therefore- mathematically better equipped for overcoming the obstacle of unlimited linear proportional reasoning. Second, because the illusion of linearity proved to be so strong with 12-13-year olds, the first study did not yield adequate information about the possible influence of self-made or given drawings on the occurrence of errors based on inappropriate proportional reasoning.

4.2 Study 2

Two-hundred-and-twenty-two 15-16-year old pupils participated in the follow-up study. Contrary to the first study, we did not administer the test twice to all pupils, but worked with three groups that were rigorously matched based on several subject characteristics. Group I (in which no special help or instructions were given), Group II (in which the pupils were instructed to make a drawing) and Group III (in which every item came with a correct drawing). We used the same 12 experimental items and the same procedure for test administration as in the first study.

The analyses of variance showed once again an extremely strong main effect of the task variable "proportionality": the overall percentages of correct responses on the proportional and non-proportional items were 93% and 17%, respectively. The hypothesis about the positive influence of the self-made or given drawings was, once again, not confirmed. Finally, as in the first study, the type of figure involved played a significant role. Pupils performed better on the non-proportional items when the figure involved was regular (a square or a circle), but as a drawback they performed worse on the proportional items about these regular figures because they sometimes started to apply non-proportional reasoning on the proportional items too.

So, both Study 1 and 2 revealed the expected alarmingly strong tendency among pupils to apply linear proportional reasoning in problem situations for which it was not suited, but they did not show the anticipated facilitating drawing effect on students' performance. It could be argued that the extremely weak results on the non-proportional items and the absence of a positive drawing effect were due to the fact that the students had approached the test with the expectation that it would consist of routine tasks only (as is frequently the case in current mathematics education). A possible additional explanation for the weak results could be that the response sheets used in these studies were not suited for measuring lengths and areas of (given) plane figures because of the lack of useful reference points for this activity on these sheets. Based on these two arguments, one could predict that the accuracy rates for the non-proportional items would increase significantly (1) if pupils would get at the beginning of the test some kind of explicit warning that not all problems in the test were standard problems, and (2) if we would make use of response sheets with (drawings made on) squared paper instead of blank paper. The third study investigates the possible influence of these two forms of "scaffolding" on pupils' solution processes and outcomes.

4.3 Study 3

Two-hundred-and-sixty 12-13-year olds and hundred-and-twenty-five 15-16-year olds participated in the study. In both age-groups students were matched in four equivalent subgroups which received one and the same paper-and-pencil test, consisting of 6 proportional and 6 non-proportional items, but the administration of the test was different in the four subgroups of pupils. In Group I -the control group- no special help was given. In Group II -the metacognitive scaffold group- the test was preceded by an introductory task that confronted pupils with a correct and an incorrect solution to a representative non-proportional item and asked them which one was the correct. In Group III -the visual scaffold group- every item came with an appropriate drawing of the problem situation made on squared paper. Finally, in Group IV both kinds of help were combined.

The study yielded small but significant effects of both kinds of scaffolds. As a result of the metacognitive scaffold, the percentage of correct responses on the non-proportional problems increased slightly from 12% to 18%. With respect to the visual support, the increase was even smaller: from 13% in the groups without visual scaffold to 17% in the groups with the visual scaffold. As a drawback of these better results on the non-proportional items in the scaffolded conditions, the pupils' results on the proportional items decreased. Apparently, the scaffolds made it easier -at least for some pupils- to discover the non-proportional nature of a problem, but as a result they sometimes began to question the correctness of the linear model for problem situations in which that model was appropriate. However, the most important result of the study is that the positive effects of the two scaffolds on pupils' solutions of the non-proportional items remained remarkably small, suggesting that students' tendency towards linear modelling is very strong, deep-rooted and resistant to change.

4.4 Study 4

While all these three studies revealed pupils' almost irresistible tendency to apply proportional reasoning in problem situations for which it was totally inappropriate, the question remains why so many pupils fell into this "proportionality trap", even after receiving visual and/or metacognitive support. In Study 4, we investigated the effect of another attempt to overcome the linearity illusion by changing the experimental setting, namely the problem formulation. In the three previous studies, all proportional and non-proportional items were presented as missing-value problems. In this problem type, three numbers (a, b and c) are given and the problem solver is asked to determine an unknown number x. In a proportional missing-problem, the unknown x is the solution of an equation of the form $a/b = c/x$. It is clear that the vast majority of the missing-value problems pupils encounter in the upper grades of the elementary school and the lower grades of secondary school, are problems for which the linear model suits perfectly. Therefore, it could be argued that pupils' extremely weak results on the non-proportional items may not be due to intrinsic difficulties with the mathematical concept involved in these problems -namely understanding the effect of a linear enlargement on area- but are merely the result of a misleading problem formulation which is associated with and therefore calls up the overlearned solution scheme and procedure of proportional reasoning. To find out this, we set up a new study in which the formulation of the problems was experimentally manipulated while keeping the intrinsic conceptual difficulties constant.

Hundred-and-sixty-four 12-13-year old pupils and hundred-and-fifty-one 15-16-year old pupils participated in the study. All pupils were administered the same paper-and-pencil test consisting

of different kinds of proportional and non-proportional items, - just as in the previous studies. In both age-groups we worked with two equivalent subgroups of pupils that were matched on an individual basis and that were given a different version of the test. For half of the pupils (the so-called missing value-group), all items were presented as missing-value problems (see, e.g., the examples given in the report of Study 1), while in the other half (the comparison group), the items were formulated as comparison problems (e.g. "Farmer Carl manured a square piece of land. Tomorrow, he has to manure a square piece of land with a side being three times as big. How much more time would he approximately need to manure this piece of land?").

The results of the analysis of variance with respect to the effects of proportionality and age on pupils' performance, confirmed the findings from the earlier studies. More interestingly, however, is that while the analysis of variance did not reveal a main influence of the problem formulation variable, a highly significant interaction effect of problem formulation and proportionality was found. As expected, the group who received the comparison problems performed significantly better on the non-proportional items than the group who received the missing-value problems (41% and 23% correct answers, respectively), but this better performance of the comparison group on the non-proportional items was parallelled with a worse score on the proportional items (i.e. 68% versus 87% correct answers in the missing-value group). Apparently, the formulation of the items used in the comparison group prevented pupils for falling into the proportionality trap, but as a result these pupils sometimes began to question the correctness of the proportional model for problem situations in which that model was appropriate - a finding that is very similar to the one obtained in our previous studies and that has been observed in several other studies about strategic and conceptual change.

So, the significantly better results of the comparison group on the non-proportional items made it clear that a significant number of pupils fail on traditionally presented non-proportional items not because of their belief in the omni-applicability of the linear model, but rather because of the association of that model with a particular type of problem formulation, in this case the missing-value type. While the effect of problem formulation was significant, it was again still rather small, as still more than half of the pupils in the comparison condition failed on the non-proportional items. This also raises the question what other aspects of the testing context affected pupils' incorrect reasoning process.

4.5 Study 5

In the last follow-up study, we investigated another possible explanation for pupils' misuse of the linear model, namely the inauthentic and unrealistic nature of the problem situations. Some evidence for this hypothetical explanation can be found in TREFFERS (1987), who realized a design experiment with sixth graders on the influence of linear enlargement on area and volume that was built around the context of "Gulliver's travels", and claimed that, in this realistic mathematics education approach, pupils have no difficulty with a problem like "How many Lilliputian handkerchiefs make one for Gulliver if you know that the length of a Lilliputian is 12 times smaller than that of Gulliver?". To test the facilitating power of making the problem context more realistic and motivating, we executed a new study.

In this study, hundred-and-fifty-two 13-14-year olds and hundred-and-sixty-one 15-16-year olds were matched in two equivalent subgroups. In both groups, a paper-and-pencil test, consisting of proportional and non-proportional scaling problems was administered. In the first group, the test was preceded by an assembly of well-chosen fragments of a film version of Gulliver's visit to the isle of the Lilliputians and all experimental items were linked to these film fragments. For instance, inspired by a fragment showing a Lilliputian occupied with filling Gulliver's wine-glass, we asked "Gulliver's wineglass has a volume of 172 800 mm^3. What's the volume of a Lilliputian wineglass?". In the second group, an equal number of mathematically isomorphic problems was presented in the form of a series of non-related traditional school problems, without any contextual support.

Contrary to our expectation, there was no positive effect of the authenticity factor on pupils' performance on the test as a whole, and on their scores on the non-proportional items in particular. On the contrary, pupils who watched the video and who received the video-related items performed even significantly worse than the other group (25% for the video versus 42% for the non-video condition). At this moment we are planning a replication study to investigate whether this unexpected finding was an artefact of our operationalization of the experimental variable (e.g., the fact that the pupils who watched the video had less time to solve the test) or if it was due to the fact that these pupils' involvement in an attractive and rich context had led them away from the necessary in depth analysis of the mathematical problem structure, instead of helping them to find it.

5 Discussion

A couple of years ago, we executed a study showing that the vast majority of pupils failed on word problems about the length and area of similar plane figures. We have conducted follow-up studies showing that several attempts to overcome these failures by changing the experimental setting did not work or had only minor success: providing drawings, providing metacognitive support, rephrasing the problem and increasing its authenticity. In our future work, we will turn from ascertaining studies to individual interviews with selected groups of pupils and then to design experiments wherein we will develop and test new instructional materials and techniques aimed at improving pupils' necessary conceptual tools and cognitive strategies to overcome this very strong and resistant illusion of linearity.

This more process-oriented research could also yield a better understanding of the relations between the illusion of linearity and the broad domain of pupils' "misconceptions" (or "pre-conceptions" or "alternative conceptions") and "illusions" in the learning of both mathematics and science. A related, but in a way more primitive misconception than the illusion of linearity, is the one of additive reasoning in situations wherein (rather) a multiplicative reasoning is applicable (for instance, in the context of a linear enlargement of a figure, pupils argue that all lengths are added to instead of multiplied by a constant. This "additive illusion" is ascertained in numerous studies and appears to affect especially elementary school pupils passing on from additive to multiplicative structures (see, e.g., HART 1981; KARPLUS, PULOS & STAGE 1983; LIN 1991; STREEFLAND 1988). A similarity with the linearity illusion is that in both cases the error results from pupils' overgeneralizations of a previously learned model beyond its proper range of application.

There exists also a similarity between the illusion of linearity and the so-called "illusion of constancy" or the intuitive rule "Same A - same B" (see, e.g., Tsamir, Tirosh & Stavy, 1998) - a rule emerging regularly when pupils formulate properties of geometrical figures (e.g. "triangles with equal angles have equal sides", "quadrilaterals with equal sides have equal angles", etc.). In the same way, many pupils spontaneously think that plane figures with the same perimeter have the same area or solids with the same surface area have the same volume. One can expect that

those pupils neither make any distinction between the scale factors appropriate for these different dimensional quantities. CASTELNUOVO & BARRA (1980) describe two classical examples of this phenomenon. The first one deals with crushing a square or rectangle made of straws. By crushing the figure only slightly, most pupils think that the area remains the same. Only when confronting them with an extreme situation (the figure is crushed almost completely), everybody can see that the area decreases by crushing the figure. A second one is the famous cylinder-problem of Galilei. Using a rectangular piece of stuff (with different length and width), one can make cylindrical (corn)sack: a wide but low one (by connecting the smallest sides of the rectangle) and a narrow but high one (by connecting the biggest sides of the rectangle). The wide but low cylinder does have the greatest volume, but many people ("except the farmers in Galilei's time") think they have the same volume because they have the same surface area.

In the field of probabilistic thinking, many well-known erroneous reasonings of pupils can be both explained in terms of the "Same A - same B" intuitive rule and of a proportionality illusion. Let's first remind the first problem de Méré propounded to Pascal: de Méré's incorrect argumentation for the equiprobability of two events was based on an equivalence of ratios, in Freudenthal's view a straightforward application of the "rule of three" de Méré learned at school. Yet another explanation for de Méré's faulty approach to this problem can be found in the theory of intuitive rules, more specially, as a manifestation of the "Same A - same B" scheme (de Méré argues: "Same proportion, thus same probability").

Recently TIROSH & STAVY (1999, p. 190) reported a quite similar example:

> The Carmel family has two children, and the Levin family has four children. Is the probability that the Carmels have one son and one daughter larger/equal to/ smaller than/ the probability that the Levins have two sons and two daughters?

The probabilities of these events could be reached by a relatively simple counting and calculation. In fact, the probability of "one boy, one girl" in the Carmel family is 1/2 while for the Levin family, the probability of "two boys, two girls" is 3/8. Thus the probability that the Carmels have one son and one daughter is larger than the probability that there are two sons and two daughters in the Levin family. The researchers presented this problem to about 40 students in grades 7 to 12. The majority of these students incorrectly argued that both probabilities are equal because "the ratio is the same, therefore the probability is the same". Remarkably, the distribution by grade showed a (slightly) increasing trend in the percentages of incorrect responses with age.

This result is very similar to the one obtained by FISCHBEIN & SCHNARCH (1996, pp. 355-356), with respect to the problem:

> In a certain town, there are two hospitals, a small one in which there are, on the average, about 15 births a day and a big one in which there are, on the average, about 45 births a day.
> The likelihood of giving birth to a boy is about 50%. (Nevertheless, there were days in which more than 50% of babies born were boys and there were days in which less than 50% if babies born were boys).
> In the small hospital one has kept a record during a year of the days in which the number of boys born was greater than 9, which represents more than 60% of the total of births in the respective hospital.
> In the big hospital, one has kept a record during a year, of the days in which there were born more than 27 boys which represented more than 60% of the births.
> In which of the two hospitals there were more such days?

In fact, the small hospital will record more days where more than 60% boys were born. The stochastic law one has to consider is the law of large numbers. As a sample size (or the number of trials) increases, the relative frequencies tend to come closer to the theoretical probability. And, on the contrary, if one considers a small sample, the relative frequencies of expected outcomes may deviate largely from the theoretical probability. Fischbein and Schnarch presented this problem to groups of 20 pupils of grades 5, 7, 9 and 11. The main misconception, in this case, "the number of days in a year on which one has recorded the birth of more than 60% boys does not depend on the sample size", increased with age in a surprisingly regular manner. Starting from 10% in grade five, it reached about 80% in grade 11. The erroneous answer was usually justified by the equality of ratios: "9/15 = 27/45 - they express both the same ratio".

Most likely, the "scheme in action" in pupils working on this kind of problems, the intuitive rule "Same A - same B" or the proportionality illusion, depends on pupils' age and school career. The first scheme is more general and even observed in very young children (for instance when solving Piagetian tasks), the second is more specific and strongly affected by schooling. Further research is needed to unravel the interaction between both schemes, its evolution with age and how this interaction can be influenced by mathematics education.

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The role of physics in introducing vectors to secondary school students

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Abstract

Despite the originally geometrical point of view adopted by mathematicians of the 19th century concerning the nature of the concept of a vector, history suggests that vectors were established as a language of mathematics and science (symbolism, terminology, and computational techniques) mainly through physics.

Physics offer intuitively suggestive situations for introducing vector notions and operations in school. Previous research by H. DEMETRIADOU & A. GAGATSIS has identified concrete and persisting difficulties among Greek high-school students (aged 15-18) concerning certain epistemological aspects of the concept of a vector. Some of them concern the confusion between notions like "vector and line segment", "sense and orientation or path", the meaning of symbols like " + " or " = ", as well as the use of vector operations.

It was also found that there is a serious difficulty concerning the differentiation between vectors as "line segments with a definite magnitude and direction (path and sense) attached to a point" (called "tied vectors" in the Greek curriculum), the prototype being the physical concept of force, and vectors as "line segments with a definite magnitude and direction (path and sense)" (called "free vectors" in the Greek curriculum), the prototype being the geometrical concept of a parallel translation.

These are related to apparently different types of addition: the parallelogram and the triangle laws respectively. From a more advanced, mathematical- epistemological point of view, these laws correspond to the distinction between "tangent vectors to a manifold" and "parallel translated tangent vectors on a manifold", the latter concept requiring more structure (i.e. the concept of a connection on a manifold) than just the ordinary differential structure of a manifold. This partly explains our experimentally confirmed result that often, secondary students do not recognize the equivalence of these laws.

We conclude that purely mathematical teaching situations are not the most appropriate means to introduce vector notions and operations. Inspired by the historical development of the subject and based on previous work concerning the students' difficulties concerning vector notions and operations, we suggest the use of real and thought experiments related to displacements, velocities analyzed as successive displacements, and forces, not only for introducing, but also for clarifying vector notions. Some results from a teaching experiment are also presented in this connection.

1 Introduction

Despite the geometrical point of view adopted by mathematicians of the 19th century, concerning the nature of a vector, history suggests that vectors were established as a language of mathematics and science mainly through physics (whether one considers symbolism, terminology, or computational techniques).

In fact, it was the interplay between pure mathematical and physical situations that influenced the emergence of vector concepts and operations. This historical influence of physics is ignored by the secondary curriculum when vectors are introduced in school. In our opinion, the vector concept is too complicated to be introduced in its abstract mathematical form in secondary education. Physics, on the contrary, offers intuitively suggestive situations not only for introducing but also for clarifying vector notions and operations in school.

Vector notions are considered as a rather marginal subject in secondary school mathematics in Greece. Young students are left for many years to form their own ideas about vector concepts on the basis of physics lessons and every day life experience, until the age of 17 or 18, in the last year of high–school. For the first time, at that age, they come in contact with vectors as a mathematical concept having both a geometrical and an algebraic character, and they are asked to use it as a tool for solving geometrical problems. On the other hand, there is a peculiar situation concerning the teaching of geometry in Greek schools. For 5 years (age 13-17) classical Euclidean geometry is systematically taught and only in the last year (age 18) are vector and analytic geometry presented.

Previous research by H. DEMETRIADOU & A. GAGATSIS ([10]) has verified that students are more successful in solving geometrical problems by euclidean methods than by vector ones, towards which they have a rather negative attitude. Similar errors are also made by younger Greek students ([6], [7], [8], [9]). Most of these errors are due to preconceptions and false ideas about vectors. Students' difficulties and strong preconceptions concerning vector quantities and operations, motion, and the distinction between a vector and a scalar have also been verified by other researchers, mainly in physics education (WARREN 1971; TROWBRIDGE & MCDERMOTT 1980, 81; WATTS & ZYLBERSZTAJN 1981; WATTS 1983; MCCLOSKEY 1983; AGUIRRE AND ERICKSON, 1984; AGUIRRE, 1988; AGUIRRE & RANKIN 1989; ECKSTEIN & SHEMESH 1989; GILBERT et al 1982; ARONS 1992; KNIGHT R.D. 1995).

Based on students' difficulties concerning vector concepts, and implicitly influenced by the historical development of the subject, we attempted a teaching experiment based on real and thought activities and situations related to displacements, velocities and forces, for both introducing, and clarifying vector notions.

2 The research

A pilot teaching procedure was performed the previous year. It concerned a class of 30 students in the second year of high-school (aged 14), before they had received any physics lessons in which vectors are introduced. The research indicated difficulties related either to the students' preconceptions, or to the nature of the concept of vector, some of which remained after teaching. These difficulties concern the confusion between notions like "vector and line segment", "direction and orientation as it is used in every day experience", the meaning of "+" and "=" signs, and the use of vector operations.

The main experiment took place during the school year 1998-99. It concerned two groups of

students in the third year of the same high- school (aged 15): the experimental group consisted of two classes (C_1, C_2) of 28 and 30 students respectively, the second being the pilot group. The researcher, who was also their mathematics teacher, taught in this group. Two other classes (C_3 , C_4) of 29 and 31 students were used as the control group. In these classes the traditional geometric and algebraic aspect of vectors, as presented in the school- book, was taught. In this connection it should be noticed that teaching this chapter is optional and in most schools it is usually omitted. The teaching in both groups started after the study of vector quantities in physics. Firstly, a questionnaire (Q_1) was given to both groups to revere their conceptions about vector notions and operations. Two other questionnaires were used, one after the end of the lessons (Q_2), and another (Q_3) after two months, to check conceptions that had been improved or difficulties and misconceptions that still remain strong.

Lessons in the experimental group concerned the introduction of vectors, their characteristic elements, symbols, and geometrical representation, equal and opposite vectors, addition, subtraction and multiplication by a number. The teaching was focused on students, in the sense that they had the opportunity to construct most of the notions, the symbolism and the operations. The lessons were taped and calendars were kept in every experimental class. The teaching was based on physical activities and situations where vector quantities were used. Displacements for the introduction of vector notions, velocities for operations between collinear vectors, and both forces and velocities analyzed as successive displacements, for the study of operations between non- collinear vectors.

3 The experiment: teaching procedures and difficulties encountered

3.1 Vector and line segment

From the very beginning we tried to clarify the distinction between vectors and line segments. The difference was focused on the element of motion, and was given by simple examples of displacements. Comparing different displacements, students realized the significance of origin and terminus points and consequently the significance of magnitude, and direction i.e. path and sense (see Appendix I). They also came in contact with the idea of opposite vectors. After these notions had been elaborated on an intuitive ground, we tried to give more formal descriptions or definitions of them. By successive questions posed by the teacher and discussions in the class, the students finally arrived at a definition. Thus a vector was defined as a new entity of both mathematical and physical character, related to a line segment, and to the concept of motion, and therefore its origin and terminus points are thus determined. A path was defined to be the line on which the vector is lying and every line parallel to it. The notion of sense was described through the movement along the line of path.

3.2 Vector and vector quantities

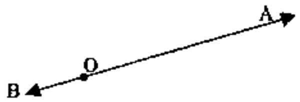
In our previous research we have identified difficulties concerning the relation between "vector" and "vector quantities". Students see only one aspect of this relation: vectors as a tool for the representation of vector quantities. We tried to present the other aspect as well, i.e. that a vector is an abstract notion whose concrete representations are vector quantities. Students were asked to formulate the concept of a vector precisely. The most convenient way was to use vector quantities as examples of this notion. They mentioned velocity, force, weight, acceleration as examples of vectors, since they are characterized by magnitude, path and sense.

3.3 Symbols and geometric representation

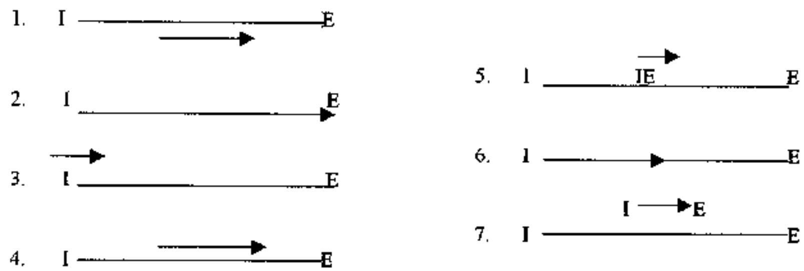
After the distinction between vectors and line segments was discussed, the need for an appropriate symbolism and geometrical representation of vectors arose naturally. Students were asked to make their suggestions, which were written on the blackboard: $IE, x. I\acute{E}, I \to E$. For every suggestion, comments were made and finally it was accepted or rejected by the majority of the class. The first two ones were very close to the notation used for line segments or straight lines and were rejected since they do not show the sense of motion. During the second lesson, the last symbol was also rejected, since it covered a lot of space.

After vectors representation had been discussed, a new symbol $\overleftarrow{BO}$ appeared in both experimental classes to show the displacement opposite to $\vec{C}$[1]

This symbol was discussed a lot, and appeared sporadically through lessons. Although it was finally rejected, two children, one at every class kept it until the end and in the final questionnaire it caused confusion to both of them.



Concerning the geometric representation of vectors, the following suggestions were made:



The students finally kept the 2nd and the 4th representation. When the teacher drew some vectors using model 4, with paths cutting each other, most of the students realized that it was complicated. It was finally left out after the second lesson.

3.4 Misconceptions about Sense and path

Several difficulties and misconceptions concern the notions of sense and path, mainly related to preconceptions about orientation used in every day experience. The following misconceptions seem to be closer to the notion of sense. However, in most cases they also have a negative influence in understanding the notion of path. The most characteristic error is that nonparallel vectors are considered to have the same path.

[1]It may be interesting to notice that this notation invented by the students is currently used in research mathematical domains such as non-commutative geometry, stochastic calculus, etc!

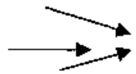
3.4.1 Sense related to circular motion

Vectors which are either successive, or with paths which could be similarly oriented as tangents to the same circle were considered as having the same sense.

Students were strongly influenced by the use of the term "sense" in physics lessons for the orientation of circular motions. A characteristic case is that of two students, one at every experiment class, who insisted that identically oriented arcs on a circle are vectors of the same sense. It is remarkable, that it took quite a long time before some other students realized that those were not vectors at all. There was a long discussion about the different points of view concerning vectors and circular motion.

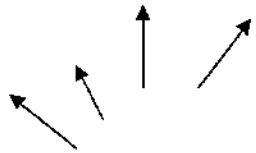
3.4.2 The concept of sense related to destination

Vectors whose terminus points are very close were considered as having the same sense.



3.4.3 The concept of sense related to points of compass

Some students consider vectors to be of the same sense when "they go up", "down", "left", "right", "southwest", etc. Or they consider two vectors of horizontal and perpendicular paths respectively, as opposite. E.g. These vectors are considered as having the same sense, since "they go up".



3.4.4 The concept of sense related to the same origin point

This could be regarded as closely related to the previous conception. These vectors are considered of the same sense, since "they go to the same place" or "they go down"

3.5 The equality relation - The "=" sign

As verified in our previous work, students consider vectors of the same magnitude to be "equal", and they use the notation: $\vec{a} = \vec{b}$ for such vectors. This misconception could partly be due to the manipulation of vectors as line segments, and partly to the presentation of equality relation in the school- books. Indeed, the use of the term "equality", as well as the use of the "=" sign for vectors, seem to be rather confusing in school mathematics. Mathematically speaking, equality of vectors is actually an equivalence relation with respect to magnitude, sense and path, which is completely different from the concept of "equality" of line segments; the latter is an equivalence relation with respect to the length of segments. However, the same word "equal" and the same symbol "=" are used to denote both equivalence relations. This is a subtle point for the clarification of which no effort has ever been made in Greek textbooks or curricula. This point was discussed in the classroom, and other examples of different types of equality in the sense of equivalence were also mentioned, like the above mentioned equality between line segments or fractions (see also MARJORAM 1966; DAVIS & SNIDER 1987). Simple situations with pairs of equal displacements, forces on the same rigid body, and velocities were used for the clarification of the subject. These vector quantities were considered equivalent because they lead to the same result. Concerning notation, students suggested the notation $|\vec{a}| = |\vec{b}|$ for vectors of equal magnitude, inspired by the symbol used for the absolute value of a number, in distinction to the notation $\vec{a} = \vec{b}$. However the misconception mentioned above was rather strong and persisted for many children, as the final questionnaire indicated.

During the lessons, the algebraic aspect of vectors was manipulated in parallel with the geometric one, especially in subjects like opposite or zero vector, operations and their properties (distributive and associative laws, etc).

3.6 Vector operations

3.6.1 Addition of collinear vectors

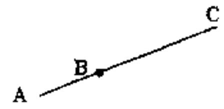
Addition between collinear vectors was presented, first through simple examples from physics, like that of two men pushing a car, and second through more complicated cases of relative motions, like that of a little ball rolling on a moving board, where velocities were analyzed as successive displacements in a unit time interval. In every case the students made the drawings of vectors to scale, and they tried to find out the relation of the magnitude and direction of components and those of the resultant. They also produced the corresponding models of vector operations and vector patterns.

A remarkable misconception about addition is related, on the one hand to the nature of this operation and on the other hand to the manipulation of vectors as line segments by the students: vector addition is considered as addition of numbers, probably because the sign of addition "+" is used with two different meanings (cf. the comments on "=" in 3.5 above). Consequently, the magnitude of the resultant vector is the sum of the magnitudes of the two vectors, even in the case of opposite or non-collinear ones. After measuring the lengths of the sides of some triangles, the students' wrong conception of vector addition led them to contradictions, since they realized the validity of the triangle inequality for the lengths of the sides of each triangle. A long discussion concerned the meaning of the symbols "+" and "=" in relations like: $\vec{F}_1 + \vec{F}_2 = \vec{F}_3$, which indicate that $\vec{F}_3$ is **equivalent**, i.e. gives **the same result** as the two others **together**. Both terms "sum" and "resultant" were used, indiscriminately.

3.6.2 Subtraction of collinear vectors- Multiplication by a number

The introduction of the subtraction of collinear vectors came naturally when adding displacements of opposite sense.

After the notation $\vec{AC}+\vec{CB} = \vec{AB}$. was used. a student suggested also: "$\vec{AC} - \vec{CB} = \vec{AB}$. Let's put "-" to show that we have a displacement and then we go back again".



Following this, other students tried to improve this notation:

2. $\vec{AC} + \overleftarrow{BC} = \vec{AB}$,
1. $\vec{AC} - \overleftarrow{BC} = \vec{AB}$,
3. $\vec{AC} - \vec{BC} = \vec{AB}$: " By "-" I indicate that $\vec{BC}$ is the opposite vector of $\vec{CB}$ "

These remarks gave the opportunity to see the subtraction of two vectors as "the addition of the opposite vector".

The introduction of multiplication by a number was presented on the basis of the addition of equal forces.

3.6.3 Addition between non-collinear vectors

Serious difficulties were found concerning the differentiation between "tied vectors" (i.e. line segments with a definite magnitude and direction attached to a point, the prototype being the physical concept of force) and "free vectors" (i.e. line segments with a definite magnitude and direction, the prototype being the geometrical concept of a parallel translation). These are related to *apparently* different types of addition: the *parallelogram* and the *triangle* laws respectively. From a more advanced mathematical - epistemological point of view, these laws correspond to the distinction between "tangent vectors to a manifold" and "tangent vectors translated in parallel on a manifold", the latter concept requiring more structure (i.e. the concept of a connection on a manifold) than just the ordinary differential structure of a manifold. The result of the pilot research that often students do not recognize the equivalence of parallelogram and triangle laws respectively may be due to this fact.

A convenient first step for establishing the equivalence of these laws seems to be connected with the commutativity property of addition. Activities were given for the verification of this property. On the one hand groups of two and three line segments as well as groups of vectors were used. In both cases, the students were asked to make them successive and find out all possible ways that this could be done. It is obvious that, for vectors where not only magnitudes and paths, but also senses were considered, the result was always the same. This verified the commutativity of vector addition.

The triangle law was verified on the basis of successive displacements on different paths, and velocities considered as successive displacements in the unit of time. Certain cases reminded some students of the parallelogram law, which they used in physics. This was the reason for studying and finally verifying the equivalence between these laws. The parallelogram law was also studied experimentally with the aid of an experimental arrangement comprising a system of two pulleys, and three weights balancing each other. By changing the weights, the students were given the opportunity to verify not only the parallelogram law, but also the inequality between the sum of magnitudes of the two components and the magnitude of the resultant. The final result was that although both methods of addition are different, they are equivalent, in the sense that they lead to the same physical result. It was also discussed that in some cases it is more convenient to use the one or the other law. More precisely, the triangle law is more convenient for successive vectors, while the parallelogram law is more convenient for vectors having the same origin.

4 Results

Concerning the analysis of the questionnaires, work is still in progress. However, the first results indicate some interesting points. The first questionnaire (Q_1) identified specific errors related to the understanding of notions like path and sense, difficulties in the distinction between line segments and vectors, and also errors concerning vector operations.

The questionnaires (Q_2) and (Q_3) indicated some improvements of the experimental group in comparison with the control group. On the other hand, it seems that some misconceptions are strong and persist after teaching. In the following we refer to some types of errors met in all classes. Classes C_1 and C_2 constitute the experimental group, while classes C_3 and C_4 constitute the control group. Concerning their performance in school, C_4 is considered to be the best class, with students having a very good level in mathematics and physics. Classes C_2 (the pilot class) and C_3 come 2nd and 3rd respectively, while C_1 has students of a rather low level in both mathematics and physics.

The numbers in the following tables correspond to percentages of errors made by the whole class.

The figures for each test exercise are given in Appendix II.

4.1 Errors related to the concept of path

Exercise 1(Q_3): *Which of the following vectors have the same path?*

Confusion between path and orientation:

C_1	C_2	C_3	C_4
11 %	10 %	43 %	24 %

4.2 Errors related to the concept of sense

Exercise 3d(Q_2): *Vectors $\vec{c}$ and $\vec{b}$ have the same sense.*

Correct Wrong Why?

Sense related to points of compass:

C_1	C_2	C_3	C_4
4 %	13 %	62 %	52 %

The following table gives the maximum percentages of errors concerning sense, met in Q_1 before teaching, compared to errors met in Q_2 after teaching. All classes were on a rather equivalent level concerning their preconceptions about sense. However, after the teaching, the experimental group showes a greater improvement.

Errors (%) related to the concept of sense

	C_1	C_2	C_3	C_4
Q_1	40	57	43	56
Q_2	11	27	62	52

4.3 Vectors regarded as line segments

Exercise 4 (Q_2): *a)* $\vec{n} = 3\vec{k}$ *b)* $\vec{m} = \vec{k}$

Correct *Wrong* *Why?*

Answers (%): "Correct" in 4 (Q_2)

	C_1	C_2	C_3	C_4
4a	14	17	38	23
4b	11	10	24	16

Exercise 5 (Q_3): *The triangles ABC are all isosceles (AB = AC). Examine which of the following cases expresses the relation between $\vec{b}$ and $\vec{c}$ for every one of these figures:*
a) $\vec{b} = \vec{c}$ *b)* $\vec{b} = -\vec{c}$ *c) Another answer. Which?*

Answer $\vec{b} = \vec{c}$

C_1	C_2	C_3	C_4
11 %	7%	36%	20%

4.4 Subtraction of collinear vectors

Exercise 6 (Q_2): *Complete the second part of every equality by the correct vector:*

a) $\vec{EA} - \vec{BA} = \cdots$ *b)* $\vec{CD} - \vec{ED} = \cdots$

As it is indicated in the answers of exercise 6 the experimental group made greater progress. We should also mention the high percentage of no answers in the control group.

Answers (%) in 6a (Q_2)

	C_1	C_2	C_3	C_4
Correct	54	67	34	29
No answer	0	10	17	19

Answers (%) in 6b (Q_2)

	C_1	C_2	C_3	C_4
Correct	57	63	31	23
No answer	7	10	21	23

4.5 Parallelogram law-triangle law

The experimental group had a better performance in manipulating the triangle law.

Exercise 6(Q_3): *Replace the question mark by the correct vector by using vectors $\vec{a}$, $\vec{b}$ or $\vec{c}$:* $\vec{a}+\vec{b}=?$

The experimental group made greater progress in this question, where the use of the triangle law is more convenient than the parallelogram law.

Answers (%) in 6 (Q_3)

	C_1	C_2	C_3	C_4
Correct	43	43	7	19
No answer	4	0	10	8

Exercise 5 (Q_2): *Replace the question mark by the correct vector:* $\vec{AB}+?=\vec{AC}$

Answers (%) in 5 (Q_2)

	C_1	C_2	C_3	C_4
Correct	86	100	69	65
No answer	0	0	10	6

4.6 Relative motion

It seems that children's strong preconceptions about motion and vector quantities are in most cases obstacles for understanding relative motion. Our groups had not been taught about such motions in physics lessons. During the experimental teaching several preconceptions appeared, especially among good students. Many of these conceptions were so strong that they led to long debates among students. Our research suggests that although the concept of relative motion is rather difficult for young children, teaching can modify some of their preconceptions.

4.6.1 Relative motion - Collinear vectors

Exercise 7 (Q_2): *The railroad car is moving rectilinearly, at a constant speed of 30 m/s from west to east. A passenger is moving along the railroad car, at a constant speed of 2 m/s. Using vectors indicate the passenger's displacement after 3 sec in the following cases:*
a) The passenger is moving from A to B. *b) The passenger is moving from B to A.*[2]

[2]The figures used in this exercise have been taken from [?].

Answers (%) in 7a (Q_2)

	C_1	C_2	C_3	C_4
Correct	68	63	0	42
No answer	11	7	38	23

Answers (%) in 7b (Q_2)

	C_1	C_2	C_3	C_4
Correct	57	47	0	39
No answer	11	20	41	29

4.6.2 Relative motion - Non collinear vectors

Exercise 9 (Q_3): A marble ball is moving across the floor of a railroad car, at a speed of 6m/s relative to the floor. When the ball starts moving, the railroad car starts also moving rectilinearly, at a constant speed of 8 m/ s. With what speed does an observer outside the railroad car see the marble ball moving?

Answers (%) in 9 (Q_3)

	C_1	C_2	C_3	C_4
Correct	36	40	14	19
No answer	11	17	17	35

Exercise 10 (Q_3): An airplane is moving horizontally with velocity $\vec{v}$ and packets of cattle food are thrown over a mountain village. There is no air resistance. Draw where a villager sees the packet moving towards, when it leaves the airplane from point A.

Answers (%) in 10 (Q_3)

	C_1	C_2	C_3	C_4
Correct	25	43	7	6
No answer	4	7	7	0

5 Discussion

Vectors are introduced, as a tool for solving geometric problems, only in the last year of Greek high school mathematics. Our previous research has verified specific and persisting difficulties among Greek students (aged 15-18), about some epistemological aspects of the concept of a vector, which have a negative influence on geometry problem solving procedures.

In our opinion, these difficulties are mainly related to the fact that vector notions are presented in their abstract form right from the beginning, without introducing them first in a more intuitive and physical way. Physics is a suitable field for introducing vector notions and operations in a more intuitive way. However, the role of physics in the development and establishment of vector notions suggested by history is ignored in the curriculum of secondary school.

The above difficulties, and indirectly the role of physics in the historical development of vector algebra, led us to an alternative teaching approach, which can help students construct basic vector notions and operations on the basis of physical and geometrical situations and activities.

Our experimental teaching gave us the opportunity on the one hand to face and discuss difficulties in depth, and on the other hand to attempt to eliminate them. Although our results presented here are mainly based on a qualitative analysis of teaching, they indicate that our approach is successful. The first results suggest that our teaching helped students in the experimental group to overcome some of their wrong conceptions. However, a more detailed description of the consequences of our teaching approach, based on a quantitative analysis of the questionnaires distributed to the students, is still in progress.

Nevertheless, we believe that the teaching approach of using intuitively physical situations, proposed here, would be helpful for a deep understanding of such notions and their construction. This understanding is necessary before vector concepts and operations are used as a tool in vector geometry.

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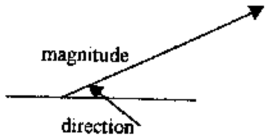
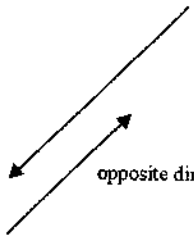
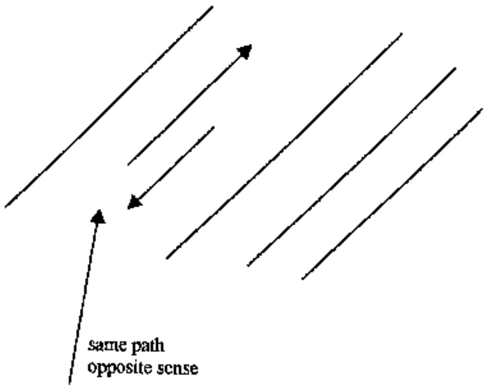
Appendix I

In English one describes a vector by two terms: Its **magnitude** and its **direction**.

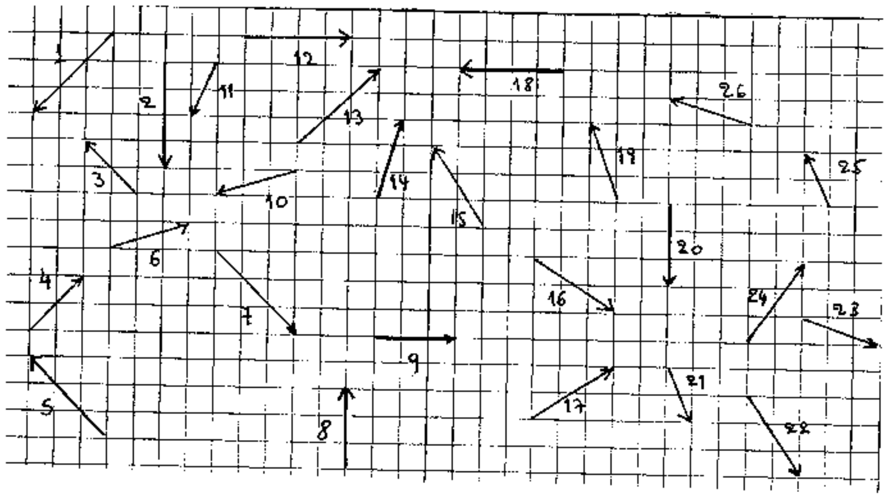




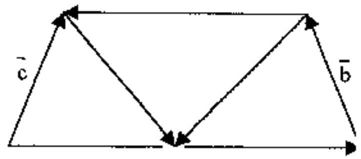
In Greek (and in French) one describes a vector by three terms: its **magnitude** (longueur), its **path** (direction) and its **sense** (sens).



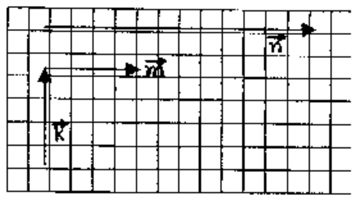
Appendix II



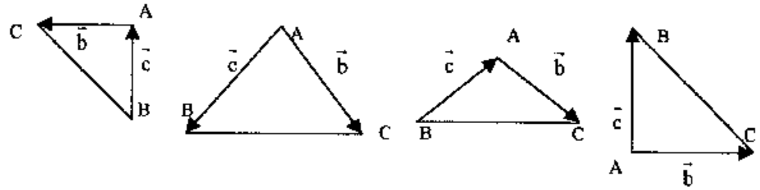
Exercise 1(Q3)



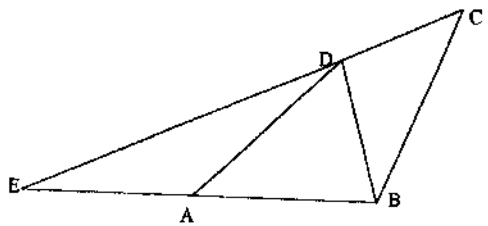
Exercise 3d(Q2)



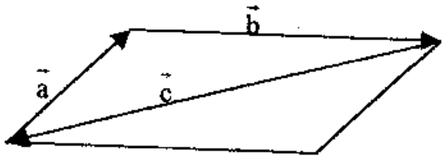
Exercise 4(Q2)



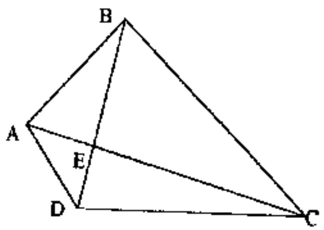
Exercise 5(Q3)



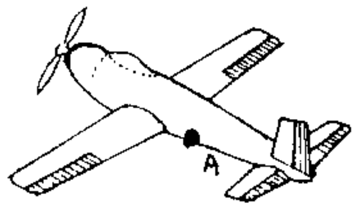
Exercise 6(Q2)



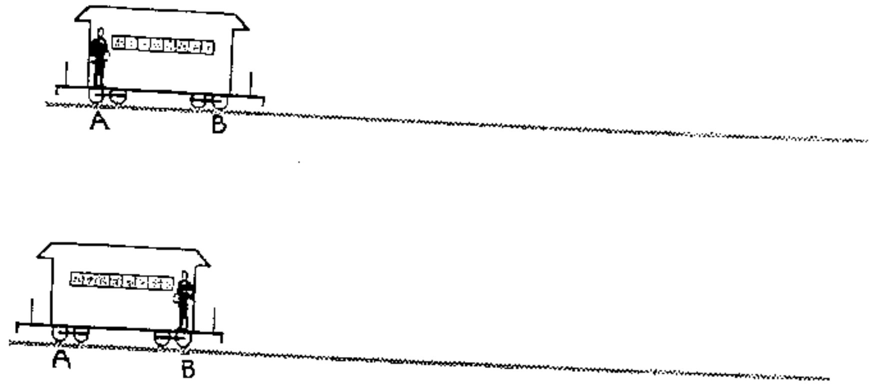
Exercise 6(Q3)



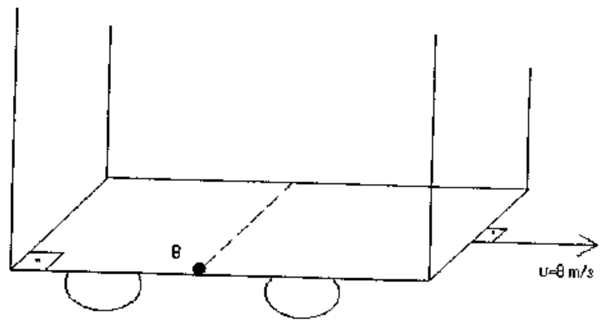
Exercise 5(Q2)



Exercise 10(Q3)



Exercise 7(Q2)



Exercise 9(Q3)

Pascal précurseur de Newton ? La chute des idoles

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Abstract

Dans les périodiques édités par les Académies et les diverses sociétés scientifiques, on découvre les Mathématiques telles qu'elles étaient faites (en public) par les plus grands savants de leur temps, dont on aperçoit parfois l'esprit étriqué ou au contraire très large. Les noms de baptême des théorèmes qui ont bercé nos études prennent corps (Rolle le ringard, Varignon le héros, Laplace le dieu vivant, etc.); des inconnus revivent, et l'on se demande pourquoi ils sont oubliés (Frenicle, connu des seuls amateurs de carrés magiques, Montmort célébré par toute l'Europe,. . .) Enfin, quelques phares de la Science descendent de leur piédestal ; c'est le cas de Chasles, l'un des plus illustres mathématiciens, avec Pythagore et Thalès, puisque son nom est dans toutes les mémoires (c'était l'homme d'une seule relation, mais tout le monde la connaît.[1])

Son aventure et sa déchéance à l'Académie des Sciences sont grandioses et incroyables : En 1867, il présenta une lettre de Blaise Pascal, tendant à prouver que ce dernier avait trouvé l'expression de la loi de la gravitation universelle bien avant Newton ! Un simple entrefilet des *Comptes Rendus* allait déclencher une polémique sans précédent, mettant en scène des scientifiques de toute l'Europe, tenant en haleine les pays concernés. Trois ans durant (de 1867 à 1869), les *Comptes Rendus* se firent l'écho, au long de centaines de pages des rebondissements de l'affaire. A la lecture des *Comptes Rendus* relatant cette affaire, on mesure l'importance de l'archivage officiel, même dans les histoires les plus douteuses, car, pour certains, l'Académie des Sciences de Paris s'était définitivement ridiculisée, alors que pour ses défenseurs au contraire, elle s'était grandie en ne cachant rien de la polémique qui grandissait en son sein.

Mais laissons la place aux textes.

[1] En tout cas en France, cocorico !

COMPTES RENDUS

HEBDOMADAIRES

DES SÉANCES

DE L'ACADÉMIE DES SCIENCES

PUBLIÉS,

CONFORMÉMENT A UNE DECISION DE L'ACADÉMIE

En date du 13 Juillet 1835,

PAR MM. LES SECRÉTAIRES PERPÉTUELS

TOME SOIXANTE-CINQUIÈME.

JUILLET - DÉCEMBRE 1867

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DES COMPTES RENDUS DES SÉANCES DE L'ACADÉMIE DES SCIENCES,

SUCCESSEUR DE MALLET-BACHELIER,

Quai des Augustins, 55

1867

Première époque : Révélation - Riposte anglaise.

Le 15 juillet 1867, la séance est présidée par Chevreul (l'Académie se réunissait tous les lundis en début d'après-midi ; le fascicule de Compte Rendu était publié le lundi matin suivant) qui invite Chasles à parler de manuscrits dont il avait fait mention quelques séances auparavant. Ce dernier annonce :

> ≪*J'ai l'honneur de mettre sous les yeux de l'Académie quelques écrits de Pascal, qui montrent qu'il s'est beaucoup occupé de la recherche des lois de l'attraction, et qu'il les a connues.*≫

Et le *Compte Rendu* en fournit même des extraits, ce qui sera fait dans toute cette histoire à chaque fois que Chasles présentera des manuscrits à l'appui de ses affirmations; voici le texte d'une lettre de Blaise Pascal au physicien Robert Boyle:

Ce 2 septembre.

MONSIEUR

Dans les mouvements célestes, la force agissant en raison directe des masses et en raison inverse du quarré de la distance suffit à tout et fournit des raisons pour expliquer toutes ces grandes révolutions qui animent l'univers. Rien n'est si beau selon moy ;

[...]

PASCAL

A M. Boyle

Stupeur dans l'assemblée ! Chasles est très estimé, on le sait collectionneur de manuscrits autographes, mais chacun semble se demander ce que recèle sa collection. Il donne un autre extrait d'une courte note attribuée à Pascal et retrouvée dans ses papiers:

> C'est par ces principes qu'on trouve que les quantités de matière du soleil, de Jupiter, de Saturne et de la terre sont entre elles comme les nombres 1, $\frac{1}{1067}$, $\frac{1}{3021}$, $\frac{1}{169282}$.

Il faut attendre la séance suivante, le 22 juillet 1867, pour que certains Académiciens donnent leur sentiment, à commencer par Duhamel :

Or peut-on supposer que Pascal eût une mesure du diamètre de la Terre plus exacte que celle que l'on connaissait en France et en Angleterre, et dont les historiens de la science n'ont pas parlé ? La comparaison des forces aux distances aurait donc plutôt éloigné Pascal de la loi qu'il énonce. Ce n'est donc pas cette comparaison qui lui en a donné l'idée. D'où lui est-elle donc venue ? [...] Comment Pascal aurait reconnu qu'une pareille force ferait décrire des ellipses ayant le Soleil pour foyer. Newton n'a pu le faire qu'après avoir établi sa belle formule entre la force centrale et certains éléments infiniment petits de la trajectoire.[...] La lettre du 2 septembre, attribuée à Pascal, semble donc inexpliquable.

Chasles répond à peine, mais donne quelques nouveaux extraits de lettres de Pascal à Boyle, étayant son propos. Il se justifie le 29 juillet 1867, peut-être face à l'étonnement (et une sourde réprobation) de ses pairs :

Je n'ai rien dit de plus, et je n'ai pas prononcé le nom de Newton, n'ayant pas pour but d'établir un parallèle entre ces deux grands génies, dignes tous deux de l'admiration et du respect des géomètres de tous les temps et de tous les pays; car la science a pour patrie le monde entier. [...] Dans une lettre inédite, Leibnitz dit que Newton possède des écrits de Pascal, et que lui-même en possède aussi. [...] Aussi je n'hésite pas à déclarer formellement qu'il ne peut y avoir aucun doute; c'est-à-dire que toutes ces pièces sont bien de la main de Pascal; que cela m'est prouvé non-seulement par le nombre de ces pièces et les sujets qu'elles traitent, mais surtout par une correspondance de dix années entre Pascal et Newton [...]

On remarquera la rhétorique classique ("j'ai pas dit ça, mais c'est vrai") et surtout le fait que Chasles cite pour sa défense une *lettre inédite*. Ce sera quasiment toujours pareil : les documents décisifs de Chasles sont toujours inédits et connus de lui seul ! C'est Prosper Faugère, grand érudit et premier éditeur des œuvres complètes de Pascal qui porte un coup que l'on aurait pu penser décisif :

[...] que la signature mise au bas de ces documents n'est pas celle de Pascal, et qu'ils sont d'une autre écriture que la sienne. [...]

Il était le mieux placé pour l'expertise, ayant détenu des manuscrits "officiels" de Blaise Pascal, prêtés par la Bibliothèque Impériale. Ce même jour, Bénard remet les choses en place :

La question est trop importante pour que l'amour-propre national des Anglais cède devant une confrontation de style, d'orthographe, d'écriture et même de papier. D'ailleurs, les documents produits par M. Chasles sont certainement fabriqués à plaisir; et même par un falsificateur assez malhabile. [...] Tout cela ne semble-t-il pas copié dans un Traité moderne de cosmographie ? On se sera contenté d'altérer grossièrement le dernier nombre. Mais comment Pascal aurait-il pu calculer le 2 janvier 1655, au plus tard, la masse de Saturne à l'aide des révolutions d'un satellite qui ne fut découvert que le 25 mars de la même année et dont les premières tables, publiées en 1659 par Huyghens, étaient encore très-imparfaites ? [...] Mais malheureusement la fraude que je prends la liberté de vous signaler doit cacher une vile perfidie. L'origine anglaise des lettres attribuées à Pascal me paraît manifeste. [...] L'auteur doit être aux aguets pour recueillir le bruit qu'elles feront en France [...]

C'est la thèse du complot (de ces perfides Anglais, bien sûr). L'argument des dates paraît lui aussi décisif et cette histoire devrait s'arrêter là, mais c'est compter sans la ténacité de ce vieux passionné de Chasles

Les Anglais attaquent, en la personne de Sir David Brewster, honorable et réputé physicien britannique, le biographe de Newton, qui écrit le 12 août 1867 :

Ayant soigneusement examiné tous les papiers et la correspondance de Sir Isaac Newton [...] je n'hésite pas à dire qu'aucune lettre de Pascal à Newton, ni aucune autre pièce contenant le nom de Pascal n'existent dans cette collection. [...] Je crois que jamais lettres n'ont été échangées entre Pascal et Newton, [...] Les lettres de Pascal à Newton datées du 20 mai 1654, et les nombreuses lettres qu'on donne comme échangées entre eux dans la même année, quand Newton avait moins de onze ans et demi, sont également forgées, car Newton n'avait nulle connaissance des sujets qui y sont traitées, s'occupant alors, d'une manière bien plus convenable à son âge, de cerfs-volants, de petits moulins et de cadrans solaires [...]

Que fait Chasles ? Il produit de nouveaux documents, qui, comme par enchantement, répondent parfaitement aux arguments de Brewster ; par exemple les extraits suivants :

Aubrey à Pascal.

Le 12 may 1654. — Je me suis rendu suivant votre désir auprès du jeune Isaac Newton, et me suis entretenu longuement avec luy. Il est fort jeune encore, car à peine a-t-il onze ans, et pourtant il raisonne fort sciemment sur les mathématiques et la géométrie [...]

18 juin. — Je vous prieray m'envoyer, s'il vous plaît, toutes les lettres de moy adressées à M. Pascal. Ce sont des souvenirs de mon enfance que je désirerois garder devers moy. [. . .]

Le problème de cette surenchère, c'est qu'elle commence à scandaliser les membres de l'Académie, car les écrits de Newton le montrent "sous un jour particulièrement odieux", selon le mot de Duhamel. Mais, est-ce la notoriété de Chasles ou sa force de persuasion? Ses contradicteurs se placeront toujours sur un terrain scientifique et historique.

Il faut imaginer ce grand personnage[1] de la fin du XIX$^{\text{ème}}$ siècle discourir, en ces termes, à la tribune ce 19 août 1867.

Un faussaire qui aurait fabriqué toutes ces Lettres, toutes ces pièces, pour prouver qu'il a existé des relations entre Pascal et Newton, aurait eu bien du talent, puisqu'il aurait fait tout à la fois du Pascal, du Newton, du Labruyère, du Montesquieu, du Leibnitz, du Malebranche, du Saint-Evremond, etc. Aussi, quelques affirmatives que soient les protestations de M. Faugère en faveur de Pascal, et de Sir David Brewster en faveur de Newton, je réitère à l'Académie l'assurance qu'elles ne font naître dans mon esprit aucun doute, et qu'elles ne me causent aucune inquiétude.

Autrement dit, il met tout son poids dans la balance. En outre, il accumule les témoins les plus prestigieux, ce qui lui vaudra les pires ennuis.

Par exemple, le 30 septembre 1867 est lue en séance une lettre adressée à l'astronome Le Verrier (membre de l'Académie, qui jouera un rôle déterminant dans la conclusion de cette histoire) par le Directeur de l'Observatoire de Glasgow, un certain Grant :

[1]Pour la petite histoire, Michel Chasles avait reçu à son baptême, dans la période post-révolutionnaire, le prénom de *Floréal*, mais un jugement du tribunal lui avait permis d'en changer alors qu'il avait une vingtaine d'années. Il ne devait pas faire bon plaisanter sur le sujet.

C'est en 1653 que fut publié l'ouvrage remarquable de Riccioli, l'Almagestum Novum. En 1659, Huyghens publia son Systema Saturnium. Ces deux ouvrages peuvent être considérés comme fournissant les meilleurs matériaux accessibles à Pascal pour former la base de ses recherches en astronomie physique. [. . .] Après avoir établi ainsi une comparaison entre les meilleurs éléments de calcul qu'on peut supposer avoir servi dans le temps de Pascal, et les éléments employés par Newton en 1687 et 1726, je vais maintenant comparer les résultats communiqués par M. Chasles à l'Académie des Sciences, avec les résultats correspondants des recherches de Newton contenus dans les éditions des Principia de 1687 et 1726.

Comparons d'abord les masses du Soleil, de Jupiter, de Saturne et de la Terre. Nous trouvons ainsi:

	Soleil.	Jupiter.	Saturne.	La Terre.
Pascal (1662).....	1	$\frac{1}{1067}$	$\frac{1}{3021}$	$\frac{1}{169282}$
Newton (1687).....	1	$\frac{1}{1100}$	$\frac{1}{2360}$	$\frac{1}{28700}$
Newton (1726).....	1	$\frac{1}{1067}$	$\frac{1}{3021}$	$\frac{1}{169382}$

L'inspection de ces nombres montrera au premier coup d'œil que l'une des deux conclusions suivantes est inévitable : ou quelque observateur inconnu a fourni à Pascal des éléments de calcul absolument identiques à ceux que Newton a obtenus en 1726 de Cassini, de Pound et Bradley, et alors Pascal a dû faire usage de la même valeur de la parallaxe solaire employée par Newton en 1726, c'est-à-dire 101/2 secondes, ou bien les chiffres communiqués à l'Académie des Sciences par M. Chasles doivent êtres de purs mensonges. La première de ces conclusions ne peut être acceptée.

Cela paraît maintenant clair : il y a eu manipulation ! Mais, vous l'avez compris, notre grand homme est au-dessus de cela, et Chasles répondra, superbe, dans la même séance :

C'est donc évidemment Newton qui, après s'être écarté, en 1687, des nombres de Pascal, qu'il connaissait, y est revenu en 1727.

Pour étayer ses affirmations, Chasles révèle une correspondance de Pascal avec un illustre savant qui fait son entrée dans cette histoire et marque l'épisode suivant.

Seconde époque : L'Italie impliquée.

Le 7 octobre 1867, toujours en réponse à la lettre de Grant, Chasles voit rouge :

Sans doute les observations faites depuis n'existaient pas; mais qu'est-ce qui prouve que Pascal n'en possédait pas qui lui permissent de faire ses calculs ?

Eh bien, heureusement je puis produire un autre ordre de documents se rapportant à ce calcul de Pascal. Et l'admiration pour Pascal s'en accroîtra encore, car *c'est à l'âge de dix-huit ans qu'il a trouvé ces nombres. C'est en 1641, en basant ses calculs sur des écrits inédits de Kepler, et des observations astronomiques que lui transmettait Galilée. C'est le témoignage de Galilée lui-même, ce sont ses propres Lettres que je vais produire. Des Lettres de Pascal et d'autres documents successifs y feront suite jusqu'à Newton lui-même, qui viendra apporter son propre témoignage.*

Et voici les lettres et la conclusion :

Florence, ce 7 juin 1641.

[...] J'ay examiné avec beaucoup de soin vos calculs des forces qui peuvent agir sur ces corps à distances égales du Soleil, de Jupiter, de Saturne et de la Terre; et ces forces donnent parfaitement la proportion de matière contenue dans ces différents corps conformément à la loi générale de la variation de la gravité, comme j'en avois eu l'idée. C'est donc par ces principes qu'on trouve que les quantités de matière du Soleil, de Jupiter, de Saturne et de la Terre sont entre elles comme les nombres 1, $\frac{1}{1067}$, $\frac{1}{3021}$, $\frac{1}{169282}$, ainsi que vous le démontrez fort bien en vostre traité. [...] Continuez-nous, je vous prie, vos nouvelles expériences. Je suis toujours très-souffrant; je n'y vois presque plus. Je suis comme toujours, Monsieur, vostre très-affectionné.

GALILÉE GALILEI

A Monsieur Pascal.

Chasles conclut en présentant des *Lettres de Pascal, d'Huyghens, de Mariotte, de Newton, du cardinal de Polignac et de Malebranche, qui s'accordent toutes à confirmer les Lettres de Galilée. Elles prouvent toutes que Pascal avait composé, en se servant des écrits de Kepler et des observations de Galilée, un petit Traité renfermant les valeurs numériques des masses et des densités des planètes, qui ont été reproduites par Newton dans l'édition de 1727 de son Livre des Principes. Telle est ma réponse aux objections, prétendues décisives, de l'éminent astronome de Glascow.*

Il y a pourtant un problème, entre autres, qui échappe pour l'instant à Chasles : il est reconnu que Galilée était aveugle à la fin de sa vie, et qu'il n'écrivait plus rien. D'après le schéma bien connu maintenant, nous allons assister à une nouvelle controverse, qui n'abattra pas pour autant le vieil homme qui balaie cet argument d'un revers méprisant à la séance du 18 novembre 1867 :

Je suis en mesure de prouver que Galilée n'a point été atteint d'une cécité complète dès la fin de 1637, mais seulement dans le dernier mois de 1641.

D'ailleurs Galilée le confirme et prévoit même avec beaucoup de pertinence :

Galilée à Pascal.

Ce 2 septembre 1641.

[...] Car quoyque je ne voye presque plus rien, je n'en parviens pas moins à déchiffrer vos escrits moy mesme, tant a sur moy de force l'amour de la science et le désir de son progrès.

Je suis votre bien affectionné serviteur

GALILÉE GALILEI

Quant à la question des observations, tout s'arrange, grâce à Boulliau

Boulliau à Huyghens.

Un de mes amis, Monsieur Pascal, qui avoit quelques relations avec Galilée, a reçu de ce dernier un instrument qui grossit prodigieusement les objets, et au moyen duquel on apperçoit près de Saturne quelque chose qui me semble extraordinaire. [...]

Huyghens à Boulliau.

Ce 2 décembre.

Dernièrement par un temps clair et magnifique, je me suis remis à observer Saturne, et non seulement j'ay revu l'anneau dont je vous avois déjà entretenu, mais j'ay découvert parfaitement le satellite que Galilée disoit avoir apperçu. Il n'y a plus de doute [...] Vostre très humble et très affectionné serviteur

CH. HUYGHENS

Huygens[2] est mouillé dans cette sale affaire ! On dirait vraiment que toute l'Europe s'est impliquée dans cette conspiration du silence, visant à déposséder notre brillant Pascal de sa découverte au profit de Newton. Il était improbable que la Hollande ne réagisse pas.

Troisième époque : La Hollande au secours de l'Italie.

Le 9 décembre 1867, Harting, astronome de l'observatoire d'Utrecht, repond depuis le Nord de l'Europe :

C'était le 25 mars, c'est à dire environ sept semaines après l'achèvement de son premier objectif, que Huyghens aperçut pour la première fois le satellite; mais les observations des jours suivants étaient nécessaires pour en établir la véritable nature. D'abord il lui attribua une révolution de seize jours et quatre heures. Ce ne fut que quelques années plus tard qu'il lui assigna un temps de révolution à peu près égal à celui qui est mentionné dans la Lettre que M. Chasles vient de faire connaître, et qui certainement est d'un faussaire, et même d'un faussaire peu habile, puisqu'il puise ses coordonnées dans les secondes éditions.

Cette attaque vient d'ailleurs après le coup de poignard de G. Govi, érudit italien, spécialiste

[2] Il ne manque que le son pour que vous puissiez entendre notre belle prononciation (uiguinsse), qui fit se pâmer d'effroi les auditeurs nordiques de cet exposé. Nous pensions naïvement nous rapprocher de la prononciation de l'époque... Finalement, la plus élémentaire courtoisie vis-à-vis de nos hôtes nous conduit à faire unpas dans leur sens et prononcer "uig-enne-se"

de Galilée comme il se doit, qui écrit une semaine auparavant, le 2 décembre 1867

> *Et d'abord. Galilée n'a jamais écrit en français. [...] Mais il y a plus : les cinq lettres sont datées de Florence; or, depuis le mois de décembre 1633, Galilée vivait près d'Arcetri, dans une villa de la famille Martellini, appelée le Giojello, où l'illustre vieillard expira le 8 janvier 1642. [...]*
> *Quant à la cécité du pauvre grand homme, elle n'était, hélas ! que trop vraie, et si M. Chasles veut bien se donner la peine de consulter la correspondance de Galilée, il verra que dès l'année 1632, ses yeux avaient été frappés d'une altération assez grave pour lui ôter le pouvoir de lire et écrire sans souffrance*

Intermède : L'année 1868

Il ne se passe rien, ou presque[3].

Le volet italien de l'affaire trouvera sa conclusion en même temps que l'affaire elle-même, lorsque l'on montrera que les lettres que possède Chasles sont indubitablement inspirées d'une édition des œuvres complètes de Galilée, établie au XIX$^{\text{ème}}$ siècle (les manuscrits ayant été transcrits), et ne peuvent donc être de la main de Galilée ou d'un de ses secrétaires.

Quatrième époque : Le réquisitoire de Le Verrier - La Vérité

C'est un certain Breton (de Champ), ingénieur ami de Le Verrier, qui dévoile une partie du pot-aux-roses, le 12 avril 1869 :

> *[...] Cet ouvrage est l'Histoire des Philosophes modernes, par Savérien, qui a paru, dans les années 1761 et suivantes, en sept volumes in-12. Dans le quatrième, qui porte la date de 1764, se trouve l'article consacré à Newton. A la suite de la partie historique de cet article vient une exposition du système du monde fondée sur la théorie de l'attraction universelle. Or cette exposition renferme non-seulement la substance, mais aussi le texte complet de la plupart des Notes et Observations relatives à ce système, qui ont été présentées à l'Académie comme étant de Pascal.*

La dernière pirouette de Chasles est semblable aux autres : c'est bien sûr Savérien qui a copié sur Pascal et cie et, des lettres (inédites...) présentées à l'Académie à la séance suivante, le 19 avril 1869, le prouvent :

[3]Des querelles "de spécialistes" sur la cécité de Galilée, entre les Italiens qui font référence à tout ce que l'on connaît du grand homme, et un Chasles dont la mauvaise foi n'a d'égale que la conviction.

> *M. Breton se prononce ainsi avec une grande assurance et une foi naïve, bien imprudente. [...] Savérien a eu connaissance et a fait usage de plusieurs des Documents qui me sont parvenus. Ces pièces se trouvaient alors dans la riche collection d'objets précieux en tous genres que possédait Mme de Pompadour. [...]*

Ce 14 mars.

MADAME LA MARQUISE,

Je vous retourne 200 lettres de Copernic, de Galilée, Descartes, Gassendi, Pascal, Malebranche, Leibnitz, Newton et autres savans du siècle passé, que vous avez bien voulu me confier. J'ai compulsé avec soin ces précieux documens, et j'en ai fait des extraits qui me seront très utiles, non-seulement pour mon histoire des progrès de l'esprit humain dans les sciences naturelles, intellectuelles et exactes, mais aussi pour une histoire des philosophes anciens et modernes, que j'ai dessin de faire.

[...]

Votre très humble, très dévoué et très obéissant serviteur,

A Madame la marquise de Pompadour SAVERIEN

Il est amusant de lire sous la plume de Chasles l'expression "foi naïve", à l'heure même où l'hypothèse d'une supercherie dont il aurait été la victime semble unanimement admise.

Il fallait alors un personnage aussi prestigieux que Chasles et membre de la même Académie pour mettre un terme à cette histoire. Ce fut Le Verrier, dont l'époustouflant réquisitoire (*Examen de la discussion soulevée au sein de l'Académie des Sciences au sujet de la découverte de l'attraction universelle*, commencé le 21 juin 1869 mais étalé sur plusieurs séances) rappelle tout ce qui a été dit et sape la dernière once de crédibilité de Chasles en mettant en évidence son système de défense. Le Verrier reprend le texte de Savérien cité par Breton (de Champ) et le met en parallèle effectivement avec le texte des manuscrits de Pascal, le 5 juillet 1869

La discussion s'envenime le 26 juillet 1869 :

> [Chasles] *Tout ce que M. Le Verrier écrit dans le Compte rendu du 5 juillet et ce qu'il vient de reproduire sur le fonctionnement de la Commission est donc dû à son imagination et est contraire à la réalité des faits. Quant à ses insinuations injurieuses sur ce que je refuse de lui dire de qui je tiens ces Documents, je ne serai point embarrassé d'y répondre quand je le jugerai à propos.*
> [Le Verrier] *M. Chasles s'écrie qu'on l'attaque, et que la dernière parole doit être réservée au droit sacré de la défense. J'accepte le principe. Mais qui donc est-ce qui attaque ici, si ce n'est M. Chasles ? Qui a donc osé dire en s'appuyant sur des papiers suspects, émanant d'une source cachée et inavouable, que Newton n'était qu'un vulgaire plagiaire qui avait soustrait à Pascal ses titres de gloire ? Et quel est celui qui se défend si ce n'est Newton, à qui nul ne peut refuser le droit de pousser à fond le débat et d'exiger qu'on y mette la même rigueur que devant un tribunal ?*

Le vieil homme est défait le 13 septembre 1869 ; après deux années de combat, il avoue avoir été l'objet d'une mystification, dans un mémoire lu à l'Académie, *Question des manuscrits de Pascal, Galilée, etc.*

> *Lorsque dans les premiers jours de juillet 1867, j'ai eu l'honneur de communiquer à l'Académie certains Documents qui prouvaient que Pascal aurait eu connaissance des lois de l'attraction et aurait même des relations avec le jeune Newton, je n'agissais pas avec précipitation; car c'était depuis 1861, en novembre, qu'un individu, se disant archiviste paléographe et faisant commerce de titres généalogiques, me procurait ces Documents étrangers à la spécialité de son commerce, de la part du possesseur qui me les faisait proposer.*
>
> *Je connaissais donc l'importance scientifique de ces Documents, je savais en outre que je ne possédais pas toute la collection; j'insistais pour qu'on me la livrât tout entière; mais on me répondait que le possesseur, qui l'avait rapportée d'Amérique, où elle avait passée en 1791, se plaisait à parcourir toutes ces Pièces, et ne voulait les livrer qu'à son loisir.*
>
> *[...] Mais bientôt, après les observations qui ont été faites à Florence sur la Lettre de Galilée du 5 novembre 1639, dont j'avais envoyé une photographie, ont éveillé mon attention, et ont commencé à m'inspirer des inquiétudes qui m'ont porté à certaines recherches et à des mesures d'information, et même à solliciter de M. le Préfet de police une surveillance, à l'effet de connaître enfin le véritable dépôt des pièces qui m'étaient vendues.*
>
> *[...] Mais on n'a trouvé chez [le vendeur] que quelques papiers blancs, provenant de registres, des plumes, un seul flacon d'encre et quelques facsimile de l'Isographie, quand j'avais espéré qu'on y trouverait la masse des Documents dont il ne m'avait livré que des copies et dont une partie considérable m'était encore due. Il a refusé d'abord de faire connaître de qui il tenait ces Documents. Et il a déclaré depuis que c'était lui qui les fabriquait.[...]*
>
> *Il y a donc là un mystère à pénétrer, et jusque-là il n'y a rien à conclure avec certitude.*

Quelle étrange fin, comme si Chasles voulait croire encore à l'incroyable...

Épilogue : le procès du faussaire

Les deux experts nommés par le tribunal pour le procès du faussaire Vrain-Lucas ont publié leurs résultats dans un livre en 1870[4]. Ce qui y est écrit est pire que ce que l'on pouvait imaginer. Vrain-Lucas avait vendu à Chasles 27.472 objets (fausses lettres, lettres anodines falsifiées,

[4]Henri Bordier & Emile Mabille, *Une fabrique de faux autographes ou récit de l'affaire Vrain-Lucas*, Paris, Léon Techener, Libraire, 1870. Cette affaire avait tant suscité d'intérêt dans le grand public, de par la personnalité des deux principaux acteurs, qu'il était logique de publier le rapport d'expertise.

livres avec faux *ex-libris*) pour environ 150 000 francs de l'époque (ça doit faire une fortune). Il écopa de 2 ans de prison et d'une amende.

On apprend à la lecture du livre que :

a) Les quelque 27000 manuscrits sont tous de la main de Vrain-Lucas. Sur tous les faux, l'écriture est d'ailleurs sensiblement la même.
b) il utilisait de vieilles feuilles vierges achetées aux puces ou volées dans les bibliothèques et une encre vieillie artificiellement,
c) toutes les lettres étaient écrites en ancien français. Comment Chasles a-t-il pu se laisser abuser quand on sait qu'il a acheté des lettres de Thalès au roi des Gaules, d'Archimède à Hiéron, d'Alexandre le Grand à Aristote, de Jeanne d'Arc au peuple de Rouen, et même de Lazare à Saint Pierre et des apôtres à Jésus !!

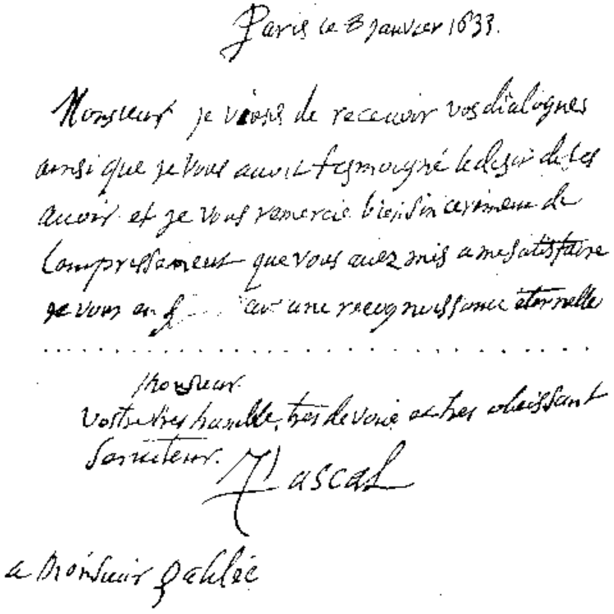
Aucune conclusion ne pourrait avoir l'intérêt des textes cités, cette histoire est suffisamment formidable pour ne pas avoir besoin de commentaire; mais, puisque vous avez lu jusqu'au bout, voici pour votre plaisir un extrait d'une des lettres achetées en toute simplicité par Chasles :

Cleopatre royne à son très amé Jules César, salut

> *Mon très amé nostre fils Cesarion va bien. J'espère que bientôt il sera en estat de supporter le voyage d'icy à Marseilles où j'ay dessin de le faire instruire tout à cause du bon air qu'on y respire et des belles choses qu'on y enseigne [...]*

On reste sans voix...

Fausse letre de Pascal à Galilée du 8 janvier 1633



Paris le 8 janvier 1633.

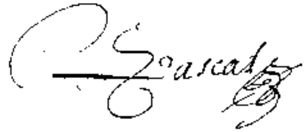
Monsieur je viens de receuoir vos dialogues ainsi que je vous auois tesmoigné le desir de les auoir et je vous remercie bien sincerement de l'empressement que vous auez mis a me satisfaire je vous en [illegible] avec une recognoissance eternelle

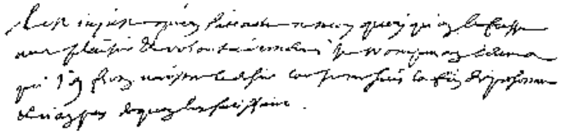
. . . .

Monsieur
Vostre tres humble, tres devoue et tres obeissant
Seruiteur. Pascal

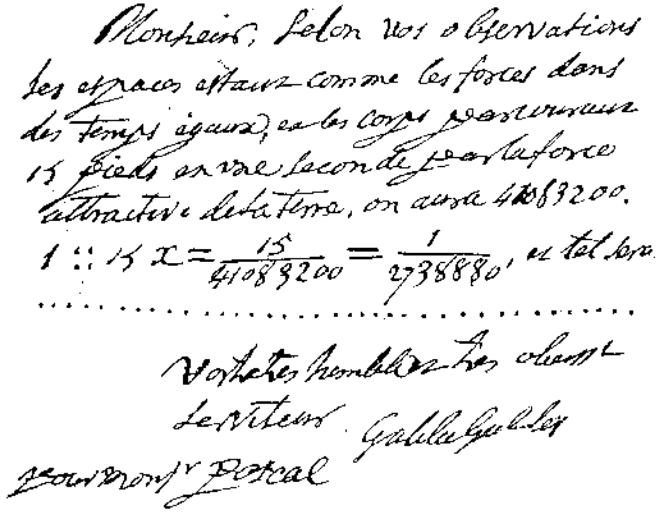
a Monsieur Galilée

Un exemple de la vraie écriture de Pascal.





Fausse lettre de Galilée à Pascal.



Monsieur, selon vos observations les espaces estant comme les forces dans des temps égaux, et les corps parcourant 15 pieds en une seconde par la force attractive de la terre, on aura 41083200.

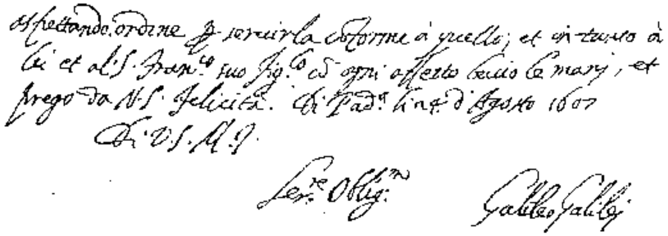
$1 :: 15\ x = \frac{15}{41083200} = \frac{1}{2738880}$, et tel sera.

. . . .

Vostre tres humble et tres obeisst
serviteur. Galileo Galilei

pour monsr Pascal

Un exemple de la vraie écriture de Galilée.



About the notion of natural logic: historical and theoretical remarks

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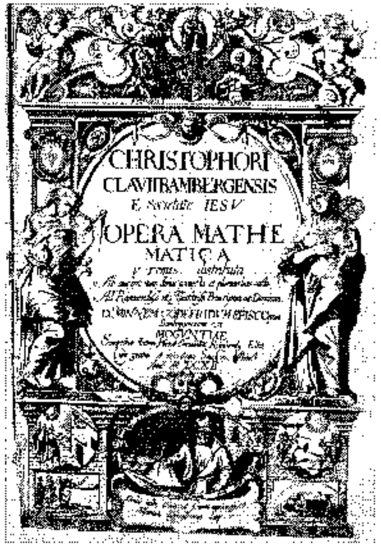
Abstract

The paper is meant as a historical and theoretical analysis of some notions we claim to characterize a "natural logic". As an example we focus our attention on the syllogistic theory. After giving a brief introduction to the classical syllogism, we examine the techniques of the syllogistic 'demostratio' with respect to the analytic and synthetic methods and we characterize them by means of elimination and introduction rules. In the second part of the paper we propose a comparison with Gentzen's Natural Deduction.



Page de titre des œuvres de Clavius publiées en 1612.

1 Introduction : Syllogisms

The syllogistic 'demostratio' has been traditionally considered to be the unique logical instrument for formalizing natural reasoning inferences. In the philosophical discussions of the XVI century a prominent role was played by the analysis of mathematical proofs. In particular the attention was focused on the *quaestio de certitudine mathematicarum* (see Piccolomini, Catena, Clavio [?] and Barozzi [?]) and the methodological symmetry between analysis and synthesis. The *resolutio* and the *compositio*, their synonyms, were described by Dupleix [?], as:

> Analytique [...] est un mot Grec dérivé d'*Analysis*, c'est-à-dire *Resolution* : qui n'est autre chose qu'un regrés ou retour d'une chose en ses principes [...] tellement que c'est le contraire de la composition [...].

The increased interest for the analysis was motivated by two events: the discussions on Euclide, from the one hand, and on Pappo and Diofanto, from the other. These two fact pushed the mathematicians to focus their attention on the method they were using. From this methodological studies the analysis acquired a new role, and was considered to be as important as the synthesis.

Following this idea, in the paper we define the syllogism in terms of mathematical rules able to expres both the synthesis and the analysis.

As it is well-known the classical syllogisms are based on four types of categorical propositions: universally affirmative (A), universally negative (E), particularly affirmative (I), particularly negative (O). The problem of how to interpret the categorical propositions correctly has been deeply discussed in Lukasiewicz [?]. We will represent these four propositions as:

A $F_1(p,q)$: Every p is q
E $F_2(p,q)$: No p is q
I $F_3(p,q)$: Some p are q
O $F_4(p,q)$: Some p are not q

Summing up we say that: $F_i(x,y)$ stands for a categorical proposition where x and y are the subject and the predicate, respectively.

A *syllogism* is an inference schema composed by two premises which share a term, known as medium term, and which carry the other two terms which will be repeated in the conclusion. Accordingly to the different positions these three terms occupy in the schema, different figures are obtained. As an example we discuss the first figure.

Let z be the medium term, x and y the terms in the major and minor premises, respectively,

1^{st} Figure

z	x
y	z
y	x

Different modes are obtained combining the four possible categorical propositions. We summarize them in the table below:

	$mode_1$	$mode_2$	$mode_3$	$mode_4$	$mode_5$	$mode_6$
$figure_1$	AA ⊢ A	EA ⊢ E	AI⊢ I	EI⊢ O	AA⊢I	EA⊢ O

where the XY ⊢ Z denotes the inference of the conclusion Z from the major and minor premises X and Y, respectively.

In order to interpret the classical syllogisms as introduction and elimination of the medium term, we represent the above inferences by means of a more general schema which has been orginally proposed in Freguglia [?] where an algebra of categories was considered as a model for the syllogisms.

Definition 1. Let $F_i(\mu,p)$, $F_j(s,\mu)$ and $F_k(s,p)$ be categorical propositions, we define a syllogism as the following transformation the validity of which is independent from the semantic truth value of the categorical propositions it consists of. Let $i,j,k \in \{1,2,3,4\}$

$$(1) \qquad \Sigma[F_i(\mu,p); F_j(s,\mu)] = F_k(s,p)$$

We call (1) σ-transformation, $F_i(\mu,p)$ and $F_j(s,\mu)$ the major and minor premises, respectively, and $F_k(s,p)$ the conclusion. This schema satisfies the modes of the first figure, similar schemata can be however given for the other figures simply changing the subject/predicate positions. As an example we give the inference in Darii first in the traditional way, then with the format proposed in def. 1:

Example 1: DARII

$$\frac{\text{Every roman is stubborn} \quad \text{Some italians are romans}}{\text{Some italians are stubborn}} \; DARII$$

Σ[Every(roman,stubborn), Some(italians,romans)] = Some(italians,stubborn)

2 Syllogisms by means of Introduction and Elimination rules

As the reader might have noticed the schema given in def. 1 simply consists in the elimination of the medium term. We can replace the above definition with a more elegant and simple one:

Definition 2 Let a and b be the major and minor premises, respectively, c the conclusion and μ the medium term shared by a and b, the syllogism is defined by the following operation:

$$(2) \qquad E\mu[a,b] = c$$

Example 2
a:= Every roman is stubborn,
b:= Some italians are romans,
c:= Some italians are stubborn,

E'roman'[a,b] = c

From an epistemological perspective we can look at the elimination rule as the *synthetic* method (the *demostratio propter quid* or deduction), see Szabø[?]. Therefore we can now define the *analysis* (i.e. the *demostratio quic* or logical induction) in terms of transformations. The analytic method starts from the results and goes back to the assumptions (see Hitikka-Remes[?]). Applying this process to the syllogism means to start from the conclusion and reconstruct the premises it has been derived from. In order to establish all the possible syllogisms with the given proposition as the conclusion, we need to use two transformations: one which gives the set of possible major premises and one which gives the set of possible minor premises.

Definition 3 Let A and B be sets of terms, $F_k(p, s)$ the given categorical proposition, we define the transformations Γ_M and Γ_m which return the major and minor premises, respectively. Let $k, i, j \in \{1, 2, 3, 4\}$

$$(3) \qquad \Gamma_M[F_k(s, p)] = F_{in}(\mu, p)$$

$$(4) \qquad \Gamma_m[F_k(s, p)] = F_{jt}(s, \mu)$$

where n, t indicates the number of $\mu \in A$ and of the $\mu \in B$, respectively. We call (3) and (4) γ-transformations.

Theorem 1 Given the two sets of categorical propositions $F_{in}(\mu, p)$ and $F_{jt}(s, \mu)$ obtained by the transformations (3) and (4), $\exists \mu \in (A \cap B)$ iff a specific transformation of the type given in (1) is found as well.

From this theorem it follows that in the same way we have formalized the Σ transformation by means of the elimination rule, we can now replace the two schemata (3) and (4) with an operation, namely the inverse of the elimination rule, viz. the introduction one.

Definition 4 Let c be the given conclusion, and a, b any of the possible categorical propositions which can be the major and minor premises, the analysis is formalized by the introduction operation:

$$(5) \qquad \mathrm{I}\mu[c] = [a,b]$$

Example 3

Let A and B be sets of terms, M and m sets of categorical propositions obtained by means of (3) and (4), respectively, and F_k(s,p) the given conclusion:

A = {girl, roman, cat}

B = {abruzzese, roman, tuscan, girl}

F_k(s,p) = Some(italians, stubborn)

M = { M_1: 'Every roman is stubborn', M_2: 'Every girl is stubborn', M_3: 'Every cat is stubborn'}

m = { m_1: 'Some italians are roman', m_2: 'Some italians are girls', m_3: 'Some italians are tuscan', m_4: 'Some italians are abruzzesi'}

applying the theorem 1 we can give an example of the transformation (1) in the following way:

1. Σ[Every(roman,stubborn),Some(italians,romans)]= Some(italians,stubborn)

2. Σ[Every(girl,stubborn),Some(italians,girls)] = Some(italians,stubborn)

Starting from the conclusion we have reconstructed the reasoning (i.e the inferences) which had brought to it.

3 Dialectic argumentation

Having both the analysis and the synthesis we can give the rules which formalize the dialection argumentations as the combinations of "analysis" and "synthesis". Human beings' argumentations are in fact made of these two different moments, one in which we ask how our claim has been deduced, and one in which we look at the consequences of our argumentation.

In order to formalize this natural way of reasoning, we use the rules below. For both the Introduction and Elimination operator, we give two types of rules to which we refer as *proper* vs. *conventional*. The former are the rules we have obtained from the transformations discussed above – 1a, 2a below; whereas the later are new ones which are introduced by convention – 1b, 2b.

Having these second rules we can proceed in our dialectic combination asking the same kind of question twice – i.e. combining the E (resp. I) operator with itself. As a constrain to correctly build the dialectic combination we require our "reasoning" to start always from either 1a or 2a; whereas 1b and 2b can be applied only as a second step. As we will see with an example, although formally all combinations of the operators E and I are possible we obtain a result only when we fix the medium term μ, i.e. when the rules combined are operating on the same term.

For reasons which will be clear soon, we include the identity operation $\mathcal{I}$ as well. For this operator as well we give two schemata considering the $\mathcal{I}$ as a function which takes one or two arguments. However in this case the different behavior of the two rules does not hold: our reasoning can start with both of them, since the two rules are both "natural".

[1a.]	E[a,b] = [c]	[1b.]	E[c] = [c]
[2a.]	I[c] = [a,b]	[2b.]	I[a,b] = [a,b]
[3a.]	$\mathcal{I}$[c] =[c]	[3b.]	$\mathcal{I}$[a,b] = [a,b]

Using the above rules we can combine the operations $\mathcal{I}$, E, I.

Examples 4 of dialectic combination.

(i) I(E$\mathcal{I}$)[a,b] = E$\mathcal{I}$[a,b] = I[c] = [a,b]

(ii) (IE)$\mathcal{I}$[a,b] = IE[a,b] = I[c] =[a,b]

where in (i) the rules are applied from the left to the right, i.e. first (3b), then (1a) finally (2a); whereas in (ii) the rules are applied from the right to the left following the same order. Therefore, the associative law holds:

$$(6) \qquad \mathrm{IEI}[a,b] = [a,b]$$

which says: "Given two categorical propositions a and b if I ask first which syllogism can have them as conclusions, then which syllogisms they can be the premises of, and finally once again from which syllogism they can be derived, the same propositions are found, i.e. I obtain the same categorical propositions I started from".

As a linguistic example we can consider the categorical propositions given in Example 3. Let μ_1 and μ_2 be two different terms in $A \cap B$, for example, the terms 'roman' and 'girl', respectively. If we combine the operators as in (i) we obtain the conclusion c, whereas the combination given in (ii) is unable to go back to the original premises.

Example 5

(i) $\mathrm{E}\mu_1\mathrm{I}\mu_1[\mathrm{M}_1,\mathrm{m}_1] = \mathrm{I}[c] = [\mathrm{M}_1,\mathrm{m}_1]$

(ii) $\mathrm{E}\mu_1\mathrm{I}\mu_2[\mathrm{M}_1,\mathrm{m}_1] = \mathrm{I}[c] = [\mathrm{M}_2,\mathrm{m}_2]$

4 Algebraic Aspects

The situation described in the previous paragraph can be better expressed introducing an explicate operation to compose $\mathcal{I}$, I and E. Simply applying the rules 1-6 we obtain the table below:

(7)

$\circ$	$\mathcal{I}$	E	I
$\mathcal{I}$	$\mathcal{I}$	E	I
E	E	E	$\mathcal{I}$
I	I	$\mathcal{I}$	I

In order to verify that this table correctly synthesize the above rules, we can look at the composition E $\circ$ I. As it results from the example (i) applying first E and then I to (a,b) we have obtained the same result which is given by the application of $\mathcal{I}$ to the same argument, i.e. E$\circ$ I $= \mathcal{I} =$ I $\circ$ E. We can therefore conclude that the following theorem holds:

Theorem 2 The combination of the σ and γ-transformations, viz. (1) and (3), (4), follows the structure of a finite commutative and idempotent group.

The commutative property (as well as the associative one) is due to the constrain we have required when speaking of the dialectic combination of the operators, viz. our argumentation can never start applying the "conventional" rules (1b and 2b).

The representation of "cognitive" processes by means of operations, and even more the interpretation of the possible combinations of these operations via an algebraic structure recall Piaget's theory, see Piaget [?]. Our proposal of considering the dialectic combination as a group based on $\mathcal{I}$, I, E is, in fact, similar to the idea of having the group of INRC as the algebraic structure behind our mental operations. Having found this algebraic structure as a model for the syllogistic theory is of particular interest because it sheds light on a class of logics which we can define as natural logic, i.e. a logic in which it is possible to *introduce* and *eliminate* logical

elements. Intuitively this definition brings into the mind besides the syllogistic theory also the natural deduction system proposed by Gentzen [?]. However, despite the similarity between the meta-operators which characterize the two systems, some important differences occur. We are going to discuss this point in the next section.

5 Gentzen's calculus and the syllogistic theory: A possible comparison

As we have mentioned if we look at the syllogisms as applications of introduction and elimination rules, it seems natural to assimilate the obtained formalization to Genzten's Natural Deduction. However, the similarity disappear as soon as we consider the composition of the operators. Before comparing the two systems, we briefly present Gentzen's calculus.

Natural Deduction is a proof-system introduced by Gentzen in order to reach a calculus closer to natural reasoning than Hilbert's assiomatic system. With this intend he has characterized a logic by means of introduction and elimination rules for each logical connective the language is built from. Simplifying:

$$\mathrm{I} * [\ldots, \mathrm{r}, \mathrm{p}, \mathrm{q}] = \mathrm{r} * \mathrm{s} \qquad \mathrm{E} * [\ldots, \mathrm{p} * \mathrm{q}] = \mathrm{r}$$

where $*$ is any connective which is introduced by means of I, and eliminated by means of E. In the introduction rule the conclusion may contain a subformula which does not occur in the premises[1]; in the elimination rule, instead, the conclusion is always a subformula of one of the premises.

Already at this point we can consider an important difference between our formalization of the syllogistic theory and this calculus. In the latter the rules eliminate or introduce connectives, whereas in the former they eliminate or introduce terms. As an example we give the introduction and elimination rules for the conjunction $\wedge$ and the conditional $\to$. Let $a_1, a_2 \in P$ – where P is a set of propositions –, and $i \in \{1, 2\}$, the logical rules for $\wedge$ and $\to$ are:

$$\frac{a_1 \quad a_2}{a_1 \wedge a_2} \wedge I \qquad\qquad \frac{a_1 \wedge a_2}{a_i} \wedge E$$

$$\frac{\begin{array}{c} a_1 \\ \vdots \\ a_2 \end{array}}{a_1 \to a_2} \to I \qquad\qquad \frac{a_1 \to a_2 \quad a_1}{a_2} \to E$$

To be noticed is the particular behavior of the introduction rule of the $\to$. The two arguments a_1 a_2 are linked to each other: the latter is a consequence of the former through a derivation.

If we compare these inferences with the rules (2) and (5) we easily see the similarity and the differences between the two systems. In particular, we notice that in both the two systems the 'synthesis' is formalized in a similar way by means of the elimination rules; whereas in the natural deduction one the 'analysis' is missing. This difference can be better understood looking at the introduction rules which are 'deductive' as well as the elimination one: from a set of premises they give back a conclusion. To facilitate the comparison we give the above rules in the format used in the previous section.

$$I \wedge [a_1, a_2] = a_1 \wedge a_2 \qquad E \wedge [a_1 \wedge a_2] = a_i$$

$$I \to [a_1, a_2] = a_1 \to a_2 \quad E \to [a_1 \to a_2, a_1] = a_2$$

[1] This is the case of the introduction rule for the $\to$ and $\vee$

Due to the differences between the introduction rules in the systems we are considering, the dialectic combination differs as well. Therefore, the algebraic structure represented in (7) cannot be used as a model of Gentzen's calculus.

Moreover, as we have said before, in the previous section it has been possible to build the finite commutative and idempotent group thanks to the combination of the standard rules with the "conventional" one. Therefore, this combination cannot be done in Gentzen system since there is no way to return the premises and because only the proper rules are given.

We now give some examples to show how the lack of the analysis moment makes the system unable to formalize natural reasoning at least in the way we have said before.

Examples 6

$$EI[a \rightarrow b, a] = I[b] = ?$$

Let a:= 'It rains', b:= 'I cannot come', the above schema becomes:

$$\dfrac{\dfrac{\textbf{If}\text{ it rains, }\textbf{then}\text{ I cannot come} \quad \text{It rains}}{\text{I cannot come}} \rightarrow E}{?} \rightarrow I$$

The question marker means that we are unable to reconstruct the reasoning which had lead to the conclusion 'I cannot come'.

6 Conclusions

In the paper we have proposed a formalization of the syllogistic theory which is able to account for both the dialectic moments: the synthesis and the analysis. The motivation behind this proposal are of two natures: a historical one which has brought us to consider the *resolutio* involved in natural reasoning as important as the *compositio*; and a physiological one – which can be traced back to Piaget's theory – which has made us thinking of the finite group as a basic and 'natural' structure in human beings' mental processes.

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Histoire des mathématiques du chaos
et épistémologie du hasard

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Abstract

Hasard ou déterminisme ? L'alternative sur laquelle les sciences se sont construites au profit d'une conception déterministe du monde excluant le hasard et "l'imprédictibilité" est actuellement mise en question aussi bien en mathématiques que dans l'étude des phénomènes.

L'exposé se propose de faire état des travaux de l'"Atelier Philo-Math" de l'IREM de Poitiers (France) sur la théorie du chaos et le statut du hasard, et sur la conception d'activités visant à permettre aux élèves :

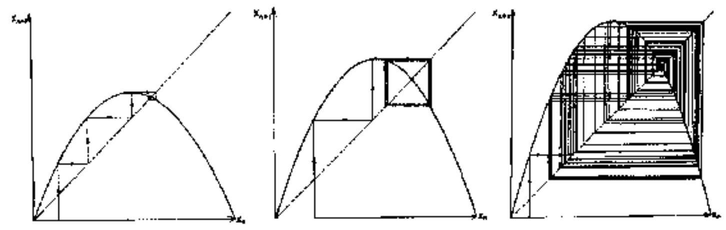
– de prendre du recul par rapport à leur conception spontanée du hasard,

– de mieux comprendre le travail de modélisation dans la "mathématisation du hasard", les limites d'un modèle probabiliste et la nécessité de recourir, pour certaines situations, à un modèle chaotique.

0. BREF HISTORIQUE DES TRAVAUX DE L'ATELIER "PHILO-MATH" SUR LE CHAOS

I. LE TRAVAIL DE RECHERCHE SUR L'HISTOIRE DES MATHÉMATIQUES DU CHAOS

II. LA CONCEPTION D'ACTIVITÉS POUR LES ÉLÈVES



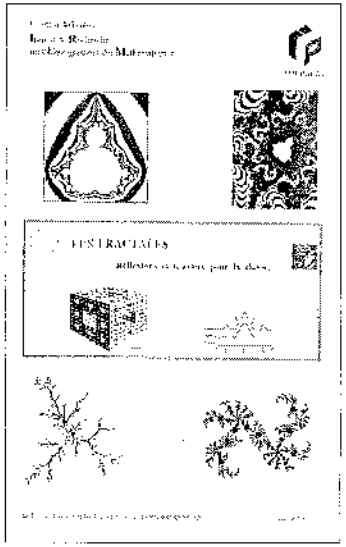
Fixed Point Period 2 Chaos

Bref historique des travaux de l'atelier "Philo-Math" sur le chaos

L'*Atelier philo-math* de l'I.R.E.M. de Poitiers est un groupe interdisciplinaire qui comporte six mathématiciens, une philosophe, ainsi que deux physiciens, et qui se réunit régulièrement depuis 8 ans, au rythme de trois ou quatre sessions d'une demi-journée par an.

Ses travaux ont pour objectif final de concevoir des activités "productrices de sens", c'est-à-dire qui aident les élèves à mieux comprendre ce qu'ils font quand ils font des mathématiques, et d'une façon générale à s'interroger sur les mathématiques elles-mêmes: le type de connaissance qu'elles constituent, leur rapport à la réalité et aux autres sciences.

Les recherches sur le chaos se situent dans le prolongement des fractales qui ont occupé l'*Atelier* de 1993 à 1995 et qui ont abouti à la production d'une brochure : LES FRACTALES. Réflexions et travaux pour la classe[1].



Les fractales permettant de modéliser beaucoup d'aspects du réel, elles fournissent, en même temps qu'un travail sur les notions mathématiques de suite, de dimension, d'homothétie, la base d'activités provoquant la réflexion sur le statut des objets mathématiques, sur le problème des rapports mathématiques-réalité, sur le statut de la vérité mathématique. C'est l'occasion de mettre l'accent sur le rôle de la modélisation, sur le problème de l'adéquation entre le modèle et la réalité.

[1]LES FRACTALES. Réflexions et travaux pour la classe. D. GAUD - J. GUICHARD - S. PARPAY - J.-P. SICRE & C. CHRÉTIEN. Brochure de 105 pages. I.R.E.M. de Poitiers. Janvier 1996.

Au terme de ce travail est apparue la nécessité d'aller plus loin, puisque les fractales peuvent être définies comme "la géométrie du chaos"[2], que l'étude de suites mathématiques, d'apparence simple, *fait apparaître des cycles de toutes sortes et même, le chaos*, – où l'on "tombe" sur des attracteurs étranges, vers quoi convergent les valeurs d'une suite chaotique; ou, si l'on suit l'évolution d'un point M pour visualiser géométriquement l'état d'un système, c'est la région de l'espace où M finit toujours par aboutir, quelles que soient les conditions initiales;

"... *certains sont même fractals, comme celui donné par le germe*
0,0101001010100010101010000101010
..."[3]; enfin, parce que le *chaos* apparaît comme une structure "naturelle" parce qu'on la retrouve dans beaucoup de phénomènes appartenant à des domaines différents.

Se trouve alors relancée la réflexion amorcée dans les travaux sur les fractales, sur ce qu'est une théorie mathématique, une théorie scientifique, sur le problème de la modélisation et le statut du modèle. L'étude de structures chaotiques permet de bien mettre en évidence qu'"*entre le modèle et la réalité*" intervient de façon incontournable "*le calcul*"[4]. C'est non seulement la possibilité de faire réfléchir les élèves sur les liens entre les phénomènes physiques et les mathématiques, mais aussi de faire un travail sur les notions de hasard, de prédictibilité, de déterminisme[5], pour démêler ce qui peut faire obstacle à la compréhension[6] et de mettre en évidence :

– que le déterminisme ne conduit pas nécessairement à du prédictible,

– qu'il n'y a pas réciprocité entre hasard et imprédictibilité, que "hasard" implique "imprédictibilité", mais qu'"imprédictibilité" n'implique pas nécessairement "hasard", puisque l'imprédictibilité peut avoir d'autres raisons, par exemple le trop grand nombre de causes,

– qu'il y a différents "types" – ou notions – de hasard et que ce n'est pas le même "type" qui est en jeu dans un modèle probabiliste et dans un modèle chaotique: et qu'il faut examiner ce qui est modélisable en matière de "phénomènes de hasard";

– que ce n'est pas parce qu'il y a chaos que l'on ne peut rien dire !

Les problèmes sont à la charnière de l'épistémologie – réflexion sur la constitution et les limites du savoir – et de la métaphysique – conception de la nature, ou structure, ou essence de tout ce qui existe et de ses principes. C'est une vieille question qui résiste : le hasard existe-t-il dans les choses ou n'est-il que l'expression des limites de la connaissance humaine, incapable de déterminer la multiplicité des causes ?

Le résultat des travaux : une brochure LES CHANTIERS DU CHAOS qui comporte :

– des repères historiques et théoriques,

– des études et des activités pour les élèves.

– les extraits des textes de référence qui ont guidé nos travaux,

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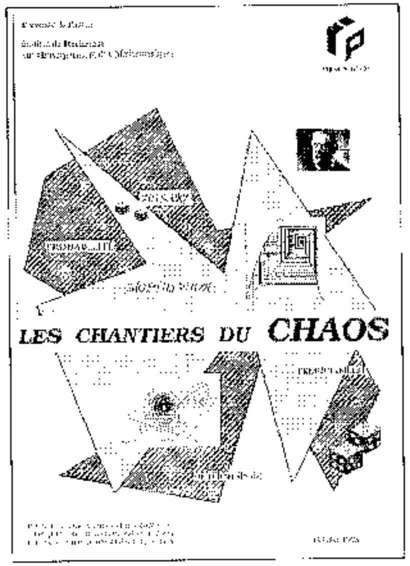
[6]Cf. LES CHANTIERS DU CHAOS o. p. II.1. QU'EST-CE QUE LE HASARD ? COMMENT LE MATHÉMATISER ?

– une bibliographie et des index des noms et des notions.



Sommaire

1. Le travail de recherche sur l'histoire des mathématiques du chaos

Il a été guidé par deux préoccupations majeures : prendre des repères sur ce qui a provoqué le recul d'une conception du monde et du savoir sur laquelle les sciences se sont construites du XVI^{ème} au XIX^{ème} siècles; et situer la rencontre chaos – fractales.

Ces points de repères historiques balisent la construction et le dépassement d'une *physique de la stabilité de l'univers* dont KEPLER énonce les lois empiriques en 1609 et d'une *mathématique du continu* dont les équations différentielles constituent l'outil, avec à la fin du XVIII^{ème} siècle LAGRANGE (1788), puis LAPLACE (1796-1812), FOURIER (1816) et CAUCHY (1820-30).

Le principe fondamental du monde et de sa connaissance est énoncé par Pierre-Simon LAPLACE dans son *Essai philosophique sur les probabilités* (1814-1825)[7]. C'est le principe du *déterminisme universel* qui vise à assurer l'objectif de la science : la possibilité de la *prévision scientifique.*

> Tous les événemens, ceux même qui par leur petitesse, semblent ne pas tenir aux grandes lois de la nature, en sont une suite aussi nécessaire que les révolutions du soleil. /.../
>
> Nous devons donc envisager l'état présent de l'univers, comme l'effet de son état antérieur, et comme la cause de celui qui va suivre. Une intelligence qui pour un instant donné, connaîtrait toutes les forces dont la nature est animée, et la situation respective des êtres qui la composent, si d'ailleurs elle était assez vaste pour soumettre ces données à l'analyse, embrasserait dans la même formule, les mouvemens des plus grands corps de l'univers et ceux du plus léger atome : rien ne serait incertain pour elle, et l'avenir comme le passé, serait présent à ses yeux. L'esprit humain offre dans la perfection qu'il a su donner à l'astronomie, une faible esquisse de cette intelligence. Ses découvertes en mécanique et en géométrie, jointes à celle de la pesanteur universelle, l'ont mis à portée de comprendre dans les mêmes expressions analytiques, les états passés et futurs du système du monde. En appliquant la même méthode à quelques autres objets de ses connaissances, il est parvenu à ramener à des lois générales, les phénomènes observés, et à prévoir ceux que des circonstances données doivent faire éclore. Tous ses efforts dans la recherche de la vérité, tendent à le rapprocher sans cesse de l'intelligence que nous venons de concevoir mais dont il restera toujours infiniment éloigné. Cette tendance propre à l'espèce humaine, est ce qui la rend supérieure aux animaux; et ses progrès en ce genre, distinguent les nations et les siècles, et fondent leur véritable gloire.

C'est ce lien pensé comme nécessaire entre déterminisme et prédictibilité qui va être remis en cause dès la fin du xix^{ème} siècle, en particulier avec MAXWELL soutenant que "... les mêmes antécédents ne coïncident jamais" (1873). Mais *le tournant décisif*, ce sont *les travaux de* POINCARÉ sur le problème des trois corps et l'instabilité de leurs trajectoires. Dans les MÉTHODES DE LA MÉCANIQUE CÉLESTE (1892-93-99), il montre que le problème de trois corps en mouvement et en interaction gravitationnelle n'est pas intégrable, et que par conséquent on ne peut calculer leurs trajectoires à long terme. C'est la première approche de la sensibilité aux conditions initiales (SCI).

[7] Introduction à la THÉORIE ANALYTIQUE DES PROBABILITÉS; PAR M. LE COMTE LAPLACE, Pair de France; Grand-Officier de la Légion-d'Honneur; Grand'Croix de l'Ordre de la Réunion; Membre de l'Institut royal et du Bureau des Longitudes de France; des Sociétés royales de Londres et de Gottingue; des Académies des Sciences de Russie, de Danemarck, de Suède, de Prusse, d'Italie, etc. SECONDE ÉDITION, REVUE ET AUGMENTÉE PAR L'AUTEUR. PARIS, Mme Ve COURCIER, Imprimeur-Libraire pour les Mathématiques et la Marine, quai des Augustins, n° 57. 1814. pp. ij-vij. – Le texte de cette introduction, qui reprend une leçon sur les probabilités donnée en 1795 à l'École Normale (le 10 mai. Cf. l'ÉCOLE NORMALE DE L'AN III. LEÇONS DE MATHÉMATIQUES. *Laplace - Lagrange - Monge*, sous la direction de J. DHOMBRES, Dunod, Paris, 1992, p. 125), était paru quelques mois auparavant chez le même éditeur sous ce titre : ESSAI PHILOSOPHIQUE SUR LES PROBABILITÉS, et il connaîtra trois autres éditions du vivant de l'auteur. Réédition PIERRE-SIMON LAPLACE - *Essai philosophique sur les probabilités* (Texte de la 5° édition, 1825) suivi d'extraits de Mémoires. Préface de René THOM, Postface de Bernard BRU, C. Bourgois Éditeur, Paris 1986.

En 1908, il consacre le chapitre IV du livre I de SCIENCE ET MÉTHODE au hasard[8], et il écrit : "*. . . que de petites différences dans les conditions initiales en engendrent de très grandes dans les phénomènes finaux*". La notion constitutive du chaos déterministe se trouve ici mise en avant bien avant sa dénomination, et avant de passer à la postérité sous la dénomination d'"*effet papillon*", par ... l'effet d'une comparaison d'Edward LORENZ dont les travaux pour modéliser les phénomènes météorologiques constituent, au début des années soixante, l'*autre tournant décisif* de l'histoire du chaos, qui *inaugure le temps des recherches et des modèles*[9]. Il utilise un système d'équations non linéaires. Les représentations graphiques des solutions ont pour limite "l'attracteur de LORENZ".

La rencontre avec la théorie des fractales s'est faite par le biais des espaces de phases[10] utilisés par les physiciens pour géométriser les états des systèmes qu'ils étudiaient. RUELLE et TAKENS montrèrent dans les années 1970 que la géométrie fractale permettait de visualiser le chaos déterministe, par les attracteurs étranges.

L'année précédente, l'astronome M. HÉNON donne une approximation numérique au problème des trois corps de POINCARÉ et contribue à éveiller l'intérêt pour le chaos déterministe. En 1976, avec le physicien Y. POMMEAU, ils construisent un attracteur qui va constituer un modèle dans l'étude du chaos dissipatif.

2. La conception d'activités pour les élèves

Deux idées ont dirigé ce travail :

– la notion de hasard est un réel obstacle pour les élèves lors de l'enseignement des probabilités ,

– le chaos déterministe pensé comme nouveau paradigme scientifique constitue une véritable rupture épistémologique[11] dont les conséquences sont pédagogiquement exploitables pour faire comprendre aux élèves que les probabilités ne constituent pas le seul moyen de mathématiser le hasard. Le chaos ouvre d'autres perspectives qui, mises en parallèle avec les probabilités, offrent l'occasion de réfléchir sur la modélisation et en particulier de revenir sur les problèmes de modélisation rencontrés dans les exercices de probabilités.

Par conséquent, pour concevoir des activités qui provoquent ces réflexions chez les élèves et les amènent à questionner leurs conceptions spontanées du hasard, nous avons pris comme repères les principaux *enjeux du chaos* qui concernent la nature des phénomènes que nous attribuons au hasard, et corrélativement la notion d'ordre, la question de la prédiction dans une théorie scientifique, et par conséquent la réflexion sur les limites du savoir scientifique.

• Le chaos conduit en effet à *repenser la notion de "hasard"* et à distinguer deux catégories dans les phénomènes aux comportements erratiques :

– ceux qui ont pour origine l'action de causes ou de forces s'exerçant *aléatoirement*, comme le mouvement des dés roulant sur la table sous l'effet des multiples chocs du brassage, ou le

[8]POINCARÉ H. *Science et méthode*, 1908. Réédition Flammarion 1934 pp. 64-94.

[9]Dans une histoire du chaos, on ne peut omettre tous les travaux de l'École russe sur les oscillateurs non linéaires. Cf. LES CHANTIERS DU CHAOS o. p. I.2. PETITE CHRONOLOGIE DU CHAOS.

[10]Espace abstrait dont les dimensions correspondent aux variables de position et de vitesse du système. Cf. LES CHANTIERS DU CHAOS o. p. I.3. LES ATTRACTEURS ÉTRANGES : DU CHAOS AUX FRACTALES.

[11]"... une véritable révolution conceptuelle" DUBOIS M., ATTEN P. & BERGÉ P. *L'ordre chaotique*. LA RECHERCHE n° 185, p, 192. Février 1987. "*. . . un événement* **historique** : le passage d'une physique qui ne considérait que les phénomènes linéaires à une physique qui a décidé de regarder les phénomènes non linéaires." DINER S. LE CHAOS, *puissance et impuissance*. Vol. 24 n° 234, p. 24. Revue du Palais de la Découverte Janvier 1996.

mouvement brownien engendré par la multitude des chocs que les particules en suspension reçoivent des molécules du fluide,

– et ceux qui sont le résultat de l'itération de lois simples, comme l'engendrement de nombres imprévisibles, bien que l'opération faisant passer d'un nombre au suivant soit strictement déterminée, et auxquels on réserve la dénomination de "chaos déterministe"[12].

• Du même coup, le chaos déterministe conduit à *repenser la notion d'ordre*, pensée comme toute notion sur "fond de son contraire" le désordre; mais... y a-t-il un désordre fondamental, premier, le chaos des Anciens, qui serait manifesté par certains phénomènes, ou bien avons-nous tendance à appeler désordre une organisation qui nous est inconnue, à laquelle nous ne nous attendions pas, comme Henri BERGSON le soutenait en 1907 dans *l'Évolution créatrice*.

• *La possibilité de la prédiction*, considérée comme une des caractéristiques d'une théorie scientifique[13] est aussi *mise en question*. Le chaos déterministe conduit à "assouplir" le concept de prédiction : entre la détermination certaine d'un phénomène ou d'un système dans la conception déterministe de type laplacien –qui constitue le concept classique de prédiction–, et l'impossibilité de dire quoi que ce soit sur les états futurs qu'engendrerait une indétermination totale, il y a place pour penser un encadrement des possibles pour ce qui concerne les systèmes sensibles aux petites variations des conditions initiales, une sorte de "cadrage prévisionnel" par discrimination du possible et de l'impossible, du transitoire et de l'essentiel - ou "naturel", c'est-à-dire lié à la nature du système en question. Ivar EKELAND l'illustre en montrant "*la mécanique du hasard*" *ou* "*le jeu du hasard et de la nécessité*" à propos de l'attracteur de LORENZ :

> /.../ il attire les trajectoires, toutes les trajectoires, quel que soit leur point de départ. D'où qu'elles partent, elles se dirigent immanquablement vers cette étroite région de l'espace qui contient l'attracteur et leurs évolutions y restent confinées. On voit bien ici la différence entre un état arbitraire du système et un état réaliste, que nous avions déjà essayé de faire sentir à propos de la météorologie. Alors que chaque point de l'espace à trois dimensions représente un état théoriquement possible, seuls certains d'entre eux sont naturels, en ce sens que l'évolution naturelle du système peut y conduire. Ces états naturels occupent une partie beaucoup plus restreinte de l'espace, représentée par l'attracteur de LORENZ. Tous les autres états sont transitoires, en ce sens que si l'on y amène le système, dès que celui-ci sera libéré et reprendra son cours normal il s'en écartera immédiatement pour ne plus jamais y revenir. En régime de croisière, il se trouve nécessairement quelque part sur l'attracteur, c'est-à-dire dans un état naturel. Il n'est pas immobile, au contraire; il poursuit indéfiniment son mouvement giratoire, quelques tours à gauche, puis quelques tours à droite, mouvement qui l'amène à explorer systématiquement tout attracteur, c'est-à-dire à visiter tous les états naturels. L'existence de l'attracteur traduit donc une distinction fondamentale à opérer entre l'immense majorité des situations, par lesquelles le système ne passera jamais de lui-même, et les quelques situations naturelles, par lesquelles le système repassera indéfiniment, quoique irrégulièrement.[14]

[12]DUBOIS M., ATTEN P. & BERGÉ P. *L'ordre chaotique*. o. p. p, 190-91, et LES CHANTIERS DU CHAOS o. p. II.2. exemples mathématiques.

[13]Caractéristique qui tient au lien interne entre la connaissance certaine et la puissance de déduction et d'action qu'elle confère, dont se sont nourris les espoirs mis dans le développement scientifique. Auguste COMTE le résumait dans la seconde leçon [II. (2)] de son COURS DE PHILOSOPHIE POSITIVE (1830-42) ; "Science d'où prévoyance; prévoyance, d'où action". Francis BACON en avait explicité le principe dans son NOVUM ORGANUM (1620). Livre I. § 3. : "La science et la puissance humaine si : correspondent dans tous les points et vont au même but; c'est l'ignorance où nous sommes de la cause qui nous prive de l'effet; car on ne peut vaincre la nature qu'en lui obéissant; et ce qui était un principe, effet ou cause dans la théorie, devient règle, but ou moyen dans la pratique." In *Œuvres philosophiques*, trad. Buchon. Paris 1836, p. 272.

[14]EKELAND I. *Le chaos*. Dominos - Flammarion, 1995, pp. 59-60.

• À partir de là, on peut conduire une réflexion éclairée sur *les limites du savoir scientifique*, qui, loin d'aboutir au scepticisme qui dénierait toute crédibilité aux sciences, est une bonne antidote au scientisme.

> Il semble que notre siècle soit celui où la science découvre ses limites : en mathématiques, le théorème de GÖDEL montre que, quel que soit le système d'axiomes adopté, il existera toujours des propositions non démontrables; en physique, les relations d'incertitude de HEISENBERG imposent une limite à ce qui peut être mesuré; et les découvertes récentes sur le chaos montrent que la prévision à long terme d'un phénomène n'est pas toujours possible. Fort heureusement, la découverte de ces limites n'a pas été ressentie par les scientifiques comme une déception, mais elle a agi au contraire comme un stimulant. L'univers apparaît une fois encore comme plus complexe et plus riche que tout ce que nous avions imaginé :
>
> *"There are more things in heaven and earth, Horatio,*
> *Than are dreamt in your philosophy."*[15]

Les activités destinées aux élèves[16] sont dans leur conception interdisciplinaires, alliant toujours travail mathématique et réflexion philosophique, que leur entrée soit :

– philosophique comme dans le dossier sur *la mathématisation du hasard,* qui commence par une recherche sur la notion de hasard pour déboucher sur une étude d'exemples mathématiques conduisant à distinguer les phénomènes déterministes et les phénomènes aléatoires, et à s'interroger sur les limites de validité d'un modèle probabiliste;

– mathématique comme dans le dossier *Chaos et fractales : approfondissement en Terminales S*; il commence par l'étude de la suite définie par $u_{n+1} = 40u_n - 13$ et $u_0 = \frac{1}{3}$ qui permet d'appréhender la notion de chaos et d'aborder les éléments essentiels de sa définition : *il y a déterminisme (il y a des lois) et il y a une extrême sensibilité aux conditions initiales.* Des exemples de phénomènes chaotiques tels que le billard, les trois aimants, la boussole... illustrent cette définition. Ce qui permet de familiariser les élèves avec l'idée que des phénomènes déterministes peuvent générer de l'imprédictibilité, c'est-à-dire l'impossibilité de prévoir leur comportement à plus ou moins long terme, et permet de revenir sur la notion de hasard à l'aide de textes philosophiques : ARISTOTE, BERGSON, LEIBNIZ, POINCARÉ, COURNOT...,[17] et sur sa mathématisation : les probabilités sont-elles toujours le meilleur outil pour mathématiser le hasard ?

Le travail en classe peut se faire en co-intervention des professeurs des disciplines concernées ou simplement en coordination des enseignements hors classe.

[15] HÉNON M. *La diffusion chaotique*. La Recherche n° 209. Avril 1989, p. 498. <Référence de la citation de SHAKESPEARE : *Hamlet* I. 5. 166-167. "Il y a plus de choses sur terre et dans le ciel, Horatio, que toute votre philosophie n'en rêve." Traduction J. HIRSCH in *Œuvres Complètes*, tome 7, Le Club français du Livre, 1968, p. 321.>

[16] Cf. LES CHANTIERS DU CHAOS o. p. II. ÉTUDES ET ACTIVITÉS POUR LES ÉLÈVES.

[17] Cf. ci-dessous : QUELQUES TEXTES DE RÉFÉRENCE SUR LA NOTION DE HASARD.

Quelques textes de référence sur la notion de hasard

Question fondamentale : *le hasard est-il objectif ou subjectif ?* = Existant dans les choses ou simple conséquence de notre ignorance ?

• Un parcours à l'aide de textes :

1. ARISTOTE (-IV° s). PHYSIQUE, livre II, chapitre 5.
Hasard et contingence : l'absence de cause nécessaire.
C'est ce qui échappe à la connaissance ⇒ pas de lois du hasard

- Cependant . . .

> joignant la rigueur des démonstrations de la science à l'incertitude du hasard, et conciliant ces choses en apparence contraires, elle peut, tirant son nom des deux, s'arroger à bon droit ce titre stupéfiant : **La Géométrie du Hasard**.

PASCAL, À LA TRÈS ILLUSTRE ACADÉMIE PARISIENNE DE SCIENCE, 1654.

2. LAPLACE. ESSAI PHILOSOPHIQUE SUR LES PROBABILITÉS (1814-1825).
Hasard et déterminisme : Le hasard ? "... l'expression de l'ignorance où nous sommes des véritables causes." Déterminisme ⇒ prédictibilité

3. COURNOT. EXPOSITION DE LA THÉORIE DES CHANCES ET DES PROBABILITÉS (1843).

Le hasard : **la rencontre de deux séries causales indépendantes**

4. POINCARÉ. SCIENCE ET MÉTHODE, CH. IV : LE HASARD, 1908.

"Petites causes, grands effets . . . "

Où l'on peut reparler sans contradiction des **"lois du hasard"** :

Vers la séparation déterminisme—prédictibilité, et le chaos comme paradigme.

5. BERGSON, L'ÉVOLUTION CRÉATRICE. 1907.

Désordre & **hasard : un ordre auquel nous ne nous attendions pas ?**

La tuile et le passant : "... je trouve un mécanisme là où j'aurais cherché, là où j'aurais dû rencontrer, semble-t-il, une intention; c'est ce que j'exprime en parlant de *hasard*."

The Mathematical School in Catania at the beginning of the 20th Century and its Influence on Didactics

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Abstract

At the beginning of this century, S. CATANIA, M. DE FRANCHIS, M. CIPOLLA, G. MARLETTA and other mathematicians taught at the University of Catania and published textbooks of mathematics at the same time. S. CATANIA wrote textbooks for secondary school, which followed PEANO's formalism in mathematics. His main work "Aritmetica razionale" (1905) simplified a lot PEANO's treatise "Aritmetica generale ed algebra elementare" (1902), which was very difficult to understand. PEANO, PIERI, and BURALI-FORTI appreciated a lot CATANIA's book, but other mathematicians -for example SCORZA and CASTELNUOVO- did not agree on Peanian approach to didactics of mathematics; there was a hard polemic on the subject. CIPOLLA published textbooks on algebra written in a traditional, but rigorous way. Textbooks on geometry were published by DE FRANCHIS and MARLETTA. In his "Geometria elementare" (1901) DE FRANCHIS considered the concepts of point and segment as fundamental, as PEANO had done in his "Principii di geometria logicamente esposti". DE FRANCHIS applied to them the idea of motion and deduced the most important properties of figures. His results were proved by using methods of the theory of transformation groups, developed by Klein and Lie. As DE FRANCHIS did, in his "Trattato di geometria elementare" (1912) MARLETTA posed points and segments at the basis of his theory and then followed VERONESE's approach to geometry.

The content of these textbooks is connected with the researches on foundations of mathematics and on geometry, which were developed in Italy at that time.

1. At the beginning of the twentieth century, the following mathematicians taught at the University of Catania: Giuseppe Zurria, Giuseppe Chizzoni, Vincenzo Mollame, Giovanni Pennacchietti, Giuseppe Lauricella, Mario PIERI and Sebastiano CATANIA. The last-mentioned wrote many textbooks of mathematics for secondary school inspired by the ideas of the Peanian school, which PIERI also belonged to what was the Peanian school, and, what was its impact on didactics?

Giuseppe PEANO (1858-1932), the founder of the so-called Peanian school, studied and taught at the University of Turin; he made fundamental contributions to mathematical logic and in 1891 founded the journal *Rivista di Matematica*, devoted to logic and the foundations of mathematics. In his *Arithmetices principia* (1889a) he stated three primitive concepts (zero, number, and successive of a number) and a group of nine axioms : four axioms describing the symbols N (natural numbers), = (equality) and $a+1$ (the successor of a number), and five further axioms, collectively known as the *Peanian axioms*.

From around 1892, PEANO embarked on a new and extremely ambitious project, namely the *Formulario Mathematico*. In his words :

> It would be of the greatest usefulness to collect and publish all the theorems now known that refer to given branches of mathematical sciences [. . .] Such a collection, which would be long and difficult in ordinary language, is made significantly easier by using the notation of mathematical logic.

This great project involved his whole group : as many as 45 mathematicians who belonged to the Peanian school collaborated on such a program in order to stimulate research in and promote the use of the axiomatic method; among them, we may mention Giovanni Vailati, Giulio Vivanti, Cesare BURALI-FORTI, Alessandro PADOA, Giovanni Vacca and Mario PIERI. According to PEANO :

> Each professor will be able to adopt this *Formulario* as a textbook, for it ought to contain all theorems and all methods. His teaching will be reduced to showing how to read the formulas, and to indicate to the students the theorems that he wishes to explain in his course.

In the foundations of geometry, he published *Principii di geometria logicamente esposti* (1889b). He did not define "point", "plane", and so on in the usual way. Instead he wrote : "The sign 1 must be read as *point*", and so on. PEANO did not prove the independence of his axioms, and coherence was an unimportant question for him. He followed Pasch's ideas in geometry and considered two primitive concepts : "point" and "segment". He proposed to deduce all the principal statements of geometry of position from such primitive concepts and some axioms inspired by Pasch's axioms. In his paper "Sui fondamenti della geometria" (1894), PEANO developed his previous ideas on geometry by introducing the notion of "congruence", which is connected with the concept of motion (two figures are congruent if a motion exists which changes one into the other). Such a concept is considered as a special affinity, characterized by suitable axioms. A relevant role is played by the concept of "flag" (a point being the origin of a half-straight line, which bounds a half-plane); a "motion" is defined by a transformation carrying flags into flags.

The Peanian school was active in didactics, and textbooks of mathematics were published by BURALI-FORTI and PEANO himself. In the pamphlet *Sulla questione III proposta dalla Mathesis* (1898) BURALI-FORTI remarked that Italian textbooks often express themselves wrongly, criticizing some recent books of practical arithmetic and elementary algebra for mathematically

unrigorous definitions such as :
"*Quantity* is everything which can be increased or decreased";
"*Unity* is each *object*, concrete or abstract, considered as isolated";
"*Number*, in general, is both the unity (number *one*) and the aggregate of unities";
"Numbers can be distinguished in *concrete* (*three horses. . .*) and in abstract (*two, three,. . .*)".

Polemics also arose between the Peanian school and other authors. For example, Pacchiani strongly criticized the textbooks published by BURALI-FORTI and Ramorino (see references). In these books, indeed, numbers are symbols, while the concept of number is introduced by means of the Peanian axioms and is not deduced from the definition of geometrical quantity. Pacchiani, in his review in *Periodico di matematica* (anno XIII, 1898, p. 201) argued for the following procedure in didactics: "to proceed from concrete to abstract"; therefore he could not agree with the point of view of BURALI-FORTI. BURALI-FORTI's answer was immediate: by a well-known logical principle "*number = abstract number*" and the science of logic has, as a consequence, that "*concrete numbers do not exist*", that is to say: "*concrete number = nothing*" (see *Periodico di matematica*, anno XIII, 1898, p. 230).

In 1903 PEANO himself published a textbook for secondary school, *Aritmetica generale ed algebra elementare*, where he used mathematical logic as an instrument for elementary arithmetic and algebra. According to him, symbols summarize language and are clearer than language; in fact, his textbook is written in a style that is quite incomprehensible for pupils. In PEANO's opinion, *Formulario* should serve as a treatise for University and his *Aritmetica generale* is the analogous handbook for secondary schools.

During the period 1890-1910, the impact of the Peanian movement on school education was weak; the books by PEANO and BURALI-FORTI were really difficult to follow for pupils and teachers, and their diffusion was low.

Sebastiano CATANIA, as we shall see, tried to make PEANO's ideas simpler and more understandable; indeed, his textbooks had many editions were relatively successful. It is remarkable that the attitude of the Peanian school to didactics changed a lot, and in the 1920s some authors belonging to PEANO's group published textbooks where intuition and experiments play a role, even if rigour is not abandoned. For example, in his *Matematica intuitiva* (1924-26) PADOA developpes a concrete, experimental method, far from logic. PEANO shows the same attitude in his textbook for primary school *Giochi di aritmetica e problemi interessanti* (1925), where exercises are interesting, stimulating, intuitive and inspired by every-day life.

Mario PIERI (1860-1913), who taught in Catania from 1900 to 1908, made remarkable contributions to axiomatics, in line with PEANO's ideas. In 1899 PIERI constructed an axiomatic system for Euclidean and Lobacevskij-Bolyai geometry based on two terms: point and motion. In 1904 he presented the first axiomatic system for complex projective geometry based on three primitive concepts: complex projective point, complex line (union of two distinct complex points) and chain (concatenation of three collinear and distinct complex points). In his 1907 paper "Sopra gli assiomi aritmetici", published in *Bollettino dell'Accademia Gioenia*, PIERI simplified PEANO's theory by reducing the number of primitive concepts to two and the number of postulates to four. Such works deeply influenced CATANIA's textbooks on algebra.

2. Sebastiano CATANIA (1853-1946) qualified for university teaching of descriptive geometry in 1883 at the University of Catania and in 1888 became professor of mathematics at the Istituto

Nautico. He started publishing textbooks in 1904, and was possibly influenced by PIERI's ideas on formalism and axiomatics.

In that period there were two main lines in didactics of mathematics: the synthetic or axiomatic method, followed by the Peanian school and by S. CATANIA, according to which rigour is the most important thing and formalism is its expression; and the analytic or intuitive method, according to which mathematics should be taught as intuitively as possible, and geometry should help in understanding arithmetic.

In 1904 CATANIA published his textbook for secondary school *Aritmetica razionale* which, in CATANIA's words, is a translation in simple language of PEANO's *Aritmetica generale ed Algebra elementare*. The reaction was not unanimous: from one side, PEANO and his group supported CATANIA's ideas; from the other, there were mathematicians such as SCORZA and VERONESE who criticized his axiomatic approach to didactics. BURALI-FORTI (*Bollettino di bibliografia* ... **7**, 1904) wrote that CATANIA's textbook is rigorous, neither trivial nor arid, but "very satisfactory"; Natucci (*Il Bollettino di Matematica* **4**, 1905) remarked that CATANIA simplified PEANO's work and published "a book corresponding to the modern didactical requirements better than many other books"; MARLETTA wrote: "Congratulations to CATANIA" in his review (*Periodico di Matematica* (3) **5**, 1907) and PIERI noticed that CATANIA's textbook shows the didactical advantages of Peanian symbolism.

However, a hard polemic arose between CATANIA and SCORZA. Gaetano SCORZA (1876-1939) graduated at the University of Pisa in 1899 and from 1902 to 1912 taught in secondary schools. From 1916 to 1921, he was professor of geometry at the University of Catania. SCORZA was one of the founders of the theory of algebras and wrote on the subject a very good and famous treatise (*Corpi numerici e algebre*, 1921). In *Nuovi Doveri* (fasc. 15 giugno 1908), SCORZA reviewed CATANIA's textbook (1906b) in a very unfavourable way, arguing that CATANIA's book was inapt for any didactical aim and, moreover, full of conceptual and grammatical mistakes and inaccuracies. "It would be necessary to raise objections about not only single proofs, but entire theories", SCORZA wrote. In his answer (1908), CATANIA maintained his argument by quoting some good reviews of his books, and some letters which he received from Michele CIPOLLA, PEANO, Francesco Gerbaldi and others.

Gerbaldi, who was professor at the University of Genoa, wrote in his letter to CATANIA : "I am very glad to see that you took advantage of PEANO's ideas, by popularizing them". And he concluded that teachers could find in CATANIA's book a good source for a rigorous teaching of Arithmetic in secondary schools.

And CIPOLLA wrote :

> I will use your Arithmetics during the scholastic year 1905-1906, since I think that it is till now the only one which meets the School's requirements and needs.

In his letter, PEANO remarked about CATANIA's attempt to draft a textbook based on his textbook *Aritmetica generale*... :

> My Arithmetic [PEANO 1903] purposed to show that it is *possible* to do symbolic Arithmetic at school. Therefore, I support your proposal, that is to say to publish a book more suitable for school...

And in other letters to CATANIA, PEANO wrote : "Your book is a work of art; therefore I can say little about its usefulness...", and:

> I am very pleased to read the new edition of your Arithmetic and Algebra and I am really amazed to see your skill to reduce many theories [...] to a form which is simple and understandable to the public.

In his answer to SCORZA, CATANIA added: "I knew very well that with my work I could strike against secular prejudices" and Professor PIERI, "who is an authority on this subject", remarked that many difficulties could arise for the circulation of his books despite their being judged in a very positive way. Such books are "fruit of many years' of conscientious work on PEANO's writings, beginning with *Arithmetices principia, nova methodo exposita*, Torino, 1889". Anyway, the old editions of CATANIA's books were sold out, and around 1910 his textbooks were a success.

3. Another polemic broke out between Natucci and CATANIA, arising from a very particular arithmetical question; but the discussion very soon involved the different points of view on didactics of mathematics. Was it better to teach rational arithmetic or intuitive arithmetic?

During the meeting of *Mathesis* (April 13th, 1913; see *Bollettino della Mathesis* **5**, 1913, p. 49-51), Ricaldone said, among other things, that CATANIA's textbook has many good qualities, "but they are too elevated for the average intelligence of the young people for whom they are intended". He recommended Palatini's book for secondary schools, which is also rigorous but clearer than CATANIA's handbook. CATANIA (1913a) reports the remark by Ricaldone and points out : "mathematical logic sometimes has a good effect [...] but R. is opposed to adopt textbooks for secondary school which are written in conformity with it": he adds in a footnote : "It is impossible to discuss a statement of this kind; I can just notice that *Mathesis* has no reason to exist if it reports such remarks".

The president of *Mathesis*, Guido CASTELNUOVO answered to CATANIA (1913a, b) that his journal published what authors sent in. And anyway, as a person and not as the president of *Mathesis*, CASTELNUOVO wrote that he was against CATANIA's didactical method. CATANIA's treatises are very rigorous, of course -CASTELNUOVO goes on- but they do not leave any place for experience and intuition. Mathematical logic has an important role in the development of science, but, he concluded, "I would be blind if I did not see that science would not be born by means of logic alone".

In his answer (1913b). CATANIA remarked that in his experience as a teacher pupils' minds start from primitive objects and then build the arithmetic or geometric conceptions by means of a logical process. Some proofs in his treatises can also be neglected; but how is it possible to build a mathematical theory just by using intuition? Giuseppe VERONESE made some considerations about the polemic of CATANIA-CASTELNUOVO (1914); his point of view was a compromise : "rational teaching of geometry must be based on a practical and experimental teaching". An equilibrium between rigour and intuition should be looked for.

It is evident that sometimes the real object of the polemic was not CATANIA, but PEANO. The discussion concerned the formalistic approach of the Peanian school to didactics in comparison with the intuitive method, and also involved a different way of making research in mathematics : the intuitive school of CASTELNUOVO, Federigo ENRIQUES, Francesco Severi in opposition to the axiomatic method supported by PEANO and his group.

4. Between 1905 and 1923 many important mathematicians happened to become professors at the University of Catania. After teaching some years in Catania, they often moved to other Universities. In that period we find at the University of Catania, Giuseppe Lauricella (mathematical physics), Carlo Severini and Guido Fubini (analysis), PIERI, SCORZA, CIPOLLA, and DE FRANCHIS (algebra and geometry). Some years later : Giuseppe MARLETTA (geometry) and Vincenzo Amato (analysis), both graduated at the University of Catania, Mauro Picone (from 1919 to 1924) and Pia Nalli (from 1926 on), two great analysts.

Among them, DE FRANCHIS, CIPOLLA, MARLETTA and Amato published textbooks for secondary school.

Michele CIPOLLA (1880-1947) graduated at the University of Palermo, after studying for one year at the Scuola Normale Superiore in Pisa. He taught several years in a secondary school before being appointed a professor. He was professor of analysis at the University of Catania from 1911 to 1923, and then moved to the University of Palermo. CIPOLLA made fundamental contributions to number theory and to the theory of finite groups as well as to the foundations of mathematics. On the latter subject, CIPOLLA published an interesting treatise (1927) collecting his lectures, which is in line with the well-known *Questioni riguardanti le matematiche elementari* (1912-14) written by ENRIQUES. CIPOLLA wrote some of his textbooks for secondary school with Vincenzo Amato (1878-1963), who graduated at the University of Catania in 1901 and became assistant of algebra and then of analytic geometry from 1901 to 1904. After teaching about thirty years at secondary school, he became professor of analysis only in 1936. At the very beginning of his career, Amato was interested in mechanics and then in the theory of groups, influenced by CIPOLLA's ideas.

We shall analyse the following textbooks by CIPOLLA and Amato, both published about 1920-25 : *Algebra elementare per il ginnasio superiore, per l'Istituto magistrale inferiore e per l'Istituto tecnico inferiore* and *Aritmetica razionale ad uso del corso superiore degl'instituti magistrali*. Here, natural numbers are introduced in a very intuitive way: How many books? How many objects?

"If the set is *single* (or *unitary*, that is to say it is constitutes by a *single* object), we answer : *one*. [...] [We answer] *two* if we exclude *one* object, then *one* object will remain" and so on. "Each of such words expresses a **natural number**, according to a meaning which is denominated **cardinal**". Then CIPOLLA and Amato define "equality" of two sets by means of classes of equivalence (two sets are "equivalent" if a bijective map between them exists; such a relation is symmetric, reflexive and transitive) and give an intuitive idea of "successor of a number", of "series of numbers", of "ordinal numbers", addition and the other operations. Fractions are introduced by employing "geometrical quantities" and by considering "homogeneous" quantities which can be added; for example n times A. $A + A + A + \dots$, is nA. Operations with rational numbers, monomials, polynomials and so on are given in the usual way.

The geometrical approach to rational numbers and the employment of classes of equivalence were adopted also by Michele DE FRANCHIS (1875-1946). He graduated at the University of Palermo, where he became assistant of F. Gerbaldi, and was professor of analytic and projective geometry at the University of Cagliari during the academic year 1905/6. Then he moved to the Universities of Parma (1906-9), Catania (1909-14) and finally Palermo (from 1914). DE FRANCHIS made research in algebraic geometry; his most important work, written together with Bagnera, concerning classification of hyperelliptic surfaces, earned the Bordin prize of 1909 from the Academy of Sciences of Paris.

DE FRANCHIS wrote an interesting textbook of geometry for secondary school published in 1909, but drafted in 1901, in which he developed a logical-deductive approach. In it, teachers have a large freedom; indeed, they can choose between fusionism and separatism and can also free pupils from the weight of certain proofs. One reads in the preface to this book:

> It must be pointed out that, in this book, plane geometry and solid geometry are united together, but it is easy to divide them: the book is indeed drafted in such a way that it is possible to treat the planimetry apart.

The book starts from two primitive elements : *point* and *segment*, in line with PEANO's ideas. From point and segment DE FRANCHIS defines all the fundamental concepts of geometry, and the definition of a plane is given by using the notion of *shadow*, as PEANO did :

> Let H and K be two figures. We call the shadow of K with respect to H the figure made by the prolongations of the segments connecting H's points with K's points; such prolongations of the segments must be considered from the side of K's points.

Then :

> A half-plane is the shadow of a straight line with respect to a point outside it.

The problem of "equality between figures" was one of the most important in didactics of mathematics of that period. Euclid considered equality as both congruence (or superposition) and equivalence (or equiextension). In order to define the congruence between figures, Euclid used the concept of rigid motion; such an approach was much criticized, even if it is postulated in the textbooks of Sannia and d'Ovidio, Faifofer, and others. Regarding the equality of figures, there were two main approaches : the *congruence* according to Hilbertian axiomatics, based on the primitive concept of the *congruence between segments and angles*, and the notion of *equality*, based on the theory of groups of transformations developed by Klein in his Erlangen program.

DE FRANCHIS chose the latter approach. He considered what we call today the group of direct isometries of space and developed geometry starting from such a group. DE FRANCHIS introduced "motion" as an element of the group of "rigid" transformations; thus a motion cannot transform an element into a proper part of it (for example a segment into a part of that segment, an angle into a part of that angle, and so on), and a motion exists which maps a segment AB onto a half straight line a, in such a way that A (or B) coincides with the origin of a and B (or A) lies on a. If a triangle has a side which lies on the origin of a half-plane, then a motion exists which leaves such a side unchanged and maps the triangle onto the half-plane. If three points of a figure -not belonging to the same straight line- are unchanged under a motion, then such a motion is the identity.

DE FRANCHIS's approach is very modern, in particular in defining equality between figures by using the theory of transformation groups.

In his *Complementi di geometria ad uso degli istituti tecnici*, written during his professorship at the University of Catania, there is an original treatment of figures in space; all figures, according to DE FRANCHIS, belong to space. He introduces the group of equality, which is nowadays called the group of direct isometries, and proves the following relevant theorem : Inverse isometries can be reduced to the composition of a symmetry with direct isometries. Since similarities -direct and inverse- are bijections of space on itself, DE FRANCHIS characterizes them as follows : Inverse similarities are the composition of a symmetry with respect to a point by direct similarities; inverse plane similarities are the composition of a symmetry with respect to an axis and direct similarities.

Interesting textbooks were also published by Giuseppe MARLETTA (1878-1944), who graduated at the University of Catania in 1901 and became professor of projective and descriptive geometry in 1926, after teaching about twenty years in secondary schools. He devoted himself to projective geometry in spaces of n dimensions; some of his results on algebraic geometry are nowadays revalued and very well considered. MARLETTA was much influenced by PIERI, whom he considered as his teacher.

In his textbooks, MARLETTA tried to write rigorously, but "in a really simple way for young pupils". His most relevant textbook is *Trattato di geometria* for secondary schools, whose first edition was published in 1911. Many editions of it were published, some of them after MARLETTA's death. In his treatise, MARLETTA considers fundamental concepts (such as points, straight lines and planes), which are already well-known by pupils in an intuitive way, that is to say as a result of very simple observations. He organizes such concepts in order to constitute the so-called "rational geometry", and posits many postulates which are not independent one of the other, since some of them are theorems. MARLETTA indeed aims to make his treatment as simple as possible, and assumes some theorems as postulates if their proofs are too difficult for pupils.

In his axiomatics, MARLETTA was influenced by Euclid on one hand and Hilbert on the other. For example, the axiom "Distinct points exist" is followed by the remark "a very little object, such as a grain of sand or the point of a needle, gives us a rough idea of what a point is".

Very original is the definition of parallel lines :

> Two straight lines are parallel if a half-plane exists which has one of them as origin and contains all points of the other one.

Such a definition allows him to develop the theory of parallels in a very simple way. As regards to equality, MARLETTA uses bijections (two figures are equal if a suitable bijection between their points exists), but he also gives the intuitive idea of equality as the overlapping of figures.

5. Around 1900-1910 several important mathematicians worked at the University of Catania and contributed to create great interest in mathematics; as a result, in 1921 a group of mathematicians founded the Circolo Matematico of Catania. Picone, SCORZA, and CIPOLLA had the most important roles in the foundation of the Circolo, and many young mathematicians, such as Niccolò Spampinato, Giorgio Aprile and Giuseppe Fichera, helped much in the organization. Unfortunately, some years after the foundation of Circolo, SCORZA moved to Naples, Picone to Pise, CIPOLLA to Palermo and the Circolo broke up.

The Circolo Matematico of Catania was devoted more to didactics than to pure mathematics; in addition, CIPOLLA and DE FRANCHIS, and later MARLETTA and Amato, had taught for many years at secondary school, before being appointed professors at the University. Therefore, didactics of mathematics was considered an interesting field at the University of Catania, where many professors of mathematics published textbooks for secondary school.

But what about the relevance of their textbooks in Italy? Sebastiano CATANIA's textbooks were adopted in schools till about 1920; then the formalistic approach was abandoned. The textbooks

published by CIPOLLA, AMATO and MARLETTA had many editions; MARLETTA's *Trattato di geometria* was published again in 1946, edited by Aprile. DE FRANCHIS's books -written just before Gentile's reform of 1923- had little circulation after the reform; DE FRANCHIS did not devote himself to making new editions of his textbooks, which became obsolete very soon with respect to scholastic programmes.

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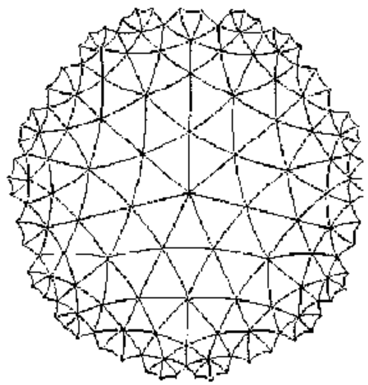

La Géométrie d'Oronce à l'attaque

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Abstract

Imaginez que vous soyez privé de tableau, craies et salle de classe; ce serait peut-être une catastrophe ! Les professeurs de Mathématiques ont peu le souci de la pratique (nos exercices d'application n'en sont pas) et, bien que "le grand livre de la nature etc.", ils seraient démunis en dehors de leur salle de cours. Ce n'est pas le cas à la Renaissance : de nombreux livres de géométrie sont divisés en "théorie" et "pratique", comme celui d'Oronce Fine (traduit en 1570). Les problèmes abordés sont ceux de la mesure d'objets distants ou inaccessibles; il est presque exclusivement fait usage du théorème "de Thalès", à l'aide d'instruments de mesure des angles.

L'usage du "baston pour mesurer" m'a permis, pour une fois, d'emmener les élèves à l'extérieur de l'école, pour mettre en pratique des connaissances jusque là assez abstraites.

Aussi-tôt qu'il y aura quelques Officiers suffisamment formés en Géométrie, le Maître de Mathématiques se portera de temps en temps avec eux sur le terrain pour les faire opérer ; ainsi ceux qui ont déjà quelque commencement, se confirmeront dans ce qu'ils sçavent, & apprendront par la suite ce qu'ils ne sçavent pas.[1]

Comment une *Géométrie*, fût-elle d'Oronce, peut-elle passer à l'attaque ? Et à l'attaque de quoi ? On ne se laissera pas impressionner par l'extrait de l'Instruction de 1720 donné ci-dessus : il ne s'agira pas de mathématiques militaires; la géométrie dont il est question ici a ceci de commun avec l'instruction des Officiers qu'il s'agit de trouver un rapport entre la table (support de la théorie) et le terrain (support de la pratique.)

Seconde question : comment une géométrie, fût-elle à l'attaque, peut-elle être d'Oronce ? Car Oronce Fine n'est pas un grand inventeur, et il est toujours délicat d'attribuer la paternité des notions mathématiques tant les querelles ont été vives sur ces sujets à toutes les époques; on reconnaît souvent des prémisses, les auteurs eux-mêmes avouent leurs dettes. Enfin, pour ce dont il est question ici, pas de grande invention, plutôt des pratiques anciennes couchées sur le papier. Oronce ne fut fier que d'un résultat : sa quadrature du cercle. Mais le mathématicien portugais Pedro Nuñes montra très vite qu'il s'était trompé, et devant l'aveuglement du professeur royal vieillissant, il publia en 1546 l'humiliant *De Erratis Orontii Finaei*, qui devait ridiculiser notre auteur pour toujours. Si le texte d'Oronce Fine peut être vu comme *sa* géométrie, c'est avant tout par son *style*.

Notre but n'est pas tant de réhabiliter Oronce que de montrer que tout texte ancien peut donner lieu à des activités en classe, surtout lorsqu'il se veut *pratique*. Puis, quand même, que la postérité est injuste en ne voyant qu'un médiocre calculateur en Fine. Citons la *Biographie* de Michaud : *Tel, à la faveur des connaissances actuelles, s'est acquis la réputation d'habile géomètre, qui n'eut peut-être pas outrepassé les travaux d'Oronce sous François 1er.*[2]

Que sait-on de l'auteur ?

Selon Emmanuel Poulle, *Fine était considéré comme un des plus grands savants du Royaume, opinion d'ailleurs conforme à l'idée qu'il se faisait de lui-même, et renforcée par sa nomination à la première chaire scientifique du Collège Royal, celle de Mathématiques.*[3] On appréciera le petit coup de pied donné au passage...

D'une manière générale, le ton des biographes suit l'époque : au temps d'Oronce Fine, il est plutôt flatteur, comme sous la plume d'André Thevet[4] qui cite cet *Archimede Dauphinois*, qui *par inclination naturelle s'adonna entre autres aux Mathematiques qui pour lors estoyent rares & comme ensevelies.* Thevet affirme même que *les Mathematiques eussent un fort long temps croupy en un pietre & pitoyable estat si du pays du Dauphiné ne fut sorti un Fine qui les eut affiné.*

Au siècle suivant, Nicéron n'est pas moins flatteur, mais à partir de Montucla[5], il en est tout autrement. Le ton devient presque méprisant (Montucla ne s'occupe certainement que des vrais mathématiciens, alors que les auteurs anciens font l'éloge d'un scientifique célèbre pour son action en faveur des mathématiques) : il fut, ainsi que Charles de Bovelles *fort au dessus de sa réputation*. Montucla reconnaît que Fine *ne fut pas inutile au rétablissement des mathématiques*, mais écrit deux fois plus de lignes au sujet de ses détracteurs Butéon et Nuñes, de vrais et solides géomètres, eux. L'époque n'est pas à une histoire "sociologique" des sciences, à l'étude de ce qui s'est vraiment fait dans l'enseignement des mathématiques, mais à la glorification des inventeurs; Oronce n'est pas de ceux-là, et lorsqu'il le croit, ce n'est pas à son avantage...

[1] *Instruction pour les Ecoles des cinq Bataillons du Regiment Royal Artillerie*, (Ordonnance du 5 février 1720, Instruction du 23 juin 1720), citée dans *Mémoires d'Artillerie, Recueillis par* M. SURIREY DE SAINT REMY, *Lieutenant du Grand-Maître de l'Artillerie de France*. Troisième édition. Paris, M. DCC. XLV. (t. I, p. 57).

[2] *Biographie universelle ancienne et moderne, ..., nouvelle édition, publiée sous la direction de M. Michaud*, Paris, C. Desplaces, 1854.

[3] Oronce Fine et l'Horloge planétaire de la Bibliothèque Sainte Geneviève. *Bibliothèque d'Humanisme et de Renaissance*, Travaux et documents, t. XXXIII, Genève, Droz, 1971.

[4] *Les vrais pourtraits et vies des hommes illustres, grecz, latins et payens recueilliz de leurs table au livre, medailles antiques et modernes.* Par André Thevet, angoumoysin, Premier Cosmographe du Roy. A Paris, par la veuve I. Kervert et Guillaume Chaudiere Rue St Jacques 1584.

[5] Jean-Etienne Montucla, *Histoire des Mathématiques*. Paris, Agasse, An VII-An X. Part. III. Liv. III., p. 574.

Le sommet est atteint avec D.E. Smith[6], qui semble faire de l'Histoire comme on donne des bons points à l'école, ou dans les tribunaux ! En effet, Oronce est rangé dans la catégorie des "écrivains mineurs", ce qui n'est pas très grave, mais qualifié de "l'un des mathématiciens les plus prétentieux de son temps et l'un des moins habiles", voilà qui confine à la calomnie. Néanmoins, dans un chapitre consacré aux instruments de géométrie[7], Smith utilise à plusieurs reprise des illustrations des ouvrages incriminés (la *Protomathesis* de 1532 et le Traité de géométrie pratique de 1556), ce qui prouve qu'Oronce était au moins digne d'illustrer le livre de Smith, ou qu'il est peut-être plus facile (et plus rapide) de lire les images que de s'attacher au contenu du texte...

Sortons de la polémique pour donner quelques indications biographiques : Oronce Fine est né en 1494 à Briançon, mais ayant perdu son père assez tôt, il part étudier à Paris. Les mathématiques sont fort peu prisées mais l'intéressent au point qu'il les enseigne au Collège de Navarre à partir de 1516. Une sombre histoire (une de ses prédictions astrologiques aurait déplu, ou il aurait été arrêté au moment de l'opposition au Concordat, ou encore fait prisonnier alors qu'il travaillait pour l'ennemi en pleine guerre d'Italie) le mène en prison de 1518 à 1524 (on ne rigolait pas à l'époque), puis sa réputation de scientifique s'accroît tant que François 1er le nomme Professeur Royal[8] en 1530. Il publie de nombreux ouvrages, dont il donne plusieurs versions (les textes changent peu mais les illustrations sont totalement revues), ce qui ne suffira pas à en faire un homme riche, puisqu'il meurt totalement désargenté en 1555. On est peu de chose...

Le texte que nous avons étudié fait partie de sa *géométrie pratique*, d'abord publiée en 1532 dans la *Protomathesis*, somme de ses connaissances de l'époque (la prison lui a t-elle permis de réfléchir à ces questions ? Il n'en dira jamais rien) où il voisine la fameuse "quadrature du cercle" qu'il aurait mieux valu qu'il n'écrivît jamais. Une deuxième édition est donnée en 1555 sous le titre *De Re & praxis geometrica libri tres*, (et en partie dans *La composition et usage du quarré géométrique* en 1566) traduit ensuite intégralement en français par Pierre Forcadel, son successeur au Collège Royal[9].

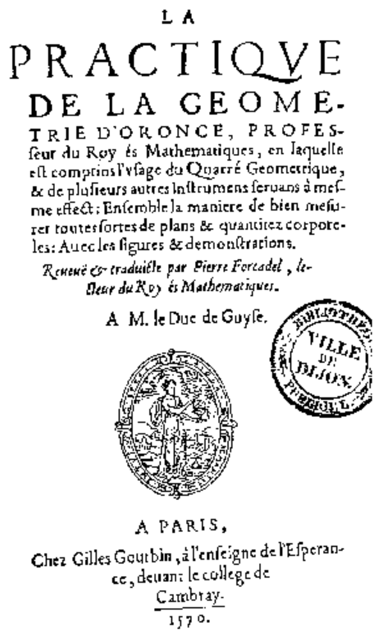
[6]David Eugen Smith, *History of mathematics*, Dover publications inc., N.Y., 1958. Vol. I, p. 308. C'est nous qui traduisons.

[7]Vol. II, *Special Topics of elementary mathematics*, p. 344.

[8]Qu'aurait dit D.E. Smith ? Que François 1er était un imbécile ?

[9]*La Practique de la géométrie d'Oronce Professeur du Roy ès mathématiques, en laquelle est comprins l'usage du Quarré Géométrique, etc.* Reveüe & traduicte par Pierre Forcadel, Lecteur du Roy és Mathématiques. A Paris chez Gilles Gourbin, 1570.

Le texte


LA PRACTIQVE DE LA GEOMETRIE D'ORONCE, PROFESseur du Roy és Mathematiques, en laquelle est comprins l'vsage du Quarré Geometrique, & de plusieurs autres Instrumens seruans à mesme effect: Ensemble la maniere de bien mesurer toutes sortes de plans & quantitez corporeles: Auec les figures & demonstrations.

Reueuë & traduicte par Pierre Forcadel, lecteur du Roy és Mathematiques.

A M. le Duc de Guyse.

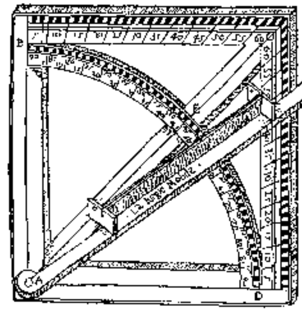
A PARIS,

Chez Gilles Gourbin, à l'enseigne de l'Esperance, deuant le college de Cambray.

1570.

Apparue dans la *Protomathesis* de 1532, la *Géométrie pratique* connut beaucoup de rééditions, ou de refontes, en particulier en ce qui concerne l'usage du "quarré géométrique" et des instruments. Nous avons consulté la première traduction française, due à Pierre Forcadel et parue en 1570 (BM Dijon, cote n° 51119). Ce livre se situe dans la tradition des géométries pratiques et des traités d'arpentage et de toisé en ce qu'il est avant tout un recueil de moyens de mesurer (longueurs, superficies puis volumes). En revanche, il n'est pas question de problèmes de construction, inscription, circonscription ou autres, qui constituent aussi une branche (euclidienne) de la géométrie pratique. Il est remarquable de constater qu'un professeur royal a pu s'intéresser à ce genre de choses, d'autant que presque tous les suivants feront de même.

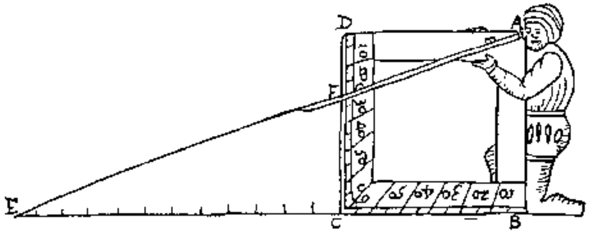
Le début du premier chapitre donne la justification de cet intérêt : *Il y a deux choses, qui en toute discipline, ont de coustume estre agreables, plaisantes & utiles à tous studieux. L'une est la facile introduction à la discipline : laquelle la voye de doctrine & le sens universel explique. L'autre est veüe* [vue] *estre le fruit colligé* [recueilli] *d'icelle discipline, compensateur agreable des travaux entrepris.* Pas de théorie sans pratique, car les fruits du travail doivent être recueillis ! Cela ne manquera pas de nous rappeler que les élèves demandent fréquemment : pour quoi faire ? Et que la compensation agréable des travaux entrepris est le cadet de nos soucis.



Ces choses étant dites, Oronce Fine ne s'embête pas avec des définitions (car il l'a déjà fait dans la *géométrie théorique*) mais explique tout de go la fabrication du premier instrument, le quarré géométrique, qui sert à effectuer des visées en vue d'utiliser des proportions. La *Protomathesis* donnait une illustration sans le quart de cercle intérieur (qui s'appelle *quadrant* et non quarré).

Il est à supposer que l'éditeur, Gilles Gourbin, qui est aussi celui de Forcadel et d'autres professeurs royaux, aura fait une économie de gravure, puisque l'on retrouve cette même illustration telle quelle dans *L'usage du quarré géométrique* de Jean de Merliers (1573). L'instrument est rudimentaire et bien connu à l'époque, il n'est pas de l'invention d'Oronce. Sa différence avec le quadrant est simple : le quarré est fixe (l'un de ses montants est à la verticale) et la règle mobile permet les visées, alors que le quadrant pivote, un viseur étant solidaire d'un de ses montants et c'est un fil à plomb qui donne l'angle par rapport à la verticale. Une utilisation de la 4$^{\text{ème}}$ proposition du livre VI des *Éléments* d'Euclide (un équivalent de notre "théorème de Thalès" ou de la propriété des triangles semblables) donne le calcul de la distance à mesurer.

Par exemple, au chapitre 3 : *Comme sont mesurées les lignes droictes, estendues en une superficie plane terrestre.*

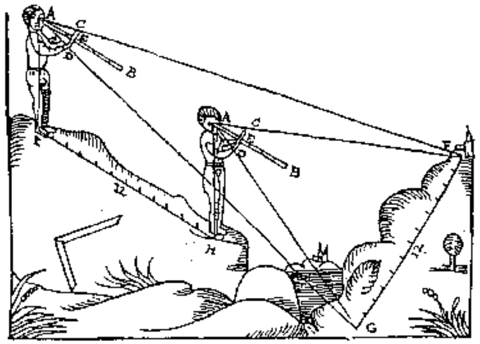


La ligne EB étant la ligne à mesurer, il suffit de poser le quarré sur le sol et d'effectuer la visée. Puis, *telle raison que a le costé du quarré ad, à la partie couppée df, icelle garde aussi la ligne donnée be, à icelluy costé ab*, autrement dit et moyennant une trahison de style, AD est à DF comme EB est à AB, ou encore $\frac{AD}{DF} = \frac{EB}{AB}$, ce qui permet (mais Oronce ne l'écrit pas) de calculer EB, connaissant la taille du montant AB et lisant la graduation sur DC. La démonstration est donnée dans le texte, elle consiste à prouver, par les *Éléments* d'Euclide, que les deux triangles ADF et EBA sont équiangles, donc proportionnels; on voit ici la supériorité de la proposition VI-4 d'Euclide sur la proposition VI-2 (notre bon vieux "Thalès") en termes d'efficacité : allez donc faire reconnaître à vos élèves une "situation de Thalès" dans les triangles ADF et ABE; il faut au moins deux applications du théorème, sous la forme que nous lui connaissons, pour établir la proportion. Reconnaissons que la pensée des triangles semblables est plus intéressante ici et que nos "figures-clés" ne sont pas assez (ou sont trop) nombreuses.

Le chapitre 6 propose *la description d'un autre instrument, avec lequel est obtenuë la longueur des lignes droictes & inaccessibles, & constituées ou eslevées orthogonellement au plan terrestre.* Il s'agit du fameux "bâton de Jacob" :

Le baston pour mesurer.

Le grand bâton est divisé en six parties égales (ou plus) et coulisse à l'intérieur du petit bâton transversal dont la longueur est égale à celle de l'une des parties du grand. La visée est des plus simples : il suffit d'aligner les extrémités C et D du petit bâton avec celles de la ligne à mesurer. Mais ça se corse : une seule visée ne suffit pas, il y aura donc un système de *double visée.*



Le principe est le suivant : l'homme est en H et effectue la visée, le petit bâton ajusté sur une graduation précise du grand. Il décale le petit bâton d'une division et se déplace en I pour pouvoir effectuer une nouvelle visée. Miracle de la géométrie : la longueur de la ligne à mesurer FG est égale à la distance IH ! Le plus beau dans ce texte est qu'Oronce ne fournit pas d'explication. Évidemment, on ne peut s'empêcher d'en chercher une (encore du Thalès) et d'ennuyer les élèves avec cette recherche.

Travail avec les élèves

Comme d'habitude, la première difficulté est pour eux de s'habituer à la typographie. Mais il en vient une seconde : les mots et les expressions sont vraiment plus compliqués que d'habitude (c'est un imprimé plutôt ancien, un des tout premiers traités de géométrie en français.) Ce qui démobilise de prime abord, car on ne peut éluder ce problème pour passer tout de suite au contenu mathématique. Cela peut nous rappeler les difficultés que représente notre propre langage pour nos élèves: la différence entre Oronce et le professeur, c'est que le texte d'Oronce est étrange pour la classe et pour le professeur. Il est donc nécessaire de travailler d'abord sur une partie facile, pour laquelle toute la difficulté résidera dans le sens à donner aux mots : l'introduction ou les instructions pour la construction du quarré peuvent jouer ce rôle de mise en train.

Les deux extraits présentés ici ont été proposés à des élèves de Première (Sciences et Technologie de Laboratoire) et de Seconde (Arts plastiques) du Lycée "Le Castel" à Dijon. Ils ont donné lieu à un travail en classe puis à des séances de mesure. Si la partie théorique a été plutôt poliment reçue mais peu appréciée, les séances de mesure ont été joyeuses ! Ce n'est pas si étonnant, nous sortons assez peu de la classe et nos exercices n'ont qu'un rapport lointain avec une quelconque mise en pratique; en outre, les voyages de classe ou les sorties au cinéma sont rarement le fait du professeur de mathématiques, alors qu'ils sont à marquer de pierres blanches dans la mémoire et l'imaginaire des classes. Il n'empêche que la sortie ne (me) suffisait pas et qu'il était nécessaire de comprendre *a priori* comment les visées allaient permettre l'estimation des longueurs. Tout ceci allait, du moins le pensais-je, convaincre les élèves de la puissance des mathématiques, de leur aptitude à mesurer le monde, donc à le décrire.

Première époque, en salle : lecture du chapitre 3 (voir plus haut). Tout se passe comme prévu, les élèves n'y comprennent rien et certains refusent même d'aller plus loin ("à quoi ça sert ? Vous êtes sûr que ça fait bien partie du programme ?") La démonstration est particulièrement difficile à comprendre, et nous devons y passer du temps. J'aurais pu mâcher le travail en donnant un glossaire et le canevas de la démonstration, mais où aurait été le plaisir ? N'est-il pas normal que la découverte ne soit pas immédiate et qu'elle dérange ? Sans aller jusqu'à laisser les élèves en échec, ne les laissons pas dans l'illusion que tout est facile et que le savoir s'acquiert sans aucun effort. L'élève n'est pas la mesure de toute chose. Allez, n'ayez pas peur: les élèves ont disposé d'un petit schéma et de quelques indications au fur et à mesure de leur avancée (un prof doit savoir rester humain de temps en temps. . .)

Les Premières ont étudié le chapitre 10, dont il n'est pas question ici (mesure de la hauteur d'un édifice à l'aide d'un seul bâton planté verticalement dans le sol, la démonstration leur semblant maintenant abordable (le plus gros avait été fait au chap. 3). Les Secondes ont eu plus de chance encore avec la fin du même chapitre, où Oronce montre comment l'utilisation d'un simple miroir permet d'estimer la hauteur d'un édifice dont le pied est accessible. Les deux classes ont terminé par la double visée, lecture commune, esquisse de démonstration en classe, puis rédaction complète de cette démonstration à la maison.

Deuxième époque, dehors[10]. Les élèves de Première disposaient du bâton (très mal fabriqué, mais tout le monde n'est pas ébéniste[11]) pour mesurer de loin l'espace entre deux pylônes du Lycée (j'avoue : novice, j'avais oublié de demander toutes les autorisations et nous avons été obligés de rester dans l'enceinte de l'établissement). Puis ils sont partis dans les frimas de janvier mesurer la hauteur d'un célèbre obélisque dijonnais à l'aide de la toise verticale, l'autre bâton ne servant à rien ici. Les élèves de Seconde devaient mesurer toutes les cotes possibles du Bastion de Guise (dernier grand ouvrage conservé des fortifications de Dijon) en vue d'une éventuelle reconstitution en maquette (qui ne fut pas construite), à l'aide du bâton, du miroir et du quarré géométrique. L'intérêt d'un bastion est qu'il permet une remise en perspective historique de ces anciennes méthodes : il ne me fut pas difficile de convaincre les élèves que les mousquets des défenseurs les auraient vite décimés s'ils avaient essayé de s'approcher du rempart ! Joyeux souvenir. . .

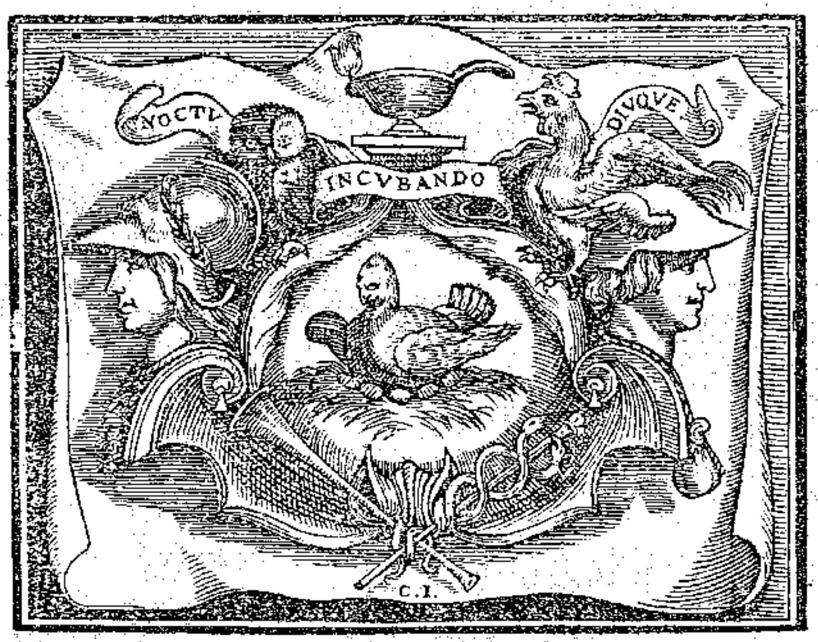
Qu'en penser ?

Par honnêteté, je dirai ce qui s'est passé en rentrant en classe : nous rendant compte de l'écart ahurissant entre les diverses mesures (plus de 20%), nous avons dû discuter de l'attitude à adopter pour décider des valeurs des longueurs. Les rigolos écartés (les deux mesures extrêmes), il a suffi d'un simple calcul de moyenne pour mettre tout le monde d'accord, la méthode des moindres carrés étant encore hors de portée ! Les mathématiques sont toutes puissantes pour mesurer le monde, certes, mais il faut éviter de les appliquer. . .

Quoi qu'il en soit, ce qui me semble évident après ces péripéties, c'est que le théorème "de Thalès" est un grand théorème, à emporter sur l'île déserte s'il n'y en a qu'un. Les élèves conservent très peu de choses de leur passage en cours de maths (sans parler des choses qui seront utilisables), mais l'idée des formes proportionnelles les marquera sans doute pour toujours : ils devaient certainement la connaître avant.

[10] Je reconnais ma dette envers Peter Ransom, dont l'atelier à l'Université d'été de Montpellier en 1993 a inspiré une bonne partie de cette activité à l'extérieur.

[11] Petite fourberie à destination de J.-M. D., de Bruxelles, qui souhaite certainement garder l'anonymat.



Emblème de l'imprimeur Meursius vers 1650, publicité de l'imprimeur Plantin, dû à P.P. Rubens, tiré de page de titre du Tractatus physico-Mathematicus de aestu maris de Th. Moretus publié en 1665.

Language engineering - The outcome of the intersection of Linguistics, Mathematics, Computer Sciences

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Abstract

The computer-based information and communication technologies are providing direct access to each and every ongoing field all over the world through a combination of written text, graphic images and sound. Language will, in some form, be at the core of the screen-based multimedia communications that are likely to become an intrinsic part of life both at the work place and at home.

As information technology keeps growing in different areas, equal opportunities for everyone to access information become a key issue. Clearly languages play an important role in this respect.

For Europe a dual challenge exists: to maintain its linguistic and cultural diversity and to ensure equal opportunities for business and citizens alike to participate fully in and share the benefits of the new information era.

Language engineering is the core of information technology and this will be the key industry of the 21st century. The information super-highways conceived today will soon carry infinite amounts of digital data, images, sounds, tables, figures, calculations and process protocols. If these data are to be intelligible, and make sense, they must be bound together by language. Without natural language processing, information remains incomprehensible.

Language, once a cultural asset only, has now become an economic commodity, too. The national language institutes have acquired new responsibilities. Now their task is to provide information from abroad available in the national language and for locally produced information to be distributed world-wide in major international languages. To train and employ more translators is not sufficient. We must also take care that the necessary language technology is being developed.

1 Current Issues in Language Engineering

Spell checkers were among the earliest successful language technology applications. They have been accepted as useful devices and are still being sold today in ever-improved versions. The majority of early machine translation systems, particularly the more sophisticated ones have not survived. SYSTRAN is still kept alive by the European Commission's translation services, but many others have disappeared without leaving a trace.

Spell-checkers don't need semantics. Even in the seventies they were based on little more than a list of the most frequent word forms, i.e. linguistic knowledge widely available or easy enough to generate from corpora.

Machine translation however needs semantics. Our understanding of word meanings or lexical semantics in the seventies was contained in dictionaries, and it was arranged in such a way that a user with some experience could understand using a great deal of implicit knowledge about the world and inductive reasoning. All assets computers do not possess, such as having the ability to draw analogies. Therefore, there is no surprise that early applications involving semantics were not very successful.

Language technology can work with two kinds of semantic information. One is rule based and presupposes a formal logical semantic analysis of the phenomenon under discussion. The other kind uses statistics and is not really semantic at all: it computes the five or ten words or word forms preceding or following the word in question and relates this information to the different translation equivalents found in parallel corpora. The German word *Schnecke* for example is translated into English either by *snail* or *slug*, where *snail* refers to the creature with a 'house' and *slug* to the one without. The rule based approach states just that and searches the German text for clues from which we can infer the correct translation equivalent. The statistic-based approach does not look at the meaning at all. It looks for words and other traces that frequently occur when *Schnecke* is translated as *snail* and for other patterns co-occurring with *Schnecke* being translated as *slug*. In the context of *slug*, we would probably find words like *vegetable* (garden), *lane*, *wet*, and various forms of *get rid of*, while in the case of *snail*, I would expect words like *table*, *course*, *wine* but also *vineyard* and *sunny*.

Today's successful applications involving semantics work with an amalgam of the rule and the statistic-based approach. The statistic approach has some attractive advantages: the data required can be generated from corpora with no or only little human intervention. As it is just an emulation of semantics, one does not have to be able to state explicitly what a lexical item means. Indeed it leaves out the entire question of meaning. Its inherent shortcoming is the rate of accuracy. Even a rate of 95 translation equivalents implies that every twentieth word in a text is mistranslated, practically, every sentence and certainly more than what most people would like to live with. On the other hand, the rule based approach can be a very expensive alternative. It presupposes something like a bilingual dictionary that would enable translation of a text correctly into an unknown foreign language. The reason why such dictionaries do not exist for human users or machines is that the explicit linguistic knowledge they would have to contain is not yet available and that this knowledge is extremely expensive to produce.

The new generation of language technology applications (monolingual and bilingual or multilingual ones) deals with semantic problems. They recognise the fact that computers cannot understand spoken or written texts in the way humans can. Therefore, these text processing systems can only emulate the human faculty of 'understanding' by a mixture of rules and probabilities. To find the right mix is less a question of theory and principles than of calibration and

learning by doing. The crucial point is the performance of an application under real life conditions. The application has to prove its cost efficiency, i.e. it must demonstrate that it can complete a task cheaper than a trained human. After two decades of experimental and pilot systems, the emphasis today is on robust applications for which there is a real market, i.e. one where users are willing to pay a fair price.

The linguistic formulated knowledge in existing grammars and dictionaries - is unsuitable for language technology tools for two reasons. First most of it is not corpus based; rather it reflects the individual linguist's (lexicographer's) competence based on a collection of data (citations); and however large their collection may be, it is permeated by a bias that cannot be avoided. Secondly, traditional grammars and dictionaries have been devised for human users, who differ substiantially from machines. Human beings use inductive reasoning and can draw analogies easily; faculties like these are taken for granted and are reflected in the traditional arrangement and presentation of processed linguistic data. Language technology tools cannot take recourse to common sense; as in this, all knowledge has to be spelled out in the form of rules, lists, and probabilities. But anyone who has gone to the sources has experienced the problem that when we start analysing language as it occurs in a corpus, we gain evidence that renders existing grammars and dictionaries as very unreliable repositories of linguistic knowledge. We discover that traditional linguistic knowledge gives us a very biased view of language, a view that has its roots in the contingency of over two thousand years of linguistic theorizing. We are so accustomed to this view that we take it as truth, as reality and not just for an interpretation of raw data. It is true that traditional grammars and dictionaries have helped us, fairly satisfactorily to overcome the linguistic problems we human beings have to deal with. But they will not be good enough for language technology applications.

That's why, however cumbersome and expensive it may be, language has to be described in a way that it is appropriate for language engineering. It has been demonstrated that monolingual and bilingual dictionaries are of no (or only little) use when it comes to automatically translating a word from one language into another in cases where there is more than one alternative. To reduce the cost of a corpus based language analysis from scratch, which is indispensable, corpus exploitation tools have to be developed. They will arrange the rough facts (including statistic driven devices for contextual analysis) and process them (with a great deal of human intervention for the semantic interpretation of data) into algorithmic linguistic knowledge and rules derived from objective data rather than individual competence. Perhaps this will result in the finding that traditional categories like noun, verbs and adjective do not, after all reflect categories useful for language processing.

2 Technological Processing of Romanian Language Data

Technologization of Romanian language is only at the beginning in spite of the fact that research has been carried out in this field for a long time. Most of the research was focussed on written language as well as speech. The first attempts to process written language started in the sixties. At that time a group of young mathematicians and engineers under the guidance of Erica Nistor Domokos started at the University of Timisoara a project on machine translation. Supported by the famous mathematician Grigore Moisil the group implemented in 1963 a prototype system of translation from French and English into Romanian. In spite of the outstanding results for the technology of that time presented in public demonstrations the lack of interest on behalf of authorities led to the abandoning of the official research in the field of machine translation.

After 1990, there was a tremendous increase of the possibilities of information and free circulations. The easier access to hardware and software advanced technologies has a positive impact on the technological field of language in general and technologization of Romanian in special. The international programmes of academic exchanges, the research stages, participations to important conferences allowed for adequate information of Romanian researchers on the advances and use of modern technologies for international and national priorities.

Linguistic engineering research focused on formalisms based on unification, on reversible methods of natural language processing. Modern linguistic theories, with computational relevance are used as instruments by more and more research groups consisting of computer sciences specialists and linguists.

We are glad to acknowledge introduction of computational linguistics and modern linguistic theory courses in the curricula of several higher education institutions in Romania (The Computer Sciences Faculty of Al. I. Cuza University of Iassy, Computer Sciences Faculty of Babes Bolyai University of Cluj-Napoca, Computer Sciences Faculty of the Department of Mathematics of the University of Bucharest, Foreign Language Faculty - Department of English of the University of Bucharest, Computer Faculty and Electronics Faculty of the Polytechnical University of Bucharest, Computer Faculty of the Technical University of Timisoara)

We have several national priorities set up in this filed regarding:

- development of computational linguistics.
- computer aided acquisition of concepts of national and European culture.
- language technologization (the setting up of computational linguistic resources.
- computer aided translation, foreign languages teaching, artistic languages presentation.
- development of multilingual computer aided services.
- development of authorial systems (systems capable to intelligently assist the utilizers to create and manage complex documents).

In an attempt to assess the most developed areas of language technology in Romania which could constitute a standing point for the later development of Romanian language technologization, the following areas have to be pin pointed:

- morpho-lexical and syntactic processing instruments (analysers and generators).
- vocal signal processing instruments (recognition, synthesis, prosody).
- spell-checkers.
- question/answer systems (friendly interfaces) on applications.
- almost complete descriptions of the Romanian morphology, both paradigmatic and derivational.
- partial syntactic descriptions of Romanian.
- vocabularies.
- dictionaries for computer-aided processing of written language.

- terminology thesauruses for various linguistic registers.
- specialises corpora.

3 Language Engineering Applications

In support of the ideas presented above I am going to provide some detailed examples of how language technology actually works in applications. I will start by introducing several facts concerning machine translation as this application is presented by Rajmund PIOTROWSKY.

The model designed to develop and implement a computer simulation of a psycho-linguistically realistic model of language behaviour is named 'linguistic automation' and consists of a hardware, an operating system computer program for natural language processing, a vastlinguistic database.

According to PIOTROWSKY, 'linguistic automation' may be conceived as a level based system. The first level is the 'linguistic information database' including linguistic and encyclopedic data with their probabilistic weights marked. This level comprises an automatic dictionary (structured as a lexicon providing information about word-forms, stems, phrases), lists of grammatical affixes, toponims, anthroponyms, abbreviations, etc. It also comprises organisation and updating programs. The second level comprises a set of functional modules (M) consisting of two subsets M_a (incorporating the analysing modules) and M_s (unifying generating modules).

$$M_a = (d, c, l_k, l, m, L_k, L, s_1, s_2, s_3)$$

where

d = module of graphemic text decoding
c = speller
l_k = module of lexical analysis of the key lexical items
l = module of lexical analysis of the lexical items in the text
m = module of autonomous morphological analysis of text words
L_k = module of lexical-morphological analysis of key lexical items
L = module of lexical-morphological analysis of text words
s = module of surface structure analysis
s_2 = module of deep topic-comment analysis of the sentence
s_3 = modules effecting semantico-pragmatic analysis of the text

$$M_s = (k, c, l', L', s'_1, s'_2, s'_3)$$

k = module of encoding
c = speller
l' = module of lexical items generation in the text
L' = module of lexical-morphological generation of lexical items
s'_1 = module of surface structure generation of the output sentence
s'_2 = module of syntactic-semantic generation of the output sentence
s'_3 = module of semantic-pragmatic generation of the target text

The third level comprises some concrete systems and subsystems of natural language processing such as: the systems and subsystems assigning 1) the analysed text to a certain language 2) word-by-word and phrase-by-phrase machine translation 3) rough lexical and morphological machine translation 4) semantic-syntactic machine translation 5) topic-comment machine translation of headlines and book titles 6) text fragmentation, compression and abstraction.

The forth level is carried out through incessant human being - computer interaction.

Taking into account that any natural language processing system has to deal with indeterminacy conditions under the form of a set of alternatives present in the database and in the algorithm blocks, language automation is endowed with an artificial brain module which has to select the appropriate decision. Similar to other control and management systems the decision making body of the linguistic automation can be described as a hierarchic structure with three levels: 1) self organisation 2) adaptation of linguistic automation to the given texts 3) selecting a suitable decision for a concrete task. The third level is very important for the development of linguistic automation and methods to work out the faults due to engineering and linguistic limitations appearing in the linguistic automation have already been put into algorithms. The methods may be interesting for specialists.

The existing polyfunctional natural language processing systems are not perfect linguistic automata. Man still plays an important part in maintaining feedback in man-machine dialogue and the interaction of linguistic automation modules. The role of man is greater in the higher linguistic automation levels, providing functional decision and synergistic organisation, than in lower, more primitive blocks. In the future, the main efforts of researchers are expected to concentrate on extending linguistic automation decision possibilities.

In order to implement various applications of natural language a lexicon is needed containing phonetic, morphologic, syntactic etc. information. The example discusses hereafter concentrates on morphological aspects of vocabulary as seen by S. COJOCARU who solves this problem appealing to two methods: static and dynamic. The standing point of the first method is the classification of Romanian language words into inflection groups. The author makes use of a 30,000 words dictionary, each word being attributed to the respective group number. To obtain a formal description of the inflection process in accordance to the above mentioned classification there have been formulated grammars with dispersed context containing rules of the form:

$$[/ | * [\#] [N_1] a_1 \urcorner b_1 a_2 \ldots a_{n-1} \urcorner b_{n-1} a_n \rightarrow a'_1 \urcorner b_1 a'_2 \ldots a'_{n-1} \urcorner b_{n-1} a'_n N_2$$

where a_i, a'_i are arbitrary words (taking into account the algebraic sense of the notion word), b_i is a non-empty word or a reserved symbol $*$, N_j indicates an inflection subparadigm. The rules are to be interpreted as follows: be w a lema word. Each sign "/" indicates a letter, which must be cut from the end of the word w. The word v, obtained from w after the letters at the end have been cut, is a stem (if N_1 exists), and N_1 is an index denoting the inflectional subparadigm containing the affixes which will be attached to the stem v to obtain the respective derivates. The substitution rule can be applied if $v = f_0 a_1 f_1 a_2 f_2 \ldots a_{n-1} f_{n-1} a_n f_n$, where f_i is an arbitrary word (including an empty one) which does not contain the words "forbidden" b_i. If there is more than one representation of this kind of the word, v is selected the first (scanning from left to right, if sign $\#$ is not present, and in the opposite direction in the opposite case. $b_i = *$ does not impose restrictions on f_i. In the indicated context parallel substitution takes place: $a_1, a_2, \ldots, a_n \rightarrow a_1, a_2, \ldots, a_n$, a new stem v being obtained, to which the affixes in the subparadigm N_2 are attached.

Thus in order to generate the inflection of a word knowing its group number it is enough to

interpret the respective rules. One or more grammatical rules can correspond to a group number. To describe the inflection system of Romanian verbs, nouns, adjectives, articles, numerals and pronouns 866 rules have been found necessary making references to 320 subparadigms. Using in some cases scanning of the lema-word from right to left we can use simple rules, context free or having only substitution contexts (not the "forbidden" ones).

For the derivation of the words that are not in the inflection groups the dynamic method is used. The inflection programs initiate a dialogue to determine the part of speech, gender (for nouns) and other additional information for more difficult cases. To carry out morphological derivation it is necessary to find out: a) the vocalic and consonantic alternances, b) the application context of alternance rules, c) affix series. The tables containing affix series, alternances and their admissible co-occurrence are the basis for the functioning of inflection programs. As the irregular words are seldom, they are described a priori and processed in a special way. Obtaining the inflections of a word we can determine the number of the inflection group thus reducing the presentation of all the words in the lexicon to the static method.

The procedures described above have been used to set up a lexicon of approximately 65,000 words which served as basis for the realisation of a spell-checker for Romanian (ROMSP). ROMSP is implemented in Borland Pascal with Objects 7.01 for MS DOS having the following components: a) the spell-checker program with a friendly dialogue, very simple even for the unexperienced utilizer; b) the morphologic derivation program of the Romanian words, necessary for updating the database; c) database management program which carries out compactization of vocabulary, its integrity upgradation and control provides similar words to the given one.

In the implementation of the vocabulary two goals have been followed: a) diminution of the database volume; b) efficient access to the items in the base.

The vocabulary is divided into pages, the table of access to pages being a hash-function stored in RAM thus diminishing considerably the number of addresses to the hard disk. Each word is presented by three components: 1) the first two letters, stored separately (elements of a page are words which begin with the same two letters); 2) the rest of the stem (theme); 3) reference to the valid affix set for this stem.

The vocabulary occupies 1.2 Mb, the checking speed being of approximately 100 words per second on a very simple computer (IBM PC 286, 12 MHz).

Last but not least I would like to bring about some issues of current interest in the development of speech technology which can be considered as one of the most important fields using interdisciplinary skills.

Speech processing is a field encompassing a great variety of technologies and applications. Many of these applications such as automatic recognition and synthesis have become traditional as outcomes of several decades of intense research. Some others although are less known or more recent, but they are important and useful. In spite of the advances in this field the outcomes are still far from what they should be. The tasks initially set up proved to be extremely difficult in time. The causes are complexity of the vocal signal as well as the difficulties encountered in its processing linked either to the recognition of its informational content (the vocal signal strongly depending on the speaker and the context of a message) its production, the transmission of this signal at distance.

In the field of speech technology, automatic speech synthesis and especially text-to-speech synthesis has a special place since it can play an important part in the man-machine interface. A text-to-speech synthesis system can provide an important range of applications in many fields from electronic mail access to various databases through standard conventional communication networks to reading systems for the blind. I would like to present just a part of a larger research carried out by an interdisciplinary research team in the Military Technical Academy in Bucharest. This team worked on a text-to-speech synthesis based on the following general text-to-speech synthesis system which has two main parts: one dealing with linguistic processing which turns input text into a phonetic and prosodic representation. The other is one of acoustic processing which generates the speech signal using specific techniques for the type of synthesis and the acoustic units chosen.

A general schemata of a text-to-speech synthesis system comprises several levels aiming at providing speech similar to natural speech. The input of the system is the text and the output is the speech.

The input message can come from various sources, utilizing various writing protocols. Linguistic processing must convert the input letters into a form that can be appropriately processed. This stage of the synthesis must detect free spaces, sentence boundaries or sometimes the end of a text sample. Next the capital letters must be detected (both the ones at the beginning of a sentence and the ones that may come up later in the text) and the most usual abbreviations must be conveyed into normal writing. The protocol at this stage must also process numbers, integers, decimal numbers, hours and dates. At the same time punctuation must be interpreted which is to be used to establish prosodic features. A morphological analysis is carried out here to each word being assigned a grammatical category; this is a result of the fact that one word may be pronounced differently in accordance to the part of speech it represents. Accents may also be analysed at this stage. The output of this module will be a sequence of letters making up words with a grammatical description accompanied by accents and other prosodic items.

The next stage converts the text into phonemic symbols which describe the way the letter sequences are to be pronounced. The schemata uses phonemic units but obviously there can be used other units making up words - syllables for example.

A note should be made of the fact that most synthesis systems, irrespective of the basic unit chosen always makes use of a hybrid system to obtain such units: a number of rules are used which take into account pronunciation and also a dictionary comprising "exceptions", i.e. words for which rules cannot be established in the regular way.

The phonetic transcript will produce as output a row of characters representing all the allophones (the pronunciation variants of the phonemes) -in case we have used phonemes as basic units- accompanied by quantitative values (duration of the acoustic units, basic frequency and its variation) which are calculated by the prosody generation module providing the intonation and accent of the words spelled out.

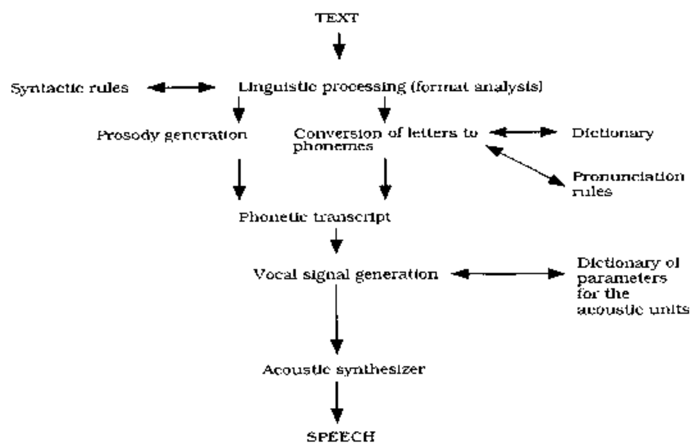
The last two modules accomplish synthesis of speech through the following operations: - information output from the previous module is decoded: - the dictionary provides the parameters corresponding to the acoustic units; - signal samples are generated using for instance LPC synthesis; - the segments thus obtained are concatenated: - the final wave form is synthesized by the acoustic synthesizer (D/A conversion, filtering, amplification).

This general system has been implemented on IBM PC using a linear prediction algorithm and here are its most important performances: compression rate 2.4kbs, real time synthesis (approximately 10% of the speech time of a statement for a 486 PC at 66MHz), good intelligibility and naturalism. The characteristics of the voice on which analysis has been carried out, previous to

undergoing synthesis, can be easily recognised.

We established syllables as the basic elements for synthesis, these segments in our opinion, offer a good compromise between the number of basic units and the number and complexity of concatenation rules for them. We still have to study prosody more thoroughly since problems are raised by the fact that, at least in Romanian, the accent is free and there are far too many exceptions to the rule. This is a problem that needs further research.

General schemata of a text-to-speech synthesis system



The conclusion of the above presented applications is that sciences such as linguistics, mathematics, computer sciences work together in achieving technological support for the future development of the global society.

The applications and facts about language engineering are pointing towards the "holistic-integrative" characteristic of modern science.

4 Some Epistemological Considerations

I would like to conclude this presentation with some considerations concerning the "epistemological" status of contemporary science, starting from some ideas offered by the work of Ilie Pârvu.

The recent sciencific development fundamentally modified one of the parameters determining the setting up of epistemological methods. Generally speaking the classical epistemological systems proceeded by taking into account a single scientific subject whose presuppositions they tried to make explicit through universalization, and based on such grounds, they defined an abstract concept of "science". The simultaneous maturation of a great number of scientific subjects nowadays generated new centres of "methodological diffusion" and philosophical problematization within science. If the "philosophical part" played by a scientific subject essentially depends on its insertion in the theoretical configuration of a certain period, then nowadays we are witnessing a multiplication of the philosophical centres of problem defining of science.

In the past, mathematics and physics (and sometimes biology), represented the "paradigms" of knowledge, providing methods and reasoning models for all subjects, at the same time providing "the extraction field" of norms and principles defining the very essence of "scientificity". In our century various other branches of science have entered the "theoretical stage", the spiritual configuration of contemporary science looking more and more similar to a "complex constellation" of subjects with methods, techniques, tools and non-homogeneous conceptual systems.

Consequently, the generation of a general epistemological interpretation from the study of present day science must be preceded by the setting up of several "regional" epistemologies by defining the status of theoretical knowledge and the methodological specificity of various groups of subjects.

I am going to review several of the most significant theoretical achievements and methodological changes which occurred in several branches of science.

Starting from a taxonomy of science by C.F. von Weizsacker I would start this review with the structural sciences, a field of knowledge creating the abstract tools needed in all branches of science. Within the structural sciences there are included not only pure and applied mathematics, but also systemic analysis, theory of information, cybernetics, theory of games considered to represent the mathematics of temporal processes or "structural theories of temporal changes" whose "assisting instrument is the computer, whose theory is itself a structural science".

The modifications at the level of structural sciences have been influenced by the development of logic. It has essentially influenced "the spiritual configuration" of contemporary science contributing alongside with mathematics both to the setting up and expansion of a way of reasoning (structural-axiomatic) and, since it provides a methodological tool necessary to the philosophical analysis of science, to the foundation of a conception on science comparable as rigour to the highest theoretical achievements of science itself. In addition to that, through its own internal elements logic has become one of the most active centres of current knowledge, generating new philosophical problems, participating in the new reconceptualisations of epistemology.

Summarising the contributions made by contemporary mathematics to the setting up of a modern epistemological conscience the following facts should be mentioned: 1) the setting up and /or operationalisation of several "ways of reasoning" with great applicability, necessary to achieve theoretical domination of complex processes and systems (structural-axiomatic, statistic, strategic, interdisciplinary reasoning); 2) through its self-reflexivity mathematics offers tools for the self-knowledge of science, for the setting up of a theory of knowledge; 3) new developments in mathematics allowed a substantial progress in the understanding of the structure and objectives of scientific theoretization; 4) by its internal evolution mathematics led to outstanding outcomes. They require re-thinking of relationships between formalism and intuitive constructions, empirical knowledge and a priori knowledge, or analytic and synthetic, making important suggestions concerning a general epistemological view on knowledge.

In the field of physics, the central subject of natural sciences, recent research has been developed, against the conceptual background set up by the theories. This influenced the "reasoning style" of contemporary physics, the theory of relativity and quantic mechanics. They have brought about a deep revolution of physic knowledge producing new sub- or supra-mathematical structures (the theory of restricted and generalised relativity), new logical onthological formalisms (quantic mechanics), or even a new "methodologic order" in theoretic construction (cosmology), proposing a new form for the scientific law, new types of theories meeting new completion standards and new criteria of "physic reality".

Out of the recent developments in psychology two research directions are of special interest for epistemology. They offer direct data for the interpretation of several basic aspects of the process of knowledge. They are the psychology of cognition and the studies on perception and representation. The theoretical models using the tools provided by the theory of information and computer simulation ("artificial intelligence"), the setting up of "universal grammars" which should define "linguistic universals", "the cognitive faculty of language" which is specific to humans, the modelling of the learning processes, they all have provided new elements for a reconceptualisation of psychology and for a deeper understanding of knowledge structure and mechanisms and for the construction of an integrated systemic model of the human being.

Important progress has been achieved lately by linguistics, a field undergoing modification of methods and concepts and setting up new cognitive objectives. As different from classical linguistics, a taxonomic empirical science, whose task was to set up the language corpus. The new trends initiated by CHOMSKY aim at going into the internal structure and the functioning mechanism of natural languages, formulating fundamental explicative models. R. MONTAGUE's theory in Universal Grammar carries out for the first time an integral semantic-syntactic analysis of natural languages with the methods of contemporary logic; this theory has had influences on several contemporary themes of epistemology such as: the logic - language relation, a priori knowledge, logical truth and analycity, etc.

However, I didn't attempt a global analysis of current science, that's why I think I can conclude at this point that the epistemological characteristics of global science is given by its "constellation of ways of reasoning" (structural-axiomatic, synthetic- integrative, evolutionist, historic, statistic, organisational, architectural, etc) which correspond to the diversification and maximum methodological-instrumental and thematic-conceptual expansion of knowledge. Contemporary science seems to have detached itself from the "methodological monism" and conceptual reductionism of other epochs admitting plurality of "methodological centres" of knowledge and the diversity of types of laws and theoretical explanations. The only unification seems to be related to the tendency of cognitive submission of "complex totalities" (highly complex phenomena and systems) present in the majority of fields of current research. The "holistic" tendencies of current research attach a great importance to the synthetic-integrative way of reasoning which became manifest in current science.

This integrative holistic tendency brought about the expansion of mathematics over all branches of research as a "reaction to a too sharp differentiation of subjects". This refers not only to the use of mathematical language or modelling techniques but also to the expansion to all fields of knowledge of the "mathematical way of reasoning" in its present aspects of functional, analogic, axiomatic, recursive, strategic, organisational, architectural, etc. ways of reasoning.

I would like to end this paper by saying that the world today is facing various problems making up a dichotomy whose terms can be formulated, as CHOMSKY puts it, as "Plato's problem" (pure knowledge) and "Orwell's problem" (social existence). The first tries to explain why we know so much in spite of the fact that we have such limited data. The second tries to explain why we know so little in spite of the multitude of available data. This is the modern paradox of knowledge as CHOMSKY suggests:

> Plato's problem, as compared to Orwell's, looks to me more profound and inciting from an intellectual point of view. But if we won't be able to understand Orwell's problem and recognise its significance in our own cultural and social life, to overcome it, the human species will have few chances to survive long enough to discover the answer to Plato's problem or to other problems inciting intellect and imagination. (Noam CHOMSKY, Knowledge of Language, 1985)

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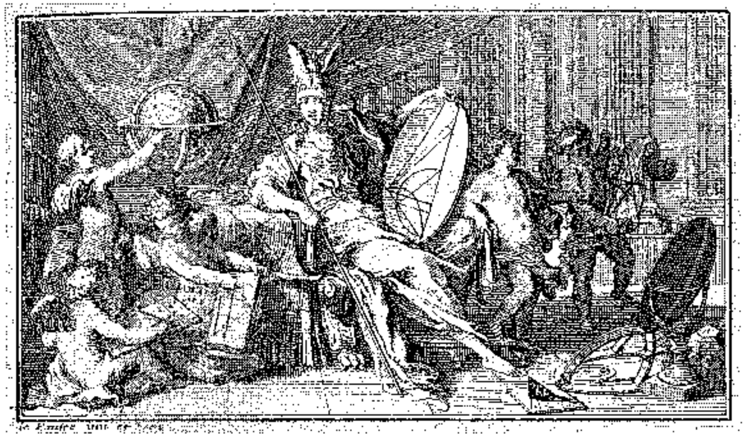

Exploring Fregean perspectives in mathematics education

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> I shall persevere until I find something that is certain - or, at least, until I find for certain that nothing is certain.
> (Descartes)

Abstract

Over the past two decades there has been a major upsurge of interest in the ideas of Gotlob FREGE (1848-1925). Two important and related themes in FREGE's writings are the logical foundations of mathematics and the importance of an appropriate conceptual notation in deriving mathematics from logic. While his thesis that mathematics is a branch of logic and his conceptual notation were deemed to fail, FREGE's perspectives concerning logic were to have a profound impact in both logical and mathematical developments in the 20th century. The goal of this lecture is to review a few of FREGE's ideas, not so much in terms of their contribution to either logic or mathematics, but rather, as the title of this lecture suggests, to try and explore Fregean perspectives for mathematics education. In particular, we explore three aspects of the link between FREGE's work and the way mathematics is taught in schools and universities : 1) the medium through which mathematics is communicated, 2) the nature of mathematical entities; and 3) the distinctiveness of the methodology of mathematics - based as it is, not on everyday ideas, but on abstract objects, exact calculation and proof.

1 Introduction

Three important and related themes in FREGE's (1948-1925) writings are: (i) the logical foundations of arithmetic; (ii) the importance of an appropriate symbolic language adequate both for any mathematical theory and simultaneously to embody all proofs within such a theory; and (iii) the need for an analysis of the meaning of words and sentences in ordinary language.

Largely ignored during his lifetime, FREGE's work has in the past three decades received considerable attention and recognition. FREGE is recognised as the father of 'linguistic philosophy', as the first philosopher to make a sharp distinction between analysis of the meaning of expressions and establishing what is true and the grounds for accepting it (DUMMETT 1981). FREGE is also celebrated as the founder of modern logic: the analysis of a proposition into function and argument(s), the truth-functional calculus, and the theory of quantification are among his fundamental contributions to the field.

FREGE's investigations into the foundations of arithmetic have both practical and theoretical significance. For example, PARSONS (1965), in assessing FREGE's theory of numbers, writes:

> It is impossible to compare Frege's *Foundations of arithmetic* with the writings on the philosophy of mathematics of Frege's predecessors - even with such great philosophers as Kant - without concluding that Frege's work represents as enormous advance in clarity and rigour. It is also hard to avoid the conclusion that Frege's analysis increases our understanding of the elementary ideas of arithmetic and that there are fundamental points that his predecessors grasped very dimly, if at all, which Frege is clear about. (p. 182).

In a similar spirit, DUMMETT (1991) claims:

> That [Frege's] philosophy of arithmetic was, indeed, fatally flawed; but it had an incontestable clarity, so that, even where it was mistaken, it pointed very precisely to where the problems lay. But it did much more than that. Frege's polemic against formalism contained a definitive refutation of that deadening philosophical interpretation of mathematics. To important questions in the philosophy of mathematics, above all those concerning the application of mathematics, the fruitfulness of deductive reasoning and the nature of mathematical necessity, his work provided, if not full-dress answers, at least sketches of what must be correct answers; later philosophers have come nowhere near his partial success in answering those questions, and have frequently failed to address them. Above all, Frege provided the most plausible general answer yet proposed to the fundamental question, 'What is mathematics', even if his answer cannot be unarguably vindicated. For all his mistakes and omissions, he was the greatest philosopher of mathematics yet to be written. (p. 321).

The goal of this paper is to review a few of FREGE's ideas, not so much in terms of their contribution to logic, or philosophy of mathematics, or philosophy of language, but rather, as its title suggests, to try and explore Fregean perspectives for mathematics education.

2 An outline of Frege's logicist program

By tracing the evolution of FREGE's logicist program (to show that arithmetic is part of logic) one might expect to gain some understanding of the processes by which mathematical theories develop. Indeed, FREGE provides an eloquent illustration of the creative mathematician.

DUMMETT (1981) considers FREGE's career divided into six periods, of which the first three correspond to setting out his project. Inspired by HADAMARD's (1945) views presented in his

Psychology of Invention in the Mathematical Field, we propose to add an initial period to the first three suggested by DUMMETT, drawing attention to the fundamental stage of unconscious work of incubation of new ideas. We have termed these four periods as *1) Incubation; 2) Preparation; 3) Illumination; and 4) Formalisation.*

2.1 Incubation

FREGE started his career as a mathematician with the presentation of his doctoral dissertation, *'On a Geometrical Representation of Imaginary Forms in the Plane'*, in 1873. In the following year he wrote his *Habilitationsschrift*, *'Methods of Calculation based on an Extension of the Concept of Magnitude'*, which would allow him the post of Privatdozent at the University of Jena. It is clear that this latter work contains the germs of FREGE's logicist insights. On the one hand, we see FREGE's underlying idea that whereas geometry can be intuited, 'quantity' cannot be an intuitive notion: "Bounded straight lines and planes enclosed by curves can certainly be intuited, but what is quantitative about them, what is common to lengths and surfaces escape our intuition". On the other hand, there is in this work an expanded notion of function related to the one he would later use extensively in his project.

In 1874, FREGE published a review of a book on arithmetic. He was largely unsympathetic to this work: he could not understand how it was that the propositions which formed the foundation of arithmetic were "lumped together without proof", while "theorems of a much more limited importance are distinguished with particular names and proved in detail". BYNUM (1972) suggests that it was probably his disappointment with this book which gave birth to FREGE's decision of setting out to explore the foundations of arithmetic.

2.2 Preparation

FREGE's investigation of the foundations of arithmetic made it necessary for him to engage in two different tasks. First, as one might expect, he focused his attention on views of other scholars on the topic: from Euclid to Newton, from Thomae to Cantor, from Leibniz to Spinosa, from Kant to Mill. In so doing, FREGE became convinced that previous approaches were untenable, and that there was a real need to find a new way out.

Second, FREGE set out to make logic more rigorous. Though attempts in this direction were already beginning to be made, namely by De Morgan and Boole in England, FREGE felt that they were entirely inadequate for his own purpose. This led him in a natural way to the development of his logical system which culminated with the publication of the *Begriffsschrift* in 1879.

The significance of this book can hardly be overlooked nowadays. It marks the beginning of modern logic, standing out as a major contribution to the field of logic and no longer as a preparatory work. But at the time, the ideas and notation that FREGE presented in this book were so revolutionary that very few people were able to appreciate them. To answer his critics, FREGE went on to defend his conceptual notation and ideas, comparing them with those of Boole, but without much success. He even saw a couple of the papers he submitted for publication rejected.

2.3 Illumination

Although disappointed and frustrated, FREGE continued to be motivated to find a solution for his original problem of the foundations of arithmetic. In the process of reviewing other authors, and finding sufficient grounds to criticise them, FREGE began to formulate his own ideas - an entirely new perspective, inspired in part by Leibniz (as it was indeed the case of his logical system).

His incisive critique of alternative views about number and his own approach were made public, in 1884, in his *Grundlagen der Arithmetic*, one of FREGE's best-known works. In this book, FREGE succeeded in lending plausibility to both his definition of number and showing that the series of natural numbers was endless. At this stage, however, FREGE presented his own project in a totally informal manner. His next step would be to provide a formal version of it.

Once more, the reaction to FREGE's work was disappointingly poor, but once more FREGE was motivated to continue to work on his project. It was during this time that FREGE presented some of his most brilliant insights concerning language. Having recognised that some of the notions he had used needed clarification, he published three influential papers, *Function and Concept*, *On Sense and Reference*, and *On Concept and Object*, in which he elaborated upon: (i) the notion of function, and the possibility of admitting functions of various levels, (ii) the distinction between sense and reference, and (iii) the distinction between concept and object.

2.4 Formalisation

Finally, in 1893, FREGE published the first volume of *Grundgesetze der Arithmetik*. In it, FREGE presented a reformulated version of his logical system, and carried out within such a system the construction of arithmetic sketched in the *Grundlagen*. He was now absolutely convinced that he had achieved his goal, but this work had no more success than his previous writings in terms of its reception and acceptance.

Despite this further disappointment, FREGE went on to write a large number of papers. It was a period of consolidation. His efforts to make his ideas known and accepted drove him to correspond with other scholars, such as Ballue, Couturat, Pasch, Peano, Hilbert, and Husserl.

Finally, in 1902, the second volume of the *Grundgesetze* was published. Here, he continued the work of formally deriving arithmetic from logic. There is also an attempt to derive the whole theory of real numbers from the same source, but this work is left uncompleted.

It is obvious that FREGE still intended to publish a third volume of the *Grundgesetze*. However, just before his second volume was published, Russell presented him with a paradox which showed that his program was inconsistent. For some years, FREGE kept up a correspondence with Russell who was visibly interested in his ideas. FREGE had finally found somebody who could appreciate his work. Until 1905, the year in which his wife died, FREGE still attempted to salvage his program, but it was in vain. Two devastating attacks on his work in the following year left him convinced that he had lost his battle. It was the beginning of the end.

3 What are these things called numbers?

One has only to read the Introduction to *Grundlagen*: to realise how important it was for FREGE to define in a precise way what numbers are:

> is it not a scandal that our science should be so unclear about the first and foremost among its objects, and one which is apparently so simple? Small hope, then, that we shall be able to say what number is. If a concept fundamental to mighty science gives rise to difficulties, then it is surely an imperative task to investigate it more closely until those difficulties are overcome. (p. IIe)

It is well to remember that FREGE was not alone in this kind of preoccupation. At roughly the same time, famous mathematicians (e.g. Dedekind, Cantor) expressed similar concerns. Their concerns are embedded in a broader movement which included the rigorisation of mathematics. In FREGE's words "that mighty academic positivistic scepticism which now prevails in Germany... has finally reached arithmetic".

Like Cantor, but unlike Dedekind, FREGE made philosophy an important partner to his discussion. According to him: "any thorough investigation of the concept of number is bound always to turn rather philosophical. It is a task which is common to mathematics and philosophy". But as BENACERRAF (1981) concedes, *Grundlagen* is first and foremost a mathematical enterprise. Philosophy comes in only as a convenient way to emphasise FREGE's point that arithmetical propositions must be proved "with the utmost rigour".

FREGE's starting point for solving the problem of what numbers are was to review the positions about the matter most commonly encountered among both philosophers and mathematicians. His analysis and refutation of other authors' views about number and numerical propositions is in three parts and amounts to over 60 pages. Here, we can do little more than restate his words in summarising his review:

> Number is not abstracted from things in the way that colour, weight and hardness are nor is it a property of things in the sense that they are. [...] Number is not anything physical, but nor is it anything subjective (an idea). Number does not result from the annexing of thing to thing. [...] The terms 'multitude', 'set' and 'plurality' are unsuitable, owing to their vagueness, for use in defining number. (1953, p. 58e)

What then is a number? FREGE turned the problem around by answering a non-linguistic question with a linguistic answer: "the content of a statement of number is an assertion about a concept". This was precisely what he meant when he stated one of the three principles - the one known as the *Context Principle* - on which he centred his investigation: *never ask for the meaning of a word in isolation, but only in the context of a proposition. (The other two principles are: (i) always separate sharply the psychological from the logical, the subjective from the objective; and (ii) never lose sight of the distinction between concept and object).*

FREGE's answer to the question is given in Part IV of the *Grundlagen*. This is divided into four subsections, each serving as a frame within which his thinking develops: (a) *every individual number is a self-subsistent object*; (b) *to obtain the concept of Number, we must fix the sense of a numerical identity*; (c) *our definition completed and its worth proved*; and (d) *infinite Numbers*. Here, we proceed by summarising his discussion in two parts, one corresponding to finite numbers and the other to infinite numbers.

3.1 Finite numbers

It is clear that in constructing his theory of natural numbers FREGE ran into considerable difficulties, and that he wanted to tell the reader about them. Hence, in his presentation, FREGE chose to incorporate how ideas and goals were formulated and refined in the course of action. For example, he began the first subsection (§55) by suggesting definitions of the expressions *the number 0 belongs to a concept, the number 1 belongs to a concept, and the number* $(n+1)$ *belongs to a concept*, but in §56, FREGE rejected them on the grounds (in part unconvincing) that in this way numbers would not be recognised as self-subsistent objects.

Next, at the beginning of the second subsection (§62), after restating his *Context Principle*, FREGE turned to the question of defining the sense in which two numbers are the same. In this he attempted to follow the road opened by Hume according to whom "when two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal". But once more he found this way not completely satisfactory: identity ought to be a general notion, and not just one which is applied to numbers. His second idea was to draw on Leibniz' definition of identity: "things are the same as each other, of which one can be substituted for the other without loss of truth". At his stage, FREGE resorted to a comparison with the idea of parallel lines and of direction of a line: "the direction of a is identical with the direction of b is to mean the same as line a is parallel to line b". But again, he noted a difficulty with such an approach: "that says nothing as to whether the proposition 'the direction of a is identical with q' should be affirmed or denied, except for the one case where q is given in the form of 'the direction of b'". He then chose a different but related path that led him to present, in §68, an explicit definition of direction, namely by defining directions as equivalence classes of lines, or as he put it as the extension of the concept parallel to a particular line.

Finally, in §68, too, he presented an analogous definition of number: "the Number which belongs to the concept F is the extension of the concept 'equinumerous to the concept F'".

Two observations are worth making at this stage. First, FREGE used the term 'extension' without having introduced it previously. In a footnote he observed that it was assumed to be known what the term meant. Whether or not he felt that there was something problematic with the notion of 'extension' is another story. The matter of fact is that, in *Grundlagen*, after showing (in §73) that "the extension of the concept 'equinumerous to the concept F' is identical with the extension of the concept 'equinumerous to the concept G' is true if and only if the proposition 'the same number belongs to the concept F as to the concept G' is also true", FREGE never used the notion again.

The second observation concerns the notion of 'equinumerous'. FREGE justly considered that such a notion may be defined in terms of one-one correlation. Moreover, he examined at some length the problem of whether or not this latter notion has anything to do with intuition, resolving the issue in favour of the doctrine of 'relation-concepts', which are part of pure logic. On the basis of the above definition, FREGE then proceeded to define the expression 'n is a Number' by stating that it is to mean the same as "there exists a concept such that n is the number which belongs to it". Then FREGE introduced the notion of the Number 0, by stating that "0 is the Number which belongs to the concept 'not identical with itself'". In describing his train of thought, FREGE mentioned that he could have used for the definition of 0 any other concept under which no objects falls, and showed further that every concept under which no object falls is equinumerous to every other concept under which no objects falls.

One would expect that FREGE would next define the Number 1. But he chose first to consider the general case of two adjacent numbers of the series of natural numbers, by introducing the definition of the expression 'n follows in the series of natural numbers directly after m' to mean the same as

> there exists a concept F, and an object falling under it x, such that the Number which belongs to the concept F is n and the number which belongs to the concept "falling under F but not identical with x" is m.

He also observed that he was not using the expression 'n is *the* Number following next after m' since he had shown neither that such an object existed nor that there was only one such an object.

Using this latter definition, FREGE went on to show that there exists a Number that follows directly after 0, namely the Number which belongs to the concept 'identical with 0', which by definition he took as being the Number 1. With this definition it is not clear that 1 is the successor of 0, but FREGE added a series of four propositions concerning the Number 1 which show that this is effectively the case.

The following, no less important, step consisted in showing that every natural number has a successor. This FREGE did by outlining the proof that "the Number which belongs to the concept 'member of the series of natural numbers ending with n' follows directly after n, given that n is a member of the series of natural numbers beginning with 0". The concept of 'member of series of natural numbers ending with a' was given in terms of 'the following of an object y after an object x in a general series', essentially the same notion that he had already defined in purely logical terms in the *Begriffsschrift*. FREGE used the expression 'n is a finite Number' to mean the same as 'n is a member of the series of natural numbers beginning with 0'.

To show that there is no last member in the series of natural numbers beginning with 0, FREGE stated that it was necessary to show that no finite number follows itself, and indicated how to prove this fact.

3.2 Infinite numbers

In contrast with his discussion of finite numbers, his presentation of infinite numbers is very short and direct. In line with the terminology he had used to define finite Numbers, he stated "the Number which belongs to the concept 'finite Number' is an infinite Number", and he used the symbol ∞_1 to denote it. Specifically, he observed that such a Number was not a finite one since it could be shown that its was a successor of itself. And he remarked:

> About the infinite Number ∞_1 there is nothing mysterious or wonderful. "The Number which belongs to the concept F is ∞_1" means no more and no less than this: that there exists a relation which correlates one to one the objects falling under the concept F with the finite Numbers. In terms of our definitions this has a perfectly clear and unambiguous sense; and that is enough to justify the symbol ∞_1 and to assure it of a meaning. [...] Any name or symbol that been introduced in a logically unexceptionable manner can be used in our enquiries without hesitation, and here our Number ∞_1 is as sound as 2 or 3. (1953, p. 96e-97e)

The title of the subsection, *infinite numbers*, and the fact that he used the symbol ∞_1 justify the suspicion that FREGE was prepared to define further infinite numbers. And yet he did not do that. On this same subject, one further point is worthy of note. As BOOLOS (1987) writes, it is somehow strange that FREGE did not define the number belonging to the concept 'identical with itself'. Hence he did not deal with the question of whether such a number would be the same as ∞_1. BOOLOS' point is that FREGE could not possibly have failed to consider such a number, and suggests that he would have regarded them as different.

Note, however, that FREGE devoted a considerable part of his subsection on infinite numbers in the *Grundlagen* to comment on the work that Cantor had published in the previous year. He praised Cantor's aims and spoke unmistakably in favour of transfinite numbers, although he openly criticised Cantor's indefinite and unclear use of the notions 'following in the succession' and 'Number' based on 'inner intuition'. He went on to say that he could anticipate how these two concepts could be made precise, but he did not offer any additional explanations.

In his later work, the *Grundgesetze*, FREGE did not take the analysis of infinite numbers much further. He even wrote in the Introduction that the propositions concerning the infinite number could have been omitted since they were not necessary for the foundation of arithmetic. Clearly, infinite numbers were not at the centre of FREGE's mathematical interests.

4 Remarks on later developments

FREGE's comments at the end of the *Grundlagen* show that he was convinced that the truths of arithmetic can be derived from logic and from logic alone. Yet he was still determined to raise this conviction to the level of absolute certainty. He regarded it as fundamental to his enterprise to provide formal proofs of all the arithmetical results he had presented in the *Grundlagen* - something he already had in mind when he developed his conceptual notation in the *Begriffsschrift*. It is worth noting that his idea was that the aim of proof is "not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another".

In the *Grundgesetze*, FREGE carried out his project in the way he had foreseen: "every 'axiom', every 'assumption', 'hypothesis', or whatever you wish to call it, upon which a proof is based is brought to light", so that there are no gaps in the chains of inference. In order to be able to meet such standards, FREGE felt it necessary to lay down previously all the primitive terms (it may be worth noting that one of the central primitive notions in FREGE's system is that of *function*) and symbols he would use, as well as all the 'axioms' to which these terms conform.

Likewise, FREGE presented the rules of inference that he would utilise to derive new results and the rules to introduce new names. Moreover, every single notion needed to define numbers was introduced previously in symbolic terms. Indeed, it is striking how meticulous FREGE was in confirming that his definitions were in agreement with the rules that he had laid out, in order to secure that the new symbols do have a referent. The crucial point for FREGE was that no symbolism brings into life a new being. Definitions are just abbreviations and one could well manage without them.

But on the whole, the path he followed to define numbers is similar to that he pursued in the *Grundlagen*. His comments in the Introduction to the *Grundgesetze* show that, while he could foresee some problems with regard to the acceptance of his Axiom V, he was convinced that nobody could erect a more durable edifice than the one he had built, nor show that his principles would lead to false conclusions.

Russell's letter informing FREGE that he had found a contradiction which could be derived from FREGE's system, namely from Axiom V, was a tremendous blow. FREGE still tried to salvage his system by presenting a different version of that axiom. But this and other possible solutions he attempted afterwards were unsuccessful. In the last years of his life FREGE judged his own

project - namely to show that numbers can be defined in purely logical terms - a complete failure. Throughout his life FREGE had dismissed Kant's view that arithmetic was synthetic *a priori*, but had accepted such a view with regard to geometry. Interestingly, he later stated that "arithmetic and geometry have developed on the same basis - a geometrical one in fact -so that mathematics in its entirety is really geometry". However, no time was left to him to explore such a radical view.

5 Implications for mathematics education

While FREGE's work is of considerable interest, its primary appeal is to philosophers, mathematicians and historians of mathematics. Our original interest in his work arose in the context of the nature of proof from precisely such a perspective. And though this was part of a larger program concerned with the position of proof in school mathematics (GARDINER & MOREIRA 1999a, 1999b; MOREIRA & GARDINER 1999) FREGE's work is so focused on subtle details that we realised only slowly the lessons for mathematics education which are implicit in his work, and in the way it was received.

The quote from DUMMETT (1991) in the Introduction acknowledges that, though FREGE's program failed, "it pointed very precisely to where the problems lay [and] did much more than that" For our purposes here, the details are in some ways less important than

- FREGE's general goal of identifying and characterising those concepts which constitute the basis of mathematics, and
- the reception which his efforts received.

Our remarks are presented under three headings, which represent important themes both for FREGE and for mathematics education.

5.1 Mathematics and language

FREGE wanted to establish mathematics on a foundation which avoided the ambiguities of ordinary language. The first (and perhaps simplest) part of his program was to establish a precise, formal language in terms of which everything else could be expressed.

FREGE struggled to make a clear distinction between *Zeichen, Sinn, and Bedeutung* - that is between a *sign* or *symbol*, its *sense* (what one understands from it), and its reference (what it denotes). Such distinctions are relevant not only in philosophy, but at all stages of education. Mathematics educators recognise the importance of deciding precisely which such distinctions need to be made, and when and how they should be made (e.g. ARZARELLO et al. 1995; BAZZINI 1998; FERRANDO 1998).

FREGE realised that Boole's symbols were too limited for a symbolic logic which included variable and quantifiers, so he had no choice but to develop his own notation. Given the profound nature of his insights into the foundations of logic and of mathematics, one could argue that the language he chose to use was largely irrelevant to the success or otherwise of his program. However, as with mathematics textbooks and courses, the language he used constituted a superficial barrier, which restricted access and sapped the determination of those who might otherwise have made sufficient progress to get to the heart of the questions FREGE was trying to address. In other words, while FREGE's commitment to the need for a precise language and symbolism was entirely sound, the language and symbolism he used hindered the understanding and acceptance by his contemporaries of his more significant logical ideas.

Today we recognise that many of the basic concepts and structures in modern logic have their origins in FREGE's work. However, the symbols we use to denote them derive from Peano - not directly from FREGE. Unlike Peano, the logical symbols which FREGE introduced ignored the linear tradition of European printing and reading, and adopted instead a 2-dimensional format. It is generally accepted that this made it more difficult for his logical ideas to be accepted: the difficulties which FREGE's notation in the *Grundgesetze* presents to the reader may explain why few scholars appear to have looked closely at his theory of real numbers. However, these difficulties arose not only because of the symbols FREGE used, but also because he failed to win over influential members of the mathematical community. Established tradition is not without its justification. New symbols and styles to overcome a naturally conservative view of what is appropriate: the ultimate fate of new ideas often depends on whether they are adopted by senior members of the relevant community, who then mediate their use to others (GAUVAIN 1998). For example, William Jones introduced the symbol π for the ratio circumference : diameter of a circle in 1706; but it was not until Euler adopted the symbol in the 1730s that its use became widespread. The radical nature of FREGE's work, and his rejection of much established practice, placed him firmly on the fringe, rather than in the centre, of contemporary mathematical culture.

The fact that FREGE wrote the *Grundlagen* in a relatively informal prose style indicates that he was aware of the difficulty. Indeed, if he had not written the *Grundlagen* in this more accessible manner, his other works may well have received even less attention than they did. But FREGE knew that, if he was to do justice to his overall goals, he had no choice but to use a precise, formal language. Whereas he compared ordinary language to the human eye, he saw his conceptual notation as like microscope, which may not help one to see "the big picture", but which is essential if one wants to uncover important, yet previously unsuspected, details.

In the context of mathematics teaching, the language we use determines how our subject is perceived and learned. We cannot (and should not try to) avoid symbols and precise language. Yet we have to recognise that learners face a substantial challenge in seeking to master the unfamiliar language of mathematics - a language that is scarcely used outside the classroom, and whose symbols require the user to observe special conventions (MORGAN 1999). Learners should not be left to stagnate in a primitive world of baby-language and naive concepts; but neither should they be prematurely swamped by meaningless terminology and rules. Hence, when a new mathematical topic, method or convention is introduced, we first seek to explain things in familiar language - initially using everyday language, then in terms of previously learned mathematical language. Learners must eventually understand that the advantages of mathematical exactness and precision are available only to those who are prepared to make this transition, who learn to respect and use its language and to breathe freely in its rarefied air. But we as teachers have to remain sensitive to the difficulties they face, and to the fact that those who fail to make this transition are likely to remain alienated from the mathematical ideas and methods which the mathematical language and symbolism encapsulates.

Reflecting on the difficulties faced by anyone who tries to understand FREGE's conceptual notation lends weight to the traditional idea that it can help to break down the process of accessing highly unfamiliar material into four stages.

- First, it is important to understand that what mathematics has to offer depends on the precise character of its terminology and language; that is, learners must accept that precise ideas and exact thinking are only possible if we restrict attention to a *mathematical world* with its associated unfamiliar language, notation and rules.

- Second, although the terminology and language may be unfamiliar, it is always important for human beings to relate the unfamiliar to the familiar, to make sense of, or grasp the meaning of, what it is that is being encapsulated in unfamiliar language or notation. In particular, examples should be given which indicate the extent to which the new and unfamiliar language sharpens important distinctions and allows one to describe and to work more accurately with important ideas (that is, distinctions and ideas whose importance can be appreciated by the learner).

- Third, one needs time and plenty of opportunity for *routine*, meaningful practice, so that the new methods and language can be incorporated into existing cognitive structures, and can be used to consolidate previous ideas and to extend one's powers of analysis and calculation.

- Finally, it is important to provide opportunities for learners to use, and to talk about, new methods and ideas in non-routine settings, so that they and their teachers become aware of outstanding conflicts, misconceptions and limitations.

5.2 The nature of mathematical objects

At the root of all FREGE's work is the fundamental question: What is the exact nature of mathematical objects? Though the way he pursued this question may not be directly relevant to the classroom, mathematics teachers face a closely related question every day. What is the status of the entities we work with in mathematics textbooks, in pupils' workbooks, and on ordinary school blackboards and white boards?

> These [and other related questions] are not just ruminations of philosophers hidden in ivory towers. The answers we give to them have a profound impact on our educational policies and research programs. Piaget's constructivism and Bourbaki's austere rigour have left their marks on our schools. (DEHAENE 1997, p. 232)

What teachers think (perhaps unconsciously) about the nature of mathematics affects the way they teach, and hence the way their pupils learn. By struggling with the kind of epistemological questions addressed by FREGE, teachers can become more aware of how questions reveal themselves in the classroom, and can begin the long process of developing a view of their subject which is consistent with the nature of the discipline.

FREGE asserted that numbers exist as "objects". By this he meant that they are *objective* - not that they are "real". FREGE rejected the empiricist view (associated with John Stuart Mill) that numbers are somehow "abstracted from experience". He also rejected the formalist conception (associated with Thomae and, later, with Hilbert) of "numbers as mere signs manipulated in conformity with certain rules specified by mathematicians" - that is, that numbers are mere creations of the human mind.

FREGE's view that numbers are objective may seem as implausible as the views he rejected. When the torpedo of Russell's paradox revealed the flawed nature of the *Grundgesetze*, FREGE accepted that his efforts to establish the objective nature of numbers had failed. Yet to the end of his life he refused to equate numbers with mere symbols. And he continued to reject the empiricist fallacy that numbers can somehow be extracted from experience. In particular, he argued that if numbers were based on sense perception, then the series of natural numbers could never be endless: this, FREGE remarked, "is not just false, it is absurd".

A final answer to the underlying question "What exactly are numbers?" continues to elude us. Yet FREGE's analysis, and subsequent work during the twentieth century, mean that we are much clearer about the nature of the question and of the extent to which we can give partial answers. The logicist approach may be epistemologically and ontologically different from the formalist and the intuitionist approaches; but each has something to teach us about the nature of mathematics. Moreover, the strengths and limitations of each approach are now more clearly understood. By studying such fundamental questions one comes to appreciate that mathematics is not quite the monolithic construction it is often thought to be. The logical structure of mathematics is indeed hierarchical, cumulative and linear: but it is more like a tree - developing downwards as well as upwards, with roots as well as branches taking part in the process of development.

5.3 Methodology of mathematics

In recent years there have been numerous claims that school mathematics should more closely reflect the adult discipline of mathematics. For example, HILTON, HOLTON & PEDERSON (1997) assert that "mathematics must be taught so that students comprehend how and why mathematics is done by those who do it successfully". To assess whether this a realistic goal -and to understand better the nature of the discipline they profess- future teachers need to confront the question: "What is it that mathematicians do when they do mathematics?".

This confronts us with a dilemma. In the context of modern mathematics education one can no longer simply declare that school mathematics constitutes an "apprenticeship", and that mere apprentices are quite different from mature mathematicians; thus, the proposition that we should teach in a way that provides high school graduates with some insight into what mathematicians do has a superficial appeal. On the other hand, the attempt in England during the late 1980s and 1990s to make "investigation" a significant part of the official curriculum only served to underline the dangers.

School curricula have to survive the process of institutionalisation; that is, curriculum content has to retain its essential form and its value even when subjected to the limitations, distortions and stereotyping of textbooks, examinations and the whole gamut of pupils, teachers and schools. Though traditional curricula are often criticised, the kind of standard topics they contain have proved themselves over the centuries to be remarkably "stable" in the face of such challenges. In contrast, the art of doing mathematics is not - and may never be - sufficiently well understood to allow us to develop a universal approach at school level which is comparably stable.

Mathematical practices - as HADAMARD (1945) showed - cannot be described in a uniform way. POLYA's classic books further illustrate the difficulty of showing what it is that mathematicians do. On the one hand, his major works (1954, 1962) are far too demanding for most potential teachers; on the other hand, his popular version (1957) has effectively encouraged many to present the process of solving mathematical problems in embarrassingly simplistic terms. Attempts to capture the spirit of mathematical problem solving in a manner which remains faithful both to mathematics and to adolescent psychology (e.g. GARDINER 1987a, 1987b) have been

largely ignored in favour of styles of exploration at school level which are antithetical to mathematics.

FREGE could scarcely be described as a typical working mathematician. But he was a very creative and productive scholar. The fact that much of his work concerns something as basic as natural numbers makes him a valuable case study for anyone who wants to understand the difficulties faced by those who struggle to construct and to communicate a novel mathematical theory. The details of FREGE's work may be appropriate and accessible only to those interested teachers (for whom we recommend the English translation of the *Grundlagen* (1959) as a starting point). But one feature of his program highlights an essential ingredient in any school mathematics curriculum which wishes to convey to high school graduates what it is that makes mathematics special.

FREGE's logicist program led to a distinctive approach to *proof*. FREGE insisted that arithmetic had not yet been properly mathematised, and he wanted to show that arithmetical facts can be derived unequivocally from basic *logical* laws, using only logical rules of inference. In one sense, his plan was to extend Euclid's vision from geometry to arithmetic. His goal was to demonstrate how arithmetical propositions may be derived from a few axioms and primitive terms, and to make explicit the logical rule which are used at each stage.

Within such a program the central focus shifts from the apparent subject matter (arithmetic) to the underlying methodology - namely *proof* itself. FREGE was concerned not so much to establish the "truth" of arithmetical propositions, as to identify the logical dependencies between propositions, and between propositions and the axioms. A simplified version of this shift of focus occurs in traditional school geometry - even if it is best experienced directly and subconsciously by the pupil rather than being made explicit at that stage.

- The initial goal is to set up a small number of geometric principles in order to analyse geometrical problems from the world around us.
- To carry this through one moves from the real world into a world of mathematical objects (ideal, *mental* triangles, rather than cardboard cut-outs), together with a hierarchy of results, and accepted rules of inference.
- The problems one then solves need to be sufficiently meaningful for pupils to engage with the task. But the details of each problem are quickly forgotten. What remains is the sense of having played the game of deducing the required result from earlier known results within the hierarchy, according the accepted rules.

If we seek to convey something of the flavour of real mathematics to those in schools, then all learners need to experience and internalise the difficulties and delights of this process - through solving problems in elementary arithmetic in a structured way, through solving equations, through calculating estimates, through reducing the solution of hard problems to easier known facts, and through angle-chasing and the derivation of important results in elementary geometry - such as the formula for the area of a triangle, or the fact that the angles in any triangle sum to two right angles (see GARDINER & MOREIRA 1999a, 1999b; MOREIRA & GARDINER 1999). There is then a chance that pupils will appreciate not only that mathematical reasoning is very different from everyday reasoning, but how it differs, and why it has to differ if it is to provide us with reliable results.

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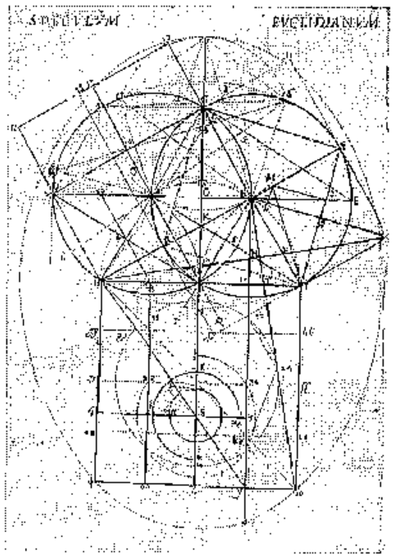

History of mathematics in a preservice program and some results

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Abstract

We refer to a preservice mathematics program, based on the history of mathematics, which we have developed and implemented during the past nine years (see PHILIPPOU & CHRISTOU 1998). First, we set the stage for the theoretical perspectives, next we elaborate on specific dimensions of the program by giving examples and describing the ways in which they were meant to function, and finally, we present the evaluation results concerning the effectiveness of the program. The efficacy beliefs[1] of the first graduates of the program, with respect to teaching of mathematics, were found to be significantly better than the respective efficacy beliefs of teachers who graduated from other programs.

[1] "Efficacy beliefs" as defined by BANDURA (1997) means "teachers beliefs in their ability to produce a desired effect" (see section 2.1) below.

1 Theoretical background

1.1 Primary preservice mathematics education

Primary school teachers often lack the connected conceptual understanding which might be defined in terms of two constructs: the nature of mathematics and the teachers' mental organization of mathematical knowledge. The former depends upon the ever-growing content of mathematics as a discipline and the latter refers to the organization of teachers' knowledge, how it is acquired, structured, and retrieved. The modern mathematics curriculum has increased demands on the part of the teachers, who are expected to select worthwhile mathematical tasks, to orchestrate classroom discourse, to seek connections that facilitate students, to deepen their mathematical understandings, to help students use technology, and to assess progress (SWAFFORD 1995).

The debate on preservice teacher mathematical education continues, though some of the earlier established concepts and perspectives have been well grounded. BROWN & BORKO (1992), for example, mention Shulmans' seven domains of cognitive schemata from which teachers draw (subject matter, specific and general pedagogical content, other related matter, the curriculum, the learners, and educational aims). LAPPAN & THEULE-LUBIENSKI (1994) focus on a shorter list of three domains that should be considered in the teacher program namely, knowledge of content, knowledge of pedagogy, and knowledge of students. The same outhors consider the integration of knowledge in these three domains with beliefs, as the main concern in the education of professional teachers.

COONEY (1994) refers to content knowledge, pedagogy and psychology of learning as the three requirements for a teacher to be an *adaptive agent* in the classroom. He suggests that a certain orientation toward knowledge and change is also necessary, since the lenses through which we see our world influence what we do. Knowledge of the content of a discipline, knowledge of the teaching of the same subject and one's orientation toward these types of knowledge are different entities. An implication for teacher education is that the development of meaning with respect to teaching any subject is fundamentally connected to the meaning one assigns to learning and teaching this subject. Prospective teachers need to learn a content in a way that is consistent with the way they are expected to teach that content.

Much of the existing practice in schools is based on the assumption that knowledge is manufactured elsewhere, acquisition is tested by another statutory agency, and the teacher is seen as simply to act as a mediator; he or she is only trusted to "deliver" the package (BURTON 1992). The new conception of teachers' education assumes that teachers need to develop a conceptual base and the orientation to become pedagogically powerful. According to COONEY (1994) this means that teachers have: a) to develop a knowledge of mathematics that permits the teaching of mathematics from a constructivist perspective, b) to develop expertise in identifying and analyzing the constraints they face in teaching and in dealing with those constraints, c) to gain experience in assessing a student's understanding of mathematics, and d) to offer teachers opportunities to translate their knowledge of mathematics into viable teaching strategies. There is a growing consensus that constructivist epistemology could be the basis for preparing teachers to teach in reform-oriented ways. According to the perspective, learning is regarded as an ongoing process of individuals trying to make sense and construct meaning, based on their experiences and interactions with the environment. In a project, driven by constructivist views, ZASLAVSKY & LEIKIN (1999) refer to goals such as enhancing teachers' mathematical and pedagogical knowledge, offering teachers opportunities to experience alternative ways of learning challenging mathematics, fostering teachers' sensitivity toward students' feelings and mathematical understanding, and promoting teachers' ability to reflect on their learning and teaching experiences.

KRAINER (1999) proposed a model for teachers' professional practice focusing on *action, reflection, autonomy*, and *networking*. Action refers to attitudes towards and competence in experimental, constructive and goal directed work. Reflection refers to attitudes and competence in self-critical work that reflects systematically on one's own actions. Autonomy refers to attitudes and competence in self-initiated, self-organized and self-determined work. Networking refers to attitudes and competence in communicative and co-operative work with increasingly public relevance. Each of the two pairs "*action-reflection*" and "*autonomy-networking*" expresses both contrast and unity, and can be seen as complementary dimensions, which have to be kept in a certain balance. The interplay between these dimensions is of great importance. In general, more reflection contributes to a higher quality of actions and a higher quality of action and autonomy promotes the quality of reflection and networking. Reflecting on others and their own teaching practices engages teachers in thinking about good teaching and the meaning that teaching has for students. Experience shows that teachers' practice is usually characterized by a lot of action and autonomy and less reflection and networking (KRAINER 1999).

Professional autonomy is equivalent to self-regulation and to one's ability to make decisions without having to be told by others. It involves the ability to exercise judgement, to make decisions by careful consideration of relevant variables, to distinguish between appropriate and inappropriate actions on the basis of well-specified criteria and standards of behavior. Professional autonomy develops as teachers construct their ideas about the content of a discipline and how it can be taught to others (CASTLE & AICHELE 1994). This means that the new experiences should develop the students' ability to challenge the old and foster new dispositions, to apply mathematical methods and symbolism, to view mathematics as a study of patterns and relationships and to enhance self-confidence. During preservice courses, students begin to envision their future role as the organizers of learning activities and develop their first models for teaching mathematics. At this stage, what the students learn is tidily connected to how they learn it; the way mathematics is taught in a preservice program is more important than the content per se. A historical and cultural approach is proposed as potentially capable of opening new perspectives and meanings on the nature of mathematics and its teaching.

1.2 History of mathematics in teacher education: Why and how?

The "law of recapitulation" received strong support for about a century. The assumption that ontogenesis recapitulates phylogenesis was behind Piaget and Garcia's interpretation of the relationship between historical and psychologic al developments in terms of the mechanisms mediating the transition through the three stages through which an idea develops (intraoperational, interoperational, and transoperational). In recent years, however, the recapitulation law is under serious criticism. The major argument derives form the emphasis on socio-cultural influences. To retrace the mental development of the race is neither possible nor desirable, because "key aspects of mental functioning can be understood within the social context in which they are embedded" (RADFORD 1997, p. 28).

JONES (1989) has seen an exaggeration in the recapitulation law; he argued though that history of mathematics coupled with up-to-date knowledge of mathematics is a significant tool in the hands of the teacher who teaches whys. The critique of the recapitul ationistic parallelism has resulted in a more realistic claim, which might be condensed in the argument that "when scrutinized, the phylogeny and the ontogeny of mathematics will reveal more than marginal similarities" (SFARD 1995, p. 15). In conclusion, we accept that similar recurrent phenomena can be retraced throughout the historical development and the individual reconstruction of knowledge. Epistemological obstacles may not be strictly intrinsic to knowledge, but we have enough evidence that students and adults frequently face learning difficulties similar to those encountered during the genesis and development of a mathematical idea. The *similarity principle* provides the ground for non-naïve use of history to facilitate learning. A commendable approach is to use history as an epistemological workshop that could change the learners' conceptions and transform the practice of teaching mathematics (BARBIN 1996; RADFORD 1997).

Several possible ways have been proposed to answer the question of how to incorporate history in teaching. AVITAL (1995), in order to break the image of mathematics as boring and difficult, draws attention to the human side of mathematics by exposing students to anecdotes and exciting stories from the lives of great mathematicians. BARBIN (1996) focuses on the potential role of problems as a means to bring to the fore the process of the construction and rectification of knowledge. Following the process of genesis of mathematics, we can foresee possible learning difficulties and create a climate of search and investigation through problems from history.

2 Teacher education and teachers' beliefs

Beliefs are conceived as the personal judgments and views that constitute one's subjective knowledge about self and the environment. Beliefs and attitudes are organized around an object or situations, predisposing one to respond in a favorable or unfavorable way; they are contextual, experientially formed, and emerge during action. RICHARDSON (1995) identifies beliefs as the teacher's own theories, which are sets of interrelated conceptual frameworks connected with action, a kind of *knowledge-in-action*. A teacher's knowledge is translated into practice through the filter of his/her own beliefs about mathematics and its teaching and learning (SWAFFORD 1995). In general, beliefs drive action while experience and reflection on action can modify beliefs, i.e., actions and beliefs interact with each other.

A preservice program needs to consider the structure of beliefs the students bring to teacher education and provide experiences that will help students overcome common myths and misconceptions about mathematics, its teaching and learning. Students should transform unexamined beliefs in relation to classroom actions into objective and reasonable beliefs. Belief systems are change resistant; change can occur only when students engage in personal explorations and are involved in powerful experiences in mathematical thinking and conceptual understanding that motivate a new perspective on students' views towards learning. This subsequently, leads to modified classroom practices, though a change in beliefs does not necessarily translate into changes in practice.

2.1 Efficacy Beliefs about Mathematics teaching

The construct of teacher efficacy grew mostly out of the work of Bandura who identified teachers efficacy as a type of self-efficacy, a cognitive process in which people construct beliefs about their capacity to perform at a given level of attainment (TSCHANNEN-MORAN, HOY & HOY 1998). Self-efficacy is a future oriented construct, which might be perceived as "beliefs in one's capabilities to organize and execute the courses of action required to produce given

attainments" (BANDURA 1997, p. 3). Feelings of competence depend on one's experience in connection to related action; teachers' efficacy beliefs about the teaching of mathematics are mostly shaped on the basis of their own content and pedagogical content knowledge. BANDURA (1997) postulated four sources of self-efficacy information: mastery experiences, physiological and emotional arousal, vicarious experience and social persuasion. These four sources contribute to both the analysis of the teaching task and the self-perception of teaching competence.

Early research in the field identified two dimensions of teacher efficacy, the personal teacher efficacy (PTE) that refers to one's feelings of his/her own capability, and the general teaching efficacy (GTE) that refers to the possibility of teachers, in general, to affect students' learning. Another dimension refers to internal versus external control of reinforcement (TSCHANNEN-MORAN et al., 1998). The former refers to the possibility of teachers controlling students' learning (Internal Control), while the latter accepts the predominance of environmental factors in learning (External Control). A teacher's sense of efficacy is a motivational factor influencing the amount of effort one will expend and the persistence he or she will show in the face of obstacles.

Numerous research studies (PAJARES 1996; TSCHANNEN-MORAN et al., 1998) have confirmed the relationship between efficacy beliefs and significant educational factors, such as professional commitment, enthusiasm, instructional experimentation, implementation of progressive and innovative methods, the level of organization, and certainly, students' affective growth and achievement. There is also some evidence relating efficacy to mathematics learning (PAJARES 1995), but the efficacy beliefs with respect to mathematics teaching is a very little researched area.

3 The preservice mathematics program

In designing the program we assumed that the mathematical background of primary school teachers could rely on an overall grasp of the nature and significance of mathematics. In the light of the foregoing discussion, we hypothesized that history of mathematics would function on one hand as a challenge and motivation. On the other hand, we hypothesized that history of mathematics would facilitate students construction mathematical meanings, develop competence and change their mathematical views. Since the major part of primary school mathematics was originated in the early historic period, the learners' Greek origin was another positive factor. On the grounds of the *similarity principle*, the evolutionary process in mathematical thinking, the solution of big problems that intrigued and inspired top mathematical minds, the study of some of the successes and understanding some failures of famous mathematicians could function as an epistemological workshop. It was expected that such tasks would motivate and attract prospective teachers, improve their conceptions about mathematics and its pedagogy, free students of misconceptions and negative attitudes, and establish a balanced relationship with the subject.

3.1 The content of the program

The program consisted of two "content courses" and one "methods course". The content courses began with prehellenic mathematics, continued with Greek mathematics, moved to Islamic and Hindu contribution and to some mediaeval and enlightenment mathematics. The course concluded with six units of contemporary mathematics (Calculus, Liberation of Geometry, Liberation of Algebra, Set Theory, Logic, and Boolean Algebra). In this section we briefly describe the content of the program, and present two examples in some details.

The work on prehellenic mathematics (numeration systems, arithmetic operations, simple problems, and geometry) aimed a) at drawing attention to the genesis of empirical mathematics, which served the needs of ancient societies, and b) at letting students realize the variety of approaches in meeting everyday social needs. Greek mathematics occupies the major part of the first course. Students' activities are organized around selected works and problems from Thales, Pythagoras, Hippocrates, Euclid, Archimedes, Eratosthenes, Apollonious, Ptolemy, Diophantus, and Pappus. At the opening, we focus on the contrast between empirical mathematics and the deductive mathematics developed by the Greek tendency to ask general questions and raise major philosophical issues. Thales provides the starting point to study similarity, Pythagoras offers many challenging examples ranging from proportions and figurative numbers, to the discovery of irrational numbers, and certainly to the great Pythagorean theorem. We study a variety of proofs and extensions of this theorem, as a means to overcome the myth that "to each problem there is always one best solution". The three classical problems of antiquity help students get the meaning of the term "solution", under certain constraints. Efforts to solve the "unsolvable" problems, which occupied the geniuses for centuries, are life experiences about the nature of mathematics.

Though Euclid is visited on various instances, the emphasis is on understanding the first model of axiomatic organization of scientific knowledge. Proposition I,1 draws attention to the level of rigor and opens the way to a respectful but intense critique of the masters. Primitive Pythagorean triples is an interesting activity and the infinitude of the primes is discussed as a model "existence theorem". We Start from Euclid's proposition (IX,21) "Let that A, B, and C be prime numbers, I say that there are more prime numbers" and continue to the modern formulation "Let $P = \{p_1, p_2, p_3, ..., p_n\}$ be a set of n prime numbers; show that there exists a prime number p_k that does not belong to $P, p_k \notin P$". For the work on the fifth postulate, see Example 1.

Archimedes's helix is studied as a means to solve the "quadrature" problem and the "trisection" problem, and his method to estimate the value of π is the departure point for a journey along the efforts to improve the accuracy of π. Eratosthenes's estimation of the circumference of the earth is mentioned as a fine application of mathematical ideas. In Ptolemy's cyclic quadrilateral, the sum of the products of opposite sides being equal to the product of the diagonals is used to compute chords and generate the equivalent of trigonometric identities. Diophantus proposition II,8 "to divide a square number into two rational numbers" is studied and related to the recently solved Fermat's last theorem.

The program includes instances of Moslem and Hindu mathematics, it continues with Leonardo Pissano, the solution of the cubic and quadric equation (Nicolo Fontana and Scipione del Ferro) and the discovery of probability (Pascal and Fermat). Open problem activities are organized on Pascal's Triangle. The unit on Calculus is an introduction to fundamental concepts and specific applications. Zeno's paradoxes provide the introduction to the concept of the "limit", Fermat's computation of the area under a curve and Barrow's method for finding the derivative are discussed, and finally we come to Leibniz-Newton's discovery of the calculus.

The "Liberation of Geometry" (see Example 2) is followed by a unit on the "Liberation of Algebra" to let students "free their minds" from the boundaries set by the Euclidean thinking. After setting the axioms, we refer to Hamilton's quaternions and turn to Cayley's matrices, as a useful example in which the commutative property as well as lot of other rules, often taken for granted,

fail to be satisfied. The unit on Set Theory includes a study of the properties of set operations (union, intersection, and complement), ordered pairs, triples and n-tables, cross-multiplication, the definition and application of binary relations, equivalence relations, and finally the definition of a function. A similar focus also continues in the unit on Mathematical Logic and on Boolean Algebra. The objective in this closing part is threefold. First, to give future teachers a taste of a formal mathematical system, second to let them realize that set algebra and propositional algebra could be united under one abstract system, and third to illustrate the connection of mathematics to electrical circuits and computers.

Example 1. The Unit on Prehellenic Mathematics concludes with a fairly complete treatment of Number Systems. It includes elements of the Babylonian, the Egyptian, the Mayan, the Chinese, the Greek, the Latin, the Decimal and the general place-value system. The structure of each system and its basic properties and operations are discussed and compared. The development of the unit proceeds to connecting the contemporary needs with the properties of the decimal system. Students gradually come to see the influence of culture in creating mathematics and the multiplicity of possible solutions to the same need. The legend about Sessa, the chess inventor, and the King's unkept promise provide a favored example with rich ideas and applications. Questions that were found to motivate the students are included. Why the King could not keep his promise to give the monk one seed wheat for the first square, two for the second, the double for the next and so on until the last square of the chess-board? What is the number of seeds requested $(1 + 2 + 2^2 + \cdots + 2^{63})$? How can this sum be found, how the method is simplified using binary numbers, how can be approximetely estimated, and how can the volume required to store this quantity be found? Extensions of the problem include: What if the request of the monk was to trible the number for each successive square? Which number system is then found? Can we find the sum using an analogous method to the one used above? Change the following numbers over to decimal notation $a_2 = 222 \cdots 2_3$ (64 times), $a_3 = 333 \cdots 3$ (64 times), and generally of the number of the form $nnn \cdots n_{n+1}$ (64 times).

Example 2. The efforts to correct Euclid provide a most challenging opportunity for the students. We begin with Euclid's main propositions on parallel lines I,27, I,28, and I,29, and proceed with some work on equivalent propositions (e. g. Playfair). Next we go to Geminus, the stoic philosopher of the last century B.C., who was one of the first to make a serious attempt to prove the fifth postulate. Examining the principles of the logical building of mathematics, Geminus concluded that "it is as futile to prove the indemonstrable as it is incorrect to assume what really requires proof" (HEATH, Vol. II, p. 227). Geminus was convinced that the fourth and the fifth postulates could be proved. He defines parallelism in terms of equidistant lines, asserting that convergent lines are not necessarily parallel (could be asymptotes). The steps of his ingenious work are followed and the arguments for its breaking down are discussed. We next go to Ptolemy's "proof", which was falsified by Proclus, and Proclus's own "proof" for which he felt so proud. We refer to Wallis and spend more time on Sacheri's work. Finally, we proceed with Bolyai and Lobachevsky to settle the issue. A selection of basic theorems of hyperbolic geometry completes the unit. Experience has shown that the students initially reject any idea of non-Eucledian geometries. The truth of the fifth postulate is taken for granted. Gradually, however, students' resistance to accept some facts of the Hyperbolic Geometry (e.g., that triangles are defective, there exist neither rectangles nor similar triangles) becomes skepticism, and eventually the students are fascinated to follow assumptions and construct proofs to the end.

4 The program evaluation

The program was evaluated with respect to students' attitudes toward mathematics and the mathematics teaching efficacy beliefs of the first graduate's of the program.

4.1 Prospective teachers attitudes

A longitudinal approach was adopted; at the beginning, the subjects were the first year primary students enrolled in the Department of Education in 1992. The students' attitudes were measured, using the same instrument, before they started the first mathematics course (Phase 1 – $P1$, $N = 162$), on the completion of the first course (Phase 2 – $P2$, $N = 137$), and at the end of the whole program in 1995 (Phase 3 – $P3$, $N = 128$). The Department of Education normally selects from among the top 25% quartile. Nonetheless, about one third of the students come from streams with only core mathematics, while about two thirds of successful candidates do not take mathematics at the entrance examinations (it is an optional subject).

Three related scales were used: The Dutton scale[1], which included eighteen items reflecting attitudes, the "justification scales" reflecting students' reasons for liking and for disliking mathematics, and a one-to-eleven point linear "self-rating" scale, reflecting the respondents feelings about mathematics. Two statistical techniques were used to detect patterns in attitude and check for significant change that might have occurred during the implementation of the program. First, the χ^2-test was applied, separately for each item of the Dutton scale, for the Justification scales, and for grouped responses of the self-rating scale (the responses were grouped into five intervals from extremely negative attitudes, to real love for mathematics). Second, the *Median Polishing Analysis* was applied for the three Phases.

4.2 Mathematics teaching efficacy beliefs

The primary teacher population of Cyprus comprises of four groups: graduates of the Pedagogical Academy (PA), graduates of the PA who have obtained a university degree (PAU), graduates of the University of Cyprus (UC), and graduates of Greek Universities (GU). A questionnaire was mailed to selected schools, and 157 were returned (about 65% of the total). The subjects were 91 (58.7%) PA graduates, 15 (9.7%) PAU, 28 (17.8%) UC graduates, and 21 (13.4%) GU graduates. About three months later, we interviewed 18 teachers (ten were UC graduates, six were PA graduates, and two were GU graduates) focusing on issues raised in specific items of the scale. The interviews were tape-recorded and qualitatively analyzed.

The instrument was a five-point Likert-type scale comprised of 28 items. PTE was measured in six different dimensions: Internal interpretation of control, external interpretation, mathematics teaching anxiety, mathematics teaching enjoyment, managing the school climate, and effectiveness of the preservice mathematics program. Four indicators, all of the external interpretation of the student's learning control, measured the general teaching efficacy dimension (GTE) (see Table 1). The negative statements were reversed and the data were analyzed for the whole sample and for each group, on the whole scale, on the sub-scales and on each component, separately. The ANOVA was used to explore possible differences between sample groups and age groups.

[1] DUTTON (1988)

5 Results and discussion

5.1 Attitudes and attitude change

An alarmingly high proportion of students bring extremely negative attitudes to Teacher Education. For instance, 24% of the students endorsed the statement "I detest mathematics and avoid using it at all times" at the entry phase. Similarly, the statements "I have never liked mathematics", "I have always been afraid of mathematics" and "I do not feel sure for myself in mathematics" were endorsed by 28%, 15%, and 47% of the students, respectively.

The χ^2-test showed significant difference ($p \leq .05$) on 14 out of 18 statements, 13 of which indicate positive change during the program implementation. The proportion of those who "detest mathematics" dropped from 24% to 12%, of those who "never liked mathematics" from 28% to 18%, while of those who "enjoy working and thinking about mathematics outside school" raised from 20% to 40%. Finally, the proportion of students who "never get tired of working with mathematics" raised from 19% to 27%.

The same pattern of responses appeared in the Self-rating scale. On entrance, 36.9% of the students located themselves in the range 1-5 indicating negative attitudes about mathematics, 20% expressed neutral views, and only 43.1% of the students' felt positively towards mathematics. In the course of the program implementation the proportion of subjects who detest mathematics dropped from 14.6% to 5.9% ($p \leq .01$), while the proportion of subjects on the positive side of the scale raised from 39.8% to 51.6%.

On entrance, students stated that they liked mathematics because "it develops mental abilities" (47%), "it is practical and useful" (39%), "it is interesting and challenging" (35%), and "it is necessary for modern life" (35%). They disliked mathematics mainly because "they were "afraid of it" (29%), due to "poor teaching" (27%), and "lack of teacher enthusiasm" (25%)". The χ^2-test showed significant change on nine out of the ten statements of the liking part of the scale and two of the disliking part, all in the positive direction. For instance, the proportion of those who like mathematics because "it is necessary for modern life" raised from 35% to 76% ($p \leq .001$), and because "it develops mental abilities" from 47% to 72% ($p \leq .001$), while significantly more students disliked mathematics due to "teachers' lack of enthusiasm", at the end of the program than at the beginning ($p \leq .001$), and fewer students continued to believe that "mathematics is never related to everyday life" ($p \leq .03$).

The *Median Polishing Analysis* was applied to responses of students' on the Dutton scale, which was partitioned into three sub-scales focusing respectively on *anxiety* from mathematics (five items), *usefulness* of mathematics (four items), and *satisfaction* from mathematics (eight items). This method partitions two-way tables into three interpretable parts, the Grand Effect (GE), which indicates the typical response across all the items, the Row Effect (RE) which tests for differences between responses in different rows (phases), and the Column Effect (CE) that reveals relative differences among the items. The results showed a low endorsement of the anxiety part (GE=21%), a rather high acceptance of the utility part (GE=41%) and a moderate agreement on the satisfaction part of the scale (GE=34%). Row effects showed consistent positive change during the three phases in all sub-scales. Specifically, a reduction was observed on the anxiety part $3.0 \Rightarrow 0.0 \Rightarrow -3.0$, meaning a positive development from one phase to the next, an increase on the utility dimension $-4.5 \Rightarrow 7.5 \Rightarrow 9.5$, showing a steady improvement of attitudes, and finally the change in the satisfaction sub-scale was from $-14.5 \Rightarrow 3.5 \Rightarrow 3.5$. Thus, an improvement was observed between the first and the final phase, in all three sub-scales, that might be due to the mathematics program.

Extracts form the interviews tend to affirm the conclusions concerning beliefs and views about mathematics and the role of the historical approach. The first three refer to students' feelings before and the fourth after exposure to the program.

- "My attitudes were extremely negative thanks to my teachers. Mathematics was for me a piece of work based on getting the right answer and I just could not do that".
- "The proper way to learn mathematics was by memorizing facts and procedures",... "any statement or answer in mathematics was either right or wrong".
- "When I entered the University I felt relief; I was happy, thinking that I had finished with mathematics. The moment I learned that the program required 3 more courses in mathematics I felt frustration. I felt that mathematics will hunt me for ever".
- "History of mathematics provided me with a variety of interesting, new, experiences. I realized that mathematics has always been and continues to be a very useful subject... I followed the efforts of people to use mathematics to solve daily problems. The course showed me that mathematics is normally a human activity. I felt more confident when I realized that even great mathematicians did mistakes as I frequently do".

5.2 Mathematics teaching efficacy

Table 1 shows indicative items from each dimension of the scale and the percentage of participants who endorsed one of the two positive alternatives, for UC graduates, PA graduates (together with PAG), and GU graduates, respectively. A variability of endorsement on items and among groups of participants is evident.

Table 1

Endorsed Proportions to selected Efficacy Items by CU, PA, and GU Subjects

	Personal Teaching Efficacy (PTE)
Internal Contrl (P-I)	11. When a child becomes better in mathematics, I believe that it was due to the variety of different ways I found to help him/her (82%, 76%, 67%)*
External Contr (P-E)	12. I feel that irrespective of my effort, I cannot teach mathematics as successfully as I can teach other subjects (93%, 81%, 86%)
Teach Anx	21. Sometimes I feel anxious that a student might ask me a question that I do not know how to answer or I cannot explain (86%, 89%, 90%)
Teaching Enjoy	30. If I were to choose one subject in a colleague's class, I would have opted for mathematics (57%, 40%, 29%)
School climate	15. When I have difficulties as to how to teach mathematics, I seek advice form experienced colleagues in my school (50%, 66%, 52%)
Preservice program	9. The preservice mathematics program, offered me the necessary basics to become an efficient mathematics teacher (50%, 25%, 28%)
	General Teaching Efficacy (GTE)
External cont. (G-E)	5. As now things really stand, the weak students cannot get the required help to get through in mathematics (50%, 43%, 19%)

* The percentages represent the positive endorsement by CY, PA, and GU graduates, respectively

Four out of five participants endorsed the positive side in six items, which reflect that teachers felt competent "to help pupils make progress even in topics of mathematics considered as difficult", "to help pupils think mathematically", "to consult experienced colleagues, when facing difficulties", "to answer the pupils' questions", "to correct pupils' assignments", and "unwillingness to give away mathematics in case they had the chance to give away one course". On the other side, the majority of participants ($\geq$ 50%) endorsed the negative side of the scale in five items reflecting "anxiety to cover the subject matter", "efficacy of the preservice program", "capability to help weak students", "possibility of weak pupils to get help", and the ability to "discipline a student who is not used to from home".

The ANOVA showed significant differences between the four sample groups (UC, PA, UPA, and GU graduates) on the general teaching efficacy dimension (GTE), on the preservice mathematics program, and on the total scale according to years of teaching service (0-5, 5-10, and $\geq$ 10, years). The first finding of this study was that the UC graduates, more than the other groups, believe that students are teachable, i.e., that students progress is controlled by internal to the school factors ($\underline{F}$ = 3.150, d.f. = 3, $\underline{p}$ = .027). Specifically, UC graduates were found to have better beliefs on each one of the four items comprising the GTE dimension of the scale. The peak of the difference was found on the item reflecting beliefs that "as the situation really stands, the weak students cannot get the required help to get through in mathematics" ($\underline{X}_{UC}$ = 3.3, $\underline{X}_{rest}$ = 2.9). UC graduates were also found to hold better beliefs that t he other three groups concerning their preservice program of mathematics education ($\underline{F}$ = 8.992, d.f. =3, p = .000). PA graduates (both sub-groups) expressed the most negative evaluations of the preservice program, while GU graduates expressed moderate feelings ($\underline{X}_{UC}$ = 3.29, X_{PA} = 2.75, X_{PAG}=2.27, X_{GU} = 2.90).

The significant difference in efficacy beliefs found according to the length of service on the total scale ($\underline{F}$ = 3.257, d.f. = 3, p= .042) means that teachers beliefs tend to get worst during the first years of service and improve later on (the mean value varie d in the form 3.59 ⇓ 3.47 ⇑ 3.65). This affirms and extends earlier findings (Hoy & Woolfolk, 1993) that efficacy feelings improve with experience. The UC graduates felt uncertain about the best procedure to adopt in teaching certain topics ($\underline{F}$=3.150, d.f. = 3, p = .021), whereas they did not, as much as the other teachers, endorse the idea that "there are children facing so many difficulties that I am unable to help". The former finding can be interpreted as an excessive responsibility of CU graduates during the first year of their employment.

Analysis of interviews

In interviews we encouraged the subjects to talk about their experiences and express their evaluations with respect to mathematics teaching. The responses were classified into three categories: as positive, neutral, or negative, on the basis of the text. We present excerpts on perceived efficacy a) to influence the learning of non-mot ivated pupils (NM), b) about the preservice program (PSP), and c) to manage the school climate (SCL).

$Q1 - NM$ How confident do you feel to help the non-motivated pupils?

Positive view I am sure that I can help all students to make progress. For the 2-3 n on-motivated or slow learners that are normally found in every class, special provisions are needed, such as to simplify activities, allow for more time, and be in close contact with parents.
Neutral view : I can help slow learners make progress, but "it is not possible for all students to reach the same level". In every class "there are two or three special cases (e.g., problematic families) for whom it is very hard to do anything". "They need special attention and I have no time".

$Q2 - NM$: In the case that a child makes progress, to whom should this success be credited?
Positive view : There are many factors that influence students' learning. The final outcome is due to a combination of joint efforts. However, in my view the teacher is the factor to be primarily credited.
Neutral view : I think that a student's progress is due to the teacher, the parents and the student himself. a) "Teacher's influence on students' learning is limited because they are at school only for about 4-5 hours a day". b) "In cases with serious problems, the teacher cannot do much".
Negative view : a) "The crucial factor is the child. quite a lot depend on him/her". b) "I think that everything depends solely on the child".

$Q1 - PSP$: How do you judge the preservice mathematics program you passed through?
Positive view : I believe it offered me all the necessary background to teach mathematics. When I was a student, I was frequently wondering whether several of the issues and ideas discussed were practically applicable, now I am convinced they are useful.
Neutral view : Most useful was the Methods course. History of mathematics helped in the sense that one appreciates the developmental nature of mathematical ideas, but I think that the tutorials could have been more profitable.
Negative view : We only did one course on teaching methods, "which was just an introduction, rather irrelevant to teaching".

$Q2 - PSR$: A frequently raised point refers to the balance between theory and practice, what do you think about that?
Positive view : I now think that the teacher needs a strong theoretical basis. We covered a wide spectrum of topics and this provides the teacher with a basis necessary to choose and create

didactical situations on his own.
Neutral view : I think it could have offered us more. There were quite a few topics, which were not useful for the primary teacher in the content courses, that caused us additional anxiety. I think we should stick to methods of teaching specific topics.
Negative view : "It paid too much attention to mathematics instead of methods of teaching mathematics, it was very poor... Instead of sets and probability, we could have done more didactics of mathematics". "We did a lot of mathematics at the high school, we should do more teaching methods instead".

$Q1 - SCL.$: How much concerned do you feel about the subject matter coverage?
Positive view : a) "I don't worry about that, though I am quite behind schedule. I do not like to rush and miss some pupils. I will finally be able to get through the subject matter". b) "In my view it is not necessary to cover all the topics. The issue is to let pupils learn what we do, so I don't worry". "I do not proceed beyond a certain point, unless I am certain that 98% of the pupils have learnt it well". The motto in our school is quality, nor quantity". "We all protested. The inspector stated that we were behind schedule, but I cannot push the children, they need to understand".
Neutral view : a) "Yes, because there is so much in the books, and I realized that the level of my pupils is not so high". b) "One has to insist on the basics, not to rush, and that worries me a lot". c) "We hurvied to cover the prescribed matter without going in depth".

$Q2 - SCL$: How do you feel when the principal or the inspector attends your class?
Positive view : "Well, it's natural not to feel as easy as when you are on your own; it may cause me some tension but not really anxiety. I have nothing to hide, I want them to get the real picture of the class, to bring possible problems on the surface". "It is a matter of self-confidence".
Neutral view : "There is a certain degree of anxiety. After all, you are under assessment, they are examining your results".
Negative view : "I feel anxious and uneasy... I think it is in my character".
Table 2 summarizes the responses on the three issues of each of the three teacher groups. The general picture seems to affirm the results found from the questionnaires. UC graduates expressed better efficacy beliefs to influence the non-motivated students and about the preservice program, while they felt relatively less efficient to manage the school climate. The latter might be a consequence of antagonism, which was developed in some of the older teachers due to the abolition of the Pedagogical Academy. There are indications that this climate is getting better. PA graduates do not feel as efficient to help the non-motivated students and they consider their preservice program as inefficient. The responses of UG graduates seem to be rather normally distributed, indicating that they hold moderate efficacy beliefs.
Table 2
A summary of classified responses in interviews

		Univ. Cyprus	P. Academy	Greek Univ.
Efficacy to influence the Non-Motivated	Positive	85%	33%	2/6
	Neutral	15%	50%	4/6
	Negative	0%	17%	2/6
Efficacy to manage the School Climate	Positive	54%	78%	4/6
	Neutral	32%	17%	2/6
	Negative	14%	5%	2/6
Efficacy of preservice program	Positive	50%	0%	2/6
	Neutral	50%	50%	4/6
	Negative	0%	50%	2/6

6 Conclusions

The results of this study provide support for three hypotheses. First, prospective teachers bring to teacher education serious misconceptions and negative attitudes towards mathematics. Second, the designed mathematics preservice program was effective to change preservice teachers' attitudes and third, the UC graduates hold better efficacy beliefs in mathematics teaching than the graduates form other institutions.

Negative feelings towards mathematics are mostly due to teachers' shortcomings, which lead to students' failures and negative attitudes. The situation calls for urgent measures, otherwise, it is highly probable that a considerable proportion of teachers will continue to view mathematics as a fixed discipline, teach along the traditional lines, influence students in non desirable directions, and perpetuate the same situation. One of the tasks of Education Departments is to break down the vicious circle so formed.

In the present study, we sought change in belief systems as a secondary goal of the teacher preparation program. History of mathematics was used throughout as the vehicle to develop mathematical understanding, though two more special environmental factors were conducive to that effort. First, the establishment of a new Department offered the chance to design a mathematics program from the beginning, and second, the historical heritage of the student population. At the end of the program, changes in the attitudes of the students were observed as evidenced by three complementary instruments and several statistical analyses. A significant change in students' responses was found in most items of the Dutton scale, on each of the three sub-scales, on the Justification scale (mostly the liking part), and on the self-rating scale. The fact that there has been non-desirable change of attitudes in two items indicates that in some cases the improvement has not reached the level to overcome deeply rooted mathematics anxiety. Most of students' feelings were formed over their entire school life and, in many cases,

were influenced by long prejudices of the social environment. It seems that some emotions in the minds of students are resistant to change; longer exposure and more challenging experiences seem to be essential in order to override them.

CU graduates more than other teachers felt that they are capable to help even the unmotivated pupils, that pupils in general are teachable, and that they were relatively satisfied with their preservice mathematics program. It should be noted, however, that the "success" of the implemented program was found to be satisfactory in comparison to the other two groups, who showed a low esteem of their own preservice programs (in absolute terms the acceptance of the program is rather moderate). The overall evaluation of the preservice program is considered, however, as positive because CU graduates were found to outperform their counterparts in almost all scale dimensions as well. This indicates that the program based on the history of mathematics was effective in improving students' attitudes and developing positive mathematics teaching efficacy beliefs.

It is surprising and it deserves further investigation that the fact that no differences were found in attitude change in terms of gender, type of high school, mathematics performance, and the family socio-cultural conditions. It would be very encouraging, if the program is really so powerful as to affect students' beliefs and efficacy feelings, irrespective of individual characteristics.

The present study did not disentangle several factors that might have been operative. One of these factors relates to the mental models that the program created in the students' minds about mathematics. Another factor is the presence of the university instructors themselves and the way they implemented these models in the classroom. A final reservation concerns the permanence of this change and its long lasting effect on actual teaching behavior. This final dimension is one of the directions in which we plan follow up and further evaluation.

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Sur les modes du savoir[2]

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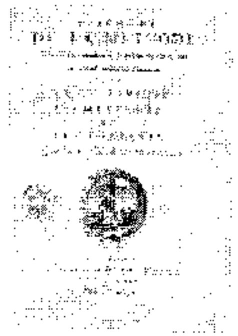
Abstract

Ce travail se penche sur le problème de la construction du savoir mathématique dans une optique culturelle. En prenant la rationalité mathématique d'une certaine époque comme partie de la rationalité culturelle dans laquelle elle se trouve inscrite, il s'agit d'investiguer comment la rationalité mathématique se forme et donne lieu à un type particulier de savoir à la lumière des pratiques de signification culturellement reconnues, pratiques qui délimitent le pensable et l'impensable, le possible et l'impossible. Le "texte" mathématique est vu ici comme un discours se déployant suivant les possibilités sémiotiques offertes par les relations issues des activités sociales des individus et la conceptualisation du monde et des objets mathématiques que ces activités permettent selon représentations sociales en place. Le cas des mathématiques Babyloniennes sert à illustrer cette approche culturelle épistémologique.



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1 Introduction

Un des débats les plus riches et intéressants du 20ème siècle concerne le rôle de l'individu et de sa culture dans la production du savoir. *Grosso modo*, on peut distinguer deux positions fort différentes dès le début du siècle. D'une part, on a le pôle offert par la psychologie expérimentale, où l'on soumet l'individu à des interrogations selon le même patron suivi en science positiviste pour interroger la nature et les animaux et qui privilégie le format de laboratoire et son emphase sur les variables, les mesures et le traitement de celles-ci. On situe le début formel de la psychologie expérimentale au moment de la création du laboratoire de Wilhelm Wundt à l'Université de Leipzig en 1879 (voir VINEY & KING 1998). D'autre part, on a l'approche anthropologique qui, au lieu de se pencher sur l'individu isolé, prend celui-ci au sein de sa culture. L'interrogation ne porte pas ici sur comment l'individu isolé arrive à connaître ce qu'il connaît et à se faire les représentations du monde qu'il détient. Il s'agit de savoir comment se forment les représentations de l'individu dans le contexte de sa culture. À vrai dire, il faudrait ajouter ici les réflexions provenant de l'épistémologie traditionnelle. Celle-ci se distingue des réflexions psychologiques et anthropologiques en ce que la connaissance est supposée ne pas pouvoir être étudiée par des techniques d'observations, que ce soit des observations expérimentales de style laboratoire psychologique ou *in situ* comme font les ethnologues.

Depuis la fin des années 70, la Didactique des Mathématiques ou Mathematics Education s'est consolidée comme une discipline plutôt liée à la psychologie qu'à l'anthropologie ou à la philosophie et comme telle elle a hérité des méthodes expérimentales de la première. Toutefois, récemment il y a une tendance de plus en plus vive à examiner les processus d'apprentissage en utilisant des outils et des points de vue liés à l'anthropologie. Sans vouloir entrer dans les détails, contentons-nous de signaler que cette tendance à l'anthropologisation de l'Éducation Mathématique va de pair avec une mise en question, dans le domaine général des sciences humaines, de la relation de l'individu et sa culture dans la construction du savoir telle que la psychologie expérimentale l'avait implicitement ou explicitement véhiculée. C'est sur ce point que nous voulons nous arrêter ici. Plus précisément, le but de cet article est d'examiner deux approches théoriques qui offrent des vues distinctes sur la relation sujet/culture dans la production du savoir.

La première approche, basée sur un point de vue évolutionniste du savoir mathématique, est celle de Raymond Wilder. La deuxième approche est celle de l'épistémologie archéologique offerte par Michel Foucault. Bien sûr, puisque les deux approches s'insèrent dans des traditions différentes, la façon même de poser le problème épistémologique culture/individu dans chacune d'elles est différente. Néanmoins, bien que les degrés d'élaboration théorique soient différents, le problème y apparaît suffisamment traité pour tirer des renseignements permettant un contraste assez clair. Notre examen des deux approches mentionnées est précédé d'une courte discussion de certains éléments qui ont servi de base à une théorie de la connaissance élaborée à partir de la philosophie de DESCARTES et qui a eu une grande influence non pas seulement dans le domaine de la philosophie, mais aussi dans celui de la psychologie. Cela permettra de mieux cerner, nous espérons, les origines historiques du courant "individualiste" de la psychologie traditionnelle et la présentation des approches retenues pour discussion dans cet article.

2 Psychologie et anthropologie : individu, culture et savoir

Un des présupposés fondamentaux au sein de la recherche épistémologique est, tel que mentionné dans l'introduction, celui de la relation individu/culture dans la construction du savoir. Une longue tradition philosophique a donné la primauté à l'individu sur la culture en

ce qui concerne la production du savoir et a donc centré les efforts sur le premier au détriment de la dernière. L'approche épistémologique qu'on peut appeler "individualiste" postule donc l'individu comme source du savoir et a son origine dans le concept du "je" qui commence à émerger lors de la Renaissance et qui trouve une expression très nette dans la philosophie du 17ème siècle (notamment dans le "je" cartésien). On sait bien que le "je" de la Renaissance s'érige en réaction contre la vue théocentrique du Moyen Âge selon laquelle l'individu est une créature de Dieu et soumis à la volonté de Celui-ci. Il apparaît lié à une nouvelle vision de l'individualité qui se met en place peu à peu à partir du 15ème siècle, lors du passage d'une pensée autoritaire et dogmatique à une pensée plus autonome, englobée par les changements produits par une mercantilisation et une étatisation croissante et l'apparition de nouvelles couches sociales riches tant en ville qu'en province. C'est dans ce contexte que l'invention d'une conscience individuelle devient possible, un contexte où l'individu s'appartient pour la première fois dans l'histoire, comme le suggère Norbert ELIAS dans son livre *La société des individus* (1991).

Ces changements au niveau de l'individualité ont des répercussions au niveau du savoir qui, pendant le Moyen Âge, avait trouvé appui dans l'Écriture divine et les textes des Pères de l'Église. Que devient donc le savoir ? Sur quoi va-t-il s'appuyer ?

La nouvelle conception du savoir s'identifie désormais au certain. Connaissance équivaut à certitude, mais il s'agit maintenant d'une certitude séculaire. Ainsi, quand DESCARTES cherche, contre l'esprit sceptique de son temps, ce qui peut être incontestable, ce sur quoi on peut être certain à coup sûr, ce qu'il trouve est le sujet qui pense et qui en pensant peut affirmer sans le moindre doute sa propre existence. Dans la quatrième partie du *Discours de la Méthode*, il dit :

> J'avais dès longtemps remarqué que, pour les mœurs, il est besoin quelquefois de suivre des opinions qu'on sait être fort incertaines (. . .) mais pour ce qu'alors je désirais vaquer seulement à la recherche de la vérité, je pensai qu'il fallait que je fisse le contraire (. . .) je me résolus de feindre que toutes les choses qui m'étaient jamais entrées en l'esprit, n'étaient non plus vrais que les illusions de mes songes. Mais aussitôt après, je pris garde que, pendant que je voulais ainsi penser que tout était faux, il fallait nécessairement que moi, qui le pensais, fusse quelque chose. Et remarquant que cette vérité : je pense, donc je suis, était si ferme et si assurée, que toutes les plus extravagantes suppositions des sceptiques n'étaient pas capables de l'ébranler, je jugeai que je pouvais la recevoir, sans scrupule, pour le principe de la philosophie, que je cherchais. (DESCARTES 1637/1967, pp. 31-32)

C'est ce principe qu'exprime DESCARTES dans la célèbre phrase "cogito, ergo sum" qui est à la base de la théorie de la connaissance du 17ème siècle. DESCARTES bien sûr est tout à fait conscient de la nouveauté qu'il est en train d'introduire. Ainsi, dans un paragraphe précédant les propos cités ci-dessus, il dit :

> Je ne sais si je dois vous entretenir des premières médiations que j'y ai faites; car elles sont si métaphysiques et si peu communes, qu'elles ne seront peut-être pas au goût de tout le monde. Et toutefois, afin qu'on puisse juger si les fondements que j'ai pris sont assez fermes, je me trouve en quelque façon contraint d'en parler. (*Op. cit.*, p. 31)

Le nouveau principe philosophique donne lieu à une tradition fondée sur une séparation nette entre la pensée (nécessairement immatérielle) et le monde matériel. L'accès au savoir n'est plus (ou en tout cas pas exclusivement) garanti par la tradition de l'Église. Avec DESCARTES, l'individu trouve, dans sa capacité à penser sa propre existence, non pas seulement l'assurance mais aussi la clé vers la vérité – vérité que maintenant il ne doit qu'à lui-même et à sa capacité de réflexion.

Cela ne veut pas dire que dans la mesure où l'on a réfléchi sur la connaissance, on n'a pas pris en compte, d'une façon ou d'une autre, le rôle de ce qui est extérieur à la pensée. Le contact de l'Europe avec d'autres cultures, que ce soit lors des campagnes de conquêtes ou lors des missions d'évangélisation, en Chine, en Afrique, en Amérique ou ailleurs, a été l'objet d'une prise de conscience que les gens d'ailleurs pensent autrement et croient à d'autres choses.

Comment est-il possible que la connaissance soit différente selon l'endroit géographique? Les philosophes des Lumières ont résolu le problème en partant de l'idée du progrès. La connaissance progresse et les différences s'expliquent par des écarts dans la ligne évolutive. Bien sûr, le rôle de l'environnement est ici très modeste. Il sert à peine à effectuer des identifications territoriales. Mais c'est ce contact avec l'"Autre", le "Différent", ou le "Primitif" comme on disait à l'époque, qui a permis de poser le problème autrement. Ainsi, à la fin du siècle dernier on a pu voir Durkheim et l'école française de sociologie se pencher sur la façon de penser des gens d'autres cultures. DURKHEIM disait de façon très claire que les concepts avec lesquels nous pensons sont des *représentations collectives* et comme tels ils ont une origine culturelle (DURKHEIM 1968, p. 621). Comme le fait une grande partie de l'anthropologie contemporaine, DURKHEIM basait sa position sur le rôle du langage dans la connaissance. En effet, selon lui, les concepts avec lesquels nous pensons couramment sont ceux consignés dans le vocabulaire d'une culture qui exprime les produits d'élaborations et d'expériences collectives qui excèdent celles de l'individu (cf. *op. cit.*, pp. 620-21).

LÉVY-BRUHL s'est intéressé à la logique sous-tendant ce qu'on appelait la "mentalité primitive" et mettait en évidence des formes de pensée très différentes de celle qu'on trouvait en Europe. En particulier, les formes de pensée se voyaient solidaires des activités communales, ce que LÉVY-BRUHL a exprimé dans sa "loi de participation" (cf. LÉVY-BRUHL 1949, 1960).

Bien que de façon très sommaire, la discussion précédente permet de situer les origines des deux courants de pensée différents du 20ème siècle, le psychologique individualiste et l'anthropologique collectiviste. Nous allons nous tourner maintenant vers deux approches spécifiques qui abordent de façon plus directe le problème qui nous intéresse ici, à savoir, la relation individu/culture dans la production du savoir.

3 Wilder et l'évolution des concepts

Dans son livre *The Evolution of Mathematical* Concepts (1968), WILDER conteste la vision d'une mathématique sans attaches culturelles et le réductionnisme historique qui présente les mathématiques comme une suite d'événements plus ou moins exactement datés. Il dit :

> Connaître l'histoire ce n'est pas assez; dates, matériel biographique, etc. sont importants, mais ils forment partie de la collection d'artefacts pour une étude de ce type. (op. cit. ix)

En fait, sa position est appuyée par une distinction qu'il propose entre histoire et évolution. Alors que dans la première l'importance est au matériel et à sa date, dans la deuxième l'importance est au *changement*.

Cette idée de base est approfondie dans un livre publié en 1981, *Mathematics as a Cultural System*, où il se donne la tâche de mieux expliquer son concept de culture. Culture apparaît ici, un peu comme chez Spengler, comme une super entité organique. Mais, chez WILDER, la culture est conçue comme une entité qui évolue à partir d'un certain nombre de principes, qu'il

convient alors de mettre au clair. Le concept d'évolution est pris dans un sens néo-darwinien (d'ailleurs une photo de l'anthropologue anglais E.B. Taylor est insérée dans le livre de 1968).

L'idée sur laquelle pivote son œuvre est celle de considérer les mathématiques comme une sous-culture de la culture générale. Ceci lui offre la possibilité de voir la première comme une entité en évolution et de s'interroger sur les lois de cette évolution.

La "lutte de survie" qui décide le destin des idées est vue à la manière d'un positivisme qui pose la survie en termes d'efficacité et de pouvoir d'explication. Il dit :

> Ce n'est pas seulement judicieux du point de vue pratique mais aussi théorique de traiter la culture comme une entité super-organique qui, comme le langage, évolue d'après ses propres 'lois'. Le jugement final concernant une théorie de la culture ne doit pas être si cela correspond avec nos propres croyances (beliefs) par rapport à la réalité, mais si cela marche mieux comme un principe d'explication et/ou comme dispositif prédictif. (WILDER 1981, p. 13)

Dans la dernière section du dernier chapitre, il donne dix lois qui gouverneraient l'évolution des concepts mathématiques, dont la suivante, qui porte le numéro 4 :

> Le degré dans lequel un concept continue à être important mathématiquement dépend du mode symbolique dans lequel il est exprimé et de sa relation avec d'autres concepts. Si un mode symbolique tend vers l'obscurité (...) alors -en supposant le concept utile- une forme plus facilement compréhensible évoluera. (...)

Par mode symbolique, WILDER n'entend pas ici les réseaux symboliques dans lesquels les signes et les concepts d'une culture se trouvent nécessairement submergés. WILDER se réfère ici au système de signes mathématiques, et donne comme exemple l'évolution de la représentation positionnelle des nombres. Selon lui, la représentation Babylonienne a été abandonnée pour une autre plus claire et mieux apte, qui a culminé finalement avec la représentation décimale infinie d'un nombre réel quelconque (1981, pp. 208 et 210). Toutefois, comme le souligne Markus, un système de signes ou une langue "peut paraître 'illogique' ou 'trop compliqué' seulement à quelqu'un qui ne la maîtrise pas. Pour le locuteur natif elle est transparente". (MARKUS 1982, p. 60). Vraisemblablement, pour le scribe Babylonien, l'akkadien ou le sumérien utilisés dans les textes mathématiques étaient aussi clairs que la terminologie 'cossiste' utilisée à la Renaissance par un maestro d'abaco.

Mais, selon nos objectifs, ce qui nous intéresse le plus chez WILDER est le rapport au savoir du sujet et de la culture qu'il propose. Il s'affiche contraire à l'idée d'une culture faite par les "grands hommes". Ce n'est pas l'individu qui, par des inventions heureuses, fait avancer la culture, car tout ce dont il avait besoin pour accomplir la nouvelle création était déjà là. L'inventeur, selon lui, dépend totalement non seulement des idées qu'il a collecté des autres mais aussi de cette poussée finale qui mène à l'invention. Cela l'amène à dire que : "En réalité, les inventions sont *collectives*, c'est-à-dire, elles sont des accomplissements culturels". (1981, p. 10: italiques dans l'original)

Pour comprendre cette position, on doit avoir présent à l'esprit l'idée de culture chez WILDER. En fait, puisque la culture évolue d'après ses propres lois, l'homme (au singulier) n'y peut rien. Il devient en quelque sorte l'exécuteur d'un destin qui lui est offert de l'extérieur. Le "grand inventeur" vient piger dans ce tout super organique qui s'ouvre à lui comme un grand réservoir ce qu'il lui faut pour accomplir l'acte d'invention. "C'est vrai, il ajoute, il [l'inventeur] fait un pas décisif; mais ceci est un jugement subjectif, puisque toutes les étapes menant à l'invention étaient aussi critiques, au sens d'être nécessaires pour l'invention finale, comme celle-là". (*op. cit.* p. 10).

On peut dire que, dans l'approche de WILDER, la relation au savoir de l'individu et à la culture reste sabotée par une vision téléologico-évolutionniste du savoir. S'il est vrai qu'il détient une position d'après laquelle les hommes (au pluriel) font la culture et que, réciproquement, en mettant à disposition de chacun de ses membres les croyances, outils, mœurs, etc. que constitue la culture, celle-ci produit chacun de ses individus, on voit mal comment peut s'orchestrer toute une diversité de pratiques culturelles de sorte que leurs produits puissent satisfaire des nécessités variées qui sont évaluées culturellement selon des critères spécifiques souvent contradictoires d'une culture à une autre. Rappelons à ce sujet ce qui disait SPENGLER :

> *Il n'y a pas une, il n'y a que des mathématiques.* Ce que nous appelons histoire de "la" mathématique, réalisation prétendue progressive d'un idéal unique et invariable, est en effet, dès qu'on écarte l'image trompeuse de la superficie historique, une variété d'évolutions qui sont achevées en soi. (SPENGLER 1917/1948, p. 70).

4 L'archéologie épistémologique foucaldienne

FOUCAULT définissait son travail des années 60 comme une quête de ce qui est au-delà de l'histoire des sciences, des connaissances et du savoir humain et qui portant la détermine. Dans une entrevue réalisée en 1968, il dit :

> Si vous voulez, l'hypothèse de travail est la suivante : l'histoire de la science, l'histoire des connaissances, n'obéit pas simplement à la loi générale du progrès de la raison, ce n'est pas la conscience humaine, ce n'est pas la raison humaine qui est en quelque sorte détentrice des lois de son histoire. Il y a au-dessous de ce que la science connaît d'elle-même quelque chose qu'elle ne connaît pas; et son histoire, son devenir, ses épisodes, ses accidents obéissent à un certain nombre de lois et de déterminations. Ces lois et ces déterminations, c'est celles-là que j'ai essayé de mettre au jour. J'ai essayé de dégager un domaine autonome qui serait celui de l'inconscient du savoir, qui aurait ses propres règles, comme l'inconscient de l'individu humain a lui aussi ses règles et ses déterminations. (FOUCAULT 1994, pp. 665-666)

Il distingue deux pôles bien délimités dans l'activité humaine. D'un côté, celui de l'expérience quotidienne, régie par ses codes fondamentaux de perception, d'utilisation du langage, des hiérarchies pratiques, etc. De l'autre côté, celui des théorisations qui tentent d'expliquer l'ordre du monde à travers, par exemple, des lois, des explications à caractère "rationnel". Et c'est entre ces deux-là qu'il situe cette région qui "délivre l'ordre dans son être même" (FOUCAULT 1966, p. 12), c'est-à-dire cette région qui explique l'ordre de l'ordre ou encore la rationalité de l'ordre.

C'est l'ordre de l'ordre qui oriente et sous-tend les différentes activités d'une culture. Dans son étude sur l'épistémé à l'âge classique, élaboré dans *Les mots et les choses*, FOUCAULT examine l'ordre du langage, celui des classifications scientifiques, celui des richesses et montre comment ces différentes activités sont toutes sous-tendues par une façon commune de représentation — une façon de représentation qui se distingue du mode de représentation médiévale. Foucault s'efforce de montrer qu'à l'époque classique, en effet, il s'agit de représenter l'identité et la différence et non plus, la similitude, comme c'était le cas au Moyen Âge.

Ce qui rend essentiellement différent l'épistémé à l'époque classique de l'épistémé au Moyen Âge est, dans l'archéologie Foucaldienne, la configuration sémiotique qui la sous-tend, le mode d'être du symbole et les représentations que celui-ci permet de former. Cette archéologie s'érige contre le courant d'objectivation basé sur la résolution de problèmes, un courant d'origine positiviste qui trouve dans la compétition des théories et dans leur faculté à résoudre une gamme plus ample de problèmes son critère d'efficacité et de supériorité. Foucault dit, non sans ironie :

> Il se peut bien —et encore ce serait à examiner— qu'une science naisse d'une autre; mais jamais une science ne peut naître de l'absence d'une autre, ni de l'échec, ni même de l'obstacle rencontré par une autre. (FOUCAULT 1966, p. 140).

L'archéologie Foucaldienne se range dans un courant opposé à celui de la résolution de problèmes; il se classe parmi les paradigmes sémiotiques du $20^{ème}$ siècle où l'on trouve Gadamer, "le dernier" Wittgenstein et d'autres. Bien qu'ayant des différences théoriques fondamentales entre eux, ces paradigmes placent le langage dans un lieu privilégié. Pour FOUCAULT, comme on la vu précédemment, un épistémé donné se révèle dans la façon dont les mots nous parlent des choses, dans l'ordre du discours. Il suggérait, en outre, que le langage appartienne à une couche archéologique différente, plus profonde, que les couches auxquelles appartiennent les richesses et les classement de la nature (1966, p. 245).

Qu'en est-il donc, chez FOUCAULT, de l'individu et de la culture dans leur rapport au savoir? Notons que, dans *Les mots et les choses*, après avoir posé la question de ce qui rend possible de penser une pensée, il dit : "À la limite, le problème qui se pose c'est celui des rapports de la pensée à la culture". (*op. cit.*, p. 64).

Certains critiques ont mentionné que l'archéologie épistémologique de FOUCAULT laisse toutefois de côté l'individu lui-même, c'est-à-dire l'individu concret. On lui reproche ce faisant de réifier le savoir. Sartre, par exemple, souligne chez FOUCAULT l'absence de l'analyse des conditions historiques de la pensée. Il dit :

> Foucault ne nous dit pas ce qui serait le plus intéressant : à savoir comment chaque pensée est construite à partir de ces conditions, ni comment les hommes passent d'une pensée à une autre. Il lui faudrait pour cela faire intervenir la praxis, donc l'histoire, et c'est précisément ce qu'il refuse. (...) [Un historien] sait qu'on ne peut pas écrire d'histoire sérieuse sans mettre au premier plan les éléments matériels de la vie des hommes, les rapports de production, la praxis. (SARTRE 1966 p. 87-88)

Sartre voyait que l'approche structuraliste se dérobait d'une réflexion sérieuse sur la praxis et les conditions historiques des individus dans la formation du savoir. Par là, les structuralistes (parmi lesquels il incluait Foucault, Lacan, Althusser et d'autres) escamotaient, selon Sartre, le problème du sujet concret. Ainsi, en parlant du sujet chez Lacan, il dit :

> S'il n'y a plus de praxis, il ne peut plus y avoir non plus de sujet. Que nous disent Lacan et les psychanalystes qui se réclament de lui? L'homme ne pense pas, il est pensé, comme il est parlé pour certains linguistes. Le sujet, dans ce processus, n'occupe plus une position centrale. Il est un élément parmi d'autres, l'essentiel étant la "couche", ou si vous préférez la structure dans laquelle il est pris et qui le constitue. (SARTRE 1966, p. 91-92).

Le manque d'insertion, dans *Les mots et les choses*, d'une catégorie historique tenant compte des individus et de leur praxis, a été perçu comme quelque chose qui menait FOUCAULT à présenter l'épistémologie comme une structure contraignante, productrice des individus, des savoirs et, comme le souligne LeBon, l'histoire même. Dans une critique virulente lancée contre FOUCAULT, elle dit : "De la structure se déduit l'histoire". (1967, p. 1318)

Bien sûr, d'autres critiques s'attaquaient au relativisme culturel qu'on discernait dans l'approche épistémologique Foucaldienne (cf. par exemple, AMIOT 1967). Quoi qu'il en soit, FOUCAULT a tenté de mieux préciser sa position, dans *L'archéologie du savoir* (1969). Il s'y efforce de prendre une distance par rapport au structuralisme avec qui la terminologie utilisée dans *Les mots et les choses* (par exemple "les quadrillages des codes ordinateurs") le rapprochait. Mais il renonce à donner à son approximation épistémologique l'orientation théorique que la critique trouvait qui lui manquait. Pour lui, il s'agissait d'éviter tout réductionnisme. D'une part, il s'agissait d'éviter de voir le savoir comme le résultat des pratiques sociales d'une culture (ce qui rapprocherait, d'après lui, le savoir à l'idéologie); d'autre part, il ne fallait pas le placer dans l'immanence d'un terrain transcendantal.

FOUCAULT trouve une solution dans la genèse réciproque de l'objet/sujet et la catégorie *d'expérience* qu'ils définissent à partir de la culture. Il entend par expérience "la corrélation, dans une culture, entre domaines de savoir, types de normativité et formes de subjectivité"[1]. La catégorie d'expérience apparaît comme une étoffe tissée à l'aide de fils provenant des vérités de la culture, le réseau du pouvoir et la subjectivation des individus. La subjectivation est telle qu'elle conditionne le pouvoir —en étant à son tour conditionnée par celui-ci— et les vérités que ce pouvoir sanctionne. Ainsi, dans une entrevue avec Duccio Trombadori, il dit :

> Everything that I have occupied myself with up till now essentially regards the way in which people in Western societies have had experiences that were used in the process of knowing a determinate, objective set of things while at the same time constituting themselves as subjects under fixed determinate conditions. (FOUCAULT 1991, p. 70)

Il importe de noter que, chez FOUCAULT, les processus de construction de l'individu lors de sa rencontre avec le savoir ainsi que les processus de construction du savoir par cet individu qui puise sa subjectivité dans l'organisation de sa culture et le réseau d'expériences que celle-ci met à sa disposition, ne sont pas des processus distribués de façon homogène. C'est FOUCAULT lui-même qui a insisté sur le fait que la culture se donne des institutions qui répartissent ou distribuent le savoir dans son intérieur, en définissant ce qu'il a appelé *la volonté de vérité*, cette relation qui érige entre les individus un système d'exclusion. En analysant Aristote, il trouve derrière le philosophe qui s'affiche désintéressé, un discours sophiste qui cache une lutte de pouvoir. Lors de sa leçon inaugurale au Collège de France, il dit :

> Or cette volonté de vérité, comme les autres systèmes d'exclusion, s'appuie sur un support institutionnel : elle est à la fois renforcée et reconduite par toute une épaisseur de pratiques comme la pédagogie, bien sûr, comme le système des livres, de l'édition, des bibliothèques, comme les sociétés savantes autrefois, les laboratoires aujourd'hui. Mais elle est reconduite aussi, plus profondément sans doute par la manière dont le savoir est mis en œuvre dans une société, dont il est valorisé, distribué, réparti et en quelque sorte attribué. (FOUCAULT 1971, pp. 19-20)

Dans l'épistémologie que FOUCAULT propose, la construction du savoir ne peut donc être comprise qu'à condition de s'interroger sur la façon dont le savoir est mis en œuvre, distribué et attribué dans une société. De plus, la relation du sujet au savoir, dans l'épistémologie foucaldienne, n'est pas, comme on a pu le constater, une relation passive. Le processus au travers

[1]FOUCAULT, *L'usage des plaisirs*, p. 10.

duquel l'individu lui-même se trouve modifié (et par là culturellement construit) par ce qu'il arrive à connaître, s'effectue au long de la "labeur accomplie afin de connaître" (cf. FOUCAULT 1991, p. 68).

FOUCAULT nous donne un exemple de cette "labeur" quand il examine l'histoire de la sexualité dans l'Antiquité (FOUCAULT 1984). Il ne s'agit pas pour lui de montrer comment le sexe était pratiqué à l'époque. Il ne s'agissait pas non plus de savoir comment on a pensé le sexe. Il s'agissait d'examiner les mécanismes producteurs d'expériences d'élaboration du soi et les chemins par lesquels passe l'individu quand celui-ci s'engage dans la "labeur" de connaissance.

Ce qui apparaît chez FOUCAULT comme "labeur" de connaissance peut naturellement être thématisé de plusieurs façons. Il peut être vu, par exemple, comme la labeur qui, en plus de produire le sujet, produit également la connaissance et la culture. Bien que l'*Histoire de la sexualité* puisse être vue comme un retour que FOUCAULT fait, vers la fin de sa vie, sur le concept de sujet qui lui a valu tant de critiques, il est resté fidèle à ses principes des années 60. On a vu qu'il présentait une vision de construction réciproque entre l'objet et le sujet, mais peut-être serait-il juste de dire qu'il s'est plutôt intéressé à la façon dont ce sujet se trouvait constitué par sa culture.

C'est pourquoi souvent il est présenté comme quelqu'un pour qui la culture avec ses pratiques, ses systèmes d'exclusion et ses formes de pouvoir étaient productrices du sujet. AMIOT, par exemple dit :

> Selon Foucault, sans aucun doute, le sujet n'est le souverain ni dans l'épistémè de la Renaissance ni dans celle de l'âge classique, et s'il apparaît à l'époque moderne, c'est sous la forme d'un produit comme le simple corrélât des nouvelles "positivités", et sans ascendance ni descendance prévisible. (AMIOT 1967, pp. 1296-97)[2]

5 Synthèse et conclusion

Nous avons présenté deux approches qui se penchent, par des raisons différentes et avec des intensités distinctes, sur le problème épistémologique général du sujet et de sa culture dans la production du savoir. L'approche que suit WILDER repose sur une conception classique de culture qu'il emprunte à l'anthropologie et qui présente une culture comme une collection d'habitudes, rites, croyances, outils, mœurs, etc. groupés par des facteurs d'associations (par exemple, des réseaux sociaux occupationnels d'ordre tribal, professionnel, etc.). (WILDER 1968, p. 18; 1981, p. 7). En suivant une perspective évolutionniste, Wilder voit la culture comme une super entité organique qui évolue selon ses propres lois. Nous avons souligné le caractère téléologique qui sous-tend la conception du savoir chez WILDER, et avons suggéré que cela ne faisait que contourner le problème de la production culturelle du savoir.

Nous avons examiné l'archéologie épistémologique proposée par FOUCAULT. L'épistémologie de FOUCAULT jette un éclairage important sur les théories de l'action ou de l'activité, comme celle de Leontiev, en ce qu'elle souligne le fait qu'action et activité se trouvent entourées par des espaces expérientiels servant de terrain fondamental à la construction du sujet et du savoir.

La discussion de ces approches permet de mettre en évidence le fait que toute approche épistémologique repose sur une conceptualisation de ce qu'on entend par individu, par culture

[2]Voir également Fay 1996, p. 52.

et par savoir. Notre allusion, dans la section 1 aux travaux de DESCARTES, d'une part, et à ceux de l'école française de sociologie des années 20, montrent clairement que ces concepts, comme tous les concepts d'ailleurs, ne sont pas des données en soi.

Il faudra, peut-être, élargir notre concept de culture et aller au delà du problème de qui construit quoi, la culture, le savoir ou l'individu, et, au lieu de voir la culture comme une entité monolithique, la voir, comme le suggère Fay (1996, p. 231), à la manière de zones interactives hétérogènes d'activité, d'opposition et d'agrément où les individus se construisent entre eux et en se construisant construisent le savoir et la culture elle-même qui, à leur tour, construisent ces individus, etc.

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Using π to Look for Obstacles of Epistemological Origin in University Students

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Abstract

We used two specific Archimedean tasks to investigate 4^{th} year mathematics students' approach to mathematical fallacies which, in one form or another, they are prone to be meeting when teaching mathematics in Portuguese secundary schools. We were particularly interested in tracing their understanding of some basic mathematical concepts comparing their performances with other groups.

In selecting the "$\pi = 3$" and the "$\pi = \lim 2\nu R$" problems, we tried to find epistemological obstacles while:

(a) exploring episodes on the history of mathematics;

(b) encompassing a wide range of basic school mathematics (geometric constructions, circumference, length, trigonometry, equation, rational, irrational and transcendent numbers);

(c) reinforcing the links between different areas of mathematics, namely arithmetic, geometry, algebra and calculus;

(d) underlying a very acute question (mathematical proof) with very simple problems;

(e) presenting problems which are not directly treated within the typical material of the University courses;

(f) bringing in geometric (visual) statements into numerical/algebraic arguments and not the other way, more conventional, around;

(g) promoting mathematical creativity;

(h) assessing mathematical reasoning and communication skills.

CONTENTS

1 The Motivation: How does one explain to a 14 years old the difference between Pi and $\sqrt{2}$?

This research of ours started from a question that was posed to us by a 14 years old boy who came home one day very intrigued with what he had just learnt about "irrational numbers".

His mathematics teacher had told him that both Pi (that, until then, he had always treated as being 3.14) and $\sqrt{2}$ were examples of irrational numbers (meaning numbers that could not be represented in the form $\frac{m}{n}$ where both m and n are integers and n is not zero) and from there she had explained that, nevertheless, one might construct geometrically a line segment whose length is $\sqrt{2}$, or $\sqrt{3}$, or .... Having taught, afterwards, these constructions and having ignored Pi, the pupil asked: *"What about Pi?"*

"Pi will, for the time being, still be seen as 3.14 *and therefore you may represent it (in an ordered straight line) close to* 3.14 ", the teacher answered.

But, this teacher was a former student at our University and we started wondering about the meaning of such an answer; we were very much interested in knowing weather her mathematics degree had given her a full understanding of Pi, weather she really knew the differences between Pi and other irrational numbers and weather she was fully aware of the importance of explaining these concepts with different emphasis according to each pupil's level and interests.

This research of ours then began with a search on historical data about Pi.

2 Pi: a never-ending story

Through the ages Pi[1] -any circumference and its diameter are directly proportional- has kept alive the interest of many, more or less famous, mathematicians.

About 4000 years ago, while in Southwest England, an ancient people built the world's known oldest circle (Stonehenge), the ancient Babylonians were using $\pi = 3\frac{1}{8}$ (the error is 1.6593×10^{-2}) and the Egyptian stone masons were approximating the area of a circle by an area of a square according to the following rule "shorten the diameter of the circle by $\left(\frac{1}{9}\right)^{th}$ to get the side of the square"[2] from where one gets $\pi = \left(\frac{16}{9}\right)^2$ (the error is 1.8901×10^{-2}).

[1]It is believed that the Welshman William Jones first introduced the symbol in the early 18^{th} century but it was Euler who generalised its use.

[2]By the Egyptian scribe Ahmose in The Rhind Papyrus, problem 48.

Others, about 2500 years ago, were content enough with $\pi = 3$ (the error is 0.14159), such is presented in the Bible (Kings 7:23).

However the first rigorous mathematical calculation of π is ascribed to Archimedes of Syracuse (~250 BC). Archimedes used a very clever geometrical scheme, based on inscribed and circumscribed polygons, to come to $3\frac{10}{71} < \pi < 3\frac{10}{70}$ (with $\pi - 3\frac{10}{71}$ the error is only 7.4758×10^{-4} while with $\pi = 3\frac{10}{70}$ the error is 1.2645×10^{-3}) following a reasoning based on the method of exhaustion but also presenting two postulates in order to guarantee the "consistency" of his subsequent reasoning, namely:

Postulate 1: The shortest route between two points is that of the segment which joints them.

Postulate 2: For two curves joining two points and convex in the same direction, the greatest length belongs to the curve that contains the other.

All over the world and through many cultures one may easily find examples of people interested in π: Tsu Ch'ung-chih (China, V^{th} century), Al-Kwarizmi (Islamic World, IX^{th} century) or unknown Indian mathematicians (India, XV^{th} century) who found the series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

with its important special case

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \ldots$$

Pedro Nunes (Portugal, XVI^{th} century), Newton (England, $XVII^{th}$ century) or Adam A. Kochanski (Poland, $XVII^{th}$ century) and J. Anastácio da Cunha (Portugal, $XVIII^{th}$ century) were among the ones who were merely using (applying) or fundamentally searching for understanding of such a "mysterious" number; looking for some decimal expansion with repetitions and thus disclosing the possible rationality of π or solving the famous *quadrature of the circle*.

Frenchmen Lambert and Legendre proved, in the late 1700s, that π is irrational (the search for repetition in π's decimal expansion was over) but some others still wondered whether *the quadrature of the circle* was possible or not; in particular whether π might be the root of some algebraic equation with integer coefficients. This search was finally settled in 1882 when the German Lindemann proved that π is transcendental.

Yet, the motivation surrounding π's characteristics did not end. With the development of computer technology in the 1950s, π has been computed to millions of digits[3]. Beyond immediate practicality mathematicians keep, nowadays, interested in studying π, for example, in relation to its normality and computer scientists are using π to test the integrity of both hardware and software.

[3]We, nevertheless, know that an approximation of π to 40 digits would be more than enough to, for example, compute the circumference of the Milky Way Galaxy to an error less than the size of a proton.

3 About a method for evaluating π: Archimedes of Syracuse

Archimedes is often considered one of the greatest mathematicians of all times. In his time, he was seen as a "wise man" and a "master". Most of his original works are lost, but his methods and achievements survived in texts, such as the ones by Pappus and Hero of Alexandria.

Archimedes used inscribed and circumscribed polygons, which lengths he easily computed, to be able to estimate π: "squeezing" it between a greater and a smaller value rather than getting only a one-way approximation.

Most of the books on History of Mathematics refer to such calculations, but because they have to rely on interpretations of Archimedes' works, one may find several approaches to the problem from where one gets, at least, two methods attributed to him.

For our particular research we came to choose one of these methods, namely the one explored by Wilbur KNORR (an algorithm to compute ratios)[4].

In his article[5], Wilbur KNORR examined a writing from Hero about the works of Archimedes. There, in particular, he attempted to demonstrate that the values of the ratios concerning the circumference and diameter of a circle, as reported by Hero, must be wrong and possibly only due to a miswriting, because:

> Archimedes could not possibly make such a great mistake, such as the case of using $\frac{211875}{67441} \approx 3.141634$ and $\frac{197888}{62354} \approx 3.173774$, both greater than π.[6]

Starting from hexagons[7] and measuring the perimeter of the circumscribed hexagon Archimedes needed to compute $\sqrt{3}$ and came to choose $\frac{265}{153} < \sqrt{3}$ and $\frac{1351}{780} > \sqrt{3}$, without further explanation[8]. These ratios have an error less than 10^{-4}($\frac{265}{153} \approx 1.7320261$, $\frac{1351}{780} \approx 1.7320518$ and $\sqrt{3} \approx 1.7320508$) which he might have considered sufficient enough to carry on the subsequent calculations.

Let us, therefore, have the hexagon circumscribed to the circle of diameter $2r$ having side $2s$ such as in the figure

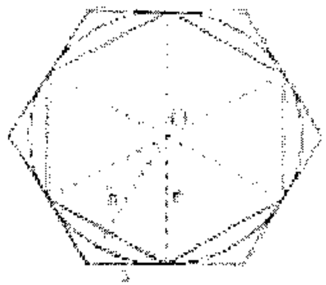
[4] We have also established some comparisons to other method presented by DÖRRIE.

[5] "Archimedes and the measurement of the circle" (1976).

[6] KNORR also refers to works of other Mathematics' historians, who most of them accepted the numbers given by Hero, namely Paul Tannery (~1880), Adriaen Metius (1625) and Ludolph van Ceulen (~1615), trying afterwards to redo the calculations to justify those values.

[7] The choice for starting with hexagons rather than, for example, with triangles or squares may be due, we suppose, to both the fact of a hexagon being clearly (visibly) nearer the circle and being much simpler to construct with ruler and compass alone.

[8] This choice of ratios was taken as if these numbers were well known. This lead to much controversy over the years, from W. W. Rouse Ball in 1908 to Sondheimer and Rogerson in 1981, and the question is not, as far as we know, yet answered. One may, nevertheless, assume that the Babylonian method for evaluating square roots was being used.



$2r$ = diameter circle
$2h$ = diameter circumscribed hexagon

One may consider the right triangle with legs s and r and hypotenuse h so that

$$r^2 + s^2 = h^2,$$

$$\frac{s}{r} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

and

$$\frac{s}{h} = \sin\frac{\pi}{6} = \frac{1}{2}$$

from where we get the numerical values in KNORR's table, such as the one below. In it, such as follows, KNORR presented the values he attributed to Archimedes, namely A, B and C so that $\frac{A}{B}$ and $\frac{A}{C}$ are, respectively, the ratios between the side of the circumscribed hexagon and the diameter of the circle, and the side of the inscribed hexagon and the diameter.

n	A	B	C	**n**	A'	B'	C'
6	153	265	306	**6**	780	1351	1560
12	153	571	$591\frac{1}{8}$	**12**	780	2911	$3013\frac{3}{4}$
24	153	$1162\frac{1}{8}$	$1172\frac{1}{8}$	**24**	780	$5924\frac{3}{4}$	
					240	1823	$1828\frac{9}{11}$
48	153	$2334\frac{1}{4}$	$2339\frac{1}{4}$	**48**	240	$3661\frac{9}{11}$	
					66	1007	$1009\frac{1}{6}$
96	153	$4673\frac{1}{2}$		**96**	66	$2016\frac{1}{6}$	$2017\frac{1}{4}$

Archimedes was therefore searching for a hexagon so that the ratio between the side and the diameter was $\frac{153}{265}$, the first value used for $\frac{1}{\sqrt{3}}$. We may then suppose that he started with a circle of radius 265, which led to a circumscribed hexagon of side 2×153, so that $\frac{A}{B} = \frac{153}{265}$. In addition, we may verify that $\frac{A}{C} = \frac{153}{306} = \frac{1}{2}$ is the ratio between the side and the diameter of the inscribed hexagon, where $C^2 = A^2 + B^2$.

Afterwards Archimedes demonstrated that, for a right triangle with legs A and B and hypotenuse C, if we bisect the vertex angle, we obtain a new right triangle whose legs satisfy the ratio $(B + C)/A$.

At this stage, Archimedes abandoned the initial geometric problem, in favour of a construction of an effective algorithm, keeping A as the side of the circumscribed hexagon and adjusting the diameter to ensure the relations[9]

[9] This "technique" of adjusting the size of the diameter for proceeding with the calculations was also used by Chinese and Japanese mathematicians dealing with the same problem.

$$A_n^2 + B_n^2 = C_n^2$$

and

$$B_{2n} = B_n + C_n.$$

Archimedes repeats this process with both initial values for $\frac{1}{\sqrt{3}}$, making five doublings, and finally used the circumscribed 96-gon with $A = 153$ and $B = 4673\frac{1}{2}$, and the inscribed 96-gon with $A = 66$ and $C = 2017\frac{1}{4}$, leading to

$$\pi < 96 \times \frac{153}{4673\frac{1}{2}} = \frac{29376}{9347} = 3 + \frac{1335}{9347} \approx 3.142826575$$

and

$$\pi > 96 \times \frac{66}{2017\frac{1}{4}} = \frac{25344}{8069} = 3 + \frac{1137}{8069} \approx 3.140909654.$$

The error is less than 2×10^{-3}, which is quite a good approximation, as we still use it nowadays.

In the end, Archimedes searched for simpler values of π, suited for practical calculations. Using $\pi < 3 + \frac{1335}{9347}$, we may notice that $\frac{9347}{1335} \approx 7.00149 > 7$ and subsequently one gets $\frac{1335}{9347} < \frac{1}{7}$, from where we eventually come with the well known $\pi < 3 + \frac{1}{7}$.

Similarly, for $\pi > 3 + \frac{1137}{8069}$ we have $\frac{8069}{1137} \approx 7.09674 < 7.1$ and therefore $\frac{1137}{8069} > \frac{10}{71}$ with suitable errors[10].

This process of using polygons with an increasing number of sides, creating a sequence that approximates the circumference, supported by the well known method of exhaustion, lead to real techniques of integration (concept of infinitesimals) allowing Archimedes to calculate areas, volumes and surface areas[11], which meant dealing with numerical results and thus developing the geometrical approaches presented in Euclid's Elements.

Basing his explanation in quite a powerful axiomatic method -which started from the two axioms we saw earlier-Archimedes calculates the circumference (C) by squeezing it between the perimeter of an inscribed n-gon (P_i), and the perimeter of a circumscribed n-gon (P_e) knowing that while Postulate 1 justified $C > P_i$, Postulate 2 served as a justification for $C < P_e$ from where we may summarize some of this data.

[10] We may also observe that, using the initial hexagons, we would obtain $\pi < 6 \times \frac{153}{265} \approx 3.46415$ and $\pi > 6 \times \frac{780}{1560} = 3$.

[11] The concept of infinitesimal would only be reinvented in the 18^{th} century by Leibnitz and Newton, 2000 years later.

$s_i = \overline{AB} = 2r\sin\alpha$
$s_e = \overline{CD} = 2r\tan\alpha,$
$P_i = ns_i = 2nr\sin\alpha$
$P_e = ns_e = 2nr\tan\alpha.$

The centre angle is $\alpha = \frac{\pi}{n}$, we may therefore easily compute the following values:

Figure	n	$\alpha = 180^o/n$	$\sin\alpha$	$\tan\alpha$	P_i	π	P_e	π
Hexagon	6	30°	0.5	0.5773	$6r$	3	$6.9276r$	3.4638
12-gon	12	15°	0.2558	0.2679	$6.2099r$	3.10495	$6.4296r$	3.2148
24-gon	24	7.5°	0.1305	0.1316	$6.2652r$	3.1326	$6.3168r$	3.1584
48-gon	48	3.75°	0.06540	0.0655	$6.2784r$	3.1392	$6.2880r$	3.1440
96-gon	96	1.875°	0.03272	0.03274	$6.2821r$	**3.14105**	$6.2854r$	**3.1427**

Archimedes' method, by its nature, allowed us to explore some hints about the mathematical potential of π alone: one may easily be dealing with arithmetic and geometry but also with calculus and algebra or numerical analysis and probability together with intuition and deduction. Therefore we found it possible to be adapted to distinct purposes namely projects involving, according to each group's previous mathematical knowledge, pupils or students and teachers or lecturers.

These facts together with the initial motivation for explaining to a 14 years old some data related to π and also our own curiosity on foreseeing mathematics teachers' reactions to π problems, lead us to posing some questionnaires to a group of future teachers of mathematics at a University in Portugal which we then compared, in some cases, with a similar group of students in England and to a group of in-training Portuguese teachers of mathematics.

4 Some data about teaching Mathematics in Portugal

World-wide growth in higher education has risen from 13 millions in the 1960s to 90 millions nowadays. In Portugal the ratio is even higher, namely 24000 (in the 1960s) to 335000 in the last academic year.

On the one hand, Portuguese universities are, at present, basically facing three types of problems:

- the ones reported in most developed countries such is the case of a huge increase in student numbers, a big diversity of institutions and a large variety of curricula and courses;

- the ones reported in some countries such is the case of the implementation of *numeri clausi* for entrance to the University system of education in general and others dealing specifically with mathematics degrees; such is the case of the belief that researchers are, by definition, good lecturers regardless their didactical preparation or even their motivation for teaching and learning. An almost institutionalised routine of assessment practice based on individual written tests appealing to memorisation and algorithmatisation techniques for evaluating the students' performance on basic or more advanced courses such as calculus, algebra, and geometry or differential equations, Galois theory and topology but never evaluating their performance in mathematics nor employing different assessment instruments which, as a whole, most of the students are expected to be teaching and to be using at secondary schools, in a quite near future, and that they do not seem to be able to "unify" nor to identify;
- the ones that, as far as we know, are not typically identified elsewhere; such is the case of most of our mathematics' students being (for many years) women, a large number of our mathematics lecturers being (quite recently) women and most of our mathematics degrees being (traditionally) teaching degrees (preparing, explicitly or not, future secondary school teachers).

On the other hand, we have been listening, in Portugal and abroad, to researchers willing to achieve success in mathematics courses through changing mathematics *curricula*: "complaining" or presenting the justification of undergraduate mathematics curriculum being topics fundamentally developed in the 18^{th} and 19^{th} centuries as opposed to the very new areas of mathematical practice (neuroscience, finances or telecommunications, for example).

5 The methodology of our research

In dealing with mathematics and teaching one gets used to listen to students using words such us "panic", "worry", "anxiety" or simply "error", "boring" and "difficult" more than to "success", "interesting", "easy" or "relevance". We, the authors of this study, were very much interested in underlying some acute questions with quite simple problems.

Besides, most of our students' experience of mathematics seems quite different from the Greek's motivation for studying it (because they wanted to); nowadays, getting a job is, in Portugal at least, one of the most relevant issues for our University students and, because we have been running short of mathematics' teachers for quite some time, mathematics teaching degrees[12] appear, in Portugal, as some of the most wanted degrees.

We basically posed two problems -The Archimedean Tasks (version one and version two)- to a group of 49 mathematics students (30 women and 19 men) who were all using some writing material as well as a ruler and a pair of compasses. By the time this investigation took place, these students, organised in groups (4 or 5 per group[13]) had already succeeded in the first two academic years of their degree: which means that they had been through all the basic math courses as well as on more advanced ones such as Hyperbolic Geometry, Numerical Analysis or Abstract Algebra. Each problem (version one and version two) was introduced to the students

[12]Mathematics Teaching degrees in Portugal are 5 years degrees. The 5^{th} year is, nevertheless, spent doing in-training service in public schools consisting of teaching normal classes of, more or less, 30 pupils.

[13]There was one group composed only by women, one group composed only by men and the others were mixed gender groups.

in a two-stage format (parts i) and ii)), which meant that part ii) was only made available to them after they had decided to finish part i) and to hand it over to the lecturers. This way, we tried to investigate the students' approach to some mathematical fallacies (in the first part-i)) related to π which, in a form or another, they are prone to be meeting in a near future and, (in the second part-ii)) and afterwards, we tried to assess their reaction to the information that these facts were wrong.

We also tried to create an informal environment of discussing the problems by means of a Mathematics Workshop which took place in a reflection-weekend (away from the University Campus) organised by these students. We have explicitly asked for intuitive as well as deductive reasoning presented in a two-column (How and Why) format answer sheet. The problems were:

The Archimedean Task -Version One

1. i) *Draw a circle and, with a pair of compasses opened to its radius, step round the circle.*

a) *How many points have you marked until you get to your starting point on the circle?*

b) *Will you then be able to conclude that the circumference is six times the length of the radius?*

1. ii) c) *How would you explain to a high school's pupil who had conducted such reasoning and come out with $\pi = 3$ that his conclusion is incorrect?*

d) *Which is your own definition of π?*

The Archimedean Task -Version Two

2. i) *The perimeter of a regular polygon inscribed in a circle of radius R might be known from: $2\nu R$, with $\nu = n \times \sin\frac{180°}{n}$ and n representing the number of sides of the polygon.*

a) *Is this a mathematically valid statement?*

b) *Show that the sequence $45 \times \sin\frac{180}{45}$, $90 \times \sin\frac{180}{90}$, $180 \times \sin\frac{180}{180}$, $360 \times \sin\frac{180}{360}$, ... seems to have a limit.*

2. ii) c) *Knowing that $\sin 0.5° \approx 0.0087265$, $\sin 1° \approx 0.017452$, $\sin 2° \approx 0.034899$, $\sin 4° \approx 0.069756$ how would you conjecture an approximation (with two decimal places) to that limit?*

d) *How would you explain to a high school's pupil, who had come out with this approximation, which is the exact value for that limit?*

6 How do Mathematics students (studying to become mathematics teachers), in Portugal, deal with π: Some typical answers

- *π is a conventional number that is difficult to define; it is transcendental.*
- *π is an irrational number;. . . .*
- *Knowing that the circumference of the circle is equal to the perimeter of the hexagon. . . .*

- *$\pi = 3$ because we can round off π to 3, for after the 3 comes a 1.*
- *From $2\pi r = 2\nu r$, we have $\pi = \nu \Rightarrow \lim \nu = \lim \pi = \pi$.*
- *. . . the value of ν has to be π.*
- *The value for the limit is an infinite expansion, so we can not determine its exact value.*
- *π is not algebraic and therefore it is not constructible.*

7 Conclusions

Through these Archimedean tasks, we have, as far as we hope, given evidence of having:

(a) sucessfully explored history of mathematics episodes in classes of University students;

(b) encompassed a wide range of basic and fundamental mathematics (geometric constructions, circumference, length, trigonometry, equation, rational, irrational and transcendent numbers);

(c) reinforced the links between different areas of mathematics, namely arithmetic, geometry, algebra and calculus;

(d) underlined a very acute question (mathematical proof) with very simple problems;

(e) presented problems which are not directly treated within the typical material of the University courses;

(f) brought in geometric (visual) statements into numerical/algebraic arguments and not the other way, more conventional, around.

About "*having been able to promote the students' mathematical creativity*", we may only say that they worked on the proposed tasks intrinsically interested for several hours, and kept asking questions on the theme for some time after the session was over, due to calendar's obligations. Besides the diversity of possible answers was attested by the endless list of solutions presented by the students and that we might, in another occasion, be able to present in some more detail.

About "*assessing mathematical reasoning and the students' understanding of the concepts involved in the tasks*", we may only summarize our feelings with one word: SHOCK!! Namely:

- About "*Basic Concepts*" - They had no idea, whatsoever, about a definition for π: *It is a number. . .* Most of the times they treated it as if it was a primitive term or a very abstract concept. The Portuguese students showed no evidence of having any instinct or intuition for this number (as opposed to the English students we also interviewed on the subject).
- About "*Knowing vs. Understanding*" - They did not have a "set" of fundamental results which they really understood; they simply had a quite large memorised "list" of results which they thought they knew and imagined that it might be applicable to the problems: *It is algebraic means it is constructible.* They misused theorems, ignored axioms and mistreated possible applications (as opposed to a less large "set" of results remembered by the English students).

- About "*Logic Reasoning*" - They were much less curious than some of the 14 years old pupils who had posed some of the initial questions: The "Why" column tended to be left aside. They were, nevertheless, more enthusiastic on treating the subject than the English students.

There is definitely a large way to be run by these students until they reach enough maturity which will allow them to teach properly the facts or to treat properly the activities dealing with number π. However the reason for this state of things might not be entirely their fault and it might very well be the case of certain changes being needed in the way they themselves are taught at the University; changes on methodology of teaching and evaluation more than changes on curricula seems to be the case for expecting better mathematical knowledge by part of the students willing to become mathematics teachers in our schools.

It came finally clear to us that the 14 years old pupil to whom we referred to, at the begining of our research, did not receive, from his teacher, a satisfactory answer to his problem because she herself did not know the answer nor knew how to explain the differences between π and $\sqrt{2}$ to all her 8^{th} grade pupils. We, the authors of this study, will go on experiencing history of mathematics episodes in our lectures and searching for epistemological obstacles in order to alter situations such as the one that we have just reported about π.

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Conflict and Compromise : the Evolution of the Mathematics Curriculum in Nineteenth Century England

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Abstract

This paper explores the social and ideological background which determined the kind of mathematics taught to different groups of people during the early part of the nineteenth century in England. These ideologies arose in and were transmitted through institutions which determined choices and decisions about what was valued as scientific knowledge. The mathematics taught in the universities and in the Public Schools was determined by a classical liberal ideology, whereas the mathematics taught in elementary schools and colleges was driven by a practical ideology of utility, democracy and social justice. The consequences of this conflict can be seen in our current school mathematics curriculum in England. Some observations are also made on the historiographical problems of the history of mathematics education.

1 Introduction

Until recently histories of mathematics paid attention only to the significant developments in mainstream mathematics and nowhere is this more obvious than in the nineteenth century. The reasons are not too difficult to determine; the recent past is more accessible, and the writing of accounts of the work of recently deceased colleagues or editing their papers was a task which coincided with the foundation of new universities and the growth of mathematics departments. Since the interests of the writers focussed almost without exception on their perception of the developments of abstract mainstream mathematics, we have the production of the 'standard' texts[1] which view mathematics as an abstract, almost 'organic' structure and totally ignore the original problem situations and creation processes and the grass-roots teaching, learning and applications in everyday life.[2,3]

This paper is an attempt to give a brief overview of some of the significant social and political movements in the nineteenth century which contributed to the development of mathematics curricula in England; particularly of the mathematics taught outside the universities leading to the formation of and divisions in our contemporary school curriculum.

2 Some remarks on Historiography

2.1 Differing views on the nature of historical enquiry

Traditionally, history has been viewed as a study of carefully delimited aspects of the past employing systematic research in all available sources. The approach can be from a social, political or economic point of view, (BARNES et al. 1996, FAY 1996) and necessarily employs a general philosophy in its interpretation. More recently, history is being regarded as a set of processes and power relations linking the past to the present, where the interpretations of events and facts are critically interrogated, the underlying assumptions are revealed, the status of texts are called into question and where groups of people and their conditions are defined and redefined by those in power.

In a similar manner, we have changes in the way history of mathematics is undertaken. 'Inductivist' history of mathematics is recognised by its tendency to see mathematics as a subject isolated from 'external' influences and as a progression of ideas which are improving and becoming more abstract and general with time, and where the events of the past are seen as instances of steps towards the present more 'perfect' structures. This kind of history tends to interpret the past in terms of modern concepts.

Formerly, the majority of research in the history of mathematics has been carried out by mathematicians, whose interpretation not only utilises technical language, but has employed a style which maintains the genre of the hypothetical-deductive style employed in mathematics itself. In this way it has been seen as an authoritative account of the events in question.

Historians, on the other hand, realise that there are many different kinds of questions about the past, giving rise to many different styles of history. The events, structures and processes

[1] For example in the first three quarters of this century like CAJORI (1893), BELL (1945), SCOTT (1958), EVES (1969), BOYER (1968) and KLINE (1972).

[2] See for example DAUBEN (1985) (pp. 397-400) and WILDER (1950, 1968, 1981).

[3] There are exceptions; see TAYLOR's work on the Mathematical Practitioners (1964 and 1966) and YELDHAM (1926 and 1936) but there is little attention to social issues.

of the past are known only through he relics of the past, which are themselves politically and conceptually loaded. In perceiving relationships between different events and conditions the historian may have to consider theories derived, for example, from economics, psychology, sociology or anthropology. Furthermore, the account is constrained by conventions of language, genre, mode, argument, and a number of other cultural and social conventions.

2.2 Facts and Events

The notion of a 'fact' is ambiguous, since it includes the sense of both 'event', (meaning the 'real' or 'imaginary' status of an occurrence), and 'a statement about an event' (where the concern is with the 'truth' or 'falsity' of an utterance) (WHITE 1995). In this sense, facts are constructed in the documents which refer to the occurrence of the events, not only by interested parties (either contemporary or recent) commenting on the events or the documents, but also by historians interested in giving what they believe is a true account of what really happened. Therefore it is the facts that are subject to revision and further interpretation, and they can even be dismissed given sufficient reasons.

This view allows us to account for the fact that historiographical consensus about any event is very difficult to achieve, since it is always open to revision from another perspective. We not only change our ideas of what the facts of a given matter are, but our notions of what a fact might be, how facts are constructed, and what criteria should be used to asses the adequacy of a given collection of facts in relation to the events which they claim to support. The relation between facts and events is always open to negotiation and reconceptualisation not because events change with time, but because we change our ways of conceptualising them.

2.3 The 'Problematique' for the Social History of Mathematics

Hence the problem is much more difficult in the history of mathematics since what we now choose to identify as mathematics has been perceived differently by people in the past, so the history of mathematics is different for different periods and cultures. In particular, for the history of Mathematics Education, we are faced with a plethora of sources which can be interpreted in various ways. For example, textbooks often have prefaces which state the authorís intention and pedagogical approach; they also may contain advertisements for books by the same or other authors, but the use of the book and the pedagogical methodology in the hands of individual teachers may be a very differnt issue. Contemporary philosophical works on the nature of mathematical knowledge influence the culture at different levels; histories of mathematics of the time tell us differnt stories, and curriculum documents produced by different institutions and interest groups relate the mathematics to the general social, political and economic milleu of the time. This sounds a daunting task, but I believe that as long as we realise that history is an interpretive activity we can cope with this over-abundance of evidence which is incomplete, fragmentary, already interpreted and politically and conceptually biased.

3 The Influences on Education in England in the late Eighteenth Century

The scientific revolution of the seventeenth century had produced significant changes in the popular views about the place of man in the physical universe, and the nature of that universe itself. The views of English philosophers like Locke, Hume, Mill and Bentham, with their theories about the nature of man, of society, and of the acquisition and purpose of knowledge began to have some far-reaching influences on political and educational ideas in the latter part of the eighteenth century.

With the growth of the population of England from six million in 1750 to nine million in 1800, there was a rapid change from urban to industrial communities with all the problems that this brings. With the hopes that industry would improve the conditions of life, the philosophy of utilitarianism was developed in the belief that society should seek to obtain and provide the "greatest good for the greatest number". At the same time, "laissez-faire" economics was promoted, with the purpose of regulating society by allowing the interplay of free forces in economics and society and monitoring their effects with minimal legislative interference. These benefits and forces, it was assumed, would be controlled by educated people making the right judgments about moral, political, and economic issues.

This belief that education was the real key to the improvement of the human condition was very strong. Philosophers emphasised human perfectibility; the idea that man is born without knowledge and becomes what education makes him, and that with the growth of knowledge and better education he is continually improving. Thus, education could do everything in influencing human beliefs, attitudes, morals and conduct. In fact, Locke's views on education were generally accepted by the "philosophes" of the enlightenment, and they appear to have had an influence on early French revolutionary politics from about 1790, and the events in France were regarded with hope and alarm by different sections of the population here in England

From the late 1700s, we see a gradual development of the professionalisation and institutionalisation of mathematics teaching. Set against the considerable changes in social organisation and the economic development of the time, the kind of mathematics, the people who taught it, and the places where it was taught all underwent significant changes.

One of my themes is that the isolation from industrial growth and technical improvement of the universities with their classical traditions was maintained and even increased, and the need for the practical applications of science and mathematics was answered by other sources and other institutions. In very broad terms, the division between the old universities and the other institutions where mathematics was taught was closely linked to the class system and the established English intellectual and social attitudes of the time. While the upper classes and the established church wanted to preserve the status quo, radical philosophy was tolerated and even fashionable among the privileged, but among the workers it was seen as sedition. This division was closely linked to the class system and the established English intellectual and social attitudes of the time.[4]

4 Early Traditions in English Mathematics Teaching

In mathematics teaching two traditions can be identified from the sixteenth century. The "Liberal" tradition was based on translations of Billingsley's Euclid (1570) evolving into a formal style which continued until Playfair's Euclid of 1792 became the standard for the next hundred years.

The other "Vocational" tradition is based on Robert RECORDE's "*Pathway to Knowledge*" where the principles of geometry were set out so they might

[4]For some general background to this period and its social, economic and political detail see SIMON (1960) and HOBSBAWN (1968).

> most aptly be applied unto practise both for the use of instruments geometrical and astronomical and also for projection of plats[5] in every kind, and therefore much necessary for all sorts of men. (RECORDE 1551)

In this way geometry and arithmetic became popularised in many self-help books where detailed explanations and exhortations to the student accompanied the examples. (FAUVEL 1989).

John Dee's "Preface" to Billingsley's Euclid contains a comprehensive description of the "Mathematical Arts" showing their universal usefulness and giving reasons for studying mathematics at all levels. In Elizabethan times mathematics (which included astronomy and astrology) was seen as the key to knowledge and the mysteries of the universe.[6] This tradition continued (TAYLOR 1964, 1966) and in the 1780s at Woolwich BONNYCASTLE was writing alternative treatments of geometry[7] intended for students with different aspirations, and Hutton's "*Course in Mathematics*" of 1798 was the practical text for the artillery and engineer cadets.

5 The English Radicals: Science as a Foundation for Education

From the 1790s onwards working people began to read the radical press, attend lectures, and learn by participation in political discussion. Organisations supporting these activities were called "corresponding societies", and provided organised and disciplined opportunities for study. "The Rights of Man" (PAINE 1798), was an attack on the established social order and its exploitation of the poor and working classes. The radicals saw the Church as the main obstacle to political reform in its reinforcement of the strong social stratification, and they replaced this indoctrination with rational education through their own schools, aiming to inform people of the reasons for their condition and the state of society and industry, and placing instruction within the reach of everyone. Teaching methods encouraged self-confidence, and the capacity for clear self expression, and the organisers realised the importance of combining systematic education with mass political agitation. Books and newssheets were shared: an individual would take a book home, read a passage and prepare a talk for the next meeting; the book would then be passed on, and the process repeated. Many subjects, including some elementary mathematics were learnt in this way. As a result, men and women became informed and critical leaders of the new working class movement, able to master and comprehend some of the most advanced political thinking. This was recognised as a threat by the establishment, and in 1799 an Act of Parliament was passed "...for the more effectual suppression of societies for seditious purposes..." (SIMON 1960, p. 183).

By 1817 there were popular demands for a rational secular education for all. PAINE had demanded the teaching of science which was directly applicable, to be regarded as the cornerstone of a rationalist philosophy. These demands were of great concern, and in 1817 a House of Lords Secret (sic) Committee reported on the unprecedented circulation of

> publications of the most seditious and inflammatory nature, marked with a peculiar character of irreligion and blasphemy, and tending not only to overturn the existing form of government and

[5]A "plat" was a plan used for building or surveying; a map or chart for finding direction or navigation; or an expalnatory table or diagram.

[6]It is interesting to note that the editions of Euclid up to Leek and Serle (1661) all contained Dee's "Preface" but after this it disappears from further editions. Thus Euclid clearly becomes part of the "Liberal" tradition.

[7]BONNYCASTLE (1810 p. ii) carries a list of practical texts of arithmetic, geometry, mensuration, astronomy, plane and spherical trigonometry, some having gone through numerous editions.

> order of society, but to root out those principles upon which alone any government or any society can be supported. (SIMON 1960, p. 131).

The Stamp Act (1817) required the registration and taxing of all newspapers and journals, and as a result, the radical newspapers were forced underground.[8]

Richard Carlile's "Address to Men of Science" (1821) also demanded a curriculum which contained reading, writing, the use of figures, elements of astronomy, geography, natural history and chemistry so that children may

> at an early period of life form correct notions of organised and inert matter, instead of torturing their minds with metaphysical and incomprehensible dogmas about religion. (Carlile 1821 p. 22)

He believed that science, best studied by observation and experiment, was the key to knowledge and freedom, and promoted a materialist psychology, and demanding social and moral education by example.

6 The Schools

In the mid eighteenth century some grammar schools existed, but few taught any mathematics; perhaps the first two books of Euclid, and some simple arithmetic. Any other kind of education was locally organised, usually by well-meaning clergymen and public benefactors. Some clergymen took private pupils and this tradition continued well into the next century.

By the late 1780s, to counter the radical political literature that was freely circulating, Sunday Schools were established for the poor, their major purpose being to indoctrinate pupils in the principles of religion and the duties of their state in life. Here, if you were lucky, it was possible to learn reading, writing, elementary arithmetic, and the catechism. However, due to the teachers' concern for the health and welfare of their pupils they unwittingly 'created thought in the unthinking masses'. (SIMON 1960, p. 183).

In the 1830s we begin to see the establishment of the English Public School system. The amount of mathematics and science taught in these schools was very variable and schools like Eton, Harrow and Rugby[9] did not appoint mathematics masters until challenged by some of the newly founded institutions. Substantial reforms were made to preserve the establishment,[10] by requiring these schools to provide an appropriate education for politicians, civil servants, the clergy, the army and the administrators of the Empire. Since most of the schoolmasters had been educated at Oxford or Cambridge, it was no surprise that the 'Liberal' ethos prevailed, and the theorems of Euclid were regarded as part of the corpus of classical literature.

[8] A significant figure in all this was James Mill (not to be confused with J.S.Mill the philosopher) who was educated at Edinburgh university and came to London in 1802 as a journalist. His political and educational theory can be found in the *Westminster Review* particularly during 1824 - 1826.

[9] Leading the reform was Dr. Thomas Arnold, appointed as Headmaster of Rugby in 1828, who reformed the school and created the ideal school for the Victorian upper middle class.

[10] The schools concerned at this time are the so-called 'nine greats': Charterhouse, Eton, Harrow, Merchant Taylors, Rugby, Shrewsbury, St Pauls, Westminster, and Winchester.

7 Non-Conformist Education and the Mechanics Institutes

From 1766 the "Lunar Society" held informal monthly meetings in Birmingham.[11] This was typical of a number of "Literary and Philosophical" Societies whose members were forward looking scientists or innovators with interests in practical applications of the new ideas of natural philosophy. Later, more radical interests developed, and they also began to encourage social and political education intending to prepare their sons for their place as leaders of the new industries.

The Private or 'Dissenting' Academies were the places where Non-conformists could be educated.[12] The earliest of these was Warrington Academy, founded in 1757, and the subjects taught had obvious practical applications. Manchester College of Arts and Sciences, founded in 1783, taught sciences and practical arts on four evenings a week. Its syllabus contained classical languages, grammar and rhetoric, mathematics (including trigonometry), mechanics, natural philosophy, (including astronomy and chemistry) English composition, French, commercial and economic geography, history, politics, writing, drawing, book-keeping and shorthand. Subjects like these became the standard curriculum, and most of the important cities of this time developed similar educational institutions. There was a great demand for applied science, and "mixed mathematics".[13] In 1786, the Manchester Academy was established, providing full-time education for students, and a permanent mathematical tutor was appointed in 1787.

The Literary and Philosophical Societies also supported the development of Mechanics Institutes, which became another focus for working class self education. They introduced science, literature and the arts; deliberately excluded politics and religion, and provided lectures, evening and day classes, and libraries. There was a substantial demand for reading scientific (and clandestinely also political) texts and reading rooms and loan systems were established. The curriculum was based on what was "useful" to workers, and lectures were related to practical applications and local engineering and manufacturing problems.[14] Advanced classes were given in a selection of subjects like Grammar, French, Latin; Science, Chemistry, Electricity; Mixed Mathematics, Algebra and Mensuration. Provision for science also meant that collections of apparatus began to be built up, and lecturers established courses, developed curricula, and wrote texts. (INKESTER 1975, Royle, 1971).

8 Military and Naval Schools

Schools of navigation had grown up in the major ports for merchants and traders, and military and naval academies provided an education for the entrants to the army and navy. Woolwich Academy, where BONNYCASTLE and Hutton taught, was founded in 1741, and the teachers there were familiar with contemporary continental texts. In 1837 the syllabus consisted of arithmetic: fractions, roots and powers, proportion, interest, permutations and combinations; algebra: arithmetic and geometric progressions, logarithms, simple, quadratic and cubic equations;

[11] Among its early members were Boulton, Watt, Priestly, Galton and Erasmus Darwin (the grandfather of Charles Darwin).

[12] Oxford and Cambridge would only admit those who were prepared to acknowledge the King as head of the Church of England.

[13] That is, practical geometry, measurement, arithmetic and sometimes fluxions. NICHOLSON (1823) is a typical text in this genre. See also COOK (1981).

[14] The Prospectus of the Sheffield Mechanics Institute (1832) states;
"The object of this Institute is to supply, at a cheap rate, to the classes of the community, those advantages of instruction, in the various branches of Science and Art which are of practical application to their diversified avocations and pursuits." (INKESTER 1975).

geometry; plane trigonometry, mensuration, surveying, conic sections; dynamics, projectiles, hydrostatics, hydraulics and fluxions. The syllabus was eventually updated to include the calculus, and other more recent aspects of applied mathematics, and a system of open competitive examinations. (RICE 1996, p. 404).

The Royal Naval Academy, founded at Portsmouth in 1722, (renamed the Royal Naval College in 1806) transferred to Greenwich in 1873. After undergoing similar problems and reorganisations to its military counterpart, from 1885 the Academy taught ballistics for gunnery and torpedo officers, mechanics and heat for engineers, and dynamics for ship construction. Thus it was that by the end of the century clear, practically focused and vocationally relevant courses had evolved for the training of military and naval personnel.

9 The Education of Girls and Women

Sometimes girls attended elementary school, but generally were only taught the most elementary skills. During the eighteenth century a few boarding schools for girls were set up which taught mathematics, science and astronomy, and by the end of the century some women were pursuing their own studies by corresponding with scientists. (Harris, 1997 p. 37). However, it was not until late nineteenth century that mathematics became firmly established in the curriculum of girls schools.[15]

No women were admitted to Oxford or Cambridge before the beginning of this century; the Victorian attitude to the mental capabilities of women, and their low social status, together ensured that any opportunities for further education were severely limited. However, this was to change slowly with the publication of the "Educational Times" in 1847, where subjects like the importance of women in society, and the qualities of women's minds were intelligently discussed. The College of Preceptors, founded in 1846, played a major role in supporting women, and from the 1860s we find a growing movement for the elimination of sex differences in education, particularly in mathematics and science. From the mid nineteenth century, higher education for women began to develop. Queens College[16] was founded in London in 1848, the Ladies College Bedford Square in 1849, and by 1878 University College became the first co-educational institution where women and men were examined together.

10 Changes in the Universities

In 1826 University College was founded with the support of those who were excluded from Oxford and Cambridge, liberal politicians, and Jeremy Bentham, the humanist philosopher. In 1828, after demands to provide a religious foundation in London, King's College was founded. In 1828 DE MORGAN was appointed the first professor of mathematics at University College. He was a thoughtful, idealistic and energetic educator whose text books and pedagogical writings show a deep concern for the problems of learning and teaching. His motives for writing *On the Study and Difficulties of Mathematics* (1831) are to help 'tutorless' students, with the areas of elementary mathematics which give most difficulty, describing their *nature* without emphasising routine operations. DE MORGAN takes the view that mathematics is a necessary

[15]Even then, mathematics was not regarded as a subject really suitable for girls neither in the 'liberal' sense nor in the 'vocational' sense. (see Harris 1997 particularly chapters 3 and 4).

[16]DE MORGAN taught at Queens college, but only for a year, apparently feeling that the ladies were not of a sufficiently high standard; and as a member of the London Mathematical Society, showed no interest in the attempts to reform the teaching of school geometry. (RICE 1996).

part of a liberal education, and that it is *useful*, being the key to other sciences. Much of his work was serialised through the "Society for the Diffusion of Useful Knowledge" (SDUK).[17]

Meanwhile WHEWELL at Cambridge, aimed to place mathematics in the curriculum of every student of the university, reinforcing the "Liberal" view:

> I believe that the mathematical study to which men are led by our present requisitions has an effect, and a very beneficial effect on their minds; but I conceive that the benefit of this effect would be greatly increased, if the mathematics thus communicated were such as to dissipate the impression, that academical reasoning is applicable only to such abstractions as space and number. (1836, p. 44).

As the century progressed, university mathematicians seemed less inclined to spend their time educating the masses; growing professionalism motivated more 'pure' mathematical interests and since, from 1850, Cambridge required a knowledge of Euclid for its entrance exams, other universities followed suit.[18]

11 The Ideological and Pedagogical Divide

By the end of the nineteenth century "laissez-faire" economics had given rise to a large number of industrial enterprises each requiring ever more specialised training. The Mechanics Institutes were one way to cater for this need, and they helped to develop ideas of economics, of the idealist possibilities of science and technology to improve everyday life, and an acute awareness of the need for appropriate training and new teaching methods. A considerable amount of their work was experimental and practical, and the mathematics required to make the machinery work efficiently was being developed alongside the craft skills of manufacture. Thus it became obvious that the traditional mathematical diet was quite inappropriate to the needs of the new industrial community and advocates of practical mathematics were designing new courses and writing new textbooks. Prominent among these was Perry,[19] a significant figure in the reform of mathematics teaching at this time. Reforms in school, however well-intentioned, were hampered by schoolmasters educated in the Oxbridge tradition, and a lack of interest from the universities.[20]

The products of industry shown in the Exhibition of 1851 were based more on the freelance initiatives of innovators than any government sponsored organisation, and it later became clear to government that economic advantage rested not only on technical education but also a good primary education. The 1870 act ensured that education up to age thirteen was available to all, and while the attempts to devise differentiated schooling on a class based system had failed, these attitudes prevailed in the secondary, technical and grammar schools that evolved. It is here that we see the ideological divide; the establishment provided for its own in continuing the liberal tradition in the Public Schools and using mathematics to control the 'gateway' to Universities where 'pure' mathematics flourished, at the same time invoking the utility of 'vocational' mathematics to train the industrial workforce in technical schools and colleges.

[17]The SDUK was founded in 1826 by Henry Brougham and other liberal politicians as an alternative to the radical press, and through its publications intended to give a 'suitable direction' to working class thinking. The *Differential and Integral Calculus* (1842) was originally published in the *Penny Cyclopaedia* in forty two weekly parts.

[18]Further discussion of the development of the mathematics curriculum and its pedagogy in the latter part of the century can be found in PRICE (1983).

[19]Perry was an engineer and his syllabus provided a new paradigm which came from outside the school tradition. (DSA 1899).

[20]Cayley as chief examiner for Cambridge entrance insisted on keeping Euclid.

Looking at the more recent past, this conflict has been compounded by issues involving pedagogy as well as style and content, but the expectations of the two ideologies can be detected in the nature and mode of presentation of the curriculum materials of today. DOWLING (1998) locates these ideologies in a detailed analysis of contemporary school texts; in this brief presentation I am attempting to show their social and historical roots.

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FIGURE 1

How did candidates pass the state examination in mathematics in the Tang dynasty (618-917)? - Myth of the "Confucian-Heritage-Culture" classroom

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Abstract

Towards the late 80s some educators began to pay attention to cultural differences that may affect the process of learning and teaching. This interest is further strengthened by the results coming out of the two recent studies sponsored by the International Association for the Study of Educational Achievement (IEA). In particular, the learning process of Asian students brought up in the tradition of the Confucian-Heritage culture (CHC) has become a much discussed issue in the past decade. In this talk we try to look at the issue from the historical angle : to investigate the curriculum in mathematics at the state university in the Tang Dynasty and to piece together a rational reconstruction of the state examination in mathematics from official records in the ancient chronicles. It is hoped that such a study will shed light on the Paradox of the Asian Learner : CHC students are perceived as using low-level, rote-based strategies in a classroom environment which should be conducive to low achievement; yet CHC students report a preference for high-level, meaning-based learning strategies and they achieve significantly better in IEA studies!

1 Prologue

In a paper on medical education, G.E. MILLER says,

> We say we want sensitive, thoughtful, analytic, independent scholars, then treat them like Belgian geese[1] being stuffed for pâté de foie gras. We reward them for compliance, rather than independence; for giving the answers we have taught them rather than for challenging the conclusions we have reached; for admiring the brilliance of purely scientific advances rather than developing greater sensitivity to the inequities in health care we have too often ignored. (MILLER 1978)

This passage, its allusion to medical education notwithstanding, sets the tone for the theme of this paper.

2 CHC classroom culture

In the past decade, with the rising number of multiethnic classrooms in many countries and with increasing international cooperation in assessing the curriculum and the academic performance of students around the world, some researchers in education take cultural differences into account in their studies. The cultural background of the students and of the teacher is seen to be a factor affecting the process of learning and teaching. See for example (CAI 1995, STEVENSON & STIGLER 1992, WATKINS & BIGGS 1996, WONG 1998).

Of particular pertinence to our present discussion is the theory put forward by a group of researchers including J.B. BIGGS (1994, 1996), F. MARTON (1976, 1996) and D.A. WATKINS (1991). What kind of teaching environment is conducive to good learning, or to what is known as the "deep" approach to learning as contrasted with the "surface" approach (MARTON 1976)? In (BIGGS 1993) the following factors are singled out: (i) varied teaching methods, student-centred activities, (ii) content presented in a meaningful context, (iii) small classes, (iv) warm classroom climate, (v) high cognitive level outcomes expected and assessed, (vi) classroom-based assessment in a non-threatening atmosphere. In their research this group came into contact with students at different levels, of different cultural backgrounds and in different parts of the world (eg. Australia, Hong Kong, Korea, Nepal, the Philippines, Sweden, and the United States). What they observed in the classrooms in the so-called "Confucian-heritage culture" is just the opposite from (i) to (vi) cited above.[2] In CHC classrooms, it appears that classes are conducted in a traditional teacher-talk/student-listen manner within a class of large size, with the austere teacher commanding an authoritative role. Assessment is carried out through strict formal examinations which exert significant influence on the student's future career. It seems that such examinations reward good memorization and industrious drilling. Therefore, the obvious conclusions drawn by Western observers of the CHC classrooms are: (1) CHC classrooms

[1]This paper is the full text of a presentation delivered in July 1999 at the troisième université d'été européene sur l'histoire et l'épistémologie dans l'éducation mathématique held at Louvain-la-Neuve and Leuven in Belgium. *La citation ci-dessus n'est qu'une allusion bienveillante, il n'y a pas de malveillance envers notre hôte!* I take this opportunity to thank our Belgian hosts for their hospitality.

[2]The nebulous term "Confucian-heritage culture", abbreviated as CHC, is not to be taken here as a term with a precise definition which clearly delineates a specific genre of culture. The term is used in a general sense to cover the cultural background of communities in mainland China, Hong Kong, Japan, Korea, Singapore, Taiwan and Vietnam. It would take many books and papers to explicate in detail and depth what CHC really means, with all its subtlety and historical evolution. That is beyond the scope of this paper and beyond the capacity of this author. I would entreat readers to take CHC classrooms to mean classrooms in the communities mentioned above, although differences already exist between them.

should be conducive to low quality outcome, viz. rote learning and low achievement; (2) CHC students are perceived as using low-level, rote-based learning strategies.

3 The Asian Learner Paradox : some clarifications

However, reality reveals a totally different picture! In the several international studies sponsored by the IEA (International Association for the Evaluation of Educational Achievement) — the FIMS in the 60s, the SIMS in 1976-1983 and the most recent TIMSS in 1993-1994 — CHC students have scored significantly higher levels of achievement than those of Western students. In his research J.B. BIGGS (1996) found that CHC students report a preference for high-level, meaning-based learning strategies.

This discrepancy between what one expects and what one witnesses has come to be known as the Asian Learner Paradox or the CHC Learner Paradox (BIGGS 1994).[3]

It is strange that a popular view is to equate Confucian learning with rote learning and with submissive learning. Let us look at some samples of what the Masters themselves had to say.

In the *Confucian Analects* (5th century B.C.) we find,

> Learning without thought is labour lost; thought without learning is perilous. (LEGGE 1960)

In the *Doctrine of the Mean* (6th-5th century B.C.) we find,

> He who attains to sincerity, is he who chooses what is good, and firmly holds it fast. To this attainment there are requisite the extensive study of what is good, accurate inquiry about it, careful reflection on it, the clear discrimination of it, and the earnest practice of it. (LEGGE 1960)

In the books by the leading neo-Confucian scholar ZHU XI (1130-1200) we find,

> In reading, if you have no doubts, encourage doubts. And if you do have doubts, resolve doubts. Only when you've reached this point have you made progress. (GARDNER 1990)

Would one call this rote-learning? submissive learning?

By reading more extensively in the books by Zhu Xi, we can perhaps understand better what appears to Western observers as rote-learning actually consists of. Zhu Xi said,

> Generally speaking, in reading, we must first become intimately familiar with the text so that its words seem to come from our own mouths. We should then continue to reflect on it so that its ideas seem to come from our own minds. Only then can there be real understanding. Still, once our intimate reading of it and careful reflection on it have led to a clear understanding of it, we must continue to question. Then there might be additional progress. If we cease questioning, in the end there'll be no additional progress. (GARDNER 1990)

[3]But personally I have certain reservations about the two outcomes described above, judging from the teaching experience I have with undergraduates in mathematics at the university. It is a paradox within a paradox! But we will put that aside in the present discussion and concentrate only on the Asian Learner Paradox.

He also elaborated further,

> Learning is reciting. If we recite it then think it over, think it over then recite it, naturally it'll become meaningful to us. If we recite it but don't think it over, we still won't appreciate its meaning. If we think it over but don't recite it, even though we might understand it, our understanding will be precarious. ... Should we recite it to the point of intimate familiarity, and moreover think about it in detail, naturally ourmind and principle will become one and never shall we forget what we have read. (GARDNER 1990)

This is an unmistakable differentiation between repetitive learning and rote learning. Contemporary researchers explain the Asian Learner Paradox based on this differentiation (BIGGS 1996, MARTON 1996).

On the other hand, modern day education in the Western world which arose in the 19th century along with the Industrial Revolution started by emphasizing the 3Rs — reading, writing and arithmetic. In a code issued by R. Lowe of the Education Department of England in 1862, specific standards for each R were explicitly stated (e.g. Standard I in Reading: Narrative monosyllables; Standard II in Writing: Copy in manuscript character a line of print; Standard IV in Arithmetic: A sum in compound rules (money)) (CURTIS 1967). The emphasis on mechanical rote learning is captured vividly in the opening sentences (which were intended as a satirical exaggeration) of the 1854 novel *Hard Times* by Charles Dickens (as words uttered by Mr. Gradgrind of Coketown)

> Now, what I want is, Facts. Teach these boys and girls nothing but Facts.
> Facts alone are wanted in life.... This is the principle on which I bring up
> my own children, and this is the principle on which I bring up these children.
> Stick to Facts, sir!"

4 Examination culture in China

4.1 Sources

Even if one can explain the Asian Learner Paradox, it remains a fact that in CHC there is a strong tradition of examination. Some even label CHC classroom as an examination-oriented classroom culture! To what extent is this true? It is commonly believed that an examination-oriented culture will hinder the learning process. Is it really that bad? Or is it a necessary evil? Or is it even beneficial to the learning process in some sense? These questions urge me to look at the issue from a historical perspective. I intend to look at an ancient period when official examination in mathematics was in its most established form — in the Tang Dynasty (618-907) — and see if it can enlighten me in my classroom teaching.

In collaboration with A. Volkov, an historian in mathematics, we attempt to piece together a picture of the state examination in mathematics in the Tang Dynasty (SIU 1999). This we do by gleaning what we can from the official records contained in certain chronicles, among which the main ones are:

(a) *Jiu Tangshu* (Old History of the Tang Dynasty), c. 941-945,

(b) *Xin Tangshu* (New History of the Tang Dynasty), c. 1044-1058,

(c) *Tang Liudian* (Six Codes of the Tang Dynasty), 738,

(d) *Tongdian* (Complete Structure of Government), c. 770-801,

(e) *Tang Huiyao* (Collection of Important Documents of the Tang Dynasty), 961.

One informative secondary source which contains the main excerpts of relevant interest in the chronicles listed above and a lot of interesting information, including lists of successful candidates in each round of examination (in the classics and literary composition rather than in mathematics) is *Dengke Jikao* (Journal on the Examinations in the Tang Dynasty) compiled by the Qing scholar XU Song in 1839.

I should at the start state clearly the attitude I adopt when consulting those ancient chronicles. I subscribe to a wider (but somewhat controversial) view of studying history as propounded by the British philosopher-historian R.G. Collingwood in *The Idea of History* (1946), "History is thus the self-knowledge of the living mind. ... For history is not contained in books or documents; it lives only, as a present interest and pursuit, in the mind of the historian when he criticizes and interprets those documents, and by so doing relives for himself the states of mind into which he inquires." Collingwood echoed the view held by the Italian philosopher B. Croce who said in *History: Its Theory and Practice* (1915), "History is living chronicle, chronicle is dead history; history is contemporary history, chronicle is past history; history is principally an act of thought, chronicle an act of will. Every history becomes chronicle when it is no longer thought, but only recorded in abstract words, which were once upon a time concrete and expressive."

4.2 Chinese Examination System inspired by the West

Let us look at the examination system in its historical context. Despite the shortcomings the system later developed, it is praised for the role it once played. Examination employed as a way to select is a very Chinese institution. According to P. Monroe in *Cyclopaedia of Education* (1931), "Written examination was probably unknown in Europe until 1702. ... Practical examinations had been employed for a long time in the medieval universities in such a subject as medicine." Dr. Sun Yat-Sen, founder of the Chinese Republic in 1911, said in *The Five-Power Constitution*,

> At present, the civil service examinations in the (Western) nations are copied largely from England. But when we trace the history further, we find that the civil service of England was copied from China. We have very good reason to believe that the Chinese examination system was the earliest and the most elaborate system in the world. (TENG 1942-43)

Indeed, Dr. Sun instituted the division of the government structure into five-power, viz. the Legislative Yuan, the Executive Yuan, the Judicial Yuan, the Examination Yuan and the Censorate. E.A. KRACKE has said,

> One of China's most significant contributions to the world has been the creation of her system of civil service administration, and of the examinations which from 622 to 1905 served as the core of the system. (KRACKE 1947)

As early as in the beginning of the 17th century, the Jesuit Father Matteo Ricci[4] reported with commendation in his journal "the progress the Chinese have made in literature and in the sciences, and of the nature of the academic degrees which they are accustomed to confer." About one and a half centuries later, another illustrious European, Voltaire (F.M. Arouet)[5], made a similar observation, "The human mind certainly cannot imagine a government better than this one where everything is to be decided by the large tribunals, subordinated to each other, of which the members are received only after several severe examinations. Everything in China regulates itself by these tribunals."

4.3 Different Types of Examination in China

The Chinese term for state examination is "keju". Literally, "ke" means "subject" and "ju" means "recommend". Combined together it means recommendation of suitable candidates (for taking up official positions) through examinations in different subjects. Some historians date the beginning of the keju system to the Sui Dynasty (581-618) when the emperor convened a state examination by decree. But some historians maintain that it started in 622 when the first Tang emperor decreed that any qualified candidates could sit for the state examination without having to be recommended by a provincial magistrate. The keju system was abolished in 1905 by an imperial edict towards the end of the last imperial dynasty in China, the Qing Dynasty (1644-1911).

In *Xin Tangshu* a section on recruitment by examinations records that there were two kinds of state examinations: (1) regular examinations held annually in the first or second lunar month for graduates of colleges and universities or for provincial candidates, (2) special examinations held by imperial decree. The second kind depended on the need at the time or on the whim of the emperor, so it covered a wide range of expertise, but could also sound rather strange. In official records one can find about a hundred of such special examinations. Just to cite a few, there were: examination on "vast erudition and great composition", examination on "deep knowledge of the ancient books and great talents in the art of teaching", examination on "having military plans with foresight and well qualifiedas a general", examination on "wisdom and good nature, rectitude and righteousness, and speaking honestly and remonstrating insistently", examination on "remarkable understanding of the art of government and suitability for administering people". A most amusing item is examination on "leading an hermetic life at Qiuyuan, not seeking fame", since logically speaking one should be awarded a degree in that if and only if one should not be! (In fact, it was recorded in *Dengke Jikao* that somebody was awarded the degree *in absentia* in 794 as he refused to receive it!) For the first kind there were initially seven subjects: examination on perfect talent, examination on classics, examination on distinguished man of letters, examination on accomplished man of letters, examination on law, examination on calligraphy and examination on mathematics. Examination on perfect talent was soon abolished, while examination on accomplished man of letters became in time the main focus enjoying the highest prestige. It was recorded in *Tongdian* that by 752, of a thousand candidates who sat for the annual examination on accomplished man of letters only one or two were awarded

[4]The quotation by Matteo Ricci can be found in his journal later compiled by Nicolas Trigault in 1615. There is a modern English translation of the journals titled "China in the Sixteenth Century : The Journals of Matthew Ricci, 1583-1610", translated by L.J. Gallagher, published by Random House, New York in 1953. There is also a modern French translation titled "Histoire de l'expédition chrétienne au royaume de la Chine, 1583-1610" translated by G. Bessiere, published by Bellarmin in 1978.

[5]"Œuvres complètes de Voltaire", t.13, 163, published by Nedeln, Lichtenstein (in the 18th century?), with a Kraus reprint in 1967.

the degree, while for instance, successful candidates for the examination on classics numbered in the tens. A source of the time said that one who passed the examination on accomplished man of letters at fifty (perhaps after many repeated attempts) was still regarded as outstanding, while one who passed the examination on classics at thirty was considered too old already! No similar data or remark is found for examination on mathematics, which serves to indicate that mathematics was accorded a lower prestige among the various subjects, only on a par with calligraphy. This becomes even more apparent when we look at the number of students enrolled at the state university. Tang institutions of higher education were divided into hierarchies. The highest institution was the School for the Sons of the State which accepted only sons of noblemen or officials from a certain rank upward. Next came the National University which accepted a similar crop with the official rank somewhat lowered. Then came the School of Four Gates which accepted besides sons of officials also a small number of sons of ordinary citizens. The three Schools of Law, Calligraphy and Mathematics accepted sons of officials of low rank and of ordinary citizens. In the early Tang Dynasty, according to *Xin Tangshu*, there were 300 students in the School for the Sons of the State, 500 students in the National University, 1300 students in the School of Four Gates, 50 students in the Law School, 30 students in the Calligraphy School and 30 students in the Mathematics School. At one time, throughout the whole empire, including the provincial colleges, there were 8000 students pursuing higher education with foreign students coming from nearby countries as well. The whole edifice of state higher education was very well-established in the Tang Dynasty.

4.4 The Annual State Examination

The culminating apex of this edifice, the annual state examination, was a gruelling experience for many. Some authors in the Tang Dynasty wrote about how candidates stood in a long queue, carrying their own stationery, supply of food and water, candles and charcoal (for preparing meals and for getting warm), waiting to be admitted to their cells, only to be searched and shouted at by guards who were stationed by the thorny hedge (an ancient analogue of barbed-wire fence) which encompassed the examination venue. Candidates were clad in flimsy clothes and shivered in the freezing weather, for they were not allowed thick clothing to prevent concealment of manuscripts. Throughout the long hours they worked on their examination scripts, the candidates were confined to their cells, in which they would prepare their own meals and take care of their own personal hygiene. In the case of failure in the examination, which was not uncommon, this gruelling experience would have to be repeated in another year, and perhaps in yet another year, ... WEI Chengyi, who was awarded the degree of accomplished man of letters in 867, once sneaked into the office of the Ministry of Rites called Nangong, which was in charge of examination affairs, and composed a poem on the wall: "Like a thousand white lotus petals, /The candles lit up the hall. /Which was filled with the peaceful rhymes /Of the Ya and the Song. /As the flame of the third candle/Flickered towards its end, /One realized it meant failure /To complete the scene of Nangong." This poem, with its trace of resignation, depicted vividly those assiduous candidates racing against time with their examination scripts by the light of the three candles allowed them to last through the night. (See Figure 1 for a humorous rendering of the scene).

Modern examinations are definitely much less gruelling than that. It would be unfair to my ancestors in the Tang Dynasty if I fail to point out that even over a thousand years ago some good modern examination procedures were already in place. In 759 the Chief Examiner LI Kui said, "The empire selects its officials for their talent. The requisite classics are displayed here. Candidates are welcome to consult them at will." This was perhaps the earliest open-book examination! In 742 the Chief Examiner WEI Zhi said, "The performance of a candidate in one single examination may not reflect his true potential, hence his previous essays should also be consulted." This was perhaps the earliest instance of assessment by project work!

5 Curriculum in mathematics in the Tang Dynasty

It is recorded in *Xin Tangshu* that the mathematics curriculum at the state university consisted of two programmes, hereafter denoted by A and B for short. For details see (SIU 1995, 1999). It suffices at this point simply to note that each programme lasted for seven years of study, with Programme A covering eight of the ten books in *Suanjing Shishu* (Ten Mathematical Manuals) and Programme B covering the remaining two. (Therefore, Programme B was a more advanced course of study. *Suanjing Shishu* was the collation of ten existing mathematical classics by LI Chunfeng at an imperial edict, and adopted as the official textbook in 656.) In each programme students must also study two more books, *Shushu Jiyi* (Memoir on Some Traditions of the Mathematical Art) and *Sandeng Shu* (Three Hierarchies of Numbers). We will come back to these two books later. Regular examinations were held throughout the seven years of study, and at the end of each year an annual examination was held. Any student who failed thrice or who had spent nine years at the Mathematics School would be discontinued. Judging from the age of admission at 14 to 19 years-old, we know that a mathematics student would sit for the state examination at around 22.

In the state examination for either programme, the candidate was examined on two types of question. The first type was described in *Xin Tangshu* as: "[The candidates should] write [a composition on] the general meaning, taking as the basic/original task a 'problem and answer'. [They should] elucidate the numbers/computations, [and] construct an algorithm. [They should] elucidate the structure/principle of the algorithm in detail." (For Programme B there was added the remark, "If there is no commentary, [the candidates should] make the numbers/computations correspond [to the right ones?] in constructing the algorithm." For an attempt to explain the latter remark, see (SIU 1999)). We will say more about this type of question in the next section. The second type was testing on quotations. Candidates had shown a line taken from either *Shushu Jiyi* or *Sandeng Shu*, with three characters covered up. Candidates had to answer what those three characters were. In to-day's terminology, this type of questions is called "fill in the blank". It is interesting to note that *Shushu Jiyi* (credited to the authorship of XU Yuein the early part of the 2nd century and commented on by ZHEN Luan in the late 6th century, although the extant version might perhaps been forged by Zhen Luan himself) is a short text with only 934 characters, which could be committed to memory with reasonable ease (not to mention that a candidate had seven years to do it!). There may well be other reasons for singling out this book for the purpose of testing on quotations, but that would be the subject of another paper. (See (VOLKOV 1994) for an interesting discussion on the content of *Shushu Jiyi*.) The book *Sandeng Shu* was lost by the Song Dynasty (960-1279). We can only surmise that it might be a text similar to *Shushu Jiyi* in this respect.

By the way, there was a reason for instituting the practice of testing on quotations. The practice was proposed by the Chief Examiner LIU Sili in 681 (in all subjects) to rectify the deficiency of a prevalent habit of candidates who only studied "model answers" to past questions instead of studying the original classics. Testing on quotations forced candidates to read (at least some) original classics. However, examination being what it is, it is prone to abuse. The setting of questions on quotations got more and more difficult and unreasonable, testing candidates on

obscure phrases, sometimes even setting up traps to confound the candidate intentionally. In response, candidates collected such obscurities and memorized them for the sole purpose of passing those unreasonable tests! The original purpose of encouraging candidates to read the original classics was totally defeated. In 728 it was decreed that quotations should be set within reasonable bounds. There is a good lesson to be learnt here about making use of examination to direct the curriculum.

6 "Re-constructed" examination questions

Let us come back to the first type of question. What are these tasks on elucidation and construction of algorithms about? Since no trace of any examination question is extant, we can only attempt to "re-construct" an examination based on the scanty and sketchy official account on the state examination in mathematics gleaned from the ancient chronicles.

Before giving such examples, it is helpful to look at a textbook and see how the author did the mathematics. We choose the prime textbook *Jiuzhang Suanshu* (Nine Chapters on the Mathematical Art, compiled between 100 B.C. and A.D. 100)[6] with commentaries by the 2nd century mathematician LIU Hui. This will also add to the stock of "circumstantial evidence" for our attempted "re-construction".

In Chapter 5 some formulae for the volume of various solids are given. In particular, Problem 17 is about that of a tunnel at the entrance of a tomb (xianchu). Mathematically speaking, a xianchu is the solid bounded by three trapeziums and two triangles on the two sides. The three trapeziums have opposite parallel sides of length a, b; a, c and b, c, the depth is h and the trapezium on top has length ℓ. (See Figure 2)

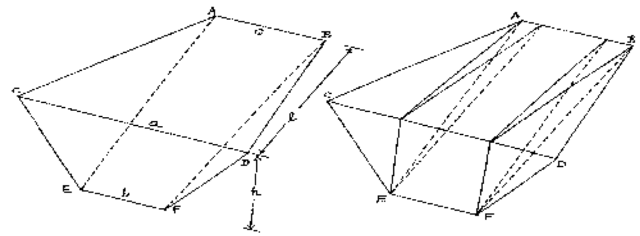

FIGURE 2

The formula for the volume of the xianchu is given in the text as

$$V = \frac{1}{6}(a+b+c)h\ell.$$

[6]The earliest translation of the JIUZHANG SUANSHU in a Western language was the German translation titled "Neun Bücher Arithmetische Technik" by Kurt Vogel, published by Friedr. Vieweg & Sohn, Braunschweig in 1968. There are two recent translations, one in English and one in French. The English translation has appeared and is titled "The Nine Chapters On the Mathematical Art : Companion and Commentary", by Kangsheng Shen, John N. Crossley and Anthony W.C. Lun, published by the Oxford University Press, Oxford in 1999. The French translation will appear in 2000 and is titled Édition critque, traduction et présentation des Neuf chapitres sur les procédures mathématiques (les débuts de l'ère commune) ainsi que des commentaires de Liu Hui (3ème siècle) et de Li Chunfeng (7ème siècle), by Karine Chemla and Shuchun Guo, in preparation by Diderot Multimedia. For the Chinese edition I follow the corrected and edited version of JIUZHANG SUANSHU by Shuchun Guo, published by Liaoning ducational Press in 1990. The original version dated back to of course much earlier times.

(In the text, numerical data are given in place of a, b, c, but the numerical data are actually generic rather than special.) Liu Hui explains in his commentaries how the volume is calculated. He dissected the xianchu into smaller pieces, each of some standard shape such as a triangular prism (qiandu), a tetrahedron of a particular type (bienao), or a pyramid with a square base (yangma). If you try to do that by yourself, you will find out that the way of dissection is different for different relations between a, b, c. For instance, if $a > c > b$, then you obtain two bienaos each of volume $(a-b)h\ell/12$, two bienaos each of volume $(c-b)h\ell/12$ and one qiandu of volume $bh\ell/2$. (See Figure 2) They add up to $(a+b+c)h\ell/6$. But if $a > b > c$, then you obtain two bienaos each of volume $(a-b)h\ell/12$, two yangmas each of volume $(b-c)h\ell/6$ and one qiandu of volume $ch\ell/2$. They also add up to $(a+b+c)h\ell/6$. In fact, Liu Hui in his commentaries treats all eight different cases except the one case $b > a = c$. The calculation is different for different ways of dissection, but the basic underlying idea is the same. Probably candidates in the examination were asked to carry out a similar explanation for other formulae on area and volume, possibly with given numerical data. Once the basic idea is understood, such a demand for elucidation is reasonable.

In the same chapter, Problem 10 is about the volume of a pavilion (fangting) with square bases. (See Figure 3)

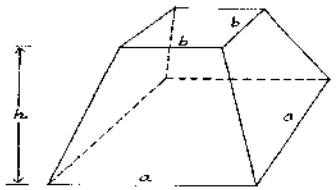

FIGURE 3

Mathematically speaking, a fangting is a truncated pyramid with square base. If a, b are the sides of the bottom and top squares respectively and h is the height, then the volume is given in the text as

$$V = \frac{1}{3}(a^2 + b^2 + ab)h.$$

Again, Liu Hui in his commentaries explains how to obtain the formula by an ingenious method of assembling blocks of standard shape (called by him "qi"). There are three kinds of "qi": cube of side a with volume a^3 (lifang, LF); pyramid of square base of side a and one vertical side of length a perpendicular to the base, with volume $\frac{1}{3}a^3$ (yangma, YM); triangular prism with isosceles right triangle of side a as base and height a, with volume $\frac{1}{2}a^3$ (qiandu, QD). He observed that the truncated pyramid is made up of one LF, four YM and four QD. (Careful readers will notice that here we require $h = b$, so that we are talking about blocks of a standard shape). He then observed that one LF makes up a cube of volume b^2h; one LF and four QD make up a rectangular block of volume abh; one LF, eight QD and twelve YM make up a rectangular block of volume a^2h. (Careful readers will notice that here we require $h = b$ and $a = 3b$ so that each corner piece is a cube formed from three YM.) In problem 15, Liu Hui further explains how to obtain the more general formula of the volume of a pyramid of rectangular base with an

arbitrary height by an infinitesimal argument (WAGNER 1979)). Altogether, three LF, twelve QD and twelve YM make up a volume $b^2h + abh + a^2h$. Hence the volume of the truncated pyramid is $\frac{1}{3}(a^2 + b^2 + ab)h$. (See Figure 4)

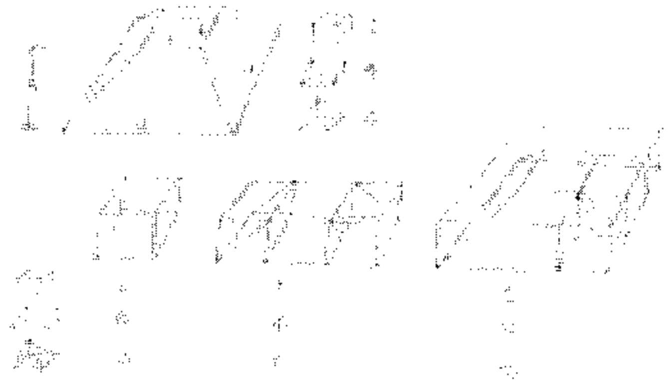

FIGURE 4

Liu Hui gave an alternative formula

$$V = \frac{1}{3}(a-b)^2h + abh$$

by another way of dissection. (See Figure 5)

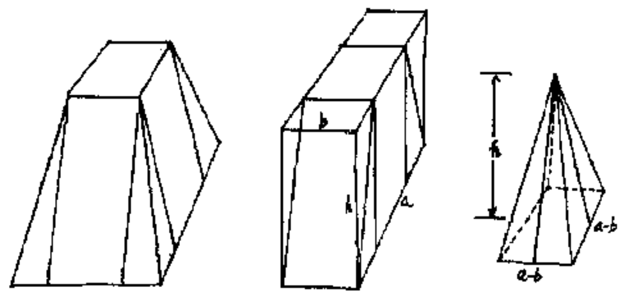

FIGURE 5

In the second explanation, there is no need to assume $h = b, a = 3b$. But it works only when the bottom and top pieces are squares.

Now, let me suggest a fictitious examination question: Compute the volume of an "oblong pavilion" of height h with bottom and top being rectangles of sides a_1, a_2 and b_1, b_2 respectively ($a_1 \neq a_2$, $b_1 \neq b_2$). If one understands the argument by Liu Hui, one can easily modify either method to arrive at the answer, which is left as an exercise for the readers. (Readers may also wish to solve the problem in a way commonly known to school pupils of today, viz. by making use of similar triangles.) The answer turns out to be

$$V = \frac{1}{3}[a_1a_2 + b_1b_2 + \frac{1}{2}(a_1b_2 + a_2b_1)]h.$$

If one merely memorizes the formula given in the textbook by heart, it is not easy to hit upon the correct formula. This is probably what is meant by "constructing a (new) algorithm". Again, such a demand is reasonable, especially when these candidates might well be facing in their subsequent career problems which are variations (e.g. with parameters changed) of the problems they learnt in the textbooks.

7 Conclusion

I have attempted to "re-construct" an examination in mathematics in the Tang Dynasty to persuade readers that the curriculum in that period was not so elementary nor was it learnt by rote. It is hard to imagine that a group of young men spent seven of their golden years in simply memorizing the mathematical classics one by one without understanding at all!

Granting that an examination is not to be passed through rote learning, what good will an examination bring?

Let us first compare the ancient Chinese examination format with a modern theory on assessment by B.S. BLOOM (BLOOM et al. 1956). The modern viewpoint includes both the formative and the summative aspects of assessment, while the ancient Chinese examination focused only on the latter function. The six major classes of taxonomy of Bloom can be matched up with the four different types of question in the ancient Chinese examination, viz. (i) testing on quotations is about knowledge, (ii) short questions are about comprehension and application, (iii) long questions (on contemporary affairs) are about analysis and synthesis, (iv) composition and poems are about evaluation.

With these varied objectives, examination can have a beneficial influence on both the student and the teacher. For the student it is good for consolidation of knowledge, enhancement of comprehension, planning of schedule of study, judgement on what is important to learn, development of learning strategies and motivation and self-perception of competence. For the teacher, besides what has been said above, it is good for monitoring the progress of the class, as a gauge of the receptivity and assimilation of the class and evaluation of the teaching. In this sense, "examination-oriented education" and "quality education" need not be a dichotomy. T.J. CROOKS says,

> As educators we must ensure that we give appropriate emphasis in our evaluations to the skills, knowledge, and attitudes that we perceive to be most important. (CROOKS 1988)

Viewed in the summative aspect, examination is a necessary evil. But viewed in the formative aspect, examination can be a useful part of the learning process. The important thing to keep in mind is not to let the assessment tail wag the educational dog! (TANG & BIGGS 1996). The demise of the examination system in Imperial China, even with its initial good intention and with its long life span of 1287 years, is a lesson to be learnt from.

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Arithmetic and algebra : can history help to close the cognitive gap ?
A proposed learning trajectory on early algebra from an historical perspective

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Abstract

Is it possible for students to self-handedly gain access to early algebra starting from informal strategies embedded in arithmetic? Can apparently fundamental differences between arithmetical and algebraic conceptions of mathematical problems be (partly) surmounted? The historical development of algebraic problem solving and algebraic symbolic language has inspired the author to develop a prototype pre-algebra learning strand on reasoning and equation solving. This article sketches the project background and gives some examples of classroom activities.

Introduction

Several algebra research projects of the last decade report on poor student performance when it comes to solving linear equations (KIERAN 1989, 1992; FILLOY & ROJANO 1989; SFARD 1991, 1996; HERSCOVICS & LINCHEVSKI 1994, 1996; BEDNARZ et al. 1996). Secondary school students often have trouble learning how to construct equations from arithmetical word problems, and how to rewrite, simplify and interpret algebraic expressions. It is conjectured that part of the problem is caused by fundamental differences between arithmetic and algebra (FILLOY & ROJANO 1989; HERSCOVICS & LINCHEVSKI 1994; BEDNARZ & JANVIER 1996; MASON 1996). Arithmetical problems, for instance, involve straightforward calculations with known numbers, whereas algebra requires reasoning about unknown or variable quantities and recognizing the difference between specific and general situations. In the transition from arithmetic to algebra there is claimed to be a discrepancy called *cognitive gap* (HERSCOVICS & LINCHEVSKI 1994) or *didactic cut* (FILLOY & ROJANO 1989), hampering manipulations of algebraic expressions.

A good starting point for an investigation into this matter could be a return to the roots. In this project we shall try to gain insight into the differences and similarities between arithmetic and algebra by looking into the historical development of algebra and learning from past experiences. Recent research on the advantages and possibilities of using and implementing history of mathematics in the classroom has led to a growing interest in the role of history of mathematics in the learning and teaching of mathematics[1]. Inspired by the HIMED (History in Mathematics Education) movement, a developmental research project called 'Reinvention of Algebra' was started at the Freudenthal Institute in 1995 to investigate which didactical means will enable students to make a smooth transition from arithmetic to early algebra. Specifically, the 'invention' of algebra from a historical perspective will be compared with possibilities of 're-invention' by the students. This paper first sketches the background of the project (part 1) and then gives a brief outline of the learning strand and some classroom impressions (part 2).

1 Background of the project

1.1 Research motive

The American Middle School Project (VAN REEUWIJK 1995) and small experiments in The Netherlands (ABELS 1994, STREEFLAND 1995) revealed a great number of accesses into algebra for relatively young learners. Ten- and eleven-year-olds have shown that they can reason algebraically in problem situations that are familiar and meaningful to them. The level of knowledge, skills and abilities of the children, and in some cases the mathematics itself, are the driving forces of the teaching-learning process. A similar observation can be made for the historical development of algebra, where both practical needs in society and internal motivation led to further progress. Given the fact that historical developments play an increasingly important role in the teaching and learning of mathematics, one of the project's aims is to investigate if history of mathematics can be a useful didactical tool. The application will be twofold: history as a guide for the hypothetical learning trajectory, and history as a rich source of mathematical problems and learning moments. Moreover, the historical development of algebra can shed light

[1]For example, work by Fauvel, Van Maanen, Kool, Arcavi, Eagle and many more; special issues *For the Learning of Mathematics 11-2* (1991) and *Mathematical Gazette 76* (1992); discussion document for an ICMI study by Fauvel and Van Maanen, *Educational Studies in Mathematics 34-3*, 255-259.

on the ruptures between arithmetical and algebraic modes of thinking.

A decade ago, the algebra working group of the W12-16 project designed a new approach of algebra for the first three years in Dutch secondary schools (Algebragroep W12-16, 1990, 1991; W12-16 COW, 1992). In this new algebra, algebraic relations play a very important role. Students develop algebraic conceptions and skills very gradually from concrete situations by switching between different forms of representation: descriptions of situations, tables, graphs and formulas. However, since the implementation of the new program it has become increasingly clear that the learning of algebraic skills like manipulation of formulas and equations still needs to be improved. Consequently, we have decided to attempt another approach. Inspired by the historical development of algebra, we will investigate accesses to algebra within the context of story problems and solving equations. Developmental research will be carried out on the teaching-learning process of the teachers as well as the groups of students involved, to determine whether the discrepancy between arithmetic and algebra can be minimized. But before going into more detail, a brief description is called for of two standpoints -on mathematics education and educational research- which are at the heart of this project.

1.2 Developmental research and Realistic Mathematics Education

Developmental research is a type of educational research whereby design of instructional material is an integrated part of the research method. In a cyclic process of anticipating and testing, new ideas on teaching and learning mathematics are developed and tried out in classroom experiments. In order to construct a hypothetical learning trajectory, educational designers can make use of heuristics such as the reinvention principle and didactical phenomenology (FREUDENTHAL 1983, 1991). Analysis of the classroom results leads to the formation of theory, which in turn is used to improve the instructional design. The completion of various cycles -in this project there have been three- will result in a product which is theoretically and empirically founded. So developmental research yields not only a new learning strand on a certain topic, but also a theory on the preferred way in which the topic should be taught and learnt. 'The preferred way' in our opinion is one according to the didactical vision of Realistic Mathematics Education (RME), which propagates the teaching and learning of mathematics as a human activity.

In agreement with the tradition of RME, the founding principles of the early algebra learning strand are:

- to create rich problem situations that are meaningful to students, either in the real world or in their mathematical experience
- to construct activities that offer opportunities for mathematizing, modeling and schematizing, not only as problem solving tools but also as a means to formalize mathematical thinking
- to choose contexts that students are familiar with to serve as frameworks of reference
- to enable students to construct their own mathematics, starting from informal knowledge and strategies and progressively building up a more formal mathematical understanding
- to instigate interactive reflection (student-student and student-teacher) and student participation in establishing algebraic conventions.

(For more information on developmental research and the RME tradition, see FREUDENTHAL 1983, 1991; TREFFERS 1978; GRAVEMEIJER 1994; VAN DEN HEUVEL-PANHUIZEN 1996).

1.3 Subject matter: algebra and arithmetic

Algebra has many faces and is therefore difficult to define. But for the sake of practicality, it is useful to distinguish four basic perspectives of school algebra: algebra as generalized arithmetic, algebra as a problem-solving tool, algebra as the study of relationships, and algebra as the study of structures. Each of these operates in a different medium, where for example letters have a specific meaning and role (USISKIN 1988). In this research project we have decided to restrict ourselves to linear relationships, formulas and equation solving. The proposed learning activities belong to the first three perspectives of school algebra as mentioned, and assume a dialectic relationship between algebra and arithmetic.

A closer look at the similarities and differences between algebra and arithmetic can help us understand some of the problems that students have with learning algebra. In bold terms, arithmetic deals with numbers and algebra with letters - letters that can stand for numbers. But the essential difference lies deeper. Several researchers (Booth 1988; KIERAN 1989, 1992; SFARD 1991, 1996) have studied problems related to the recognition of mathematical structures in algebraic expressions. Kieran speaks of two conceptions of mathematical expressions: *procedural* (concerned with operations on numbers, working towards an outcome) and *structural* (concerned with operations on mathematical objects) (or *operational* and *structural* respectively, SFARD 1996). The contrasting natures of algebra and arithmetic in this respect will be discussed in connection with the theoretical conjectures later in this paper.

And yet there is a definite interdependency: algebra relies heavily on arithmetic operations and arithmetic expressions are sometimes treated algebraically. And word problems have always been and still are a part of mathematics that algebra and arithmetic have in common. A summary of the historical development of algebra[2] can shed more light on how algebra has its roots in arithmetic.

1.4 Historical development of algebra

It is generally accepted to distinguish three periods in the development of algebra (oversimplifying, of course, the complex history in doing so!), according to the different forms of notation: rhetorical, syncopated and symbolic (see also table 1)[3]. From ancient times until about 500 years ago, with the exception of Diophantus and a number of other mathematicians who used abbreviations and symbols, both the problem itself and the solution process were mostly written in only words (*rhetorical* notation). Early algebra was a more or less sophisticated way of solving word problems. A typical rule used by the Egyptians and Babylonians for solving problems on proportions is the Rule of Three: given three numbers, find the fourth. Such problems are commonly classified as arithmetic, but in situations where numbers do not represent specific concrete objects and where operations are required on unknown quantities, we can speak of algebraic problems. Another commonly used method for solving word problems is called the Rule of False Position, first used systematically by Diophantus (TROPFKE 1980). According to

[2]The historical overview is confined to the research topic and therefore "algebra" will be limited here to early algebra, in particular the field of algebraic notation, word problems and linear equations.

[3]The classification of algebra into rhetorical, syncopated and symbolic algebra first appeared in G.H.F. Nesselmann. *Die Algebra der Griechen*, Berlijn 1842 (STRUIK 1990, p. 78).

this rule one is to assume a certain value for the solution, perform the operations stated in the problem, and depending on the error in the answer, adjust the initial value using proportions. Although the Rule of False Position is generally not said to be an algebraic algorithm, its wide acceptance and perseverance even after the invention of symbolic algebra indicate it was and can still be a very effective problem solving tool.

	rhetoric	syncopated	symbolic
written form of the problem	only words	words and numbers	words and numbers
written form in the solution method	only words	words and numbers; abbreviations and mathematical symbols for operations and exponents	words and numbers; abbreviations and mathematical symbols for operations and exponents
representation of the unknown	word	symbol or letter	letter
representation of given numbers	specific numbers	specific numbers	letters

Table 1: characteristics of the 3 types of algebraic notation

Depending on the number concept of each civilization as well as the mathematical problem, the unknown could be a quantity or a measure and was denoted by words like "heap" (Egyptian), "length" or "area" (Babylonian, Greek), "thing" or "root" (Arabic), "cosa", "res" or "ding(k)" (Western). The solution was given in terms of instructions and calculations, with no explanation or mention of rules. The unknowns were treated as if they were known; reasoning about an undetermined quantity apparently did not form a conceptual barrier. For instance, in the case of problems that we would nowadays represent by linear equations of type $x + \frac{1}{n}x = a$, the unknown quantity x was conveniently split up into n equal parts.

Diophantus (ca. 250 AD) invented shortened notations (*syncopated* algebra) which enabled him to rewrite a mathematical problem into an 'equation' (abbreviated form). He systematically used abbreviations for powers of numbers and for relations and operations. In his equations he used the symbol ς to denote the unknown and additional unknowns were derived from it. TROPFKE (1980) explains that this change from representing the unknown by words to symbols really persevered only once the symbols were also used in the calculations. He gives two arguments to indicate that Diophantus appears to have been the first mathematician to do so. Firstly, Diophantus performed arithmetic operations on powers of the unknowns, carrying out additions and subtractions of like terms self-evidently without explicitly stating any rules. And secondly, he explained the method and purpose of adding and subtracting like terms on both sides of an equation. (TROPFKE 1980, p. 378).

After Diophantus there were other practitioners of syncopated algebra. In India (7^{th} century AD) words for the unknown and its powers -which were extended in a systematic way- were abbreviated to the first or the first two letters of the word. Additional unknowns were named after different colors. In Arabic algebra (9^{th} century AD) powers of the unknown were also built up consecutively, using the terms for the second and third power of the unknown as base. In abbreviated form, the first letter of these words was written above the coefficient. In Western Europe (13^{th} century) there were minor differences in the technical terms between Italy and Germany, and only in the second half of the 14^{th} century the words "res" and "cosa" were shortened to r and s respectively. In the middle of the 16th century Stifel introduced consecutive letters for unknowns and stated arithmetical rules using these letters. From there Buteo, Bombelli, Stevin, Recorde (see figure 1) and many others developed a system to symbolize powers of unknowns and formulate equations. (TROPFKE 1980, pp. 377-378). Recorde introduced the equals-sign in print, saying: "And to avoid the tedious repetition of these words: is equal to: I will set as I do often in work use, a pair of parallels, or Gemowe lines of one length, thus: =, because no 2 things, can be more equal." [4] (EAGLE 1995, p. 82).

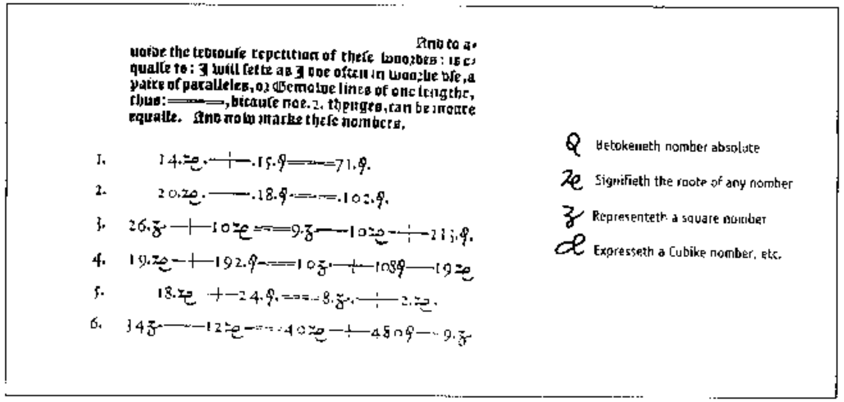

Figure 1: Algebraic notations in Western Europe
Date: Recorde (1557), *The Whetstone of Witte*
Source: EAGLE (1995), *Exploring Mathematics through History*

In the rhetorical and syncopated periods we see a certain degree of standardization. Routine solving procedures were based on the specific numerical properties of standard problems. Diophantus, Arabic mathematicians and the mathematicians in Western Europe contributed a variety of general methods of solving indeterminate, quadratic and cubic equations. But with the lack of a suitable language to represent the *given* numbers in the problem, it was still a difficult task to write the procedures down legibly. In a few isolated cases geometrical identities were expressed algebraically (with variables instead of numbers) but nonetheless written in full sentences. Syncopated notation did not (yet) enable mathematicians to take algebra to a higher level: the level of generality. It is important that students experience this limitation themselves in order to appreciate the value and power of modern mathematical notation.

The development of algebraic notation in the 16^{th} century was a process still instigated by problem solving (see also RADFORD 1995). In 1591 Viète introduced a system for denoting the unknown as well as given numbers by capital letters, resulting in a new number concept: "algebraic number concept" (HARPER 1987). The signs and symbols became separated from that what they represent (a context-bound number) and symbolic algebra became a mathematical object in its own right. For a Vietan solution to a typical Diophantine problem, see figure 2 below. A few decades later Descartes proposed the use of small letters as we do nowadays: letters early in the alphabet for given numbers, and letters at the end of the alphabet for unknowns. With the creation of this new language system, earlier notions of the "unknown" had to be adjusted. The first objective had always been to uncover the value of the unknown, but in the new symbolic algebra the unknown served a higher purpose, namely to express generality.

[4] Gemowe lines mean twin lines, as in Gemini (EAGLE 1995).

Algebra as generalized arithmetic was a fact, and in its new role algebra detached itself from arithmetic.

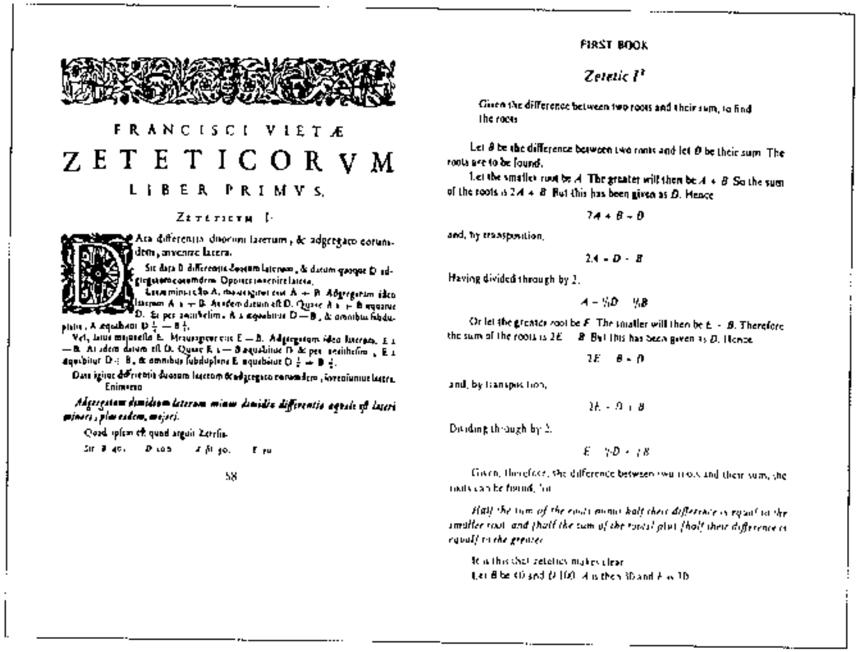

Figure 2: Symbolic algebra
Date, left: Viète (1593), *Zeteticorum Liri Quinque*; date, right: Witmer (1983)
Source, left: Latin text from F. van Schooten's edition, p. 42 (Leiden, 1646, reprint in Hofman, 1970); source, right: English translation in Witmer (1983), p. 83-84.

The historical development of equations in particular shows that, no matter how revolutionary, symbolic algebra was not a necessity for the existence of equations. That is, if we allow other forms of notation than the conventional symbolic one. As a matter of fact, linear equations were very common in Egypt, and the Babylonians already knew how to solve equations of the first, second and third degree. In order to solve with the method of elimination the following system of equations (given in modern notation):

$$\begin{aligned} x + \tfrac{1}{4}y &= 7 \\ x + y &= 10 \end{aligned}$$

the first equation was multiplied by 4 and the second equation was then subtracted from the first, which gave $3x = 18$. Hence $x = 6$, and from the second equation it followed that $y = 4$.

In a very different part of the world a systematic treatment of solving equations developed in ancient China. Just like the ancient civilizations, the Chinese lacked a notational system of writing problems down in terms of the unknowns, but the computational facilities of the rod numeral system enabled them to surpass the rest of the world in equation solving. The *Jiu zhang suanshu* or 'Nine Chapters on the Mathematical Art' (206 BC to 220 AD) is the oldest book known until now that contains a method of solving any system of n simultaneous linear equations with n unknowns, with worked-out examples for $n = 2,3,4$ and 5. This was done using the method *fang cheng* (calculation by tabulation), writing the coefficients down or organizing them on the counting board in a tabular form and then performing column operations on it (much like the Gauss elimination method of a matrix). The general application of the *fang cheng* method led quite naturally to negative numbers and some rules on how to deal with them, which is in great contrast with the late acceptance of negative numbers in other parts of the world.

Diophantus certainly demonstrated a pursuit of generality of method, but his first concern was to find a (single) solution for each problem. The *Arithmetica* (ca. 250 AD) is a collection of about 150 specific numerical problems that exemplify a variety of techniques for problem solving. Diophantus distinguished different categories and systematically worked through all the possibilities, reducing each problem to a standard form. Negative solutions were not accepted, and if there was more than one solution, only the largest was stated. He solved linear equations in one unknown by expressing the unknown and the given numbers in terms of their sum, difference and proportion. If a problem contained several unknowns, he expressed all the unknowns in terms of only one of them, thereby dealing with successive instead of simultaneous conditions. Diophantus is also known for his treatment of indeterminate equations: equations of the second degree and higher with an unlimited amount of rational solutions. Once again the general method involved reducing the problem to one unknown and finding a single solution.

The Arabs also played an important role in the historical development of equation solving. Although the boundaries of this research project have been set at (systems of) linear equations, their achievements on quadratic and cubic equations deserve mentioning. An influential book on Arabic algebra is al-Khwarizmi's *Hisab al-gabr wa-l-muqabala* (early 10^{th} century). It contains a clear exposition of the solutions of six standard equations, followed by a collection of problems to illustrate how all linear and quadratic equations can be reduced to these standard forms. Al-Khwarizmi also gave geometric proofs and rules for operations on expressions, including those for signed numbers, even though negative solutions were not accepted at that time. But as far as the difficulty of the problems and the notations are concerned, the book remained behind compared to the work of Diophantus; everything was written in words, even the numbers. The Arabs did not succeed at solving cubic equations algebraically, but in the 11^{th} century AD Omar Khayyam presented a well-known yet incomplete treatise on solving cubic equations with geometric means.

Arabic algebra became known in the Western world in the 12^{th} century, when al-Khwarizmi's work was translated by Robert of Chester. Two centuries later, mathematical textbooks on arithmetic and algebra were very common in certain parts of Europe, and equation solving (even of the third and fourth degree) had become a regular subject in the Italian abbacus schools. In 1545 Cardano presented the solution of the general cubic equations by means of radicals. After the invention of symbolic algebra, equation solving developed very rapidly and soon found new applications in other areas of mathematics.

1.5 History in mathematics education

We would not plead for the use of history of mathematics in mathematics education if we did not believe that history has something extra to offer. It can benefit students, teachers, curriculum developers and researchers in different ways. Students can see the subject in a new light, they will have a notion of processes and progress, they will learn about social and cultural influences, to name just a few advantages (FAUVEL 1991). Teachers may find that information on the development of a mathematical topic makes it easier to tell, explain or give an example to students. It also helps to sustain the teacher's interest in mathematics. And history of mathematics can give the educational developer or researcher more insight into the subject matter and perhaps even the learning process.

Another argument for using history in education is the so-called Biogenetic Law popular at the beginning of this century. The Biogenetic Law states that mathematical learning in the individual (philogenesis) follows the same course as the historical development of mathematics itself (ontogenesis). However, it has become more and more clear since then that such a strong statement cannot be sustained. A short study of mathematical history is sufficient to conclude that its development is not as consistent as this law would require. Freudenthal also warns against unthinkingly accepting the Biogenetic Law in the following passage on 'guided reinvention':

> Urging that ideas are taught genetically does not mean that they should be presented in the order in which they arose, not even with all the deadlocks closed and all the detours cut out. What the blind invented and discovered, the sighted afterwards can tell how it should have been discovered if there had been teachers who had known what we know now [......] It is not the historical footprints of the inventor we should follow but an improved and better guided course of history. (FREUDENTHAL 1973, pp. 101, 103).

In other words, we can still find history helpful in designing a hypothetical learning trajectory and use parts of it as a guideline. HARPER (1987), for example, argues that algebra students pass through consecutive stages of equation solving, using more sophisticated strategies as they become older, in a progression similar to the historical evolvement of equation solving. He pleads for more awareness of these levels of algebraic formalism in algebra teaching.

There are different ways of implementing history in educational design. First of all, history can be used as a designer guide. Milestones in the development of mathematics are indications of conceptual obstacles. We can learn from the ways in which these obstacles were conquered, sometimes by attempting to travel the same course but at other times by deliberately using a different approach. 'Reinvention' does not mean following the path blindly. On the contrary, it means that developers need to be selective and should attempt to set out a learning trajectory in which learning obstacles and smooth progress are in balance. History can set an example but also a non-example. And secondly, we can choose between a direct and an indirect approach, bringing history into the open or not. Learning material can be greatly enriched by integrating historical solution methods and pictures and fragments taken from original sources, but in some situations it may be more appropriate that only the teacher knows the historical background.

Having decided to use history of mathematics as a source of inspiration for both the researcher and the students, it has become an important issue to find out in this project what the effect is. We aim to determine:

- how the historical development of algebra compares with the individual learning process of the student following the proposed learning program;
- whether or not historical problems and texts indeed help students to learn algebraic problem solving skills.

1.6 'Reinvention of algebra'

In order to facilitate the 'reinvention of algebra' in the classroom, we need to find out where the historical development of algebra indicates accesses from arithmetic into algebra. Historically, word problems form an obvious link between arithmetic and algebra. Although algebra has made it much simpler to solve word problems in general, it is remarkable how well specific cases of such mathematical problems were dealt with before the invention of algebra, using arithmetical procedures. Some types of problems are even more easily solved without algebra! One important characteristic of algebra, the ability to reason with unknown or variable quantities, can be trained within an arithmetical context. Another possible access is based on notation use, for instance by comparing the historical progress in symbolization and schematization with that of modern students. Thirdly, we could study old textbooks on early algebra in order to learn more about how algebra was understood and applied just after it became accepted.

Despite the clear bond between algebra and arithmetic shown by the historical development of algebra, one look at a schoolbook is enough to realize that they still seem to be separate worlds. Decades ago it was already clear that inconsistencies between arithmetic and algebra can cause great difficulties in early algebra learning. The difficulty of algebraic language is often underestimated and certainly not self-explanatory: "Its syntax consists of a large number of rules based on principles which, partially, contradict those of everyday language and of the language of arithmetic, and which are even mutually contradictory." (FREUDENTHAL 1962, p. 35). He then says:

> The most striking divergence of algebra from arithmetic in linguistic habits is a semantical one with far-reaching syntactic implications. In arithmetic 3 + 4 means a problem. It has to be interpreted as a command: add 4 to 3. In algebra 3 + 4 means a number, viz. 7. This is a switch which proves essential as letters occur in the formulae. a + b cannot easily be interpreted as a problem. (FREUDENTHAL 1962, p. 35)

The two interpretations (arithmetical and algebraic) of the sum 3+4 in the citation above correspond with the terms procedural and structural used by Kieran.

We also need to consider how notation and concept formation are related. SFARD (1991) conjectures that symbolic algebra is equivalent to a structural conception of algebra and consequently more advanced in terms of concept development than rhetoric algebra, which corresponds with an operational approach. However, this view is not commonly shared. Radford argues that the categorization rhetoric –syncopated– symbolic is the result of our modern conception of how algebra developed, and that it is often mistaken for a gradation of mathematical abstraction (RADFORD 1997). When the development of algebra is seen from a socio-cultural perspective instead, syncopated algebra was not an intermediate stage of maturation but it was merely a technical matter. As Radford explains, the limitations of writing and lack of book printing quite naturally led to abbreviations and contractions of words. Perhaps modern day students do naturally shorten their notations (from a context-bound to a general mathematical language), but it

has yet to be decided whether this process implies a better understanding of letter use.

1.7 Cognitive gap

In recent years, much research has been done on difficulties that students have in translating word problems into algebraic equations, and it has produced an abundance of new conjectures. In the transition from arithmetic to algebra there is a discrepancy known as the *cognitive gap* (HERSCOVICS & LINCHEVSKI 1994) or the *didactic cut* (FILLOY & ROJANO 1989). There are differences regarding the interpretation of letters, symbols, expressions and the concept of equality. For instance, in arithmetic, letters are usually abbreviations or units, whereas algebraic letters are stand-ins for variable or unknown numbers. FILLOY & ROJANO (1989) as well as LINCHEVSKI & HERSCOVICS (1996) point out a rupture in the learning process of equation solving. Operating on an unknown requires another notion of equality. In the transfer from a word problem (arithmetic) to an equation (algebraic), the meaning of the equal sign changes from announcing a result to stating equivalence. And when the unknown appears on both sides of the equality sign instead of one side, the equation can no longer be solved arithmetically (by inverting the operations one by one). Matz (1979) and Davis (1975), for example, have done research on students' interpretation of the expression $x + 3$. Students see this as a process (adding 3) rather than a final result that stands by itself. They have called this difficulty the "process-product dilemma". SFARD (1996) has compared discontinuities in student conceptions of algebra with the historical development of algebra. She writes that syncopated algebra is linked to an operational conception of algebra, whereas symbolic algebra corresponds with a structural conception of algebra.

DA ROCHA FALCÃO (1996) suggests that the disruption between arithmetic and algebra is contained in the approach to problem-solving. Arithmetical problems can be solved directly, possibly with intermediate answers if necessary. Algebraic problems, on the other hand, need to be translated and written in formal representations first, after which they can be solved. MASON (1996, p. 23) formulates the problem as follows: 'Arithmetic proceeds directly from the known to the unknown using known computations; algebra proceeds indirectly from the unknown, via the known, to equations and inequalities which can then be solved using established techniques'.

Summarizing the theoretical background of the research project described above, we aim to determine how a bottom-up-approach (starting from informal methods that students already use) towards algebra can minimize the discrepancy between arithmetic and algebra. We will investigate which early algebra activities can help students to proceed more naturally from the arithmetic they are familiar with to new algebraic territories, and how procedural and structural properties in both algebra and arithmetic can become more connected. In our attempt to investigate possible accesses from arithmetic into algebra from a historical perspective, we will look into the past for contexts (topics), types of mathematical problems, mathematical ways of thinking, solving procedures, notations, and suitable sources. The historical development of algebra indicates certain courses of evolution that the individual learner can reinvent. Ideally, the student will acquire a new attitude towards problem solving by developing certain (pre-)algebraic tools: a good understanding of the basic operations and their inverses, an open mind to what letters and symbols mean in different situations, and the ability to reason about (compare and relate) (un)known quantities. The study will be based on data collected through lesson observations, two written assessment tests made by the students at the end of each booklet, student workbooks and student and teacher questionnaires.

The principal aim of the research project is to find answers to questions like:

- are there moments in the learning process when students overcome a part of the discrepancy between arithmetic and algebra, and why?
- what is the effect of integrating the history of algebra in the learning strand the students?
- which type of shortened notations do children use naturally, and how does it compare with the historical development of algebraic notations?
- is there an acceptable compromise between intuitive, inconsistent symbolizations and formal algebraic notations?
- how can students actively take part in the process of fine tuning notations and establishing (pre-) algebraic conventions?
- to what extent and in what way can students become aware of different meanings of letters and symbols?

The next part of the article gives an outline of the learning strand and reports on a few classroom results from the most recent try-out.

2 Learning strand and classroom results

2.1 Proposed learning program: an outline

The historical development of algebra has inspired us to base the core learning material on word- or story-problems. The early rhetorical phase of algebra finds itself in-between arithmetic and algebra, so to speak: an algebraic way of thinking about unknowns combined with an arithmetic conception of numbers and operations. Babylonian, Egyptian, Chinese and early Western algebra was primarily concerned with problem solving situated in every day life, but mathematical riddles and recreational problems were common too. Fair exchange, money, mathematical riddles and recreational puzzles are rich contexts for developing handy solution methods and notation systems, and they are also appealing and meaningful for students. The natural preference and aptitude for solving word problems arithmetically will form the basis for the first half of the learning strand, whereby students' own informal strategies will be adequately fit in. The barter context in particular appears to be a natural, suitable setting to develop (pre-) algebraic notations and tools such as a good understanding of the basic operations and their inverses, an open mind to what letters and symbols mean in different situations, and the ability to reason about (un)known quantities. The transfer to a more algebraic approach will be instigated by the guided development of algebraic notation, especially the change from rhetorical to syncopated notation, as well as a more algebraic way of thinking. It will be interesting to determine whether the evolvement of intuitive notations used by the learner show similarities with the historical development of algebraic notation. Several original texts will be integrated to illustrate the inconvenience of syncopated notations and the value of our modern symbols, and different historical sources will be used to let students compare ancient solving methods like the Rule of False Position with modern techniques.

Outline of the mathematical content:

- restriction problems: problems with two variables and one or two conditions

- reverse calculations: practicing with inverse operations and arrow diagrams
- comparing quantities: reasoning with given barter relations
- progressive formalization of symbol use: discussing conflicts of notations and the changing role of symbols (eg. letters, equal sign)
- informal algebra: Rule of Three, reasoning about unknowns, Rule of False Position
- linear equations in one unknown and two unknowns.

The learning strand currently consists of two consecutive booklets at primary school level (*Change* and *Barter trade*, totaling 25 lessons), and two consecutive booklets at secondary school level (*Fancy Fair* and *Time travelers*, totaling 15 lessons). The learning strand can be split up into two parts, but ideally it is treated as one complete lesson series.

2.2 Classroom impressions

In the spring of 1999, the booklets *Change* and *Barter trade* were tested in two primary school classes, grade 6, consisting of 18 and 23 students. The 25 pre-algebra lessons were given by the regular teacher, based on explanatory notes in the teacher guide and occasional talks with the researcher. Approximately one-third of the lessons was observed and recorded; some lessons were videotaped. In early summer 1999, the third booklet *Fancy Fair* was tried out in two first year classes of secondary school; one of these classes also tested the last booklet. The regular teacher gave all the lessons according to guidelines in the teacher guide. However, since the booklet *Time travelers* had never been tested before, the teacher was guided more closely during the last lesson series. The next few paragraphs are meant to give an idea of what kind of solutions and discussions occurred; there has not been time yet to analyze the data with reference to the research questions.

The first topic in the primary school part of the program is problems with restrictions. The students are given a list with prices of 20 different candy bars (ranging between 5 and 95 cents), and are asked to write down what can be bought for precisely 1 guilder (100 cents). Many students realize that the answer will require a lot of paper and decide to use abbreviations. Immediately there is an opportunity to talk about effective mathematical notation (letters, syllables, operator symbols, tabular forms). In the next question, students are asked to comment on a disagreement between two imaginary students: "I found all the possibilities for 1 guilder!" one says; "But you can never know that for sure!", the other says. In one of the classes this activity instigated a lively discussion on the total number of possibilities, along these lines:

Several students working on their own reckon it is possible to know for sure, but it will take a long time.

Observer: 'How do you know you haven't missed one out?'

A girl replies that in that case you are doing it wrong. Another girl replies that she would start at the top of the list, take one item and check all the possibilities, and then take the next item from the top of the list, and so on.

Class discussion. The teacher asks for answers: some students give a numerical answer.

Teacher: 'How do you know there are so many?'

Student: 'At some time there will be an end to all the possibilities'.

The class investigates all the possibilities in combination with potato chips; there are too many to write down.

Teacher: 'How many possibilities altogether, do you think?'

A boy replies: 400. He then explains: he compared the problem with a comment the teacher made a week earlier, that there are as many as 520 possible simple sums with the first 20 natural numbers! And so, he concludes, there must be at least 400 in this case.

Other students then suggest more than 1000 possibilities, but they would like to hear the exact number from the author of the booklet!

This example illustrates how an open problem can lead to higher level thinking (reasoning about solvability) and can invite students to strike up other mathematical knowledge.

Another typical restriction problem in the first paragraph is situated in a money context. It is split up into two parts:
1. *how many quarters and dimes do you get for a coin worth 2.5 guilders*
2. *if the total number of coins is 13, how many dimes and how many quarters are there.*

Figure 3 shows how one student thought up a useful strategy for part 1. In general, students use a trial-and-error method and do not think of supportive notations like a table to structure their attempts. It also does not occur to them or disturb them that they might miss out some solutions this way.

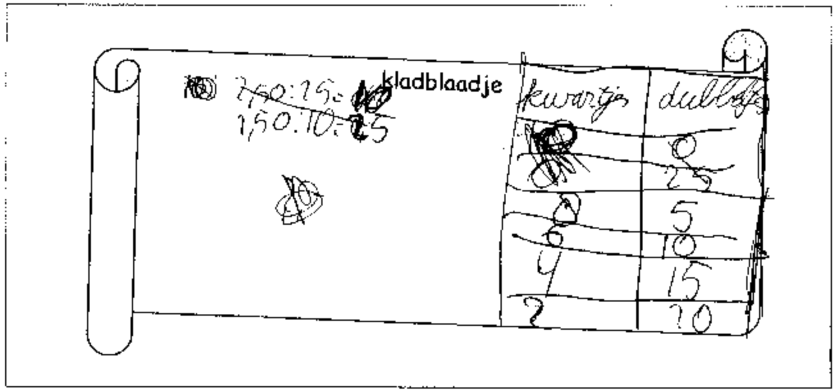

Figure 3: combinations of coins totaling 2.5 guilders

Restriction problems also appear in the third paragraph, for example:

1. riddles on age: *Mom is 5 times as old as John; but she is also 28 years older than him.*
2. Diophantine problems on sum and difference: *split the number 150 up into two numbers such that the difference between those numbers is 65* (Figure 4).

Figure 4a shows a spontaneous student strategy, where the given number is halved (75 and 75) and then the given difference is evenly allocated to the two numbers (75 + 32.5 and 75 - 32.5). In the other class the number line method as shown in figure 4b was introduced by the teacher as an alternative - more visual - strategy. Diophantine problems are handled again in the secondary school program, but this time with the intention to solve them using a linear equation in one unknown: *call the smaller number s, then the bigger one is s + 30, and the sum 2s + 30 = 100.* In this way students get a chance to reflect on the effectiveness of an informal and formal strategy of problem solving.

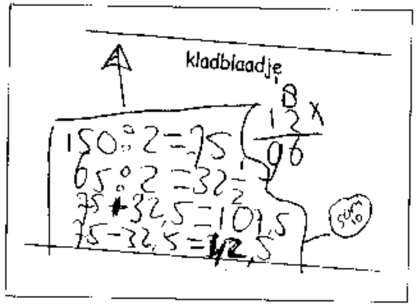

Figure 4a: halving the sum and the difference

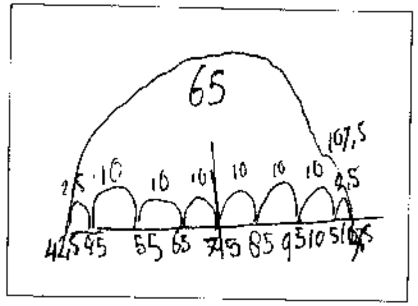

Figure 4b: mirrored jumps on the number line

Towards the end of the first booklet, there is a paragraph on reversing a string of calculations to find the initial number. 'Guess my number' goes as follows: one student thinks up a number and tells another student something like 'do it times 3, then add 5, subtract 2 and divide by 2, and you get 4: what was the number?' It is a successful activity: students enjoy it and they can do it at their own level and pace. Moreover, the teacher can make up many variations to practice even fractions and percentages in a playful way that takes little time. The last paragraph is a historical application of reverse calculations, organized around an original problem by Chistianus van Varenbraken (1532) (translated and summarized):

> A hermit prays to Saint Paul 'Double the amount of money in my purse and I will give you 6 pennies', and the saint complies. The hermit does the same when he comes to Saint Peter and Saint Francis. In the end, when his prayers have all been heard, the hermit has no money left. The question is, how much money did the hermit have at the start?[5]

The initial plan was to give students the original text along with some explanatory notes, and then ask them to solve the problem. This task turned out to be too complex even for a colleague designer, and irrelevant for the learning process besides. Looking for a way to visualize the problem and make it dynamic, we found the solution: instead of solving the problem on paper, the situation should be acted out in a short play. The teacher appoints 4 students to play the roles of the hermit and the three saints, and the other students in the class have to solve the problem. The play can be repeated indefinitely with different outcomes, enabling all students to catch on. The students then see the author's own solution in the booklet, where he merely gives the answer and checks that it is correct. Two higher order questions in this paragraph are:

[5] Source: Koot (1988).

1. *suppose you have to solve a similar problem, whereby the hermit has 38 pennies in the end instead of none; does the author's solution help you solve it?*

2. *what is the minimum number of pennies that the hermit needs to have at the start in order to make a profit?*

One clear outcome of the questionnaire is that students really enjoy acting. By literally *doing* the problem, it comes alive. Considering the problem's original purpose, 'a matter of delight' as the Van Varenbraken says, this activity is a good example of an appropriate reproduction of history.

The content of *Fancy Fair*, the first secondary school booklet, is concerned primarily with solving systems of two equations in two unknowns. The fancy fair attractions are represented by iconic markers; some of these have a fixed price and others are not yet priced. In order to concur with the primary school program, the booklet begins with expressions (equations) for trading markers fairly and suitable notations to represent these trade expressions. In the third paragraph students perform reverse calculations to determine the price of the markers. The program then moves on to pairs of combinations of markers for a given price: an iconic system of equations (see figure 5a and 5b). The problems can all be solved informally, by comparing the numbers of markers and reasoning about them. The problem in figure 5a requires determining the difference between the two combinations of markers and the prices ('subtracting'), and comparing again.

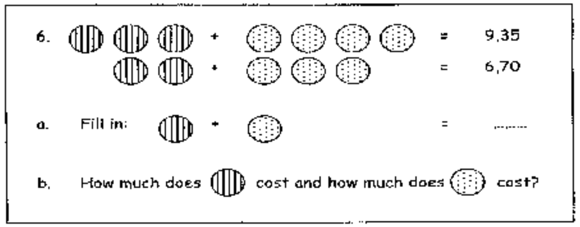

Figure 5a: solving an iconic system of equations by determining the difference

The problem in figure 5b is based on interchanging repeatedly one striped marker for a checkered one - raising the price by 50 cents in doing so - until only one type of marker remains.

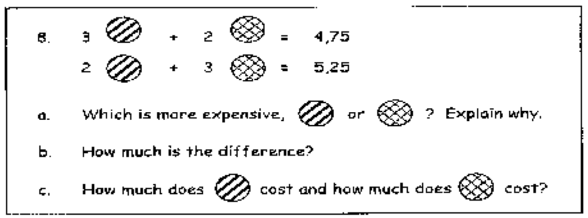

Figure 5b: solving an iconic system of equations by repeatedly interchanging one marker for another

In the final booklet historical problems are embedded in a story about two 13-year-old children who visit different countries in different eras and discover mathematics from the past. For example, there is a paragraph on the Rule of Three, another on the Rule of False position and the last paragraph deals with Diophantine problems. The Rule of False position is initiated by a well-known fish problem by Calandri (1491): *The head of a fish weighs 1/3 of the whole fish, his tail weighs 1/4 and its body weighs 300 grams. How much does the whole fish weigh?* Students are asked to estimate the weight first and then solve it using a rectangular bar (see Figure 6), after which they study the solution method of Calandri (see Figure 7).

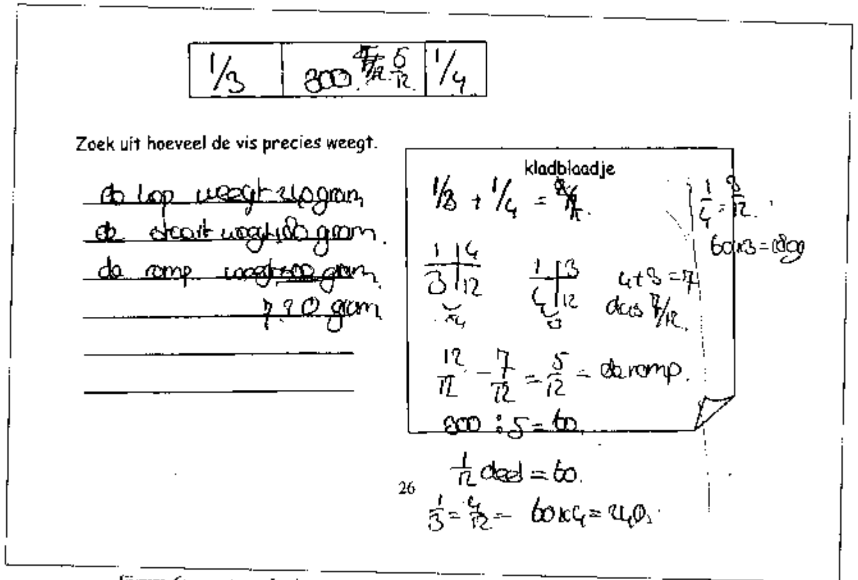

Figure 6: a rectangular bar to represent the fish, and the calculation of the weight

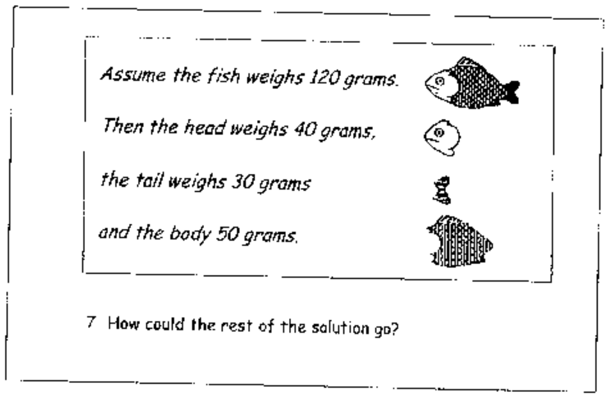

Figure 7: part of Calandri's solution[6]

The paragraph also includes some reflective questions:

[6]Problem and pictures originate from OFIR, R. & ARCAVI, A. (1992).

1. *why does Calandri choose 120 to start with?*
2. *what name would you give to this Rule of False Position?*
3. *what do you think of this method?*

It is really remarkable that not one student remarks that 120 is a ridiculous number to begin with. considering that the body of the fish already weighs 300 grams! The second question triggered very little response in the classroom but perhaps the student notebooks will reveal more.

2.3 Final remarks

In the next few months the research data will be searched for unequivocal, concrete indications that will help to answer the research questions. Nevertheless, at this time we would like to put forward a few conjectures regarding student attitude based on the observation of lessons. At primary school level, students are not trained to make notes or draft work as an aid to problem solving; they want to and try to solve even complex reasoning problems mentally. In secondary school they learn that they should distill the information, but they often don't. Students have trouble formulating their solution strategy; they are sometimes unwilling to write down an explanation to their answer, believing that the solution itself is more important how they got it. Especially at primary school we see a very passive attitude towards problem solving; students tend to wait for the teacher to give them a clue rather than investigating for themselves. The activities in the learning program seem to challenge the boys more than the girls. The effect of historical elements in the classroom at primary school level is disappointing; students are not as interested in the mathematical heritage as we expected and ancient solution strategies have not really stimulated students to attain a more critical attitude.

To finish off, here is just a note of precaution to the reader. The classroom results presented in this article serve to illustrate the kind of activities the proposed learning program can activate; they are by no means a representative selection of what the moderate student can accomplish. Similarly it must be clearly understood that the conjectures on student attitude are still subject to change if the final data analysis proves differently.

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"Si les mathématiques m'étaient contées..."

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Abstract

J'aborde ici la question suivante : peut-on rendre les seuils épistémologiques plus accessibles à la plupart des élèves ? Très souvent ceux-ci demeurent "pseudostructurels" (au sens de Sfard), c'est-à-dire qu'ils tendent à sous-évaluer les aspects sémantiques pour rester au niveau syntaxique. Ils perçoivent les mathématiques essentiellement comme un ensemble de symboles plus ou moins vides, qu'il faut savoir manier pour réussir à l'école et dans la vie. Mon travail, s'agissant d'élèves de 14 à 19 ans, vise à montrer qu'il est intéressant de proposer par moments, pour véhiculer certaines notions mathématiques de base, des langages moins structurés et symboliques que ceux utilisés le plus souvent. La proposition que je fais dans la suite consiste en une approche des ensembles infinis à l'aide d'une pièce de théâtre.

1 Introduction

Cet article présente une pièce de théâtre au contenu mathématique, qui est le résultat d'un travail fait dans la classe de première section B de l'Istituto d'arte "Max Fabiani" de Gorizia pendant l'année scolaire 1998-99.

Le but est lancer la proposition qu'on puisse utiliser de temps en temps le langage théâtral pour élaborer des projets d'enseignement permettant aux élèves de s'approprier plus aisément les aspects sémantiques des notions mathématiques étudiées. En effet la recherche en didactique et la pratique de l'enseignement montrent que souvent les étudiants se contentent de rester au niveau syntaxique, c'est-à-dire estiment avoir compris dès qu'ils sont capables de manier des symboles suffisamment bien que pour réussir. Ils restent, pour le dire avec le mot utilisé par Arzarello, Bazzini, Chiappini, en citant les travaux de Sfard [1], "pseudostructurels".

En regardant les élèves plus faibles en maths, on peut observer qu'un des obstacles qui rend difficile l'acquisition du sens est le langage formalisé à travers lequel les mathématiques sont traditionellement véhiculées. Ce qu'on propose d'habitude pour améliorer la situation est un enseignement moyennant le "problem solving" qui permet aux élèves de reconstruire le savoir [3]. Je trouve que cette méthode n'est pas toujours aisément praticable. Il y a des parties des mathématiques qui se prêtent moins bien à être traitées de cette façon, étant donné le fort degré d'abstraction qui les caractérise et l'applicabilité non immédiate. Un de ces sujet me paraît la notion d'ensemble infini. J'ai voulu essayer de rendre ce concept "plein de sens" [4] en l'approchant par une métaphore moyennant le langage théâtral.

Il y a un autre commentaire important. La réforme des maths modernes avait élevé les ensembles au rôle de langage privilegié et universel dans l'enseignement des mathématiques. Il me semble que la réaction à cette réforme a justement porté la didactique à s'occuper des sujets qui avaient été sous-estimés. Aujourd'hui, par contre, lorsqu'on nomme les ensembles on se trouve parfois face à un auditoire froid. Je tiens à souligner qu'à mon avis il n'y a pas de "bonnes" et des "mauvaises" branches des mathématiques, mais seulement des méthodes plus ou moins efficaces de les utiliser dans la transmission du savoir.

2 Pourquoi le théâtre?

L'idée d'écrire ce texte théâtral m'est venue pendant les vacances de Noël 1997. J'étais à Rome chez mon ami Giuliano Spirito. Il venait de m'offrir une copie de son dernier bouquin : "La grammatica dei numeri" [2], dans lequel j'ai trouvé la métaphore de l'hôtel infini par David Hilbert. L'image m'a tout de suite fascinée, et j'ai decidé d'en faire une pièce pour des élèves de première ou seconde du cycle supérieur italien (étudiants de 14 à 16 ans). Au début donc l'idée est née de façon quasi artistique, sous l'effet d'une suggestion. Cela n'empêche qu'en y revenant par après j'ai trouvé qu'elle a des justifications et des retombées pédagogiques et didactiques.

Un peu partout dans le monde, les mathématiques sont perçues comme une discipline rigide, qui n'offre pas de possibilités d'y mettre quelque chose de personnel, qu'on comprend seulement si on est naturellement doué, si on a pour ainsi dire "la bosse des maths" [4]. En Italie la situation est peut-être encore plus grave pour des raisons à mon avis assez évidentes [6]. L'école appelée supérieure, dans laquelle les élèves ont de 14 à 19 ans, est en gros encore réglée par une réforme née en 1923, qui porte le nom de "Riforma Gentile" en l'honneur de son promoteur, le philosophe Giovanni Gentile, Ministre de l'Education pendant le premier gouvernement Mussolini. Dans cette réforme la philosophie et le latin sont les matières considérées indispensables à la formation de la classe dirigeante, tandis que les disciplines scientifiques sont

négligées. Les mathématiques sont présentées essentiellement comme un outil au service des sciences expérimentales et par conséquent trouvent place surtout dans les écoles techniques et beaucoup moins dans les lycées, les seules écoles qui à l'époque permettaient l'entrée à l'université. Ces dernières années, le nouveau Ministre de l'Education, Luigi Berlinguer, a entamé une réforme de l'école supérieure. Sans entrer dans les détails des nouvelles propositions, ni dans la méthode de travail adoptée qui personnellement ne me paraît pas tout à fait adéquate à cette tâche difficile, il faut reconnaître qu'il y a une plus grande attention à la formation scientifique. Mon impression est que, cette fois-ci, dans le but de rendre l'école plus proche de la réalité, on pousse surtout à l'informatisation et aux mathématiques appliquées, en risquant d'oublier d'autres aspects de la discipline.

Dans ce contexte, proposer aux élèves une pièce de théâtre de contenu mathématique assez abstrait me paraît une façon d'insinuer que les mathématiques sont aussi autre chose. Cela non seulement auprès des élèves, mais, si on parvient à presenter la pièce en public, aussi auprès des collègues d'autres disciplines et auprès des parents. On agirait ainsi en même temps sur plusieurs fronts différents :

- présenter une idée mathématique dénuée de sa veste technique,
- permettre aux étudiants de s'amuser en faisant des mathématiques,
- travailler de façon multidisciplinaire et par objectifs en investiguant des aspects de la notion envisagée liés à d'autres parties du savoir comme les arts figuratifs, l'architecture, la musique, la peinture pendant le travail de mise au point de la mise en scène,
- faire de la divulgation mathématique.

3 La pièce

HOTEL ALEPH
Pièce en un acte

Personnages
Le professeur de mathématique
Susy
Des camarades de classe de Susy
Françoise (copine de Susy dans le rêve)
Louis, garçon de chambre (voix au téléphone)
Le garçon de café
Le réceptionniste
Premier client
Équipe de 13 joueurs
L'entraîneur de l'équipe
Groupe "infini" de clients
Le porte-parole du groupe

INDICATIONS SCÉNIQUES

La scène se présente divisée en deux.
A gauche une classe.
A droite une entrée d'un hôtel avec un café annexé.
Au début, la partie à gauche est éclairée, tandis qu'à droite il fait sombre.

SCÈNE DE GAUCHE

Les élèves arrivent en discutant entre eux et ils s'asseyent.
Le professeur entre lui-aussi.

Le professeur

Bonjour à tous. Ça va ? Très bien, je vois que vous êtes tous là aujourd'hui. Tant mieux. Je vais vous expliquer une notion qui n'est pas très facile. Avez-vous bien compris les fonctions ? Les fonctions injectives, surjectives, les bijections ?

Un élève

Ce n'est pas immédiat. Avec mon cahier ouvert devant moi, je parviens à faire correctement les exercices, sinon je confonds . . . d'habitude les relations fonctionnelles non injectives et les fonctions injectives.

Le professeur

Il se peut que tu n'aies pas encore étudié comme il faut. Pour la prochaine fois, tâche de faire les exercices sans avoir les définitions devant toi, pour vérifier si finalement elles sont entrées dans ta tête. Aujourd'hui je continue mon cours en traitant un concept nouveau, l'infini dénombrable.

L'enseignant commence son cours magistral très structuré et formalisé, dos à la classe (qui s'ennuie) et ne comprend quasi rien. Au dernier rang Susy s'endort. Elle commence à rêver. La lumière diminue petit à petit dans la partie gauche jusqu'à ce que seule Susy soit faiblement éclairée. Des bulles de savon, sortant derrière sa tête appuyée sur le banc, indiquent que ce qui se passera dorénavant réprésente son rêve. En même temps la partie droite de la scène s'éclaire progressivement. Pendant le rêve de Susy, le professeur continue sa leçon à voix basse, comme un bruit de fond. Durant ce changement de scène, et puis comme musique de fond dans le café de l'hôtel, on entend la "Musique de π", c'est-à-dire un des arragements qu'on peut faire en "traduisant en musique" un nombre assez élévé de chiffres de π. La lampe est un ruban de Moebius.

SCÈNE DE DROITE

Susy (au café avec son amie Françoise)

Alors Françoise, ça fait longtemps qu'on ne s'est plus vue, qu'est-ce que tu racontes ?

Françoise

Je viens de me disputer avec mes parents. La barbe ! Tout le temps avec cette histoire de mauvaises notes. Je leur ai dit qu'en hiver mon cerveau est gelé, mais maintenant c'est le printemps et il va falloir inventer un argument différent. Mais causons d'autre chose s'il te plaît.

Susy

Ça va, tu as raison. Mes parents aussi sont des casse-pieds. J'en ai vraiment ras-le-bol de leurs histoires stupides. Mais, à propos de stupidité . . . as-tu vu ces imbéciles ? Ils affichent deux pancartes contradictoires : COMPLET et CHAMBRES À LOUER.

Françoise

Ne sois pas si sévère. Ils sont peut-être tout simplement distraits. Ils ont sans doute oublié de retirer un avis lorsqu'ils ont accroché l'autre. Allons le dire au réceptionniste.

Le garçon arrive prendre la commande.

Le garçon

Mesdames désirent ?

Susy

Je prends une blanche et toi ?

Françoise

Moi-aussi. Deux blanches, s'il vous plaît.

Le garçon

Très bien. J'arrive tout de suite.
(et il s'en va chercher les bières)

Susy (à Françoise)

Je vais parler au réceptionniste de cette histoire de pancartes.

Elle se lève et va vers la réception, pas très loin.

Excusez-moi, monsieur, il me semble que vous avez une pancarte de trop. Vous devriez vous en occuper. Il est clair que l'hôtel ne peut pas avoir des chambres à louer tout en étant complet.

Le réceptionniste

Là ma petite, vous vous trompez. C'est justement pour cela que cet hôtel est renommé. Il a été bâti de façon telle que, même étant complet, il a toujours des chambres disponibles.

Susy (perplexe)

Cela semble impossible.

Le réceptionniste

On voit bien que vous ne savez pas qui est l'architecte. Il s'agit de David Hilbert, un des meilleurs.

Susy

Je ne comprends pas.

Le réceptionniste

Le secret est d'avoir un nombre infini de chambres.

Susy (En aparté)

Ça alors!

(Puis au réceptionniste)

Merci pour les explications.

Elle revient à la table de café où son amie l'attend en buvant sa bière, qui a été servie entretemps par le garçon.

Françoise

Assieds-toi, viens boire ta bière. Qu'est-ce qu'il a dit? Est-il amoureux de toi ou quoi? Tu ne revenais plus ...

Susy

Je crois plutôt qu'il se fiche de moi! Il dit que l'architecte, David Je-ne-sais-plus-quoi, très connu et patati et patata, aurait construit un bâtiment avec une infinité de chambres et donc les deux avis ne seraient pas contradictoires.

Françoise

Je n'y pige pas grande chose, moi.

Susy

Moi non plus.

Arrive un client qui s'approche de la réception.

Premier client

Excusez-moi, je viens de lire que vous avez des chambres à louer. En réalité j'ai lu aussi que l'hôtel est complet, et n'ayant pas tout-à-fait compris, je suis venu me renseigner. J'aimerais bien rester trois nuits.

Le réceptionniste

En effet, toutes les chambres sont occupées, mais ne vous inquiétez pas, je peux en libérer une pour vous, il suffit que vous attendiez deux heures. Cela vous convient?

Premier client

Bien sûr, merci. Je ne suis pas pressé. Puis-je vous laisser mes bagages, pendant que je me promène au centre-ville ?

Le réceptionniste

Certes. Vous pouvez laisser vos bagages dans la salle à côté.

(puis au téléphone)

Allo, Louis?

. . . .

Nous avons un nouveau client. Comme je lui donne la numéro 1, s'il te plaît, veux-tu dire à tous les aux autres de se déplacer à la chambre suivante.

. . . .

Merci beaucoup. Au revoir.

Le réceptionniste raccroche.

Françoise

As-tu entendu? Comment a-t-il fait? On ne peut pas libérer la 1 en ne chassant personne!

Susy (en réfléchissant)

... Attends, peut-être que je commence à comprendre. Écoute: que se passerait-il si l'hôtel avait, disons, dix chambres et était complet?

Françoise

Il se passerait qu'il n'y aurait plus de place pour personne, comme dans tous les hôtels "normaux".

Susy

On ne pourrait pas libérer la 1, comme ici. Si on disait à tous les clients qui logent à l'auberge de se déplacer à la chambre suivante, le client de la 1 irait à la 2, celui de la 2 à la 3, et ainsi de suite ... ... , jusqu'à celui de la 10 qui ne trouverait plus de place, car il n'y a pas une onzième chambre. Compris?

Françoise

Oui, mais en quoi est-ce différent ici?

Susy

Mais, parce qu'ici il n'y a pas de dernière chambre, puisqu'elles sont en nombre infini!

Un groupe de treize touristes se présente à la réception. Un d'entre eux s'adresse au réceptionniste tandis que les autres continuent à chuchoter entre eux.

L'entraîneur de l'équipe de volley-ball

Excusez-nous, monsieur. Nous regrettons d'arriver comme ça sans réservation. Nous sommes une équipe de volley-ball. Nous jouerons demain. Au secrétariat ils ont oublié de réserver les places nécessaires. Il nous faudrait treize chambres. Le match de demain est très important pour la qualification et il faut absolument que nous nous reposions tous tranquillement cette nuit pour bien trouver la concentration. Si vous pouviez nous aider!

Le réceptionniste

Ne vous inquiétez pas. Vos chambres seront prêtes dans deux heures. Vous savez, l'hôtel étant complet, c'est un peu long de libérer de la place. Si vous pouvez vous permettre une promenade, je vous conseille le Jardin Cantor, juste derrière la Place Zermelo-Fraenkel. Il est vraiment magnifique.

L'entraîneur

Je vous remercie pour le conseil. Une promenade est justement ce qu'il nous faut pour nous remettre du voyage.

Françoise (qui a soigneusement écouté)

Et maintenant? Qu'est-ce qu'il va faire? Il a déjà assigné la chambre 1. La situation s'embrouille.

Susy

Je crois que non. Si tu réfléchis bien, il suffit que les gens qui occupent les chambres à partir de la deuxième se déplacent de treize places en avant, non ?

En même temps on entend le réceptionniste parler au téléphone.

Le réceptionniste

Louis, c'est de nouveau moi. Il y a treize nouveaux clients. S'il te plaît, déplace les clients à partir de la deuxième, de treize places en avant.

. . . . . .

Merci, tu es formidable! La gestion de cet hôtel n'est pas facile. Il est clair que le fait d'avoir toujours des places disponibles permet de gagner davantage. Bon . . . je dois te quitter. Le client de la 1 approche. Est-elle prête?

. . . . . .

À tout à l'heure. Ciao.

Premier client

Bonjour. Je sais que je suis un peu à l'avance. La chambre est-elle déjà prête?

Le réceptionniste

Bien sûr. Voilà les clefs.

Tout d'un coup, on entend un grand bruit. Une multitude de personnes s'approche.

Susy

Que se passe-t-il maintenant ? Dis-donc, quel bruit!

On voit une personne qui parle au réceptionniste, et derrière elle un tas de gens bruyants.

Le réceptionniste (en s'adressant au porte-parole de ce groupe énorme)

Attendez. Attendez. Oui j'ai compris. Vous êtes très nombreux. Il n'y a pas de quoi s'inquiéter. La particularité de cet hôtel, ce qui le rend vraiment unique, est justement le fait qu'on n'est jamais obligé de renvoyer quelqu'un. Donc, dites-moi traquillement, combien êtes-vous au juste?

Le porte-parole du groupe

Nous sommes une infinité.

Françoise (qui a écouté attentivement)

Ça alors! Comment peut-il les loger tous? Il est clair que cette fois-ci l'astuce de déplacer les clients d'un certain nombre de places en avant ne marchera plus.

Le réceptionniste

Très bien, mais vous devrez attendre deux heures environ. Cela vous convient ?

Le porte-parole du groupe

Oui, on n'espérait pas mieux. À quatre heures alors. Nous irons manger un bout. Merci beaucoup. À tantôt.

Susy

Je suis curieuse de voir comment il se débrouillera maintenant.

Françoise

Il suffit d'écouter. Il va sans doute appeler son copain du service des chambres.

Le réceptionniste (au téléphone)

Allo, Louis ? Nous avons un groupe vraiment nombreux. Une infinité de touristes.

. . . .

Oui, fais comme on nous a expliqué au cours de la formation. Demande aux gens déjà logés s'ils veulent bien se déplacer à la chambre qui porte le numéro double.

. . . .

Je leur ai dit de passer vers quatre heures de l'après-midi.

. . . .

C'est bien. D'accord. Ciao.

Susy

C'est vrai, enfin . . . logique! De cette façon-ci toutes les chambres impaires seront libérées d'un seul coup!

À ce moment le professeur, dans la scène de gauche, vient de terminer sa leçon. La lumière revient à gauche et s'affaiblit à droite jusqu'à s'éteindre tout-à-fait. Le prof. finalement face à la classe, se rend compte que Susy a dormi tout le temps. Il va vers elle et la réveille.

Le professeur (à voix suffisamment haute pour réveiller l'élève)

Mademoiselle!

(sur un ton normal)

La leçon était plaisante, je vois. Alors voilà, c'est à toi de résumer pour tes copains la notion d'infini dénombrable.

Susy

Hum ... infini? ... Justement ... Imaginez que vous avez un hôtel avec un nombre infini de chambres et vous verrez que des phénomènes vraiment bizarres peuvent apparaître.

Le professeur

Qu'est-ce que les hôtels viennent faire dans cette histoire? Tu te moques de moi, pas vrai?

Susy (l'air embêté)

Si vous voulez bien me laisser continuer, vous verrez que je suis sérieuse.

Le professeur (gêné)

Ça va, écoutons.

(en aparté)

Je veux voir quelle histoire fantaisiste elle va sortir cette fois!

Susy

Je venais de dire ... un hôtel normal ... il n'est pas complet ou bien il l'est, et alors il n'y a plus de chambres à louer. Si par contre nous supposons que nous avons un hôtel avec une infinité de chambres, tout se passe autrement.

Façon de parler, bien entendu! Des hôtels comme-ça n'existent pas.

Mais, revenons sur nos pas ... un hôtel infini pourrait être complet tout en ayant des chambres à louer. Attention: je dis "à louer", je ne dis pas "libres", ce qui ne revient pas au même dans ce cas-ci. Écoutez: si, l'hôtel étant complet, il arrivait un nouveau client, il suffirait de dire à tous ceux qui y sont logés de se déplacer à la chambre suivante, puisqu'il n'y a pas de dernière chambre. Si, à la place d'une seule personne, devait arriver un groupe, alors il suffirait de déplacer les gens d'autant de chambres en avant qu'il en faut de libres. Et même si on suppose l'arrivée d'un groupe infini de clients, en perfectionnant l'astuce, c'est-à-dire en déplaçant chacun à la chambre de numéro double, on aurait d'un coup une infinité de chambres libres, toutes les impaires! Extraordinaire, n'est-ce pas?

Drrinnn! On entend la sonnette qui annonce la récréation.

Le professeur (presque fâché)

Bravo! Et tu penses t'en sortir comme ça à l'examen? Photocopie les notes de tes camarades et étudie attentivement pour apprendre à t'exprimer avec le langage formalisé qui caractérise la discipline.

(puis à toute la classe)

Au revoir, à lundi prochain.

Des camarades s'approchent de Susy.

Un copain

Je ne sais pas ce qu'il en pense au juste, il avait une de ces têtes! Moi en tout cas, ton histoire ... chapeau! Ça faisait quasi une heure que je tâchais de comprendre l'idée de fond cachée parmi ses flèches ... Et puis en trois minutes, avec ton hôtel, il me semble que finalement j'y pige quelque chose.

Une copine

Moi aussi j'ai commencé à percevoir un sens. Ce n'est pas de la foutaise, en tous cas. Faudra y réfléchir à fond.

Un autre copain

Certes, l'humanité est bizarre ... Comme si on n'avait pas assez de problèmes concrets, les guerres, la famine, les maladies et cetera, on va s'occuper de l'infini ... Allez, laissons tomber, et allons boire un café. Après la pause nous avons le cours de philo, et là aussi on va se casser la tête avec des choses folles.

FIN

4 Conclusions

Entre l'image de l'hôtel et un ensemble ayant la propriété qu'il existe une bijection entre l'ensemble même et une des ses parties non triviales, il y a un saut important, dans lequel se situe ce qu'on appelle le seuil épistémologique [3].

Dans une théorie mathématique, un concept peut être caractérisé directement par des axiomes, c'est-à-dire se trouver au départ d'un système déductif ou bien être introduit par une définition à une place plus ou moins éloignée des axiomes. Dans le premier cas les axiomes nous donnent souvent une idée assez claire de ce qu'on peut ou on ne peut pas faire avec l'objet introduit. Au cas où, par contre, la définition se trouve assez éloignée des axiomes, tout en étant obligatoirement caractérisante l'objet mathématique qu'elle introduit à l'intérieur de la théorie dans laquelle elle se situe, elle n'est souvent pas assez éclairante pour notre esprit. Je veux dire qu'il ne suffit pas de connaître la définition, savoir ce que c'est l'objet mathématique introduit. D'habitude on commence à posséder mentalement l'objet après l'avoir utilisé un certain nombre de fois. Quand on enseigne par chantiers de problèmes [3], les notions sont introduites de façon instrumentale et les définitions viennent après.

Ici la proposition est de créer une image mentale avant de donner la définition. En utilisant une métaphore, on bâtit une image sur laquelle la définition peut s'enraciner dans le but de permettre au concept de s'épanouir. Je crois que si les élèves ont devant eux deux approches tout à fait différentes de la notion, dont une est figurée et hors du contexte usuel des mathématiques, il leur sera plus facile de franchir le seuil épistémologique. Il me semble que l'importance d'utiliser le théâtre soit double. D'un côté, ce travail est motivant car inhabituel et de l'autre, le fait de devoir mettre en scène oblige à passer et repasser sur l'image figurée de façon telle qu'elle devienne très familière et presque naturelle.

En réalité je ne peux malheureusement pas être plus profonde en ce qui concerne cette deuxième affirmation. Mon idée initiale de proposer un travail multimédia présentant la pièce jouée par les élèves n'a pas pu être réalisée à temps. Mes élèves et moi-même serions très contents d'apprendre que quelqu'un, quelque part ailleurs, l'a fait, peut-être avant nous.

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Les fonctions continues sont-elles toujours différentiables ?
Le cas de Philippe Gilbert (1873)

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Abstract

Aujourd'hui on apprend tout au début d'un cours d'analyse qu'une fonction continue n'est pas toujours différentiable comme on le démontre par la valeur absolue. L'idée de différentiabilité est une idée clef de notre mathématique. Il est surprenant que cette idée ne fut développée que dans la deuxième moitié du 19ème siècle. L'histoire de cette notion fondamentale est liée à celle d'un théorème souvent appelé théorème d'AMPÈRE disant qu'une fonction continue est différentiable sauf en des points isolés.


André Marie Ampère (1775-1836)

1 Le théorème dit d'Ampère

En 1806 un jeune mathématicien, nommé André Marie AMPÈRE (1775–1836), alors répétiteur à l'École polytechnique, publiait un mémoire sous le titre *Recherches sur quelques points de la théorie des fonctions dérivées qui conduisent à une nouvelle démonstration de la série de Taylor, et à l'expression finie des termes qu'on néglige lorsqu'on arrête cette série à un terme quelconque*. AMPÈRE avait été étudiant à l'École polytechnique et ses travaux sont influencés par les idées de son maître, LAGRANGE; en particulier pour AMPÈRE les séries forment la base du calcul différentiel.

Après avoir introduit la fonction

$$F(x, i) := \frac{f(x+i) - f(x)}{i}$$

pour une fonction réelle non-constante $f(x)$ *et une quantité non-nulle arbitraire* i, AMPÈRE constate :

> Je me propose d'abord de démontrer que la fonction de x et de i
>
> $$\frac{f(x+i) - f(x)}{i}$$
>
> . . ., ne peut devenir ni nulle ni infinie pour toutes les valeurs de x, lorsqu'on fait $i = 0, \ldots$; il résultera nécessairement de cette démonstration, que
>
> $$\frac{f(x+i) - f(x)}{i}$$
>
> se réduit, quand $i = 0$, à une fonction de x. . . . nous la représenterons comme cet illustre mathématicien [LAGRANGE], par $f'(x)$, et notre premier but sera d'en démontrer l'existence.
>
> (AMPÈRE 1806, 148sq)

Qu'est-ce que différentier une fonction?

Au début du $19^{\text{ème}}$ siècle trois conceptions guidaient les réponses données à cette question :

• L'école de Leibniz (les frères Bernoulli, l'Hôpital, Euler, Carnot, . . .) travaillait avec des infiniment petits. Le quotient différentiel $\frac{dy}{dx}$ d'une fonction est le rapport d'un incrément infiniment petit dx de la variable indépendante x l'incrément infini petit dy causé par dx (d'où s'ensuit que la fonction $y(x)$ doit être continue). La valeur de ce quotient se calcule selon des règles (comme les axiomes de Bernoulli et l'Hôpital : une grandeur qui est augmentée d'une grandeur qui est infiniment plus petite qu'elle n'est pas augmentée). Dans cette optique on n'avait pas de problème d'existence avec la dérivée parce que le quotient différentiel peut toujours se former. Le vrai problème est de calculer sa valeur.

• La théorie des limites était proposée dans le $18^{\text{ème}}$ siècle par d'Alembert qui se considérait comme le successeur de NEWTON; elle était reprise par S. LACROIX dans son *Traité de calcul différentiel et de calcul intégral* (1797) et se trouve au moins au niveau des termes (peut-être pas des choses) chez CAUCHY et d'autres. Mais il faut voir que les idées de ces auteurs n'étaient pas les nôtres; cf. le passage suivant du *Cours d'analyse* (1821) de CAUCHY :

> Quelques fois, tandis qu'une ou plusieurs variables convergent vers des limites fixes, une expression qui renferme ces variables converge à la fois vers plusieurs limites différentes les unes des autres. (CAUCHY II4, 26)

Il donne les exemples suivants (la variable x tend vers zéro) :

> $\lim A^x = 1$ et $\lim \sin x = 0$ sont des cas où il y a une seule limite, *tandis que l'expression* $\lim\left(\left(\frac{1}{x}\right)\right)$ *admet deux valeurs, savoir "plus infini" et "moins infini", et*
>
> $$\lim\left(\left(\sin\frac{1}{x}\right)\right)$$
>
> *une infinité de valeurs comprises entre les limites* $- -1$ *et* $+1$. (CAUCHY II4, 26)

Autrement dit quelques cas de non-existence d'une limite au sens moderne sont remplacés chez CAUCHY par l'idée d'une multitude de limites (ce que revient à nos points d'accumulation).

• La théorie de LAGRANGE qui fonde le calcul différentiel et intégral par les séries était assez populaire au débu du $19^{\text{ème}}$ siècle (AMPÈRE, Arbogast, Méray [plutard]). Selon LAGRANGE différentier c'est chercher la fonction dérivée. On trouve cette fonction à l'aide de la série dont il est le premier coéfficient (d'où vient le terme "dérivée"). Parce que chaque fonction est à développer dans une série sauf à des points isolés on n'a aucun problème d'éxistence avec la dérivée.

> Je vais d'abord démontrer que, dans la série résultante du développement de la fonction $f(x+i)$, il ne peut se trouver aucune puissance fractionnaire de i, à moins qu'on ne donne à x des valeurs particulières. (LAGRANGE IX, 22)

Par un malentendu remarquable AMPÈRE a été considéré jusqu'à nos jours comme étant le premier à avoir démontré l'existence de la dérivée d'une fonction continue en général (c'est-à-dire en-dehors de singularités isolées) un énoncé considéré comme évident d'un point de vue intuitif (cf. La citation de GILBERT en bas). Si on analyse soigneusement le mémoire d'AMPÈRE on comprend qu'AMPÈRE avait l'intention de démontrer le théorème des accroisements finis, qui est –comme on le sait depuis LAGRANGE– un outil important pour démontrer la formule du reste dans la série de Taylor. Le terme "existence" veut dire que la dérivée existe dans le sens que ses valeurs ne sont ni nulles ni infinies (sauf en des points isolés); on parle même aujourd'hui de variables évanouissantes! Cette sorte d'existence est garantie par le fait que $f(x)$ est supposée non-constante c'est-à-dire que $f(x+i)$ est différent de $f(x)$ pour "presque" tout x; par conséquent

$$\left(\frac{h}{k}\right) = \frac{f(x+i) - f(x)}{i} \neq 0, \neq \infty$$

et donc $f'(x) \neq 0, \neq \infty$ pour presque tout i.

Toute une tradition importante de l'enseignement supérieur français –celui de l'École polytechnique– était fondée plus tard sur le "théorème d'AMPÈRE". Je cite comme exemple Jean–Marie–Constant DUHAMEL (1797–1872) :

> Cela posé, il est facile de démontrer que, sous ces conditions, pour toute valeur de x, il existe une limite finie pour le rapport des accroisements infiniment petits correspondants h et k; c'est-à-dire qu'il ne peut y avoir que des valeurs exceptionelles de x pour lesquelles ce rapport croisse ou décroisse indéfiniment. (DUHAMEL 1856, 94sq)

La démonstration en est la suivante :

> Soit f une fonction continue dans l'intervalle $[x_0, X]$ et soit $y_0 = f(x_0)$ et $Y = f(X)$. On divise l'intervalle $[x_0, X]$ en n parties de même longueur telle que $X - x_0 = nh$. Soit $K_1 = f(x_0+h) - f(x_0), K_2, K_3 \ldots$ les différences analogues.
>
> $$\frac{Y - y_0}{X - x_0} = \frac{\frac{K_1}{h} + \frac{K_2}{h} + \ldots + \frac{K_n}{h}}{n},$$
>
> c'est-à-dire que le rapport invariable des accroissements finis de x et de y, quand on passe de x_0 à X, est la moyenne arithmétique des rapports $\frac{K_1}{h}, \frac{K_2}{h}, \ldots, \frac{K_n}{h}$ quelque soit le nombre entier n. Si maintenant on fait croître n indéfiniment, les termes de ces rapports tendront vers zéro, et leur moyenne arithmétique étant toujours égale à la quantité finie $\frac{Y-y_0}{X-x_0}$, il est impossible qu'ils tendent tous vers zéro, ou qu'ils croissent tous indéfiniment. (DUHAMEL 1856, 94sq)

Les conditions posées par DUHAMEL sont la continuité de la fonction $f(x)$ et sa monotonie par morceaux, sa démonstration est "quasi-algébrique" au sens où elle cache le vrai problème (les increments dépendent des valeurs de x, leur nombre tend vers l'infini) par sa notation. Ce faux théorème est conservé dans les manuels provenants de l'École polytechnique jusqu'en 1868 (Serret *Cours de calcul différentiel et intégral*); il se trouve même dans la nouvelle édition du livre de DUHAMEL déjà cité et qui fut effectuée par son neveu Joseph Bertrand [1822–1900] en 1874. Dix ans après les choses étaient reglées dans notre sens, comme on l'apprend par exemple de Jean HOUEL [1823–1886] (le correspondant de DARBOUX; cf. infra!) :

> 202. L'existence de la dérivée d'une fonction suppose essentiellement que la fonction soit continue. Mais la continuité de la fonction n'est pas, réciproquement, une condition suffisante de l'existence d'une dérivée, et l'on peut exprimer par les signes de l'Analyse une infinité de fonctions continues n'ayant pas de dérivée. (HOUEL 1878, 143sq)

2 La situation après 1850

Un des premiers qui se soient occupés du problème de la différentiabilité –mais seulement d'une manière implicite– était Bernard RIEMANN (1820–1864) en 1854. Dans son Habilitationsschrift *Sur la représentation d'une fonction arbitraire par des séries trigonométriques*, un ouvrage d'une importance séculaire inspiré par Dirichlet, son maître à Berlin et publié en 1868 –il construit plusieurs fonctions bizarres. Pour nous la plus intéressante est la suivante : Soit $\Delta(x) := x - e(x)$, où $e(x)$ est le nombre entier le plus proche du nombre réel x (si $x = z \pm \frac{1}{2}$ avec un entier z on définit $\Delta(x) = 0$). Par conséquent la fonction $f(x)$ a des solutions de continuité pour toutes les valeurs $x = z + \frac{1}{2}$; elle y fait des sauts de hauteur $\frac{1}{2}$.

Alors on forme la série

$$f(x) = \sum_{n}^{\infty} \frac{\Delta(nx)}{n^2}.$$

Cette fonction est continue à l'exception des valeurs $x = \frac{p}{2q}$, où p et q sont des entiers sans diviseurs communs, où elle fait des sauts de hauteur $\frac{\pi^2}{8q^2}$. De plus elle peut s'intégrer; par conséquent sa fonction primitive $F(x)$ est une fonction continue qui n'est pas différentiable pour les valeurs $x = \frac{p}{2q}$, un fait qui n'est pas exprimé par RIEMANN. Il est difficile de dire si RIEMANN a bien compris cet aspect (on a besoin d'un théorème publié en 1875 par Gaston DARBOUX (1842–1917) disant qu'une fonction dérivée ne peut pas avoir des sauts d'une hauteur finie).
L'intérêt pour les fonctions bizarres, justifié par RIEMANN d'une manière explicite dans l'introduction de son mémoire, se trouvait encore une fois chez un de ses disciples : Hermann Hankel (1839–1873). Hankel, qui est mort jeune comme RIEMANN, était quelqu'un qui s'occupait de préférence des thèmes novateurs comme la géométrie projective, les nombres complexes comme structure abstraite et les fonctions bizarres. En 1870, il publiait à Tübingen un mémoire concernant le "principe de la condensation des singularités". À l'aide de son principe il construisait beaucoup d'exemples bizzares. J'en cite un :
La fonction $f(x) = x \sin\frac{1}{x}$ est continue en $x = 0$ si on définit $f(0) = 0$. Mais elle n'est pas différentiable en ce point; par conséquent $f(x)$ possède en $x = 0$ une singularité du point de vue différentiel. L'idée de Hankel est de transporter cette singularité en chaque point rationnel. Pour cela il définit la série suivante :

$$g(x) = \sum \frac{1}{n^s}(\sin nx\pi)(\sin \frac{1}{\sin nx\pi}).$$

Si $s > 2$ la série converge uniformément et sa limite $g(x)$ possède –selon Hankel– une singularité en chaque point rationnel. Soit $x = \frac{p}{q}$ (p et q étant des entiers sans diviseur commun). Alors on retrouve q comme valeur de la variable de sommation n, de sorte qu'on a $\sin \pi p = 0$. Par conséquent $g(\frac{p}{q}) = f(0)$ et g possède la même singularité en $\frac{p}{q}$ comme f en 0.

L'idée de Hankel est plausible mais il y a des problèmes techniques : en sommant la série, il se peut que quelques termes se compensent et que la singularité disparaîsse. Ce point faible était critiqué par GILBERT et d'autres; le principe lui-même sera amélioré plus tard par Ulisse DINI (1845–1918) et Georg CANTOR (1845–1918). De plus il y a une limitation évidente pour la condensation des singularités : seul les nombres rationnels sont possibles comme singularités. Autrement dit : on ne peut pas construire à l'aide de ce principe une fonction continue non-différentiable pour tous x réel! Le mémoire de Hankel était un "programme" écrit à l'occasion de l'anniversaire du roi de Württemberg, il fut imprimé en quelques exemplaires. Par conséquent il est surprenant que ce mémoire fut connu à Paris (DARBOUX en faisait un compte-rendu pour son Bulletin) et à Louvain (GILBERT en faisait une critique détaillée). Cela montre que ce mémoire était d'un grand intérêt parce qu'il traitait une question actuelle de l'époque.

Le premier exemple du type général fut publié par Karl WEIERSTRAß (1815–1897), le maître des analysistes allemands. Le 18 Juillet 1872, WEIERSTRAß parlait à l'Académie de Berlin *des fonctions continues d'une seule variable réelle, qui n'ont pas un quotient différentiel bien déterminé pour une valeur quelconque de la variable*. Peut-être la découverte de WEIERSTRAß était motivée indirectement par GILBERT! J. MAWHIN cite une lettre par H.A. SCHWARZ à WEIERSTRAß du 20 juin 1872 disant que

> dans l'un des derniers numéros des Nouvelles Annales, un certain Gilbert, si je me ne trompe, affirme de nouveau l'absurdité qu'il est tout à fait évident qu'une fonction continue possède une dérivée; mais, sans doute, ne peut-on en faire le reproche à un mathématicien français de province, puisque Bertrand débute son Traité par une prétendue démonstration de cette affirmation. (MAWHIN 1992, 373)

Le contenu de l'intervention de WEIERSTRAß n'est connu que par un mémoire de Paul DU BOIS-REYMOND (1831–1889) paru en 1875. Selon DU BOIS–REYMOND qui cite une lettre de WEIERSTRAß l'exemple du dernier était le suivant :

$$f(x) = \sum_n b^n \cos(a^n x\pi)$$

où a est un entier positif impair, b est un nombre réel entre 0 et 1 et $ab > 1 + \frac{3}{2}\pi$. La fonction $f(x)$ est continue parce que la série est majorée par $\sum_n b^n$ (critère de WEIERSTRAß) mais elle n'est différentiable en aucun point. Le point important est qu'il y a des suites $h \to 0+$ et $k \to 0-$ tel que les restes $r_n(x)$ des quotients de différences croissent sans limite en valeur absolue mais avec un signe opposé. Autrement dit : La fonction $f(x)$ possède en chaque point un "pic" :

La démonstration donnée par WEIERSTRAß est un modèle pour le nouveau style en analyse –en particulier pour la distinction nette entre les qualités locales et les qualités globales ou uniformes. Autrement dit on fait attention à l'interdépendance des ϵ, des δ et des x. D'autre part l'intuition ne compte plus dans l'analyse "arithmétisée". Si on atteint par la voie analytique un résultat qui est en contradiction avec l'intuition on conclut : c'est l'intuition qui trompe! Tant pis pour l'intuition. Le domaine des calculs s'est complètement séparé de celui des intuitions.

Même en Allemagne la découverte par WEIERSTRAß n'était pas acceptée sans discussion; comme le prouve par exemple un mémoire d'un mathématicien de Karlsruhe qui s'appellait Christian WIENER. WIENER se proposait de faire voir que l'exemple de WEIERSTRAß n'était pas dépourvu de dérivée dans tous les points réels et qu'il n'était pas inaccessible à l'intuition (WIENER était un géomètre qui travaillait surtout sur la géométrie descriptive). WEIERSTRAß a répondu en montrant que WIENER n'avait pas bien compris la nouvelle idée de "différentiablité" (l'exemple de WEIERSTRAß admet dans certains points des dérivées unilatères infinies). On comprend bien que les nouvelles idées étaient partagées par un petit cercle de chercheurs et non pas par toute la communauté mathématique. Même Leo KÖNIGSBERGER

(1837–1921), un disciple très fidèle de WEIERSTRAß, avait des difficultés avec les idées de son maître : un fait qui est démontré par l'inclusion du théorème d'AMPÈRE dans son livre sur les fonctions elliptiques de 1875. On connait une lettre de WEIERSTRAß à DU BOIS–REYMOND dans laquelle il se plaint de l'ignorance de son disciple.

Le fait que WEIERSTRAß avait construit un contre-exemple était par contre connu à Paris (mais pas à Louvain). C'est démontré par une lettre écrite par G. DARBOUX à son co-éditeur J. HOUEL concernant GILBERT :

> Quant à Gilbert, le grand Belge, nous avons grand besoin d'agir avec prudence et il faut bien choisir notre moment pour lui assener un coup terrible et dont le grand Belge ne puisse se relever. Il attaque Hankel. C'est bien... Quand Hankel aura répondu d'une manière victorieuse, je n'en doute pas, nous arriverons à la rescousse et gare à Gilbert. Nous aurons la partie d'autant plus belle qu'à Berlin, il y a aussi des géomètres pointus et que WEIERSTRAß a lu un article sur les fonctions qui n'ont pas de dérivée. Je répondrai la démonstration de Gilbert que j'affirme fausse, sans l'avoir vue... je vous promets que nous l'assomerons. (GISPERT 1983, 86)

Il semble que l'intérêt pour les recherches de l'école allemande n'était pas du tout développé en France -DARBOUX en étant presque la seule exception (d'autres exceptions étaient J. Tannéry [1848–1910] et Jean HOUEL). En Belgique la situation était peut-être un peu différente grâce à LAMARLE et GILBERT. Je vais revenir sur ce point.

DARBOUX lui-même a publié en 1875 un *Mémoire sur les fonctions discontinues* qui était lui aussi écrit dans le nouveau style. DARBOUX critique entre autre la pratique des manuels français qui définissent la continuité par les valeurs intermédiaires. De plus il donne un nouvel exemple d'une fonction continue sans dérivée qui est plus simple que celui de WEIERSTRAß. Il s'agit de la fonction

$$f(x) = \sum_n \frac{\sin((n+1)!x)}{n!}$$

Cet exemple est mentionné déjà avant dans deux conférences faites par DARBOUX devant la Société mathématique de France (18.3.73 et 28.1.74).

En somme on peut constater que le problème de la différentiabilité des fonctions continues faisait l'objet d'une certain attention dès le début des années 1870. D'un part nous rencontrons les innovateurs comme WEIERSTRAß, DARBOUX et d'autres qui veulent aller au bout des nouvelles idées, d'autre part il y avait les conservatifs adhérant à des idées venant de l'époque d'AMPÈRE et de CAUCHY. Considérons ces derniers.

3 Le théorème d'Ampère en Belgique

Le premier à s'être occupé du théorème d'AMPÈRE en Belgique était Anatole–Henri-Ernest LAMARLE. En 1855, il publiait un mémoire intitulé *Étude appronfondie sur les deux équations fondamentales*.

$$\lim \frac{f(x+h) - f(x)}{h} = f'(x) \text{ et } dy = f'(x) \cdot \Delta x.$$

LAMARLE était un partisan des idées de NEWTON. Pour lui l'analyse a une base cinématique c'est-à-dire que son idée fondamentale est la vitesse instantanée. Une fonction est représentée par une courbe qui est la trajectoire d'un point qui se déplace. À chaque moment du déplacement le point possède une vitesse bien déterminée. Du point de vue analytique cela veut dire que la dérivée doit exister en chaque point de la courbe -sauf à des points exceptionnels où on rencontre un changement brusque du mouvement. En passant je veux noter que les courbes sans tangentes ont été utilisées pour la première fois en physique aux environs de 1900 par L. BOLTZMANN pour décrire le mouvement brownien.

LAMARLE a traduit ses idées cinématiques dans le langage des limites (dans la tradition de D'ALEMBERT [un autre partisan de NEWTON], LACROIX et CAUCHY). Il faut dire que les analyses données par LAMARLE sont assez précises -plus précises que les autres analyses de son époque. Il écrit :

> L'objet du chapitre 1er est d'établir l'équation fondamentale
>
> $$\lim \frac{f(x+h)-f(x)}{h} = f'(x).$$
>
> La fonction $y = f(x)$ étant supposée continue, il est visible que si l'on fait décroître indéfiniment l'accroisement h, le rapport
>
> $$\frac{f(x+h)-f(x)}{h} = \frac{\Delta y}{\Delta x}$$
>
> se trouve assujetti à subir l'une ou l'autre des cinq conditions suivantes :
>
> 1. Demeurer constant;
>
> 2. Converger vers une limite constante ou nulle;
>
> 3. Croître sans limites;
>
> 4. Osciller sans fin entre plusieurs limites distinctes;
>
> 5. Converger vers une limite qui dépend de la valeur attribuée à la variable x et change avec avec cette valeur.
>
> On démontre aisément qu'abstraction faite du cas particulier où la fonction y est linéaire et où la condition (1) se réalise d'une manière permanente chacune des trois premières conditions n'est jamais possible que pour certaines valeurs de la variable conservant entre elles des écarts déterminés. (LAMARLE 1855, 4)

L'analyse donnée par LAMARLE est précieuse en particulier parce qu'il traite des dérivées unilatères (à ma connaissance LAMARLE était le premier mathématicien qui ait publié cette idée [conçue à l'époque aussi par BOLZANO dans son *Functionenlehre*]) et même les grandeurs assez sophistiquées

$$L = \limsup\{\frac{f(x+h)-f(x)}{h}\} \text{ et } l = \liminf\{\frac{f(x+h)-f(x)}{h}\}$$

une idée qui sera reprise plus tard par U. DINI dans son fameux manuel de 1878. Même aujourd'hui les dérivées de DINI –il y en a quatre– sont un instrument important pour analyser les fonctions du point de vue de leur différentiabilité. Les grandeurs L et l sont des fonctions de x; les donnent des limites pour les oscillations du quotient des différences si h tend vers zéro. LAMARLE croyait que L et l *varient continûment avec* x (LAMARLE 1855, 37) d'où s'ensuit :

> Concluons en conséquent que, il n'est aucun intervalle dans toute l'étendue duquel le rapport $\frac{\Delta y}{\Delta x}$ puisse osciller sans fin entre deux limites distinctes, à mesure que h converge vers zéro. (LAMARLE 1855, 83)

De plus LAMARLE conclut que la dérivée $f'(x)$ de la fonction f est elle-même continue. En effet la dernière conclusion sera vraie si $L(x)$ est continue ce qui est démontré dans la théorie de DINI. La faute de LAMARLE était donc de croire que $L(x)$ [ou $l(x)$] soit continu ce qui n'est pas vrai en général.

D'une manière analogue LAMARLE excluait aussi les autres possibilités de divergence pour les quotients de différences pour conclure :

> Toute valeur de x, pour laquelle les limites des rapports $\frac{f(x+h)-f(x)}{h}$ et $\frac{f(x)-f(x-h)}{h}$ cessent d'être égales, constitue une des valeurs isolées.... (LAMARLE 1855, 49)

Il semble que LAMARLE fut la victime de sa croyance à l'évidence intuitive. Si il n'était pas tellement convaincu de la différentiablité des fonctions continues LAMARLE disposait d'un bagage technique nécessaire pour répondre à la question de manière négative. Peut-être aurait-il découvert un contre-exemple comme celui de WEIERSTRAß.

Conclusion : Même en les mathématiques on ne voit que ce que l'on sait!

Anatole–Henri–Ernest Lamarle

Né à Calais le 16.9.1806

1825 – 27 École polytechnique

1827 – 28 École des ponts et chaussées

1838 – 68 Professeur à l'université de Gand

1847 Associé de l'Académie Royale de Belgique

Mort à Douai le 14.3.1875

(Pour l'influence de LAMARLE sur le dévéloppement de l'analyse en Belgique on peut consulter l'article de BOCKSTAELE 1966; pour sa biographie de DE TILLY 1879).

Les idées de LAMARLE furent reprises par Ph. GILBERT au début des années 1870. GILBERT avait pris connaissance du travail de HANKEL sur la condensation des singularités et il était sûr que ce travail contenait des fautes. Comme je l'ai déjà indiqué c'est vrai. La faute de GILBERT était qu'il concluait que les résultats de HANKEL étaient aussi faux –en particulier ses exemples de fonctions bizarres comme la fonction continue qui n'est pas différentiable dans un ensemble dense (chez HANKEL : ce sont les nombres rationnels). Par conséquent GILBERT ne critiquait pas seulement les méthodes de HANKEL mais il voulait démontrer l'impossibilité de l'exemple de HANKEL. Pour y arriver il voulait améliorer le travail de LAMARLE.

Le commentaire fait par DARBOUX est intéressant

> Gilbert est inoui. Il raisonne comme un Peau Rouge. Il dit, le raisonnement de H. est faux (accordé) et il conclut, la proposition de H. est donc fausse. (GISPERT 1983, 86)

GILBERT lui-même donnait une motivation intéressante à son mémoire :

> C'est qu'en effet, l'existence de la dérivée dans une fonction continue $f(x)$ se traduit, géométriquement, par l'existence de la tangente en un point quelconque de la courbe continue qui est la figuration géométrique de cette fonction, et s'il nous est possible de concevoir qu'en certains points singuliers, même très rapprochés, la direction de la tangente soit parallèle à l'axe des x ou à l'axe des y, ou soit même tout à fait indéterminée, nous ne pouvons comprendre qu'il en soit ainsi dans toute l'étendue d'un arc de courbe, si petit qu'on le suppose d'ailleurs. De là, la tendance à regarder l'existence de la dérivée, dans une fonction continue, comme inutile à démontrer. (GILBERT 1873, II)

Autrement dit GILBERT était convaincu que l'analyse se trouve dans une harmonie préétablie avec l'intuition : Ce qui n'est pas possible d'un point de vue intuitif n'est pas possible au niveau analytique.

GILBERT procède comme LAMARLE : en excluant par un raisonnement par l'absurde "toutes" les possibilités de divergence, il croyait démontrer la convergence des quotients de différences par presque tous les points d'un intervalle. Il y a quelques différences mineures en comparaison de LAMARLE mais cela ne vaut pas la peine de les étudier ici.

Louis - Philippe Gilbert

Né à Beauraing le 7.2.1832
Étude des mathématiques à Louvain sous Pagani
Diplômé à Louvain en juillet 1855
Professeur associé à Louvain en octobre 1855
Membre associé de l'Académie Royale de Belgique en 1867
1872 Première édition du "Cours d'Analyse"
6.12.1873 Gilbert se retire de l'Académie
Mort à Louvain le 4.2.1892

GILBERT travaillait sur les fondements de l'analyse, sur la géométrie différentielle et sur l'optique théorique. Il s'occupait aussi de la cause catholique, du procès de Galilée, de la rotation de la terre (étude mathématique du barogyroscope [un instrument pour mesurer la rotation de la terre inventé par GILBERT]), de la géographie de l'Afrique (*L'Afrique inconnue. Récits et aventures des voyageurs modernes au Soudan oriental* [8 éditions entre 1863 et 1884]). Il semble que GILBERT ait eu une grande influence sur la communauté mathématique belge. DARBOUX écrivait à son confident HOUEL le 30. Mai 1872 : CATALAN *et* GILBERT *sont pour le moment les* CHASLES – BERTRAND – SERRET – LIOUVILLE – BONNET – DELAUNAY – MANNHEIM *de la Belgique.* (ZERNER 1991, 308)

Pour savoir plus sur GILBERT on peut consulter les articles de Jean MAWHIN 1989 et 1992.

Mais depuis les travaux de LAMARLE les temps ont changés. En 1855 une "démonstration" du théorème d'AMPÈRE ne faisait pas scandale, en 1872 plusieurs chercheurs se sentaient scandalisés par le même sujet. WEIERSTRAß qui n'aimait pas la publicité laissait la parole à un des ses nombreux disciples : Hermann Amandus SCHWARZ (1843–1921) alors à Zürich et plus tard (en 1892) successeur de son maître à Berlin. Celui publiait en 1873 un *exemple d'une fonction continue non-différentiable* et il écrivait aussi une lettre à GILBERT dans laquelle il informait le dernier de sa découverte. La fonction construite par Schwarz n'est pas un exemple de fonction dépourvue de dérivée pour tous les nombres réels; elle est non-différentiable seulement dans un sous-ensemble dense des réels. Par conséquent son exemple est à comparer à celui donné par RIEMANN. La construction en est la suivante :

Soit $E(x)$ le plus grand nombre entier qui est inférieur ou égal au nombre réel x. Alors on définit la fonction

$$f(x) = E(x) + \sqrt{x - E(x)}$$

pour tous les réels positifs x.

La courbe définie par cette fonction est composée des arcs du graphe de la fonction racine carré.

Évidemment cette fonction possède des points anguleux pour tous les entiers positifs : la dérivée de gauche en est $\frac{1}{2}$, celle de droite est infini. L'idée principale est maintenant la même que celle de HANKEL : multiplier ces singularités à l'aide d'une série convergente. Soit

$$F(x) = \sum_{n=0}^{\infty} \frac{f(2^n x)}{2^{2n}}.$$

On démontre que cette série converge uniformément. D'où s'ensuit que $F(x)$ est une fonction continue. Mais la dérivée de droite est infinie pour tous les arguments de la forme

$$x = \frac{p}{2^q}$$

avec des entiers positifs p et q. Parce que ces x forment un ensemble dense dans les réels la fonction $F(x)$ est un contre-exemple contre l'hypothèse qu'une fonction continue ne peut posséder une dérivée infinie dans un sous-ensemble dense de son domaine de définition.

Bien entendu l'exemple de WEIERSTRAß montre de plus que la distinction des cas de non-convergence donnée par LAMARLE et reprise par GILBERT n'est pas du tout complète. Autrement dit le comportement des quotients de différences finies peut être beaucoup plus compliqué que ne l'avaient pensé LAMARLE et GILBERT.

GILBERT a publié une *Rectification au sujet d'un mémoire précédent* en 1875; il a aussi incorporé l'exemple de SCHWARZ dans les nouvelles éditions de son *Cours d'analyse* (à partir de la quatrième édition de 1878). La même année est paru le fameux manuel de DINI, le premier qui traitait la nouvelle ananlyse d'une manière systématique et rigoureuse. Donc la question était tranchée contre LAMARLE, GILBERT et beaucoup d'autres. Pour conclure je veux résumer quelques points importants.

4 Conclusions

Le développement de la notion de "différentiabilité" se peut décrire comme suit : On trouve le mot "differentiabilis" pour la première fois dans les écrits de Leibniz. Pour celui-ci toutes les grandeurs variables sont différentiables par opposition aux grandeurs constantes qu'on ne peut pas différentier. Par conséquent l'idée de différence est strictement liée à l'idée de changement pour Leibniz et ses successeurs, un changement qui obéit une loi et qui ne connait pas de sauts. Après Leibniz le mot "différentiable" a disparu pendant longtemps (à ma connaissance); on trouve quelques idées qui vont dans cette direction dans les discussions sur la corde vibrante p.e. chez d'ALEMBERT. C'est EULER qui a, pour une grande partie, donné à l'analyse sa forme moderne, en particulier EULER a mis la fonction au centre de cette discipline : L'objet de l'analyse sont les fonctions réelles et leurs qualités. D'ou s'ensuit qu'il faut interpréter la différentiation en terme de fonction. LAGRANGE l'a fait en parlant de la fonction dérivée. Autrement dit : la dérivée d'une fonction doit être une fonction aussi. Pour nous c'est tout à fait évident mais au début du $19^{ème}$ siècle cette décision posait quelques difficultés. Ces problèmes sont liés aux restrictions nécessitées par la notion de fonction. Selon DIRICHLET (1837) une fonction réelle doit être univoque (à chaque x il y a seulement un $y(x)$) et finie (infini est exclu comme valeur). Par conséquent la fonction dérivée doit posséder ces qualités : cela mène à l'exclusion de quelques cas singuliers comme des points anguleux et des tangentes parallèles à l'axe des y (souvent on a exclu aussi les tangentes parallèles à l'axe des x!).

La première discussion nette de ces cas singuliers se trouve chez COURNOT dans son *Traité élémentaire de la théorie des fonctions et du calcul infinitésimal* (tome premier de 1841). Il est intéressant de remarquer que COURNOT ne parle pas de non-existence de la fonction dérivée mais des *solutions de continuité* dans les *jarrets* et dans les *points saillants*. (COURNOT 1841, 61)

COURNOT était un partisan des infiniment petits; dans le langage des infiniment petits le terme "non-existence" ne faisant pas grand sens. Même en 1875 DARBOUX écrit :

> Il y a donc deux dérivées ou plûtot la dérivée n'existe pas toutes les fois que $f(x_0 + h)$ est différent de $f(x_0 - h)$. (DARBOUX 1875, 93)

Notre notion moderne de "différentiabilité" est née dans la deuxième moitié du 19ème siècle par une combinaison de la théorie des limites d'une part et une orientation stricte vers les qualités locales d'autre part. Le langage des limites menait à une distinction fine des cas de divergence ou de non-existence en particulier en relation avec les limites unilatères. Maintenant on n'a plus seulement les cas singuliers des jarrets et des points salliants mais aussi la divergence par oscillation (fameux exemples : $f(x) = \sin\frac{1}{x}$ et $f(x) = x\sin\frac{1}{x}$ [qu'on trouve chez CAUCHY] et même des situations beaucoup plus compliquées comme celle du monstre de WEIERSTRAß). Il a fallu du temps pour explorer tous ces possibilités.

Il est bien connu que la pensée de CAUCHY avait tendance à être globale : pour celui-ci la continuité était une continuité sur un intervalle (par conséquent il s'agit d'une continuité uniforme d'un point de vue moderne), la convergence d'une série de fonctions est aussi uniforme. De plus il est fort remarquable que CAUCHY soit toujours assez rigoureux en regard de la continuité (c'est-à-dire qu'il parle explicitement de cette hypothèse s'il a besoin d'elle) et de ses conséquences (par exemple le théorème des valeurs intermédiaires), mais qu'il ne parle que très peu de la différentiabililité (bien entendu le terme ne se trouve jamais chez CAUCHY). On a l'impression que pour CAUCHY la différentiabilité était une conséquence de la continuité; les cas de non-différentiabilité sont des cas de solutions de continuité (cf. par exemple CAUCHY II,4 332sq).

L'attention qu'on a donné de plus en plus à la différentiabilité est l'expression d'une part des nouvelles possibilités techniques qu'on avait acquises pendant le 19ème siècle et d'autre part d'une nouvelle orientation concernant les fondements des mathématiques : on ne s'intéresse plus uniquement aux exemples typiques de fonctions etc. mais on se s'interroge sur les cas exceptionnels. POINCARÉ a commenté le nouveau style comme suit :

> Autrefois quand on inventait une fonction nouvelle, c'était en vue de quelque but pratique; aujourd'hui on les invente tout exprès pour mettre en défaut les raisonnements de nos pères, et on n'en tirera que cela. (POINCARÉ 1889, 131)

C'était aussi l'heure où notre entendement moderne de quantificateurs est né avec l'idée qu'un seul contre-exemple suffit pour falsifier un énoncé de type général. Mais Abel en 1826

parlait des "exceptions" et non pas des "contre-exemples" quand il critiquait le théorème de CAUCHY sur les séries des fonctions continues. Cet aspect est lié aussi avec la naissance de la théorie des ensembles en particulier avec le point de vue extentional : une notion est caractérisée par la classe des exemples qui sont subordinés à cette notion. Tout ça est clairement montré par un travail de P. DU BOIS–REYMOND de 1875 *Essai d'une classification des fonctions arbitraires des arguments réels d'après leurs changements dans les intervalles les plus petits.* Ici l'auteur décrit des ensembles de fonctions définis par des qualités des fonctions. L'ensemble le plus général c'est "la fonction sans hypothèse spécifique" (c'est-à-dire sans qualité explicite) suivi par "la fonction intégrable" et "la fonction continue". Après la fonction continue vient "la fonction différentiable" qui est définie de la manière suivante :

> Une fonction est nommée différentiable dans un intervalle, où elle possède un quotient différentiel fini et bien déterminé pour chaque valeur de l'argument. (DU BOIS–REYMOND 1875, 72)

Déjà dans le contexte de la continuité DU BOIS–REYMOND avait remarqué :

> L'hypothèse de la continuité n'a rien à faire avec l'existence du quotient différentiel non pas seulement pour un seul point. C'est un des résultats les plus émouvants des mathématiques récentes qu'une fonction se peut être continues dans tous les points d'un intervalle sans avoir un quotient différentiel dans un point quelconque de cet intervalle. (DU BOIS–REYMOND 1875, 27)

Pour démontrer son énoncé DU BOIS-REYMOND utilise le montre de WEIERSTRAß. Mais on apperçoit que même DU BOIS-REYMOND a une certaine tendance à considérer des notions globales.

Une définition strictement locale de la différentiabilité se trouve trois ans plus tard chez DINI. Même après cette définition beaucoup de manuels l'ignorent; c'est seulement vers la fin du siècle qu'on a pris la coutume de formuler *expris verbis* des hypothèses sur la différentiabilité.

Comme conclusion je veux souligner trois points :

1. Il n'y a pas de développement d'une notion mathématique (même d'une importance centrale); ce qui se développe est un réseau de notions (dans lequel il y a des notions plus ou moins importantes).

2. Les erreurs mathématiques peuvent être enracinées dans un entendement différent sur les fondements des mathématiques. Souvent il vaut la peine de les analyser plus de proche.

3. Durant le 19ème siécle le rôle de l'intuition dans les mathématiques a changé de manière radicale. Avant ce siècle, l'idée de l'existence en mathématique était strictement liée à l'intuition ("sans intuition pas d'existence") après ce siècle l'intuition n'a plus d'importance officielle. Elle peut s'en servir comme source d'idées –un outil heuristique- mais pas comme base d'existence. Cette dernière est définie maintenant par l'absence des contradictions.

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La construction et la validation de la connaissance chez Stevin

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Abstract

Le fécond changement subit au 16ème siècle par le concept de nombre et dû à l'unification du traitement des quantités discrètes et continues, n'a été possible que grâce à une profonde transformation des méthodes et des critères de construction, et de validation, des objets mathématiques. Les travaux théoriques de SIMON STEVIN (1548-1620) ont contribué, de façon primordiale, à la réalisation d'une telle transformation, produisant une rupture tant épistémologique que méthodologique avec la Mathématique ancienne. Dans cet article, nous analysons les arguments que STEVIN utilise dans *L'Arithmétique* pour démontrer les propositions dérivées de sa conception opératoire de nombre, et nous montrons le caractère innovateur de ces arguments.

DE

THIENDE

Leerende door onghehoorde lichticheyt allen rekeningen onder den Menschen noodich vallende, afveerdighen door heele ghetalen sonder ghebrokenen.

Beschreven door SIMON STEVIN *van Brugghe.*

TOT LEYDEN,
By Christoffel Plantijn.
M. D. LXXXV.

Introduction

Au cours du XVII^{ème} siècle, l'histoire des Mathématiques a subit de dramatiques transformations qui ont inauguré l'ère moderne. Tout d'abord, la création mathématique cartésienne a permis le traitement unifié de l'algèbre et de la géométrie, nécessaire au futur développement du Calcul. Ensuite, l'apparition de l'Algèbre généralisée, comprenant les processus infinis, a marqué le début de la modélisation mathématique du monde physique. Plus important encore a été l'introduction de la méthode analytique, point de départ de la science moderne.

Au XVI^{ème} siècle se sont produits les changements conceptuels relatifs aux objets mathématiques et à leurs opérations qui ont rendu possibles les résultats innovateurs obtenus au XVII^{ème} siècle. Cependant, les transformations survenues à la fin de la Renaissance ne sont pas encore entièrement étudiées et comprises par les historiens des Mathématiques et des sciences et donc pas valorisées à leur juste mesure. Un de ces changements importants concerne la notion de *nombre*, sa signification et ses opérations; il a été initié par le grand ingénieur et mathématicien flamand SIMON STEVIN (1548-1620).

Les travaux théoriques de STEVIN ont contribué, de façon capitale, à l'accomplissement d'une rupture aussi bien épistémologique que méthodologique avec la Mathématique ancienne.

STEVIN est connu pour avoir introduit le système décimal de numération dans la culture Occidentale. Le système de STEVIN inclut notamment le traitement des fractions de l'unité qui est encore employé aujourd'hui. Afin de procurer des fondements théoriques à son arithmétique décimale, STEVIN a dû mettre à jour le concept grec de nombre. Il proposa spécialement une nouvelle formulation permettant de relier théoriquement *nombre* et *grandeur* par l'intermédiaire de processus de mesure. Cela a donné une identité aussi bien aux fractions décimales qu'à un important ensemble d'expressions numériques que les anciens avaient explicitement exclues du domaine numérique.

Les méthodes et les critères de construction et de validation des objets mathématiques chez les Grecs.

Il convient de rappeler que l'opposition *continu-discret*, toujours présente dans la cosmologie grecque, atteignait aussi les concepts de base des Mathématiques. ARISTOTE définit la *quantité* comme étant une catégorie de la pensée telle que, si elle est discrète -et par conséquent dénombrable-, il s'agit d'un *nombre*; par contre, si la quantité est continue -et, donc mesurable-, on est face à une *grandeur*. L'opération qui permet l'identification et la classification des quantités est la *division* : une quantité qui ne peut être divisée qu'un nombre fini de fois est un nombre, et à la limite d'une telle division se trouve l'*unité*. Mais, si la quantité peut être divisée indéfiniment -sans perdre son essence- il s'agit d'une *grandeur*, auquel cas la division n'est pas bornée. Ceci implique qu'il n'existe pas d'unité naturelle de mesure.

Poursuivant la tradition aristotélienne, EUCLIDE établit les définitions suivantes dans *Les Éléments* :

> *Définition VII-1* : L'**unité** est ce suivant quoi chacune des choses existantes est dite **une**
>
> *Définition VII-2* : Le **nombre** est une multitude composée d'unités.
>
> *Définition VII-3* : Un nombre est **une partie d'un autre nombre**, le plus petit du plus grand, quant le premier mesure le deuxième.

De cette façon, EUCLIDE suggère l'origine (concrète, appartenant au monde réel) de l'unité, aussi bien que le principe de génération du nombre par l'itération d'unités, et la décomposition

des nombres par la partition de nombres.

De ces définitions et du traitement opératoire qu'EUCLIDE attribue aux nombres dans *Les Éléments*, il découle que, pour l'arithmétique grecque :

> L'unité (numérique) n'est pas un nombre
>
> Les fractions de l'unité ne sont pas des nombres.

Le domaine numérique est donc constitué, chez les Grecs, des nombres entiers, positifs bien sûr, l'unité n'y appartenant pas.

Les grandeurs, à leur tour, constituent un territoire distinct de celui des nombres, un territoire irréductible. Par conséquent, l'étude qui leur est consacrée, la Géométrie, n'a aucun rapport avec l'Arithmétique. En fait, dans *Les Éléments* d'EUCLIDE les livres traitant des nombres (7, 8, 9 et 10) sont complètement indépendants des livres géométriques : jamais, dans un même livre, les mots *nombre* et *grandeur* n'apparaissent ensemble, à l'exception de la proposition 5 du Livre 10, où EUCLIDE énonce la propriété qu'ont les grandeurs commensurables de se comporter "comme les nombres".

Il n'y a aucune définition de grandeur chez EUCLIDE. Néanmoins, il apparaît clairement qu'il utilise tacitement la définition d'ARISTOTE[1] lorsqu'il affirme :

> *Définition V-1* : Une grandeur est **une partie d'une autre grandeur**, la plus petite de la plus grande, quant la première mesure la deuxième. [Les Éléments]

Remarquons que cette définition reproduit la définition VII-3, que nous avons discuté plus haut, le mot *nombre* étant remplacé par *grandeur*.

On reconnaît donc chez EUCLIDE que *nombre* et *grandeur* sont des objets différents dès leurs origines, que leurs traitements théoriques sont dissemblables et que leurs domaines sont disjoints et restreints par rapport aux domaines actuels.

Quant aux méthodes de raisonnement des Grecs, rappelons que la Mathématique euclidienne est l'archétype de la structure hypothético-déductive de la science; cette organisation suppose que tous les éléments qui la composent sont liés par des relations d'*antécédent-conséquent* constituant une chaîne déductive parfaite. Pour échapper à la régression infinie, il faut supposer l'existence de certaines vérités -évidentes d'elles mêmes- appelées *axiomes* et *postulats*. Une fois acceptée l'évidence des axiomes et des postulats, ainsi que toutes les règles de l'enjeu logique, la vérité des propositions est assurée à l'intérieur de la structure. Cela veut dire qu'il n'est jamais nécessaire de sortir de la structure théorique pour démontrer les théorèmes.

Grâce à cette organisation, les Grecs ont développé le système théorique inébranlable que nous connaissons aujourd'hui. Pour aboutir à cette construction, la Mathématique grecque laisse de côté le réalisme et l'empirisme de l'épistémologie aristotélienne en limitant leur participation au seul démarrage de la chaîne logique, c'est-à-dire, à l'établissement de l'évidence des axiomes et des postulats. Une fois qu'EUCLIDE définit l'unité numérique en invoquant le référent du monde physique (comme dans "*ce suivant quoi chacune des choses existantes est dite une*"), tous les concepts et les résultats numériques sont validés à l'intérieur de la théorie. C'est la force de la vérité nécessaire, tirée de la déduction logique, qui procure de la stabilité à la structure.

Voici, sommairement, les impasses de la Mathématique ancienne que STEVIN doit surmonter afin de fournir un fondement théorique au système décimal :

[1] "Ce qui est divisible par deux ou par plus de parties aliquotes" [Métaphysique 1020a, 5].

- Un domaine numérique restreint aux nombres entiers positifs.
- Une unité numérique qui n'est pas un nombre et dont les fractions ne sont pas non plus des nombres.
- Un divorce, aussi bien conceptuel qu'opératoire, entre nombres et grandeurs.
- Une façon de raisonner -dans la tradition axiomatico-déductive- où il est impossible d'ignorer.

Les méthodes et les critères de construction et de validation des objets mathématiques chez STEVIN

Contrairement à la tradition grecque, STEVIN accepte la possibilité de diviser l'unité numérique et d'attribuer aux fractions de l'unité la propriété d'être des *nombres*. Ce fait constitue le point capital sur lequel STEVIN doit diriger toute la force de son raisonnement. Cela est compréhensible dès lors que le but de son travail est de convaincre les savants de l'époque de la justesse et la convenance de la représentation décimale des nombres -rompus-, sans violenter les principes théoriques des anciens.

L'ouvrage mathématique de SIMON STEVIN, qui contient les plus importants apports théoriques à cette science, s'intitule *L'Arithmétique*, un volume publié en français en 1585. Dans ce texte STEVIN présente une extension du concept de nombre qui n'est possible qu'au dépens d'une rupture explicite avec la conception euclidienne. A l'aide de ce nouveau concept de nombre, STEVIN cherche à donner des fondements théoriques aux procédures du calcul arithmétique, nécessaires à la représentation décimale qu'il vient de développer[2].

Le référent concret des objets

STEVIN commence sa dissertation par les définitions suivantes :

> Définition I : L'arithmétique est la science des nombres. [STEVIN, 1585, p. 1]

(à ce stade, aucune rupture ne s'observe encore).

> Définition II : Nombre est cela par lequel s'explique la quantité de chacune chose. [STEVIN, 1585, p. 1]

Ce qui aboutit à :

> Définition X : Nombre rompu est partie ou parties de nombre entier. [STEVIN, 1585, p. 3]

Chez STEVIN, contrairement à ce que la Mathématique grecque préconise, le nombre est associé à la quantité sans que l'opposition entre discret et continu fasse partie du concept. Dès sa deuxième définition, STEVIN supprime la dichotomie continu-discret en tant que propriété définissant la quantité. En effet, il nie explicitement que les nombres soient, dans leur essence, des quantités discrètes :

[2]La même année (1585), STEVIN publie un bref fascicule intitulé *La Disme* traitant la notation décimale et son arithmétique, qui inclut le traitement des fractions décimales.

QUE NOMBRE N'EST POINT QUANTITE DISCONTINUE[3]

Le nombre en tant qu'entité isolée est, d'après STEVIN, "continu" dans le sens aristotélique, c'est-à-dire, qu'il est possible, dans la plupart des cas, de le diviser indéfiniment sans qu'il perde son essence; à la limite, il hérite la propriété de continuité ou de discontinuité de la "chose" (sa quantité) qu'il quantifie. Par exemple, si l'on parle de 2 hommes, le 2 est discret parce que son référent concret -l'ensemble d'hommes- est discret; tandis que si l'on parle de 2 kilomètres, le 2 est continu puisqu'il fait référence à une distance (ou à une longueur) continue. Dorénavant, continu et discret cessent d'être des catégories ontologiques. La discussion sur ce point s'éloigne du domaine des Mathématiques, puisque le fait d'être continu ou discret devient une propriété circonstancielle imputable uniquement aux objets quantifiés. Ceci marque une première différence entre STEVIN et la science grecque : *une façon de valider les objets mathématiques qui a besoin de faire constamment appel aux référents concrets afin de discerner s'il s'agit d'objets continus ou discontinus.*

Cette ligne d'argumentation, souvent utilisée par STEVIN, a le caractère extra-logique qui lui impose le référent concret dont STEVIN a besoin pour conférer une réalité aux nombres. Examinons cet argument lors de sa première apparition, lorsque STEVIN affirme que l'unité est un nombre :

> La partie est de mesme matiere qu'est son entier,
> l'unité est partie de multitude d'unitez.
> Ergo l'unité est de mesme matiere qu'est la multitude d'unitez
> Mais la matiere de la multitude d'unitez est nombre
> Doncques la matiere d'unité est nombre. [STEVIN, 1585, p. 1]

Qui le nie, continue STEVIN, *fait comme celui qui nie qu'une pièce de pain soit du pain.* Nous reviendrons sur ce point à plusieurs reprises par la suite.

L'existence opératoire des objets

Le concept de nombre est édifié par STEVIN sur la constatation d'un isomorphisme opératoire entre nombres et grandeurs, c'est-à-dire, que le nombre se développe à partir de la considération selon laquelle il est possible d'opérer avec lui exactement de la même façon qu'avec les quantités continues. *Nous trouvons ici la façon de valider les objets mathématiques qui domine l'argumentation stevinienne : les opérations définissent (et ainsi attribuent) l'existence des objets.* Dans le cas qui nous intéresse, l'essence du nombre est portée par ses opérations.

Voici un passage de *L'Arithmétique* où STEVIN, argumentant en faveur de la division de l'unité, utilise cette façon de valider l'existence des nombres; il affirme que nier la divisibilité de l'unité revient à étouffer la *nature du nombre*, dont l'essence se manifeste par les opérations arithmétiques qu'on peut réaliser avec lui :

> L'unité est divisible en parties (vrai est qu'ils [les anciens] les nient, mais mille leurs distinctions ne sont pas suffisantes, de pouvoir ainsi opprimer la nature du nombre, qu'elle ne manifeste par force son essence, es arithmétiques opérations de plusieurs auteurs, comme entre autres par l'absolue partition de l'unité de la 33 question du 4 livre, & la 12, 13, 14, 15 questions du cinquième livre du Prince des Arithméticiens Diophante[4]). [STEVIN, 1585, p. 2]

[3]STEVIN, 1585, p. 2 (en majuscules dans l'original)

[4]Voir HEATH (1964). Ces propositions sont nommées IV 31, V 9, 10, 11 y 12: Elles commencent toutes avec la phrase : "pour diviser l'unité en deux parties (ou nombres)" ou bien, "pour diviser l'unité en trois nombres".

L'affirmation précédente confère au nombre une existence opératoire, c'est-à-dire que ce sont les opérations que nous pouvons réaliser sur les nombres qui déterminent leur nature.

Le fait que les résultats des opérations algébriques réalisées sur les nombres soient à leur tour des nombres constitue un pas très important vers l'extension du domaine numérique. Cette affirmation n'obéit pas aux règles grecques d'homogénéité des grandeurs. Chez les Grecs, il est interdit d'opérer avec des grandeurs dont la nature est inégale; ainsi, l'addition d'aires et de volumes ou de lignes et d'aires n'a pas de sens. De même, une grandeur de la quatrième puissance n'a pas non plus de signification. Dans la géométrie grecque, une grandeur de la première puissance représente un segment, une de la deuxième puissance, une surface, et une grandeur de la troisième puissance représente un volume; en conséquence, il n'y a aucun besoin de puissances supérieures à trois.

STEVIN, inspiré par la nature opératoire des nombres, ne trouve aucune difficulté à accepter :

QUE NOMBRES QUELCONQUES PEUVENT
Etre Nombres carrés, cubiques,&c. Aussi que racine quelconque est nombre[5]

Dont la démonstration directe se base sur l'exposition des grandeurs géométriques qui représentent ce genre de nombres (nous reviendrons sur ce point plus tard). Cette affirmation conduit STEVIN à définir un *nombre algébrique* comme étant une *quantité ou une multitude composée de quantités* [Définition XIX, p. 6], peu importe que les quantités en question portent une puissance supérieure à trois (ou inférieure à un) ou que leurs puissances soient différentes entre elles.

Afin de compléter le cadre qui élimine le problème d'homogénéité dimensionnelle, STEVIN définit le polynôme (multinome) algébrique :

> Multinomie algebraique est un nombre consistant en plusieurs diverses quantitez. [STEVIN, 1585, Définition XXVI, p. 6]

et il présente un exemple :

> Comme 3③ + 5② - 4① + 6 s'appelle multinomie algebraique [*Ibidem*]

La raison de fond qui nourrit la conviction de STEVIN sur le caractère numérique des nombres algébriques est la nature opératoire du nombre : tous les résultats des opérations qui se réalisent avec des nombres sont, à leur tour, des nombres[6].

Le raisonnement par l'absurde appuyé sur le contexte concret

Afin de gagner l'acceptation de son nouveau concept, STEVIN reconnaît qu'il doit montrer la supériorité de sa définition par rapport à la définition traditionnelle. C'est ainsi qu'il commence son argumentation par l'affirmation suivante :

QUE L'UNITE EST NOMBRE[7]

[5]STEVIN, 1585, p. 8, En majuscules dans l'original
[6]Sur les arguments de STEVIN en faveur du caractère opératoire du nombre, voir WALDEGG 1993.
[7]STEVIN 1585 p. 1 (en majuscules dans l'original)

STEVIN conteste le point de vue traditionnel selon lequel l'unité n'est pas un nombre mais le "principe" générateur du nombre, un rôle équivalent au rôle du point (géométrique) par rapport au segment de droite. Il élabore deux arguments contre la perspective ancienne : l'un est philosophique, l'autre pseudo-mathématique. L'argumentation philosophique -que nous avons aperçue dans une citation précédente[8]- fait référence au caractère ontologique de l'unité, concluant que la négation de l'unité en tant que nombre équivaut à nier qu'un morceau de pain soit du pain.

D'autre part, l'argument pseudo-mathématique est le suivant :

> Si du nombre donné l'on ne soubstraie nul nombre, le nombre donné demeure
> Soit trois le nombre donné, & du mesme soubstrayons un
> que n'est point nombre, comme tu veux.
> Doncques le nombre donné demeure, c'est-à-dire qu'il y restera encore trois, ce qui est absurde
> [STEVIN, 1585, p. 1]

Les deux argumentations font partie d'un même discours démonstratif : *un raisonnement par l'absurde appuyé sur le recours au contexte concret* : en niant la proposition en question on aboutit à une incohérence dont la constatation est à la portée de quiconque.

L'appel à l'analogie

STEVIN affirme que l'exclusion de l'unité du genre nombre retrouvée dans la Mathématique de l'antiquité, est due à la volonté des anciens de trouver le "principe" ou la "cause" du nombre. Il reconnaît qu'il s'agit là de la méthode utilisée par les philosophes dans de telles discussions et ajoute que, dans le cas des grandeurs géométriques comme la longueur, l'aire et le volume, le principe évident est le "point". C'est dans ce sens de "principe" que l'on cherche un principe pour le nombre. STEVIN, plutôt que d'argumenter sur la pertinence de rechercher les causes dans un traité de Mathématiques, accepte le défi philosophique et élabore une critique de la solution donnée dans le passé. Voilà un autre raisonnement typique de la pensée de STEVIN dont le fondement principal se trouve dans l'*analogie*.

Les nombres et les grandeurs, poursuit STEVIN, ont tant de choses en commun qu'ils pourraient paraître presque identiques; par conséquent, il y a quelque chose dans le nombre qui doit correspondre à ce que le point est par rapport aux grandeurs. Pour les Grecs, l'unité est le principe du nombre de même que le point est le principe de la grandeur et ceci, affirme STEVIN, est à la source de toutes les difficultés. [Cf. STEVIN, 1585, p. 2]. L'analogie proposée par les Grecs, selon STEVIN, présente deux défauts fondamentaux : d'abord, l'unité est une partie du nombre tandis que le point n'est pas une partie de la droite[9]. Ensuite, l'unité est divisible mais le point ne l'est pas, comme STEVIN démontre dans la citation que nous avons étudiée plus haut[10].

[8]La partie est de mesme matiere qu'est son entier,
l'unité est partie de multitude d'unitez.
Ergo l'unité est de mesme matiere qu'est la multitude d'unitez
Mais la matiere de la multitude d'unitez est nombre
Doncques la matiere d'unité est nombre [STEVIN, p. 1].

[9]EUCLIDE, dans sa définition du Livre I, suit l'analyse d'ARISTOTE. Un point constitue la frontière d'un segment ou divise un segment en deux parties. Mais un segment n'est pas fait de points [EUCLIDE, Livre I, Définition 3].

[10]L'unité est divisible en parties (vrai est qu'ils [les anciens] les nient, mais mille leurs distinctions ne sont pas suffisantes, de pouvoir ainsi opprimer la nature du nombre, qu'elle ne manifeste par force son essence, es arithmé-

Ainsi, STEVIN conclut que l'unité n'est pas au nombre ce que le point est au segment. Cependant, puisqu'il soutient que nombre et grandeur sont à ce point semblables que "*ils paraissent presque identiques*", la question suivante se pose : qu'est qu'il y a pour le nombre qui soit équivalent au point pour la droite? STEVIN répond :

> Je dit que c'est 0 (qui se le dict vulgairement Null, & que nous nommons commencement en la suivante 3 définition) ce que ne tesmoignent pas seulement leurs parfaictes & générales communautez, mais aussi leurs irréfutables effects. [STEVIN, 1585, p. 2]

Les "communautés" que STEVIN établi entre le zéro et le point sont les suivantes :

1. Point et zéro ne sont ni segment ni nombre (respectivement) mais ils y sont attachés.
2. Ni le point ni le zéro ne peuvent être divisés en parties.
3. Une infinité de points ne fait pas de segment de même qu'une infinité de zéros ne fait pas de nombre.
4. Ajouter un point ou un zéro au segment ou au nombre (respectivement) n'accroît pas sa quantité.

Concernant cette dernière analogie, STEVIN présente le raisonnement suivant (figure 1) :

> Mais si l'on concède que AB soit prolongée jusques au poinct C ainsi que AC soit une continue ligne, alors AB s'augmente par l'aide du poinct C; Et semblablement si l'on concède que D 6 soit prolongé jusques en E 0, ainsi que DE 60 soit un continue nombre faisant soixante, alors D 6 s'augmente par l'aide du nul 0... [STEVIN, 1585, p. 2]

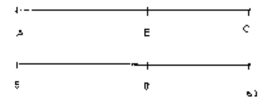

FIGURE 1

De cette façon, la ressemblance est encore valable.

L'exposition des grandeurs

STEVIN est convaincu que grâce à ses arguments, l'unité ne joue plus le rôle central dans l'analyse de la quantité. Le rôle spécial qu'ARISTOTE donne à l'unité et qu'EUCLIDE incorpore dans sa Mathématique, repose sur des arguments philosophiques. Ces arguments, à leur tour, sortent de la distinction qualitative entre nombre et grandeur en termes de continuité.

La définition euclidienne de nombre ("*une multitude composée d'unités*") fait penser davantage à un "nombre pur", de telle sorte que les rapports entre deux ou plus de nombres peuvent être conçus indépendamment des quantités auxquelles ils font référence; par exemple, pour

tiques opérations de plusieurs auteurs, comme entre autres par l'absolue partition de l'unité de la 33 question du 4 livre, & la 12, 13, 14, 15 questions du cinquième livre du Prince des Arithméticiens Diophante) [*L'Arithmétique*, p. 2].

déterminer que 10 est un nombre entier, paire, moitié de 20, double de 5, etc. il n'est pas nécessaire d'imaginer les quantités (la chose énumérée) que ces nombres représentent.

Un problème qui surgit au moment d'étendre la définition de nombre aux quantités continues est que, contrairement à ce que l'on trouve pour les quantités discrètes, il n'y a pas d'"unité naturelle" : on affecte un nombre à une grandeur au moyen d'une unité conventionnelle, à savoir, celle qui a été choisie comme unité de mesure. Dans ce cas, lorsqu'on opère sur les nombres identifiés aux quantités (continues et discontinues) il n'est pas claire si l'on opère sur des "vrais nombres", sur des symboles, ou sur les "quantités dénombrées", ce que Le Teneur reproche à STEVIN [Cf. JONES, 1978]. Le manque d'une unité naturelle entraîne une difficulté à débarrasser le nombre du contexte qui permet d'opérer sans avoir besoin de faire appel à la quantité correspondante. Conscient de cette difficulté, STEVIN définit le nombre arithmétique comme étant celui qui s'obtient après abstraction de la quantité dont il provient :

> Définition IV : Nombre arithmétique est celuy qu'on explique sans adjective de grandeur [STEVIN, 1585, Def. 4, p. 3]

Une grande partie du travail de STEVIN dans *L'Arithmétique* est consacrée à montrer que le domaine des nombres est homogène, étant donné que les nombres sont indépendants de leur genèse et que, de cette façon, on peut opérer sur eux sans aucune référence aux quantités d'où ils proviennent. De sorte que, par exemple, le nombre 9, que nous pouvons imaginer comme étant associé à une quantité linéaire, peut être interprété aussi comme l'aire d'un carré, sans perdre ou modifier ses propriétés relationnelles par rapport aux autres nombres, ni même ses propriétés opératoires.

Il est clair que l'unité cesse d'avoir le caractère privilégié qu'elle a dans la Mathématique grecque, en tant que principe générateur du nombre. C'est la raison pour laquelle STEVIN doit renoncer à la possibilité d'avoir un principe de génération absolu; dorénavant il sera toujours obligé de chercher un support en dehors des Mathématiques, situé dans le contexte concret : la quantité de chaque chose.

STEVIN, en ouvrant un si vaste spectre aux nombres, soulève les préoccupations suivantes : ces nouvelles entités sont-elles cohérentes avec la définition initiale de nombre? Représentent-elles vraiment des "quantités des choses"? STEVIN se consacre donc à la tâche de montrer que, de la même façon qu'il y a un nombre pour chaque grandeur (résultat de la mesure de cette grandeur), il existe une grandeur pour chaque nombre. Pour y aboutir, STEVIN a besoin de montrer que l'identification grecque qui restreint certaines grandeurs géométriques aux puissances d'un nombre (x^2 est une aire et x^3 un volume, sans aucune autre possibilité) n'est pas unique.

Un exemple de ce raisonnement est le suivant : supposons que nous ayons un segment de longueur 2 (figure 2). Si nous dessinons un carré sur le segment, l'aire sera $2^2 = 4$ et si maintenant nous construisons un cube sur le carré d'aire 4, son volume sera $2^3 = 8$. Jusqu'alors ce raisonnement coïncide avec le raisonnement grec : la première puissance est linéaire, la deuxième est carré et la troisième est cubique. Si maintenant nos empilons 2 de ces cubes, le volume du prisme résultant sera $2^4 = 16$ de sorte qu'on a conféré un sens géométrique a la quatrième puissance ainsi qu'à toutes les puissances suivantes [Cfr. STEVIN, 1585, pp. 4-5].

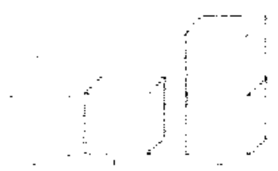

FIGURE 2

De la même façon, STEVIN expose les grandeurs géométriques associées à des puissances de nombres fractionnaires. Par exemple, partons d'un segment de longueur 1/2 (figure 3) et construisons le carré correspondant qui aura une aire $(1/2)^2 = 1/4$, et le cube qui aura un volume $(1/2)^3 = 1/8$. En divisant le cube en deux nous aurons un prisme dont le volume es $(1/2)^4 = 1/16$, de même que pour toutes les puissances fractionnaires suivantes [Ibidem].

FIGURE 3

Avec cette méthode, STEVIN veut montrer qu'il est toujours possible de produire une grandeur qui représente un nombre donné, de la même façon qu'il est toujours possible d'exhiber un nombre associé à une grandeur donnée. À travers cette identification fonctionnelle, STEVIN prétend apporter un fondement a l'homogénéité du domaine numérique et à la nature de son "nombre arithmétique".

Les processus infinis

STEVIN accepte l'existence théorique des expansions décimales infinies en tant que conséquence de l'existence opératoire des nombres. En fait, il propose une méthode (basée sur l'algorithme de la division) permettant de s'en approcher indéfiniment [cf. STEVIN, 1585, p. 210]. La citation suivante, relative aux grandeurs incommensurables, montre la position de STEVIN à cet égard :

> ... Mais combien ce théorème est véritable[11], toutesfois nous ne pouvons cognoistre par telle experience, l'incommensurance de deux grandeurs proposée; Premièrement parce qu'à cause de l'erreur de nos yeux et mains (qui ne peuvent parfaitement veoir et partir) nous iugerions à la fin que tous grandeurs, tant incommensurables que commensurables, fussent commensurables. Au second, encore qu'il nous fust possible, de soubsttraire par action, plusieurs cent mille fois la moindre grandeur de la maieure, et le continuer plusieurs milliers d'années, toutesfois (estant les deux nombres proposez incommensurables) l'on travailleroit éternellement, demeurant tousisours ignorants de ce qui à la fin en pourroit encore avenir; Ceste, **manière donc de cognition n'est pas légitime**, ainsi position de l'impossible, à la fin d'ainsi aucunement declairer, ce qui consiste véritablement en la Nature. [STEVIN, *Traité des incommensurables grandeurs*, p. 215)]

[11] STEVIN fait référence à la Deuxième Proposition du Livre X des *Éléments* d'EUCLIDE : "Si de deux quantités distinctes données, on reste toujours la plus petite de la plus grande, et le reste ne mesure jamais la quantité précèdente, Telles quantités sont incommensurables".

Dans cette affirmation, STEVIN montre aussi bien sa position empiriste que la prééminence de l'opération : il suppose que la connaissance est extraite de la Nature, à travers les opérations que le sujet peut effectuer et *conclure*.

La validation de la connaissance

Quand STEVIN caractérise les objets théoriques par les opérations qu'on réalise sur eux, il se retrouve dans la nécessité de faire appel aux arguments exogènes afin de valider ses résultats. Tout au long de son ouvrage mathématique STEVIN passe constamment des arguments numériques aux arguments géométriques et aux arguments physiques. Lorsqu'il définit le nombre en fonction de "la quantité de chaque chose" et résume son essence aux opérations qu'on réalise sur lui. STEVIN crée une série d'objets (résultats des opérations) pour lesquels il n'existe pas de structure théorique pour les valider. En conséquence, STEVIN a besoin de faire appel à de divers contextes, soit en présentant des grandeurs géométriques associées à ces objets, soit en s'appuyant sur "la quantité de la chose" que ces objets représentent. Bien que suffisamment cohérente pour être formalisable, la théorie de STEVIN est subordonnée aux contenus donnés, extra-logiques, d'où sa faiblesse structurale du point de vue formel.

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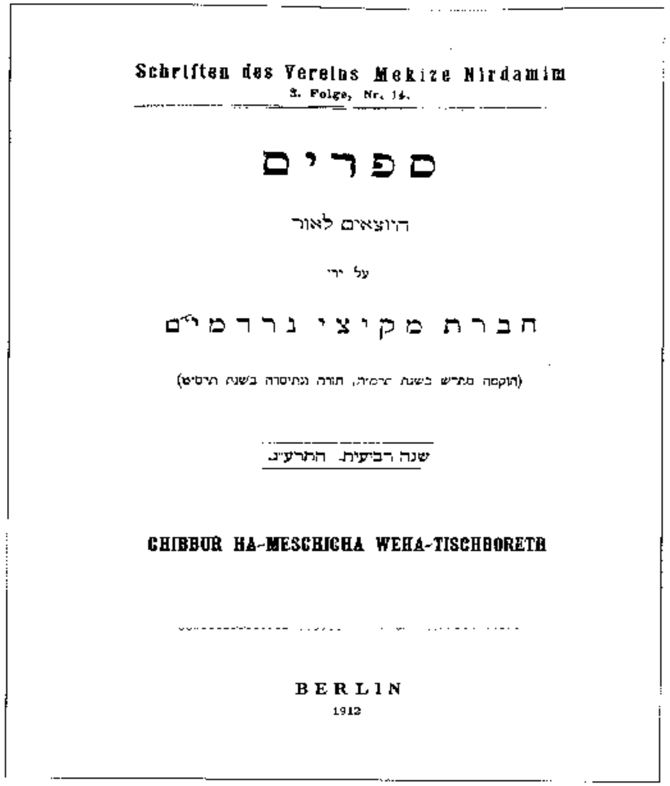
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3. Folge, Nr. 14.

ספרים

היוצאים לאור

על ידי

חברת מקיצי נרדמים

שנה רביעית התרע״ב

CHIBBUR HA-MESCHICHA WEHA-TISCHBORETH

BERLIN
1912

FIGURE 1 *Treatise on Measurement and Calculation*

Elementary School Teachers meet Abraham bar Hiyya Ha-Nassi

WINICKI-LANDMAN Greisy[1] (Israël)

Abstract

Abraham bar Hiyya was a Spanish Jewish mathematician from the XII Century. Between the books he wrote, the following may be mentioned: (a) Hibbur ha-Meshihah veha-Tishboret (Treatise on Measurement and Calculation), translated into Latin as Liber Embadorum and, (b) Yesod ha-Tebunah u-Migdal ha-Emunah (The Foundation of Understanding and the Tower of Faith) which is the first encyclopedia in Hebrew.

Since there are very few sources written in Hebrew, we decided to use some passages from these two books during a History of Mathematics course. This is a course designed for Israeli pre-service elementary school teachers as well as a part of a professional development program for in-service teachers. Many teaching techniques are implemented during this course but only lately we adopted an approach learnt from our French colleagues: to study ancient primary sources. One of the main difficulties faced was to find appropriate sources: contents appropriate for the learners and sources written in a language known by those learners.

The workshop proposed here will present the activities designed for this course and the learners' reactions and comments.

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Abraham bar Hiyya was a Spanish Jewish mathematician from the XII Century. Among the books he wrote, the following may be mentioned: (a) *Hibbur ha-Meshihah ve-ha-Tishboret* (Treatise on Measurement and Calculation), translated into Latin as Liber Embadorum and, (b) *Yesod ha-Tebunah u-Migdal ha-Emunah* (The Foundations of Understanding and the Tower of Faith) which is the first encyclopedia in Hebrew. He also wrote books on Astronomy - *Zurath ha-Haretz ve-Tavnith ha-Shamaim* and *Heshbon Mahalkhoth ha-Kokhavim* - and calendar calculations - *Sefer ha-Hibbur*. He can be considered the father of Hebrew mathematics.

Since there are very few mathematical sources written in Hebrew, we decided to use some passages from his books during a history of mathematics course. This is a course designed for Israeli pre-service elementary school teachers as well as a part of a professional development program for in-service teachers. Many teaching techniques are implemented during this course but lately we adopted an approach learnt from our French colleagues: to study ancient primary sources. One of the main difficulties was to find appropriate sources: appropriate contents for the students and sources written in a language known by them.

This paper presents the activity designed within the framework of this course as well as some of the students' reactions and comments.

1 The activity

Four passages from Bar Hiyya were chosen, following two didactic considerations: a) diversity of content (geometry problems and arithmetic problems) b) diversity of role in the mathematical discourse (i.e. problems, demonstrations, algorithms). Two passages were taken from the book *Treatise on Measurement and Calculation*, one of them dealing with the calculation of the area of a triangle, and the other one explaining why we calculate the area of a circle the way we do. The other two passages were taken from the book *The Foundations of Understanding and the Tower of Faith*. One is part of the introduction and the other one deals mainly with arithmetic calculations.

The main aims of the activity were:

a) To have the learners read mathematical texts;

b) To have the learners learn from original mathematical texts;

c) To have Hebrew speakers as well as Arabic speakers read mathematics written in the Middle Ages and to have them compare the terminology used then with the terminology used nowadays;

d) To have the learners analyze the teaching methods employed by the writer, to discuss them and to compare these methods to the "modern" ones.

The materials used for the activity may constitute a proof of an existence statement: there *are* Hebrew scientific manuscripts and they are of considerable importance because they "testify to the part played by the Jews in the vast enterprise of transferring Greek science, by way of the Islamic world, to the European nations" (SARFATTI, 1968). In that sense, another aim of this activity was

e) To expose the learners to a not very known aspect of Jewish history: its contribution to the development of mathematical knowledge during the Middle Ages.

At the beginning of the activity and without any introduction, the students were organized in small groups and each group was presented with a different Hebrew passage. Each group had: a) to read the text and to make sense of it, b) to summarize it and to present it to the others, and finally, c) to try to figure it out when that text was written.

The presentation of each small group was followed by a general discussion of the passage. Part of the didactic analysis of the passages was conducted according to Swetz's ideas. He wrote: "An examination and analysis of didactic trends in historical material can take place along several lines:

1. The organization of material, the sequential ordering of topics and specific problems;
2. The use of an instructional discourse and techniques of motivation contained within the discourse;
3. A use of visual aids; diagrams, illustrations, and colors, to assist in the grasping of concepts on the part of the learner;
4. The employ of tactile aids, either directly or by reference, to clarify a mathematical concept." (SWETZ, 1996)

TEXT I (from *Treatise on Measurement and Calculation* cfr. Figure 1)
Bar Hiyya classified triangles according to their sides. In this passage he treats the triangle with "changing" sides (what we call a scalene triangle, that is, with three inequal sides).

חיבור המשיחה והתשבורת

ג.

משלש מתחלף הצלעות.

והמשלש הזה יכול להיות נצב הזויות או מרויח הזויות או מחודד כל הזויות. ואני מבאר בראשונה דרך המחדד בכל הזויות. והדמיון למשלש הזה משלש **אבג** אשר לצלע **אב** ממנו י״ג אמות וצלע **בג** יש בו י״ד אמות וצלע **אג** הוא ט״ו אמות ואנו רוצים לדעת שיעור תישבורת המשלש הזה ואין אנו יכולים לדעת תשבורתו אלא מן עמודיו, אנו צריכים בהוצאת העמוד במשלש הזה לדעת גבול מעמד העמוד מן התושבת כי העמוד במשלש הזה אינו נופל על מחצית התושבת כאשר היה מנהגו במשלש השוה בצלעותיו וגם במשולש השוה בשוקיו בעת שהוצאנו את העמוד בין השוקיים אל הצלע המתחלפת אבל במשולש הזה הוא נוטה בכל צלעיו ממחצית התושבת אל צד אחד. והצד הארוך מגבול מעמדו אנו קוראים לו מעמד הארוך. ומפני זה אנו צריכים תחלה להוציא נקודת גבול מעמד העמוד על התושבת ולדעת צד מעמד הארוך ומרחקו וצד המעמד הקצר ומרחקו ואחר כך נבא לדעת אורך העמוד. והדרך הזה הוא על הענין הזה: אם נרצה במשלש **אבג** אשר מסרנו לך להוציא עמוד בין צלעי **אב אג** על התושבת **בג** אשר ארכה י״ד אמה אנו צריכים תחלה להגביל את מעמד העמוד ואם נבא להוציא המעמד הארוך נקח מרובע הצלע הארוך משני הצלעות המקיפות את זוית הראש במשלש אשר נוציא ביניהם את העמוד והוא צלע **אג** אשר אורכו ט״ו אמה ונחבר אל המרובע הזה מרובע התושבת ויהיו שני המרובעים האלה תכ״א אמות, נוציא מהם מרובע צלע **אב** הנשאר והוא הצלע הקצר ומרובעו קס״ט וישאר בידך רנ״ב. נחלק את הנותר לשנים ויהיה מחציתו קכ״ו, וחלק המחצית הזאת על התושבת אשר הוא י״ד ויהיה חלוקת ט׳ והוא מרחק גבול מעמד העמוד מן הצלע הארוך.

ואם נרצה לדעת המעמד הקצר נקח מרובע הצלע הקצר אשר בו י״ג עם מרובע התושבת אשר הוא י״ד ויהיו שני המרבעים שס״ה, נוציא מהם מרבע הצלע האחר יהיה רכ״ה וישאר ק״מ נחלק אותו לשנים ויהיה מחציתו ע׳ ויחלק המחצית הזאת על התושבת ויהיה חלוקה ה׳ והוא מרחק גבול העמוד מן הצלע הקצר.

ועל הדרך הזה היינו עושים על כל צלע וצלע אלו היינו באים להוציא עליו עמוד ואחר נדע גבול מעמד העמוד נבא לדעת אורך העמוד ודרך ידיעתו היא כך: נרבע הצלע ונוציא ממרובעו מרבע המעמד הנדבק בו ונקח גדר הנשאר והוא אורך העמוד. כאילו היינו מרבעים את הצלע הקצר במשלש הזה והוא צלע **אב** אשר הוא י״ג ומרובעו קס״ט ונוציא ממנו מרובע המעמד הנדבק בו והוא הקצר אשר הוא ה׳ ומרובעו כ״ה וישאר קמ״ד וגדר המספר הזה הוא אורך העמוד והוא י״ב. וכן אם היינו מרבעים את הצלע הארוך והוא ט״ו היה מרובעו רכ״ה. כיון שהיינו משליכים ממנו מרובע המעמד הארוך והוא ט׳ ומרובעו פ״א ונשאר קמ״ד כאשר נשאר מן הצלע הקצר. וגדר המספר הזה הוא י״ב והוא אורך העמוד. ורבוע העמוד הזה בחצי התושבת, אשר היא י״ד ומחציתה ז׳ הוא פ״ד והוא תשבת המשלש הזה.

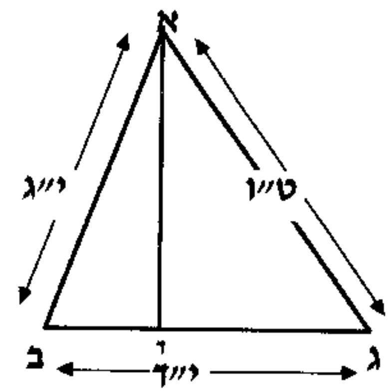
ואם תבוא לדעת האות על הוצאת העמוד הזה בא ועיין בצורת המשלש הזה אשר אצייר לך עתה. ודע כי מרובע צלע **אב** אשר היא מיתר זוית חדה כאשר הוא במשלש הזה פוחת ממרובע הצלע **אג** וצלע **בג** אשר היא התושבת בכדי רביע **גד** אשר הוא המרחק האחר בכל **בג** אשר היא התושבת פעמיים כאשר מפורש בחכמת השיעור. ואות לזה ידענו כי העמוד לעולם יפול על התושבת על זוית נצבה בכל משלש מחדד הזויות כגון עמוד אד בצורה אשר עשינו, וצלע **אג** הוא אלכסון וצלע **אד** הוא העמוד וצלע **גד** והוא מעמד הארוך מהתושבת. וידענו כי מרובע צלע **אג** שוה למרובע **אד** ומרובע **גד** כל אחת בפני עצמה, כענין כל אלכסון.......

* *The learners identified the problem discussed by Bar Hiyya:*

ABC is a triangle and AB = 13 cubits, BC = 14 cubits, AC = 15 cubits.
Calculate the area of the triangle.
It is important to note here that Bar Hiyya does not use Latin letters for the triangle's vertices, but Hebrew letters: Aleph (instead of A), Beth (instead of B) and so on. He does not use Hindu-Arabic numerals either, but Hebrew letter-numerals. So, instead of "13 cubits" he writes "yod - guimmel amot". The following table presents the numerical value of the Hebrew letters.

1	aleph	א
2	bet	ב
3	gimmel	ג
4	dalet	ד
5	hey	ה
6	vav	ו
7	zayin	ז
8	chet	ח
9	tet	ט
10	yud	י
20	caf	כ
30	lamed	ל
40	mem	מ
50	nun	נ
60	samech	ס
70	ayin	ע
80	peh	פ
90	tzadi	צ
100	kuf	ק
200	resh	ר
300	shin	ש
400	tav	ת

* *Without trying to solve the problem themselves, the learners went on reading and tried to follow Bar Hiyya's rhetorical solution:*
Bar Hiyya suggests drawing the altitude AD and establishes that the length of CD is:



$$CD = \frac{15^2 - 13^2 + 14^2}{\frac{2}{14}}.$$

Then, he claims that :

$$\begin{aligned} AD^2 &= AC^2 - DC^2 \Rightarrow \\ AD &= \sqrt{(AC - DC)(AC + DC)} \Rightarrow \\ AD &= \sqrt{6 \times 24} = 12. \end{aligned}$$

From here, Bar Hiyya concludes that the area of the triangle is 84 squared cubits:

$$S(\Delta ABC) = \frac{14 \times 12}{2} = 84.$$

* *The learners tried to verify if Bar Hiyya's solution is correct by comparing its numerical result with the result they got by using the Pythagorean Proposition twice.*
Working with h - the length of AD - as unknown, and with x - the length of BD - they wrote:

ΔABD is a right triangle. So, $AD^2 = AB^2 - BD^2 \Rightarrow h^2 = 13^2 - (14 - x)^2$

ΔACD is a right triangle too. So, $AD^2 = AC^2 - CD^2 \Rightarrow h^2 = 15^2 - x^2$.

From here they wrote the equation:

$$\begin{aligned} & 13^2 - (14 - x)^2 = 15^2 - x^2 \Rightarrow \\ & 13^2 - 14^2 + 2 \cdot 14 \cdot x - x^2 = 15^2 - x^2 \Rightarrow \\ & 2 \cdot 14 \cdot x = 15^2 - 13^2 + 14^2 \Rightarrow \\ & x = \frac{15^2 - 13^2 + 14^2}{2 \cdot 14} = \frac{(15 - 13)(15 + 13) + 14^2}{2 \cdot 14} = \frac{2 \cdot 2 + 14}{2} = 9. \end{aligned}$$

And from here, they concluded that

$$AD^2 = 15^2 - 9^2 \Rightarrow AD^2 = 225 - 81 = 144 \Rightarrow AD = 12.$$

Finally, they could calculate $S(\Delta ABC)$ and got that $S(\Delta ABC) = \frac{14 \times 12}{2} = 84$, the same result and the same calculation Bar Hiyya asked them to do!!!.
* *The learners were eager to know details about the text and the author.*
They found it hard to believe that this text was written in the XII Century by a Spanish Rabbi who lived in Barcelona. They were encouraged to look up details about him and his time with the help of the Internet.
* *The learners were engaged in a deeper analysis of the text from a pedagogical point of view.*
The discussion focused mainly on Bar Hiyya's technical language and the comments he added in order to make the text an exemplary learning text book. For example, the term *altitude* was not used by Bar Hiyya. He used the term *column* or *pillar*. Some of the learners pointed out that it is a more convenient term for the concept. Others were strongly against the idea because they believed that student would wrongly believe that a "mathematical column" is always vertical.

Another term used by Bar Hiyya was *gader*. The Hebrew speakers did not understand the meaning of this word in this mathematical context, since the term's popular meaning is *fence* or *limit*. The Arabic speakers thought about the term *jidhr* which is used as *square root*. A discussion about the use of this term may be found also in (EFROS 1969, p. 128).
Re-reading the text with this meaning in mind, they conjectured that Bar Hiyya borrowed the term from Arabic sources. This point led us to discuss interactions between the Jews and the Arabs during the Muslim Period in Spain.

Bar Hiyya's language is elegant and very clear. Moreover, his explanations and considerations enable the reader to generalize the method to calculate the area of a triangle determined by the length of its sides. So, in this case, his teaching method relies on the use of a clear example that enables its generalization in a very transparent way. We thought it was not by chance that Bar Hiyya chooses such a triangle (13, 14, 15 - an Heronian triangle) to present his ideas. All the calculations are easy and the numbers "nice". One of the students asked if Bar Hiyya knew about Heron's formula to compute the area of a triangle defined by its three sides. In this passage, Bar Hiyya asks to compute one height and the two parts of the base, so he does not use the Heronian formula here. But some pages later, when Bar Hiyya summarizes the chapter on triangular figures, he describes rhetorically a general method to calculate the area of a triangle without having to calculate any height. The method described is Heron's. He also exemplifies its use by means of an "nice" case, the 6-8-10 triangle. All through the chapter, Bar Hiyya does not explain how his results are obtained and he declares he is aware of that.

חיבור המשיחה והתשבורת

וכבר הודעתיך דרך מדות המשלשות בכל חלקיהם וכל מיניהם והודעתיך המדה אשר משלש נצב הזוית פתיחו בה נגד מדת מרוח הזווית וכן חוי יורע כי לפרוד הזויות מדה שהוא פתיחו בה, והוא, כי מיחד כל זוית חדה מרובעו הוא פוחת ממרובעי שתי הצלעות המקיפות לה כבדי רבוע הקו אשר הוא תושבת לעמוד בקו מרחקי גבול העמוד מנקודת הזוית החדה פעמים.

ואני הודיעך דך הענין הזה בהוצאת העמוד במשלש מתחלף הצלעות הסחדד ובחשבון הזה אתה יכול לחשב כל משלש, כי אין לך משלש שאין לו זוית חדה. ואם אתה סומך על החשבון ההוא בעמודי המשלשות ותשבורתם והיה מספיק לך ואין אתה צריך ענין אחר

ואעפ״י שהדבר כן וכל הדרכים האלה אשר הודיתיך הם נכונים ונכוחים למבין ומוציאים אלחשבון אמתי, אתה מוצא דרך לחשבון המשלשות ומדידתן שאינו מצריך אל העמוד והוא החשבון אשר קראו לו חשבון המותרות.

והדרך הזה הוא שתהיה ידע מחצית כל אחת מצלעי המשלש ותקבוץ את המציות האלה, ותדע מותר כללם על כל צלע וצלע ותשמור המותרות האלה ותמנה אחת מהם בשני והמספר הנקבץ מנה אותו במותר השלישי, ואשר יכנס בידך מן החשבון הזה מנה אותו בכל כלל המחציות אשר קבצת ויהיה המספר הזה מרובע תשבורת המשלש, ואם תוציא גדר המספר הזה תמצא התשבורת.

והדמיון לחשבון הזה משלש אשר צלעו האחת י׳ והשניה ח׳ והשלישית ו׳ , אם אתה מקבץ לשלש חציי הצלעות האלה יהיו י״ב, ומותר י״ב על הצלע האחת ב׳ ועל השנית ד׳ ועל השלישית ו׳. ואתה בא ומנה ב׳ אשר הם מותרהאחת בדי אשר הם מותר הצלע השנית והיה ח׳ , ומנה ח׳ בו׳ אשר הם הצלע השלישית ויהיה מ״ח, ושוב ומנה מ״ח בי״ב אשר הוא כלל המחציות ויהיה הכל תקע״ו. ודע כי המספר הזה הוא מרובע התשבורת, ואם אתה מוציא את גדרו והוא כ״ד תמצא תשבורת המשלש. והחשבון הזה בנוי על יסודי חכמת השיעור והאות עליו מפורשת שם ואין אנו יכולין לאומרה בכאן ואין אנו צריכין אליה במקום הזה כ״ב מפני שראית שאמת הוא החשבון הזה מן המנין אשר אמרתי לך

From the above we concluded, that one of the messages from Bar Hiyya to these future teachers may be: teach mathematics by means of clear examples, examples that explain and examples that naturally engage the learner in a process of generalization and justification.

TEXT II (from *Treatise on Measurement and Calculation*)

In this chapter Bar Hiyya describes how to compute the area of a circular field and the area of circular segments.

חיבור המשיחה והתשבורת

החלק הרביעי.

בפירוש מדידת קרקעות בצורת העגול התמים או על צורת העגול הפגום אשר הוא מחצית עיגול או מרבה על המחצית או פוחת מן המחצית.

א.

ואתה יודע את תשבורת העגול התמים אם אתה יודע את אלכסונו והוא הנקרא קוטר בלשון ערבי, ותכפול אותו שלושה פעמים ושביעית פעם ויהיה אורך הקו הסובב, ואחר כך הוי מרבע את מחצית הקוטר במחצית הקו הסובב ותמצא תשבורת העגול.

ואות לאורך הקו הסובב ג׳ פעמים ושביעית פעם במידת הקוטר כגון **אבגדהו**, והקוטר שלה הוא **אזד** ומחציתו על הנקודה **ז** והוא מרכז העגולה. וידענו כי קו **או** הישר שוה לקו **אז** ולקו **אב** כי שלושתם יצאו מציר העגולה **בזו** והוא **א**, וכן עגולת **בזו** מצד האחד מראה כי קו **גד** וקו **גב** שוים אליהם וכן קו **דה** וקו **הו** הישרים כי כמדת זה כן מידת זה. נמצא כמדת אז חצי הקוטר נמנו ששה ישרים בתוך העגול מששים והמוקף בששה מקומות והם ג׳ במידת הקוטר כלו. וידענו כי עגולת **אב** שהיא קשת גדולה מקו הישר **אב** שהוא מיתר לקשת כי כל קשת מוסיף על מיתרו. ושערו המדקדקים וקבצו אלה המותרות מכל קשת וקשת מאלו הששה ועלו למדת שביעית הקוטר.

The students identified the purpose of the chapter and the main concepts: circle, area of the circle, circumference (surrounding line) and diameter. Bar Hiyya explains that instead of diagonal

of a circle he uses *koter* (diameter), a term he borrowed from the Arabic. He presents the well known rule:

$$\text{Circumference of a circle} = \text{its diameter} \times (3 + 1/7).$$

Then he presents his rule to calculate the area of a circle:

$$\text{Area of the circle} = (\text{half its circumference}) \times (\text{half its diameter}).$$

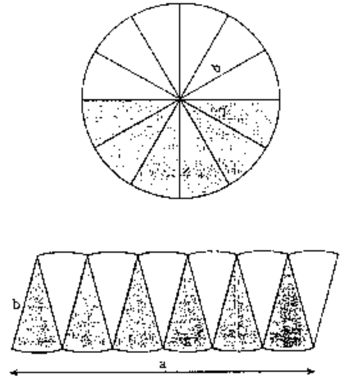
Following his statement, he comments on how you can deduce the second rule from the first one.

חיבור המשיחה והתשבורת

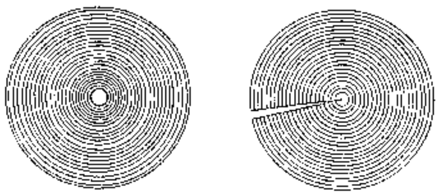
ואחר שידענו הקו הסובב והקוטר אנו יודעים תשבורת כל העגולה שהיא מחצית הקוטר במחצית הקו הסובב. והאות על התשבורת הזה ידענו אם תפתח שטח העול מצד אחד ותישר כל הקוים הסובבים מקו החיצוני עד המרכז יתפשטו המקיפים שטח העגול ויחזרו לקוים ישרים מתמעטים והולכים עד שחוזרים אל נקודה אחת והיא נקודת המרכז כגון הקו **אבגד דהוז חטי** שציירתי וכן החיצון גדול מכולם, ואשר לפנים ממנו קטן ממנו וגדול מאשר לפנים ממנו וכן הולכים עד הנקודה ובזה נולדה לנו צורת המשלש. ותשבורת המשלש כבר בארנו, הוא כדי העמוד בחצי התושבת וזהו מחצית הקוטר במחצית הקו המקיף והוא בצורה אשר עשינו עמוד **אב** במחצית קו **ג**.

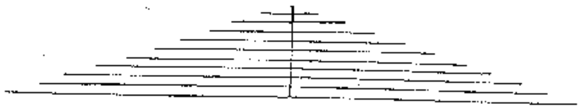
והדמיון לענין הזה עגולה שיש בקוטר שלה י״ד ואתה תכפול אותה שלושה פעמים ושביעית ויהיה מ״ד, והיא מדת הקו הסובב. ואם אתה כופל חצי הקוטר והוא ז׳ במחצית הקו הסובב והוא כ״ב יהיה המנין הנקבץ קנ״ד והוא תשבורת העגול.

The students read it and draw the following diagram:



This diagram appears in the school textbook as a part of the activities about the area of the circle. But... this is not what Bar Hiyya was telling them to do!!! Bar Hiyya demonstration may be illustrated as follows:

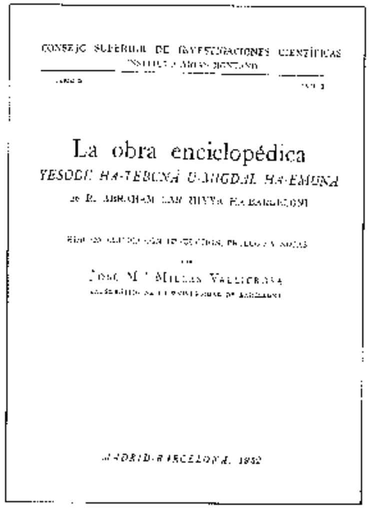




$$S(\Delta) = \frac{b}{2} \times h = \frac{2\pi r}{2} \times r = \pi r^2$$

This presentation was followed by a very enlightening discussion about the idea and purpose of proof in mathematics and in mathematics teaching.

TEXT III (from *The Foundations of Understanding and the Tower of Faith*)



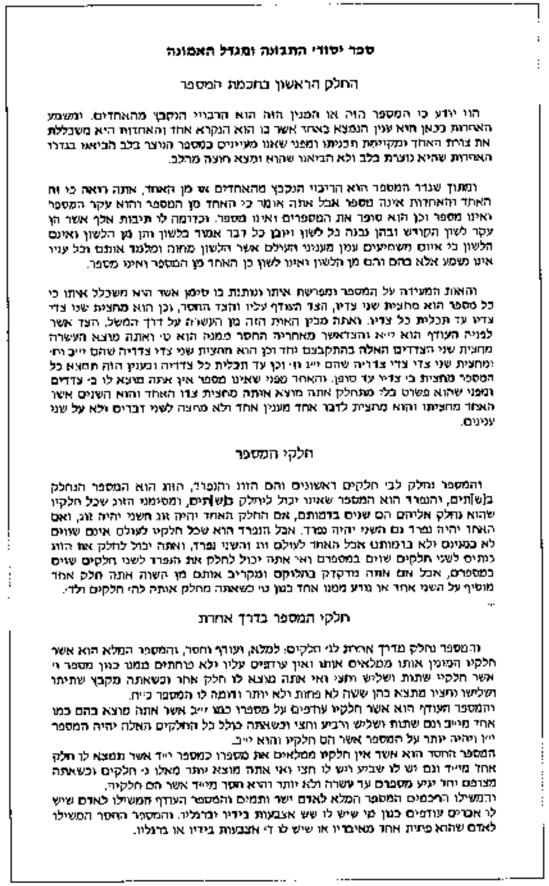
CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS

La obra enciclopédica

YESODÉ HA-TEBUNÁ U-MIGDAL HA-EMUNÁ

MADRID-BARCELONA, 1952

According to SARFATTI (1969), the main goal of this book was to clarify the main terms used in the different sciences, to present their definitions and also to present general rules (p. 90). It is a sort of dictionary for scientific terms organized by topic rather than alphabetic order. The book is divided in two parts: a) *The foundations of understanding*, b) *The tower of faith*. The first part is divided in four foundations (mathematics, physics, politics and metaphysics). The mathematics foundation is composed of three pillars which are: arithmetic, geometry, music. The arithmetic pillar is divided in two parts: the theory of the number (pure arithmetic) and the theory of computations (logistics). The texts presented to the students belong to the theory of the number.



ספר יסודי התבונה ומגדל האמונה

החלק הראשון בחכמת המספר

הוי יודע כי המספר הזה או המנין הזה הוא הרבוי הנקבץ מהאחדים. ומשמע האחדות כאן הוא ענין הנמצא באחד אשר בו הוא הנקרא אחד והאחדות היא משכללת את צורת האחד ומקיימת תכניתו ומפני שאנו מעיינים במספר הנוצר בלב הביאנו בגדרו האחדות שהיא נוצרת בלב ולא הביאנו שהוא ומצא חוצה מהלב.

ומתוך שגדר המספר הוא הריבוי הנקבץ מהאחדים או מן האחד, אתה רואה כי זה האחד והאחדות אינה מספר אבל אתה אומר כי האחד מן המספר והוא עקר המספר ואינו מספר וכן הוא סופר את המספרים ואינו מספר. וכדומה לו תיבות אלף אשר הן עקר לשון הקודש ובהן נבנה כל לשון ויובן כל דבר אמור בלשון והן מן הלשון ואינם הלשון כי איום חשחיעים עמן מעניני העולם אשר הלשון מחוה ומלמד אותם וכל ענין אינו נשמע אלא בהם והם מן הלשון ואינו לשון כן האחד מן המספר ואינו מספר.

והאות המעידה על המספר ומפרשת איתו ונותנת בו סימן אשר הוא משכלל אותו כי כל מספר הוא מחצית שני צדיו, הצד הנוסף עליו והצד החסר, וכן הוא מחצית שני צדי צדיו עד תכלית כל צדיו. ואתה מבין האות הזה מן העשורה על דרך המשל, הצד אשר לפניה העודף הוא י״א והצד אשר מאחריה החסר ממנה הוא ט׳ ואתה מוצא העשרה מחצית שני הצדדים האלה בהתקבצם יחד וכן הוא מחצית שני צדי צדדיה שהם י״ב וח׳ ומחצית שני צדי צדי צדדיה שהם י״ג וז׳ וכן עד תכלית כל צדדיה ובענין הזה תמצא כל המספר מחצית ב׳ צדיו עד סופן. והאחד מפני שאינו מספר אין אתה מוצא לו ב׳ צדדים ומפני שהוא פשוט בלי מתחלק אתה מוצא אותה מחצית צדו האחד והוא השנים אשר האחד מחציתו והוא מחצית לדבר אחד מענין אחד ולא מחצה לשני דברים ולא על שני ענינים.

חלקי המספר

והמספר נחלק לב׳ חלקים ראשונים והם הזוג והנפרד, הזוג הוא המספר הנחלק בשתים, והנפרד הוא המספר שאינו יכול ליחלק בשתים, ומסימני הזוג שכל חלקיו שהוא נחלק אליהם הם שנים בדמותם, אם החלק האחד יהיה זוג השני יהיה זוג, ואם האחד יהיה נפרד גם השני יהיה נפרד. אבל הנפרד הוא שכל חלקיו לעולם אינם שוים לא בענינים ולא בדמותם אבל האחד לעולם זוג והשני נפרד, ואתה יכול לחלק את הזוג בנתים לשני חלקים שוים במספרם ואי אתה יכול לחלק את הנפרד לשני חלקים שוים במספרם, אבל אם אתה מדקדק בחלוקם ומקריב אותם מן השוה אתה חלק אחד מוסיף על השני אחד או גורע ממנו אחד כגון ט׳ כשאתה מחלק אותה לה׳ חלקים ולד׳.

...

חלקי המספר בדרך אחרת

והמספר נחלק מדרך אחרת לג׳ חלקים: למלא, ועודף וחסר, והמספר המלא הוא אשר חלקיו המזומן אותו ממלאים אותו ואין עודפים עליו ולא נחסרים ממנו כגון מספר ו׳ אשר חלקיו שתות ושליש וחצי ואי אתה מוצא לו חלק אחר וכשאתה מקבץ שתיתו ושלישו וחציו תמצא בהן ששה לא פחות ולא יותר ודומה לו המספר כ״ח.

והמספר העודף הוא אשר חלקיו עודפים על מספרו כמו י״ב אשר אתה מוצא בהם כמו אחד מי״ב וגם שתות ושליש ורביע וחצי וכשאתה כולל כל החלקים האלה יהיה המספר ט״ז ויהיה יותר על המספר אשר הם חלקיו והוא י״ב.

המספר החסר הוא אשר אין חלקיו ממלאים את מספרו כמספר י״ד אשר תמצא לו חלק אחד מי״ד וגם יש לו שביע ויש לו חצי ואי אתה מוצא יותר מאלו ג׳ חלקים וכשאתה מצרפם יחד יגיע מספרם עד עשרה ולא יותר והוא חסר מי״ד אשר הם חלקיה.

והמשילו החכמים המספר המלא לאדם ישר ותמים והמספר העודף המשילו לאדם שיש לו אברים עודפים כגון מי שיש לו שש אצבעות בידיו וברגליו. והמספר החסר המשילו לאדם שהוא פחות אחד מאבריו או שיש לו ד׳ אצבעות בידיו או ברגליו.

This passage deals with the Pythagorean ideas of number: number is always connected with counted things and number is the substance of all things. A number is, according to Bar Hiyya - who must be considered a Neo-Pythagorean like Nicomachus - a multitude composed of units. It follows that **1** cannot be considered a number. Reading this passage presented an excellent opportunity to look for Bar-Hiyya's descriptions of known terms like *odd numbers, even numbers, prime numbers, composite numbers, perfect number*, etc.

TEXT IV (from *The Foundations of Understanding and the Tower of Faith*)

ספר יסודי התבונה ומגדל האמונה

החלק השני בחכמת החשבון

וכל אשר הקדמנו למעלה מצורת המנין וגדריו הוא מן החכמה השכלית ומעניינת במנין מהדרך שהוא עומד בלב. וחכמת החשבון אשר אנו באים אתה לדבר עליה בחלק הזה מעיינת במנין כדרך שהוא נופל על כל עניני העולם הזה ועל כל עסקיו כי אין לך דבר בעולם הזה שאין המנין משמש ומונה אותו וחופף עליו מכל צדדיו. וכל המנין נכלל משנים עשר שמות, תשעה מהם חוזרין חלילה עליו ושלשה מהם בונים את מעלותיו. והתשעה שמות הם האחדים והם המעלה הראשונה מהם עד תשעה והגי שמות הן לגי מעלות המנין אחר האחדים והם העשרות והמאות והאלפים. ואלו הארבע הם עקר מעלות המנין אשר הם חוזרים עליו ומקיפין אותו כי כל מעלה אשר עולה על האלפים שמה תצוב משלושת המעלות אשר לפניה אלפי אלפים ועשרות אלפי אלפים ומאות אלפי אלפים ואלפי אלפי אלפים וכן למעלה מזה.

...

וחכמי המנין מתעסקים בחשבון על ענינים רבים, ראשונה הם צריכים לחשוב מנין במנין, ושנית לחלק מנין אל מנין ושלישית לדעת קצב מנין ממנין, ורביעית לפחות מנין מן מנין, וחמישית לספות מנין אל מנין, ושישית להשלים מנין במנין ושביעית להשיב מנין אל מנין, ובזה הענין הם צריכים לחלק את האחד ולעיין בחלקיו איך הם נחשבים זה בזה ונשלמים זה מזה, ואנו נותנים כללות לכל אחד מעניינים האלה בדרך החבור הזה. והמבין צורת הענינים האלה נקרא מבין בתכמת החשבון, והממהר בחשבון ומקבץ אותו נקרא זריז במלאכת החשבון ואינו נקרא על זה חכם בה ומבין וזה דרך הבינה יכול אדם ללומדה וללמדה בספר והמלאכה והזריזות אינה נלמדת מספר כי בינה ממעשה הלב והמלאכה ממעשה ידים.

...

דרך חלוק החלקים

אתה מקים ראשונה מספר שיהיה מוכיח ואליו תקיש כל חשבונך כגון האומר לך חלק גי שמיניות על בי חמישיות, אתה חושב ראשונה שמות החלקים והם חי על הי והנה המוכיח מי וקח גי שמיניות מי והם טייו ובי חמישיותיו והם ייו ונחלק טייו על ייו הוא חלק גי שמיניות על בי חימישיות והוא אחד חסר חצי שמינית.

This passage deals with the ratio of fractions and it is presented by Bar Hiyya after the passage that deals with division of numbers (natural numbers) and the passage that deals with the calculation of a certain part of a number. Bar Hiyya uses the term *moneh* (counts) for "divides". For example he may say: 3 counts 12 (similar to the Greek "3 measures 12"), and : 5 does not count 12. He uses the term "division" when the dividend is bigger that the divisor and the term "ratio" when the dividend is smaller than the divisor.

Before the students read Bar Hiyya's passage, they were asked to describe how would they explain a young child how to divide fractions. All of them followed the same procedure: "Tell them that to divide by a/b is to multiply by b/a". This is not a surprising fact since this is the usual procedure described in school textbooks.

Then, they were allowed to read Bar Hiyya's passage. He says:

> To find the ratio between 3/8 by 2/5, you may have a "helping" number, like 40. 3 eighths of 40 is 15 and 2 fifths of 40 is 16. So, the ratio between 3/8 and 2/5 is the ratio between 15 and 16.

The students found again. that Bar Hiyya's explanations are transparent and his example explains not only *how* but mainly *why* you divide fractions the way you do. From this text we learn again that to teach is not only to tell a story but to know when to shut down.

Summary

This paper described an experience in which teachers were involved in a worthwhile mathematical task in which the context was provided by the analysis of primary sources in Hebrew. We would like to encourage more teachers to think in this direction and to invite them to share primary sources which they have found with other teachers.

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Aspects linguistiques de l'Épistémologie et de l'Éducation des Mathématiques

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Abstract

La linguistique structurale depuis la 'révolution Chomskienne' est saturée de conséquences importantes pour les questions classiques de la nature du savoir mathématique et de son acquisition, surtout si l'on adopte de manière conséquente la position de regarder les mathématiques comme un domaine de communication, ou plus précisément, comme une famille de registres linguistiques. En fait, l'importance des phénomènes discursifs dans l'éducation mathématique a été l'objet de nombreuses études depuis les années 70. Cependant, il nous semble que la philosophie des mathématiques et (surtout) la didactique des mathématiques n'ont pas envisagé les plus profonds aspects épistémologiques du travail Chomsky sur la compétence linguistique (qui d'ailleurs s'inspire de manière essentielle des mathématiques), ce qui s'explique certainement par l'absence de théories suffisamment explicites et cohérentes des relations entre les mathématiques et les langues naturelles.

Dans cette communication, j'expliquerai les contours d'une telle théorie, ou les mathématiques (de la maternelle à l'université, de la théorie pure aux applications) seront décrites dans des termes purement linguistiques. On se limitera à traiter les idées fondamentales (registre, transformation etc.) de cette description, pour discuter en plus de détails les conséquences possibles pour l'épistémologie de l'enseignement des mathématiques.

1 Introduction

Les relations et les interactions entre langues et mathématiques sont le sujet de nombreux articles et livres dans la littérature moderne sur la didactique et l'épistémologie des mathématiques. Dans la plupart de ces études, il s'agit du rôle joué par le langage employé dans les communications touchant aux mathématiques : dans les salles de classe (p.e. DURKIN & SHIRE 1991; PIMM 1994), dans les textes pédagogiques et scientifiques (p.e., MORGAN 1998), dans les discussions des chercheurs (p.e., ERNEST 1998), etc.; surtout, il s'agit en première ligne de l'usage des langues naturelles dans ces contextes divers, même si cet usage est lié aux spécificités de la matière des mathématiques. A côté de ces efforts du *mainstream*, le langage mathématique interne, propre aux mathématiques, a été ocassionellement traité, surtout dans un sens métaphorique ('les mathématiques sont une langue', cf. p.e. PIMM 1987 ou SCHWEIGER 1994), et même comme un langage construit à base des métaphores (LAKOFF & NÚÑEZ 1997).

Le but de ce travail est de décrire l'usage linguistique des mathématiques dans des termes qui sont et linguistiques dans un sens général, et propres à saisir les spécificités des mathématiques de manière à élucider son épistémologie. Ce faisant, nous partirons des données suivantes :

- Notre analyse se réduit à décrire les mathématiques comme phénomène communiqué, donc les questions classiques de l'ontologie sont pour ainsi dire écartées par avance;
- Nous renonçons à problématiser le genre des mathématiques comme tel, supposant un certain niveau de consentement sur l'existence d'un pratique homogène des 'mathématiques modernes' (en Anglais, *mainstream mathematics*);
- Également nous ne nous engageons point dans les débats sur les paradigmes linguistiques (linguistique structurale et socio-linguistique) auxquelles nous puisons les notions conceptuels pour notre analyse; nous nous bornerons à les expliciter.

En somme, notre but est pragmatique plutôt que dogmatique : de fournir des notions analytiques pour l'enseignant et pour le chercheur en quête d'une compréhension des aspects linguistiques des mathématiques.

2 Définitions fondamentales : langue, langage, registre

Les catégories Saussuriennes de *langue et parole* (DE SAUSSURE 1967) pénetrent toute la linguistique moderne. Les deux forment ce phénomène hétérogène du *langage*, comprenant les aspects physiques, physiologiques et psychologiques de la communication entre êtres humains. Tandis que *parole* comprend les manifestations individuelles et observables de cette communication, c'est à dire les 'textes' (écrites, orales etc.) qui constituent la communication concrète – *langue* peut être grossièrement caractérisé comme les structures de savoir qui sont à la base de la production de textes, telles que la syntaxe, la sémantique etc. Une définition plus formelle, très utile pour nos buts, est donnée par GIRSDANSKY (1963, p. 3) :

> Language is a set of arbitrary symbols (words) which are placed in orderly relationship with one another according to conventions accepted and understood by the speakers, for the transmission of messages.
>
> Une langue est un ensemble de symboles (mots) arbitraires qui sont placés en relations ordonnées d'après des conventions acceptées et comprises par les interlocuteurs, dans le but de transmettre des messages.

Traditionnellement, le sujet des sciences linguistiques se définit comme l'étude de *langues naturelles* (telle que le Français, le Danois etc.) à partir des évidences présentes en usage concret, en *parole*. Avec une compréhension suffisamment générale de la définition de *langue* que nous venons de citer, il se voit aisément que ce domaine d'enquête peut être élargie de manière à comprendre aussi les *langues formelles* telle que nous les trouvons p.e. dans la *parole* des mathématiciens. En effet, JACOBSON (1970, p. 15) note que

> formalized languages... are artificial transforms of natural language, in particular, of its written variety.
>
> Les langues formelles sont... des transformés artificielles de la langue naturelle, surtout dans sa variété écrite.

Bien que la situation se montre quelque plus complexe en usage concret des mathématiques, nous verrons qu'une partie de la syntaxe de la *langue* des mathématiques peut effectivement être conçue de cette manière.

Néanmoins, des précisions sur les théories de *parole* sont nécessaires pour placer notre étude d'une langue des mathematiques dans son propre contexte linguistique. Comme il n'y a pas de parole sans interlocuteurs, et même une société d'interlocuteurs avec une langue commune, il n'y a pas de communication (en parole) cohérente sans un certain cadre d'usage déterminé par le contexte ; p.e., une leçon en classe, un exercice militaire, un débat entre hommes politiques etc. Ces systèmes de conventions d'usage propre à des groupes d'interlocuteurs placés dans certaines situations, sont nommés communément des *registres* linguistiques (voir HALLIDAY et al. 1964, pp. 87-94). Nous avons donc des registres professionnels (liés à la communication dans la cadre d'une certaine profession), des registres d'enfants (p.e. liés a certains jeux), des registres religieux (p.e. la rhétorique des oraisons) etc.; dans tous les cas cités, la *langue* peut bien évidemment être la même, la différence étant marqué par son usage pour des buts différents.

3 La révolution Chomskienne

Les années 50 et 60 ont vu l'essor de la linguistique structurale, promue surtout par la 'linguistique de transformation' de Zellig Harris, et par la théorie des 'grammaires génératrices' de son étudiant, Noam Chomsky.

Il est intéressant de constater que notre idée de puiser aux études des langues naturelles pour notre analyse des mathématiques, est historiquement précédée par l'influence directe et indirecte des mathématiques dans la génèse de la linguistique structurale. Cette influence est très claire dans toute l'œuvre de Harris, où elle est souvent directe et explicite :

> The interest here is... in formulating in a mathematical system precisely those properties sufficient and necessary to characterize the whole of natural language and its unique power.
>
> Le point d'intérêt ici est... de formuler dans un système mathématique exactement les propriétés qui sont nécessaires et suffisantes pour caractériser la totalité de la langue naturelle et sa puissance unique. (HARRIS 1970, p. 603)

Voici un résumé simplifié d'une partie (concernant la syntaxe) caractéristique de la stratégie de Harris pour ce faire (HARRIS 1970, pp. 533-577) : toute phrase se dérive par *transformations* d'un ensemble fini de *phrases de base*. Les transformations forment une structure algébrique –un semigroupe– sous composition, et l'étude de la construction de phrases se réduit à analyser la structure de ce semigroupe, en particulier de décider s'il possède un ensemble fini de générateurs.

L'idée d'utiliser des notions mathématiques pour décrire les structures linguistiques est également fondamental dans (CHOMSKY 1957) qui contient la démonstration (mathématique) du fait bouleversant que les langues naturelles sont infinies non seulement en nombre de phrases possibles, mais aussi –à grands traits– en nombre de 'règles' qui sont nécessaires et suffisantes pour produire toutes les phrases correctes (et seulement celles-ci). D'autant plus revolutionnaire était donc l'idée des 'grammaires génératrices', qui

> attempts to characterize in the most neutral possible terms the knowledge of the language that provides the basis for actual use of language by a speaker-hearer.
> cherche à caractériser dans les termes les plus neutres les connaissances qui sont à la base de l'usage concrète de la langue par un interlocuteur. (CHOMSKY 1965, p. 9)

Donc, revenant aux catégories Saussuriennes, on cherche à isoler les aspects de *langue* qui, en écartant un amas de faits (p.e. lexicologiques) de genre plus ou moins arbitraires, sont fondementales dans la production de *parole*. Dans les termes de Chomsky, ces aspects constituent *l'état initial (initial state ou universal grammar)* de la faculté linguistique, la partie de nos connaissances qui –par leur complexité même, dont nous venons de faire la mention– ne saurait être apprises à partir de l'expérience nécessairement finie que l'on puisse acquérir avec un langage quelconque, mais qui d'autre part nous permet en principe d'acquérir n'importe quelle langue naturelle. Cet état initial étant nécessairement une partie innée du cerveau humain, on a affaire ici avec l'aspect le plus controversé de la théorie de Chomsky – bien que personne n'ait su réfuter l'argument que, sans cette hypothèse, la compétence linguistique de l'homme serait un paradoxe épistémologique des plus manifestes.

Il est d'autant plus intéressant pour nous que Chomsky suggère que d'autres compétences relatées pourraient être analogues, et même dérivées, de la compétence linguistique innée :

> Assuming the language faculty to be a distinguished component of the mind, it has a genetically-determined initial state S_0^L ... But the mind has other ressources as well, with other capacities, and thus a more general initial state S_0^M for formation of concepts – for example, in the course of theory construction in the advanced sciences.
> Supposant que la faculté linguistique soit un composant distinct de l'esprit humain, il a un état initial S_0^L qui est génétiquement determiné... Mais l'esprit humain a d'autres capacités, donc un état initial S_0^M plus général, pour former des notions – p.e. dans la construction des théories des sciences avancées. (CHOMSKY 1991)

Nous avons donc en particulier à traiter de la partie de S_0^M où le M signifie 'mathématiques modernes' – et, surtout, de ses relations avec S_0^L.

4 Syntaxe universelle des mathémathiques

Pour toute classe d'objets mathématiques –comme les nombres réels, les ensembles, les polyèdres– il y a des *opérations* naturelles par lesquelles un ou plusieurs objets sont transformés en un autre objet. De même, nous avons des *relations* qui nous permettent de comparer deux objets – par exemple, un objet et l'objet obtenu par une opération. Cette structure, dans un sens vague 'universelle' pour les mathématiques, a été partiellement formalisée dans la théorie des catégories. Ici, nous esquisserons une formalisation dans l'esprit de la linguistique structurale. Pour un exposé plus détaillé, voir (WINSLØW 1998).

Notons d'abord que pour le moment nous nous occuperons uniquement de la syntaxe des parties 'symboliques' de la communication mathématique, c'est-à-dire des séquences de symboles (incluant les représentations par diagrammes géométriques) qui apparaissent en pratique

au milieu des explications verbales écrites ou parlés selon la situation. Dans la section suivante, nous allons traiter la communication mathématique dans son ensemble.

Les séquences de symboles contiennent explicitement trois catégories de 'syntaxe' : *objets*, *relations* et *opérateurs*. Les *objets* sont seuls à pouvoir constituer une séquence complète, telle que '$\sum_n 1/n$' ou '2'. Les symboles représentants des *relations* sont toujours placés entre deux objets, comme dans '$2 \leq \sum_n 1/n$'. Finalement, les *opérateurs* se placent de deux manières différentes : *entre deux objets* comme dans '2+3=5', où le complément '=5' indique le résultat de l'opérateur '+' agissant sur la paire d'objets '2' et '3'; et de différentes manières en *combinaison* (avant, au dessus, au dessous...) avec l'objet (ou les objets) auquel agit l'opérateur, comme dans

$$\sqrt{9}, \beta^2, e^{-}, \sin(x), f(x,y,z), V^*, \text{Hom}(V,\mathbb{C}), \text{etc.}$$

Ces séquences d'objets avec opérateurs représentent, par définition, de nouveaux objets (composés) qui se placent dans toutes les positions possibles pour un objet. Les séquences *de base* sont de forme O (*séquence d'objet*, contenant un seul objet) et O_1RO_2 (*séquence simple*, objet-relation-objet).

Implicitement, on trouve une quatrième catégorie des *transformations*.

Les plus importantes sont les transformations induites par un opérateur (que nous pouvons, en analogie avec la syntaxe de transformation de HARRIS (1970), appeler les *transformation simples*). Celles-ci agissent *simultanément* sur objets et relations, de manière à transformer une séquence entière. Par exemple, l'inversion additive des nombres réels transforme la relation '$\leq$' en '$\geq$', donc, le résultat de cette transformation de la séquence '$2 \leq \sum_n 1/n$' est '$-2 \geq -\sum_n 1/n$'. Plus généralement, la transformation induite par l'opérateur T agit sur les séquences de base ainsi :

$$O \to TO; \quad O_1\, R\, O_2 \to TO_1\, TR\, TO_2,$$

où TR désigne la relation transformé de R par T.

Les transformations simples sont *unaires* comme elles agissent sur une seule séquence. D'autres transformations unaires sont celles de *réduction* (p.e. réduction de transitivité dans certains contextes : $O_1RO_2RO_3 \to O_1RO_3$, etc.).

Pour construire les séquences à plusieurs relations, il nous faut introduire les transformations *binaires de juxtaposition* et *de combinaison*. Si S_1 et S_2 sont des séquences quelconques, O est un objet, et R_1 et R_2 sont des relations, la juxtaposition de S_1R_1O et OR_2S_2 a pour résultat la séquence $S_1R_1OR_2S_2$. Comme un exemple, notons le passage

$$0 = 0 - 0 < 1 + 1, 1 + 1 \leq \sum_n 1/n \to 0 = 0 + 0 < 1 + 1 \leq \sum_n 1/n.$$

De même, la combinaison des séquences $S_1R_1S_2$ et $S_3R_2S_4$ induite par un opérateur T à deux arguments a pour résultat la séquence $T(S_1,S_3)T(R_1,R_2)T(S_2,S_4)$ où $T(R_1,R_2)$ est la relation induite par R_1 et R_2 sous T (cette induction n'étant évidemment pas possible pour n'importe quelle combinaison de relations et opérateur). Par exemple,

$$1 < 2, 3 \leq \sum_n 1/n \to 1 + 3 \leq 2 + \sum_n 1/n.$$

Comme c'est aussi le cas en syntaxe de transformation, on a besoin seulement de tranformations unaires et binaires pour générer les séquences d'une complexité arbitraire. Notons néanmoins qu'une répétition de la même transformation peut bien être infinie dans le cas traité ici.

De cette manière, nous avons indiqué un mécanisme pour générer un grand nombre de séquences de symboles correctes. Une indication du contexte sera nécessaire pour fournir des règles plus précises p.e. des transformations induites par les opérateurs, en vue d'assurer que seules les séquences correctes soient générées.

L'universalité des quatre entités de syntaxe que nous venons d'examiner –objets, relations, opérateurs, transformations– se prolonge bien sûr dans des aspects de sémantique. Ils sont brièvement indiqués dans le schéma suivant (WINSLØW 1999) :

	Représentation	**Structure**
État	Object	Relation
Processus	Opérateur	Transformation

5 Contours des registres mathématiques

L'usage mathématique combine deux éléments : une langue de symboles régularisé par le contexte mais aussi par des principes universels (décrits au-dessus), et une langue naturelle comme le Français ou le Danois. Il est d'une importance capitale pour donner une image fidèle des registres mathématiques de tenir compte de *l'interaction* des deux langues dans l'usage mathématique. En fait, ôter d'un texte mathématique toute expression symbolique le rend absurde et du point de vue grammatical et du point de vue sémantique, tandis que la seule langue symbolique ne saurait –en dépit des efforts faits par certains logiciens –rendre tout le sens même des plus simples raisonnements mathématiques.

Commençons par l'exemple suivant d'un texte classique (CARTAN 1961, p. 75) :

> Soit Γ le bord orienté d'un compact K contenu dans un ouvert D et soit $f(z)$ une fonction holomorphe dans D. Alors
> $$\int_{\Gamma} f(z)dz = 0\ldots$$

Il se voit aisément que les séquences de symboles 'γ', 'K', 'D', '$f(z)$' et '$\int_{\Gamma}(f(z)dz = 0$' remplacent des éléments verbaux dans la structure d'une phrase également représenté par : 'Soit *Pierre* l'ami d'un marin. *Paul*, embarqué au navire *Napoléon*, et soit *Pierrot* un clown engagé au *Napoléon*. Alors *l'ami de Pierre voyage avec un clown*.' Hors un certain clou logique, la dernière phrase n'a certes pas beaucoup de sens, mais elle est parfaitement grammaticale. Les séquences d'objets remplacent des nom propres, tandis que la séquence simple '$\int_{\Gamma} f(z)dz = 0$' tient lieu d'une phrase complète, la relation correspondant grossièrement au verbe fini. Abstraction faite de ces remplacements, la structure des deux phrases reste la même.

Il est, en effet, possible d'énumérer les règles qui déterminent, en grands traits, comment insérer dans la structure d'une phrase de langue naturelle des séquences de langue symbolique pour arriver à des phrases correctes pour le registre *mainstream* des mathématiques modernes (WINSLØW 1999, §6). La pointe est essentiellement que les noms propres (et leurs équivalents ainsi que les noms après certains articles) se remplacent avec les séquences d'objets, tandis que les propositions principales précédées de manière arbitraire (mais non-vaine) se remplacent par les séquences contenant une ou plusieurs relations.

Nous avons toujours à rendre compte de l'usage de l'inventaire verbal dans les registres mathématiques, comme celui-ci est clairement assez restreint; bien qu'aucune tournure ne soit

exclue d'avance, il est impossible de rencontrer dans un texte mathématique des expressions tel que 'humeur', 'ma mère est malade' etc., en tous cas dans leur sens quotidien. Le registre mathématique est 'fermée' dans le sens qu'il ne permet pas les références externes, donc, ne saurait exprimer les opinions, les sentiments, etc. (ne comptant pas, ici, la possibilité de registre mixte, tel que l'usage dans les applications des mathématiques). Dans ce sens, les mathématiques sont, selon l'expression de JACOBSON (1970, en vue de la musique), *un langage qui se signifie soit-même* (cf. aussi ROTMAN 1988).

Le texte mathématique crée sa propre sémantique par des processus de nomenclature très explicites. Retournant à notre example du texte de Cartan, tous les termes 'bord orienté', 'un compact', 'contenu', 'un ouvert', 'une fonction holomorphe' doivent être explicitement définis par le contexte (typiquement par les parties antérieures du texte) pour être employés dans le présent; ils font alors, et seulement alors, partie de l'inventaire sémantique du registre, avec un sens très différent du sens dans tout autre registre, s'ils ne sont pas exclusivement créés pour l'usage du registre (tel que l'adjectif 'holomorphe'). Cette possibilité de créer *ad hoc* une sémantique est à la racine de l'extrême flexibilité du registre mathématique. Elle est, d'ailleurs, différente d'un autre processus, beaucoup plus local, de nomenclature, par lequel un sens soit assigné aux symboles (dans l'exemple, 'Soit Γ le bord orienté...'), mais qui ne crée pas une partie du registre proprement dit.

Finalement, les registres mathématiques se retrouvent rarement en usage isolé hors des salles de classes ou de recherche. Par contre, on va souvent trouver les mathématiques dans des contextes appliqués; donc, le registre est *mixte*, comme le registre du contexte (par exemple de la physique, du commerce, etc.) se mêle avec le registre mathématique utilisé pour représenter (modéliser) une situation décrite par le premier registre. Dans ces situations, il est utile sinon nécessaire de séparer ce mélange –d'isoler ce qui est dit ou écrit en langage mathématique– pour l'analyse du discours, ou plus modestement, pour donner un sens à l'usage apparemment bizarre de symboles au milieu d'un discours verbal.

6 L'analyse du discours mathématique

Notre analyse jusqu'ici peut bien sembler assez éloignée des problèmes réels de l'usage communicatif des mathématiques – en classe aussi bien qu'ailleurs. C'est dans le discours cohérent –la *parole*– plutôt que dans les détails de syntaxe que les difficultés du langage mathématique se montrent les plus urgentes.

L'analyse grammaticale se termine traditionnellement par la phrase entière. Pour saisir le déroulement sémantique d'un texte comprenant plus d'une phrase, il est indispensable de traiter aussi la relation entre ces phrases et leur contribution à la totalité. Ceci est le point de départ d'une discipline nouvelle (STUBBS 1983, p. 15) :

> Connected discourse is clearly not random. People are able to distinguish between a random list of sentences and a coherent text, and it is the principles which underlie this recognition of coherence which are the topic of study for discourse analysis.
>
> Le discours cohérent n'est évidemment pas arbitraire. Nous sommes capables de distinguer une liste arbitraire de phrases d'un texte cohérent, et ce sont les principes qui se trouve derrière cette reconnaissance de cohérence qui sont le sujet pour l'analyse du discours.

Cette cohérence est déterminée grossièrement par l'information transmise, reçue et interprétée au cours de la communication.

Les phrases d'un texte mathématique sont centrées autour de certains ensembles sémantiques représentés surtout par les séquences symboliques dont la cohérence est typiquement établie par les parties des phrases usant de l'inventaire verbal. Nous distinguons, pour notre analyse, les ensembles suivants :

1. L'ensemble *primaire*, représenté par les phrases, et surtout les séquences symboliques, du texte,
2. L'ensemble *secondaire*, constitué par le 'contexte'; p.e. dans un texte pédagogique, le contexte d'un extrait quelconque est souvent défini dans l'ensemble précédent du texte,
3. L'ensemble *tertiaire*, constitué par le savoir uni des *agents* du discours, c'est-à-dire des interlocuteurs (voir ci-dessous).

Notons que ces ensembles sont d'habitude successivement plus larges, l'ensemble primaire étant le seul à être directement 'visible' à fleur du texte; seulement, dans certaines situations, l'ensemble secondaire (le contexte) n'est pas entièrement contenu dans le tertiaire, ce qui est susceptible de donner lieu à des difficultés au cours du discours (et à se manifester ainsi dans le texte). Il est aussi important de noter que les ensembles doivent être conçus comme des entités *dynamiques*, donc, qui changent au cours du discours.

En considérant un texte (dans le sens général) mathématique, nous observons un ensemble grandissant d'éléments (formules, concepts, allégations, théorèmes etc.) qui, à un point du texte donné, constituent l'ensemble primaire. Il est donc vide au début du texte, et sa capacité explanateur dépend de la manière dont nous l'aurons délimité. Si, p.e., notre texte est constitué par un théorème et sa démonstration –et non pas, disons, par la dernière moitié de la démonstration– l'ensemble primaire donnera souvent une image déjà assez significative du déroulement superficiel des messages communiqués.

Toutefois, n'importe quel discours mathématique a lieu dans un contexte assez restreint de définitions, théorèmes, méthodes etc.; dans la surface du texte, nous observons de temps en temps l'introduction d'un élément secondaire dans l'ensemble primaire. Il est capital pour la compréhension du texte d'avoir une image assez précise de cet ensemble, ce qui rend l'analyse du discours oral beaucoup plus difficile que dans le cas des textes écrits, où l'ensemble secondaire est souvent explicite dans le texte précédent l'extrait considéré.

L'ensemble tertiaire est, en contraste avec le secondaire, dépendant des interlocuteurs. Pour les textes écrits, il faut d'ailleurs comprendre comme 'interlocuteur muet' aussi le 'lecteur imaginé' par les auteurs, comme le discours est souvent visiblement dirigé vers celui-ci (et ferait typiquement peu de sens conçu comme un monologue sans destinataire). Cet ensemble est pour ainsi dire la limite du domaine qui saurait apparaître à la surface du texte; l'introduction d'éléments tertiaires non-partagés par tous les interlocuteurs est une source fréquente de difficultés communicatives, bien qu'elle soit essentielle pour le discours pédagogique où l'ensemble tertiaire pourait souvent être compris comme représentant 'le savoir du professeur' (ou de l'auteur). En général, l'ensemble tertiaire d'un groupe donné d'interlocuteurs est difficile à déterminer, et même la détermination partielle demande une observation longue et variée du discours du groupe. Cet ensemble est pour ainsi dire 'le spectre nécessaire' de l'analyse du discours mathématique : indispensable pour la compréhension d'un texte non-trivial, mais bien caché sous la surface textuelle.

Cette analyse procède donc de la manière suivante : identifier les interlocuteurs et le contexte (après une lecture rapide du texte), suivre le déroulement du texte au niveau de l'ensemble primaire, observer les introductions d'éléments secondaires et (possiblement) tertiaires non secondaires et les transformations des éléments primaires; décrire au moyen de ceci le flux

d'informations et les aspects globaux de l'événement discoursif. (Pour un exemple simple mais assez détaillé du processus, voir WINSLØW 1998, §3.3.6).

Le rôle pour cette analyse de notre description du registre et de la syntaxe universelle reste à être explicité (bien qu'il soit partiellement évident dans l'analyse de n'importe quel texte suivant les idées ci-dessus). Il est lié aux aspects les plus délicats du développement du sens dans le cours d'un discours mathématique : l'interprétation et la transformation d'informations données dans l'ensemble primaire. Un texte mathématique peut être plein de sens et de dynamique et pourtant être constitué essentiellement d'une succession de transformations unitaires d'une seule séquence simple – p.e., le texte représentant la solution d'une équation bicarrée par complément des carrés. Un texte verbal procédant de manière pareille est peu vraisemblable hors des genres comme le théâtre absurde... D'autre part, pour les textes plus compliqués, le sens est typiquement représenté par (ou peut être reconstruit comme) un amas de transformations comportant la combinaison d'un grand nombre de séquences symboliques de tous les ensembles. C'est de la structure de cet amas de transformations que dépend notre compréhension du texte, comme d'ailleurs celle des interlocuteurs, et c'est un trait caractéristique du registre que la transformation linguistique des éléments soit aussi significative pour son usage.

7 Conséquences pour l'épistémologie

Qu'est-ce que le savoir mathématique et comment l'acquière-t-on? Il me semble que la compréhension du savoir mathématique comme une compétence discursive contribuera à éclairer d'une lumière nouvelle ces questions du moment que nous avons décrit cette compétence au niveau de *langue* (comme la syntaxe universelle) et au niveau de *parole* (registre et dynamique du discours). Ce faisant nous avons à la fois donné substance à la thèse que le savoir mathématique soit une extension de notre faculté de langage, et signalé de quelle manière les problèmes des épistémologies des langues et des mathématiques sont interdépendants. Dans les deux cas, nous avons à faire avec une pratique plutôt qu'avec un corpus d'objets avec certaines qualités. Selon le philosophe Resnik,

> ... in mathematics the primary subject matter is not the individual mathematical objects but rather the structures in which they are arranged. The objects of mathematics... are themselves atoms, structureless points...
>
> ... en mathématique, la matière n'est pas les objets mathématiques individuels, mais plutôt les structures dans lesquelles ils sont situés. Les objets des mathématiques... sont en eux-mêmes des atomes, des points dépourvus de structure.... (RESNIK 1997, p. 201)

Notre discussion précédente nous éloigne d'un pas de plus des objets comme la matière principale du savoir mathématique : elle n'est même pas les relations entre objets (ou points) situés dans une structure inerte, mais plutôt la dynamique du changement de ces relations – c'est-à-dire, la structure transformable de la syntaxe et du discours. D'être savant des mathématiques ne dépend pas seulement de la connaissance des relations fixes entre objets (tel que l'arrangement des nombres, ou un corpus de théorèmes), mais surtout de la compétence de manipuler ces relations (tel que dans la pratique de l'arithmétique ou dans l'invention et dans la démonstration des théorèmes). L'analogie ici avec la faculté de langue me semble très persuasive; le savoir lexical et la connaissance reçus d'un amas fini de phrases toutes faites sont nécessaires mais bien loin d'être suffisants pour la participation au discours d'une langue naturelle. L'importance de la maîtrise d'une structure transformable des séquences symboliques, ainsi que de sa relation avec la syntaxe de la langue naturelle dont le registre se sert, nous montre que cette analogie n'est pas une coïncidence mais une conséquence de la parenté proche entre les deux formes de compétences expressives.

La compétence mathématique est donc essentiellement de nature linguistique et elle est acquise de manière analogue à la faculté linguistique : par la participation au discours (dans un sens large, comprenant aussi la lecture de textes) basée sur un registre mathématique. Il est donc trivial que l'acquisition est partiellement un processus de socialisation; mais c'est une faute, de nos jours trop commune, de conclure qu'elle est pour cette raison arbitraire ou bien entièrement dépendante d'un milieu de socialisation, c'est-à-dire du groupe d'interlocuteurs dans lequel elle a lieu. Il n'est pas entièrement vrai (en effet, essentiellement faux) que le savoir mathématique et le résultat d'un accord explicite, ou que

> ... objective mathematical knowledge is to be found socially in the interrelations and interaction of... texts and persons within the culture and institution of mathematics.
>
> ... le savoir mathématique objectif se trouve socialement dans les interrelations et les interactions des... textes et individus à l'intérieur de la culture et de l'institution des mathématiques. (ERNEST 1998, p. 244)

L'argument d'Ernest et d'autres pour refuser essentiellement l'objectivité (dans le sens habituel) de tout savoir mathématique est précisement le processus discursif dans lequel ce savoir se laisse observer – mais qui ne le crée pas pour cela dans sa totalité. L'analogie avec le cas de la faculté de langue naturelle nous montre qu'il n'y a pas de nécessité qu'une compétence d'ordre linguistique soit entièrement formée par la participation au discours; remplaçant le savoir mathématique par les connaissances linguistiques l'extrait ci-dessus contredit clairement l'existence de l'état initial qui ne réside certainement pas dans des institutions ou dans des 'cultures de l'institution de langue naturelle'. Il y a des éléments de notre faculté de langue qu'il ne nous est pas donnée de changer; hors notre appareil physique d'articulation, il y a dans notre constitution mentale des structures qui la déterminent partiellement. Comme nous venons de souligner la relation intime entre cette faculté et l'usage mathématique, nous voyons que la question du savoir mathématique objectif ne se réduit point à l'analyse des institutions ou des 'cultures' – de même que la linguistique ne se laisse pas concevoir comme un coin de la sociologie.

Un argument principal pour la non-existence d'une objectivité invariante du discours mathématique a été le développement historique des formes acceptées de démonstrations de théorèmes. Sans doute, on observe au cours de l'histoire, p.e., de l'analyse infinitésimale, des changements assez importants de la perception d'un raisonnement correct et aussi, bien que moins prononcé, l'abandon de résultats autrefois conçus comme bien étayés. Toutefois, la possibilité d'observer de tels changements et de les concevoir comme stades commensurables de théories, nous montre que nous n'avons aucunement à faire avec des 'changements de paradigmes' au sens de KUHN (1962), où toute la base du savoir ancien est soudainement renversée. Aussi, il n'appartient au fait pas à une communauté de chercheurs en mathématiques de changer de façon abrupte la structure des transformations permissibles, bien que nous ayons des tentatives partielles telle que l'école intuitionniste, qui a justement échoué par manque de continuité et de commensurabilité avec les registres du *mainstream*. D'ailleurs, les langues naturelles se développent de manière pareille; le français d'un Molière ou d'un Pascal est différent, mais loin d'être incommensurable, du français contemporain, tandis que l'introduction de langues 'artificielles' comme l'espéranto ne nous donnera justement jamais une langue 'naturelle'. On pourrait objecter ici que les mathématiques ressemblent peut être plus à l'espéranto qu'au français, étant plutôt une langue articifielle que naturelle; mais il me semble que cette objection est tout

simplement sans évidence dans la pratique contemporaine et historique des mathématiques. Jamais ne fut-il décidé d'ériger de telle ou telle manière ce bâtiment de savoir qui nous est connu sous ce nom; et surtout ce bâtiment est de manière irréversible contigu aux langues naturelles dont il se sert. En effet, les mathématiques remplissent beaucoup des conditions normalement posées dans les définitions des langues naturelles (p.e. MORAVSCIK 1983), à l'exception, bien sûr, d'avoir la forme orale comme le médium primaire.

D'un autre côté, l'utilité des mathématiques comme moyen pour décrire à peu près tous les phénomènes du monde physique, et aussi un grand nombre de phénomènes sociaux, est souvent citée comme évidence d'une objectivité inhérente aux mathématiques. L'argument me semble capital mais aussi plein de dangers. Il faut résister à la tentation de conclure que cette objectivité réside donc 'hors de la sphère humaine', et démontrer qu'au contraire elle vient de notre incapacité à décrire et même à concevoir des relations extérieures de nous-mêmes sans nous servir des moyens d'expression qui nous sont fournis par les langues. L'invariance et l'objectivité de notre conception du monde sont imposées par celles de nos langues et elles ne sont point plus étendues que l'objectivité de nos langues; notre compréhension du monde physique est en effet bien moins stable que les structures mathématiques par lesquelles nous l'exprimons. Ici, KUHN (1962) est à sa place pour nous convaincre que notre description, et aussi notre construction partielle, de l'univers physique et social, sont limitées et induites par la liberté actuelle de choisir parmi les moyens. C'est aussi un point cardinal pour WITTGENSTEIN (1969, §5.6) :

> Die Grenzen meiner Sprache bedeuten die Grenzen meiner Welt.
>
> Les limites de ma langue sont les limites de mon univers.

Cela se comprend aussi (par nous) au pluriel, dans le sens social; il est peu étonnant alors que les partie de notre 'langue' (au sens général) qui s'occupent des quantités et des formes –comme avant tout les mathématiques– nous semblent bien utiles pour décrire et former les aspects quantitatifs du monde. La régularité intrinsèque de la langue ne doit pas être confondue avec les incidents de son usage.

Le caractère transformable du langage mathématique a aussi des conséquences pour l'interprétation du concept de vérité dans l'usage des mathématiciens et des philosophes. Dire qu'une proposition mathématique est 'vraie' est une manière de dire qu'elle est 'bien formée' ou 'correctement dérivée', dans le sens que nos critères pour sa vérité portent typiquement sur les transformations d'un discours (ici, d'une démonstration). Si cette chaîne de transformations, ainsi que son point de départ, sont reconnus par nous comme corrects, nous affirmons la correction de la proposition avec autant de sûreté que nous le pouvons pour la forme d'une phrase dans notre langue maternelle. Dans les deux cas, l'idiosyncrasie joue un rôle relativement petit, bien que dans le discours mathématique, les divergences entre le savoir d'un individu, l'ensemble secondaire et l'ensemble tertiaire donnent fréquemment lieu à des efforts de raccommodage – non pas des 'points de vues', mais des 'connaissances'. Dans le cas de l'usage d'une langue naturelle, p.e. dans les registres de débat politique, on ne trouve pas cette corrélation intime entre syntaxe, structure de discours et sémantique; les divergences sur le contenu ne sont pas primairement d'ordre linguistique. Cette différence s'explique égalment au niveau de la signification : comme nous l'avons déjà noté dans notre discussion du registre mathématique, celui-ci est fermé et donc le discours est dépourvu de références extérieures – ce qui restreint aussi la portée de son concept de vérité.

8 Conséquences pour l'éducation et l'enseignement

Il me semble essentiel que l'enseignement général des mathématiques soit conçu dans sa totalité comme un mouvement vers un but de perfection qui pourrait être atteint à différents degrés, toujours partiels. Notre analyse des registres et du discours des mathématiques nous permet d'abord de formuler avec plus de précision ce but et donc la direction générale du 'vecteur' de l'éducation mathématique, puis d'analyser brièvement son implantation aux niveaux différents.

Pour la direction générale, l'analogie avec l'acquisition d'une langue étrangère est très utile. Quoique lointain au début de l'apprentissage, le but principal déterminant la direction de l'enseignement est bien sûr la compétence de comprendre et de s'exprimer comme un adulte ayant cette langue pour langue maternelle. A ceci se joint des moyens qui sont aussi des buts partiels, comme l'enseignement de la culture et des littératures associées à la langue en question. Retenons comme mots clés : *compétence adulte, langue maternelle, culture, littératures.* A priori, ils ne semblent peut être pas très éclairants dans le contexte de l'apprentissage mathématique. D'ailleurs, par quel 'mathématicien idéal' aurons nous un modèle pour la compétence 'adulte', pour ne pas dire 'de langue maternelle'? Nous aurons à opérer avec un *ideal speaker* (CHOMSKY 1965) dans les deux cas, mais le besoin de préciser ses attributs est peut être plus grand dans le cas présent. Dans l'usage des mathématiques, la compétence pour participer au discours me semble étroitement liée à la faculté de suivre et de produire les transformations qui constituent, comme nous l'avons vu, l'élément central de la dynamique interne de ce discours. Notre *ideal speaker* est donc complètement libre aux jeux de transformations dans n'importe quel contexte sensé des mathématiques. Il connaît aussi la littérature, donc, possède le savoir jusqu'ici obtenu, en étant capable de le situer dans son contexte historique et culturel. De plus, il lui est possible de communiquer et d'appliquer son savoir hors du contexte protégé des mathématiques pures.

Comment peut on s'y rendre, puisqu'il est clair que l'on n'atteint jamais ce but idéal? Pour les calculs comme pour les raisonnements d'un certain genre, il est certes possible de développer des facultés basées sur les recettes, sur les méthodes toutes faites, mais où l'élément transformable des opérations est pour ainsi dire donné d'avance. Évidemment, ces facultés ont peu de valeur de transfert au-delà des situations strictement analogues. Par contre, l'étude de telles 'recettes' peut fournir des *exemples* importants de la navigation de transformation du mathématicien dans une structure mathématique. En effet, la 'littérature' des mathématiques est pleine de tels exemples 'modèles'; par exemple, la preuve de l'irrationalité de la racine de 2 est un cas classique et exemplaire de l'argument indirect, et elle est aussi par sa signification historique une partie de la 'culture' associée aux mathématiques. Si l'on réussit à élargir cet argument aux racines d'un nombre non-carré quelconque, on aurait déjà une expérience valable pour des exploits plus avancés. D'autres exemples plus élémentaires sont les algorithmes de multiplication et de division (pour l'enseignement primaire) dont l'importance pratique a certes diminué avec l'introduction des calculateurs, mais qui sont néanmoins des véhicules possibles pour l'acquisition des transformations associées à ces opérations (qui sont, d'ailleurs, à leur tour, nécessaires pour la compréhension de notre premier exemple). L'importance pour l'apprenti de formuler pour lui-même les hypothèses, les arguments et les contre-exemples est analogue aux principes modernes de la pédagogie linguistique :

> Rules that the child discovers are more important and carry greater weight than practice. Concept attainment and hypothesis testing are more likely paradigms in language teaching than response strength through rote memory and repetition.

> Les règles découvertes par l'enfant même sont plus importantes et ont plus de poids que la pratique. La réalisation des concepts et l'épreuve d'hypothèses sont des paradigmes plus prometteurs pour l'enseignement des langues que la faculté de répondre par cœur et par répétition. (JACOBOVITS 1970, p. 15)

Nous n'avons ici qu'à remplacer le mot 'langues' pour avoir un manifeste bien sensé d'une éducation des mathématiques modernes.

Retournons à la question des significations possibles de 'langue maternelle' dans le contexte des compétences mathématiques. Puisque celles-ci sont, comme nous l'avons vu, intégrées aux compétences linguistiques générales, il n'y a pas de raison théorique qu'elles soient acquises comme celles d'une langue secondaire. Effectivement, des rudiments des registres des mathématiques élémentaires, tels que les notions de quantité et de forme, sont présents même dans les stades les plus primitifs de l'acquisition des langues naturelles (voir p.e. USISKIN 1997, pp. 234f). On peut compter parmi les conclusions les plus manifestes des études de l'éducation mathématique élémentaire qu'il est favorable à sa réusite de profiter de ces éléments déjà intériorisés et qu'il est substantiellement plus difficile, quoique possible, d'acquérir l'arithmétique comme des processus détachés de l'usage connu de la langue maternelle.

Considérons ensuite le problème de l'âge idéal pour commencer l'initiation aux aspects centraux du registre mathématique (le raisonnement logique par discours de transformation). On trouve un problème parallèle dans le 'facteur d'âge' dans l'apprentissage d'une langue secondaire, où l'analyse se résume ainsi (KRESHEN et al. 1979) :

> ... adults and older children in general initially acquire the second language faster than young children (older-is-better-for-rate-of acquisition), but child second language acquirers will usually be superior in terms of ultimate attainment (younger-is-better-in-the-long-run).
>
> ... les adultes et les enfants plus âgés en général sont plus prompts à acquérir une langue secondaire que les jeunes enfants (plus-âgé-est-mieux-pour-la-rapidité-d'acquisition), tandis que les enfants apprenant une langue secondaire sont d'habitude supérieurs au sens du résultat final (plus-jeune-est-mieux-à-la-longue).

On serait tenté de conclure par analogie que l'introduction des éléments centraux du registre devait être effectuée aussi rapidement que possible. Toutefois nous nous heurtons ici aux limites posées par la capacité cognitive de l'enfant, dont nous informent Piaget et son école avec une documentation écrasante :

> On voit donc ce qu'est la déduction formelle : elle consiste à tirer les conséquences, non pas d'un fait d'observation directe, ou d'un jugement auquel on adhère sans réserve... mais d'un jugement que l'on assume simplement... C'est cette déduction dont nous situons l'âge vers 11-12 ans. (PIAGET 1924, p. 82)

Puisque la déduction formelle -les transformations effectuées sur un ensemble d'éléments donnés- est tellement fondamentale dans le discours 'adulte' des mathématiques, nous nous trouvons donc face à une des tensions les plus importantes dans les débats sur l'éducation mathématique.

Face à ces problèmes, la tentation est grande de diviser et de compartimenter l'enseignement des mathématiques selon des buts utilitaires, et de renoncer ainsi à communiquer le registre dans sa totalité. L'absurdité de la conception instrumentale des mathématiques pour l'enseignement général est bien illustré avec l'analogie d'une présentation semblable d'une langue (ou d'un registre de langue) naturelle, où l'on peut bien pour des propos très spécifiques enseigner un inventaire des phrases toutes faites qui sont simplement expliquées une à une. L'enseignement parallèle des mathématiques est pourtant réalisé dans maintes écoles, comme un inventaire des procédures à accomplir en présence de certaines tâches. On évite par là les difficultés présentées par la compréhension d'un discours abstrait et de transformation, mais on perd aussi toute la force expressive du registre.

Pour l'enseignant, il me semble essentiel d'avoir conscience de ces tensions partiellement inévitables entre les besoins du but final de compétence discursive et les conditions d'apprentissage pour les jeunes enfants. Bien que Bruner (1960) a peut être été un peu trop optimiste à déclarer que l'on peut enseigner n'importe quoi aux enfants de n'importe quel âge, il a raison d'insister sur le fait que l'enseignant doit posséder et communiquer les idées fondamentales de son sujet, et peut être même les 'personnaliser'. Cela implique que l'enseignant doit, au plus haut degré possible, être un *ideal speaker* dans le sens défini plus tôt, au moins pour les contextes qu'il enseigne.

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